Traitement subjectif de l'incertitude dans les décisions individuelles

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Traitement Subjectif de l'Incertitude dans les Décisions Individuelles

Disentangling Beliefs and Attitudes towards Uncertainty in Individual Decision Making

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Notice

This dissertation begins with a summary in French. Then, Chapter 1 introduces the main concepts and models that are used in the other chapters. Chapter 2 (joint work with Bram DRIESEN [a] and Peter P. WAKKER [a]) is dedicated to modeling risk and ambiguity attitudes through marginal utility. In Chapter 3 (joint chapter with Laure CABANTOUS [b]), we propose and implement tools to analyze how decision makers combine experts' judgments. Chapter 4 (joint work with Mohammed ABDELLAOUI [c] and Peter P. WAKKER [d]) studies how combining Bayesian beliefs and willingness-to-bet allows for analyzing attitude towards uncertainty. Chapter 5 investigates the robustness of several choice-based techniques for eliciting subjective probabilities. Chapter 6 discusses and concludes.

La thèse comporte tout d'abord un résumé détaillé en langue française. Le reste de la thèse est rédigé en langue anglaise. Le chapitre 1 introduit les principaux concepts et modèles utilisés dans la thèse. Le chapitre 2, travail en commun avec Bram DRIESEN [a] et Peter P. WAKKER [a], étudie la modélisation de l'attitude face au risque et à l'ambiguïté via l'utilité marginale. Le chapitre 3 est issu d'une collaboration avec Laure CABANTOUS [b]. Il propose et met en œuvre des outils pour analyser comment les décideurs prennent en compte les jugements de probabilité donnés par des experts. Le chapitre 4, fruit d'une collaboration avec Mo-

hammed ABDELLAOUI[c] et Peter P. WAKKER[d], combine l'analyse des probabilités subjectives avec celle des consentements à parier pour comprendre et mesurer l'attitude des décideurs en situation d'incertitude. Le chapitre 5 étudie plus en profondeur les techniques d'élicitation des probabilités subjectives. Le chapitre 6 conclut.

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Contents

Résumé de la Thèse

0.1. Introduction

0.1.1. Qu'est-ce que l'incertitude ?

Knight (1921) a introduit la distinction entre incertitude mesurable, autrement appelée risque, et incertitude non-mesurable, souvent appelée simplement incertitude : la première incertitude désigne donc le cas où il existe une mesure de probabilité sur les évènements possibles, tandis qu'il n'en existe pas dans le second cas. Cette distinction constitue la première tentative de décrire et de définir l'incertitude mais a donné lieu à différents travaux complémentaires ou critiques. Nous allons ainsi présenter une brève topographie de l'incertitude.

Incertitude et probabilité objective : C'est au XVIème siècle que Jerome Cardan a donné la première intuition de la définition des probabilités comme un ratio du nombre de cas favorables sur le nombre de cas possibles. La vision fréquentiste associe les probabilités aux limites des fréquences quand le nombre d'observations tend vers l'infini. Quand il existe de telles probabilités correspondant à des propriétés objectives du monde extérieur, alors nous nous retrouvons dans ce que Knight appelle le risque.

Incertitude et probabilité subjective : A l'opposé de la vision fréquentiste, l'approche subjectiviste considère qu'il n'existe de probabilité que relativement à un individu faisant face à un événement donné. Cette vision permet de définir des probabilités même s'il n'est pas possible d'effectuer un calcul fréquentiste. Par exemple, les décisions d'un individu peuvent révéler ses croyances. Savage (1954) a établi les conditions suffisantes pour que les choix révèlent l'existence de telles probabilités. Puisque l'incertitude est ainsi mesurable, ne serait-ce que subjectivement, elle peut aussi être qualifiée de risque. Mais reste-t-il alors une incertitude qui ne se réduise pas au risque ?

Incertitude sans probabilité subjective : Ellsberg (1961) propose des exemples dans lesquels les axiomes de Savage sont violés, et par conséquent, ne permettent pas de définir de probabilités subjectives. Il désigne par *ambiguïté* de telles situations. Toutefois, comme nous le verrons dans cette thèse, il n'est pas certain qu'il n'existe pas de probabilités subjectives. Des généralisations du résultat de Savage tentent d'étendre la définition des probabilités afin de couvrir des situations que les axiomes de Savage ne prenaient pas en compte (voir sous-section 0.1.4).

Incertitude radical : Suivant Keynes (1936), l'approche post-keynésienne explique que la vraie incertitude signifie que l'on ne sait tout simplement rien, et que par conséquent les approches traditionnelles, fondées sur la description de tous les événements possibles et toutes leurs conséquences, sont limitées. Dans ces situations, les post-keynésiens expliquent que le décideur peut préférer ne pas décider ou suivre ses *« esprits animaux »* (Davidson, 1991).

Par la suite nous nous contenterons d'appeler *risque* toute situation où *les probabilités sont connues*. Dans ces cas nous nous réfèrerons explicitement à ces probabilités objectives. Quand les *probabilités ne sont pas connues*, nous nous réfèrerons à des événements ou des ensembles de probabilités possibles. L'*incertitude* comprend aussi bien le risque que ces autres cas. Nous utiliserons parfois le terme ambiguïté pour désigner de manière un peu abusive des situations où aucune probabilité objective n'existe, mais où des probabilités subjectives peuvent être définies. Ceci se justifiera par le fait que même s'il existe des probabilités (subjectives), celles-ci apparaissent moins certaines au décideur. Nous allons présenter à présent le principal modèle de décision en situation d'incertitude.

0.1.2. Utilité Espérée Subjective (SEU)

Par la suite, nous utiliserons les notations suivantes. Tout d'abord les *conséquences* pour le décideur seront toujours exprimées comme des éléments de $\mathbb{R}$. Quand les probabilités sont connues, alors le décideur fera face à des *loteries* (souvent notées ℓ), c'est-à-dire à des distributions de probabilités sur les conséquences. Nous désignerons par L *l'ensemble des loteries*. Dans ce chapitre, nous utiliserons souvent des *loteries simples*, qui peuvent s'écrire $(p_1{:}x_1,\dots,p_n{:}x_n)$ où les p_i sont des probabilités qui se somment à 1 et où le décideur reçoit x_i avec une probabilité p_i pour tout $i=1,\dots,n$. Les loteries binaires seront désignées typiquement par xpy (obtenir x avec une probabilité p et y sinon).

Lorsque les probabilités ne sont pas connues, alors il existera des *états de la nature* dont un seul se révèlera vrai. S est l'*univers* ou *ensemble des états de la nature*. Nous parlerons d'*événements* pour désigner des sous-ensembles de S. Par simplicité dans cette première section, l'univers sera fini et s'écrira donc $S=\{1,\dots,m\}$ avec m fini. Un *acte* sera alors une fonction qui associe à chaque élément de S une conséquence et sera représenté par f, g… Comme S est fini, de tels actes peuvent s'écrire comme des éléments de $\mathbb{R}^m$, $(x_1,\dots,x_m)$ où x_i est la conséquence associée à l'état i. Les actes binaires seront parfois écrits xEy, où le décideur obtient la conséquence x si l'événement $E\subseteq S$ se réalise et y sinon. Un acte ou une loterie qui donne uniquement la conséquence x sera désigné par cette conséquence. $\succcurlyeq$ désigne la *relation de préférence* sur l'ensemble des actes ou des loteries.[1]

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[1] Il est possible de définir rigoureusement, en présence d'une mesure de probabilité (objective) sur S, l'ensemble des loteries comme un sous-ensemble des actes afin que la relation de préférence soit définie sur un seul ensemble (homogène). Nous ne détaillerons pas ici.

Lorsque les probabilités sont connues, la première intuition a été de penser que les décisions se faisaient en fonction de l'espérance mathématique. Nicolas Bernoulli a alors suggéré le paradoxe suivant. Pourquoi ne sommes-nous prêts qu'à payer un montant limité pour jouer à un jeu qui nous permettrait de gagner 2^n euros si une pièce tombe sur pile au $n^{ième}$ lancer alors que l'espérance mathématique de ce jeu est infinie. Daniel Bernoulli (1738) a alors suggéré que ce n'était pas l'espérance de gain mais l'espérance d'utilité qui entre en considération. Il a ainsi introduit le modèle d'*utilité espérée* qui a ensuite été axiomatisé par von Neumann & Morgenstern (1944). Dans ce modèle, chaque loterie est représentée par $(p_1 : x_1, \ldots, p_n : x_n) \longmapsto \sum_{i=1}^{n} p_i u(x_i)$, où u est une *fonction d'utilité,* continue et strictement croissante, définie sur $\mathbb{R}$ et unique à une transformation affine près.

Savage (1954) a produit la première axiomatisation de ce modèle lorsque les probabilités ne sont pas connues. Il a donné des conditions suffisantes pour que la relation de préférence sur les actes permette de définir une *mesure de probabilité* (une fonction P définie de $\mathcal{P}(S)$, *l'ensemble des parties de S*, vers [0,1] pour laquelle P(S)=1 et pour tout couple d'événements E et F tels que $E \cap F = \emptyset$, $P(E \cup F) = P(E) + P(F)$), telle que les préférences du décideur soient représentées par une fonctionnelle d'utilité espérée. En effet dans le cadre des hypothèses de Savage, les préférences sont représentées par l'*utilité espérée subjective*, i.e. $(x_1, \ldots, x_m) \longmapsto \sum_{i=1}^{n} P(i) u(x_i)$, où u est une fonction d'utilité unique à une transformation affine près et où P est une mesure de probabilité unique. Il est important de noter que Savage a ainsi fourni une première fondation au concept de probabilité subjective.

0.1.3. Les paradoxes

Allais (1953) a suggéré une célèbre violation de la théorie d'utilité espérée. Le tableau suivant décrit les loteries.

Tableau 0.1.1 : Le paradoxe de Allais

	p=89%	P=10%	p=1%
ℓ_1	€1,000,000	€1,000,000	€1,000,000
ℓ_2	€1,000,000	€5,000,000	€0
ℓ_3	€0	€1,000,000	€1,000,000
ℓ_4	€0	€5,000,000	€0

ℓ_1, …, ℓ_4 désignent quatre loteries. Pour la plupart des gens, $\ell_1 \succ \ell_2$ mais $\ell_3 \prec \ell_4$. Dans le cadre de l'utilité espérée, avec u(0)=0 et u(1)=1 (les conséquences sont en millions d'euros):

$$0.89\times u(1)+0.10\times u(1)+0.01\times u(1) > 0.89\times u(1)+0.10\times u(5)+0.01\times u(0),$$

et

$$0.89\times u(0)+0.10\times u(1)+0.01\times u(1) < 0.89\times u(0)+0.10\times u(5)+0.01\times u(0).$$

Ceci implique la contradiction suivante : $0.11>0.10\times u(5)$ et $0.11<0.10\times u(5)$. Il est clair dans le tableau que la première paire de loterie ne diffère de la seconde que pour ce qui arrive avec une probabilité de 89% et que dans ce cas, la conséquence est commune aux deux loteries dans chaque paire. Dans un modèle séparable comme SEU, où ce qui est commun n'importe pas, une telle inversion des préférences n'est pas possible. Pourtant, l'intuition derrière ce paradoxe réside dans le fait que le changement de conséquence entre ℓ_1 et ℓ_3 a un fort impact car il transforme une *loterie dégénérée* (un montant certain) en une loterie non dégénérée (donc risquée). Par contre, la même modification apportée à ℓ_2 (devenant ℓ_4), une modification de loterie risquée à loterie risquée, ne change donc pas sa nature et a moins d'impact. Le paradoxe de Allais est aussi vérifié quand les probabilités ne sont pas connues (MacCrimmon & Larsson 1979 ; Tversky & Kahneman 1992).

Ellsberg (1961) a proposé un autre paradoxe que ne peut expliquer l'utilité espérée subjective. Il sera désigné par la suite sous le terme *paradoxe d'Ellsberg à deux couleurs.* Soient deux urnes : la première contient 50 boules rouges et 50 boules noires. La seconde contient 100 boules, rouges ou noires mais dans une proportion inconnue. S'ils peuvent gagner en tirant une boule rouge, la plupart des gens vont préférer tirer dans la première urne. Idem s'ils peuvent gagner en tirant une boule noire. Dans le modèle d'utilité espérée, de tels choix révèlent que la probabilité de tirer une boule rouge (respectivement noire) est plus grande dans l'urne connue que dans l'urne inconnue. Sachant que les probabilités dans l'urne connue sont de 1/2 pour les deux couleurs, alors la somme des probabilités subjectives dans l'urne inconnue est inférieure à 1.

Une seconde version de ce paradoxe (dite *paradoxe d'Ellsberg à trois couleurs*) utilise une seule urne, contenant 30 boules rouges (R) et 60 boules jaunes (J) ou noires (N). La plupart des gens vont préférer pour gagner 100€ parier qu'ils tireront une boule rouge plutôt qu'une boule noire car ils sont sûrs que la probabilité d'avoir une boule rouge est 1/3 alors que la probabilité d'avoir une boule noire est inconnue. Inversement, ils préfèreront parier qu'ils tireront une boule noire ou jaune (probabilité connue de 2/3) plutôt qu'une boule rouge ou jaune (probabilité inconnue). Par conséquent dans le cadre de l'utilité espérée subjective, $P(R)>P(N)$ mais $P(R)+P(J)<P(N)+P(J)$.

Dans la prochaine sous-section nous allons présenter différents modèles visant à intégrer ces paradoxes.

0.1.4. Généralisations de SEU

Sophistication probabiliste : Machina & Schmeidler (1992) ont proposé une définition plus robuste des probabilités subjectives en modifiant les axiomes de Savage, afin de prendre en compte les comportements tels que le paradoxe de Allais. Ils en déduisent un modèle appelé sophistication probabiliste, où les choix sont toujours déterminés par une distribution de probabilité subjective sur les conséquences, mais où l'utilité espérée peut être violée. Chew & Sagi (2006a) ont fourni une axiomatisation plus générale des probabilités subjectives que nous utiliserons dans les sections 0.4 et 0.5.

Utilité espérée de Choquet (CEU) : Il s'agit dans ce modèle proposé par Schmeidler (1989) de garder la structure de l'utilité espérée mais en acceptant que les probabilités ne soient pas additives. Par simplicité, et parce que nous n'utiliserons que ce cas dans ce chapitre, nous allons en présenter la formulation pour les actes binaires :

$$xEy \longmapsto W(E)u(x)+(1-W(E))u(y),$$

où $x \geq y$, u représente toujours la fonction d'utilité (toujours définie à une fonction affine près) mais W(E) est maintenant une fonction définie telle que $W(\emptyset)=0$, $W(S)=1$ et $W(A) \leq W(B)$ pour tout $A \subseteq B$. W n'a pas à être additive. L'équivalent de CEU quand les probabilités sont connues est l'*utilité à dépendance de rang (RDU)*, $xpy \longmapsto w(p)u(x)+(1-w(p))u(y)$ où $x \geq y$ et p est une fonction de transformation de probabilités strictement croissante sur [0,1], avec $w(0)=0$ et $w(1)=1$. Tversky & Kahneman (1992) ont introduit une version de ces modèles qui dépend du signe, c'est-à-dire w et W sont différentes selon qu'il s'agisse de gains ou de pertes par rapport à un point de référence. Ce modèle est appelé Théorie Cumulative des Prospects (CPT). Tous ces modèles permettent de prendre en compte les paradoxes de Allais et Ellsberg.

Les modèles de type multi-prior : Le modèle Maximin de Gilboa & Schmeidler (1989) permet d'intégrer le paradoxe d'Ellsberg en représentant les préférences par la considération de la pire possibilité parmi un ensemble de distribution de probabilités possibles (les priors). Des extensions de ce modèle ont été proposées par Ghirardato et al. (2004), Gajdos et al. (2007) et Maccheroni et al (2006).

Les modèles à deux étages : Klibanoff, Marinacci & Mukerji (2005) ont suggéré un modèle à deux étages, où le décideur a une distribution de probabilités subjectives (dites *croyances de second ordre*) sur les distributions possibles. Ils considèrent en outre que le décideur ne réduit pas ces deux distributions en une distribution moyenne mais maximise une fonctionnelle d'utilité espérée, où les probabilités sont les croyances de second ordre et où les conséquences sont les utilités espérées obtenues quand chacune des distributions possibles se réalise.

0.1.5. L'approche de la thèse

Dans les sections suivantes nous allons présenter les apports de la thèse par rapport à la littérature que nous venons d'étudier. Il s'agira de comprendre dans quelles conditions les probabilités subjectives peuvent exister, mais aussi comment les observer et comment séparer ce qui relève de l'attitude de ce qu'on peut considérer être des croyances. Chacune des sections suivantes correspond à un chapitre de la thèse, et chaque chapitre constitue un travail indépendant.

0.2. Modéliser l'aversion au risque et à l'ambiguïté par l'utilité marginale décroissante

Nous avons vu dans la section précédente que Bernoulli (1738) expliquait le paradoxe de St Petersburg par l'introduction d'une fonction d'utilité. Son intuition était que la valeur psychologique associée aux montants monétaires n'est pas linéaire mais concave, c'est-à-dire que chaque nouvelle hausse de la richesse apporte un supplément d'utilité de plus en plus petit. L'*aversion au risque* désigne le fait qu'un individu préfère avoir de manière certaine l'espérance mathématique d'une loterie que la loterie elle-même. Dans le cadre de l'utilité espérée, l'aversion au risque est équivalente à la décroissance de l'utilité marginale. Des différences d'utilité marginale sont aussi utilisées dans d'autres modèles pour capter l'attitude face à l'ambiguïté ou face à la résolution de l'incertitude. Ainsi Kreps & Porteus (1978) modélisent la préférence pour une résolution de l'incertitude le plus tôt possible en laissant l'utilité marginale varier selon la date de résolution. Klibanoff, Marinacci & Mukerji (2005) représentent l'aversion à l'ambiguïté par une utilité plus concave dans l'incertitude qui détermine la 'vraie' distribution que dans le risque.

Le chapitre 2 de la thèse utilise une relation dite d'arbitrage, issue des préférences, qui permet de caractériser l'utilité marginale et propose ensuite de nouvelles axiomatisations plus générales des modèles précédemment évoqués. En outre, cette relation est utilisée pour établir des conditions permettant d'observer les attitudes (face au risque et à l'ambiguïté) via l'utilité marginale et même de les comparer pour des décideurs différents. Nous allons présenter maintenant ces résultats mais pour cela, il nous faut légèrement modifier notre cadre de travail. Les conséquences restent des nombres réels mais maintenant l'ensemble des états de la nature S peut être infini. Un acte, noté f (ou g), associe donc pour tout état $s \in S$ la conséquence f(s). Un acte sera dit *simple* s'il comporte un nombre fini de conséquences. L'acte f pour lequel on aura remplacé les conséquences par α pour tout $s \in E$ avec $E \subseteq S$ s'écrira αEf.

0.2.1. Cohérence en arbitrage et Utilité Espérée

Définissons tout d'abord la *relation d'arbitrage (tradeoff)*. Pour quatre conséquences $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, nous écrirons :

$$\alpha\beta \sim^{*} \gamma\delta$$

dès lors qu'il existe deux actes simples f et g et un événement E tels que

$$\alpha Ef \sim \beta Eg \qquad \text{et} \qquad \gamma Ef \sim \delta Eg$$

Pour interpréter cette relation, supposons que $\alpha > \beta$: cela signifie que le décideur compense le fait que f soit moins intéressant que g si E ne se réalise pas, en demandant plus si E se réalise. De la même manière, il fait le même arbitrage en demandant γ au lieu de δ dans la seconde indifférence. En d'autres termes, ce que lui amène α par rapport à β est équivalent à ce que lui apporte γ par rapport à δ. Ce sont deux arbitrages équivalents.

L'intuition derrière cela est que ce qui se passe lorsque E ne se réalise pas, sert d'étalon pour mesurer des arbitrages. Si on veut un modèle avec séparabilité, c'est-à-dire où l'évaluation psychologique des conséquences est indépendante des événements, alors il faut souhaiter que l'équivalence entre deux arbitrages ne soient pas remise en cause par d'autres observations. En effet, dans le cadre de l'utilité espérée subjective, cette relation est équivalente avec $u(\alpha)-u(\beta) =$

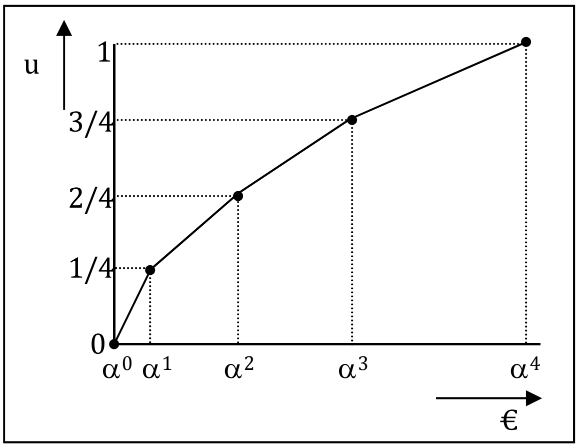
$u(\gamma) - u(\delta)$. Il est donc nécessaire qu'une fois une telle relation obtenue, elle ne puisse être contredite pour d'autres actes ou d'autres événements.

C'est ce que fait la condition de *cohérence en arbitrage (tradeoff consistency)* définie par le fait que toute augmentation d'une seule conséquence dans une relation $\alpha\beta \sim^* \gamma\delta$ brise cette relation. Köbberling & Wakker (2003, 2004) montrent que sous certaines hypothèses et pour un ensemble fini d'états de la nature, cette condition est nécessaire et suffisante à l'existence de l'utilité espérée. Le chapitre 2 de la thèse fournit une généralisation de ce résultat pour S infini.

La relation d'arbitrage peut permettre de mesurer l'utilité (voir Wakker & Deneffe 1996). En effet, en mesurant une suite α^j telle que pour un événement E non-nul et pour deux actes f et g :

$\alpha^j Ef \sim \alpha^{j-1} Eg$ pour tout j = 1, …, n,

on obtient la *suite standard* suivante $\alpha^0, \ldots, \alpha^n$ avec $\alpha^{j+1}\alpha^j \sim^* \alpha^1\alpha^0$ pout tout j. Chaque élément de la suite est à la même distance du précédent que du suivant dans l'espace des utilités, ce qui permet en normalisant l'utilité (et si $\alpha^1 > \alpha^0$), que $u(\alpha^i) = i/n$ pour tout i. Le graphique suivant représente l'utilité déduite d'une suite avec 4 éléments.



Graph 0.2.1 : L'utilité mesurée par une suite standard

Nous pouvons d'ores et déjà remarquer que la relation d'arbitrage peut nous informer sur la forme de u. Si les α^i s'éloignent les uns des autres quand i grandit, alors la fonction est concave. De cela découle l'idée suivante : supposons que pour trois conséquences $\alpha<\beta<\gamma\in\mathbb{R}$, nous ayons $\alpha\beta\sim^*\beta\gamma$. Si β est plus proche de α que de γ, alors l'utilité doit être concave (si c'est le cas pour tous les trios de conséquences ainsi reliés par $\sim^*$). Puisque la décroissance de l'utilité marginale correspond à l'aversion au risque dans le cadre de l'utilité espérée, alors la relation d'arbitrage permet d'observer l'attitude face au risque ; et si pour un individu, β est plus proche de α que pour un autre alors son utilité est plus concave et il prendra moins de risque.

Expliquons d'où vient la nécessité de déterminer des conditions pour observer l'attitude face au risque. Lorsque les probabilités ne sont pas connues, il n'est pas possible de définir et d'observer l'aversion au risque par la préférence pour l'espérance mathématique sur un acte, puisque celle-ci dépend des probabilités subjectives et donc dépend de ce que pense le décideur. Il existe toutefois une définition observable de l'attitude face au risque par la *quasi-concavité des préférences* (voir Chateauneuf & Tallon 2002 ; Debreu & Koopmans 1982), i.e. pour tout acte f, g, $f\sim g$ implique $\lambda f+(1-\lambda)g\succcurlyeq f$ où $\lambda f+(1-\lambda)g$ est l'acte qui donne pour chaque s $\lambda f(s)+(1-\lambda)g(s)$. Toutefois, il ne semble pas exister de telle condition pour comparer l'attitude de deux décideurs, à moins de faire l'hypothèse qu'ils aient les mêmes croyances.

0.2.2. Comparaison d'attitude face au risque

Pour comparer l'attitude de deux décideurs face au risque quand les probabilités sont connues, il est possible de comparer leurs équivalents certains pour une loterie donnée. Celui qui a le plus faible est le plus averse au risque. Quand les probabilités ne sont pas connues, Yaari (1969) a montré que définir l'aversion comparative (« A est plus averse au risque que B ») via l'équivalent certain impliquait des deux décideurs d'avoir la même mesure de probabilité.

Le théorème 2.4.1 du chapitre 2 utilise les relations d'arbitrage des deux agents pour observer lequel des deux est le plus averse, puisque l'aversion au risque est assimilable à la décroissance de l'utilité marginale dans le cadre de l'utilité espérée. Ce théorème donne donc l'équivalence des trois propositions suivantes dès lors que les deux décideurs, A et B, sont des maximisateurs d'utilité espérée dans le risque (avec respectivement $\succcurlyeq^A$, $\sim^{*A}$, u^A et P^A pour A et $\succcurlyeq^B$, $\sim^{*B}$, u^B et P^B pour B) :

(i) u^A est plus concave que u^B et $P^A=P^B$;

(ii) Pour tout acte, B a un équivalent certain au moins aussi grand que A ;

(iii) $\alpha\beta\sim^{*B}\beta\gamma$ et $\alpha\beta'\sim^{*A}\beta'\gamma$ implique que $\beta'\leq\beta$; en outre, $P^A = P^B$.

Ce résultat rend aisément observable l'attitude de deux décideurs. Rappelons que (iii) désigne une utilité marginale qui décroit plus vite pour A que pour B. Nous allons à présent nous intéresser à un seul décideur, faisant face à des situations où les probabilités sont parfois connues, et parfois non. Nous pourrons ainsi étudier son attitude face à la non-connaissance des probabilités, son attitude face à l'ambiguïté.

0.2.3. Représentation de l'attitude face à l'ambiguïté

Nous avons introduit dans la sous-section 4.0.1 de ce résumé le modèle de Klibanoff, Marinacci et Mukerji (2005) (désignés par KMM ci-après). Nous allons ici considérer ce type de modèle à deux étages. Dans cette sous-section et la suivante, les actes seront *à la Anscombe-Aumann*, c'est-à-dire des fonctions de S vers L, l'ensemble des loteries sur les conséquences (voir Anscombe et Aumann 1963). Ainsi pour l'acte f, f(s) sera une loterie $\ell\in L$ spécifique. Nous appellerons *acte de premier étage* les actes f tels que chaque loterie f(s) est dégénérée (i.e. ne donne qu'une conséquence avec certitude). Les actes f tels que $f(s)=\ell$ pour tout s seront appelés *actes de deuxième étage*, ou abusivement, loterie.

Ainsi selon l'état s qui se réalise, l'acte génère une loterie spécifique. Le décideur ne connaît donc pas avec précision le risque auquel il fait face mais possèdera une distribution de probabilité subjective sur la distribution objective qui apparaitra. Neilson (1993) a ainsi modélisé l'ambiguïté. Pour KMM, le décideur a de même une distribution de probabilité subjective mais sur les distributions subjectives possibles (en termes de notation, leurs actes de second-ordre correspondent à nos actes de premier étage). Toutefois, chacune de ces distributions peut être répliquée par une loterie (voir leur Lemme 1) et même remplacée par l'équivalent certain de la loterie correspondante (cf. leurs Définition 2 et Hypothèse 3).

Dans notre modèle nous utilisons aussi une telle *rétro-induction* : le décideur est indifférent entre tout acte f et l'acte de premier étage obtenu en remplaçant chaque loterie f(s) par son équivalent certain. D'après le théorème 2.5.3 du chapitre 2 :

(i) Le décideur vérifie les trois hypothèses suivantes :
   (a) La rétro-induction est satisfaite ;
   (b) Ses préférences restreintes aux actes de premier étage $\succcurlyeq^1$ sont représentées par une fonctionnelle d'utilité espérée subjective avec u^1 et P ;
   (c) Ses préférences restreintes aux actes de deuxième étage $\succcurlyeq^2$ sont représentées par une fonctionnelle d'utilité espérée nommée Eu^2.

(ii) Les préférences $\succcurlyeq$ sont représentées par

$$f \mapsto \int_S \varphi\left(Eu^2\big(f(s)\big)\right) dP(s).$$

Dans ce modèle, nous allons voir que φ représente l'attitude face à l'ambiguïté. L'*aversion à l'incertitude* a été définie par Schmeidler (1989) et Gilboa & Schmeidler (1989) comme la quasi-concavité des préférences, i.e., pour tout acte f, g, $f\sim g$ implique pour tout $\lambda\in[0,1]$, $\lambda f+(1-\lambda)g \succcurlyeq f$ où $\lambda f+(1-\lambda)g$ est l'acte qui donne pour chaque s la loterie $\lambda f(s)+(1-\lambda)g(s)$. Dans le cadre de ce modèle, cette définition observable est équivalente à la définition d'*aversion à l'ambiguïté lisse* (*smooth ambiguity aversion*) de KMM. Le théorème 2.5.5 du chapitre 2 montre alors que sous les conditions (i)(b) et (i)(c) ci-avant, les 2 propositions suivantes sont équivalentes :

(i) La fonction φ est concave ;

(ii) $\alpha\beta \sim^{*2} \beta\gamma$ et $\alpha\beta' \sim^{*1} \beta'\gamma$ implique que $\beta' \leq \beta$.

En outre, sous certaines conditions de richesse que nous ne détaillerons pas ici et si la condition (i)(a) est satisfaite, alors les deux propositions précédentes sont aussi équivalentes à :

(iii) l'aversion à l'incertitude ;

(iv) l'aversion à l'ambiguïté lisse.

Nous pouvons d'ores et déjà remarquer qu'il n'y a pas besoin de vérifier la rétro-induction pour observer l'attitude à travers la condition (ii). La sous-section suivante va s'intéresser à la comparaison d'attitudes entre décideurs.

0.2.4. Comparaison d'attitude face à l'ambiguïté

En termes de notation, nous allons réutiliser les notations précédentes en ajoutant en indice le décideur, A ou B. KMM proposent de définir qu'un individu A est plus averse à l'ambiguïté lisse qu'un individu B quand ils ont les mêmes croyances $P^A=P^B$ sur S par la condition suivante pour tout acte f et loterie ℓ : $f \succcurlyeq^A \ell \Rightarrow f \succcurlyeq^B \ell$. Cette condition implique aussi que les deux agents ont les mêmes préférences ($\succcurlyeq^{2A}$ et $\succcurlyeq^{2B}$) sur les loteries. En supposant que les préférences des deux décideurs A et B vérifient les hypothèses (i)(b) et (i)(c) évoquées dans les deux derniers résultats, alors les deux propositions suivantes sont équivalentes :

(i) φ^A est une transformation concave de φ^B et $u^{2A} = u^{2B}$;

(ii) Si $\alpha\beta \sim^{*1B} \beta\gamma$ et $\alpha\beta' \sim^{*1A} \beta'\gamma$, alors $\beta' \leq \beta$; en outre, $\sim^{*2A} = \sim^{*2B}$.

Si en outre la rétro-induction est satisfaite pour les deux décideurs, si les mêmes conditions de richesse que dans la section précédente sont vérifiées et enfin si $P^A=P^B$ alors ces propositions sont aussi équivalentes à :

(iii) A est plus averse à l'ambiguïté lisse que B.

Un autre théorème du chapitre 2 permet d'observer et de comparer la concavité des fonctions φ des deux décideurs, quelles que soient leurs croyances (les mesures P^A et P^B peuvent être différentes) et leurs utilités pour les loteries (l'utilité u^{2A} peut être différente de u^{2B}).

0.2.5. Discussion

Pour conclure cette section, nous pouvons tout d'abord remarquer que les relations d'arbitrage permettent de rendre observables et comparables les attitudes des décideurs. Cette relation permet de capturer l'utilité marginale de différents décideurs ou du même décideur dans des situations différentes. Or, tous les modèles utilisés dans cette section représentent les attitudes via des utilités marginales différentes. En outre, l'intérêt de cette relation réside dans ce qu'elle n'est pas influencée par les croyances des décideurs et permet donc de les exclure de l'analyse.

Toutefois, ces résultats peuvent être réinterprétés de manière différente, dès lors que l'on constate que nous avons défini des conditions, qui justement excluent de l'analyse tous les aspects du risque et ne conservent que l'utilité marginale. Le chapitre 2 discute ainsi de l'opportunité d'utiliser des modèles réduisant ainsi toutes attitudes aux conséquences. Dans les sections suivantes, nous considérerons des modèles où l'attitude en situation d'incertitude est décrite via l'influence à la fois des conséquences et des probabilités.

0.3. Agrégation des jugements d'experts

Dans de nombreuses situations, un décideur doit demander l'avis d'experts pour obtenir une évaluation du risque auquel il fait face. Selon les cas, les experts peuvent être d'accord et donner une évaluation précise du risque (« la probabilité de perdre x€ est p »), mais ils peuvent être parfois imprécis (« la probabilité de perdre x€ est comprise entre p−r et p+r ») ou en désaccord (« la probabilité de perdre est p−r » selon A mais « elle est de p+r » selon l'expert B). Dans cette section, nous désignerons le premier cas par *risque*, le second par *ambiguïté imprécise* (A^i), et le troisième par *ambiguïté conflictuelle* (A^c). Que faisons-nous dans ces cas ambigus ? Considérons nous le pire comme certain ou n'agissons-nous qu'en fonction de la probabilité moyenne ? Percevons-nous différemment la situation conflictuelle de la situation ambiguë ? Le chapitre 3 de la thèse expose une méthode pour étudier de telles situations et rapporte les résultats d'une étude expérimentale.

0.3.1. Qu'en dit la littérature ?

L'agrégation des jugements d'experts est un sujet très étudié dans la littérature de théorie de la décision. Une première approche vise à déterminer des règles mathématiques (telles que la règle de Bayes) pour obtenir une évaluation unique malgré des avis multiples (par exemple Genest 1984 ; Winkler 1968 ; Clemen & Winkler 1993). Il est aussi possible de faire collaborer, discuter, débattre les experts afin d'obtenir un consensus (par exemple Dalkey, 1969 ; Delbecq, Van de Ven & Gustafson 1975). Une troisième approche ambitionne à comprendre comment un agent associe dans son esprit les différents avis (par exemple Sniezek & Buckley 1995) ou comment il les associe dans ses choix (par exemple Du & Budescu 2005). C'est à cette dernière approche que se rattache la présente étude.

Parallèlement, de nombreuses études expérimentales se sont intéressées aux situations ambiguës (sans référence à des experts). Aux vues de ces études, l'attitude face à l'ambiguïté dépend des conséquences (gain ou perte) mais aussi du niveau absolu des conséquences et du niveau des probabilités (cf Cohen, Jaffray & Said 1985, 1987; Hogarth & Einhorn 1990; Lauriola & Levin 2001; Viscusi & Chesson 1999).

Des résultats similaires ont été trouvés lorsque les possibilités de gains et de pertes sont adossées à des événements (dont les probabilités ne sont pas connues). Ne pas connaître les probabilités de manière certaine engendre deux types d'effets : plus de pessimisme et une moindre sensibilité aux changements de niveau de vraisemblance (cf. Kilka & Weber 2001 ; Tversky & Fox 1995 ; Tversky & Wakker 1995 ; Wakker 2004). La plupart de ces dernières études font d'ailleurs appel au modèle CPT comme cadre d'analyse, ce dernier permettant de capter les changements d'attitudes en fonction du signe des conséquences et du niveau de probabilité. C'est aussi ce que nous allons faire dans la section suivante.

0.3.2. Cadre théorique

Par la suite, nous ne considérerons que des pertes (c'est-à-dire les conséquences appartiennent à $\mathbb{R}^-$). L'ambiguïté imprécise sera représentée par un intervalle de probabilité [p−r,p+r] et l'ambiguïté conflictuelle par un ensemble de deux possibilités {p−r,p+r}. Une loterie sera représentée par xpy, x[p−r,p+r]y ou x{p−r,p+r}y selon que le décideur se retrouve en situation de risque, d'ambiguïté imprécise ou d'ambiguïté conflictuelle (par convention $x<y<0$). Une *croyance révélée* associée à [p−r,p+r] (respectivement {p−r,p+r}) sera une probabilité q telle que il existe des conséquences $x<y$ et z appartenant à $\mathbb{R}^-$ vérifiant x[p−r,p+r]y∼z (respectivement x{p−r,p+r}y∼z) et xqy∼z. Nous écrirons $[p-r,p+r]\approx^R q$ (respectivement $\{p-r,p+r\}\approx^R q$).

Le décideur exhibera une moindre sensibilité en situation d'ambiguïté imprécise qu'en situation de risque si la condition suivante est vérifiée :

Si $[p-r,p+r]\approx^R q$ et $[p'-r,p'+r]\approx^R q'$, alors $|q-q'|\leq|p-p'|$.

Cette condition signifie que le décideur réagit moins à un changement de vraisemblance en situation ambiguë qu'en situation risquée. La condition suivante représente l'aversion à l'ambiguïté imprécise :

If $[p-r,p+r]\approx^R q$, then $q\geq p$.

Ne disposant d'aucune autre information que l'intervalle de probabilité, le décideur n'aime pas l'ambiguïté s'il agit comme si la 'vraie' probabilité était inférieure à la moyenne. Des conditions similaires peuvent être exprimées pour l'ambiguïté conflictuelle.

A présent, nous allons faire l'hypothèse que toutes les loteries ambiguës ou non peuvent être représentées par CPT. Soit :

$$xpy \longmapsto w(p)u(x) + \big(1 - w(p)\big)u(y),$$

$$x[p-r,p+r]y \longmapsto W^i([p-r,p+r])u(x) + \Big(1 - W^i([p-r,p+r])\Big)u(y),$$

et

$$x\{p-r,p+r\}y \longmapsto W^c(\{p-r,p+r\})u(x) + \big(1 - W^c(\{p-r,p+r\})\big)u(y),$$

avec u la fonction d'utilité, w la fonction de transformation de probabilité et W^i et W^c les fonctions de pondération en situation ambiguë. En conséquence, il est possible de décomposer les poids de décision ainsi :

$$x[p-r,p+r]y \longmapsto w\big(q^c([p-r,p+r])\big)u(x) + \Big(1 - w\big(q^c([p-r,p+r])\big)\Big)\,u(y),$$

et

$$x\{p-r,p+r\}y \longmapsto w\big(q^c(\{p-r,p+r\})\big)u(x) + \Big(1 - w\big(q^c(\{p-r,p+r\})\big)\Big)\,u(y)$$

La fonction w conjointement avec l'utilité capture l'attitude face au risque. Les fonctions q^c et q^i représentent les croyances révélées. Ce sont elles que nous allons étudier pour comprendre l'attitude face à l'ambiguïté des décideurs.

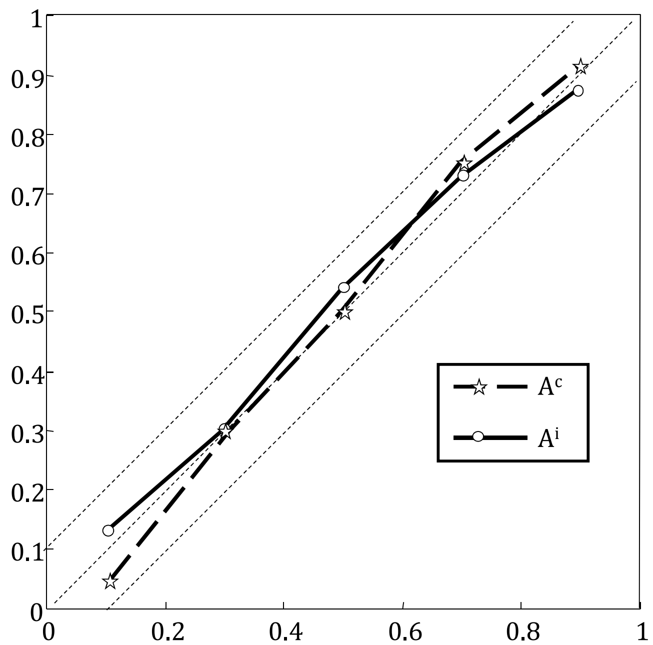
0.3.3. L'expérience

Une expérience a été conduite en octobre 2006 auprès de 61 élèves de l'ENSAM, confrontés aux différents types de choix suivants :

- Dans le premier cas, deux experts sont d'accord sur une probabilité précise de perte. Ce type de loterie peut s'écrire xpy. En sont déduits des équivalents certains (z tel que z∼xpy) à partir de choix hypothétiques et avec une méthode de bissection. En faisant varier x et y entre 0 et −1000, et en conduisant une optimisation non linéaire sur les équivalents certains, nous pouvons obtenir une estimation paramétrique de u et de w.
- Dans le second cas, les deux experts sont d'accord sur un intervalle de probabilité. Pour 5 valeurs de p (respectivement 0.1, 0.3, 0.5, 0.7, 0.9), les équivalents certains de −1000[p−0.1,p+0.1]0 ont été élicités. Avec $t \sim -1000[p-0.1,p+0.1]0$, on peut déduire $w(q^i([p-0.1,p+0.1])) = -u(t)$ puis $q^i([p-0.1,p+0.1]) = w^{-1}(-u(t))$.
- Le même processus a été conduit avec les situations d'ambiguïté conflictuelle −1000{p−0.1,p+0.1}0 (avec les mêmes valeurs de p).

Enfin, des indices ont été calculés pour mesurer l'aversion à l'ambiguïté et la moindre sensibilité par rapport au risque. En prenant une estimation linéaire de q^i (ou q^c) sur le centre des intervalles ou ensembles (p dans [p−0.1,p+0.1] ou dans {p−0.1,p+0.1}), il est possible de comparer la droite ainsi obtenue avec la diagonale car celle-ci représenterait un agent qui utilise toujours le centre p dans sa décision, et qui est donc totalement neutre à l'ambiguïté. La pente donne un indice de sensibilité en ce qu'elle indique la discriminabilité en situation ambiguë relativement au risque. L'élévation de la droite correspond enfin à l'aversion à l'ambiguïté en ce qu'elle mesure l'attractivité relative des loteries ambiguës.

0.3.4. Les résultats



Graphique 0.3.1 : Les croyances révélées.

En abscisse sont représentés les centres des intervalles étudiés ([0,0.2], [0.2,0.4], [0.4,0.6], [0.6,0.8], [0.8,1]) ou des ensembles correspondants ({0,0.2}…). Les ordonnées représentent les croyances révélées. Elles diffèrent significativement des centres (c'est-à-dire 0.1,…0.9) pour les cas [0,0.2], {0,0.2} et [0.8,1] (tous ces résultats ainsi que les suivants sont issus de tests de Student appariés). C'est pour ces mêmes situations extrêmes (lorsque les centres sont 0.1 ou 0.9) que les valeurs sont significativement différentes entre A^i et A^c. Par exemple, lorsque les deux experts sont d'accord pour dire que la probabilité de perte est comprise entre

0 et 20%, l'individu médian agit comme si la 'vraie' probabilité est de 19%. Le pessimisme l'emporte et l'individu agit comme si le pire était presque sûr. A l'inverse, lorsque un des experts annoncent une probabilité de 0 et l'autre de 20% alors l'individu médian semble considérer que la 'vraie' probabilité est de 6%. Il semble ainsi être plus influencé par le caractère extrême d'un expert disant qu'il n'y a aucun risque. Ceci le pousse à l'optimisme. Il y a donc préférence pour le conflit dans cette situation par rapport à l'imprécision. Par contre, la préférence est inversée (préférence pour l'imprécision par rapport au conflit) lorsqu'un des experts annonce que la perte est sûre.

Pour chaque sujet et pour chaque contexte ambigu, une estimation paramétrique a permis de déterminer un indice d'aversion à l'ambiguïté (l'élévation moyenne des croyances révélées, i.e. si la courbe précédente est significativement au dessus de la diagonale), et un indice de sensibilité (i.e. si la pente des croyances révélées est ou non différente de 1, cas qui représenterait la même sensibilité en situation ambiguë qu'en situation de risque). Il s'est avéré que l'aversion à l'ambiguïté n'est significative que dans le cas imprécis. Dans ce même cas, les sujets ont aussi montré une moindre sensibilité par rapport au risque. Inversement, ils exhibent plus de sensibilité à l'ambiguïté conflictuelle qu'au risque. Les deux indices sont en tout cas significativement différents entre les deux situations ambiguës, révélant par la même que l'implémentation de l'ambiguïté a un impact non nul sur les résultats.

0.3.5. Conclusion de l'étude

Différents points de l'étude peuvent être discutés. Le choix du modèle, tout d'abord, parmi les différents modèles que nous avons présentés dans la première section, nous a semblé justifié par la nécessaire prise en compte d'une attitude face à l'ambiguïté qui varie selon le niveau de probabilité. La méthode d'élicitation, reposant sur des équivalents certains puis sur des estimations paramétriques a été déterminée après une étude pilote. Dans cette dernière, nous obtenions (via un processus de bissection) directement la probabilité q telle que $xqy \sim x[p-r,p+r]y$. Mais il s'est avéré que les sujets ne considéraient pas les conséquences et perdaient ainsi un aspect important du choix. Enfin, des choix hypothétiques ont été utilisés parce que la mise en place d'une procédure incitative était difficilement compatible avec la volonté de contextualiser l'expérience (faire référence aux experts) et de contrôler les niveaux de croyances donnés par les experts.

Pour conclure nous pouvons présenter ainsi les principaux résultats de notre étude.

- Il existe des situations significatives où le décideur n'est pas neutre à l'ambiguïté, particulièrement pour les niveaux de vraisemblance proche des extrêmes (0 et 100%).
- Le contexte conflictuel diffère significativement de l'ambiguïté imprécise, car il génère moins de pessimisme (donc moins d'aversion à l'ambiguïté) mais une plus forte sensibilité aux changements de vraisemblance.
- Une position extrême d'un expert (annonçant une perte impossible ou certaine) influence plus fortement le décideur qu'un expert donnant comme évaluation une probabilité intermédiaire.

L'influence du type d'incertitude est un résultat important de la littérature, avec les concepts de sources d'incertitude et de dépendance de l'attitude face à la source. La section suivante rapporte les résultats du chapitre 4 qui met en lumière de tels résultats.

0.4. Croyances Bayésiennes et consentement à parier pour étudier l'ambiguïté

Le chapitre 4 de la thèse s'intéresse aux situations où le décideur fait face à des situations décrites par des événements. Nous allons rapporter ici le modèle utilisé et les résultats de l'étude expérimentale visant à tester ce modèle. Il s'agit de déterminer dans quelles conditions le décideur peut agir en fonction de *croyances Bayésiennes* (ou probabilité subjectives), puis d'étudier le consentement à parier sur des événements avec la même croyance Bayésienne afin de mettre en évidence les déterminants de l'attitude face à l'ambiguïté.

0.4.1. La rencontre de deux concepts

Le modèle que nous allons utiliser s'appuie sur deux concepts fondamentaux : la sophistication probabiliste et les sources d'incertitude.

Tout d'abord, rappelons que les préférences d'un individu satisfont la sophistication probabiliste s'il existe une distribution de probabilité telle que l'individu est indifférent entre deux actes générant la même distribution de probabilité sur les conséquences. Par exemple, si un décideur pense que la probabilité qu'il pleuve est égale à la probabilité que la température soit inférieure à 20°C, alors il doit être indifférent entre gagner 500€ s'il pleut et gagner 500€ si la température est effectivement sous la barre des 20°C. Machina & Schmeidler (1992) ont proposé une axiomatisation de la sophistication probabiliste dans un cadre Savagien.

Plus récemment, Chew & Sagi (2006a) ont utilisé le concept d'échangeabilité pour fournir une nouvelle fondation à la sophistication probabiliste. L'*échangeabilité*, suggérée par Ramsey (1926) et de Finetti (1937) sous les noms respectifs de « *neutralité éthique* » et d'« *équivalence* », est définie ainsi : les événements disjoints E et F sont échangeables si en permutant les conséquences affectées à E et à F, on ne change pas la valeur de l'acte en termes de préférence. Pour les actes binaires, cela peut s'écrire, pour deux conséquences x et y, $xEy \sim xFy$. L'intérêt du résultat de Chew & Sagi est de donner des conditions nécessaires et suffisantes à la sophistication probabiliste beaucoup plus légère que les axiomatisations préalables en se basant sur le concept d'échangeabilité.

Le second concept est celui de *source d'incertitude*. Une source d'incertitude est un ensemble d'événements générés par un même mécanisme incertain. Par exemple, les événements liés au cours du CAC40 correspondent à une source d'incertitude, tandis que les événements décrivant la température à Tokyo en constituent une autre. L'importance du concept de source d'incertitude a été mis en évidence par différents travaux de Amos Tversky dans les années 1990 (Fox & Tversky 1998 ; Heath & Tversky 1991 ; Tversky & Kahneman 1992 ; Tversky & Fox 1995 ; Tversky & Wakker 1995). Le principal résultat de ces travaux expérimentaux et théoriques ainsi que de Kilka & Weber (2001) réside dans le fait que l'attitude des décideurs dépend de la source d'incertitude. En effet celle-ci affecte leur capacité à discriminer entre les niveaux de vraisemblance et certaines sources sont préférées à d'autres. Nous verrons plus loin comment tout cela peut être représenté.

L'étude exposée au chapitre 4 vise à concilier ces deux concepts de la manière suivante: un décideur sera supposé vérifier la sophistication probabiliste à l'intérieure de chaque source, mais pas entre les sources. Il s'agit en effet d'accepter que le décideur puisse avoir un comportement différent pour deux événements auxquels il affecte la même probabilité subjective mais qui appartiennent à deux sources différentes. Dans notre démarche, nous partirons de différentes sources et nous vérifierons tout d'abord que la sophistication probabiliste est vérifiée à l'intérieure de ces sources. Ensuite, nous représenterons l'attitude du décideur face à cette source. Notons que Chew & Sagi (2006b) utilisent une approche duale pour fournir une définition des sources à partir des préférences. Ils classent tous les événements pour lesquels la sophistication probabiliste (restreinte à ces événements) est satisfaite dans une même source.

0.4.2. Le paradoxe d'Ellsberg à 2 couleurs

Reprenons maintenant le paradoxe d'Ellsberg à deux couleurs pour présenter les intuitions de la démarche que nous venons d'exposer. Nous avons vu que ce paradoxe conduit à une violation des propriétés des probabilités subjectives dans le cadre de SEU. Il s'agit plus généralement d'une violation de la sophistication probabiliste car ce paradoxe révèle que deux événements complémentaires (« tirer une boule rouge dans l'urne inconnue » et « tirer une boule noire dans l'urne inconnue ») semblent tous les deux avoir une probabilité strictement inférieure à 1/2 car la plupart des personnes préfèrent parier sur les événements avec probabilité connue de 1/2 (« tirer une boule rouge dans l'urne connue » et « tirer une boule noire dans l'urne connue ») que sur ces événements avec probabilité inconnue.

Remarquons que chaque urne est un mécanisme d'incertitude différent ; la première fournie des événements avec une probabilité sûre, la seconde est plus ambiguë. Ainsi les événements concernant l'urne connue appartiennent à une source, les événements issus de l'urne inconnue constituant une autre source. Par conséquent, rien ne prouve que la sophistication probabiliste n'est pas vérifiée à l'intérieure de chaque urne. Par exemple, elle sera satisfaite si l'agent est indifférent entre parier sur « rouge » ou sur « noire » dans une urne donnée. Cela signifierait que les événements « rouge dans l'urne connue », « noire dans l'urne connue », « rouge dans l'urne inconnue », et « noire dans l'urne inconnue » ont tous une probabilité de 1/2, mais que l'attitude diffère entre les urnes.

En effet, les préférences du paradoxe d'Ellsberg peuvent alors se réinterpréter simplement comme une préférence pour l'urne connue. C'est un résultat classique dans la littérature sur les sources d'incertitude que d'obtenir que les agents préfèrent parier sur les événements pour lesquels ils disposent de plus d'informations. En considérant que l'urne connue représente le risque et que l'autre urne est ambiguë, le paradoxe d'Ellsberg correspond à de l'aversion à l'ambiguïté, ce qui n'est pas incompatible avec la notion de croyance Bayésienne, si cette dernière est définie à l'intérieure de chaque source.

0.4.3. Sources uniformes

Les différents concepts ayant été introduits et les intuitions exposées, nous allons à présent définir avec précision l'outil que nous proposons pour étudier l'ambiguïté. Il s'agit du concept de *source uniforme*. Une source d'incertitude sera dite uniforme si la sophistication probabiliste est vérifiée entre événements de cette source. Le terme uniforme capture l'idée que l'ambiguïté au sein de cette source apparaît comme uniforme, homogène au décideur et que par conséquent son attitude ne sera pas influencée par autre chose que la probabilité de l'événement ; en d'autres termes, le décideur a une connaissance uniforme sur la source et ne pense pas détenir plus d'information pour un événement que pour un autre.

Considérons à présent une partition de S en événements échangeables, c'est-à-dire une partition de S en $\{E_1,\ldots,E_n\}$ telle que pour tout acte $(E_1:x_1,\ldots,E_n:x_n)$, l'agent est indifférent entre cet acte et un acte obtenu par permutation d'un x_i et d'un x_j pour n'importe quelle paire de i et j (prenant des valeurs de 1 à n). Une fois une telle partition obtenue, il est possible de générer une source en prenant toutes les unions possibles de ces événements. A partir du résultat de Chew & Sagi (2006a) on sait alors que cette source est uniforme. Notons que la distribution de probabilité subjective sur les E_is est uniforme : $P(E_i)=1/n$ pour tout i de 1 à n.

Un contre-exemple trouve sa source dans le paradoxe d'Ellsberg à 3 couleurs. En effet, l'urne contenant 30 boules rouges et 60 boules noires ou jaunes n'est pas uniforme, l'agent disposant de plus d'informations pour certains événements que pour d'autres. Chew & Sagi (2006b) proposent de considérer deux sources, l'une contenant les événements avec probabilité connue, l'autre les événements avec probabilité inconnue. Cette seconde source ne couvre pas l'espace des événements et il n'est donc pas possible de définir une partition de S en événement échangeables. Ce type de situation ne peut pas être traité avec la méthode exposée dans le chapitre 4.

0.4.4. Description des attitudes face à l'ambiguïté

Nous ne considérerons dans la suite de cette section uniquement des actes binaires avec conséquences positives. Dans ce cadre, lorsque les probabilités sont connues, les modèles RDU et CPT sont équivalents et peuvent s'écrire (avec $x \geq y$) :

$$xpy \longmapsto w(p)u(x)+(1-w(p))u(y).$$

Pour le même type d'actes mais lorsque les probabilités ne sont pas connues, les modèles CEU, CPT et Maximin ont une formulation commune (voir Ghirardato & Marinacci 2001):

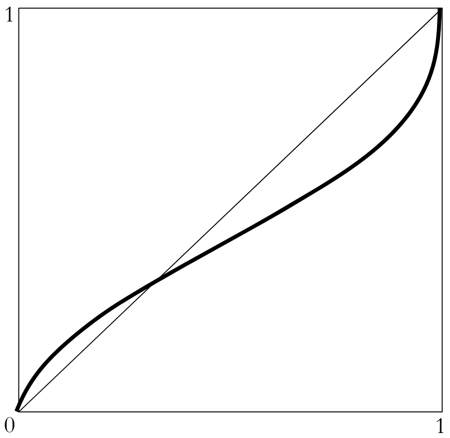
$$xEy \longmapsto W(E)u(x)+(1-W(E))u(y).$$

La fonction d'utilité est supposée la même dans les deux cas, probabilités connues ou non. Pour des événements appartenant à une source uniforme donnée, puisque la sophistication probabiliste est satisfaite, alors ceci peut être réécrit :

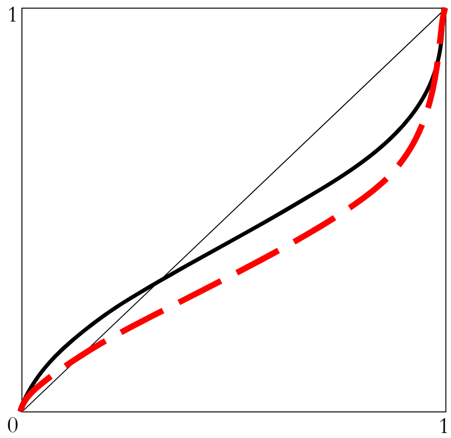
$$xEy \longmapsto w(P(E))u(x)+(1-w(P(E)))u(y),$$

où w est la fonction de transformation de probabilité spécifique à la source en question.

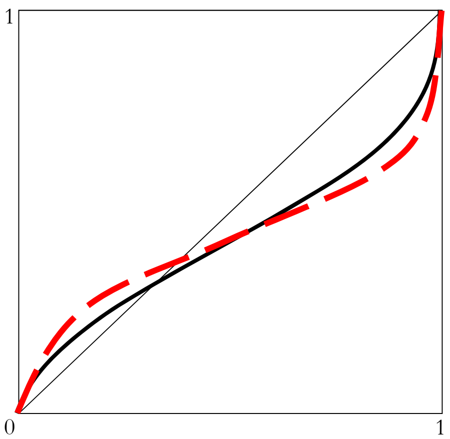
Dans le cadre de ce modèle, en élicitant la fonction w quand les probabilités sont connues et pour différentes sources uniformes, il est alors possible d'étudier l'impact précis des différentes sources par rapport au risque. Exposons en effet comment ces fonctions peuvent être analysées. Lorsque les probabilités sont connues, w est généralement en forme de S inversé (surévaluation des petites probabilités et sous-évaluation des grandes), et ceci est généré par une sensibilité au changement de niveau de vraisemblance plus forte vers les extrêmes (0 et 100%) que pour les probabilités intermédiaires. En d'autres termes, l'agent réagit plus à une variation de probabilité entre 0 et 1% (ou entre 99% et 100%) qu'entre 49 et 50%. En outre, w dans le risque (et pour les gains) a tendance à être en dessous de la diagonale, représentant ainsi le pessimisme des agents qui agissent comme si leur chance de gagner était inférieure à la probabilité. Ceci est équivalent à une faible attractivité des loteries par rapport aux gains certains. La courbe en noire du graphique 0.4.1 représente w quand les probabilités sont connues. Elle combine les deux effets, S inversé et tendance à être en dessous de la diagonale.



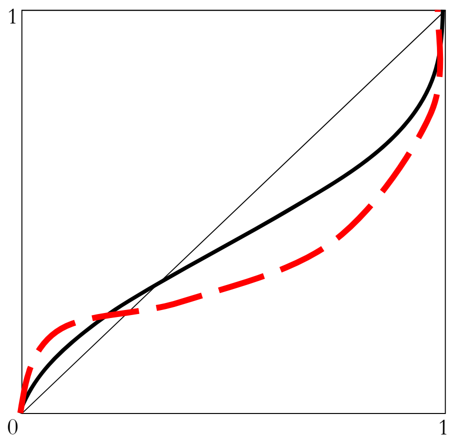
Graph a : w dans le risque



Graph b : Aversion à l'ambiguïté



Graph c : Perte de sensibilité liée à l'ambiguïté



Graph d : Perte de sensibilité et aversion à l'ambiguïté

Graph 0.4.1 : Attitudes face au risque et à l'ambiguïté à l'ambiguïté

Lorsque les probabilités ne sont pas connues, le paradoxe d'Ellsberg nous suggérait que les agents sont moins attirés par les paris que lorsque les probabilités sont connues. Cette baisse d'attractivité des paris, ou de manière équivalente cette hausse du pessimisme des agents, devrait se traduire par une fonction de transformation des probabilités en dessous de celle du risque, comme si un événement E avec une probabilité subjective de P(E)=p était pénalisé par rapport à une probabilité objective égale à p lorsqu'il s'agit de parier sur l'un ou l'autre. Il s'agit d'aversion à l'ambiguïté. Le graphique 0.4.1.b représente en pointillé rouge une telle fonction w pour une source uniforme (avec probabilités inconnues) par rapport à la fonction de pondération w du risque en noir.

Un second effet de l'ambiguïté est la baisse de la discriminabilité entre niveau de vraisemblance. En effet, s'il est déjà difficile lorsque les probabilités sont connues de distinguer cognitivement entre 49 et 50% de chances de gagner, cela est d'autant plus difficile lorsqu'il s'agit de probabilité subjective, qui sont avant tout des niveaux de vraisemblance dans l'esprit du décideur. Ce phénomène se traduit par une plus forte réaction aux changements par rapport à la certitude ou l'impossibilité, mais à une moindre sensibilité pour toutes les probabilités intermédiaires. Le graphique 0.4.1.c représente cette situation avec le même code couleur que 0.4.1.b.

Enfin, lorsque les deux effets de l'ambiguïté sont combinés (aversion à l'ambiguïté et moindre sensibilité à l'incertitude qu'au risque), on obtient le graphique 0.4.1.d. Afin d'obtenir des indices pour ces phénomènes, les différentes transformations de probabilité (hors bornes 0 et 1) seront approximées par des droites. La comparaison des pentes nous permettra d'obtenir une évaluation relative des sensibilités, et les élévations des droites donneront une indication sur l'attractivité des différentes sources et du risque. Les prochaines sous-sections présentent la méthode et les résultats d'une étude expérimentale permettant de tester l'uniformité de différentes sources et d'obtenir les transformations de probabilités pour ces sources et pour le risque.

0.4.5. Méthode expérimentale

Une expérience a été conduite en janvier et février 2006 auprès de 62 étudiants de l'ENSAM à Paris. Tous ont reçu un paiement fixe de 20€ pour leur participation (en moyenne 1h30). 31 sujets faisaient partie du groupe dit « hypothétique » et n'ont rien reçu d'autres. En outre, parmi les 31 sujets du groupe dit « Réel », l'un d'entre eux a été tiré au hasard et un de ces choix (aussi tiré au hasard) a été appliqué réellement.

L'expérience traitait trois sources d'incertitude : le CAC40, la température à Paris et la température dans un pays étranger lointain, le relevé des valeurs devant avoir lieu le 31 mai 2006. Il s'agissait tout d'abord d'obtenir une partition de l'espace des événements en événement échangeables pour chaque source. Nous allons exposer la méthode utilisée pour la température à Paris, car elle est similaire pour les trois sources. Le sujet se voyait proposer de gagner 1000€ si la température était supérieure à 20°C (par exemple) ou de gagner la même somme si la température était inférieure ou égale à 20°C. S'il préférait le premier pari, alors la température seuil était augmentée et les questions étaient posées avec la nouvelle température. Ceci permettait de déterminer deux événements (A_2^1=« la température sera inférieure à $a_{1/2}$°C » et A_2^2=« la température sera supérieure à $a_{1/2}$°C ») telle que la permutation des conséquences entre les événements n'influent pas sur les préférences. Chaque événement était ensuite décomposé en deux nouveaux sous-événements tels que parier sur l'un ou l'autre n'avait pas plus de valeur pour le sujet. Le schéma suivant indique le processus.

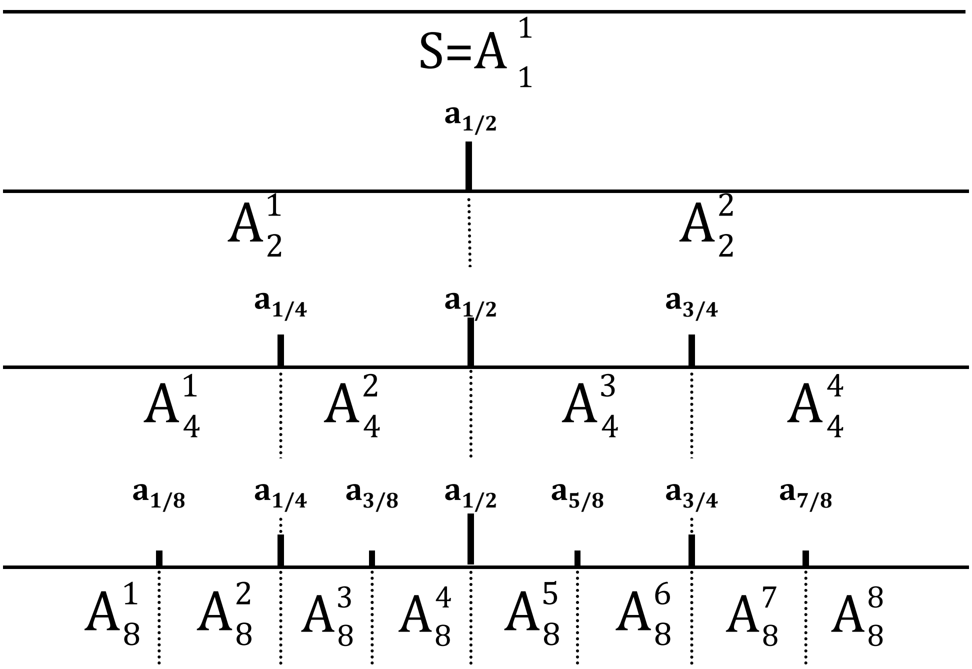


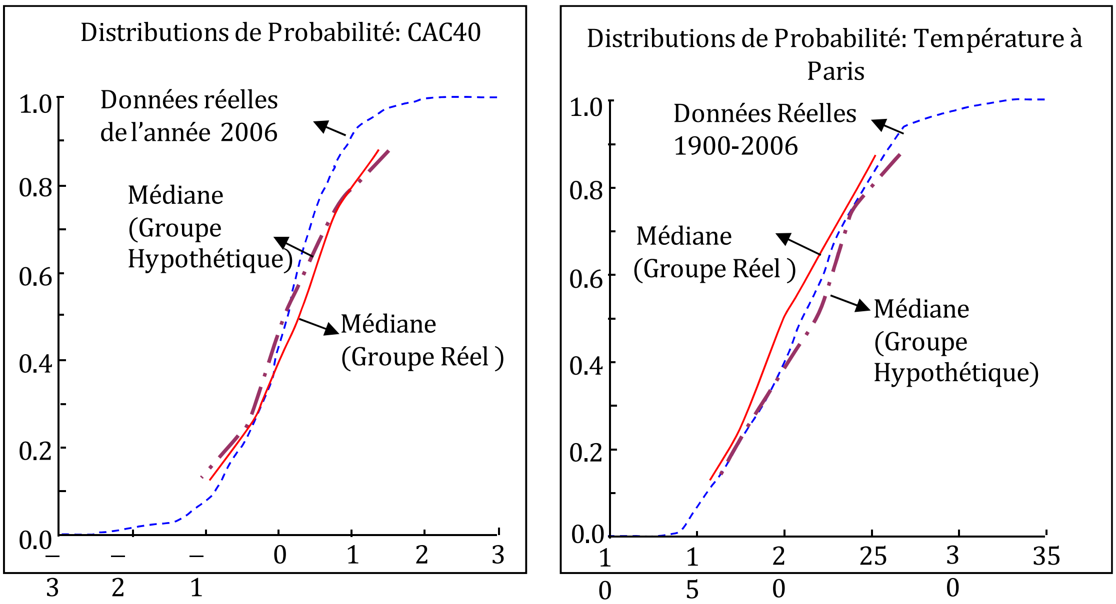
Schéma 0.4.2 : Décomposition de S

Pour vérifier que la partition ainsi obtenue était une partition en événement échangeable, et pour garantir ainsi l'uniformité de la source, nous testions si A_8^1 et A_8^8, A_8^2 et A_8^7, ou encore A_4^2 et A_4^3 étaient bien échangeables.

Une fois une telle partition construite, et pouvant considérer que chaque événement se voyait associer une probabilité 1/8 par les sujets, nous recherchions le consentement à parier sur des événements de probabilité subjective 1/8, 2/8,... 7/8. A cela étaient ajoutés des consentements à parier sur des probabilités objectives équivalentes. A partir de ces derniers, u et w dans le risque étaient élicités. En utilisant la même fonction d'utilité u, les transformations de probabilités dans les trois sources étaient déduites.

0.4.6. Résultats sur les probabilités subjectives

Commentons à présent les résultats concernant les distributions de probabilité pour les trois sources d'incertitude. Pour le groupe Hypothétique et pour la température à l'étranger seulement, un test d'échangeabilité conduit à rejeter l'hypothèse de source uniforme (avec un seuil de significativité de 5%). L'analyse s'arrête donc ici pour cette source (pour le groupe Hypothétique). Les graphiques suivants représentent les différentes distributions de probabilité pour le CAC40 et la température à Paris.

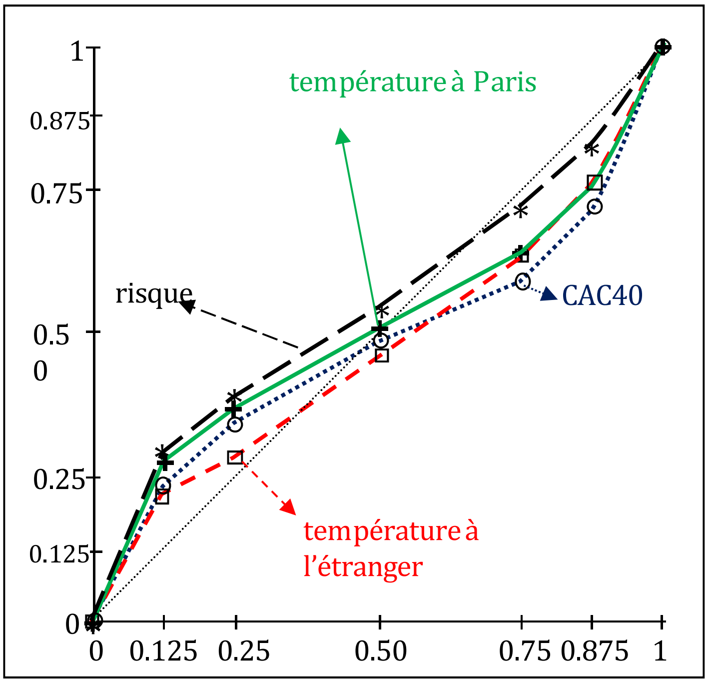


Graphs 0.4.3 : Distributions de probabilité subjectives et données réelles

Nous pouvons remarquer que les valeurs médianes obtenues ne sont pas aberrantes par rapport aux données réelles. La calibration des deux groupes est même particulièrement bonne pour la température à Paris, alors qu'il y a une tendance notable des deux groupes (Réel et Hypothétique) à surestimer la probabilité des hausses du CAC40. Notons qu'il est plutôt normal pour des élèves ingénieurs vivant à Paris pendant l'année scolaire de mieux maîtriser les températures que les variations de la Bourse.

0.4.7. Résultats sur l'attitude face à l'ambiguïté

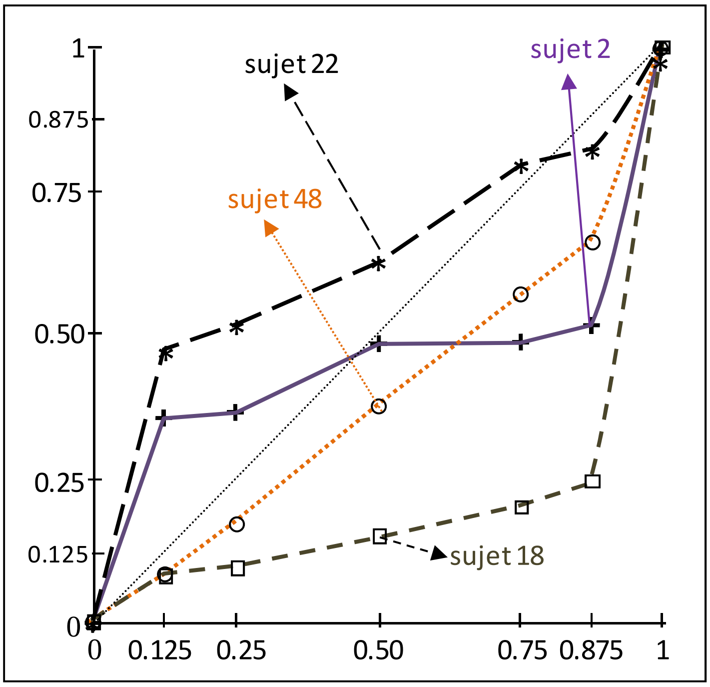
Nous pouvons à présent étudier les fonctions de transformation de probabilités élicitées pour les différentes sources. Pour simplifier la présentation, nous ne traiterons ici que le Groupe Réel.



Graph 0.4.4 : Transformations de probabilité moyennes pour le groupe Réel

Les graphiques ci-dessus représentent les fonctions de transformation des probabilités moyennes pour les trois sources et le risque. Il est important de noter tout d'abord qu'une analyse de la variance des différentes fonctions w entre les sources permet d'accepter que l'attitude dépend de la source pour les probabilités 0.125, 0.25, 0.75 et 0.875. En d'autres termes, les sujets de l'expérience ont une attitude qui diffère significativement selon la source, confirmant ainsi l'intérêt de ce concept.

De même une analyse de la variance sur l'indice de pessimisme (révélant l'élévation des différentes courbes) conduit à conclure que la source influe significativement sur l'attractivité des paris. Ce résultat confirme l'intuition issue du paradoxe d'Ellsberg. Les comparaisons deux à deux des indices de sensibilité confirment la moindre sensibilité en situation d'ambiguïté qu'en situation de risque pour les sources CAC40 et température à Paris.



Graph 0.4.5 : Transformations de probabilité pour la température à Paris

Le schéma précédent représente des résultats individuels de 4 sujets différents. Il s'agit de remarquer ici qu'il est possible de comparer l'attitude des agents face aux probabilités (sur une source donnée), quelque soient leurs croyances sur les événements. Par exemple, le sujet 18 a un consentement à parier beaucoup plus faible que les autres. Le sujet 2 exhibe la plus faible sensibilité aux niveaux de vraisemblance, c'est-à-dire son consentement à parier (ou à s'assurer) ne variera que peu même lorsque ses croyances varient fortement.

0.4.8. Prédictions

Dans la lignée des résultats précédents, il est aussi possible de décomposer pour un sujet donné en plusieurs éléments sa *prime d'incertitude* pour un acte.

Cette prime d'incertitude correspond à la différence entre la valeur espérée (subjective) de l'acte et l'équivalent certain que le sujet lui associe. Cette prime peut se décomposer en une *prime de risque* qui correspond à la prime d'incertitude si la probabilité avait été objective et une *prime d'ambiguïté* qui correspond au surcroit de prime engendré par le fait que la probabilité ne soit pas connue avec certitude. Dans le tableau suivant, nous calculons ces primes pour un acte donnant 40000€ si E se réalise (et rien sinon) où E est un événement concernant la température à Paris. Nous ferons les hypothèses simplificatrices que les deux sujets ont la même fonction d'utilité $u(x)=x^{0.88}$ et que tous les deux ont la même croyance.

Tableau 0.4.6 : Calcul des primes d'incertitude, de risque et d'ambiguïté

	Sujet 2 P(E)=0.125	Sujet 48 P(E)=0.125	Sujet 2 P(E)=0.875	Sujet 48 P(E)=0.875
w(P(E))	0.35	0.08	0.52	0.67
P(E)×40000	5000	5000	35000	35000
Equivalent certain	12133	2268	19026	25376
Prime d'incertitude	−7133	2732	15974	9624
Prime de risque	−4034	2078	5717	−39
Prime d'ambiguïté	−3099	654	10257	9663

Ainsi quand P(E)=0.125, la prime d'incertitude pour le sujet 48 s'explique principalement par son attitude générale face au risque, la prime d'ambiguïté ne représentant que moins d'un quart de la prime totale. Par contre pour une très grande probabilité (0.875), ce même sujet serait prêt à prendre un risque si la probabilité était objective (sa prime de risque est légèrement positive) mais il devient très averse à l'ambiguïté ce qui explique sa prime d'incertitude élevée. Le tableau permet aussi de comparer les deux sujets et de comprendre si leur comportement (d'assurance, par exemple) provient de l'attitude face au risque ou de l'attitude face à l'ambiguïté : pour la probabilité 0.875, l'ambiguïté a un impact assez proche sur le sujet 2 et sur le sujet 48 mais le sujet 2 est moins joueur dans le risque pour les grandes probabilités ce qui explique que sa prime d'incertitude est plus élevée.

0.4.9. Conclusion de l'étude

Plusieurs limites de cette étude peuvent être mentionnées ici. Tout d'abord le fait que la méthode développée ne permet pas de prendre en compte le cas de sources où l'uniformité est violée (cf. les remarques sur le paradoxe d'Ellsberg à 3 couleurs). Sur le point de vue expérimental, le processus incitatif n'est pas sans défaut car le sujet peut penser qu'il existe une stratégie plus efficace que dire la vérité, même si cette stratégie potentielle n'est pas évidente par la complexité et le nombre important de questions de l'expérience.

Pour conclure, nous avons souhaité par cette étude montrer que la définition de sources uniformes d'incertitude permettait de réconcilier les résultats traditionnels en situations d'ambiguïté comme le paradoxe d'Ellsberg à deux couleurs, avec le concept de croyance Bayésienne, puis de comprendre de manière précise les attitudes face aux différentes sources. En outre, nous avons développé des outils pour représenter visuellement ces attitudes et les étudier afin de permettre des prédictions en termes de comportement assurantiel par exemple. Enfin il peut être noté que des travaux récents en neuroéconomie tendent à confirmer l'importance du concept de sources (Hsu et al. 2005 ; Camerer 2007).

0.5. Une méthode robuste d'élicitation des probabilités subjectives

Dans le chapitre 5 de la thèse, l'accent est mis sur la technique d'élicitation des probabilités subjectives que nous venons de décrire dans la section précédente. L'élicitation de probabilités subjectives est utilisée en analyse de la décision dans le but de recueillir des avis d'experts (cf. section 0.3). En économie comportementale comme en économie expérimentale, il est parfois nécessaire d'éliciter ce que pensent les participants d'une expérience (voir par exemple Nyarko & Schotter 2002). Dans la littérature de psychologie, sont souvent utilisées des probabilités dites *jugées*, c'est-à-dire directement exprimées par le sujet comme étant sa croyance. L'approche *des préférences révélées*, dominante en économie, est plus compatible avec des probabilités *fondées sur les choix* : le sujet doit révéler par ses choix ce qu'il pense, pour être sûr qu'il a intérêt à dire la vérité. Nous allons présenter différentes méthodes fondées sur les choix, qui permettent d'obtenir les probabilités subjectives. Pour cela nous ferons l'hypothèse que l'agent dont on veut connaître les croyances maximise son espérance de gain. Notons au passage que cette section ne traitera que de gains.

0.5.1. Quatre techniques d'élicitation des probabilités subjectives

La probabilité canonique : La première de ces méthodes revient à déterminer la probabilité p telle que $xpy \sim xEy$ (voir par exemple Raiffa 1968 p110, Wright 1988, Holt 2006). Si l'agent a une probabilité subjective pour E et ne considère que son espérance de gain, alors il doit choisir p tel que $p=P(E)$. Par exemple, s'il pense qu'il va pleuvoir demain avec une probabilité de 1/3, alors il doit être indifférent entre gagner 100€ s'il pleut et gagner 100€ avec 1 chance sur 3.

L'équivalent certain : Imaginons que nous proposions à l'agent un pari où il peut gagner 1€ s'il pleut et qu'il est prêt à payer 33 centimes pour ce pari. Si nous faisons l'hypothèse que ces 33 centimes correspondent à son espérance de gain, alors cela signifie que sa probabilité subjective est 1/3. Formellement, la méthode de l'équivalent certain revient à déterminer $c \sim 1E0$, pour déduire $P(E)=c$.

Les règles de scores (scoring rule) : Selon Winkler (1969), cette règle de scores est une fonction de paiement qui dépend de la probabilité rapportée par l'agent et de la survenance ou non de l'événement ; elle sert à la fois à l'inciter à être honnête et permet aussi de mesurer sa performance afin de l'aider à s'améliorer. La plus connue est la règle quadratique qui peut s'écrire ainsi $[1-(1-r)^2]E[1-r^2]$, lorsque l'agent dit que l'événement E a une probabilité r. Un agent qui maximise la valeur espérée de ce qu'il peut toucher avant d'indiquer sa croyance cherche donc à maximiser $P(E)[1-(1-r)^2]+(1-P(E))[1-r^2]$; les conditions nécessaires et suffisantes pour cela sont $r=P(E)$ et $-2<0$, cette dernière étant trivialement satisfaite.

La méthode d'échangeabilité : Il s'agit ici de la méthode que nous avons décrite dans la section précédente pour obtenir les probabilités subjectives. Elle consiste à partitionner l'ensemble des états de la nature en 2, puis 4, puis 8 événements échangeables, c'est-à-dire tels qu'une permutation entre les conséquences laisse l'agent indifférent. Au premier niveau, cela revient donc à déterminer E tel que $xEy \sim x(S-E)y$. Ensuite il faut déterminer $F \subset E$ tel que $xFy \sim x(E-F)y$, puis continuer sur l'événement complémentaire $(S-E)$ et de nouveau sur les événements obtenus. D'une partition en n événements échangeables, on peut déduire la distribution de probabilité subjective, puisque chacun de ces événements est censé avoir une probabilité de 1/n. Là encore, un agent ne considérant que son espérance de gain ne doit être indifférent qu'entre parier sur des événements de même probabilité, ce qui justifie cette méthode.

0.5.2. Les quatre techniques face aux déviations de comportements vis-à-vis de l'espérance mathématique

Nous avons vu dans la section 0.1 que la plupart des agents ne décident pas en fonction de l'espérance mathématique. Le paradoxe de St Petersburg a conduit à l'ajout d'une fonction d'utilité représentant l'attitude face aux conséquences. Le paradoxe de Allais et celui d'Ellsberg (à deux couleurs) ont conduit à introduire des modèles plus généraux encore que l'utilité espérée. Dans le chapitre 5, l'effet de chacun des paradoxes est détaillé sur chaque technique. Par simplicité, nous allons simplement considérer ici l'effet global de ces paradoxes.

Pour représenter ces trois paradoxes, nous allons considérer des loteries avec probabilités connues et des actes dont tous les événements appartiennent à la même source d'incertitude (cf. section 0.4). Nous utiliserons alors le modèle de la section 0.4 en n'utilisant ici w (fonction continue et strictement croissante) que lorsque les probabilités sont objectives, et avec φ la fonction telle que $w \circ \varphi$ désigne la transformation de probabilité (aussi continue et strictement croissante) de la source d'incertitude considérée (avec $x \geq y \geq 0$) :

$$xpy \longmapsto w(p)u(x)+(1-w(p))u(y)$$

et

$$xEy \longmapsto w\circ\varphi(P(E))u(x)+(1-w\circ\varphi(P(E)))u(y).$$

Ainsi u représente la fonction d'utilité (nécessitée par le paradoxe de St Petersburg), w la fonction de pondération des probabilités dans le risque (pour prendre en compte le paradoxe de Allais) et φ l'attitude liée au fait que la probabilité ne soit pas objective (attitude face à l'ambiguïté selon la terminologie de la section précédente, l'exemple le plus célèbre d'une telle attitude étant le paradoxe d'Ellsberg).

Commençons par étudier ce que devient la méthode des probabilités canoniques dans un tel cadre. Déterminer la probabilité p telle que $xpy \sim xEy$ revient à trouver $p=\varphi(P(E))$. La probabilité canonique capture donc la probabilité subjective mais aussi l'attitude face à l'ambiguïté. Considérons le paradoxe d'Ellsberg. Pour un individu préférant parier sur les boules rouges (ou noires) dans l'urne connue ($p=1/2$) plutôt que dans l'urne inconnue (événement sans probabilité objective), cela signifie que ses probabilités canoniques pour les événements « rouge dans l'urne inconnue » et « noire dans l'urne inconnue » sont inférieures toutes les deux à 1/2. Outre que ceci violerait l'additivité des probabilités subjectives, il n'est pas impossible que cet individu pense que la probabilité de ces deux événements soit vraiment 1/2 mais que ces préférences s'expliquent par son aversion à l'ambiguïté. Ainsi les probabilités canoniques ne permettent pas de distinguer entre ce qui relève de la croyance et ce qui relève de l'attitude face à l'absence d'information.

Pour ce qui est de la méthode de l'équivalent certain, $c \sim 1E0$ correspond dans notre modèle à $c=u^{-1} \circ w \circ \varphi(P(E))$. Il est toutefois imaginable de déterminer avec des probabilités connues les fonctions u et w, pour nettoyer l'équivalent certain c de leurs effets respectifs. Cependant, la fonction φ n'est pas déterminable avec des probabilités connues (puisqu'elle est générée par l'ambiguïté). En conclusion, l'équivalent certains capturent tous les éléments d'attitudes, et même s'il est possible d'en corriger certains, distinguer ce qui relève strictement de la croyance ne semble pas possible.

Les règles de score sont elles aussi évidemment touchées par ces déviations des comportements par rapport à l'espérance de gain puisqu'elles ont été définies dans le cadre d'un agent ne s'intéressant qu'à la valeur espérée. Etudions donc la règle de score quadratique avec une légère modification (nous allons simplement ajouter 1 à la conséquence sur E), pour être sûr que la conséquence si E se réalise soit toujours plus grande que si E ne se réalise pas. Offerman et al. (2007) suggère cela pour pouvoir représenter aisément la valeur de la règle dans un modèle à dépendance de rang. Rappelons en effet que nous avons défini notre modèle pour xEy avec $x \geq y \geq 0$. La règle peut alors s'écrire $[2-(1-r)^2]E[1-r^2]$, lorsque l'agent dit que l'événement E a une probabilité r. L'agent va donc choisir r pour maximiser :

$$w \circ \varphi(P(E))u(2-(1-r)^2)+(1-w \circ \varphi(P(E)))u(1-r^2).$$

Il est évident que la condition nécessaire à cette maximisation va maintenant intégrer l'influence de w, φ et u. Pour débiaiser la probabilité reportée, Offerman et al. (2007) propose de déterminer une fonction R(p), correspondant à la valeur r rapportée par l'agent qui pourrait toucher $[2-(1-r)^2]$ avec une probabilité p et $[1-r^2]$ sinon. Ils montrent que R(p) permet de prendre en compte l'effet de w et de u. Ainsi, si l'agent rapporte comme probabilité pour E la valeur r_E, alors $R(p)=r_E$ pour une valeur p donnée implique que $p=\varphi(P(E))$. Dans tous les cas, ce qui est obtenu comprend encore l'attitude face à l'ambiguïté.

Nous savons déjà, grâce à la section précédente, que la méthode d'échangeabilité permet dans le cadre du modèle considéré de distinguer entre attitudes et probabilités subjectives. En effet, la recherche d'événements échangeables dans une source donnée permet de ne pas utiliser de probabilité objective. Ainsi $xEy \sim x(S-E)y$ (avec $x \geq y \geq 0$) implique $w \circ \varphi(P(E))=w \circ \varphi(1-P(E))$, en fixant $u(x)=1$ et $u(y)=0$ (car u est définie à une fonction affine près). En conséquence, si P représentant la mesure de probabilité (additive), $P(E)=1/2$. Il est important de noter que c'est le choix entre deux actes incertains (non-dégénérés) et se référant à une seule source d'incertitude qui permet cela.

La conclusion de cette analyse est donc que la méthode d'échangeabilité, même si elle peut ne pas fonctionner dans certains cas comme le paradoxe d'Ellsberg à 3 couleurs (cf. sous-section 0.4.3), semble toutefois mieux à même de distinguer entre croyance et attitudes. Cette observation a nécessité toutefois plusieurs hypothèses, comme le fait qu'il existe une distribution de probabilité (additive) cohérente avec les choix des décideurs. Ceci n'est pourtant pas anodin et c'est pourquoi nous avons conduit une étude expérimentale visant à tester cette hypothèse ainsi que d'autres propriétés que nous allons à présent décrire.

0.5.3. Méthode et buts de l'expérience

L'expérience a été conduite de mars à mai 2005 ; 52 élèves de l'Ecole Normal Supérieure de Cachan y ont participé. Nous pouvons présenter 4 principaux objectifs de cette étude expérimentale.

Faisabilité et Fiabilité : Si l'expérience décrite dans la section 0.4 nous suggère que la méthode d'échangeabilité est applicable, il est positif de confirmer cette faisabilité par une nouvelle étude. Les sources utilisées étaient la température à Paris, le cours Euro contre Dollar et la variation journalière du CAC40. Il était indiqué au sujet que les valeurs seraient relevées 4 semaines après leur participation. Nous avons donc élicité trois distributions de probabilité subjectives par sujet en obtenant pour chacune des trois sources une partition en 8 événements échangeables. En outre, nous avons testé la fiabilité des réponses en répétant certaines questions à la fin de l'expérience.

Prédictibilité : Est-il possible de prédire à partir de la distribution de probabilité obtenue de nouveaux choix du sujet ? Nous avons pour cela estimé pour chaque sujet deux événements censés avoir une probabilité de 1/3 et nous avons testé leur échangeabilité.

Additivité et Effet de Découpage : De nombreuses études en psychologie ont révélé que les croyances (estimées par des probabilités jugées) des individus ne sont pas additives ; ainsi, Tversky & Koehler (1994) ont proposé d'expliquer la sous-additivité des probabilités jugées par leur Théorie du Support, c'est-à-dire, la croyance dépend de la présentation de l'événement, plus ou moins précise. Ainsi un événement plus général peut avoir une probabilité jugée plus petite que la somme des probabilités de chacun des sous-événements précis qui le composent. Dans la même logique, Starmer & Sugden (1993) et Humphrey (1995) ont introduit le concept d'*effet de découpage des événements*[2] *(EDE)* défini par le fait que deux événements incompatibles sont plus attractifs dans les paris que leur réu-

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[2] appelé par les auteurs « Event-Splitting Effect ».

nion. Puisque nous faisons ex ante l'hypothèse d'additivité, il est nécessaire de tester la validité de ce type de phénomène qui peut la contredire.

Dépendance à la source : Puisque c'est l'attitude face à l'ambiguïté (l'attitude spécifique à une source donnée) qui semblait être mieux distinguée des croyances par la méthode d'échangeabilité que par les autres techniques d'élicitation des probabilités subjectives, nous avons testé cette hypothèse en proposant aux sujets des paris sur des événements des différentes sources mais pour lesquels nous avions déterminé qu'ils affectaient une même probabilité subjective. Nous proposions aussi une loterie avec les mêmes conséquences et la même probabilité de gain mais cette fois-ci objective.

0.5.4. Résultats

Nous avons obtenu 156 distributions de probabilité subjective, avec un résultat intéressant en termes de fiabilité. Lors de la répétition de certaines questions déjà posées, nous avons pu mesurer la constance des choix des sujets. Or celle-ci est fortement influencée par la proximité (ou non) avec ce que nous avons déterminé comme étant l'indifférence. En d'autres termes, lorsque la question répétée demande de choisir entre deux événements préalablement déterminés comme échangeables, alors la seconde réponse ne coïncide avec la première que dans un peu plus de 50% des cas. Ceci correspond d'ailleurs à la notion d'indifférence, où l'agent est censé choisir équivalemment une option ou l'autre. Par contre, le nombre de réponses identiques augmente fortement dès que nous nous éloignons de l'indifférence avec par exemple 100% de réponses identiques lorsque pour la température, les événements proposés diffèrent de plus de 2°C des événements échangeables.

Les tests de prédictibilité n'ont pas conduit à rejeter l'hypothèse que les événements estimés comme échangeables l'étaient réellement (les probabilités de rejet des tests de Student appariés étant respectivement de 0.4957, 0.2356 et 0.3409 pour la température, le cours de l'euro et le CAC40).

Pour tester l'EDE, en reprenant la notation du schéma 0.4.2, nous avons cherché à voir si $A_4^1 \cup A_4^4$ et $A_4^2 \cup A_4^3$ étaient bien échangeables. En effet chaque événement est l'union de deux événements incompatibles de probabilité 1/4, mais le premier événement n'est pas convexe, il comprend deux parties, alors que le second est convexe et peut être représenté 'en un bloc'. L'EDE doit alors jouer en faveur du premier événement, sa non-convexité le rendant plus attirant. C'est en effet la tendance que nous avons trouvé même si l'EDE n'était significatif à 5% (par des tests de Student appariés) que pour le cours de l'Euro. Cela renvoie donc à une limite potentielle de la méthode, où la description des événements compte beaucoup. Notons toutefois que lors de l'élicitation même de la distribution, seule l'échangeabilité d'événements convexes est utilisée.

Enfin, il s'agissait de parier sur quatre événements, le premier avec une probabilité objective de 1/4, les autres avec une probabilité subjective identiquement de 1/4 et provenant des trois sources. Chaque sujet devait indiquer un ordre de préférence entre les quatre possibilités, puis recommencer avec une probabilité de 1/2 et enfin de 7/8 (identique pour les 4 événements à chaque fois). Pour chaque classement, un test de Friedman comparant les rangs respectifs des différentes sources et du risque rejette l'hypothèse nulle d'indifférence entre les sources pour une significativité de 5%. Ceci confirme que l'attitude dépend de la source. En regardant la position du risque par rapport aux sources incertaines, les résultats suggèrent une attitude face à l'ambiguïté représentée par le graphique 0.4.1.d pour la température et 0.4.1.c pour le cours de l'euro et le CAC40.

0.5.5. Conclusion de l'étude

Nous devons rappeler ici une limite inhérente aux techniques d'élicitation basée sur les choix (limite qui est valable pour les 4 techniques décrites dans cette section). Si l'agent voit en l'événement une utilité intrinsèque, ou s'il a des intérêts propres à sa réalisation, alors ces techniques peuvent être biaisées. Ainsi un vendeur de glace peut préférer un acte qui lui donne de l'argent s'il pleut un jour donné, car cet acte fonctionnera comme une sorte d'assurance, couvrant son manque-à-gagner lié au temps. Il est donc nécessaire de vérifier que l'agent dont on veut récolter les préférences évalue les conséquences de manière indépendante des événements et n'a pas d'intérêts particuliers à la survenance d'un événement.

En outre, il doit être noté que contrairement aux trois autres techniques, la méthode d'échangeabilité nécessite de pouvoir découper l'ensemble des états de la nature facilement, prélevant une partie d'un événement et le greffant sur un autre afin d'obtenir leur échangeabilité. Les trois autres techniques partent des événements et recherchent leur probabilité. La méthode de l'échangeabilité part d'un niveau de partition (2, 4, 8…) et détermine les événements. Comment faire si l'ensemble des états de la nature ne comprend que « il pleut » et « il ne pleut pas » et si ces deux événements ne sont pas échangeables ? Comment obtenir une partition en événements échangeables ? Une possibilité reviendrait à composer cet ensemble des états de la nature avec un mécanisme externe générant uniformément des nombres appartenant à [0,1). Ainsi les événements deviendraient par exemple « il pleut et le nombre tiré appartient à [0,p) ». Modifier p permettrait de faire varier la vraisemblance de l'événement afin d'atteindre l'échangeabilité. L'étude de la faisabilité d'un tel mécanisme est laissée pour des recherches futures.

Le principal but du chapitre 5 était de comparer la robustesse théorique des différentes méthodes d'élicitation des probabilités subjectives dans le cadre d'un modèle qui prend en compte les déviations comportementales par rapport à la maximisation de l'espérance de gain. Dans ce cadre, nous avons observé que la méthode d'échangeabilité semblait plus efficace pour séparer l'attitude face à l'ambiguïté des croyances. Ensuite une étude expérimentale visait à tester en pratique cette méthode et à la confronter avec ces points faibles potentiels, tel l'effet de découpage des événements. Nous avons pu établir ainsi parallèlement à ses avantages, l'existence de limites à son utilisation.

0.6. Conclusion générale

Le but principal de la thèse a été de fournir des éléments, théoriques et expérimentaux, qui permettent de comprendre, d'observer et de mesurer à la fois les attitudes face à l'incertitude mais aussi les croyances des décideurs lorsque les probabilités ne sont pas connues. Les modèles utilisés n'ont pas de vocation normative en ce sens qu'ils violent différents principes de rationalité. Pour cela, l'utilité espérée reste le modèle de référence. Toutefois, ces modèles permettent de comprendre et de représenter de manière plus exacte les décisions. Le chapitre 6 expose en outre des limites potentielles de ces modèles en soulevant quelques récents paradoxes suggérés par la littérature (Birnbaum 2007, Machina 2007). Enfin sont suggérées quelques pistes de recherche future, visant à compléter les résultats de la thèse, comme des améliorations de la méthode d'élicitation des probabilités subjectives ou encore l'analyse de l'effet de nouvelles informations, effet autant sur les croyances (sont-elles révisées de manière Bayésienne ?) que sur l'attitude (l'information augmente-t-elle la confiance du décideur dans ses croyances ?).

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Chapter 1.
Introduction

"I will suggest, as does Ellsberg, that subjective probability judgments relating to various processes are not strictly comparable."

Fellner (1961)

1.1. Introductory examples

Imagine you have to choose between a bet that gives you €1000 with probability 1/2 (and nothing otherwise) and a sure gain of €500: if you prefer the bet to its expected value, then you like taking risk (at least in this situation), you are a *risk seeker*. If you prefer the sure gain, then you dislike risk and you are *risk averse*. *Risk neutrality* is defined as being indifferent between getting a lottery and receiving its expected value for sure. This simple choice shows us that observing preferences can enable us to determine an agent's attitude. It then makes it possible to study whether, for instance, attitude towards risk depends on probabilities or not. But in most real life situations we do not know the exact probabilities of the risks we are facing; we do not often base our decisions on a precise description of the likelihood of each consequence.

Consequently, let us replicate the previous choice situation in a context in which probabilities are not clearly known. You have now to choose between winning €500 for sure and €1000 if it rains in Paris tomorrow (and nothing otherwise). Let us assume that you prefer €500 for sure: does it mean that you are risk averse or that you simply think that it will not rain (or at least that the probability of rain is quite low)? Analyzing attitudes when probabilities are unknown generates new issues: how can we distinguish between beliefs ("I think that the probability of rain is p") and the attitude of the agent ("I dislike taking risk")?

We could think that a way of deriving beliefs from choices would consist in comparing the two previous bets, i.e. winning €1000 with probability 1/2 and winning €1000 if it rains tomorrow in Paris. But let us assume that you prefer the probability bet: does it mean that you think the probability of rain is less than 1/2 or that you believe rain is at least as possible as no rain but you do not like betting when you are not totally sure of the probability? Most people prefer bets on known probabilities to bets on vague or unknown ones; we will see in subsections 1.2.3 and 1.4.2 below that this phenomenon was highlighted by Ellsberg (1961). As a consequence, these three examples reveal that when probabilities are unknown, choices come from a complex mixture of beliefs, attitudes towards risk but also attitudes towards the knowledge about the risk we encounter. This constitutes the core issue of this dissertation: how can we distinguish between beliefs (subjective probabilities), risk attitude (the taste for taking risk) and ambiguity attitude (the impact of not knowing precisely the risk).

Of course, the examples we described were only used so as to reveal the intuitions underlying our topic. However, this reasoning can easily be applied to real-life examples as in the case with global warming, for instance. Let us seek advice from two groups of experts found on Wikipedia[3]. The first one says that there is a *"90 percent certainty that global warming is caused by man's burning of fossil fuels"*. The second group of experts tells us that *"Detailed examination of current climate data strongly suggests that current observations do not correlate with the*

[3] See http://en.wikipedia.org/wiki/Scientific_opinion_on_climate_change (June 2007)

assumptions or supportable projections of human-induced greenhouse effects". This conflicting set of opinions may make us hesitate: Should we apply the Kyoto protocol? Should we decrease our consumption of fossil fuels? Are ecological policies for decreasing our carbon dioxide emission really useful? When we have to base our decisions on experts' opinions, such disagreements are common and we have to face conflicting or imprecise descriptions of the situations. In this dissertation, we will report the results of an experimental study that describes how experts' judgments are combined and, consequently, what determines our decision in those situations.

As a conclusion to these examples, we hope that the main issues of studying individual decisions under uncertainty (or at least the intuitions behind these issues) appear more transparent for the reader. Obtaining beliefs and describing attitudes will constitute the main topics of this work. But before presenting our results, let us introduce explicitly what uncertainty is, how it is modeled through probability, and the limitations of this model. Section 1.2 thus presents the concept uncertainty as it appears in the literature. Then, section 1.3 describes the main model and section 1.4 its limits. Section 1.5 and 1.6 are dedicated to new models of uncertainty. Section 1.7 concludes.

1.2. Risk, uncertainty and ambiguity

Knight (1921) clearly distinguishes between risk or "*measurable uncertainty"*, in which probabilities exist and *"unmeasurable uncertainty"* also simply called *"uncertainty"*. Since this first distinction, the notions of uncertainty, risk and even ambiguity have been abundantly debated. The following subsections will provide a basic topography of the notion of uncertainty taken, in a very broad sense, as everything that is not sure, everything that we cannot assert with probability 0 or 1.

1.2.1. Uncertainty and 'objective probabilities'

In his *Liber de Ludo Aleae* (1663, but written a century before), Jerome Cardan[4] is the first to define probability as a ratio of favorable elementary events on possible results. Nowadays, we use probabilities as a measure of uncertainty in many decisions and the standard frequentist viewpoint assimilates probabilities with frequencies for future events. According to Poirier (1988) *"frequentists interpret probability as a property of the external world, i.e. the limiting relative frequency of the occurrence of an event as the number of suitably defined trials goes to infinity"*. Those probabilities are often seen as "objective": for instance the probability of a ball stopping on a given number in a roulette game is 1 (one favorable elementary event, the given number) over 37 (possible numbers).

Uncertainty, in which probabilities are known, clearly belongs to the Knightian "measurable uncertainty", risk. But does it exist? *"Probability is always a subjective notion, inasmuch as it is the measure of uncertainty felt by a given person facing a given event. 'Objective probability' is a meaningless notion"* (de Finetti 1974). The next subsection is dedicated to this viewpoint.

1.2.2. Uncertainty and 'subjective probabilities'

Let us try to understand the subjective view of probabilities. Subjectivists think that probability is above all a degree of belief, measuring a person's knowledge about an event. Jacob Bernoulli (1713) firstly introduces this subjective view in his concept of subjective degree of certainty. But his subjectivism is related to the fact that for him, everything that will occur is objectively certain, determined by God (Hacking 1971). The subjectivism of Ramsey, de Finetti and Savage is different in essence. According to them, there is neither such determinism, nor frequencies that would be probabilities: *"We are never entitled to predict future frequencies with certainty, […] since that would only be legitimate under*

[4] See Bernstein (1996) for the history of risk and probability.

some deterministic hypothesis. If we accepted such a deterministic hypothesis, no question of probability would exist." (de Finetti 1974).

Probabilities are subjective because they depend on our knowledge and on our reasoning. For Borel (1924), Ramsey (1926) and de Finetti (1931), probabilities could be revealed by choices. Savage (1954) then provided an axiomatization (presented in section 1.3) of subjective probabilities based on preferences. He gives conditions under which decisions are based on subjective probabilities. Since it allows measuring uncertainty, this part of uncertainty on which there are subjective probabilities may be viewed as belonging to risk. For a rational man, whose preferences satisfy Savage's axioms, uncertainty can be reduced to risk. That is why Ellsberg (1961) asks: *"Are there uncertainties that are not risks?"*

1.2.3. Uncertainty without 'subjective probabilities'

Ellsberg proposes different choice situations, under which Savage's axioms are violated. We will extensively discuss them later but we can already explain the main one. Consider an urn, whose content is known (50 black balls and 50 red balls), and another, in which the proportion of red and black balls is unknown. Most people prefer betting on red (black) balls in the first urn to betting on red (black) balls in the unknown urn. This behavior suggests that drawing a red (black) ball from the first urn is more likely than drawing a red (black) ball in the second urn. Hence, subjective probabilities in the first urn are violated because their sum cannot be equal to one[5].

Ellsberg refers to this situation, in which probabilities are vaguely known or even unknown, as *ambiguous*. Uncertainty is then often divided into risk (in which there are objective or subjective probabilities) and ambiguity (in which no probability exists). However, this distinction is not obvious: most people do believe that red and black balls are equally likely in the unknown urn but they prefer the bet on

[5] P("The ball is red")<0.5 and P("The ball is black")<0.5 imply P("The ball is red or black")<1.

'sure' probabilities. Fellner (1961) suggests that "*subjective probability judgments relating to various processes are not strictly comparable"*. He does not deny the existence of probabilistic beliefs; he just explains *"distortion of probabilities as a reaction to uncertainty"*. This ambiguity may be high or not and may distort more or less subjective probabilities, which can still exist. In other words, Ellsberg's example may be equally interpreted either as a proof of distortion or as a proof of nonexistence of subjective probabilities.

We shall mention that preference-based definitions of ambiguity have also been developed, for instance in Epstein & Zhang (2001) or Ghirardato & Marinacci (2001). We will present them in subsection 1.6.3 and see how they may fail to clearly characterize ambiguity without error.

1.2.4. Radical uncertainty

Radical uncertainty corresponds to situations, in which we do not even know what can possibly happen. That is what Keynes calls uncertainty: *"The sense in which I am using the terms is that [...] there is no scientific basis on which to form any calculable probability whatever. We simply do not know."* (Keynes, 1936). Post-Keynesians think that this radical uncertainty corresponds to the *"true uncertainty"*. Under radical uncertainty, decision makers may prefer not to decide or may *"follows their 'animal spirit'.[...] Post Keynesians believe that this behavior is sensible and understandable only in a world where uncertainty is distinguished from probabilistic concepts"* (Davidson, 1991).

1.2.5. Definitions

First, the current dissertation will not follow the Post-Keynesian approach, i.e. it will not deal with radical uncertainty, because our primary topic is the relation between uncertainty and probabilistic beliefs. Then, because ambiguity does not totally rule out probabilities (that may exist and be distorted), we need to clarify some definitions. The definitions we are proposing do not exactly match the current literature for the reason that the literature itself is controversial. We will denote by *risk* the cases in which *probabilities are 'known'*. In choice problems, this will correspond to *'given'* or *'objective' probabilities* (e.g. winning €x with probability p). Risk is a subset of *uncertainty*. In some other uncertain situations, *probabilities are 'unknown'* or *'vaguely known'*. These situations will sometimes be referred to as *ambiguity* (mostly in chapters 3 and 4) and they will be represented either by *sets* or *intervals of possible probabilities* (e.g. the probability of winning is between p and p') or by *events* (e.g. you can win if it rains). The following section presents the principal model of behavior under uncertainty and how subjective probability can be derived from preference.

1.3. Subjective Expected Utility

1.3.1. Introduction to Subjective Expected Utility

Uncertainty and risk being defined, we have now to understand how behaviors are modeled. We will refer throughout this section to a decision maker called 'you' and your *preference relation* over a set of choice objects (goods, lotteries, acts…) will be denoted by $\succcurlyeq$, as usual. For any two choice objects a and b, $a \succcurlyeq b$ means that a is at least as preferred as b. If not $a \succcurlyeq b$, we will write $b \succ a$. The case where $a \succcurlyeq b$ and $b \succcurlyeq a$ will be denoted by $a \sim b$. We will say that a real-valued function V *represents* an agent's preferences $\succcurlyeq$ if $a \succcurlyeq b$ is equivalent to $V(a) \geq V(b)$. We will sometimes use $\preccurlyeq$ and $\prec$ defined such that $a \succcurlyeq b \Leftrightarrow b \preccurlyeq a$ and $a \succ b \Leftrightarrow b \prec a$.

The first and most used representation under uncertainty is Savage's (1954) *Subjective Expected Utility (SEU)*. In order to introduce Savage's model, let us develop three key elements, the three 'pillars' of this model[6].

First Pillar (State Space): First of all, SEU is built on a *state space* framework: all the possible *states of the world* that can occur are known and only one such state will be true. Savage (1954) said that a state of the world is *"a descrip-*

[6] The presentation through those three key elements is inspired by Pr Simon Grant's lecture in Paris in May 2007.

tion of the world, leaving no relevant aspect undescribed". The state space S, also called *the universe* or *the world*, contains all the states of the world. An *event* is a subset of S, e.g. the event $E \subseteq S$. It is thus an element of the *set of all subsets of S,* denoted *2^S*. The *empty set* is denoted $\emptyset$. Depending on the state that occurs, you face a consequence that belongs to C, the *space of outcomes*. You must choose between *acts*, which are mapping from S to C. An *act* f associates a consequence $f(s) \in C$ to each s from S. In this chapter, we will always work on finite-outcome acts, that can be denoted $(E_1{:}x_1,\ldots,E_m{:}x_m)$ for some positive integer m where x_i is the outcome if E_i obtains. For a finite S of cardinality n and a given ordering of these n events, these acts can be equivalently written as members of C^n, e.g. $(x_1,\ldots,x_n)$. An act that matches with f on an event E and with g otherwise is denoted by fEg. The same rule can be used with several disjoint events, e.g. fEgFh gives f on E, g on F and h otherwise. *Constants acts* will be designated by their unique outcome. The preference relation $\succcurlyeq$ is defined over the set of such acts. An event E is *null* if you are indifferent between any two acts that only differ on the outcome on E; otherwise the event is *nonnull*. It is worth nothing that this way of describing the choice situation must not be neglected. It rules out radical uncertainty. You (as the decision maker) fully understand and fully know what can arrive; you have a complete description of the possible states of the world and you know that you know everything, i.e. you do not take into account that "something else" could occur. When uncertainty is modeled like that, there is no place for unforeseen contingencies (see Dekel et al. 1998).

Second Pillar ("Subjective Probability"): The second key element of SEU is constituted by *Probabilistic Sophistication*: you are said to be probabilistically sophisticated if your behavior is consistent with a subjective probability distribution over outcomes. Formally, a *probability measure* is a function P defined over 2^S that associates $P(E) \in [0,1]$ to each event E with $P(S)=1$ and for all events E and F such that $E \cap F = \emptyset$, $P(E \cup F) = P(E) + P(F)$. For any act f, we can define the probability measure over 2^C that is induced by P, i.e. the function P_f such that $P_f(C') = P(\{s : f(s) \in C'\})$ for all subset of consequences $C' \subseteq C$. Note that P_f gives the subjective probability distribution over outcomes associated with act f. Probabilistic sophistication holds if there exists a probability measure P over S such that for all acts f and g, $P_f=P_g$ implies f~g. The probability distribution P is often described as representing beliefs but what your choices reveal may not match your conscious thoughts. In addition, what your choices reveal may also not correspond to additive probabilities. In fact, allowing for non-additivity is a generalization of this model (see subsection 1.5.2).

Third Pillar ("Utility"): Eventually, you are supposed to be an *expected utility* maximizer with respect to your subjective probabilities. Let us first study the simple situation, when probabilities are known. Under risk, a choice object is a *lottery* $(p_1{:}x_1,\ldots,p_n{:}x_n)$ with $p_i\in[0,1]$ for all i from 1 to n and with the p_is summing to 1. Imagine for instance that you can win €2^n if the first tail of a coin appears at the n^{th} toss. Since the probability of the event "tail at the n^{th} toss" is $1/2^n$, the expected value of this lottery is infinite. But nobody would agree to pay more than a few Euros to play. This is known as the St Petersburg paradox. Daniel Bernoulli (1738) argues that agents do not maximize their expected gain but focus on the satisfaction or the *'utility'* they can expect from their choices. Consequently, the representation of the preference relation over the lottery set comes $(p_1{:}x_1,\ldots,p_n{:}x_n)\longmapsto\sum_{i=1}^{n}p_i\,u(x_i)$ where u is the *utility function* defined over the outcome space.

Von Neumann & Morgenstern (1944) present the first axiomatization of this model. With L the set of finite-outcome lotteries (lotteries that can be written $\ell=(p_1{:}x_1,\ldots,p_n{:}x_n)$ for some n finite) over an outcome set X, *Expected Utility* is equivalent to the following 3 axioms:

AXIOM A1 *(Weak Ordering):* the preference relation $\succcurlyeq^l$ over L is transitive and complete

AXIOM A2 *(Independence):* for all lotteries ℓ,ℓ',ℓ'', and $\alpha\in[0,1]$, $\ell\succcurlyeq^l\ell'$ implies $\alpha\ell+(1-\alpha)\ell''\succcurlyeq^l\alpha\ell'+(1-\alpha)\ell''$

AXIOM A3 *(Jensen Continuity):* for all lotteries ℓ,ℓ',ℓ'', if $\ell\succ^l\ell'$ then there exist $\alpha,\beta\in(0,1)$ such that $\alpha\ell+(1-\alpha)\ell''\succ^l\ell'$ and $\ell\succ^l\beta\ell'+(1-\beta)\ell''$.

Moreover, the utility function is unique up to an affine transformation.

Combining the state space framework, probabilistic sophistication and expected utility enables us to obtain subjective expected utility. SEU holds if there exist a utility function u (unique up to an affine transformation) and a subjective probability measure P such that f≽g if and only if the expectation of u(f(.)) using P is higher than the expectation of u(g(.)) (using the same probability measure P). Savage (1954) provides axioms that are necessary and sufficient for SEU. Let us now go through these axioms.

1.3.2. Savage's axiomatization of SEU

We are going to describe each axiom and its usefulness, focusing on the existence of probabilities. This subsection is based on Savage (1954) of course, but also on the presentations by Fishburn (1970) and Machina & Schmeidler (1992).

AXIOM P1 *(Weak Ordering):* ≽ is a *weak order*, i.e. it is transitive and complete.

This axiom is obvious and widely used in the literature because it is necessary for the representation theorem. If there exists a real-valued function V that represents ≽ (i.e. f≽g iff V(f)≥V(g)) then ≽ must be a weak order. Indeed, assume that V represents ≽ and take any two acts f and g. Among the real numbers V(f) and V(g), three situations may occur: V(f)≥V(g), V(g)≥V(f) or both. As a consequence, we must have f≽g or g≽f or both, which means that ≽ is complete. Assume that for any three acts f, g and h, f≽g and g≽h. Consequently, V(f)≥V(g) and V(g)≥V(h) and therefore V(f)≥V(h). This is equivalent to f≽h: transitivity holds.

AXIOM P2 *(Sure-Thing Principle):* For all events E and acts f, g, h and k, fEg≽kEg implies fEh≽kEh.

This second axiom is really intuitive and may look innocuous but in fact, it constitutes a weak point that is violated in some famous paradoxes (see section 1.4 below). It simply says that what is sure does not matter for you. Choosing between fEg and kEg means that whatever you choose, you will have g when E does not occur. If you prefer fEg to kEg, this should mean that when E occurs, you prefer f to k. As a consequence, remark that for the two new acts fEh and kEh, you are also sure to obtain h when E does not occur and if you are consistent, if the "Sure-Thing Principle" applies, you should still prefer f to k on E, and thus fEg≽kEg should imply fEh≽kEh. This axiom implies that your choices are independent from what is sure.

AXIOM P3 *(Eventwise Monotonicity):* For all outcomes x and y, non-null events E and act g(.), x≽y iff xEg≽yEg.

Assume you prefer outcome x to outcome y (for sure) and consider an act g and an event E; if you can choose between xEg and yEg, that only differ on E, it seems rational to prefer xEg. Eventwise monotonicity means that tastes for outcomes remain constant no matter the event they are associated to.

AXIOM P4 *(Weak Comparative Probability):* For all events A, B and outcomes x'≻x and y'≻y, x'Ax≽x'Bx implies y'Ay≽y'By.

Imagine that we want to know if you think that an event A is more likely than an event B. We know that you like money and hence, that you prefer winning €20 to nothing; thanks to P3, we also know that your tastes are not influenced by events. If you prefer 20A0 to 20B0, we can infer that A is more likely than B according to your choice. But if we want this technique to be reliable, we need to be sure that your decision will not change if we propose for instance €30 instead of €20 or €2 instead of nothing (i.e. you still prefer 30A2 to 30B2): if that is the case, your choice still reveals the same beliefs. This axiom ensures that we can derive your belief from your choices without any contradiction and that we can infer an ordering of events in terms of likelihood. But we have to be sure that you like at least one outcome more than another:

AXIOM P5 *(Nondegeneracy):* There exist outcomes x and y such that x≻y.

We cannot infer anything about your beliefs if you do not care about the rewards that we are proposing to you. That is why nondegeneracy is so important.

Axioms from P1 to P5 enable us to define a *qualitative probability*. It is a binary relation $\succcurlyeq^P$ on events that satisfies the following four properties (for all events A, B and $C \subseteq S$):

Weak Ordering; (1.3.1)

$S \succ^P \emptyset$; (1.3.2)

$A \succcurlyeq^P \emptyset$; (1.3.3)

If $A \cap C = B \cap C = \emptyset$, then $A \succcurlyeq^P B$ iff $A \cup C \succcurlyeq^P B \cup C$. (1.3.4)

As we suggested when we commented P4, the qualitative probability $\succcurlyeq^P$ will be defined using comparison between simple bets: for some events A and B, A is revealed more likely than B or equivalently, $A \succcurlyeq^P B$ whenever there exists some outcomes $x \succ y$ such that $xAy \succcurlyeq xBy$. $A \succcurlyeq^P B$ and not $B \succcurlyeq^P A$ define $A \succ^P B$. $A \succcurlyeq^P B$ and $B \succcurlyeq^P A$ defines $A \sim^P B$. Thanks to P5, we know that there is at least one outcome that is strictly preferred to another (let us call them $\bar{x}$ and $\underline{x}$ respectively). P4 ensures that we cannot find inconsistencies depending on the outcomes we used: it is thus sufficient to determine the qualitative probability for the two outcomes, between which you are not indifferent. Hence, for all events $A,B \subseteq S$, $\bar{x}A\underline{x} \succcurlyeq \bar{x}B\underline{x} \Leftrightarrow A \succcurlyeq^P B$. Completeness of $\succcurlyeq$ implies that we can always compare $\bar{x}A\underline{x}$ and $\bar{x}B\underline{x}$ for all $A,B \subseteq S$ and transitivity of the preference relation implies that $A \succcurlyeq^P B$ & $B \succcurlyeq^P C \Rightarrow \bar{x}A\underline{x} \succcurlyeq \bar{x}B\underline{x}$ & $\bar{x}B\underline{x} \succcurlyeq \bar{x}C\underline{x} \Rightarrow \bar{x}A\underline{x} \succcurlyeq \bar{x}C\underline{x} \Rightarrow A \succcurlyeq^P C$. Under P4 and P5, P1 implies weak ordering of $\succcurlyeq^P$ (Eq. 1.3.1). By definition of constant acts, $\bar{x} \succ \underline{x}$ means $\bar{x}S\underline{x} \succ \bar{x}\emptyset\underline{x}$ and thus not $\emptyset \succcurlyeq S$. Eq. 1.3.2 follows from this result and from completeness. For all $A \subseteq S$, if A is null then by definition $\bar{x}A\underline{x} \sim \underline{x}$ or equivalently, $\bar{x}A\underline{x} \sim \bar{x}\emptyset\underline{x}$. Therefore, $A \succcurlyeq \emptyset$. If A is not null, P3 implies $\bar{x}A\underline{x} \succcurlyeq \underline{x}A\underline{x}$, which can be rewritten $\bar{x}A\underline{x} \succcurlyeq \bar{x}\emptyset\underline{x}$. Eq. 1.3.3 follows.

Last we are going to see that P2 is central in the elaboration of Eq. 1.3.4. Let us note first that this last condition is the qualitative equivalent to additivity of a probability measure. Secondly, let us remark that we are going to use this axiom only on two-outcome acts because we defined the qualitative probability only on this family of simple bets. Assume first that for some A, B and $C \subseteq S$, $A \cap C = B \cap C = \emptyset$

and $A \succcurlyeq^P B$. We have thus $\bar{x}A\underline{x} \succcurlyeq \bar{x}B\underline{x}$. Note that, if C occurs, you are sure to get $\underline{x}$. The Sure-Thing Principle tells us that your preference should be unchanged if you are now sure of getting $\bar{x}$ if C occurs. Indeed, P2 implies

$$\bar{x}A\underline{x} \succcurlyeq \bar{x}B\underline{x} \quad \Leftrightarrow \quad \bar{x}A{\cup}C\underline{x} \succcurlyeq \bar{x}B{\cup}C\underline{x}$$

and thus,

$$A \succcurlyeq^P B \quad \Leftrightarrow \quad A \cup C \succcurlyeq^P B \cup C.$$

However, a qualitative probability is not sufficient for defining a unique probability measure. Assume that $2^S=\{\emptyset,A,S-A,S\}$ and that $S \succ^P A$, $A \succcurlyeq^P S-A$ and $S-A \succcurlyeq^P \emptyset$. $\succcurlyeq^P$ is a qualitative probability but there exist an infinite number of probability measures that agree with $\succcurlyeq^P$: P(A) may take any value in [0.5,1) because this is sufficient to ensure that $0 \leq 1-P(A) \leq P(A) < 1$. It however suffices to add some richness conditions on the state space to obtain a unique probability measure that agrees with the qualitative probability. The next axiom is dedicated to this purpose.

AXIOM P6 *(Small Event Continuity):* For any acts $f \succ g$ and outcome x, we can find a finite partition $\{A_1,\ldots,A_n\}$ of S such that for all $i,j \in \{1,\ldots,n\}$, $f \succ xA_ig$ and $xA_jf \succ g$

P6 means that we can always find very small events such that even if we modify an act in order to put the most (least) preferred outcome on it, you will not change your preferences. Take an event B such that $B \succ^P \emptyset$. It means that $\bar{x}B\underline{x} \succ \underline{x}$. We can find a partition $\{A_1,\ldots,A_n\}$ of S such that for all i, $\bar{x}B\underline{x} \succ \bar{x}A_i\underline{x}$ and thus such that $B \succcurlyeq^P A_i$. Savage says that the qualitative probability is *"fine"* when this condition is fulfilled. It is also *"tight"* when it satisfies the following condition for all events B and C: if for all $E \succ^P \emptyset$ and $F \succ^P \emptyset$ such that $B \cap E = C \cap F = \emptyset$, $B \cup E \succcurlyeq^P C$ and $C \cup F \succcurlyeq^P B$, then $B \sim^P C$. Savage shows that P6 induces tightness. Hence, $\succcurlyeq^P$ is a fine and tight qualitative probability. This is sufficient for the existence of a subjective probability distribution (see Niiniluoto 1972 and Wakker 1981 for the proof).

Moreover, P3 and P2 separate outcomes from events and events between themselves. They play a major role in the construction of the utility function. This result can be proved in two steps: first by showing that the agent is indifferent between any two acts that generate the same probability measure over outcomes and then by showing that von Neumann & Morgenstern's (1944) axioms are satisfied on these probability distributions over outcomes. (Note that A1 is straightforward but A2 and A3 are harder to obtain).

That is how we get the following result: under P1-P6, there exist a probability measure P and a utility function u such that any finite-outcome act f is represented by $V(f)=\sum_{i=1}^{n} P\big(f^{-1}(x_i)\big)u(x_i)$ where $\{x_1,\ldots,x_n\}$ is the outcome set of f. Savage gives also an axiom P7 that enables to deal with acts with an infinite number of outcomes.

The two next subsections will be dedicated to other axiomatizations of SEU that will slightly change the framework. The first one deals with acts that are mapped from a (finite) state space to a set of lotteries instead of sure consequence. The second will still use Savage's acts but will transfer the proof from the event domain to the outcome domain by replacing the richness of the state space by the richness of the outcome set.

1.3.3. Anscombe & Aumann's axiomatization

Anscombe & Aumann (1963) propose a simple derivation of SEU from expected utility under risk. By using acts that map states to lotteries instead of general outcomes, they allow for mixture of acts that were not possible for general outcomes. The presentation that follows is inspired from Fishburn (1970) and Kreps (1988). Let us consider a *finite state space* $S=\{1,\ldots,n\}$ and the set L of simple lotteries over an outcome set X. Acts are mappings from S to L: with $\ell_1,\ldots,\ell_n \in L$, $f:S \longrightarrow L$ can be written $(\ell_1,\ldots,\ell_n)$. It gives lottery ℓ_i if state i obtains. $\mathfrak{F}$ is the set of such maps.

When $f=(\ell_1,\ldots,\ell_n)$ and $g=(\ell'_1,\ldots,\ell'_n)$ and $\alpha \in [0,1]$, a mixed act $\alpha f+(1-\alpha)g$ is defined by $(\alpha f+(1-\alpha)g)(i)= \alpha\ell_i+(1-\alpha)\ell'_i$. Let us now reuse von Neumann & Morgenstern axioms on $\mathfrak{F}$ (instead of using them on L):

AXIOM AA1 *(Weak Ordering):* the preference relation $\succcurlyeq$ over $\mathfrak{F}$ is transitive and complete

AXIOM AA2 *(Independence):* for all acts f, g, h$\in\mathfrak{F}$ and $\alpha\in[0,1]$, f$\succcurlyeq$g implies $\alpha f+(1-\alpha)h\succcurlyeq\alpha g+(1-\alpha)h$

AXIOM AA3 *(Jensen Continuity):* for all acts f, g, h$\in\mathfrak{F}$, if f$\succ$g then there exist $\alpha,\beta\in(0,1)$ such that $\alpha f+(1-\alpha)h\succ g$ and $f\succ\beta g+(1-\beta)h$.

These axioms are equivalent to expected utility under each state. However, there is no reason for utility to be constant across states. This corresponds to *state-dependent preferences*, whose representation is: $f\longmapsto\sum_{s\in S}U_s(\ell_s)$ where $f=(\ell_1,\dots,\ell_n)$ and U_s is an expected value function that depends on state s. As a consequence, if we want you to have a state-independent utility, we need an axiom that says that if you prefer a lottery ℓ to another lottery ℓ' when state i obtains (what you get when other states occur remaining constant), you must also have the same preference between ℓ and ℓ' when another state j (j≠i) occurs. Axiom AA5 achieves this goal and axiom AA4 ensures that at least one state is not null (recall that a state s is non-null when you are not indifferent between at least two acts that differ only on s).

AXIOM AA4 *(Nondegeneracy):* there exist f and g from $\mathfrak{F}$ such that f$\succ$g.

AXIOM AA5 *(State-Independence):* If $f=(\ell_1,\dots,\ell_n)$and $g=(\ell'_1,\dots,\ell'_n)$ from $\mathfrak{F}$ and $\mathfrak{H}$, $\mathfrak{H}'$ from L are such that

$$(\ell_1,\dots,\ell_{s-1},\mathfrak{H},\ell_{s+1},\dots,\ell_n)\succ(\ell'_1,\dots,\ell'_{s-1},\mathfrak{H}',\ell'_{s+1},\dots,\ell'_n)$$

for some s, then for all non-null s'$\in$S

$$(\ell_1,\dots,\ell_{s'-1},\mathfrak{H},\ell_{s'+1},\dots,\ell_n)\succ(\ell'_1,\dots,\ell'_{s'-1},\mathfrak{H}',\ell'_{s'+1},\dots,\ell'_n)$$

AA1-AA5 holds if and only if SEU holds, i.e. $\forall f\in\mathfrak{F}$, $f\longmapsto\sum_{s\in S}P(s)U(\ell_s)$. It is noteworthy that probabilities are generated thanks to the hypothesis of expected utility over lotteries (implied by A1-A3 over constant acts), which is replicated over acts and not anymore by the existence of a fine and tight qualitative probability. The last axiomatization we are presenting directly uses the richness of the outcome set (and not the richness of a set of lotteries over the outcome set).

1.3.4. Another axiomatization with richness of outcomes

The following result comes from Köbberling & Wakker (2003, 2004). This result will be used in chapter 2; that is why we want to present it now. For simplicity, let us restrict our presentation to a simple case: the state space[7] $S=\{1,\ldots,n\}$ has n elementary states of the world and the outcome set is now $\mathbb{R}$. Acts are maps from S to $\mathbb{R}$. They will be denoted by $f=(x_1,\ldots,x_n)\in\mathbb{R}^n$ (where x_i is the outcome on state i) and the binary act that gives outcome x on event E, a subset of $\{1,\ldots,n\}$, and y otherwise will be still denoted by xEy. Constant acts will be still denoted by their outcome.

The preference relation $\succcurlyeq$ needs to satisfy weak ordering since we know that weak ordering is implied by the existence of a representation. Let us assume nondegeneracy: for some event E and some outcomes x and y, $x\succ xEy\succ y$. This avoids the partition of S containing only null events and S. We then assume *continuity*: for all act f, $\{g: g\succcurlyeq f\}$ and $\{g: f\succcurlyeq g\}$ are closed subsets of $\mathbb{R}^n$. It prevents small changes of outcomes from creating large changes in preferences. Under the previous assumptions and assuming SEU, continuity of $\succcurlyeq$ is equivalent to continuity of the utility function. Eventually $\succcurlyeq$ also verifies *monotonicity*: if for all i, $x_i\geq y_i$ then $(x_1,\ldots,x_n)\succcurlyeq(y_1,\ldots,y_n)$ and if for all i, $x_i>y_i$ then $(x_1,\ldots,x_n)\succ(y_1,\ldots,y_n)$. It simply means that the more you have, the happier you are.

The previous assumptions seem technical and are not really controversial. We have thus defined a framework such that we can derive expected utility from a simple axiom: Tradeoff consistency. Let us first define a *tradeoff (TO) relation* $\sim^*$. For outcomes x, y, x' and y', we will write that $xy\sim^*x'y'$ whenever we can find an

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[7] A generalization of Köbberling & Wakker's (2003, 2004) result to general acts over a finite or infinite state space is provided in chapter 2.

event E and two acts f and g such that xEf~yEg and x'Ef~y'Eg. This means that going from y to x is worth going from y' to x'.

AXIOM *(TO consistency):* Any change in a ~* breaks the relationship

i.e. xEf~yEg & x'Ef~y'Eg & xFh~yFk ⟹ x'Fh~y'Fk

This axiom says that if we found that going from y to x or from y' to x' on E compensates having f instead of g when E does not occur, then we must not found any other cases, in which going from y to x is the same tradeoff as going from y' to x'' where x'' is different from x'. If the tradeoff xy is worth x'y', then it must not be worth x''y' (or x'y'') for some other x'' or y''. Under the previous assumptions, TO-consistency is equivalent to SEU (with u, a continuous and strictly increasing utility function and P, an additive probability measure). Let us give a sketch of the proof when n=2.

Assuming SEU, xy~*x'y' and xy~*x'y''. These TO-relationships imply that for some events G, F∈{1,2} and some outcomes z, z', t and t', xGz~yGt, x'Gz~y'Gt, xFz'~yFt' and x'Fz'~y''Ft'. S and ∅ are neglected because monotonicity implies that TO-consistency is trivially satisfied. Under subjective expected utility, these indifferences imply respectively:

$$P(G)u(x)+(1-P(G))u(z)=P(G)u(y)+(1-P(G))u(t) \quad (1.3.5)$$

$$P(G)u(x')+(1-P(G))u(z)=P(G)u(y')+(1-P(G))u(t) \quad (1.3.6)$$

$$P(F)u(x)+(1-P(F))u(z')=P(F)u(y)+(1-P(F))u(t') \quad (1.3.7)$$

$$P(F)u(x')+(1-P(F))u(z')=P(F)u(y'')+(1-P(F))u(t') \quad (1.3.8)$$

Recall that G and F are not ∅ and that under SEU, nondegeneracy implies that each state has a non-null probability.

Eq. 1.3.5 & Eq. 1.3.6=>$u(x)-u(y)=u(x')-u(y')=(1-P(G))/P(G)(u(z)-u(t))$

Eq. 1.3.7 & Eq. 1.3.8=>$u(x)-u(y)=u(x')-u(y'')=(1-P(F))/P(F)(u(z')-u(t'))$

Thus, we obtain u(y')=u(y'') and thus y'=y'' because u is a strictly increasing function. SEU implies TO-consistency.

Let us now deal with the most difficult part of the proof: how does TO-consistency imply SEU? To understand the mechanism behind the proof, remark that TO-consistency implies a weaker axiom:

AXIOM *(Hexagonal condition):* For all real numbers x_0,y_0,x_1,y_1,x_2,y_2,
$(x_0,y_1)\sim(x_1,y_0)$ & $(x_0,y_2)\sim(x_1,y_1)$ & $(x_1,y_1)\sim(x_2,y_0) \Rightarrow (x_1,y_2)\sim(x_2,y_1)$

When n=2, TO-consistency can be rewritten:

$$xGz\sim yGt \quad \& \quad x'Gz\sim y'Gt \quad \& \quad xFz'\sim yFt' \quad \Rightarrow \quad x'Fz'\sim y'Ft'.$$

The hexagonal condition obviously corresponds to the case F=G=E, x=y' and z=t'.

Let us now start the elaboration by using this hexagonal condition. Under our assumptions and with n=2, this condition is known to be equivalent to an additive representation. We have indeed to show that we can build an additive function V such that $V(x,y)=V_1(x)+V_2(y)$. Let us choose some $x_0=y_0$ that will characterize the origin of the indifference curve graph. Similarly $x_1>x_0$ defines a unit of good in state 1. Let us define a unit y_1 in state 2 such that $(x_0,y_1)\sim(x_1,y_0)$. In the graph, those two acts are on the same indifference curve. Let us fix $V_1(x_0)=V_2(y_0)=0$ and $V_1(x_1)=1$. As a consequence $V_2(y_1)=1$. Find x_2 and y_2 such that $(x_2,y_0)\sim(x_1,y_1)$ and $(x_0,y_2)\sim(x_1,y_1)$. (Remark also that it implies $x_2x_1\sim^*x_1x_0$ and $y_2y_1\sim^*y_1y_0$.) Hence, additivity would imply $V_1(x_2)=V_2(y_2)= V_1(x_1)+V_2(y_1)=2$. However, we have to check that our measure is consistent: indeed, $V_1(x_2)+V_2(y_1)=3$ and $V_1(x_1)+V_2(y_2)=3$ should imply $(x_1,y_2)\sim(x_2,y_1)$. This is guaranteed by the Hexagonal Condition; Figure 1.3.1 represents this condition. Note that we could carry on, by constructing two sequences (x_i) and (y_j) such that $x_{i+1}x_i\sim^*x_ix_{i-1}$ and $y_{j+1}y_j\sim^*y_jy_{j-1}$ where the indexes designate the utility of the outcome, and that we would never find any inconsistencies in these utilities.

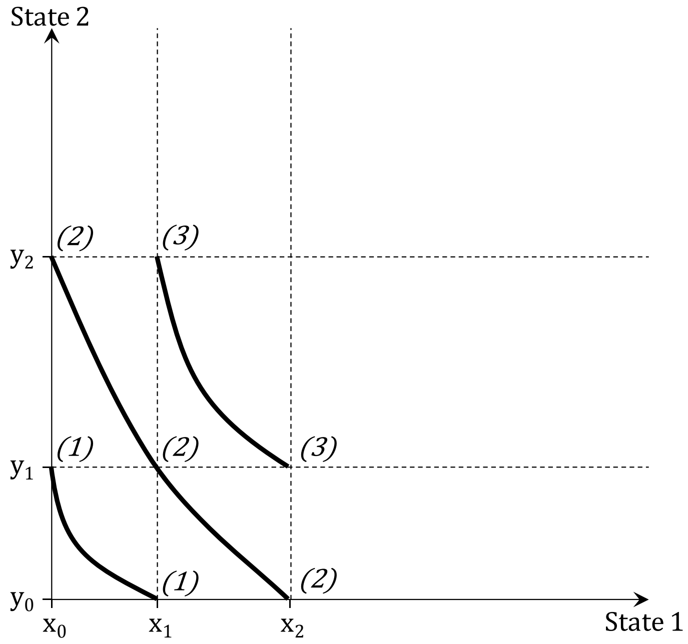


Figure 1.3.1: Indifference Curves and the Hexagonal Condition

Wakker (1989) shows how the complete representation can be obtained. Once it is reached, we need to decompose V_1 and V_2 such that $V_1/p=V_2/(1-p)=U$. In other words, V_1 and V_2 must be proportional and the constant ratio V_2/V_1 determines the odds in favor of the first event $p/(1-p)$. This proportionality is what TO-consistency brings in addition to the Hexagonal Condition.

It is remarkable that subjective probabilities are generated by regularities in attitudes towards outcomes and not by conditions on the state space. TO-consistency generates both additivity that separates states between themselves and proportionality that defines the odds. A dual approach is proposed by Abdellaoui & Wakker (2005), where consistency requirements are applied on likelihood (and thus events) instead of tradeoffs (and outcomes), or by Abdellaoui (2002) under risk where such requirements are applied on probabilities.

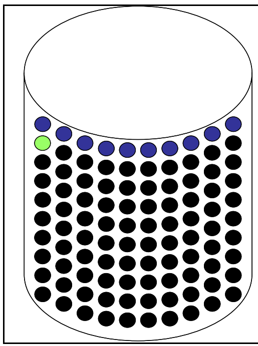
1.4. Paradoxes

This section presents the two most common violations of SEU.

1.4.1. The Allais paradox

Allais (1953) proposes a paradox that clearly proves the limitations of expected utility models. This paradox is initially enunciated under risk but is then generalized under uncertainty by MacCrimmon & Larsson (1979) and Tversky & Kahneman (1992). Let us start with the original paradox and consider the four lotteries (ℓ_1, ℓ_2, ℓ_3 and ℓ_4) described in table 1.4.1. The first column represents the probability levels used in the paradox through an urn containing 89 black balls, 10 blue balls and 1 green ball.

Table 1.4.1. Allais Paradox



	p=89%	P=10%	p=1%
ℓ_1	€1,000,000	€1,000,000	€1,000,000
ℓ_2	€1,000,000	€5,000,000	€0
ℓ_3	€0	€1,000,000	€1,000,000
ℓ_4	€0	€5,000,000	€0

Allais showed that for most people, $\ell_1 \succ \ell_2$ but $\ell_3 \prec \ell_4$. Under expected utility, with u(0)=0 and u(1)=1 (outcomes are expressed in millions of €):

$$.89\times u(1)+.10\times u(1)+.01\times u(1) \quad > \quad .89\times u(1)+.10\times u(5)+.01\times u(0)$$

and

$$.89\times u(0)+.10\times u(1)+.01\times u(1) \quad < \quad .89\times u(0)+.10\times u(5)+.01\times u(0)$$

imply $.11>.10\times u(5)$ and $.11<.10\times u(5)$, a contradiction. To understand what fails, let us define the lotteries $\ell_5=(10/11{:}5,1/11{:}0)$ and $\ell_6=(1{:}0)$. Note that:

$$\ell_1=0.11\times\ell_1+0.89\times\ell_1$$

$\ell_2 = 0.11 \times \ell_5 + 0.89 \times \ell_1$

$\ell_3 = 0.11 \times \ell_1 + 0.89 \times \ell_6$

$\ell_4 = 0.11 \times \ell_5 + 0.89 \times \ell_6$

According to the Independence axiom, not($\ell_2 \succcurlyeq \ell_1$) implies not($\ell_5 \succcurlyeq \ell_1$) but not($\ell_3 \succcurlyeq \ell_4$) implies not($\ell_1 \succcurlyeq \ell_5$). We can conclude that under completeness, the Allais paradox is a violation of the Independence axiom. We can already infer that SEU will be violated because Anscombe & Aumann's axiomatization is based on the same axiom.

Generalizations of the Allais paradox by MacCrimmon & Larsson (1979) and Tversky & Kahneman (1992) consist in showing that the same certainty effect appears when probabilities are unknown. In a Savagean framework, if we replace probabilities by events K (blacK), B (Blue) and G (Green) and if we denote by 0 the constant act that gives 0 no matter which event occurs,

f=(K:1,B:1,G:1)=fKf

and

g=(K:1,B:5G:0)=fKg,

then we have fKf≻fKg and 0Kf≺0Kg, which violates the Sure Thing Principle. Indeed, the outcome if K obtains is equal in the two options of the first (second) choice and should not influence this choice. However, changing this common consequence change the preference. Intuitively, f is a constant act and amending it has a great impact because the obtained act becomes risky. That is what is often called the *certainty effect*. However, does it mean that subjective probabilities do not exist? That is not sure, because a violation of what we presented as the 3rd pillar of SEU does not mean that the second one (probabilistic sophistication) must be rejected. Indeed, this paradox violates the Sure Thing Principle for acts with 3 outcomes. But remember that we did not use more than 2 outcomes when we proved the existence of a fine and tight qualitative probability. As a consequence, some restriction of the Sure-Thing Principle to 2-outcome acts may not be violated by the Allais paradox and may still make it possible to prove probabilistic sophistication. We will discuss this in section 1.5.1.

1.4.2. The Ellsberg paradox

The second paradox, due to Ellsberg (1961), that we will be discussing now is much more a problem for the existence of probabilities. Let us consider two urns: the first one contains 50 black balls and 50 red balls (this define probability 1/2), whereas we do not know the proportion of red (R) and black (K) balls among the eight balls contained in the second urn.

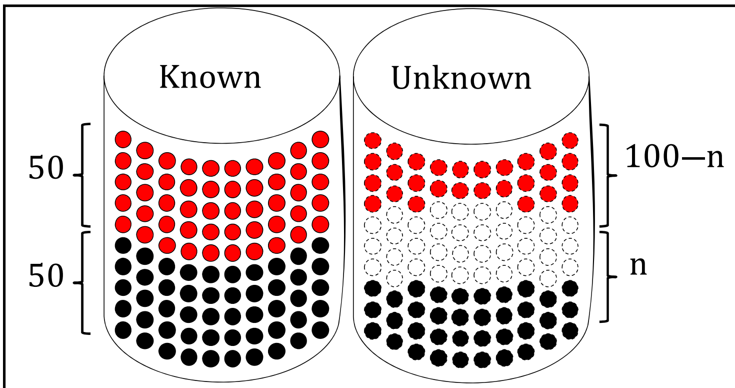
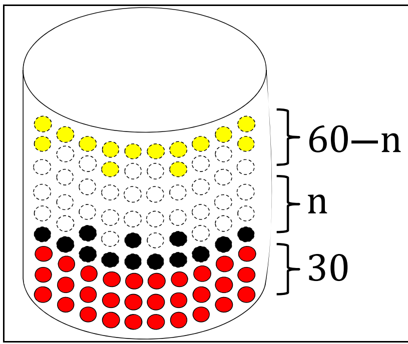


Figure 1.4.2. Two-Color Ellsberg Paradox

Whatever the color, most people prefer bets that deal with the known urn to bets on the unknown urn. Formally, for some positive outcome x, $x(1/2)0 \succ xR0$ and $x(1/2)0 \succ xK0$ implies $1/2 \times u(x) > P(R) \times u(x)$ and $1/2 \times u(x) > P(K) \times u(x)$. As a consequence, $P(K)+P(R)<1$. To better understand the Ellsberg paradox, let us study another version that matches better with the Savagean framework.

One urn contains 90 balls: 30 Red balls (R), and 60 balls that can be yellow (Y) or black (K). Table 1.4.3. displays the acts; the proportion of black balls is referred to as n in the picture:

Table 1.4.3. Three-color Ellsberg Paradox



	R	K	Y
f	€1,000	€0	€0
g	€0	€1,000	€0
f'	€1,000	€0	€1,000
g'	€0	€1,000	€1,000

The common findings are that $f \succ g$ but $f' \prec g'$. Intuitively, in act f the exact probability of winning is known (1/3) but we do not exactly know it in act g (between 0 and 2/3). Inversely, in act g' the exact probability of winning is known (2/3) but we do not exactly know it in act f' (between 1/3 and 1). Under SEU with outcomes in thousands of Euros and $u(1)=1$ and $u(0)=0$, we can first see that $P(R)>P(K)$ but $P(R)+P(Y)<P(K)+P(Y)$, a contradiction. Remark that $f=f(R\cup K)0$, $g=g(R\cup K)0$, $f'=f(R\cup K)1$ and $g'=g(R\cup K)1$.

$$f(R\cup K)0 \succ g(R\cup K)0$$

and

$$f(R\cup K)1 \prec g(R\cup K)1$$

constitute a violation of the Sure Thing Principle. Having €1,000 or not on Y makes f and g alternatively ambiguous. This is a second example, in which we are influenced by a common consequence of two acts and in which this common consequence determines our choice. What is a real issue for us is that this three color Ellsberg paradox violates the Sure Thing Principle for acts having only two consequences. And it thus violates what we needed for getting subjective probabilities. With Allais, we lost Expected Utility. With Ellsberg, we are now loosing Subjective Probabilities.

The next sections are now going to propose generalizations of SEU that are compatible with these two paradoxes.

1.5. Generalizing SEU

The three subsections of this section will propose several generalizations that accommodate at least one of these paradoxes with at least one of the two major elements of SEU: probability or utility.

1.5.1. Probability without utility

If we want to keep subjective probabilities in a way that does not violate the Allais paradox, we already saw that we may have to restrict the Sure-Thing Principle to two-outcome prospects only. Machina & Schmeidler (1992) propose *"a more robust definition of subjective probability"* than Savage. They asked: *"What does it take for choice behavior that does not necessarily conform to the expected utility hypothesis to nonetheless be based on probabilistic beliefs?"* They axiomatized probabilistic sophistication by removing P2 and replacing P4 by:

AXIOM P4* *(Strong Comparative Probability):* For all disjoint events A and B, outcomes $\bar{x}\succ\underline{x}$ and $\bar{y}\succ\underline{y}$, and acts g and h,

$$\bar{x}A\underline{x}Bg \succcurlyeq \bar{x}B\underline{x}Ag \quad\Longrightarrow\quad \bar{y}A\underline{y}Bh \succcurlyeq \bar{y}B\underline{y}Ah$$

Note first that $\bar{x}A\underline{x}Bf$ means $\bar{x}$ on A, $\underline{x}$ on B and f on S−A−B. Obviously, this axiom implies Weak Comparative Probability that permitted us to be sure that if the agent prefers to bet on an event than on another, this should be the case whatever the outcomes are. Intuitively, this axiom has the same meaning but it must hold, whatever the outcomes are and whatever there is on the rest of the state space. In other words, what the agent is certain to have (act f on S−A−B in the first pair or act g on S−A−B in the second pair of prospects) must not influence likelihood perception. It is thus transparent that it is violated by the three-color Ellsberg paradox in which 1R0K0≻1K0R0 but 1R0K1≺1K0R1. But how can we see that this axiom is compatible with the Allais paradox?

Grant, Polak & Özsoy (2007) propose to decompose P4* into P4 and the following P2':

AXIOM P2' *(Two-outcome Sure-Thing Principle):* For all disjoint events A and B, outcomes $\bar{x}\succ\underline{x}$, and acts g and h,

$$\bar{x}A\underline{x}Bg \succcurlyeq \bar{x}B\underline{x}Ag \quad\Longrightarrow\quad \bar{x}A\underline{x}Bh \succcurlyeq \bar{x}B\underline{x}Ah$$

Recall that the Sure-Thing Principle says that for all events E and acts f, g, h and k, fEg≽kEg implies fEh≽kEh. P2' means that the Sure-Thing Principle must only hold for events f and k that only differ on two outcomes. It is obvious that P4*⟹P2' because P2' is the restriction of P4* with $\bar{x}=\bar{y}$ and $\underline{x}=\underline{y}$ as P4 is the restriction of P4* with $A\cup B=S$. Grant, Polak & Özsoy (2007) also shows that P4&P2'$\Longrightarrow$P4*.

We can remark that the Allais paradox, i.e. $1B1G1 \succ 5B0G1$ and $1B1G0 \prec 5B0G0$ violates the Sure-Thing Principle but does not violates the Two-Outcome Sure-Thing Principle because there are alternatively 3 outcomes on B and G. Hence, the Allais paradox is compatible with P1, P2', P3, P4, P5 &P6 (or equivalently P1, P3, P4*, P5 &P6).

What interests us is that P2' (instead of P2) suffices to prove Eq. 1.3.4 (additivity of qualitative probability). We had to show that if $A\cap C=B\cap C=\emptyset$, then $A\succcurlyeq^P B$ iff $A\cup C\succcurlyeq^P B\cup C$. Assume that for some A, B and C such that $A\cap C=B\cap C=\emptyset$, $A\succcurlyeq^P B$. It is equivalent to:

$$\bar{x}(A-B)\underline{x}(B-A)\bar{x}(A\cap B)\underline{x}C\underline{x} \quad \succcurlyeq \quad \bar{x}(B-A)\underline{x}(A-B)\bar{x}(A\cap B)\underline{x}C\underline{x}\,.$$

P2' is sufficient to ensure that this is equivalent to

$$\bar{x}(A-B)\underline{x}(B-A)\bar{x}(A\cap B)\bar{x}C\underline{x} \quad \succcurlyeq \quad \bar{x}(B-A)\underline{x}(A-B)\bar{x}(A\cap B)\bar{x}C\underline{x}$$

and thus to $A\cup C\succcurlyeq^P B\cup C$.

Machina & Schmeidler (1992) show that under P1, P2', P3, P4, P5 &P6 (or equivalently P1, P3, P4*, P5 &P6) there exists a probability measure such that the agent is indifferent between two acts that have the same probability distribution over outcomes. Moreover, these axioms are equivalent to a functional representation of preferences that satisfies first order stochastic dominance with respect to the probability measure.

Chew & Sagi (2006a) propose a derivation of probabilistic sophistication with weaker axioms. They provide axioms that are necessary and sufficient for the basic version of probabilistic sophistication (without stochastic dominance) that they expressed with these words: *"As long as the decision maker is indifferent between two acts that induce the same lottery, it seems reasonable to conclude that she is probabilistically sophisticated"*. They start from Ramsey (1926) and de Finetti's (1937) first attempts to characterize subjective probability using *"ethically* *neutral"* or *"exchangeable"* events. Given the state-space S and an algebra Σ on S, acts are maps from Σ to X, the outcome set. The preference relation is a weak order and at least two acts lead to a strict preference.

DEFINITION *(Exchangeability):* For any disjoint A,B∈Σ, A≈B (A and B are exchangeable) if ∀x,x'∈X and act f, xAx'Bf ~ x'AxBf

Two events are exchangeable if any payoff permutation between these two events does not change the preference value of the prospect. Chew & Sagi base their definition of a qualitative probability on exchangeability:

DEFINITION *(Comparative Likelihood):* For any A,B∈Σ, $A \succcurlyeq^C B$ whenever there exists E⊆(A−B) such that E≈(B−A). (E is called the comparison event.)

For two disjoint events A and B, A is revealed more likely than B whenever A contains a subevent that is exchangeable with B. The previous definition just allows comparing non-disjoint events A and B by applying the same logic to the differences of A and B. Note that this definition directly generates Eq. 1.3.4 of qualitative probability (additivity). Assume that for some events $A \succcurlyeq^C B$. Take now A∪C and B∪C such that A∩C=B∩C=∅. It is obvious that the comparison of A and B also works for A∪C and B∪C. Then $A \succcurlyeq^C B \Leftrightarrow A \cup C \succcurlyeq^C B \cup C$. Chew & Sagi then add the three following axioms:

AXIOM A *(Event Archimedean Property):* Any sequence of pairwise disjoint and nonnull events $\{e_i\}_{i=0}^{n}$ such that $e_i \approx e_{i+1}$ for every i from 0 to n−1 is necessarily finite.

AXIOM C *(Completeness of $\succcurlyeq^C$):* Given any two-disjoint events, one of the two must contain a subevent that is exchangeable with the other, i.e. $\succcurlyeq^C$ is complete.

AXIOM N *(Event Nonsatiation):* For any pairwise disjoint A, B and E∈Σ with E nonnull, A≈B implies that no subevent of A is exchangeable with B∪E.

$S \succ^C \emptyset$ (Eq. 1.3.2) and $A \succcurlyeq^C \emptyset$ (Eq. 1.3.3) are induced by this last axiom and nondegeneracy (at least one f and one g such that $f \succ g$). Axiom C directly assumes that the likelihood relation is complete and transitivity is a non-trivial consequence of the three axioms. Eq. 1.3.1 of qualitative probability is thus satisfied. We know that the qualitative probability must be fine and tight so as to get the probability measure: axioms A and N provide the requisite conditions . Chew and Sagi conclude that axioms A, C and N are satisfied if and only if there exists a unique solvable and finitely additive probability measure that agrees with $\succcurlyeq^C$.

Nevertheless, this axiomatization is still not compatible with the Ellsberg paradox. $1R0K0 \succ 1K0R0$ and $1R0K1 \prec 1K0R1$ means that R and K are not exchangeable and that they cannot contain a subevent that is exchangeable with the other. Consequently, axiom C is violated.

As a conclusion of this subsection, we saw that we can accommodate subjective probability with the Allais paradox. But what would a generalization of SEU that allows for both Allais and Ellsberg paradoxes look like?

1.5.2. Utility without probability

In this subsection, we will not look for the conditions for the existence of subjective probabilities but we will try to understand what occurs if we relax our hypothesis such that Allais and Ellsberg paradoxes are taken into account. We will answer this question by modifying the main axioms of subsections 1.3.3 and 1.3.4. Throughout this subsection, a finite state space is considered, i.e. $S=\{1,\ldots,n\}$. But first of all, we need to define what *comonotonic* acts are: f and g are comonotonic if for no s and t in S, $f(s) \succ f(t)$ and $g(t) \succ g(s)$. It means that there exists a permutation $\{s_1,\ldots,s_n\}$ of the n states of the world such that $f(s_1) \succcurlyeq \ldots \succcurlyeq f(s_n)$ and $g(s_1) \succcurlyeq \ldots \succcurlyeq g(s_n)$. Note that we can define the set of all acts h that also satisfy $h(s_1) \succcurlyeq \ldots \succcurlyeq h(s_n)$. The n! such sets are called *comoncones*. Remark that in the Allais and Ellsberg paradoxes, acts are not comonotonic. That is why restricting properties, which were violated by those paradoxes, to comonotonic acts allows a derivation of a more general model that is compatible with such behaviors.

Let us now replace in Anscombe-Aumann's approach (subsection 1.3.3) AA5 by the following axiom (keeping AA1 (Weak Ordering), AA3 (Jensen Continuity), AA4 (Non-degeneracy)):

AXIOM AA5' *(Monotonicity):* ∀ acts f and g, if f(s)≽g(s) ∀ s∈S, then f≽g

and restrict AA2 to comonotonic acts:

AXIOM AA2' *(Comonotonic Independence):* AA2 applies only to acts that belong to the same comoncone.

Schmeidler (1989) proves that these five axioms are equivalent to *Choquet Expected Utility (CEU)*:

$$(\ell_1, \dots, \ell_n) \mapsto \sum_{i=1}^{n} \pi_{s_i} U(\ell_{s_i})$$

where U(.) is the expected utility of a lottery, s_i is defined such that $\ell_{s_1} \succcurlyeq \dots \succcurlyeq \ell_{s_n}$, and π_{s_i} is a *decision weight* that is derived from a function W (called *weighting function*) with $\pi_{s_i} = W(\{s_1, \dots, s_i\}) - W(\{s_1, \dots, s_{i-1}\})$ and $\pi_{s_1} = W(\{s_1\})$. W(S)=1, W(∅)=0 and W is an increasing function: for two sets of state spaces A⊆B, W(A)≤W(B). Straightforwardly, if W is additive, then we are back to SEU.

However, even if this derivation of CEU is robust to the Allais paradox that is defined on events, it is still violated for lotteries. Indeed, in this framework, constant acts are lotteries and are obviously comonotonic. The Independence axiom holds for lotteries and thus is still violated by the Allais paradox for lotteries.

Let us use the framework that we defined in subsection 1.3.4. Assume now that acts are mapping from the same S but directly to outcomes. Recall that the preference relation is a continuous, monotone, nondegenerate weak order. Köbberling & Wakker (2003) showed that under these conditions:

AXIOM *(Comonotonic TO Consistency):* if a TO relation αβ∼*γδ is elicited in a comoncone, then there does not exist α'≠α such that α'β∼*γδ in the same or in another comoncone.

is equivalent to the CEU representation

$$(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} \pi_{s_i} u(x_{s_i})$$

where π_{s_i} are defined as previously and u is unique up to an affine transformation.

Under CEU, the Ellsberg paradox (like the Allais paradox) only means that decision weights are not additive. Choquet Expected Utility corresponds to the generalization of Rank Dependent Utility (RDU) from risk to uncertainty. Indeed, under risk, Quiggin (1981) proposed a generalization of Expected Utility that accommodates the Allais paradox. A lottery $\ell=(p_1: x_1, \ldots, p_m: x_m)$ where $x_i \geqslant x_{i+1}$ for all i from 1 to m-1 is represented by:

$$(p_1: x_1, \ldots, p_m: x_m) \mapsto \sum_{i=1}^{m} \omega_i u(x_i)$$

where ω_i is a decision weight that is defined such that $\omega_1=w(p_1)$ and $\omega_i=w(p_1+\ldots+p_i)-w(p_1+\ldots+p_{i-1})$. The weighting function w transforms probabilities such that w(0)=0, w(1)=1 and w is increasing. If w is linear, RDU corresponds to EU.

The weighting function usually exhibit subadditivity, i.e. w is inverse-S shaped. This corresponds to a strong sensitivity to departure from 0 and 1, and consequently it accommodates Allais's certainty effect quite well. Tversky & Kahneman (1992) propose a sign dependent version of these models: Cumulative Prospect Theory. Under their model, utility functions and weighting functions are different for gains and for losses.

As a conclusion to this subsection, we can say that CEU (or CPT) matches observed behavior very well but loses sight of our main topic of concern, the existence of subjective probability. Is it possible to reconcile subjective probability with the Ellsberg paradox?

1.5.3. Sources of uncertainty

Let us try to understand where the Ellsberg paradox comes from. A first intuition may be found in Keynes's *Treatise on Probabilities* (1921). He wrote:

"The magnitude of the probability of an argument [...] depends upon a balance between what may be termed the favourable and the unfavourable evidence; a new piece of evidence which leaves the balance unchanged, also leaves the probability of the argument unchanged. But it seems that there may be another respect in which some kind of quantitative comparison between arguments is possible. This comparison turns upon a balance, not between the favourable and the unfavourable evidence, but between the absolute amounts of relevant knowledge and of relevant ignorance respectively."

The determination of the probability of an event is based on a balance between multiple pieces of evidence but the total amount of evidence is also crucial in the confidence one can have in the assessed probability. Fellner (1961) says that we cannot compare all probabilities; he explains the *"distortion of Subjective Probabilities as a reaction to uncertainty"*. This intuition was formalized and studied by Tversky in the 90s using the concept of sources of uncertainty (Heath & Tversky 1991; Tversky & Kahneman 1992; Tversky & Fox 1995; Tversky & Wakker 1995).

A *source of uncertainty* is a set of events that are generated by a common mechanism of uncertainty (e.g. events about the temperature in Paris constitute a source, election results are another). We could think of sources as set of events for which the amount of evidence is perceived as similar for the agent. In the two-color Ellsberg paradox, each urn constitutes a source. In the three-color Ellsberg paradox, the events whose probability is given (because of knowing that there are 20 red balls) constitute a source. The events whose probability is vague (because we do not know the proportion of black and yellow balls) constitute another source.

Chew & Sagi (2006b) propose an axiomatization of sources such that probabilistic sophistication holds inside each source. Indeed, recall that for general probabilistic sophistication, they assume that all events are comparable (Axiom C): for every pair of events, one must contain a subevent that is exchangeable with the other. Now, they do not assume that for all events but they define a source as a maximal set of comparable events. As a conclusion, under some richness and topological conditions, probabilistic sophistication holds for each source.

This result constitutes the current frontier of the research on subjective probability. Chapters 4 and 5 will further analyze this direction. SEU, CEU/CPT and probabilistic sophistication will be the basis of the major part of this dissertation. That is why they have been extensively presented. Nevertheless, we must not neglect that many other models are proposed to represent attitudes under uncertainty. We will often have to refer to them. The next section is dedicated to them.

1.6. Alternative models

1.6.1. The multiple priors approach

Throughout this subsection, we will describe models that represent behaviors through the attitudes towards a set of possible probability distribution that would be in the agent's mind. Let us assume again that acts are maps from states to lotteries over X and that the preference relation satisfies weak ordering (AA1), continuity (AA3), nondegeneracy (AA4) and monotonicity (AA5'). Gilboa & Schmeidler (1989) showed that these axioms together with

AXIOM AA2'' *(Certainty Independence):* for all acts f and g, constant act ℓ and all $\alpha \in (0,1)$, $f \succcurlyeq g \Leftrightarrow \alpha f+(1-\alpha)\ell \succcurlyeq \alpha g+(1-\alpha)\ell$

and

AXIOM AA6 *(Uncertainty Aversion):* for all acts f and g and all $\alpha \in [0,1]$, $f \sim g \Rightarrow \alpha f+(1-\alpha)g \succcurlyeq g$

are equivalent to

$$(\ell_1, \dots, \ell_n) \mapsto \min_{P \in C} \left(\sum_{s \in S} P(s) U(\ell_s) \right)$$

where C is a uniquely determined convex set of priors. This model is called *Maxmin Expected Utility (MEU)*.

It is clear that Certainty Independence is a weakening of AA2. Under this model, the agent has a set of priors and uses the worst case to make her decision. This pessimism is implied by the assumption of uncertainty aversion. Ghirardato et al. (2004) remove this assumption in order to propose a more general model. Then, they propose a special case that authorizes both uncertainty aversion and uncertainty seeking, which they call *α-MEU*:

$$(\ell_1, \dots, \ell_n) \mapsto \alpha \min_{P \in C} \left(\sum_{s \in S} P(s) U(\ell_s) \right) + (1 - \alpha) \max_{P \in C} \left(\sum_{s \in S} P(s) U(\ell_s) \right)$$

MEU corresponds to the case where $\alpha=1$; $\alpha=0$ is the opposite case (optimism or risk seeking). However, when C is not a singleton, there may be no α that corresponds to SEU.

An alternative approach is proposed by Gajdos et al. (2007) in a slightly different framework since the subjective set of priors is replaced by a given family of possible distributions. They also provide a general model (yet assuming uncertainty aversion), whose a special case is a linear combination of MEU and SEU. As a consequence, contrarily to α-MEU, this model does not rule out Bayesians that would think that all possible distributions are equally likely. Furthermore, it is also directly observable in experiments without further assumptions while measuring other multiple prior models need to assume that the subjective set of priors matches a given external set of possible distributions (e.g. Potamites & Zhang 2007).

Eventually, Maccheroni et al (2006) weaken the certainty independence axiom of Gilboa & Schmeidler (1989) to obtain what they call a *"Variational Representation of Preferences"*. Indeed, the preference relation satisfies weak ordering

(AA1), continuity (AA3), nondegeneracy (AA4) and monotonicity (AA5'), uncertainty aversion (AA6) and

AXIOM AA2''' *(Weak Certainty Independence):* for all acts f and g, all constant acts ℓ and ℓ' and all $\alpha\in(0,1)$,

$$\alpha f+(1-\alpha)\ell \succcurlyeq \alpha g+(1-\alpha)\ell \Rightarrow \alpha f+(1-\alpha)\ell' \succcurlyeq \alpha g+(1-\alpha)\ell'$$

if and only if it can be represented by

$$(\ell_1, \dots, \ell_n) \mapsto \min_{P\in\Delta(\Sigma)} \left(\sum_{s\in S} P(s)U(\ell_s) + c\,(P) \right)$$

where $\Delta(\Sigma)$ is the set of all possible probability distribution over Σ (an algebra over S) and c a function from $\Delta(\Sigma)$ to $[0,+\infty]$. The function c is a function that associates a weight to each probability distribution. MEU is the special case where $c(P)=+\infty\ \forall\ P\notin C$ and $c(P)=0\ \forall\ P\in C$ for some subset C of $\Delta(\Sigma)$.

Even if these models have been proposed to better describe behaviors under uncertainty (and above all to accommodate the Ellsberg paradox), a main limitation of this family of model is that they are violated by both the Allais paradox under risk and its generalization under uncertainty. Recall that we derived from the Allais paradox that

$$0.11\times\ell_1+0.89\times\ell_1 \succcurlyeq 0.11\times\ell_5+0.89\times\ell_1$$

and

$$0.11\times\ell_1+0.89\times\ell_6 \preccurlyeq 0.11\times\ell_5+0.89\times\ell_6$$

which straightforwardly contradict Weak Certainty Independence and consequently its strengthening, Certainty Independence.

1.6.2. The multi-stage approach

Kreps & Porteus (1978) suggest that non-neutrality towards the timing of the resolution of uncertainty should be modeled through non-reduction of several-stage lotteries. Nau (2006) and Ergin & Gul (2004) propose to use the same approach so as to model ambiguity attitude. In the first stage, a first event occurs and determines which unambiguous probability distribution will apply on the second stage. Non-reduction of these two-stage lotteries is reinterpreted as ambiguity aversion. Klibanoff, Marinacci & Mukerji (2005), denoted *KMM* hereafter, provide an endogenous decomposition of acts between the two-stages.

AXIOM KMM1: Expected Utility holds over lotteries;

AXIOM KMM2: Subjective Expected Utility holds over second-order acts (that associate a possible probability measure with a consequence);

AXIOM KMM3: Preferences over acts f and g are consistent with preferences over second-order acts f^2 and g^2 where f^2 (g^2) affects the certainty equivalent of the lottery generated by f, when a possible distribution occurs, to this possible distribution.

According to KMM, these three axioms imply the reinterpreted endogenous version of Kreps & Porteus' result:

$$f \mapsto \sum_{P \in \Delta(\Sigma)} \mu(P) \times \varphi\left(\sum_{s \in S} u\big(f(s)\big) \times P(s)\right)$$

where φ is unique up to an affine transformation (for a given u) and represents ambiguity attitudes.

Ambiguity neutrality, i.e. lottery reduction in this framework, corresponds to φ being linear, while φ being concave generates ambiguity aversion. It is possible to derive from this model similar concepts as those that are used under risk, like the Arrow-Pratt coefficient. KMM also suggest a version of their model that is robust to the Allais paradox, by replacing EU under risk by RDU.

As a conclusion of this approach, Halevy (2007) provides experimental evidence that tends to show some correlation between the non-reduction of compound lotteries and ambiguity aversion.

1.6.3. Preference-based definitions of ambiguity

To complete our overview of the representations of behaviors under uncertainty, we must focus on the literature that tries to provide a definition of ambiguity: how can we determine that a decision maker is facing ambiguity?

A first attempt is due to Ghirardato & Marinacci (2001). Assuming that agents are expected value maximizers under risk, they proposed to say that a decision maker is ambiguity averse whenever there exists a hypothetical decision maker who has the same attitude under risk and who is ambiguity neutral such that the first agent prefers more certainty than the hypothetical one under uncertainty. The main limit of this original and interesting proposition is that it assumes a specific behavior under risk and does not encompass the Allais paradox.

Epstein & Zhang (2001) built a list of 4 desiderata which must be fulfilled by a definition of ambiguity: it should be expressed in terms of preferences, it must be model-free, explicit and constructive ("given an event, it should be possible to check whether or not it is ambiguous"), and eventually it should be consistent with probabilistic sophistication on unambiguous acts. This last requirement aims to capture the Knightian distinction between measurable and unmeasurable uncertainty.

DEFINITION *(Unambiguous Event):* An event T is unambiguous if (a) for all disjoint subevents A and B of S-T, act f, and outcomes x, y, z and t

$$zTxAyBf \succcurlyeq zTyAxBf \quad \Longrightarrow \quad tTxAyBf \succcurlyeq tTyAxBf$$

and (b) if the same condition is satisfied on S−T.

This definition means that what is sure on T (or S−T) must not influence comparison of likelihoods (comparison of having a good outcome on A or on B) in its complement. An ambiguous event (for instance the Yellow event of the three-color Ellsberg paradox) has such an impact. Under some conditions, they show that probabilistic sophistication holds on the set of unambiguous events.

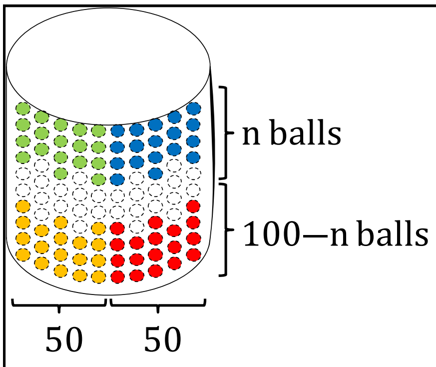


Figure 1.6.1. A violation of Epstein & Zhang's definition of ambiguity

Wakker (2006) provides some critical arguments against this definition. Let us derive the following examples from Wakker's (2006). Assume that an urn contains 100 balls. We do not know the proportion of balls that have a cool or a warm color but we know that cool-color balls are equally shared between green and blue and that half of the warm-color balls are red, the others being orange. In other words, there is an even integer n≤100 such that n/2 balls are Green (G), n/2 balls are Blue (B), (100−n)/2 balls are orAnge (A) and (100−n)/2 are Red (R). Intuitively, we would say that (in addition to the null event and the universe) events G∪A, B∪A, G∪R and B∪R are unambiguous because they have probability 1/2 for sure. On the contrary, events G, B, A, R, B∪G, R∪A, G∪R∪A, G∪R∪B, G∪B∪A and R∪B∪A seem ambiguous because their probability depends on n.

Let us try to apply Epstein & Zhang's definition with the outcome set X={0,20}. We can reasonably assume that for you, 0G20B0R0~0G0B20R0 (in both acts, the probability of winning is unknown) but 20G20B0R0≺20G0B20R0 because the second act gives you €20 with probability 1/2 while the first one's probability of winning is n/100 (vague probability). The definition works well and identifies G as ambiguous. But it seems even more plausible that you do not care about betting on red or on orange balls, or similarly on green or on blue balls because they have the same probability ((100−n)/200 and n/200 respectively). So, let us assume that this is the case and that you have to choose between 0(R∪A)0B20G and 0(R∪A)20B0G. You should be indifferent between the two acts, because your probability is n/200 (and thus vague) in both cases. If you have to choose now between 20(R∪A)0B20G and 20(R∪A)20B0G, you may be also indifferent because the two acts display the same vague probability of winning, 1−n/200. And we could repeat the same reasoning for $(R∪A)^C$ and obtain that z(B∪G)20R0A~z(B∪G)0R20A for any z. Hence, according to Epstein & Zhang, R∪A and B∪G are unambiguous. Their definition fails in capturing ambiguity of events

(R∪A and B∪G) containing subevents (R and A, G and B respectively) that are not ambiguous with respect to them.

Furthermore, this definition may even fail to capture unambiguous events. Assume that we add 90 blue balls in Ellsberg's three-color urn (see figure 1.6.2). This urn now contains 90 blues balls (P(B)=1/2), 30 red balls (P(R)=1/6), n≤60 black balls (P(K)=n/180) and 60-n yellow balls (P(Y)=(60-n)/180). It is clear that event B is unambiguous. Now, you have to choose between 0B20R0Y0 and 0B0R20Y0. Your probability of winning €20 is 1/6 for sure in the first act and [0,1/3] in the second one. For such small probabilities, you may be ambiguity seeking even if you are ambiguity averse for higher probabilities. This assumption would be completely consistent with the experimental literature on ambiguity (e.g. Hogarth & Einhorn 1990). Assume thus that 0B20R0Y0≻0B0R20Y0.

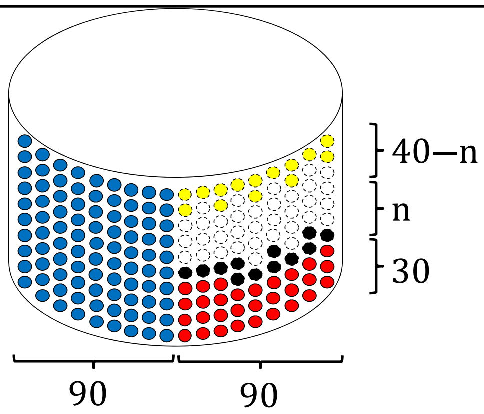


Figure 1.6.2. Another violation of Epstein & Zhang's definition of ambiguity?

Now you must choose between 20B20R0Y0 and 20B0R20Y0 and therefore, between winning with probability 2/3 or [1/2,5/6]. For these higher probabilities, you may have become ambiguity averse and 20B20R0Y0≺20B0R20Y0. As a consequence B is ambiguous according to Epstein & Zhang's definition. If ambiguity attitudes depend on probability level, their definition may erroneously identify unambiguous events as being ambiguous. We must repeat that this assumption is not at all arbitrary and that experimental findings corroborate it. Moreover, chapters 3, 4 and 5 of this dissertation will provide three different experimental studies that support it.

Finally, we must report that an alternative definition is due to Chew & Sagi (2006b), who suggest an exchangeability-based definition of ambiguity that satisfies Epstein & Zhang's four desiderata. Events are said unambiguous if they belongs to a source whose envelop is the universe. Yet, they agree that their definition fails if several sources have the universe as envelop. We can conclude that even if those definitions are promising, they do not catch the diversity of ambiguity and ambiguity attitudes. That is why we will keep our basic definitions, risk being associated with objective (or external or given or known) probabilities and ambiguity including all other uncertain situations (events with unknown probability or sets of possible probabilities…).

1.7. Conclusion

Our introductory examples showed that distinguishing between beliefs, attitudes towards risk and attitudes towards ambiguity is not obvious. The different attempts that have been proposed in the literature do not completely reach this goal. Despite of its limits, SEU remains the most used model because of its simplicity. And yet, in a descriptive viewpoint, it is not convincing. We have thus to take into account these limits, and the different questions and issues that appear in this introductory chapter: do subjective probabilities (always) exist? Can they be compatible with Allais and Ellsberg paradoxes? How can we observe and describe attitudes towards ambiguity?

Let us define some desiderata that will guide our approach in contributing to the literature on uncertainty. They are based on our analysis of SEU. We do not argue that they have a general value, but they may help the reader to understand the theoretical choices that will be made in the following chapters.

D1 (Bayesianism): The model must be consistent with Bayesian beliefs, i.e. with additive subjective probabilities, or at least it must make it possible to test for their existence.

Savage's main contribution is an axiomatization of subjective probabilities. We saw that there are still debates about their existence and that several models remove them. One of our goals consists in testing their existence.

D2 (Robustness): The model must be robust to Allais or/and Ellsberg paradoxes.

This desideratum simply means that the model must better describe behavior than SEU.

D3 (Observability/Measurability): The model must be directly testable or, even better, directly measurable.

This desideratum contains a methodological issue: we want our models to satisfy the Popperian criterion of falsifiability. Furthermore we would like to be able to observe and measure behaviors directly. SEU was simply testable and measurable. It is not obvious that all its generalizations are (e.g. we said that measuring multiple prior models may imply unobservable assumptions about those priors).

D4 (Prediction): The model must allow predictions.

Eventually, a model in economics and management is not really interesting if everything depends on everything. We have to remember that a model is more valuable if it permits us to predict decisions, attitude and behavior. SEU is still very used because of its powerful ability to assess simple attitudes (risk aversion/risk seeking) and to forecast decisions through combinations of beliefs and attitudes.

The four next chapters will work on models that aim to fulfill these desiderata. They will provide theoretical arguments and/or experimental evidences in favor of them, eventually assuming that the best way to evaluate a model remains to *let the data speak*.

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Chapter 2.
Modeling
Risk and Ambiguity Aversion
through Diminishing Marginal Utility

Abstract.

This chapter presents a general technique for comparing concavity of different utility functions under expected utility when probabilities need not be known. The technique is used to generalize several classical results on risk and ambiguity. We obtain: (a) Yaari's between-person comparisons of risk aversion without his restriction that the persons should have identical beliefs; (b) Kreps & Porteus' preference for the timing of the resolution of uncertainty without their commitment to violations of reduction of compound lotteries and while allowing for unknown (subjective) probabilities; (c) Klibanoff, Marinacci, & Mukerji's (absolute) smooth ambiguity aversion without their commitment to violations of reduction of compound lotteries and without need to use (the theoretical construct of) subjective probability as an input in the preference condition; (d) Klibanoff, Marinacci, & Mukerji's comparative results on ambiguity aversion are generalized similarly, where we further do not need the restriction that the different decision makers have the same risk attitude or have the same second-order subjective probabilities over what the true first-order probability measure is. Our results shed new light on the properness of modeling risk and ambiguity attitudes through utility.

2.1. Introduction

Bernoulli (1738) introduced expected utility and concave utility to explain risk aversion. Since then, risk aversion and diminishing marginal utility have commonly been equated in the literature. Indexes of risk aversion, such as the absolute and relative Arrow-Pratt indexes, refer exclusively to marginal utility. Kreps & Porteus (1978) explained preference for early resolution of uncertainty by letting marginal utility depend on the time of resolution; Klibanoff, Marinacci & Mukerji (2005), Nau (2006), and Neilson (1993) explained ambiguity attitudes such as exhibited by the Ellsberg paradox by letting marginal utility be different under ambiguity than under risk.

In this chapter, we consider a preference-based tradeoff relation to analyze marginal utility and we apply it to the models of the afore-mentioned references. First, we obtain new and more general axiomatizations of the corresponding models. Second, we can derive properties of those models in a more general and a more easily observable way. Our tradeoff relation puts marginal utility very central, and thus sheds new light on the question to what extent marginal utility can capture attitudes towards the timing of the resolution of uncertainty and towards ambiguity.

Section 2.2 gives basic definitions and section 2.3 presents the tradeoff relation and states the main results. An application of these results to Yaari's (1969) comparative risk aversion appears in section 2.4. Section 2.5 axiomatizes a two-stage model trough the tradeoff relation and derives some concavity results. Comparative results for the two-stage model are in section 2.6, and section 2.7 compares our results to related results in the literature. Section 2.8 contains a discussion and section 2.9 concludes.

2.2. Definitions

We first give some basic definitions. S denotes the *state space*, which can be finite or infinite. Exactly one state $s \in S$ is true, the other states are not true, and it is uncertain which state is the true one. Events will be particular subsets of S. For the purposes of this chapter it is convenient to work with sources, which are varying collections of subsets of S. We will assume that they are algebras. An *algebra* is a collection of subsets of S containing $\varnothing$ and S and closed under complement taking and finite unions and intersections. We assume that $\mathcal{A}$ is an algebra of subsets of S, the elements of which are called *events*. All sources considered later will be subalgebras of $\mathcal{A}$.

The *outcome set* $\mathcal{C}$ is a nondegenerate subinterval of $\mathbb{R}$, with *outcomes* being monetary. Outcomes are denoted by Greek letters α, β, etc. Preferences will be defined over a set X of *prospects*. Prospects, denoted x, y, ..., and called acts in Savage (1954), are measurable mappings from states to outcomes; x assigns outcome x(s) to each state $s \in S$. *Measurability* means that the inverse of every subinterval of $\mathcal{C}$ is contained in $\mathcal{A}$. Readers not interested in measure theory may assume that $\mathcal{A}$ contains all subsets of S, in which case all functions from S to $\mathcal{C}$ are measurable so that all measurability considerations are trivially satisfied and can be ignored.

Each outcome α is identified with the constant prospect x with $x(s)=\alpha$ for all s. We assume that all finite-valued measurable mappings from S to $\mathcal{C}$ (called *simple* prospects) are contained in X. $(E_1{:}x_1, \ldots, E_n{:}x_n)$ denotes the simple prospect assigning x_j to each s in E_j; the E_js partition S. Other than that, we allow X to be almost any subset of the set of measurable mappings from S to $\mathcal{C}$. One more restriction on X, to ensure that all expected utilities considered hereafter are well-defined and finite, will be added later.

A *preference relation* of a *decision maker* is a binary relation over X, denoted by $\succcurlyeq$. The notation $\succ$, $\sim$, $\preccurlyeq$, and $\prec$ is as usual. *Expected utility* (*EU*) holds if there exist a (finitely additive) *probability measure* P on $\mathcal{A}$, and a strictly increasing *utility function* U: $\mathbb{R} \to \mathbb{R}$, such that $EU(x) = \int_S U(x)dP$ (the *expected utility* of prospect x) is well-defined and finite for all x in X, and we have $x \succcurlyeq y$ if and only if $EU(x) \geq EU(y)$. EU implies that $\succcurlyeq$ is a *weak order*, i.e. it is *complete* (for all x, y, $x \succcurlyeq y$ or $y \succcurlyeq x$) and transitive.

A *certainty equivalent* of a prospect x is an outcome α such that $\alpha \sim x$. Under EU with continuous strictly increasing utility, $\alpha = U^{-1}(EU(x))$ is unique. For outcome α, event E, and prospect x, $\alpha_E x$ denotes the prospect assigning outcome α to each state in event E and the same outcome as x to each state off E. Thus, given that β can denote a constant prospect, $\alpha_E\beta$ denotes the two-outcome prospect (E:α, not-E:β).

An event E is *null* if $\alpha_E x \sim \beta_E x$ for all prospects x and outcomes α and β, and *nonnull* otherwise. Under EU, an event E is null if and only if $P(E) = 0$. We assume *nondegeneracy* throughout, implying that there exists a nonnull event E for which the complement is also nonnull. Under EU nondegeneracy is equivalent to $0 < P(E) < 1$. We also assume *monotonicity*, i.e. $x \succcurlyeq y$ if $x(s) \geq y(s)$ for each state s, with $x \succ y$ if $x(s) > y(s)$ for each state s. Monotonicity implies that $\alpha \succcurlyeq \beta$ if and only if $\alpha \geq \beta$ for all outcomes α, β. EU implies monotonicity.

In the rest of this section we define technical conditions. This part can be skipped by readers interested primarily in empirical implications. We avoid infinite-dimensional topological complications by restricting continuity to finite-dimensional subspaces: $\succcurlyeq$ is *continuous* if, for every partition $(E_1,\ldots,E_n)$ of S, the preference relation restricted to prospects $(E_1{:}x_1,\ldots E_n{:}x_n)$ satisfies the usual Euclidean continuity. Under EU, continuity of preference can be seen to be equivalent to continuity of utility.

To avoid undefined or infinite EU values, and to express this requirement in directly observable preference conditions, we define truncations of prospects, following Wakker (1993). All truncation conditions defined hereafter are trivially satisfied if all prospects $x \in X$ are bounded in the sense that there exists an outcome preferred to all x(s) and one less preferred than all x(s). This includes the case where X contains only the simple prospects. Readers only interested in those special cases may skip the following definitions concerning truncations. For prospect x and outcome μ, $x \wedge \mu$, the *above truncation* of x at μ, assigns x(s) to s whenever $x(s) \preccurlyeq \mu$ and it assigns μ to s whenever $x(s) \succ \mu$. For prospect x and outcome η, $x \vee \eta$, the *below truncation* of x at η, assigns x(s) to s whenever $x(s) \succcurlyeq \eta$ and it assigns η to s whenever $x(s) \prec \eta$. X is *truncation-closed* if all (above and below) truncations of all of its prospects are contained in X. *Truncation continuity* holds if, whenever $x \succ y$ for a simple prospect y, then $x \wedge \mu \succ y$ for some outcome μ and, whenever $x \prec y$ for a simple prospect y, then $x \vee \eta \prec y$ for some outcome η.

It is sometimes convenient if probability measures are countably additive. A probability measure is *countably additive* if $A_j \downarrow \varnothing$ ($A_j \supset A_{j+1}$ and the intersection of these sets is empty) implies that $P(A_j)$ converges to 0. It is equivalent to the probability of a countable disjoint union being the sum of the individual probabilities whenever that countable union is contained in the algebra. The condition is most useful if the algebra $\mathcal{A}$ is a sigma-algebra (an algebra closed under countable unions). A preference condition necessary and sufficient for countable additivity is *set-continuity* (similar to Wakker 1993, section 4):

If $\beta < \gamma$, $A_j \downarrow \varnothing$, and $x \succ \beta$

then, for some natural number J, $x \succ \gamma_{A_j}\beta$ for all $j > J$. (2.2.1)

STRUCTURAL ASSUMPTION 2.2.1 [*Decision under Uncertainty*]. S is a state space endowed with an algebra $\mathcal{A}$ of subsets called events, and $\mathcal{C}$, a subinterval of $\mathbb{R}$, is the outcome set. X, the set of prospects, contains all simple measurable mappings from S to $\mathcal{C}$, and possibly some other measurable mappings from S to $\mathcal{C}$. $\succcurlyeq$ is a binary relation on X. Truncation closedness holds, and for every prospect there exists a certainty equivalent. Nondegeneracy holds. □

We will sometimes assume that objective probabilities of the events are available, in which case by common assumptions of decision under risk the prospect can be identified with the probability distribution generated over the outcomes. Then $\alpha_p\beta$ denotes the probability distribution yielding α with probability p and β with probability $1-p$.

STRUCTURAL ASSUMPTION 2.2.2 [*Decision under Risk*]. Structural Assumption 2.2.1 holds. An (objective) probability measure is given on $\mathcal{A}$. The preference value of a prospect depends only on the probability distribution generated over the outcomes, and every finite probability distribution over outcomes is generated by some prospect. □

2.3. A tool for analyzing utility

The following definition can be used to elicit equalities of utility differences which, given the usual uniqueness of utility up to level and unit, completely determines utility. We write

$$[\alpha; \beta] \sim^* [\gamma; \delta] \text{ or } \alpha\beta \sim^* \gamma\delta \tag{2.3.1}$$

if

$$\alpha_E x \sim \beta_E y \tag{2.3.2}$$

and

$$\gamma_E x \sim \delta_E y \tag{2.3.3}$$

for some nonnull event E and simple prospects x and y. The intuitive interpretation of the condition is, in short, that receiving α instead of β can offset the same outcome-pattern over other events as receiving γ instead of δ, so that the former is as big an improvement as the latter. Köbberling & Wakker (2004) present a detailed discussion of the interpretation and intuition of this relation.

LEMMA 2.3.1. Under EU, $\alpha\beta \sim^* \gamma\delta$ if and only if $U(\alpha) - U(\beta) = U(\gamma) - U(\delta)$. □

By measuring a sequence of indifferences (with E a nonnull event)

$$\alpha^j{}_E x \sim \alpha^{j-1}{}_E y \quad j = 1, \ldots, n, \qquad (2.3.4)$$

we obtain a sequence

$$\alpha^0,\ldots,\alpha^n \text{ with } \alpha^{j+1}\alpha^j \sim^* \alpha^1\alpha^0 \text{ for all j.} \qquad (2.3.5)$$

Such a sequence is called a *standard sequence*. By Lemma 2.3.1 it is equally-spaced in utility units. By normalizing $U(\alpha^0) = 0$ and $U(\alpha^n) = 1$, we get $U(\alpha^j) = j/n$ for all j. Thus, n indifferences yield n−1 data points of utility. We can then obtain the graph of U (Wakker & Deneffe 1996), even if we do not know the subjective probabilities of the relevant events, as in Figure 2.3.1.

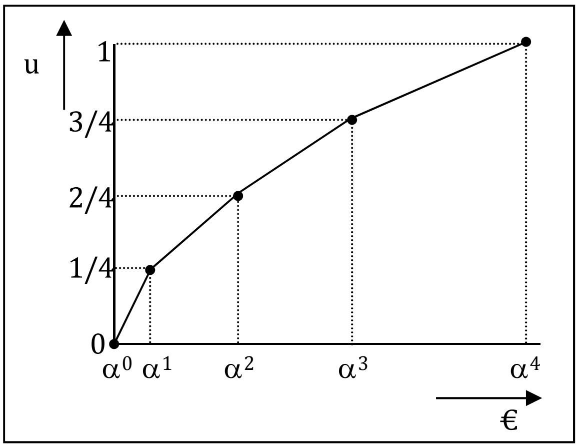

Figure 2.3.1. Utility graph derived from ~* observations

The following consistency condition for utility measurement is obviously necessary for EU to hold, given that utility is strictly increasing. It implies that standard sequences as in Eq. 2.3.5, when elicited from different events E and from different auxiliary prospects x and y, should agree and give the same utility graph in Figure 2.3.1.

DEFINITION 2.3.2. *Tradeoff consistency* holds if improving an outcome in any $\alpha\beta \sim^* \gamma\delta$ relationship breaks that relationship. □

Whenever derived concepts such as $\sim^*$ above are used in a preference foundation, it should be verified that the corresponding preference conditions can be restated directly in terms of empirical primitives. Tradeoff consistency can indeed be reformulated in this manner, simply by substituting the definition of $\sim^*$, as:

$$\alpha_E x \sim \beta_E y \quad \& \quad \alpha'_F f \sim \beta_F g \quad \&$$

$$\gamma_E x \sim \delta_E y \quad \& \quad \gamma_F f \sim \delta_F g$$

$$\text{implies } \alpha' = \alpha. \qquad (2.3.6)$$

Thus, the condition can be used in preference foundations (Köbberling & Wakker 2004, p. 142). Following preference conditions expressed in terms of $\sim^*$ can similarly be restated directly in terms of preferences. For brevity, we do not make those restatements explicit in what follows.

For finite state spaces Köbberling & Wakker (2003, 2004) showed that tradeoff consistency is not only necessary, but also sufficient, for EU, providing a preference foundation of EU. The following theorem provides a generalization to general, possibly infinite, state spaces.

THEOREM 2.3.3. Under the Structural Assumption 2.2.1, the following two statements are equivalent.

(i) Expected utility holds with continuous strictly increasing utility.

(ii) $\succcurlyeq$ satisfies:
(a) weak ordering;
(b) monotonicity;
(c) continuity;
(d) truncation continuity;
(e) tradeoff consistency.

In Statement (i), the probabilities are unique and the utility function is unique up to level and unit. The probability in Statement (i) is countably additive if and only if set-continuity holds. □

Köbberling & Wakker (2004) showed that instead of tradeoff consistency in (e) in the theorem, we can also impose transitivity of $\sim^*$. Utility need not be bounded in the above theorem. Given that utility is determined only up to level and unit, we will throughout equate utility functions, and for instance write $U^1 = U^2$ when in fact these functions are only in the same interval-scale class, so that $U^1 = \tau + \sigma U^2$ for a real τ and a positive σ. The following corollary adapts the above result to decision under risk.

COROLLARY 2.3.4. Under the Structural Assumption 2.2.2, the following two statements are equivalent.

(i) Expected utility holds with continuous strictly increasing utility and the (subjective) probabilities used in the EU model of Theorem 2.3.3 identical to the objective probabilities.

(ii) $\succcurlyeq$ satisfies:
  (a) weak ordering;
  (b) monotonicity;
  (c) continuity;
  (d) truncation continuity;
  (e) tradeoff consistency.

The utility function in Statement (i) is unique up to level and unit. □

The above corollary can be considered an alternative to the well-known preference foundation of EU by von Neumann and Morgenstern (1944) with continuity of utility added, with the possibility added to deal with nonsimple probability distributions, and, relative to Fishburn (1970, Theorem 10.1), without a restriction to bounded utility. Although the main condition to be used in the von Neumann-Morgenstern theorem, independence, is very appealing, we have presented the above alternative version because we need the $\sim^*$ relation for later purposes. We will use this relation here for the main topic of interest in this chapter, diminishing marginal utility. The above theorem further illustrates that the $\sim^*$ relation captures the essence of expected utility.

Risk aversion is defined as preference for expected values over prospects. This definition refers to expected value, which can only be defined if probabilities over the state space S are available. This happens for instance for decision under risk and for decision under uncertainty if EU holds. For decision under risk, the condition is directly observable because the probabilities are objectively given. Then it is well known that, under EU, risk aversion holds if and only if utility is concave.

For uncertainty with EU, the analysis of risk aversion is more complex. First, probabilities in EU reflect subjective beliefs that are not directly observable and, hence, cannot be easily used as inputs in preference foundations. Hence, Statement (ii) in the following theorem is not an easily observable preference condition in this strict sense. A second complication is that it is less plausible to assume that all probability distributions over outcomes are available in the domain of preference, for instance if S is finite, which complicates the proof somewhat.

Quasi-concavity of preferences in terms of outcome mixing provides an alternative characterization of risk aversion for unknown probabilities (Chateauneuf & Tallon 2002; Debreu & Koopmans 1982). This condition is appealing for decision under uncertainty because there it is directly observable, unlike risk aversion, in that it does not use theoretical constructs as inputs. Formally, for any $0 \leq \lambda \leq 1$ and prospects x, y, $\lambda x + (1-\lambda)y$ is the prospect that assigns to each s the outcome $\lambda x(s) + (1-\lambda)y(s)$. *Quasi-concavity* holds if $\lambda x + (1-\lambda)y \succcurlyeq x$ whenever $x \sim y$. We can also obtain a directly observable preference condition for risk aversion under uncertainty using $\sim^*$. The following theorem presents the various results.

THEOREM 2.3.5. Assume that all assumptions and results of Theorem 2.3.3 hold. Then the following four statements are equivalent.

(i) Utility is concave;

(ii) Risk aversion holds;

(iii) Quasi-concavity holds;

(iv) If $\alpha\beta \sim^* \beta\gamma$, then $\beta \leq (\alpha + \gamma)/2$.

□

Note that each triple α^{i-1}, α^i, α^{i+1} in Eq. 2.3.5 entails a test of condition (iv) above, so that n indifferences in Eq. 2.3.4 give n−2 tests of the condition. This illustrates that preference conditions in terms of $\sim^*$ are easily observable. Also note that condition (iv) transparently corresponds with concavity of U in Figure 2.3.1, where $\alpha^{j+1}\alpha^j \sim^* \alpha^j\alpha^{j-1}$ with $\alpha_j < (\alpha_{j+1}+\alpha_{j-1})/2$ for all j.

We next derive a comparative result using the $\sim^*$ relation. We are not aware of similar comparative results using quasi-concavity or other conditions. In the following result, as throughout this chapter, superscripts indicate indexes and not powers.

THEOREM 2.3.6. Assume that for both j = 1 and j = 2: $\mathcal{A}^j$ is a subalgebra of $\mathcal{A}$; the outcome set is $\mathcal{C} \subset \mathbb{R}$; $\succcurlyeq^j$ is a preference relation over the prospects that are measurable with respect to $\mathcal{A}^j$; $\sim^{*j}$ refers to the corresponding tradeoff relation; all assumptions and results of Theorem 2.3.3 hold with algebra $\mathcal{A}^j$, subjective probability P^j on $\mathcal{A}^j$, and utility function U^j. Then the following two statements are equivalent:

(i) $U^1 = \varphi \circ U^2$ for a concave transformation φ;

(ii) If $\alpha\beta \sim^{*2} \beta\gamma$ and $\alpha\beta' \sim^{*1} \beta'\gamma$, then $\beta' \leq \beta$.

□

Following sections will present many applications of Theorem 2.3.6. The following figure illustrates how the condition in Statement (ii) above can directly be tested empirically. In step 1, the value of β is elicited that gives indifference. Next, in step 2, where β is used as an input, the value of α is elicited to give indifference. We have thus obtained $\alpha\beta \sim^{*2} \beta\gamma$. Then we similarly measure ε and α´ to give the indifferences in steps 3 and 4, implying $\alpha'\beta \sim^{*1} \beta\gamma$. Then $\beta' \leq \beta$ if and only if $\alpha \leq \alpha'$ because utility is strictly increasing.

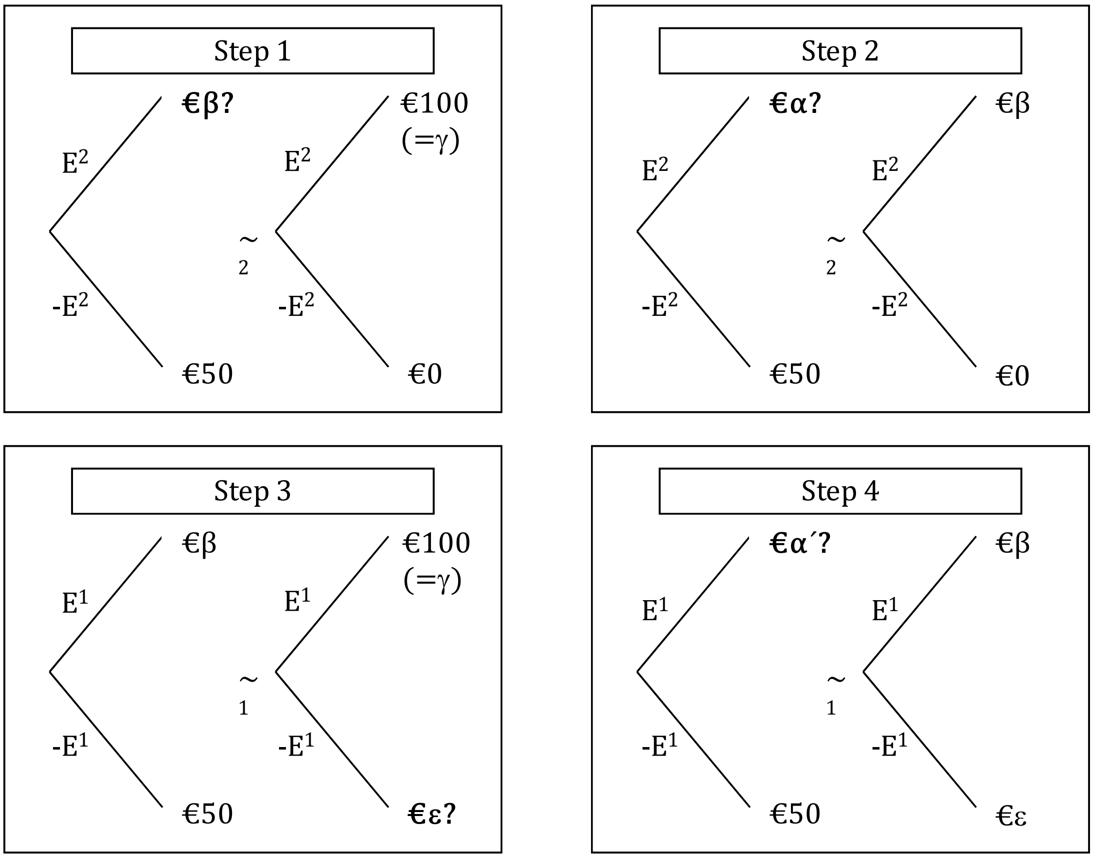

Figure 2.3.2. The observability of Statement (ii) in Theorem 2.3.6.

The special case where $\succcurlyeq^1$ and $\succcurlyeq^2$ are preferences of the same decision maker and $\mathcal{A}^1$ and $\mathcal{A}^2$ are different sources yields a within-person between-source comparison, similar to the Ellsberg two-color paradox. The special case where $\succcurlyeq^1$ and $\succcurlyeq^2$ concern different decision makers and $\mathcal{A}^1 = \mathcal{A}^2$ concerns a between-person within-source comparison.

2.4. Yaari's comparative risk aversion results

It is well known that for decision under risk, decision maker 2 has a more concave utility function than decision maker 1, as in Statement (i) of Theorem 2.3.6, if and only if decision maker 2's risk premium (difference between expected value and certainty equivalent) exceeds that of decision maker 1 for every prospect. This condition is more difficult to handle for uncertainty for reasons as discussed above: subjective probabilities are not easily observable so that expected value and risk premium are so neither. Yaari (1969) showed that comparisons are still possible in terms of certainty equivalents (he used an equivalent formulation in terms of acceptance sets) under a restrictive condition. In other words, he showed equivalence of Statements (i) and (ii) below.

THEOREM 2.4.1. Assume that for both j = 1 and j = 2: the outcome set is the nondegenerate interval $\mathcal{C} \subset \mathbb{R}$; $\succcurlyeq^j$ is a preference relation over the prospects; all assumptions and results of Theorem 2.3.3 hold with algebra $\mathcal{A}^j = \mathcal{A}$, subjective probability P^j on $\mathcal{A}^j$, and utility function U^j. Then the following three statements are equivalent:

(i) $U^1 = \varphi \circ U^2$ for a concave transformation φ and $P^1 = P^2$;

(ii) For each prospect, the certainty equivalent for $\succcurlyeq^2$ is at least as large as that for $\succcurlyeq^1$;

(iii) If $\alpha\beta \sim^{*2} \beta\gamma$ and $\alpha\beta' \sim^{*1} \beta'\gamma$, then $\beta' \leq \beta$; further, $P^1 = P^2$.

□

Our results have brought a separation of beliefs and tastes. Under EU, probabilities are commonly taken to reflect beliefs, and utilities reflect attitudes towards risk. A typical feature of our condition in Statement (ii) of Theorem 2.3.6 is that it obtained a comparative risk-attitude result irrespective of what the beliefs are. Yaari's (1969) result did not obtain such a separation of risk attitude and belief. As he showed, the risk attitudes of different decision makers could only be compared through his condition for the special case of identical beliefs.

2.5. Two-stage models

This section turns to multi-stage models where in the first stage probabilities may be unknown but in the second stage probabilities are known. Thus, all following models can be considered to be special cases of Anscombe and Aumann's (1963) model. L denotes the set of *lotteries*, where lotteries (typically denoted by ℓ) are probability distributions over $\mathcal{C}$. A typical prospect x maps a state space S to L, assigning a lottery x(s) to each state s. $(E_1{:}\ell_1, \ldots, E_m{:}\ell_m)$ denotes a prospect with $E_1, \ldots, E_m$ partitioning the state space S, and each ℓ_i designating a lottery. *First-stage prospects* x have their outcomes depend only on first-stage events, i.e., all lotteries x(s) are degenerate. *Second-stage prospects* x have their outcomes depend only on the second-stage uncertainty, i.e. there exists a lottery ℓ such that $x(s)=\ell$ for all $s \in S$. First-stage prospects are identified with the corresponding mappings from S to $\mathcal{C}$, and second-stage prospects are identified with the lottery that they generate. The restrictions of the preference relation $\succcurlyeq$ to first- and second-stage prospects are denoted $\succcurlyeq^1$ and $\succcurlyeq^2$. We will impose the richness assumptions of preceding sections on first- and second-stage prospects. We do not need further richness assumptions. In particular, we need not assume the presence of every (measurable) allocation of lotteries to states s. This point will be crucial in the application to the model by Klibanoff, Marinacci, and Mukerji (2005). The following assumptions will be assumed sometimes, but not always.

ASSUMPTION 2.5.1 [Within-Source EU]. For both j = 1 and j = 2, $\succcurlyeq^j$ satisfies all assumptions and results of Theorem 2.3.3/Corollary 2.3.4 with respect to P^1 and U^1 for $\succcurlyeq^1$ and U^2 for $\succcurlyeq^2$. □

ASSUMPTION 2.5.2 [Backward Induction]. The preference value of prospect x is not affected if each lottery x(s) is replaced by its $\succcurlyeq^2$ certainty equivalent, i.e. an outcome α_s with $\alpha_s \sim^2 x(s)$. □

By backward induction, a preference between any pair of prospects can be derived from the preference between the first-stage prospects that result after the substitutions of certainty equivalents. This implies, in particular, that $\succcurlyeq$ inherits weak ordering from $\succcurlyeq^1$. EU^2 denotes expected utility with respect to U^2 and the objectively given probabilities of stage 2.

THEOREM 2.5.3. Let the Structural Assumptions 2.2.1 and 2.2.2 hold for $\succcurlyeq^1$ and $\succcurlyeq^2$, respectively. Then the following two statements are equivalent.

(i) There exist continuous strictly increasing functions U^2 and φ, and a probability measure P on S, such that prospects are evaluated through

$$x \mapsto \int_S \varphi\left(EU^2(x(s))\right) dP(s);$$

(ii) Assumption 2.5.2 holds and, for each $j = 1,2$, $\succcurlyeq^j$ satisfies:

(a) weak ordering;

(b) monotonicity;

(c) continuity;

(d) truncation continuity;

(e) tradeoff consistency.

□

Statement (i) above means that Assumptions 2.5.1 and 2.5.2 hold. Several papers have studied conditions that imply concavity of φ in the above theorem. They all need richness assumptions at least as strong as the following one.

ASSUMPTION 2.5.4 [Richness]. Under the within-source EU Assumption 2.5.1, there exists an event $E \subset S$ with $0 < P(E) = p < 1$ such that for all α and β in the image of U^2, there exists a prospect $(E{:}\ell_1, \text{not-}E{:}\ell_2)$ in the preference domain with $EU^2(\ell_1)=\alpha$ and $EU^2(\ell_2)=\beta$. □

Preference for second-stage resolution (PSR) of uncertainty holds if, for each first-stage prospect $\alpha_E\beta$ and second-stage prospect $\alpha_p\beta$ with $P(E) = p$, $\alpha_E\beta \preccurlyeq \alpha_p\beta$. In practice, preference for first-stage resolution of uncertainty is more plausible and more interesting. Nevertheless, to be consistent with some later results on ambiguity, where concavity rather than convexity of φ is interesting, we analyze preference for second-stage resolution and the implied concavity of φ in our theorems. The analysis of preference for first-stage resolution of uncertainty is completely analogous, with the above preference reversed and with convexity of φ rather than concavity.

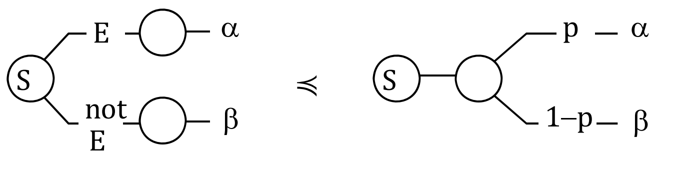

Figure 2.5.1. P(E) = p. Preference for second-stage resolution of uncertainty.

Preference for second-stage resolution can be interpreted as an aversion to mean-preserving spreads in terms of second-stage prospects because the second-stage prospect $\alpha_p\beta$ is a mix of the two (degenerate) second-stage prospects that can arise from the first-stage prospect $\alpha_E\beta$. In other words, and assuming $\alpha \succcurlyeq \beta$, in the first-stage prospect the second-stage probability at the good outcome α is 1 with probability P(E) = p and 0 with probability 1−p, whereas in the second-stage prospect it is p with certainty. The condition can be reinforced into a general aversion to mean-preserving spreads for mixtures of second-stage prospects (Ergin & Gul 2004), but we will not pursue this point. The condition is equivalent to $p\varphi(U^2(\alpha)) + (1-p)\varphi(U^2(\beta)) \leq \varphi(p(U^2(\alpha) + (1-p)(U^2(\beta)))$ which, given sufficient richness of U^2's image, is equivalent to concavity of φ (Lemma 2.A.1).

An alternative preference condition can be obtained if we use an *alternative-outcome interpretation*, taking the EU values of the second-stage prospects as outcomes. Consider the lotteries x(s) in prospect x, their EU^2 values[8], and then the P-weighted average of those EU^2 values; call this value $EV(EU^2)$. It is the value of the prospect x that would result if φ were the identity function, and it is the expected value of the prospect under the alternative interpretation. Klibanoff, Marinacci, & Mukerji (2005), abbreviated *KMM* hereafter, introduced *smooth ambiguity aversion*: any lottery ℓ with $EU^2(\ell) = EV(EU^2)$ is preferred to the original prospect x. This condition is risk aversion in the traditional sense under the alternative-outcome interpretation. Given sufficient richness, the risk aversion mentioned is equivalent to concavity of φ.

The two above conditions, PSR and smooth ambiguity aversion, used the first-stage subjective probabilities as inputs. The following condition does not need such inputs. For $0 \leq \lambda \leq 1$, and lotteries ℓ, ℓ', $\lambda\ell + (1-\lambda)\ell'$ denotes a probabilistic mixture defined in the usual way. For prospects x, y, and $0 \leq \lambda \leq 1$, the probabilistic mixture of prospects $\lambda x + (1-\lambda)y$ is defined through statewise mixing, assigning lottery $\lambda x(s) + (1-\lambda)y(s)$ to each s. Given EU preferences over second-stage prospects, the probabilistic mixture of prospects just defined is equivalent to outcome mixing under the alternative-outcome interpretation. Gilboa & Schmeidler (1989) and Schmeidler (1989) defined *uncertainty aversion*: $\lambda x + (1-\lambda)y \succcurlyeq x$ whenever x

[8] generating the "alternative" first-stage prospect assigning the real number $EU^2(x(s))$ to each state s).

$\sim$ y. The condition is equivalent to quasi-concavity under the alternative-outcome interpretation. We now summarize various ways to get concavity of φ.

THEOREM 2.5.5. Assume the conditions and assumptions of Theorem 2.3.6 for $\succcurlyeq^1$ and $\succcurlyeq^2$. The following two statements are equivalent:

(i) φ is concave;

(ii) Statement (ii) of Theorem 2.3.6 holds.

If we further have Assumptions 2.5.2 and 2.5.4, then the following three statements are also equivalent to the above two statements:

(iii) Smooth ambiguity aversion holds;

(iv) PSR (preference for second-stage resolution of uncertainty) holds;

(v) Uncertainty aversion holds.

□

In the above theorem, the condition in (ii) is more generally applicable than those in (iii), (iv), and (v). It shares with condition (v) the advantage over the conditions (iii) and (iv) that it does not need subjective first-stage probabilities as inputs, which is important if the first-stage probabilities are not objectively given.

2.6. Comparative results for two-stage models

We now turn to comparative results. KMM proposed the following condition. The condition is only defined for decision maker $\succcurlyeq^A$ and $\succcurlyeq^B$ with same first-stage beliefs $P^A = P^B$; then $\succcurlyeq^A$ is *more smooth-ambiguity averse* than decision maker $\succcurlyeq^B$ if

$$x \succcurlyeq^A \ell \Rightarrow x \succcurlyeq^B \ell \tag{2.6.1}$$

for all prospects x and second-stage prospects ℓ. As pointed out by KMM, this condition implies that $\succcurlyeq^A$ and $\succcurlyeq^B$ are identical on lotteries, as follows by substituting lotteries for x. Although for general preference relations the resulting condition need not imply identical preferences, it does so for nondegenerate preferences that maximize EU, such as $\succcurlyeq^{A2}$ and $\succcurlyeq^{B2}$. Given the same preferences over lotteries, we may restrict Eq. 2.6.1 to degenerate lotteries ℓ, i.e. sure outcomes, these being the same for $\succcurlyeq^A$ and $\succcurlyeq^B$. Thus, comparative smooth ambiguity aversion amounts to the requirement of identical preferences over lotteries plus Yaari's certainty-equivalent condition (ii) in Theorem 2.4.1.

THEOREM 2.6.1. For both $\succcurlyeq^A$ and $\succcurlyeq^B$, assume the conditions and assumptions of Theorem 2.5.3, with the notation $\succcurlyeq^{1A}$, $\succcurlyeq^{2A}$, $\succcurlyeq^{1B}$, and $\succcurlyeq^{2B}$ as before. The following two statements are equivalent:

(i) φ^A is a concave transformation of φ^B and $U^{2A} = U^{2B}$;

(ii) If $\alpha\beta \sim^{*1B} \beta\gamma$ and $\alpha\beta' \sim^{*1A} \beta'\gamma$, then $\beta' \leq \beta$; further, $\sim^{*2A} = \sim^{*2B}$.

If we further have Assumptions 2.5.2 and 2.5.4 for both $\succcurlyeq^A$ and $\succcurlyeq^B$, and if $P^A = P^B$, then the following statement is also equivalent to the above two statements:

(iii) A is more smooth-ambiguity averse than B.

□

We are not aware of similar comparative results using uncertainty aversion. The above results compared ambiguity attitudes only under the restrictive assumption of identical preferences over lotteries. We next show how ambiguity attitude can be compared without this restriction.

THEOREM 2.6.2. For both $\succcurlyeq^A$ and $\succcurlyeq^B$, assume the conditions and assumptions of Theorem 2.5.3, with the notation $\succcurlyeq^{1A}$, $\succcurlyeq^{2A}$, $\succcurlyeq^{1B}$, and $\succcurlyeq^{2B}$ as before. The following two statements are equivalent:

(i) φ^A is a concave transformation of φ^B;

(ii) If $\alpha\beta \sim^{*1B} \beta\gamma$ and $\alpha\beta' \sim^{*1A} \beta'\gamma$, then $\beta' \leq \bar{\beta}$ whenever $\bar{\beta} \sim^{2A} \alpha_p\gamma$ and $\beta \sim^{2B} \alpha_p\gamma$ for some p.[9] □

In the above theorem, identical preferences over lotteries can be imposed by adding the requirement of $\sim^{*2A} = \sim^{*2B}$, or by requiring that always $\bar{\beta} = \beta$ in Condition (ii) above. Thus, the restriction of identical preferences under risk is optional in the above theorem. The condition $\bar{\beta} \sim^{2B} \alpha_p\gamma$ and $\beta \sim^{2B} \alpha_p\gamma$ in Statement (ii) above ensures that $\bar{\beta}$ has the same position relative to α and γ in terms of U^{2A} as β has in terms of U^{2B}. In other words, Statement (ii) in Theorem 2.6.2 is Statement (ii) of Theorem 2.3.6 reformulated for the alternative-outcome interpretation.

2.7. Alternative two-stage models

We show how the above results generalize a number of classical multi-stage results in the literature. All of these models make the within-source EU Assumption 2.5.1 and we will maintain this assumption in the following discussion. The

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[9] There always exists a probability p with $\beta \sim^{2B} \alpha_p\gamma$ because, by Lemma 2.3.1, $U^{2B}(\beta)$ is between $U^{2B}(\alpha)$ and $U^{2B}(\gamma)$, and (unless the trivial case of $U(\alpha) = U(\gamma)$, when $\beta = \bar{\beta} = \alpha = \gamma$ and p can be anything) p is uniquely determined by $U^{2B}(\beta) = pU^{2B}(\alpha) + (1-p)U^{2B}(\gamma)$. Then by continuity of U^{2B} there exists a $\bar{\beta}$; $\bar{\beta}$ is unique because U^{2B} is strictly increasing.

first study that used a representation as in Statement (i) of Theorem 2.5.3 was Kreps & Porteus (1978) (they only considered bounded utility). They obtained the equivalence of Statements (i) and (iv) in Theorem 2.5.5. Their result was applied to intertemporal decision making, assuming that the first-stage uncertainty was resolved before the second-stage uncertainty. Then preference for first-stage resolution amounts to a preference for an early resolution of uncertainty. Statement (ii) in Theorem 2.5.5 is the equivalence resulting from our Theorem 2.3.3. Section 2.6 has provided comparative extensions of Kreps & Porteus' results, where one decision maker has a stronger preference for late (or early) resolution of uncertainty than another decision maker. Under Assumptions 2.5.2 and 2.5.4, Statement (ii) can be used as an alternative to the condition of Kreps & Porteus. It, however, extends to cases where Assumptions 2.5.2 and 2.5.4 need not hold.

For nonexpected utility it is desirable that Assumption 2.5.2, necessitating a violation of the reduction of compound lotteries, be relaxed. It is well known that one of some desirable dynamic-decision principles has to be abandoned under nonexpected utility (Hammond 1988). Machina (1989) strongly argued for abandoning consequentialism rather than the reduction of compound lotteries or dynamic consistency. Karni & Safra (1989) argued for abandoning, in modern terminology, dynamic consistency. Our analysis in terms of $\sim^*$ leaves all these options open, and does not commit to the violation of reduction of compound lotteries.

Kreps & Porteus (1978) considered multi-stage models; their results follow from repeated application of the two-stage results. They formulated their result for the special case of decision under risk, where the probability measure P in the first-stage is also objective and given beforehand, so that it can easily be used as input for obtaining testable preference conditions. For decision under uncertainty, where P is subjective and has to be derived from choice in the revealed preference approach, their condition of PSR is not easy to observe. Then an equality $P(E) = p$ can only be derived from observed choice, which need not always be easy to do. For exampe, if $S = (s_1,s_2\}$ and $P(s_1) = \sqrt{0.5}$, then we are not aware of a finite number of observed preferences within the model assumed that can reveal this probability.[10] Our conditions are directly observable even if first-stage probabilities are subjective.

Neilson (1993) (brought to our attention by a reference in KMM) presented the equivalence of Statements (i) and (iii) in Theorem 2.5.5. To obtain the representation in Theorem 2.5.3, he imposed the von Neumann-Morgenstern axioms on lotteries, and Savage's (1954) axioms on first-stage prospects, where he added a weak continuity axiom to imply continuity of utility; it also implies boundedness of utility. He applied his result to ambiguity, with the first-stage events ambiguous. Then smooth ambiguity aversion and concavity of φ can be interpreted as ambiguity aversion. In this approach the first-stage probabilities P are not objective or known but are subjective and must be derived from choice. Then the first-stage probabilities P in smooth ambiguity aversion are not easy to observe, which makes smooth ambiguity aversion not so easy to observe empirically. This problem is similar to the problem of PSR. In addition, smooth ambiguity aversion uses the utility function U^2, which must similarly be derived from preference.[11] We have relaxed these restrictions and have added comparative results.

Theorem 2.5.3 provides a complete preference foundation for the basic decision model of KMM.[12] These authors indicated that such a foundation was possible but for brevity did not elaborate on it. We can present the entire preference

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[10] We recall that the objective probabilities of the second stage involve different utilities so that they cannot be used for calibration purposes.

[11] This problem can be mitigated. Assume that there are a maximal outcome M and a minimal outcome m, and normalize $U^2(M) = 1$ and $U^2(m) = 0$. We can replace every second-stage prospect by the prospect (p:M, 1–p:m) equivalent to it, and the EU^2 value of that second-stage prospect then is p, which makes EU^2 relatively easy to observe so that it can be used in behavioral preference axioms. In this manner, utilities are converted into probabilities and smooth ambiguity aversion, i.e. risk aversion with respect to second-stage EU, can be seen to be stronger than a preference for first-stage resolution of uncertainty.

[12] These authors assumed countably additive probability measures. This condition can be ensured by adding set-continuity as in Theorem 2.3.3.

foundation and at the same time maintain brevity because we use the same $\sim^*$ tool in all our results. Whereas in KMM's model utility must be bounded for each prospect, we also allow for prospects with unbounded utility.

KMM obtained the equivalence of Statements (i) and (iii) in Theorem 2.5.5. A major step forward in KMM's analysis was that through subtle interpretations of the concepts involved they opened a new way to analyze ambiguity. We discuss their interpretations in some detail. At the outset, the uncertainty in the second stage in their model concerns a Savagean state space T rather than given probabilities as in our approach. The uncertainty in the first stage concerns what is the appropriate subjective second-stage probability measure to use for the Savage state space; this uncertainty entails a psychologically realistic modeling of ambiguity. Thus, in our notation, every state $s \in S$ specifies what the subjective probability measure over T is. First-stage prospects (called second-order acts in their model) assign outcomes contingent on what the appropriate second-stage subjective probability measure on T is. KMM assume that the two-stage decomposition is endogenous. By nevertheless assuming preferences between first-stage prospects to be available, they make it possible to have this endogenous two-stage decomposition observable. Thus, their model becomes considerably more broadly applicable while at the same time coming close to our psychological perceptions of ambiguity. By assuming the second-stage probabilities over T to be completely specified by the first-stage states, KMM achieve that subjective probabilities over the state space T can be treated as known probabilities in the second stage (KMM, Lemma 1, Definition 2, and Assumption 3). This leads to the paradoxical but extremely useful result that the Savagean second-stage uncertainty can be treated as objective risk, as in the second stage of the Anscombe-Aumann model as we do. All aspects of ambiguity are controled in the first stage. The following remark prepares for explaining our generalization of the required richness of KMM's model.

REMARK 7.1 [Richness in KMM]. Not all mappings from S to L need to be available in the preference domain of KMM. They do assume that all first-stage prospects (called second-order acts) are available. They also assume that there is a subalgebra of events on T that have objective probabilities (being the same under all second-stage probability measures P^2 considered) and that is rich enough to generate all probability distributions over outcomes, so that there is also richness for the second-stage prospects. They, further, assume that $T = \Omega\times(0,1]$ where the second component (0,1] generates these objective probabilities, leading to an Anscombe-Aumann-like two-stage decomposition of T. This composition only serves to calibrate risk attitude and should not be confused with our Anscombe-Aumann decomposition. The second stage of our decomposition concerns the whole space T.

Using Assumption 2.5.2, we can generate the preference values of all mappings from S to L in the KMM model. KMM impose some richness assumptions on the set of prospects, i.e. two-stage prospects with nondegenerate Savagean prospects/lotteries assigned to first-stage states of nature, with thought experiments involving two different first-stage subjective probability measures with disjoint supports. In this manner, they derive the richness of Assumption 2.5.4 in the proof of their Proposition 1. □

The contribution of Theorem 2.3.5 (Statement (ii) in Theorem 2.5.5) to the ambiguity-aversion characterization of KMM is that it, again, avoids the need to use subjective probabilities or second-stage utilities as inputs in a preference condition, and that it does not need Assumptions 2.5.2 and 2.5.4. All it needs is that all simple first-stage prospects (“second-order acts”) are available, and those second-stage prospects in the model of KMM that have their outcomes depend only on events with known probabilities (called lotteries by KMM). In particular, we do not impose any richness on the objects of primary interest in the KMM model, i.e. the mappings from Ω to outcomes (i.e. Savagean acts). We do not need the richness assumptions in the second paragraph of Remark 7.1. Thus, we leave complete flexibility regarding the important Savagean acts. Also, we require no commitment to a violation of reduction of compound lotteries. We hope that these generalizations can enhance the applicability of KMM’s approach to ambiguity.

We also generalized KMM's results on comparative ambiguity aversion. We did not use the subjective first-stage probabilities as inputs and, in particular, need not assume that these are the same for different decision makers. We also did not require same preferences over lotteries. Thus, ambiguity attitude can be analyzed as a component completely independent from beliefs and risk attitudes.

Nau (2006) assumed the two-stage decomposition of Figure 2.5.1 to be exogenous. He considered a state-dependent generalization of EU for first- and second-stage prospects, and then used a local version of PSR (called local uncertainty aversion) to characterize concavity of his analog of φ. He also expressed this condition in terms of generalized Pratt-Arrow measures. State-dependent versions of our results could be obtained by using event-dependent relations $\sim^*_E$ in Eqs. 2.3.1-2.3.3, but we will not pursue this point here. Grant, Kajii, & Polak (2001) used a two-stage model as in theorem 2.5.3 in a game-theory context.

2.8. Discussion

A Discussion Assuming that within-source EU (Assumption 2.5.1) is valid. We first discuss the above results for readers who accept EU as an appropriate model for $\succcurlyeq^1$ and $\succcurlyeq^2$, which was the assumption underlying our analysis and the literature cited above. It then is very convenient that we can analyze risk attitudes, intertemporal attitudes, and even ambiguity attitudes, using techniques to analyze utility, i.e., techniques that are well known in traditional economic analyses. Thus, KMM wrote: "One advantage of this model is that the well-developed machinery for dealing with risk attitudes can be applied as well to ambiguity attitudes" (p. 1849). See also Neilson (1993, p. 7). We have presented a convenient tool to analyze marginal utility independently from beliefs. We applied it to some classic results where we obtained conditions that are more general and more easily observable.

A Discussion Assuming that within-source EU (Assumption 2.5.1) is not valid. For readers who believe that EU is not an appropriate model for risk and uncertainty (say the intended applications are descriptive), the interpretation of our preference axiomatizations will be different than the interpretations considered above. In Theorem 2.3.5 it may be felt that the preference condition in Statement (ii) (always preferring a prospect less than its expected value) intuitively concerns aspects of risk attitudes, but our condition in Statement (iv) does less so. The latter condition seems to concern primarily the value of outcomes rather than attitudes towards risk. The fact that Statements (ii) and (iv) are equivalent under EU but are not perceived to be so intuitively, then turns into an argument against EU. This intuition was the basis for the development of many nonexpected utility models (Schoemaker 1982; Schmeidler 1989). Similar observations can be made for Yaari's model in Theorem 2.4.1.

Note that when probabilities are known, Wakker (1994) gives tradeoff conditions characterizing concavity of the utility function even when probabilities are distorted and Abdellaoui (2002) applies a similar tradeoff consistency on probabilities to axiomatize a nonexpected utility model and to give concavity/convexity conditions for the probability distortion.

In the intertemporal model of Kreps & Porteus in section 2.5, the conditions of PSR and smooth ambiguity aversion intuitively capture the attitude towards the timing of the resolution of uncertainty. For our condition in Statement (ii) of Theorem 2.5.5, the intuitive relation with time is less clear. Our condition does not need the Backward Induction Assumption 2.5.2 and, more strongly, it does not seem to involve any of the dynamic structure in the model at all. Prospects with outcomes depending both on first- and second-stage events, even if available, need not be considered, and it does not matter whether the first-stage events really come first or not. Yet, under EU, our condition is equivalent to conditions considered in the literature. It may be felt that EU for $\succcurlyeq^1$ and $\succcurlyeq^2$ deprives the conditions used in the literature from their intuitive content. Under this view, such an intuitive content could better be modeled through different models, such as with anxiety as an additional component of utility if rationally relevant. If preference for early or late resolution of uncertainty and the violation of reduction of compound lotteries are driven by irrational factors, then it may be more suited to use irrational nonEU models instead of the within-source EU Assumption 2.5.1.

In the models of ambiguity in sections 2.5-2.7, it may likewise be felt that, whereas conditions advanced in the literature intuitively concern attitudes towards ambiguity, our conditions in terms of $\sim^*$ do not and they exclusively concern the value of outcomes. For example, our condition already implies concavity of φ if we maintain the richness of outcomes but consider only one nondegenerate first-stage event E. That is, first-stage events, which should capture all ambiguity, hardly play any role in our preference condition and the condition is based almost exclusively on outcomes. From this perspective, the within-source EU Assumption 2.5.1, by equating our conditions with the intuitive conditions put forward in the literature, seems to deprive those intuitive conditions from their intuitive content. It suggests that ambiguity is better not analyzed in terms of the utility of outcomes, and is better analyzed through functions that apply to events such as the nonadditive measures of Schmeidler (1989) and the set of multiple priors in Gilboa & Schmeidler (1989).

The criticism of EU advanced here can be compared to that advanced by Rabin (2000). He derived unacceptable implications of taking utility as index of risk attitude. We have added undesirable implications of EU when utility is used to model ambiguity. The criticisms described here add to the violations of within-source EU revealed by Allais' (1953) paradoxes.

2.9. Conclusion

We have presented a convenient tool for analyzing marginal utility. It led to many generalizations of classical results and to alternative preference foundations and interpretations. Readers who have doubts about the appropriateness of EU to model risk and ambiguity attitudes can test their confidence in EU by inspecting if the alternative preference conditions put forward in this chapter can convey the same intuition as preceding preference conditions used in the literature. If they do, then EU is appropriate. If they do not, then EU must be questioned. For example, if the reader expects that smooth ambiguity aversion is found empirically in the two-stage model of section 2.5, but that the standard sequences in Eq. 2.3.5 will be the same and will exhibit the same utility graphs (Figure 2.3.1) for stage 1 and stage 2 uncertainty, then this amounts to EU not being valid.

Appendix – Proofs

We begin with two lemmas that will be useful for the elaboration of our main results.

LEMMA 2.A.1. Let $f : I \to \mathbb{R}$ be continuous, with $I \subset \mathbb{R}$ an interval. Then f is concave if and only if, for every $\alpha,\beta \in I$, there exists a $p_{\alpha,\beta}$ with $0 < p_{\alpha,\beta} < 1$ and $f\big(p_{\alpha,\beta}\alpha + (1-p_{\alpha,\beta})\beta\big) \geq p_{\alpha,\beta}f(\alpha) + (1-p_{\alpha,\beta})f(\beta)$.

PROOF. This follows from Hardy, Littlewood, & Pòlya (1934, Observation 88). We will only need the case where $p_{\alpha,\beta} = p$ is independent of α,β. □

LEMMA 2.A.2. Let $I \subset \mathbb{R}$ be an interval. A continuous and strictly increasing function $f : I \to \mathbb{R}$ is concave if and only if $[f(\alpha) - f(\beta) = f(\beta) - f(\gamma) \Rightarrow \beta \leq (\alpha+\gamma)/2]$.

PROOF. The condition between brackets is equivalent to midpoint concavity, which is the case of Lemma 2.A.1 with $p_{\alpha,\beta} = 0.5$ for all α,β, and is equivalent to concavity of f by Lemma 2.A.1. □

PROOF OF LEMMA 2.3.1. Follows from substitution of EU (Köbberling & Wakker 2004, Observation 1). □

PROOF OF THEOREM 2.3.3.

PART I [The Implication (i) ⇒ (ii)]. This follows from substitution. For simple prospects and conditions (a)-(c) and (e), it was established by Köbberling & Wakker (2003, Corollary 10). For general prospects and condition (e), it was established by Wakker (1993, Lemma 1.8, Corollary 2.14, and section 4.4).

PART II [The Implication (ii) ⇒ (i)].

Assume (ii). We first restrict attention to simple prospects, for which we will not use truncation continuity. For finite state spaces the result was obtained by Köbberling & Wakker (2003). The extension to all simple prospects for a general state space is routine.

The extension of the representation to general, possibly unbounded, prospects follows from Theorem 2.5 in Wakker (1993). Note here that EU is a special case of Wakker's CEU (Choquet expected utility), and that Wakker's step equivalent assumption is implied by the existence of certainty equivalents.

PART III [Further Results]. The uniqueness results follow from Köbberling & Wakker (2003, Corollary 10). The set-continuity is necessary and sufficient for countable additivity follows from substitution, similar to Wakker (1993, section 4.1; for additive measures we only need Wakker's (1993) Eq. 4.2, and only for $A = \varnothing$). □

PROOF OF COROLLARY 2.3.4. The only thing to be added to the proof of Theorem 2.3.3 is that subjective probabilities must agree with objective probabilities. This follows first for equally likely events in n-fold partitions with objective probability 1/n, which because of symmetry should also have subjective probability 1/n. Then it follows for every event with rational probability j/n by comparing to a union of j events with probability 1/n. It follows for events with real probabilities from the previous result plus monotonicity of both probability and preference. Note that the latter extension does not use countable additivity. □

PROOF OF THEOREM 2.3.5. (i) ⇒ (ii) is exactly as for decision under risk.

For (ii) ⇒ (i) assume (i). By nondegeneracy, there exists an event E with $0 < P(E) = p < 1$. Risk aversion implies $\alpha_E\beta \preccurlyeq (p\alpha + (1-p)\beta)$ (the latter taken as degenerate) and thus, under EU, $U(p\alpha+(1-p)\beta) \geq p \times U(\alpha) + (1-p) \times U(\beta)$. By Lemma 2.A.1, U is concave.

(i) ⇔ (iii) is due to Debreu & Koopmans (1982).

(i) ⇔ (iv) follows from Lemmas 2.3.1 and 2.A.2 □

PROOF OF THEOREM 2.3.6. Express outcomes in U^1 units, and apply Theorem 2.3.5 to φ instead of U with φ such that $U^2 = \varphi \circ U^1$. □

PROOF OF THEOREM 2.4.1. (i) ⇔ (ii) is by Yaari (1969). (i) ⇔ (iii) is by Theorem 2.3.6, with equality of probabilities added. □

PROOF OF THEOREM 2.5.3. For (i) ⇒ (ii), assume (i). EU with P^j and U^j represents $\succcurlyeq^j$ for j = 1 (because φ is strictly increasing) and 2 (with $U^1 = \varphi \circ U^2$), which by Theorem 2.3.3 implies Conditions (a)-(d) in Statement (ii). Assumption 2.5.2 follows because all α^i have the same EU^2 value as the prospects they replace.

We now assume (ii) and derive (i). Theorem 2.3.3 applied to both $\succcurlyeq^1$ and $\succcurlyeq^2$ implies that there exist continuous and strictly increasing functions U^1 and U^2 and a probability measure P such that first stage prospects $(E_1^1{:}x_{11}, \ldots, E_m^1{:}x_{m1})$ are evaluated through $\sum_{i=1}^m P(E_i)U^1(x_{i1})$, and second-stage prospects are evaluated through

EU^2, being EU with respect to U^2. Define the continuous strictly increasing $\varphi = U^1 \circ (U^2)^{inv}$.

Consider any prospect x. By continuity and strict increasingness of U^2, we can obtain α_s such that $\alpha_s \sim^2 x(s)$ (i.e., these have the same EU^2 value) for all s. By backward induction, x ~ y with $y(s) = \alpha_s$ for all s. Thus, the evaluation of x must equal:

$$\int_S U^1(y(s))\,dP(s) \;=\; \int_S \varphi\big(U^2(y(s))\big)\,dP(s) \;=\; \int_S \varphi\big(EU^2(x(s))\big)\,dP(s). \;\square$$

PROOF OF THEOREM 2.5.5. (i) ⇔ (ii) is by Theorem 2.3.6.

(i) ⇔ (iii) is due to KMM (Proposition 1) In our setup, the derivation is as follows. (i) ⇒ (iii) is the traditional risk aversion implication. For (iii) ⇒ (i), smooth ambiguity aversion applied to event E from Assumption 2.5.4 implies that $\varphi(p\alpha+(1-p)\beta) \geq p\varphi(\alpha) + (1-p)\varphi(\beta)$ which, by Lemma 2.A.1, implies (i).

For (i) ⇔ (iv), make Assumption 2.5.2, and assume that $0 < P(E) = p < 1$ for some first-stage event E. Take some arbitrary $U^2(\alpha)$ and $U^2(\beta)$. Then $\alpha_E\beta \preccurlyeq \alpha_p\beta$ implies $p\varphi\big(U^2(\alpha)\big) + (1-p)\varphi\big(U^2(\alpha)\big) \leq \varphi\big(pU^2(\alpha) + (1-p)U^2(\alpha)\big)$. By Lemma 2.A.1, φ is concave.

(i) ⇔ (v) follows from the equivalence (i) ⇔ (iii) of Theorem 2.3.5 under the alternative-outcome interpretation. □

PROOF OF THEOREM 2.6.1. For (i) ⇔ (ii), we note that $\sim^{*2A} = \sim^{*2B}$ is equivalent to U^{2A} and U^{2B} having same equalities of utility differences, which, given that they are defined on an interval and are continuous and strictly increasing, is equivalent to them being the same in the sense of being in the same interval class.

By Theorem 2.3.5, the first part of condition (ii) is equivalent to U^{1A} being more concave than U^{1B}. Because $U^{1A} = \varphi^{A}\circ U^{2A}$, $U^{1B} = \varphi^{B}\circ U^{2B}$, and $U^{2A} = U^{2B}$, it is equivalent to φ^{A} being more concave than φ^{B}.

(i) ⇔ (iii) is due to KMM (Theorem 2), and follows from the equivalence of (i) and (ii) in Theorem 2.4.1 under the alternative-outcome interpretation. □

PROOF OF THEOREM 2.6.2. First, φ^{A} and φ^{B} being continuous and strictly increasing functions, there exist a continuous and strictly increasing function Ψ such that $\varphi^{A}=\Psi\circ\varphi^{B}$. Let us fix $U^{2A}(\alpha)=U^{2B}(\alpha)=a$, $U^{2A}(\gamma)=U^{2B}(\gamma)=c$ and $U^{2A}(\beta')=b'$. Then $\bar{\beta} \sim^{2A} \alpha_{p}\gamma$ and $\beta \sim^{2B} \alpha_{p}\gamma$ imply $U^{2B}(\beta)=U^{2A}(\bar{\beta})=b$ (where $b=pa+(1-p)c$). Statement (ii) can be rewritten:

$\varphi^{B}(a) - \varphi^{B}(b) = \varphi^{B}(b) - \varphi^{B}(c)$ and $\Psi\circ\varphi^{B}(a) - \Psi\circ\varphi^{B}(b') = \Psi\circ\varphi^{B}(b') - \Psi\circ\varphi^{B}(c)$ imply $\varphi^{B}(b')\leq\varphi^{B}(b)$. By Lemma 2.A.2, Ψ must be concave. The theorem follows. □

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Chapter 3.
Combining Experts' Judgments: A Choice-Based Study

Abstract

For many decisions, from environmental policy to strategic investments of firms, decision makers seek advice from experts. Hence, a major issue concerns the aggregation of multiple opinions by decision makers. This chapter provides an experimental study of the impact on decision of imprecision and conflict in experts' probabilistic forecasts. Three contexts are defined: when experts agree on a precise probability, when they agree on an interval of probability (imprecise ambiguity) and when they disagree (conflicting ambiguity), each of them having a different estimation. We use Cumulative Prospect Theory to describe behavior when a unique probability is given and certainty equivalents under (conflicting or imprecise) ambiguity are corrected for risk attitudes so as to obtain the impact of conflict and imprecision. Our results show that decision makers do not always use the mean estimation, even without any knowledge about the reliability of the sources. Furthermore, both the informational context (conflicting or imprecise ambiguity) and the probability level strongly influence the decision. Eventually, extremeness of advisors seems to have an impact on decision.

3.1. Introduction

When facing a new problem, a decision maker may have to seek advice from experts. For instance, as consumers of fossil fuels or activities emitting carbon dioxide, we may have an impact on global warming and we have to choose whether or not we carry on the same way of life. We can find information and advice from different groups of experts on Wikipedia[13]. The first group says that there is *"90 percent certainty that global warming is caused by man's burning of fossil fuels"*. The second group of experts tells us that *"Detailed examination of current climate data strongly suggests that current observations do not correlate with the assumptions or supportable projections of human-induced greenhouse effects"*. Now, we have to decide: Should we apply the Kyoto protocol? Should we decrease our consumption of fossil fuels? Are ecological policies for decreasing our carbon dioxide emission really useful?

This chapter aims at understanding how decisions are made in such cases. Without any information about the reliability of the sources, what do we do? On which probability do we base our decision? Do we decide by taking into account our fuel consumption having an impact on the environment with a probability that belongs to [0%,90%]? Or do we consider that the probability is either 0 or 90%? Or eventually, do we use the mean probability 45% or another linear combination of the two boundaries? This chapter addresses these questions. It defines a choice-based method to deal with them and presents the results of an experimental study that compares these three possibilities so as to identify the main characteristics of decision processes when the decision maker asks experts for probability judgments.

In decision analysis, an important field of research concerns combinations of experts' judgments. Budescu (2006) distinguishes three approaches in this literature: the two first orientations are mostly normative and consist in providing a

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[13] See http://en.wikipedia.org/wiki/Scientific_opinion_on_climate_change for much more advices than the two ones we are using.

unique judgment, either by computation or by group discussion (see also Clemen & Winkler 1999). On the contrary, the third approach is mainly descriptive and prevails in the behavioral decision literature. Our study belongs to this last research direction and tries to understand what it takes for a complex informational setting to be reduced to a single probability. Do we really reduce important information such as experts' imprecision or even experts' disagreement into a single probability? It thus seems relevant to test whether or not the information structure constitutes a key element in the decision process.

The impact of conflict and/or imprecision does not just concern managers' decision-making, it also affects their communication. When a rumor or a critical report threaten a firm's existence by describing a potential risk and when the managers of the firm have reliable information that contradicts this risk, will they induce the same effect by enhancing imprecision as by focusing on conflict in risk evaluations? Will they be equivalently trusted? Such questions directly concern risk communication. For instance Slovic (1993) highlights that open debates among experts may decrease the trust the public have in their risk evaluations. More recently, Dean and Shepherd (2007) study the effect of conflict and consensus in risk communication about genetically modified food.

In order to provide some answers to these various questions, we develop a technique and conduct an experimental study with three contexts: when experts agree on a precise probability, when they agree on an (imprecise) interval of probability and when they disagree, each of them having different estimations. We first introduce the concept of revealed belief (the probability that a decision maker's choices reveal as being associated to an ambiguous risk description) and then we define some behavioral conditions and indexes characterizing attitudes towards ambiguity. Eventually, we elicit both the revealed beliefs and the indexes in an experiment.

Based on this experiment, we claim that:

i) Under both imprecise and conflicting ambiguity, agents do not simply use the mean estimation. There are significant cases, in which agents act as if the 'true' probability were closer to one expert's estimation than the other's (or to one of the two boundaries of the interval) even without any information on experts' respective reliability.

ii) The information structure (imprecision/conflict) has a significant impact on attitudes. Depending on the probability level, presenting the information as being imprecise or conflicting may lead to different behaviors.

iii) Between two conflicting assessments, an extreme probability (0 or 100%) has a higher impact than an intermediate probability. When an expert gives an extreme judgment ("the event is sure" or "the event is impossible") then (s)he has a stronger effect on the decision, as if this expert was more trustworthy or more reliable.

The structure of the article is as follows. Section 3.2 consists of a review of the literature. Section 3.3 sets up the theoretical framework. Section 3.4 describes the experimental design. The key results are presented in section 3.5, section 3.6 discusses and section 3.7 concludes.

3.2. Review of the literature

Aggregation of experts' opinions represents a key issue in decision analysis and consequently, several orientations have been studied so as to deal with it. The first research direction aims at developing formal models for combining assessments from multiple sources into a unique probability. Those models are based either on axioms (e.g. Stone 1961, Genest 1984), or directly on Bayes' rule (e.g. Winkler 1968, Clemen & Winkler 1993). The second approach is behavioral and consists in finding how group discussion or interaction between experts can lead to a consensual evaluation; the Delphi method (Dalkey, 1969) or the Nominal Group Technique (Delbecq, Van de Ven & Gustafson 1975), for instance, are based on a first individual evaluation followed by a discussion and then by another individual assessment. In the Delphi method, first evaluations remain anonymous while they are presented by their author in the Nominal Group Technique. Clemen & Winkler (1999) provide a review of the literature about these two first (mathematical and behavioral) approaches. Eventually, the third and last part of this literature on aggregation of multiple sources is descriptive and study how a unique decision maker combines information. Studies are based on judgments (e.g. Sniezek & Buckley 1995) or on choices (e.g. Du & Budescu 2005). This chapter clearly wants to contribute to this choice-based descriptive literature on combining judgments of experts.

We can more generally relate our study to the literature on ambiguity. Since Ellberg (1961) the impact of ambiguity (or vaguely known probabilities) on choices has been well-documented (cf. Camerer and Weber 1992 for a review of the literature). Contrary to what the Subjective Expected Utility framework says (Savage 1954), there is much evidence that ambiguity affects decision-making in some systematic ways: decision makers usually exhibit ambiguity aversion for low probabilities of loss and large probabilities of gain but become ambiguity seeking for large probabilities of loss and small probabilities of gain (e.g., Cohen, Jaffray and Said 1985, 1987; Hogarth and Einhorn 1990; Lauriola and Levin 2001; Viscusi and Chesson 1999).

In addition, recent experimental research on ambiguity has shown that decision-makers are sensitive to the sources of ambiguity (Cabantous 2007; Smithson 1999). In the literature, ambiguity is commonly implemented by either providing the participants with ranges of probabilities (cf. Budescu et al. 2002; Cohen, Jaffray and Said 1985; Ho, Keller and Keltika 2002) or by providing them with conflicting probabilistic estimates (cf. Einhorn and Hogarth 1985; Kunreuther, Meszaros and Spranca 1995; Viscusi and Chesson 1999 for examples of expert disagreement as a source of ambiguity). Those two implementations of ambiguity are usually assumed to be equivalent. Smithson (1999) however has recently shown that decision-makers disentangle these two sorts of ambiguity and are most of the time averse to conflict: they tend to exhibit a preference for imprecise ambiguity (i.e., ranges of probability) over conflicting ambiguity (i.e. disagreement over the probability value of an uncertain target event).

In a model with nonlinear probability weighing, such as Cumulative Prospect Theory (CPT) (Tversky and Kahneman 1992), the finding that attitude towards ambiguity depends on the location of the probability implies that the weighting function is more "inverse-S shape" for events with vaguely known probability (i.e. ambiguous events) than for their counterparts with precisely known probability (i.e. risky events). We will indeed use CPT because it allows attitudes to depend on probabilities. In so doing, it accommodates the pattern of behaviors to ambiguity observed in most empirical research. Tversky and Wakker (1995) formalize preference-based conditions for the "less sensitivity to uncertainty than to risk" effect and their equivalent formulation under CPT. In addition, assuming source dependence and greater subadditivity of the weighting functions for uncertainty than for risk have been established as the way to study the effects of various sources of uncertainty on decision weights (e.g., Abdellaoui 2000; Abdellaoui, Vossman and Weber 2005; Kilka and Weber 2001; Tversky and Fox 1995; Tversky and Wakker 1995; Wakker 2004). A novelty of our research is that it extends this literature on decomposition of decision weights so as to bridge the gap with both the experimental literature about ambiguity represented by sets or intervals of probabilities and the literature about aggregation of probability judgments.

Note that there exist alternative models for representing attitudes towards ambiguity, above all models with multiple priors. Maxmin Expected Utility (Gilboa & Schmeidler 1989) represents behaviors by the worst possible case (in terms of expected utility) among a subjective set of priors. This corresponds to ambiguity aversion. Some generalizations of this model allow for ambiguity seeking (α-MEU of Ghirardato et al 2004) or for a linear combination of the worst case and the average one (Gajdos et al, 2007; in their model, the set of possible probability distribution is yet objective). Even if we will not use this family of models in the analysis of our experiment, we will discuss about them regarding our results in section 3.6.

3.3. Theoretical framework

3.3.1. Behavioral definitions

For simplicity of presentation we restrict the present treatment to a single domain of outcomes and we consider that the objects of choice are binary prospects on the outcome set $\mathbb{R}^-$ (non-mixed negative binary prospects). This article focuses on losses because vagueness of probabilistic information is quite common in the loss domain (e.g. insurance decision or all the examples we gave in introduction). Moreover, only a few studies have looked at probability weighing, beliefs and ambiguity attitudes in this domain (e.g., Etchart-Vincent for probability weighing). We assume that the decision-maker's preferences on prospects are represented by a binary preference relation. As usual, $\succcurlyeq$ denotes weak preference, $\sim$ and $\succ$ respectively denote indifference and strict preference among binary prospects.

We note xpy the usual "risky" binary prospect yielding the outcome x with probability p and the outcome y (with 0>y>x) with probability (1−p). We then define two special cases of ambiguity: imprecise ambiguity (A^i) and conflicting ambiguity (A^c). Imprecise ambiguity, where the uncertain target event is characterized by an imprecise probability (i.e. a probability interval) is probably the most common operationalization of ambiguity in the literature (e.g., Budescu et al. 2002). In this article, we denote x[p−r,p+r]y an A^i prospect that gives x with an imprecise probability that belongs to the interval [p−r,p+r] and y (with 0>y>x) otherwise. The other typical way to implement ambiguity is to provide the participants with conflicting probability estimates (e.g. Viscusi and Chesson 1999). We denote x{p−r,p+r}y the A^c prospect which gives x with a probability which can be either (p−r) or (p+r) and y (with 0>y>x) otherwise. Throughout, r will be assumed as fixed and strictly positive. Studying the impact of variations of r is left for future research. The sets

$$\Delta^{\mathrm{i}} = \{[\mathrm{p}-\mathrm{r}, \mathrm{p}+\mathrm{r}]: \mathrm{r} \leq \mathrm{p} \leq 1-\mathrm{r}\}$$

and

$$\Delta^{\mathrm{c}} = \big\{\{\mathrm{p}-\mathrm{r}, \mathrm{p}+\mathrm{r}\}: \mathrm{r} \leq \mathrm{p} \leq 1-\mathrm{r}\big\}$$

represent the two different ambiguous contexts.

DEFINITION *(Revealed Belief):* A revealed belief q is a probability such that the certainty equivalent for a risky prospect xqy is equal to the certainty equivalent for the A^i (A^c) prospect x[p−r,p+r]y (x{p−r,p+r}y). Formally, we write [p−r,p+r]$\approx^R$q whenever there exist x<y and z from $\mathbb{R}^-$ such that x[p−r,p+r]y~z and xqy~z. Similarly {p−r,p+r}$\approx^R$q whenever there exist x<y and z from $\mathbb{R}^-$ such that x{p−r,p+r}y~z and xqy~z.

The binary relation $\approx^R$ constitutes a useful tool to study attitudes towards ambiguity since it allows defining several testable preference conditions, analogous to the ones Wakker (2004) introduces (see also Tversky and Fox 1995; Tversky and Wakker 1995). By analogy with researches on weighting functions (e.g., Wu and Gonzalez 1996 and 1999), this chapter focuses on two noticeable physical features of revealed beliefs: their degree of curvature and their degree of elevation. In addition, it considers that each characteristic reflects a specific psychological process at play when decision makers evaluate uncertain gambles: the degree of curvature measures the decision maker's degree of sensitivity whereas the degree of elevation reflects the decision maker's perception of attractiveness of the lottery (Gonzalez and Wu 1999).

We first focus on the degree of curvature of the revealed beliefs. Eq. 3.3.1 (resp. 3.3.2) defines the testable preference conditions for *less sensitivity to A^i (resp. A^c) than to risk*[14].

If [p−r,p+r]$\approx^R$q and [p'−r,p'+r]$\approx^R$q', then |q−q'|≤|p−p'|. (3.3.1)

If {p−r,p+r}$\approx^R$q and {p'−r,p'+r}$\approx^R$q', then |q−q'|≤|p−p'|. (3.3.2)

These two equations mean that A^i and A^c revealed beliefs vary less than the attached intervals. Typically, this indicates that a decision maker reacts less to a change in the probability level when the probabilities are ambiguous than when

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[14] Note that for a given 1>r>0, revealed beliefs are defined over [r,1−r]. This domain does not contain 0 and 1. Because we automatically stay away from the bounds and only deal with intermediate probabilities, we do not use any boundary constant (unlike Tversky & Wakker 1995).

they are precise. This is the reason why these equations define less sensitivity to ambiguity than to risk. On the contrary, when the revealed beliefs vary more than the attached intervals, opposite inequalities hold, and decision makers exhibit *more sensitivity to ambiguity (A^i, A^c) than to risk.* Furthermore, as decision makers might disentangle the two sources of ambiguity and set up different certainty equivalents for A^i and A^c gambles, A^i and A^c revealed beliefs can also differ in terms of sensitivity. Eq. 3.3.3 defines the testable preference condition for *less sensitivity to A^i than to A^c*. Note that the inverse inequality defines *more sensitivity to A^i than to A^c*.

If $[p-r,p+r]\approx^R q$, $[p'-r,p'+r]\approx^R q'$, $\{p-r,p+r\}\approx^R h$ and $\{p'-r,p'+r\}\approx^R h'$,

then $|q-q'|\leq|h-h'|$. (3.3.3)

The second noticeable physical feature of revealed beliefs is their degree of elevation usually referred to as the degree of ambiguity aversion. The next equations are concerned with this effect and define respectively "*ambiguity aversion under A^i*" and "*ambiguity aversion under A^c*" for negative prospects.

If $[p-r,p+r]\approx^R q$, then $q\geq p$. (3.3.4)

If $\{p-r,p+r\}\approx^R q$, then $q\geq p$. (3.3.5)

Similarly, "*more ambiguity aversion under A^i than under A^c*" is defined by:

If $[p-r,p+r]\approx^R q$ and $\{p'-r,p'+r\}\approx^R q'$, then $q\geq q'$. (3.3.6)

In the loss domain indeed, when a revealed belief q of an A^i prospect, giving x with $\{p-r:p+r\}$, is greater (smaller) than midpoint probability p, this means that the decision-maker finds the A^i prospect less attractive (more attractive) than a risky prospect that gives x with probability p. This should lead him/her to exhibit ambiguity aversion (ambiguity seeking). Note that we use the midpoint p, that is the simple arithmetic mean of $[p-r,p+r]$ and $\{p-r,p+r\}$, to define the degree of ambiguity aversion/seeking of revealed beliefs. Since no information about experts' competence is available, the midpoint p, which is the solution of lottery reduction when uniform partition holds, is indeed a useful benchmark to which we can compare revealed beliefs.

3.3.2. Representation

We assume Cumulative Prospect Theory (Tversky and Kahneman 1992) for risky and ambiguous contexts, with a single utility function. According to CPT, the value of a prospect xpy with $x<y<0$ is:

$$xpy \longmapsto w(p)u(x) + \big(1 - w(p)\big)u(y)$$

where, u(.) is the value function satisfying $u(0)=0$, and w(.), called the probability weighting function, is a continuous and strictly increasing function from [0,1] to [0,1] satisfying $w(0)=0$ and $w(1)=1$. Similarly, we define the values of A^i and A^c prospects as follows:

$$x[p-r,p+r]y \longmapsto W^i([p-r,p+r])u(x) + \Big(1 - W^i([p-r,p+r])\Big)u(y)$$

and

$$x\{p-r,p+r\}y \longmapsto W^c(\{p-r,p+r\})u(x) + \big(1 - W^c(\{p-r,p+r\})\big)u(y)$$

where W^i and W^c are the weighting functions for A^i and A^c prospects.

Under these assumptions we know that there exists a unique revealed belief for each element of Δ^ior Δ^c. There therefore exists a unique function q^i from Δ^i to [0,1] such that $[p-r,p+r]\approx^R q$ is equivalent to $q^i([p-r,p+r])=q$ and there also exists a similar function q^c on Δ^c such that $\{p-r,p+r\}\approx^R q$ is equivalent to $q^c(\{p-r,p+r\})=q$.

The CPT value for imprecisely ambiguous prospects can thus be rewritten:

$$x[p-r,p+r]y \longmapsto w\big(q^c([p-r,p+r])\big)u(x) + \Big(1 - w\big(q^c([p-r,p+r])\big)\Big)u(y)$$

Finally, if the prospect is a A^c prospect, its CPT value is given by:

$$x\{p-r,p+r\}y \longmapsto w\big(q^c(\{p-r,p+r\})\big)u(x) + \Big(1 - w\big(q^c(\{p-r,p+r\})\big)\Big)u(y)$$

Knowing the value function u (defined under risk) and the individual probability weighting function w of a participant (defined under risk as well), the A^i and A^c revealed beliefs can be deduced from the certainty equivalents for A^i and A^c prospects respectively. To complement the non-parametric analysis, which highlights the impact of probability levels on revealed beliefs, the study also relies on a regression line to characterize the general properties of revealed beliefs. We use a linear approximation of the functions q^i and q^c in order to define the sensitivity and ambiguity aversion indexes. These indexes are directly adapted from Kilka and Weber (2002).

First, we determine two values a and b for each context such that

$q^i([p-r,p+r])$ is approximated by a^i+b^ip

and

$q^c(\{p-r,p+r\})$ is approximated by a^c+b^cp.

Then, b is considered as a sensitivity index (since this slope measures the decision-maker's sensitivity to changes in probability) and the index of ambiguity aversion is defined as the average elevation (a+b/2) of the estimation. Because the linear estimation goes from 0 to 1, the value of the estimation at p=½ gives a good estimate of the elevation of the function. We can therefore determine the degree of ambiguity aversion of the revealed beliefs by assessing the departure of the ambiguity aversion index, a+b/2, from the benchmark ½. Note that those indexes will not only enable us to study attitudes under each kind of ambiguity but also to compare together the degrees of sensitivity and aversion under A^i and A^c. (Appendix furthers the explanations of those indexes.)

3.4. Method

3.4.1. Participants

The participants in this study were 61 post-graduate students (60 men, 1 woman, median age = 22) in civil engineering at the Ecole Nationale Supérieure d'Arts et Métiers (ENSAM) in Paris, France. They were invited by email to participate in a study on decision-making, and guaranteed a 10€ flat participation fee. None of them had already participated in an experiment in decision making.

3.4.2. Procedure

The experiment was conducted in the form of computer-based individual interview sessions, using software specifically developed for the experiment. The experimenter and the participant were seated in front of a laptop and the experimenter entered the participant's statements into the computer after clear confirmation. After a brief explanation of the task, where the participants were asked to assume their own role and give their own preferences and a series of three trial choices, the experiment started. On average, the participants required about 30 minutes to complete the experiment. There was absolutely no time pressure, the participants were given the time they needed and encouraged to think carefully about the questions.

3.4.3. Stimuli

We designed the experiment to estimate participants' certainty equivalents (CEs) for three kinds of negative binary prospects: conventional risky prospects, imprecisely ambiguous (A^i) prospects and, conflictingly ambiguous (A^c) prospects (see Table 3.4.1).

In Table 3.4.1 below, the first ten prospects are risky prospects of the form xpy. For instance, prospect 1 is a risky prospect yielding the outcome −1000€ with probability 10% and the outcome 0€ with probability 90%. The five next prospects are A^i prospects with probability intervals. Prospect 11 for instance, is an A^i prospect, of the form x[p−r,p+r]y, that gives the outcome −1000€ with (a) probability belonging to the range 0% and 20%. Last, prospects 16 to 20 are A^c prospects are of the form x{p−r,p+r}y. They give x with probability which can be either (p−r) or (p+r) and y (with y>x) otherwise. Prospect 20 for instance gives the outcome −1000€ with probability that is either 80% or 100% and 0 otherwise. It is noteworthy that the 20 prospects are such that the probabilities varied all over the probability interval [0,1]. In addition, and so as to simplify matters, in all A^i and A^c prospects we fixed the width of the probability interval 2r to 20.

Table 3.4.1: The twenty prospects

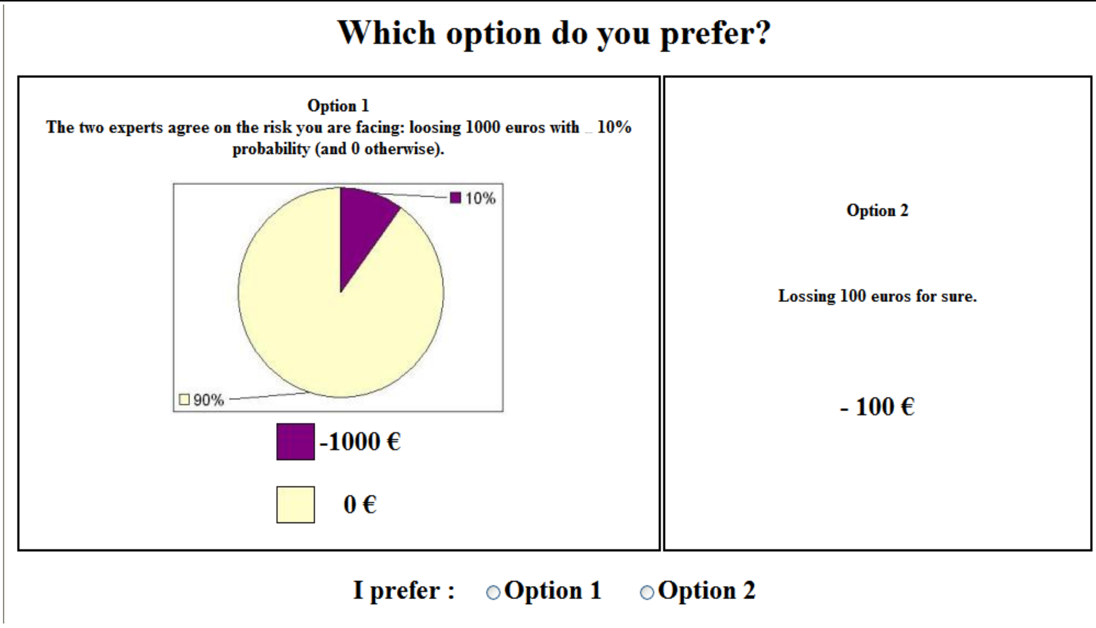
#	Context	p	x	y	#	Context	p−r	p+r	x	y
1	Risk	10	−1000	0	11	A^i	0	20	−1000	0
2	Risk	30	−1000	0	12	A^i	20	40	−1000	0
3	Risk	50	−1000	0	13	A^i	40	60	−1000	0
4	Risk	70	−1000	0	14	A^i	60	80	−1000	0
5	Risk	90	−1000	0	15	A^i	80	100	−1000	0
6	Risk	50	−500	0	16	A^c	0	20	−1000	0
7	Risk	50	−500	−250	17	A^c	20	40	−1000	0
8	Risk	50	−750	−500	18	A^c	40	60	−1000	0
9	Risk	50	−1000	−500	19	A^c	60	80	−1000	0
10	Risk	50	−1000	−750	20	A^c	80	100	−1000	0

#: prospect number

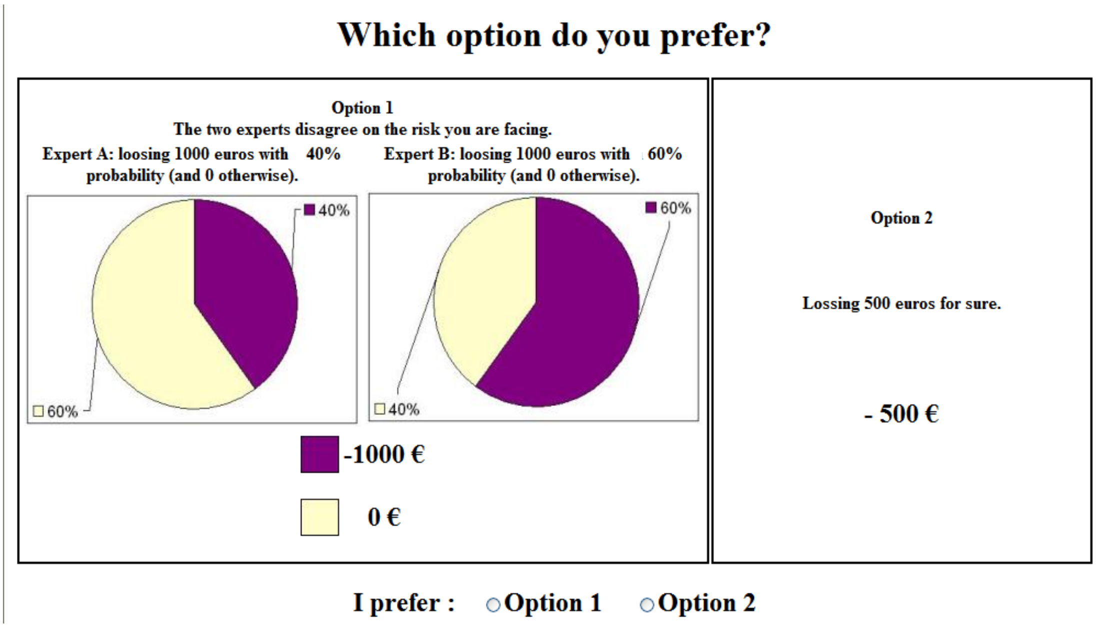
To estimate subjects' CEs for the twenty prospects, we constructed a bisection-like process. Such a method does not require the participants to state a precise value such that they would be indifferent between losing that amount for sure and playing a two-outcome negative lottery. It involves choices only, and is therefore easier for the participants to answer than the direct matching method. Moreover, choice method has been found to generate more reliable data (Bostic et al., 1990). With a bisection-like process, from 3 to 7 choices between a given prospect and a sure loss are required to estimate the CE of a prospect. The CE of a prospect is then determined by computing the average of the highest sure loss accepted and the lowest sure loss rejected.

In this experiment, each trial started with a choice between a prospect and its expected value. Figure 3.4.2 illustrates the task the participants were presented with. To simplify the participants' task, the risky, A^i and A^c screenshots had exactly the same structure: option 1 (the prospect) was systematically displayed at the left-hand side, option 2 (the sure loss) was displayed at the right-hand side of the computer screen and, whatever the informational context, x was in purple and y in yellow. In the risky context (screenshot A), we used a typical pie with a fixed line to provide the participants with a visual representation of the task. For these risky prospects the participants, who were told they had received advice from two independent experts, could read: "The two experts agree on the risk you are facing: loosing X euros with p% probability (and loosing €0 otherwise)."

In the A^i context, the participants could read the following "The two experts agree on the risk you are facing: loosing X euros with probability belonging to the range (p−r)% and (p+r)% (and loosing €0 otherwise)." In addition, to help the participants understand A^i prospects, we introduced a dynamic pie. This means that the program made the size of the pie vary slowly between (p−r) and (p+r). Eventually, screenshot B displays the typical choice-task in the A^c context. In that context, we introduced two different fixed pies to make clear to the participants that the two sources of information did not have the same estimate of the probability of the loss and, we told them that "The two experts disagree on the risk you are facing. Expert A: loosing X euros with (p−r)% probability (and 0 otherwise). Expert B: loosing X euros with (p+r)% probability (and 0 otherwise)."



Screenshot A: Risky context



Screenshot B: A^c context

Figure 3.4.2: Screenshots of typical choice tasks

In addition to this series of about 100 choices (i.e., 20 prospects times a number of choices between 3 and 7), we introduced 6 choice questions, at the end of the questionnaire, to check the reliability of the data. The participants were asked to give their preference for the following six choice questions: prospects 1-3-16-18-11-13 vs. their expected value. We then can check for the consistency of the answers the respondents gave to the six questions for which we have two statements per subject.

The sequence of presentation of the twenty prospects (prospects 1-18-10-12-4-16-7-15-11-3-20-9-14) was chosen to have questions with different contexts alternating and with different magnitudes of losses and different probability levels. It was the same for all the subjects who thus completed exactly the same questionnaire. The program did not enforce dominance and allowed the participants to modify their answer after confirmation if they wish.

3.4.4. Elicitation technique

In this experiment, five risky prospects of the form x.50y and five risky prospects of the form −1000p0, where the probability p of losing −1000€ varied from 10 to 90 were used to simultaneously elicit parametric estimations of the value function u(.) and of the probability weighting function w(.). We used the five A^i prospects and the five A^c prospects, with the normalization conditions $u(-1000)=-1$ and $u(0)=0$, to estimate the decision weights under imprecise ambiguity (W^i) and under conflicting ambiguity (W^c). Note that under the representation previously assumed (see 3.3.2), $x[p-r,p+r]y\sim z$ is equivalent to $W^i([p-r,p+r])=-u(z)$ and $x\{p-r,p+r\}y\sim z$ is also equivalent to $W^c(\{p-r,p+r\})=-u(z)$. This means that decision weights are equal to the utility of the certainty equivalents. Then, to proceed with the analysis, revealed beliefs can be computed using the following equivalence:

$$
\begin{aligned}
& W^i([p-r,p+r]) &=& \quad -u(z)\\
\Leftrightarrow\quad & w(q^i([p-r,p+r])) &=& \quad -u(z)\\
\Leftrightarrow\quad & q^i([p-r,p+r]) &=& \quad w^{-1}(-u(z))
\end{aligned}
$$

and

$$
\begin{aligned}
& W^c(\{p-r,p+r\}) &=& \quad -u(z)\\
\Leftrightarrow\quad & w(q^c(\{p-r,p+r\})) &=& \quad -u(z)\\
\Leftrightarrow\quad & q^c(\{p-r,p+r\}) &=& \quad w^{-1}(-u(z)).
\end{aligned}
$$

Consequently, knowing w, we can deduce revealed beliefs from decision weights. We will thus be able to study revealed beliefs for several levels of probability. Lastly, we will use these values to obtain a linear approximation of q^i and q^c so as to compute the sensitivity indexes and the ambiguity aversion indexes.

3.5. Results

3.5.1. Data reliability

In this article reliability refers to participants' stability (or consistency) for the six questions that were presented twice (prospects 1-3-16-18-11-13 in Table 3.4.1). Across questions the mean reliability rate is 77.32%. This means that on average about 3/4 of the participants gave the same answer when the identical choice task was presented twice. Table 3.5.1 gives the consistency rate for each question. A Friedman test reveals that the consistency rate does not significantly depend on the informational context (χ^2_2=2.15; p=0.341). Similarly, a Cochran test for dichotomous data shows that reliability does not significantly depend on the question (χ^2_5=9.98; p=0.076). The overall picture thus suggests that participants were consistent in their responses and that the elicited preferences are reliable.

Table 3.5.1. Consistency check

Context		Risk		A^c		A^i
Prospect number	1	3	16	18	11	13
Number of consistent subjects	42	54	45	44	51	47
Consistency rate	69%	89%	74%	72%	84%	77%

3.5.2. Utility function

For each participant, the utility function and the probability transformation function were simultaneously obtained from the ten certainty equivalents under risk using standard nonlinear least square regression (Levendberg-Marquadt algorithm). Parametric estimation of the utility function in the loss domain was conducted using the power functional form $u(x)=-(-x)^{\beta}$, $x\leq 0$. The median is 1.13, the mean 1.26 and the standard deviation 0.52. A two-tailed t-test on the mean estimate β reveals that it is significantly greater than 1 (t_{60}=3.99; p=0.000) indicating concavity of the utility functions. Though one might expect to obtain a convex utility function, it is noteworthy that in the loss domain, results on utility functions tend to be rather mixed. Recent experimental studies for instance have reported convex utility functions but have also show that, at the individual level, there are always some subjects exhibiting concave utility functions (e.g., Abdellaoui, Bleichrodt and Paraschiv 2007; Abdellaoui 2000; Tversky and Kahneman 1992; Fennema and Van Assen 1999; Etchart-Vincent 2004). Abdellaoui, Bleichrodt and L'Haridon (2007) for instance have reported linear utility functions for losses between 0 and −10,000€; and in Abdellaoui, Bleichrodt and Paraschiv (2007), the utility function is convex between 0 and −100,000FF (0 and −15,000€). More generally, in the loss domain, two phenomena generate different effects: one effect, called diminishing sensitivity (Tversky and Kahneman 1992) implies convexity of the utility function and is strongly related to the numerosity effect (see Köbberling et al. 2007), but the neoclassical decreasing marginal utility – the second effect - generates concavity. Our results therefore suggest that for small amounts (between 0 and −1000€), the impact of diminishing marginal utility exceed the impact of diminishing sensitivity.

3.5.3. Weighting function

Parametric estimations of individual weighting functions were conducted using Goldstein and Einhorn's (1987) two-parameter specification, $w(p)=\delta p^{\gamma}/(\delta p^{\gamma})+(1-p)^{\gamma}$. This specification has been frequently employed in recent experimental studies (e.g., Latimore et al. 1992; Tversky and Fox 1995; Abdellaoui 2000; Etchart-Vincent 2004) because it provides a clear separation between two physical properties of the function, elevation and curvature, each of which is captured independently by a parameter (Gonzalez and Wu 1999). The δ parameter mainly controls the elevation of the function and thus the attractiveness of the gamble, whereas the γ parameter essentially governs the curvature of the function and captures the decision-makers' ability to discriminate between probabilities. Table 3.5.2 gives the median and mean estimates of the parameters.

Table 3.5.2. Summary statistics for parameters of the weighting function

Parameter	Median	Mean	SD
δ	0.72	0.75	0.33
γ	0.73	0.86	0.49

A two-tailed *t*-test shows that δ is significantly smaller than 1 ($t_{60}=-6.00$; $p=0.000$). This indicates that the probability weighting function exhibits a small degree of elevation and reflects the fact that on average the participants perceived the negative risky gambles as attractive ones. Although such a small degree of elevation may be surprising in the loss domain, Abdellaoui (2000) obtained a similar result with δ = 0.84; and Etchart-Vincent (2004) reported δ smaller than 1 for both small and large losses (δ = 0.84 and δ = 0.85 respectively). Concerning the curvature of the probability weighting function, the estimate of γ is significantly smaller than 1 ($t_{60}=-2.24$; $p=0.029$, two-tailed *t*-test), indicating that the probability weighting function exhibits the usual inverse S-shape. This estimate of γ is in accordance with previous empirical estimates in the loss domain: Abdellaoui (2000) for instance reported γ = 0.65 and Etchart-Vincent found γ = 0.836 and γ =0.853 for small and large losses respectively.

3.5.4. Revealed beliefs

One novelty of this study is that estimated degrees of beliefs are not "judged probabilities" (i.e., a subjective probability given through a direct judgment) but revealed beliefs (i.e., a belief component derived from choices). In this article, participants' beliefs are indeed determined through choices and directly inferred from certainty equivalents using Wakker's (2004) theorem. Table 3.5.4 reports the revealed beliefs' mean and median values (as well as the standard deviations) of the revealed beliefs in the two ambiguous contexts (called q^i and q^c). It also gives the results of two-tailed *t*-tests with midpoint probabilities.

Table 3.5.4. Mean, Median (SD) values for revealed beliefs

Midpoint Probability		Revealed Belief q^c	q^i
0.1	Mean	0.06^{***} (AS)	0.19^{***} (AA)
0.3		0.31	0.33
0.5		0.49	0.53
0.7		0.73	0.73
0.9		0.90	0.86^{**} (AS)
0.1	Median	0.04 (0.07)	0.13 (0.16)
0.3		0.30 (0.15)	0.31 (0.15)
0.5		0.50 (0.19)	0.54 (0.13)
0.7		0.75 (0.13)	0.73 (0.15)
0.9		0.92 (0.08)	0.88 (0.11)

**:p<0.05 ; **:p<0.01 ; ***:p<0.001.*
AA/AC: Ambiguity Aversion/ Ambiguity Seeking

Patterns depicted in Table 3.5.4 show that for medium probabilities, revealed beliefs do not differ from midpoint probabilities. In such cases, revealed beliefs are almost equal to p, the probability of the risky loss, leading participants to be "neutral to ambiguity" (cf. W^c and W^i are not different from w). However, such neutrality to ambiguity is no more present when participants are exposed to ambiguous losses with extremes probabilities. This is true in particular in the A^i context, where the revealed belief associated with the lowest range of probability is significantly above the corresponding midpoint probability, indicating that participants acted "as if" the probability of the A^i loss was higher than the probability of the risky loss (inducing ambiguity aversion). On the contrary, the A^i revealed belief associated with the highest range of probability is significantly below the corresponding midpoint probability, inducing ambiguity seeking behavior.

Concerning the A^c context, the series of two-tailed *t*-test reveals that revealed beliefs associated with medium probability losses are not significantly different from the midpoint probability. This indicates, once again, that ambiguity does not have any impact for medium probabilities losses (neutrality to ambiguity) but does affect extreme probability losses. More specifically, the participants are ambiguity seeking for low probability losses but are ambiguity neutral for high probability losses – $q^c(\{0,0.2\})$ is significantly below p=0.1 but $q(\{0.8,1\})$ is not significantly different from midpoint probability.

To complement the analysis of the impact of ambiguity on revealed beliefs, we also tested for differences between the two revealed beliefs. The series of *t*-tests for paired samples reported in Table 3.5.5 confirm previous findings on decision weights. They show again that for extreme events, where ambiguity has an impact on revealed beliefs, the kind of ambiguity matters. For instance, for very unlikely losses, q^i is significantly greater than q^c, reflecting a net preference for A^c (over A^i). For very likely losses, the kind of ambiguity also matters but the respective effects of A^i and A^c on revealed-beliefs are reversed: q^i is significantly smaller than q^c, suggesting that participants prefer A^i over A^c (conflict aversion) when facing very likely losses.

Table 3.5.5. Revealed beliefs:
results of two-tailed paired *t*-tests

Midpoint probability	Revealed beliefs $q^c - q^i$
0.1	$t_{60}=-6.37^{***}$ (CS)
0.3	$t_{60}=-0.91$
0.5	$t_{60}=-1.43$
0.7	$t_{60}=0.05$
0.9	$t_{60}=3.5^{***}$ (CA)

**:p<0.05; **:p<0.01; ***:p<0.001.*
CA/CS: Conflict Aversion/ Seeking

Figure 3.5.3 (below) illustrates these results graphically. It first shows that for medium probabilities, revealed beliefs are not different from midpoint probabilities. This means that ambiguity has no impact on revealed beliefs associated with medium probabilities. Second, the figure makes clear that for extreme probability losses (i.e., very likely and very unlikely losses), where ambiguity has an impact on revealed beliefs, the source of ambiguity does matter. The figure indeed shows that whereas q^i starts above the 45° (leading to ambiguity aversion), crosses the line near 0.9 and ends below the 45° diagonal (leading to ambiguity seeking); q^c starts below the 45° line (leading to ambiguity seeking) and tends to finish above it (reflecting a tendency to ambiguity aversion). Third, the figure also clearly depicts the finding that even if both the q^i and q^c revealed beliefs belong to the range [p−r,p+r], represented by the two parallel dashed lines above and below the 45° line, they do not look like a constant linear combination of the two end points of the range or set of probabilities. This finding will be confirmed by the analysis of the sensitivity indexes in paragraph 4.5.

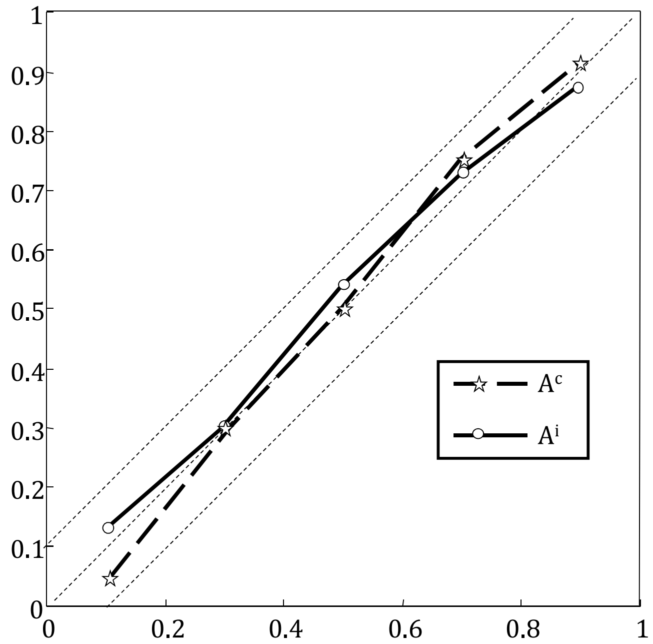


Figure 3.5.3: Revealed beliefs (q^i and q^c median values)

3.5.5. Indexes of sensitivity and ambiguity aversion

This subsection proceeds with the analysis conducted in 3.5.4 and tries to understand something of the causes of participants' attitude to ambiguity by analysing the sensitivity index and the ambiguity aversion index (see 3.3.2). Participants' non neutrality to ambiguity can indeed result from two distinct but complementary mechanisms (see Wakker 2004): they can exhibit a dispreference (or a preference) for ambiguity because they consider that ambiguous gambles are inherently less (or more) attractive than risky gambles (cf. ambiguity aversion). But, their reaction to ambiguous gambles can also result from a more "cognitive" effect of vaguely known probabilities on their ability to discriminate between different levels of likelihood (cf. sensitivity index). Table 3.5.6 (below) reports the mean and median values of the sensitivity and ambiguity aversion indexes we obtained using linear optimization: $q^i([p-r,p+r])$ is approximated by a^i+b^i*p and $q^c(\{p-r,p+r\})$ by a^c+b^c*p.

First, a series of two-tailed *t*-test on the ambiguity aversion index, which measures the global elevation of revealed beliefs, indicates that A^i generates significant ambiguity aversion ($a^i+b^i/2$ is significantly higher than ½; $t_{60}=2.94$; $p=0.005$). In the loss domain indeed, the higher the index, the more ambiguity averse the participants are. These *t*-tests also show that contrary to A^i, the A^c context does not induce any specific effect ($a^c+b^c/2=0.50$; $t_{60}=0.04$; $p=0.97$). An additional *t*-test for paired sample confirms that participants are significantly more ambiguity averse under A^i than under A^c ($a^i+b^i/2>a^c+b^c/2$; $t_{60}=2.75$; $p=0.008$). In this experiment, thus, A^i clearly generates higher beliefs than risk and A^c do. Since the participants were presented with negative outcome, this finding indicates that participants found, on average, the A^i prospects less attractive than the A^c and risky prospects.

Table 3.5.6: Ambiguity aversion and Sensitivity indexes
mean, median (SD) values and results of two-tailed *t*-test

Index of	Comparison to	A^c Mean	A^c Median (SD)	A^i Mean	A^i Median (SD)
Ambiguity aversion (a+b/2)	1/2	0.50	0.50 (0.06)	0.53^{**}	0.53 (0.08)
Sensitivity(b)	1	1.05^{*}	1.04 (0.20)	0.87^{***}	0.94 (0.27)

*: $p<0.05$; **: $p<0.01$; ***: $p<0.001$.

Second, the analysis reveals that the two sensitivity indexes are significantly different from 1. This indicates that both sources of ambiguity had an impact on participants' discriminability. There is nevertheless a key difference between the two sensitivity indexes: while the sensitivity index is significantly smaller than 1 in the A^i context ($t_{60}=-3.84$; $p=0.000$), it is significantly higher than 1 in the A^c context ($t_{60}=2.07$, $p=0.042$). This finding suggests that A^i decreases the participants' ability to distinguish among various levels of likelihoods (by comparison with their ability to discriminate between precise probabilities). The effect of A^i on revealed beliefs therefore corresponds to "less sensitivity under imprecise ambiguity than under risk". On the other hand, the finding that the sensitivity index is greater than 1 in the A^c means that the participants are more sensitive to changes in conflicting probabilities than they are to changes in precise probabilities. This "over-sensitivity" phenomenon results from a strong sensitivity to extreme cases (i.e., cases where one expert says that the loss is sure and cases when one expert says it is impossible). An additional *t*-test (for paired sample) confirms that both indexes are significantly different from each other ($t=6.83$; $p=0.000$). We can therefore conclude that, in this experiment, the participants are less sensitive to changes of probability levels when receiving imprecise probabilities of the form "both sources agree that the probability of loosing belongs to the interval [p−r,p+r]" than when they face an A^c situation where one source of information says that the probability of the target event is p−r but the other source says it is p+r.

To conclude, the analysis interestingly reveals that the results we obtained for decision weights and revealed beliefs can be explained by i) the negative impact of imprecision on the attractiveness of prospects and ii) by the opposite impacts of imprecise and conflicting ambiguities on sensitivity. In other words, under A^c, the "non-neutrality" towards ambiguity is mainly due to a stronger sensitivity; but under A^i, it results from the combined effects of imprecise probability on both the attractiveness of the gamble (i.e., ambiguity aversion) and on participants' ability to discriminate between different levels of likelihood (i.e., weaker sensitivity than under risk).

3.6. Discussion

The experimental design we used to study the properties of decision weights and revealed beliefs might raise some objections as it did not involve any real incentive mechanism. In addition to Camerer and Hogarth (1999)'s argument that for simple tasks (such as a certainty equivalent task without any performance measure) real incentives do not systematically make any difference, there is another reason for this methodological choice: in this study, the use of real incentives would have confounded the description of the informational contexts by introducing strategic interaction between the subject and the experimenter. Consider for instance an experiment in which a subject receives x€ as an initial endowment and then is asked for his/her certainty equivalent of the prospect −x[0.6,0.8]0. The subject can anticipate that an experimenter facing his/her budgetary constraint will minimize the cost of the experiment by implementing the worst case. Consequently, the subject may consider [0.6,0.8] as being 0.8 for sure. This kind of anticipations would have prevented us from studying the effects of ambiguity on decision weighs and beliefs.

There exist several alternatives in the literature for this kind of problems: the first one consists in using balls and urns with an adequate display but we wanted to study the impact of the informational contexts (imprecision and conflict of experts). Some experimental displays used participants as experts (e.g. Budescu &Yu 2007) but cannot control for experts beliefs: agreement or not, imprecision or not, width of the estimations. As a conclusion of this point, we chose hypothetical choices so as to control for amounts (significant losses), for contexts and for probabilities.

The experimental design might raise a second critique: in this research, revealed beliefs are derived from certainty equivalents, whereas in Abdellaoui et al. (2005), choice-based probabilities are directly obtained by finding indifference between a risky and an uncertain prospect. Since revealed beliefs and choice-based probabilities should be equivalent assuming transitivity of preferences, it could be asked why the same technique was not applied here. The answer to that question is that during a pilot study, it appeared that asking participants for choice-based probability made them focus on the probability dimension (see Tversky, Sattath and Slovic 1988 for the effects on preferences of the response scale used). As a result, they tended to systematically compute the midpoint of the ambiguous probabilities [p−r,p+r] and {p−r,p+r}; and the averaging strategy ended up to be very common. Consequently, we introduced a certainty equivalents task to allow the participants to consider the two dimensions of the choice. It is noteworthy that this methodological strategy also contributes to prevent subjects from easily guessing what the main purpose of the experiment was.

In the literature review, we introduced an alternative family of models: multiple priors. We would like to stress three points that explain why we did not use them. First, the only model that is directly observable is Gajdos et al.'s (2007) one because they postulate the existence of a given family of possible distributions while other multiple prior models have a nonobservable subjective set of priors.

Indeed, estimations, which assume that the subjective set corresponds to the given one, are quite abusive (e.g.. Then, multiple prior models are founded on Expected Utility under risk while we found significant probability weighting, i.e., significant deviation from a linear treatment of probabilities. Eventually, we found that attitudes (ambiguity aversion or ambiguity seeking) depend both on probability and on contexts while these models assume a constant attitude (e.g. Gilboa & Schmeidler 1989; Gajdos et al 2007). In our data, we can observe significant these changes of behaviors even at the certainty equivalent level.

3.7. Conclusion and further research

The purpose of this chapter was to investigate the potential effects on decision of imprecision or conflict of experts. These situations were modeled through Imprecise Ambiguity or A^i (where the decision maker learns that the probability of the uncertain target event belongs to a probability interval) and, Conflicting Ambiguity or A^c (where the decision-maker receives precise but different estimates of the likelihood of an uncertain target event). To achieve its objective, the chapter first provided a general framework based on the Cumulative Prospect Theory for studying revealed beliefs under different informational contexts. By providing a coherent framework that is able to accommodate the pattern of behavior under ambiguity observed in most experimental studies, this chapter contributes to the literature on ambiguity (Camerer and Weber 1992; Ellsberg 1961). Another contribution of the chapter is to extend Wakker (2004)'s revealed-preference study of decision weights and beliefs to two specific kinds of uncertain contexts which, even though they are common operationalizations of ambiguity in the experimental literature on ambiguity, have been neglected in the literature on decomposition of decision weights. The chapter therefore also contributes to the literature on decision weights (e.g. Abdellaoui et al. 2005) by extending its scope of investigation to new informational contexts.

Eventually, let us return to the series of claims stated in the introduction to assess the contributions of the research.

i) Agents do not use the mean estimation.

Research on attitude towards ambiguity has speculated that nonneutrality to ambiguity (i.e. ambiguity aversion or ambiguity seeking) results from the fact that decision-makers probability judgments of ambiguous events are different from the precise probability of their risky counterpart (i.e., the midpoint of the range of probability). Budescu et al. (2002) for instance have suggested that decision-makers' probability judgments under ambiguity are a weighted combination of the two end points of the range of probability. To estimate participants' attitude to ambiguity, they estimated, for each participant, a single "probability vagueness coefficient". In the loss domain, for instance, if the estimated probability vagueness coefficient of a participant is below ½ (resp. above), this means that the participant gives more weight to the upper bound of the probability interval and then, is ambiguity averse (resp. ambiguity seeking).

One limitation of that approach is that it cannot capture the common finding that attitude towards ambiguity depends on the location of the probability (Camerer and Weber 1992; Viscusi and Chesson 1999). In this article, we therefore adopted a different viewpoint: we introduce the notion of revealed belief to allow the weighted combination of the two end points to vary along the probability interval. Our experiment confirms the need for such an approach as it shows that the weighted combination of the two end points depends on the location of the midpoint probability. In the A^c context for instance, revealed beliefs for very unlikely events are above the midpoint probability (i.e., more weight is given to the upper bound of the probability interval) but they are below the midpoint probability for very likely events (i.e. weight is given to the lower bound of the probability interval).

It is noteworthy that the highest sensitivity of decision weights for extreme probabilities we observed in this experiment is in line with previous research on decision weights. Wu and Gonzalez (1996, 1999) in particular highlight that diminishing sensitivity (i.e. sensitivity decreases when the distance from the reference points "impossibility" and "certainty" increases) affects both decision weights under risk and uncertainty. As a consequence, it is more likely to observe significant changes in sensitivity near those reference points than for medium probability events. For such events indeed the distance from the reference points is higher and thus the sensitivity to changes in likelihood is smaller.

ii) The information structure (imprecision/conflict) has a significant impact on the agents' attitudes

Until Smithson (1999), the experimental literature on ambiguity has assumed that the type of ambiguity, conflict or imprecision, does not matter. In this research, we experimentally tested this assumption and we compared revealed beliefs in two different sorts of ambiguity commonly used in the literature: imprecise ambiguity (A^i) and conflicting ambiguity (A^c). Our experimental results support Smithson (1999) as they make clear that decision-makers do not equivalently react to the two kinds of ambiguity. We indeed found that the way extreme probabilities are weighted significantly depends on the kind of ambiguity.

In particular, tests on the A^i and A^c revealed beliefs strongly suggest that the A^i and A^c revealed beliefs could be modelled as slightly different non-additive linear combinations of the upper and lower bounds of the probability set (or range): the A^i revealed belief function would tend to be inverse S-shape (sub-additive function) but the A^c revealed belief function would rather have an S-shaped form.

Lastly, analysis of the ambiguity aversion and sensitivity indexes highlighted the fact that implementing ambiguity through imprecision decreases participants' discriminability and makes them more pessimistic while conflicting ambiguity generates "over-sensitivity". These results, all pointing in the same direction, therefore strongly suggest that ambiguity does not correspond to a unique, homogeneous set but congregates informational contexts that are differently treated by decision makers and induce different responses. In this article, by stressing the impact of the source of ambiguity (i.e., imprecision or conflict) on revealed beliefs we therefore contributed to further the analysis of source dependency (Tversky and Fox 1995, Tversky and Wakker 1995, Kilka and Weber 2001).

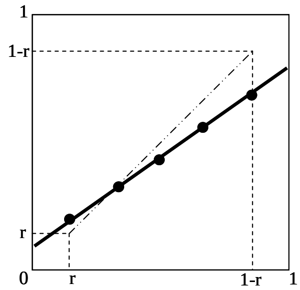
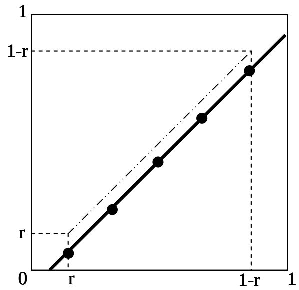
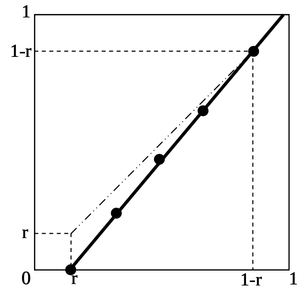
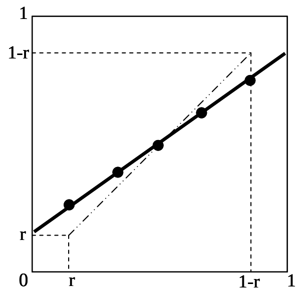
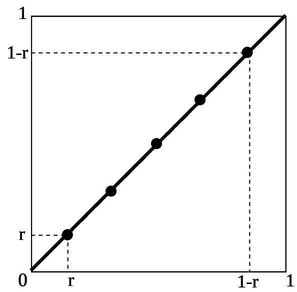
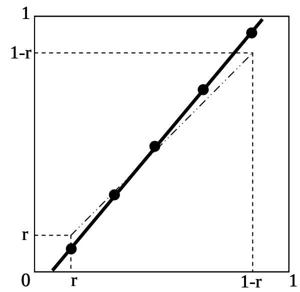
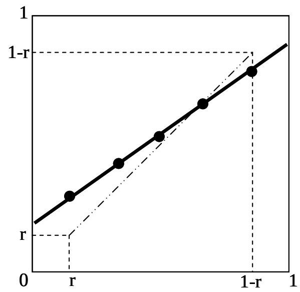
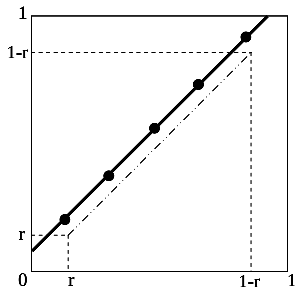
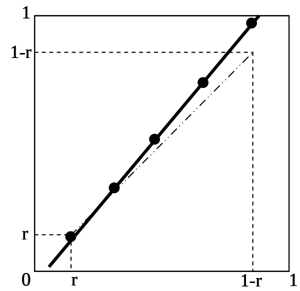
iii) Between two conflicting probability assessments, an extreme probability (0 or 100%) has a higher impact than an intermediate one.

Under ambiguity, we found that conflict decreases revealed beliefs for low probabilities and increases them for high probabilities. This result should be associated with recent findings in the literature: Budescu & Yu (2007) find that decision makers' confidence is positively correlated with extremeness of advisors (an advisor that says the probability is 0 or 1). It seems that extremeness gives more weight to the expert in terms of advisor's confidence. It is noteworthy that this effect counterbalances the usual "less sensitivity under ambiguity" effect. The reveal belief for {0.8,1} being not significantly different from the mean evaluation can be explained by the combination of the less sensitivity effect and of the extremeness effect that cancel each other. Further research to disentangle those two effects has still to be conducted.

Appendix

Table 3.A.1 (below) – based on Wakker (2004) – visually presents the indexes of sensitivity and ambiguity aversion and illustrates how the combination of the two different psychological processes combine together to create(s) a non additive revealed-belief exhibiting some elevation.

Table 3.A.1. Representations of the degrees of sensitivity and ambiguity aversion of revealed beliefs

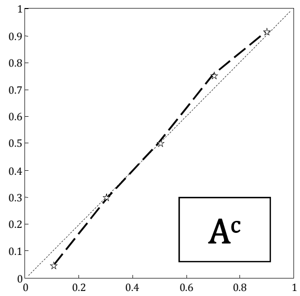
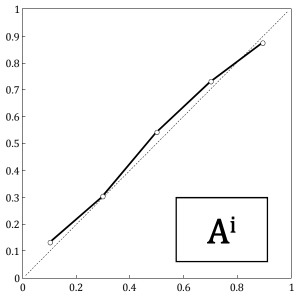
	Less sensitivity to ambiguity than to risk b<1	Same sensitivity to ambiguity as to risk b=1	More sensitivity to ambiguity than to risk b>1
Ambiguity Seeking in the loss domain a+b/2<1/2			
Ambiguity Neutrality a+b/2=1/2			
Ambiguity Aversion a+b/2>1/2			

The box in the middle of the table depicts a revealed-belief without any ambiguity aversion or ambiguity seeking and with the same sensitivity to ambiguity as to risk. The rows above and below the neutrality row then depict the preference or dispreference for ambiguous lotteries (over risky lotteries) that could arise, independently of any effect of ambiguity on the ability to discriminate between different levels of likelihoods. The interpretation of the attractiveness obviously depends on the domain of the outcome. In the loss domain, a shift-down of the revealed belief ($a+b/2<1/2$) reflects ambiguity seeking because the revealed-belief for the ambiguous lottery is below the midpoint probability p at all levels. On the contrary a shift-up of the revealed belief ($a+b/2>1/2$) (in the loss domain) traduces the fact that the participant(s) considers the probability of losing with the ambiguous lottery is larger than the probability of losing with the risky lottery at all levels. The participant thus exhibits ambiguity aversion. The opposite interpretation holds in the gain domain.

By moving now from the column in the middle to the left-hand column or the right-hand column, we consider another kind of deviation: b, the slope of the function q, measures the decision-maker's sensitivity to changes in probabilities. b equals 1 reflects the fact that the participant exhibits exactly the same sensitivity to ambiguity as to risk: ambiguity does not affect the his/her ability to distinguish among various likelihood levels. On the contrary, when ambiguity affects the participant's discriminability, b is different from 1. In that case, the participant is said to have less sensitivity to ambiguity than to risk when $b<1$ (right-hand column) and to have more sensitivity to ambiguity than to risk when $b>1$ (left-hand column).

According to subsection 3.5.5, there is significant ambiguity aversion under imprecise ambiguity only and revealed beliefs exhibit significantly less sensitivity to A^i than to risk and more sensitivity to A^c than to risk. Table 3.A.2 displays these results:

Table 3.A.2. Our results

	Less sensitivity to ambiguity than to risk $b<1$	Same sensitivity to ambiguity as to risk $b=1$	More sensitivity to ambiguity than to risk $b>1$
Ambiguity Seeking in the loss domain $a+b/2<1/2$			
Ambiguity Neutrality $a+b/2=1/2$			
Ambiguity Aversion $a+b/2>1/2$			

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Chapter 4.
Combining Bayesian Beliefs and Willingness to Bet to Analyze Attitudes towards Uncertainty

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Abstract

Many deviations from Bayesianism have been found for choices under uncertainty with unknown probabilities (“ambiguity”). General choice-based models and qualitative tests have been developed in the economic literature. However, tractable quantitative measurements of uncertainty- and ambiguity-attitudes provided so far were based on introspective psychological inputs, such as the anchor probabilities in the influential venture theory of Einhorn and Hogarth. We provide a choice-basis for the psychological approaches, thus introducing their tractable quantitative measurements into economic models. To do so, we identify proper sources of uncertainty, comprising events generated by a common mechanism and with a uniform degree of ambiguity. For such sources we can define choice-based probabilities and then (source-dependent) probability transformations. The latter display attitudes towards uncertainty and ambiguity in tractable graphs. We demonstrate the implementability of our approach in an experiment. The prevailing phenomena towards uncertainty and ambiguity that we find comprise both aversion and insensitivity.

4.1. Introduction

Whereas most analyses of decision under uncertainty assume that risks can be quantified through probabilities, most uncertainties in economics concern one-shot events for which no probabilities are available (Keynes 1921, Knight 1921). De Finetti (1931), Ramsey (1931), and Savage (1954) subsequently showed that probabilities can still be defined for such one-shot events. These probabilities are derived from observed willingness to bet, and express subjective ("Bayesian") degrees of belief rather than objective frequentist data. Decisions are derived using the expected utility model. Despite the early criticisms by Allais (1953) and Ellsberg (1961), the Bayesian expected utility model has become the standard tool for analyzing decision under uncertainty in economics.

Recent developments in behavioral and experimental economics have, however, demonstrated the need to introduce more realistic and sophisticated models that do account for Allais' and Ellsberg's criticisms. While Machina & Schmeidler (1992) demonstrated, in their probabilistic sophistication model, that Bayesian beliefs can still be reconciled with non-Bayesian decision attitudes as in the Allais paradox, the problems posed by the Ellsberg paradox are more fundamental and have usually been taken to imply that beliefs must be non-Bayesian. This chapter shows that a recent generalization of Machina & Schmeidler (1992) by Chew & Sagi (2006a,b) can be used to nevertheless reconcile the Ellsberg paradox with Bayesian beliefs in many settings of interest to economic analyses. This reconciliation can be used to greatly facilitate the analysis of uncertainty and ambiguity. We need not develop new mathematical results to achieve our goals, but we do develop new concepts to obtain our goals.

The first step to obtain our reconciliation is to distinguish between different sources of uncertainty. A source of uncertainty concerns a group of events that is generated by a common mechanism of uncertainty. In Ellsberg's (1961) classical two-color paradox, one source of uncertainty concerns the color of a ball drawn randomly from an urn with 50 black and 50 red balls, and another source concerns the color of a ball drawn randomly from an urn with 100 black and red balls in unknown proportion. Alternatively, one source of uncertainty can concern the Dow

Jones index, and another source the Nikkei index. Whereas probabilistic sophistication is usually violated across sources, as first demonstrated by the Ellsberg paradox (explained later), within single sources it is often satisfied.[15] Then, and this is contrary to what has commonly been thought, probabilities can still be comparable across sources. Key in this comparability, and the resulting reconciliation with Bayesian beliefs, is to allow for different decision attitudes rather than different beliefs across different sources.

The usefulness of distinguishing between different sources of uncertainty was first advanced by Tversky in the early 1990s (Heath & Tversky 1991; Tversky & Kahneman 1992; Tversky & Fox 1995). Hsu et al. (2005) found different brain activities for different sources of uncertainty. Several studies have demonstrated that decision behavior can be affected by the degree to which an agent feels competent regarding the source (Fox & Tversky 1998, Kilka & Weber 2001). Grieco & Hogarth (2007) re-examined a recent study by Camerer & Lovalla (1999), separating a belief component (overconfidence) from a general decision-component (perceived competence). They found that the excess entry of entrepreneurial activities is primarily driven by the decision-component rather than by beliefs.

Probabilistic sophistication within a source entails a uniform degree of ambiguity for that source (Wakker 2007), which is why we call such sources *uniform*. For such sources we can obtain probabilities that are not introspective as the anchor-probabilities of Einhorn & Hogarth (2005, 2006), Kilka & Weber (2001), and others, but that are entirely choice-based. Thus, by identifying uniform sources, we give a economic (revealed-preference) basis to the psychological approaches, and we introduce the tractability of these psychological approaches into the models popular in the economic literature today.

With choice-based probabilities available, we can define source-dependent probability transformations. Thus we can completely capture attitudes towards

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[15] This combination of phenomena is supported by empirical evidence in Fox & Clemen (2005) and Halevy (2007).

uncertainty and ambiguity through convenient graphs. Quantitative measurements of uncertainty attitudes through choice-based weighting functions were presented before by Abdellaoui, Vossman, & Weber (2005) and Diecidue, Wakker, & Zeelenberg (2007). These, however, concerned general weighting function that do not achieve the tractability of the psychological measurements and that become intractable for large state spaces.[16] In our approach, the dimensionality of general weighting functions is reduced, which makes them tractable.

Uniform sources can resolve a well known problem of classical Bayesian decision models. This problem concerns the impossibility to distinguish between symmetry based on information and symmetry based on absence of information (Aragones, Gilboa, Postlewaite, & Schmeidler 2005). In our approach, we can make this distinction while maintaining Bayesian beliefs, by allowing different decision attitudes for different sources. All uniform sources satisfy the symmetry mentioned, but parameters that we will introduce in this chapter can distinguish between different levels of information for different sources. They will, for instance, demonstrate that there is more information for risk than for the ambiguous sources considered.

In an experimental implementation, we first introduce a new method for measuring subjective probabilities. By using Chew & Sagi's (2006a) exchangeability, our method need not commit to any decision model. In this way we can identify uniform sources while allowing for different decision attitudes across sources. Unlike traditional measurements of beliefs from the psychological literature that are based on introspective or hypothetical judgments, our method is based on revealed preferences implemented through real incentives. We, therefore, call the obtained probabilities choice-based. We applied our method to sources of uncertainty taken from daily life, where we tested for uniformity. In five of the six cases considered, uniformity was satisfied. The one violation that we found was similar to the three-color Ellsberg paradox, where more than one mechanism of uncertain-

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[16] For a state space S a general weighting function requires 2^S-2 evaluations and is of a higher cardinality.

ty is involved. In the five cases where uniformity was verified, the obtained probability estimates were well calibrated relative to objective statistics that became available later, supporting the validity of our elicitation method.

Our measurements of weighting functions can be applied to Choquet expected utility of Gilboa (1987) and Schmeidler (1989) and prospect theory of Tversky & Kahneman (1992), where we completely capture attitudes towards uncertainty and ambiguity for all prospects. Because we only use binary prospects, for which virtually all nonexpected utility models coincide, such as the multiple priors model (Gilboa & Schmeidler 1989), our findings are also relevant for those other nonexpected utility models.

Most traditional methods for measuring utility are essentially based on expected utility, and the resulting utilities are distorted by deviations from expected utility. We use a method introduced by Abdellaoui, Bleichrodt, & l'Haridon (2007) that gives correct utilities also if expected utility is violated. Whereas the measurement of Bayesian beliefs and utility would suffice to completely capture decisions under uncertainty under the traditional expected utility, additional measurements are needed to capture decisions under uncertainty for the modern nonexpected utility models. Those additional measurements are provided through our source-dependent probability transformations.

With Bayesian beliefs, utility, and source-dependent probability transformations measured, we can completely predict the behavior under uncertainty of agents. We illustrate this point in an example concerning the homebias (French & Poterba 1991), with investors systematically preferring domestic stocks to foreign stocks beyond beliefs (subjective probabilities) or tastes (utilities). The bias is explained by the different uncertainty attitudes displayed in our graphs. These attitudes reflect interactions between beliefs and tastes that cannot be captured by expected utility and that are typical of the modern non-Bayesian nonexpected utility models.

The main difference with alternative approaches in the recent economic literature is that our approach is descriptive, whereas most of the recent economic approaches have primarily been normatively oriented. They, for instance, usually assume expected utility for given probabilities, an assumption that is empirically violated. In this sense our approach is closer to the descriptive approach of Einhorn & Hogarth (1985), Hogarth & Einhorn (1990), and a number of follow-up papers, summarized hereafter as Hogarth et al. These authors developed an influential psychological theory of ambiguity, with tractable graphs similar to the ones that we will introduce later. This chapter can be interpreted as a decision-theoretic foundation of Hogarth et al.'s model and the probabilities used therein (Hogarth & Einhorn 1990, p. 799). Section 4.10 will demonstrate how our approach can indeed be used to predict decisions. The phenomena that will come out as major components of uncertainty attitudes from our revealed-preference based data, aversion and likelihood insensitivity, agree with the psychological phenomena advanced by Hogarth et al.

The organization of this chapter is as follows. Section 4.2 presents preliminaries, and §4.3 informally introduces sources of uncertainty, showing that they lead to a reconciliation of the two-color Ellsberg urn with Bayesian beliefs. The particularly tractable uniform sources are introduced in §4.4, with the ways to model different attitudes towards uncertainty and ambiguity displayed in §4.5. Section 4.6 presents indexes of aversion and insensitivity towards uncertainty. Section 4.7 describes the method of our experiment, and §4.8 gives results on the Bayesian concepts of subjective probabilities and utilities. Results on the novel, non-Bayesian, concepts of this chapter, concerning attitudes towards uncertainty and ambiguity, are in §4.9. Section 4.10 demonstrates how predictions about decision making can be derived in a convenient manner using our approach. Section 4.11 contains a discussion and §4.12 concludes.

4.2. Preliminaries

In our theoretical analysis, we assume the usual model of decision under uncertainty of Savage (1954). Here S denotes a *state space*. One state is true, the other states are not true, and the decision maker does not know for sure which state is true. Subsets of S are *events*. The *outcome set* is $\mathbb{R}^+$, designating nonnegative amounts of money. $(E_1{:}x_1,\ldots,E_n{:}x_n)$ denotes an *act*, which is a mapping from S to $\mathbb{R}^+$. The E_js are events that partition S, the x_js are outcomes, and the act assigns x_j to each state in event E_j. Acts take only finitely many different values ($n \in \mathbb{N}$). $\succcurlyeq$ denotes the *preference relation* of a decision maker over the acts. We assume *weak ordering* throughout, i.e. $\succcurlyeq$ is complete and transitive. The symbols $\succ$ (strict preference), $\sim$ (indifference or equivalence), and $\preccurlyeq$ and $\prec$ (reversed preferences) are as usual. For each act, the *certainty equivalent* is the sure amount that is indifferent to the act. *Expected utility* holds if there exist a *utility function* $U : \mathbb{R}^+ \to \mathbb{R}$ and a probability measure P on S such that preferences maximize $(E_1{:}x_1,\ldots,E_n{:}x_n) \mapsto \sum_{j=1}^{n} P(E_j)U(x_j)$, the *expected utility* of the act.

In the analyses of nonexpected utility for uncertainty and ambiguity, we will only need two-outcome acts with gains. We use xEy as shorthand for (E:x, S–Y:y). For these acts, virtually all static and transitive nonexpected utility theories known today use the following evaluation. These theories include: (a) Kahneman & Tversky's (1979) original prospect theory; (b) Luce & Fishburn's (1991) and Tversky & Kahneman's (1992) new prospect theory; (c) Quiggin's (1981), Gilboa's (1987), and Schmeidler's (1989) rank-dependent utility theory; (d) Gul's (1991) disappointment theory, (e) Wald's (1950) and Gilboa & Schmeidler's (1989) multiple priors model. The major theory not incorporated is the one of Maccheroni, Marinacci, & Rustichini (2006). Hence, the results of this chapter apply to virtually all nonexpected utilities known today. This convenient feature of binary acts was demonstrated most clearly by Ghirardato & Marinacci (2001). Luce (1991) and Miyamoto (1988) also used the generality of binary acts. Any of these theories can be used to derive predictions from our findings for acts with more than two outcomes (Gonzalez & Wu 2003).

We first define nonexpected utility for uncertainty, when no probabilities need to be given. A *weighting function* W assigns to each event E a number W(E) between 0 and 1, such that:

(i) $W(\emptyset) = 0$ and $W(S) = 1$;

(ii) $E \supset F$ implies $W(E) \geq W(F)$.

Nonexpected utility holds for binary acts if there exist a strictly increasing *utility function* $U : \mathbb{R} \to \mathbb{R}$ and a weighting function W such that preferences maximize

$$\text{for } x \geq y,\ xEy \mapsto W(E)U(x) + \big(1 - W(E)\big)U(y). \qquad (4.2.1)$$

Obviously, if $x < y$ then we interchange E and its complement S–E.

For calibrations of likelihoods of events, we arbitrarily fix a "good" and a "bad" outcome. We take these to be 1000 and 0, the values used in the experiment reported later. A *bet on event* E designates the act 1000E0, yielding 1000 if E and nil otherwise. E and F are *revealed equally likely*, denoted E ~ F, if 1000E0 ~ 1000F0. We next consider a stronger condition.

DEFINITION 4.2.1. Two disjoint events E_1 and E_2 are *exchangeable* if exchanging the outcomes under the events E_1 and E_2 does not affect the preference value of an act, i.e., always $(E_1{:}x_1, E_2{:}x_2, \ldots, E_n{:}x_n) \sim (E_1{:}x_2, E_2{:}x_1, \ldots, E_n{:}x_n)$. A partition $(E_1, \ldots, E_n)$ is *exchangeable* if all of its elements are mutually exchangeable. □

Exchangeability of events implies that they are equally likely. Exchangeable partitions were called uniform by Savage (1954), and they played a central role in his analysis. We will use his term uniform for a slightly different and more general concept.

Probabilistic sophistication holds if there exists a probability measure P on S such that for each act $(E_1{:}x_1, \ldots, E_n{:}x_n)$ the only relevant aspect for its preference value is the probability distribution $(p_1{:}x_1, \ldots, p_n{:}x_n)$ that it generates over the outcomes, where $p_j = P(E_j)$ for all j. In other words, two different acts that generate the same probability distribution over outcomes are equivalent in terms of $\succcurlyeq$. The probability of an event then captures everything relevant for preference evaluations. Probabilistic sophistication maintains the probability measure P from expected utility but does not restrict the decision model over probability distributions, and this model may deviate from expected utility. In this chapter we also maintain the assumption that outcomes are state-independent. For state-dependent generalizations of probabilistic sophistication, see Grant & Karni (2004).

Under probabilistic sophistication, revealed equal likelihood is not only necessary, but also sufficient for exchangeability. That is, two events are exchangeable if and only if they have the same probability. Under probabilistic sophistication, all events in an exchangeable partition $(E_1,\ldots,E_n)$ have probability 1/n.

4.3. Reconciling the Ellsberg two-color paradox with Bayesian beliefs

Because the two-color Ellsberg paradox is well known, we use it to introduce the main concepts of this chapter. We will demonstrate that this paradox can be reconciled with Bayesian beliefs.

EXAMPLE 4.3.1 [Ellsberg two-color paradox]. Imagine a "known" urn containing 50 R_k (Red from known) and 50 B_k (Black from known) balls, and an "unknown" urn containing 100 R_u (Red from unknown) and B_u (Black from unknown) balls in unknown proportions. One ball is drawn randomly from each urn. People commonly prefer to bet on colors from the known urn, i.e. $1000R_k0 \succ 1000R_u0$ and $1000B_k0 \succ 1000B_u0$.[17] Under probabilistic sophistication we have

$$1000R_k0 \succ 1000R_u0 \Rightarrow P(R_k) > P(R_u), \quad (4.3.1)$$

and

$$1000B_k0 \succ 1000B_u0 \Rightarrow P(B_k) > P(B_u). \quad (4.3.2)$$

[17] This holds also if the color to bet on is their own choice, so that there is no reason to suspect unfavorable compositions of the urn.

However, Eqs. 4.3.1 and 4.3.2 generate a contradiction in view of $P(R_k) + P(B_k) = 1 = P(R_u) + P(B_u)$. The left-hand large probabilities cannot sum to the same as the right-hand small probabilities. □

The above reasoning has traditionally been interpreted as a falsification of the Bayesian modeling of beliefs. A first step in the reconciliation of the above example with Bayesian beliefs concerns the distinction between different sources of uncertainty. The uncertainties regarding the unknown urn in the above Ellsberg example constitute one source, and the uncertainties regarding the known urn constitute another source. The decision maker has a general dislike for the former source relative to the latter..

For convenience, we will assume that sources are algebras, which means that they contain S, ∅, they contain the complement of each of their elements, and they contain the union of each pair of their elements. Then they also contain every finite union and intersection of their elements. Extensions to domains other than algebras are left to future works.

It is conceivable that people do not have different beliefs for the known and the unknown urn, but that instead they have different tastes for, and different decision attitudes towards the two sources. People may simply *dislike* compositions being kept secret, also if their beliefs about levels of likelihoods of favorable outcomes are the same. In general, people may dislike unknown probabilities relative to known probabilities for reasons beyond perceived differences of likelihood (Fox & Tversky 1995). Hence, we no longer accept the implications in Eqs. 4.3.1 and 4.3.2. Because of the symmetry in the known urn, B_k and R_k are exchangeable. Similarly, B_u and U_u are exchangeable for the unknown urn. These exchangeabilities suggest that all the events in question have subjective probability 0.5. Indeed, it is hard to argue that the belief in a color from the unknown urn would be anything other than fifty-fifty. The exchangeability used in this discussion is central in Chew & Sagi's (2006a, 2006b) model.

Several authors have argued that beliefs should still be quantified through probabilities 0.5 in the unknown Ellsberg urn. For example, Smith (1969) wrote:

"I grant the right of a man to have systematic and deliberate preferences for rewards based on dice game contingencies over the same rewards based on Dow-Jones stock price contingencies. ... But if he insists also that he is less than certain that the Dow-Jones average will either rise or not rise by five points tomorrow, then so far as I am concerned he is now making a "mistake." He is entitled to his tastes, but not to any new definitions of probability. Fortunately, this is not what subject violators do. They merely violate the axioms, without the necessity for the probabilistic interpretation [probabilistic interpretation refers to models that abandon Bayesian beliefs].

[...] As I see it, it is much more plausible to say that violators in "nonstandard process" contingencies, such as the stock price example, suffer utility losses (or gains) relative to what is experienced in less controversial "standard process" contingencies, such as dice games." (p. 325)

A similar viewpoint, and many other citations, are in Winkler (1991). These authors would usually seek to model ambiguity as an extra attribute of outcomes, quantified through utility, a classical integral of which evaluates acts. We will not follow this route, but leave utility unaffected, following Hogarth & Einhorn (1990) in this respect:

"The view adopted here is that the value of an outcome received following a choice made under certainty does not differ intrinsically from the value of the same outcome received following a choice made under risk or uncertainty." (p. 708)

We will use probability transformations rather than modifications of probabilities (beliefs) or utilities ("tastes") to capture ambiguity attitudes. These probability transformations need not reflect beliefs, in agreement with the views of Smith, Winkler, and others, as they need not reflect tastes, in agreement with the views of Hogarth et al. We prefer to interpret them as interactions between beliefs and tastes that take place at the level of decision attitudes and that are typical of nonexpected utility.

4.4. Uniform sources

We call a source *uniform* if probabilistic sophistication holds with respect to that source. Formally, this means that there exists a probability measure on the events of the source such that the preference value of each act $(E_1{:}x_1,\ldots,E_n{:}x_n)$ with all events E_j from the source depends only on the probability distribution that it generates over outcomes. Such probability distributions are denoted $(p_1{:}x_1,\ldots,p_n{:}x_n)$ with p_j the probability $P(E_j)$, and are called *lotteries*. Under uniformity, P will usually denote the relevant probability measure on the source without further mention.

If a finite partition $(E_1,\ldots,E_n)$ is exchangeable then the generated source (consisting of unions of events from that partition) is uniform. Chew & Sagi (2006a,b) showed that, under some regularity and richness conditions[18], a source is uniform if and only if the following conditions hold.

$E \sim F$ implies that E and F are exchangeable (holds for all uniform partitions). (4.4.1)

For each pair of disjoint events, one contains a subset that is exchangeable with the other. (4.4.2)

For each n there exists an exchangeable n-fold partition. (4.4.3)

[18] Satisfied if the probability measure is atomless and countably additive on a sigma algebra.

This result shows that uniformity is a natural extension of exchangeability from finite sources to rich (continuum) structures.

For a rich uniform source, we can elicit probabilities to any desired degree of precision using a bisection method and Eq. 4.4.2 (see section 4.7). We can, for example, partition S into two equally likely events E_1^1 and E_2^1 that then must have probability 0.5. We next partition E_1^1 into two equally likely events E_1^2 and E_2^2 that must both have probability ¼, and we partition E_2^1 into two equally likely events E_3^2 and E_4^2 that also have probability ¼. We continue likewise. This method will be used in the experiment described later. We will then test some implications of the displayed equations. The most well-known example of a nonuniform source is the Ellsberg three-color example.

EXAMPLE 4.4.1 [Ellsberg three-color paradox]. Assume that an urn contains 30 R (red) balls, and 60 B (black) and Y (yellow) balls in unknown proportion. People prefer betting on R to betting on B, which in classical analyses is taken to imply that $P(R) > P(B)$. People also prefer betting on [B or Y] to betting on [R or Y], usually taken to imply $P(B) + P(Y) > P(R) + P(Y)$, and then implying $P(B) > P(R)$ which contradicts the inequality derived before. The (ambiguity of the) urn is not uniform, and events have different effects and interactions in different configurations, with the weight of Y particularly high in the presence of B but low in the absence of B.

It may be argued that the three-color urn events concern the intersections of events from two different uniform sources (Chew & Sagi 2006b; Ergin & Gul 2004). The events of the color being R or not-R can be embedded in a rich uniform (and unambiguous) source. Whether the color is B or Y involves another random mechanism concerning the composition of the urn, which in turn can be embedded in a rich and uniform source. Our technique would then assign probability 1/3 to the color R, and probabilities $2/3 \times 1/2 = 1/3$ to the other two colors. This is the most plausible Bayesian belief corresponding with the behavior of the decision maker. □

We next consider an implication of probabilistic sophistication (with probability measure P) that will be useful for the analysis of ambiguity for uniform sources in the next subsection. Under probabilistic sophistication, there exists a function w such that[19]

$$W(\cdot) = w(P(\cdot)) . \qquad (4.4.4)$$

We call w the *(choice-based) probability transformation (function)*. We call w a *weighting function* only if the probabilities are objective and extraneously given. Thus, a probability transformation w depends on the source considered, but the weighting function w exclusively concerns the special case of given probabilities.

Under usual regularity conditions, w(0) = 0, w(1) = 1, and w is continuous and strictly increasing. Substituting Eq. 4.4.4 in Eq. 4.2.1 results in the following evaluation of lotteries, capturing all nonexpected utility theories for risk used today. Writing p for P(E) and xpy for (p:x, 1−p:y), preferences maximize

$$\text{for } x \geq y,\ xpy \mapsto w(p)U(x) + \big(1-w(p)\big)U(y). \qquad (4.4.5)$$

If $x < y$ then we interchange p and 1−p.

4.5. Uncertainty attitudes as modeled in the literature

Figure 4.5.1 depicts the main properties of W and probability transformations w (c.f. Hogarth & Einhorn 1990, Figure 1). The x-axis designates probabili-

[19] The implication can be derived as follows. If P(E) = P(F), then 1000E0 ~ 1000F0. Substituting Eq. 4.2.1 shows that then W(E) = W(F). Thus, equality of P implies equality of W. It is well known that then Eq. 4.4.4 follows.

ties p, which need not be objective but can be choice-based. The y-axis designates weights w(p), i.e. transformed probabilities. Fig. 4.5.1a displays expected utility with a linear probability transformation. Fig. 4.5.1b displays a convex probability transformation, where all probabilities are underweighted. For later purposes we give one of the many equivalent ways to define convexity:

w is *convex* if $w(q+p)-w(p) \leq w(q+p+\varepsilon)-w(p+\varepsilon)$

for all $q \geq 0$ and $\varepsilon \geq 0$. (4.5.1)

Convexity of w implies low values of w, leading to low weights for good outcomes and enhancing risk aversion. The more convex w is, the more risk averse the person is and the more he will pay to obtain insurance. The overweighting of unfavorable outcomes is also called *pessimism*.

Figure 4.5.1. Shapes of probability transformations

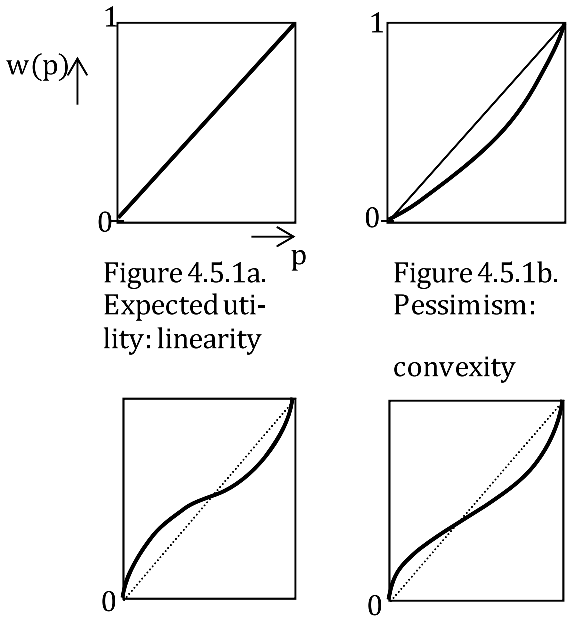

Figure 4.5.1a. Expected utility: linearity

Figure 4.5.1b. Pessimism: convexity

Figure 4.5.1c. likelihood insensitivity: inverse-S

Figure 4.5.1d. **Common finding**

Fig. 4.5.1c displays an inverse-S shaped probability transformation w. Formally, it is often defined as concave in a left region and convex thereafter. Alternative definitions, called subadditivity, are in Tversky & Fox (1995) and Tversky &

Wakker (1995). Because w is steep and generates large differences for both high probabilities associated with the least favorable outcomes (see above) and for low probabilities associated with the most favorable outcomes, such functions lead to an extremity-orientedness of overweighting the best and worst outcomes and underweighting the intermediate outcomes.

The inverse-S shaped probability transformations reflect a lack of sensitivity towards intermediate changes in likelihood, so that all intermediate likelihoods are moved in the direction of fifty-fifty and the jumps from certainty to uncertainty are overweighted. Hence, this phenomenon is also called *likelihood insensitivity*. It enhances both the long-shot effect of risk seeking for low-probability high-gain options, as in gambling, and risk aversion for small probabilities of unfavorable outcomes as in insurance. That is, it resolves the classical economic paradox of the coexistence of gambling and insurance (Hogarth & Einhorn 1990, p. 800; Tversky & Kahneman 1992. p. 316). It also suggests that decisions will not be influenced much by the updating of probabilities. These phenomena will be illustrated by Example 4.10.2. Likelihood insensitivity resembles regression to the mean. It is, however, not a statistical artefact, but a perceptual phenomenon that occurs in actual decisions. Fig. 4.5.1d combines the two deviations from expected utility, pessimism and likelihood insensitivity.

For general uncertainty we cannot draw graphs because the x-axis consists of general events. (Resolving this problem for many cases will be a major contribution of this chapter; see later.) The relevant properties of weighting functions can, however, be defined analogously. For instance,

$$\text{W is } \textit{convex} \text{ if } W(A \cup B) - W(B) \leq W(A \cup B \cup E) - W(B \cup E) \qquad (4.5.2)$$

whenever all unions are disjoint, which naturally extends Eq. 4.5.1 to uncertainty. Under usual richness, Eq. 4.5.2 agrees with Eq. 4.5.1 if $W(\cdot) = w(P(\cdot))$ for a probability measure P, illustrating once more that Eq. 4.5.2 is the natural analog of Eq. 4.5.1. Eq. 4.5.2 can be seen to be equivalent to the conventional definition of convexity ($W(A \cup B) + W(A \cap B) \geq W(A) + W(B)$). *Concavity* results if the inequalities in Eqs. 4.5.1 and 4.5.2 are reversed. Inverse-S can be defined as concavity for all events less likely than some threshold event, and convexity for all more likely events. (It can also be defined in terms of subadditivity.) The properties for uncertainty generate the same phenomena as for risk, with convexity related to pessimism and inverse-S to extremity-orientedness and likelihood insensitivity. Empirical evidence suggests that uncertainty displays phenomena similar as risk does, but to a more pronounced degree (Hansen, Sargent, & Tallarini 1999; Hogarth & Einhorn 1990; Kilka & Weber 2001).

Comparative concepts can be defined, with one weighting function being more convex or more inverse-S shaped than another. It can be done in a within-person way (this person is more averse to investing in foreign stocks than in home-country stocks; cf. Fox & Tversky 1995, p. 162) and in a between-person way (Mr. A is more averse to investing in Dutch stocks than Mr. W). Formal definitions and results are in Tversky & Fox (1995), Kilka & Weber (2001), Prelec (1998), and Tversky & Wakker (1995). Illustrations are in §4.10.

Ambiguity is often taken as what uncertainty comprises beyond risk. Ambiguity attitudes can be examined in our analysis by applying the comparative concepts to ambiguous sources versus given probabilities. More general comparisons, between different sources that are all ambiguous, are possible this way. This will be illustrated in §§4.9 and 4.10.

Theoretical studies in the economic literature, often normatively oriented, have focused on aversion to risk and ambiguity, i.e. the phenomenon illustrated in Fig. 4.5.1b (Dow & Werlang 1992; Mukerji & Tallon 2001; and many others). It is often explained by a rational suspicion in social interactions that is transferred to games against nature (Morris 1997, §3.) Empirical studies have found that the phenomenon in Fig. 4.5.1c plays an important role too, both for risk (Abdellaoui 2000; Bleichrodt & Pinto 2000; Gonzalez & Wu 1999) and for ambiguity (Einhorn & Hogarth 1986; Di Mauro & Maffioletti 2002; Tversky & Fox 1995; Wu & Gonzalez 1999).

4.6. Indexes of aversion and insensitivity

The approach taken in this chapter will be not to commit to any of the phenomena described in the preceding section. Instead, we will empirically observe what uncertainty attitudes are, comprising both risk and ambiguity, basing our analysis on a general decision theory for uncertainty. We will show that general properties of ambiguity attitudes can be inferred visually from graphs without any restriction imposed on what these properties are. Convenient quantitative indexes will then be introduced for the two main properties suggested by preceding empirical studies and illustrated in Figure 4.5.1, and we will let the "data speak."

We can obtain global quantitative indexes of pessimism and likelihood insensitivity using linear regression, illustrated in Figure 4.5.2. Assume that the regression line of the probability transformation on the open interval (0,1) is $p \mapsto c + sp$, with c the intercept and s the slope. Let $d = 1 - c - s$ be the distance from 1 of the regression line at p=1, i.e. the "dual intercept." We propose

$a = c + d$ (= 1–s) as an index of likelihood insensitivity, (4.5.3)

and

$b = d - c$ (= 1 – s – 2c) as an index of pessimism. (4.5.4)

These indexes can be considered to be special cases of more general indexes used by Kilka & Weber (2001; they used the term subadditivity for our term likelihood insensitivity), which in turn were based on tests by Tversky & Fox (1995).[20] Similar quantitative indexes were considered for somewhat different contexts by Pre-

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[20] The summary of the data that we propose here amounts to finding the best-fitting neo-additive weighting functions, an appealing family examined by Chateauneuf, Eichberger, & Grant (2005). These functions are linear in the middle and discontinuous at 0 and 1. They are tractable but capture the main deviations from Bayesianism. For this reason, the boundary points p=0 and p=1 should not be incorporated in the regression. Kilka & Weber (2001) and Tversky & Fox (1995) proposed measures for general weighting functions that, for neo-additive weighting functions, agree with the measures that we propose.

lec (1998) and Tversky & Wakker (1995). An elaborate discussion and theoretical analysis of these measures, and of alternative measures, is outside the scope of this chapter, and is a topic for future research.

Figure 4.5.2. Quantitative indexes of pessimism and likelihood insensitivity

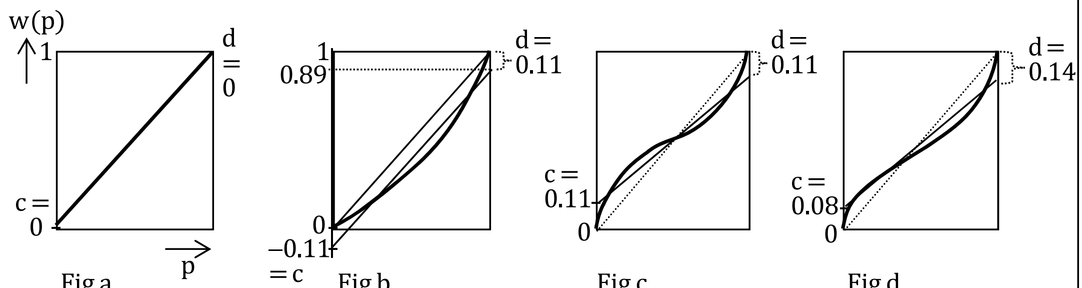


Fig.a. Insensitivity index a: 0; pessimism index b: 0.

Fig.b. Insensitivity index a: 0; pessimism index b: 0.22.

Fig.c. Insensitivity index a: 0.22; pessimism index b: 0.

Fig.d. Insensitivity index a: 0.22; pessimism index b: 0.06.

Alternative ways to obtain graphs as above have been considered in the literature. Several authors considered a "two-stage model" for this purpose (Einhorn & Hogarth 1985; Fox & Tversky 1998; Kilka & Weber 2001; Wu & Gonzalez 1999). Quantitative direct judgments of probabilities are then elicited verbally, and these are on the x-axis. Decision weights are derived from these judged probabilities in graphs similar to those above. For many economists it will be problematic that such judged probabilities are not based on revealed preference but on introspection. Further, the direct judgments themselves will not be additive and they comprise part of the deviation from Bayesianism that, accordingly, is not captured by the graphs of the probability transformations. Other authors generated uncertainty by giving subjects probability intervals for events (Cabantous 2005; Curley & Yates 1989; Di Mauro & Maffioletti 2002; Dolan & Jones 2004; Maffioletti & Santoni 2005; a behavioral foundation is in Hellman 2007). Then graphs as above can be displayed by taking arithmetic midpoints of those probability intervals on the x-axis. One drawback is, obviously, that the status of these midpoints again is not clear. Another drawback is that uncertainty usually cannot be fully captured through probability intervals, so that the applicability of this approach is limited.

EXAMPLE 4.5.1 [Ellsberg two-color paradox continued]. We elaborate on Example 4.3.1. We assume that $P(R_k) = P(B_k) = P(R_u) = P(B_u) = 0.5$. For the events R_k and B_k, the weight is $w_k(0.5) = 0.4$, and for events R_u and B_u, the weight is $w_u(0.5) = 0.3$. Under nonexpected utility (Eq. 4.2.1), the preferences in Eqs. 4.3.1 and 4.3.2 hold, but the probabilities of the four colors are not different. It is, instead, the decision attitude generated by different weighting that explains the different preferences. Note that our reconciliation does not need payments contingent on both drawings, unlike some models discussed in §4.11. □

4.7. Experimental method

We now describe the experiment to measure the concepts described above, in particular the source-dependent probability transformations.

Subjects.—N=62 students (54 male, 8 female) of the Ecole Nationale Supérieure d'Arts et Métiers (one of the leading French engineering schools) participated, all living in or near Paris. They were mathematically sophisticated and well acquainted with probability theory, but had no training in economics or decision theory. They were sampled through an e-newsletter and an internet-based registration.

Stimuli; unknown probabilities.—We considered three sources of uncertainty with unknown probabilities, first concerning the French Stock Index (CAC40) (how much it would *change* on a given day), then concerning the temperature in Paris, and, finally, concerning the temperature in a randomly drawn remote country (different for each subject). All these events concerned one fixed day (May 31, 2005) about three months after the experiment. For each subject and each source we first elicited boundary values $b_0 < b_1$ such that according to the subject there was "almost no chance" that the value to be observed would be outside the interval $[b_0,b_1]$. These bounds served only in graphical presentations for the subjects, and to help them get familiar with the stimuli, and their actual values do not play any role in our analysis. To obtain b_0 and b_1, we made subjects choose between bets with probability 1/1000 and bets on the events "$< b_0$" and "$> b_1$", using $1000 as the prize to be won. We write $E_1^1 = (-\infty,\infty)$.[21]

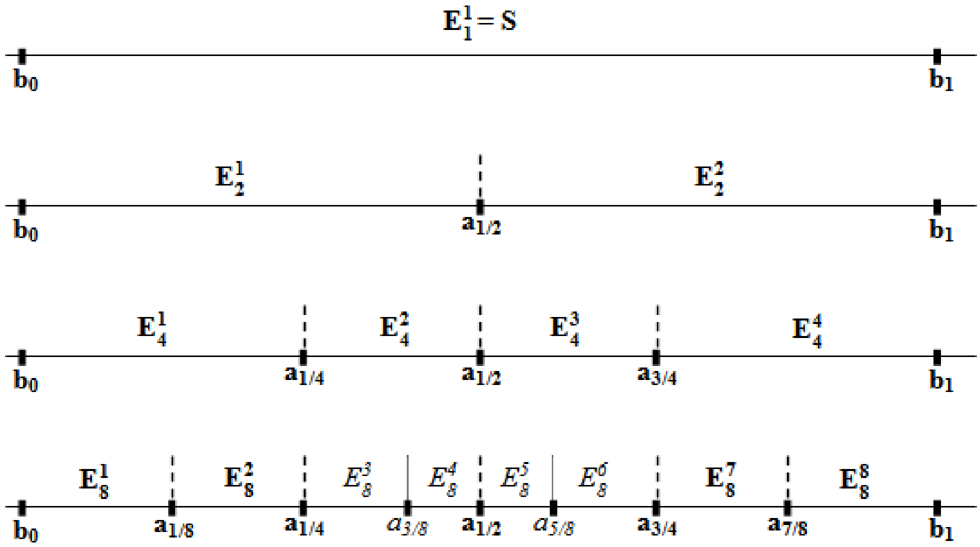
We next determined $a_{1/2}$ such that $b_0 < a_{1/2} < b_1$ and $(-\infty,a_{1/2}] \sim [a_{1/2},\infty)$. We write $E_2^1 = (-\infty,a_{1/2}]$ and $E_2^2 = [a_{1/2},\infty)$, which under uniformity yields the twofold exchangeable partition $\{E_2^1, E_2^2\}$. We determined $a_{1/4}$ such that $b_0 < a_{1/4} < a_{1/2}$ and $E_4^1 = (-\infty,a_{1/4}] \sim [a_{1/4},a_{1/2}] = E_4^2$, and we determined $a_{3/4}$ such that $a_{1/2} < a_{3/4} < b_1$ and $E_4^3 = [a_{1/2},a_{3/4}] \sim [a_{3/4},\infty) = E_4^4$. Under uniformity, it yields the fourfold exchangeable partition $\{E_4^1, E_4^2, E_4^3, E_4^4\}$. We, finally, determined $a_{1/8}$ and $a_{7/8}$ such that $E_8^1 = (-\infty,a_{1/8}] \sim [a_{1/8},a_{1/4}] = E_8^2$ and $E_8^7 = [a_{3/4},a_{7/8}] \sim [a_{7/8},\infty) = E_8^8$. We did not measure $a_{3/8}$ and $a_{5/8}$ so as to reduce the burden of the subjects, and because the literature on risk and uncertainty suggests that the most interesting phenomena occur at extreme values. In other words, we did not determine the middle events of the exchangeable partition $\{E_8^1, \ldots, E_8^8\}$. Thus, we have ended up with $a_{1/8},a_{1/4},a_{1/2},a_{3/4},a_{7/8}$ and the corresponding intervals

$$E_j^i = [a_{(i-1)/j}, a_{i/j}] .$$

In the notation E_j^i, the lower index j indicates the level, i.e. the number of events in the related exchangeable partition under uniformity, and the upper index is the number of the event in a left-to-right reading. In the notation $a_{i/j}$, the subscript i/j designates the probability of not exceeding $a_{i/j}$ under uniformity. E_j^i can be divided into E_{2j}^{2i-1} and E_{2j}^{2i}. According to this notation, we can write $a_0 = -\infty$ and $a_1 = \infty$. The values b_0 and b_1 are approximations of a_0 and a_1 sometimes used in graphs and displays, where infinity obviously cannot be depicted. Figure 4.7.1 displays the design.

[21] For simplicity, we do not express in notation that temperature is physically bounded below.

Figure 4.7.1. Decomposition of the universal



The italicized numbers and events in the bottom row were not elicited

Having measured these values, we carried out a second and third measurement of $a_{1/2}$, the second one (re)measuring $a_{1/2}$ as the midpoint of $[a_{1/4},a_{3/4}]$, and the third (re)measuring it as midpoint of the real axis. The value $a_{1/2}$ is important because the other measurements of events are derived from it, which is why we measured $a_{1/2}$ extensively.

Next, to test exchangeability, we asked subjects two choice questions, each time choosing between bets on intervals (receiving €1000 for the interval and nil otherwise), where the pairs of intervals to choose from were:

$(-\infty,a_{1/8}]$ versus $[a_{7/8},\infty)$ and $[a_{1/8},a_{1/4}]$ versus $[a_{3/4},a_{7/8}]$

We measured certainty equivalents for bets on intervals, for the intervals $(-\infty,a_{1/8}]$, $(-\infty,a_{1/4}]$, $(-\infty,a_{1/2}]$, $(-\infty,a_{3/4}]$, $(-\infty,a_{7/8}]$, and $[a_{1/2},\infty)$.

Stimuli; known probabilities.—After the CAC40 elicitations, we measured certainty equivalents of lotteries yielding a prize of €1000 with probability 1/8, 1/4, 1/2, 3/4, and 7/8, respectively, and nil otherwise. We explained that the probabilities were generated by random numbers from a computer, with which our subjects were well acquainted. We followed the same procedure here as for unknown probabilities, so as to treat the source of known probability similarly as the sources of unknown probabilities. After the Paris-temperature elicitations, we measured certainty equivalents of fifty-fifty lotteries yielding the following pairs of prizes: (0,500), (500, 1000), (250,500), (500,750), and, finally, (750,1000).

Measuring indifferences.—All indifferences were elicited through repeated choices and bisection until a satisfactory degree of precision had been reached. In each case, the second choice of the bisection was repeated later as a consistency check. No matching questions were used. Although bisection is more time consuming than matching, it has been found to provide more reliable results (Bostic, Herrnstein, & Luce 1990). When measuring the midpoint of an interval [a,b], we always started with $a/3 + 2b/3$ and then $2a/3 + b/3$ as the first two choice questions, and only then continued with usual bisection. Certainty equivalents were always measured using traditional bisection, starting with the expected value.

Procedure.—Each subject was interviewed individually. The interview consisted of five minutes of instructions, 10 minutes of practice questions, and 70 minutes of experimental questions, interrupted for small breaks and cakes when deemed desirable. After finishing all questions pertaining to one source, we did not immediately start with the next source, but asked intermediate questions eliciting risk attitudes so as to prevent that subjects continued to think of one source when dealing with the next one.

Motivating subjects.—All subjects received a flat payment of €20. For the *hypothetical treatment* (n=31), all choices were hypothetical. For the *real treatment* (n=31), real incentives were implemented through the random lottery incentive system in addition to the flat payment. One of the 31 subjects was randomly selected at the end, and one of his choices was selected randomly to be played for real. Given that the high prize was usually €1000, and that subjects would usually choose the more likely gain, the expected value of the subject selected exceeded 1000/2 = €500, and the expected gain (in addition to the €20) per subject in the real treatment exceeded 500/31 ≈ €16.

The money earned could be collected about three months later, after the uncertainty had been resolved. The subjects in the hypothetical treatment did not know that later a real-incentive treatment would follow with other subjects.

Pilots.—Pilots were done with 18 subjects, to determine which sources and which incentive system to use in the real experiment. The pilots suggested that randomized and mixed orders of presentation, with choice questions pertaining to one source or aiming at one indifference question not asked in a row, were tiring and confusing for subjects. Hence we grouped related questions together in the real experiment.

Analysis.—Unless stated otherwise, all statistical tests hereafter concern t-tests with $\alpha = 0.05$ as level of significance. For fitting the data of the risky questions we used a method introduced by Abdellaoui, Bleichrodt, & L'Haridon (2007). We first used the certainty equivalents of the fifty-fifty lottery regarding the prize-pairs (0,500), (500, 1000), (250,500), (500,750), and (750,1000) to optimally fit Eq. 4.2.1 with as free parameters the decision weight at p = 0.5, i.e. w(0.5), and the power ρ of utility in $U(x) = x^{\rho}$. With the utility function thus determined, we used the certainty equivalents of the lottery that yielded prize €1000 with probabilities 1/8, 1/4, 1/2, 3/4, and 7/8 to determine the decision weights of these probabilities, with CE ~ 1000_p0 implying the equality $w(p) = CE^{\rho}/1000^{\rho}$ for all relevant p.

We similarly used the certainty equivalents of the events $(-\infty,a_{1/8}]$, $(-\infty,a_{1/4}]$, $(-\infty,a_{1/2}]$, $(-\infty,a_{3/4}]$, and $(-\infty,a_{7/8}]$, and the power utility function obtained above to determine the W values of these events, with CE ~ 1000_E0 implying the equality $W(E) = CE^{\rho}/1000^{\rho}$.

4.8. Results on subjective probability and utility (Bayesian results)

Uniformity.—We can define subjective probabilities only for uniform sources and, hence, we first discuss tests of uniformity for the sources considered. For these tests there were no irregularities in the answers that subjects supplied, so that we used the whole sample to carry out our tests. The third measurement of $a_{1/2}$ (as midpoint of $(-\infty,\infty)$) was identical to the first measurement, and served as a reliability test. Pairwise t-tests never rejected the null hypothesis of equal values (for neither treatment nor for the whole group), and the correlations exceeded

0.85 for all three sources and both treatments. These results suggest that the measurements were reliable.

The most refined level of partitioning for which we obtained observations concerned the eight-fold partition of the events E_8^i, which we observed for i = 1, 2, 7, 8. The equivalences $E_8^1 \sim E_8^2$ and $E_8^7 \sim E_8^8$ hold by definition. Assuming transitivity of indifference, it suffices to verify the equivalence $E_8^2 \sim E_8^7$ to obtain equivalence of all E_8^i available. For no case did a binomial test reject the null hypothesis of indifference between bets on E_8^2 and E_8^7. The choices between E_8^1 and E_8^8 serve as an extra test of exchangeability joint with transitivity of indifference. Again, a binomial test never rejected indifference.

We made no observations of the eight-fold partition $\{E_8^i\}$ between $a_{1/4}$ and $a_{3/4}$, but in this region we can test exchangeability for the four-fold partition $\{E_4^i\}$. Given the equivalences $E_4^1 \sim E_4^2$ and $E_4^3 \sim E_4^4$ that hold by definition, and transitivity of indifference, it suffices to verify the indifference $E_4^2 \sim E_4^3$. We did not directly test choices between bets on E_4^2 and E_4^3. Our second measurement of $a_{1/2}$, as midpoint of $[a_{1/4}, a_{3/4}]$, entails a test of the equivalence $E_4^2 \sim E_4^3$ though. The correlations between the first and second measurement of $a_{1/2}$ exceeded 0.75 for all three sources and both treatments as well as the whole group, exceeding 0.90 in all but one case. Pairwise t-tests never rejected the null hypothesis of equal values of $a_{1/2}$ (for neither treatment nor for the whole group) with one exception: For the hypothetical group and foreign temperature the difference was significant ($t_{30} = 2.10$, $p = 0.04$).

Another test of exchangeability can be derived from comparing the certainty equivalents of bets on events E_2^1 to those on events E_2^2. Under exchangeability, these should all be the same. Pairwise t-tests never rejected the null hypothesis of equal values (for neither treatment nor for the whole group), with correlations of approximately 0.5 and more. Hence these tests do not reject exchangeability.

The tests suggest that exchangeability is least satisfied for foreign temperature with hypothetical choice, with no violations found for the other five cases. Because our techniques have been developed primarily for uniform sources, we will report our analyses of risk and ambiguity attitudes only for the five remaining cases in what follows. We use the term *case* to specify both the source and the treatment (real incentives or hypothetical choice).

Subjective Probabilities.—Figures 4.8.1 and 4.8.2 display median subjective probability estimations for the real and hypothetical treatments contrasted with historical frequencies. The medians are always derived from the medians of the $a_{i/j}$. Figure 4.8.1 displays the median subjective probability distribution functions for CAC40. Both curves show that our subjects were optimistic in the sense that they considered increases of the index to be more probable than decreases. The figure also displays the real probability distribution over the year 2006.[22] Our subjects expected extreme, primarily positive, changes to be more likely than they were in 2006.

Figure 4.8.2 displays the median subjective distribution function for Paris temperature. The historical distribution for the time considered (May 31, 1 PM) has been added too. The curves are very well calibrated. Our subjects are apparently better acquainted with temperature volatility than with stock volatility. The data also suggest that subjects did not expect higher temperatures than the historical distribution over the past century. They were apparently not influenced by the effects of global warming. We do not report the subjective probabilities for foreign cities because the cities were different for different subjects so that this distribution did not concern the same random event for all subjects.

[22] Details are as follows. The distribution is based on 254 days. The estimates concern increase rates from 5:30 PM one day until 1 PM the next day (the time period considered in our experiment), which can be estimated as (daily rates to the power 19.5/24) – 1.

Figure 4.8.1. Probability distributions for CAC40

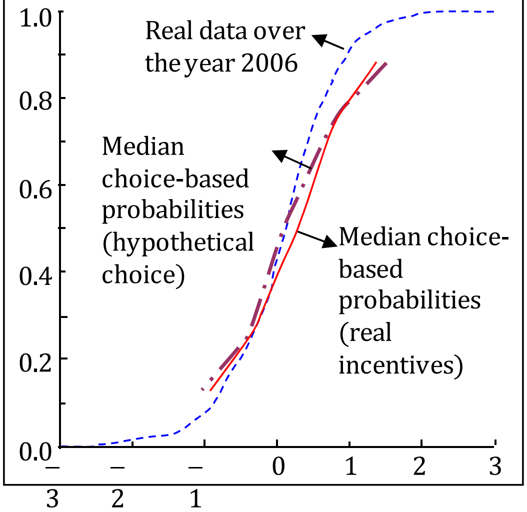
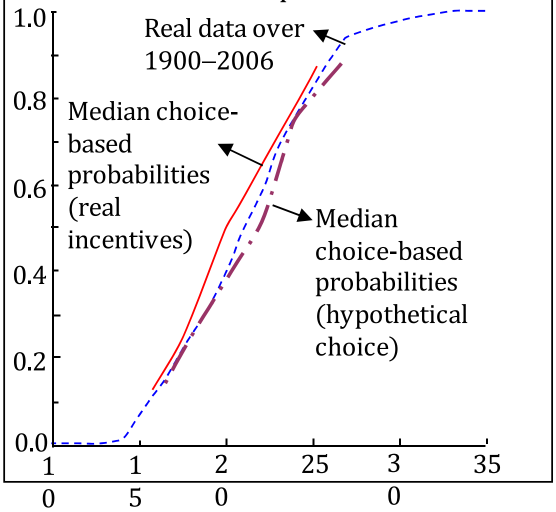


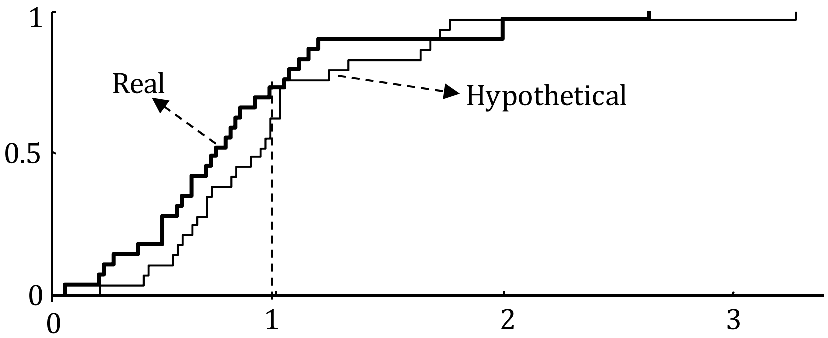
Figure 4.8.2. Probability distributions for Paris temperature



Utility.—For the certainty-equivalence measurements used to analyze risk attitudes, one subject was removed from the group with hypothetical choice because he always chose the sure option, suggesting that he did not seriously consider the choice options. To avoid introducing a bias towards risk seeking (the subject removed was obviously the most risk averse one), we also removed the most risk seeking subject from this group. In the group of real incentives, we similarly removed one subject who always chose the safe option and one subject who always chose the risky option, for similar reasons. Thus, 4 subjects were removed and 58 subjects remained, 29 in each treatment.

The certainty equivalents (statistics not reported) suggest risk seeking for low probabilities and risk aversion for moderate and high probabilities, with more risk aversion for the real treatment than for the hypothetical treatment. All these findings agree with common findings in the literature (Abdellaoui 2000; Bleichrodt & Pinto 2000; Camerer & Hogarth 1999; Gonzalez & Wu 1999), and will be confirmed by the parametric estimations given hereafter. Figure 4.8.3 displays the empirical distribution of the individual powers of utility. The majority of powers is below 1, suggesting concavity in agreement with common findings (61.2% for the hypothetical treatment and 72.4 % for the real treatment). Median, mean, and standard deviations are 0.9244, 1.0078, and 0.5911 for the hypothetical treatment and 0.7458, 0.8495, and 0.5594 for the real treatment. The powers of utility were lower for the real-incentive group than for the hypothetical group, but the difference was not significant. A lower power entails more concavity which will generate more risk aversion again (given a fixed weighting function), in agreement with the common finding of more risk aversion for real incentives.

Figure 4.8.3. Cumulative distribution of powers



4.9. Results on uncertainty and ambiguity for uniform sources (non-Bayesian results)

This section reports results on source dependence, describing the attitudes found. It does so only for the five cases where uniformity is satisfied, i.e. where the techniques of this chapter apply. The only case where uniformity was violated, foreign temperature with hypothetical choice, will not be analyzed further.

4.9.1. Overall results

The following figures display probability transformations. In each figure, part a displays probability transformations obtained from the raw data through linear interpolation, and part b displays the best-fitting function from Prelec's (1998) compound invariance family

$$w(p) = \left(exp(-(-ln(p))^{\alpha})\right)^{\beta}. \qquad (4.9.1)$$

The parameters α and β have meanings similar to our parameters a and b (Eqs. 4.5.3 and 4.5.4), respectively, but less clearly so, depending on numerical aspects of the functional. The correlations between a and α, and b and β, were strong. The statistical results for α and β were similar to those for a and b, but with less power, and we will not report them.

The indexes a and b were calculated for each individual and each source. The correlations between the individual parameters a for sensitivity and b for pessimism were not significantly different from 0 in all cases, suggesting that these concern two independent components in risk and ambiguity attitudes.

The parameters displayed in Figures 4.9.1 and 4.9.2 are calculated to fit the group averages, and will not be used in statistical analyses. Their orderings agree with all qualitative findings made below. Note how these figures compactly present much information. Together with utility and subjective probabilities they completely capture attitudes towards uncertainty, exactly quantified, for three or four sources or persons at the same time. They make it possible to immediately and visually compare these attitudes. In particular, through comparisons with graphs for given probabilities, they immediately reveal attitudes towards ambiguity.

The hypothetical-treatment curves (Figure 4.9.2) are similar to those of the real-payment treatment (Figure 4.9.1), but hypothetical choices were subject to more noise. All curves display the basic inverse-S shape of Fig. 4.5.1d with low probabilities overweighted and high probabilities underweighted. Most observed points w(p) deviate significantly from linearity, i.e. the null hypothesis $w(p) = p$ is usually rejected if we take the whole sample joining hypothetical choice and real incentives, except at $p= 0.5$, in agreement with inverse-S. For hypothetical choice and real incentives separately, for about half the observed points the deviation is not significant. The insensitivity parameter a was significantly higher for real incentives than for hypothetical choice for CAC40 and foreign temperature, and marginally so for Paris temperature ($p = 0.053$). The pessimism parameter b was not different for the two treatments.

Figure 4.9.1. Average probability transformations for real payment

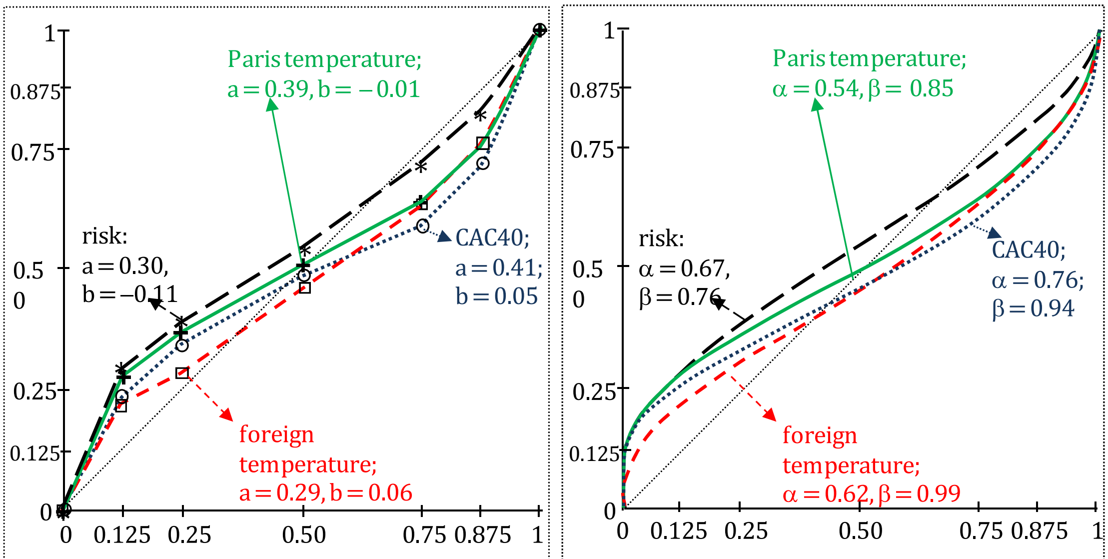


Fig. a. Raw data and linear interpolation.   Fig. b. Best-fitting $(exp(-(-ln(p))^{\alpha}))^{\beta}$.

Regarding source dependence of probability transformation, no significant differences are found for different sources under hypothetical payment, and we focus on real payment. We first consider probability transformations w(p) at single probabilities p. With risk included, a repeated-measures analysis of variance (corrected by the Huynh-Feldt ε) finds significant source dependence for w(p) and real payment except at p=0.5. Figure 4.9.1 shows that there is source preference (less pessimism and higher curve) for risk over all other sources. Indeed, paired t-tests for risk against each of the three sources indicate that the values w(p) are significantly higher for risk than for foreign temperature at all probabilities (i.e. ambiguity aversion at all probabilities), for CAC40 at p=0.125 and p > 0.50 and for Paris temperature at p>0.5 (i.e. ambiguity aversion for high probabilities). If we exclude risk, then the analysis of variance finds significant source dependence for p = 0.25. The figure suggests source preference for Paris temperature over CAC40 and foreign temperature, and more pronounced inverse-S for CAC40 than for foreign temperature, but the differences between the curves at the various probabilities were not significant except for Paris against foreign temperature at p < 0.5.

Figure 4.9.2. Average probability transformations for hypothetical payment

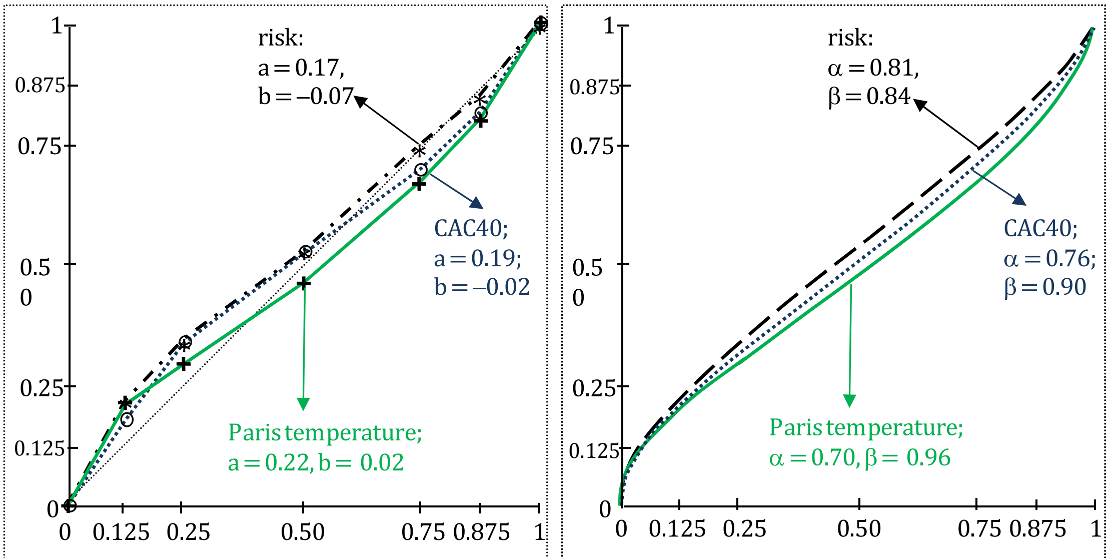


Fig. a. Raw data and linear interpolation.

Fig. b. Best-fitting $(exp(-(-ln(p))^{\alpha}))^{\beta}$.

We next consider tests of pessimism and likelihood insensitivity through the global parameters a and b. A repeated-measures analysis of variance (corrected by the Huynh-Feldt ε) reveals a clear source dependence of the pessimism index b. The sensitivity parameter appears to be non-significantly different across sources at 5% once the Huynh-Feldt correction is applied. In pairwise t-tests the pessimism index b for risk is significantly lower than for all sources with unknown probabilities. For the sources with unknown probabilities, the parameters b do not differ significantly. Parameter a, the index of likelihood insensitivity, is significantly lower for foreign temperature than for CAC40 and Paris temperature. The other differences were not significant.

4.9.2. Results at the individual level

To illustrate that our techniques can be used at the individual level, Figure 4.9.3 displays the curves for the four sources of one individual, subject 2 from the real-payment treatment. This subject thought long and seriously about each question, and the interview took almost two hours. He exhibits source preference for all sources over foreign temperature. Further, risk is less likelihood insensitive than CAC40 and Paris temperature. In the raw data, the subject slightly violates monotonicity for CAC40, showing that there is noise in the data.

Figure 4.9.3. Probability transformations for subject 2 for real payment

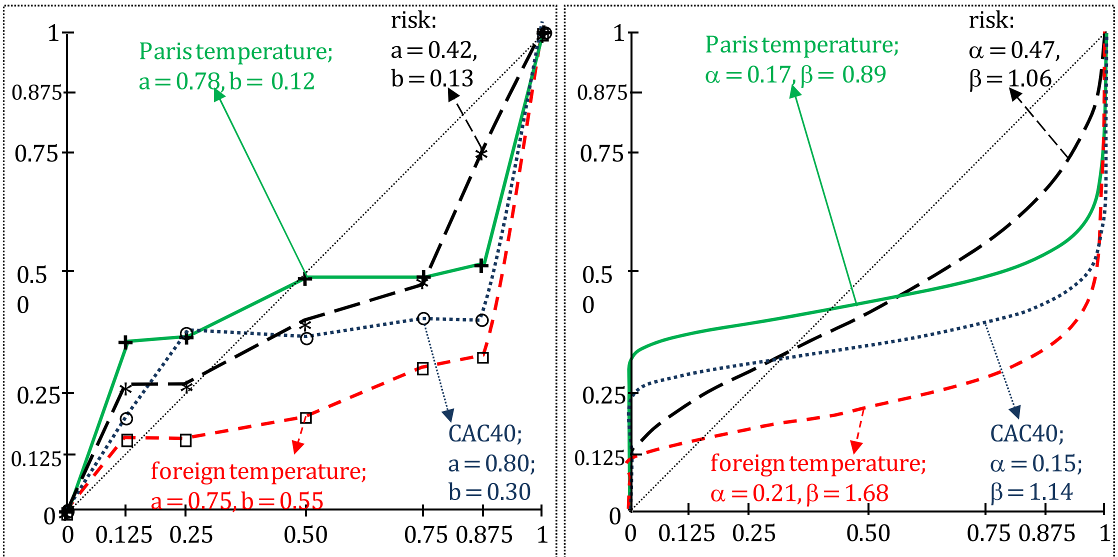


Fig. a. Raw data and linear interpolation.

Fig. b. Best-fitting $(exp(-(-ln(p))^{\alpha}))^{\beta}$.

Behavioral implications are that the subject will be more prudent, invest less, and take more insurance, for foreign temperature events than for the other events. The subject will be more open to long shots for Paris temperature and CAC40 than for risk but, on the other hand, will also rather insure for Paris temperature and CAC40 than for risk. An updating of probabilities will affect the subject less for Paris temperature and CAC40 than it will for risk under usual gradual changes of moderate likelihood.

Figure 4.9.4. Probability transformations for Paris temperature and 4 subjects for real payment

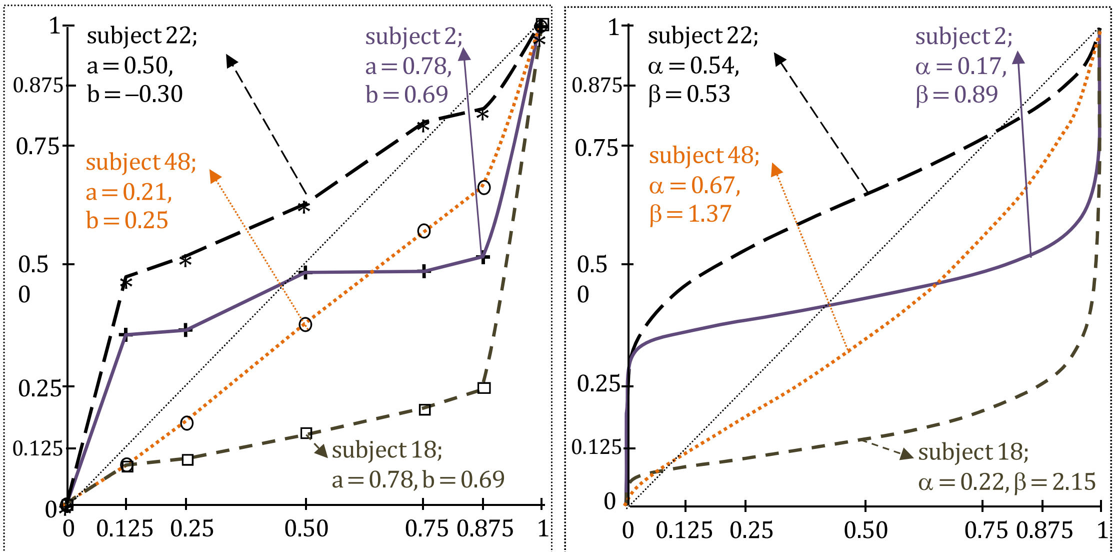


Fig. a. Raw data and linear interpolation.

Fig. b. Best-fitting $(exp(-(-ln(p))^{\alpha}))^{\beta}$.

Figures 4.9.1 and 4.9.3 concerned a within-person comparison of different attitudes towards uncertainty for different sources, which we take as the main novelty initiated by the Ellsberg paradoxes. We can also use probability transformations and the above indexes of pessimism and likelihood insensitivity for the—more traditional—between-person comparisons of uncertainty attitudes. Figure 4.9.4 displays some comparisons. We selected four subjects for the purpose of an illustration with clearly distinct curves. All curves concern the same source, being Paris temperature. The lowest curve (subject 18) is more pessimistic than all other subjects. This subject will buy more insurance, for instance. The dark middle curve (subject 2) clearly displays more pronounced likelihood insensitivity than the dashed curve that is close to linear (subject 48). Hence, simultaneous gambling and insurance is more likely to be found for subject 2 than for subject 48, and subject 2's decisions will be influenced less by new information (updating probabilities) than those of subject 48.

In general, there was more variation in the individual parameter estimates for the ambiguous sources than for risk. It is not surprising, indeed, that risk is perceived more homogeneously across individuals than ambiguity. The pessimism-parameters b were strongly correlated over different sources (usually exceeding 0.8), suggesting that this parameter depends primarily on the individual and less on the source. The sensitivity-parameters a were less strongly correlated over different sources (usually about 0.65), suggesting that this parameter depended more on the source of uncertainty than b.

4.9.3. Results regarding ambiguity

Ambiguity attitudes are usually taken to reflect the differences between sources with unknown probabilities and sources with known probabilities. We can infer those comparisons from comparing the curves for risk with the other curves, in Figures 4.9.1-4.9.3. Those comparisons have been discussed above, with mostly the risk-curves dominating the other curves suggesting aversion to ambiguity.

4.10. Using our approach to derive predictions

The following examples illustrate applications of our techniques.

EXAMPLE 4.10.1 [Homebias; Within-Person Comparisons]. Consider options yielding \$40000 or nil, as follows.

Foreign-option: (Favorable Foreign temperature: 40000, otherwise: 0).

Paris-option: (Favorable Paris temperature: 40000, otherwise: 0);

We assume that, both for Paris temperature and for foreign temperature, subject 2 (living in Paris) considers favorable and unfavorable temperatures to be equally likely, and that his utility function on the domain relevant for this example is well approximated by $U(x) = x^{0.88}$. Wheras under expected utility this information would completely determine the preference values of the options considered, under nonexpected utility we need more information. This information is captured in Figure 4.9.3, leading to the following predictions.

Because the decision weight of outcome 40000 is 0.20 for foreign temperature, the certainty equivalent for the foreign option is $U^{-1}(0.20 \times U(40000)) =$ \$6424. We use the term *uncertainty premium* as the analog of risk premium, referring to the context of uncertainty with unknown probabilities. Assuming probability 0.50, the uncertainty premium for the foreign option is \$20000 − \$6424 = \$13576. For risk with known probability p = 0.50, the decision weight is 0.40, giving a certainty equivalent of \$14121 and a risk premium of \$5879. Subject 2 exhibits ambiguity aversion for foreign temperature because he evaluates the choice-based probability 0.50 lower than the objective probability 0.50. We interpret the difference between the uncertainty premium and the risk premium, \$13576 − \$5879 = \$7697, as an *ambiguity premium*.

Table 4.10.1 gives similar calculations for Paris temperature, for which subject 2 exhibits considerably more favorable evaluations and is even ambiguity seeking, with a negative ambiguity premium. Subject 2 exhibits a strong homebias for temperature-related investments. This bias cannot be ascribed to beliefs or tastes because they are the same for the investments in Paris and foreign temperature. The homebias is explained by the different uncertainty attitudes displayed in Figure 4.9.3.

Table 4.10.1. Calculations for Subject 2

	Paris temperature	Foreign temperature
Decision weight	0.49	0.20
Expectation	20000	20000
Certainty equivalent	17783	6424
Uncertainty premium	2217	13576
Risk premium	5879	5879
Ambiguity premium	–3662	7697

□

EXAMPLE 4.10.2 [Less likelihood Sensitivitity, and More Gambling & Insurance; Between-Person Comparisons]. Consider an option (Favorable Paris temperature: 40000, otherwise: 0). Assume that there are eight exhaustive and mutually exclusive Paris-temperature events that are equally likely according to both Subjects 2 and 48. We assume, for clarity of exposition, that the utility function on the domain relevant for this example is well approximated with $U(x) = x^{0.88}$ for both subjects.

Table 4.10.2. Calculations for Paris Temperature

	Subject 2, p=0.125	Subject 48, p=0.125	Subject 2, p=0.875	Subject 48, p=0.875
Decision weight	0.35	0.08	0.52	0.67
Expectation	5000	5000	35000	35000
Certainty equivalent	12133	2268	19026	25376
Uncertainty premium	–7133	2732	15974	9624
Risk premium	–4034	2078	5717	–39
Ambiguity premium	–3099	654	10257	9663

We consider two cases.

CASE 1. Assume that one of the eight events is favorable and seven are unfavorable, so that the choice-based probability at 40000 is 0.125. Figure 4.9.4 shows that the favorable event has weight 0.35 for subject 2, yielding certainty equivalent \$12133. The columns in Table 4.10.2 with p = 0.125 give this number, and several other results that were calculated similarly as in Table 4.10.1.

CASE 2. Assume that seven of the eight events are favorable and one is unfavorable, so that the choice-based probability at 40000 is 0.875. The right two columns in Table 4.10.2 give results for this case.

Subject 2 has a higher certainty equivalent for p = 0.125 than subject 48, but a lower one for p = 0.875. Thus, at the same time he exhibits more proneness to gambling (small probability at favorable outcome as in Case 1) and to insurance (small probability at unfavorable outcome as in Case 2) than Subject 48. Both the risk and the ambiguity attitudes contribute to these differences between the two subjects, as the premiums show.

It is interesting to consider the changes in evaluations if the number of favorable events changes from one (Case 1) to seven (Case 2). Subject 2 exhibits little sensitivity to this big change in likelihood. His certainty equivalent of the investment changes only by approximately \$7000 and does not even double, whereas the certainty equivalent of subject 48 changes drastically and is increased 11-fold. We can conclude that subject 48 exhibits considerably more sensitivity to likelihood changes than Subject 2 in the domain considered here.

Subjects 2 and 48 have the same beliefs, as argued by Smith (1969) and Winkler (1991), and the same tastes, as argued by Hogarth & Einhorn (1990). Their different behavior is generated by differences in uncertainty and ambiguity attitudes, displayed in the graphs in Figure 4.9.4.

□

4.11. Discussion

In our approach to uncertainty, three components are needed to describe decision under uncertainty for uniform sources: (a) utility of outcomes; (b) Bayesian beliefs for each source; (c) the source-dependent probability transformations. Component (c) comprises the deviations from Bayesianism in a tractable manner, generating Allais' and Ellsberg's paradoxes, the homebias, ambiguity aversion, and other deviations from expected utility. Attitudes towards ambiguity can be measured by comparing (c) for known and unknown probabilities. We next discuss some details of these measurements, and then discuss some other issues.

Measuring utility.—Our utility measurements are valid for virtually all presently existing models. In particular, they are not distorted by violations of expected utility, contrary to traditional methods based on the latter theory. Abdellaoui, Barrios, & Wakker (2007) review the implications of nonexpected utility for utility measurement.

Measuring subjective probabilities.—This chapter has introduced a new way to empirically measure subjective probabilities. Unlike traditional psychological elicitations (Manski 2004; McClelland & Bolger 1994), we have elicited subjective probabilities exclusively from revealed choices with real incentives implemented. And unlike the proper scoring rules that are popular in experimental economics today, our measurements do not require the assumption of expected value maximization (Camerer 1995, pp. 592-593; Nyarko & Schotter 2002) so that they are not biased by empirical violations thereof. They do not even require the assumption of expected utility maximization and are virtually theory-free. They are based exclusively on elementary revelations of equal likelihood and exchangeable partitions as put forward in the theoretical counterpart to our empirical measurement, being Chew & Sagi (2006a). A generalization of proper scoring rules to violations of expected value and expected utility is in Offerman et al. (2006). Their approach obtains decision weights under uncertainty as functions of decision weights under risk, where the latter need not be additive as are our choice-based probabilities so that they comprise part of the ambiguity attitude. Abdellaoui, Vossman, & Weber (2005) also analyzed general decision weights under uncertainty as functions of decision weights under risk. They used the term choice-based probability to refer to such functions that, again, did not have to be additive. Unlike Offerman et al. they did not use proper scoring rules but they carried out a full decision analysis to elicit these values.

The choice-based probabilities that we derive need not reflect the subjective beliefs held by the decision maker. The actual subjective beliefs of the decision maker can deviate from Bayesianism,and according to some authors such deviations are even rational. These deviations may generate (part of) the nonadditivity comprised in the source-dependent probability transformation. The choice-based probabilities are, however, the best Bayesian beliefs to reflect the information held by the decision maker, as simply follows from the symmetry implied by exchangeability. They will be useful in normative applications where the decision maker is, for instance, an expert whose observed decisions we want to use as a basis for our own decisions if we take Bayesianism of belief as a normative desideratum.

Uniformity.—Consistency checks of probability measurements yielded consistent subjective probabilities in five out of six cases, with one violation of uniformity in only one of the six cases considered, the case of foreign temperature with hypothetical choice.

We chose Savage's (1954) term uniform instead of exchangeable for two reasons. First, it is slightly more general than exchangeability when imposed on finite sources, not requiring that all states of nature be equally likely, so that a different term had to be chosen. Second, the condition suggests a uniform ambiguity of the source where, once two events have been revealed equally likely, they become completely substitutable in every relevant aspect. It is immaterial what their precise location and configuration is relative to other events. There are also some formal differences between our concept of uniformity and Chew & Sagi's (2006b) concept of homogeneity. The main difference is that our sources span the whole state space S and are not conditioned on subevents of S. We prefer to separate the static concept of uniformity from dynamic issues regarding conditioning.

Reducing complexity.—General nonexpected utility models have many parameters to assess, which make predictions intractable without proper restrictions. For general weighting functions, not only for every singleton event (as for Bayesian probabilities), but for every subevent, a weight has to be chosen, which for large and infinite state spaces quickly becomes intractable. Our approach greatly simplifies this complexity. We identify uniform sources, and for each have to add one function, the probability transformation, to what is required for Bayesian analyses (utilities and probabilities). Such a procedure remains within tractability bounds for a large class of sources.

Source comparisons.—The Ellsberg paradoxes have mostly been interpreted as evidence showing that people are more averse to unknown probabilities and ambiguity than to known probabilities. This chapter contributes to a line of research that extends this interpretation: People behave differently towards different sources of uncertainty, also if none of these sources concern known probabilities (Tversky & Fox 1995).

Uncertainty is a rich domain where many kinds of incomplete information with many different characteristics can be found, and many different kinds of phenomena can be discovered. Thus, our data show that besides a general tendency to be more or less averse to some source of uncertainty (of which ambiguity aversion is a special case), another dimension of uncertainty attitudes concerns whether people are more or less sensitive to likelihood information about sources. For uniform sources the latter dimension corresponds with more or less pronounced inverse-S shaped probability transformations. The two dimensions mentioned are also central in the psychological works of Hogarth et al., referenced in §4.1. Other studies finding this phenomenon include Curley & Yates (1989), Dolan & Jones (2004), Di Mauro & Maffioletti (2002) Fox & Tversky (1998), and Wu & Gonzalez (1999).

Multistage models of uncertainty.—We next discuss some promising multistage models of uncertainty that have been introduced recently, and for which methods for empirical measurement remain to be developed. These models present alternative ways to reconcile beliefs in the two-color Ellsberg paradox, adapting

Kreps & Porteus' (1978) two-stage model from intertemporal preference to uncertainty.[23] These papers assume Bayesian beliefs in each stage, so that such beliefs can hold both for the known and for the unknown urn in the Ellsberg example. Ambiguity is modeled by assuming that the multiplication rule for conditional probability is violated, so that in this respect they still abandon Bayesian beliefs.

Klibanoff, Marinacci, & Mukerji (2005), Nau (2006), and Neilson (1993) assume expected utility at each stage, and different uncertainty or ambiguity attitudes at different stages are modeled through different utilities of outcomes. In our approach, uncertainty and ambiguity attitudes are modeled through functions that directly operate on the uncertain events, such as the weighting functions in Schmeidler's Choquet expected utility and prospect theory. The latter may be more natural from a psychological perspective. Unlike the aforementioned expected-utility approaches, Allais-type violations of expected utility are allowed within sources[24], which is desirable for descriptive purposes. Existing empirical evidence that aversion or attraction to ambiguity depends on events, in particular on whether the events are likely or unlikely (Tversky & Fox 1995; also supported by our data), can then be accommodated. Further, utility then is not linked inextricably to risk and ambiguity attitudes so that psychological interpretations of utility as an index of well being are not ruled out a priori (Mandler 2006).

Ergin & Gul (2004) assume general probabilistic sophistication (uniform sources) at each stage. As we do, they interpret events at different stages as different sources of uncertainty, using the term issue instead of source. They abandon the probability-multiplication rule and, in this sense, deviate from Bayesian beliefs. The paper closest to our way of reconciling the two-color Ellsberg paradox with Bayesian beliefs is Chew & Sagi (2006b). They avoid sequential decisions as we

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[23] The same functional form appeared in Grant, Kajii, & Polak (2001) in an application to game theory.

[24] Kilbanoff, Marinacci, & Mukerji (2005, end of §2) suggest a generalization of their model where Allais-type choices can be allowed for in their second-stage (unambiguous) events. Such choices cannot, however, appear in their first-stage events, the events that comprise ambiguity.

do, and extend their derivation of probabilistic sophistication from Chew & Sagi (2006a) to so-called Dynkin systems or lambda systems of events. They then introduce small worlds that, apart from some formal differences, play a role similar to sources in this chapter. This chapter has taken their model as point of departure.

Real incentives.—We used both a real-incentive treatment and a hypothetical-choice treatment so as to investigate the effects of real incentives when examining uncertainty and ambiguity attitudes. Throughout, we find more aversion, and less noise, for real incentives, in agreement with other studies (Hogarth & Einhorn 1990; Keren & Gerritsen 1999), and in agreement with findings in other domains (Camerer & Hogarth 1999). Our finding supports the principle that real incentives should be implemented whenever possible.

The random-lottery incentive system.—The random-lottery incentive system has become the almost exclusively used incentive system for individual choice in experimental economics (Holt & Laury 2002).[25] We used a form where not for each subject one choice is played for real, but only for some randomly selected subjects. This form was also used by Harrison, Lau, & Williams (2002). Two studies examined whether there was a difference between this form and the original form where each subject is paid, and did not find a difference (Armantier 2006, p. 406; Harrison, Lau, & Rutström 2007).

In the pilot study we asked subjects which form of the random lottery incentive system would motivate them better, the one that we later implemented (with a high possible payment), or a traditional form paying one randomly selected choice for each subject, in which case prizes will be moderate. The subjects expressed a clear preference for the single-large prize system that accordingly was

[25] A detailed discussion is at

http://people.few.eur.nl/wakker/miscella/debates/randomlinc.htm.

implemented in our experiment.[26] The expected gain per subject, €16, is in agreement with common payments for experimental subjects used in the traditional random-lottery incentive system and in other contexts. The data for the real-incentive treatment were of higher quality than the data for the hypothetical-choice treatment, confirming the effectiveness of the incentive system used.

Real incentives and chaining.—It is well known that real incentives can be problematic in chained experiments (Harrison 1986). Because in our construction of the $a_{i/j}$'s with one $a_{i/j}$ obtained influencing the questions asked next, one may be concerned about it being advantageous for subjects to not answer according to their true preferences in a question but instead to seek to improve the stimuli that will occur in future questions.

We organized our chaining of the $a_{i/j}$'s as follows so as to minimize the chaining problem for our real incentives. First, our subjects did not know about this chaining. In addition, we paid attention during the interviews, all done individually, to whether subjects were aware of this chaining. No interview suggested any such awareness. The indifference values used in follow-up questions were midpoints of intervals, so that these values had not occurred before and could not be recognized. Second, even if subjects would know that this chaining took place, they would not know how this was done, so that they would not know in which direction to manipulate their choices. Even for someone who knows the actual organization of the $a_{i/j}$'s (such as the reader), it is not clear in which direction to manipulate answers so as to improve future stimuli.

As regards the chained bisection method used to measure indifferences, we compared the questions used there with consistency-check questions that were not part of a chained procedure. We found no differences, which again suggests no strategically-driven biases. The parameters found for utility and probability transformation, and the discrepancies found between real and hypothetical choice, are

[26] In the decision actually played, the subject preferred a certainty equivalent of €400 to the chance mentioned, and this is what he received.

all in agreement with common findings in the literature, and the subjective probabilities elicited are very well calibrated. These findings further suggest that our data were not distorted by strategic considerations generated by chaining.

An application of our approach.—In a neuroeconomic study, Hsu et al. (2005) considered three sources of uncertainty, being a card-deck with known composition, temperature in New York, and temperature in Dushanbe (Tajikstan). They estimated judged probabilities, in a role similar to our choice-based probabilities, to be 0.5 for the respective events. As explained by Camerer (2007, sections 3.3 and 4), they then estimated what they called probability weighting functions and what we call probability transformations. Not only did they do within-person comparisons as in Figures 4.9.1 and 4.9.3, finding more pessimism for the more ambiguous source, but also between-person comparisons as in Figures 4.9.2 and 4.9.4. They compared two groups of patients with different brain lesions. The "frontal" patients transformed probabilities less than the other group.

Topics for future research.—An obviously important topic for future research concerns the development of tractable tools for analyzing and measuring attitudes for sources that are not uniform. A further development of theory regarding the phenomena found, such as likelihood insensitivity, and properties of the measures a and b for sensitivity and pessimism, is also desirable.

We have reported basic tests of exchangeability, restricting attention to two outcomes so as to focus on the likelihood aspects of decision making. We also restricted attention to noncomposite events. More elaborate tests, for instance regarding composite events and more general outcomes, are planned for future research. Empirical violations of uniformity can then be expected that are not based on intrinsic non-uniformity, but on perceptual biases. For example, convex unions of intervals may be underestimated relative to nonconvex unions because, in the terminology of Tversky & Koehler (1994), the former may be perceived as implicit unions and the latter as explicit unions. Machina's (2004) almost objective events constitute an extreme case of "nonconvex" events. Those events concern for instance whether the 10^{th} digit of temperature is odd or even. Similarly, events related to extreme values (such as E_2^1) may be perceived differently than events related to intermediate values (such as $E_4^2 \cup E_4^3$). The purpose of this chapter has been to demonstrate the conceptual usefulness of sources of uncertainty, and the tractable way in which sources allow for the analysis of attitudes towards uncertainty and ambiguity.

4.12. Conclusion

We have demonstrated that uniform sources and source-dependent probability transformations can be used to analyze uncertainty and ambiguity in a tractable manner. Some examples that have traditionally been put forward as demonstrations that beliefs cannot be modeled through subjective probabilities, can be reconciled with subjective probabilities after all by properly identifying different sources of uncertainty. We demonstrated the implementability of our approach in an experiment.

For most sources of uncertainty choice-based (subjective) probabilities existed (in particular, for all under real incentives). For those we showed how graphs can easily be drawn that capture everything relevant about uncertainty attitude, with exact quantifications provided. This approach is, to the best of our knowledge, the first one that can obtain such exact and complete quantifications, entirely choice-based, empirically.

The phenomena that we observed in the data confirm descriptive theories of ambiguity put forward in the psychological literature (Einhorn & Hogarth 1985; Tversky & Fox 1995). Besides the important component of ambiguity aversion, extensively studied in theoretical economic analyses, and often taken as normative, we also find likelihood insensitivity, a cognitive component that underlies the coexistence of gambling and insurance. We hope that our study will enhance the operationalizability of theories of uncertainty and ambiguity.

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Chapter 5.
A Robust Choice-Based Technique for Eliciting Subjective Probabilities

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Abstract

In the literature, several choice-based techniques have been proposed to elicit subjective probabilities: scoring rules, certainty equivalents or canonical probability. All of them are equivalent when the agent who has to express her belief is an expected value maximizer. However, we will observe that these techniques are not robust to some of the most common deviations from expected value (St Petersburg, Allais and Ellsberg paradoxes). Assuming the existence of additive beliefs, we describe a robust technique (the exchangeability method), which is based on subsequent decompositions of the state space into equally likely events. The feasibility of this technique is then confirmed in an experiment, in which subjacent assumptions of the exchangeability method are tested.

"But it seems that the method of betting permits us in the majority of cases a numerical evaluation of probabilities that has exactly the same characteristics as the evaluation of prices by the method of exchange"

(Borel 1924).

5.1. Introduction

Several fields of the literature in economics and management are concerned by probability assessment. In decision analysis, decision makers often have to seek advice from experts for a probabilistic description of the risk they are facing. Experimental economists sometimes need to know what subjects believe (e.g. Nyarko & Schotter 2002). In behavioral decision making, testing whether subjective probabilities exist and are updated using Bayes' rules is a topic of major interest. Even if subjective probability assessment may be done through simple direct judgment, a particular credence is given to probabilities derived from choices. The former viewpoint is developed in the psychological literature, in which belief measurements are based on introspective judgments; the latter viewpoint is defined as the revealed preference approach and is prevalent in economics. Through choices, agents have incentives to tell the truth. This is why several choice-based methods have been proposed to elicit beliefs. Let us successively present four techniques.

The first and simple method consists in asking a person for her *canonical probability* of event E, i.e. the probability p such that she is indifferent between winning €x with probability p or if E occurs (e.g. Raiffa 1968 p110, Wright 1988, Holt 2006). A second method is based on the *certainty equivalent* of the prospect "winning €1 if E occurs". Under expected value, it is obvious that the subjective probability must be equal to the certainty equivalent (Chesley 1978 tests this method). The most used elicitation techniques are *proper scoring rules* (and above all the *quadratic scoring rule*). The agent should give her subjective probability and receive a score that depends on the assessed probability and on the event that occurs. The score, which can correspond to a monetary payment, is computed such that the agent has an incentive to tell the truth if she wants to maximize her expected gain.

The last technique is based on a bisection process, i.e. subsequent partition of the state space into two subevents that the agent is indifferent to bet on. We can find the intuition of this method in Ramsey (1926) or Fellner (1961), and a complete description of a judgment-based equivalent in Raiffa (1968) (see Chesley 1978 or Wright 1988 for some tests of the judged version). Chapter 4 presented a choice-based implementation of this technique, which will be referred to as the *exchangeability method* because it is founded on the Ramsey-de Finetti's basic idea of event exchangeability. More recently, Chew & Sagi (2006a) derived the existence of probabilistic beliefs from this concept. Eventually, the four above-mentioned techniques are supposed to give the same result when applied to a person that is an expected value maximizer.

However, it is well-known that most agents' behavior under uncertainty does not match with expected value. First of all, Bernoulli (1738) proposes to accommodate the St Petersburg paradox (that will be described in the next section) by introducing expected utility, in which the expected satisfaction rather than the expected gain is maximized. Then, Allais (1953) highlights what he called the *"distortion of objective probabilities"*, and especially the fact that departure from certainty has a higher impact than an equivalent change in intermediate probabilities. Finally, Ellsberg (1961) shows that people would rather bet on known probabilities than on vague, imprecise, or even unknown probabilities. This third and latest deviation from expected value is a crucial point in probability assessment since it obviously deals with events having unknown probabilities. From these three deviations, the ability of the different techniques to elicit Bayesian beliefs (beliefs verifying probability laws) is examined.

By comparing these methods under the three just mentioned behavioral violations of expected value, we show that the exchangeability method is more robust to the three paradoxes than the alternative techniques, especially to the two-color Ellsberg paradox. However, the exchangeability method is crucially based on the hypothesis that subjective probabilities satisfy additivity. However, several studies show that (judged) probabilities exhibit subadditivity. For instance, Teigen (1974) showed that the judged probability of a union of disjoint events may be lower than the sum of the judged probability of each event. Fox & Tversky (1998) found similar subadditivity of beliefs. Tversky & Koehler (1994) proposed their Support Theory in order to explain non-additivity of beliefs: beliefs are influenced by the description of the concerned events and different descriptions of the same event can lead to different beliefs.

We thus have to test the existence of additive probabilities that some previous results find to be a potential Achilles' heel. This is why we first implement the exchangeability method in an experiment, which challenged the hypothesis of additive probabilities. Some tests are also built to provide further information about reliability and predictability. Eventually, our data confirm Ellsberg-type behavior (differing treatments between risk and uncertainty), on which the exchangeability method is supposed to do better than alternative techniques.

Section 5.2 presents the elicitation techniques and confronts them with the St Petersburg, Allais and Ellsberg paradoxes. Section 5.3 describes the method used in the experiment while section 5.4 exposes the results. Section 5.5 discusses and concludes.

5.2. Comparing the robustness of the elicitation techniques

5.2.1. Elicitation techniques under expected value

Throughout section 5.2, only decision models that include a subjective probability distribution over the *state space* (denoted by S) will be considered. The simplest model among these is *expected value*. To begin with its presentation, we must define the *binary act* xEy: it gives $x \in \mathbb{R}^+$ if an event $E \subseteq S$ occurs and another positive real outcome y otherwise. We restrict our presentation to outcomes from $\mathbb{R}^+$ for the sake of simplicity. For some events from S, probabilities are *'known'* or *'objective'.* Acts on such events will be written xpy, they yield $x \in \mathbb{R}^+$

with probability p and $y \in \mathbb{R}^+$ otherwise[27]. For all x and E, the constant act xEx (or equivalently xpx) will be referred to as x. The nondegenerate preference relation on the set of acts is denoted by $\succcurlyeq$ (with the usual strict preference $\succ$ and indifference $\sim$). In this first part, we will assume that the decision maker is an expected value maximizer: her preferences over the set of acts are represented by $P(E)x+(1-P(E))y$. P is a *subjective probability measure* and assigns $P(E) \in [0,1]$ to each event E with $P(S)=1$; for all events E and F such that $E \cap F = \emptyset$, $P(E \cup F)=P(E)+P(F)$. For acts with known probability (acts that can be rewritten as xpy), the representation function obviously becomes $px+(1-p)y$. Lastly, an elicitation technique will be said to be *robust* or *unbiased* if it would allows us to find the subjective probability of an event.

The first elicitation technique consists in finding the probability p, for each event E and for some nonzero outcome x, such that $xp0 \sim xE0$. Under expected value, $px=P(E)x$. Consequently, the subjective probability of E is equal to the *canonical probability* p.

The second technique, *the certainty equivalent method*, is based on the determination of c such that $c \sim xE0$. Under expected value, $c=P(E)x$ and thus $P(E)=c/x$. This can be simply implemented using $x=1$, such that $P(E)=c$.

Scoring rules (the third challenger in our robustness comparison) are built on a score (the payoff) that is determined as a function of the reported probability r and of the occurring of the event E. According to Winkler (1969), *"a payoff function which depends on the assessor's stated probabilities and on the event which actually occurs may be used (1) to keep the assessor honest or (2) […] to evaluate assessors and to help them to become "better" assessors."* Formally, a decision maker assessing that the probability of E is r will receive an act $x_r E y_r$, where x_r and

[27] Sarin & Wakker (1997) proposed such a framework, in which some events have known probabilities, and expressed one main condition for consistency: if events E and F have the same known probability p, then $xEy \sim xFy$ for all x and y. It induces that when both acts are rewritten xpy and xpy, then we cannot have $xpy \succ xpy$.

y_r depend on r. For instance, the *quadratic scoring rule* (the most used scoring rule) corresponds to the act:

$$[1-(1-r)^2]E[1-r^2].$$

Under expected value, telling the truth is the optimal strategy. Indeed, maximizing $P(E)[1-(1-r)^2]+(1-P(E))[1-r^2]$ with respect to r gives the following necessary and sufficient conditions: $P(E)=r$ and $-2<0$ (which is trivially satisfied).

Finally, let us present the *exchangeability method*. According to Chew & Sagi (2006a), events A and B are *exchangeable* if permuting the outcomes between these two events does not change the preference value of a prospect. The exchangeability method thus consists in partitioning the state space S into events E and (S−E) such that for some nonnull x, $xE0\sim x(S-E)0$: the agent bets indifferently on E or its complement. Under expected value, $P(E)x=(1-P(E))x$ implies that $P(E)=1/2$. Splitting E into F and (E−F) such that $xF0\sim x(E-F)0$ enables us to determine that each event F and (E−F) has probability 1/4. Carrying on this procedure completes the distribution. However this method requires rich state space: it must be possible to remove a small "part" of an event to add it to another one in order to obtain equally likely events.

OBSERVATION 5.2.1: Under Expected Value, the four techniques would allow us to robustly elicit subjective probability. □

5.2.2. First deviation: St Petersburg paradox

Daniel Bernoulli (1738) proposed expected utility as an explanation to the St Petersburg Paradox that was suggested by his cousin Nicolas Bernoulli. The story is the following: why does any player refuse to pay more than a limited (finite) amount of money €x to play a game that gives 2^n if the first head side of a coin appears in the n^{th} toss, while this game has an infinite expected value? Daniel Bernoulli argues that players maximize some *expected utility* instead of the expected value, taking into account the satisfaction (i.e. the psychological value) they can get instead of the monetary value of outcomes. As seen in chapter 1, von Neumann & Morgenstern (1944) axiomatized this model with known probability and Savage (1954) generalized expected utility to unknown probabilities. Using the same framework as in the previous subsection, it will now assumed that the decision maker is an expected utility maximizer: her preferences over the set of acts are represented by $P(E)u(x)+(1-P(E))u(y)$ with P the subjective probability function and u the *utility function*. It will assumed throughout that $u(0)=0$.

Under Expected Utility, the canonical probability method still works: $x_p0 \sim x_E0$ implies that $pu(x)=P(E)u(x)$ and thus $P(E)=p$. This is also true for the exchangeability method, for which $P(E)u(x)=(1-P(E))u(x)$ still implies that $P(E)=1/2$. Obviously, the certainty equivalent method is now biased, because $c \sim x_E0$ implies that c is no longer equal to P(E) but $c=u^{-1}(P(E)u(x))$. Using $x=1$ and fixing $u(1)=1$, $c=u^{-1}(P(E))$. A decision maker who is risk averse will give a certainty equivalent lower than her subjective probability. Indeed, assuming that $u(x)=x^a$ with $a<1$ (risk aversion), and that $P(E)=q$ $(0 \leq q \leq 1)$, $E0 \sim q^{(1/a)}$. So, if the certainty equivalent is used to elicit the subjective probability, the decision maker's true belief is now underestimated: $q^{(1/a)}<q$.

Now, as regards the scoring rules, they are based on the maximization of the expected gain. It is thus obvious that, under expected utility, the obtained probability is biased. To be more specific, Offerman et al (2007) prove that the reported probability r satisfies

$$r = \frac{P(E)}{P(E) + \left(1 - P(E)\right)\frac{u'(1 - r^2)}{u'(1 - (1 - r)^2)}}.$$

Murphy & Winkler (1970) already showed that, if the subjective probability P(E) is lower (higher) than 1/2, the reported probability r is higher (lower) than P(E). So, small probabilities are overestimated and large probabilities are underestimated.

OBSERVATION 5.2.2: Under expected utility, only the canonical probability method and the exchangeability method are unbiased. □

5.2.3. Second deviation: Allais paradox

In a famous paradox, Allais (1953) presents a violation of expected utility (and consequently of expected value) with known probabilities. MacCrimmon & Larsson (1979) and Tversky & Kahneman (1992) generalize this paradox to unknown probabilities. For instance, Tversky and Kahneman interview 156 money managers about daily variations of the Dow Jones index, using the following three cases: L (variation strictly lower than 30), M (variation between 30 and 35) and H (strictly larger than 35). It appears that:

$$(L{:}\$25000, M{:}\$25000, H{:}\$25000) \succ (L{:}\$25000, M{:}0, H{:}\$75000)$$

(5.2.1) (5.2.2)

in 68% of answers, but that:

$$(L{:}\$0, M{:}\$25000, H{:}\$25000) \prec (L{:}\$0, M{:}\$0, H{:}\$75000)$$

(5.2.3) (5.2.4)

for 77% of the subjects. Under expected utility, these preferences imply:

$$P(H)u(75000) > [P(M)+P(H)]u(25000)$$

and

$$P(L)u(25000)+P(H)u(75000) < u(25000),$$

which result in the following contradictory inequality:

$$P(L)u(25000) < P(L)u(25000).$$

The intuition under this paradox is that removing the opportunity of winning \$25000 if L occurs has a greater impact if it is applied on certainty (from prospect 5.2.1 to prospect 5.2.3) than if it is applied on an (already) uncertain act (from prospect 5.2.2 to prospect 5.2.4). More generally, this phenomenon is known as diminishing likelihood sensitivity (Wu & Gonzalez 1999): the closer to impossibility or certainty a variation of likelihood is, the stronger the reaction is. This implies that most people overweight the less likely events and underweight the more likely ones (Fox & Tversky 1995).

Non-expected utility theories are developed in order to take into account the Allais paradox. Generalizations of expected utility accommodating this kind of behavior include Schmeidler's (1989) Choquet Expected Utility, Tversky & Kahneman's (1992) Cumulative Prospect Theory, Gilboa & Schmeidler's (1989) Maxmin Expected Utility. In our framework and according to Ghirardato & Marinacci (2001), these theories coincide with a common representation. From now, it will thus be assumed that that the decision maker's preferences over the set of acts are represented by a non-expected utility functional that assigns $W(E)u(x)+(1-W(E))u(y)$ to act xEy where $x \geq y$. W is the *weighting function* with $W(\emptyset)=0$, $W(S)=1$ and $W(A) \leq W(B)$ for all $A \subseteq B$. The weighting function may be additive or not. Note that every positive binary act can be written as xEy for some event E and outcomes $x \geq y$: this is important because the representation depends on which event the highest consequence is affected to. Since this chapter deals with probability elicitation, we will also assume that decision makers are *probabilistically sophisticated*, i.e. that preferences satisfy first order stochastic dominance with respect to a subjective probability measure. As a consequence, W(E) can be rewritten w(P(E)).

Under this model, it is obvious that the certainty equivalent c of xE0 is still different from P(E): it is now equal to $u^{-1}(w(P(E))u(x))$. As regards to scoring rules, it is not sure that the highest outcome is associated to E because outcomes depend on the reported probability. To ensure that the outcome on E is larger than the outcome on S−E and that the above-mentioned representation can be directly applied, Offerman et al (2007) point out that we can add a unit if E occurs such that the rule becomes:

$[2-(1-r)^2]E[1-r^2]$

This act always gives the highest outcome on E, and r is such that (according to Offerman et al 2007):

$$r = \frac{w(P(E))}{w(P(E)) + \left(1 - w(P(E))\right)\frac{u'(1-r^2)}{u'(2-(1-r)^2)}}.$$

On the contrary, the canonical probability still corresponds to the true belief: w(p)u(x)=w(P(E))u(x) implies P(E)=p. The exchangeability method does not suffer from this paradox either: w(P(E))u(x)=w(1−P(E))u(x) still implies that P(E)=1/2.

OBSERVATION 5.2.3: Under the postulated non-expected utility model and probabilistic sophistication, the canonical probability method and the exchangeability method are unbiased. □

Offerman et al (2007) propose a method that corrects the quadratic scoring rule for the two previous biases, by applying the scoring rules to known probabilities, in order to evaluate the bias at each probability. Knowing the reported probability of event E and removing the bias, they obtained the belief. In other words, they elicit a function R such that R(p) is the reported probability of

$[1-(1-r)^2]p[1-r^2]$.

Second, they show that if $R(p)=r_E$ where r_E is the reported probability of

$[1-(1-r)^2]E[1-r^2]$,

then P(E)=p.

5.2.4. Third deviation: Ellsberg paradox

Ellsberg (1961) proposes a paradox that has already been extensively discussed in this dissertation: an urn contains 50 red balls and 50 black balls (these events will be referred as 1/2), while another urn contains red (R) and black (B) balls in unknown proportion. When people are asked for indicating an urn such that they can win €x if a red (black) ball is drawn in the urn they choose, most of them prefer the known urn. Consequently $x(1/2)0 \succ xR0$ and $x(1/2)0 \succ xB0$. Under the three previously assumed models, these preferences imply P(R)<1/2, P(B)<1/2 and thus P(R)+P(B)<1: this is a violation of the additivity property of probabilities. It may be argued that additive subjective probabilities do not exit or that beliefs are not additive. However, it seems that most people do believe that the probability of drawing a red (black) ball in the unknown urn is one half, even if they prefer to bet on the urn about which they have more information. This was already suggested by Fellner (1961).

This dependency on information implies the necessity of defining a *source of uncertainty*, which is a set of events that are generated by a common mechanism of uncertainty. The necessity of this concept was introduced in Heath & Tversky (1991), Tversky & Kahneman (1992), Tversky & Fox (1995) and Tversky & Wakker (1995). For instance, football games correspond to events that belong to a first source of uncertainty, while results of basketball games belong to another source. The football fan has high knowledge about the chance of winning of a given football team, but very little knowledge for a basketball team. A decision maker having the same level of knowledge for all the events from one source is indifferent between betting on equally likely events from this source. For instance, considering the source of uncertainty that corresponds to the events of the unknown urn, most people are indifferent between betting on red or black balls. Their preferences within the unknown urn are consistent with additive probabilities (P(R)=P(B)=0.5), but attitudes that differ across urns explain the Ellsberg paradox.

As a consequence, agents can be probabilistically sophisticated within a source of uncertainty, even if their attitudes differ across sources. In the previous subsection decision weights W(E) were decomposed into a probabilistic belief P(E) and a general weighting function w(.) that was independent from the source. But many experimental findings are consistent with source dependent weighting functions, i.e. source dependent attitudes (Dolan & Jones 2004; Fox & See 2003; Fox & Tversky 1998; Kilka & Weber 2001). The same dependency result has been found in chapter 4 with choice-based probabilities instead of judged probabilities.

Let us slightly modify the framework that was previously assumed in order to add sources of uncertainty. Unlike Chew & Sagi (2006b) who propose an endogenous definition of sources of uncertainty (see chapter 1), only exogenously defined sources (like events concerning the French stock index CAC40) will be considered here (as in chapter 4). For the simplicity of presentation, each of the m sources will be represented by a different state space S_i, for i=1,…,m. Doing so, intersections of events from two different sources, e.g. the three-color Ellsberg paradox, are rule out. This paradox is the replication of a similar mechanism of the two-color problem, but with only one urn containing 30 red balls, as well as 60 black and yellow balls in unknown proportion. The event “a red ball is drawn” and its complementary event belong to a first source of uncertainty, a second source is generated by not knowing the proportion of black and yellow balls. The event “the ball is black” is thus at the intersection between the two sources. Even if ruling out such cases limits the generality of the model we intend to introduce, it still takes into account significant deviations like the two-color Ellsberg paradox.

The same canonical representation as in the previous section is still assumed, but probabilistic sophistication only holds in each source, i.e. on each S_i, but not between sources, i.e. between different S_is. In other words, the decision maker’s preferences within each source are only explained by her subjective probability distribution over outcomes, while her preferences across sources depend on likelihood considerations as well as on her attitude towards each source. Probabilistic sophistication restricted to sources is recently appeared in the literature (Chew & Sagi 2006b, Ergin & Gul 2004).

So, we keep on assuming non-expected utility, but probabilistic sophistication is now restricted to each source. As a consequence, the preferences over the set of acts xEy (with $x \geq y$), where E belongs to a source S_i, are represented by $w_i(P(E))u(x)+(1-w_i(P(E)))u(y)$. The *source dependent probability transformation function* of source i is denoted w_i. Let us consider that source S_0 contains all events the probability of which is known and rewrite w_0 without its lower index. The weighting function when probabilities are known is thus denoted w. Because w and w_i are one-to-one functions from [0,1] to [0,1], the decision weights can be rewritten: $w_i(P(E))=w(\varphi_i(P(E)))$, where w is the weighting function when probabilities are known and the one-to-one function φ_i describes the specific attitude towards source S_i. Similar decomposition of decision weights into three functions can be found in Fox & Tversky (1998) or Wakker (2004), even if they do not assumed that beliefs are represented by additive probabilities.

Let us come back to our elicitation techniques and begin with the canonical probability p such that $xpy \sim xEy$. It is clear that $p=\varphi_i(P(E))$ (considering that $E \in S_i$). Consequently, the canonical probability does not only contain a belief but also an attitude component that is generated by source i. The Ellsberg paradox clearly suggests this point. The certainty equivalent of xEy is equal to $u^{-1}(w(\varphi_i(P(E)))u(x))$. It should be noted that, even if obtaining u and w is possible when probabilities are known, correcting the certainty equivalent from risk attitude will not be sufficient to obtain the subjective probability. The combination of belief and attitude towards source S_i remains.

Now, regarding the scoring rules, the reported probability[28] of the quadratic scoring rule is given by r such that

$$r = \frac{w\left(\varphi_i\left(P(E)\right)\right)}{w\left(\varphi_i\left(P(E)\right)\right) + \left(1 - w\left(\varphi_i\left(P(E)\right)\right)\right)\frac{u'(1-r^2)}{u'(2-(1-r)^2)}}.$$

The corrected version of Offerman et al. (2007) suffers from the same limitation as canonical probability. Under the assumed model, with R(p) (resp. r_E) the reported probability of $[1-(1-r)^2]p[1-r^2]$ (resp. $[1-(1-r)^2]E[1-r^2]$), $r_E=R(p)$ implies $p=\varphi_i(P(E))$.

None of these methods is able to correct for attitude towards sources. On the contrary, the exchangeability method still works. For events E and S_i-E that belong to source i, $xE0 \sim x(S_i-E)0$ induces

$$w(\varphi_i(P(E)))u(x)=w(\varphi_i(1-P(E)))u(x)$$

[28] Recall that a 1 unit is added if E occurs, to ensure that the outcome on E is always higher than on its complement.

and thus $\varphi_i(P(E))=\varphi_i(1-P(E))$. As a consequence, φ_i being a one-to-one function implies that $P(E)=P(S_i-E)=1/2$. Equivalently, in the two-color Ellsberg paradox, indifference between bets on red or black balls in the unknown urn can be interpreted as a subjective probability of 1/2 for both events. Note that richness has to be assumed for each source, i.e. if $xE0 \succ xF0$ for some disjoint E and F from source i, a small subevent e can always be removed from E to be added to F so that $E-e$ and $F \cup e$ are equally likely. This assumption may be seen as a too strong limitation and will be discussed later.

OBSERVATION 5.2.4: Among the four techniques under consideration, the exchangeability method is the only unbiased one under the model we chose to encompass the St Petersburg, Allais and (two-color) Ellsberg paradoxes. □

This result generates several questions: even if the exchangeability method is theoretically safer and sounder, is it yet feasible? Its good properties are due to two main hypotheses: additivity of subjective probabilities and source dependent behaviors. Could these hypotheses be empirically supported? The two following sections are devoted to the presentation of an experimental study that aims at giving some answers to these questions.

5.3. Method

5.3.1. Subjects

Fifty-two subjects (25 women and 27 men) participated to the experiment during March-May 2005. All participants were studying in Economics and Management (27 subjects) or Social Sciences (25 subjects) at Ecole Normale Supérieure de Cachan (France). They were enrolled thanks to posters and presentations at the beginning of their courses. None of them knew the true goal of the experiment. They were only told that the experimenter wanted to collect their choices in an uncertain framework. The computer-based experiment was conducted through individual interviews, using software specifically developed for the experiment. Each participant was seated in front of a screen in the presence of the experimenter, who entered the participant's statements into the computer and submitted it after clear confirmation.

5.3.2. Elicitation technique

Consider a source S_A and then determine two complementary events such that they are revealed equally likely. Let us denote those events A_2^1 and A_2^2. They are such that for some x, $xA_2^1 0 \sim xA_2^2 0$. From this 2-fold partition of S, a 4-fold one can be generated by splitting each of these two events into two equally likely sub-events, i.e. by finding A_4^1, A_4^2,A_4^3 and A_4^4 satisfying $A_4^1 \cap A_4^2 = \emptyset$, $A_4^3 \cap A_4^4 = \emptyset$, $xA_4^1 0 \sim xA_4^2 0$ and $xA_4^3 0 \sim xA_4^4 0$. An 8-fold partition of S can be done by splitting A_4^1, A_4^2,A_4^3 and A_4^4 in the same way. If probabilistic sophistication holds in this source, the A_j^is will constitute an exchangeable partition of the state space, i.e. a partition the events of which are all exchangeable. The whole subjective probability distribution that is associated to the source can thus be inferred. Indeed, as seen in chapter 4, the events of an exchangeable n-fold partition have the same subjective probability: 1/n. The following picture describes the process and the notations.

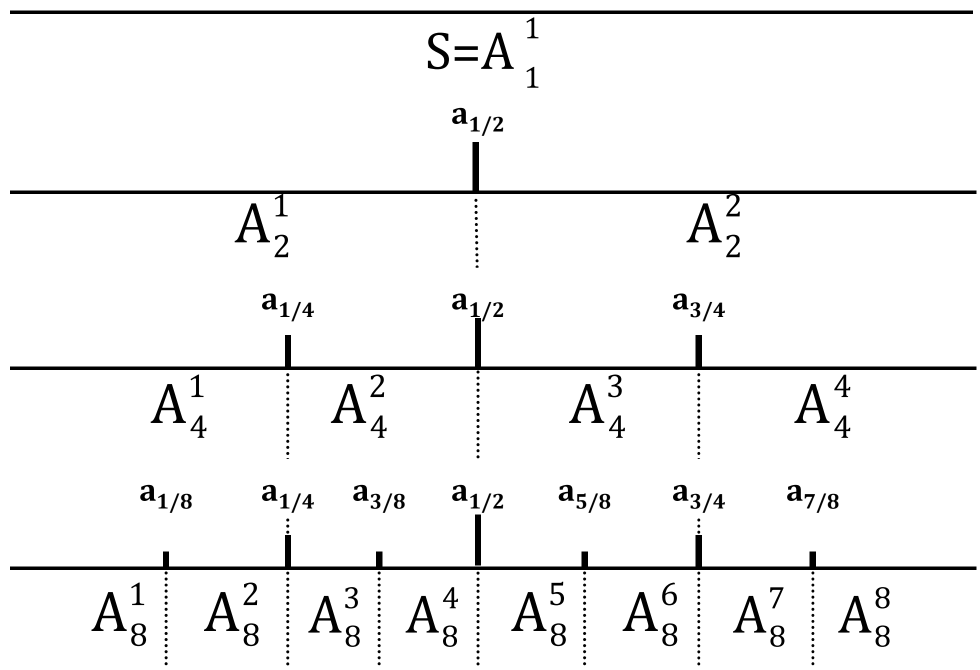

Figure 5.3.1: Decomposition of the state space

Let us explain the intuition that underlies the notation A_j^i: the lower index j indicates in how many exchangeable events the state space is split. The upper index i is the number of the event in a left-to-right reading. Each A_j^i is then divided into A_{2j}^{2i-1} and A_{2j}^{2i}. We may remark that i/j gives the cumulative probability of the right-hand boundary, which is called $a_{i/j}$. As a consequence, $A_j^i = \left(a_{(i-1)/j}, a_{i/j}\right]$, except for $A_j^1 = \left(-\infty, a_{1/j}\right]$ and $A_j^j = \left(a_{(i-1)/j}, +\infty\right)$.

5.3.3. Implementation

The experiment begins with the presentation of the sources of uncertainty, some calibration questions and several trials. Three sources are used: the temperature in Paris (this source will be referred to as S_T, and the corresponding events as $T_j^i = \left(t_{(i-1)/j}, t_{i/j}\right]$), the Euro/Dollar exchange rate (with events $E_j^i = \left(e_{(i-1)/j}, e_{i/j}\right] \in S_E$) and the French stock index CAC40 (with $C_j^i = \left(c_{(i-1)/j}, c_{i/j}\right] \in S_C$). Note that the generic notation S_A and A_j^i will be kept for general comments. For each source, uncertainty resolution occurs exactly four weeks after the experiment.

The experiment starts with a 15 minutes calibration task. For each variable, calibration consists in asking the subject for b_0 and b_1 such that she thinks that there is almost no chance that the value of the variable will be out of the interval $[b_0,b_1]$. The result of the calibration task has strictly no impact on the state space, which remains $(-\infty,+\infty)$[29], but is necessary for graphical reasons and to avoid influencing the participant's belief. Indeed, the state space is displayed as a graduated ruler, and $[b_0,b_1]$ determines the part of $(-\infty,+\infty)$ that is drawn on the screen. After this calibration task, the main part of the experiment begins with no time pressure. On the average, this part lasts 50 minutes.

After the training/calibration, we first determine $t_{1/2}$ to obtain T_2^1 and T_2^2. The first question is built on those b_0 and b_1 that are previously determined for this particular source, and the subject is proposed to bet either on $(-\infty,(b_0+b_1)/2]$ or on $((b_0+b_1)/2,+\infty)$. Then the determination of indifferences is done through a bisection process. After this first elicitation step, $e_{1/2}$, $c_{1/2}$, $t_{1/4}$, $e_{3/4}$, $c_{1/4}$, $t_{1/4}$,…, $t_{1/8}$, $e_{3/8}$, $c_{7/8}$… are determined, switching between sources and probability levels, until an 8-fold partition is obtained for the three sources. In order to introduce diversity, x is randomly drawn between €130, €140 and €150. On the screen (see figure 5.3.2), the position of bets on the right-hand event and on the left-hand one is randomly mixed across questions.

[29] Note that the real minima are not -∞ but -273.15°C, 0$ and -100% for Paris temperature, Euro/dollar and CAC40 respectively.

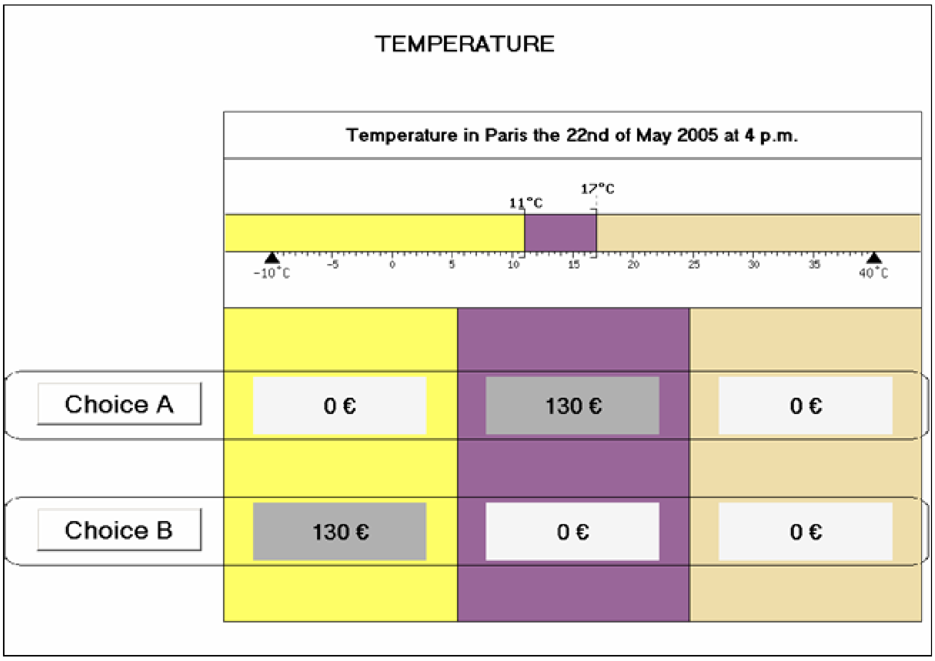

Figure 5.3.2: Screenshot

5.3.4. Tests

Once the partition is obtained, tests are implemented. We want to provide some further information about the reliability of this technique and to challenge the two major hypotheses Observation 5.2.4 is based on: additivity of beliefs and Ellsberg-type behavior.

A test of reliability is implemented by repeating the fourth choice that has been made during the bisection process of $t_{1/2}$, $t_{1/4}$ and $t_{3/4}$ (resp. $e_{1/2}$, $e_{1/4}$, $e_{3/4}$, and $c_{1/2}$, $c_{1/4}$ and $c_{3/4}$) and by computing the rate of identical answers across questions. Then, a topic of interest is the possibility of inferring new exchangeable events from the obtained probability distribution: it is done using the subjective probabilities to determine $t'_{1/3}$ and $t'_{2/3}$, and then testing whether $T_3^1 = (-\infty, t'_{1/3}]$ and $T_3^2 = (t'_{1/3}, t'_{2/3}]$ are revealed equally likely. The same thing is done for the two other sources. This test is implemented using a bisection process, in order to determine a new indifference and to compare the obtained boundary with the theoretical one. In other words, testing the exchangeability of T_3^1 and T_3^2 is done by splitting $T_3^1 \cup T_3^2$ into two equally likely subevents with a common boundary called $t_{1/3}$, and by comparing it with the theoretical value $t'_{1/3}$.

Tversky & Koehler (1994) point out that subadditivity of subjective probabilities may come from the description of each event. In the same vein, Starmer & Sugden (1993) and Humphrey (1995) show the so-called event-splitting effect, i.e. the fact that two incompatible sub-events look more attractive than their union. In our experiment, exchangeability is confronted with unions of events by checking whether $T_4^1 \cup T_4^4$ and $T_4^2 \cup T_4^3$ are also exchangeable. This test compares a convex event ($T_4^2 \cup T_4^3$) with a non convex one ($T_4^1 \cup T_4^4$). If probabilistic sophistication holds, these events will be equally likely but the event-splitting effect predicts that the non convex event will be more attractive than the convex one. We of course implement similar tests with Euro/Dollar exchange rate and with CAC40. Alike predictability, subadditivity is tested by eliciting two exchangeable events, namely $T_4^1 \cup T_4^4 = (-\infty, t_{1/4}] \cup (t'_{3/4}, +\infty)$ and $T_4^2 \cup T_4^3 = (t_{1/4}, t'_{3/4}]$, such that $t'_{3/4}$ has then to be compared with the original value $t_{3/4}$.

Eventually, the crucial point of Observation 5.2.4 is that behavior depends on the source. It is thus necessary to test for violations of probabilistic sophistication across sources, i.e. if an event from one source can be strictly preferred to an equally likely event from another source. Chapter 4 achieves it using willingness-to-bet. Here, preferences are elicited between bets on some event A_j^i from one source and bets on the same event from another source. The participants are thus asked for their preferences among bets, either on an event with known probability p=1/4, or on T_4^1, E_4^1, or C_4^1. They thus have to rank 140p0, $140T_4^1 0$, $140E_4^1 0$ and $140C_4^1 0$. This question is replicated with p=1/2, T_2^1, E_2^1 and C_2^1, and then with p=7/8, $S-T_8^8$, $S-E_8^8$ and $S-C_8^8$.

5.3.5. Real incentives

There is no flat payment in this experiment. During the presentation, the participants are told that some questions will be played for real. First, a subject will be randomly drawn and then, one of her answers (except the answers for the ranking between bets on the different sources[30]) will be played for real. Then the

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[30] See the last paragraph of the previous subsection.

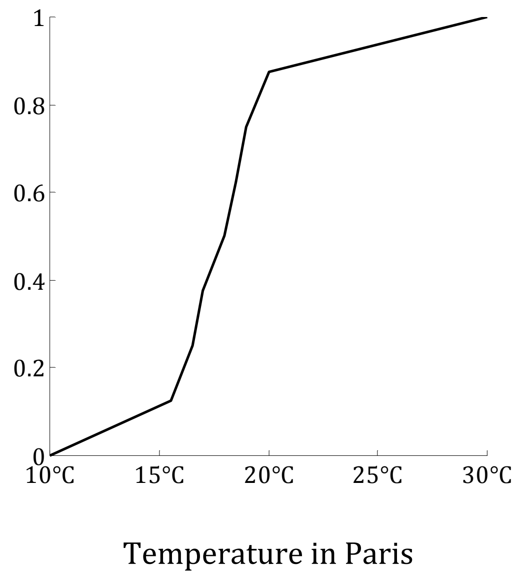
same process will be applied for another subject. Subjects are told that at most one question per subject will be drawn and that the process will stop as soon as four subjects won. This implies that at most €600 will be distributed to the subjects. Four weeks after each experiment, the values taken by the variables are recorded. After the last record, the payments are determined.[31] It could be argued that, even if real incentives are implemented, the process may not be incentive compatible because of the presence of chaining between questions and because of the use of bisection (Harrison 1986). However, there exist several strong arguments suggesting that these problems did not appear. First, the randomization of questions between sources of uncertainty makes the chaining unclear. Second, in our experiment, no simple alternative strategy exists that dominates telling the truth for sure, even for a participant that completely knows the elicitation process. Moreover, we argue that telling the truth is the simplest strategy for any subject, who decides to maximize her gains and minimize the cognitive cost implied by the experiment. Eventually, the reliability questions, which are not chained with later questions, prove consistency of the data (see next subsection).

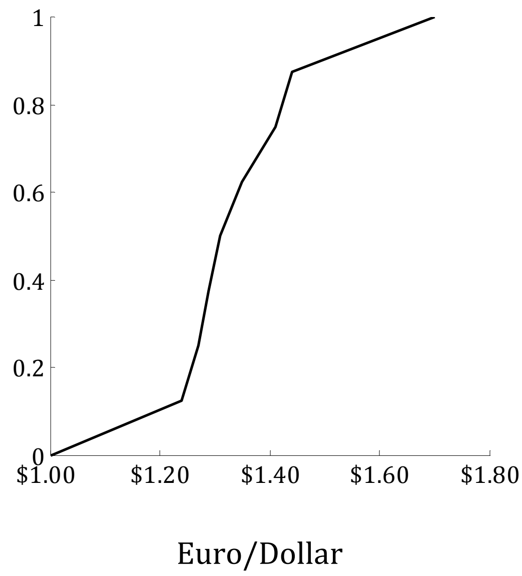
5.4. Results

5.4.1. Subjective probabilities and reliability

Through the experiment, 156 probability distributions are obtained. Because each distribution concerns a particular day, mean or median results are not interesting. To give an idea of what is obtained, the following figure displays the probability distributions of Subject 26.

[31] In June 2005, two participants received €140, and two received €150.





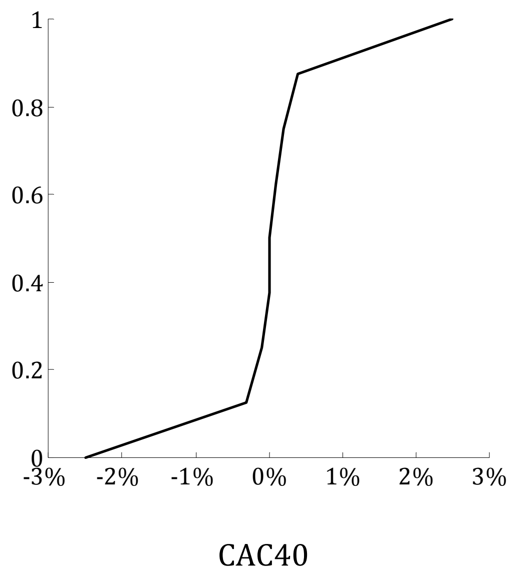


Figure 5.4.1: Subjective probability distributions of Subject 26

One of our topics of interest is the reliability of the method. For 70.51% of the repeated questions, the second answer is identical to the first one. Those questions concern the three sources and the determination of $a_{1/2}$, $a_{1/4}$ and $a_{3/4}$. According to Cochran tests, reliability is not significantly different across question types $a_{1/2}$, $a_{1/4}$ or $a_{3/4}$ (χ^2_2=2.16, p=0.34), nor across sources (χ^2_2=0.53, p=0.77). To further analyze reliability, the rate of identical second answers has to be related to the distance from indifference: when a participant is indifferent between two bets, she is supposed to randomly choose on which events she bets. Thus if the repeated questions only concern those indifferent bets, reliability will be close to

50%. This is why the distance from the indifference is measured as the difference between the common boundary of the two events on which the participant can bet, and the previously obtained value $t_{1/2}, \ldots, c_{3/4}$. Units are 0.5°C, 0.01€ and 0.1% for temperature in Paris, Euro/Dollar exchange rate and CAC40 respectively. The following graphs display the proportion of consistent second answers; abscises represent the distance from indifference.

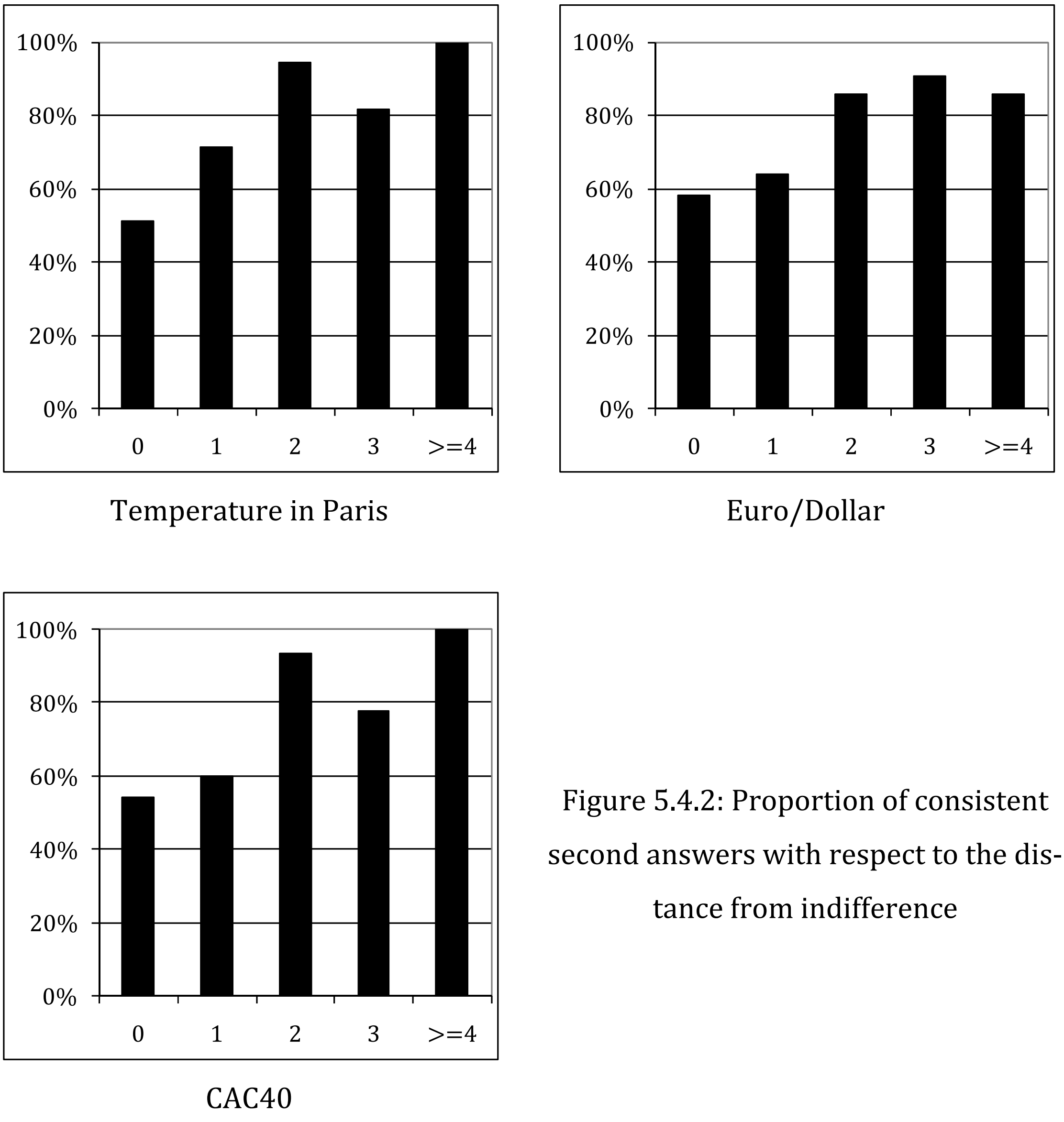


Figure 5.4.2: Proportion of consistent second answers with respect to the distance from indifference

Because any departure from indifference hugely increases the rate of identical second answers, our results strongly suggest that the exchangeability method is able to catch true indifferences, and therefore, under our assumptions, subjective probabilities.

5.4.2. Testing predictability

Are we able to predict beliefs thanks to the elicited subjective probabilities? Our exchangeability method aims at extracting beliefs from choices. But in fact, we do not know whether what we obtain really corresponds to beliefs or not. Nonetheless, a probabilistically sophisticated agent is supposed to act in a consistent way with respect to a subjective probability distribution. This is why a way of evaluating the exchangeability method consists in predicting subjective probabilities of events and then testing for consistency of the agent's behavior with respect to those probabilities. Figure 5.4.3 schematizes the implementation of our predictability test.

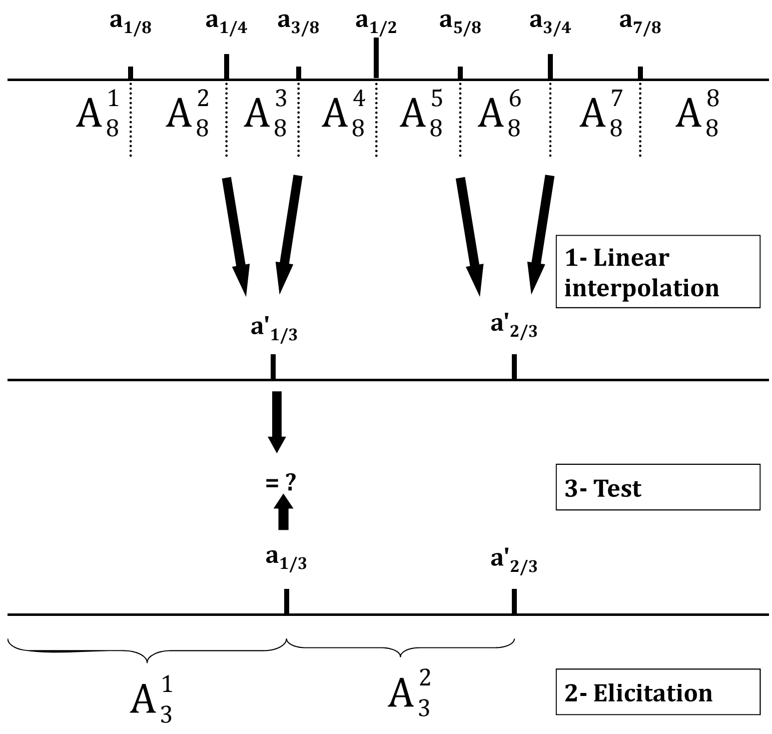


Figure 5.4.3: Testing predictability

A linear interpolation gives theoretical values $a'_{1/3}$ and $a'_{2/3}$. Then $a_{1/3}$ is elicited such that $A_3^1 = (-\infty, a_{1/3}]$ and $A_3^2 = (a_{1/3}, a'_{2/3}]$ are revealed equally likely and is finally confronted with $a'_{1/3}$. However, $a'_{1/3}$ and $a'_{2/3}$ have to be corrected for some approximations: because of the graduations of the rulers (0.5°C, \$0.01 and 0.1%), we only estimate a'_p such that p is as close as possible to 2/3, given the scale units, and then $a'_{p/2}$ is computed. Eventually, $a'_{p/2}$ is rounded off with the scale units and compared with the elicited boundary. Table 5.4.4 gives some statistics.

Table 5.4.4: Testing predictability, the results

	Temperature	Euro / Dollar	CAC40
Difference mean	0.1442°C	−0.0038€	−0.0346%
Correlation	0.9366	0.9511	0.8089
p–values (paired t-test)	0.4957	0.2356	0.3409
N	52	52	52

For all sources, p-values are higher than 5%. High correlations confirm that this acceptation of the null-hypothesis is not caused by noise in the data. As a conclusion of this test, we can say that participants' behavior looks consistent with respect to the elicited probability distribution.

5.4.3. Exchangeability and Event-Splitting Effect

Let us challenge our method with the Achilles' heels of subjective probability elicitation, namely non-additivity of beliefs and sensitivity to the description of the events. The test deals with two events, one being a convex union of disjoint exchangeable subevents, the other one a non-convex union of such events. Does Event-Splitting Effect (ESE) occur and induce a violation of exchangeability? For each source, two events $(-\infty, a_{1/4}] \cup (a'_{3/4}, +\infty)$ and $(a_{1/4}, a_{1/2}] \cup (a_{1/2}, a'_{3/4}] = (a_{1/4}, a'_{3/4}]$ are elicited such that they are revealed equally likely. Then, $a'_{3/4}$ and the original value $a_{3/4}$ are compared. Table 5.4.5 displays the results.

Table 5.4.5: Exchangeability versus ESE

	Temperature	Euro / Dollar	CAC40
$a'_{3/4}-a_{3/4}$	0.3800	0.0170	0.0500
Correlation	0.9258	0.9306	0.6017
p–values (paired t-test)	0.1200	0.0177	0.5241
N	50	46	42

Let us first explain why there are less than 52 observations. A few participants change their mind during the experiment, and their beliefs change especially when they do not have enough knowledge about the source. Those who think they have overestimated $a_{1/4}$ clearly always preferred $(-\infty, a_{1/4}] \cup (a'_{3/4}, +\infty)$ even when $a'_{3/4}$ tends to infinity. This is the case when the new belief coincides with a subjective probability of $(-\infty, a_{1/4}]$ higher than 1/2. This problem appears once for the temperature in Paris but five times for the stock index, which subjects were less familiar with. In order to compensate the bias generated by the absence of those observations, the same number of participants exhibiting the opposite behavior (i.e. participants who think they have underestimated $a_{1/4}$) are removed. This is why ten subjects are missing for the stock index. Note that $a'_{3/4}-a_{3/4}>0$ indicates that $A_4^1 \cup A_4^4$ is preferred to $A_4^2 \cup A_4^3$ and consequently, that the non-convex events appears more attractive. This corresponds to ESE and happens for the three sources, even if it reaches significance only for the exchange rate source. We can conclude that, even with the exchangeability method, belief can be manipulated through the description of the events: a non-convex event appears to have more support than a convex one. However, we could argue that the original elicitation does not suffer from such limitation because it consists in comparing similar convex events.

5.4.4. Testing source dependency

Let us recall that probabilistic sophistication is assumed under each source of uncertainty but violated for the whole state space, according to Ellsberg paradox. Since the interest of this method (and the non-robustness of alternative methods) is grounded on violations of probabilistic sophistication across sources, a test, which aims at comparing events that are supposed to have the same subjective probability, is implemented. This test is built on four bets, one on an event from each source and a bet with an explicit probability of winning. Each bet is displaying the same (subjective) probability distribution over outcomes and participants are asked for ranking them. A Friedman test on ranks shows significantly different ranks between the three sources and the known probability source for each of the three probabilities under consideration ($p=0.016$ at probability 1/4, $p=0.000$ at probabilities 1/2 and 7/8). Removing known probabilities from the analysis, subjects are still significantly influenced by the source at probability 1/4 ($p=0.0352$) but not at higher probabilities.

Table 5.4.6: Comparison of ranks between a source of uncertainty and the known probability source

(p-values, Wilcoxon test)	1/4	1/2	7/8
Temperature	0.001 (AS)	0.001 (AA)	0.000 (AA)
Euro/Dollar	0.002 (AS)	0.341 (AN)	0.000 (AA)
CAC40	0.000 (AS)	0.122 (AN)	0.000 (AA)

The former table displays the results of a signed rank test (Wilcoxon test) comparing the rank of each source with the rank of the known probability source at the three probability levels under consideration. Assuming that the three sources are more *ambiguous* than the source with objective probability, such a comparison highlights ambiguity attitude. If a subject bets on an ambiguous event rather than on a known probability, she is ambiguity seeking (AS). The opposite preference characterizes ambiguity aversion (AA). Indifference between those bets means ambiguity neutrality (AN). Our results clearly show that ambiguity aversion increases with probability, which is consistent with previous literature on ambiguity (e.g. Hogarth & Einhorn 1990).

As a conclusion, recall that source dependency is the phenomenon that makes the other choice-based methods that were considered in section 5.2 biased. This is how the current results support the exchangeability method.

5.5. Discussion and conclusion

5.5.1. Discussion

A first limitation of all choice-based elicitation techniques under consideration is that they all assume Savage's separation between utilities and consequences. Thus, they do not work when utilities are state-dependent, i.e. when the decision maker associates an intrinsic utility to the states of the world. For instance, even if you think that the probability of raining tomorrow is one half, you may not be indifferent between winning an umbrella if it rains and winning an umbrella if it does not rain. This issue also occurs when the decision maker has "stakes" in an event. Assume that an ice cream seller must choose between winning €1000 if the temperature is higher than 20°C and winning the same amount if the temperature is lower than 20°C. She must prefer the second gamble, not because she thinks the event is more likely, but because she wants to cover a possible loss. Karni (1999) proposes a method to deal with state-dependent preferences. However, the choice-based techniques that are discussed in this chapter, and above all the exchangeability technique, remain valid if choice situations, in which the decision maker has either state-dependent preferences or "stakes" in the events, are excluded.

Subadditivity of beliefs constitutes another limitation of our study. However, if beliefs are not additive, separating non-additive attitude from non-additive beliefs seems to be impossible without assuming some parametric functional, either for the probability distribution or for a weighting function that represents this attitude. Clearly, additivity of probabilities was introduced to allow the disentangling of beliefs and attitudes towards sources. It is thus important to carry on the analysis of belief additivity. To this respect, two distinct orientations are left for future studies. First, what tests should be conducted to challenge the additivity hypothesis of the exchangeability method? We conducted such a kind of test by confronting exchangeability with ESE, but various tests would bring some more information. Second, what is the relation between subadditive judged probabilities and the subjective probabilities we obtained with the exchangeability technique (and their built-in additivity)?

However, a strong argument can be put forward in favor of our additive beliefs: *"Objective rules of coherence (the axioms and theorems of probability theory) must be strictly obeyed in any subjective probability evaluation. Coherence is necessary to prevent substantial contradictions, such as the possibility of incurring sure losses as a result of an action"* (de Finetti 1974). As a subjective probability evaluation, exchangeability guarantees a built-in kind of additivity that may be useful to prevent actions that could be based on this evaluation from violating rationality rules.

Finally, it could be argued that alternative methods work for discrete events, while the exchangeability technique requires richness of the state space. A solution to this problem is suggested by Chew & Sagi (2006b). They propose to use an auxiliary draw (a random number from [0,1)) that is used to transform the "poor" state space into a rich one. They apply it to the Ellsberg two-color paradox: instead of betting on red or black balls, the decision maker is asked for her preferences on events such that "a red ball is drawn and the random number belongs to [0,p)". If she bets on this event rather than on its complement, then varying p makes the event less attractive. However, a limitation could be the cognitive cost implied by such a compounded source. Such events that mix a probability and an event could be difficult to understand for the agent. The evaluation of the feasibility of this procedure is left for future research.

5.5.2. Conclusion

The first contribution of this chapter is to show that the exchangeability method, which is a way of eliciting probabilities without making explicit reference to them, is a probability elicitation technique that is robust to the three most-known paradoxes in decision making. The key issue in belief elicitation consists in separating beliefs from attitudes. Indeed, decisions under uncertainty cumulate various effects that make the extraction of beliefs from choices more difficult.

First of all, outcomes are not considered linearly by the decision makers and it is thus necessary to correct for utility. Second, attitude towards risk does not reduce to this attitude towards outcomes but also comes from the subjective treatment of probabilities. Models like cumulative prospect theory capture it through a probability weighting function. If the effect of this weighting is ignored, then elicitation of subjective probabilities could imply an overestimation of small probabilities and an underestimation of large ones.

The last effect that should be separated from belief is the source-dependent attitude. Indeed, agents do not act in the same way when probabilities are known or unknown; and, even if we only consider unknown probabilities, they act differently depending on the knowledge they have about the events under consideration. From expected value to a source dependent and probabilistically sophisticated cumulative prospect theory, the impact of all these deviations on the various elicitation techniques was investigated. Our conclusion is that the exchangeability technique better ensures the separation of beliefs from attitudes towards outcomes, probabilities and knowledge about the events than the alternative methods.

An experiment was run in order to investigate the empirical accuracy of the exchangeability method, namely its reliability and its ability to predict. Even if it was possible to violate exchangeability using event-splitting effect, our data confirm that the notion of source of uncertainty constitutes a major concept to understand behavior under uncertainty. As a consequence, they precisely reinforce the arguments in favor of the exchangeability method, which is able to deal with source dependency.

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Chapter 6.
General Conclusion

This last chapter is dedicated to a general and critical overview of this dissertation. The first section summarizes every chapter and confronts them with the desiderata posited in the introduction. Section 6.2 discusses the main findings, the methodology and the main assumptions of the dissertation. At last, further research that should follow this work is announced in section 6.3.

6.1. Are the goals reached?

Let us first recall the desiderata we defined in Chapter 1. Indeed, after having presented the main model of individual choice under uncertainty, i.e. Subjective Expected Utility (SEU), we exposed and discussed different characteristics that an alternative model should have to be at least as interesting as SEU.

D1 (Bayesianism): The model must be consistent with Bayesian beliefs, i.e. with additive subjective probabilities, or at least it must make it possible to test for their existence.

The existence of subjective probabilities remains a major topic because they are widely used in the economic literature to represent beliefs. One of the main interests of SEU consists in representing beliefs through additive probabilities.

D2 (Robustness): The model must be robust to Allais or/and Ellsberg paradoxes.

These two paradoxes are the main limitations of SEU. This is why allowing for them is necessary for a new model to improve the descriptive abilities of SEU.

D3 (Observability/Measurability): The model must be directly testable or, even better, directly measurable.

Falsifiability of a model is an important requirement in science and its measurability is useful for instance in decision analysis or in order to calibrate an economic model.

D4 (Prediction): The model must allow predictions.

We do not only want to represent a decision maker's behavior in a given situation, but we would also like being able to predict future decisions or attitudes thanks to our analysis.

We wanted these desiderata to be guidelines for the dissertation. We have now to evaluate each chapter with regards to them.

6.1.1. Modeling risk and ambiguity aversion through diminishing marginal utility

Chapter 2 develops a preference-based tool characterizing marginal utility, and uses it to provide axiomatizations for models such as SEU or the smooth model of ambiguity of Klibanoff, Marinacci & Mukerji (2005). This tool, called tradeoff relation, permits to simply represent a wide range of models and it is used in chapter 2 to define necessary and sufficient conditions for two-stage models. The main advantage of such an approach is to enhance simple preference conditions that are equivalent to risk or ambiguity attitudes in those models.

Bayesianism: Chapter 2 first studies Subjective Expected Utility (SEU) and consequently, Bayesianism is assumed. A two-stage model is then introduced, in which even if (subjective and objective) probabilities are considered at each stage, they are associated to a different utility function. Such models encompass Bayesianism at each stage while allowing for violations of reduction of compound lotteries.

Robustness: The two-stage model studied in sections 2.5 and 2.6 of chapter 2 accommodates the Ellsberg paradox by dealing with the uncertainty on the right probability distribution. This model is thus robust with regards to the Ellsberg paradox. However, the assumption of expected utility applying at each stage is violated by the Allais paradox. Remember that this paradox can be explained by a certainty effect: departures from certainty (e.g. a sure gain x becomes a lottery that yields x with probability 99% and 0 otherwise) have a higher impact than equivalent changes in intermediate probabilities (e.g. a lottery that yields x with probability 50% and 0 otherwise becomes a new lottery that gives the same amount x with probability 49% and 0 otherwise). Hence, the perception of certain amounts differs from the perception of lotteries, i.e. sure amounts are not perceived as special cases of lotteries. This is why reducing behavior under risk to marginal utility deprives intuitive contents of risk attitudes. Similarly, the results of chapter 2 can be reinterpreted to show that EU-based models restrict all attitudes to (differences of) marginal utilities and thus lose their intuitive contents. For instance, it may be felt that tradeoff conditions mostly catch the value of outcomes while being supposed to characterize risk or ambiguity attitude. As a conclusion, robustness of models with respect to the Allais paradox seems even more crucial after this chapter and is satisfied in all other contributions of the dissertation.

Observability: One of the main topics of chapter 2 is to provide observable conditions that characterize or compare attitudes towards risk and ambiguity. The tradeoff conditions are easily testable and do not make use of any reference to a theoretical construct like subjective probability. Indeed, several contributions in the literature on ambiguity define attitude through references to expected value (or expected utility) of acts, which are based on subjective probabilities and are thus not directly observable. Chapter 2 aims at avoiding such references. Moreover, the tradeoff conditions we define do not need the restriction that the different decision makers have the same belief or that they have the same behavior when probabilities are known. We can conclude that this chapter clearly contributes to increase the observability of EU-based models.

Prediction: The models under consideration associate attitudes towards risk or ambiguity to (differences of) marginal utility. We propose tools that only characterize marginal utility to observe a decision maker's behavior. Under these models, it is then easy to deduce attitude and predict behavior from simple observations of tradeoff relations.

After this first theoretical contribution to the EU theory, the other chapters are dedicated to analyzing and eliciting non-EU models for behavior under uncertainty.

6.1.2. Combining experts' judgments

In chapter 3, we implement an experiment in order to understand how agents incorporate experts' judgment into their decision making. This experiment is based on the revealed belief associated to a given ambiguous context, i.e. the probability q such that the agent makes the same decision when facing a loss with probability q or the same loss with the ambiguous probability set. We implement two ambiguous situations: experts either agree on an imprecise probability interval $[p-r,p+r]$ or disagree and each of them gives his/her own judgment (this is represented by a set $\{p-r,p+r\}$). Observations of revealed beliefs enable us to understand the impact of conflict or imprecision of experts.

Bayesianism: If we consider that Bayesians having no information about experts' respective reliability should have a uniform distribution over the ambiguous probability set and thus, should use the mean distribution, our experiment reveals that subjects exhibit significant discrepancies with respect to Bayesianism. However, we do not look for subjective probabilities. We can only conclude that even if agents think as Bayesians, they however distort the mean distribution in their choices.

Robustness: Our method is founded on Cumulative Prospect Theory in order to take account of deviations from EU like the Allais paradox. We indeed find significant nonlinear probability weighing, suggesting that such a model is necessary. The model is also robust to the Ellsberg paradox and aims at measuring Ellsberg-type phenomena through indexes describing the impact of ambiguity. For instance, when probabilities are imprecise, we find that prospects are significantly less attractive than when probabilities are clearly defined.

Observability: We assume CPT because it is easily observable, but once this is accepted, revealed beliefs are estimated without any further assumption. For example, we assume neither ambiguity aversion nor that revealed beliefs should be a constant linear combination of the two extreme possible probabilities. Our model allows for attitudes depending on probability levels. Then, we conduct an experiment, in which we measure the impact of different types of ambiguity, namely imprecise and conflicting ambiguity.

Prediction: Finally, an interesting feature of revealed beliefs is that they can be directly interpreted in terms of prediction. For example, the median revealed beliefs of the ambiguous contexts [0,.20] and {0,.20} are equal to .19 and .06 respectively. This means that the median agent takes more risks when the experts disagree than when they agree on the mean estimation .10 or when they agree but give an imprecise statement.

6.1.3. Exchangeability and source dependence

In chapters 4 and 5, we work on the combination of a general non-EU representation for binary acts and probabilistic sophistication restricted to sources of uncertainty. Probabilistic beliefs hold inside each collection of events generated by a common mechanism of uncertainty, and attitudes depend on these collections. In two experiments, we obtain subjective probabilities through the exchangeability method, i.e. through subsequent partitions of the state space into events the agent is indifferent to bet on. Then, we combine those probabilities with willingness-to-bet so as to elicit source-dependent probability transformation functions that describe attitude towards uncertainty.

Bayesianism: Our model is based on the existence of uniform sources, i.e. sources of uncertainty in which probabilistic sophistication holds. By testing exchangeability, from which probabilistic sophistication is derived, we provide new arguments in favor of Bayesianism.

Robustness: The combination of a non-EU representation and uniform sources of uncertainty permits us to get a model that accommodates traditional violations of SEU: the Allais paradox and the (two-color) Ellsberg paradox. The source-dependent probability transformations encompass these phenomena through attitudes depending on probabilities and on knowledge about probabilities. We also observe in chapter 5 that the elicitation technique for subjective probabilities we use is robust to the Allais and (two-color) Ellsberg paradoxes unlike common alternative techniques. However, the three-color Ellsberg paradox remains an issue because it is based on a non-uniform state space.

Observability: An interesting particularity of our model consists in being directly observable as we show in two experiments. Moreover, we quantify Bayesian beliefs and several psychological phenomena such as pessimism and likelihood insensitivity.

Prediction: Eventually, we dedicate a section of chapter 4 to show how our measurements induce computable predictions on agents' behavior. In chapter 5, we also test whether the elicited subjective probabilities well predict new choices.

6.1.4. A few comments on the desiderata

When we posited the 4 desiderata in chapter 1, we gave some intuitions or arguments for them. However, we did not say that every study should follow them or that they have a particular value. They just expressed some goals and the different studies we conducted were oriented so as to reach them. Of course, these studies suffered from limitations and we already presented some of them. Moreover, the next section is dedicated to several critical points, which we wish to develop.

6.2.Discussion

6.2.1.Beliefs and choice-based probabilities

There are several cases in which the choice-based technique for eliciting subjective probabilities we propose in chapters 4 and 5 (and also the other techniques discussed in chapter 5) may be not relevant. We have already introduced some examples in the previous chapter but let us enunciate the main issues:

- *When events have an intrinsic value:* for instance, E is the event "it rains" and the agent does not like rain. If she is willing to pay a high amount of money for an act that makes her win when E occurs, does it mean that she thinks E is very likely or that the act would be a kind of insurance that would compensate the disutility of the rain? In such cases, preferences are said to be *state-dependent.* If both the utility part and the belief part of the representation function are defined on the state space, the traditional choice-based methods we discussed in chapter 5 do not apply anymore.
- *When the decision maker has 'stakes' in the events:* this case is really close to the previous one. It appears when the agent, from whom we want to elicit beliefs, has some interests in the event occurring. For example, you cannot elicit beliefs about the future sales from the sales manager using bets if you do not take into account that her wages may depend on those sales. It is even more difficult if the consequences are also nonmonetary, e.g. career progression or reputation. The problem is even more complex here: a bet on bad sales has an insurance effect while a bet on good sales is a way of appearing sure and confident. The induced preferences are also state-dependent.
- *When uncertainty is resolved long after the elicitation:* a choice-based technique is founded on the fact that the agent faces consequences of

her decisions. However, if the resolution takes place in the far future, how can the agent be paid? Revealed preferences would remain hypothetical.

- *When resolution is itself uncertain:* How much would you bet on the existence of extraterrestrial life? This is typically a situation, in which we do not know if we will know someday.

We can first conclude that in several significant situations, choice-based techniques may be irrelevant. Alternatively, judged probabilities may remain possible. We can always ask somebody for what she thinks. However, we do not know whether or not she is answering seriously. Furthermore, we are never sure whether she is telling the truth or whether she wants us to think she believes what she says (e.g. recall or argument about career progression and reputation; it also works for judged probabilities).

Furthermore, we would like to come back to a term we use in this dissertation: "belief". We use it to designate subjective probabilities in chapter 2, and it is associated to "revealed" and then "Bayesian" in chapters 3 and 4. We have to add that we never know if what we elicit matches with what people think. In the revealed preference approach that prevails in economics, beliefs are inferred from choices. The first limitation is thus the link between what an agent does and what she really thinks. Furthermore, even if acts and thoughts go together, what does exactly correspond to a belief? Is it the decision weight? Maybe it is the revealed belief, which contains attitude towards ambiguity but does not include attitude towards risk. Should beliefs only be additive subjective probabilities? We do not (and even cannot) have a straight answer.

6.2.2. Limits of rank dependent models

In section 1.4 of our introduction, we present paradoxes that violate EU or SEU through the independence axiom or the Sure Thing Principle. We show then that rank dependent models, such as Rank Dependent Utility (RDU), Choquet Expect Utility (CEU) and Cumulative Prospect Theory (CPT), are robust to these paradoxes. In chapters 3, 4 and 5, we work on two-outcome acts and used models that are equivalent to RDU/CEU/CPT. The use of weighting functions suggests that a direct extension of our work to general acts would go through rank-dependence. But are there violations of the rank-dependent models?

Let us first study some tests that were conducted under risk (with known probabilities). Birnbaum (2007) decomposes the Allais paradox into violations of two possible properties: coalescing and Restricted Branch Independence. Coalescing means that branches of a lottery yielding to a common consequence can be combined by adding their probabilities. According to Restricted Branch Independence, if two gambles have a common branch (i.e. a common consequence and probability on this consequence), changing the common consequence does not affect the preference between the gambles.

Thus

$$1 \succ (.10{:}5, .89{:}1, .01{:}0)$$

$\Leftrightarrow$(by coalescing and transitivity)

$$(.10{:}1, .89{:}\boldsymbol{1}, .01{:}1) \succ (.10{:}5, .89{:}\boldsymbol{1}, .01{:}0)$$

$\Leftrightarrow$(by Restricted Branch Independence)

$$(.10{:}1, .89{:}\boldsymbol{0}, .01{:}1) \succ (.10{:}5, .89{:}\boldsymbol{0}, .01{:}0)$$

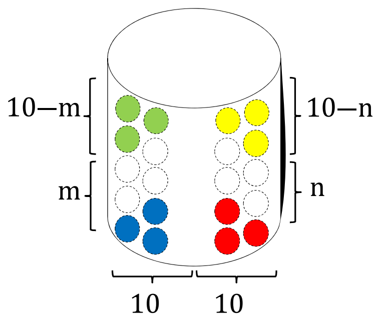
$\Leftrightarrow$(by coalescing and transitivity)

$$(.11{:}1, .89{:}0) \succ (.10{:}5, .90{:}0).$$

The Allais paradox may come from a violation of one or both of these two properties. If RDU allows for violations of Restricted Branch Independence, it yet assumes coalescing. Birnbaum (2007) shows that coalescing is also violated in experiments. Note that he runs a series of tests of rank dependence (e.g. Birnbaum et al. 1999; Birnbaum 2004), in which several properties of RDU like stochastic dominance are violated.

Moreover, Machina (2007) suggests that there may also be Ellsberg paradoxes for CEU. He gives the following example. An urn contains 20 balls. 10 are either blue or green. The others are either yellow or red. Four acts are displayed in the following table.

Table 6.2.1: The Modified Ellsberg Paradox?



	Y	R	B	G
f	€0	€200	€100	€100
g	€0	€100	€200	€100
f'	€100	€200	€100	€0
g'	€100	€100	€200	€0

Machina argues that according to Informational Symmetry, $f \succ g$ should be equivalent to $g' \succ f'$ and $f \prec g$ to $g' \prec f'$. Indeed, we do not have more information about the proportion of yellow balls than about the proportion of green balls. Consequently, (f and g') and (g and f') are respectively symmetric.

Under CEU, if $f \succ g$ and $g' \succ f'$ (the symmetric reasoning applies for $f \prec g$ and $g' \prec f'$) with $u(0)=0$, we would have

$$W(R)\times u(200)+[W(B\cup R\cup G)-W(R)]\times u(100) > W(B)\times u(200)+[W(B\cup R\cup G)-W(R)]\times u(100) \quad (6.2.1)$$

and

$$W(R)\times u(200)+[W(B\cup R\cup Y)-W(R)]\times u(100) < W(B)\times u(200)+[W(B\cup R\cup Y)-W(B)]\times u(100) \quad (6.2.2)$$

$$(6.2.1)\Rightarrow \quad W(R)\times[u(200)-u(100)] > W(B)\times[u(200)-u(100)] \quad (6.2.3)$$

$$(6.2.2)\Rightarrow \quad W(R)\times[u(200)-u(100)] < W(B)\times[u(200)-u(100)] \quad (6.2.4)$$

Eqs 6.2.3 and 6.2.4 are contradictive. Hence, what Machina calls the Modified Ellsberg Paradox constitutes a violation of CEU. As a consequence, a necessary condition for CEU to accommodate informational symmetry is $f \sim g$ and $f' \sim g'$. It can be justified by the fact that in a decumulative viewpoint, f and g (f' and g') associate consequences to events with the same information level: €200 on R or B (R or B), 100 € on $B \cup R \cup G$ ($B \cup R \cup Y$). However, Machina highlights alternative non-cumulative notions of ambiguity that could justify $f \succ g$ and $g' \prec f'$. CEU fails to capture these ambiguities.

As a conclusion, Machina or Birnbaum's work raises some major issues:

- Is cumulative vision of ambiguity validated by experiment? On the contrary, is this paradox that is only suggested by Machina confirmed by data? We cannot confer the same weight to it as to Allais and Ellsberg paradoxes if it is not established by observations.
- Which level of generality should we reach so as to accommodate these paradoxes? What does it take for a model to be compatible with all these new paradoxes?
- Which minimal consistency rules should we require in a model? For instance, Birnbaum (2007) compares RDU with models that allow for violations of coalescing and stochastic dominance that could appear as major rationality requirements.

In order to conclude, we must recall that we only use binary prospects in chapters 3, 4 and 5. We have already explained that for those acts, the model we use is not only CEU/CPT but is also consistent with MEU and other models. The paradoxes mentioned in this section do not apply on binary acts. It is though true that they raise doubts on the ability of rank-dependence to correctly model behaviors for general acts with more than two outcomes.

6.2.3. Normative, prescriptive or descriptive models

In this part, we would like to compare the model developed in chapter 4 with SEU. Particular credence has been given to SEU as a normative theory because it prevents from inconsistencies (e.g. Kahneman & Tversky 1979 p277; Savage 1954 p97-104). Let us focus on an argument in favor of SEU given by Hammond (1988). Assume that a decision maker prefers fEh to gEh for some g, f, h and E. She learns that the event E will occur: we could think that she would still prefer f to g. This rationality property is called dynamic consistency (for any acts,

$fEh \succ gEh$ implies $fEh \succ^E gEh$ where $\succ^E$ is the preference when the agent knows that E obtains).

Let us introduce another consistency property, called consequentialism: $fEg \sim^E f$. When E obtains, the agent does not care about what she could have had otherwise. A non-consequentialist decision maker agrees to pay for something she knows she will not have. That is why we may want to exclude this type of irrational behavior. If consequentialism holds, dynamic consistency is required by rationality because it is possible to build a money pump on a consequentialist but dynamically inconsistent agent: assume $fEh \succ gEh$ and $fEh \prec^E gEh$. By consequentialism, $fEh \sim^{S-E} gEh$. The agent is initially willing to pay to have fEh instead of gEh. If she learns that E occurs, she is now willing to pay to have gEh back. If E does not obtain, she accepts to have gEh instead of fEh for free. As a conclusion, she has the same *ex post* act as the *ex ante* act, but she pays for that.

Now, let us assume that an agent's preference does not satisfy the Sure-Thing Principle (remember that this principle states that for any acts f, g, h and h' and event E, $fEh \succcurlyeq gEh$ implies $fEh' \succcurlyeq gEh'$). Consequently, there exist at least four acts such that $fEh \succ gEh$ and $fEh' \prec gEh'$. If preferences when E has occurred are complete, dynamic consistency implies $fEh \succ^E gEh$ and $fEh' \prec^E gEh'$ while consequentialism implies $fEh \sim^E f$, $gEh \sim^E g$, $fEh' \sim^E f$ and $gEh' \sim^E g$. We thus have $f \succ^E g$ and $f \prec^E g$. We can conclude that consequentialism and dynamic consistency imply the Sure Thing Principle. This result is due to Hammond (1988)[32].

If we do think that a normative model must satisfy rationality and thus consequentialism and dynamic consistency, then the models we used in chapter 3, 4 and 5 must not be used in normative or prescriptive applications because they are based on weakened version of the Sure Thing Principle. Nonetheless, we did not work on them for normative or prescriptive purposes but we aimed at describing behaviors.

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[32] Weakened versions of dynamic consistency that allow for non-EU models are proposed for instance by Hanany & Klibanoff (2007), Sarin & Wakker (1998) and Tallon & Vergnaud (2002).

Moreover, these good descriptive abilities may even be interesting when they are combined with the prescriptive advantages of SEU. Bleichrodt, Pinto & Wakker (2001) suggest that a descriptive use of Prospect Theory can improve the prescriptive use of EU, by avoiding biases. Through the same logic, the model developed in chapter 4 and the robust elicitation technique for subjective probabilities (chapter 4 and 5) allow us to find unbiased probabilities that could be used in a prescriptive setting.

6.3. Further Research

6.3.1. Ambiguity and new paradoxes

In Chapter 1, we present Epstein & Zhang's (2001) definition of ambiguity and we give two examples (based on Wakker 2006), in which it could lead to mischaracterize ambiguous or unambiguous events. In subsection 6.2.2 of the current chapter, we study Machina's (2007) modified Ellsberg paradox. None of them has already been tested and it seems necessary to confront them with data. This would constitute a further step in apprehending ambiguity.

In addition, some paradoxes are based on two contradicting preferences and it appears interesting to study whether or not this contradiction is caused or avoided by the direct comparison of the two choice situations, e.g. the Ellsberg paradox is stronger when the agent knows the two urns according to Fox & Tversky (1995).

The results of such an experiment should be related to the common findings on previous paradoxes. This comparison would endeavor to obtain a ranking among paradoxes so as to evaluate each model in terms of violation rate. Choosing a model could then be based on a more conscious tradeoff between tractability and representativeness.

6.3.2. Exchangeability and Incentive compatibility

A main point that we discuss in chapters 4 and 5 is that our technique might not satisfy incentive compatibility because of chained questions. Improving it thus remains a major topic of research. We can already put forward three directions that should be studied. The first one would consist in asking a high number of systematic questions so as to find some indifference. We can also try to hide the chaining by a complex mixing of questions. The last direction is based on assuming a parametric form for the probability distribution and/or for the source-dependent probability transformation so as to minimize the number of questions. These different possibilities must be carefully studied and compared in terms of efficiency and assumptions.

For instance, in a work in progress with Mohammed Abdellaoui and Lætitia Placido, we test if subjects' behavior is consistent with a uniform subjective probability distribution in Ellsberg's unknown urn and we elicit probability transformation functions that depend on urn. In other words, we apply techniques developed in chapter 4 except that we test a particular subjective probability distribution instead of completely inferring Bayesian beliefs from choices. Thus, we avoid chaining and improve incentive compatibility of our experiment.

6.3.3. Beliefs, decision weights and updating

In a descriptive viewpoint, it should be interesting to study how exchangeability-based probabilities are updated. Do they satisfy Bayes' rule? And what about the decision weights? Through the combination of willingness-to-bet and subjective probabilities, the impact of learning on decision weights can be decomposed into belief updating and changes in attitude.

In chapter 4, we obtain that probability transformation functions exhibit more pessimism under uncertainty with unknown probability than under risk. Keynes (1921) wrote:

"The magnitude of the probability of an argument [...] depends upon a balance between what may be termed the favourable and the unfavourable evidence;

a new piece of evidence which leaves the balance unchanged, also leaves the probability of the argument unchanged. But it seems that there may be another respect in which some kind of quantitative comparison between arguments is possible. This comparison turns upon a balance, not between the favourable and the unfavourable evidence, but between the absolute amounts of relevant knowledge and of relevant ignorance respectively."

Keynes distinguished the evidence in favor of an argument and the total amount of evidence. The first one was related to the likelihood, the second one to what Keynes called the *"weight"* of an event. It is clear that the methods we proposed in chapter 4 and 5 can separate the likelihood part from the impact of the *"weight"*. Assume that E is an independently and identically distributed event and that an agent has a certainty equivalent equal to €y for a bet that would make her win €x if E occurs in period 2 (and nothing otherwise). She learns that E does not obtain in period 1 and being Bayesian, she updates her beliefs. But even if there is less evidence in favor of E (its likelihood is lower), the event has more *"weight"*. An increase of weight may make the agent less pessimistic (see the impact of unknown probability with respect to known probability in chapters 3, 4 and 5). A lower likelihood decreases the certainty equivalent; less pessimism increases it. Put differently, the negative evidence can decrease the subjective probability but increase the elevation of the probability transformation. That follows Keynes' intuition and the method developed in chapter 4 can further analyze the combination of the two effects.

6.3.4. Experts, ambiguity and health

In all the experiments of this dissertation, outcomes are monetary. Studying the impact of alternative outcomes on ambiguity attitudes would provide further information on its determinant. Such nonmonetary outcomes could be well-being, environment or health. For instance, if outcomes are quality-adjusted life-years (QALY, an index measure that combines life duration and quality of life), what does ambiguity attitude become? How does an agent react to ambiguity when probabilities are over her health status?

Furthermore, we could complete our analysis of combination of experts' judgments (chapter 3) in the same direction. What is the impact of conflicting judgments when they are medical advices? Before having surgery, most inpatients like to have several opinions, but how do they decide when opinions differ? In other words, does disagreement between doctors lead to more or less surgery? The fact that monetary outcomes are replaced by health or/and life may change attitude towards ambiguity and this has to be studied.

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Résumé de la thèse :

La thèse porte sur la théorie de la décision individuelle en situation d'incertitude. Elle vise à comprendre, décrire et représenter les décisions, en différenciant ce qui provient des croyances du décideur de ce qui se rattache à son attitude face à l'incertitude. Elle est principalement composée de quatre contributions complémentaires. La première est théorique et caractérise l'attitude face au risque et à l'ambiguïté via l'utilisation de relations d'arbitrage portant sur les conséquences, dans la lignée des modèles de type utilité espérée, représentant l'aversion au risque et/ou à l'ambiguïté par la décroissance de l'utilité marginale. Le reste de la thèse s'appuie sur des modèles généralisant l'utilité espérée où les probabilités sont transformées par le décideur. La deuxième contribution est alors expérimentale et s'intéresse au décideur disposant d'avis d'experts lui indiquant le risque encouru. Il est proposé une méthode, basée sur l'observation des choix, pour étudier comment le décideur combine les avis à sa disposition. Cette méthode est appliquée pour comparer des situations où les experts donnent une évaluation imprécise du risque à des situations où leurs évaluations du risque encouru sont conflictuelles. Le troisième travail introduit le concept de source uniforme d'incertitude, c'est-à-dire d'ensemble d'événements générés par un même mécanisme d'incertitude et pour lesquels il existe une mesure de probabilité subjective. Une expérience est conduite dans laquelle de telles probabilités subjectives sont obtenues. Est ensuite étudié le consentement à parier des individus sur des événements de probabilité (subjective) équivalente mais provenant de sources différentes. La dernière contribution revient sur la méthode d'obtention des probabilités subjectives et la compare (théoriquement) aux autres méthodes. Sa faisabilité et ses limites sont ensuite étudiées dans une nouvelle expérience.

Mots clés : incertitude, ambiguïté, probabilité subjective, décision, croyances, utilité non-espérée.

Abstract:

This dissertation deals with the theory of decision making under uncertainty. It aims at describing and modeling decisions in order to disentangle beliefs and attitudes towards uncertainty. It is made of four main contributions. The first one is theoretical and characterize risk and ambiguity attitude through tradeoff relations. Indeed, expected utility and some of its generalization represents risk (and ambiguity) aversion through decreasing marginal utility. The tradeoff relation is defined on consequences and allows us to compare concavity of different utility functions under expected utility when probabilities need not be known. The other parts of the dissertation are based on nonexpected utility models including probability weighting functions. The second contribution is an experimental study of how decision makers combine experts' probability judgments. A new choice-based method is proposed and applied to the comparison of two typical situations. In the first one, the experts give an imprecise evaluation of the risk and in the second one, they disagree and each of them gives his/her own evaluation. The third work is based on uniform sources of uncertainty, i.e. set of events that pertain to a similar mechanism of uncertainty and on which a probability measure exists. In an experiment, such probabilities are elicited. Then willingness-to-bet on events having the same subjective probabilities but from different sources are obtained. The last work is specifically dedicated to the technique for eliciting subjective probability. It observes that this technique is more robust than several well-known techniques and provides new evidence regarding its feasibility.

Keywords: uncertainty, ambiguity, subjective probability, decision, beliefs, nonexpected utility.