Etude de Quelques Problèmes d'Estimation Statistique en Finance

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Résumé

Cette thèse traite plusieurs problèmes de finance statistique et se compose de quatre parties.

Dans la première partie, on étudie la question de l'estimation de la persistance de la volatilité à partir d'observations discrètes d'un modèle de diffusion sur un intervalle $[0, T]$, où T est un temps objectif fixé. Pour cela, on introduit un mouvement brownien fractionnaire d'indice de Hurst H dans la dynamique de la volatilité. On construit une procédure d'estimation du paramètre H à partir des données haute fréquence de la diffusion. On montre que la précision de notre estimateur est $n^{-1/(4H+2)}$, où n est la fréquence d'observation et on prouve son optimalité au sens minimax. Ces considérations théoriques sont suivies d'une étude numérique sur données simulées et données financières.

La seconde partie de la thèse traite de la problématique du bruit de microstructure. Pour cela, on considère les observations à la fréquence n et avec erreur d'arrondi α_n tendant vers zéro, d'un modèle de diffusion sur un intervalle $[0, T]$, où T est un temps objectif fixé. On propose dans ce cadre des estimateurs de la volatilité intégrée de l'actif dont on montre que la précision est $\max(\alpha_n, n^{-1/2})$. On obtient par ailleurs des théorèmes centraux limites dans le cas de diffusions homogènes. Cette étude théorique est ici aussi suivie d'une étude numérique sur données simulées et données financières.

On établit dans la troisième partie de cette thèse une caractérisation simple des espaces de Besov et on l'utilise pour démontrer de nouvelles propriétés de régularité pour certains processus stochastiques. Cette partie peut paraître déconnectée des problèmes de finance statistique mais a été inspiratrice pour la partie 4 de la thèse.

On construit dans la dernière partie de la thèse un nouvel indice de bruit de microstructure et on l'étudie sur des données financières. Cet indice, dont le calcul se base sur les p-variations de l'actif considéré à différentes échelles de temps, peut être interprété en terme d'espaces de Besov. Comparé aux autres indices, il semble posséder plusieurs avantages. En particulier, il permet de mettre en évidence des phénomènes originaux comme une certaine forme de régularité additionnelle dans les échelles les plus fines. On montre que ces phénomènes peuvent être partiellement reproduits par des modèles de bruit de microstructure additif ou de diffusion avec erreur d'arrondi. Néanmoins, une reproduction fidèle semble nécessiter soit une combinaison de deux formes d'erreur, soit une forme sophistiquée d'erreur d'arrondi.

Abstract

This Ph.D dissertation treats several problems of statistical finance and consists of four parts.

In the first part, we study the question of the estimation of the volatility persistence in a discretely observed diffusion model on $[0, T]$, where T is a fixed objective time. For that purpose, we introduce a fractional Brownian motion with Hurst parameter H in the dynamic of the volatility. From the high frequency data of the diffusion, we build an estimation procedure for the parameter H. We show that the accuracy of our estimator is $n^{-1/(4H+2)}$, where n denotes the observation frequency and we prove its optimality in a minimax sense. These theoretical considerations are followed by a numerical study, both on simulated and financial data sets.

The second part of the dissertation treats a problem of microstructure noise. We consider the observations at the frequency n and with round-off error α_n tending to zero, of a diffusion model on $[0, T]$, where T is a fixed objective time. We construct in this framework estimators of the integrated volatility of the asset whose accuracy is proved to be $\max(\alpha_n, n^{-1/2})$. Moreover, we obtain central limit theorems in the case of a homogeneous diffusion. These theoretical study is also followed by a numerical study, both on simulated and financial data sets.

We state in the third part of the dissertation a simple characterization of Besov spaces and we use it to prove new regularity properties for some stochastic processes. This part may seem disconnected from statistical finance problems but has been inspiring for the fourth part of the dissertation.

We build in the last part of the dissertation a new microstructure noise index and we study it on financial data sets. This index, whose computation is based on the p-variations of the considered asset at different time scales, can be interpreted thanks to Besov smoothness spaces. Compared with other indexes, it seems to have several virtues. In particular, it gives rise to original phenomena such as some kind of additional regularity in the finest scales. We show that these phenomena can be partially reproduced using models involving additive errors or rounding errors. Nevertheless, an accurate reproduction seems to require either both kinds of error together or some unusual form of rounding error.

Contents

Introduction

1 Introduction

L'objectif de ce travail est d'apporter des éléments de réponse à des questions de finance de marché sous l'angle de la statistique des processus. On s'impose donc un cadre de travail cohérent avec les faits stylisés de marché : celui d'observations discrètes d'un modèle de diffusion à volatilité stochastique, voir Shephard [105] et l'annexe E pour la motivation d'un tel modèle. Dans ce contexte, on répond aux questions de finance statistique suivantes :

- A partir d'observations discrètes de prix, comment estimer la persistance de la volatilité ? On considère des modèles dans lesquels la persistance est caractérisée par un paramètre et on donne un estimateur de ce paramètre.

- Comment estimer la volatilité intégrée d'un actif dans les petites échelles ? On se place dans des modèles diffusifs perturbés par du bruit de microstructure, notamment par des erreurs d'arrondi, et on présente des estimateurs de la volatilité intégrée.

- Comment mesurer l'intensité du bruit de microstructure ? On propose un nouvel indice de mesure et on étudie son comportement sur des données de marché et dans des modèles théoriques.

Pour répondre à la première question, on étudie dans la première partie de ce travail le problème de l'estimation de la persistance de la volatilité à partir d'observations discrètes d'un modèle de diffusion sur un intervalle $[0, T]$, où T est un temps objectif fixé. Pour cela, on introduit un mouvement brownien fractionnaire d'indice de Hurst $H \in [1/2, 1]$ dans la dynamique de la volatilité. On construit une procédure d'estimation du paramètre H à partir des données haute fréquence de la diffusion. On montre que la précision de notre estimateur est $n^{-1/(4H+2)}$, où n est la fréquence d'observation et on prouve son optimalité au sens minimax. Ces considérations théoriques sont suivies d'une étude numérique sur données simulées et données financières. Ce travail s'inspire notamment des articles de Gloter et Hoffmann [55] et Comte et Renault [32].

Le second problème est traité dans la deuxième partie de ce travail. On propose des estimateurs de la volatilité intégrée d'un actif à partir d'observations discrètes à fréquence n et avec erreur d'arrondi α_n tendant vers zéro, d'une diffusion sur un intervalle $[0, T]$, où T est un temps objectif fixé. On montre que la précision de nos estimateurs est $\max(\alpha_n, n^{-1/2})$ et on obtient par ailleurs des théorèmes centraux limites dans le cas de diffusions homogènes. Cette étude théorique est ici aussi suivie d'une étude numérique sur données simulées et données financières. On prolonge donc dans cette partie les travaux de Jacod [70], Delattre et Jacod [36] et Delattre [35].

On aborde la troisième question dans les deux dernières parties de ce travail. On établit dans la troisième partie une caractérisation simple de certains espaces de régularité : les espaces de Besov. En se basant sur ce résultat, on construit dans la quatrième partie un nouvel indice de bruit de microstructure qui associe à une période de sous échantillonnage donnée une mesure de régularité. Comparé aux autres indices, il semble posséder plusieurs avantages. En particulier, il permet de mettre en évidence des phénomènes originaux comme une certaine forme de régularité additionnelle dans les échelles les plus fines. On montre que ces phénomènes peuvent être partiellement reproduits par des modèles de bruit de microstructure additif ou de diffusion avec erreur d'arrondi. Néanmoins, une reproduction fidèle semble nécessiter soit une combinaison de deux formes d'erreur, soit une forme sophistiquée d'erreur d'arrondi.

2 Première partie : estimation de la persistance de la volatilité

La volatilité n'est pas une quantité observable. Néanmoins, les approximations empiriques nous conduisent à soulever le problème de la persistance de la volatilité, voir l'annexe E. On souhaite donc répondre statistiquement aux questions suivantes :

- A partir des observations de prix, peut-on avoir de l'information sur la régularité de la volatilité et si oui, peut-on quantifier cette information ?
- Comment « estimer » la régularité de la volatilité ?

La réponse à la première question conduit au nombre de données de prix nécessaires pour pouvoir espérer obtenir de l'information sur la régularité de la volatilité. La réponse à la seconde permet d'accéder effectivement à cette information. On cherchera bien sûr à répondre à la seconde question de manière optimale. Pour une précision fixée, on souhaite accéder à l'information sur la régularité de la volatilité avec le nombre minimum de données préconisé par la première question.

Pour tenter de résoudre le problème de l'estimation de la régularité de la volatilité, il est nécessaire de construire un modèle statistique suffisamment riche. Pour cela on considère le modèle statistique suivant. On considère l'échantillon

$$Y^n = \{Y_{i/n},\ i = 0, \dots, nT\}$$

issu de l'observation haute fréquence du modèle

$$Y_t = y_0 + \int_0^t \sigma_s \,\mathrm{d}B_s,\ y_0 \in \mathbb{R},\quad t \in [0,T],$$

$$\sigma_t = \Phi\Big(\int_0^t a(t,u)\mathrm{d}W_u^H + f(t)\xi_0\Big).$$

Dans ce modèle, T est un temps objectif fixé, Φ, a et f sont des fonctions déterministes, ξ_0 est une variable aléatoire $\mathcal{F}_0-$mesurable, B est un mouvement brownien et W^H est un mouvement brownien fractionnaire d'indice de Hurst $H \in [1/2, 1]$, indépendant de B. Le mouvement brownien fractionnaire est un processus gaussien à accroissements stationnaires. Pour $H = 0.5$, c'est simplement le mouvement brownien. Pour $H > 0.5$, les accroissements du mouvement brownien fractionnaire sont positivement corrélés. Son utilisation permet donc d'introduire de la persistance dans le processus de volatilité. Notre cadre inclut en particulier le modèle de volatilité stochastique à mémoire longue de Comte et Renault [32]. La version stationnaire du modèle de Comte et Renault est obtenue en prenant

$$\Phi(x) = \exp(x),\ a(t,u) = \gamma\exp(-k[t-u]),\ f(t) = \gamma\exp(-kt),\ \xi_0 = \int_{-\infty}^0 \exp(ku)\mathrm{d}W_u^H.$$

Pour cette spécification, Comte et Renault montre que $\mathrm{Cov}[\sigma_{t+h}, \sigma_t]$ est de l'ordre de $|h|^{-(1-2d)}$ quand h tend vers l'infini, avec $d = H - 1/2$. Ainsi, le processus de volatilité est alors un processus à mémoire longue avec paramètre de longue mémoire $d = H - 1/2$. Notons bien que la notion de mémoire longue est une notion ergodique. Ainsi, nous lui préférons la notion de régularité ou de persistance, plus adaptée à notre asymptotique haute fréquence.

Comme le montrent les graphes suivants, l'indice H quantifie la régularité du mouvement brownien fractionnaire.

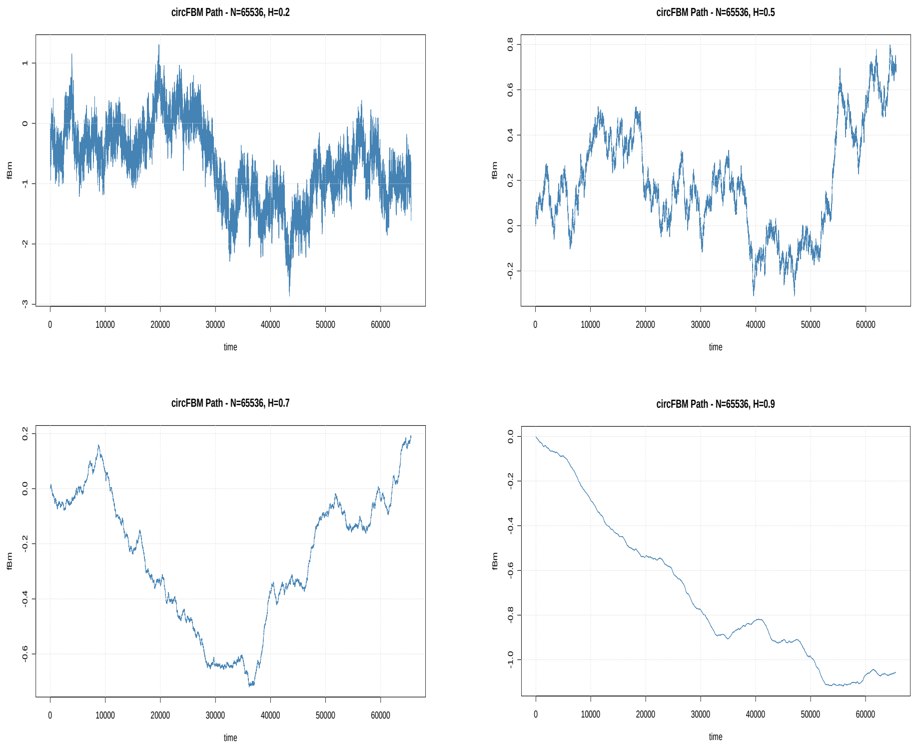

FIG. 1 – *Trajectoires de mouvement brownien fractionnaire pour H=0.2, 0.5, 0.7 et 0.9.*

Ainsi, il va aussi caractériser la régularité de notre processus de volatilité. Ceci est formalisé dans la proposition I.1 grâce aux espaces de Besov $\mathcal{B}^s_{p,\infty}([0,T])$, où s est un indice de régularité et p représente la norme dans laquelle on mesure cette régularité, voir l'annexe A pour les définitions. On montre en particulier dans la proposition I.1 que presque sûrement, notre volatilité appartient à l'espace de Besov $\mathcal{B}^H_{2,\infty}([0,T])$ et n'appartient pas à l'espace de Besov $\mathcal{B}^{H+\varepsilon}_{2,\infty}([0,T])$. Ainsi, on formalise que la régularité de notre trajectoire (au sens des espaces de Besov) est exactement H. Répondre à nos deux questions est donc équivalent dans notre modèle à

- Donner une borne inférieure pour l'estimation du paramètre H.
- Construire un estimateur « optimal » de ce paramètre.

L'appartenance à un espace de Besov se traduit par des propriétés des coefficients d'ondelette d_{jk} du processus de volatilité, avec

$$d_{jk} = \int \sigma_t \psi_{jk}(t)\mathrm{d}t, \ j \in \mathbb{N}, \ k = 0, \ldots, \lfloor T(2^j - 1) \rfloor,$$

où $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$ avec ψ une ondelette mère. En effet, on prouve une propriété d'échelle asymptotique

$$\frac{Q_{j+1}}{Q_j} \approx 2^{-2H},$$

avec

$$Q_j = \sum_k d_{jk}^2.$$

A partir d'estimateurs des coefficients d'ondelette, cette propriété nous permet donc de construire un estimateur du coefficient H.

$$\widehat{H}_{j,n} = -\frac{1}{2}\log_2 \frac{\widehat{Q}_{j+1,n}}{\widehat{Q}_{j,n}}.$$

Le niveau j optimal est choisi en utilisant les règles de l'estimation adaptative des fonctionnelles quadratiques. En utilisant les travaux de Gloter et Hoffmann [54], [55], on montre sous hypothèses de régularité les résultats suivants.

Résultat 1 *Pour tout compact $\mathcal{H} \subset (1/2, 1)$, la suite $n^{1/4H+2}(\widehat{H}_n - H)$ est bornée en probabilité, uniformément sur $\mathcal{H}$.*

Résultat 2 *Pour tout compact $\mathcal{H} \subset (1/2, 1)$ d'intérieur non vide, la vitesse de convergence $n^{1/(4H+2)}$ est une borne inférieure de convergence au sens minimax.*

Ainsi, notre estimateur est optimal au sens minimax et sa précision est $n^{-1/(4H+2)}$.

On ne peut accéder à des observations directes de la volatilité, seule une approximation est possible (en utilisant un schéma d'Euler par exemple). Ainsi, nous sommes face à un problème d'estimation à partir de données bruitées. Ceci explique l'obtention d'une vitesse d'estimation optimale non standard. Bien entendu, la précision de notre estimateur est bien plus mauvaise que la vitesse paramétrique classique $n^{-1/2}$. Cependant, elle reste polynômiale, ce qui signifie qu'à partir d'un nombre raisonnablement grand de données, estimer est concevable. Remarquons que ceci est loin d'être le cas dans de nombreux modèles statistiques. Citons par exemple certains problèmes inverses où l'on prouve l'optimalité de vitesses logarithmiques, voir Cavalier *et al.* [24]. Si l'on compare notre vitesse d'estimation à la vitesse non paramétrique classique $n^{-s/(1+2s)}$, où s est la régularité de la fonction estimée, estimer le paramètre $H \in (1/2, 1)$ est « aussi difficile » que d'estimer une fonction de régularité $s \in (1/4, 1/2)$.

Les expériences sur données simulées confirment que pour des échantillons raisonnablement grands, estimer correctement le paramètre est possible. L'estimation du paramètre H sur de grandes séries de contrats futures américains et européens conduit à ne pas rejeter l'hypothèse d'une volatilité diffusive ($H = 1/2$). De plus, pour « augmenter le nombre de données », des méthodes d'agrégation entre actifs financiers sont envisageables. Nous en proposons une dans le chapitre 2 de la partie 1. On utilise cette procédure d'agrégation sur plusieurs petits échantillons (2 semaines de données) d'action composant l'indice SBF 120. L'estimation du paramètre H pour une action donnée n'a alors pas de sens, le nombre d'observation étant trop petit. Cependant, cette procédure nous incite à envisager une valeur d'ensemble du paramètre H significativement supérieure à 0.5 pour les données equity considérées.

3 Deuxième partie : volatilité intégrée et erreur d'arrondi

S'attachant à explorer des modèles adaptés aux données financières, on traite dans la seconde partie de la thèse une question liée à la présence de bruit de microstructure dans ces données. Si le cadre des processus de diffusion s'est largement imposé en mathématiques financières, à la fois théoriquement et pratiquement, nous avons accès de nos jours à des données haute fréquence très propres, dont l'étude montre indéniablement que les modélisations de type diffusion ne sont pas robustes à toutes les échelles. Dans les hautes fréquences, cet écart entre prix théorique diffusif et prix observé s'appelle le bruit de microstructure. Ce bruit de microstructure est dû en particulier au fait que les prix sont discrets (variation tick par tick) et à l'existence d'un spread bid-ask. Ce bruit de microstructure complique la problématique de l'estimation de quantités d'intérêt à partir des données observées.

Parmi ces quantités d'intérêt, l'une des plus pertinentes, intervenant dans les formules de prix, de couverture et de gestion des risques est la volatilité intégrée de l'actif. Considérons pour $t \in [0, 1]$ un processus d'Ito de la forme

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t.$$

On souhaite estimer les volatilités intégrées absolue et relative :

$$\int_0^1 \sigma_t^2 \mathrm{d}t \text{ et } \int_0^1 X_t^{-2} \sigma_t^2 \mathrm{d}t.$$

Supposons d'abord que l'on observe parfaitement les données à la fréquence n,

$$(X_{i/n}, i = 0, \dots, n).$$

Dans ce cas, les estimateurs par variation quadratique

$$\sum_{i=1}^{n} (X_{i/n} - X_{(i-1)/n})^2,$$

et

$$\sum_{i=1}^{n} \big(\log(X_{i/n}) - \log(X_{(i-1)/n})\big)^2,$$

estiment respectivement les deux formes de volatilité intégrée que nous avons définies, voir Jacod et Protter [74] et Barndorff-Nielsen et Shephard [16].

En présence de bruit de microstructure, les estimateurs précédents sont la plupart du temps non convergents. Une solution consiste alors à « jeter » des données. Cependant, en supposant que l'on a une donnée par seconde et que cinq minutes est la période minimale pour rendre le bruit insignifiant, il faut jeter 299 données sur 300, ce qui est difficilement acceptable du point de vue du statisticien. Ainsi, l'estimation de la volatilité intégrée dans ce cadre de données bruitées a suscité un grand intérêt, voir en particulier Zhang [113], Zhang, Mykland, et Aït-Sahalia [114], Hansen et Lunde [58], Bandi et Russell [12], Aït-Sahalia, Mykland et Zhang [4], Gloter et Jacod [56]. Différents estimateurs sont comparés par Andersen, Bollerslev et Meddahi [10], Bandi, Russel et Yang [13], Gatheral et Oomen [46].

Dans la plupart de ces travaux, on observe aux instants i/n, $i = 0, \dots, n$, un prix logarithmique $\tilde{Y}_{i/n}$ composé d'un prix logarithmique théorique diffusif $\tilde{X}_{i/n}$, contaminé par un bruit de microstructure additif $\varepsilon^n_{i/n}$. Ainsi

$$\tilde{Y}_{i/n} = \tilde{X}_{i/n} + \varepsilon^n_{i/n}.$$

Néanmoins, alors que le fait que les prix soient discrets est probablement la cause la plus évidente de bruit de microstructure, cette caractéristique des prix n'entre pas dans le cadre des modèle précédents.

On s'intéresse donc à un modèle permettant d'obtenir des prix discrets et un comportement diffusif à basse fréquence. La manière la plus naturelle et la plus simple d'obtenir le comportement recherché est probablement de considérer le modèle de diffusion avec erreur d'arrondi, introduit et étudié par Delattre et Jacod [36] et Delattre [35]. Soit α_n une suite positive tendant vers zéro, on considère un processus $(X_t)_{t\in[0,1]}$ de la forme

$$X_t = x_0 + \int_0^t \sigma(X_s, s)\mathrm{d}W_s + \int_0^t a_s \mathrm{d}s,$$

et on observe l'échantillon

$$\{Y_{i/n} = X_{i/n}^{(\alpha_n)}, i = 0, \ldots, n\},$$

où

$$X_{i/n}^{(\alpha_n)} = \alpha_n \lfloor X_{i/n}/\alpha_n \rfloor.$$

On obtient donc des prix discrets et dans les basses fréquences un comportement diffusif, l'erreur d'arrondi devenant alors négligeable devant la variation du prix. De plus, il est frappant d'observer à quel point les graphes de prix « ressemblent » à des graphes de diffusions avec erreurs d'arrondi.

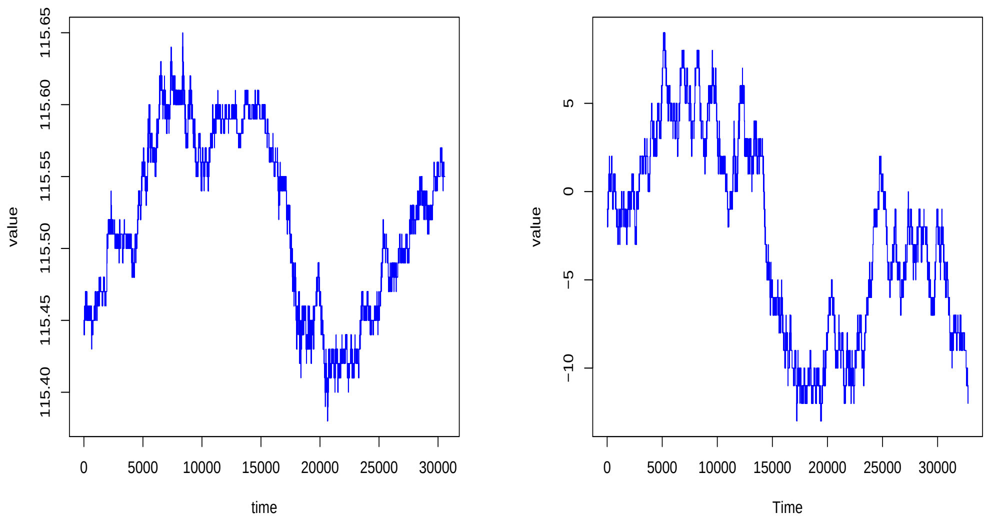

FIG. 2 – *Contrat Bund, 06/05/2007, une donnée par seconde (gauche),* $\lfloor 20W_t \rfloor$*, frequence* $n = 2^{15}$ *sur [0,1](droite).*

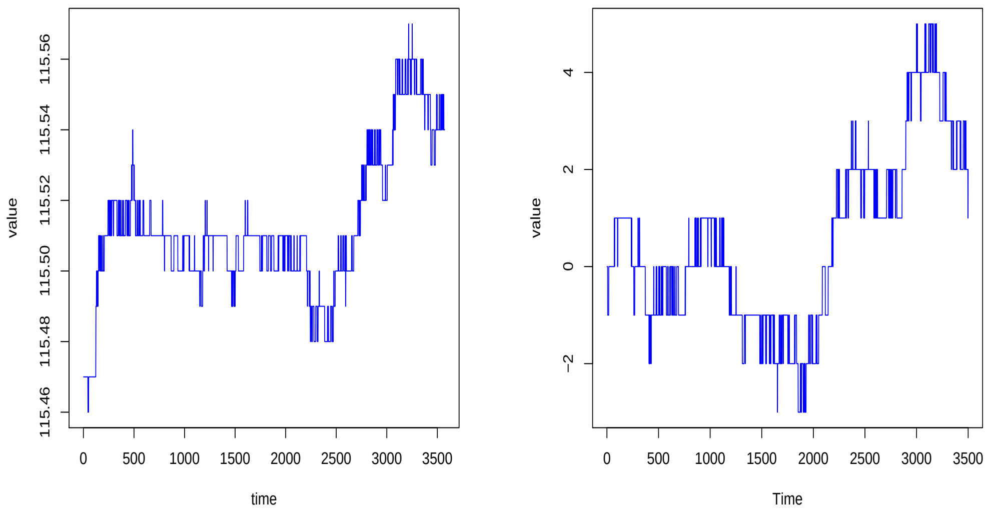

FIG. 3 – *Contrat Bund, 06/05/2007, de 10 am à 11 am, heure de Paris, une donnée par seconde (gauche),* $\lfloor 20W_t \rfloor$*, frequence* $n = 2^{15}$ *sur [0,0.1] (droite).*

A partir de l'échantillon observé, on souhaite donc estimer la volatilité intégrée (dans les deux sens) de l'actif. Notons que Li et Mykland [86] ont montré que des estimateurs classiques en présence de bruit de microstructure, comme l'estimateur de Zhang, Mykland et Aït-Sahalia [114], ne sont pas robustes en présence d'erreur d'arrondi. On illustre ceci sur des simulations dans le chapitre 3 de la partie 2. Pour estimer la volatilité intégrée, l'idée naturelle est d'utiliser la variation quadratique du processus. Cependant, pour $p > 0$, on montre qu'asymptotiquement

$$\mathbb{E}_{\sigma(X_{(i-1)/n})}[|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|^p] \approx \alpha_n^p \mathbb{E}_{\sigma(X_{(i-1)/n})}[|\lfloor U + \beta_n^{-1}\sigma(X_{(i-1)/n})Y \rfloor|^p],$$

où U est une variable suivant une loi uniforme sur $[0,1]$, indépendante de X, Y est une variable gaussienne standard, indépendante de X et U et $\beta_n = \alpha_n\sqrt{n}$. Ainsi, si $\beta_n \to +\infty$,

$$\begin{aligned}\mathbb{E}_{\sigma(X_{(i-1)/n})}[|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|^p] &\approx \alpha_n^{p-1}\mathbb{E}_{\sigma(X_{(i-1)/n})}[|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|] \\ &\approx \alpha_n^p \beta_n^{-1}(2/\pi)^{1/2}\sigma(X_{(i-1)/n}).\end{aligned}$$

On voit donc que le fait de considérer un incrément à la puissance p ne fait pas apparaître de puissance p dans la volatilité, sauf pour $p = 1$. C'est pourquoi, plutôt que d'utiliser la variation quadratique, on a recours à la variation d'ordre 1 pour construire des estimateurs des volatilités intégrées. On note λ la volatilité intégrée relative, $\widetilde{\lambda}_n$ son estimateur et $r_n = \alpha_n \vee n^{-1/2}$. Sous hypothèses de régularité, dans le cas où $\sigma(x,t)$ est de la forme $g_1(x)g_2(t)$, on montre le résultat suivant.

Résultat 3 *La suite*

$$r_n^{-1}(\widetilde{\lambda}_n - \lambda)$$

est tendue.

Par une méthode de compensation originale utilisant le comportement des coefficients d'ondelette des fonctions d'un mouvement brownien, on obtient les théorèmes centraux limites suivants pour l'estimateur compensé $\widehat{\lambda}_n$ dans le cas $\sigma(x,t) = \sigma(x)$.

Résultat 4 *Soit B un mouvement brownien standard, indépendant de toutes les autres variables,*

$$\begin{aligned}
&\textit{si } \beta_n \to 0, && r_n^{-1}(\widehat{\lambda}_n - \lambda) \to_{\mathcal{L}s} \sqrt{2}(\pi-2)^{1/2} \int_0^1 X_t^{-1}\sigma(X_t)^2 \mathrm{d}B_t,\\
&\textit{si } \beta_n \to \beta > 0, && r_n^{-1}(\widehat{\lambda}_n - \lambda) \to_{\mathcal{L}s} 2 \int_0^1 X_t^{-1}\sigma(X_t)[\Delta_\beta(X_t)]^{1/2} \mathrm{d}B_t,\\
&\textit{si } \beta_n \to +\infty, && r_n^{-1}(\widehat{\lambda}_n - \lambda) \to_{\mathcal{L}s} \frac{2}{\sqrt{3}} \int_0^1 X_t^{-1}\sigma(X_t) \mathrm{d}B_t.
\end{aligned}$$

La fonction Δ_β est définie dans la partie 2. Les convergences précédentes sont des convergences en loi stable dont on peut facilement déduire des convergences en loi classique, voir la partie 2. Des résultats analogues aux deux précédents sont démontrés pour l'estimation de la volatilité intégrée absolue.

Enfin, à partir d'une conjecture sur le comportement des erreurs d'arrondi en présence de bruit additif, on construit des estimateurs qui semblent robustes à la forme d'erreur

$$Y_{i/n} = (X_{i/n} + \varepsilon_i^n)^{(\alpha_n)} \text{ ou } \log(Y_{i/n}) = \big(\log(X_{i/n}) + \varepsilon_i^n\big)^{(\alpha_n)}$$

où les ε_i^n sont des variables centrées iid de variance constante.

On utilise nos estimateurs sur données equity du marché euronext. Notre jeu de données contient deux semaines particulièrement intéressantes pour notre étude. En effet, la taille officielle du tick a changé la deuxième semaine. Les résultats des estimations semblent montrer que nos estimateurs sont plus robustes au changement de tick que l'estimateur de Zhang, Mykland et Aït-Sahalia [114]. Cette étude nous conduit par ailleurs à envisager des modélisations du type

$$Y_{i/n} = (X_{i/n} + \zeta_i^n)^{(\alpha_n)},$$

avec $\zeta_i^n = \tilde{\zeta}_{i/n}$, où $\tilde{\zeta}$ est un processus à temps continu de type diffusion fractionnaire avec indice de Hurst $H > 1/2$.

4 Troisième partie : p-variations d'ordre 1 et espaces de Besov

La troisième partie de la thèse porte sur un thème déjà évoqué et a priori assez éloigné des problématiques de finance statistique : les espaces de Besov. Nous avons vu que les espaces de Besov sont utiles pour définir de manière précise la régularité d'un signal. Par exemple, si presque sûrement, les trajectoires du mouvement brownien sur $[0,1]$ sont h hölderiennes pour tout $h < 1/2$, pour $1 \leq p < +\infty$, elles appartiennent p.s. à l'espace de Besov $\mathcal{B}^{1/2}_{p,+\infty}[0,1]$ et pour $\varepsilon > 0$, p.s. elles n'appartiennent pas à l'espace de Besov $\mathcal{B}^{1/2+\varepsilon}_{p,+\infty}[0,1]$. Ainsi, la « vraie » régularité du mouvement brownien est caractérisée par la valeur 1/2 grâce aux espaces de Besov. Ces espaces, qui apparaissent naturellement en analyse numérique, ont connu un engouement particulier en statistique par le développement des techniques d'ondelette en estimation non paramétrique, voir Donoho *et al.* [39]. En effet, on peut définir la norme Besov d'une fonction en terme de coefficients d'ondelette, voir Meyer [95], Cohen [28].

Cependant, Cieselski, Kerkyacharian et Roynette [26] ont montré pour $0 < s < 1$, $1 \leq p, q \leq \infty$ et $s > 1/p$, l'équivalence de la norme usuelle sur l'espace de Besov $\mathcal{B}^s_{p,q}[0,1]$ avec une norme basée sur les coefficients de la fonction considérée dans la base de Schauder. Plus précisément, leur caractérisation se base sur la $p-$variation dyadique d'ordre 2, c'est à dire pour une fonction f sur $[0,1]$, la quantité

$$\sum_{k=1}^{2^j} |-f(\{2k\}2^{-(j+1)}) + 2f(\{2k-1\}2^{-(j+1)}) - f(\{2k-2\}2^{-(j+1)})|^p.$$

Nous montrons que l'espace de Besov $\mathcal{B}^s_{p,q}[0,1]$ peut en fait être défini en fonction d'une quantité plus simple : la p-variation dyadique d'ordre 1, c'est à dire

$$V_j^p(f) = \sum_{k=1}^{2^j} |f(k2^{-j}) - f(\{k-1\}2^{-j})|^p.$$

On a le résultat suivant.

Résultat 5 *Soit* $0 < s < 1$, $s > 1/p$, $1 \leq p, q \leq \infty$. *La norme usuelle sur l'espace de* $\mathcal{B}^s_{p,q}([0,1])$ *est équivalente à la norme définie par*

$$\|f\| = \max\big\{|f(0)|, \big(\sum_{j\geq 0} 2^{jq(s-1/p)}\{V_j^p(f)\}^{q/p}\big)^{1/q}\big\}.$$

On montre de plus un autre résultat qui permet d'inclure les espaces de Besov avec $p < 1$. On note

$$\nu_p(f) = \sup\Big\{ \sum_{k=0}^{m-1} |f(t_{k+1}) - f(t_k)|^p,\ 0 = t_0 < t_1 < \ldots < t_m = 1,\ m = 1, 2, \ldots \Big\}.$$

Résultat 6 *Soit $f : [0,1] \to \mathbb{R}$. Si $\nu_p(f) < +\infty$, f appartient à l'espace de Besov $\mathcal{B}^{1/p}_{p,\infty}([0,1])$.*

Les deux résultats précédents nous permettent d'obtenir quelques nouvelles propriétés de régularité pour des processus stochastiques.

Ainsi, ce chapitre peut paraître déconnecté de nos problématiques de finance statistique. Cependant, les p-variations sont en fait des quantités bien connues des économètres de la finance, notamment dans le cas $p = 2$: on parle alors de « realized volatility ». Nous utilisons ce lien entre espaces de Besov, régularité de processus et quantités d'intérêt en finance pour définir un nouvel indice de microstructure dans la dernière partie de la thèse.

5 Quatrième partie : un nouvel indice de microstructure

Une manière de définir le bruit de microstructure peut être la suivante : une forme d'irrégularité dans les hautes fréquences qui disparaît dans les basses fréquences. La question est alors la suivante : comment mesurer cette présence d'irrégularité en fonction de la fréquence ? Soit N un entier positif. Considérons l'observation avec période 2^{-N} d'un prix (ou d'un prix logarithmique) sur $[0, T]$

$$\{Y_{k2^{-N}}, k = 0, \ldots, T2^N\}.$$

Un moyen d'étudier la présence ou non de bruit de microstructure est de tracer le *signature plot*, c'est à dire la fonction

$$q \to \sum_{k=0}^{T(2^{N-q}-1)} |Y_{(k+1)2^{q-N}} - Y_{k2^{q-N}}|^2$$

avec $q \in [0, N]$. Le *signature plot* a été popularisé par Andersen *et al.* [9]. Dans le cadre de la théorie des semi-martingales continues, cette quantité converge quand N tend vers l'infini vers la variation quadratique de Y sur $[0, T]$. Ceci n'est plus vrai en présence de bruit de microstructure. Le *signature plot* a alors tendance à surestimer la variation quadratique. Ainsi, si le *signature plot* est plat, cela indique qu'il n'y a pas de bruit de microstructure. En présence de bruit, la courbe n'est plus plate (voir le graphe 4). Remarquons que cette mesure n'est pas absolue, on regarde simplement l'allure de la courbe.

En s'appuyant sur la notion de régularité de processus, nous proposons une alternative au *signature plot*. Pour $p > 0$ on considère la fonction $q \longrightarrow S_q^p$ avec

$$S_q^p = \frac{1}{p}\{1 + \log_2(\frac{V_{q+1}^p}{V_q^p})\}$$

et

$$V_q^p = \sum_{k=0}^{T(2^{N-q}-1)} |Y_{(k+1)2^{q-N}} - Y_{k2^{q-N}}|^p$$

La quantité S_q^p peut être heuristiquement interprétée comme une mesure de régularité associée à la fréquence 2^{N-q}. En effet, en utilisant le résultat 5, si $t \longrightarrow Y_t$ appartient à $\mathcal{B}_{p,\infty}^s([0,1])$ et n'appartient pas à $\mathcal{B}_{p,\infty}^{s+\varepsilon}([0,1])$, on peut espérer que

$$V_q^p \sim c2^{(q-N)(ps-1)},$$

et ainsi

$$S_q^p \sim s.$$

Notre indice peut donc être vu comme la régularité Besov associée à la fréquence 2^{N-q}. Cet indice est absolu : une régularité de type brownienne correspond à une valeur de l'indice de 1/2. Voici des courbes typiques de l'indice et d'un *signature plot* pour $p = 2$.

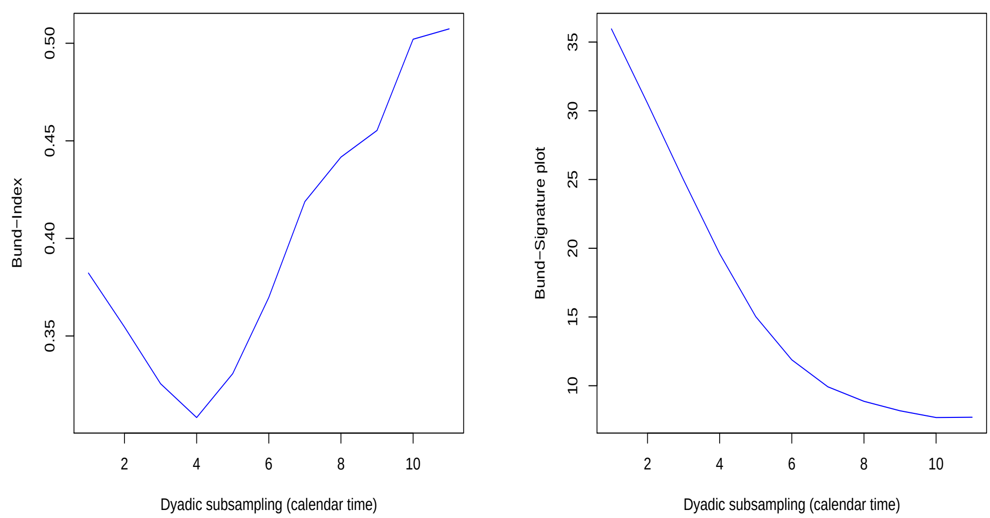

FIG. 4 – *Indice (gauche) et signature plot (droite) pour le Bund, données seconde par seconde issues de l'agrégation des mois de Octobre 2005, Novembre 2005, Février 2006, Octobre 2006, Novembre 2006 et Février 2007.*

Les conclusions sur nos données sont donc les suivantes.

Résultat 7 *(sur données de marché)*

- *On atteint une régularité diffusive à partir d'une période de sous-échantillonnage de 15-20 minutes.*
- *La régularité décroît en allant vers les échelles plus fines (15 minutes à 10 secondes).*
- *L'indice détecte une régularité additionnelle dans les échelles les plus fines (période 10 secondes à 1 seconde), qui n'apparaît pas du tout dans le signature plot.*

Les deux premières observations sont intuitives : le bruit est important dans les hautes fréquences et apporte de l'irrégularité. Du point de vue théorique, on obtient le résultat suivant.

Résultat 8 *Des modèles de bruit de microstructure additif iid ou de bruit d'arrondi permettent de reproduire les 2 premières observations empiriques.*

La dernière observation empirique est nettement plus surprenante. Un moyen de le comprendre est d'ajouter de l'autocorrélation dans un modèle de bruit additif. Par rapport au cas iid, cette corrélation apporte un supplément de régularité dans les très hautes fréquences qui disparaît rapidement.

Résultat 9 *Des modèles de bruit de microstructure additif autocorrélé permettent de reproduire les 3 observations empiriques.*

On parvient aussi à expliquer ce résultat par une forme sophistiquée d'erreur d'arrondi forçant notre processus à sauter moins souvent (voir graphe 3). Soit Y le prix observé et X le prix théorique. Pour un paramètre d fixé et une erreur d'arrondi α, on définit alors le modèle suivant.

– Si $X_{i/n}^{(\alpha)} \neq Y_{(i-1)/n}$, alors, si $|X_{i/n} - X_{i/n}^{(\alpha)}| < d$, $Y_{i/n} = Y_{(i-1)/n}$ sinon, $Y_{i/n} = X_{i/n}^{(\alpha)}$,
– si $X_{i/n}^{(\alpha)} = Y_{(i-1)/n}$, alors, $Y_{i/n} = X_{i/n}^{(\alpha)}$.

Ainsi, le prix reste constant sur de petits intervalles de temps plus nombreux qu'en cas d'erreur d'arrondi classique. On apporte donc de la régularité additionnelle.

Enfin, l'utilisation de l'indice avec $p = 1$ nous incite de nouveau à considérer des modèles du type

$$Y_{i/n} = (X_{i/n} + \zeta_i^n)^{(\alpha_n)},$$

avec $\zeta_i^n = \tilde{\zeta}_{i/n}$, où $\tilde{\zeta}$ est un processus à temps continu de type diffusion fractionnaire avec indice de Hurst $H > 1/2$.

6 Expériences numériques

Les parties 1, 2 et 4 de ce travail sont illustrées à la fois par des simulations et des expériences sur données financières. Les simulations sont réalisées grâce au logiciel R. Les expériences sur données financières ont pu être réalisées grâce aux données de *fixed income* fournies par l'équipe FIRST-ETG (Fixed Income Research Strategic Team-Electronic Trading Group) de BNP-Paribas et aux données action fournies par Crédit Agricole Cheuvreux, Groupe CALYON. Plus précisément, on utilise les données suivantes :

- Les prix *bid*, *mid* et *last* du contrat future Bund traité sur le marché Eurex, seconde par seconde, entre juillet 2005 et août 2007,
- Les prix *last* des *30 Year US Treasury Bonds Futures*, *10 Year US Treasury Notes Futures* et *5 Year Treasury Notes Futures*, traités sur le *Chicago Board of Trade*, seconde par seconde, entre mai 2005 et août 2007,
- Les valeurs *bid* du taux de change *spot* Euro/ US Dollar, du 15 août 2005 au 1er octobre 2005, seconde par seconde, à partir de la base de données Reuters,
- Les prix *bid* et *last* de plusieurs actions du SBF 120 traitées sur Euronext, seconde par seconde, entre le 12 février 2007 et le 23 février 2007.

7 Quelques repères bibliographiques sur la statistique des processus pour des données haute fréquence

Le développement des problèmes de discrétisation pour des données hautes fréquence date du milieu des années 80 (Dacunha-Castelle et Florens [33], Florens [42]). La régularité des modèles (propriété LAN ou LAMN) et la construction d'estimateurs paramétriques sont abordés par Donhal [38], Genon-Catalot et Jacod [48], [49], Yoshida [112], Kessler [78], [79], Bibby et Sørensen [19], Jacobsen [69], Kessler et Sørensen [80]. Des modèles plus généraux sont traités par Genon-Catalot *et al.* [50], [51] (volatilité stochastique), Aït-Sahalia et Mykland [3] (temps d'observation aléatoires), Delattre et

Jacod [36] et Delattre [35] (erreur d'arrondi), Gloter et Jacod [56] et Aït-Sahalia, Mykland et Zhang [114] (erreur additive), Jacod [73] et Aït-Sahalia et Jacod [1], [2] (présence de sauts, semi-martingales d'Ito générales).

En comparaison, la statistique non-paramétrique des diffusions pour données haute fréquence a été moins étudiée. On citera les travaux de Florens [43], Genon-Catalot, Laredo et Picard [52], Comte, Genon-Catalot et Rozenholc [30], Hoffmann [60], [61], [62] et Jacod [72].

Liste des travaux ayant contribué à la rédaction de la thèse

- M. Rosenbaum, *Estimation of the volatility persistence in a discretely observed diffusion model*, à paraître dans *Stochastic Processes and Their Applications*;

- M. Rosenbaum, *Integrated volatility and round-off error*, soumis à *Bernoulli*;

- M. Rosenbaum, *First order p-variations and Besov spaces*, soumis à *Statistics and Probability Letters*;

- M. Rosenbaum, *A new microstructure noise index*, document de travail.

Part I

Estimation of the volatility persistence in a discretely observed diffusion model

Abstract

We consider the stochastic volatility model

$$\mathrm{d}Y_t = \sigma_t \,\mathrm{d}B_t,$$

with B a Brownian motion and σ of the form

$$\sigma_t = \Phi\Big(\int_0^t a(t,u)\mathrm{d}W_u^H + f(t)\xi_0 \Big),$$

where W^H is a fractional Brownian motion, independent of the driving Brownian motion B, with Hurst parameter $H \geq 1/2$. This model allows for persistence in the volatility σ. The parameter of interest is H. The functions Φ, a and f are treated as nuisance parameters and ξ_0 is a random initial condition. For a fixed objective time T, we construct from discrete data $Y_{i/n},\ i = 0, \dots, nT$, a wavelet based estimator of H, inspired by adaptive estimation of quadratic functionals. We show that the accuracy of our estimator is $n^{-1/(4H+2)}$ and that this rate is optimal in a minimax sense. Numerical results on both simulated and financial data sets are also given.

Keywords: Stochastic volatility models; Discrete sampling; High frequency data; Fractional Brownian motion; Scaling exponent; Adaptive estimation of quadratic functionals; Wavelet methods.

Note

The first chapter of this part is based on a paper to appear in *Stochastic Processes and Their Applications*. A numerical study of the estimator, both on simulated and financial data sets, is presented in the second chapter. I am grateful to Renaud Drappier, Jean-Marc Duprat, Patrick Guével, Nicolas Michon, Ian Solliec and Sebouh Takvorian from BNP-Paribas for providing and discussing the financial data.

Chapter 1

Theoretical results

1 Introduction

1.1 Stochastic volatility and volatility persistence

Since the celebrated model of Black and Scholes, the behavior of financial assets is often modelled by processes of type

$$\mathrm{d}S_t = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}B_t,$$

where S is the price (or the log-price) of the asset, B a Brownian motion and μ a drift process. The volatility coefficient σ represents the fluctuations of S and plays a crucial role in trading, option pricing and hedging. It is well known that stochastic volatility models, where the volatility is a random process, are a way to deal with the endemic time-varying volatility and to reproduce various stylised facts observed on the markets, see Shephard [105], Barndorff-Nielsen, Nicolato and Shephard [14]. Among these stylised facts, there are many arguings about volatility persistence. This presence of memory in the volatility has in particular consequences for option pricing, see Ohanissian, Russel and Tsay [99], Taylor [106], Comte, Coutin and Renault [29]. Hence, continuous time dynamics have been introduced to capture this phenomenon, see Comte and Renault [32], Comte, Coutin and Renault [29] or Barndorff-Nielsen and Shephard [15]. However, in statistical finance, the question of volatility persistence has been mostly treated with discrete time models, see among others Breidt, Crato and De Lima [22], Harvey [59], Andersen and Bollerslev [8], Robinson [102], Hurvich and Soulier [66], Teyssière [107]. Concurrently, statistical methods to detect this volatility persistence have been specifically developed for these models, see Hurvich, Moulines and Soulier [64], Deo, Hurvich and Lu [37], Hurvich and Ray [65], Lee [84], Jensen

[76]. In this chapter, our objective is to build for continuous time models, a statistical program allowing to recover information about the volatility persistence.

1.2 A diffusion model with fractional stochastic volatility

We consider a class of diffusion models whose volatility is a non-linear transformation of a stochastic integral with respect to fractional Brownian motion. Recall that a fractional Brownian motion $(W_t^H, t \in \mathbb{R})$, with Hurst parameter $H \in (0, 1]$ is a self-similar centered Gaussian process with covariance function

$$\mathbb{E}[W_t^H W_s^H] = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}).$$

If $H > 1/2$, the use of fractional Brownian motion (fbm for short) is a way to allow for persistence. Indeed, its increments are then stationary, positively correlated and the value of the Hurst parameter quantifies the presence of so-called long-memory between them, see Mandelbrot and Van Ness [87], Taqqu [40]. We define on a rich enough probability space $(\Omega, \mathcal{A}, \mathbb{P})$ a Brownian motion B, a fractional Brownian motion W^H, independent of B, with unknown Hurst parameter $H \in (1/2, 1)$ and a random variable ξ_0, measurable with respect to the sigma algebra generated by $(W_t^H, t \leq 0)$. We fix an objective time $T > 0$ and we consider the one-dimensional stochastic process Y defined by

$$Y_t = y_0 + \int_0^t \sigma_s \, \mathrm{d}B_s, \;\; y_0 \in \mathbb{R}, \quad t \in [0, T], \tag{I.1}$$

where y_0 is deterministic and σ is another one-dimensional stochastic process of the form

$$\sigma_t = \Phi\Big(\int_0^t a(t,u) \mathrm{d}W_u^H + f(t)\xi_0\Big). \tag{I.2}$$

The functions Φ, a and f are deterministic and unknown. Since we only consider continuously differentiable integrands, the stochastic integral with respect to the fractional Brownian motion W^H with $H \in (1/2, 1)$ is simply defined as a pathwise Riemann-Stieltjes integral. In particular, this definition gives that for a continuously differentiable real function g,

$$\int_0^t g(u) \mathrm{d}W_u^H = -\int_0^t g'(u) W_u^H \mathrm{d}u + g(t) W_t^H.$$

This framework is an extension of the long memory stochastic volatility model introduced in mathematical finance by Comte and Renault [32]. We retrieve the volatility function used by Comte and Renault in [32] taking

$$\Phi(x) = \exp(x), \;\; a(t,u) = \gamma \exp(-k[t-u]), \;\; \xi_0 = 0,$$

where k and γ are positive constant parameters. Its stationary version, that is the exponential of a long memory fractional Ornstein-Uhlenbeck process, is obtained taking the same specification for Φ and a and

$$f(t) = \gamma \exp(-kt),\ \xi_0 = \int_{-\infty}^{0} \exp(ku) \mathrm{d}W_u^H,$$

see Cheridito *et al.* [25] for details. For the preceding specification of the volatility process, Comte and Renault have shown in [32] that $\mathrm{Cov}[\sigma_{t+h}, \sigma_t]$ is of order $|h|^{-(1-2d)}$ as h tends to infinity, with $d = H - 1/2$. Hence, not only the logarithm of the volatility but the volatility process itself entails long memory with long memory parameter $d = H - 1/2$.[1]

Remark also that in the limit case $H = 1/2$, under smoothness assumptions on Φ, letting

$$a = 1,\ f = 0,\ g = (\Phi^2)' \circ \Phi^{-1} \text{ and } h = (\Phi^2)'' \circ \Phi^{-1},$$

we equivalently have

$$\mathrm{d}\sigma_t^2 = h(\sigma_t^2)\mathrm{d}t + g(\sigma_t^2)\mathrm{d}W_t.$$

Thus, we (partially) retrieve the standard stochastic volatility diffusion framework, see for example Hull and White [63], Melino and Turnbull [91] or Musiela and Rutkowski [96] for a more exhaustive study.

For $I \subseteq \mathbb{R}$, we denote by $\mathcal{C}^k(I)$ the set of all deterministic k times differentiable function from I to $\mathbb{R}$. The assumptions on a, Φ, f and ξ_0 in model (I.1)-(I.2) are the following.

Assumption A.

(i) For all $t \in [0, T]$, $u \to a(t, u) \in \mathcal{C}^2([0, T])$, with bounded derivative uniformly in t.

(ii) For all $u \in [0, T]$, $t \to a(t, u) \in \mathcal{C}^2([0, T])$, with bounded derivatives uniformly in u.

(iii) $t \to f(t) \in \mathcal{C}^2([0, T])$.

(iv) For $p > 0$, $\mathbb{E}[e^{p|\xi_0|}] < \infty$.

(v) There exist $0 \leq \beta_1 < \beta_2 \leq T$ such that $\inf_{u \in [\beta_1, \beta_2]} a^2(u, u) > 0$.

[1] Note that if $\{X_t, t \in \mathbb{Z}\}$ is a stationary long memory Gaussian process, the fact that $\Phi(X_t)$ is also a long memory process with the same memory parameter is not always true. This is the case provided the linear term in the Hermite expansion of $\Phi(X_t)$ does not vanish. In every instance, a nonlinear transform $\Phi(X_t)$ can not "increase" the memory of X_t.

Assumption B.

(i) $x \to \Phi(x) \in \mathcal{C}^2(\mathbb{R})$.

(ii) For some $c_1 > 0$, $c_2 > 0$ and $\gamma \geq 0$, $|(\Phi^2)'(x)| \geq c_1 |x|^\gamma \mathbb{1}_{|x| \in [0,1]} + c_2 \mathbb{1}_{|x|>1}$.

(iii) For some $c_3 > 0$, $|(\Phi^2)''(x)| \leq c_3 e^{|x|}$.

1.3 Statistical model and results

We consider model (I.1)-(I.2). For technical convenience (see section 2.1.2), we take $T \geq 3$. We observe the diffusion at the sampling frequency n. This means that we observe

$$Y^n = \{Y_{i/n},\ i = 0, \ldots, nT\}.$$

For simplicity, we assume throughout the chapter $n = 2^N$. We study the problem of the inference of H based on Y^n .

A rate $v_n \to 0$ is said to be achievable over $\mathcal{H} \subset (1/2, 1)$ if there exists an estimator $\widehat{H}_n = \widehat{H}_n(Y^n)$ such that the normalized error

$$\{v_n^{-1}(\widehat{H}_n - H)\}_{n \geq 1} \tag{I.3}$$

is bounded in probability, uniformly over $\mathcal{H}$. The rate v_n is moreover a lower rate of convergence on $\mathcal{H}$ if there exists $C > 0$ such that

$$\liminf_{n\to\infty} \inf_F \sup_{H \in \mathcal{H}} \mathbb{P}[v_n^{-1}|F - H| \geq C] > 0, \tag{I.4}$$

where the infimum is taken over all estimators $F = F(Y^n)$. We prove in this chapter that the rate $v_n(H) = n^{-1/(4H+2)}$ is optimal in a minimax sense. This means that (I.3) and (I.4) agree with $v_n = v_n(H)$. We also exhibit an optimal estimator based on the behaviour of the wavelet coefficients of the process σ^2.

Theorem I.1 *Under assumptions A and B, the rate $v_n(H) = n^{-1/(4H+2)}$ is achievable over every compact set $\mathcal{H} \subset (1/2, 1)$. Moreover, the estimator $\widehat{H}_n$ explicitly constructed in section 2.2 achieves the rate $v_n(H)$.*

Our next result shows that, under an additional restriction on the non-degeneracy of the model and on the initial condition, this result is indeed optimal.

Assumption C.
The variable ξ_0 is deterministic. Moreover, for some $c_4 > 0$, $c_5 > 0$, $c_4 \neq c_5$ and $c_6 > 0$, we have $c_4 \leq |\Phi(x)| \leq c_5$ and $|\Phi'(x)| \leq c_6$.

Theorem I.2 *Under assumptions A, B and C, the rate $v_n(H) = n^{-1/(4H+2)}$ is a lower rate of convergence over every compact set $\mathcal{H} \subset (1/2, 1)$ with non empty interior.*

1.4 Discussion

• Contrary to other works, we do not consider intrinsically discrete data, but discretely observed data from an underlying continuous time process. Thus, as the objective time T is fixed, the dynamic between two data depends on the sampling frequency. This approach largely differs from those based on ergodic properties. In our context, the available information quantity does not increase because of longer observation period but because of higher sampling frequency. The estimation rates are naturally different according to the approaches. Compare our accuracy with the rate $n^{-(2/5-\varepsilon)}$ obtained by Hurvich, Moulines and Soulier in an ergodic context, see [64].

• Through this model, we aim at showing that we can recover the smoothness of the volatility from historical data. The following proposition, whose proof is given in appendix A, shows that the Hurst parameter can be interpreted as a regularity parameter thanks to Besov smoothness spaces (see appendix A).

Proposition I.1 *(Smoothness of the volatility process). For large enough T, under assumptions A and B, in model* (I.1)-(I.2)*,*

(i) Almost surely, the trajectory of $t \to \sigma_t^2$ belongs to the Besov space $\mathcal{B}^H_{2,\infty}([0,T])$ but, for all $q < \infty$, a.s. it does not belong to $\mathcal{B}^H_{2,q}([0,T])$.

(ii) For all $s < H$, almost surely, the trajectory of $t \to \sigma_t^2$ belongs to the Besov space $\mathcal{B}^s_{\infty,\infty}([0,T])$ but, if moreover there exists $c > 0$ such that $|(\Phi^2)'(x)| > c$, then, a.s. it does not belong to $\mathcal{B}^H_{\infty,\infty}([0,T])$.

• With the point of view of the estimation the local Hölder index of a process (in our case, this is equal to the parameter H), theorem I.1 remains true in a slightly more general setting. Consider the model

$$Y_t = y_0 + \int_0^t \sigma_s \, \mathrm{d}B_s, \; y_0 \in \mathbb{R}, \quad t \in [0,T], \tag{I.5}$$

with $\sigma_t = \Phi(Z_t)$. Here Φ verifies assumption B and $(Z_t, t \in [0,T])$ is a continuous time process such that for all $(s,t) \in [0,1]^2$, $s \leq t$,

$$Z_t - Z_s = a(s)(Z'_t - Z'_s) + (t-s)f(s) + h(t,s) + [v(t) - v(s)]\xi_0$$

where

- $a \in \mathcal{C}^1([0,T])$ and there exist $0 \leq \beta_1 < \beta_2 \leq T$ such that $\inf_{u\in[\beta_1,\beta_2]} a^2(u) > 0$ and $\mathbb{P}[\forall u \in [\beta_1,\beta_2], Z_u^2 = 0] = 0$.
- $(Z'_t, t \in [0,T])$ is a centered Gaussian process, independent of B, such that $Z'_0 = 0$ and for all $t \geq 0$ and $h > 0$,

$$\mathbb{E}[(Z'_{t+h} - Z'_t)^2] = \mathbb{E}[Z'^2_h] \text{ and } \mathbb{E}[Z'^2_h] = h^{2H}(1 + g(h)h^{1/2}),$$

  with $H \in (1/2, 1)$ and $g \in \mathcal{C}^4([0,T])$.
- $f : [0,T] \to \mathbb{R}$ is a random function such that for $p > 0$,

$$\sup_{s\in[0,T]} \mathbb{E}[\,|f(s)|^p] < \infty.$$

- $h : [0,T]^2 \to \mathbb{R}$ is a random function such that for $p > 0$ and $(t,s) \in [0,T]^2$

$$\mathbb{E}[\,|h(t,s)|^p] \leq c_p(t-s)^{3p/2} \text{ and } \mathbb{E}[\sup_{t\in[0,T]} e^{p|h(t,0)|}] < \infty.$$

- $v \in \mathcal{C}^2([0,T])$.
- ξ_0 is a random variable, independent of B, such that for $p > 0$, $\mathbb{E}[e^{p|\xi_0|}] < \infty$.

This general setting includes various Gaussian processes with stationary increments and local Hölder index equal to H, see for example Istas and Lang [68]. The following proposition enables us to work in the general setting of model (I.5) for the proof theorem I.1.

Theorem I.3 *(General formalism for theorem I.1)*

(i) The formalism of model (I.5) *includes model* (I.1)-(I.2).

(ii) In model (I.5)*, theorem I.1 holds for the estimation of the parameter H.*

Hence, we only prove theorem I.3 and theorem I.2.

• The accuracy $v_n(H)$ is slower by a polynomial order than the usual $n^{-1/2}$ of regular parametric models. This rate of convergence seems to be characteristic of high frequency parametric inference from noisy data in presence of fractional Brownian motion. Indeed, this rate is also found by Gloter and Hoffmann [54] in the high frequency inference of the finite dimensional parameter θ in the model

$$\mathrm{d}Y_t = \sigma_t \,\mathrm{d}B_t, \quad \sigma_t = \Phi(\theta, W_t^H) \tag{I.6}$$

and in the high frequency estimation of the Hurst parameter in the model

$$Y_i^n = \sigma W_{i/n}^H + a(W_{i/n}^H)\xi_i^n, \tag{I.7}$$

where a is an unknown function and ξ_i^n a centered white noise, see [55]. In a sense, our approach is a generalization of both (I.6) and (I.7) for the estimation of the parameter H.

• In practice, a usual way to estimate the regularity, or the long memory parameter, of the volatility of an asset is to build a volatility proxy[2] from the prices, and then to use classical method for regularity estimation or long memory detection. Although linked with the preceding practice, the method we give in this chapter is mathematically rigorous, and in some sense optimal. The optimal rates of convergence are quite slow, but not catastrophic. Hence, our result shows that getting accurate enough information about the smoothness of the volatility process is possible, but compulsorily requires a large amount of data. This is not surprising. Indeed, the volatility is not observed and any pointwise approximation of it is very noisy. For an illustration of this, see the numerical results in chapter 2.

1.5 Organisation of the chapter

In section 2, we present our estimation method for the volatility Hurst parameter. Section 3 states the main propositions which lead to theorems I.3 and theorem I.2. The proof of theorem I.3 (i) is given in section 4. We prove in sections 5, 6 and 7 the results stated in section 3 concerning the upper bound whereas theorem I.3 (ii) is proved in section 8. We end with the proof of theorem I.2 in section 9. The proof of proposition I.1 is given in appendix A.

2 Estimation strategy

2.1 Estimation of the Hurst parameter: preliminaries

2.1.1 Estimation of H from direct observation of a fractional Brownian motion

Imagine we observe high frequency data

$$\{\sigma W_{i/n}^H, i = 0, \dots, n\},$$

[2]Such proxies are often based on the absolute or quadratic variation of the log prices, with sampling period higher than 10 minutes to avoid microstructure noise effects.

where σ is an unknown constant and W_t^H a fractional Brownian motion. Then, we can recover the Hurst parameter at the parametric rate $n^{-1/2}$. Indeed, we can use as follows local properties of the trajectory of the fractional Brownian motion, see Istas and Lang [68], see also Berzin and Leon [18], Lang and Roueff [82]. Let $s = (s_0, \dots, s_p) \in \mathbb{R}^{p+1}$ be such that

$$\text{for } \ k = 0, \dots, p-1: \ \sum_{i=0}^{p} s_i \, i^k = 0 \ \text{ and } \ \sum_{i=0}^{p} s_i \, i^p \neq 0.$$

The integer $p = m(s)$ is called the order of the difference. For instance, the usual difference $s = (-1, 1)$ is of order 1 and $s = (1, -2, 1)$ is of order 2. For such sequence s and $i = 0, \dots, n - m(s) - 1$, we define for a function $f : [0, 1] \to \mathbb{R}$, the generalized difference

$$\Delta_{i,n} f = \sum_{j=0}^{m(s)} s_j f\big(\frac{i+j}{n}\big).$$

Consider

$$V_n(H) = \sum_{i=0}^{n-m(s)-1} (\Delta_{i,n} W^H)^2.$$

Istas and Lang [68] show that for $m(s) > 1$, there exists a constant $L_{s,H} > 0$ such that[3]

$$n^{2H-1} V_n(H) = L_{s,H} + \frac{1}{\sqrt{n}} \xi_n,$$

with ξ_n bounded in probability. Then, an estimator achieving the rate $n^{-1/2}$ is for example[4]

$$\widehat{H} = \frac{1}{2}\big(1 + \log_2 \frac{V_{\lfloor n/2 \rfloor}(H)}{V_n(H)}\big).$$

Beyond fractional Brownian motion, the problem of estimating the local Hölder index of a process has been largely studied in the Gaussian context, see in particular Istas and Lang [68] and Lang and Roueff [82].

2.1.2 Estimation of H from noisy observation of a fractional Brownian motion

Consider now model (I.7). Recovering the Hurst parameter in this context of noisy data is more difficult. Indeed, Gloter and Hoffmann show in [55] that the statistical structure of model (I.7) is significantly modified by the noise. They prove that the rate $n^{-1/(4H+2)}$

[3] The condition $m(s) > 1$ is necessary for $H > \frac{3}{4}$, if $H \leq \frac{3}{4}$, one can take $m(s) = 1$.

[4] Note that if σ is known, estimator with accuracy $n^{-1/2}(\log n)^{-1}$ can be built, see Cœurjolly [27].

is optimal to estimate H in the minimax sense of (I.3) and (I.4). Their estimation strategy is based on the behavior of the energy levels of the fractional Brownian motion that reflects the Besov properties of the trajectories. Pick a mother wavelet ψ with 2 vanishing moments. Hence, the wavelet support has a minimal length of 3, see Daubechies [34]. For j and k positive integers, let

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k),\ d_{jk} = \int \psi_{jk} W_s^H \mathrm{d}s \text{ and } Q_j = \sum_k d_{jk}^2.$$

The sequence of energy levels $(Q_j, j \geq 0)$ has the following scaling property[5]:

$$\frac{Q_{j+1}}{Q_j} = 2^{-2H} + o(1) \text{ as } j \to +\infty. \tag{I.8}$$

Gloter and Hoffmann [55] construct estimators $\widehat{d_{jk}^2}$ of the d_{jk}^2 up to a maximal resolution level $J_n = \lfloor \frac{1}{2}\log_2(n) \rfloor$. Setting

$$\widehat{Q}_j = \sum_k \widehat{d_{jk}^2}, \tag{I.9}$$

one obtain a sequence of estimators:

$$\widehat{H}_{j,n} = -\frac{1}{2}\log_2 \frac{\widehat{Q}_{j+1,n}}{\widehat{Q}_{j,n}}, \quad j = 1, \ldots, J_n. \tag{I.10}$$

The final estimator is $\widehat{H}_{J_n^*,n}$ where the optimal resolution level J_n^* is defined following the rules of adaptive estimation of quadratic functionals

$$J_n^* = \max\big\{j = 1, \ldots, J_n,\ \widehat{Q}_{j,n} \geq \frac{2^j}{n}\big\}. \tag{I.11}$$

We adapt the preceding strategy in this chapter.

2.2 Construction of an estimator

We build in this section our estimator in the general setting of model (I.5).

2.2.1 An Euler scheme-type transformation

By an Euler scheme-type transformation, we boil down the problem to a regression model. Indeed, we have

$$z_i^n = n(Y_{(i+1)/n} - Y_{i/n})^2 = \sigma_{i/n}^2 + \xi_i^n, \tag{I.12}$$

[5] For the moment, we do not specify the meaning of $o(\cdot)$.

with

$$\xi_i^n = n\big[\int_{\frac{i}{n}}^{\frac{i+1}{n}} (\sigma_t^2 - \sigma_{i/n}^2)\mathrm{d}t + \big(\int_{\frac{i}{n}}^{\frac{i+1}{n}} \sigma_t \,\mathrm{d}B_t\big)^2 - \int_{\frac{i}{n}}^{\frac{i+1}{n}} \sigma_t^2 \mathrm{d}t\big].$$

Conditional on the fbm W^H and up to negligible terms, the ξ_i^n are martingale increments with variance of order 1.

2.2.2 Estimation of the energy levels

Let ψ be a mother wavelet with 2 vanishing moments and support $[0,T]$. Let

$$d_{jk} = \int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}} \sigma_t^2 \psi_{jk}(t)\mathrm{d}t \text{ and } Q_j = \sum_k d_{jk}^2.$$

By proving a scaling-type property on the energy levels analogous to (I.8), we can follow the strategy of section 2.1.2. The main difficulty lies here in the non-linearity introduced by the function Φ^2. We now present the estimation of the energy levels. To get rid of boundary effects, without any loss of generality in our asymptotic framework, we do not take into account the wavelets ψ_{jk} whose support is not totally included in $[0,T]$. We have

$$d_{jk} = \sum_{l=0}^{T2^{N-j}-1} \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t^2 \psi_{jk}(t)dt.$$

A first natural estimator of d_{jk} is

$$\widetilde{d_{jk}} = \sum_{l=0}^{T2^{N-j}-1} z_{k2^{N-j}+l}^n \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t.$$

Let

$$M_{k,l,t} = \big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{t} \sigma_u \mathrm{d}B_u\big)^2 - \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{t} \sigma_u^2 \mathrm{d}u.$$

From (I.12), we have the following decomposition:

$$\widetilde{d_{jk}} - d_{jk} = b_{jk} + e_{jk} + f_{jk},$$

with

$$b_{jk} = \sum_{l=0}^{T2^{N-j}-1} \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)(\sigma_{k2^{-j}+l2^{-N}}^2 - \sigma_t^2)dt,$$

$$e_{jk} = n \sum_{l=0}^{T2^{N-j}-1} \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t \; M_{k,l,\frac{k}{2^j}+\frac{l+1}{2^N}},$$

$$f_{jk} = n \sum_{l=0}^{T2^{N-j}-1} \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} (\sigma_t^2 - \sigma_{k2^{-j}+l2^{-N}}^2)\mathrm{d}t.$$

In order to estimate d_{jk}^2 accurately enough, we can not use $\widetilde{d_{jk}}^2$ because the remaining term e_{jk}^2 has to be compensated. The other terms are negligible.
Conditional on W^H, $(M_{k,l,t}, t \geq 0)$ is a continuous local martingale. Let $\tilde{\mathbb{E}}$ denote the expectation conditional on the path of the volatility. Then, by the independence of the Brownian increments,

$$\tilde{\mathbb{E}}[e_{jk}^2] = n^2 \sum_{l=0}^{T2^{N-j}-1} \big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t\big)^2 \tilde{\mathbb{E}}[M_{k,l,\frac{k}{2^j}+\frac{l+1}{2^N}}^2].$$

Let

$$N_{k,l,t} = \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{t} \sigma_u \mathrm{d}B_u.$$

By Ito's formula,

$$M_{k,l,t} = 2\int_0^t \sigma_u N_u \mathbb{1}_{\{u \geq \frac{k}{2^j}+\frac{l}{2^N}\}} \mathrm{d}B_u.$$

Let

$$a_{j,k,l}^2 = \tilde{\mathbb{E}}[M_{k,l,\frac{k}{2^j}+\frac{l+1}{2^N}}^2] = 2\big(\tilde{\mathbb{E}}\big[(Y_{k2^{-j}+(l+1)2^{-N}} - Y_{k2^{-j}+l2^{-N}})^2\big]\big)^2.$$

We need to compensate $a_{j,k,l}^2$, so we estimate it by

$$\widehat{a_{j,k,l}^2} = \big(\frac{\sqrt{2}}{h(n)} \sum_{p=0}^{h(n)} (Y_{k2^{-j}+(l+1+p)2^{-N}} - Y_{k2^{-j}+(l+p)2^{-N}})^2\big)^2,$$

where $h(n) = \lfloor n^{1/2} \rfloor$. Let

$$\nu_{jk} = n^2 \sum_{l=0}^{T2^{N-j}-1} \big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t\big)^2 a_{j,k,l}^2,$$

$$\bar{\nu}_{jk} = n^2 \sum_{l=0}^{T2^{N-j}-1} \big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t\big)^2 \widehat{a_{j,k,l}^2}.$$

Finally we define

$$\widehat{d_{jk}^2} = \widetilde{d_{jk}}^2 - \bar{\nu}_{jk} \text{ and } \widehat{Q}_j = \sum_k \widehat{d_{jk}^2}.$$

We thus obtain our estimator $\widehat{H}_{J_n^*,n}$ of H with the same specifications as in (I.10) and (I.11).

3 The behavior of the energy levels

We present here the steps that enable us to prove theorem I.3 (ii) and theorem I.2.

3.1 Upper bound

We work in the general setting of model (I.5). Let

$$d_{jk} = \int \sigma_t^2 \psi_{jk}(t) \mathrm{d}t \text{ and } Q_j = \sum_k d_{jk}^2.$$

We write c for a constant depending on Φ, a, f, v, H, ψ and continuous in its arguments.

Proposition I.2 *(Limit of the energy levels). In model* (I.5)*, there exists a constant* $c(\psi) > 0$*, depending on* ψ *and* H*, continuous in its arguments, and* $c > 0$ *such that*

$$\mathbb{E}\big[\,|2^{2jH} Q_j - c(\psi) \int_0^T a(u)^2 \{(\Phi^2)'(Z_u)\}^2 du|\,\big] \leq c 2^{-j/2}.$$

More precisely, proposition I.2 enables us to obtain the following result.

Proposition I.3 *(Scaling property). In model* (I.5)*, we have*

(i) For all $\varepsilon > 0$*, there exist* j_0 *and* $r > 0$ *such that for all* $j \geq j_0$*,*

$$\mathbb{P}[2^{2jH} Q_j \geq r] \geq 1 - \varepsilon,$$

(ii) For all $\varepsilon > 0$*, there exist* j_0 *and* $M > 0$ *such that for all* $j \geq j_0$*,*

$$\mathbb{P}\big[2^{j/2} \sup_{l \geq j} \big|\frac{Q_{l+1}}{Q_l} - 2^{-2H}\big| \geq M\big] \leq \varepsilon.$$

Finally, we have the following result for the estimator.

Proposition I.4 *(Deviation of the estimator).*
Let $j_n(H) = \lfloor (2H+1)^{-1} \log_2(n) \rfloor$ *and* $\mathcal{H}$ *be a compact set included in* $(1/2, 1)$*. In model* (I.5)*, for all* $H \in \mathcal{H}$*,* $J_n \geq j_n(H)$ *and for any* $L > 0$*, the sequence*

$$\{n 2^{j_n(H)/2} \sup_{J_n \geq j \geq j_n(H) - L} 2^{-j} |\widehat{Q}_{j,n} - Q_j|\}$$

is bounded in probability, uniformly over $\mathcal{H}$*.*

We then prove in section 8 that proposition I.3 and proposition I.4 together imply theorem I.3 (i).

3.2 Lower bound

For the lower bound, we work in model (I.1)-(I.2). Let $\mathbb{P}_f^n$ denote the law of the data $Y^n = \{Y_{i/n},\ i = 0, \dots, nT\}$ conditional on $W^H = f$. The key point of the lower bound is the following.

Proposition I.5 *(Distance in total variation). Under assumptions A, B and C, there exists $c > 0$ such that*

$$\|\mathbb{P}_f^n - \mathbb{P}_g^n\|_{TV}^2 \leq cn\|f - g\|_2^2,$$

where $\|\cdot\|_{TV}$ denotes the distance in total variation and $\|\cdot\|_2$ the usual L^2 norm of functions on $[0, T]$ with respect to the Lebesgue measure.

Proposition I.5 together with proposition 5 of Gloter and Hoffmann [55] imply the lower bound.

4 Proof of theorem I.3 (i)

We show here than we can prove theorem I.1 under the general formalism of model (I.5).

4.1 Notation

In all the proofs, we repeatedly use the notation c for constants depending on H, ψ and the functions defined in model (I.1)-(I.2) or model (I.5), continuous in their arguments, and that may vary from line to line. We write the symbol $=$ also for almost sure equality and for a function g, we set $\|g\|_\infty = \sup_t |g(t)|$. Finally, $\partial_i^j f(u, t)$ denotes the j-*th* derivative of f with respect its i-*th* variable.

4.2 Proof of theorem I.3 (i)

For $s \leq t$

$$\int_0^t a(t, u)\mathrm{d}W_u^H - \int_0^s a(s, u)\mathrm{d}W_u^H$$

is equal to $a(s, s)(W_t^H - W_s^H) + (t - s)f(s) + h(t, s)$ with

$$f(s) = \int_0^s \partial_1 a(s, u)\mathrm{d}W_u^H,$$

and

$$h(t, s) = \frac{(t - s)^2}{2} \int_0^s \partial_1^2 a(\theta_1[t, s], u)\mathrm{d}W_u^H + R(t, s),$$

where

$$R(t,s) = \int_s^t [a(t,u)-a(s,u)]\mathrm{d}W_u^H + \int_s^t [a(s,u)-a(s,s)]\mathrm{d}W_u^H$$
$$= (t-s)\int_s^t \partial_1 a(\theta_2[t,s],u)\mathrm{d}W_u^H + \int_s^t (u-s)\partial_2 a(s,\theta_3[u,s])\mathrm{d}W_u^H.$$

Here θ_1, θ_2 and θ_3 are deterministic functions with values in $[0,T]$. Using assumption A, we get that all the preceding integrands are deterministic, continuously differentiable with respect to the variable u and uniformly bounded with respect to all variables. Hence, in our case, the Riemann-Stieltjes integral with respect to the fbm coincides almost surely with the Wiener integral with respect to the fbm. Consequently, for $f \in \mathcal{C}^1([0,T])$ and $g \in \mathcal{C}^1([0,T])$,

$$\mathbb{E}[\int_0^T f(u)\mathrm{d}W_u^H \int_0^T g(u)\mathrm{d}W_u^H] = H(2H-1)\int_0^T \int_0^T f(s)g(t)|s-t|^{2H-2}\mathrm{d}s\mathrm{d}t, \qquad \text{(I.13)}$$

see for example Norros *et al.* [97]. Hence, using the fact that the preceding stochastic integral are Gaussian variables together with assumption A, we easily get that for $(t,s)\in[0,T]^2$, $s \le t$ and $p > 0$

$$\mathbb{E}[\,|\frac{(t-s)^2}{2}\int_0^s \partial_1^2 a(\theta_1[t,s],u)\mathrm{d}W_u^H + R(t,s)|^p] \le c_p(t-s)^{p(1+H)}.$$

For $p > 0$ and $t > 0$, let

$$V_t = \int_0^t [a(t,u)-a(0,0)]\mathrm{d}W_u^H,\ \tilde{V} = \sup_{t\in[0,T]} |pV_t| \text{ and } \nu = \sup_{t\in[0,T]} \mathbb{E}[(pV_t)^2].$$

We now prove that $\mathbb{E}[e^{\tilde{V}}] < \infty$. The process $(pV_t, t \ge 0)$ is a Gaussian process starting from 0 with continuous trajectories, so we can use Dudley's entropy bound. There exists a universal constant c such that

$$\mathbb{E}[\tilde{V}] \le c\int_0^{d(0,T)} \sqrt{\log N(T,d,\varepsilon)}\mathrm{d}\varepsilon,$$

where $d^2(s,t) = p^2\mathbb{E}[|V_t - V_s|^2]$ and $N(T,d,\varepsilon)$ is the minimal number of balls of radius ε needed to recover $[0,T]$. Since

$$\mathbb{E}[|V_t - V_s|^2] \le c|t-s|^{2H},$$

we easily get that $N(T,d,\varepsilon)$ is less than $cT\varepsilon^{-1/H}$. Hence, we get $\mathbb{E}[\tilde{V}] < \infty$. We now use Borell's inequality: for $\lambda > \mathbb{E}[\tilde{V}]$,

$$\mathbb{P}[\tilde{V} \ge \lambda] \le 2e^{-\frac{1}{2}(\lambda-\mathbb{E}[\tilde{V}])^2/\nu}.$$

As

$$\mathbb{E}[e^{\tilde{V}}] = \int_0^{+\infty} \mathbb{P}[e^{\tilde{V}} \geq \lambda]\mathrm{d}\lambda,$$

we get

$$\mathbb{E}[e^{\tilde{V}}] \leq c + 2\int_{e^{\mathbb{E}[\tilde{V}]}}^{+\infty} e^{-\frac{1}{2}(\log\lambda - \mathbb{E}[\tilde{V}])^2/\nu}\mathrm{d}\lambda \leq c + 2\int_0^{+\infty} e^{\mathbb{E}[\tilde{V}]+u-u^2/2\nu}\mathrm{d}u.$$

Finally, suppose that on $[\beta_1, \beta_2]$,

$$t \to \int_0^t a(t,u)\mathrm{d}W_u^H + f(t)\xi_0$$

is equal to zero with positive probability. This implies that $t \to \int_0^t a(t,u)\mathrm{d}W_u^H$ belongs to $\mathcal{C}^1([\beta_1, \beta_2])$ with positive probability. This is absurd, see the proof of proposition I.1 in appendix A.

5 Proof of proposition I.2

From now, and until the end of the proof of theorem I.3 (*ii*), we work in model (I.5).

5.1 Technical lemmas

We establish here several useful lemmas. We apply here ideas of Gloter and Hoffmann [54], initially developed for generalized differences. We first prove 2 lemmas on the expectation and covariance of the wavelet coefficients for the stochastic integral. Let

$$\beta_{jk} = \int_0^T Z_t\psi_{jk}(t)\mathrm{d}t, \quad \beta'_{jk} = \int_0^T Z'_t\psi_{jk}(t)\mathrm{d}t, \quad F(t) = \int_0^t \psi(u)du.$$

We have the following lemma.

Lemma I.1 *For all positive integers j, k,*

$$\beta_{jk} = a(k2^{-j})\beta'_{jk} + 2^{-2j}R_{jk},$$

and

$$\mathbb{E}[\beta'^2_{jk}] = 2^{-j(1+2H)}\{c(\psi) + 2^{-j/2}R'_{jk}\},$$

with

$$c(\psi) = \mathbb{E}\big[\big(\int_0^T F(t)dW_t^H\big)^2\big] > 0,$$

where $(W_t^H, t \geq 0)$ is a fractional Brownian motion and $\mathbb{E}[|R_{jk}|^p + |R'_{jk}|^p] \leq c_p$ for $p > 0$.

Proof. The coefficient β_{jk} is equal to

$$2^{-j/2}\int_0^T \psi(v)Z_{(k+v)2^{-j}}\mathrm{d}v.$$

The two vanishing moments of the wavelet easily give the first assertion of the lemma. Using that

$$2Z'_{(k+u)2^{-j}}Z'_{(k+v)2^{-j}} = Z'^2_{(k+u)2^{-j}} + Z'^2_{(k+v)2^{-j}} - (Z'_{(k+u)2^{-j}} - Z'_{(k+v)2^{-j}})^2, \qquad \text{(I.14)}$$

together with the vanishing moments of the wavelet, we get that $\mathbb{E}[\beta'^2_{jk}]$ is equal to

$$\begin{aligned}&-2^{-(j+1)}\int_0^T\int_0^T \psi(u)\psi(v)2^{-j2H}|u-v|^{2H}\mathrm{d}u\mathrm{d}v\\&-2^{-(j+1)}\int_0^T\int_0^T \psi(u)\psi(v)2^{-j(2H+1/2)}|u-v|^{2H+1/2}g(|u-v|2^{-j}).\end{aligned}$$

Hence,

$$\mathbb{E}[\beta'^2_{jk}] = -2^{-(j+1)}\int_0^T\int_0^T \psi(u)\psi(v)2^{-j2H}|u-v|^{2H}\mathrm{d}u\mathrm{d}v + 2^{-j(3/2+2H)}R'_{jk},$$

with $|R'_{jk}|\leq c$. Then we easily show that

$$-\int_0^T\int_0^T \psi(u)\psi(v)|u-v|^{2H}\mathrm{d}u\mathrm{d}v = 2H(2H-1)\int_0^T\int_0^T F(u)F(v)|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v.$$

We conclude using (I.13). □

Lemma I.2 *(Decorrelation of the wavelet coefficients).*
There exists c such that, for all j,k,k',

$$|\mathbb{E}[\beta'_{jk}\beta'_{jk'}]| \leq 2^{-j(1+2H)}c(1+|k-k'|)^{2H-4}.$$

Proof. For $k\geq k'+T+1$, let $m_{k,k',u,v} = 2^{-j}(k-k'+u-v)$. We have

$$\begin{aligned}\mathbb{E}[\beta'_{jk}\beta'_{jk'}] =&2^{-j}\int_0^T\int_0^T \psi(u)\psi(v)\mathbb{E}[Z'_{(k+u)2^{-j}}Z'_{(k'+v)2^{-j}}]\mathrm{d}u\mathrm{d}v\\ =&-2^{-(j+1)}\int_0^T\int_0^T \psi(u)\psi(v)\mathbb{E}[Z'^2_{(k-k'+u-v)2^{-j}}]\mathrm{d}u\mathrm{d}v\\ =&-2^{-(j+1)}\int_0^T\int_0^T \psi(u)\psi(v)|m_{k,k',u,v}|^{2H}\\ &-2^{-(j+1)}\int_0^T\int_0^T \psi(u)\psi(v)g(m_{k,k',u,v})|m_{k,k',u,v}|^{2H+1/2}\mathrm{d}u\mathrm{d}v.\end{aligned}$$

For the first term, we make a fourth order Taylor expansion of $x \to (x+u-v)^{2H}$ around point $k-k'$. Thanks to the two vanishing moments of the wavelet, we get that the first term is less than $c2^{-j(1+2H)}(|k-k'|-T)^{2H-4}$. For the second term, we first make a fourth order Taylor expansion of g around point $(k-k')2^{-j}$. The results follows thanks to expansions of $x \to (x+u-v)^{2H+1/2}$ up to order 4, 3, 2 and 1. □

Lemma I.3 *Let $\xi:\ [0,T] \to \mathbb{R}$ be a deterministic bounded function. Define*

$$\Sigma_j(\xi) = 2^j \sum_{k=0}^{T(2^j-1)} \{2^{j2H}\beta_{jk}^2 - c(\psi)2^{-j}a(k2^{-j})^2\}\xi_{k2^{-j}}. \tag{I.15}$$

Then,

$$\mathbb{E}[\Sigma_j(\xi)^2] \leq c\|\xi\|_\infty^2 2^j.$$

Proof. We have

$$\begin{aligned}\Sigma_j(\xi) = 2^j &\sum_{k=0}^{T(2^j-1)} 2^{j2H}a(k2^{-j})^2(\beta_{jk}'^2 - \mathbb{E}[\beta_{jk}'^2])\xi_{k2^{-j}} \\ &+ 2^j \sum_{k=0}^{T(2^j-1)} 2^{j2H}(2^{-4j}R_{jk} + 2^{-2j}2^{-j(1/2+H)}R_{jk}' + 2^{-j(3/2-2H)}R_{jk}'')\xi_{k2^{-j}},\end{aligned}$$

with $\mathbb{E}[|R_{jk}|^p + |R_{jk}'|^p + |R_{jk}''|^p] \leq c$, for $p>0$. Hence $\mathbb{E}[\Sigma_j(\xi)^2]$ is less than

$$c2^j\|\xi\|_\infty^2 + c2^{2j}\mathbb{E}\Big[\sum_{k,k'=0}^{T(2^j-1)} 2^{j4H}\{\beta_{jk}'^2 - \mathbb{E}[\beta_{jk}'^2]\}\{\beta_{jk'}'^2 - \mathbb{E}[\beta_{jk'}'^2]\}\xi_{k2^{-j}}\xi_{k'2^{-j}}\Big].$$

Let $Y_k = \beta_{jk}'^2/\mathbb{E}[\beta_{jk}'^2] - 1$. The preceding inequality can be written

$$\mathbb{E}[\Sigma_j(\xi)^2] \leq c2^j\|\xi\|_\infty^2 + c2^{2j}2^{j4H}\sum_{k,k'=0}^{T(2^j-1)} \mathbb{E}[Y_kY_{k'}]\mathbb{E}[\beta_{jk}'^2]\mathbb{E}[\beta_{jk'}'^2]\xi_{k2^{-j}}\xi_{k'2^{-j}}.$$

We now apply Mehler's formula and we get

$$\begin{aligned}\mathbb{E}[\Sigma_j(\xi)^2] &\leq c2^j\|\xi\|_\infty^2 + 2^{2j}2^{j4H}\|\xi\|_\infty^2 2\sum_{k,k'=0}^{T(2^j-1)} \mathrm{Cov}(\beta_{jk}', \beta_{jk'}')^2 \\ &\leq c2^j\|\xi\|_\infty^2 + c2^{2j}2^{j4H}\|\xi\|_\infty^2 \sum_{k,k'=0}^{T(2^j-1)} 2^{-2j(1+2H)}(1+|k-k'|)^{4(H-2)} \\ &\leq c2^j\|\xi\|_\infty^2 + c\|\xi\|_\infty^2 \sum_{k=0}^{T(2^j-1)}\sum_{i=0}^{+\infty}(1+i)^{4(H-2)} \leq c2^j\|\xi\|_\infty^2.\end{aligned}$$

□

Lemma I.4 *Assume that $\xi : [0,T] \to \mathbb{R}$ is bounded and vanishes outside the interval $[k2^{-j_0}, k'2^{-j_0}] \subset [0,T]$ for some $k, k', j_0 \geq 1$, $k \neq k'$. Then, there exists $c > 0$ such that for $j \geq j_0$,*

$$\mathbb{E}[\Sigma_j(\xi)^2] \leq c\|\xi\|_\infty^2 |k' - k| 2^{j-j_0}.$$

Proof. As $\xi_{z2^{-j}}$ is different from zero only if $k2^{-j_0} \leq z2^{-j} \leq k'2^{-j_0}$, there are less than $|k-k'|2^{j-j_0}+1$ admissible values for z. Hence we easily get that $\mathbb{E}[\Sigma_j(\xi)^2]$ is less than

$$c|k'-k|2^{j-j_0}|k'-k|2^{-j_0}\|\xi\|_\infty^2 + c\sum_{z,z'}^{T(2^j-1)} |\xi_{z2^{-j}}||\xi_{z'2^{-j}}|(1+|z-z'|)^{4(H-2)}.$$

By similar computations on the series as in proof of lemma I.3, we get

$$\mathbb{E}[\Sigma_j(\xi)^2] \leq c|k'-k|2^{j-j_0}\|\xi\|_\infty^2 + c\|\xi\|_\infty \sum_z |\xi_{z2^{-j}}|.$$

The result follows. □

We now decompose the function $t \to (\{\Phi^2\}')^2(Z_t)$ in a wavelet basis with support $[0,T]$. Thus, we use the same wavelet as before but in another context. We have the following lemma.

Lemma I.5 *(Decomposition in a wavelet basis). Let $\Gamma = (\{\Phi^2\}')^2$. Let ϕ be the scaling function associated to ψ. We write $\phi_{0k}(t) = \phi(t-k)$,*

$$c_k = \int \Gamma(Z_t)\phi_{0k}(t)\,dt \text{ and } c_{jk} = \int \Gamma(Z_t)\psi_{jk}(t)\,dt.$$

Then,

$$\Gamma(Z_t) = \sum_{k=0}^{r} c_k\phi_{0k}(t) + \sum_{j=0}^{+\infty}\sum_{k=0}^{T(2^j-1)} c_{jk}\psi_{jk}(t),$$

where r is a constant value depending on T and with

$$\mathbb{E}[c_0 + \cdots + c_r] \leq c, \quad \mathbb{E}[c_{jk}^2] \leq c2^{-j(1+2H)}.$$

Proof. We have

$$\begin{aligned} c_{jk} &= 2^{-j/2}\int_0^T \psi(u)\Gamma(Z_{2^{-j}(k+u)})\mathrm{d}u \\ &= 2^{-j/2}\int_0^T \psi(u)[\Gamma(Z_{2^{-j}(k+u)}) - \Gamma(Z_{2^{-j}k})]\mathrm{d}u \\ &= 2^{-j/2}\int_0^T \psi(u)[Z_{2^{-j}(k+u)} - Z_{2^{-j}k}]\Gamma'(\eta)\mathrm{d}u, \end{aligned}$$

with η a random value between $Z_{2^{-j}k}$ and $Z_{2^{-j}(k+u)}$. By the continuity of the sample path of $t \to Z_t$, we know there exists a random point θ between $k2^{-j}$ and $(k+u)2^{-j}$ such that $\eta = Z_\theta$. Thus, we have

$$c_{jk}^2 \leq c2^{-j} \int_0^T \psi^2(u)[Z_{2^{-j}(k+u)} - Z_{2^{-j}k}]^2 \{\Gamma'(r[\theta])\}^2 \mathrm{d}u,$$

with

$$r(\theta) = a(0)Z'_\theta + \theta f(0) + h(\theta, 0) + [v(\theta) - v(0)]\xi_0.$$

As Z' is a Gaussian process, we easily get

$$\mathbb{E}[(Z_{2^{-j}(k+u)} - Z_{2^{-j}k})^4] \leq c2^{-j4H}.$$

Using Assumption B, we have

$$\mathbb{E}[c_{jk}^2] \leq c2^{-j(1+2H)} \int_0^T \psi(u)^2 \{\mathbb{E}[e^{c|r(\theta)|}]\}^{1/2} \mathrm{d}u.$$

Let $p > 0$ and $\tilde{Z} = \sup_{t \in [0,T]} |pZ'_t|$. By the same arguments as in the proof of theorem I.3 (i), we prove that $\mathbb{E}[e^{\tilde{Z}}] < \infty$. Hence, using the hypothesis of model (I.5) and Cauchy-Schwarz inequality, we get $\mathbb{E}[c_{jk}^2] \leq c2^{-j(1+2H)}$. By a Taylor expansion, we get $\mathbb{E}[c_0 + \cdots + c_r] \leq c$. □

Lemma I.6 *Let Γ be as in lemma I.5. We have*

$$\mathbb{E}\big[\big|2^j \sum_k \{2^{j2H}\beta_{jk}^2 - c(\psi)2^{-j}a(k2^{-j})^2\}\Gamma(Z_{k2^{-j}})\big|\big] \leq c2^{j/2}.$$

Proof. We know from lemma I.5 that

$$\Gamma(Z_t) = \sum_{k=0}^{r} c_k \phi_{0k}(t) + \sum_{j=0}^{+\infty} \sum_{k=0}^{T(2^j-1)} c_{jk}\psi_{jk}(t).$$

Let

$$S_j(\Gamma) = 2^j \sum_k \{2^{j2H}\beta_{jk}^2 - c(\psi)2^{-j}a(k2^{-j})^2\}\Gamma(Z_t).$$

We can write

$$S_j(\Gamma) = \sum_{k=0}^{r} c_k \Sigma_j(\phi_{0k}) + \sum_{j_1=0}^{+\infty} S_{j,j_1},$$

with

$$S_{j,j_1} = \sum_{k=0}^{T(2^j-1)} c_{j_1 k}\Sigma_j(\psi_{j_1 k}).$$

For $k = 0$ to r, $\mathbb{E}[\,|c_i \Sigma_j(\phi_{0i})|\,] \le c 2^{j/2}$, by lemma I.3. Now we prove that

$$\mathbb{E}[\sum_{j_1=0}^{+\infty} |S_{j,j_1}|\,] \le c 2^{j/2}.$$

If $j_1 \le j$, by lemma I.4,

$$\mathbb{E}[\,|S_{j,j_1}|\,] \le c \sum_{k=0}^{T(2^{j_1}-1)} 2^{-j_1(1+2H)/2} \big(\mathbb{E}[\Sigma_j(\psi_{j_1 k})^2]\big)^{1/2} \le c 2^{j_1(1/2-H)} 2^{j/2}.$$

Because $H > 1/2$, we have

$$\sum_{j_1=0}^{j} \mathbb{E}[\,|S_{j,j_1}|\,] \le c 2^{j/2}.$$

If $j < j_1$, $\psi_{j_1 k}$ has support $[k2^{-j_1}, (k+T)2^{-j_1}]$, so $\Sigma_j(\psi_{j_1 k}) = 0$ unless there exists $i \in [0, T(2^j - 1)]$ such that $i2^{-j} \in [k2^{-j_1}, (k+T)2^{-j_1}]$, that is

$$k 2^{j-j_1} \le i \le (k+T) 2^{j-j_1}.$$

Thus, there are less than $c2^j$ possible values for i and moreover, for such i, the sum defining $\Sigma_j(\psi_{j_1 k})$ is reduced to one single term, so, combining this result with lemma I.1, we get

$$\mathbb{E}[\Sigma_j(\psi_{j_1 k})^2] \le c \|\psi_{j_1 k}\|_\infty^2 \le c 2^{j_1}$$

and

$$\mathbb{E}[\,|S_{j,j_1}|\,] \le c \sum_{k=0}^{T(2^j-1)} 2^{-j_1(1+2H)/2} 2^{j_1/2} \le c 2^j 2^{-j_1 H}.$$

Finally

$$\sum_{j_1=0}^{+\infty} \mathbb{E}[\,|S_{j,j_1}|\,] = \sum_{j_1=0}^{j} \mathbb{E}[\,|S_{j,j_1}|\,] + \sum_{j_1=j+1}^{+\infty} \mathbb{E}[\,|S_{j,j_1}|\,] \le c 2^{j/2}.$$

□

Lemma I.7 *(Riemann's approximation).*
Let $H(x,t) = a(x)^2 \Gamma(Z_t)$. We have

$$\mathbb{E}[\,|\int_0^T H(t,t)\,dt - 2^{-j} \sum_{k=1}^{2^j T} H(k2^{-j}, k2^{-j})|\,] \le c 2^{-j/2}.$$

Proof. We easily get that

$$\mathbb{E}[|\int_0^T H(t,t)\mathrm{d}t - 2^{-j}\sum_{k=1}^{2^jT} H(k2^{-j},k2^{-j})|]$$

is smaller than

$$\sum_{k=1}^{2^jT}\int_{(k-1)2^{-j}}^{k2^{-j}} a(t)^2\mathbb{E}[\,|(Z_t - Z_{k2^{-j}})\Gamma'(Z_{\theta[t,k2^{-j}]})|\,] + |a(k2^{-j})^2 - a(t)^2|\mathbb{E}[\,|\Gamma(Z_{k2^{-j}})|\,]\mathrm{d}t,$$

with $\theta[t,k2^{-j}]$ a random value between t and $k2^{-j}$. The same arguments as in the proof of lemma I.5 gives that it is less than $c(2^{-j}+2^{-jH})$. □

5.2 Proof of proposition I.2

Let

$$\tilde{\beta}_{jk} = \int \psi_{jk}(t)\Phi^2(Z_t)\mathrm{d}t.$$

Using the first vanishing moment of ψ, we have

$$\tilde{\beta}_{jk} = (\Phi^2)'(Z_{k2^{-j}})\beta_{jk} + 2^{-j/2}\int \psi(t)(Z_{(k+u)2^{-j}} - Z_{k2^{-j}})^2(\Phi^2)''(Z_{\theta[k2^{-j},(k+1)2^{-j}]}).$$

with $\theta[k2^{-j},(k+1)2^{-j}]$ a random value between $k2^{-j}$ and $(k+1)2^{-j}$. So, $\tilde{\beta}_{jk}^2$ is equal to

$$\Gamma(Z_{k2^{-j}})\beta_{jk}^2 + 2^{-j}2^{-4jH}X_{jk} + 2^{-j(1/2+H)}2^{-j/2}2^{-2jH}Y_{jk},$$

with $\mathbb{E}[|X_{jk}|^p + |Y_{jk}|^p] \leq c_p$, for $p>0$. Hence

$$\begin{aligned}&\sum_k \{2^{j2H}\tilde{\beta}_{jk}^2 - c(\psi)2^{-j}a(k2^{-j})^2\Gamma(Z_{k2^{-j}})\}\\ =&\sum_k \{2^{j2H}\beta_{jk}^2 - c(\psi)2^{-j}a(k2^{-j})^2\}\Gamma(Z_{k2^{-j}}) + 2^{-j/2}W_{jk},\end{aligned}$$

with $\mathbb{E}[|W_{jk}|^p] \leq c_p$, for $p>0$. We finally get proposition I.2 by lemma I.6 and lemma I.7.

6 Proof of proposition I.3

We begin by the proof of (i). With the notation of model (I.5), there exists $\eta > 0$ such that

$$c(\psi)\int_0^T a^2(u)\{(\Phi^2)'(Z_u)\}^2\mathrm{d}u \geq \eta\int_{\beta_1}^{\beta_2}\{(\Phi^2)'(Z_u)\}^2\mathrm{d}u.$$

Let

$$\zeta = \eta \int_{\beta_1}^{\beta_2} \{(\Phi^2)'(Z_u)\}^2 \mathrm{d}u.$$

Suppose there exists $\varepsilon > 0$ such that for all $r > 0$, $\mathbb{P}[\zeta \leq r] \geq \varepsilon$. Since $\zeta \geq 0$, $\mathbb{P}[\zeta = 0] \geq \varepsilon$. By assumption B, this implies $Z_u = 0$ on $[\beta_1, \beta_2]$ with positive probability, which is absurd by the assumptions on model (I.5). Then, for $\varepsilon > 0$, there exists $r > 0$ such that

$$\mathbb{P}[\zeta \geq 2r] \geq 1 - \varepsilon.$$

By Markov's inequality, we have

$$\mathbb{P}\big[2^{2jH} Q_j \notin [\zeta - r, \zeta + r]\big] = \mathbb{P}[\,|2^{2jH} Q_j - \zeta| > r] \leq c \frac{2^{-j/2}}{r}.$$

Thus,

$$\sum_{j \geq 0} \sup_H \mathbb{P}\big[2^{2jH} Q_j \notin [\zeta - r, \zeta + r]\big] < +\infty.$$

Then, by Borel Cantelli's lemma, for large enough j a.s.

$$2^{2jH} Q_j \geq \zeta - r.$$

We now prove (ii). Let $\varepsilon > 0$, r and j_0 associated by proposition I.3 (i) and $j \geq j_0$. We have

$$\begin{aligned}\mathbb{P}\big[2^{j/2} \sup_{l \geq j} \big|\frac{Q_{l+1}}{Q_l} - 2^{-2H}\big| \geq M\big] &= \mathbb{P}[\sup_{l \geq j} |Q_{l+1} - 2^{-2H} Q_l| \geq M Q_l 2^{-j/2}] \\ &\leq \varepsilon + \mathbb{P}[\sup_{l \geq j} |Q_{l+1} - 2^{-2H} Q_l| \geq M 2^{-j/2} 2^{-2lH} r] \\ &\leq \varepsilon + \sum_{l \geq j \geq j_0} \mathbb{E}[\,|Q_{l+1} - 2^{-2H} Q_l|\,] 2^{2lH} 2^{j/2} (Mr)^{-1}.\end{aligned}$$

Let

$$L = c(\psi) \int_0^T a^2(u,u) \{(\Phi^2)'[Z_u]\}^2 \mathrm{d}u.$$

The quantity $\mathbb{E}[\,|Q_{l+1} - 2^{-2H} Q_l|$ is equal to

$$\mathbb{E}[\,|Q_{l+1} - 2^{-2(l+1)H} L + 2^{-2(l+1)H} L - 2^{-2H} Q_l|\,] \leq c 2^{-l(2H+1/2)}.$$

Eventually,

$$\mathbb{P}\big[2^{j/2} \sup_{l \geq j} \big|\frac{Q_{l+1}}{Q_l} - 2^{-2H}\big| \geq M\big] \leq \varepsilon + c \sum_{l \geq j \geq j_0} 2^{-l/2} 2^{j/2} (Mr)^{-1}.$$

For large enough M, this can be made arbitrarily small.

7 Proof of proposition I.4

With the notation of section 2.2.2, we have

$$\widehat{Q_j} - Q_j = \sum_k b_{jk}^2 + \sum_k f_{jk}^2 + \sum_k b_{jk} f_{jk} + \sum_k d_{jk} b_{jk} + \sum_k d_{jk} f_{jk}$$

$$+ \sum_k (e_{jk}^2 - \bar{\nu}_{jk}) + \sum_k e_{jk} f_{jk} + \sum_k b_{jk} e_{jk} + \sum_k d_{jk} e_{jk} + \sum_k \nu_{jk} - \bar{\nu}_{jk}.$$

Following Gloter and Hoffmann [55], it is enough to prove

$$\sup_{J_n \geq j \geq j_n(H)-L} \sup_{H \in [H_-, H_+]} 2^{-j/2} \mathbb{E}[\,|\widehat{Q}_{j,n} - Q_j|\,] \leq c n^{-1}.$$

Now we bound the 10 terms one by one.

• Term 1: let $V_{tl} = \sigma_t^2 - \sigma_{k2^{-j}+l2^{-N}}^2$. We have

$$\mathbb{E}[b_{jk}^2] = \sum_{l=0}^{T(2^{N-j}-1)} \sum_{l'=0}^{T(2^{N-j}-1)} \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \int_{\frac{k}{2^j}+\frac{l'}{2^N}}^{\frac{k}{2^j}+\frac{l'+1}{2^N}} \psi_{jk}(t)\psi_{jk}(t')\mathbb{E}[V_{tl}V_{tl'}]\mathrm{d}t\mathrm{d}t'$$

$$\leq c2^j \sum_l \sum_{l'} \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \int_{\frac{k}{2^j}+\frac{l'}{2^N}}^{\frac{k}{2^j}+\frac{l'+1}{2^N}} \big(\mathbb{E}[V_{tl}^2]\mathbb{E}[V_{tl'}^2]\big)^{1/2}\mathrm{d}t\mathrm{d}t'.$$

Moreover, for $t \in [k2^{-j} + l2^{-N}, k2^{-j} + (l+1)2^{-N}]$,

$$V_{tl} = (Z_t - Z_{k2^{-j}+l2^{-N}})\Phi^{2'}\{Z_{k2^{-j}+(l+v)2^{-N}}\}$$

with $v \in [0,1]$. By assumption B and the same arguments as previously, $\mathbb{E}[V_{tl}^2] \leq c2^{-2NH}$. Hence $\mathbb{E}[b_{jk}^2] \leq c2^{-j}n^{-1}$.

• Term 2 and term 3 follow easily with the same order.

• Term 4: as in lemma I.5, we easily prove that $\mathbb{E}[d_{jk}^2] \leq c2^{-j(1+2H)}$ and then, because $j \geq \frac{1}{2H+1}\log_2(n)$, $\mathbb{E}[\,|d_{jk}b_{jk}|\,] \leq c2^{-j/2}n^{-1}$.

• Term 5 follows as term 4 with the same order.

• Term 6: we argue first conditional on the path of the volatility. We write $\tilde{\mathbb{E}}$ for the expectation conditional on the path of the volatility. Because of the independence of the Brownian increments and because the variables are centered, we have

$$\tilde{\mathbb{E}}\big[\big(\sum_k e_{jk}^2 - \nu_{jk}\big)^2\,\big] = \sum_k \tilde{\mathbb{E}}[(e_{jk}^2 - \nu_{jk})^2] \leq c \sum_k \tilde{\mathbb{E}}[e_{jk}^4 + \nu_{jk}^2].$$

Let

$$M_l = \Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t \,\mathrm{d}B_t\Big)^2 - \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t^2 \mathrm{d}t.$$

Because the variables M_l, $l = 0, \dots, T(2^{N-j}-1)$ are centered and independent, we get that $\tilde{\mathbb{E}}[e_{jk}^4]$ is equal to

$$\sum_{l=0}^{T(2^{N-j}-1)} \sum_{l'=0}^{T(2^{N-j}-1)} n^4 \Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t \int_{\frac{k}{2^j}+\frac{l'}{2^N}}^{\frac{k}{2^j}+\frac{l'+1}{2^N}} \psi_{jk}(t)\mathrm{d}t\Big)^2 \tilde{\mathbb{E}}[M_l^2 M_{l'}^2].$$

Indeed the product of terms of power 3 with terms of power 1 are equal to zero. But, we have the following equality in law

$$M_l^2 \overset{\mathcal{L}}{=} \Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t^2 \mathrm{d}t\Big)^2 (Z^2-1)^2,$$

with Z a standard Gaussian variable. Hence,

$$\tilde{\mathbb{E}}[M_l^4] = c\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t^2 \mathrm{d}t\Big)^4.$$

Now, we have

$$\mathbb{E}\Big[\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t^2 \mathrm{d}t\Big)^4\Big] \leq \iiiint \big(\mathbb{E}[\sigma_{t_1}^8]\mathbb{E}[\sigma_{t_2}^8]\mathbb{E}[\sigma_{t_3}^8]\mathbb{E}[\sigma_{t_4}^8]\big)^{1/4} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3 \mathrm{d}t_4.$$

Moreover, there exists $\theta \in [0, T]$ such that,

$$\sigma_t^2 = \Phi^2(Z_t) = \Phi^2(0) + \Phi^{2'}(Z_\theta) Z_t.$$

This leads to $\mathbb{E}[\sigma_t^8] \leq c$. Hence $\mathbb{E}[e_{jk}^4] \leq cn^{-2}$. We have

$$\mathbb{E}[\nu_{jk}^2] = 4n^4 \sum_{l=0}^{T(2^{N-j}-1)} \sum_{l'=0}^{T(2^{N-j}-1)} \Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t \int_{\frac{k}{2^j}+\frac{l'}{2^N}}^{\frac{k}{2^j}+\frac{l'+1}{2^N}} \psi_{jk}(t)\mathrm{d}t\Big)^2$$
$$\mathbb{E}\Big[\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t^2 \mathrm{d}t\Big)^2 \Big(\int_{\frac{k}{2^j}+\frac{l'}{2^N}}^{\frac{k}{2^j}+\frac{l'+1}{2^N}} \sigma_t^2 \mathrm{d}t\Big)^2\Big].$$

In the same way as for $\mathbb{E}[e_{jk}^4]$, we get $\mathbb{E}[\nu_{jk}^2] \leq cn^{-2}$.

• Term 7: in the preceding proof, we have shown $\mathbb{E}[e_{jk}^2] \leq cn^{-1}$ and so we obtain $\mathbb{E}[\,|f_{jk}e_{jk}|\,] \leq c2^{-j/2}n^{-1}$.

• Term 8 follows exactly as term 7.

• Term 9: we argue first conditional on the path of the volatility. Because of the independence of the Brownian increments and because the variables are centered, we have

$$\tilde{\mathbb{E}}\big[\big(\sum_k e_{jk}d_{jk}\big)^2\big] = \sum_k d_{jk}^2 \tilde{\mathbb{E}}[e_{jk}^2].$$

Again because of the independence of the Brownian increments and because the variables are centered, we have

$$d_{jk}^2\tilde{\mathbb{E}}[e_{jk}^2] = c\int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}} \psi_{jk}(t_1)\sigma_{t_1}^2 \mathrm{d}t_1 \int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}} \psi_{jk}(t_2)\sigma_{t_2}^2 \mathrm{d}t_2$$
$$\sum_l n^2\big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t_3)\mathrm{d}t_3\big)^2\big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_{t_3}^2 \mathrm{d}t_3\big)^2.$$

So, we get

$$\mathbb{E}[d_{jk}^2e_{jk}^2] = cn^2\sum_l \big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t_3)\mathrm{d}t_3\big)^2$$
$$\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \mathbb{E}\big[\int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}} \psi_{jk}(t_1)\sigma_{t_1}^2\mathrm{d}t_1\int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}} \psi_{jk}(t_2)\sigma_{t_2}^2\sigma_{t_3}^2\sigma_{t_4}^2\mathrm{d}t_2\big]\mathrm{d}t_3\mathrm{d}t_4.$$

Because of the vanishing moment of the wavelet, we have

$$\begin{aligned}
&\mathbb{E}\big[\int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}} \psi_{jk}(t_1)\sigma_{t_1}^2\mathrm{d}t_1\int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}} \psi_{jk}(t_2)\sigma_{t_2}^2\sigma_{t_3}^2\sigma_{t_4}^2\mathrm{d}t_2\big]\\
=&\mathbb{E}\big[\int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}} \psi_{jk}(t_1)\int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}} \psi_{jk}(t_2)V_{t_10}V_{t_20}\sigma_{t_3}^2\sigma_{t_4}^2\mathrm{d}t_2\mathrm{d}t_1\big]\\
\leq& c2^j2^{-2j}\big(\mathbb{E}[V_{t_10}^4]\mathbb{E}[V_{t_20}^4]\big)^{1/4} \leq c2^{-j}2^{-j2H}.
\end{aligned}$$

Consequently, $\mathbb{E}[d_{jk}^2e_{jk}^2] \leq cn^{-1}2^{-3j}$, but, as $j \geq \dfrac{\log_2 n}{3}$, $\mathbb{E}[d_{jk}^2e_{jk}^2] \leq cn^{-2}$.

• Term 10: let $X = \big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t\,\mathrm{d}B_t\big)^2$ and $X_i = \big(\int_{\frac{k}{2^j}+\frac{l+i}{2^N}}^{\frac{k}{2^j}+\frac{l+i+1}{2^N}} \sigma_t\,\mathrm{d}B_t\big)^2$. Then,

$$\nu_{jk} = 2\sum_{l=0}^{2^{N-j}-1} n^2\big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t\big)^2\big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_u^2\mathrm{d}u\big)^2,$$
$$\bar{\nu}_{jk} = 2\sum_{l=0}^{2^{N-j}-1} n^2\big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t\big)^2\big(\frac{1}{h}\sum_{i=0}^{h} X_i\big)^2,$$

where $h = h(n) = \lfloor n^{1/2} \rfloor$. The term $\nu_{jk} - \bar{\nu}_{jk}$ is equal to

$$2 \sum_{l=0}^{2^{N-j}-1} n^2 \big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t) \mathrm{d}t \big)^2 \big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_u^2 \mathrm{d}u + \frac{1}{h} \sum_{i=0}^{h} X_i \big) \big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_u^2 \mathrm{d}u - \frac{1}{h} \sum_{i=0}^{h} X_i \big).$$

We argue first conditional on the path of the volatility. We have

$$\begin{aligned} &\tilde{\mathbb{E}}\big[\big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_u^2 \mathrm{d}u - \frac{1}{h} \sum_{i=0}^{h} X_i \big)^2 \big] \\ \le & c\tilde{\mathbb{E}}\big[\big(\frac{1}{h} \sum_{i=0}^{h} \{X_i - \tilde{\mathbb{E}}[X_i]\} \big)^2 \big] + c\tilde{\mathbb{E}}\big[\big(\frac{1}{h} \sum_{i=0}^{h} \tilde{\mathbb{E}}[X_i] - \tilde{\mathbb{E}}[X] \big)^2 \big], \end{aligned}$$

with the following equality in law

$$X_i - \tilde{\mathbb{E}}[X_i] \overset{\mathcal{L}}{=} \big(\int_{\frac{k}{2^j}+\frac{l+i}{2^N}}^{\frac{k}{2^j}+\frac{l+i+1}{2^N}} \sigma_t^2 \mathrm{d}t \big)(Z^2 - 1),$$

with Z a standard Gaussian variable. Now,

$$\mathbb{E}\big[\big(\int_{\frac{k}{2^j}+\frac{l+i}{2^N}}^{\frac{k}{2^j}+\frac{l+i+1}{2^N}} \sigma_t^2 \mathrm{d}t \big)^2 \big] \le c 2^{-2N}.$$

Then, by independence of the Brownian increments and because the variables are centered,

$$\tilde{\mathbb{E}}\big[\big(\frac{1}{h} \sum_{i=0}^{h} \{X_i - \tilde{\mathbb{E}}[X_i]\} \big)^2 \big] = \frac{1}{h^2} \sum_{i=0}^{h} \tilde{\mathbb{E}}[(X_i - \tilde{\mathbb{E}}X_i)^2] \le \frac{c}{h} 2^{-2N}.$$

For the other term, $\mathbb{E}\big[\big(\frac{1}{h} \sum_{i=0}^{h} \tilde{\mathbb{E}}X_i - \tilde{\mathbb{E}}X \big)^2 \big]$ is equal to

$$\begin{aligned} &\mathbb{E}\big[\big(\frac{1}{h} \sum_{i=0}^{h} \int_{\frac{k}{2^j}+\frac{l+i}{2^N}}^{\frac{k}{2^j}+\frac{l+i+1}{2^N}} \sigma_t^2 \mathrm{d}t - \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t^2 \mathrm{d}t \big)^2 \big] \\ = & \frac{1}{h^2} \sum_{i=0}^{h} \sum_{g=0}^{h} \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \mathbb{E}[(\sigma_{u+i2^{-N}}^2 - \sigma_u^2)(\sigma_{v+g2^{-N}}^2 - \sigma_v^2)] \mathrm{d}u\mathrm{d}v \\ \le & \frac{c}{h^2} \sum_{i=0}^{h} \sum_{g=0}^{h} \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} (ig)^H 2^{-2NH} \mathrm{d}u\mathrm{d}v \le c 2^{-2N(1+H)} h^{2H}. \end{aligned}$$

Eventually,

$$\mathbb{E}\big[\big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_u^2 \mathrm{d}u - \frac{1}{h}\sum_{i=0}^{h} X_i\big)^2\big] \le c\frac{n^{-2}}{\sqrt{n}}.$$

We easily check that the term

$$\mathbb{E}\big[\big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_u^2 \mathrm{d}u + \frac{1}{h}\sum_{i=0}^{h} X_i\big)^2\big]$$

is less than cn^{-2} and finally $\mathbb{E}[\,|\nu_{jk} - \bar{\nu}_{jk}|\,] \le cn^{-1}2^{-j/2}$, because $j \le \dfrac{\log_2 n}{2}$.

8 Proof of theorem I.3 (ii)

We now prove that proposition I.3 and I.4 together imply theorem I.3 (ii). Following lemma 1 of Gloter and Hoffmann [55], we easily obtain that for all positive ε, there exist n_0 and $M > 0$, such that for all $n \ge n_0$,

$$\mathbb{P}[n^{1/(4H+2)}|\widehat{H}_n - H| \ge M] \le \varepsilon. \tag{I.16}$$

With no loss of generality, we may demand $\widehat{H} \le C$, with $C > 2$ a constant value, by considering $\widetilde{H} = \widehat{H}\mathbb{1}_{|\widehat{H}|\le C}$. Let $\varepsilon > 0$, n_0, M associated by (I.16). For $n \ge n_0$, if $(C-1)n^{1/(4H+2)} > M$, we have

$$\mathbb{P}[\widehat{H}_n \ge C] \le \mathbb{P}[n^{1/(4H+2)}|\widehat{H}_n - H| \ge (C-1)n^{1/(4H+2)}] \le \varepsilon.$$

Let $n_0^* \ge n_0$ such that $(C-1)n_0^* \ge M$. For all $n \le n_0^*$,

$$n^{1/(4H+2)}|\widetilde{H}_n - H| \le (C+1)(n_0^*)^{1/(4H+2)}.$$

Let $M_1 = \max\{M, (C+1)(n_0^*)^{1/(4H+2)}\}$. For all n,

$$\mathbb{P}[n^{1/(4H+2)}|\widetilde{H}_n - H| \ge M_1] \le \varepsilon.$$

9 Proof of theorem 2

9.1 Proof of proposition I.5

We observe

$$\big\{Y_{i/n} = y_0 + \int_0^{i/n} \Phi\big(\int_0^s a(s,u)\mathrm{d}W_u^H + f(s)\xi_0\big)\mathrm{d}B_s, \quad i = 1, \dots, nT\big\}.$$

Without loss of generality, we set here $\xi_0 = 0$. Consider the equivalent sample

$$\{Z_{i/n} = Y_{i/n} - Y_{(i-1)/n},\quad i = 1,\dots,nT\}.$$

Conditional on $W^H = f$, $Z_{i/n}$ is a centered Gaussian variable with variance γ_i where

$$\gamma_i = \int_{(i-1)/n}^{i/n} \Phi^2\Big(\int_0^s a(s,u)\mathrm{d}f_u\Big)\mathrm{d}s.$$

Moreover, conditional on W^H, the observations are independent. We define by $K(\mu,\nu) = \int(\log\frac{\mathrm{d}\mu}{\mathrm{d}\nu})\mathrm{d}\mu \le +\infty$ the Kullback-Leibler divergence between two probability measures μ and ν. We recall the classical Pinsker's inequality $\|\mu-\nu\|_{TV} \le \sqrt{2}K(\mu,\nu)^{1/2}$. Let $\mathbb{P}_f^n$ be the law of the sample conditional on $W^H = f$, let

$$\beta_i = \int_{(i-1)/n}^{i/n} \Phi^2\Big(\int_0^s a(s,u)\mathrm{d}g_u\Big)\mathrm{d}s.$$

We have

$$\|\mathbb{P}_f^n - \mathbb{P}_g^n\|_{TV} \le \sqrt{2}K(\mathbb{P}_f^n,\mathbb{P}_g^n)^{1/2}.$$

By classical computations, we get

$$K(\mathbb{P}_f^n,\mathbb{P}_g^n) = \frac{1}{2}\sum_{i=1}^{nT}\big(-\log\frac{\gamma_i}{\beta_i} - 1 + \frac{\gamma_i}{\beta_i}\big).$$

By assumption C, we have $(c_4/c_5)^2 \le \gamma_i/\beta_i \le (c_5/c_4)^2$. Let $a = (c_4/c_5)^2$, $b = (c_5/c_4)^2$ and $c \ge 1/2$. Consider

$$z(x) = \log x - 1 + 1/x - c(x-1)^2,\ x \in [a,b].$$

We have $z(a) = \log a - 1 + 1/a - c(a-1)^2$, so, if $c \ge \frac{\log a - 1 + 1/a}{(a-1)^2}$, we have $z(a) \le 0$. Take

$$c = c^* = \max\big(\frac{1}{2}, \frac{\log a - 1 + 1/a}{(a-1)^2}\big).$$

Hence z is non positive on $[a,b]$, consequently, $K(\mathbb{P}_f^n,\mathbb{P}_g^n)$ is less than

$$\begin{aligned}
K(\mathbb{P}_f^n,\mathbb{P}_g^n) &\le c\sum_{i=1}^{nT}\big(\frac{\beta_i}{\gamma_i} - 1\big)^2\\
&\le cn^2\sum_{i=1}^{nT}\Big(\int_{(i-1)/n}^{i/n}\big|\Phi\big(\int_0^s a(s,u)\mathrm{d}f_u\big) - \Phi\big(\int_0^s a(s,u)\mathrm{d}g_u\big)\big|\mathrm{d}s\Big)^2\\
&\le cn\int_0^T\big|\int_0^s a(s,u)\mathrm{d}f_u - \int_0^s a(s,u)\mathrm{d}g_u\big|^2\mathrm{d}s\\
&\le cn\int_0^T\big|a(s,s)[f(s)-g(s)] + \int_0^s \partial_2 a(s,u)[g(u)-f(u)]\mathrm{d}u\big|^2\mathrm{d}s\\
&\le cn\|f-g\|_2^2.
\end{aligned}$$

Chapter 2

Numerical results

1 Introduction

Getting an accurate estimation of the volatility Hurst parameter with a small number of data is hopeless, this is the teaching of the lower bound proved in the preceding chapter. Nevertheless, the estimation rates remain polynomial, which is far better than in numerous statistical problem. For example, logarithmic rates of convergence are proved to be optimal in some inverse problems, see Cavalier *et al.* [24]. The rate $n^{-1/(4H+2)}$ can also be compared with the usual optimal rate in non parametric estimation $n^{-s/(1+2s)}$, where s denotes the smoothness of the estimated function. The two rates are equal for $s = 1/(4H)$ and so, roughly speaking, estimating the volatility Hurst parameter for $H \in (1/2, 1)$ is, in some sense, as difficult as estimating a function with regularity between 1/2 and 1/4. Hence, estimation procedures are conceivable as soon as we get a "reasonably big" number of data. For example, financial data can be available in large amounts. Moreover, the number of data can be increased using aggregation techniques between financial assets. Note that the fact that T is fixed and that we consider a "high frequency" asymptotic does not mean we can only consider $T = 1$ day. The value of T might be in the order of magnitude of years. Nevertheless, the sampling period has to be so that the discretized process lives at the "diffusive scale" (that is in general a sampling period bigger than 10 minutes), in order to avoid microstructure noise effects. In this section, we draw histograms of our estimator on simulated data and compute it on fixed income and equity data.

2 One numerical illustration

We present here some numerical results in the model

$$Y_t = \int_0^t \exp(W_s^H)\,\mathrm{d}B_s, \quad t \in [0,1], \tag{I.17}$$

where W^H is a fractional Brownian motion with Hurst parameter H, independent of the driving Brownian motion B. We observe

$$Y^n = \{Y_{i/n},\ i = 0, \ldots, n\}. \tag{I.18}$$

In the following, we draw histograms of the estimator for $H = 0.5$ and $n = 2^{20}$ and for $H = 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9$ and $n = 2^{16},\ 2^{10}$. We also graphically illustrate in section 3 the importance of the compensator.

2.1 Simulation method

First we simulate the points

$$\{W_{i/n}^H,\ i = 0, \ldots, n\},$$

of the sample path of a fractional Brownian motion using the exact method of Wood and Chan, see [111] for details. Then, from the given points of the sample path of the volatility process

$$\{\exp(W_{i/n}^H),\ i = 0, \ldots, n\},$$

the trajectory of the process is simulated by a Euler scheme. For computational restrictions, for $H \neq 1/2$, n is smaller than 2^{16} in our simulations of the fractional Brownian motion.

2.2 The case $H = 0.5$

Even if the case $H = 0.5$ is not, strictly speaking, included in our model, it is of course interesting to study the empirical behavior of our estimator for this value of the Hurst parameter. For $H = 0.5$, we are able to rapidly simulate a big number of data. Thus, we draw 10000 simulations of model (I.17) – (I.18) for $n = 2^{20}$ and $H = 0.5$. Note that in that case, $n^{-1/(4H+2)} = 0.03125$. We give the histogram of the estimator.

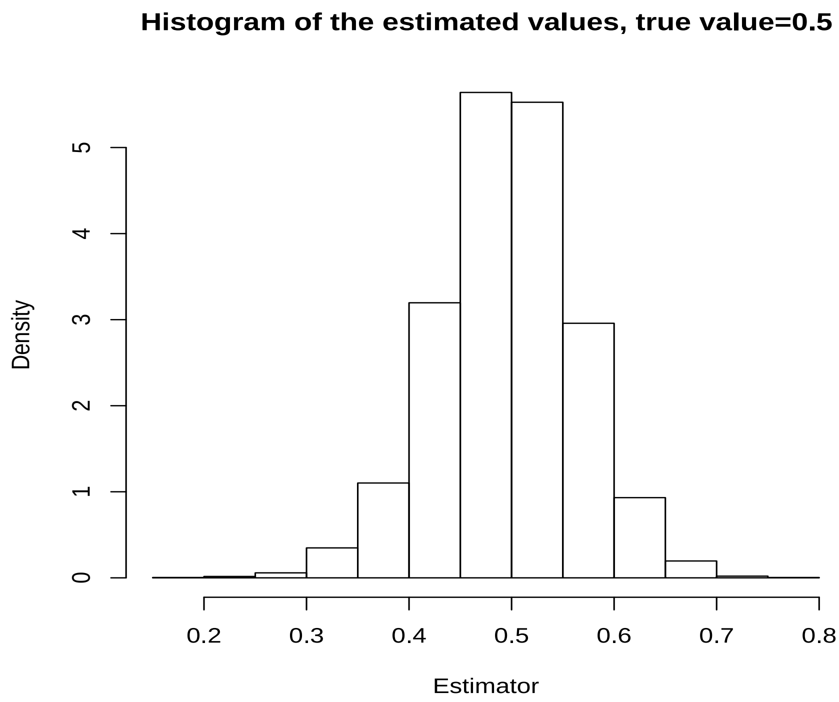

Figure I.1: *Histogram of the estimator from 10000 simulations of model* (I.17) – (I.18)*,* $n = 2^{20}$*,* $H = 0.5$*.*

Comments on figure I.1. We see that for a big number of data, we obtain a quite accurate estimation. Moreover, from this graph, we may think that a central limit theorem holds for our estimator.

We now consider the case where $n = 2^{16}$ ($n^{-1/(4H+2)} = 0.0625$). This number can be reasonable in the context of financial data. Take for example a sampling period of 10 minutes and financial data which are available in continuous. The value $n = 2^{16}$ corresponds to 455 days of data. From now, we truncate our estimator, that is we set the estimator to 0.5 if the estimated value is smaller than 0.5 and to 1 if it is bigger than 1. We obtain the following histogram for 10000 simulations of model (I.17) – (I.18) for $H = 0.5$ and $n = 2^{16}$.

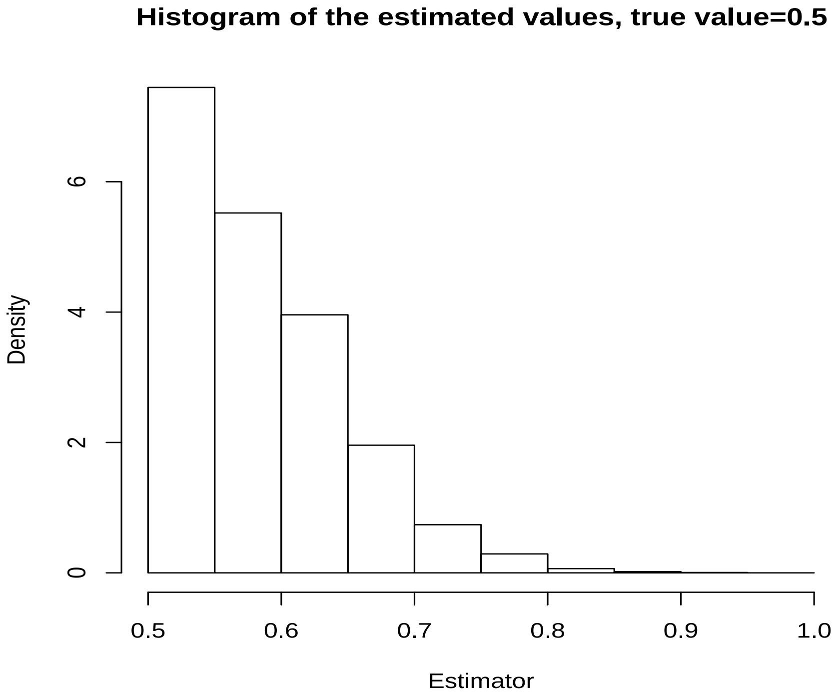

Figure I.2: *Histogram of the estimator from 10000 simulations of model* (I.17) – (I.18)*,* $n = 2^{16}$*,* $H = 0.5$*.*

Comments on figure I.2. Hence, for $H = 0.5$ and $n = 2^{16}$, the estimation remains quite sharp. In particular, one can build for model (I.17) – (I.18) a test of the following null hypothesis and alternative.

$$\mathcal{H}_0 \text{ (Diffusive volatility) } : H = 0.5, \quad \mathcal{H}_1 \text{ (Highly persistent volatility) } : H > 0.7.$$

We define the acceptance zone by $\{\widehat{H} < 0.7\}$. This test is convergent and for $n = 2^{16}$ it has an empirical level of 5%.

2.3 The case $H \in (0.5, 1)$

We now draw the same histograms for $n = 2^{16}$ and $H = 0.6, 0.7, 0.8, 0.9$. Note that $n^{-1/(4H+2)}$ is so respectively equal to $0.080, 0.099, 0.118$ and 0.138.

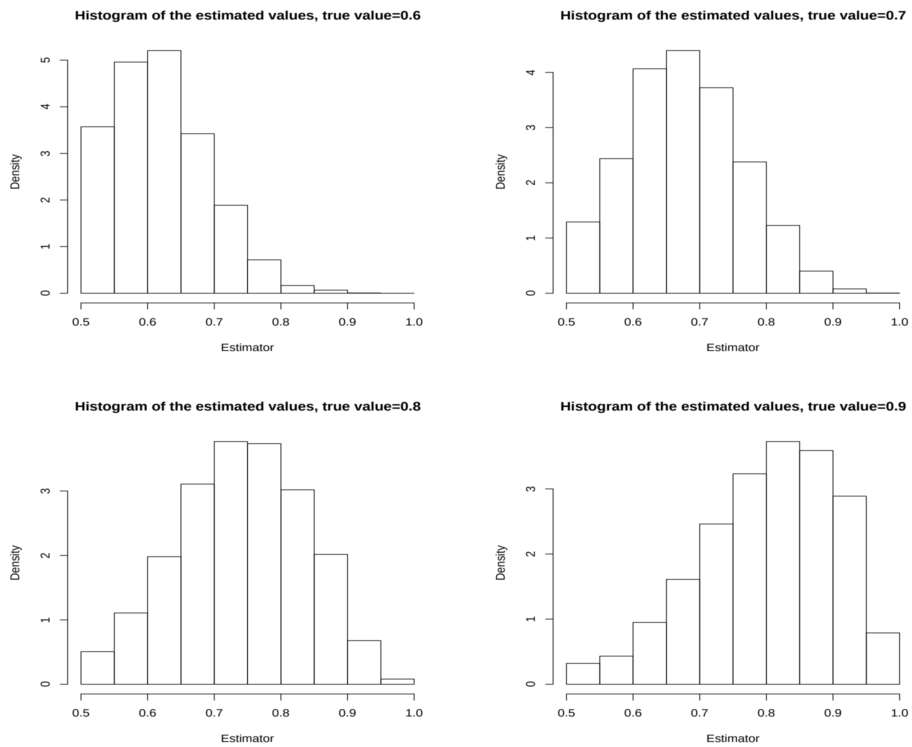

Figure I.3: *Histograms of the estimator from 10000 simulations of model* (I.17) – (I.18), $n = 2^{16}$ *and* $H = 0.6,\ 0.7,\ 0.8,\ 0.9$.

Comments on figure I.3. We clearly see on the simulations that the empirical distribution of the estimator is shifted to the right as H goes from 0.6 to 0.9. We also notice that the accuracy of the estimation is decreasing as H goes from 0.6 to 0.9. This is in agreement with our theoretical results. Considering the preceding test of hypothesis, we have the following empirical result for the power of the test.

$$\mathbb{P}_{H=0.8}[\widehat{H} \geq 0.7] = 0.67, \quad \mathbb{P}_{H=0.9}[\widehat{H} \geq 0.7] = 0.84.$$

3 The importance of the compensator

We present here the results obtained in the preceding simulations ($n = 2^{16}$ and $H = 0.6,\ 0.7,\ 0.8,\ 0.9$) without using any compensator (this means that in the construction of $\widehat{H}$, we use $\widetilde{d_{jk}}$ instead of $\widehat{d_{jk}}$ in equation (I.9)).

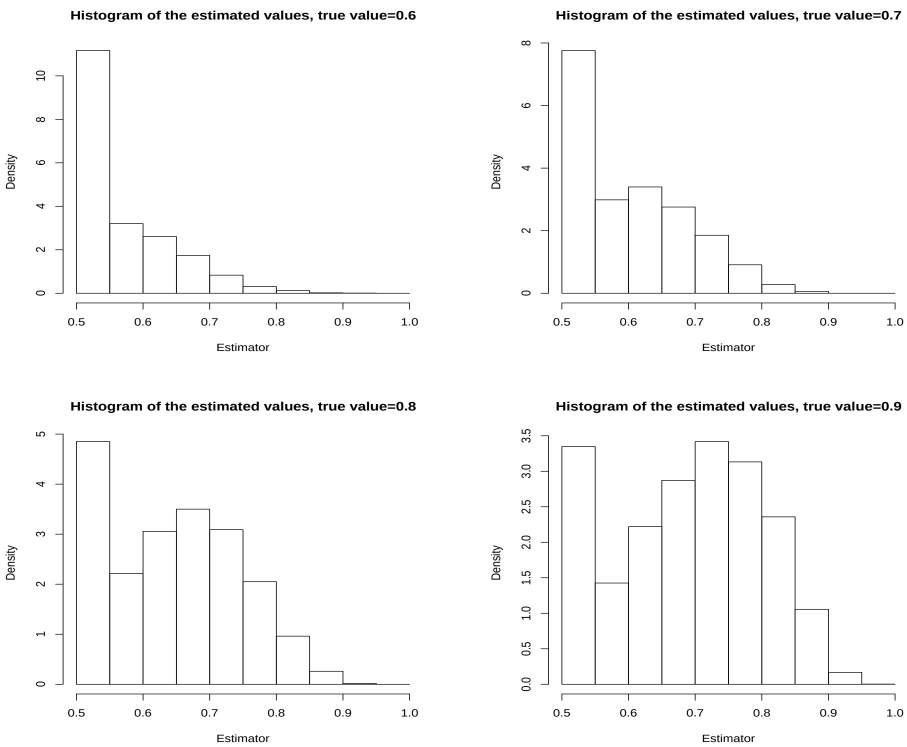

Figure I.4: *Histograms of the non compensated estimator from 10000 simulations of model* (I.17) – (I.18)*,* $n = 2^{16}$ *and* $H = 0.6,\ 0.7,\ 0.8,\ 0.9.$

Comments on figure I.4. The estimations do not make sense anymore. Indeed, without compensator, the volatility approximation is much more noisy and so appears less regular. That is why the histograms are shifted to the left.

4 Interest rate data

Interest rate data are very correlated. Thus, an aggregation procedure as those presented in section 5 appears quite dubious. Consequently, in this section, we use large data sets to be able to give some information on the regularity of the volatility. We work here on future contracts. We use the 30 Year US Treasury Bonds Futures, 10 Year US Treasury Notes Futures and 5 Year Treasury Notes Futures, from the Chicago Board of Trade market. Our data set goes from 2005-07-27 to 2007-08-24. We take the beginning of the day at 9 am and the end of the day at 19.00 local time. Data are stuck together from a day to another such that the last value of a day corresponds to the first value of the following day. These three assets are of course extremely correlated. This is shown by the following graphs of the three contracts, with a sampling period of one data per hour.

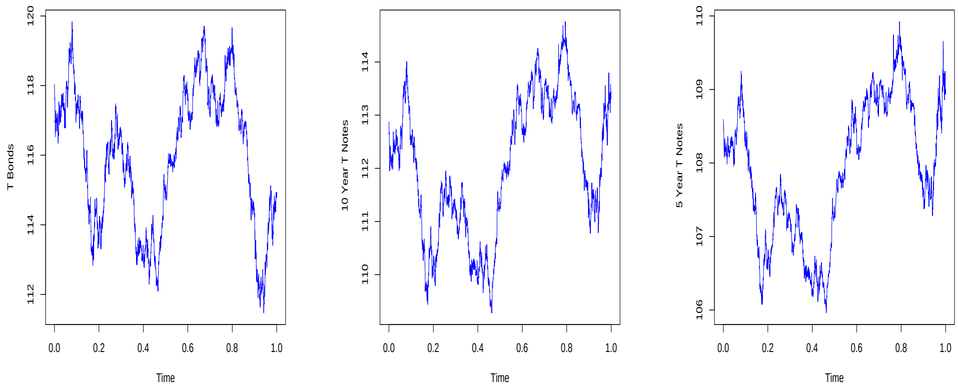

Figure I.5: *The 30 Year US Treasury Bonds Futures, 10 Year US Treasury Notes Futures and 5 Year Treasury Notes Futures from 2005-07-27 to 2007-08-24, sampling period of one data per hour.*

Comments on figure I.5. Although the assets are very close in term of general behavior, we will see that using high frequency data leads to different values of our estimator from an asset to another.

We also give results for the Bund future contract from the EUREX market, The Bund contract is a 10 Year future contract on German Bonds. Our data set goes from 2005-07-27 to 2007-08-24. Once again, we take the beginning of the day at 9 am and the end of the day at 19.00 local time and data are stuck together from a day to another such that the last value of a day corresponds to the first value of the following day. Below follows the graph of the contract, from 2005-07-27 to 2007-08-24, with a sampling period of one data per hour.

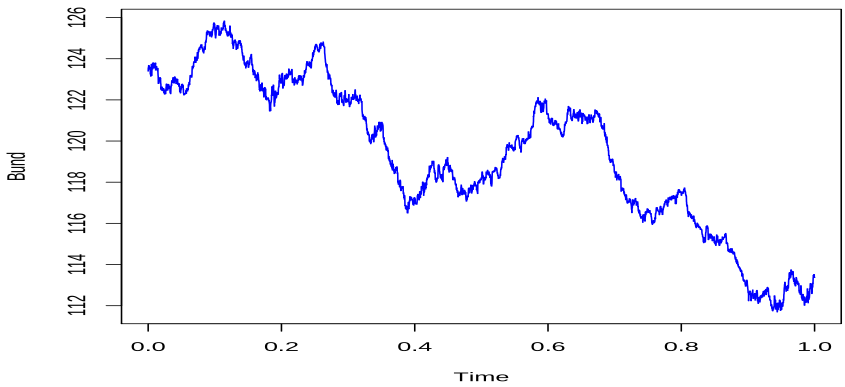

Figure I.6: *The Bund contract from 2005-07-27 to 2007-08-24, sampling period of one data per hour.*

Note that the Bund contract is the most traded future contract in Europe. For illustration, the following tabular gives the number of traded contracts of the different assets we consider for the month of May 2007.

Asset	Number of traded contracts
Bund	26,001,313
30 Year US Treasury Bonds Futures	10,578,521
10 Year US Treasury Notes Futures	34,343,898
5 Year US Treasury Notes Futures	15,673,429

Figure I.7: *Number of traded contracts of the different assets for the month of May 2007.*

We choose a sampling period of 20 minutes. We obtain the following results for the estimation of the volatility Hurst parameter on our data set and on the set of the logarithm of the data.

Asset	Estimation	Estimation on log data
Bund	0.48	0.50
30 Year US Treasury Bonds Futures	0.62	0.50
10 Year US Treasury Notes Futures	0.87	0.81
5 Year US Treasury Notes Futures	0.60	0.52

Figure I.8: *Estimation of the volatility Hurst parameter for the Bund, 30 Year US Treasury Bonds Futures, 10 Year US Treasury Notes Futures and 5 Year US Treasury Notes Futures, from 2005-07-27 to 2007-08-24, sampling frequency of 20 minutes.*

Comments on figure I.8. Based on these estimations, we can not reject the hypothesis of a diffusive behavior of the volatility ($H = 0.5$). The value obtained for the 10 Year US Treasury Notes Futures is quite surprising. It could be linked to the fact that it is the most liquid of the three US contracts we consider.

5 A suggestion of aggregation procedure

5.1 The technique

We work here with a small value of n. It is not possible to get any trustworthy estimation if n is too small. Nevertheless, if we get several independent set of size n, although the individual estimations do not make sense, some information could perhaps be read in the histogram obtained from the estimation of the volatility Hurst parameter. In particular, we could be able to say whether or not we can reject the hypothesis of a diffusive volatility and whether or not we can reject the hypothesis of a highly persistent volatility (big value of H).

5.2 Simulation

For each value of H, we compute 10000 simulations of model (I.17)−(I.18), with $n = 2^{10}$. We consider $H = 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9$ ($n^{-1/(4H+2)} = 0.177,\ 0.206,\ 0.236,\ 0.264,\ 0.290$). Note that for $n = 2^{10}$, the estimator can not always be computed, for example when a logarithm is applied to a negative value (this phenomenon almost never occurs for $n \geq 2^{16}$). The percentage of impossible computations in our simulations is given in the following tabular.

H	0.5	0.6	0.7	0.8	0.9
Percentage of impossible computation	18%	17%	19%	22%	26%

Figure I.9: *Percentage of impossible computation of the estimator in model* (I.17) − (I.18)*, 10000 simulations.*

We now give the histograms of the computed values.

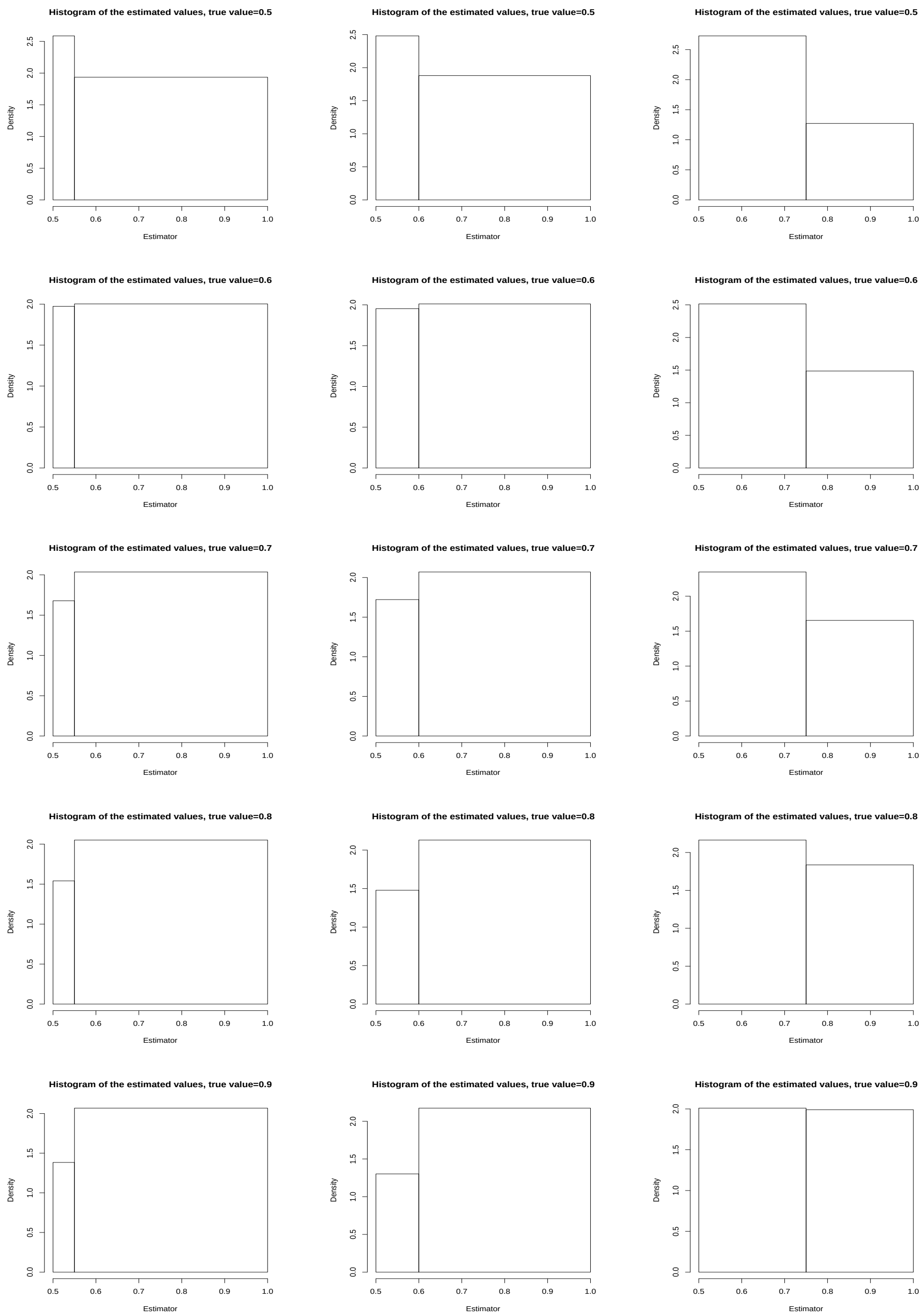

Figure I.10: *Histograms of the estimator from 10000 simulations of model* (I.17)−(I.18), $n = 2^{10}$ *and* $H = 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9$, *break point*=$0.55,\ 0.6,\ 0.75$.

Comments on figure I.10. Hence, from a sufficient number of independent sample of model (I.17)−(I.18) with $n = 2^{10}$, we can clearly distinguish between the hypothesis of a diffusive volatility and the hypothesis of a highly persistent volatility. For that, one can for example draw the histograms of the estimator with two classes and break point 0.55.

To be able to compare with the results obtained on equity data in next section, we compute the same histograms with only 20 simulations.

H	0.5	0.6	0.7	0.8	0.9
Percentage of impossible computation	15%	20%	20%	25%	35%

Figure I.11: *Percentage of impossible computation of the estimator in model* (I.17) − (I.18), *20 simulations.*

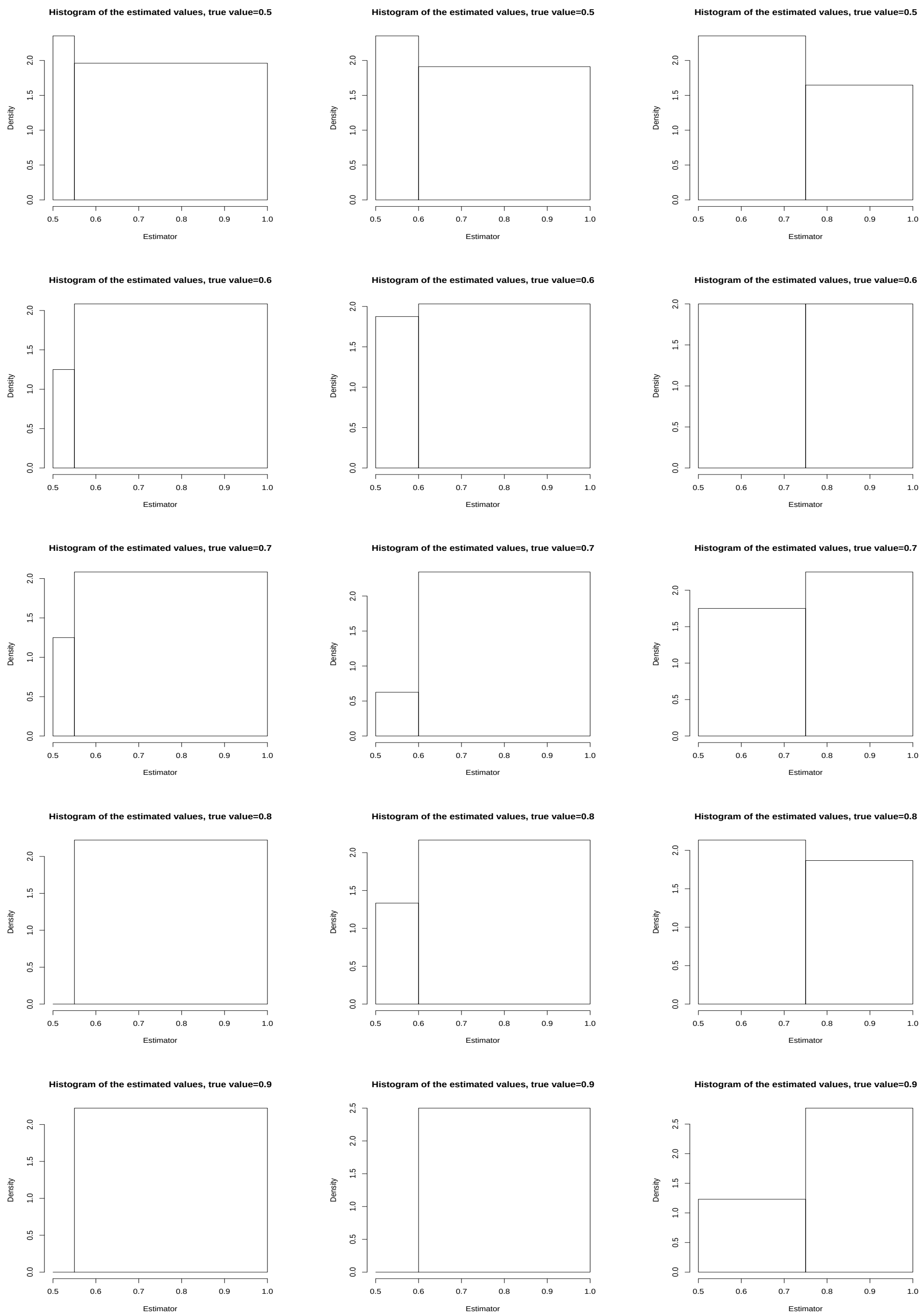

Figure I.12: *Histograms of the estimator from 20 simulations of model* (I.17) − (I.18), $n = 2^{10}$ *and* $H = 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9$*, break point*$=0.55,\ 0.6,\ 0.75.$

Comments on figure I.12. These graphs show that even if the number of data is extremely small for our purpose and if the number of samples is also quite small, using the histograms, we, roughly, may be able to distinguish between the hypothesis of diffusive volatility and the hypothesis of highly persistent volatility. Naturally, such a result is to be taken with caution.

5.3 Application: equity data

We consider 20 assets of the SBF 120 index, traded on Euronext, from 2007-02-12 to 2007-02-23. These assets are from several sectors of activity. Thus, they are quite independent but we suppose that their volatility has the same regularity. We take the beginning of the day at 9.30 am and the end of the day at 17.00 Paris time (the whole trading days starts at 9.00 am and ends at 17.40 Paris time). For each asset, the daily data are stuck together such that the last value of a day corresponds to the first value of the following day. We use a quite small subsampling in order to get 2^{10} data per asset (263 seconds). We compute our estimator on these data sets. We obtain the following results, where the value "NA" signifies that the estimation was not possible.

Asset	Estimation	Estimation on log data
Alcatel-Lucent	0.66	0.69
AXA	0.64	0.63
BNP-Paribas	0.64	0.65
CFF	0.59	0.59
Crédit Agricole	0.81	0.80
Danone	NA	NA
Essilor	0.80	0.80
Eurazeo	0.79	0.78
GDF	NA	NA
Icade	0.65	0.59
Imerys	0.63	0.62
LVMH	0.76	0.75
Renault	NA	NA
Safran	0.67	0.68
Sanofi-Aventis	NA	NA
Steria	1.00	NA
Total	NA	NA
Valeo	0.71	0.59
Veolia environnement	0.58	0.61
Wendel	0.92	0.88

Figure I.13: *Estimation of the volatility Hurst parameter on the equity data set.*

The associated histograms of the estimator follows.

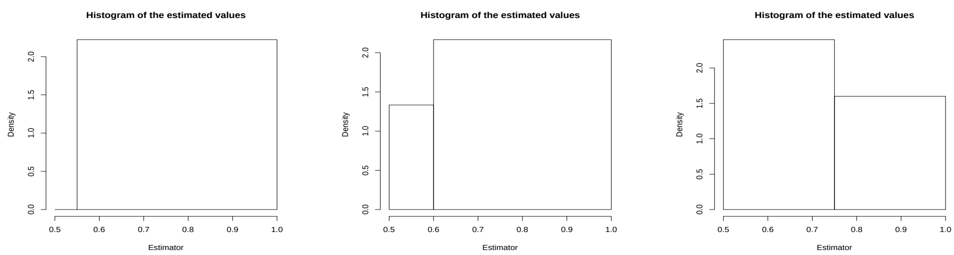

Figure I.14: *Histograms of the estimator on the equity data set.*

Comments on figure I.13 and figure I.14. Individually, these results do not make any sense. However, they seem to indicate that based on these assets, the “global” regularity of the volatility for the period 2007-02-12 to 2007-02-23 on the French equity market was significantly bigger than 0.5 and that we can not reject the hypothesis of a highly persistent volatility. For example, no estimation is under 0.55. Of course, one would need a basket with a larger number of assets to give more accurate conclusions.

Part II

Integrated volatility and rounding error

Abstract

We consider a microstructure model for a financial asset, allowing for prices discreteness and for a diffusive behavior at large sampling scale. This model, introduced by Delattre and Jacod, consists in the observation at the high frequency n, with round-off error α_n, of a diffusion on a finite interval. We give from this sample estimators of the absolute and relative integrated volatilities of the asset. Our method is based on variational properties of the process. We prove the accuracy of our estimation procedures is $\alpha_n \vee n^{-1/2}$. Using compensated estimators, limit theorems are obtained. Numerical results on both simulated and financial data sets are also given.

Keywords: Diffusion models; Integrated volatility; High frequency data; Rounding error; Microstructure noise; Variation methods.

Note

The first chapter of this part is based on a paper submitted to *Bernoulli*. It treats the case of an homogeneous diffusion coefficient. The second chapter extends the results of the first one to the case of an inhomogeneous diffusion coefficient. A numerical study of the estimators, both on simulated and financial data sets, is presented in the third chapter. I am grateful to Charles-Albert Lehalle from Crédit Agricole Cheuvreux, Groupe CALYON for providing and discussing the data.

Chapter 1

Homogeneous case

1 Introduction

Nowadays, a massive amount of high frequency financial data is available. This large quantity of data has paradoxically complicated some problems in statistical finance. Among them, one of the most relevant is the estimation of the integrated volatility of an asset. To fix ideas, let us consider for $t \in [0, 1]$ an Ito process of the form

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t,$$

where W_t is a Brownian motion, μ_t the drift process and σ_t^2 the instantaneous volatility. We wish to estimate the absolute integrated volatility

$$\int_0^1 \sigma_t^2 \mathrm{d}t$$

and the relative integrated volatility[1]

$$\int_0^1 X_t^{-2} \sigma_t^2 \mathrm{d}t.$$

Assume first that we observe the data with frequency n, that is the sample

$$(X_{i/n}, i = 0, \dots, n).$$

In this case, a common convergent estimator of the integrated volatility, with rate $n^{-1/2}$ and feasible asymptotic theory, is given by the realized volatility, that is for the absolute integrated volatility

$$\sum_{i=1}^{n} (X_{i/n} - X_{(i-1)/n})^2$$

[1]Note that the usual notion of integrated volatility refers to the relative integrated volatility.

and for the relative integrated volatility

$$\sum_{i=1}^{n} \big(\log(X_{i/n}) - \log(X_{(i-1)/n}) \big)^2,$$

see Jacod and Protter [74], Barndorff-Nielsen and Shephard [16], see also Meddahi [94], Gonçalves and Meddahi [57].

However, it is a well known fact that high frequency financial data do not behave like an Ito process. In the literature, this gap is often considered to be a "contamination" of the true, theoretical price and is called microstructure noise. This microstructure noise increases with the sampling frequency and is due to several reasons, one of the most obvious being prices discreteness.

The first solution to get rid of this noise is to sample our data at larger period. But, if we imagine we get a data every second and that we consider five minute as the finest period we can tolerate to make the noise insignificant, we throw away a lot of data, what is hardly acceptable. Consequently, dealing with these high frequency noisy data has become a challenging issue. Many recent papers treat this problem, especially in the purpose of estimating the integrated volatility, see in particular Zhang [113], Zhang, Mykland, and Aït-Sahalia [114], Hansen and Lunde [58], Bandi and Russell [12], Aït-Sahalia, Mykland and Zhang [4], Gloter and Jacod [56]. For a comparison between several estimators, see Andersen, Bollerslev and Meddahi [10], Bandi, Russel and Yang [13], Gatheral and Oomen [46].

In most of these works, we observe at time i/n, $i = 0, \dots, n$, a log-price $\tilde{Y}_{i/n}$ composed of a true, theoretical log-price $\tilde{X}_{i/n}$, coming from the classical continuous time financial theory, contaminated by an additive microstructure noise $\varepsilon^n_{i/n}$, that is

$$\tilde{Y}_{i/n} = \tilde{X}_{i/n} + \varepsilon^n_{i/n},$$

where $\tilde{X}_t$ is for example an Ito process. In these additive microstructure noise models, the developed technologies often aims at reducing the impact of the noise.

Nevertheless, although prices discreteness is largely accepted as one of the main reasons for microstructure noise, these models rarely allow for it, see Large [83] for a model considering discrete prices. In this chapter, we study the problem of the estimation of the absolute and relative integrated volatilities in a model of diffusion with round-off error. Indeed, this model allows both for prices discreteness and for a usual diffusive behavior at large sampling period.

2 Model and Results

2.1 Description of the model

We consider the model of a diffusion observed with round-off error. Let α_n be a positive sequence tending to zero as n goes to infinity and $\beta_n = \alpha_n\sqrt{n}$. On a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$, we consider a one-dimensional Brownian semimartingale $(X_t)_{t\in[0,1]}$, taking values in an open interval (ν, μ), $-\infty \leq \nu < \mu \leq +\infty$, of the form

$$X_t = x_0 + \int_0^t \sigma(X_s)\mathrm{d}W_s + \int_0^t a_s \mathrm{d}s, \tag{II.1}$$

where $(W_t)_{t\in[0,1]}$ is a $(\mathcal{F}_t)$-standard Brownian motion, $(a_t)_{t\in[0,1]}$ a progressively measurable process with respect to $(\mathcal{F}_t)_{t\in[0,1]}$, $x \to \sigma(x)$ a real deterministic function and x_0 a real constant. We observe the sample

$$(X_{i/n}^{(\alpha_n)}, i = 0, \dots, n), \tag{II.2}$$

where

$$X_{i/n}^{(\alpha_n)} = \alpha_n \lfloor X_{i/n}/\alpha_n \rfloor.$$

Thus, $X_{i/n}^{(\alpha_n)}$ is the observation of $X_{i/n}$ with round-off error α_n. This model has already been studied by Delattre and Jacod [36] when β_n tends to a constant finite value and by Delattre [35] in the other cases. Based on the sample (II.2), our goal is to estimate the random parameters

$$\theta = \int_0^1 \sigma^2(X_s)\mathrm{d}s$$

and

$$\lambda = \int_0^1 X_s^{-2}\sigma^2(X_s)\mathrm{d}s.$$

Note that for the Black-Scholes specification of the model

$$\sigma(X_s) = \sigma X_s,$$

the problem of the estimation of the constant parameter λ has been partially treated by Li and Mykland [85] in the case where β_n tends to zero. We denote by $\mathcal{C}^k(I)$ the set of k times continuously differentiable functions on $I \subseteq \mathbb{R}$. We write $\mathcal{C}_b^k(I)$ if all the derivatives are bounded. We will consider the following assumptions.

Assumption A.

$$\sup_{n\geq 0}\alpha_n(\log n)^2<\infty.$$

Assumption A1.
There exists $\rho>0$ such that $\sup_{n\geq 0}\alpha_n^{1-\rho}(\log n)^2<\infty$.

Assumption B.

(i) For all $x\in(\nu,\mu)$, $\sigma(x)>0$,

(ii) $x\to\sigma(x)\in\mathcal{C}^2\big((\nu,\mu)\big)$,

(iii) $\int_0^1 a_s^2\mathrm{d}s<+\infty$, almost surely.

2.2 First estimators

Our estimation method is based on the theory of wavelet methods for quadratic functionals estimation, see for example Gayraud and Tribouley [47]. Throughout the chapter, for $k\in\mathbb{N}$ and $j\in\mathbb{N}$, we set

$$\mathbb{1}_{jk}(s)=\mathbb{1}_{(\frac{k}{2^j},\frac{k+1}{2^j}]}(s),\ \psi(s)=-\mathbb{1}_{[0,1/2]}(s)+\mathbb{1}_{(1/2,1]}(s),\ \psi_{jk}(s)=2^{j/2}\psi(2^js-k).$$

We define the coefficients c_{j_0k}, e_{j_0k}, $j_0\in\mathbb{N}$, $k\in[0,2^{j_0}-1]$ and d_{jk}, f_{jk} $j\in\mathbb{N}$, $k\in[0,2^j-1]$ by

$$c_{j_0k}=2^{j_0/2}\int\mathbb{1}_{j_0k}(s)\sigma(X_s)\mathrm{d}s,\ d_{jk}=\int\psi_{jk}(s)\sigma(X_s)\mathrm{d}s,$$

$$e_{j_0k}=2^{j_0/2}\int\mathbb{1}_{j_0k}(s)X_s^{-1}\sigma(X_s)\mathrm{d}s,\ f_{jk}=\int\psi_{jk}(s)X_s^{-1}\sigma(X_s)\mathrm{d}s.$$

Hence, the c_{j_0k} and d_{jk} are the coefficients of $s\to\sigma(X_s)$ in the Haar basis. The e_{j_0k} and f_{jk} are the coefficients of $s\to X_s^{-1}\sigma(X_s)$. Consequently, we have[2]

$$\theta=\sum_{k=0}^{2^{j_0}-1}c_{j_0k}^2+\sum_{j=j_0}^{+\infty}\sum_{k=0}^{2^j-1}d_{jk}^2,\quad\lambda=\sum_{k=0}^{2^{j_0}-1}e_{j_0k}^2+\sum_{j=j_0}^{+\infty}\sum_{k=0}^{2^j-1}f_{jk}^2.$$

With the following convention, remaining in force throughout the chapter

$$1/X_{(i-1)/n}^{(\alpha_n)}=\alpha_n^{-1}\text{ if }X_{(i-1)/n}^{(\alpha_n)}=0,$$

[2] We will see in the following that we can suppose that almost surely the functions $s\to\sigma(X_s)$ and $s\to X_s^{-1}\sigma(X_s)$ belong to $\mathbb{L}^2[0,1]$.

we set

$$\widehat{c}_{j_0k} = \sqrt{\frac{\pi}{2}} \frac{2^{j_0/2}}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{1}_{j_0k}(i/n) |X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|$$

and

$$\widehat{e}_{j_0k} = \sqrt{\frac{\pi}{2}} \frac{2^{j_0/2}}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{1}_{j_0k}(i/n) \frac{|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|}{X_{(i-1)/n}^{(\alpha_n)}}.$$

Hence, $\widehat{c}_{j_0k}$ can be seen as a rescaled local average of the increments of the rounded diffusion in a window of size 2^{-j_0}. We define our first estimators $\widetilde{\theta}_n$ and $\widetilde{\lambda}_n$ of θ and λ by

$$\widetilde{\theta}_n = \sum_{k=0}^{2^{j_0(n)}-1} \widehat{c}_{j_0(n)k}^{\,2}, \quad \widetilde{\lambda}_n = \sum_{k=0}^{2^{j_0(n)}-1} \widehat{e}_{j_0(n)k}^{\,2}$$

with $j_0(n) = \lfloor \log_2(\alpha_n^{-1} \wedge \sqrt{n}) \rfloor$.

2.3 Convergence in probability

We set $r_n = \alpha_n \vee n^{-1/2}$. We have the following theorems.

Theorem II.1 *(Absolute integrated volatility). In model* (II.1)-(II.2)*, under assumptions A and B, the sequence*

$$r_n^{-1}(\widetilde{\theta}_n - \theta)$$

is tight.

Theorem II.2 *(Relative integrated volatility). In model* (II.1)-(II.2)*, under assumptions A and B, if* $\nu \geq 0$*, the sequence*

$$r_n^{-1}(\widetilde{\lambda}_n - \lambda)$$

is tight.

2.4 Compensated estimators

It is probably not possible to get some central limit theorems with the previous estimators (see the proofs for details). We introduce compensated estimators. We set

$$Q_j = \sum_{k=0}^{2^j-1} d_{jk}^2, \quad Q_j^{'} = \sum_{k=0}^{2^j-1} f_{jk}^2.$$

We define

$$\widehat{d}_{jk} = \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{jk}(i/n) |X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|$$

and

$$\widehat{f}_{jk} = \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{jk}(i/n) \frac{|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|}{X_{(i-1)/n}^{(\alpha_n)}}.$$

We denote by $\mathcal{S}$ the set of all triples $\big(a, (j_{1,n}), (j_{2,n})\big)$ where a is a real number with $0 < a < 1$ and $(j_{1,n})$, $(j_{2,n})$ are two sequences of integers such that

$$\sup_n \alpha_n^{1-a} (\log n)^2 < \infty,\ r_n 2^{2j_{2,n} - j_{1,n}} \to 0,\ r_n^{-1} 2^{j_{1,n}/2} (\alpha_n^2 \log n + 1/n) \to 0,$$
$$r_n 2^{j_{1,n}} \to 0,\ r_n^{-1} 2^{-3j_{1,n}/2} \to 0,\ 2^{j_{2,n} - j_{1,n}} \to 0,\ r_n^{-1} 2^{-(j_{1,n} + j_{2,n}/2)} \to 0.$$

Under assumption A1, the set $\mathcal{S}$ is not empty. For example, if we take $j_{1,n} = \lfloor \log_2(r_n^{-3/4}) \rfloor$ and $j_{2,n} = \lfloor \log_2(r_n^{-2/3}) \rfloor$, then $\big(\rho, (j_{1,n}), (j_{2,n})\big) \in \mathcal{S}$. For $S = \big(a, (j_{1,n}), (j_{2,n})\big) \in \mathcal{S}$, we set

$$\widehat{Q}_{j_{2,n}} = \sum_k \widehat{d}_{j_{2,n}k}^{\,2},\quad \widehat{Q}_{j_{2,n}}^{'} = \sum_k \widehat{f}_{j_{2,n}k}^{\,2}$$

and we consider

$$R_n(S) = \sum_{j=j_{1,n}}^{\lfloor (1+a)\log_2 r_n^{-1} \rfloor} 2^{j_{2,n} - j} \widehat{Q}_{j_{2,n}},\quad R_n^{'}(S) = \sum_{j=j_{1,n}}^{\lfloor (1+a)\log_2 r_n^{-1} \rfloor} 2^{j_{2,n} - j} \widehat{Q}_{j_{2,n}}^{'}.$$

For the absolute integrated volatility, for $S = \big(a, (j_{1,n}), (j_{2,n})\big) \in \mathcal{S}$, our final estimator of θ is

$$\widehat{\theta}_n(S) = \sum_{k=0}^{2^{j_1(n)}-1} \widehat{c}_{j_{1,n}k}^{\,2} + R_n(S).$$

We also define

$$\overline{\lambda}_n(S) = \sum_{k=0}^{2^{j_1(n)}-1} \widehat{e}_{j_{1,n}k}^{\,2} + R_n^{'}(S).$$

We finally set for estimating the relative integrated volatility λ

$$\widehat{\lambda}_n(S) = \overline{\lambda}_n(S)(1 - \alpha_n).$$

2.5 Convergence in law

We state in this section some limit theorems. In our context, it is convenient to use the notion of *stable convergence in law*, see Rényi [100], Aldous and Eagleson [6], Jacod and Shiryaev [75], Jacod [71].

Definition II.1 *A sequence of variable $(X_n)_{n\in\mathbb{N}}$ converges stably in law to a variable X ($X_n \to_{\mathcal{L}s} X$) if X is defined on an appropriate extension $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P})$ and if for any $\mathcal{F}$-measurable bounded variable Y and any bounded continuous function g, $\mathbb{E}[Yg(X_n)] \to \bar{\mathbb{E}}[Yg(X)]$.*

For $\beta > 0$, we define the function Δ_β by

$$\Delta_\beta(x) = \lim_n \mathbb{E}[n^{-1/2}(\sum_{i=1}^{n} Z_i)^2],$$

with

$$Z_i = \beta(\pi/2)^{1/2}|\lfloor\{U + \beta^{-1}\sigma(x)W_{i-1}\} + \beta^{-1}\sigma(x)(W_i - W_{i-1})\rfloor| - \sigma(x),$$

where W is a Brownian motion and U a uniform random variable on $[0,1]$, independent of W. From Delattre [35], we get that the function Δ_β is well defined. We have the following theorems.

Theorem II.3 *(Convergence in law). In model* (II.1)-(II.2)*, under assumptions A1 and B, for $S \in \mathcal{S}$, we have the following stable convergences in law, where B is a standard Brownian motion, independent of $\mathcal{F}$.*

$$\text{if } \beta_n \to 0, \qquad r_n^{-1}(\widehat{\theta}_n(S) - \theta) \to_{\mathcal{L}s} \sqrt{2}(\pi-2)^{1/2}\int_0^1 \sigma(X_t)^2 \mathrm{d}B_t,$$

$$\text{if } \beta_n \to \beta > 0, \qquad r_n^{-1}(\widehat{\theta}_n(S) - \theta) \to_{\mathcal{L}s} 2\int_0^1 \sigma(X_t)[\Delta_\beta(X_t)]^{1/2} \mathrm{d}B_t,$$

$$\text{if } \beta_n \to +\infty, \qquad r_n^{-1}(\widehat{\theta}_n(S) - \theta) \to_{\mathcal{L}s} \frac{2}{\sqrt{3}}\int_0^1 \sigma(X_t) \mathrm{d}B_t.$$

Let $\mathcal{N}$ denote a standard Gaussian variable. From theorem II.3, we deduce the following corollary.

Corollary II.1 *In model* (II.1)-(II.2)*, under assumptions A1 and B, for $S \in \mathcal{S}$, we have the following convergences in law (in the classical sense).*

$$\text{if } \beta_n \to 0, \qquad \frac{1}{\sqrt{2}}(\pi-2)^{-1/2}\Big[\int_0^1 \sigma(X_t)^4 \mathrm{d}t\Big]^{-1/2} r_n^{-1}(\widehat{\theta}_n(S) - \theta) \to_{\mathcal{L}} \mathcal{N},$$

$$\text{if } \beta_n \to \beta > 0, \qquad \frac{1}{2}\Big[\int_0^1 \sigma(X_t)^2 \Delta_\beta(X_t) \mathrm{d}t\Big]^{-1/2} r_n^{-1}(\widehat{\theta}_n(S) - \theta) \to_{\mathcal{L}} \mathcal{N},$$

$$\text{if } \beta_n \to +\infty, \qquad \frac{\sqrt{3}}{2}\Big[\int_0^1 \sigma(X_t)^2 \mathrm{d}t\Big]^{-1/2} r_n^{-1}(\widehat{\theta}_n(S) - \theta) \to_{\mathcal{L}} \mathcal{N}.$$

Theorem II.4 *(Convergence in law). In model* (II.1)-(II.2)*, under assumptions A1 and B, if $\nu \geq 0$, for $S \in \mathcal{S}$, we have the following stable convergences in law, where B*

is a standard Brownian motion, independent of $\mathcal{F}$.

$$\text{if } \beta_n \to 0, \qquad r_n^{-1}(\widehat{\lambda}_n(S) - \lambda) \to_{\mathcal{L}s} \sqrt{2}(\pi - 2)^{1/2} \int_0^1 X_t^{-1}\sigma(X_t)^2 \mathrm{d}B_t,$$

$$\text{if } \beta_n \to \beta > 0, \qquad r_n^{-1}(\widehat{\lambda}_n(S) - \lambda) \to_{\mathcal{L}s} 2 \int_0^1 X_t^{-1}\sigma(X_t)[\Delta_\beta(X_t)]^{1/2} \mathrm{d}B_t,$$

$$\text{if } \beta_n \to +\infty, \qquad r_n^{-1}(\widehat{\lambda}_n(S) - \lambda) \to_{\mathcal{L}s} \frac{2}{\sqrt{3}} \int_0^1 X_t^{-1}\sigma(X_t) \mathrm{d}B_t.$$

Corollary II.2 *In model* (II.1)-(II.2)*, under assumptions A1 and B, if $\nu \geq 0$, for $S \in \mathcal{S}$, we have the following convergences in law (in the classical sense).*

$$\text{if } \beta_n \to 0, \qquad \frac{1}{\sqrt{2}}(\pi - 2)^{-1/2}\Big[\int_0^1 X_t^{-2}\sigma(X_t)^4 \mathrm{d}t\Big]^{-1/2} r_n^{-1}(\widehat{\lambda}_n(S) - \lambda) \to_{\mathcal{L}} \mathcal{N},$$

$$\text{if } \beta_n \to \beta > 0, \qquad \frac{1}{2}\Big[\int_0^1 X_t^{-2}\sigma(X_t)^2 \Delta_\beta(X_t) \mathrm{d}t\Big]^{-1/2} r_n^{-1}(\widehat{\lambda}_n(S) - \lambda) \to_{\mathcal{L}} \mathcal{N},$$

$$\text{if } \beta_n \to +\infty, \qquad \frac{\sqrt{3}}{2}\Big[\int_0^1 X_t^{-2}\sigma(X_t)^2 \mathrm{d}t\Big]^{-1/2} r_n^{-1}(\widehat{\lambda}_n(S) - \lambda) \to_{\mathcal{L}} \mathcal{N}.$$

3 Discussion

3.1 Comments on the results

• The model is obviously built to face the problem of prices discreteness. Indeed, market prices increments have to be multiples of the tick size. It is also striking to see how high frequency financial data[3] do look like diffusions with round-off error, see the following figures of the Bund contract and of rounded diffusions.

[3] In the integrated volatility literature, authors mostly used mid-quote prices. Note that if one assume that the theoretical price lies between the bid price and the ask price, and that the bid-ask spread is constant equal to one tick, then the bid price is almost surely the right measure of the rounded theoretical price.

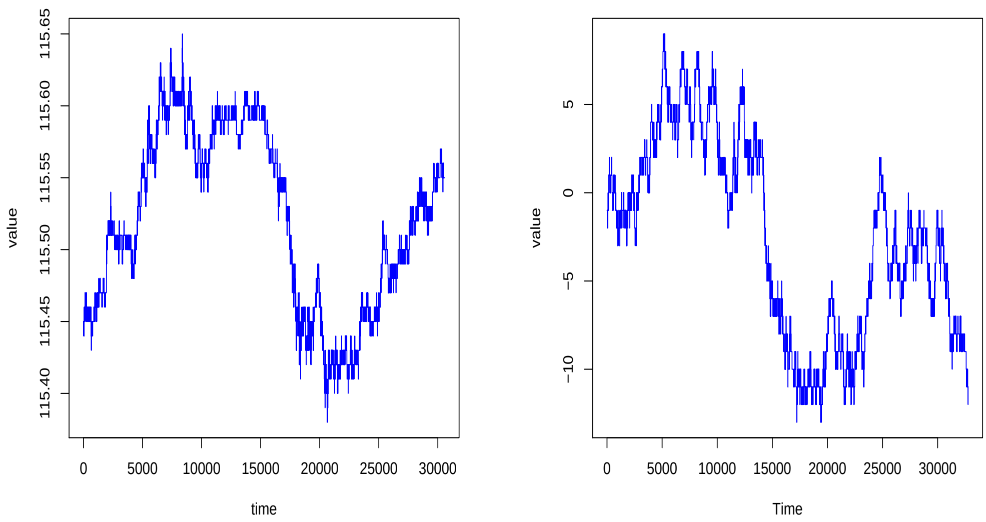

Figure II.1: *Bund contract, 2007-05-06, one data every second (left),* $\lfloor 20W_t \rfloor$, *frequency* $= 2^{15}$ *on [0,1](right).*

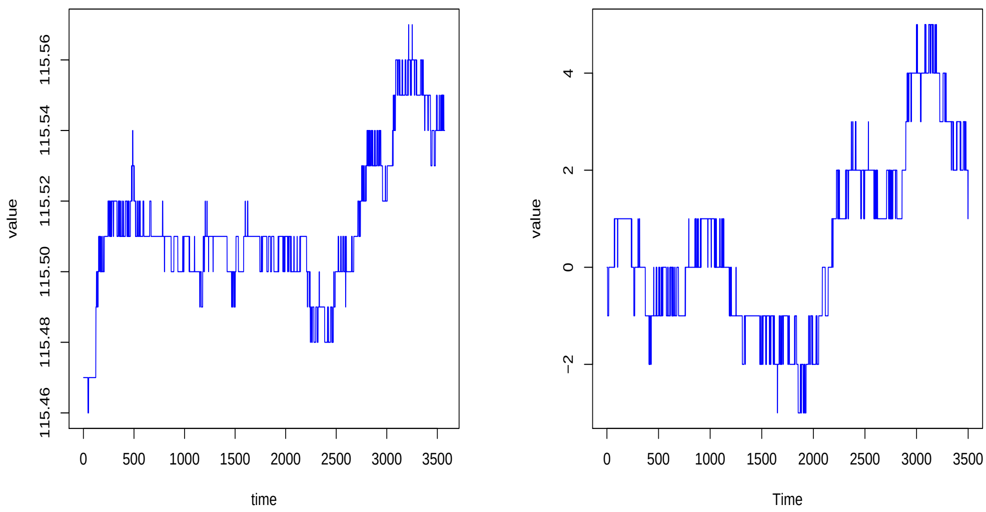

Figure II.2: *Bund contract, 2007-05-06, from 10 am to 11 am, Paris time, one data every second (left),* $\lfloor 20W_t \rfloor$, *frequency* $= 2^{15}$ *on [0,0.1] (right).*

Moreover, in our model, if the sampling period becomes big, the round-off error becomes insignificant. According to the theory and the empirical studies, this is also the case on the markets, indeed low frequency financial data can often appear like data coming from a diffusion process. Hence, this model is relevant because it is clearly linked with market observations and financial theory.

• Our point of view is different from those of an additive microstructure noise. We do not make assumptions on the difference between the observed log-price and the theoretical log-price but on the observed price itself. Hence, our method is not a denoising method, we directly use the properties of the noisy data. Moreover, Li and Mykland [86] have proved that estimators built for additive noise, like the two scales estimator of Zhang, Mykland and Aït-Sahalia [114], are not robust in the case of a "quite big" rounding error.

• The estimation rate is the same as those obtained by Delattre [35] for other procedures on this model. In particular, if the round-off error is smaller than $n^{-1/2}$, we find the classical parametric rate. Compare our accuracy with the rate $n^{-1/4}$ obtained by Zhang in an additive context.

• Under assumptions, theorem II.1 and theorem II.2 remain true for an inhomogeneous diffusion. The proofs being slightly more technical in this context, this case is described in next chapter.

3.2 Intuition for the results and important ideas

To give some intuition for the result, important ideas used in the proofs and to explain why methods based on the quadratic variation do not work here, we recall and explain an inspiring result of Delattre [35] when β_n tends to infinity.

3.2.1 The behavior of the p-variations

Let h be the density of a standard Gaussian variable and

$$\gamma_p(\sigma,\beta)=\int_0^1 \mathrm{d}u\int_{\mathbb{R}}\mathrm{d}y h(y)|(\beta u+\sigma y)^{(\beta)}|^p.$$

Delattre has shown in [35] that if β_n tends to infinity, that is if the round-off error is quite big, for $p>0$, we have that

$$\alpha_n^{-p}\beta_n n^{-1}\sum_{i=1}^n |X_{i/n}^{(\alpha_n)}-X_{(i-1)/n}^{(\alpha_n)}|^p-\beta_n^{1-p}\int_0^1\gamma_p(\sigma(X_s),\beta_n)\mathrm{d}s$$

tends to zero in probability. The stable convergence in law of this sequence normalized by α_n^{-1} has also been proved in [35].

3.2.2 Remarks and explanations

The point is to remark that if $p=1$,

$$\beta_n^{1-p}\int_0^1 \gamma_p(\sigma(X_s),\beta_n)\mathrm{d}s=(2/\pi)^{1/2}\int_0^1 \sigma(X_s)\mathrm{d}s$$

and that if $p>0$,

$$\beta_n^{1-p}\int_0^1 \gamma_p(\sigma(X_s),\beta_n)\mathrm{d}s-(2/\pi)^{1/2}\int_0^1 \sigma(X_s)\mathrm{d}s$$

tends to zero in probability. Hence, the estimation of

$$\int_0^1 \sigma^2(X_s)\mathrm{d}s,$$

seems more complicated than the estimation of

$$\int_0^1 \sigma(X_s)\mathrm{d}s.$$

As a matter of fact, in the case β_n tends to infinity, even if we consider a power $p\neq 1$ of the increments, $\sigma(X_t)^p$ does not appear in the limit. We give now an intuition for this quite surprising fact in a non rigorous argument where we introduce several important ideas and show that

$$\mathbb{E}_{\sigma(X_{(i-1)/n})}[|X_{i/n}^{(\alpha_n)}-X_{(i-1)/n}^{(\alpha_n)}|^p]\approx \alpha_n^p\beta_n^{-1}(2/\pi)^{1/2}\sigma(X_{(i-1)/n}),$$

where $\mathbb{E}_{\sigma(X_{(i-1)/n})}$ denotes the expectation conditional on $\sigma(X_{(i-1)/n})$. We define the fractional part of X_t by $\{X_t\}=X_t-\lfloor X_t\rfloor$. First we have to remark that

$$X_{i/n}^{(\alpha_n)}-X_{(i-1)/n}^{(\alpha_n)}=\alpha_n\lfloor\{X_{(i-1)/n}/\alpha_n\}+(X_{i/n}-X_{(i-1)/n})/\alpha_n\rfloor. \tag{II.3}$$

Kosulajeff [81] and Tukey [110] have established that when α is small, $\{X/\alpha\}$ is almost independent of X and follows a uniform law on $[0,1]$. More precisely, the following result has been shown by Delattre and Jacod [36].

Lemma II.1 *(The fractional part of a variable) Let k be a function on $\mathbb{R}$, $\mathcal{C}^r$ $(r\geq 1)$, integrable with integrable derivatives. Let f be a function on $\mathbb{R}\times[0,1]$, $\mathcal{C}^r$ in the first variable and such that for $0\leq l\leq r$, $M_l=\sup_x\int_0^1|\frac{\partial^l}{\partial x^l}f(x,u)|du<+\infty$. Then*

$$\Big|\int_{\mathbb{R}} k(x)[f(x,\{x/\alpha\})-\int_0^1 f(x,u)du]dx\Big|\leq(2\alpha)^r\sup_{0\leq l\leq r}M_l\sup_{0\leq l\leq r}\int_{\mathbb{R}}|\frac{\partial^l}{\partial x^l}k(x)|dx.$$

Thus, since

$$X_{i/n} - X_{(i-1)/n} \approx \sigma(X_{(i-1)/n})(W_{i/n} - W_{(i-1)/n}),$$

we have

$$\mathbb{E}_{\sigma(X_{(i-1)/n})}[|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|^p] \approx \alpha_n^p \mathbb{E}_{\sigma(X_{(i-1)/n})}[|\lfloor U + \beta_n^{-1}\sigma(X_{(i-1)/n})Y\rfloor|^p],$$

where U is a uniform variable on $[0,1]$, independent of X and Y is a standard Gaussian variable, independent of X and U. Hence, if $\beta_n \to +\infty$,

$$\mathbb{E}_{\sigma(X_{(i-1)/n})}[|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|^p] \approx \alpha_n^{p-1} \mathbb{E}_{\sigma(X_{(i-1)/n})}[|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|].$$

We conclude our argument using the simple but nice fact that if U is a uniform variable on $[0,1]$ and Z a random variable, independent of U, with a density with respect to the Lebesgue measure,

$$\mathbb{E}[|\lfloor U + Z\rfloor|] = \mathbb{E}[|Z|].$$

4 Proofs

We successively prove theorem II.1, theorem II.3, theorem II.2 and theorem II.4. In all the proofs, we use the previously defined notation. For technical reasons, we suppose without loss of generality that for given j, $n2^{-j}$ is a positive integer. In the following, c and c_p denote constants not depending on n, j, k and that may vary from line to line.

4.1 Preliminaries for the proofs of theorem II.1 and theorem II.3

4.1.1 Localization procedure

We recall here a localization procedure used for example in Delattre [35]. It will enable us to replace assumption B by a much stronger assumption in the proofs of theorem II.1 and theorem II.3. We fix two sequences $(\nu_q)_{q\geq 1}$ and $(\mu_q)_{q\geq 1}$ such that (ν_q) is strictly decreasing to ν and (μ_q) is strictly increasing to μ and a sequence of functions $\chi_q : \mathbb{R} \to [0,1]$ such that $\chi_q \in \mathcal{C}_b^2(\mathbb{R})$ and

$$\chi_q(x) = 1 \text{ on } [\nu_q, \mu_q] \text{ and } \chi_q(x) = 0 \text{ on } (-\infty, \nu_{q+1}] \cup, [\mu_{q+1}, +\infty).$$

For $q \in \mathbb{N}$, we set

$$\sigma_q : x \to \sigma(x)\chi_q(x) + \big(1 - \chi_q(x)\big)$$

and

$$T_q = \inf\{t \in [0,1], X_t \leq \nu_q \text{ or } X_t \geq \mu_q \text{ or } \int_0^t a_s^2 \mathrm{d}s \geq q\} \wedge 1.$$

Under assumptions B, T_q tends almost surely to 1 and $\mathbb{P}(T_q = 1) \to 1$ as $q \to +\infty$. Let $(W_t^q, t \geq 0)$ be defined by $W_t^q = W_{(T_q+t)\wedge 1} - W_{T_q}$ and $(Y_t^q)_{t\geq 0}$ be the solution of

$$\mathrm{d}Y_t^q = \sigma_q(Y_t^q)\mathrm{d}W_t^q, \quad Y_0^q = X_{T_q}.$$

Consider now the process $(X_t^q)_{t\in[0,1]}$ defined by $X_t^q = X_t$ for $t \in [0, T_q]$ and $X_t^q = Y_{t-T_q}^q$ for $t \in (T_q, 1]$. This process satisfies

$$\mathrm{d}X_t^q = \sigma_q(X_t^q)\mathrm{d}W_t + a_t^q \mathrm{d}t,$$

where $a_t^q = a_t$ for $t \in [0, T_q]$ and $a_t^q = 0$ for $t \in (T_q, 1]$. The process X^q coincides with the initial process X on $[0, T_q]$. Hence it is enough to prove theorem II.1 and theorem II.3 for the processes X^q, for all $q \in \mathbb{N}$ and so it is enough to prove theorem II.1 and theorem II.3 under assumption B' instead of assumption B, with assumption B' defined the following way.

Assumption B'.

(i) There exists $c > 0$ such that for all $x \in \mathbb{R}$, $\sigma(x) \geq c$,

(ii) $x \to \sigma(x) \in \mathcal{C}_b^2(\mathbb{R})$,

(iii) $\sup_{\omega\in\Omega} \int_0^1 a_s^2 \mathrm{d}s < +\infty$.

4.1.2 Change of probability

Under assumption B', by Girsanov theorem, we can construct a probability $\mathbb{P}'$ on $(\Omega, \mathcal{F})$, absolutely continuous with respect to $\mathbb{P}$ and a Brownian motion under $\mathbb{P}'$, $(W_t', t \geq 0)$ such that

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t' + \frac{1}{2}\sigma(X_t)\frac{\partial}{\partial x}\sigma(X_t)\mathrm{d}t.$$

Assumptions B' holds for this representation. We define the following supplementary hypothesis.

Assumption C.

$$a_t = \frac{1}{2}\sigma(X_t, t)\frac{\partial}{\partial x}\sigma(X_t, t).$$

The convergence in probability and the stable convergence in law being preserved by absolutely continuous change of probability, it is consequently enough to prove theorem II.1 under assumptions A, B' and C and theorem II.3 under assumptions A1, B' and C. Under assumptions B' and C, $X_t = h(W_t)$ with $h : x \to S^{-1}\big(x + S(x_0)\big)$ and

$$S : x \to \int_0^x \frac{1}{\sigma(y)}\mathrm{d}y.$$

For the rest of proof of theorem II.1 and theorem II.3, without loss of generality, we suppose $x_0 = 0$. Note that X is an homogeneous Markov process with transition densities

$$p_t(x,y) = \sigma(y)^{-1}(2\pi t)^{-1/2}\exp[-(2t)^{-1}\big(S(y)-S(x)\big)^2].$$

Moreover, the following inequalities hold, see for example Delattre and Jacod [36].

$$\int |\frac{\partial^{i+j}}{\partial x^i \partial x^j} p_t(x,y)|\mathrm{d}y \leq ct^{-(i+j)/2}, \quad i+j \leq 2, \tag{II.4}$$

$$\int |\frac{\partial^{i}}{\partial x^i} q_t(x,y)||y|^p\mathrm{d}y \leq c_p t^{p/2}, \quad i \leq 2, \tag{II.5}$$

with $q_t(x,y) = p_t(x,x+y)$. We now give the proofs of theorem II.1 and theorem II.3.

4.2 The behavior of the sampling functions

We give in this section a key proposition for the proofs of theorem II.1 to II.4. As in Delattre [35], we consider the following assumption.

Assumption D. Let $(x,u,y) \to f_n(x,u,y)$ be a sequence of real functions on $\mathbb{R}\times[0,1]\times\mathbb{R}$. The sequence f_n verifies assumption D if the functions f_n are twice continuously differentiable with respect to the first variable and if there exists $\gamma > 0$ such that for all $n \geq 1$,

(i) $|f_n(x,u,y)| \leq \gamma(1+\beta_n^2)(1+|y|^\gamma)$,

(ii) $\int_0^1 |f_n(x,u,y)|\mathrm{d}u \leq \gamma(1+|y|^\gamma)$,

(iii) $|\frac{\partial^i}{\partial x^i} f_n(x,u,y)| \leq \gamma(1+\beta_n^2)(1+|y|^\gamma),\ i=1,2$,

(iv) $\int_0^1 |\frac{\partial^i}{\partial x^i} f_n(x,u,y)|\mathrm{d}u \leq \gamma(1+|y|^\gamma),\ i=1,2.$

Notation.
For some sequences of real functions $x \to g_n(x)$ on $\mathbb{R}$ and $(x,u,y) \to f_n(x,u,y)$ on $\mathbb{R}\times[0,1]\times\mathbb{R}$, we define

$$V^{jk}(n,g_n) = \frac{2^{j/2}}{n}\sum_{i=1}^n \mathbb{1}_{jk}(i/n) g_n(X_{(i-1)/n})$$

and

$$V^{jk}(n,f_n) = \frac{2^{j/2}}{n}\sum_{i=1}^n \mathbb{1}_{jk}(i/n) f_n(X_{(i-1)/n},\{X_{(i-1)/n}/\alpha_n\},\sqrt{n}[X_{i/n}-X_{(i-1)/n}]).$$

Let h_σ be the density of a centered Gaussian variable with variance σ^2. For a real function $(x,u,y) \to f(x,u,y)$ on $\mathbb{R} \times [0,1] \times \mathbb{R}$, we set

$$mf(x,u) = \int_{\mathbb{R}} h_{\sigma(x)}(y) f(x,u,y)\mathrm{d}y, \;\; Mf(x) = \int_0^1 mf(x,u)\mathrm{d}u.$$

The following proposition is a general result on the behavior of the sampling functions.

Proposition II.1 *(Behavior of the sampling functions)*
Let $(x,u,y) \to f_n(x,u,y)$ be a sequence of real functions on $\mathbb{R} \times [0,1] \times \mathbb{R}$ satisfying assumption D. Under assumptions A, B' and C,

$$\mathbb{E}[(V^{jk}(n,f_n) - 2^{j/2} \int_0^1 \mathbb{1}_{jk}(s) M f_n(X_s)\,ds)^2] \leq c r_n^2,$$

for $0 \leq j \leq \lfloor \log_2 r_n^{-1} \rfloor$ and $0 \leq k \leq 2^j - 1$. This holds for $0 \leq j \leq \lfloor (1+\rho)\log_2 r_n^{-1} \rfloor$ under assumptions A1, B' and C.

4.3 Proof of proposition II.1

In this proof, we widely use the methods and results developed by Delattre in [35]. We set ρ to zero if only assumptions A, B' and C are satisfied and write $\mathbb{E}_{\mathcal{F}_t}$ for the conditional expectation with respect to $\mathcal{F}_t$.

4.3.1 Fundamental decomposition

Notation.
Let $s_{jk} = [2^{-j}nk+1, \ldots, 2^{-j}n(k+1)]$. We use the following notation.

$$m_n f_n(x,u) = \int q_{1/n}(x,y) f_n(x,u,\sqrt{n}y)\mathrm{d}y, \quad M_n f_n(x) = \int_0^1 m_n f_n(x,u)\mathrm{d}u,$$

$$\bar{m_n} f_n(x) = m_n f_n(x, \{x/\alpha_n\}) - M_n f_n(x), \quad l_i^n f_n(x) = \int p_{i/n}(x,y) \bar{m_n} f_n(y)\mathrm{d}y.$$

We set

$$f_{i+1}^n = f_n(X_{i/n}, \{X_{i/n}/\alpha_n\}, \sqrt{n}[X_{(i+1)/n} - X_{i/n}]), \;\; \eta_i^n(f_n) = f_i^n - M_n f_n(X_{(i-1)/n}),$$

$$\delta_i^n(f,l) = \sum_{z=i}^{n \wedge (i+l-1)} (\mathbb{E}_{\mathcal{F}_{i/n}}[\eta_z^n(f)] - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[\eta_z^n(f)])$$

and

$$\mathcal{M}^n_{jk}(f_n,l) = \frac{2^{j/2}}{n}\sum_{i=1}^{n}\mathbb{1}_{jk}(i/n)\delta^n_i(f_n,l),$$
$$H^n_{jk}(f_n,l) = \frac{2^{j/2}}{n}\sum_{i\in s_{jk}}[\bar{m}_n f_n(X_{i/n}) - \bar{m}_n f_n(X_{(i-1)/n})]$$
$$- \frac{2^{j/2}}{n}\sum_{i\in s_{jk}} l^n_{(n-i)\wedge(l-1)} f_n(X_{(i-1)/n})\mathbb{1}_{2\leq(n-i)\wedge(l-1)},$$
$$K^n_{jk}(f_n,l) = \frac{2^{j/2}}{n}\sum_{i\in s_{jk}}\sum_{z=1}^{(n-i-1)\wedge(l-2)}[l^n_z f_n(X_{i/n}) - l^n_z f_n(X_{(i-1)/n})].$$

Remark that for given n and $z \in s_{jk}$,

$$\mathcal{M}^n_y = \frac{2^{j/2}}{n}\sum_{i=1}^{y}\mathbb{1}_{jk}(i/n)\delta^n_i(f_n,l)$$

is a $(\mathcal{F}_t)$-martingale in y. The following fundamental decomposition will be constantly used.

Proposition II.2 *(Fundamental decomposition)*

$$V^{jk}(n,f_n) - 2^{j/2}\int_0^1 \mathbb{1}_{jk}(s)Mf_n(X_s)ds = \mathcal{M}^n_{jk}(f_n,l) + V^{jk}(n, M_n f_n - Mf_n)$$
$$+ V^{jk}(n,Mf_n) - 2^{j/2}\int_0^1 \mathbb{1}_{jk}(s)Mf_n(X_s)ds - H^n_{jk}(f_n,l) - K^n_{jk}(f_n,l).$$

Proof. We have

$$\delta^n_i(f_n,l) = \eta^n_i(f_n) - M_n f_n(X_{i/n}) + M_n f_n(X_{(i-1)/n}) - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[f^n_i] + \mathbb{E}_{\mathcal{F}_{i/n}}[f^n_{i+1}]$$
$$- \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[\eta^n_{i+1}(f_n)] + \sum_{z=i+2}^{n\wedge(i+l-1)}(\mathbb{E}_{\mathcal{F}_{i/n}}[\eta^n_z(f_n)] - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[\eta^n_z(f_n)]).$$

Using that

$$\mathbb{E}_{\mathcal{F}_{i/n}}[f^n_{i+1}] = \int q_{1/n}(X_{i/n},y)f_n(X_{i/n},\{X_{i/n}/\alpha_n\},\sqrt{n}y)\mathrm{d}y,$$

we get

$$\delta_i^n(f_n,l) = \eta_i^n(f_n) + \bar{m_n} f_n(X_{i/n}) - \bar{m_n} f_n(X_{(i-1)/n}) - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}\left[\mathbb{E}_{\mathcal{F}_{i/n}}[\eta_{i+1}^n(f_n)]\right]$$
$$+ \sum_{z=i+2}^{n\wedge(i+l-1)} \left(\mathbb{E}_{\mathcal{F}_{i/n}}\left[\mathbb{E}_{\mathcal{F}_{(z-1)/n}}[\eta_z^n(f_n)]\right] - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}\left[\mathbb{E}_{\mathcal{F}_{(z-1)/n}}[\eta_z^n(f_n)]\right]\right)$$
$$= \eta_i^n(f_n) + \bar{m_n} f_n(X_{i/n}) - \bar{m_n} f_n(X_{(i-1)/n}) - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[\bar{m_n} f_n(X_{i/n})]$$
$$+ \sum_{z=i+2}^{n\wedge(i+l-1)} (\mathbb{E}_{\mathcal{F}_{i/n}}[\bar{m_n} f_n(X_{(z-1)/n})] - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[\bar{m_n} f_n(X_{(z-1)/n})]).$$

Since

$$\mathbb{E}_{\mathcal{F}_{i/n}}[\bar{m_n} f_n(X_{(z-1)/n})] = \int p_{(z-1-i)/n}(X_{i/n}, y)\bar{m_n} f_n(y)\mathrm{d}y,$$

we obtain

$$\delta_i^n(f_n,l) = \eta_i^n(f_n) + \bar{m_n} f_n(X_{i/n}) - \bar{m_n} f_n(X_{(i-1)/n})$$
$$- l_1^n f_n(X_{(i-1)/n}) + \sum_{z=2}^{(n-i)\wedge(l-1)} [l_{z-1}^n f_n(X_{i/n}) - l_z^n f_n(X_{(i-1)/n})].$$

Thus,

$$\delta_i^n(f_n) = \eta_i^n(f_n) + \bar{m_n} f_n(X_{i/n}) - \bar{m_n} f_n(X_{(i-1)/n})$$
$$- l_{(n-i)\wedge(l-1)}^n f_n(X_{(i-1)/n})\mathbb{1}_{2\le(n-i)\wedge(l-1)} + \sum_{z=1}^{(n-i-1)\wedge(l-2)} [l_z^n f_n(X_{i/n}) - l_z^n f_n(X_{(i-1)/n})].$$

We finally get

$$V^{jk}(n,f_n) - V^{jk}(n,M_n f_n) = \frac{2^{j/2}}{n}\sum_{i=1}^{n} \mathbb{1}_{jk}(i/n)\eta_i^n(f_n)$$
$$= \mathcal{M}_{jk}^n(f_n,l) - H_{jk}^n(f_n,l) - K_{jk}^n(f_n,l).$$

□

4.3.2 Technical lemmas

We prove here some useful lemmas. In particular, they will enable us to control the different terms of the decomposition. We begin with a usual Riemann approximation.

Lemma II.2 *(Riemann approximation) Let $f \in \mathcal{C}_b^1$ and*

$$A_n = \frac{2^{j/2}}{n}\sum_{i=1}^{n} \mathbb{1}_{jk}(i/n) f(X_{i/n}) - 2^{j/2}\int_0^1 \mathbb{1}_{jk}(s) f(X_s)\,ds.$$

Then,

$$\mathbb{E}[A_n^2] \leq c2^{-j}n^{-1}.$$

Proof. Let

$$\xi_n^i = 2^{j/2}\int_{(i-1)/n}^{i/n}[\mathbb{1}_{jk}(s)f(X_s) - \mathbb{1}_{jk}(i/n)f(X_{i/n})]\mathrm{d}s.$$

We have

$$|\xi_n^i| \leq 2^{j/2}\int_{(i-1)/n}^{i/n}|f(X_s) - f(X_{i/n})|\mathrm{d}s.$$

Since $f \in \mathcal{C}_b^1$, using the Burkholder-Davis-Gundy inequality, we get $\mathbb{E}[(\xi_n^i)^2] \leq c2^jn^{-3}$. Now, $A_n = \sum_{i=1}^n \xi_n^i$ with $n2^{-j}$ terms in the sum. Thus,

$$\mathbb{E}[A_n^2] \leq \sum_{i=1}^n\sum_{i'=1}^n(\mathbb{E}[(\xi_n^i)^2]\mathbb{E}[(\xi_n^{i'})^2])^{1/2} \leq c2^{-j}n^{-1}.$$

□

The following lemma is a consequence of assumption D together with lemma II.1 and inequalities (II.4) and (II.5). Details can be found in Delattre [35].

Lemma II.3

$$|m_nf_n(x,u)| \leq c(1+\beta_n^2), \tag{II.6}$$

$$\int_0^1|\frac{\partial^i}{\partial x^i}m_nf_n(x,u)|du + |\frac{\partial^i}{\partial x^i}M_nf_n(x)| \leq c, \quad 0 \leq i \leq 2, \tag{II.7}$$

$$|M_nf_n(x) - Mf_n(x)| \leq cn^{-1/2}, \tag{II.8}$$

$$|l_i^nf_n(x)| \leq c\alpha_n^2(1+n/i). \tag{II.9}$$

We end this section with the following bounds for H_{jk}^n and K_{jk}^n.

Lemma II.4

$$|H_{jk}^n(f_n,l)| \leq c2^{j/2}n^{-1} + c2^{j/2}\alpha_n^2[1+n2^{-j}(l-1)^{-1} + (\log n)\mathbb{1}_{k=2^j-1}],$$

$$|K_{jk}^n(f_n,l)| \leq c2^{j/2}\alpha_n^2\log n.$$

Proof. From inequality (II.6) and (II.9), we get

$$n|H_{jk}^n(f_n,l)| \leq c2^{j/2}(1+\beta_n^2) + c2^{j/2}\alpha_n^2\sum_{i\in s_{jk}}n[(l-1)^{-1} + (n-i)^{-1}\mathbb{1}_{2\leq(n-i)}].$$

We also have

$$n|K^n_{jk}(f_n,l)| = 2^{j/2}\sum_{z=1}^{n}\sum_{i\in s_{jk}} \mathbb{1}_{1\leq z\leq (n-i-1)\wedge(l-2)}[l^n_z f_n(X_{i/n}) - l^n_z f_n(X_{(i-1)/n})]$$
$$\leq c2^{j/2}\alpha_n^2\sum_{z=1}^{n}(1+n/z) \leq c2^{j/2}\alpha_n^2 n\log n.$$

□

4.3.3 End of the proof of proposition II.1

For the proof of theorem II.1, that is until the end of section 4.4, we take $l = n$ and omit this index in the notation. We now bound the different terms of the fundamental decomposition. By lemma II.4, as $0 \leq j \leq \lfloor (1+\rho)\log_2 r_n^{-1}\rfloor$, we get

$$|H^n_{jk}(f_n) + K^n_{jk}(f_n)| \leq c2^{j/2}(n^{-1} + \alpha_n^2 \log n) \leq cr_n.$$

Inequality (II.7) together with lemma II.2 on Riemann approximation give

$$\mathbb{E}[|V^{jk}(n, Mf_n) - 2^{j/2}\int_0^1 \mathbb{1}_{jk}(s)Mf_n(X_s)|^2] \leq c2^{-j}n^{-1}.$$

Inequality (II.8) gives

$$|V^{jk}(n, M_nf_n - Mf_n)| \leq 2^{-j/2}n^{-1/2}.$$

We now turn to the approximation term $\mathcal{M}^n_{jk}(f_n)$. We have

$$\mathbb{E}[\mathcal{M}^n_{jk}(f_n)^2] = \frac{2^j}{n^2}\sum_{i\in s_{jk}}\mathbb{E}[\delta^n_i(f)^2].$$

From the results of Delattre [35, chap 7-8], we can show that

$$\mathbb{E}[\delta^n_i(f)^2] \leq c(n\alpha_n^2 + (1+\beta_n^2)(1+\alpha_n(n/i)^{1/2}).$$

Since $\sum_{i=1}^{n} i^{-1/2} \leq 2\sqrt{n}$, we have

$$\mathbb{E}[\mathcal{M}^n_{jk}(f_n)^2] \leq c\alpha_n^2 + c(1+\beta_n^2)(n^{-1} + 2^{j/2}\alpha_n n^{-1})$$
$$\leq c(\alpha_n^2 + n^{-1} + 2^{j/2}\alpha_n n^{-1} + 2^{j/2}\alpha_n^3). \quad \text{(II.10)}$$

Putting all the inequalities together, we obtain proposition II.1.

4.4 Proof of theorem II.1

Using the remark on the change of probability in section 4.1.2, the following proposition implies theorem II.1.

Proposition II.3 *(L^1 convergence, absolute integrated volatility) Let $\widetilde{\theta}_n$ be the estimator defined in section 2.2. Under assumptions A, B' and C,*

$$\mathbb{E}[|\widetilde{\theta}_n - \theta|] \le cr_n,$$

with c a constant not depending on n.

4.4.1 Proof of proposition II.3

We consider here $f_n(x,u,y) = (\pi/2)^{1/2}\beta_n|\lfloor u + \beta_n^{-1}y\rfloor|$. In that case,

$$Mf_n(X_s) = \sigma(X_s).$$

We begin with a lemma on the behavior of the wavelets coefficients. Let c_{j_0k}, d_{jk} and $\widehat{c}_{j_0k}$ defined in section 2.2. Thanks to the vanishing moment of ψ, we easily get the following result.

Lemma II.5

$$c_{j_0k}^2 \le c2^{-j_0}, \quad \mathbb{E}[d_{jk}^2] \le c2^{-2j}.$$

Let

$$Z_j = \sum_{k=0}^{2^j-1} \mathcal{M}_{jk}^n(f_n)c_{jk}, \quad \tilde{Z}_j = \sum_{k=0}^{2^j-1} K_{jk}^n(f_n)c_{jk}.$$

We have the following lemma.

Lemma II.6 *Let $0 \le j \le \lfloor(1+\rho)\log_2 r_n^{-1}\rfloor$, then*

$$\mathbb{E}[|Z_j| + |\tilde{Z}_j|] \le cr_n.$$

Proof. We have $Z_j = Z_{j,1} + Z_{j,2}$ with

$$Z_{j,1} = \sum_{k=0}^{2^j-1} 2^{j/2}\Big(\int_{k/2^j}^{(k+1)/2^j} [\sigma(X_s) - \sigma(X_{k2^{-j}})]\mathrm{d}s\Big)\mathcal{M}_{jk}^n,$$

$$Z_{j,2} = \frac{1}{n}\sum_{k=0}^{2^j-1} \sigma(X_{k2^{-j}})\mathrm{d}s \sum_{i\in s_{jk}} \delta_i.$$

We easily get $\mathbb{E}[|Z_{j,1}|] \leq cr_n$. For $Z_{j,2}$, we have

$$\mathbb{E}[|Z_{j,2}|^2] = \frac{1}{n^2} \sum_{k=0}^{2^j-1} \sum_{k'=0}^{2^j-1} \sum_{i \in s_{jk}} \sum_{i' \in s_{jk'}} \mathbb{E}[\sigma(X_{k2^{-j}})\sigma(X_{k'2^{-j}})\delta_i \delta_i'].$$

For $i \neq i'$, conditioning by $\mathcal{F}_{\max(i,i')-1/n}$, we get

$$\mathbb{E}[\sigma(X_{k2^{-j}})\sigma(X_{k'2^{-j}})\delta_i \delta_i'] = 0.$$

Hence,

$$\begin{aligned}\mathbb{E}[|Z_{j,2}|^2] &= \frac{1}{n^2} \sum_{k=0}^{2^j-1} \mathbb{E}\big[\sigma(X_{k2^{-j}})^2 \mathbb{E}_{\mathcal{F}_{k2^{-j}}}[\sum_{i \in s_{jk}} \delta_i^2]\big] \\ &= 2^{-j} \sum_{k=0}^{2^j-1} \mathbb{E}\big[\sigma(X_{k2^{-j}})^2 \mathbb{E}_{\mathcal{F}_{k2^{-j}}}[\mathcal{M}_{jk}^{n\ 2}]\big] \leq cr_n^2.\end{aligned}$$

For $\tilde{Z}_j$, recall that

$$K_{jk}^n(f_n) = \frac{2^{j/2}}{n} \sum_{i \in s_{jk}} \mathbb{1}_{jk}(i/n)\tilde{\delta}_i,$$

with

$$\tilde{\delta}_i = \sum_{z=1}^{n-i-1} [l_z^n f_n(X_{i/n}) - l_z^n f_n(X_{(i-1)/n})]$$

and that

$$l_z^n f_n(X_{i/n}) = \mathbb{E}_{\mathcal{F}_{i/n}}[\bar{m}_n f_n(X_{(i+z)/n})].$$

The same method gives the result. □

We now end the proof of proposition II.3. Let $j_0 = \lfloor \log_2(r_n^{-1}) \rfloor$. From proposition II.1 and (III.2), we can write

$$\begin{aligned}\widehat{c}_{j_0k} = c_{j_0k} &+ \mathcal{M}_{j_0k}^n(f_n) + V^{j_0k}(n, M_n f_n - Mf_n) \\ &+ V^{j_0k}(n, Mf_n) - 2^{j_0/2} \int_0^1 \mathbb{1}_{j_0k}(s) Mf_n(X_s) \mathrm{d}s - H_{j_0k}^n(f_n) - K_{j_0k}^n(f_n),\end{aligned}$$

and

$$\mathbb{E}[|\widehat{c}_{j_0k} - c_{j_0k}|^2] \leq cr_n^2.$$

We have

$$\mathbb{E}[|\widetilde{\theta}_n - \theta|] \leq c\mathbb{E}\big[\sum_{j=j_0+1}^{+\infty} \sum_k d_{jk}^2 + |\sum_k c_{j_0k}\mathcal{R}_k^n|\big] + c(r_n + 2^{j_0} r_n^2)$$

with

$$\mathcal{R}_k^n = V^{j_0k}(n, M_nf_n - Mf_n) + V^{j_0k}(n, Mf_n) - 2^{j_0/2}\int_0^1 \mathbb{1}_{j_0k}(s)Mf_n(X_s)\mathrm{d}s - H_{j_0k}^n(f_n).$$

By lemma II.5, we have

$$\mathbb{E}\big[\sum_{j=j_0}^{+\infty}\sum_k d_{jk}^2\big] \leq c2^{-j_0}.$$

Moreover, using preceding computations, it is easy to see that

$$\mathbb{E}[|\sum_k c_{j_0k}\mathcal{R}_k^n|] \leq c(2^{j_0}n^{-1} + 2^{j_0}\alpha_n^2 + n^{-1/2} + \alpha_n^2\log n).$$

The result follows.

4.5 Proof of theorem II.3

In this proof, assumptions A1, B' and C are in force for α_n and X.

4.5.1 Compensator

We have

$$\sum_k \widehat{c}_{j_0k}^{\,2} - \int_0^1 \sigma(X_s)^2\mathrm{d}s = \sum_k(\widehat{c}_{j_0k} - c_{j_0k})^2 + 2\sum_k c_{j_0k}(\widehat{c}_{j_0k} - c_{j_0k}) - \sum_{j\geq j_0}\sum_k d_{jk}^2.$$

The central limit theorems will be derived from the double product term. If, as previously, we choose j_0 such that 2^{j_0} is of order r_n^{-1}, re-normalized by r_n^{-1}, the two other terms do not tend to zero. Hence, we can either choose $2^j > r_n^{-1}$ and compensate the first term or choose $2^j < r_n^{-1}$ and compensate the last term. The first method is classical in quadratic functionals estimation. However, it seems difficult here. Indeed, a compensator of $\sum_k(\widehat{c}_{jk} - c_{jk})^2$ requires an accurate enough estimation of the function $x \to \sigma(x)$. Consequently, we compensate the last term. This is unusual, but possible in our specific setting. A one by one estimation of the coefficients d_{jk} can not be sufficient as a compensator. Indeed, the error between the coefficient d_{jk}^2 and its estimation is of the same order as the error between the coefficient c_{jk}^2 and its estimation. That is why we use here the following scaling property of the wavelet coefficients, whose proof can be done exactly the same way as the proof of proposition I.2.

Lemma II.7 *Let*

$$Q_j = \sum_k d_{jk}^2,\ G(u) = \int_0^u \psi(u)du \text{ and } c(\psi) = \int_0^1 G^2(u)du.$$

We have

$$\mathbb{E}[|2^j Q_j - c(\psi)\int_0^1 h'(W_u)^2 du|] \leq c2^{-j/2}.$$

The following lemma shows that our method enable us to estimate the remaining coefficients accurately enough.

Lemma II.8 *Let* $S = [a, (j_{1,n}), (j_{2,n})] \in \mathcal{S}$. *Then,*

$$r_n^{-1}\big[\sum_k (\widehat{c}_{j_{1,n}k} - c_{j_{1,n}k})^2 - \sum_{j \geq j_{1,n}} \sum_k d_{jk}^2 + R_n(S)\big] \to 0.$$

Proof. We want to compensate

$$\sum_{j=j_{1,n}}^{\lfloor (1+a)\log_2 r_n^{-1} \rfloor} Q_j.$$

We know that for big enough j and $j_{2,n} \leq j$, Q_j is close to $2^{j_{2,n}-j}Q_{j_{2,n}}$. Hence we estimate the preceding quantity by

$$\sum_{j=j_{1,n}}^{\lfloor (1+a)\log_2 r_n^{-1} \rfloor} 2^{j_{2,n}-j}\widehat{Q}_{j_{2,n}},$$

with

$$\widehat{Q}_{j_{2,n}} = \sum_k \widehat{d}_{j_{2,n}k}^{\,2},$$

for appropriate $j_{1,n}$ and $j_{2,n}$. Let

$$U_n = \sum_{j=j_{1,n}}^{\lfloor (1+a)\log_2 r_n^{-1} \rfloor} Q_j - \sum_{j=j_{1,n}}^{\lfloor (1+a)\log_2 r_n^{-1} \rfloor} 2^{j_{2,n}-j}\widehat{Q}_{j_{2,n}}$$

and $Y = c(\psi)\int_0^1 h'(W_u)^2 \mathrm{d}u$. We have

$$U_n = \sum_{j=j_{1,n}}^{\lfloor (1+a)\log_2 r_n^{-1} \rfloor} (Q_j - 2^{-j}Y) + 2^{-j}(Y - 2^{j_{2,n}}Q_{j_{2,n}}) + 2^{j_{2,n}-j}(Q_{j_{2,n}} - \widehat{Q}_{j_{2,n}}).$$

Using the same arguments as for proposition II.1, for $j_{1,n} \leq j \leq \lfloor (1+a)\log_2 r_n^{-1} \rfloor$, we get

$$\mathbb{E}[|\widehat{d}_{jk} - d_{jk}|^2] \leq c r_n^2.$$

Hence, we also obtain

$$\mathbb{E}[|d_{jk}||\widehat{d}_{jk} - d_{jk}|] \leq c 2^{-j} r_n.$$

Consequently, we have

$$\mathbb{E}[|U_n|] \leq c(2^{-3j_{1,n}/2} + 2^{-(j_{1,n}+j_{2,n}/2)} + 2^{2j_{2,n}-j_{1,n}} r_n^2 + 2^{j_{2,n}-j_{1,n}} r_n).$$

As $a > 0$, it is clear that

$$r_n^{-1} \sum_{j > \lfloor (1+a)\log_2 r_n^{-1} \rfloor} Q_j \to 0.$$

□

4.5.2 Limit theorems

We begin with a fundamental lemma.

Lemma II.9 *Let*

$$C_{j_n} = \sum_k c_{j_n k}(\widehat{c}_{j_n k} - c_{j_n k}).$$

if $2^{-j_n} + r_n^{-1} 2^{j_n/2}(n^{-1} + \alpha_n^2 \log n) + 2^{j_n} r_n \to 0$, *then we have the following convergences in stable law, where* B *is a standard Brownian motion, independent of* $\mathcal{F}$.

$$\begin{aligned}
&\text{if } \beta_n \to 0, && r_n^{-1} C_{j_n} \to_{\mathcal{L}\mathrm{s}} \frac{1}{\sqrt{2}}(\pi - 2)^{1/2} \int_0^1 \sigma(X_t)^2 \mathrm{d}B_t, \\
&\text{if } \beta_n \to \beta > 0, && r_n^{-1} C_{j_n} \to_{\mathcal{L}\mathrm{s}} \int_0^1 \sigma(X_t)[\Delta_\beta(X_t)]^{1/2} \mathrm{d}B_t, \\
&\text{if } \beta_n \to +\infty, && r_n^{-1} C_{j_n} \to_{\mathcal{L}\mathrm{s}} \frac{1}{\sqrt{3}} \int_0^1 \sigma(X_t) \mathrm{d}B_t.
\end{aligned}$$

Proof. Let $f_n(x,u,y) = (\pi/2)^{1/2} \beta_n |\lfloor u + \beta_n^{-1} y \rfloor|$,

$$q_i^n = \frac{1}{n} f_n(X_{(i-1)/n}, \{X_{(i-1)/n}/\alpha_n\}, \sqrt{n}[X_{i/n} - X_{(i-1)/n}])$$

and $z_i^n = q_i^n - \int_{(i-1)/n}^{i/n} M f_n(X_s)$. We have

$$\sum_k c_{jk}(\widehat{c}_{jk} - c_{jk}) = \sum_k \left[2^{j/2} \int \mathbb{1}_{jk}(s) \sigma(X_s) \mathrm{d}s\right] \left[2^{j/2} \sum_{i \in s_{jk}} z_i^n\right] = T_1 + T_2 + T_3 + T_4,$$

with

$$T_1 = \sum_k \big[2^{j/2} \int \mathbb{1}_{jk}(s)[\sigma(X_s) - \sigma(X_{k2^{-j}})]\mathrm{d}s\big] \big[2^{j/2} \sum_{i \in s_{jk}} z_i^n\big],$$

$$T_2 = \sum_i \sigma(X_{(i-1)/n}) q_i^n - \int \sigma(X_s) M f_n(X_s) \mathrm{d}s,$$

$$T_3 = \sum_i \int_{(i-1)/n}^{i/n} [\sigma(X_s) - \sigma(X_{(i-1)/n})] M f_n(X_s) \mathrm{d}s,$$

$$T_4 = \sum_k \sum_{i \in s_{jk}} [\sigma(X_{k2^{-j}}) - \sigma(X_{(i-1)/n})] z_i^n.$$

We now prove that $r_n^{-1}(T1 + T2 + T3)$ tends to zero in probability. We write $\mathcal{M}_{jk}(l_n)$ for $\mathcal{M}_{jk}(f_n, l_n)$. We have $T_1 = T_{11} + T_{12}$ with

$$T_{11} = \sum_k \big[2^{j/2} \int \mathbb{1}_{jk}(s)[\sigma(X_s) - \sigma(X_{k2^{-j}})]\mathrm{d}s\big] \mathcal{M}_{jk}(l_n).$$

We have

$$\mathbb{E}[|T_{12}|] \leq c 2^{-j/2} r_n + c 2^{j/2}(n^{-1} + \alpha_n^2 \log n + \alpha_n^2 n 2^{-j}/l_n).$$

For T_1, we take $l_n = \lfloor n/\log n \rfloor$, hence $r_n^{-1}\mathbb{E}[|T_{12}|]$ tends to zero. We set $F(x) = \sigma[h(x)]$. The term T_{11} can be written

$$T_{11} = \sum_k \sum_{i \in s_{jk}} \big[2^j \int \mathbb{1}_{jk}(s)(W_s - W_{k2^{-j}}) F'(W_{k2^{-j}}) \frac{\delta_i(l_n)}{n} \mathrm{d}s\big] + 2^{j/2} \sum_k R_k \mathcal{M}_{jk}(l_n),$$

with $\mathbb{E}[|R_k|^2] \leq 2^{-4j}$. Following Delattre [35, chap 7-8], there exists $\tilde{\delta}_i(l_n)$ such that for $l_n = \lfloor n/\log n \rfloor$

$$\mathbb{E}[|\delta_i(l_n) - \tilde{\delta}_i(l_n)|^2] \leq c n r_n^2 / \log n,$$

$$\mathbb{E}[\tilde{\delta}_i(l_n)^2] \leq c(1 + \beta_n^2)(1 + \alpha_n [n/i]^{1/2}).$$

Hence,

$$\mathbb{E}[\mathcal{M}_{jk}(l_n)^2] = \frac{2^j}{n^2} \sum_{i \in s_{jk}} \mathbb{E}[\delta_i(l_n)^2] \leq c r_n^2.$$

Consequently, we obtain that the expectation of the second term of T_{11} is less than $2^{-j/2}r_n$. The first term can be written $A_1 + A_2 + A_3 + A_4 + A_5$ with,

$$A_1 = \sum_k \sum_{i \in s_{jk}} 2^j \int_{k2^{-j}}^{(i-1)/n} (W_s - W_{k2^{-j}}) F'(W_{k2^{-j}}) \frac{\delta_i(l_n)}{n} \mathrm{d}s,$$

$$A_2 = \sum_k \sum_{i \in s_{jk}} 2^j \int_{(i-1)/n}^{i/n} (W_s - W_{k2^{-j}}) F'(W_{k2^{-j}}) \frac{\delta_i(l_n)}{n} \mathrm{d}s,$$

$$A_3 = \sum_k \sum_{i \in s_{jk}} 2^j \int_{i/n}^{(k+1)2^{-j}} (W_s - W_{i/n}) F'(W_{k2^{-j}}) \frac{\delta_i(l_n)}{n} \mathrm{d}s,$$

$$A_4 = \sum_k \sum_{i \in s_{jk}} 2^j [(k+1)2^{-j} - i/n](W_{i/n} - W_{(i-1)/n}) F'(W_{k2^{-j}}) \frac{\delta_i(l_n)}{n} \mathrm{d}s,$$

$$A_5 = \sum_k \sum_{i \in s_{jk}} 2^j [(k+1)2^{-j} - i/n](W_{(i-1)/n} - W_{k2^{-j}}) F'(W_{k2^{-j}}) \frac{\delta_i(l_n)}{n} \mathrm{d}s.$$

We easily get that $\mathbb{E}[A_1^2 + A_5^2] \leq c2^{-j}r_n^2$. For A_2, we have

$$\mathbb{E}[|A_2|] \leq c\frac{2^{j/2}}{n}\big(\sup_i\{\mathbb{E}[\delta_i(l_n)^2]\}\big)^{1/2} \leq c2^{j/2}(1/n + \alpha_n^2).$$

We now turn to A_3. We write here δ_i for $\delta_i(l_n)$. We easily obtain that $\mathbb{E}[A_3^2]$ is equal to

$$2^{2j} \sum_k \sum_{\substack{i \in s_{jk} \\ i' \in s_{jk}}} \int_{i/n}^{(k+1)2^{-j}} \int_{i'/n}^{(k+1)2^{-j}} \mathbb{E}[F'(W_{k2^{-j}})^2 (W_s - W_{i/n})(W_{s'} - W_{i'/n}) \frac{\delta_i}{n}\frac{\delta_{i'}}{n}] \mathrm{d}s\mathrm{d}s'.$$

We consider the quantity

$$u_i = \mathbb{E}_{\mathcal{F}_{i/n}}[F'(W_{k2^{-j}})^2 (W_s - W_{i/n})(W_{s'} - W_{i'/n}) \frac{\delta_i}{n}\frac{\delta_{i'}}{n}].$$

Suppose that $i \geq i'$ and $s' > i/n$. Then

$$\begin{aligned} u_i &= F'(W_{k2^{-j}})^2 \frac{\delta_i}{n}\frac{\delta_{i'}}{n} \mathbb{E}_{\mathcal{F}_{i/n}}[(W_s - W_{i/n})(W_{s'} - W_{i'/n})] \\ &= F'(W_{k2^{-j}})^2 \frac{\delta_i}{n}\frac{\delta_{i'}}{n} \mathbb{E}[(W_s - W_{i/n})(W_{s'} - W_{i/n})]. \end{aligned}$$

Suppose that $i \geq i'$ and $s' \leq i/n$. Then

$$u_i = F'(W_{k2^{-j}})^2 \frac{\delta_i}{n}\frac{\delta_{i'}}{n} \mathbb{E}_{\mathcal{F}_{i/n}}[(W_s - W_{i/n})(W_{s'} - W_{i'/n})] = 0.$$

Finally,

$$\mathbb{E}[A_3^2] \leq c2^j \sum_k \sum_{i \in s_{jk}} \int_{i/n}^{(k+1)2^{-j}} \int_{i/n}^{(k+1)2^{-j}} \mathbb{E}[F'(W_{k2^{-j}})^2 (\frac{\delta_i}{n})^2] \mathrm{d}s\mathrm{d}s'.$$

Hence,

$$\mathbb{E}[A_3^2] \le c 2^{-j} r_n^2.$$

For A_4, consider the function ζ defined on $[0,1]$ by $\zeta(t) = 1-t$ and $\zeta_{jk}(t) = 2^{j/2}\zeta(2^j x - k)$.

$$\begin{aligned} A_4 = & \sum_k \sum_{i \in s_{jk}} 2^j[(k+1)2^{-j} - i/n] F'(W_{k2^{-j}})(W_{i/n} - W_{(i-1)/n}) \frac{\delta_i(l_n) - \tilde{\delta}_i(l_n)}{n} \mathrm{d}s \\ & + \sum_k F'(W_{k2^{-j}}) \sum_{i \in s_{jk}} 2^{-j/2} \zeta_{jk}(i/n)(X_{i/n} - X_{(i-1)/n}) \sigma(X_{(i-1)/n})^{-1} \frac{\tilde{\delta}_i(l_n)}{n} \mathrm{d}s + R, \end{aligned}$$

with $\mathbb{E}[|R|] \le c(1/n + \alpha_n^{3/2})$. Using that the functions f_n verifies in our case

$$|f_n(x,u,y)| \le c(1+\beta_n)(1+|y|),$$

following Delattre [35, chap 6], we can show that the quantity

$$\sqrt{n}(X_{i/n} - X_{(i-1)/n}) \sigma(X_{(i-1)/n})^{-1} \tilde{\delta}_i(l_n)(1+\beta_n)^{-1}$$

can be written $g_n(X_{(i-1)/n}, \{X_{(i-1)/n}\alpha_n\}, \sqrt{n}[X_{i/n} - X_{(i-1)/n}])$. The function g_n verifies assumption D. Hence, as $Mg_n(x) = 0$, using the same arguments as in the proof of proposition II.1, we can prove that

$$\mathbb{E}[|\frac{1}{n} \sum_{i \in s_{jk}} \zeta_{jk}(i/n) g_n(X_{(i-1)/n}, \{X_{(i-1)/n}\alpha_n\}, \sqrt{n}[X_{i/n} - X_{(i-1)/n}])|^2] \le c r_n^2.$$

Consequently

$$\mathbb{E}[|\sum_k F'(W_{k2^{-j}}) \sum_{i \in s_{jk}} 2^{-j/2} \zeta_{jk}(i/n)(X_{i/n} - X_{(i-1)/n}) \frac{\tilde{\delta}_i(l_n)}{n} \mathrm{d}s|] \le c 2^{j/2} r_n^2.$$

For the first term, we use that

$$\frac{1}{n^2} \sum_{i=1}^n \mathbb{E}[|\delta_i(l_n) - \tilde{\delta}_i(l_n)|^2] \le c r_n^2 / \log n$$

and finally

$$\begin{aligned} \mathbb{E}[| \sum_k \sum_{i \in s_{jk}} 2^j[(k+1)2^{-j} - i/n](W_{i/n} - W_{(i-1)/n}) F'(W_{k2^{-j}}) \frac{\delta_i(l_n) - \tilde{\delta}_i(l_n)}{n} \mathrm{d}s|] \\ \le c r_n (\log n)^{-1/2}. \end{aligned}$$

We now turn to T_4.

$$-T_4 = \sum_k \sum_{i \in s_{jk}} [\sigma(X_{(i-1)/n}) - \sigma(X_{k2^{-j}})](\frac{\delta_i(l_n)}{n} + r_i + R_i),$$

with

$$\begin{aligned} r_i &= \frac{1}{n}[\bar{m_n} f_n(X_{i/n}) - \bar{m_n} f_n(X_{(i-1)/n})] \\ &- \frac{1}{n}\sum_{z=1}^{n-i-1}[l_z^n f_n(X_{i/n}) - l_z^n f_n(X_{(i-1)/n})] - \frac{1}{n} l_{n-i}^n f_n(X_{(i-1)/n}) \mathbb{1}_{i \leq n-2} \end{aligned}$$

and $|R_i| \leq cn^{-3/2}$. We easily get

$$\begin{aligned} &\mathbb{E}[|\sum_k \sum_{i \in s_{jk}} [\sigma(X_{(i-1)/n}) - \sigma(X_{k2^{-j}})] \frac{\delta_i(l_n)}{n}|^2] \\ &\qquad = \mathbb{E}[|\sum_k \sum_{i \in s_{jk}} [\sigma(X_{(i-1)/n}) - \sigma(X_{k2^{-j}})]^2 (\frac{\delta_i(l_n)}{n})^2|] \leq c2^{-j} r_n^2. \end{aligned}$$

The second term of the decomposition can be written

$$\sum_k \sum_{i \in s_{jk}} [\sigma(X_{(i-1)/n}) - \sigma(X_{k2^{-j}})] r_i = B_1 + B_2 + B_3$$

with

$$\begin{aligned} B_1 &= -\sum_k \sum_{i \in s_{jk}} [\sigma(X_{i/n}) - \sigma(X_{(i-1)/n})] r_i, \\ B_2 &= \sum_k [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{k2^{-j}})] \sum_{i \in s_{jk}} r_i, \\ B_3 &= -\sum_k \sum_{i \in s_{jk}} [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_i)] r_i. \end{aligned}$$

Using lemma II.4, we get

$$\mathbb{E}[|B_2|] \leq 2^{j/2}(\frac{1}{n} + \alpha_n^2 \log n).$$

For B_1 we consider the decomposition $B_1 = B_{11} - B_{12}$ with

$$\begin{aligned} B_{11} &= \sum_i [\sigma(X_{i/n}) - \sigma(X_{(i-1)/n})] z_i^n, \\ B_{12} &= \sum_i [\sigma(X_{i/n}) - \sigma(X_{(i-1)/n})](\frac{\delta_i(l_n)}{n} + R_i). \end{aligned}$$

Using the same method as for A_4, we get $\mathbb{E}[|B_{12}|] \leq cr_n(\log n)^{-1/2}$. We have for the other term

$$B_{11} = \sum_i (X_{i/n} - X_{(i-1)/n})\sigma'(X_{(i-1)/n})q_i^n + \sum_i (X_{i/n} - X_{(i-1)/n})\sigma'(X_{(i-1)/n})\frac{\sigma(X_{(i-1)/n})}{n} + R$$

with $\mathbb{E}[|R|] \leq c/n$. Hence, we easily get that $\mathbb{E}[|B_{11}|] \leq cr_n n^{-1/2}$. We now treat B_3. The quantity

$$\sum_k \sum_{i\in s_{jk}} [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{i/n})]\frac{1}{n}[\bar{m}_n f_n(X_{i/n}) - \bar{m}_n f_n(X_{(i-1)/n})]$$

can be written

$$\sum_k \sum_{i\in s_{jk}} [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{i/n})]\frac{1}{n}\bar{m}_n f_n(X_{i/n}) - \sum_k \sum_{i\in s_{jk}} [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{i-1/n})]\frac{1}{n}\bar{m}_n f_n(X_{(i-1)/n}) + \sum_k \sum_{i\in s_{jk}} [\sigma(X_{i/n}) - \sigma(X_{(i-1)/n})]\frac{1}{n}\bar{m}_n f_n(X_{(i-1)/n}).$$

This is equal to

$$-\sum_k [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{k2^{-j}})]\frac{1}{n}\bar{m}_n f_n(X_{k2^{-j}}) + \sum_k \sum_{i\in s_{jk}} (X_{i/n} - X_{(i-1)/n})\sigma'(X_{(i-1)/n})\frac{1}{n}\bar{m}_n f_n(X_{(i-1)/n}) + R$$

with $\mathbb{E}|R| \leq cr_n^2$. Eventually this term is equal to

$$R' + \sum_k \sum_{i\in s_{jk}} (X_{i/n} - X_{(i-1)/n})\sigma'(X_{(i-1)/n})\frac{1}{n}\bar{m}_n f_n(X_{(i-1)/n}) + R,$$

and $\mathbb{E}|R'| \leq c2^{j/2}r_n^2$. The quantity

$$\sqrt{n}(X_{i/n} - X_{(i-1)/n})\sigma'(X_{(i-1)/n})\bar{m}_n f_n(X_{(i-1)/n})(1+\beta_n)^{-1}$$

can be written $g_n(X_{(i-1)/n}, \{X_{(i-1)/n}\alpha_n\}, \sqrt{n}[X_{i/n} - X_{(i-1)/n}])$. This function verifies assumption D. Hence, as $Mg_n(x) = 0$, we obtain

$$\mathbb{E}[|\sum_k \sum_{i\in s_{jk}} (X_{i/n} - X_{(i-1)/n})\sigma'(X_{(i-1)/n})\frac{1}{n}\bar{m}_n f_n(X_{(i-1)/n})|] \leq cr_n^2.$$

We now treat

$$\sum_k \sum_{i\in s_{jk}} [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{i/n})]\frac{1}{n}\sum_{z=1}^{(n-i-1)\wedge(l_n-2)} [l_z^n f_n(X_{i/n}) - l_z^n f_n(X_{(i-1)/n})].$$

This can be written

$$\begin{aligned}&\sum_k \sum_z \sum_{i\in s_{jk}} [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{i/n})]\frac{1}{n} l_z^n f_n(X_{i/n})\\ &-\sum_k \sum_z \sum_{i\in s_{jk}} [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{i-1/n})]\frac{1}{n} l_z^n f_n(X_{(i-1)/n})\\ &+\sum_k \sum_z \sum_{i\in s_{jk}} [\sigma(X_{i/n}) - \sigma(X_{(i-1)/n})]\frac{1}{n} l_z^n f_n(X_{(i-1)/n}).\end{aligned}$$

Hence, it is equal to

$$\begin{aligned}&-\sum_k \sum_z [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{k2^{-j}})]\frac{1}{n} l_z^n f_n(X_{(i-1)/n})\\ &+\sum_k \sum_z \sum_{i\in s_{jk}} (X_{i/n} - X_{(i-1)/n})\sigma'(X_{(i-1)/n})\frac{1}{n} l_z^n f_n(X_{(i-1)/n}) + R,\end{aligned}$$

with $\mathbb{E}|R| \leq c\alpha_n^2 \log n$. This is finally equal to

$$R' + \sum_z \sum_k \sum_{i\in s_{jk}} (X_{i/n} - X_{(i-1)/n})\sigma'(X_{(i-1)/n})\frac{1}{n} l_z^n f_n(X_{(i-1)/n}) + R,$$

with $\mathbb{E}|R'| \leq c2^{j/2}\alpha_n^2 \log n$. The quantity

$$\sqrt{n}(X_{i/n} - X_{(i-1)/n})\sigma'(X_{(i-1)/n}) l_z^n f_n(X_{(i-1)/n})(\alpha_n^2[1 + n/z])^{-1}$$

can be written $g_n(X_{(i-1)/n}, \sqrt{n}[X_{i/n} - X_{(i-1)/n}])$. This function verifies assumption D. Hence, as $Mg_n(x) = 0$,

$$\mathbb{E}[|\sum_k \sum_{i\in s_{jk}} (X_{i/n} - X_{(i-1)/n})\sigma'(X_{(i-1)/n})\frac{1}{n} l_z^n f_n(X_{(i-1)/n})| \leq cn^{-1}\alpha_n^2(1 + n/z).$$

Eventually,

$$\mathbb{E}[|\sum_z \sum_k \sum_{i\in s_{jk}} (X_{i/n} - X_{(i-1)/n})\sigma'(X_{(i-1)/n})\frac{1}{n} l_z^n f_n(X_{(i-1)/n})|] \leq c\alpha_n^2(1 + \log n).$$

It is also clear that

$$\mathbb{E}\big[|[\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{i/n})]\frac{1}{n} l_{n-i}^n f_n(X_{(i-1)/n})|\big] \leq c\alpha_n^2(1 + \log n).$$

We now turn to T_3. Let $c_\eta = (2/\pi)^{1/2}$. We have

$$T_3 = c_\eta \sum_i \int_{(i-1)/n}^{i/n} [\sigma(X_s) - \sigma(X_{(i-1)/n})]\sigma(X_s)\mathrm{d}s.$$

We can write

$$\begin{aligned} T_3 = c_\eta \sum_i & \int_{(i-1)/n}^{i/n} [\sigma(X_s) - \sigma(X_{(i-1)/n})]^2 \mathrm{d}s \\ + c_\eta \sum_i & \int_{(i-1)/n}^{i/n} [\sigma(X_s) - \sigma(X_{(i-1)/n})]\sigma(X_{(i-1)/n})\mathrm{d}s. \end{aligned}$$

Ito's formula gives

$$\begin{aligned} T_3 = c_\eta \sum_i & \int_{(i-1)/n}^{i/n} \mathrm{d}s\ \sigma(X_{(i-1)/n}) \int_{(i-1)/n}^{s} \sigma'(X_t)\sigma(X_t)\mathrm{d}W_t \\ + c_\eta \sum_i & \int_{(i-1)/n}^{i/n} \mathrm{d}s\ \sigma(X_{(i-1)/n}) \int_{(i-1)/n}^{s} \frac{1}{2}\sigma'(X_t)^2\sigma(X_t) + \frac{1}{2}\sigma(X_t)^2\sigma''(X_t)\mathrm{d}t + R, \end{aligned}$$

with $\mathbb{E}[|R|] \leq c/n$. Finally, we obtain

$$T_3 = R + c_\eta \sum_i \int_{(i-1)/n}^{i/n} \mathrm{d}s\ \sigma(X_{(i-1)/n}) \int_{(i-1)/n}^{s} \sigma'(X_t)\sigma(X_t)\mathrm{d}W_t + R',$$

with $\mathbb{E}[|\tilde{R}'|] \leq c/n$. Let

$$\eta_i = \int_{(i-1)/n}^{i/n} \mathrm{d}s\ \sigma(X_{(i-1)/n}) \int_{(i-1)/n}^{s} \sigma'(X_t)\sigma(X_t)\mathrm{d}W_t.$$

For $i' < i$, $\mathbb{E}_{\mathcal{F}_{i'}}[\eta_i] = 0$. Hence, for given n, $M_i^n = \sum_{j=1}^{i} \eta_j^n$ is a martingale. Consequently,

$$\mathbb{E}[(M_i^n)^2] = \sum_{i=1}^{n} \mathbb{E}[\eta_i^2].$$

Since,

$$\eta_i^2 \leq \frac{1}{n} \int_{(i-1)/n}^{i/n} \mathrm{d}s\ (\sigma(X_{(i-1)/n}) \int_{(i-1)/n}^{s} \sigma'(X_t)\sigma(X_t)\mathrm{d}W_t)^2,$$

we get

$$\mathbb{E}[(M_i^n)^2] \leq c/n^2.$$

We conclude the proof applying the results of Delattre [35, chap 2] to the term $T2$. □

The proof of theorem II.3 follows using the result on the compensator.

4.6 Preliminaries for the proofs of theorem II.2 and theorem II.4

We use here the same localization procedure and change of probability as for theorem II.1 and theorem II.3. Thus we get that we can prove theorem II.2 and theorem II.4 under assumptions B' and C instead of assumption B. Nevertheless, we have to take into account the fact that for $q \in \mathbb{N}$, $X^q \geq \nu_q > 0$ on $[0, T_q]$ only. Since $T_q \to 1$, a.s. and $\mathbb{P}(T_q = 1) \to 1$, we easily get that $\mathbb{P}'(T_q = 1) \to 1$. Hence it is in fact enough to prove for theorem II.2 that for any $q \in \mathbb{N}$, for all $\varepsilon > 0$, there exists $M > 0$ such that

$$\sup_{n \geq n_0} \mathbb{P}'[r_n^{-1}(\widetilde{\lambda}_n - \lambda) \geq M \cap \Omega_q] \leq \varepsilon,$$

with $\Omega_q = \{\omega \in \Omega, T_q(\omega) = 1\}$ and n_0 be such that $\nu_q - \alpha_{n_0} > 0$. For theorem II.4, we just have to prove that for any $q \in \mathbb{N}$ the stable convergence in law in restriction to Ω_q holds: for any $\mathcal{F}$-measurable bounded variable Y, vanishing outside Ω_q, and any bounded continuous function g,

$$\mathbb{E}_{\mathbb{P}'}[Yg(r_n^{-1}(\widehat{\lambda}_n - \lambda)]) \to \bar{\mathbb{E}}_{\bar{\mathbb{P}}'}[Yg(Z)],$$

with Z the appropriate limit variable defined in theorem II.4. Let $0 \leq T \leq 1$ be a stopping time such that there exists a constant $K > 0$ satisfying $X_t \geq K$ for all $t \in [0, T]$. Let $\phi \in \mathcal{C}_b^2(\mathbb{R})$ be such that $\phi(x) = x$ on $[K, +\infty)$ and $\inf_{x \in \mathbb{R}} \phi(x) > 0$. It is finally enough to prove the two following propositions.

Proposition II.4 *Let*

$$\dot{\lambda} = \int_0^1 \phi(X_s)^{-2} \sigma^2(X_s) ds,$$

$$\widehat{e}_{j_0 k} = \sqrt{\frac{\pi}{2}} \frac{2^{j_0/2}}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{1}_{j_0 k}(i/n) \frac{|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|}{\phi(X_{(i-1)/n}^{(\alpha_n)})},$$

and

$$\widetilde{\dot{\lambda}}_n = \sum_{k=0}^{2^{j_0(n)}-1} \widehat{e}_{j_0(n)k}^{\,2}, \text{ with } j_0(n) = \lfloor \log_2 r_n^{-1} \rfloor.$$

Under assumptions A, B' and C, the sequence

$$r_n^{-1}(\widetilde{\dot{\lambda}}_n - \dot{\lambda})$$

is tight.

We use the notation of proposition II.4 for the second proposition.

Proposition II.5 *Let $S = \big(a, (j_{1,n}), (j_{2,n})\big) \in \mathcal{S}$, we define*

$$\widehat{\dot{f}}_{jk} = \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{jk}(i/n) \frac{|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|}{\phi(X_{(i-1)/n}^{(\alpha_n)})},$$

$$\widehat{\dot{Q}}_{j_{2,n}} = \sum_k \widehat{\dot{f}}_{jk}^{\,2}, \quad \dot{R}_n(S) = \sum_{j=j_{1,n}}^{\lfloor (1+a)\log_2 r_n^{-1} \rfloor} 2^{j_{2,n}-j} \widehat{\dot{Q}}_{j_{2,n}}$$

and

$$\widehat{\dot{\theta}}_n(S) = (1-\alpha_n)\Big\{ \sum_{k=0}^{2^{j_1(n)}-1} \widehat{\dot{e}}_{j_{1,n}k}^{\,2} + \dot{R}_n(S) \Big\}.$$

Under assumptions A1, B' and C, we have the following stable convergences in law, where B is a standard Brownian motion, independent of $\mathcal{F}$.

$$\text{if } \beta_n \to 0, \qquad r_n^{-1}(\widehat{\dot{\lambda}}_n(S) - \dot{\lambda}) \to_{\mathcal{L}s} \sqrt{2}(\pi-2)^{1/2} \int_0^1 \phi(X_t)^{-1}\sigma(X_t)^2 \mathrm{d}B_t,$$

$$\text{if } \beta_n \to \beta > 0, \qquad r_n^{-1}(\widehat{\dot{\lambda}}_n(S) - \dot{\lambda}) \to_{\mathcal{L}s} 2 \int_0^1 \phi(X_t)^{-1}\sigma(X_t)[\Delta_\beta(X_t)]^{1/2} \mathrm{d}B_t,$$

$$\text{if } \beta_n \to +\infty, \qquad r_n^{-1}(\widehat{\dot{\lambda}}_n(S) - \dot{\lambda}) \to_{\mathcal{L}s} \frac{2}{\sqrt{3}} \int_0^1 \phi(X_t)^{-1}\sigma(X_t) \mathrm{d}B_t.$$

4.7 Proof of proposition II.4

We work for $n \geq n_0$ with n_0 such that $K - \alpha_{n_0} > 0$. The following proposition implies proposition II.4.

Proposition II.6 *(L^1 convergence, relative integrated volatility) Under assumptions A, B' and C,*

$$\mathbb{E}[|\widetilde{\dot{\lambda}}_n - \dot{\lambda}|] \leq c r_n,$$

with c a constant not depending on n.

4.7.1 Proof of proposition II.6

Let

$$\dot{e}_{jk} = 2^{j/2} \int \mathbb{1}_{jk}(s)\phi(X_s)^{-1}\sigma(X_s)\mathrm{d}s$$

and

$$\widehat{\dot{e}}_{jk}^{\,*} = \sqrt{\frac{\pi}{2}} \frac{2^{j_0/2}}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{1}_{j_0k}(i/n) \frac{|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|}{\phi(X_{(i-1)/n})}.$$

We easily get the result in the same way as in the absolute integrated volatility case remarking that

$$\widehat{\dot{e}}_{jk} = \widehat{\dot{e}}_{jk}^{\,*}(1+Y),$$

with $|Y| \leq c\alpha_n$. Hence, using proposition II.1, we obtain

$$\mathbb{E}[|\widehat{\dot{e}}_{jk} - \dot{e}_{jk}|^2] \leq c\mathbb{E}[|\widehat{\dot{e}}_{jk} - \widehat{\dot{e}}_{jk}^{\,*}|^2] + c\mathbb{E}[|\widehat{\dot{e}}_{jk}^{\,*} - \dot{e}_{jk}|^2] \leq cr_n^2.$$

and

$$\sum_k \dot{e}_{jk}(\widehat{\dot{e}}_{jk} - \dot{e}_{jk}) = c\sum_k \dot{e}_{jk}(\widehat{\dot{e}}_{jk}^{\,*} - \dot{e}_{jk}) + Z,$$

with $\mathbb{E}|Z| \leq c\alpha_n$. The result follows.

4.8 Proof of proposition II.5

We give a sketch of proof of this theorem. We have the following lemma.

Lemma II.10 *Let*

$$\dot{E}_{j_n} = \sum_k \dot{e}_{j_n k}(\widehat{\dot{e}}_{j_n k} - \dot{e}_{j_n k}).$$

if $2^{-j_n} + r_n^{-1}2^{j_n/2}(n^{-1} + \alpha_n^2 \log n) + 2^{j_n} r_n \to 0$, *then we have the following convergences in stable law, where* B *is a standard Brownian motion, independent of* $\mathcal{F}$.

$$\text{if } \beta_n \to 0, \qquad r_n^{-1}\dot{E}_{j_n} \to_{\mathcal{L}s} \frac{1}{\sqrt{2}}(\pi-2)^{1/2}\int_0^1 \phi(X_t)^{-1}\sigma(X_t)^2 \mathrm{d}B_t,$$

$$\text{if } \beta_n \to \beta > 0, \quad r_n^{-1}\dot{E}_{j_n} \to_{\mathcal{L}s} \beta\int_0^1 \frac{\phi'(X_s)\sigma(X_s)^2}{2\phi(X_s)^2}ds + \int_0^1 \phi(X_t)^{-1}\sigma(X_t)[\Delta_\beta(X_t)]^{1/2}\mathrm{d}B_t,$$

$$\text{if } \beta_n \to +\infty, \quad r_n^{-1}\dot{E}_{j_n} \to_{\mathcal{L}s} \int_0^1 \frac{\phi'(X_s)\sigma(X_s)^2}{2\phi(X_s)^2}ds + \frac{1}{\sqrt{3}}\int_0^1 \phi(X_t)^{-1}\sigma(X_t)\mathrm{d}B_t.$$

Proof. In this case, we have by analogy with section 4.5.2

$$f_n(x,u,y) = (\pi/2)^{1/2}\frac{\beta_n|\lfloor u + y/\beta_n\rfloor|}{\phi(x-\alpha_n u)}.$$

Let

$$\tilde{f}_n(x,u,y) = (\pi/2)^{1/2}\frac{\beta_n|\lfloor u + y/\beta_n\rfloor|}{\phi(x)}.$$

We have

$$\sum_k \dot{e}_{jk}(\widehat{\dot{e}}_{jk} - \dot{e}_{jk}) = \sum_k \dot{e}_{jk}(\widehat{\dot{e}}_{jk}^{\,*} - \dot{e}_{jk}) + Z,$$

with $\mathbb{E}[|Z|] \leq c\alpha_n$. Hence, we easily get the result when β_n tends to zero. We also have

$$\sum_k \dot{e}_{jk}(\widehat{\dot{e}}_{jk} - \dot{e}_{jk}) = \sum_k \dot{e}_{jk}(\widehat{\dot{e}}_{jk} - 2^{j/2}\int \mathbb{1}_{jk}(s) M f_n(X_s)\mathrm{d}s)$$
$$+ \sum_k \dot{e}_{jk}(2^{j/2}\int \mathbb{1}_{jk}(s)[M f_n(X_s) - M\tilde{f}_n(X_s)]\mathrm{d}s).$$

A bias is induced by the second term if β_n does not tend to zero. If $\beta_n \to \beta > 0$,

$$\sqrt{n}[M f_n(X_s) - M\tilde{f}_n(X_s)] \approx \beta\frac{\phi'(X_s)\sigma(X_s)}{2\phi(X_s)^2},$$

and if $\beta_n \to \infty$,

$$\alpha_n^{-1}[M f_n(X_s) - M\tilde{f}_n(X_s)] \approx \frac{\phi'(X_s)\sigma(X_s)}{2\phi(X_s)^2}.$$

We conclude the proof of theorem II.4 using that if U_n tends to U in probability on Ω and Y_n tends to Y in stable law, then (U_n, Y_n) tends to (U, Y) in stable law. $\square$

Chapter 2

Inhomogeneous case

1 Model and results

We consider in this chapter the case of a one-dimensional Brownian semimartingale $(X_t)_{t\in[0,1]}$, taking values in an open interval (ν,μ), $-\infty \leq \nu < \mu \leq +\infty$, of the form

$$X_t = x_0 + \int_0^t \sigma(X_s, s)\mathrm{d}W_s + \int_0^t a_s \mathrm{d}s, \tag{II.11}$$

where $(W_t)_{t\in[0,1]}$ is a $(\mathcal{F}_t)$-standard Brownian motion, $(a_t)_{t\in[0,1]}$ a progressively measurable process with respect to $(\mathcal{F}_t)_{t\in[0,1]}$, $(x,t)\rightarrow\sigma(x,t)$ a real deterministic function and x_0 a real constant. We observe the sample

$$(X_{i/n}^{(\alpha_n)}, i = 0, \ldots, n), \tag{II.12}$$

More particularly, we suppose that $(x,t)\rightarrow\sigma(x,t)$ is of the form

$$\sigma(x,t) = g_1(x)g_2(t).$$

This specification covers several classical models, see for example Musiela and Rutkowski [96]. Moreover, note that introducing the function g_2 enables to give (conditional on the path of the volatility) results for some stochastic volatility models where the volatility is independent of the driving Brownian motion. We suppose throughout this chapter that assumption A is in force. We replace assumption B by assumption $\tilde{B}$, defined the following way.

Assumption $\tilde{B}$.

(i) For all $x \in (\nu,\mu)$, $g_1(x) > 0$ and $g_1 \in \mathcal{C}^2\big((\nu,\mu)\big)$,

(ii) For all $t \in [0,1]$, $g_2(t) > 0$ and $g_2 \in \mathcal{C}^2([0,1])$,

(iii) $\int_0^1 a_s^2 \mathrm{d}s < +\infty$, almost surely.

The wavelet coefficients, their estimators, and the estimators $\widetilde{\theta}_n$ and $\widetilde{\lambda}_n$ are defined the same way as in the homogeneous case, replacing $\sigma(X_s)$ by $\sigma(X_s, s)$.

Theorem II.5 *(Absolute integrated volatility). In model* (II.11)-(II.12)*, under assumptions A and $\tilde{B}$, the sequence*

$$r_n^{-1}(\widetilde{\theta}_n - \theta)$$

is tight.

Theorem II.6 *(Relative integrated volatility). In model* (II.11)-(II.12)*, under assumptions A and $\tilde{B}$, if $\nu \geq 0$, the sequence*

$$r_n^{-1}(\widetilde{\lambda}_n - \lambda)$$

is tight.

We only give the proof of theorem II.5, the proof of theorem II.6 being deduced in the same way as in the homogeneous case. In all this proof, we follow Delattre [35], adapting and extending his results to our context. We first present the localization procedure and the change of probability in section 2.1 and 2.2. Then we give in section 2.3 results on the transition densities, necessary to obtain a lemma analogous to lemma II.3 in the homogeneous case. Preliminary results are stated in section 2.4 and we write the fundamental decomposition in the inhomogeneous context and bound some of its terms in section 2.5. The last two sections are dedicated to the control of the main term of the decomposition. The proof of theorem II.5 follows in the same way as in the homogeneous case.

2 Proof of theorem II.5

2.1 Localization procedure

We fix $a_0 \in (\nu, \mu)$, two sequences $(\nu_q)_{q\geq 1}$ and $(\mu_q)_{q\geq 1}$ such that (ν_q) is strictly decreasing to ν and (μ_q) is strictly increasing to μ and a sequence of functions $\chi_q : \mathbb{R} \to [0,1]$ such that $\chi_q \in \mathcal{C}_b^2(\mathbb{R})$ and

$$\chi_q(x) = 1 \text{ on } [\nu_q, \mu_q] \text{ and } \chi_q(x) = 0 \text{ on } (-\infty, \nu_{q+1}] \cup, [\mu_{q+1}, +\infty).$$

For $q \in \mathbb{N}$, we set

$$\sigma_q : (x,t) \to \sigma(x,t)\chi_q(x) + \big(1-\chi_q(x)\big)\sigma(a_0,t)$$

and

$$T_q = \inf\{t \in [0,1], X_t \leq \nu_q \text{ or } X_t \geq \mu_q \text{ or } \int_0^t a_s^2 \mathrm{d}s \geq q\} \wedge 1.$$

Remark that $\sigma_q(x,t)$ is of the form $g_{1,q}(x)g_{2,q}(t)$ with $g_{2,q} = g_2$ and

$$g_{1,q}(x) = g_1(x)\chi_q(x) + \big(1-\chi_q(x)\big)g_1(a_0).$$

Under assumption $\tilde{B}$, T_q tends almost surely to 1 and $\mathbb{P}(T_q = 1) \to 1$ as $q \to +\infty$. Let $(W_t^q, t \geq 0)$ be defined by $W_t^q = W_{(T_q+t)\wedge 1} - W_{T_q}$ and $(Y_t^q)_{t\in[0,1-T_q]}$ be the unique solution of

$$\mathrm{d}Y_t^q = \sigma_q(Y_t^q, t+T_q)\mathrm{d}W_t^q, \quad Y_0^q = X_{T_q}.$$

Consider now the process $(X_t^q)_{t\in[0,1]}$ defined by $X_t^q = X_t$ for $t \in [0,T_q]$ and $X_t^q = Y_{t-T_q}^q$ for $t \in (T_q, 1]$. This process satisfies

$$\mathrm{d}X_t^q = \sigma_q(X_t^q, t)\mathrm{d}W_t + a_t^q \mathrm{d}t,$$

where a_t^q is defined by $a_t^q = a_t$ for $t \in [0,T_q]$ and $a_t^q = 0$ for $t \in (T_q, 1]$. The process X^q coincides with the initial process X on $[0,T_q]$. Hence it is enough to prove theorem II.5 for the processes X^q, for all $q \in \mathbb{N}$ and so it is enough to prove theorem II.5 under assumption $\tilde{B}'$ instead of assumption $\tilde{B}$, with assumption $\tilde{B}'$ defined the following way.

Assumption $\tilde{B}'$.

(i) There exists $c > 0$ such that for all $x \in \mathbb{R}$, $g_1(x) > c$ and $g_1 \in \mathcal{C}_b^2(\mathbb{R})$,

(ii) For all $t \in [0,1]$, $g_2(t) > 0$ and $g_2 \in \mathcal{C}^2([0,1])$,

(iii) $\sup_{\omega\in\Omega} \int_0^1 a_s^2 \mathrm{d}s < +\infty$.

2.2 Change of probability

We work under assumption $\tilde{B}'$. For the inhomogeneous case, we have to deal with an additional term in Girsanov theorem. We define

$$S_t : x \to \int_0^x \frac{1}{\sigma(y,t)}\mathrm{d}y.$$

By Ito's formula, we have

$$g_2(t)S_t(X_t) = \int_0^t g_2(t)\mathrm{d}W_t - \frac{1}{2}\int_0^t g_1'(X_t)g_2^2(t)\mathrm{d}t + \int_0^t \frac{a_t}{g_1(X_t)}\mathrm{d}t + \int_0^{x_0} \frac{1}{g_1(y)}\mathrm{d}y.$$

Let

$$W'_t = W_t + \int_0^t \frac{1}{2\sigma(X_s,s)}[2a_s - \sigma(X_s,s)\frac{\partial}{\partial x}\sigma(X_s,s) - 2g'_2(s)S_s(X_s)g_1(X_s)]\mathrm{d}s.$$

Since g_1 is bounded, using a Novikov's type criterium, see Revuz-Yor p.323 [101], by Girsanov theorem, we can construct a probability $\mathbb{P}'$ on $(\Omega, \mathcal{F})$, absolutely continuous with respect to $\mathbb{P}$ such that W'_t is a Brownian motion under $\mathbb{P}'$ and

$$\mathrm{d}X_t = \sigma(X_t,t)\mathrm{d}W'_t + \Big[\frac{1}{2}\sigma(X_t,t)\frac{\partial}{\partial x}\sigma(X_t,t) + g'_2(t)S_t(X_t)g_1(X_t)\Big]\mathrm{d}t,\ X_0 = x_0.$$

We now consider the following hypothesis.

Assumption $\tilde{B}''$.

(i) There exists $c > 0$ such that for all $x \in \mathbb{R}$, $g_1(x) > c$ and $g_1 \in \mathcal{C}_b^2(\mathbb{R})$,

(ii) For all $t \in [0,1]$, $g_2(t) > 0$ and $g_2 \in \mathcal{C}^2([0,1])$,

(iii)
$$a_t = \frac{1}{2}\sigma(X_t,t)\frac{\partial}{\partial x}\sigma(X_t,t) + g'_2(t)S_t(X_t)g_1(X_t).$$

The convergence in probability being preserved by absolutely continuous change of probability, it is consequently enough to prove theorem II.5 under assumptions A and $\tilde{B}''$. Thus, equation (II.11) admits a unique solution, $X_t = h(W_t,t)$, with

$$h : (x,t) \to S_t^{-1}\big(x + S_0(x_0)\big).$$

Note that

$$h(x,t) = K^{-1}(g_2(t)x)$$

and

$$\frac{\partial h(x,t)}{\partial t} = x g'_2(t) g_1\{K^{-1}\big(g_2(t)x\big)\}$$

with

$$K(x) = \int_0^x \frac{1}{g_1(y)}\mathrm{d}y.$$

The inhomogeneous case requires intermediary computations to prove some technical lemmas that can be found or partially found in Delattre [35] in the homogeneous case. We present them in next sections.

2.3 Transition densities

For the rest of proof of theorem II.5, without loss of generality, we suppose $x_0 = 0$. We easily obtain that

$$p_{s,t}(x,y) = \big(2\pi(t-s)\big)^{-1/2}\sigma(y,t)^{-1}\exp\big\{\frac{-\big(S_t(y)-S_s(x)\big)^2}{2(t-s)}\big\}.$$

Let $q_{s,t}(x,y) = p_{s,t}(x,x+y)$. After tedious but straightforward computations, using in particular that for $p > 0$,

$$\mathbb{E}_{X_s=x}[|S_t(X_t)-S_s(x)|^p] \le c|t-s|^{p/2}$$

and

$$\mathbb{E}_{X_s=x}[|X_t-X_s|^p] \le c|t-s|^{p/2} + c|x|^p|t-s|^p,$$

one can show the following lemma.

Lemma II.11 *For $0 \le i+j \le 2$ and $p > 0$,*

$$\begin{aligned}
|\frac{\partial^{i+j}}{\partial x^i \partial y^j} p_{s,t}(x,y)| &\le c p_{s,t}(x,y)|t-s|^{-(i+j)/2}\big\{1 + |\frac{S_t(y)-S_s(x)}{|t-s|^{1/2}}|^{i+j}\big\},\\
|\frac{\partial^i}{\partial x} q_{s,t}(x,y)| &\le c q_{s,t}(x,y)(1+x^2+y^2+(yx)^2+y^4|t-s|^{-2}),\\
\int |\frac{\partial^{i+j}}{\partial x^i \partial y^j} p_{s,t}(x,y)|dy &\le c|t-s|^{-(i+j)/2},\\
\int |y|^p|\frac{\partial^i}{\partial x} q_{s,t}(x,y)|dy &\le c(1+|x|^{4+p})|t-s|^{p/2},\\
|\frac{\partial}{\partial t} q_{t,t+u}(x,y)| &\le c q_{t,t+u}(|y| + |u(x+y)|).
\end{aligned}$$

We now define for $t > s$ and $p > 0$

$$\begin{aligned}
m_{x,s,t} &= \sigma(x,t)[S_t(x)-S_s(x)]\\
h_{x,s,t}(y) &= \{2\pi(t-s)\sigma(x,t)\}^{-1/2}\exp\{-(y+m_{x,s,t})^2[2(t-s)\sigma(x,t)^2]^{-1}\},\\
g(x,y,t) &= \frac{1}{2}\sigma'(x,t)\sigma(x,t)^{-1}(y^2\sigma(x,t)^{-2}-2)y,\\
b_{s,t}(x,y) &= |q_{s,t}(x,y) - h_{x,s,t}(y)\big(1+|t-s|^{1/2}g(x,\frac{y}{|t-s|^{1/2}})\big)|,\\
f_{s,t,p}(x,y) &= b_{s,t}(x,y)(1+|\frac{y}{|t-s|^{1/2}}|^p).
\end{aligned}$$

We have the following lemma.

Lemma II.12 *For* $|y| \leq |t-s|^{1/3}$,

$$b_{s,t}(x,y) \leq ch_{x,s,t}(y)|t-s|(1+|x|^2)\big\{|\frac{y}{|t-s|^{1/2}}|^4 + \exp[c|x|]\big\}.$$

Moreover,

$$\int_{\mathbb{R}} f_{s,t,p}(x,y)\,dy \leq c|t-s|(1+|x|^{p+7})\exp[c|x|].$$

Proof. We have

$$\frac{q_{s,t}(x,y)}{h_{x,s,t}(y)} - 1 - |t-s|^{1/2} g(x, \frac{y}{|t-s|^{1/2}}, s, t) = T_1 + T_2 T_3 + T_4$$

with

$$\begin{aligned}
T_1 &= \frac{\sigma(x,t)}{\sigma(x+y,t)} - 1 + \frac{\sigma'(x,t)}{\sigma(x,t)} y \\
T_2 &= T_1 - \frac{\sigma'(x,t)}{\sigma(x,t)} y \\
T_3 &= \exp\Big\{\frac{-[S_t(x+y) - S_s(x)]^2 + (y + m_{x,s,t})^2/\sigma(x,t)^2}{2(t-s)}\Big\} - 1 \\
T_4 &= T_3 - \frac{\sigma'(x,t)y^3}{2(t-s)\sigma(x,t)^3}.
\end{aligned}$$

We easily get that $|T_1| \leq cy^2$. We now turn on the term T_4. Let

$$z = -[S_t(x+y) - S_s(x)]^2 + [(y + m_{x,s,t})/\sigma(x,t)]^2.$$

We have

$$z = -\big[\big((y+m_{x,s,t})/\sigma(x,t)\big) - \big(y^2\sigma'(x,t)/2\sigma(x,t)^2\big) + cy^3\big]^2 + \big((y+m_{x,s,t})/\sigma(x,t)\big)^2.$$

Since $y \leq |t-s|^{1/3}$,

$$\big|z - [y^3\sigma'(x,t)/\sigma(x,t)^3]\big| \leq c(y^4 + y^2|x(t-s)|).$$

Let $z' = z/\big(2(t-s)\big)$. Using that $|\exp(u) - 1 - u| \leq u^2 \exp|u|$, for $|y| \leq |t-s|^{1/3}$, we get

$$|\exp(z') - 1 - \frac{\sigma'(x,t)y^3}{2(t-s)\sigma(x,t)^3}| \leq c(y^4|t-s|^{-1} + y^2|x|) + z'^2 \exp|z'|.$$

Consequently,

$$|T_4| \leq c|t-s|(1+|x|^2)\big\{|\frac{y}{|t-s|^{1/2}}|^4 + c\exp[c|x|]\big\}.$$

We prove the same way that

$$|T_2T_3| \leq c|t-s|(1+|x|^2)\big\{|\frac{y}{|t-s|^{1/2}}|^4 + c\exp[c|x|]\big\}.$$

The first inequality of the lemma follows. Let $p>0$ the quantity

$$\int_{|y|\leq|t-s|^{1/3}} f_{s,t,p}(x,y)\mathrm{d}y$$

is less than

$$c\int_{\mathbb{R}} h_{x,s,t}(y)|t-s|(1+|x|^2)\big\{|\frac{y}{|t-s|^{1/2}}|^{p+4} + c\exp[c|x|]\big\}\mathrm{d}y.$$

This can be rewritten

$$c|t-s|(1+|x|^2)\{\exp[c|x|] + \mathbb{E}[|Z|^{p+4}]\},$$

where Z is a Gaussian variable with mean $-m_{x,s,t}|t-s|^{-1/2}$ and variance $\sigma(x,t)^2$. Thus, we obtain

$$\int_{|y|\leq|t-s|^{1/3}} f_{s,t,p}(x,y)\mathrm{d}y \leq c|t-s|(1+|x|^2)\exp[c|x|].$$

Using Markov's inequality, we easily have

$$\int_{|y|\geq|t-s|^{1/3}} q_{s,t}(x,y)(1+|\frac{y}{|t-s|^{1/2}}|^p)\mathrm{d}y \leq c(1+|x|^{p+6})|t-s|$$

and

$$\int_{|y|\geq|t-s|^{1/3}} h_{x,s,t}(y)|t-s|^{1/2}(1+|\frac{y}{|t-s|^{1/2}}|^{p+3})(1+|x|)\mathrm{d}y \leq c(1+|x|^{p+7})|t-s|.$$

The result follows. □

2.4 Preliminary results

Assumption $\tilde{D}$. Let $(x,u,y)\to f_n(x,u,y)$ be a sequence of real functions on $\mathbb{R}\times[0,1]\times\mathbb{R}$. The sequence f_n verifies assumption $\tilde{D}$ if the functions f_n are twice continuously differentiable with respect to the first variable and if there exists $\gamma>0$ such that for all $n\geq 1$,

(i) $|f_n(x,u,y)| \leq \gamma(1+\beta_n)(1+|y|^\gamma)$,

(ii) $\int_0^1 |f_n(x,u,y)|\mathrm{d}u \leq \gamma(1+|y|^\gamma)$,

(iii) $|\frac{\partial^i}{\partial x^i} f_n(x,u,y)| \leq \gamma(1+\beta_n)(1+|y|^\gamma),\ i=1,2,$

(iv) $\int_0^1 |\frac{\partial^i}{\partial x^i} f_n(x,u,y)|\mathrm{d}u \leq \gamma(1+|y|^\gamma),\ i=1,2.$

Remark that it is in fact enough to consider β_n in (i) and (iii) instead of β_n^2. Indeed, we do not want to show a general result as in the homogeneous case and we will mainly consider $f_n(x,u,y) = \beta_n|\lfloor u + y/\beta_n\rfloor| \leq c(\beta_n + |y|)$. Let

$$h_\sigma^m(y) = \frac{1}{\sqrt{2\pi}\sigma}\exp\big(\frac{(y-m)^2}{2\sigma^2}\big).$$

We use the following notation.

$$m_{s,t}f(x,u,t) = \int_{\mathbb{R}} h_{\sigma(x,t)}^{-|t-s|^{-1/2}m_{x,s,t}}(y)f(x,u,y)\mathrm{d}y,\quad M_{s,t}f(x,t) = \int_0^1 m_{s,t}f(x,u,t)\mathrm{d}u$$

$$mf(x,u,s) = m_{s,s}f(x,u,s),\quad Mf(x,s) = \int_0^1 m_{s,s}f(x,u,s)\mathrm{d}u$$

$$m_nf_n(x,u,t) = \int q_{t,t+1/n}(x,y)f_n(x,u,\sqrt{n}y)\mathrm{d}y,\quad M_nf_n(x,t) = \int_0^1 m_nf_n(x,u,t)\mathrm{d}u.$$

We also define

$$\bar{m}_nf_n(x,t) = m_nf_n(x,\{x/\alpha_n\},t) - M_nf_n(x,t)$$

and

$$l_i^nf_n(x,t) = \int p_{t,t+i/n}(x,y)\bar{m}_nf_n(y,t)\mathrm{d}y.$$

From now, we denote by $P(x)$ any polynomial in x. We have the following lemma.

Lemma II.13 *Let f_n be a sequence verifying assumption $\tilde{D}$. The following inequalities hold.*

$$|m_nf_n(x,u,t)| \leq c(1+\beta_n)P(|x|), \tag{II.13}$$

$$\int_0^1 |\frac{\partial^i}{\partial x^i}m_nf_n(x,u,t)|du + |\frac{\partial^i}{\partial x^i}M_nf_n(x,t)| \leq cP(|x|),\quad 0\leq i\leq 2, \tag{II.14}$$

$$\int_0^1 |m_nf_n(x,u,t) - m_{t,t+1/n}f_n(x,u,t)|du \leq cn^{-1/2}P(|x|)\exp[c|x|], \tag{II.15}$$

$$|M_nf_n(x,t) - M_{t,t+1/n}f_n(x,t)| \leq cn^{-1/2}P(|x|)\exp[c|x|], \tag{II.16}$$

$$|l_i^nf_n(x,t)| \leq c\alpha_n^2P(|x|)(n/i), \tag{II.17}$$

$$|l_i^nf_n(x,t)| \leq c\alpha_nP(|x|)(n/i)^{1/2}. \tag{II.18}$$

Proof. The two first results are easily obtained in the same way as in the homogeneous case from assumption $\tilde{D}$ together with lemma II.11. The quantity

$$\int_0^1 |m_nf_n(x,u,t) - m_{t,t+1/n}f_n(x,u,t)|\mathrm{d}u$$

is smaller than

$$c\int_0^1\int_{\mathbb{R}} b_{t,t+1/n}(x,y)|f_n(x,u,n^{1/2}y)|\mathrm{d}y\mathrm{d}u$$

$$+c\int_0^1\int_{\mathbb{R}} h_{x,t,t+1/n}n^{-1/2}|g(x,n^{1/2}y)f_n(x,u,n^{1/2}y)|\mathrm{d}y\mathrm{d}u.$$

The third and fourth inequalities follow using assumption $\tilde{D}$ and lemma II.12. The last results are obtained from lemma II.1 on the fractional part of a variable. □

2.5 Decomposition

Notation.

Let $s_{jk} = [2^{-j}nk + 1, \ldots, 2^{-j}n(k+1)]$. We set

$$f_{i+1}^n = f_n(X_{i/n}, \{X_{i/n}/\alpha_n\}, \sqrt{n}[X_{(i+1)/n} - X_{i/n}]),\ \eta_i^n(f_n) = f_i^n - M_n f_n(X_{(i-1)/n}, (i-1)/n),$$

$$\delta_i^n(f) = \sum_{z=i}^{n\wedge(i+n-1)} (\mathbb{E}_{\mathcal{F}_{i/n}}[\eta_z^n(f)] - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[\eta_z^n(f)])$$

and

$$\begin{aligned}
\mathcal{M}_{jk}^n(f_n) &= \frac{2^{j/2}}{n}\sum_{i=1}^n \mathbb{1}_{jk}(i/n)\delta_i^n(f_n),\\
H_{jk}^n(f_n) &= \frac{2^{j/2}}{n}\sum_{i\in s_{jk}} [\bar{m}_n f_n(X_{i/n}, i/n) - \bar{m}_n f_n(X_{(i-1)/n}, (i-1)/n)]\\
&\quad - \frac{2^{j/2}}{n}\sum_{i\in s_{jk}} l^n_{(n-i)\wedge(n-1)} f_n(X_{(i-1)/n}, (i-1)/n)\mathbb{1}_{2\le(n-i)\wedge(n-1)},\\
K_{jk}^n(f_n) &= \frac{2^{j/2}}{n}\sum_{i\in s_{jk}}\sum_{z=1}^{(n-i-1)\wedge(n-2)} [l_z^n f_n(X_{i/n}, i/n) - l_z^n f_n(X_{(i-1)/n}, (i-1)/n)].
\end{aligned}$$

For some sequences of real functions $(x,t)\to g_n(x,t)$ on $\mathbb{R}$ and $(x,u,y)\to f_n(x,u,y)$ on $\mathbb{R}\times[0,1]\times\mathbb{R}$, we define

$$V^{jk}(n,g_n) = \frac{2^{j/2}}{n}\sum_{i=1}^n \mathbb{1}_{jk}(i/n) g_n(X_{(i-1)/n}, (i-1)/n)$$

and

$$V^{jk}(n,f_n) = \frac{2^{j/2}}{n}\sum_{i=1}^n \mathbb{1}_{jk}(i/n) f_n(X_{(i-1)/n}, \{X_{(i-1)/n}/\alpha_n\}, \sqrt{n}[X_{i/n} - X_{(i-1)/n}]).$$

As in the homogeneous case, we get

$$\delta_i^n(f_n) = \eta_i^n(f_n) + \bar{m}_n f_n(X_{i/n}, i/n) - \bar{m}_n f_n(X_{(i-1)/n}, (i-1)/n)$$
$$- l^n_{(n-i)\wedge(n-1)} f_n(X_{(i-1)/n}, (i-1)/n) \mathbb{1}_{2\leq(n-i)\wedge(n-1)}$$
$$+ \sum_{z=1}^{(n-i-1)\wedge(n-2)} [l^n_z f_n(X_{i/n}, i/n) - l^n_z f_n(X_{(i-1)/n}, (i-1)/n)].$$

The following proposition follows.

Proposition II.7 *(Fundamental decomposition)*

$$V^{jk}(n, f_n) - 2^{j/2}\int_0^1 \mathbb{1}_{jk}(s) M f_n(X_s, s)\, ds = \mathcal{M}^n_{jk}(f_n) + V^{jk}(n, M_n f_n - M f_n)$$
$$+ V^{jk}(n, M f_n) - 2^{j/2}\int_0^1 \mathbb{1}_{jk}(s) M f_n(X_s, s)\, ds - H^n_{jk}(f_n) - K^n_{jk}(f_n).$$

Using that $X_t = S_t^{-1}(W_t)$ together with the change of variable formula, we easily show that for $p > 0$, $\mathbb{E}[|X_t|^p] < +\infty$ and $\mathbb{E}[e^{p|X_t|}] < +\infty$. Thus, thanks to lemma II.13, as in the homogeneous case, we obtain

$$(\mathbb{E}[|K^n_{jk}(f_n)|] + \mathbb{E}[|H^n_{jk}(f_n)|]) \leq c 2^{j/2}(1/n + \alpha_n^2 \log n).$$

and

$$\mathbb{E}[|V^{jk}(n, M_n f_n - M f_n)|^2] \leq c 2^{-j} n^{-1}.$$

From a Riemann-type approximation, we get

$$\mathbb{E}[|M f_n(X_s, s) - M_{(i-1)/n, i/n} fn(X_{i/n}, i/n)|^2] \leq c 2^{-j}(1/n + \alpha_n^2)$$

and so

$$\mathbb{E}[|V^{jk}(f_n) - 2^{j/2}\int_0^1 \mathbb{1}_{jk}(s) M f_n(X_s, s)|^2] \leq c(1/n + \alpha_n^2).$$

For the last term, we have

$$\mathbb{E}[\mathcal{M}^n_{jk}(f_n)^2] = \frac{2^j}{n^2}\sum_{i=1}^n \mathbb{1}_{jk}(i/n)\mathbb{E}[\delta_i^n(f_n)^2].$$

We control the $\delta_i^n(f_n)$ in next section.

2.6 Approximation of delta

To control the $\delta_i^n(f_n)$, the idea is to build approximation of them $\tilde{\delta}_i^n(f_n)$ and to control $\tilde{\delta}_i^n(f_n)$ and $\delta_i^n(f_n) - \tilde{\delta}_i^n(f_n)$. We redefine $m_{x,s,t}$ and $h_{x,s,t}$ by

$$m_{x,s,t} = \sigma(x,s)[S_t(x) - S_s(x)]$$
$$h_{x,s,t}(y) = \{2\pi(t-s)\sigma(x,s)\}^{-1/2} \exp\{-(y+m_{x,s,t})^2[2(t-s)\sigma(x,s)^2]^{-1}\}.$$

First, we begin with useful lemmas.

Lemma II.14

$$\int_{\mathbb{R}} |\frac{\partial^2}{\partial x \partial y} p_{s,t}(x,y) + h''_{x,s,t}(y-x)|(1 + |\frac{y-x}{|t-s|^{1/2}}|^p)\, dy \le c|t-s|^{-1/2} P(|x|) \exp[c|x|].$$

Proof. We have

$$\frac{\partial^2}{\partial x \partial y} p_{s,t}(x,y) = \frac{p_{s,t}(x,y)}{\sigma(x,s)\sigma(y,t)}\Big(\frac{1}{|t-s|} + \frac{S_t(y)-S_s(x)}{|t-s|} - (\frac{S_t(y)-S_s(x)}{t-s})^2\Big).$$

Consequently,

$$|\frac{1}{p_{s,t}}\frac{\partial^2}{\partial x \partial y} p_{s,t}(x,y) - \frac{1}{|t-s|\sigma(x,s)\sigma(y,t)} + \frac{[(y-x)/\sigma(x,s)]^2}{|t-s|^2\sigma(x,s)\sigma(y,t)}|$$
$$\le c\big(1 + \frac{|y-x|}{|t-s|} + x^2 + \frac{(y-x)^4}{(t-s)^2} + \frac{|y-x|^3}{(t-s)^2} + (y-x)^2|x|\big).$$

We also have

$$h''_{x,s,t}(y-x) = -\frac{h_{x,s,t}(y-x)}{\sigma^2(x,s)}\Big(\frac{1}{|t-s|} - \frac{(y-x+m_{x,s,t})^2}{\sigma(x,s)^2|t-s|^2}\Big).$$

and so

$$|\frac{h''_{x,s,t}(y-x)}{h_{x,s,t}(y-x)} + \frac{1}{|t-s|\sigma(x,s)\sigma(y,t)} - \frac{\big[(y-x)/\sigma(x,s)\big]^2}{|t-s|^2\sigma(x,s)\sigma(y,t)}|$$
$$\le c\big(1 + \frac{|y-x|^3}{(t-s)^2} + \frac{(1+|x|)|y-x|}{|t-s|} + \frac{|x|(y-x)^2}{|t-s|} + x^2|y-x| + x^2\big).$$

Hence, if $|y-x| \le |t-s|^{1/3}$, we get

$$|\frac{1}{p_{s,t}}\frac{\partial^2}{\partial x \partial y} p_{s,t}(x,y) + \frac{h''_{x,s,t}(y-x)}{h_{x,s,t}(y-x)}| \le \frac{c}{|t-s|^{1/2}}\big(1 + x^2 + (1+|x|)|\frac{y-x}{|t-s|^{1/2}}|^3\big).$$

We have

$$|\frac{1}{p_{s,t}}\frac{\partial^2}{\partial x \partial y} p_{s,t}(x,y)| \le \frac{c}{|t-s|}(1 + x^2 + |\frac{y-x}{|t-s|^{1/2}}|^2)$$

and using lemma II.12, we easily obtain for $|y-x| \leq |t-s|^{1/3}$

$$|\frac{p_{s,t}(x,y)}{h_{x,s,t}(y)} - 1| \leq c|t-s|^{1/2}(1+|\frac{y-x}{|t-s|^{1/2}}|^4)(1+x^2)\exp[c|x|].$$

Thus, we get for $|y-x| \leq |t-s|^{1/3}$,

$$|\frac{\partial^2}{\partial x \partial y} p_{s,t}(x,y) + h''_{x,s,t}(y-x)| \leq c\frac{h_{x,s,t}(y-x)}{|t-s|^{1/2}}(1+x^2)\exp[c|x|](1+|\frac{y-x}{|t-s|^{1/2}}|^6).$$

Afterwards, we get that for $p > 0$,

$$\int_{|y-x|\leq|t-s|^{1/3}} |\frac{\partial^2}{\partial x \partial y} p_{s,t}(x,y) + h''_{x,s,t}(y-x)|(1+|\frac{|y-x|}{|t-s|^{1/2}}|^p)\mathrm{d}y \leq \frac{cP(|x|)\exp[c|x|]}{|t-s|^{1/2}}.$$

After simple computations for the case $|y-x| > |t-s|^{1/3}$, we obtain the result. □

Lemma II.15 *For some positive constants* c, c_1 c_2 c_3 c_4 *and* c_5, *we following inequality holds*

$$\begin{aligned}
&|x-x_0|^{-1}|t-s||h''_{x_0,s,t}(y-x) - h''_{x,s,t}(y-x)| \\
&\leq ch^m_{c|t-s|^{1/2}}\big(1+|\frac{y-m}{c|t-s|^{1/2}}|^4\big) + c\mathbb{1}_{|y|\leq(c_5|x_0|+c_6|x|)|t-s|}(1+x^4(t-s)^2) \\
&+ c|t-s|^{1/2}\big(h^{(c_1|x_0|+c_2|x|)|t-s|}_{c|t-s|^{1/2}} + h^{(c_3|x_0|+c_4|x|)|t-s|}_{c|t-s|^{1/2}}\big)\big(1+x^4(t-s)^2+|\frac{y}{c|t-s|^{1/2}}|^4\big).
\end{aligned}$$

Proof. We have

$$\begin{aligned}
|\frac{\partial}{\partial m} h^{m''}_\sigma(y)\big| &\leq ch^m_\sigma(y)\frac{1}{\sigma^3}\big(1+|\frac{y-m}{\sigma}|^3\big) \\
|\frac{\partial}{\partial \sigma} h^{m''}_\sigma(y)\big| &\leq ch^m_\sigma(y)\frac{1}{\sigma^3}\big(1+|\frac{y-m}{\sigma}|^4\big).
\end{aligned}$$

Thus, we have

$$|h^{m''}_\sigma(y) - h^{\tilde{m}''}_{\tilde{\sigma}}(y)| \leq c(E_1+E_2),$$

where

$$\begin{aligned}
E_1 &= |m-\tilde{m}|h^{m^*}_\sigma(y)\frac{1}{\sigma^3}\big(1+|\frac{y-m^*}{\sigma}|^4\big) \\
E_2 &= |\sigma-\tilde{\sigma}|h^m_{\sigma^*}(y)\frac{1}{\sigma^{*3}}\big(1+|\frac{y-m}{\sigma^*}|^4\big),
\end{aligned}$$

with $\min(m,\tilde{m}) \leq m^* \leq \max(m,\tilde{m})$ and $\min(\sigma,\tilde{\sigma}) \leq \sigma^* \leq \max(\sigma,\tilde{\sigma})$. We easily get that

$$E_2 \leq c|\frac{x-x_0}{t-s}|h^{-m_{x_0,s,t}}_{c|t-s|^{1/2}}\big(1+|\frac{y+m_{x_0,s,t}}{c|t-s|^{1/2}}|^4\big)$$

and we bound E_1 splitting $\mathbb{R}$ in three intervals. □

We finally recall a result about the fractional part of a variable that can be found in Delattre [35].

Lemma II.16 *Let g be a function on $\mathbb{R} \times [0,1]$ such that $x \to g(x,u)$ is absolutely continuous. Thus, $\frac{\partial}{\partial x} g(x,u)$ exists $dx \otimes du$ almost everywhere. Assume that*

$$\int_{\mathbb{R}} \int_0^1 |\frac{\partial}{\partial x} g(x,u)| du dx < \infty.$$

Then, for any positive α,

$$\int_{\mathbb{R}} g(x, \{x/\alpha\}) dx = \int_{\mathbb{R}} \int_0^1 g(x,u) du dx + \alpha \int_{\mathbb{R}} \int_0^1 \chi(u, \{x/\alpha\}) \frac{\partial}{\partial x} g(x,u) du dx.$$

with $\chi(u,v) = v - \mathbb{1}_{v>u}$.

Let $f_{n_x}(u,y) = f_n(x,u,y)$. We define

$$\tilde{\delta}_i(f_n) = \eta_i^n(f_n) + m f_n(X_{i/n}, \{X_{i/n}/\alpha_n\}, i/n) - m f_n(X_{(i-1)/n}, \{X_{(i-1)/n}/\alpha_n\}, (i-1)/n)$$

$$+ L^{n-2} f_{n_{X_{(i-1)/n}}} \big(\sigma(X_{(i-1)/n}, (i-1)/n), \beta_n, , X_{(i-1)/n}, \{X_{i/n}/\alpha_n\}, \sqrt{n}(X_{i/n} - X_{(i-1)/n}), i/n\big).$$

with

$$L^k \phi(\sigma, \beta, x, u, y, s) = \sum_{j=1}^k l_j \phi(\sigma, \beta, x, \{u + y/\beta\}, s, s + j/n) - l_j \phi(\sigma, \beta, x, u, s, s + j/n),$$

$$l_j \phi(\sigma, \beta, x, u, s, t) = \mathbb{E}[m_\sigma \phi(\{u + \sigma(W_j - n^{1/2}[S_t(x) - S_s(x)])/\beta\})] - M_\sigma \phi,$$

$$m_\sigma \phi(u) = \mathbb{E}[\phi(u, \sigma W_1)],$$

and

$$M_\sigma \phi = \int_0^1 m_\sigma \phi(u) \mathrm{d}u.$$

We have the following result.

Lemma II.17

$$|\sum_{j=1}^k [l_j^n f_n(x+y, i/n) - l_j^n f_n(x, (i-1)/n)] - L^k f_{n_x}\big(\sigma(x, (i-1)/n), \beta_n, x, \{x/\alpha_n\}, \sqrt{n} y, i/n\big)|$$

$$\leq c P(|y| + |x|) \exp[c|x|] \alpha_n \sqrt{k} (1 + n y^2).$$

Proof. First, using a Taylor expansion, we have that

$$|l_j^n f_n(x_0, i/n) - l_j^n f_n(x_0, (i-1)/n)| \leq c P(|x_0|)(1/n + \alpha_n/\sqrt{n}).$$

Hence, $l_j^n f_n(x_0 + y_0, i/n) - l_j^n f_n(x_0, (i-1)/n)$ is equal to

$$\int_{x_0}^{x_0+y_0} \int_{\mathbb{R}} \frac{\partial}{\partial x}(x,y) p_{i/n,(i+j)/n}(x,y) \bar{m}_n f_n(y, (i+j)/n) \mathrm{d}y + R,$$

with $|R| \leq cP(|x_0|)(1/n + \alpha_n/\sqrt{n})$. On the other hand,

$$l_j f_{n_{x_0}}\big(\sigma(x_0,(i-1)/n), \beta_n, x_0, \{(x_0+y_0)/\alpha_n\}, i/n, (i+j)/n\big) \\ - l_j f_{n_{x_0}}\big(\sigma(x_0,(i-1)/n), \beta_n, x_0, \{x_0/\alpha_n\}, i/n, (i+j)/n\big)$$

is equal to

$$-\int_{x_0}^{x_0+y_0}\int_{\mathbb{R}} h'_{x_0,i/n,(i+j)/n}(y-x)\big(mf_n(x_0,\{y/\alpha_n\},(i-1)/n) - Mf_n(x_0,(i-1)/n)\big)\mathrm{d}y\mathrm{d}x.$$

Consider now

$$|\int \frac{\partial}{\partial x}p_{s,t}(x,y)\bar{m}_n f_n(y,t)\mathrm{d}t + \int h'_{x_0,s,t}(y-x)\big(m_n f_n(x_0,\{y/\alpha_n\},t) - Mf_n(x_0,t)\big)\mathrm{d}y\mathrm{d}x|.$$

By lemma II.16, the quantity

$$\int \frac{\partial}{\partial x}p_{s,t}(x,y)\bar{m}_n f_n(y,t)\mathrm{d}y$$

is equal to

$$\alpha_n \int_{\mathbb{R}}\mathrm{d}y\int_0^1 \Big(\frac{\partial}{\partial x}p_{s,t}(x,y)\frac{\partial}{\partial y}m_n f_n(y,u,t) + m_n f_n(y,u,t)\frac{\partial^2}{\partial x\partial y}p_{s,t}(x,y)\Big)\chi(u,\{y/\alpha_n\})\mathrm{d}u$$

and

$$\int h'_{x_0,s,t}(y-x)\big(m_n f_n(x_0,\{y/\alpha_n\},t) - Mf_n(x_0,t)\big)\mathrm{d}y$$

is equal to

$$\alpha_n \int_{\mathbb{R}} h''_{x_0,s,t}(y-x)\int_0^1 mf_n(x_0,u,t)\chi(u,\{y/\alpha_n\})\mathrm{d}u\mathrm{d}y.$$

We have

$$\int_{\mathbb{R}}\mathrm{d}y|\frac{\partial}{\partial x}p_{s,t}(x,y)|\int_0^1 |\frac{\partial}{\partial y}m_n f_n(y,u,t)|\mathrm{d}u \leq cP(|x|)|t-s|^{-1/2}.$$

Hence, as $\sum_{i=1}^k 1/\sqrt{i} \leq 2\sqrt{k}$, it is enough to show that the following inequality holds.

$$\int_{\mathbb{R}}\mathrm{d}y\int_0^1 \mathrm{d}u|m_n f_n(y,u,t)\frac{\partial^2}{\partial x\partial y}p_{s,t}(x,y)) + mf_n(x_0,u,t)h''_{x_0,s,t}(y-x)| \\ \leq cP(|x|+|x_0|)\exp[c|x_0|](|t-s|^{-1/2} + |t-s|^{-1}n^{-1/2} + |t-s|^{-1}|x-x_0|).$$

Using lemma II.11, we have

$$\int_{\mathbb{R}}\mathrm{d}y\int_0^1 \mathrm{d}u|m_n f_n(y,u,t) - m_n f_n(x_0,u,t)||\frac{\partial^2}{\partial x\partial y}p_{s,t}(x,y)| \\ \leq c\int_{\mathbb{R}}(|y-x|+|x-x_0|)|\frac{\partial^2}{\partial x\partial y}p_{s,t}(x,y)|\big(P(|y|)+P(|x_0|)\big)\mathrm{d}y \\ \leq cP(|x|+|x_0|)(|t-s|^{-1/2} + |t-s|^{-1}|x-x_0|).$$

Using again lemma II.11, we easily get

$$\int_{\mathbb{R}} \mathrm{d}y \int_0^1 \mathrm{d}u |m_n f_n(x_0,u,t) - m_{t,t+1/n} f_n(x_0,u,t)| |\frac{\partial^2}{\partial x \partial y} p_{s,t}(x,y)| \leq \frac{cP(|x_0|)\exp[c|x_0|]}{|t-s|n^{1/2}}.$$

By lemma II.14 and since

$$\int_0^1 \mathrm{d}u |m_{t,t+1/n} f_n(x_0,u,t)| \leq cP(|x_0|),$$

$$\int_{\mathbb{R}} \mathrm{d}y \int_0^1 \mathrm{d}u |m f_n(x_0,u,t)| |\frac{\partial^2}{\partial x \partial y} p_{s,t}(x,y)) + h''_{x,s,t}(y-x)| \leq \frac{cP(|x|)\exp[c|x|]}{|t-s|^{1/2}}.$$

Finally, using lemma II.15, we get

$$\int_{\mathbb{R}} \mathrm{d}y \int_0^1 \mathrm{d}u |m f_n(x_0,u,t)| |h''_{x,s,t}(y-x) - h''_{x_0,s,t}(y-x)| c|x-x_0| \frac{P(|x|+|x_0|)}{|t-s|}).$$

The result follows. □

As in Delattre [35], we finally get the following lemma.

Lemma II.18

$$\mathbb{E}[|\delta_i(f_n) - \tilde{\delta}_i(f_n)|^p] \leq c(1 + \beta_n^{2p-1} n^{-1/2} + \beta_n^p).$$

2.7 Control of the approximation

We now control $\tilde{\delta}_i(f_n)$. We have the following lemma where

Lemma II.19 *Let ϕ verifying assumption $\tilde{D}$. We have*

$$|L^k \phi(\sigma,\beta,x,u,y,s)| \leq c_\varepsilon \frac{\beta^{1+\varepsilon}}{\sigma^{2+\varepsilon}} |y| M_\sigma |\phi|,$$

$$\int_0^1 \big(L^k \phi(\sigma,\beta,x,u,y,s)\big)^2 \leq \frac{1}{3} (M_\sigma |\phi|)^2 \frac{\beta^2 y^2}{\sigma^4},$$

$$\int_0^1 \big(\frac{\partial}{\partial x} L^k \phi(\sigma(x,t),\beta,x,u,y,s)\big)^2 \leq cP(|x|) \beta_n^2 y^2.$$

Proof. We define $\bar{h}_\sigma = \sum_{a \in \mathbb{Z}} h^0_\sigma(a+u)$ and $u \to \bar{h}_\sigma(u) \mathbb{1}_{[0,1]}(u)$ be the density of the fractional part of a centered Gaussian variable with variance σ^2. Let $d_{n,x,s,t} = n^{1/2}\big(S_t(x) - S_s(x)\big)$. We have

$$l^n_j \phi(\sigma,\beta,x,u,s,t) = \int_0^1 \bar{h}_{\sqrt{j}\sigma/\beta}(v) \big[m_\sigma \phi(\{u+v-\sigma d_{n,x,s,t}/\beta\})\big] - M_\sigma \phi \mathrm{d}v.$$

Using the change of variable formula, the 1-periodicity of $\bar{h}_\sigma$ and the fact that for $z \in \mathbb{Z}$, $\{x+z\} = \{x\}$, we obtain

$$l_j^n \phi(\sigma, \beta, x, u, s, t) = \int_0^1 [\bar{h}_{\sqrt{j}\sigma/\beta}(v - u + \sigma d_{n,s,t,x}/\beta) - 1] m_\sigma \phi(v) \mathrm{d}v.$$

we now use the Fourier decomposition of $\bar{h}_\sigma$. We have

$$\bar{h}_\sigma(u) = \sum_{\lambda \in \mathbb{Z}} e^{-2\pi^2\lambda^2\sigma^2} e^{-i2\pi\lambda u}.$$

Consequently, $L^k \phi(\sigma, \beta, x, u, y, s)$ is equal to

$$\sum_{\lambda \in \mathbb{Z}^*} \hat{m}_\sigma \phi(\lambda)(e^{-2i\pi\lambda y/\beta} - 1)e^{-2i\pi\lambda u} \sum_{j=1}^{k} e^{-2\pi^2\lambda^2\sigma^2 j/\beta^2 + 2i\pi\lambda\sigma d_{n,x,s,s+j/n}/\beta},$$

with $\hat{m}_\sigma \phi(\lambda) = \int_0^1 m_\sigma \phi(v) e^{2i\pi\lambda v} \mathrm{d}v$. Using the orthogonality properties of the Fourier basis together with the following inequality

$$\sum_{j=1}^{k} e^{-2\pi^2\lambda^2\sigma^2 j/\beta^2} = (1 - e^{-2\pi^2 k\lambda^2\sigma^2/\beta^2})/(e^{-2\pi^2\lambda^2\sigma^2/\beta^2} - 1),$$

after computations, the same method as in Delattre [35, chap 6] gives the result. □

We set

$$\Delta_i^n f_n(\beta_n, x, u, y) = \eta_i^n(f_n) + m f_n(x, u, i/n) - m f_n(x, u, (i-1)/n)$$
$$+ L^{n-2} f_{n_x}\big(\sigma(x, (i-1)/n), \beta_n, x, u, y, i/n\big).$$

As a result of the preceding lemma, we have the following inequalities.

Lemma II.20

$$|\Delta_i^n f_n(\beta_n, x, u, y)| \leq c(1 + \beta_n^2) P(|y|),$$
$$\int_0^1 \big(\Delta_i^n f_n(\beta_n, x, u, y)\big)^2 \leq c(1 + \beta_n^2) P(|y|),$$
$$\int_0^1 \big(\frac{\partial}{\partial x} \Delta_i^n f_n(\beta_n, x, u, y)\big)^2 \leq c(1 + \beta_n^2) P(|y|) P(|x|).$$

Hence, using the preceding lemma together with lemma II.1, the same computations as in Delattre [35, chap 8] lead to the following final result.

Lemma II.21

$$\mathbb{E}[\tilde{\delta}_i(f_n)^2] \leq c(1 + \beta_n^2)(1 + \alpha_n \sqrt{n/i}).$$

The completion of the proof of theorem II.5 follows as in the homogeneous case.

Chapter 3

Numerical results

1 Introduction

In this section, we present numerical results for our estimators of the integrated volatility. We give simple simulations results for the absolute integrated volatility in section 2. We then focus in section 3 on the relative integrated volatility estimation in classical financial models. For exploratory purpose, in addition to the rounding error, we also introduce an additive microstructure noise and adapt our estimators to it. Our results are compared to those obtained with the two scales estimator of Zhang, Mykland and Aït Sahalia. We work on equity data in section 4. Our equity data set is particularly relevant because the legal tick size has changed inside it.

2 Absolute integrated volatility

We consider here two simple models where we estimate the absolute integrated volatility. We choose these two models because the value of the absolute integrated volatility is not random.

2.1 Constant diffusion coefficient

We consider the following model

$$X_t = \sigma W_t, t \in [0,1], \tag{II.19}$$

where (W_t) a Brownian motion. We observe

$$\{X_{i/n}^{(\alpha_n)},\ i = 0, \dots, n\}. \tag{II.20}$$

Below is given a sample path of such a model.

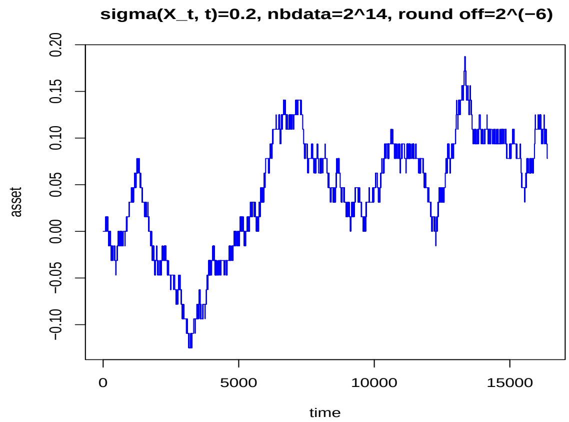

Figure II.3: *Sample path of* $t \to X_t^{(\alpha)}$ *for* $\sigma = 0.2$, $n = 2^{14}$ *and* $\alpha_n = 2^{-6}$.

Our goal is to estimate the parameter σ^2. We consider the previously defined estimators but, we manually set the parameters j_0, a, j_1 and j_2 in their definitions. Hence we use $\widetilde{\theta}_n(j_0)$ and $\widehat{\theta}_n(a, j_1, j_2)$. Remark that in this model,

$$\int_0^1 \sigma\psi_{jk}(t) = 0.$$

Consequently, $\widetilde{\theta}_n(0)$ is theoretically one of the best of our estimators. We take $\sigma = 0.2$, $n = 2^{14}$ and $\alpha_n = 2^{-6}$. So, the theoretical optimal level is $j_0 = 6$. Nevertheless, the fact that in this model the d_{jk} are equal to zero make that estimators built at lower levels are even better. We first give the empirical distribution of the absolute realized volatility estimator obtained from 500 independent simulations of model (II.19)-(II.20).

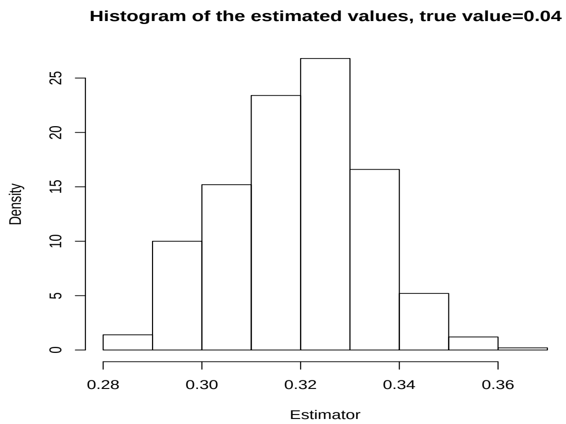

Figure II.4: *Histogram of the absolute realized volatility estimator for* $\sigma = 0.2$, $n = 2^{14}$ *and* $\alpha_n = 2^{-6}$, *500 simulations.*

Comments on figure II.4. Of course, the naïve realized volatility estimator totally over estimates the value of σ^2.

We now give the empirical distribution of the estimators of the estimators $\widetilde{\theta}_n(6)$, $\widehat{\theta}_n(0.5, 5, 3)$ and $\widetilde{\theta}_n(0)$ obtained from the same 500 simulations of model (II.19)-(II.20).

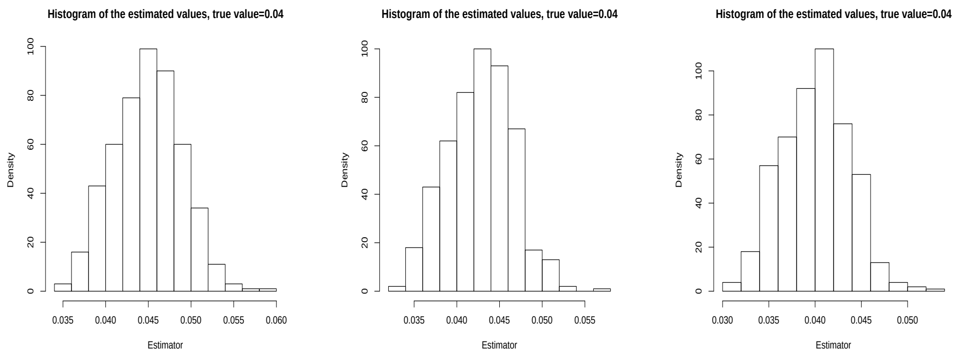

Figure II.5: *Histograms of* $\widetilde{\theta}_n(6)$, $\widehat{\theta}_n(0.5, 5, 3)$ *and* $\widetilde{\theta}_n(0)$ *for* $\sigma = 0.2$, $n = 2^{14}$ *and* $\alpha_n = 2^{-6}$, *500 simulations.*

We have the following results for the empirical mean and standard deviation of the considered estimators.

	mean	standard deviation
$\widetilde{\theta}_n(6)$	0.0449	0.00396
$\widehat{\theta}_n(0.5, 5, 3)$	0.0427	0.00382
$\widetilde{\theta}_n(0)$	0.0400	0.00361

Figure II.6: *Empirical mean and standard deviation of* $\widetilde{\theta}_n(6)$, $\widehat{\theta}_n(0.5, 5, 3)$ *and* $\widetilde{\theta}_n(0)$ *for* $\sigma = 0.2$, $n = 2^{14}$ *and* $\alpha_n = 2^{-6}$, *500 simulations.*

Comments on figure II.5 and figure II.6. The results are far better than those obtained with the realized volatility estimator. As theoretically explained, in this model, $\widetilde{\theta}_n(0)$ is the best of these estimators.

2.2 Deterministic diffusion coefficient

We consider the following model

$$X_t = \int_0^t \sigma(t) \mathrm{d}W_t, t \in [0, 1], \tag{II.21}$$

where (W_t) a Brownian motion and $\sigma(t) = 1 - t$. We observe

$$\{X_{i/n}^{(\alpha_n)},\ i = 0, \ldots, n\}. \tag{II.22}$$

Below is given a sample path of such a model.

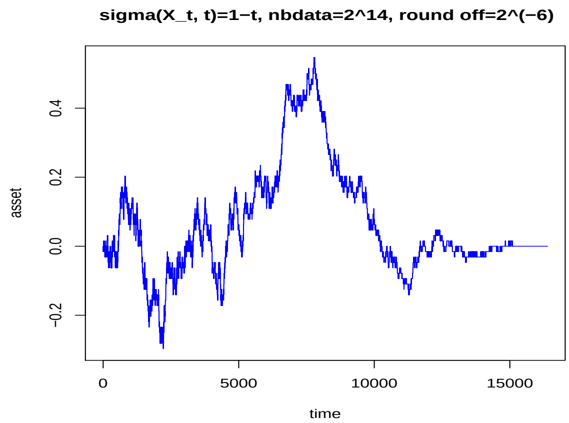

Figure II.7: *Sample path of* $t \to X_t^{(\alpha)}$ *for* $\sigma(t) = 1 - t$, $n = 2^{14}$ *and* $\alpha_n = 2^{-6}$.

Our goal is to estimate the parameter

$$\int_0^1 \sigma^2(t) \mathrm{d}t.$$

We first give the empirical distribution of the absolute realized volatility estimator obtained from 500 independent simulations of model (II.21)-(II.22).

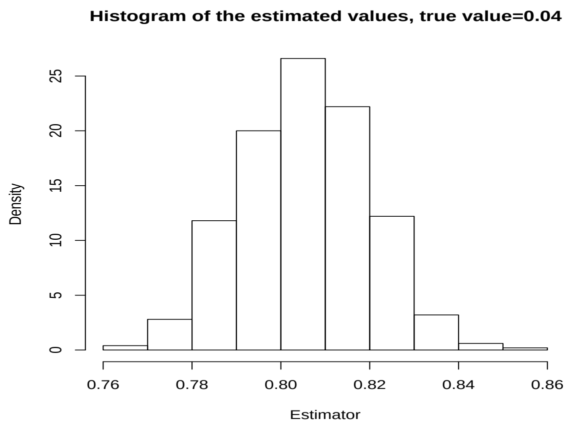

Figure II.8: *Histogram of the absolute realized volatility estimator for* $\sigma = 0.2$*,* $n = 2^{14}$ *and* $\alpha_n = 2^{-6}$*, 500 simulations.*

Comments on figure II.8. Once again, the naïve realized volatility estimator totally over estimates the value of σ^2.

We now give the empirical distribution of the estimators $\widetilde{\theta}_n(6)$, $\widehat{\theta}_n(0.5, 5, 3)$ and $\widetilde{\theta}_n(0)$ obtained from the same 500 simulations of model (II.19)-(II.20). Note that in this model of inhomogeneous diffusion, we have no theoretical result for a compensated estimator such as $\widehat{\theta}_n(0.5, 5, 3)$. Moreover, the d_{jk} are no longer equal to zero and so the estimator $\widetilde{\theta}_n(0)$ has no reason to be a good estimator.

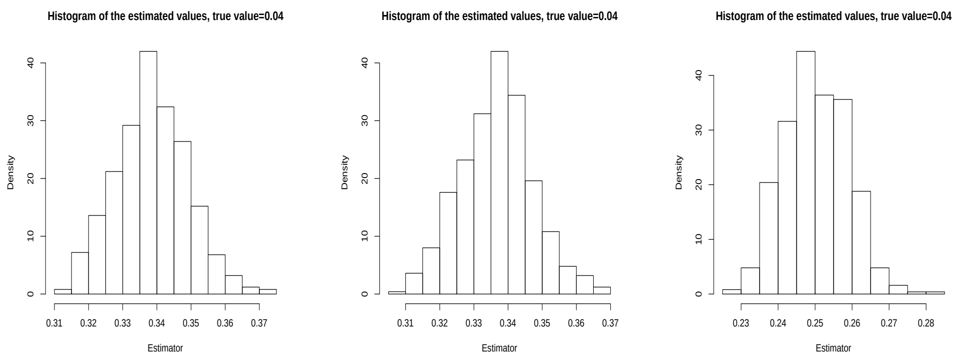

Figure II.9: *Histograms of* $\widetilde{\theta}_n(6)$*,* $\widehat{\theta}_n(0.5, 5, 3)$ *and* $\widetilde{\theta}_n(0)$ *for* $\sigma = 0.2$*,* $n = 2^{14}$ *and* $\alpha_n = 2^{-6}$*, 500 simulations.*

We have the following results for the empirical mean and standard deviation of the considered estimators.

	mean	standard deviation
$\widetilde{\theta}_n(6)$	0.3385	0.01062
$\widehat{\theta}_n(0.5, 5, 3)$	0.3366	0.01064
$\widetilde{\theta}_n(0)$	0.2500	0.00861

Figure II.10: *Empirical mean and standard deviation of* $\widetilde{\theta}_n(6)$*,* $\widehat{\theta}_n(0.5, 5, 3)$ *and* $\widetilde{\theta}_n(0)$ *for* $\sigma = 0.2$*,* $n = 2^{14}$ *and* $\alpha_n = 2^{-6}$*, 500 simulations.*

Comments on figure II.9 and figure II.10. Here, the results of $\widetilde{\theta}_n(6)$ and $\widehat{\theta}_n(0.5, 5, 3)$ are far better than those obtained with the realized volatility estimator. As theoretically explained, in this model, $\widetilde{\theta}_n(0)$ is not a good estimator.

3 Relative integrated volatility

3.1 ZMA estimator

We will compare our results to those obtained with the estimator of Zhang, Mykland and Aït Sahalia [114]. So, following the presentation of Bandi et al.[13], we first rapidly recall the construction and properties of this estimator. Consider a trading day of length 1 and $n+1$ equispaced observed logarithmic prices over $[0,1]$ and write

$$p_{t_j} = p^*_{t_j} + \eta_{t_j}$$

and $\varepsilon_{t_j} = \eta_{t_j} - \eta_{t_{j-1}}$. The quantity p^*_t represents an "unobservable equilibrium log price" of the form

$$p^*_t = \int_0^t \sigma_t \mathrm{d}W_t,$$

where σ_t is a regular enough stochastic volatility process and (W_t) a Brownian motion. We suppose that the η are iid and independent of (p^*_t). The goal is to estimate

$$\int_0^1 \sigma_t^2 \mathrm{d}t.$$

Consider now q non overlapping sub-grids $\mathcal{G}^i$ of the original grid of observation with $i = 1 \dots q$. The first sub-grid starts at t_0 and takes every q arrival times, i.e. $\mathcal{G}^1 = (t_0,\ t_{0+q},\ t_{0+2q}\dots)$. The second sub-grid starts at t_1 and also takes every q arrival times, i.e. $\mathcal{G}^1 = (t_1,\ t_{1+q},\ t_{1+2q}\dots)$ and so on. Given the i-th sub-grid, the corresponding realized variance estimator is defined as

$$\widehat{V}^i = \sum_{t_j, t_{j+1} \in \mathcal{G}^i} (p_{t_{j+1}} - p_{t_j})^2.$$

The two scales estimator of Zhang, Mykland and Aït Sahalia is defined by

$$\widehat{V} = \frac{\sum_{i=1}^q \widehat{V}^i}{q} - \frac{n-q+1}{qn} \sum_{i=0}^{n-1} (p_{t_{i+1}} - p_{t_i})^2.$$

The first term is an average of the realized variance estimators obtained thanks to the subsamples and the second term is a bias correction. Zhang, Mykland and Aït Sahalia show that, if n and q tend to infinity with q/n tends to zero and q^2/n tends to infinity, the estimator is consistent. Specifically, if $q = cn^{2/3}$, we have

$$n^{1/6}(\widehat{V} - V) \to_{\mathcal{L}} \left(8c^{-2}(\mathbb{E}[\eta^2])^2 + c4Q/3\right)^{1/2} \mathcal{N}(0,1),$$

with

$$Q = \int_0^1 \sigma_t^4 \mathrm{d}t.$$

The constant c can be selected in order to minimize the estimator limit variance. This minimization leads to

$$q^* = \big(\frac{3(\mathbb{E}[\varepsilon^2])^2}{Q}\big)^{1/3} n^{2/3}.$$

We can estimate $\mathbb{E}[\varepsilon^2]$ by

$$\frac{\sum_{i=0}^{n-1}(p_{t_{i+1}} - p_{t_i})^2}{n}.$$

Finally, in the following, we define as the ZMA oracle estimator the estimator obtained replacing Q by its approximation by the Riemann sum on the non noisy data. The ZMA estimator is obtained replacing Q by its estimation by the renormalized realized quarticity at a large sampling scale. Note that this estimator has been improved by he multi scales estimator of Zhang [113], that achieves the optimal accuracy of $n^{-1/4}$.

3.2 Models and simulation parameters

We consider now models of the form

$$\mathrm{d}\log X_t = a_t \mathrm{d}t + \sigma_t \mathrm{d}W_t,\ X_0 = x_0, t \in [0, T], \tag{II.23}$$

where (W_t) a Brownian motion. We observe

$$\big\{(X_{iT/n} + \gamma\xi_i^n)^{(\alpha_n)},\ i = 0, \dots, n\big\}, \tag{II.24}$$

where the (ξ_i^n) are independent centered Gaussian variable with variance 1. In all our simulations we draw 500 independent sample path and we take

$$T = 1/512,\ x_0 = 1,\ \alpha_n = 0.01,\ n = 2^{14}.$$

With this set of parameters, the theoretical optimal level j_0 is between 6 and 7. We use the estimators $\widetilde{\lambda}_n(j_0)$ and $\widehat{\lambda}_n(a, j_1, j_2)$. In the following we call "Black-Scholes" estimator the estimator given by $\widetilde{\lambda}_n(0)$. We treat here the case of a Black-Scholes model. The case of non correlated Heston model and of a correlated Heston model, even if not covered by our theoretical results are also treated. The conclusions for these two models being closed to those obtained in the Black-Scholes case, they are relegated to appendix B. For the three types of models, we also treat in appendix B the cases where we observe

$$\{(\exp[\log X_{iT/n} + \gamma\xi_i^n])^{(\alpha_n)},\ i = 0, \dots, n\}.$$

Note that at the time scale of one day, as soon as X is quite constant, the two ways of modeling the noise impact are very close.

3.3 Additional estimators and estimation of the variance of the noise

From an estimator $\widetilde{\lambda}_n(j_0)$ or $\widehat{\lambda}_n(a, j_1, j_2)$, we call "robust estimator" the estimator

$$2\big(\widetilde{\lambda}_n(j_0)\big)' - \widetilde{\lambda}_n(j_0)$$

or

$$2\big(\widehat{\lambda}_n(a, j_1, j_2)\big)' - \widehat{\lambda}_n(a, j_1, j_2),$$

where the estimators with a prime denote the estimators obtained with the same algorithm as for the estimators without a prime, but considering only one out of every two data. The reason why we introduce these estimators is that in the presence of this additive noise, following the same kind of intuitive argument as in section 3.2 of chapter 1, we may conjecture that, instead of estimating the relative integrated volatility, we in fact estimate with our algorithms

$$\int X_t^{-2}\sigma_t^2 \mathrm{d}t + 4n\gamma^2 \int X_t^{-2}\mathrm{d}t.$$

Consequently, we can think that the robust estimators are "robust to the noise". Moreover one can probably estimate γ^2 using

$$\widehat{\gamma^2} = \frac{\widetilde{\lambda}_n(j_0) - \big(\widetilde{\lambda}_n(j_0)\big)}{2n\widehat{\int X_t^{-2}}\mathrm{d}t},$$

where the integral is estimated by its Riemann sums with the rounded data.

3.4 Results

Here, in the general case (including Heston models), the value of the integrated volatility varies from one sample path to another. To compare our estimators, we use the relative bias, standard deviation and MSE, that is the empirical counterparts of

$$\mathbb{E}\Big[\frac{\widehat{\sigma^2} - \sigma^2}{\sigma^2}\Big],\quad \mathrm{sd}\Big[\frac{\widehat{\sigma^2} - \sigma^2}{\sigma^2}\Big],\quad \Big(\mathbb{E}\Big[\Big(\frac{\widehat{\sigma^2} - \sigma^2}{\sigma^2}\Big)^2\Big]\Big)^{1/2}.$$

where $\widehat{\sigma^2}$ is the estimator and, abusing notation slightly, σ^2 represents the Riemann sum approximation of $\int_0^1 \sigma_t^2 \mathrm{d}t$. For simplicity we work in the Black-Scholes model, the results of the two other models leading to the same conclusions. We take here $a_t = 0$ and $\sigma_t = 2$.

3.4.1 Pure Rounding: $\gamma = 0$

We have the following results in the pure rounding case.

	Relative bias	Relative standard deviation	Relative MSE
Realized volatility	7.23688	0.72558	7.27309
Oracle ZMA	1.01581	0.17052	1.03
ZMA	0.97101	0.19122	0.98963
$j_0 = 7$	0.26473	0.11688	0.28934
$j_0 = 6$	0.13774	0.10378	0.1724
$j_0 = 5$	0.07287	0.09759	0.12172
$j_0 = 4$	0.04	0.09402	0.10209
Black-Scholes	0.00857	0.0918	0.09211
Robust $j_0 = 7$	0.2664	0.15641	0.30885
Robust $j_0 = 6$	0.13667	0.13864	0.19458
Robust $j_0 = 5$	0.07098	0.12964	0.14769
Robust $j_0 = 4$	0.03943	0.12435	0.13033
Robust Black-Scholes	0.00743	0.12037	0.12048
$j_1 = 6$, $j_2 = 6$	0.37677	0.13231	0.39928
$j_1 = 6$, $j_2 = 5$	0.1987	0.11222	0.22815
$j_1 = 6$, $j_2 = 4$	0.15316	0.10611	0.18626
$j_1 = 6$, $j_2 = 3$	0.14162	0.10422	0.17577
$j_1 = 5$, $j_2 = 5$	0.19885	0.11221	0.22827
$j_1 = 5$, $j_2 = 4$	0.10473	0.10247	0.14645
$j_1 = 5$, $j_2 = 3$	0.08088	0.09851	0.12738
$j_1 = 4$, $j_2 = 4$	0.10475	0.10247	0.14646
$j_1 = 4$, $j_2 = 3$	0.05628	0.09607	0.11126

Figure II.11: *Estimation of the relative integrated volatility in model* (II.23)-(II.24), *case* $\gamma = 0$, *Black-Scholes.*

Comments on figure II.11. The Black-Scholes estimator seems to be the best one. This is once again lead to the fact the relative volatility function is constant and so that the coefficients f_{jk} are equal to zero. Note that we also obtain good results in the Heston models. The realized volatility and ZMA are not robust in the pure rouding case. This phenomenon is largely explained in Li and Mykland [86].

3.4.2 Additive noise: non robust estimation

We now introduce the additive noise. We give the graphs of the empirical bias of the estimators $\widetilde{\lambda}_n(j_0)$ with respect to standard deviation of the noise.

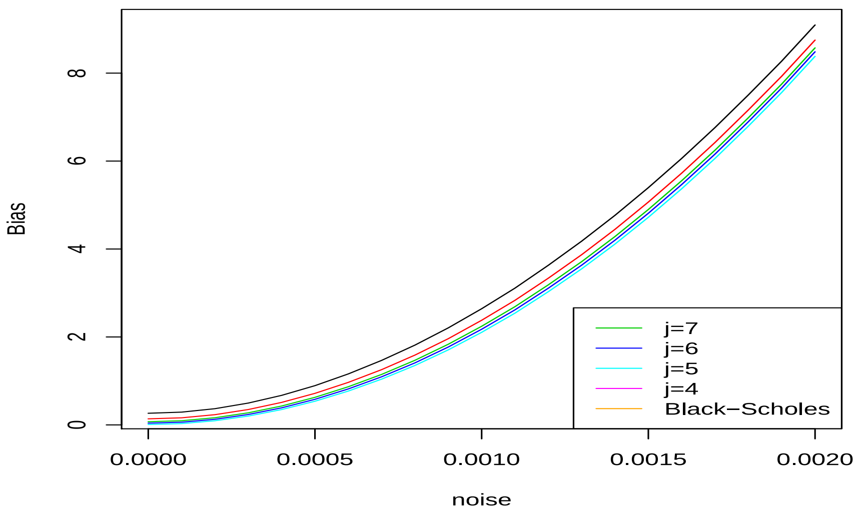

Figure II.12: *Bias of the non robust estimators in model* (II.23)-(II.24)*, Black-Scholes.*

Comments on figure II.12. We see that our estimators are good only in the case of pure rounding. This is logical: the additive noise adds jumps in the observed process and our estimators over estimate the true value of the integrated volatility.

3.4.3 Additive noise: robust estimation

We give the empirical bias and standard deviation (between parenthesis) obtained with our simulations for the robust estimations.

	Robust $j_0 = 7$	Robust $j_0 = 6$	Robust $j_0 = 5$	Robust $j_0 = 4$	Robust B-S
$\gamma = 0$	0.2664 (0.15641)	0.13667 (0.13864)	0.07098 (0.12964)	0.03943 (0.12435)	0.00743 (0.12037)
$\gamma = 0.0001$	0.26525 (0.14832)	0.13604 (0.13187)	0.07137 (0.12351)	0.03974 (0.11873)	0.00772 (0.11529)
$\gamma = 0.0002$	0.26104 (0.14623)	0.13361 (0.13102)	0.06974 (0.12436)	0.03875 (0.12072)	0.00709 (0.11737)
$\gamma = 0.0003$	0.2618 (0.15469)	0.13497 (0.13909)	0.07143 (0.13193)	0.03967 (0.12755)	0.0084 (0.12423)
$\gamma = 0.0004$	0.26315 (0.16039)	0.13762 (0.14476)	0.07408 (0.13727)	0.04171 (0.13349)	0.01069 (0.12933)
$\gamma = 0.0005$	0.26125 (0.17131)	0.13477 (0.15394)	0.07107 (0.14606)	0.03848 (0.14223)	0.0087 (0.13834)
$\gamma = 0.0006$	0.2562 (0.1838)	0.13071 (0.16733)	0.06663 (0.15699)	0.03487 (0.15248)	0.00533 (0.14874)
$\gamma = 0.0007$	0.25202 (0.19335)	0.127 (0.17587)	0.06342 (0.16398)	0.03169 (0.15965)	0.00272 (0.15618)
$\gamma = 0.0008$	0.24355 (0.19947)	0.1184 (0.18314)	0.05491 (0.17314)	0.02335 (0.16861)	-0.00487 (0.16525)
$\gamma = 0.0009$	0.24772 (0.2086)	0.11935 (0.19165)	0.0551 (0.18206)	0.02343 (0.17883)	-0.005 (0.17435)
$\gamma = 0.001$	0.2528 (0.22837)	0.12177 (0.2105)	0.05802 (0.1997)	0.02612 (0.19593)	-0.00153 (0.19147)
$\gamma = 0.0011$	0.25084 (0.249)	0.12077 (0.23183)	0.05759 (0.21902)	0.02468 (0.21445)	-0.00292 (0.21067)
$\gamma = 0.0012$	0.25311 (0.2707)	0.12107 (0.25317)	0.05705 (0.23977)	0.0244 (0.23453)	-0.0036 (0.23023)
$\gamma = 0.0013$	0.25471 (0.28627)	0.12169 (0.2704)	0.05601 (0.25718)	0.02262 (0.25174)	-0.00541 (0.2458)
$\gamma = 0.0014$	0.25812 (0.30028)	0.12248 (0.28736)	0.05637 (0.27728)	0.02253 (0.27077)	-0.00625 (0.26383)
$\gamma = 0.0015$	0.26338 (0.32344)	0.12453 (0.31215)	0.05621 (0.30007)	0.02222 (0.2934)	-0.00679 (0.28697)
$\gamma = 0.0016$	0.27697 (0.34987)	0.13413 (0.33258)	0.06408 (0.32117)	0.0295 (0.31398)	-0.00048 (0.3076)
$\gamma = 0.0017$	0.28658 (0.37468)	0.13839 (0.35713)	0.06763 (0.34739)	0.03236 (0.33965)	$-7e-05$ (0.33344)
$\gamma = 0.0018$	0.28706 (0.38899)	0.1362 (0.37447)	0.06447 (0.36584)	0.02785 (0.35726)	-0.00421 (0.35114)
$\gamma = 0.0019$	0.30472 (0.42009)	0.15093 (0.40393)	0.07846 (0.3964)	0.04065 (0.38858)	0.00673 (0.38179)
$\gamma = 0.0020$	0.3051 (0.4494)	0.14683 (0.43417)	0.07232 (0.42791)	0.03424 (0.41954)	-0.00091 (0.4132)

Figure II.13: *Estimation of the relative integrated volatility by robust estimators in model* (II.23)-(II.24) *with additive noise, Black-Scholes.*

This leads to the following graph.

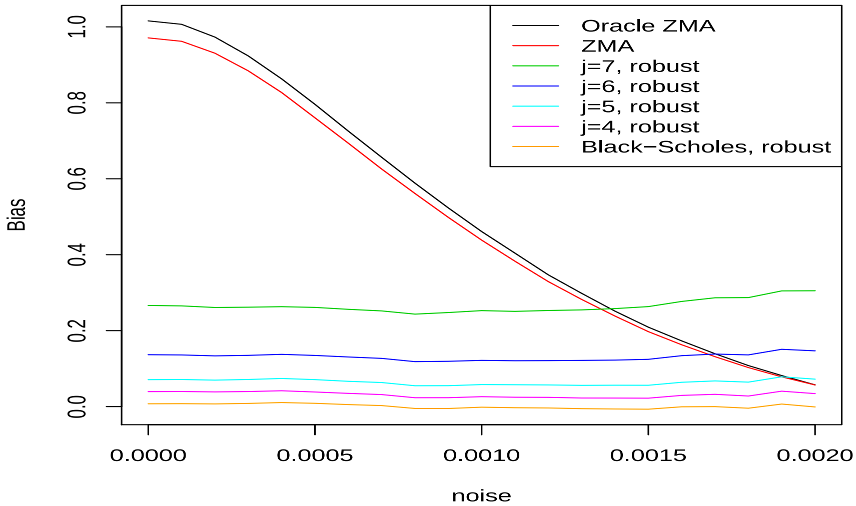

Figure II.14: *Bias of the robust estimators in model* (II.23)-(II.24)*, Black-Scholes.*

Comments on figure II.13 and figure II.14. It is important to remark that the same kind of graphs are obtained for the Heston models. So, our robust estimators seem to be robust to the noise. As explained in Li and Mykland [86], the robustness of the ZMA estimator increases with the variance of the noise. Indeed, if the noise is big, the effect of the rounding error becomes negligible regarding the effect of the additive error.

3.5 Estimation of γ

We end with the results of the estimation of γ. We give the average of the estimations and their standard deviation (between parenthesis).

	Robust $j_0 = 7$	Robust $j_0 = 6$	Robust $j_0 = 5$	Robust $j_0 = 4$	Robust B-S
$\gamma = 0$	$9.814e-05$ (0.00011946)	$9.586e-05$ (0.00011156)	$9.307e-05$ (0.00010778)	$8.958e-05$ (0.00010481)	$8.881e-05$ (0.00010368)
$\gamma = 0.0001$	0.00012034 (0.00012632)	0.00011799 (0.00011814)	0.00011505 (0.00011367)	0.00011203 (0.0001102)	0.00011099 (0.00010867)
$\gamma = 0.0002$	0.00020113 (0.00013262)	0.00019095 (0.00012784)	0.00018699 (0.00012155)	0.00018239 (0.0001193)	0.00018091 (0.00011689)
$\gamma = 0.0003$	0.00031076 (0.00012292)	0.00029822 (0.00011567)	0.00029095 (0.00011172)	0.00028786 (0.00010604)	0.00028421 (0.00010454)
$\gamma = 0.0004$	0.00042826 ($9.307e-05$)	0.00041074 ($8.638e-05$)	0.0004011 ($8.274e-05$)	0.00039618 ($8.066e-05$)	0.0003917 ($7.815e-05$)
$\gamma = 0.0005$	0.00053909 ($8.14e-05$)	0.00051851 ($7.681e-05$)	0.00050681 ($7.421e-05$)	0.00050127 ($7.248e-05$)	0.00049494 ($7.046e-05$)
$\gamma = 0.0006$	0.00064771 ($7.677e-05$)	0.00062394 ($7.284e-05$)	0.00061104 ($7.032e-05$)	0.00060453 ($6.807e-05$)	0.00059723 ($6.602e-05$)
$\gamma = 0.0007$	0.00075336 ($7.387e-05$)	0.00072699 ($7.073e-05$)	0.00071283 ($6.799e-05$)	0.00070582 ($6.605e-05$)	0.00069765 ($6.448e-05$)
$\gamma = 0.0008$	0.00085793 ($7.091e-05$)	0.00083009 ($6.817e-05$)	0.00081506 ($6.588e-05$)	0.00080748 ($6.41e-05$)	0.00079859 ($6.276e-05$)
$\gamma = 0.0009$	0.00095856 ($7.112e-05$)	0.00092987 ($6.847e-05$)	0.00091457 ($6.594e-05$)	0.00090663 ($6.448e-05$)	0.00089734 ($6.308e-05$)
$\gamma = 0.001$	0.00105929 ($7.054e-05$)	0.00103019 ($6.824e-05$)	0.00101387 ($6.654e-05$)	0.00100577 ($6.515e-05$)	0.00099598 ($6.379e-05$)
$\gamma = 0.0011$	0.00115951 ($7.166e-05$)	0.00112971 ($6.906e-05$)	0.00111312 ($6.723e-05$)	0.00110498 ($6.589e-05$)	0.00109503 ($6.481e-05$)
$\gamma = 0.0012$	0.00126054 ($7.363e-05$)	0.00123048 ($7.078e-05$)	0.00121394 ($6.909e-05$)	0.00120573 ($6.768e-05$)	0.00119575 ($6.668e-05$)
$\gamma = 0.0013$	0.00135938 ($7.24e-05$)	0.00132926 ($7.026e-05$)	0.00131296 ($6.914e-05$)	0.00130484 ($6.768e-05$)	0.00129491 ($6.668e-05$)
$\gamma = 0.0014$	0.00145762 ($7.191e-05$)	0.0014281 ($7.053e-05$)	0.00141216 ($6.971e-05$)	0.00140416 ($6.85e-05$)	0.0013943 ($6.746e-05$)
$\gamma = 0.0015$	0.00155538 ($7.397e-05$)	0.00152689 ($7.245e-05$)	0.00151156 ($7.128e-05$)	0.00150372 ($7.01e-05$)	0.00149398 ($6.914e-05$)
$\gamma = 0.0016$	0.0016515 ($7.483e-05$)	0.00162438 ($7.238e-05$)	0.00160976 ($7.148e-05$)	0.00160226 ($7.053e-05$)	0.00159269 ($6.929e-05$)
$\gamma = 0.0017$	0.00174797 ($7.593e-05$)	0.00172238 ($7.376e-05$)	0.00170834 ($7.317e-05$)	0.00170108 ($7.239e-05$)	0.00169202 ($7.12e-05$)
$\gamma = 0.0018$	0.00184548 ($7.672e-05$)	0.00182124 ($7.513e-05$)	0.00180778 ($7.45e-05$)	0.001801 ($7.377e-05$)	0.00179208 ($7.253e-05$)
$\gamma = 0.0019$	0.00193948 ($7.812e-05$)	0.00191659 ($7.626e-05$)	0.00190381 ($7.576e-05$)	0.00189754 ($7.53e-05$)	0.00188913 ($7.382e-05$)
$\gamma = 0.0020$	0.00203733 ($7.889e-05$)	0.00201607 ($7.733e-05$)	0.00200425 ($7.678e-05$)	0.00199836 ($7.619e-05$)	0.00199037 ($7.468e-05$)

Figure II.15: *Estimation of the standard deviation of the noise in model* (II.23)-(II.24), *Black-Scholes.*

Comments on figure II.15. We see that our estimations are really sharp. This is also true for the Heston models, see appendix C.

4 Equity Data

4.1 The data

We now use our estimators on equity data. We work on five stocks traded on Euronext. For clarity we just present here the results obtained for BNP-Paribas. The results of Danone, Total, Gaz de France and Renault are relegated to appendix B. For the five stocks, our data set goes from 2007-02-12 to 2007-02-23. We take $2^{14}+1$ data per day from 10.00, Paris Time, with a sampling period of one data per second[1]. Our data set is particularly relevant because because a change of the tick size took place on Euronext between these two weeks. Every day, we compute on our data (last traded prices and bid prices) the ZMA and Black-Scholes estimators and the estimators $\widetilde{\lambda}_n(j^*)$ and $\widetilde{\lambda}_n(j^*+1)$, where j^* denotes the theoretical optimal level, and the robust form of the three last estimators. We also compute the Garman-Klass estimator.

4.2 Garman-Klass estimator

The Garman-Klass estimator is a common market measure of the integrated volatility introduced by Garman and Klass [45]. There are several form of this estimator. We define it the following way

$$GK = 0.5\big(\log(M/m)\big)^2 - 0.39\big(\log(c/o)\big)^2,$$

where, M, m, c and o represent respectively the highest, lowest, opening and closing price for the considered day. Garman and Klass proved that this estimator is optimal in a mean-variance sense among a certain class of estimator, see [45].

4.3 Results

We now give the results for BNP-Paribas

- Tick first week=0.05.
- Tick second week=0.01.
- Volume on the two weeks=51370913.

[1] Note that we consider a day as an interval $[0,1]$.

4.3.1 Volatility estimation with the last traded price

We begin with the use of the last traded price.

	Day 1	Day 2	Day 3	Day 4	Day 5
ZMA	0.00014622	0.00012181	0.00013461	0.00024745	0.00019534
Garman-Klass	1.113e-05	8.87e-06	2.21e-05	5.053e-05	3.518e-05
j^*	1.166e-05	7.32e-06	1.204e-05	5.956e-05	3.623e-05
j^*+1	1.21e-05	7.78e-06	1.277e-05	6.095e-05	3.8e-05
Black-Scholes	1.06e-05	6.8e-06	9.73e-06	5.162e-05	3.238e-05
Robust j^*	3.031e-05	2.004e-05	3.176e-05	0.00012815	7.882e-05
Robust j^*+1	3.164e-05	2.129e-05	3.354e-05	0.00013123	8.031e-05
Robust Black-Scholes	2.759e-05	1.881e-05	2.57e-05	0.00011401	7.19e-05

Figure II.16: *Estimation of the integrated volatility with the last traded price for BNP-Paribas, first week.*

	Day 6	Day 7	Day 8	Day 9	Day 10
ZMA	6.735e-05	3.205e-05	2.964e-05	0	4.841e-05
Garman-Klass	2.671e-05	1.501e-05	1.151e-05	9.695e-05	7.297e-05
j^*	9.27e-06	9.06e-06	6.23e-06	8.73e-06	1.674e-05
j^*+1	1.003e-05	9.62e-06	6.72e-06	9.69e-06	1.995e-05
Black-Scholes	6.84e-06	5.76e-06	2.96e-06	7.01e-06	1.128e-05
Robust j^*	2.315e-05	1.415e-05	1.501e-05	2.053e-05	3.265e-05
Robust j^*+1	2.529e-05	1.52e-05	1.634e-05	2.234e-05	3.688e-05
Robust Black-Scholes	1.713e-05	1.065e-05	7.58e-06	1.655e-05	2.498e-05

Figure II.17: *Estimation of the integrated volatility with the last traded price for BNP-Paribas, second week.*

4.3.2 Volatility estimation with the bid price

We now use of the bid price. Recall that if one assume that the theoretical price lies between the bid price and the ask price, and that the bid-ask spread is constant equal to one tick, then the bid price is almost surely the right measure of the rounded theoretical price.

	Day 1	Day 2	Day 3	Day 4	Day 5
ZMA	4.373e-05	2.124e-05	2.892e-05	6.111e-05	3.429e-05
Garman-Klass	1.534e-05	7.46e-06	1.825e-05	4.436e-05	2.595e-05
j^*	1.56e-06	5.6e-07	1.14e-06	6.04e-06	2.01e-06
j^*+1	1.74e-06	6.1e-07	1.43e-06	6.59e-06	2.32e-06
Black-Scholes	1.15e-06	3.6e-07	9.7e-07	4.25e-06	1.39e-06
Robust j^*	4.3e-06	1.54e-06	2.97e-06	1.511e-05	5.45e-06
Robust j^*+1	4.89e-06	1.65e-06	3.79e-06	1.632e-05	6.34e-06
Robust Black-Scholes	3.09e-06	9.3e-07	2.57e-06	1.093e-05	3.76e-06

Figure II.18: *Estimation of the integrated volatility with the bid price for BNP-Paribas, first week.*

	Day 6	Day 7	Day 8	Day 9	Day 10
ZMA	2.296e-05	1.396e-05	0	0	0
Garman-Klass	2.578e-05	1.382e-05	1.027e-05	9.774e-05	6.809e-05
j^*	3.16e-06	2.56e-06	3.55e-06	2.92e-06	5.03e-06
j^*+1	3.69e-06	2.93e-06	4e-06	3.36e-06	5.77e-06
Black-Scholes	2.07e-06	1.52e-06	8.9e-07	2.32e-06	3.54e-06
Robust j^*	8.48e-06	5.38e-06	8.12e-06	8.15e-06	1.264e-05
Robust j^*+1	9.98e-06	6.31e-06	8.71e-06	9.4e-06	1.472e-05
Robust Black-Scholes	5.62e-06	3.5e-06	2.17e-06	6.47e-06	9.02e-06

Figure II.19: *Estimation of the integrated volatility with the bid price for BNP-Paribas, second week.*

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4.4 Comments on figures II.17 to II.19

• Our estimated values are lower using the bid price than the last traded price. This is logical: the bid price varies less than the last traded price.

• The first week, our non robust estimators are of the order of magnitude of the Garman-Klass estimator when considering the last traded price whereas the ZMA estimator is of the order of magnitude of the Garman-Klass estimator when considering the bid price. This phenomenon is not so clear the second week.

• The ZMA estimations of the second week are all smaller than the smallest ZMA estimation of the first week (except day 6 with the bid price). This is not true for our non robust estimators. Hence, it seems that the change of tick size have had a more important impact on the ZMA estimator than on ours.

• Surprisingly the robust volatilities are bigger than the non robust volatilities. This never happens in our Monte Carlo simulations. Moreover, it occurs for all the considered stocks. Consider the model

$$\mathrm{d}X_t = a_t\mathrm{d}t + \sigma_t\mathrm{d}W_t,\ X_0 = x_0, t \in [0, T],$$

where we observe

$$\big\{(X_{iT/n} + \zeta_i^n)^{(\alpha_n)},\ i = 0, \ldots, n\big\},$$

Assuming the noise is independent of the process, we saw that we may think that we in fact estimate with the non robust estimators

$$\int X_t^{-2}\sigma_t^2\mathrm{d}t + n\mathbb{E}[(\zeta_{i+1}^n - \zeta_i^n)^2]\int X_t^{-2}\mathrm{d}t.$$

Hence, our numerical results implies

$$\mathbb{E}[(\zeta_{i+1}^n - \zeta_i^n)^2] < \frac{1}{2}\mathbb{E}[(\zeta_{i+2}^n - \zeta_i^n)^2].$$

This inequality can not be obtained if the ζ_i^n are independent or follow an AR(1) model. But, if we suppose there exists a continuous noise process $\tilde{\varepsilon}_t$ such that for all n and i, $\zeta_i^n = \tilde{\varepsilon}_{i/n}$, if $\tilde{\varepsilon}_t = cB_t^H$, with B_t^H a fractional Brownian motion[2], then

$$\zeta_{i+1}^n - \zeta_i^n = v(W_{(i+1)/n}^H - W_{i/n}^H)$$

and

$$\zeta_{i+2}^n - \zeta_i^n = v(W_{(i+2)/n}^H - W_{i/n}^H).$$

Hence the inequality is true as soon as $H > 1/2$. In such a model, we see that the non robust estimators are in fact better that the robust estimators and that we would obtain more accurate results increasing the sampling frequency so that the term

$$n\mathbb{E}[(\zeta_{i+1}^n - \zeta_i^n)^2]\int X_t^{-2}\mathrm{d}t.$$

becomes negligible. This point is further discussed in part IV.

[2] A stationary process like a fractional Ornstein-Uhlenbeck process is probably more suitable but the theoretical part of this section remaining to be done, we only focus on very simple examples.

Part III

First order p-variations and Besov spaces

Abstract

Based on the notion of first order dyadic p-variation, we give a new characterization of Besov spaces $\mathcal{B}^s_{p,q}([0,1])$ for $0 < s < 1$, $1 \leq p, q \leq +\infty$ and $s > 1/p$. We also give results in the case where $p < 1$. Hence we provide simple tools that enable us to derive new regularity properties for the trajectories of various continuous time stochastic processes.

Keywords: Besov spaces; Schauder basis; p-variation; Wavelet; Gaussian processes; Markov chains; Lévy processes.

Note

This part is based on a paper submitted to *Statistics and Probability Letters*. This small part is quite on its own with respect to the other of the thesis, but has lead to ideas used in the fourth part of the thesis.

1 Introduction

Besov spaces are a natural framework to study the smoothness of the sample paths of a continuous time random process, see Ciesielski, Kerkyacharian and Roynette [26] (CKR for short). For $s > 0$ and $1 \le p, q \le +\infty$, the Besov spaces $\mathcal{B}^s_{p,q}([0,1])$ are usually defined in term of modulus of continuity, see appendix C for definitions. Using the Schauder basis, CKR have proved that for $s > 1/p$, the usual Besov norm on $\mathcal{B}^s_{p,q}([0,1])$ is equivalent to a norm based on the *second order dyadic p-variation*, that is, for a real function f on $[0,1]$ and $0 < p < +\infty$[3], the quantity

$$\sum_{k=1}^{2^j} |-f(\{2k\}2^{-(j+1)}) + 2f(\{2k-1\}2^{-(j+1)}) - f(\{2k-2\}2^{-(j+1)})|^p,$$

see appendix C for details. This result has been used to obtain regularity properties of the sample paths of some stochastic processes such as Brownian motion, fractional Brownian motion, Lévy stable processes and stochastic integrals, see Roynette [103], Ciesielski, Kerkyacharian and Roynette [26]. The local time of the Brownian motion and of Lévy stable processes have been studied by Boufoussi and Roynette [21] and Boufoussi and Ouknine [20].

In this part, we simplify the results of CKR and we extend them using either *first order dyadic p-variations* or *first order general p-variations*. For a real function f on $[0,1]$ and $0 < p < +\infty$, the first order dyadic p-variation $V_j^p(f)$ is defined by

$$V_j^p(f) = \sum_{k=1}^{2^j} |f(k2^{-j}) - f(\{k-1\}2^{-j})|^p$$

and the first order general p-variation $\nu_p(f)$ by

$$\nu_p(f) = \sup\Big\{ \sum_{k=0}^{m-1} |f(t_{k+1}) - f(t_k)|^p,\ 0 = t_0 < t_1 < \ldots < t_m = 1,\ m = 1, 2, \ldots \Big\}.$$

The behavior of this kind of objects has been studied for many stochastic processes, see section 3. Our objective is to be able to derive Besov smoothness properties for the sample paths of various continuous time processes such as Gaussian processes or Markov processes.

[3] The case $p = +\infty$ will be treated considering throughout the part the following usual modification: for a sequence real $(a_j)_{j\ge 0}$, $(\sum_j |a_j|^p)^{1/p} = \sup_j |a_j|$ for $p = +\infty$.

We give in section 2 a new characterization of Besov spaces $\mathcal{B}^s_{p,q}([0,1])$ for $0 < s < 1$, $1 \le p,q \le \infty$, $s > 1/p$, and a useful result for the case $p < 1$. Section 3 contains new smoothness properties for the trajectories of various processes derived from our results. Some proofs are relegated to section 4.

2 New characterization of Besov spaces

We state in this section two theorems that we widely use in section 3 to obtain Besov smoothness properties for the sample paths of stochastic processes. We refer to appendix C for the definition of the usual norm (or quasi-norm) $\|\cdot\|_{\mathcal{B}^s_{p,q}([0,1])}$ on $\mathcal{B}^s_{p,q}([0,1])$.

2.1 Results

Theorem III.1 *Let $0 < s < 1$, $s > 1/p$, $1 \le p,q \le \infty$*[4]*. The usual norm on $\mathcal{B}^s_{p,q}([0,1])$ is equivalent to the norm defined by*

$$\|f\| = \max\big\{|f(0)|, \big(\sum_{j\ge 0} 2^{jq(s-1/p)}\{V^p_j(f)\}^{q/p}\big)^{1/q}\big\}.$$

Theorem III.2 *Let $f : [0,1] \to \mathbb{R}$ be a Borel function and $0 < p < \infty$. If $\nu_p(f) < +\infty$, then f belongs to $\mathcal{B}^{1/p}_{p,\infty}([0,1])$.*

2.2 Discussion

• The result of theorem III.1 is obviously very close to the results of CKR. It simply shows that, to derive some Besov regularity of a trajectory, it is often enough to consider first order dyadic p-variation. A striking example is provided by the Brownian motion. The Besov regularity of the Brownian motion has been studied in particular by Roynette [103]. Let $(B_t)_{t\ge 0}$ be a Brownian motion. Using the Lévy construction of the Brownian motion, Roynette has proved the following proposition, where $a.s.$ means almost surely.

Proposition III.1 *(Roynette [103])*

- *If $0 < s < 1/2$, for $1 \le p < \infty$ and $1 \le q \le \infty$, a.s., $(t \to B_t) \in \mathcal{B}^s_{p,q}([0,1])$.*
- *For $1 \le p < \infty$, a.s., $(t \to B_t) \in \mathcal{B}^{1/2}_{p,\infty}([0,1])$. Moreover, for $p > 2$, there exists a constant $c_p > 0$ such that a.s., $\|B\|_{\mathcal{B}^{1/2}_{p,\infty}([0,1])} > c_p$.*
- *For $2 < p < \infty$ and $1 \le q < \infty$, a.s., $(t \to B_t) \notin \mathcal{B}^{1/2}_{p,q}([0,1])$.*

[4] Note also that we directly get from CKR that for $0 < s < 1 + 1/p$, $1 \le p,q \le \infty$ and f càdlàg, $\|f\|_{\mathcal{B}^s_{p,q}([0,1])} \le c \max\big\{|f(0)|, \big(\sum_{j\ge 0} 2^{jq(s-1/p)}\{V^p_j(f)\}^{q/p}\big)^{1/q}\big\}$.

- *If* $s > 1/2$, *for* $2 < p < \infty$ *and* $1 \leq q \leq \infty$, *a.s.,* $(t \to B_t) \notin \mathcal{B}^s_{p,q}([0,1])$.

Thanks to theorem III.1, we have a very simple proof of the results of Roynette. Indeed, using convergence L^2 convergence, Markov inequality and Borel-Cantelli lemma, we classically have that for $1 \leq p < \infty$, a.s.,

$$2^{j(p/2-1)} \sum_{k=0}^{2^j-1} |B_{(k+1)2^{-j}} - B_{k2^{-j}}|^p \to \mathbb{E}[|\eta|^p],$$

where η denotes a standard Gaussian variable. Hence, by theorem III.1, the positivity of $\mathbb{E}[|\eta|^p]$ and obvious embeddings, we have the result. We extend this result to more general Gaussian processes in proposition III.2.

- A corollary of theorem III.1 is that for a function f on $[0,1]$ and $\alpha > 0$,

$$\sup_{j\geq 0} 2^{j\alpha} \sum_{k=1}^{2^j} |-f(\{2k\}2^{-(j+1)}) + 2f(\{2k-1\}2^{-(j+1)}) - f(\{2k-2\}2^{-(j+1)})|^p < \infty$$

implies that

$$\sup_{j\geq 0} 2^{j\alpha} \sum_{k=1}^{2^j} |f(\{k+1\}2^{-j}) - f(k2^{-j})|^p < \infty.$$

This can be used to prove the finiteness of the supremum of the renormalized first order dyadic p-variation. Indeed, it is interesting to remark that for some processes, it is quite easy to prove the first of these two properties whereas a direct proof of the second one appears difficult. An example is provided by simple branching process wavelet series, see Brouste [23].

- Another corollary of theorem III.1 is the following.

Corollary III.1 *Let* $\{X_t, t \in [0,1]\}$ *be a continuous process such that for* $p > 1$, $1/p \leq s < 1$ *and* Y *an almost surely finite and positive random variable,*

$$2^{j(ps-1)} V^p_j(X) \to Y \text{ in probability.}$$

Then, for any $\varepsilon > 0$, *a.s.* $(t \to X_t) \in \mathcal{B}^{s-\varepsilon}_{p,\infty}([0,1])$ *and a.s.* $(t \to X_t) \notin \mathcal{B}^{s+\varepsilon}_{p,\infty}([0,1])$.

Example: Let X be a 1-dimensional continuous Ito-semimartingale of the form

$$X_t = X_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s,$$

with B a Brownian motion and where the volatility process is itself an Ito-semimartingale (not necessarily continuous), see Jacod [73] for definitions. Under the assumptions of Jacod [73][5], for $1 \leq p < \infty$, we have the following convergence in probability

$$2^{j(p/2-1)} V_j^p(X) \to \int_0^1 \int_{\mathbb{R}} |x|^r (2\pi\sigma_u^2)^{1/2} \mathrm{e}^{x^2/(2\sigma_u^2)} \mathrm{d}x \mathrm{d}u.$$

The proof of this result is given in Jacod [73]. Hence, we get that for any $\varepsilon > 0$ and $2 \leq p < \infty$ a.s. $(t \to X_t) \in \mathcal{B}_{p,\infty}^{1/2-\varepsilon}([0,1])$ and a.s. $(t \to X_t) \notin \mathcal{B}_{p,\infty}^{1/2+\varepsilon}([0,1])$. This seems to complete some results of Roynette [103] on the stochastic integral.

• The result of theorem III.2 is in fact a corollary of an analogue of theorem III.1 for distributional Besov spaces $\tilde{\mathcal{B}}_{p,q}^s$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$, as defined in Triebel [108], [109]. Details are given in the proof of theorem III.2.

3 New regularity properties for stochastic processes

In this section we present regularity properties for various processes. Our goal is to give an overview of the kind of results one can easily obtain thanks to theorem III.1 and theorem III.2. From now on, we write $\mathcal{B}_{p,q}^s$ for $\mathcal{B}_{p,q}^s([0,1])$.

3.1 Gaussian processes

We first give a very simple characterization for some Gaussian processes for belonging to the Besov spaces $\mathcal{B}_{p,q}^s$.

Proposition III.2 *Let $\{X_t, t \in [0,1]\}$ be a zero mean Gaussian process with stationary increments. Let $\sigma(h) = (\mathbb{E}[(X_{t+h} - X_t)^2])^{1/2}$. Assume that for some $0 < r < 1$ and $0 < \alpha < \infty$,*

$$\lim_{h \to 0} \sigma(h)/h^r = \alpha.$$

Then, we have the following result:

- *If $0 < s < r$, for $1 \leq p < \infty$ and $1 \leq q \leq \infty$, a.s., $(t \to X_t) \in \mathcal{B}_{p,q}^s$.*
- *For $1 \leq p < \infty$, a.s. $(t \to X_t) \in \mathcal{B}_{p,\infty}^r$. Moreover, for $p > 1/r$, there exists a constant $c_p > 0$ such that a.s., $\|X\|_{\mathcal{B}_{p,\infty}^r} > c_p$.*
- *For $2 \leq p < \infty$, $rp > 1$ and $1 \leq q < \infty$, a.s., $(t \to X_t) \notin \mathcal{B}_{p,q}^r$,*

[5]The assumptions on the process are very weak but technical so that we do not give them here.

- *If $s > r$, for $2 \leq p < \infty$, $rp > 1$ and $1 \leq q \leq \infty$, a.s.,* $(t \to X_t) \notin \mathcal{B}^s_{p,q}$.

This proposition can for example be applied to fractional Brownian motion with Hurst parameter H in $(0,1)$. Indeed, we have in this case $\sigma(h) = h^H$.
We now present results for Baxter and Gladyshev processes, see Baxter [17] and Gladyshev [53] for details and examples. Recall the following definitions.

Definition III.1 *(Baxter process)*
A real valued Gaussian process $\{X_t, t \in [0,1]\}$ *is called a Baxter process if*
- $m(t) = \mathbb{E}[X_t]$ *has a bounded first derivative,*
- $r(s,t) = \mathbb{E}[X_t X_s] - m(s)m(t)$ *is continuous on* $[0,1]^2$ *and has uniformly bounded second derivatives on* $[0,1]^2 - \Delta$, $\Delta = \{(t,t),\ t \in [0,1]\}$.

Definition III.2 *(Gladyshev process)*
A real valued Gaussian process $\{X_t, t \in [0,1]\}$ *is called a Gladyshev process if*
- $m(t)$ *has a bounded first derivative,*
- $r(s,t)$ *is continuous on* $[0,1]^2$*, has second derivatives on* $[0,1]^2 - \Delta$ *and* $|\partial^2 r(s,t)/\partial s \partial t| \leq L/|t-s|^\gamma$ *for some constants* $L > 0$ *and* $0 < \gamma < 2$,
- *the expression* $|r(t,t) - 2r(t,t-h) + r(t-h,t-h)|/h^{2-\gamma}$ *converges uniformly on* $[0,1]$ *to some function* $f(t)$ *as* $h \to 0$.

We have the following propositions.

Proposition III.3 *Let* $\{X_t, t \in [0,1]\}$ *be a Baxter process. Let* $D^+(t)$ *and* $D^-(t)$ *denote respectively the left and right derivatives of* $s \to r(s,t)$ *at point* t *and* $f(t) = D^-(t) - D^+(t)$. *Then, a.s.,* $(t \to X_t) \in \mathcal{B}^{1/2}_{2,\infty}$. *Moreover, if* $\int_0^1 f(t)dt \neq 0$, *for all* $\varepsilon > 0$, *a.s.,* $(t \to X_t) \notin \mathcal{B}^{1/2+\varepsilon}_{2,\infty}$.

Proof. Directly from theorem III.1 together with theorem 1 in [17]. □

Proposition III.4 *Let* $\{X_t, t \in [0,1]\}$ *be a Gladyshev process of index* $\gamma \in (0,2)$.

- *A.s.,* $(t \to X_t)$ *belongs to* $\mathcal{B}^{\gamma/2}_{2,\infty}$.
- *If* $\int_0^1 f(t)dt \neq 0$, $\gamma > 1$ *and* $1 \leq q < \infty$, *a.s.,* $(t \to X_t) \notin \mathcal{B}^{\gamma/2}_{2,q}$.
- *If* $\int_0^1 f(t)dt \neq 0, \gamma \geq 1$ *and* $\varepsilon > 0$, *a.s.,* $(t \to X_t) \notin \mathcal{B}^{\gamma/2+\varepsilon}_{2,\infty}$.

Proof. Directly from theorem III.1 together with theorem 1 in [53]. □

3.2 Markov processes

Proposition III.5 *Let $\{X_t, t \in [0,1]\}$ be a strong Markov process. Denote the transition probability function of X_t by $P_{s,t}(x, dy)$. For any $h \in [0,1]$ and $a > 0$, define*

$$\alpha(h,a) = \sup\{P_{s,t}(x, \{y : |x-y| \geq a\}), 0 \leq s \leq t \leq (s+h) \wedge 1\}.$$

Let $\beta \geq 1$, $\gamma > 0$. If there exist constants $a_0 > 0$ and $K > 0$ such that, for all $h \in [0,1]$ and $a \in (0, a_0]$,

$$\alpha(h,a) \leq K h^{\beta} a^{-\gamma}.$$

then, for all $\gamma/\beta < p < \infty$, $(t \to X_t)$ belongs to $\mathcal{B}^{1/p}_{p,\infty}$.

Proof. directly from theorem III.2 and theorem 1.3 in [88]. □

Remark: It is proved in [88] that for symmetric stable Lévy processes with index $\alpha \in (0,2]$, the condition $p > \gamma/\beta$ applies with $\gamma/\beta = \alpha$.

3.3 Functionals of Lévy processes

The Besov regularity of Lévy processes has been studied by CKR [26], in the case of stable processes of index β, $1 < \beta < 2$, $p \geq 1$, and by Schilling [104] in the general case[6]. We give here other results related to Lévy processes.

Proposition III.6 *Let $\{X_t, t \in [0,1]\}$ be a real valued stable process of index β, $1 < \beta \leq 2$. For $0 < \gamma < (\beta-1)/2$, we define H_t, its fluctuating continuous additive functional of order γ by*

$$H_t = \frac{1}{\Gamma(-\gamma)} \int_0^{\infty} y^{-1-\gamma}(L_t^{-y} - L_t^0)\,dy,$$

where $\{L(t,x), t \geq 0\}$ is the local time at x for X. Let $p_0 = (\beta-1)/(\beta-1-\gamma)$. Then for all $p_0 < p < \infty$, a.s., $(t \to H_t)$ belongs to $\mathcal{B}^{1/p}_{p,\infty}$ and for $\varepsilon > 0$, a.s. $(t \to H_t)$ does not belong to $\mathcal{B}^{1/p_0+\varepsilon}_{p_0,\infty}$.

Proof. Directly from theorem III.2 and theorem 4.3 in [41]. □

Proposition III.7 *Let $f(x) = x^{\delta}$ with $\delta \in (\{3-e\}/\{e-1\}, 1)$. Let $\{X_t, t \in [0,1]\}$ be a symmetric real-valued $\alpha-$stable Lévy process with $\alpha \in (0,\delta)$. We define $Y_t = X_{f(t)}$ for $t \in [0,1]$. For any $\alpha/\delta < p < \infty$, $(t \to Y_t)$ belongs a.s. to $\mathcal{B}^{1/p}_{p,\infty}$.*

Proof. Directly from theorem III.2 and remark 1 in [89]. □

[6]Note that for $p < 1$, we can give simple proofs of some results of Schilling [104] on Lévy processes. We just use theorem III.2 together with classical results on the p-variation of Lévy processes (for stable processes, theorem 2 in Fristedt and Taylor [44], for general Lévy processes, theorem 2 in Monroe [93]).

4 Proofs

4.1 Proof of theorem III.1

In this proof, c denotes a positive constant that may vary from line to line. Let $1 \leq p, q \leq \infty$ and $s > 1/p$. The fact that $\|f\|_{\mathcal{B}^s_{p,q}}$ is smaller than $c\|f\|$ is obvious because of proposition C.1 together with a convexity inequality. For the other inequality, we use the development of f in the Schauder basis, see appendix C. Let $f \in \mathcal{B}^s_{p,q}([0,1])$, $1 \leq p < \infty$, , $1 \leq q \leq \infty$ and $s > 1/p$.

$$V_n^p(f) = \sum_{r \leq 2^n} |f(\{r+1\}2^{-n}) - f(r2^{-n})|^p.$$

Consider the Schauder basis (ϕ_{jk}) and the coefficients of the expansion of f defined as in proposition C.1. We have

$$V_n^p(f) \leq 2^{p-1} \sum_{r \leq 2^n} \big| \sum_{j=0}^{n} \sum_{k=1}^{2^j} f_{jk}[\phi_{jk}(\{r+1\}2^{-n}) - \phi_{jk}(r2^{-n})] \big|^p + 2^{p-1} f_1^p 2^{n(1-p)}.$$

For given j, as $j \leq n$, there exists a unique integer $k = k_{j,r,n}$ such that $[\phi_{jk}(\{r+1\}2^{-n}) - \phi_{jk}(r2^{-n})] \neq 0$. Let

$$\widetilde{V}_n^p(f) = \sum_{r \leq 2^n} \big| \sum_{j=0}^{n} \sum_{k=1}^{2^j} f_{jk}[\phi_{jk}(\{r+1\}2^{-n}) - \phi_{jk}(r2^{-n})] \big|^p.$$

Since $|\phi_{jk}(\{r+1\}2^{-n}) - \phi_{jk}(r2^{-n})| \leq 2^{j/2-n}$, we have $\widetilde{V}_n^p(f) \leq M_n$ with

$$M_n = \sum_{r \leq 2^n} \big(\sum_{j=0}^{n} 2^{j/2-n} |f_{jk_{j,r,n}}| \big)^p.$$

Let $\varepsilon = p - \lfloor p \rfloor$, we have

$$M_n \leq 2^{-n\lfloor p \rfloor} \sum_{j_1,\dots,j_{\lfloor p \rfloor}} 2^{(j_1+\dots+j_{\lfloor p \rfloor})/2} \sum_{r \leq 2^n} |f_{j_1 k_{j_1,r,n}}| \cdots |f_{j_{\lfloor p \rfloor} k_{j_{\lfloor p \rfloor},r,n}}| \{ \sum_{j=0}^{n} 2^{j/2-n} |f_{jk_{j,r,n}}| \}^{\varepsilon}.$$

By Hölder inequality, we obtain

$$M_n \leq 2^{-n\lfloor p \rfloor} \sum_{j_1,\dots,j_{\lfloor p \rfloor}} 2^{(j_1+\dots+j_{\lfloor p \rfloor})/2} M_n^{\varepsilon/p} \prod_{i=1}^{\lfloor p \rfloor} \big(\sum_{r \leq 2^n} |f_{j_i k_{j_i,r,n}}|^p \big)^{1/p}.$$

Thus,

$$
\begin{aligned}
M_n^{1-\varepsilon/p} &\le 2^{-n\lfloor p\rfloor} \sum_{j_1,\dots,j_{\lfloor p\rfloor}} 2^{(j_1+\cdots+j_{\lfloor p\rfloor})/2} \prod_{i=1}^{\lfloor p\rfloor} \big(2^{n-j_i} \sum_k |f_{j_i k}|^p\big)^{1/p} \\
&\le 2^{-n\lfloor p\rfloor} \big\{ \sum_{j=0}^{n} 2^{j/2} \big(2^{n-j} \sum_k |f_{jk}|^p\big)^{1/p} \big\}^{\lfloor p\rfloor}.
\end{aligned}
$$

If $q = +\infty$, as $s < 1$, using proposition C.1 we easily obtain $M_n \le c2^{n(1-sp)}$. The result follows using that $V_n^p(f) \le M_n + c2^{n(1-p)}$. If $1 \le q < +\infty$, we have

$$
2^{nq(s-1/p)} M_n^{q/p} \le \big\{ \sum_{j=0}^{n} 2^{(j-n)(1-s)} 2^{j(-1/2+s-1/p)} \big(\sum_k |f_{jk}|^p\big)^{1/p} \big\}^q .
$$

Let γ be such that $0 < \gamma < 1 - s$. Then, using Hölder inequality, we get

$$
2^{nq(s-1/p)} M_n^{q/p} \le c \sum_{j=0}^{n} 2^{(j-n)q\gamma} 2^{qj(-1/2+s-1/p)} \big(\sum_k |f_{jk}|^p\big)^{q/p}. \tag{III.1}
$$

Moreover,

$$
V_n^p(f)^{q/p} \le c(M_n^{q/p} + 2^{nq(1/p-1)}).
$$

The result follows using proposition C.1 and remarking that the series in j in (III.1) is n-th term of the Cauchy product of two convergent series. The case $p = +\infty$ is treated the same way.

4.2 Proof of theorem III.2

We denote by $\mathcal{S}'$ the space of all tempered distributions, by $\mathcal{C}^r$ the set of all compactly supported r-times continuously differentiable real functions and by $\mathcal{W}^r$ the set of all couple (ψ_0, ψ) of Daubechies scaling function and mother wavelet, both in $\mathcal{C}^r$, see Daubechies [34] for details. For $s \in \mathbb{R}$ and $0 < p, q \le \infty$, let $\sigma_p = (1/p - 1)_+$ and $r(s,p) = \max(s, \sigma_p - s)$. Let $b^s_{p,q}$ be the set of all sequences $\lambda = \{\lambda_{jk} \in \mathbb{C},\ j \in \mathbb{N}, k \in \mathbb{Z}\}$ such that

$$
\|\lambda | b^s_{p,q}\| = \big(\sum_{j=0}^{\infty} 2^{jq(s-1/p)} \big\{ \sum_{k\in\mathbb{Z}} |\lambda_{jk}|^p \big\}^{q/p} \big)^{1/q} < \infty.
$$

We now define the distributional Besov space $\tilde{\mathcal{B}}^s_{p,q}$, see Triebel [109].

Definition III.3 *Let $(\psi_0, \psi) \in \mathcal{W}^r$, with $r > r(s,p)$. Then, $f \in \mathcal{S}'$ is an element of $\tilde{\mathcal{B}}^s_{p,q}$ if and only if it can be represented as*

$$
f = \sum_{\substack{j\in\mathbb{N},\\ k\in\mathbb{Z}}} \lambda_{jk} \psi_{jk}, \tag{III.2}
$$

with $\|\lambda|b^s_{p,q}\| < \infty$, *convergence in* $\mathcal{S}'$ *and* $\psi_{jk}(x) = \psi(2^{j-1}x - k)$. *Furthermore, if* $f \in \tilde{\mathcal{B}}^s_{p,q}$, *then the representation* (III.2) *is unique with* $\lambda_{jk} = 2^j\langle f, \psi_{jk}\rangle$. *Equipped with* $\|f\|_{\tilde{\mathcal{B}}^s_{p,q}} = \|\lambda(f)|b^s_{p,q}\|$, $\tilde{\mathcal{B}}^s_{p,q}$ *is a quasi-Banach space.*

Remark that for duality reasons, the notation $\langle f, \psi_{jk}\rangle$ makes sense as soon as $f \in \tilde{\mathcal{B}}^s_{p,q}$ and $g \in \mathcal{C}^r$ with $r > -s + \sigma_p$. This definition coincides with the definition given in appendix C for $s > \sigma_p$. Next proposition will enable us to give an analogous of theorem III.1 for distributional Besov spaces.

Proposition III.8 *(see Kerkyacharian and Picard [77]) Let* $(\psi_0, \psi) \in \mathcal{W}^r$ *for given* $r \geq 1$. *Then there exists a function* θ *in* $\mathcal{C}^r$, *whose support is included in the support of* ψ, *such that* $\psi(x) = \theta(x) - \theta(x - 1/2)$.

For $f \in \tilde{\mathcal{B}}^s_{p,q}$ and g a compactly supported function in $\mathcal{C}^r$ with $r > s - \sigma_p$, we classically set

$$\langle f_a, g\rangle = \langle f, g_{-a}\rangle, \langle f^\lambda, g\rangle = |\lambda|\langle f, g^{1/\lambda}\rangle \text{ and } \langle f^\lambda_a, g\rangle = \langle (f^\lambda)_a, g\rangle,$$

with $g_a(x) = g(x - a)$ and $g^\lambda(x) = g(x/\lambda)$. Let $(\psi_0, \psi) \in \mathcal{W}^r$ and θ be the function associated to ψ by proposition III.8. For $j \in \mathbb{N}$, we define $\tilde{V}^p_{j,\psi}(f)$ by

$$\tilde{V}^p_{0,\psi}(f) = \sum_k |\langle f, \psi_{0k}\rangle|^p \text{ and } \tilde{V}^p_{j+1,\psi}(f) = 2^{-p}\sum_k |\langle f^{2^j}_{-k} - f^{2^j}_{-(k+1/2)}, \theta\rangle|^p.$$

We now deduce the following proposition[7].

Proposition III.9 *Let* f *be a tempered distribution such that* $f \in \tilde{\mathcal{B}}^s_{p,q}$ *for given* $s \in \mathbb{R}$ *and* $0 < p, q \leq \infty$. *Let* $(\psi_0, \psi) \in \mathcal{W}^r$, *with* $r > r(s,p)$. *Then*

$$\|f\|_{\tilde{\mathcal{B}}^s_{p,q}} = \big(\sum_{j\geq 0} 2^{jq(s-1/p)}\{\tilde{V}^p_{j,\psi}(f)\}^{q/p}\big)^{1/q}.$$

From proposition III.9, obvious computations lead to theorem III.2.

4.3 Proof of proposition III.2

We write η for a standard Gaussian variable. Let $2 \leq p < \infty$. We first prove the following almost sure equality which slightly extend a result from Marcus and Rosen [90]:

$$\lim_{j\to\infty} 2^{j(ps-1)}V^p_j(X) = \alpha^p\mathbb{E}[|\eta|^p].$$

[7] Note that proposition III.9 can be easily extended to distributional Besov spaces with functional smoothness using the results of Almeida [7].

Then, we obtain the proposition applying theorem III.1. We adapt here the method used by Marcus and Rosen [90]. Consider our process $\{X_t, t \in [0,1]\}$. We will write here V_j^p for $V_j^p(X)$. Let q be the conjugate exponent of p. Let S^j denote the set of all sequences of cardinal 2^j of the unit ball of l_q and U^j be a countable dense subset in S^j. For $(u^j = \{u_k^j, k = 1, \ldots, 2^j\}) \in U^j$, we set

$$H(u^j) = 2^{j(s-1/p)} \sum_{k=1}^{2^j} u_k^j (X_{(k+1)/2^j} - X_{k/2^j}).$$

Let

$$Z_j = 2^{j(s-1/p)} \{V_j^p\}^{1/p} - \mathbb{E}[2^{j(s-1/p)} \{V_j^p\}^{1/p}].$$

Remarking that

$$\sup_{u_j \in U_j} H(u^j) = 2^{j(s-1/p)} \big(\sum_{k=1}^{2^j} |X_{(k+1)/2^j} - X_{k/2^j}|^p \big)^{1/p},$$

we can apply Borell inequality and we obtain that for $t > 0$

$$\mathbb{P}[|Z_j| > t] \leq 2e^{-t^2/(2\nu_j^2)},$$

where $\nu_j^2 = \sup_{u^j \in U^j} \mathbb{E}[H(u^j)^2]$. As $p \geq 2$, by Jensen inequality, we get

$$\nu_j^2 \leq 2^{2j(s-1/p)} \big(\mathbb{E}\big[\sum_{k=0}^{2^j-1} |X_{(k+1)/2^j} - X_{k/2^j}|^p \big] \big)^{2/p} \leq 2^{2js} (\mathbb{E}[|\eta|^p])^{2/p} \sigma^2(2^{-j}) \leq c,$$

with c a constant finite value. Hence for $t > 0$

$$\mathbb{P}[|Z_j| > t] \leq 2e^{-t^2/(2c)}.$$

Let $M_j = \mathbb{E}[2^{j(s-1/p)} \{V_j^p\}^{1/p}]$, we have

$$M_j \leq c 2^{j(s-1/p)} \mathbb{E}[\{V_j^p\}^{1/p}] \leq c 2^{js} (\mathbb{E}[|\eta|^p])^{1/p} \sigma(2^{-j}) \leq c(\mathbb{E}[|\eta|^p])^{1/p}.$$

Choose some convergent subsequence $\{M_{j_i}\}_i$ of $\{M_j\}_j$ with limit $\bar{M}$. Then, using Borel-Cantelli lemma, we get that a.s.

$$\lim_{i \to \infty} 2^{j_i(s-1/p)} \{V_{j_i}^p\}^{1/p} = \bar{M}.$$

Since $\mathbb{E}[2^{j(ps-1)} V_j^p] \leq c$, we can deduce that

$$\lim_{i \to \infty} \mathbb{E}[2^{j_i(ps-1)} V_{j_i}^p] = \bar{M}^p.$$

Now, it is clear that

$$\lim_{j \to \infty} \mathbb{E}[2^{j(ps-1)} V_j^p] = \alpha^p \mathbb{E}[|\eta|^p].$$

Hence, $\bar{M}^p = \alpha^p \mathbb{E}[|\eta|^p]$ and the bounded sequence (M_j) has a unique limit point $\bar{M}$.

Part IV

A new microstructure noise index

Abstract

We introduce a new microstructure noise index for financial data. This index, whose computation is based on the p-variations of the considered asset or rate at different time scales, can be interpreted thanks to Besov smoothness spaces. We study the behavior of our new index on empirical data. It gives rise to phenomena that a classical signature plot is unable to detect. In particular, on our data set, it enables us to separate the sampling frequencies in three zones: no microstructure noise in the low frequencies, an increasing microstructure noise frow the low to the high frequencies, and some kind of additional regularity in the finest scales. Then, we investigate the index from a theoretical point of view, under various contexts of microstructure noise, trying to reproduce the facts observed on the data. We show that this can be partially done using models involving additive correlated errors or rounding error. Nevertheless, an accurate reproduction seems to require either both kinds of error together or some unusual form of rounding error.

Keywords: Microstructure noise; High frequency data; Diffusion models; Rounding error; Besov spaces.

Note

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1 Introduction

1.1 Microstructure noise and signature plot

Microstructure noise is usually defined as what makes the observed price differ from the "theoretical price" at fine scales. This "theoretical price" is often modeled by a semi-martingale, as suggested by the classical financial theory. In the literature, the reasons why this noise occurs are in particular the bid-ask spread and the discreteness of the prices. Imagine we observe a financial asset Y_t on a time interval $[0, T]$. We want to know if there is any microstructure noise in our data. To answer this question, we define microstructure noise as a form of irregularity in the high frequency data that disappears in the lower frequencies. Consider our sample

$$\{Y_{k\Delta}, k = 0, \dots, T/\Delta\},$$

where Δ denotes the minimum time to wait to be able to record a new data (in our data, Δ will be equal to one second). For simplicity we take $\Delta = 2^{-N}$ with N a positive integer. A usual way to study the presence of some kind of irregularity in the high frequencies is to compute the signature plot, that is the function

$$q \to Z_q,$$

with $q \in [0, N]$ and[8]

$$Z_q = \sum_{k=0}^{T(2^{N-q}-1)} |Y_{(k+1)2^{q-N}} - Y_{k2^{q-N}}|^2 \quad \text{(absolute signature plot)}$$

or

$$Z_q = \sum_{k=0}^{T(2^{N-q}-1)} |\log(Y_{(k+1)2^{q-N}}) - \log(Y_{k2^{q-N}})|^2 \quad \text{(relative signature plot).}$$

The signature plot has been made popular by Andersen *et al.* [9]. As soon as Y (or $\log Y$) is a continuous Ito semi-martingale, for fixed T and given q, this quantity converges in probability as N tends to infinity to the quadratic variation of Y (or $\log Y$) on $[0, T]$, see Jacod [73] for a general study of this type of quantities. Hence, in the low frequencies, where there is no microstructure noise, the signature plot of an asset is quite flat as soon as the number of data is sufficient. On the contrary, it is not flat in the presence of noise. A drawback of this measure is that it is not absolute, one just observes where the signature plot is flat and where it is not.

[8] The question of the "best" of the two measure is probably not so relevant for high frequency data on a short period. Indeed, $\log(Y_{(i+1)/n}) - \log(Y_{i/n}) \sim \frac{Y_{(i+1)/n} - Y_{i/n}}{Y_{i/n}}$ and so, it is almost proportional to $Y_{(i+1)/n} - Y_{i/n}$ as soon as the variation of the price on the period is not too big.

1.2 A new microstructure noise index

In this part, we propose an alternative to the signature plot. For a given real $p > 0$, we now consider the functions

$$q \rightarrow S_q^p \text{ and } q \rightarrow S_q^{'p}$$

with

$$S_q^p = \frac{1}{p}\{1 + \log_2\big(\frac{V_{q+1}^p}{V_q^p}\big)\} \quad \text{(absolute microstructure index)},$$

and

$$S_q^{'p} = \frac{1}{p}\{1 + \log_2\big(\frac{V_{q+1}^{'p}}{V_q^{'p}}\big)\} \quad \text{(relative microstructure index)},$$

where

$$V_q^p = \sum_{k=0}^{T(2^{N-q}-1)} |Y_{(k+1)2^{q-N}} - Y_{k2^{q-N}}|^p$$

and

$$V_q^{'p} = \sum_{k=0}^{T(2^{N-q}-1)} |\log(Y_{(k+1)2^{q-N}}) - \log(Y_{k2^{q-N}})|^p.$$

Using Besov smoothness spaces, we will see in section 2.1 that the microstructure indexes S_q^p and $S_q^{'p}$ may be considered as regularity measures associated to the subsampling frequency 2^{N-q}. Hence, if the function $q \rightarrow S_q^p$ (or $q \rightarrow S_q^{'p}$) is constant, we say that, in our sense, there is no microstructure noise in the data. Practically speaking, we may hope to observe a constant regularity for sufficiently low frequencies and a more specific behavior for high frequencies, reflecting the presence of microstructure noise. Note that close quantities are used in Aït-Sahalia and Jacod [1] in the context of jumps.

We give in section 2 an interpretation of our new indexes in term of Besov spaces and explain their virtues. In section 3, we study them on financial data: the Bund future contract from the Eurex market and the Euro/US Dollar exchange rate, from the Reuters database. What we learn from the data is that the regularity remains constant for large sampling scales (bigger than 15-20 minutes), is decreasing when going to the finer scales (from 15 minutes to 10 seconds) and is increasing when going to the finest scales (from 10 seconds to 1 second). Then, we show in section 4 and in section 5 that models involving additive correlated noise or rounding error enable to reproduce most of the facts observed on the data. Nevertheless, we will see in section 6 that the use of complementary microstructure functions seems to indicate that an accurate reproduction requires either both kinds of error together or some unusual form of rounding error. Supplementary definitions and graphs can be found in appendix D.

2 Properties of the index

2.1 Interpretation of the index

In this section, we aim at showing why the quantity S_q^p can be viewed as a regularity measure. For that purpose, we consider here that T is equal to one and we use Besov smoothness spaces $\mathcal{B}^s_{p,\infty}([0,1])$ as defined in part III. Following theorem III.1, under (strong) assumptions, if $t \to Y_t$ belongs to $\mathcal{B}^s_{p,\infty}([0,1])$ and does not belong to $\mathcal{B}^{s+\varepsilon}_{p,\infty}([0,1])$ for any positive ε, we may hope that asymptotically

$$V_q^p \simeq c2^{(q-N)(ps-1)},$$

with c a positive constant value. Consequently, we easily see that in this case,

$$S_q^p \simeq s.$$

Thus, we can interpret the preceding index S_q^p the following way: based on the subsampled data at period $m = 2^q$, in term of Besov spaces $\mathcal{B}^s_{p,\infty}([0,1])$, the regularity s of the underlying continuous time process suggested by the data is S_q^p.

2.2 Virtues of the index

• The index is model free. We do not have to suppose that the underlying process follows a diffusion process or a "noisy" diffusion process. On the contrary, the signature plot really makes sense only in the context of diffusion type processes. For example, for fixed T, as N goes to infinity, the quadratic variation of a fractional Brownian motion with Hurst parameter $H > 1/2$ converges almost surely to zero, whereas, for any $p > 0$, the microstructure index converges almost surely to H.

• The value of the index depends on the subsampling period and consequently we can hope to discriminate some sampling periods thanks to the index. For example, if we fix a reference sampling period, e.g. 1 hour, and we compute the associated index S_{ref}, we could say that the microstructure noise occurs at the sampling period $2^q < 1$ hour if S_q^p is significantly different from S_{ref}.

• In the case of a continuous semi-martingale, for any value of p, the value of the index S_q^p is asymptotically equal to 1/2 (convergence in probability). Hence, if one consider now that the continuous semi-martingale is the reference case, the index is in some sense absolute: the farther from 1/2 the index is, the more the noise is important.

• The calculation of the index is based on well known financial quantities. Indeed, it can be seen for $p = 2$ as a function of two realized volatilities considered at different time scales.

From now, although we mainly find on our data regularities smaller than 1/2, for simplicity, except in section 6, we will always take $p = 2$.

3 Empirical study

We compute the index on a financial asset and an exchange rate.

• The Bund future contract from the Eurex market, every second, for October 2005, November 2005, February 2006 and October 2006, November 2006, February 2007 (we skip the months of December and January to avoid some particular effects due to the roll of the contract and to the beginning of the year). We fix the start of a day at 9 am CET and the end of a day at 7 pm CET[9] (once again, we want to avoid particular effects due to the beginning or the end of the day). We consider in this section that the value at time t is the last traded value. Results for the bid and midquote prices are given in appendix D. Each day, we compute

$$\tilde{V}_q^2 = \sum_{k=0}^{T(2^{N-q}-1)} |Y_{(k+1)2^{q-N}} - Y_{k2^{q-N}}|^2$$

and

$$\tilde{V}_q'^2 = \sum_{k=0}^{T(2^{N-q}-1)} |\log(Y_{(k+1)2^{q-N}}) - \log(Y_{k2^{q-N}})|^2.$$

Then V_q^2 (resp. $V_q'^2$) is the sum over each day of the $\tilde{V}_q^2$ (resp. $\tilde{V}_q'^2$). Note that the bund is an asset which is mostly traded an a penny. Consequently, the impact of the bid-ask spread on the microstructure noise may not be very important. Graphs for the Bund contract are given in figures I.6, II.1 and II.2.

• The Euro/US Dollar exchange rate, every second, bid prices from the Reuters database, from 15/08/2005 to 01/10/2005.

The results are the following for dyadic subsamplings $m = 2^k$ seconds, $k = 0, \ldots 12$. (The value 1 on the x axis corresponds to $k = 0$).

[9]Precisely, starting from 9 am we take $2^{15} + 1$ data per day.

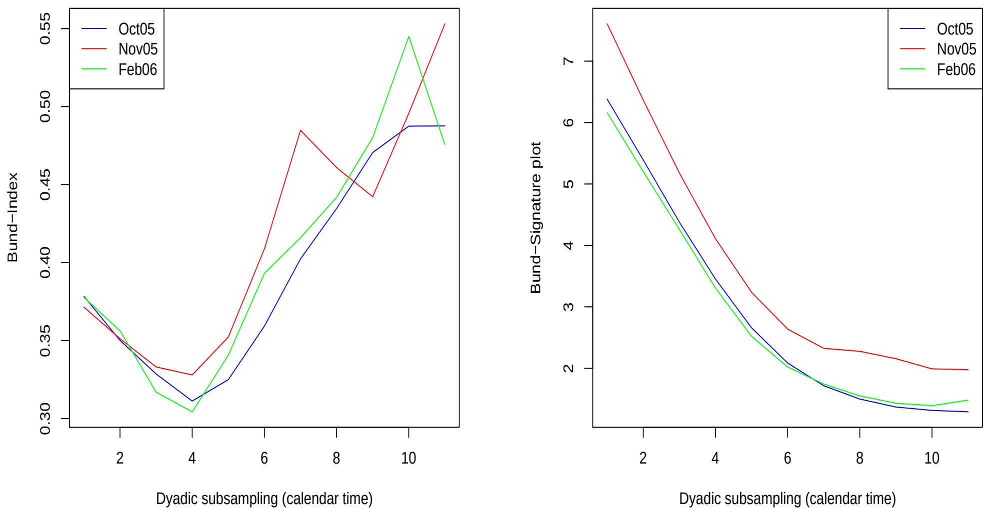

Figure IV.1: *Microstructure noise index (left) and signature plot (right) for the Bund, Oct 05, Nov 05, Feb 06.*

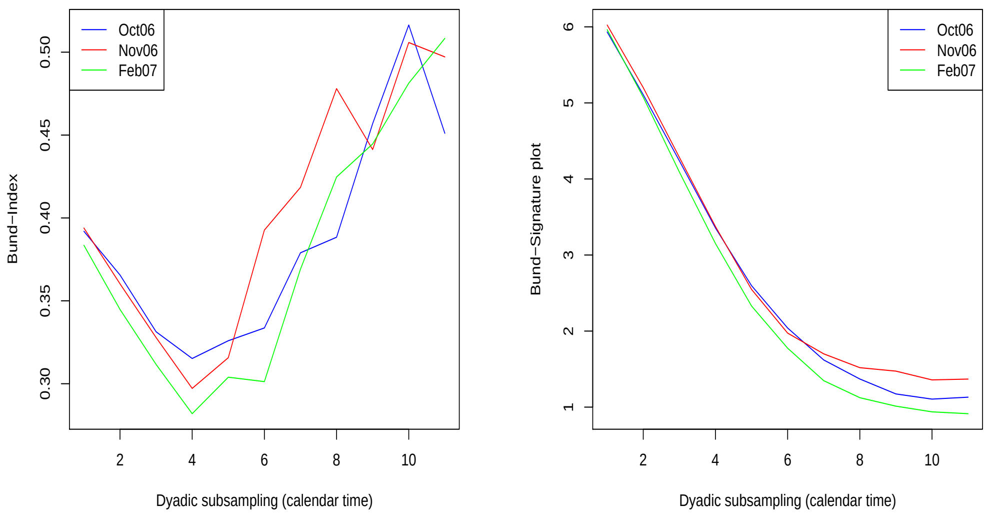

Figure IV.2: *Microstructure noise index (left) and signature plot (right) for the Bund, Oct 06, Nov 06, Feb 07.*

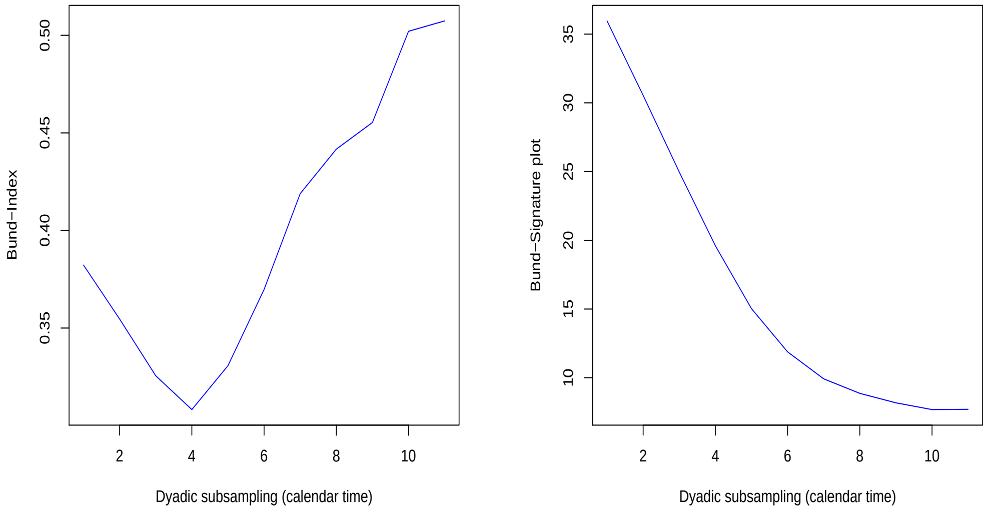

Figure IV.3: *Microstructure noise index (left) and signature plot (right) for the Bund, aggregated data: Oct 05, Nov 05, Feb 06, Oct 06, Nov 06, Feb 07.*

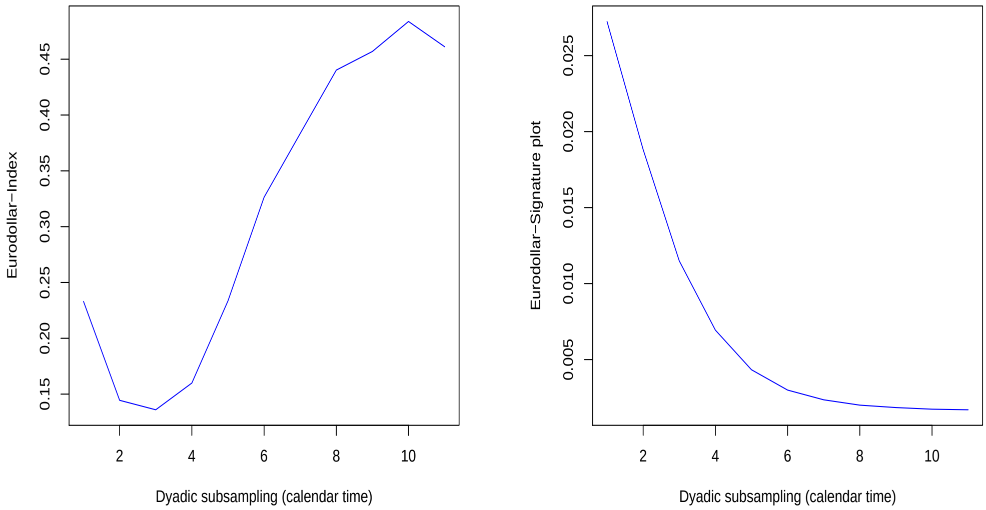

Figure IV.4: *Microstructure noise index (left) and signature plot (right) for the Bid Eurodollar 15/08/2005-01/10/2005.*

The results for the relative microstructure index are very close and can be found in appendix D.

Thus, we observe an under diffusive behavior ($S_q^2 < 1/2$) in the very high frequencies and a regularity which is close to 1/2 in the lower frequencies. It is also surprising that the graphs are decreasing for q between 0 and 3. This particular behavior does not appear at all in the signature plots and can be linked to some ultra high frequency persistence in the prices.

The curves show that the regularity is lower in the high frequencies what can be considered as microstructure noise effects. Thus, in our approach, we can conclude from these graphs that there is a mainly decreasing microstructure noise in the Bund and the Euro/US Dollar exchange rate if the subsampling period is lower than 2^{10} seconds.

Now the question is to explain this behavior of the graphs. The idea is to consider models which behave as the classical financial models (e.g. diffusions) at the low frequencies but differently in the high frequencies. We study in next section our index in the classical framework of additive microstructure noise. Then we study models with rounding error. Note that our goal in this part is not to study complex continuous time dynamic for the theoretical price. We aim at focusing on the impact of the noise on the observed price.

4 Models with additive noise

4.1 Introduction

Models with additive microstructure noise have been widely studied, see in particular Zhang [113], Zhang, Mykland, and Aït-Sahalia [114], Hansen and Lunde [58], Bandi and Russell [12], Aït-Sahalia, Mykland and Zhang [4], Andersen, Bollerslev and Meddahi [10], Gloter and Jacod [56]. We observe here

$$Y_{k\Delta} = c\exp(X_{k\Delta} + \varepsilon_k^\Delta),\ k = 0, \ldots, T/\Delta,$$

where $X_{k\Delta}$ is the "theoretical log-price", c is an initial condition (we take $c = 115$ to be coherent with the Bund data) and ε_k^Δ is an additive centered noise with variance V^2, independent of X. For simplicity, we suppose

$$X_{k\Delta} = \sigma W_{k\Delta},$$

where W is a Brownian motion and σ is equal to 10^{-2} (this value is chosen in order to be coherent with the Bund data). A case of stochastic volatility model on X is treated in appendix D. We work under the following specifications for the noise.

$(M1)$ The ε_k^Δ are iid Gaussian variables.

$(M2)$ The ε_k^Δ follow an $AR(1)$ model $\varepsilon_k^\Delta = r\varepsilon_{k-1}^\Delta + \xi_k^\Delta$, where the ξ_k^Δ are iid centered Gaussian variables.

$(M3)$ The ε_k^Δ follow a fractional Gaussian noise[10] with Hurst parameter $H = 0.1$.

$(M4)$ The ε_k^Δ follow a fractional Gaussian noise with Hurst parameter $H = 0.9$.

Model $(M2)$ and model $(M4)$ introduce positive correlations between the noise terms. It is a possible way to add some "ultra high frequency regularity" and so to try to reproduce the fact that the microstructure index is decreasing for q between 0 and 3.

4.2 Graphical overview

To get a first idea of the behavior of the absolute microstructure index and of the absolute signature plot in such models, we give some graphs for the four preceding models. For each model, we compute 50 simulations with $T = 1$ and $\Delta = 2^{-19}$ and give the average absolute microstructure index and signature plot for V equal to

$$0, 4.10^{-5}, 8.10^{-5}, 12.10^{-5}, 16.10^{-5}, 2.10^{-4}.$$

Moreover, we take $r = 0.8$ for model $(M2)$.

[10] Recall that a fractional Gaussian noise $(\varepsilon_k^\Delta,\ k \geq 0)$ is defined by $\varepsilon_k^\Delta = W_{k+1}^H - W_k^H$, where $(W_t^H,\ t \geq 0)$ is a fractional Brownian motion with Hurst parameter H.

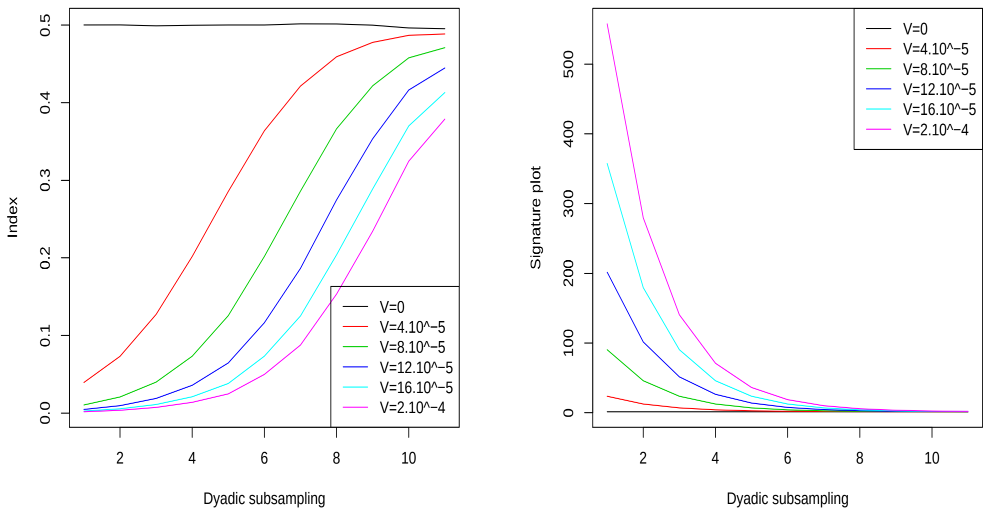

Figure IV.5: *Model M1, microstructure noise index (left) and signature plot (right).*

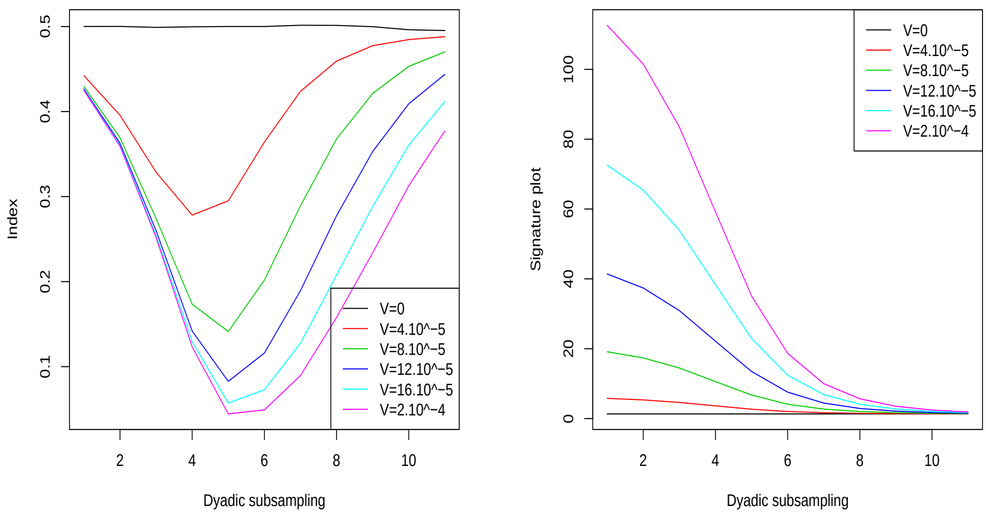

Figure IV.6: *Model M2, microstructure noise index (left) and signature plot (right).*

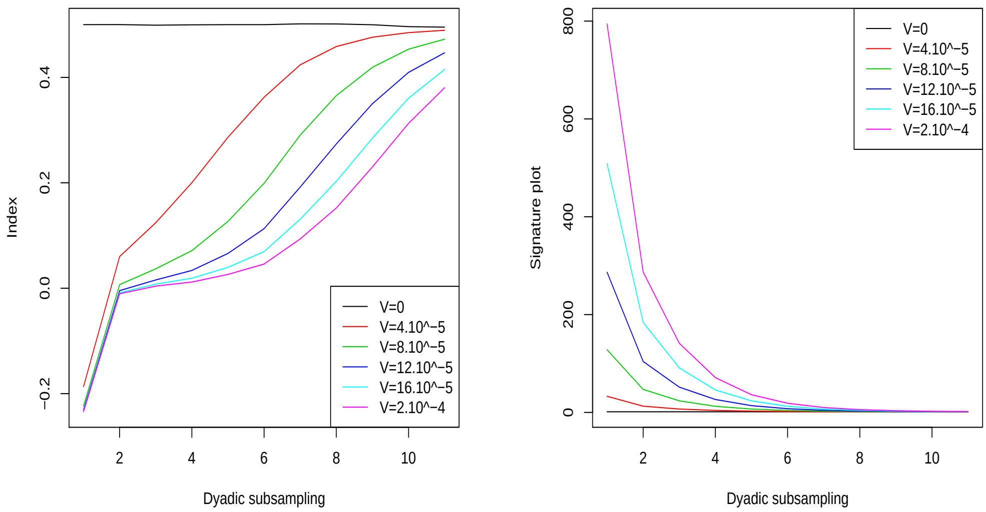

Figure IV.7: *Model M3, microstructure noise index (left) and signature plot (right).*

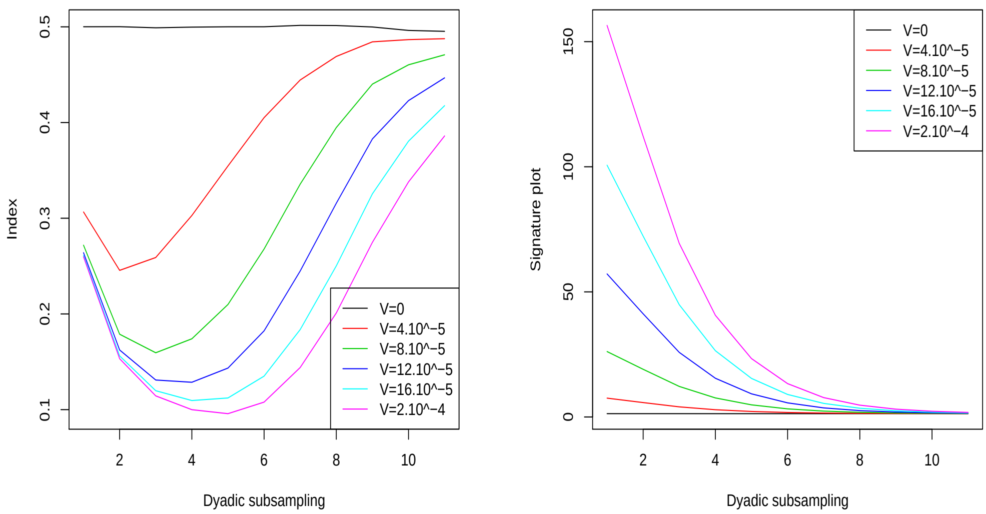

Figure IV.8: *Model M4, microstructure noise index (left) and signature plot (right).*

4.3 Dependent noise: theoretical considerations

In this section, we investigate model $(M2)$ on a theoretical point of view. For simplicity, we work here with the log prices. So, we consider the model where the observed log prices $Y_{k\Delta}$ are given by

$$Y_{k\Delta} = X_{k\Delta} + \varepsilon_k^{\Delta}, k = 0, \ldots, T/\Delta,$$

where $X_t = \sigma W_t$ and the ε_k^{Δ} are independent of X and are the Gaussian stationary solution of an $AR(1)$ model

$$\varepsilon_k^{\Delta} = r\varepsilon_{k-1}^{\Delta} + \gamma\zeta_k^{\Delta},\ k \geq 1$$

with $0 \leq r < 1$. The ζ_k^{Δ} are iid centered Gaussian variable with variance 1, such that ζ_k^{Δ} is independent of the past of the ε_j^{Δ}, $j < k$ and ε_0^{Δ} is appropriately chosen. This kind of framework has been investigated by Aït-Sahalia, Mykland and Zhang [5]. They build in [5] estimators of the integrated volatility for such models, using modification of the two scales of Zhang, Mykland and Aït-Sahalia [114] and of the multiscales estimator of Zhang [113]. Using computations on the realized volatility, our objective is to give an approximate equation of the microstructure noise index in this context.

4.3.1 Asymptotic for T and Δ

We want to study the signature plot and the microstructure index in the low and the high frequencies. For the low frequencies, we need to have enough time between two data so that the microstructure noise disappears. Hence, it is natural to suppose for the low frequencies $\Delta = \Delta_n$ tends to infinity and so $T = T_n$ tends to infinity. On the contrary, we need close data in the high frequencies. Hence, in that case, we will suppose $\Delta = \Delta_n$ tends to zero.

4.3.2 First asymptotic for the noise: constant variance of the noise

Our noise sequence is defined at the finest scale Δ. We first study asymptotics where the variance of the noise does not depend on Δ.

Realized volatility Let m be a positive integer and

$$RV(T, \Delta, m) = \sum_{i=0}^{T(m\Delta)^{-1}} (Y_{im\Delta} - Y_{(i-1)m\Delta})^2.$$

We also define some positive sequences T_n and Δ_n. We have the following results.

Theorem IV.1 *In model $(M2)$, there exist N_1 and N_2 such that $\mathbb{E}[N_1^2 + N_2^2] < +\infty$ and*

$$RV(T,\Delta,m) = \sigma^2 \sum_{i=1}^{T(m\Delta)^{-1}} (W_{im\Delta} - W_{(i-1)m\Delta})^2 + 2\gamma^2 \frac{1-r^m}{1-r^2} T(m\Delta)^{-1} + T^{1/2}(m\Delta)^{-1/2} N_1 + N_2 T^{1/2}.$$

If T is fixed, as Δ_n tends to zero,

$$RV(T,\Delta_n,m) = 2\gamma^2 \frac{1-r^m}{1-r^2} T(m\Delta_n)^{-1} + \mathcal{O}_p(m\Delta_n)^{-1/2}.$$

If T_n, Δ_n, $T_n\Delta_n^{-1}$ tend to infinity and $T_n^{1/2}\Delta_n^{-3/2}$ tends to $\beta \geq 0$,

$$T_n^{1/2}(2\sigma^2 m\Delta_n)^{-1/2}(T_n^{-1} RV(T_n,\Delta_n,m) - \sigma^2) \to_{\mathcal{L}} \sqrt{2}(\sigma^2 m)^{-1/2}\gamma^2\beta \frac{1-r^m}{1-r^2} + \mathcal{N}(0,1).$$

If T_n, Δ_n, $T_n\Delta_n^{-1}$, $T_n^{1/2}\Delta_n^{-3/2}$ tend to infinity,

$$T_n^{-1} RV(T_n,\Delta_n,m) = \sigma^2 + 2\gamma^2 \frac{1-r^m}{1-r^2}(m\Delta_n)^{-1} + \mathcal{O}_p\big((m\Delta_n/T_n)^{1/2}\big).$$

Proof. Abusing notation slightly, we write ε_k for ε_k^Δ.

$$\begin{aligned} Y_{(i+1)m\Delta} - Y_{im\Delta} &= X_{(i+1)m\Delta} - X_{im\Delta} + \varepsilon_{(i+1)m} - \varepsilon_{im} \\ &= X_{(i+1)m\Delta} - X_{im\Delta} + (r^m - 1)\varepsilon_{im} + \gamma \sum_{j=0}^{m-1} r^j \zeta_{m(i+1)-j} \\ &= X_{(i+1)m\Delta} - X_{im\Delta} + (r^m - 1)\varepsilon_{im} + \gamma \sqrt{\frac{1-r^{2m}}{1-r^2}}\, v_{i,m} \end{aligned}$$

where the $v_{i,m}, i \geq 0$ are iid centered Gaussian variable with variance 1, such that $v_{i,m}$ is independent of the past of the ε_j^n, $j \leq im$. Let

$$\mu_{i,m} = (\mathbb{E}[\varepsilon_i^2])^{-1/2} \varepsilon_{im}.$$

Since $E[\mu_{i,m}\mu_{j,m}] = r^{m|i-j|}$, by Mehler's formula, we have

$$\mathbb{E}[(\mu_{i,m}^2 - 1)(\mu_{j,m}^2 - 1)] = 2r^{2m|i-j|}.$$

Hence,

$$\mathbb{E}\big[\big(\sum_{i=1}^{T(m\Delta)^{-1}} \mu_{i-1,m}^2 - 1\big)^2\big] \leq cT(m\Delta)^{-1}.$$

Consequently,

$$\sum_{i=1}^{T(m\Delta)^{-1}} \varepsilon_{(i-1)m}^2 = \mathbb{E}[\varepsilon_i^2]T(m\Delta)^{-1} + T^{1/2}(m\Delta)^{-1/2}R_1,$$

with $E[R_1^2] \leq c$. We also easily get that

$$\sum_{i=1}^{T(m\Delta)^{-1}} (X_{im\Delta} - X_{(i-1)m\Delta})\varepsilon_{(i-1)m} = cR_2T^{1/2},$$

with $E[R_2^2] \leq c$ and

$$\sum_{i=1}^{T(m\Delta)^{-1}} (X_{im\Delta} - X_{(i-1)m\Delta})v_{i,m} = cR_3T^{1/2},$$

with $E[R_3^2] \leq c$. Using conditional expectations, we also obtain

$$\sum_{i=1}^{T(m\Delta)^{-1}} \varepsilon_{(i-1)m}v_{i-1,m} = T^{1/2}(m\Delta)^{-1/2}R_4.$$

with $E[R_4^2] \leq c$. Finally, using that $\mathbb{E}[\varepsilon_{im}^2] = \gamma^2(1-r^2)^{-1}$, we obtain

$$\begin{aligned} RV(T,\Delta,m) = \sigma^2 & \sum_{i=1}^{T(m\Delta)^{-1}} (W_{im\Delta} - W_{(i-1)m\Delta})^2 \\ & + 2\gamma^2\frac{1-r^m}{1-r^2}T(m\Delta)^{-1} + T^{1/2}(m\Delta)^{-1/2}R_5 + R_6T^{1/2}, \end{aligned}$$

with $E[R_5^2 + R_6^2] \leq c$. The result follows in the case where T is fixed and Δ tends to zero. For the second kind of asymptotics, as Δ/T tends to zero, Lindeberg condition holds in order to prove that

$$2^{-1/2}T_n^{-1/2}(m\Delta_n)^{1/2} \sum_{i=1}^{T_n(m\Delta_n)^{-1}} \big\{(m\Delta_n)^{-1}(W_{im\Delta_n} - W_{(i-1)m\Delta_n})^2 - 1\big\} \to_{\mathcal{L}} \mathcal{N}(0,1).$$

□

Microstructure noise index For the microstructure noise index, we have the following corollary of the preceding theorem.

Corollary IV.1 *In model $(M2)$, if T is fixed, as Δ_n tends to zero,*

$$S_q'^2 = \frac{1}{2}\log_2(1 + r^{2^q}) + \mathcal{O}_p\big((2^q\Delta_n)^{1/2}\big).$$

If T_n, Δ_n, $T_n\Delta_n^{-1}$ tend to infinity and $T_n^{1/2}\Delta_n^{-3/2}$ tend to $\beta \geq 0$,

$$S_q'^2 = \frac{1}{2} + \mathcal{O}_p\big((2^q\Delta_n/T_n)^{1/2}\big).$$

If T_n, Δ_n, $T_n\Delta_n^{-1}$, $T_n^{1/2}\Delta_n^{-3/2}$ tend to infinity,

$$S_q'^2 = \frac{1}{2} - \frac{\gamma^2(1-r^{2^q})^2\Delta_n^{-1}}{2^{q+1}(1-r^2)} + \mathcal{O}_p\big((2^q\Delta_n/T_n)^{1/2} + (2^q\Delta_n)^{-2}\big).$$

Remark that in the case where T is fixed and Δ_n tends to zero, we in fact have

$$2S_q'^2 = 1 \underbrace{-1}_{\text{due to the noise}} + \underbrace{\log_2(1+r^{2^q})}_{\text{due to the correlation}} + \mathcal{O}_p\big((2^q\Delta_n)^{1/2}\big).$$

Hence we reproduce the decreasing behavior at the beginning of the graphs.

Parameters estimation Under the preceding specification, we have the following corollary.

Corollary IV.2 *Let*

$$\widehat{r} = 2^{2S_0'^2} - 1$$

and

$$\widehat{\gamma^2} = T^{-1}\Delta V_1'^2.$$

In model $(M2)$, if T is fixed, as Δ_n tends to zero, the sequences $\Delta_n^{-1/2}(\widehat{r} - r)$ and $\Delta_n^{-1/2}(\widehat{\gamma^2} - \gamma^2)$ are tight.

4.3.3 Second asymptotic for the noise: variance of the noise depending on the finest scale

It can also be natural to suppose that the ε_i^Δ are of order $\Delta^{1/2}$. Indeed, our noise process is built at the finest period and under this specification, $\varepsilon_{(i+1)}^\Delta - \varepsilon_i^\Delta$ is of order $\Delta^{1/2}$, which is the order of magnitude of the increment of the Brownian motion on $[i\Delta, (i+1)\Delta]$. This asymptotic is in particular justified by assets for which the signature plot does not explode but is quite flat around $q = 0$, see Hansen and Lunde [58] for such examples.

Realized volatility Similar computations as in the preceding theorem lead to the following result.

Theorem IV.2 *In model (M2), if* $\gamma^2 = \gamma^2(\Delta) = \eta^2\Delta$*, there exist* N_1 *and* N_2 *such that* $\mathbb{E}[N_1^2 + N_2^2] < +\infty$ *and*

$$RV(T,\Delta,m) = \sigma^2 \sum_{i=1}^{T(m\Delta)^{-1}} (W_{im\Delta} - W_{(i-1)m\Delta})^2 + 2\eta^2 \frac{1-r^m}{1-r^2} Tm^{-1} + m^{-1/2}(\Delta T)^{1/2} N_1 + (\Delta T)^{1/2} N_2.$$

Hence, under this specification for the noise, the signature plot does not explode around $q = 0$. An important remark is that in this framework, if T is fixed, as Δ_n tends to zero, from V'_0, V'_1 and V'_2, one can estimate σ^2, r and η^2 with accuracy $\Delta_n^{1/2}$.

Microstructure noise index For the microstructure noise index, we have the following corollary of the preceding theorem.

Corollary IV.3 *In model (M2), if* $\gamma^2 = \gamma^2(\Delta_n) = \eta^2\Delta_n$ *and if* T *is fixed, as* Δ_n *tends to zero, we have*

$$S_q'^2 = \tilde{S}_q'^2 + \mathcal{O}_p\big(2^{-q/2}(\Delta_n T)^{1/2}\big)$$

with

$$\tilde{S}_q'^2 = \frac{1}{2} + \frac{1}{2}\log_2\Big\{\frac{2^q + (1-r^{2^{q+1}})\eta^2(\sigma^2)^{-1}(1-r^2)^{-1}}{2^q + 2(1-r^{2^q})\eta^2(\sigma^2)^{-1}(1-r^2)^{-1}}\Big\}.$$

Hence, the approximate noise index $\tilde{S}_q'^2$ only depends on q, r and the noise signal ratio $\eta^2(\sigma^2)^{-1}(1-r^2)^{-1}$. Moreover, $\tilde{S}_q'^2 \leq 1/2$. Note that if $\eta^2 = 0$, $\tilde{S}_q'^2 = 1/2$ and if $q = 0$ and $r = 0$,

$$\tilde{S}_0'^2 = \frac{1}{2} + \frac{1}{2}\log_2\Big\{\frac{1+\eta^2(\sigma^2)^{-1}}{1+2\eta^2(\sigma^2)^{-1}}\Big\}.$$

We also have the following relation

$$\frac{\eta^2(\sigma^2)^{-1}}{1-r^2} = \frac{1-2^{2\tilde{S}_0'^2-1}}{(2^{2\tilde{S}_0'^2}-1-r)(1-r)}. \tag{IV.1}$$

To make sense, if $\tilde{S}_0'^2 < 1/2$, the preceding equation implies

$$r \leq 2^{2\tilde{S}_0'^2} - 1.$$

We present now the graphs of the function $\tilde{S}_q'^2$ for $\tilde{S}_0'^2 = 0.37$, $r = (0, 0.1, 0.3, 0.4, 0.5, 0.6)$ and $\eta^2(\sigma^2)^{-1}(1-r^2)$ associated by (IV.1).

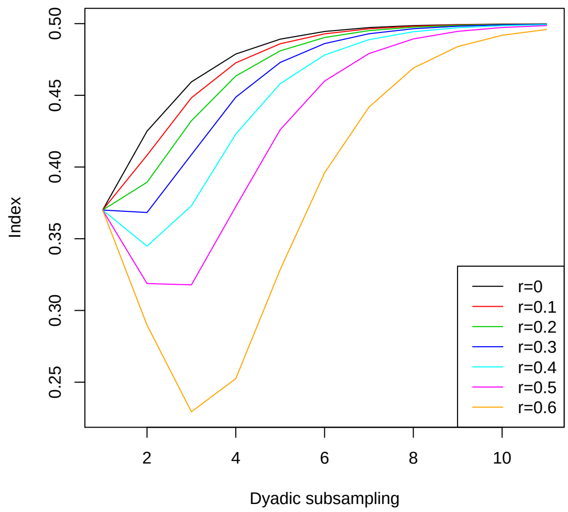

Figure IV.9: *Approximate noise index* $\tilde{S}'^2_q$.

Approximation of the Microstructure noise index In this section, we just aim at giving approximations of the noise index from a practical point of view in the case $\gamma^2 = \gamma^2(\Delta) = \eta^2\Delta$. For small q, the signature plot shows that

$$\sigma^2 \ll 2\eta^2 \frac{1 - r^{2^q}}{1 - r^2} 2^{-q}$$

and for big q,

$$\sigma^2 \gg 2\eta^2 \frac{1 - r^{2^q}}{1 - r^2} 2^{-q}.$$

We have the following result.

Corollary IV.4 *If* $\sigma^2 \to 0$ *while the other parameters are fixed,*

$$\tilde{S}'^2_q = \frac{2^{q-1}\sigma^2}{\eta^2} \left(\frac{1 - r^2}{1 + r^{2^q}}\right) + \frac{1}{2}\log_2(1 + r^{2^q}) + o(\sigma^2).$$

If $q \to +\infty$ *while the other parameters are fixed,*

$$\tilde{S}'^2_q = \frac{1}{2} - \frac{\eta^2}{(1 - r^2)2^{q+1}\sigma^2}\{1 - r^{2^q}(2 - r^{2^q})\} + o(2^{-q}).$$

Note that we have in the first case

$$\tilde{S}'^2_q = \frac{1}{2} \underbrace{-\frac{1}{2} + \frac{2^{q-1}\sigma^2}{\eta^2}}_{\text{due to the noise}} \underbrace{+\frac{1}{2}\log_2(1 + r^{2^q}) - \frac{2^{q-1}\sigma^2}{\eta^2}\left(\frac{r^{2^q} + r^2}{1 + r^{2^q}}\right)}_{\text{due to the correlation}} + o(\sigma^2),$$

and in the second case

$$\tilde{S}_q'^2 = \frac{1}{2} \underbrace{- \frac{\eta^2}{2^{q+1}\sigma^2}}_{\text{due to the noise}} + \underbrace{\frac{\eta^2}{(1-r^2)2^{q+1}\sigma^2}\{r^{2^q}(2-r^{2^q})-r^2\}}_{\text{due to the correlation}} + o(2^{-q}).$$

For illustration, we give the graphs of the two preceding approximations of $\tilde{S}_q'^2$ for $\frac{\sigma^2}{\eta^2}(1-r^2) = 0.05$ and $r = (0, 0.2, 0.4, 0.6, 0.8)$.

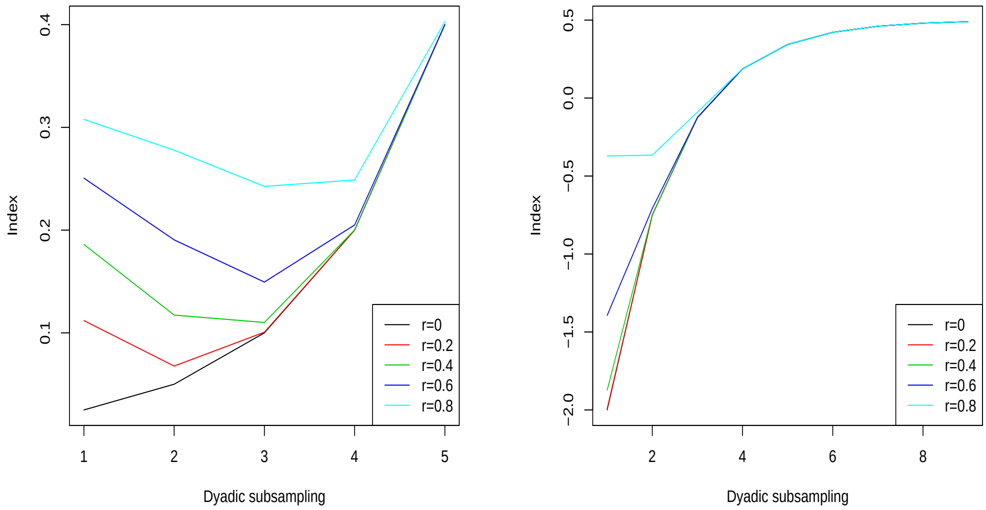

Figure IV.10: *High frequency approximation (left) and low frequency approximation (right).*

5 Models with rounding error

5.1 Introduction

Models with rounding error have been in particular studied by Delattre and Jacod [36], Delattre [35], Li and Mykland [85], [86], Large [83] and in part III. A model with rounding error is a way to get discrete prices and a diffuse behavior at large sampling scale. Moreover, it is striking to see how rounded diffusions visually look like tick by tick financial data, see figure II.1 and II.2. We define the following notation: $x^{(0)} = x$ and for $\beta \neq 0$, $x^{(\beta)} = \beta \lfloor x/\beta \rfloor$. We now observe

$$Y_{k\Delta} = (c \exp\{X_{k\Delta} + V\varepsilon_k^{\Delta}\})^{(\alpha)},\ i = 0, \ldots, T/\Delta,$$

where $X_{k\Delta}$ is the "theoretical log price", $V\varepsilon_k^\Delta$ is an additive centered noise with variance V^2, independent of X, and α is the rounding level, equal to 10^{-2} for the Bund. As previously, we take $c = 115$. We suppose

$$X_k^\Delta = \sigma W_k^\Delta,$$

where W is a Brownian motion and σ is equal to 10^{-2}. A case of stochastic volatility model on X is treated in appendix D. We work under the following specifications for the noise.

$(M'1)$ The ε_k^Δ are iid Gaussian variables.

$(M'2)$ The ε_k^Δ follow an $AR(1)$ model ε_k^Δ=r$\varepsilon_{k-1}^\Delta + \xi_k^\Delta$, where the ξ_k^Δ are iid centered Gaussian variable.

$(M'3)$ The ε_k^Δ follow a fractional Gaussian noise with Hurst parameter $H = 0.1$.

$(M'4)$ The ε_k^Δ follow a fractional Gaussian noise with Hurst parameter $H = 0.9$.

$(M'5)$ The ε_k^Δ are iid with law $\sqrt{12}\mathcal{U}[0,1]$, where $\mathcal{U}[0,1]$ denotes the uniform law on $[0,1]$.

5.2 Graphical overview

We give in this section the graphs corresponding to those in section 4.2.

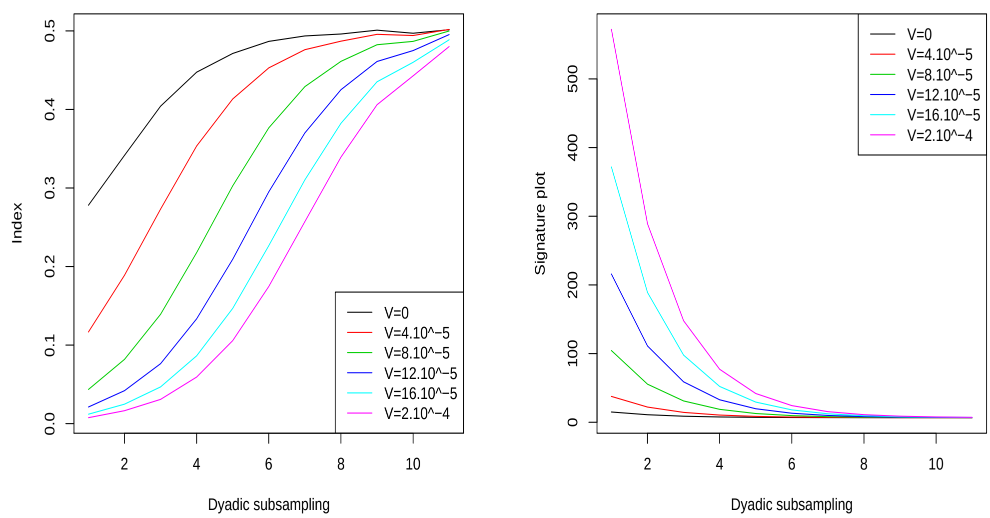

Figure IV.11: *Model M'1, Microstructure noise index (left) and signature plot (right).*

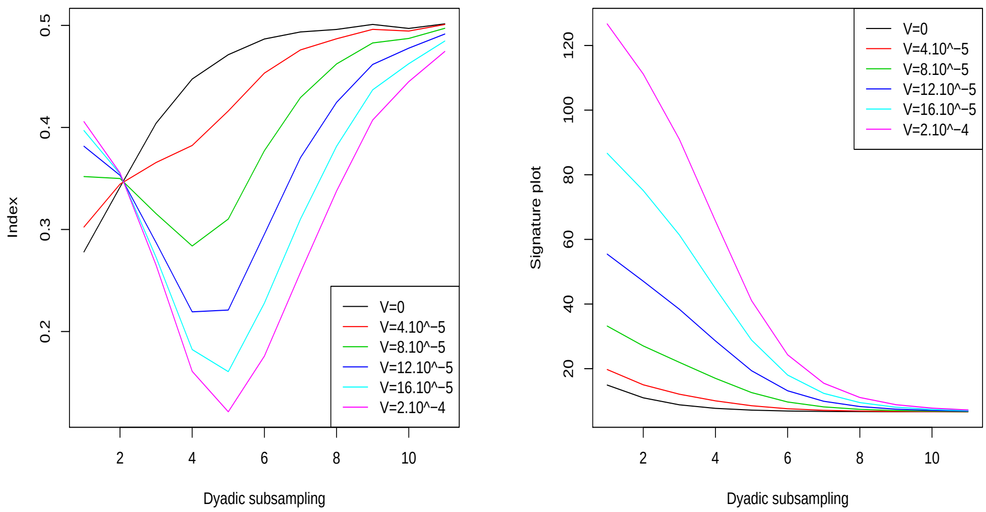

Figure IV.12: *Model M'2, Microstructure noise index (left) and signature plot (right).*

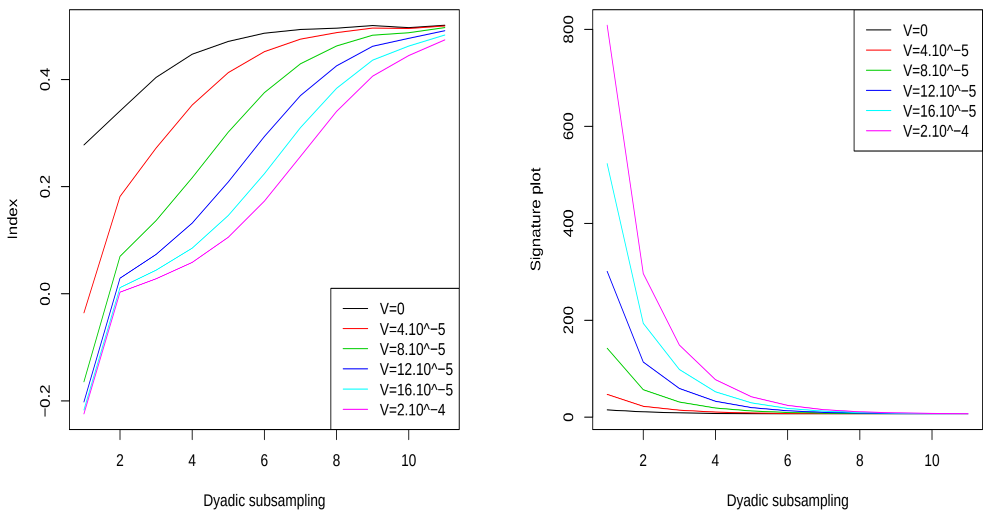

Figure IV.13: *Model M'3, Microstructure noise index (left) and signature plot (right).*

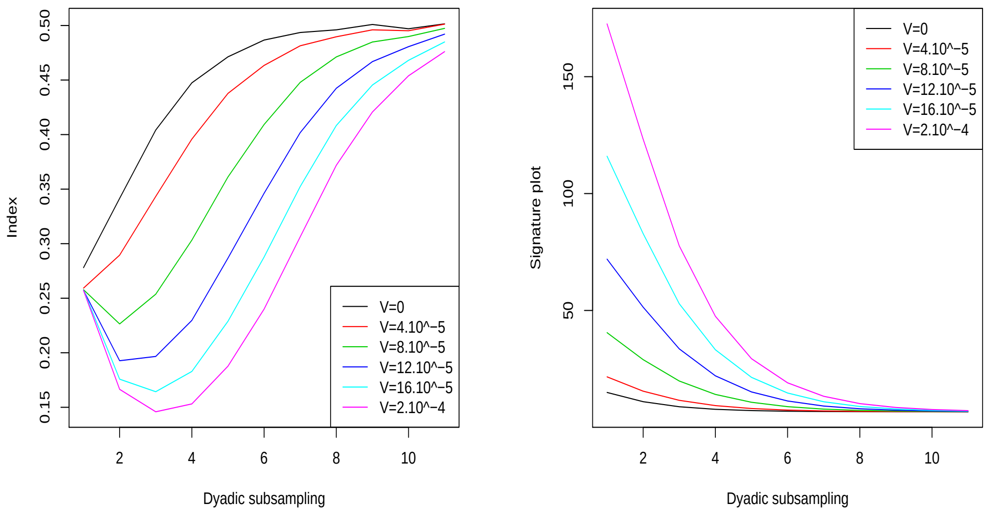

Figure IV.14: *Model M'4, Microstructure noise index (left) and signature plot (right).*

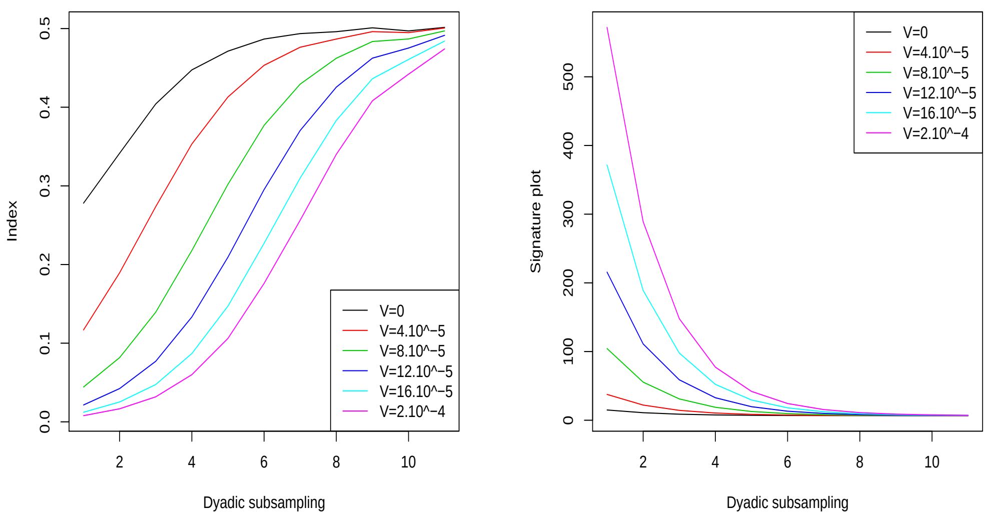

Figure IV.15: *Model M'5, Microstructure noise index (left) and signature plot (right).*

5.3 Pure rounding: theoretical considerations

Consider the model where the observed (absolute) price $Y_{k\Delta}$ is equal to $X_{k\Delta}^{(\alpha)}$, where X is a regular enough underlying diffusion process. Let m be a subsampling period. In the high frequencies (m small), the rounded theoretical price increment

$$X^{(\alpha)}_{m(k+1)\Delta} - X^{(\alpha)}_{mk\Delta}$$

largely differs from the quantity $X_{m(k+1)\Delta} - X_{m\Delta}$ whereas in the low frequencies (m big) the two quantities are very close. This difference appears in our index. We now describe theoretically the behavior of the absolute signature plot and of the absolute microstructure index in the model of a diffusion observed with rounding error. We take here $T = 1$ and $\Delta = 1/n$. Let α_n be a positive sequence tending to zero as n goes to infinity. On a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$, we consider $(X_t)_{t\in[0,1]}$ of the form

$$X_t = x_0 + \int_0^t \sigma(X_s, s)\mathrm{d}W_s + \int_0^t a_s \mathrm{d}s, \tag{IV.2}$$

where $(W_t)_{t\in[0,1]}$ is a $(\mathcal{F}_t)$-standard Brownian motion, $(a_t)_{t\in[0,1]}$ a progressively measurable process with respect to $(\mathcal{F}_t)_{t\in[0,1]}$, $(x, y) \to \sigma(x, t)$ a real deterministic function and x_0 a real constant. We observe the sample

$$(X^{(\alpha_n)}_{i/n}, i = 0, \ldots, n), \tag{IV.3}$$

where

$$X^{(\alpha_n)}_{i/n} = \alpha_n \lfloor X_{i/n}/\alpha_n \rfloor$$

and α_n tends to zero. We denote by $\mathcal{C}^k(I)$ the set of k times continuously differentiable functions on $I \subseteq \mathbb{R}$. We suppose that $\sigma(x, t) = g_1(x)g_2(t)$, with g_1 and g_2 two positive functions such that $g_1 \in \mathcal{C}^2(\mathbb{R})$ and $g_2 \in \mathcal{C}^1([0, 1])$. Let $\beta_n = \alpha_n\sqrt{n}$,

$$\gamma(\sigma, \beta) = \int_0^1 \mathrm{d}u \int_{\mathbb{R}} \mathrm{d}y h(y)\big((\beta u + \sigma y)^{(\beta)}\big)^2,$$

where h denotes the density of a standard Gaussian variable and

$$\tilde{V}^2_n = \sum_{k=0}^{n-1} |X^{(\alpha_n)}_{(k+1)/n} - X^{(\alpha_n)}_{k/n}|^p.$$

From Delattre [35] and part II, we have the following result.

Theorem IV.3 *In model* (IV.2)-(IV.3)*,*

$$\tilde{V}^2_n = \int_0^1 \gamma(\sigma(X_t, t), \beta_n)\, dt + \mathcal{O}_p(\alpha_n \vee n^{-1/2}).$$

Moreover, as β_n tends to zero,

$$\gamma(\sigma(X_t, t), \beta_n) \sim \sigma(X_t, t)^2 + \beta_n^2/6,$$

and as β_n tends to infinity,

$$\gamma(\sigma(X_t,t),\beta_n) \sim \sqrt{\frac{2}{\pi}}\sigma(X_t,t)\beta_n.$$

From the preceding theorem, we immediately obtain the following corollary.

Corollary IV.5 *Assume that our data are generated by model* (IV.2)-(IV.3) *and suppose that $\sigma(X_t,t)=\sigma$. If $\beta_n \to +\infty$,*

$$S_q^2 \to_p 1/4.$$

If $\beta_n \to 0$,

$$S_q^2 = 1/2 - \frac{\alpha_n^2 n 2^{-q}}{12\sigma^2} + \mathcal{O}_p(n^{-1/2}2^{-q/2}) + o(\alpha_n^2 n 2^{-q}).$$

Let α be the rounding error on our data. The case β_n tends to infinity corresponds to the situation where $\alpha(n2^{-q})^{1/2}$ is "big", that is small values of Δ and the case β_n tends to zero corresponds to the situation where $\alpha(n2^{-q})^{1/2}$ is "small", that is big values of Δ.

6 Complementary microstructure functions

6.1 Theoretical considerations

We present in this section some results based on the behavior of the 1-variation V_q^1. Theoretical results on the 1-variation for discretely observed Ito semi-martingale are given in Jacod [73]. The 1-variation is less commonly used than the quadratic variation but, in our context, it has the advantage to be less impacted by the rounding effect than the quadratic variation. Indeed, we have the following theorem, which is a very specific property of the rounding error.

Theorem IV.4 *In model* (IV.2)-(IV.3),

$$\frac{1}{\sqrt{n}}\tilde{V}_n^1 = \sqrt{\frac{2}{\pi}}\int_0^1 \sigma(X_t,t)\,dt + \mathcal{O}_p(\alpha_n \vee n^{-1/2}).$$

This result has a very interesting consequence for the microstructure noise index.

Corollary IV.6 *Assume that our data are generated by model* (IV.2)-(IV.3)*, then, for all q, as $n \to +\infty$,*

$$S_q^1 = 1/2 + \mathcal{O}_p(\alpha_n \vee n^{-1/2}).$$

This point is very particular. Indeed, in the case of an additive noise, roughly speaking, S_q^1 and S_q^2 have the same behavior. This is no longer the case for rounding error. We illustrate the point by the following graphs which are the averages of 50 computations of S_q^1 and S_q^2, in model $(M1)$ with $V = 4.10^{-5}$ and $(M'1)$ with $V = 0$.

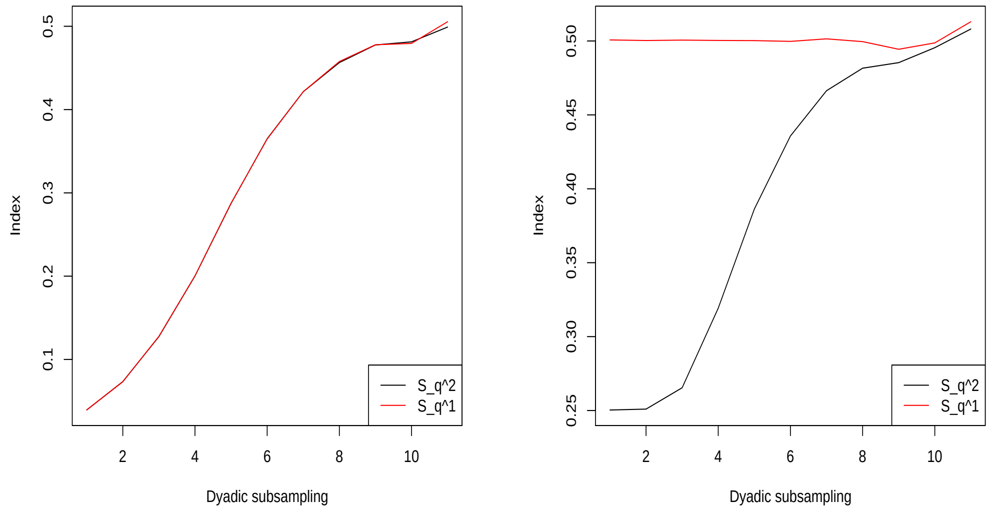

Figure IV.16: *Model M1, microstructure noise indexes S_q^1 and S_q^2 (left) and model M'1, microstructure noise indexes S_q^1 and S_q^2 (right).*

6.2 Data analysis

We give here the graphs of $q \rightarrow S_q^1$ and $q \rightarrow (\pi/2)^{1/2}(n2^{-q})^{-1/2}V_q^1$ for the Bund, last traded price. The graphs for the bid price can be found in appendix D. Note that in the context of rounding, one may think that the appropriate quantity to consider is the bid price. Indeed, if one assume that the theoretical price lies between the bid price and the ask price, and that the bid-ask spread is constant equal to one tick, then the bid price is almost surely the right measure of the rounded theoretical price.

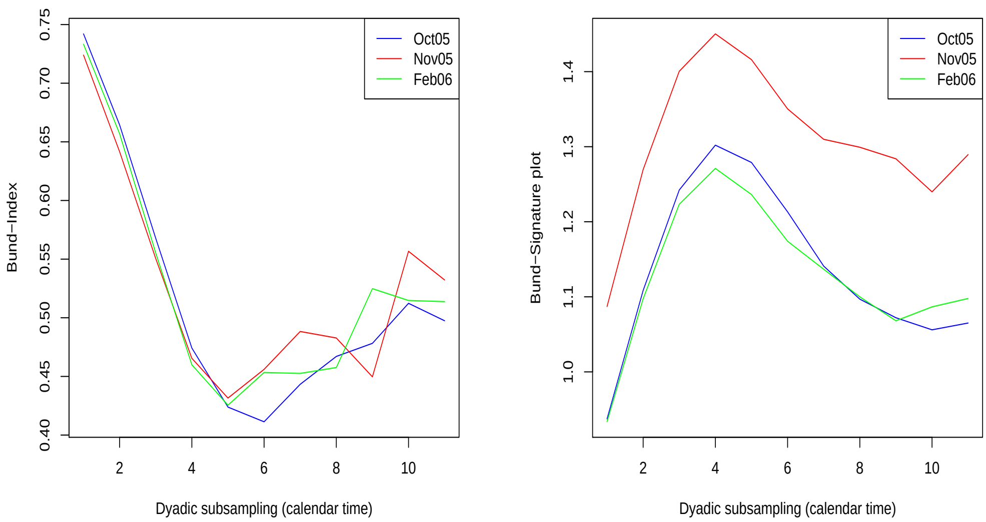

Figure IV.17: *Microstructure noise index* S_q^1 *(left) and signature plot for* $p = 1$*(right), for the Bund, last traded price, Oct 05, Nov 05, Feb 06.*

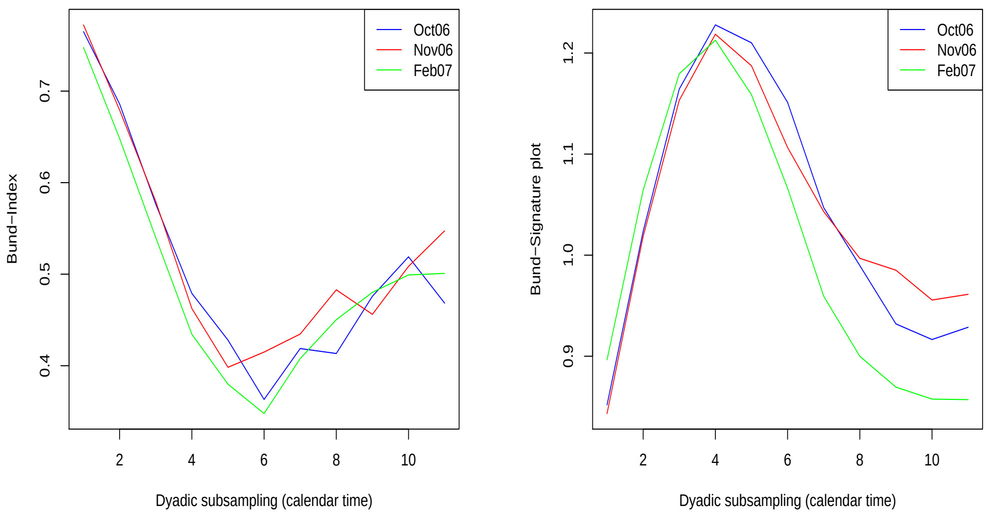

Figure IV.18: *Microstructure noise index* S_q^1 *(left) and signature plot for* $p = 1$*(right), for the Bund, last traded price, Oct 06, Nov 06, Feb 07.*

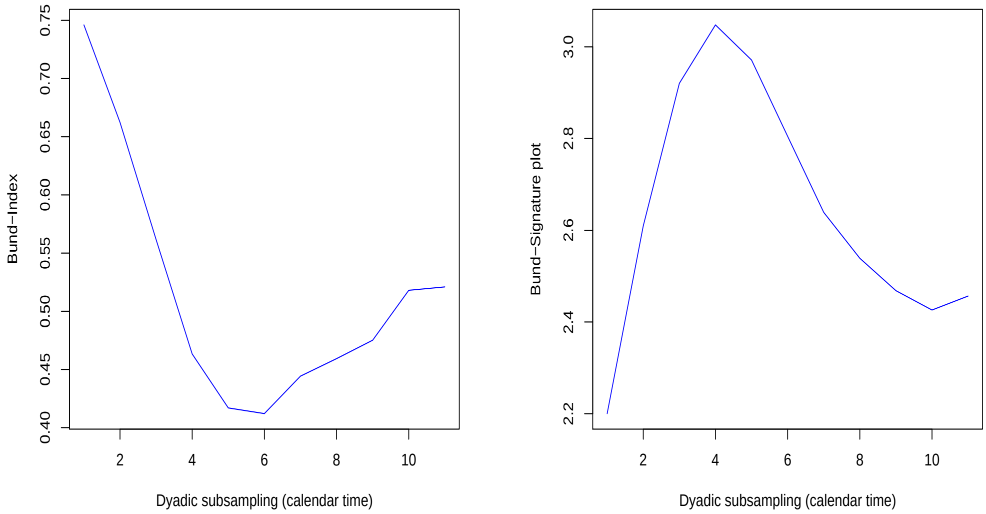

Figure IV.19: *Microstructure noise index* S_q^1 *(left) and signature plot for* $p = 1$*(right), for the Bund, last traded price, aggregated data: Oct 05, Nov 05, Feb 06, Oct 06, Nov 06, Feb 07.*

6.3 Comments, perspectives and conjectures

6.3.1 Rounding and additive error

In this section, we use conjectures on the behavior of the 1-variation under rounding and additive error. The theoretical part remains to be done. The behavior of the 1-variation based signature plot is quite surprising. Indeed, a similar shape as those of the realized volatility based signature plot could have been expected. Note that the increasing part of the graph is even more pronounced for the bid prices. To understand this fact, we consider that there are two sources of error. First an additive error and then a rounding error. Let α_n be a sequence tending to zero. We consider the following toy model for the observed price:

$$Y_{i/n} = (\sigma W_{i/n} + V\varepsilon_i^n)^{(\alpha_n)},\ i = 0, \ldots, n,$$

where the ε_i^n are centered Gaussian variables such that $\gamma_\varepsilon^{n,m} = \mathbb{E}[(\varepsilon_{mi}^n - \varepsilon_{m(i-1)}^n)^2]$ and

$$\gamma_{\varepsilon,W}^{n,m} = \mathbb{E}[(W_{mi/n} - W_{m(i-1)/n})(\varepsilon_{mi}^n - \varepsilon_{m(i-1)}^n)]$$

does not depend on i. Let

$$\mathcal{V}(m,n) = (\pi/2)^{1/2}\sqrt{\frac{m}{n}}\sum_{i=0}^{n/m-1}|Y_{(i+1)m/n} - Y_{im/n}|.$$

Assume that as $n \to +\infty$, $\mathcal{V}(m,n)$ converges in probability to its expectation if there was no rounding error, that is

$$\mathcal{V}(m,n)^2 = (\sigma^2 + V^2\frac{n}{m}\gamma_\varepsilon^{n,m} + 2V\sigma\sqrt{\frac{n}{m}}\gamma_{\varepsilon,W}^{n,m})\big(1 + \mathcal{O}_p(1)\big).$$

We want to reproduce the behavior of the 1-variation based signature plot. We first focus on the increasing part of the graph. First note that we can not obtain it if $\gamma_{\varepsilon,W}^{n,m} = 0$ and if $\gamma_\varepsilon^{n,m} = c$. Suppose that $\gamma_{\varepsilon,W}^{n,m} = 0$. We need that the following inequality holds

$$\gamma_\varepsilon^{n,m} < \frac{1}{2}\gamma_\varepsilon^{n,2m}.$$

This inequality can not be obtained if the ε_i^n follow an AR(1) model. But, if we suppose there exists a continuous noise process $\tilde{\varepsilon}_t$ such that for all n, m and i, $\varepsilon_{mi}^n = \tilde{\varepsilon}_{mi/n}$, if $\tilde{\varepsilon}_t = cB_t^H$, with B_t^H a fractional Brownian motion[11], then,

$$\varepsilon_{mi}^n - \varepsilon_{m(i-1)}^n = c(B_{mi/n}^H - B_{m(i-1)}^H)$$

and

$$\varepsilon_{2mi}^n - \varepsilon_{2m(i-1)}^n = c(B_{2mi/n}^H - B_{2m(i-1)}^H).$$

Hence the inequality is true as soon as $H > 1/2$. We think the whole behavior of the curve can be reproduce using a fractional Brownian motion, correlated with the driving Brownian motion. Let (B_t^H) be the Brownian motion defined by

$$B_t^H = c_H\Big\{\int_{-\infty}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}]\mathrm{d}B_s + \int_0^t (t-s)^{H-1/2}\mathrm{d}B_s\Big\},$$

with (B_t) a Brownian motion of the form $\rho W_t + \sqrt{1-\rho^2}W_t'$, where (W_t') is a Brownian motion, independent of (W_t) and $|\rho| < 1$. In that case,

$$\gamma_{\varepsilon,W}^{n,m} = \frac{c_H}{H+1/2}\rho(m/n)^{H+1/2}.$$

So, we may conjecture that under this specification, if $\rho < 0$,

$$\mathcal{V}(m,n)^2 = (\sigma^2 + c_1(m/n)^{2H-1} - c_2(m/n)^H)\big(1 + \mathcal{O}_p(1)\big),$$

[11]Once again, a stationary process like a fractional Ornstein-Uhlenbeck process is probably more suitable but the theoretical part of this section remaining to be done, we only focus on very simple examples.

with $c_1 \geq 0$ and $c_2 \geq 0$. Such a specification seems to enable to reproduce the curve of $q \rightarrow S_q^1$. Of course one can also obtain the decreasing part of the graph using an additive noise such that the variance of its increments does not depend on n. It is important to remark also that the increasing behavior of the beginning of the 1-variation based signature plot is not incompatible with a decreasing realized volatility based signature plot.

6.3.2 Model with uncertainty zone

A drawback of the previous model is that for the very high frequencies, because of the local time of the semi-martingale, the observed price "jumps a lot". One can probably deal with this fact using for example a non centered noise. We present here a perhaps more natural model to treat the local time problem, allowing for ultra high frequency persistence. The observed price is Y and the theoretical price is X. We define our model as follows. For a fixed parameter d,

- if $X_{i/n}^{(\alpha)} \neq Y_{(i-1)/n}$, then, if $|X_{i/n} - X_{i/n}^{(\alpha)}| < d$, $Y_{i/n} = Y_{(i-1)/n}$ else, $Y_{i/n} = X_{i/n}^{(\alpha)}$,
- if $X_{i/n}^{(\alpha)} = Y_{(i-1)/n}$, then $Y_{i/n} = X_{i/n}^{(\alpha)}$.

This model is quite similar to that used in Large [83]. Then, we obtain the following graphs for the microstructure noise indexes and signature plots for the model where the theoretical price is

$$X_t = c \exp \sigma W_t,$$

with c and σ defined as previously (average over 50 simulations).

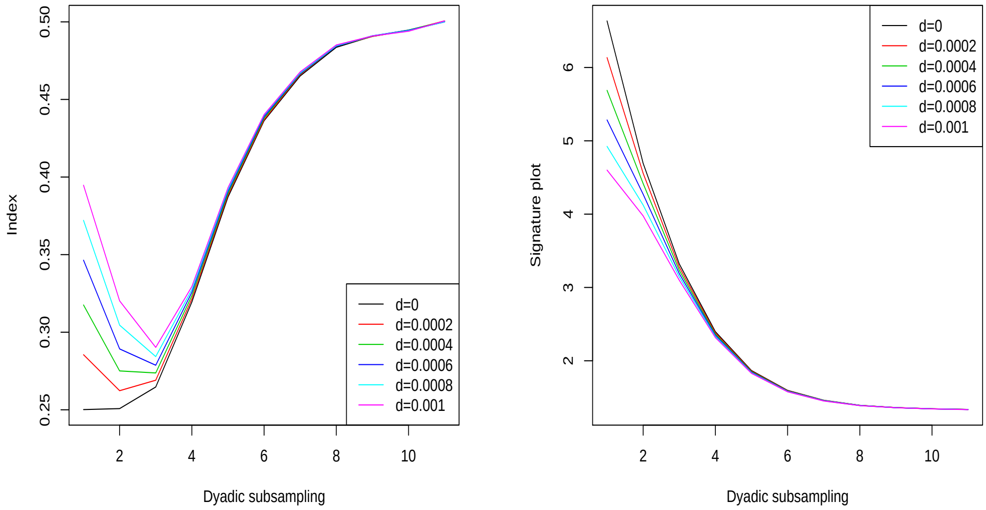

Figure IV.20: *Microstructure noise index* S_q^2 *(left) and signature plot for* $p = 2$*(right).*

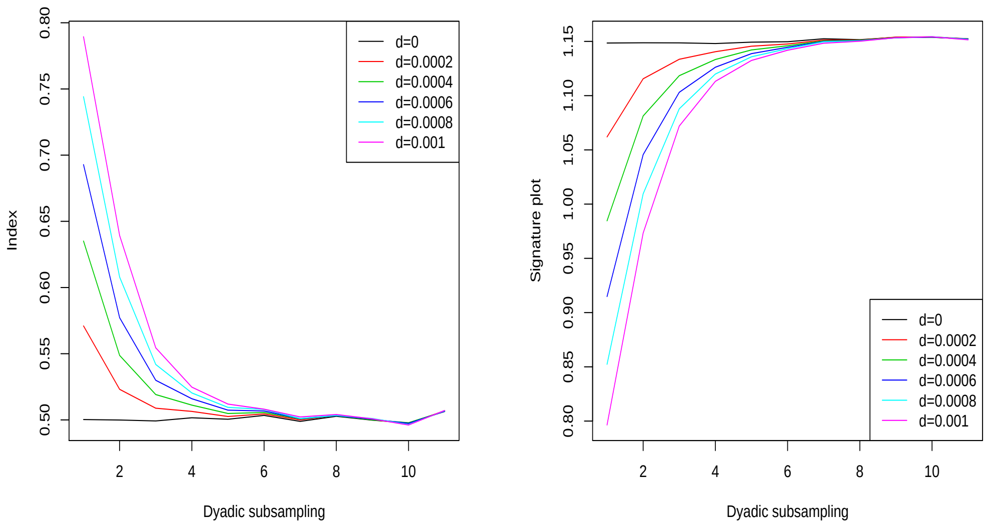

Figure IV.21: *Microstructure noise index* S_q^1 *(left) and signature plot for* $p = 1$*(right).*

The decreasing part of the 1-variation based signature plot could probably be obtained introducing correlation between the price increments and the size of the uncertainty zone.

Appendix A

Appendix of part I

1 Proof of proposition I.1

The link between Besov spaces and Gaussian processes has been largely studied, see in particular Ciesielski, Kerkyacharian and Roynette [26] and Nualart and Ouknine [98]. We give here some simple proofs for our case. Let (ϕ, ψ) be a well chosen wavelet basis. For f a real function on $\mathbb{R}$, we set

$$\alpha_{0k} = \int f(x)\phi_{0k}(x)\mathrm{d}x, \quad \beta_{jk} = \int f(x)\psi_{jk}(x)\mathrm{d}x.$$

Recall that in term of wavelets coefficients, the Besov space $\mathcal{B}^s_{p,q}(\mathbb{R})$, with $s \in [0,1]$, $1 \leq p, q < \infty$ is the space of all functions f such that the following norm is finite

$$\|f\|_{\mathcal{B}^s_{p,q}} = \|\alpha_{0.}\|_{l_p} + \big[\sum_j \big(2^{j(s-1/p+1/2)}\|\beta_{j.}\|_{l_p}\big)^q\big]^{1/q},$$

where

$$\|\beta_{j.}\|_{l_p} = \big(\sum_k |\beta_{jk}|^p\big)^{1/p}.$$

If p or q is equal to ∞, then the corresponding norm in p or q is replaced by the sup norm. For details, we refer to Meyer [95] and Cohen [28]. Here we say that a $f \in \mathcal{B}^s_{p,q}([0,T])$ if there exists g such that $g \in \mathcal{B}^s_{p,q}(\mathbb{R})$ and the restriction of g to $[0,T]$ is equal to f.

First, we show that the trajectory of $t \to \sigma_t^2$ belongs a.s. to $\mathcal{B}^H_{2,\infty}([0,T])$. It is enough to prove that $\sup_j 2^{2jH}Q_j < \infty$. We know that for all positive ε, there exist j_0 and $M > 0$ such that

$$\mathbb{P}\big[2^{j/2}\sup_{l\geq j\geq j_0}\big|\frac{Q_{l+1}}{Q_l} - 2^{-2H}\big| \geq M\big] \leq \varepsilon.$$

This implies that

$$\mathbb{P}\big[\exists j_0, \exists M, 2^{j/2} \sup_{l\geq j\geq j_0} \big|\frac{Q_{l+1}}{Q_l} - 2^{-2H}\big| \leq M\big] = 1.$$

Let $u_j = 2^{2jH}Q_j$. For such j_0, for all $j \geq j_0$, $|u_{j+1}/u_j| \leq 1 + M2^{-j/2}$. Thus, $\log u_{j+1} - \log u_j \leq \log(1+M2^{-j/2}) \leq M2^{-j/2}$ and $\log u_n \leq c$. Hence the trajectory belongs a.s. to $\mathcal{B}^H_{2,\infty}([0,T])$. Nevertheless, it does not belong to $\mathcal{B}^H_{2,q}([0,T])$, $q < \infty$. As a matter of fact, for all ε positive, there exist j_0 and $r > 0$ such that for all $j \geq j_0$, $\mathbb{P}[2^{2jH}Q_j \geq r] \geq 1-\varepsilon$. So, almost surely,

$$\sum_{j=0}^{+\infty}(2^{2jH}Q_j)^q = +\infty.$$

The fact that for $s < H$, the trajectory belongs almost surely to $\mathcal{B}^s_{\infty,\infty}([0,T])$ is clear by Kolmogorov's criterion and preceding calculations on the expectations. We now prove that it does not belong to $\mathcal{B}^H_{\infty,\infty}([0,T])$. Suppose that almost surely, there exists c such that for all $(s,t) \in [\beta_1, \beta_2]$,

$$\big|\Phi^2\big(\int_0^t a(t,u)\mathrm{d}W^H_u\big) - \Phi^2\big(\int_0^s a(s,u)\mathrm{d}W^H_u\big) + [f(t) - f(s)]\xi_0\big| \leq c|t-s|^H.$$

Because there exists $c > 0$ such that for all x, $|(\Phi^2)'(x)| > c$, this implies

$$\big|\int_0^t a(t,u)\mathrm{d}W^H_u - \int_0^s a(s,u)\mathrm{d}W^H_u + [f(t) - f(s)]\xi_0\big| \leq c|t-s|^H.$$

Ito's formula gives

$$|W^H_t a(t,t) - W^H_s a(s,s) + R(t,s)| \leq c|t-s|^H,$$

with

$$R(t,s) = [f(t) - f(s)]\xi_0 - \int_s^t \partial_2 a(t,u)W^H_u\mathrm{d}u - \int_0^s \partial_2[a(t,u) - a(s,u)]W^H_u\mathrm{d}u.$$

For fixed $\varepsilon > 0$ and $|t-s|$ small enough,

$$(t-s)^{1-H}\big|\frac{R(t,s)}{t-s}\big| \leq \varepsilon$$

and consequently,

$$\big|\frac{(W^H_t - W^H_s)a(t,t)}{(t-s)^H} - \frac{W^H_s[a(s,s) - a(t,t)]}{(t-s)^H}\big| \leq c + \varepsilon.$$

Eventually, we get for $|t-s|$ small enough

$$\big|\frac{W^H_t - W^H_s}{(t-s)^H}\big| \leq \frac{c+2\varepsilon}{\inf_{x\in[\beta_1,\beta_2]} a(x,x)},$$

which is absurd as a.s. the fbm is H Hölderian on no interval, see Arcones [11].

Appendix B

Appendix of part II

1 Simulations results

We consider models of the form

$$\mathrm{d}\log X_t = a_t \mathrm{d}t + \sigma_t \mathrm{d}W_t,\ X_0 = x_0, t \in [0,T], \tag{B.1}$$

where (W_t) a Brownian motion. We observe

$$\big\{(X_{iT/n} + \gamma\xi_i^n)^{(\alpha_n)},\ i = 0,\dots,n\big\}, \tag{B.2}$$

or

$$\big\{(\exp[\log X_{iT/n} + \gamma\xi_i^n])^{(\alpha_n)},\ i = 0,\dots,n\big\}. \tag{B.3}$$

Here the (ξ_i^n) are independent centered Gaussian variable with variance 1. In all our simulations we draw 500 independent sample path and we take

$$T = 1/512,\ ,x_0 = 1,\ \alpha_n = 0.01, n = 2^{14}.$$

We consider the previously defined Black-Scholes model, the Heston model and the correlated Heston model. The Heston models are defined the following way

$$\mathrm{d}\log X_t = (\mu - \nu_t/2)\mathrm{d}t + \sigma \mathrm{d}W_t,$$
$$\mathrm{d}\nu_t = \kappa(\beta - \nu_t)\mathrm{d}_t + \eta\nu_t^{1/2}\mathrm{d}B_t$$

where $\nu_0 = 4$, $\nu_t = \sigma_t^2$, $\mu = 0.05$, $\kappa = 5$, $\beta = 0.04$, $\eta = 0.5$ and $\rho = 0$ in the non correlated case and $\rho = -0.5$ in the non correlated case, where ρ is the correlation coefficient between the two Brownian motions.

1.1 Model (B.1)-(B.2): Heston

1.1.1 Pure Rounding: $\gamma = 0$

	Relative bias	Relative standard deviation	Relative MSE
Realized volatility	7.32718	0.74143	7.36453
Oracle ZMA	1.03627	0.16232	1.04888
ZMA	0.99802	0.19425	1.01671
$j_0 = 7$	0.26178	0.11912	0.28756
$j_0 = 6$	0.13517	0.10757	0.17269
$j_0 = 5$	0.06867	0.09922	0.12058
$j_0 = 4$	0.03467	0.09538	0.10139
Black-Scholes	0.00253	0.09182	0.09176
Robust $j_0 = 7$	0.26212	0.14894	0.30141
Robust $j_0 = 6$	0.13266	0.13333	0.188
Robust $j_0 = 5$	0.06531	0.12404	0.14008
Robust $j_0 = 4$	0.032	0.1206	0.12466
Robust Black-Scholes	0.00067	0.11662	0.1165
$j_1 = 6$, $j_2 = 6$	0.3735	0.13398	0.39676
$j_1 = 6$, $j_2 = 5$	0.19763	0.11799	0.23012
$j_1 = 6$, $j_2 = 4$	0.15113	0.10992	0.18681
$j_1 = 6$, $j_2 = 3$	0.1392	0.10823	0.17626
$j_1 = 5$, $j_2 = 5$	0.19776	0.11795	0.2302
$j_1 = 5$, $j_2 = 4$	0.10165	0.10425	0.14553
$j_1 = 5$, $j_2 = 3$	0.07698	0.10058	0.12658
$j_1 = 4$, $j_2 = 4$	0.10169	0.10425	0.14556
$j_1 = 4$, $j_2 = 3$	0.05156	0.09824	0.11086

Figure B.1: *Estimation of the relative integrated volatility in model* (B.1)-(B.2)*, case* $\gamma = 0$*, Heston.*

1.1.2 Additive noise: non robust estimation

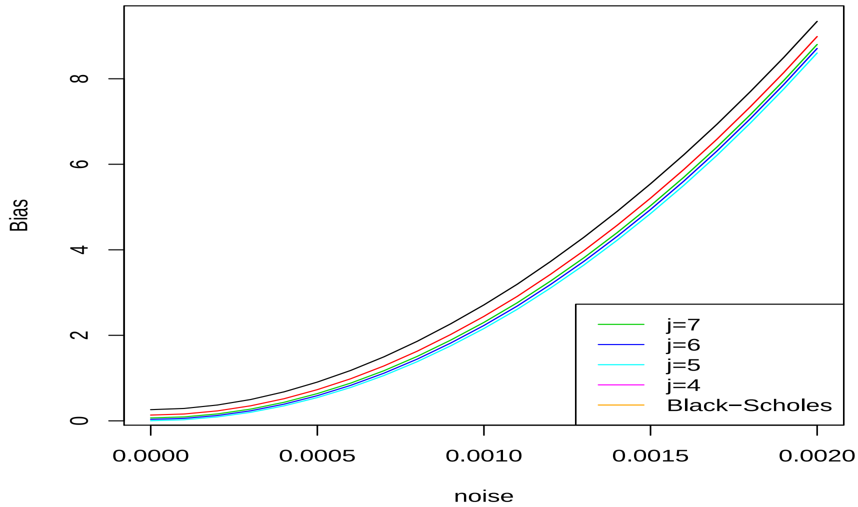

Figure B.2: *Bias of the non robust estimators in model* (B.1)-(B.2)*, Heston.*

1.1.3 Additive noise: robust estimation

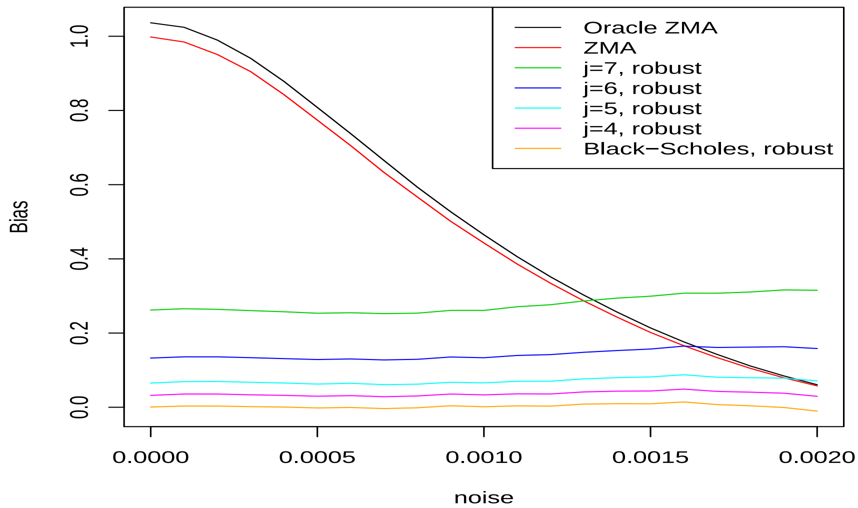

Figure B.3: *Bias of the non robust estimators in model* (B.1)-(B.2)*, Heston.*

	Robust $j_0 = 7$	Robust $j_0 = 6$	Robust $j_0 = 5$	Robust $j_0 = 4$	Robust B-S
$\gamma = 0$	0.26212 (0.14894)	0.13266 (0.13333)	0.06531 (0.12404)	0.032 (0.1206)	0.00067 (0.11662)
$\gamma = 0.0001$	0.26556 (0.14709)	0.13596 (0.13073)	0.06909 (0.12273)	0.0357 (0.1187)	0.00334 (0.11442)
$\gamma = 0.0002$	0.26394 (0.15156)	0.13591 (0.13413)	0.0695 (0.12534)	0.03568 (0.12133)	0.00325 (0.11646)
$\gamma = 0.0003$	0.26051 (0.14879)	0.13369 (0.13256)	0.06735 (0.12397)	0.03356 (0.11982)	0.00167 (0.11617)
$\gamma = 0.0004$	0.25742 (0.15548)	0.13113 (0.13866)	0.06538 (0.12898)	0.03209 (0.12468)	0.00054 (0.12122)
$\gamma = 0.0005$	0.25348 (0.16881)	0.12866 (0.15069)	0.06253 (0.13978)	0.02984 (0.13517)	−0.0018 (0.13036)
$\gamma = 0.0006$	0.25483 (0.18318)	0.13024 (0.16304)	0.06461 (0.15179)	0.03137 (0.14759)	−0.00054 (0.14275)
$\gamma = 0.0007$	0.25245 (0.19216)	0.12762 (0.1727)	0.06099 (0.15974)	0.02848 (0.15506)	−0.00348 (0.14971)
$\gamma = 0.0008$	0.25366 (0.20385)	0.12931 (0.18593)	0.06218 (0.17194)	0.03047 (0.16582)	−0.00138 (0.16101)
$\gamma = 0.0009$	0.26113 (0.21712)	0.13544 (0.20117)	0.0673 (0.18798)	0.03564 (0.18124)	0.00409 (0.17693)
$\gamma = 0.001$	0.26114 (0.23832)	0.13352 (0.21914)	0.06582 (0.20581)	0.03342 (0.19981)	0.00139 (0.196)
$\gamma = 0.0011$	0.27098 (0.25822)	0.13963 (0.23851)	0.06999 (0.22346)	0.03619 (0.21778)	0.0038 (0.21255)
$\gamma = 0.0012$	0.27642 (0.27641)	0.14199 (0.25425)	0.07034 (0.23997)	0.03584 (0.23475)	0.00311 (0.23035)
$\gamma = 0.0013$	0.28663 (0.29187)	0.1482 (0.2693)	0.0765 (0.25661)	0.04137 (0.25155)	0.00853 (0.24682)
$\gamma = 0.0014$	0.29422 (0.3123)	0.15292 (0.29)	0.07992 (0.27815)	0.04337 (0.27301)	0.00967 (0.26811)
$\gamma = 0.0015$	0.29937 (0.33352)	0.157 (0.31436)	0.08181 (0.304)	0.04389 (0.29979)	0.00943 (0.29339)
$\gamma = 0.0016$	0.30749 (0.34981)	0.16483 (0.3283)	0.08755 (0.3168)	0.04891 (0.31195)	0.01423 (0.30545)
$\gamma = 0.0017$	0.30741 (0.37566)	0.16119 (0.35124)	0.08096 (0.34017)	0.04278 (0.33478)	0.00728 (0.32767)
$\gamma = 0.0018$	0.31072 (0.40493)	0.16214 (0.37972)	0.07966 (0.36792)	0.04066 (0.36317)	0.00393 (0.35564)
$\gamma = 0.0019$	0.3163 (0.43034)	0.16322 (0.41001)	0.07855 (0.39896)	0.03792 (0.39437)	−0.00057 (0.38579)
$\gamma = 0.0020$	0.31534 (0.46493)	0.15831 (0.44357)	0.0711 (0.43249)	0.02953 (0.42692)	−0.01031 (0.41902)

Figure B.4: *Estimation of the relative integrated volatility by robust estimators in model* (B.1)-(B.2) *with additive noise, Black-Scholes.*

1.1.4 Estimation of γ

	Robust $j_0 = 7$	Robust $j_0 = 6$	Robust $j_0 = 5$	Robust $j_0 = 4$	Robust B-S
$\gamma = 0$	0.00010015 (0.00011748)	$9.628e-05$ (0.00011161)	$9.342e-05$ (0.00010735)	$9.138e-05$ (0.00010442)	$8.971e-05$ (0.00010178)
$\gamma = 0.0001$	0.00011882 (0.00012777)	0.00011487 (0.00011877)	0.00011204 (0.00011406)	0.00011044 (0.00011013)	0.00010814 (0.00010859)
$\gamma = 0.0002$	0.00019379 (0.00013982)	0.00018713 (0.00012829)	0.00018275 (0.00012336)	0.00017988 (0.00012059)	0.00017813 (0.00011808)
$\gamma = 0.0003$	0.00030867 (0.0001215)	0.00029827 (0.00010914)	0.00029179 (0.00010438)	0.00028867 (0.00010041)	0.0002863 ($9.723e-05$)
$\gamma = 0.0004$	0.00042714 ($9.827e-05$)	0.00041118 ($8.867e-05$)	0.00040205 ($8.419e-05$)	0.0003973 ($8.003e-05$)	0.00039294 ($7.785e-05$)
$\gamma = 0.0005$	0.0005392 ($8.937e-05$)	0.00051892 ($7.947e-05$)	0.00050789 ($7.616e-05$)	0.00050127 ($7.335e-05$)	0.00049557 ($7.161e-05$)
$\gamma = 0.0006$	0.00064518 ($7.955e-05$)	0.00062146 ($7.229e-05$)	0.0006085 ($6.963e-05$)	0.00060141 ($6.772e-05$)	0.00059497 ($6.607e-05$)
$\gamma = 0.0007$	0.0007508 ($7.423e-05$)	0.00072478 ($6.866e-05$)	0.00071091 ($6.533e-05$)	0.00070307 ($6.368e-05$)	0.00069592 ($6.215e-05$)
$\gamma = 0.0008$	0.0008552 ($6.992e-05$)	0.00082705 ($6.498e-05$)	0.00081227 ($6.224e-05$)	0.0008036 ($6.049e-05$)	0.0007957 ($5.909e-05$)
$\gamma = 0.0009$	0.00095639 ($6.962e-05$)	0.00092724 ($6.529e-05$)	0.00091185 ($6.273e-05$)	0.00090281 ($6.14e-05$)	0.00089447 ($5.996e-05$)
$\gamma = 0.001$	0.00105745 ($6.982e-05$)	0.00102758 ($6.61e-05$)	0.00101145 ($6.404e-05$)	0.00100264 ($6.278e-05$)	0.00099397 ($6.136e-05$)
$\gamma = 0.0011$	0.00115574 ($7.252e-05$)	0.00112595 ($6.918e-05$)	0.00110992 ($6.693e-05$)	0.00110126 ($6.58e-05$)	0.00109238 ($6.427e-05$)
$\gamma = 0.0012$	0.00125595 ($7.238e-05$)	0.00122636 ($6.952e-05$)	0.00121028 ($6.781e-05$)	0.00120171 ($6.728e-05$)	0.00119264 ($6.601e-05$)
$\gamma = 0.0013$	0.00135342 ($7.325e-05$)	0.00132434 ($7.064e-05$)	0.00130821 ($6.925e-05$)	0.00129978 ($6.858e-05$)	0.00129066 ($6.744e-05$)
$\gamma = 0.0014$	0.00145148 ($7.382e-05$)	0.00142306 ($7.125e-05$)	0.00140731 ($6.965e-05$)	0.00139924 ($6.907e-05$)	0.00139027 ($6.796e-05$)
$\gamma = 0.0015$	0.0015494 ($7.376e-05$)	0.00152134 ($7.191e-05$)	0.00150617 ($7.072e-05$)	0.00149847 ($7.024e-05$)	0.00148964 ($6.897e-05$)
$\gamma = 0.0016$	0.00164585 ($7.521e-05$)	0.00161835 ($7.311e-05$)	0.00160382 ($7.23e-05$)	0.00159649 ($7.167e-05$)	0.00158767 ($7.018e-05$)
$\gamma = 0.0017$	0.0017434 ($7.656e-05$)	0.00171723 ($7.441e-05$)	0.00170356 ($7.365e-05$)	0.00169638 ($7.297e-05$)	0.00168784 ($7.167e-05$)
$\gamma = 0.0018$	0.00184099 ($7.777e-05$)	0.001816 ($7.573e-05$)	0.00180311 ($7.496e-05$)	0.00179633 ($7.446e-05$)	0.00178819 ($7.35e-05$)
$\gamma = 0.0019$	0.00193729 ($7.896e-05$)	0.00191398 ($7.72e-05$)	0.00190201 ($7.616e-05$)	0.00189577 ($7.571e-05$)	0.001888 ($7.48e-05$)
$\gamma = 0.0020$	0.00203505 ($7.812e-05$)	0.00201327 ($7.657e-05$)	0.00200228 ($7.578e-05$)	0.00199649 ($7.539e-05$)	0.00198903 ($7.448e-05$)

Figure B.5: *Estimation of the standard deviation of the noise in model* (B.1)-(B.2), *Heston.*

1.2 Model (B.1)-(B.2): correlated Heston

1.2.1 Pure Rounding: $\gamma = 0$

	Relative bias	Relative standard deviation	Relative MSE
Realized volatility	7.34336	0.70561	7.37712
Oracle ZMA	1.03693	0.15396	1.04827
ZMA	0.99427	0.18074	1.01054
$j_0 = 7$	0.27058	0.12081	0.29628
$j_0 = 6$	0.14258	0.10865	0.1792
$j_0 = 5$	0.07657	0.10246	0.12782
$j_0 = 4$	0.04212	0.09956	0.10801
Black-Scholes	0.01018	0.09575	0.0962
Robust $j_0 = 7$	0.26687	0.14753	0.30487
Robust $j_0 = 6$	0.1359	0.13115	0.18877
Robust $j_0 = 5$	0.06895	0.12267	0.14061
Robust $j_0 = 4$	0.03532	0.11922	0.12423
Robust Black-Scholes	0.00353	0.114	0.11394
$j_1 = 6$, $j_2 = 6$	0.38359	0.13575	0.40686
$j_1 = 6$, $j_2 = 5$	0.20459	0.1171	0.23568
$j_1 = 6$, $j_2 = 4$	0.15874	0.11081	0.19353
$j_1 = 6$, $j_2 = 3$	0.14652	0.10933	0.18275
$j_1 = 5$, $j_2 = 5$	0.20472	0.11712	0.2358
$j_1 = 5$, $j_2 = 4$	0.10996	0.10699	0.15335
$j_1 = 5$, $j_2 = 3$	0.0847	0.10387	0.13395
$j_1 = 4$, $j_2 = 4$	0.10999	0.107	0.15338
$j_1 = 4$, $j_2 = 3$	0.05865	0.10252	0.11802

Figure B.6: *Estimation of the relative integrated volatility in model* (B.1)-(B.2)*, case* $\gamma = 0$*, correlated Heston.*

1.2.2 Additive noise: non robust estimation

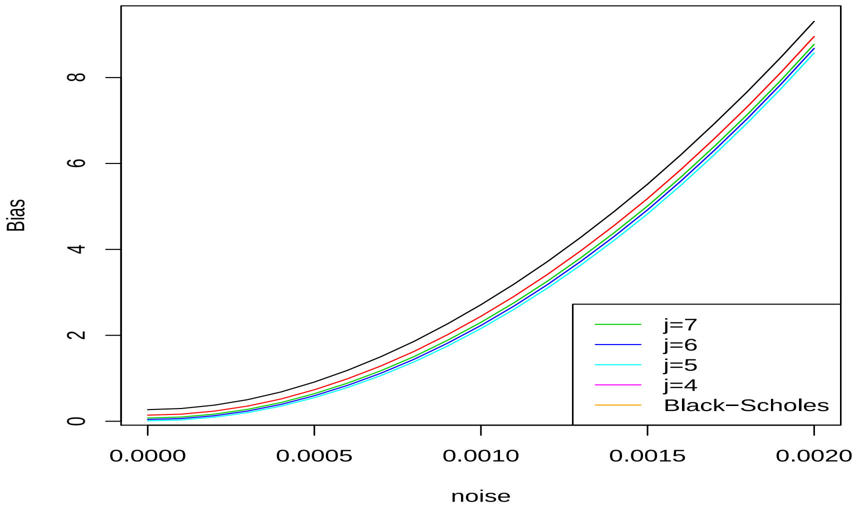

Figure B.7: *Bias of the non robust estimators in model* (B.1)-(B.2)*, correlated Heston.*

1.2.3 Additive noise: robust estimation

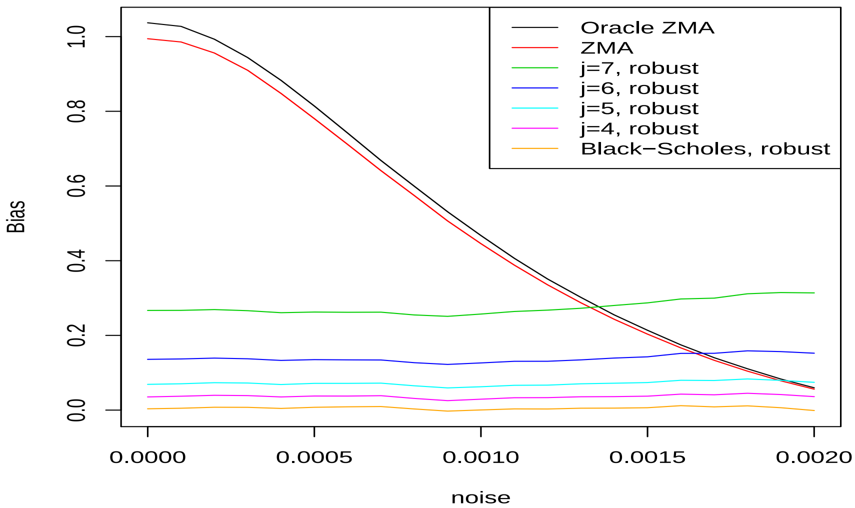

Figure B.8: *Bias of the non robust estimators in model* (B.1)-(B.2)*, correlated Heston.*

	Robust $j_0 = 7$	Robust $j_0 = 6$	Robust $j_0 = 5$	Robust $j_0 = 4$	Robust B-S
$\gamma = 0$	0.26687 (0.14753)	0.1359 (0.13115)	0.06895 (0.12267)	0.03532 (0.11922)	0.00353 (0.114)
$\gamma = 0.0001$	0.26705 (0.14872)	0.13696 (0.13122)	0.07052 (0.12229)	0.03717 (0.11875)	0.00503 (0.1134)
$\gamma = 0.0002$	0.26914 (0.14977)	0.13923 (0.13373)	0.07341 (0.12465)	0.03963 (0.12142)	0.00764 (0.11712)
$\gamma = 0.0003$	0.26617 (0.15275)	0.13742 (0.13618)	0.07255 (0.12645)	0.03882 (0.12306)	0.00741 (0.11846)
$\gamma = 0.0004$	0.2608 (0.16036)	0.13304 (0.14257)	0.06847 (0.13332)	0.03526 (0.13045)	0.00451 (0.12625)
$\gamma = 0.0005$	0.26253 (0.16773)	0.13513 (0.1479)	0.07158 (0.1378)	0.0377 (0.13443)	0.00755 (0.12988)
$\gamma = 0.0006$	0.2619 (0.18028)	0.13452 (0.15764)	0.07158 (0.14606)	0.03759 (0.14211)	0.00886 (0.13692)
$\gamma = 0.0007$	0.26219 (0.18973)	0.1342 (0.16868)	0.07219 (0.15778)	0.0386 (0.15385)	0.00955 (0.14728)
$\gamma = 0.0008$	0.25478 (0.20446)	0.12695 (0.18104)	0.0651 (0.17043)	0.03126 (0.16571)	0.00309 (0.16018)
$\gamma = 0.0009$	0.25103 (0.21933)	0.12233 (0.19711)	0.05945 (0.18569)	0.02535 (0.18019)	−0.00265 (0.17449)
$\gamma = 0.001$	0.2572 (0.23613)	0.12634 (0.21183)	0.06252 (0.19975)	0.02925 (0.19421)	0.00042 (0.18809)
$\gamma = 0.0011$	0.26404 (0.24925)	0.13067 (0.22355)	0.06639 (0.21204)	0.03315 (0.20621)	0.00327 (0.20058)
$\gamma = 0.0012$	0.26765 (0.27752)	0.13075 (0.25295)	0.06689 (0.23974)	0.03341 (0.23331)	0.00301 (0.22665)
$\gamma = 0.0013$	0.27265 (0.29084)	0.1345 (0.26611)	0.07044 (0.25467)	0.03572 (0.24906)	0.0051 (0.2421)
$\gamma = 0.0014$	0.28015 (0.30983)	0.13938 (0.28387)	0.072 (0.27223)	0.03608 (0.26648)	0.00534 (0.25896)
$\gamma = 0.0015$	0.28713 (0.32726)	0.14268 (0.302)	0.07378 (0.29172)	0.03733 (0.28615)	0.00636 (0.2792)
$\gamma = 0.0016$	0.29756 (0.35082)	0.15184 (0.32547)	0.07983 (0.31614)	0.04278 (0.31076)	0.01188 (0.30295)
$\gamma = 0.0017$	0.29976 (0.36928)	0.15206 (0.34691)	0.07924 (0.33869)	0.04102 (0.33277)	0.00871 (0.32501)
$\gamma = 0.0018$	0.31146 (0.38501)	0.15872 (0.36363)	0.0834 (0.35588)	0.04507 (0.34957)	0.01138 (0.34238)
$\gamma = 0.0019$	0.31473 (0.4102)	0.1566 (0.38927)	0.07961 (0.37972)	0.0417 (0.37453)	0.0064 (0.36588)
$\gamma = 0.0020$	0.31384 (0.44138)	0.15238 (0.41936)	0.07433 (0.41279)	0.03611 (0.40796)	−0.00108 (0.39755)

Figure B.9: *Estimation of the relative integrated volatility by robust estimators in model* (B.1)-(B.2) *with additive noise, correlated Heston.*

1.2.4 Estimation of γ

	Robust $j_0 = 7$	Robust $j_0 = 6$	Robust $j_0 = 5$	Robust $j_0 = 4$	Robust B-S
$\gamma = 0$	$9.$0.0001013 (0.0001182)	$9.774e-05$ (0.0001136)	$9.512e-05$ (0.00011038)	$9.422e-05$ (0.00010699)	$9.083e-05$ (0.0001054)
$\gamma = 0.0001$	0.0001267 (0.0001301)	0.00012141 (0.00012283)	0.00011878 (0.00011759)	0.00011586 (0.00011448)	0.00011358 (0.00011171)
$\gamma = 0.0002$	0.0001961 (0.00013802)	0.00018768 (0.00013089)	0.00018363 (0.00012501)	0.00018119 (0.00012311)	0.00017994 (0.00011863)
$\gamma = 0.0003$	0.00030921 (0.00012391)	0.00029586 (0.00011678)	0.00028964 (0.00010956)	0.0002856 (0.00010864)	0.00028184 (0.00010565)
$\gamma = 0.0004$	0.00043009 ($9.545e-05$)	0.00041223 ($8.919e-05$)	0.00040305 ($8.425e-05$)	0.00039813 ($8.274e-05$)	0.00039319 ($7.947e-05$)
$\gamma = 0.0005$	0.00054203 ($7.857e-05$)	0.00052023 ($7.402e-05$)	0.00050833 ($7.043e-05$)	0.00050258 ($6.953e-05$)	0.00049603 ($6.726e-05$)
$\gamma = 0.0006$	0.00064899 ($7.648e-05$)	0.0006245 ($7.243e-05$)	0.00061083 ($6.842e-05$)	0.00060422 ($6.74e-05$)	0.0005963 ($6.501e-05$)
$\gamma = 0.0007$	0.00075387 ($7.106e-05$)	0.00072696 ($6.861e-05$)	0.00071165 ($6.571e-05$)	0.00070398 ($6.549e-05$)	0.00069573 ($6.319e-05$)
$\gamma = 0.0008$	0.0008585 ($6.854e-05$)	0.00082992 ($6.58e-05$)	0.0008139 ($6.348e-05$)	0.00080597 ($6.304e-05$)	0.00079692 ($6.131e-05$)
$\gamma = 0.0009$	0.00096239 ($6.891e-05$)	0.00093289 ($6.625e-05$)	0.00091648 ($6.408e-05$)	0.00090824 ($6.334e-05$)	0.00089866 ($6.218e-05$)
$\gamma = 0.001$	0.00106229 ($6.923e-05$)	0.00103212 ($6.662e-05$)	0.00101532 ($6.442e-05$)	0.00100654 ($6.396e-05$)	0.00099702 ($6.288e-05$)
$\gamma = 0.0011$	0.00116117 ($6.888e-05$)	0.00113094 ($6.597e-05$)	0.00111404 ($6.403e-05$)	0.00110521 ($6.357e-05$)	0.00109571 ($6.251e-05$)
$\gamma = 0.0012$	0.00125994 ($7.083e-05$)	0.00123033 ($6.798e-05$)	0.00121328 ($6.621e-05$)	0.00120454 ($6.548e-05$)	0.00119501 ($6.417e-05$)
$\gamma = 0.0013$	0.00135869 ($7.043e-05$)	0.00132905 ($6.749e-05$)	0.00131209 ($6.603e-05$)	0.00130372 ($6.55e-05$)	0.00129415 ($6.418e-05$)
$\gamma = 0.0014$	0.00145586 ($7.183e-05$)	0.00142683 ($6.802e-05$)	0.00141061 ($6.704e-05$)	0.0014026 ($6.645e-05$)	0.00139314 ($6.499e-05$)
$\gamma = 0.0015$	0.00155212 ($7.264e-05$)	0.00152414 ($6.878e-05$)	0.00150849 ($6.802e-05$)	0.00150064 ($6.77e-05$)	0.00149121 ($6.642e-05$)
$\gamma = 0.0016$	0.00164939 ($7.349e-05$)	0.00162193 ($7.05e-05$)	0.00160719 ($6.958e-05$)	0.00159965 ($6.93e-05$)	0.00159023 ($6.827e-05$)
$\gamma = 0.0017$	0.00174723 ($7.389e-05$)	0.00172093 ($7.134e-05$)	0.00170668 ($7.048e-05$)	0.00169952 ($7.015e-05$)	0.00169051 ($6.899e-05$)
$\gamma = 0.0018$	0.00184254 ($7.601e-05$)	0.00181789 ($7.335e-05$)	0.00180454 ($7.239e-05$)	0.00179763 ($7.176e-05$)	0.00178905 ($7.042e-05$)
$\gamma = 0.0019$	0.00193925 ($7.706e-05$)	0.00191632 ($7.439e-05$)	0.00190384 ($7.334e-05$)	0.00189714 ($7.28e-05$)	0.0018889 ($7.182e-05$)
$\gamma = 0.0020$	0.00203686 ($7.805e-05$)	0.00201553 ($7.567e-05$)	0.00200376 ($7.484e-05$)	0.00199734 ($7.434e-05$)	0.00198956 ($7.33e-05$)

Figure B.10: *Estimation of the standard deviation of the noise in model* (B.1)-(B.2), *correlated Heston.*

1.3 Model (B.1)-(B.3): Black-Scholes

1.3.1 Pure Rounding: $\gamma = 0$

	Relative bias	Relative standard deviation	Relative MSE
Realized volatility	7.23688	0.72558	7.27309
Oracle ZMA	1.01581	0.17052	1.03
ZMA	0.97101	0.19122	0.98963
$j_0 = 7$	0.26473	0.11688	0.28934
$j_0 = 6$	0.13774	0.10378	0.1724
$j_0 = 5$	0.07287	0.09759	0.12172
$j_0 = 4$	0.04	0.09402	0.10209
Black-Scholes	0.00857	0.0918	0.09211
Robust $j_0 = 7$	0.2664	0.15641	0.30885
Robust $j_0 = 6$	0.13667	0.13864	0.19458
Robust $j_0 = 5$	0.07098	0.12964	0.14769
Robust $j_0 = 4$	0.03943	0.12435	0.13033
Robust Black-Scholes	0.00743	0.12037	0.12048
$j_1 = 6$, $j_2 = 6$	0.37677	0.13231	0.39928
$j_1 = 6$, $j_2 = 5$	0.1987	0.11222	0.22815
$j_1 = 6$, $j_2 = 4$	0.15316	0.10611	0.18626
$j_1 = 6$, $j_2 = 3$	0.14162	0.10422	0.17577
$j_1 = 5$, $j_2 = 5$	0.19885	0.11221	0.22827
$j_1 = 5$, $j_2 = 4$	0.10473	0.10247	0.14645
$j_1 = 5$, $j_2 = 3$	0.08088	0.09851	0.12738
$j_1 = 4$, $j_2 = 4$	0.10475	0.10247	0.14646
$j_1 = 4$, $j_2 = 3$	0.05628	0.09607	0.11126

Figure B.11: *Estimation of the relative integrated volatility in model* (B.1)-(B.3)*, case* $\gamma = 0$*, Black-Scholes.*

1.3.2 Additive noise: non robust estimation

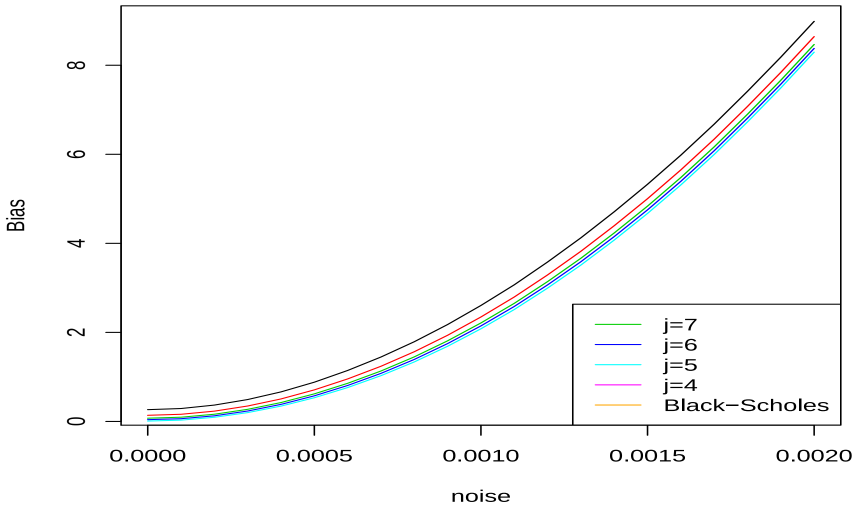

Figure B.12: *Bias of the non robust estimators in model* (B.1)-(B.3)*, Black-Scholes.*

1.3.3 Additive noise: robust estimation

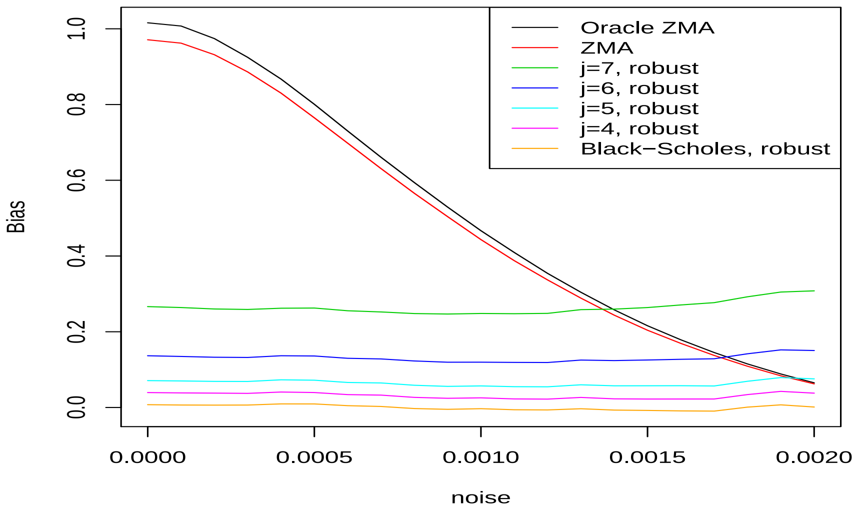

Figure B.13: *Bias of the non robust estimators in model* (B.1)-(B.3)*, Black-Scholes.*

	Robust $j_0 = 7$	Robust $j_0 = 6$	Robust $j_0 = 5$	Robust $j_0 = 4$	Robust B-S
$\gamma = 0$	0.2664 (0.15641)	0.13667 (0.13864)	0.07098 (0.12964)	0.03943 (0.12435)	0.00743 (0.12037)
$\gamma = 0.0001$	0.26395 (0.14761)	0.13476 (0.13082)	0.07009 (0.12244)	0.03852 (0.11758)	0.00655 (0.11433)
$\gamma = 0.0002$	0.26013 (0.1487)	0.13271 (0.13318)	0.06891 (0.12628)	0.03798 (0.1222)	0.00627 (0.11872)
$\gamma = 0.0003$	0.25889 (0.15415)	0.1321 (0.13845)	0.0687 (0.13152)	0.03728 (0.12733)	0.00646 (0.12396)
$\gamma = 0.0004$	0.26213 (0.15966)	0.13665 (0.14352)	0.0731 (0.13639)	0.04084 (0.13275)	0.00949 (0.12898)
$\gamma = 0.0005$	0.2626 (0.16803)	0.13601 (0.15159)	0.07205 (0.14427)	0.03951 (0.14024)	0.00938 (0.13645)
$\gamma = 0.0006$	0.25545 (0.18104)	0.12992 (0.16544)	0.06609 (0.15618)	0.03402 (0.15188)	0.00482 (0.14799)
$\gamma = 0.0007$	0.2523 (0.19188)	0.12824 (0.17573)	0.06481 (0.16361)	0.03269 (0.15886)	0.0027 (0.15515)
$\gamma = 0.0008$	0.24803 (0.20166)	0.12285 (0.18538)	0.0587 (0.17501)	0.02669 (0.17032)	-0.00275 (0.16703)
$\gamma = 0.0009$	0.24681 (0.20527)	0.11956 (0.18884)	0.05575 (0.18054)	0.02429 (0.17669)	-0.00482 (0.17398)
$\gamma = 0.001$	0.24826 (0.22322)	0.11964 (0.20686)	0.05697 (0.19751)	0.02548 (0.19331)	-0.00312 (0.19075)
$\gamma = 0.0011$	0.24765 (0.24499)	0.11904 (0.22796)	0.05493 (0.21547)	0.02265 (0.21187)	-0.00593 (0.20894)
$\gamma = 0.0012$	0.24852 (0.26447)	0.11878 (0.24773)	0.05464 (0.23507)	0.02215 (0.22943)	-0.00641 (0.22556)
$\gamma = 0.0013$	0.2584 (0.27814)	0.12523 (0.26271)	0.05994 (0.25277)	0.02655 (0.24741)	-0.00328 (0.2431)
$\gamma = 0.0014$	0.25981 (0.29192)	0.12385 (0.27901)	0.05729 (0.27)	0.02285 (0.26519)	-0.00685 (0.26057)
$\gamma = 0.0015$	0.2639 (0.31857)	0.12543 (0.30641)	0.05737 (0.2963)	0.02234 (0.29005)	-0.00774 (0.28393)
$\gamma = 0.0016$	0.27079 (0.34049)	0.12727 (0.32619)	0.05754 (0.31715)	0.0224 (0.31127)	-0.00896 (0.30664)
$\gamma = 0.0017$	0.27684 (0.36293)	0.12865 (0.34604)	0.05682 (0.33744)	0.02246 (0.3318)	-0.00952 (0.32716)
$\gamma = 0.0018$	0.29233 (0.38348)	0.14183 (0.36732)	0.06924 (0.35776)	0.03411 (0.35205)	0.00096 (0.34713)
$\gamma = 0.0019$	0.30486 (0.40595)	0.15215 (0.39133)	0.07901 (0.38361)	0.04256 (0.37613)	0.00701 (0.37119)
$\gamma = 0.0020$	0.30802 (0.43741)	0.15046 (0.42342)	0.07562 (0.41394)	0.03792 (0.40696)	0.00128 (0.40186)

Figure B.14: *Estimation of the relative integrated volatility by robust estimators in model* (B.1)-(B.3) *with additive noise, Black-Scholes.*

1.3.4 Estimation of γ

	Robust $j_0 = 7$	Robust $j_0 = 6$	Robust $j_0 = 5$	Robust $j_0 = 4$	Robust B-S
$\gamma = 0$	$9.814e-05$ (0.00011946)	$9.586e-05$ (0.00011156)	$9.307e-05$ (0.00010778)	$8.958e-05$ (0.00010481)	$8.881e-05$ (0.00010368)
$\gamma = 0.0001$	0.00012107 (0.00012549)	0.00011825 (0.00011796)	0.00011588 (0.00011299)	0.00011295 (0.00010955)	0.00011231 (0.00010805)
$\gamma = 0.0002$	0.00020273 (0.00013386)	0.00019202 (0.00012985)	0.00018765 (0.00012436)	0.00018373 (0.00012066)	0.00018147 (0.00011933)
$\gamma = 0.0003$	0.0003128 (0.00012298)	0.00030149 (0.0001143)	0.00029428 (0.00011003)	0.00028969 (0.0001071)	0.00028663 (0.00010394)
$\gamma = 0.0004$	0.00042871 ($9.53e-05$)	0.00041163 ($8.915e-05$)	0.00040219 ($8.63e-05$)	0.00039733 ($8.465e-05$)	0.00039331 ($8.234e-05$)
$\gamma = 0.0005$	0.00053875 ($8.389e-05$)	0.0005189 ($7.995e-05$)	0.00050761 ($7.857e-05$)	0.00050217 ($7.728e-05$)	0.00049631 ($7.591e-05$)
$\gamma = 0.0006$	0.00064853 ($8.24e-05$)	0.00062504 ($8.063e-05$)	0.00061217 ($7.919e-05$)	0.00060586 ($7.81e-05$)	0.00059895 ($7.69e-05$)
$\gamma = 0.0007$	0.00075327 ($8.261e-05$)	0.00072699 ($8.159e-05$)	0.00071303 ($8.023e-05$)	0.00070623 ($7.918e-05$)	0.00069908 ($7.841e-05$)
$\gamma = 0.0008$	0.00085697 ($8.366e-05$)	0.00082933 ($8.336e-05$)	0.00081474 ($8.294e-05$)	0.00080741 ($8.227e-05$)	0.00079969 ($8.17e-05$)
$\gamma = 0.0009$	0.00095943 ($8.465e-05$)	0.00093105 ($8.535e-05$)	0.00091565 ($8.539e-05$)	0.0009078 ($8.504e-05$)	0.00089969 ($8.488e-05$)
$\gamma = 0.001$	0.0010609 ($8.839e-05$)	0.00103156 ($8.921e-05$)	0.00101522 ($8.963e-05$)	0.0010072 ($8.96e-05$)	0.0009987 ($8.96e-05$)
$\gamma = 0.0011$	0.00116122 ($9.481e-05$)	0.00113135 ($9.571e-05$)	0.001115 ($9.603e-05$)	0.001107 ($9.628e-05$)	0.00109847 ($9.635e-05$)
$\gamma = 0.0012$	0.00126226 ($9.963e-05$)	0.00123197 (0.0001005)	0.00121553 (0.0001012)	0.00120747 (0.00010134)	0.00119896 (0.00010161)
$\gamma = 0.0013$	0.00135995 (0.00010319)	0.00133032 (0.00010456)	0.00131412 (0.00010607)	0.00130615 (0.0001064)	0.00129804 (0.00010669)
$\gamma = 0.0014$	0.00145941 (0.00010746)	0.00143039 (0.00010956)	0.00141453 (0.00011149)	0.00140673 (0.00011211)	0.00139865 (0.00011277)
$\gamma = 0.0015$	0.00155701 (0.00011507)	0.00152877 (0.00011753)	0.00151342 (0.00011952)	0.00150589 (0.00012021)	0.00149806 (0.00012081)
$\gamma = 0.0016$	0.00165404 (0.00012208)	0.00162717 (0.00012469)	0.0016125 (0.00012667)	0.00160516 (0.00012764)	0.00159781 (0.00012823)
$\gamma = 0.0017$	0.00175134 (0.0001288)	0.00172585 (0.00013163)	0.00171198 (0.00013377)	0.00170471 (0.0001349)	0.00169764 (0.00013554)
$\gamma = 0.0018$	0.00184757 (0.00013442)	0.00182329 (0.00013729)	0.00181001 (0.00013934)	0.0018031 (0.00014063)	0.00179648 (0.00014117)
$\gamma = 0.0019$	0.00194332 (0.00014063)	0.00192033 (0.00014378)	0.00190764 (0.00014621)	0.00190122 (0.00014742)	0.00189527 (0.000148)
$\gamma = 0.0020$	0.00203994 (0.00014784)	0.00201857 (0.00015092)	0.00200668 (0.00015304)	0.00200078 (0.00015435)	0.00199532 (0.00015496)

Figure B.15: *Estimation of the standard deviation of the noise in model* (B.1)-(B.3), *Black-Scholes.*

1.4 Model (B.1)-(B.3): Heston

1.4.1 Pure Rounding: $\gamma = 0$

	Relative bias	Relative standard deviation	Relative MSE
Realized volatility	7.32718	0.74143	7.36453
Oracle ZMA	1.03627	0.16232	1.04888
ZMA	0.99802	0.19425	1.01671
$j_0 = 7$	0.26178	0.11912	0.28756
$j_0 = 6$	0.13517	0.10757	0.17269
$j_0 = 5$	0.06867	0.09922	0.12058
$j_0 = 4$	0.03467	0.09538	0.10139
Black-Scholes	0.00253	0.09182	0.09176
Robust $j_0 = 7$	0.26212	0.14894	0.30141
Robust $j_0 = 6$	0.13266	0.13333	0.188
Robust $j_0 = 5$	0.06531	0.12404	0.14008
Robust $j_0 = 4$	0.032	0.1206	0.12466
Robust Black-Scholes	0.00067	0.11662	0.1165
$j_1 = 6$, $j_2 = 6$	0.3735	0.13398	0.39676
$j_1 = 6$, $j_2 = 5$	0.19763	0.11799	0.23012
$j_1 = 6$, $j_2 = 4$	0.15113	0.10992	0.18681
$j_1 = 6$, $j_2 = 3$	0.1392	0.10823	0.17626
$j_1 = 5$, $j_2 = 5$	0.19776	0.11795	0.2302
$j_1 = 5$, $j_2 = 4$	0.10165	0.10425	0.14553
$j_1 = 5$, $j_2 = 3$	0.07698	0.10058	0.12658
$j_1 = 4$, $j_2 = 4$	0.10169	0.10425	0.14556
$j_1 = 4$, $j_2 = 3$	0.05156	0.09824	0.11086

Figure B.16: *Estimation of the relative integrated volatility in model* (B.1)-(B.3), *case* $\gamma = 0$, *Heston.*

1.4.2 Additive noise: non robust estimation

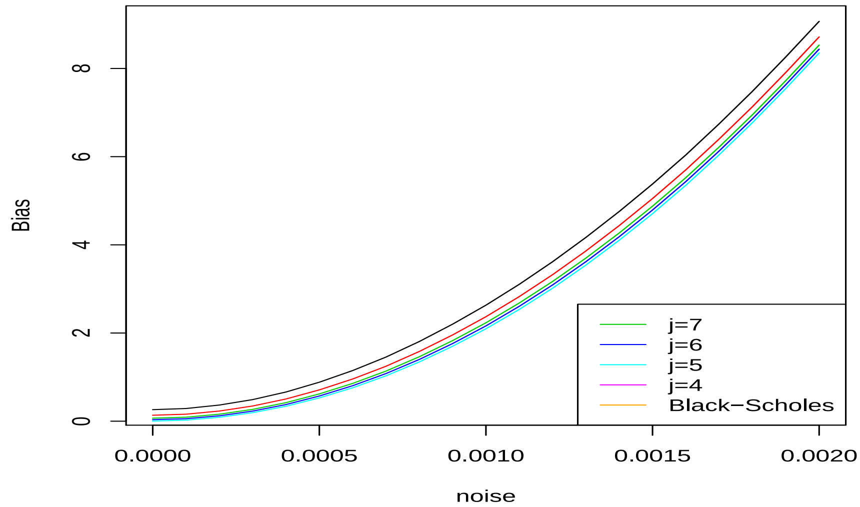

Figure B.17: *Bias of the non robust estimators in model* (B.1)-(B.3)*, Heston.*

1.4.3 Additive noise: robust estimation

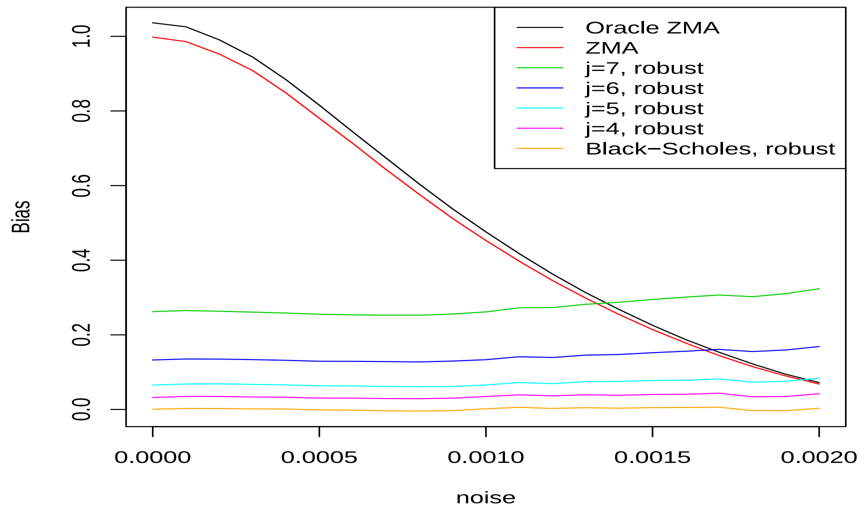

Figure B.18: *Bias of the non robust estimators in model* (B.1)-(B.3)*, Heston.*

	Robust $j_0 = 7$	Robust $j_0 = 6$	Robust $j_0 = 5$	Robust $j_0 = 4$	Robust B-S
$\gamma = 0$	0.26212 (0.14894)	0.13266 (0.13333)	0.06531 (0.12404)	0.032 (0.1206)	0.00067 (0.11662)
$\gamma = 0.0001$	0.26505 (0.14648)	0.13526 (0.13033)	0.06826 (0.12207)	0.03504 (0.11811)	0.00268 (0.11377)
$\gamma = 0.0002$	0.26323 (0.14954)	0.13498 (0.13266)	0.06857 (0.12466)	0.0348 (0.12057)	0.00256 (0.11556)
$\gamma = 0.0003$	0.26088 (0.15196)	0.13356 (0.13486)	0.06722 (0.12602)	0.03341 (0.12141)	0.00158 (0.11718)
$\gamma = 0.0004$	0.25838 (0.15387)	0.13172 (0.1369)	0.06573 (0.12745)	0.03267 (0.12335)	0.00102 (0.1198)
$\gamma = 0.0005$	0.25523 (0.16787)	0.1292 (0.14942)	0.06343 (0.13962)	0.03053 (0.13445)	-0.00113 (0.13083)
$\gamma = 0.0006$	0.25363 (0.17414)	0.12894 (0.15625)	0.06299 (0.1457)	0.03031 (0.14127)	-0.00187 (0.13707)
$\gamma = 0.0007$	0.25266 (0.19351)	0.1283 (0.17251)	0.06167 (0.15965)	0.02923 (0.1547)	-0.00372 (0.14992)
$\gamma = 0.0008$	0.25264 (0.20224)	0.12738 (0.18232)	0.06094 (0.16854)	0.02888 (0.16226)	-0.00439 (0.15739)
$\gamma = 0.0009$	0.25585 (0.21449)	0.12967 (0.19689)	0.06151 (0.18383)	0.03024 (0.17738)	-0.00301 (0.17243)
$\gamma = 0.001$	0.26143 (0.23328)	0.1334 (0.21441)	0.06573 (0.20079)	0.03466 (0.19461)	0.00185 (0.19008)
$\gamma = 0.0011$	0.27249 (0.25094)	0.14132 (0.23214)	0.07217 (0.21714)	0.03931 (0.21206)	0.00578 (0.20772)
$\gamma = 0.0012$	0.27315 (0.26626)	0.13929 (0.24894)	0.06914 (0.23637)	0.03632 (0.23216)	0.00277 (0.22831)
$\gamma = 0.0013$	0.28185 (0.28598)	0.14565 (0.26679)	0.07476 (0.25463)	0.03946 (0.25004)	0.0047 (0.24652)
$\gamma = 0.0014$	0.2873 (0.2944)	0.14731 (0.27532)	0.075 (0.26263)	0.03795 (0.2587)	0.00335 (0.25494)
$\gamma = 0.0015$	0.29485 (0.31021)	0.15212 (0.28986)	0.07758 (0.27904)	0.04019 (0.27455)	0.00493 (0.27084)
$\gamma = 0.0016$	0.30119 (0.33377)	0.1561 (0.31382)	0.07858 (0.30291)	0.04061 (0.29798)	0.00533 (0.29326)
$\gamma = 0.0017$	0.30672 (0.36446)	0.16103 (0.34259)	0.08171 (0.33336)	0.04357 (0.32874)	0.00604 (0.32376)
$\gamma = 0.0018$	0.30223 (0.38902)	0.15513 (0.36763)	0.07299 (0.35663)	0.03403 (0.35204)	-0.00309 (0.34704)
$\gamma = 0.0019$	0.3104 (0.41281)	0.15946 (0.39139)	0.07559 (0.3788)	0.03473 (0.37489)	-0.00313 (0.36996)
$\gamma = 0.0020$	0.32352 (0.44138)	0.16852 (0.42088)	0.08348 (0.41102)	0.04224 (0.40584)	0.00302 (0.4011)

Figure B.19: *Estimation of the relative integrated volatility by robust estimators in model* (B.1)-(B.3) *with additive noise, Heston.*

1.4.4 Estimation of γ

	Robust $j_0 = 7$	Robust $j_0 = 6$	Robust $j_0 = 5$	Robust $j_0 = 4$	Robust B-S
$\gamma = 0$	0.00010015 (0.00011748)	$9.628e-05$ (0.00011161)	$9.342e-05$ (0.00010735)	$9.138e-05$ (0.00010442)	$8.971e-05$ (0.00010178)
$\gamma = 0.0001$	0.00011953 (0.0001267)	0.00011362 (0.00011949)	0.00011095 (0.00011472)	0.00010997 (0.00011024)	0.0001078 (0.00010863)
$\gamma = 0.0002$	0.00019296 (0.00013869)	0.00018622 (0.00012881)	0.00018272 (0.00012301)	0.00018018 (0.00012011)	0.00017826 (0.00011754)
$\gamma = 0.0003$	0.0003056 (0.00012403)	0.0002958 (0.00011227)	0.00028922 (0.00010833)	0.00028679 (0.0001034)	0.00028433 (0.00010034)
$\gamma = 0.0004$	0.00042249 ($9.978e-05$)	0.00040768 ($8.984e-05$)	0.00039882 ($8.593e-05$)	0.00039376 ($8.306e-05$)	0.00038973 ($8.1e-05$)
$\gamma = 0.0005$	0.00053447 ($8.981e-05$)	0.00051468 ($8.333e-05$)	0.00050372 ($8.051e-05$)	0.00049743 ($7.813e-05$)	0.00049241 ($7.533e-05$)
$\gamma = 0.0006$	0.00064118 ($8.218e-05$)	0.00061758 ($7.849e-05$)	0.00060514 ($7.631e-05$)	0.00059791 ($7.54e-05$)	0.00059211 ($7.416e-05$)
$\gamma = 0.0007$	0.00074504 ($8.358e-05$)	0.00071921 ($8.117e-05$)	0.00070569 ($7.876e-05$)	0.00069799 ($7.811e-05$)	0.00069189 ($7.726e-05$)
$\gamma = 0.0008$	0.00084864 ($8.439e-05$)	0.00082139 ($8.361e-05$)	0.00080664 ($8.205e-05$)	0.00079827 ($8.156e-05$)	0.00079167 ($8.108e-05$)
$\gamma = 0.0009$	0.00095013 ($8.782e-05$)	0.00092143 ($8.867e-05$)	0.00090634 ($8.783e-05$)	0.00089734 ($8.785e-05$)	0.00089026 ($8.781e-05$)
$\gamma = 0.001$	0.00104934 ($9.324e-05$)	0.00101989 ($9.461e-05$)	0.00100416 ($9.398e-05$)	0.00099509 ($9.415e-05$)	0.00098758 ($9.429e-05$)
$\gamma = 0.0011$	0.00114736 ($9.779e-05$)	0.00111777 ($9.988e-05$)	0.00110205 ($9.956e-05$)	0.00109327 ($9.978e-05$)	0.00108561 (0.00010012)
$\gamma = 0.0012$	0.00124689 (0.00010156)	0.00121735 (0.00010491)	0.00120155 (0.00010515)	0.0011927 (0.00010539)	0.00118493 (0.00010624)
$\gamma = 0.0013$	0.00134465 (0.00010709)	0.00131541 (0.00011043)	0.00129947 (0.00011101)	0.00129129 (0.00011133)	0.00128374 (0.00011244)
$\gamma = 0.0014$	0.00144238 (0.00011293)	0.00141378 (0.00011664)	0.00139821 (0.00011743)	0.00139032 (0.00011811)	0.00138292 (0.00011913)
$\gamma = 0.0015$	0.001539 (0.00012001)	0.00151105 (0.00012364)	0.00149608 (0.00012473)	0.0014884 (0.00012527)	0.0014812 (0.00012635)
$\gamma = 0.0016$	0.00163514 (0.00012692)	0.00160816 (0.000131)	0.00159386 (0.00013257)	0.0015865 (0.00013324)	0.00157944 (0.00013438)
$\gamma = 0.0017$	0.00173239 (0.00013476)	0.00170607 (0.00013896)	0.00169236 (0.00014074)	0.00168518 (0.00014139)	0.00167862 (0.00014259)
$\gamma = 0.0018$	0.00182978 (0.00014086)	0.00180444 (0.00014536)	0.00179169 (0.0001471)	0.00178491 (0.00014784)	0.00177857 (0.00014891)
$\gamma = 0.0019$	0.00192569 (0.00014705)	0.00190179 (0.00015159)	0.00188967 (0.00015345)	0.00188346 (0.00015432)	0.00187751 (0.00015549)
$\gamma = 0.0020$	0.00201967 (0.00015286)	0.00199736 (0.0001575)	0.00198586 (0.00015952)	0.00198007 (0.00016036)	0.00197458 (0.00016148)

Figure B.20: *Estimation of the standard deviation of the noise in model* (B.1)-(B.3), *Heston.*

1.5 Model (B.1)-(B.3): correlated Heston

1.5.1 Pure Rounding: $\gamma = 0$

	Relative bias	Relative standard deviation	Relative MSE
Realized volatility	7.34336	0.70561	7.37712
Oracle ZMA	1.03693	0.15396	1.04827
ZMA	0.99427	0.18074	1.01054
$j_0 = 7$	0.27058	0.12081	0.29628
$j_0 = 6$	0.14258	0.10865	0.1792
$j_0 = 5$	0.07657	0.10246	0.12782
$j_0 = 4$	0.04212	0.09956	0.10801
Black-Scholes	0.01018	0.09575	0.0962
Robust $j_0 = 7$	0.26687	0.14753	0.30487
Robust $j_0 = 6$	0.1359	0.13115	0.18877
Robust $j_0 = 5$	0.06895	0.12267	0.14061
Robust $j_0 = 4$	0.03532	0.11922	0.12423
Robust Black-Scholes	0.00353	0.114	0.11394
$j_1 = 6$, $j_2 = 6$	0.38359	0.13575	0.40686
$j_1 = 6$, $j_2 = 5$	0.20459	0.1171	0.23568
$j_1 = 6$, $j_2 = 4$	0.15874	0.11081	0.19353
$j_1 = 6$, $j_2 = 3$	0.14652	0.10933	0.18275
$j_1 = 5$, $j_2 = 5$	0.20472	0.11712	0.2358
$j_1 = 5$, $j_2 = 4$	0.10996	0.10699	0.15335
$j_1 = 5$, $j_2 = 3$	0.0847	0.10387	0.13395
$j_1 = 4$, $j_2 = 4$	0.10999	0.107	0.15338
$j_1 = 4$, $j_2 = 3$	0.05865	0.10252	0.11802

Figure B.21: *Estimation of the relative integrated volatility in model* (B.1)-(B.3), *case* $\gamma = 0$, *correlated Heston.*

1.5.2 Additive noise: non robust estimation

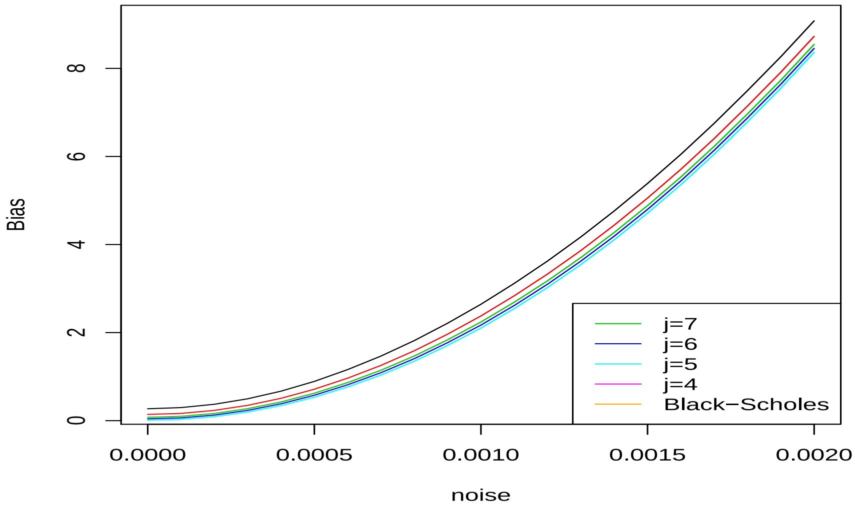

Figure B.22: *Bias of the non robust estimators in model* (B.1)-(B.3)*, correlated Heston.*

1.5.3 Additive noise: robust estimation

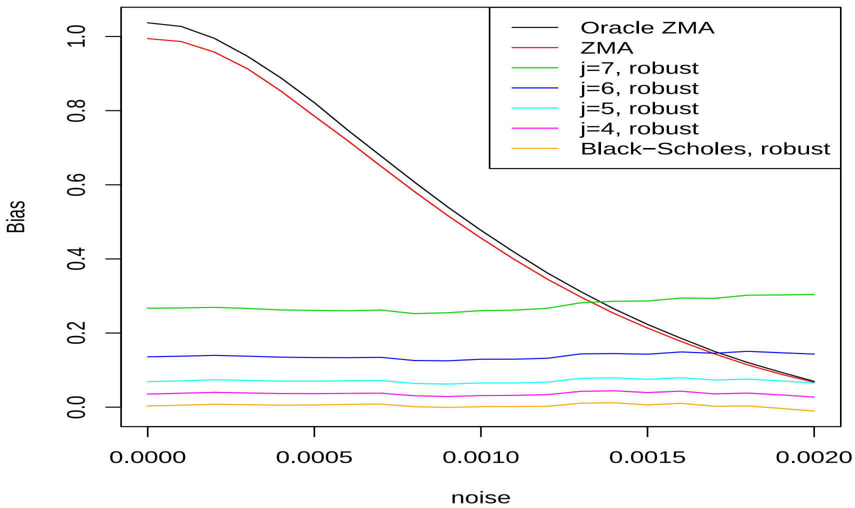

Figure B.23: *Bias of the non robust estimators in model* (B.1)-(B.3)*, correlated Heston.*

	Robust $j_0 = 7$	Robust $j_0 = 6$	Robust $j_0 = 5$	Robust $j_0 = 4$	Robust B-S
$\gamma = 0$	0.26687 (0.14753)	0.1359 (0.13115)	0.06895 (0.12267)	0.03532 (0.11922)	0.00353 (0.114)
$\gamma = 0.0001$	0.26778 (0.14849)	0.13758 (0.13154)	0.07106 (0.12254)	0.03762 (0.11894)	0.00547 (0.11386)
$\gamma = 0.0002$	0.26924 (0.14816)	0.13975 (0.1325)	0.07397 (0.12346)	0.04008 (0.12025)	0.00797 (0.11588)
$\gamma = 0.0003$	0.26642 (0.15204)	0.1376 (0.13542)	0.07208 (0.12623)	0.03833 (0.123)	0.00682 (0.11828)
$\gamma = 0.0004$	0.2624 (0.15954)	0.13495 (0.1425)	0.07042 (0.13293)	0.03684 (0.12983)	0.00547 (0.12527)
$\gamma = 0.0005$	0.2611 (0.16976)	0.1339 (0.14947)	0.07026 (0.13904)	0.03655 (0.13556)	0.00602 (0.13179)
$\gamma = 0.0006$	0.26018 (0.17888)	0.13359 (0.15763)	0.07127 (0.14692)	0.03731 (0.14253)	0.00758 (0.13725)
$\gamma = 0.0007$	0.26198 (0.19095)	0.13446 (0.16979)	0.07183 (0.15946)	0.03775 (0.15596)	0.00837 (0.14991)
$\gamma = 0.0008$	0.25243 (0.20764)	0.12588 (0.18342)	0.06398 (0.17238)	0.03086 (0.168)	0.00134 (0.16176)
$\gamma = 0.0009$	0.25472 (0.21683)	0.12513 (0.19255)	0.06245 (0.18147)	0.02878 (0.17633)	−0.00073 (0.17158)
$\gamma = 0.001$	0.2605 (0.23345)	0.12931 (0.20877)	0.06536 (0.19745)	0.03131 (0.1913)	0.00146 (0.18657)
$\gamma = 0.0011$	0.26177 (0.24211)	0.12949 (0.21653)	0.06529 (0.20507)	0.03181 (0.19839)	0.00161 (0.19293)
$\gamma = 0.0012$	0.26686 (0.26773)	0.13204 (0.2438)	0.0676 (0.23194)	0.03396 (0.22628)	0.00235 (0.21987)
$\gamma = 0.0013$	0.28167 (0.2837)	0.14386 (0.25914)	0.07783 (0.24732)	0.04271 (0.24085)	0.01075 (0.23485)
$\gamma = 0.0014$	0.28581 (0.3013)	0.14462 (0.27878)	0.07909 (0.26798)	0.04418 (0.26243)	0.01202 (0.25565)
$\gamma = 0.0015$	0.28648 (0.31846)	0.14288 (0.29558)	0.07529 (0.28672)	0.03945 (0.28213)	0.00633 (0.27642)
$\gamma = 0.0016$	0.29443 (0.33577)	0.14903 (0.31398)	0.07938 (0.30442)	0.0432 (0.29809)	0.01046 (0.29318)
$\gamma = 0.0017$	0.2936 (0.36132)	0.14551 (0.33765)	0.07339 (0.32923)	0.03594 (0.32257)	0.00256 (0.31719)
$\gamma = 0.0018$	0.30209 (0.387)	0.15074 (0.36376)	0.07567 (0.35493)	0.03799 (0.34818)	0.00317 (0.34303)
$\gamma = 0.0019$	0.3028 (0.40523)	0.14686 (0.38328)	0.07108 (0.37497)	0.03316 (0.36794)	−0.00315 (0.36277)
$\gamma = 0.0020$	0.30428 (0.43331)	0.14331 (0.40891)	0.0652 (0.40048)	0.02727 (0.39408)	−0.01052 (0.38755)

Figure B.24: *Estimation of the relative integrated volatility by robust estimators in model* (B.1)-(B.3) *with additive noise, correlated Heston.*

1.5.4 Estimation of γ

	Robust $j_0 = 7$	Robust $j_0 = 6$	Robust $j_0 = 5$	Robust $j_0 = 4$	Robust B-S
$\gamma = 0$	0.0001013 (0.0001182)	$9.774e-05$ (0.0001136)	$9.512e-05$ (0.00011038)	$9.422e-05$ (0.00010699)	$9.083e-05$ (0.0001054)
$\gamma = 0.0001$	0.00012634 (0.00012961)	0.00012168 (0.00012269)	0.00011792 (0.00011805)	0.00011641 (0.00011409)	0.00011395 (0.00011175)
$\gamma = 0.0002$	0.00019537 (0.00013746)	0.0001853 (0.00013242)	0.00018146 (0.00012674)	0.00017896 (0.00012486)	0.00017824 (0.00011997)
$\gamma = 0.0003$	0.00030774 (0.00012432)	0.00029439 (0.00011814)	0.00028861 (0.00011173)	0.00028447 (0.00011134)	0.0002813 (0.00010826)
$\gamma = 0.0004$	0.00042568 (0.00010223)	0.00040805 ($9.609e-05$)	0.00039895 ($9.189e-05$)	0.00039426 ($9.046e-05$)	0.00038983 ($8.783e-05$)
$\gamma = 0.0005$	0.00053766 ($8.595e-05$)	0.00051654 ($8.153e-05$)	0.00050483 ($7.904e-05$)	0.00049913 ($7.825e-05$)	0.00049327 ($7.666e-05$)
$\gamma = 0.0006$	0.0006449 ($8.655e-05$)	0.00062057 ($8.4e-05$)	0.0006068 ($8.178e-05$)	0.00060019 ($8.153e-05$)	0.00059314 ($8.001e-05$)
$\gamma = 0.0007$	0.00074797 ($8.663e-05$)	0.00072175 ($8.503e-05$)	0.00070671 ($8.369e-05$)	0.00069943 ($8.419e-05$)	0.00069201 ($8.302e-05$)
$\gamma = 0.0008$	0.00085375 ($8.955e-05$)	0.00082538 ($8.823e-05$)	0.0008094 ($8.75e-05$)	0.00080131 ($8.839e-05$)	0.00079348 ($8.755e-05$)
$\gamma = 0.0009$	0.00095534 ($9.193e-05$)	0.00092648 ($9.107e-05$)	0.00091014 ($9.097e-05$)	0.00090194 ($9.15e-05$)	0.00089371 ($9.139e-05$)
$\gamma = 0.001$	0.00105418 ($9.396e-05$)	0.00102465 ($9.322e-05$)	0.00100806 ($9.326e-05$)	0.00099966 ($9.362e-05$)	0.00099141 ($9.402e-05$)
$\gamma = 0.0011$	0.0011544 ($9.791e-05$)	0.00112449 ($9.776e-05$)	0.0011077 ($9.805e-05$)	0.00109906 ($9.865e-05$)	0.00109083 ($9.897e-05$)
$\gamma = 0.0012$	0.00125219 (0.00010452)	0.00122257 (0.00010464)	0.00120576 (0.00010545)	0.00119729 (0.00010626)	0.00118935 (0.00010681)
$\gamma = 0.0013$	0.00134817 (0.00010851)	0.00131881 (0.00010909)	0.00130238 (0.00011015)	0.00129426 (0.00011125)	0.00128639 (0.00011193)
$\gamma = 0.0014$	0.00144617 (0.00011326)	0.0014173 (0.00011396)	0.00140085 (0.00011561)	0.00139267 (0.00011681)	0.00138489 (0.00011753)
$\gamma = 0.0015$	0.00154376 (0.00011867)	0.00151567 (0.00011958)	0.00149993 (0.00012157)	0.00149202 (0.00012273)	0.00148455 (0.00012378)
$\gamma = 0.0016$	0.00163986 (0.00012475)	0.00161259 (0.00012595)	0.00159753 (0.00012785)	0.00158994 (0.00012893)	0.00158247 (0.00013016)
$\gamma = 0.0017$	0.00173776 (0.00013183)	0.00171161 (0.00013346)	0.00169743 (0.00013544)	0.00169028 (0.00013663)	0.00168313 (0.00013795)
$\gamma = 0.0018$	0.0018342 (0.00014043)	0.00180937 (0.00014227)	0.00179611 (0.00014393)	0.00178917 (0.000145)	0.00178243 (0.00014628)
$\gamma = 0.0019$	0.00192998 (0.000147)	0.00190675 (0.00014918)	0.00189406 (0.00015108)	0.00188741 (0.00015201)	0.00188129 (0.0001535)
$\gamma = 0.0020$	0.00202642 (0.00015397)	0.00200484 (0.00015623)	0.00199301 (0.0001581)	0.00198665 (0.00015907)	0.00198102 (0.00016038)

Figure B.25: *Estimation of the standard deviation of the noise in model* (B.1)-(B.3), *correlated Heston.*

2 Other equity data

2.1 Danone

- Tick first week=0.1.
- Tick second week=0.01.
- Volume on the two weeks=11689473.

2.1.1 Volatility estimation with the last traded price

	Day 1	Day 2	Day 3	Day 4	Day 5
ZMA	0.00011063	0.00011574	0.00020393	0.00026631	0.00016592
Garman-Klass	1.061e-05	1.145e-05	3.491e-05	7.916e-05	9.99e-06
j^*	2.62e-06	4.01e-06	6.73e-06	2.07e-05	8.4e-06
j^*+1	2.9e-06	4.14e-06	7.3e-06	2.192e-05	1.008e-05
Black-Scholes	2.45e-06	3.41e-06	5.95e-06	1.956e-05	6.62e-06
Robust j^*	7.4e-06	1.056e-05	1.802e-05	5.425e-05	2.284e-05
Robust j^*+1	8.21e-06	1.09e-05	1.987e-05	5.707e-05	2.666e-05
Robust Black-Scholes	6.97e-06	8.88e-06	1.605e-05	5.151e-05	1.867e-05

Figure B.26: *Estimation of the integrated volatility with the last traded price for Danone, first week.*

	Day 6	Day 7	Day 8	Day 9	Day 10
ZMA	0	4.504e-05	0	0	8.06e-05
Garman-Klass	5.738e-05	6.384e-05	5.378e-05	0.00010272	3.74e-05
j^*	1.418e-05	5.38e-06	5.58e-06	3.08e-06	3.6e-06
j^*+1	1.53e-05	6.13e-06	6.67e-06	3.78e-06	4.78e-06
Black-Scholes	7.71e-06	3.6e-06	3.15e-06	1.98e-06	2.22e-06
Robust j^*	3.619e-05	1.403e-05	1.436e-05	8.5e-06	1.041e-05
Robust j^*+1	3.912e-05	1.605e-05	1.699e-05	1.056e-05	1.392e-05
Robust Black-Scholes	2.035e-05	9.56e-06	8.2e-06	5.47e-06	6.25e-06

Figure B.27: *Estimation of the integrated volatility with the last traded price for Danone, second week.*

2.1.2 Volatility estimation with the bid price

	Day 1	Day 2	Day 3	Day 4	Day 5
ZMA	0	0	6.114e-05	9.694e-05	1.511e-05
Garman-Klass	1.063e-05	1.147e-05	3.491e-05	9.123e-05	1.374e-05
j^*	2.1e-07	8.8e-07	1.28e-06	5.59e-06	2.1e-07
j^*+1	2.9e-07	1.02e-06	1.43e-06	6.11e-06	2.7e-07
Black-Scholes	6e-08	4.6e-07	8.9e-07	4.32e-06	1.5e-07
Robust j^*	6.4e-07	2.01e-06	3.17e-06	1.5e-05	5.3e-07
Robust j^*+1	8.9e-07	2.19e-06	3.84e-06	1.61e-05	6.5e-07
Robust Black-Scholes	2e-07	1.13e-06	2.32e-06	1.152e-05	3.7e-07

Figure B.28: *Estimation of the integrated volatility with the bid price for Danone, first week.*

	Day 6	Day 7	Day 8	Day 9	Day 10
ZMA	0	0	0	0	0
Garman-Klass	6.151e-05	4.622e-05	4.895e-05	9.159e-05	4e-05
j^*	4.79e-06	1.74e-06	2.88e-06	2.18e-06	1.64e-06
j^*+1	5.47e-06	2.18e-06	3.82e-06	3.07e-06	2.06e-06
Black-Scholes	2.7e-06	1.28e-06	1.17e-06	1.13e-06	9.4e-07
Robust j^*	1.253e-05	4.93e-06	5.83e-06	6.52e-06	4.53e-06
Robust j^*+1	1.423e-05	6.16e-06	6.96e-06	9.16e-06	5.66e-06
Robust Black-Scholes	7.18e-06	3.62e-06	3.09e-06	3.36e-06	2.61e-06

Figure B.29: *Estimation of the integrated volatility with the bid price for Danone, second week.*

2.2 Total

- Tick first week=0.05.
- Tick second week=0.01.
- Volume on the two weeks= 77985275.

2.2.1 Volatility estimation with the last traded price

	Day 1	Day 2	Day 3	Day 4	Day 5
ZMA	0.00024142	0.00023181	0.00031501	0.00036887	0.00030766
Garman-Klass	1e-05	1.114e-05	1.028e-05	7.237e-05	9.18e-06
j^*	1.794e-05	2.249e-05	4.032e-05	3.956e-05	3.785e-05
j^*+1	1.906e-05	2.419e-05	4.133e-05	4.113e-05	4.023e-05
Black-Scholes	1.687e-05	1.989e-05	3.62e-05	3.736e-05	3.297e-05
Robust j^*	4.686e-05	5.938e-05	0.00010019	9.195e-05	8.609e-05
Robust j^*+1	4.979e-05	6.335e-05	0.00010299	9.639e-05	8.96e-05
Robust Black-Scholes	4.418e-05	5.265e-05	9.032e-05	8.867e-05	7.767e-05

Figure B.30: *Estimation of the integrated volatility with the last traded price for Total, first week.*

	Day 6	Day 7	Day 8	Day 9	Day 10
ZMA	0	3.53e-05	6.647e-05	4.982e-05	3.939e-05
Garman-Klass	2.883e-05	1.752e-05	1.866e-05	2.946e-05	3.986e-05
j^*	9.82e-06	5.71e-06	1.366e-05	7.44e-06	6.73e-06
j^*+1	1.112e-05	6.49e-06	1.543e-05	8.07e-06	7.58e-06
Black-Scholes	5.73e-06	4.56e-06	6.69e-06	5.74e-06	5.04e-06
Robust j^*	2.313e-05	1.432e-05	2.67e-05	1.901e-05	1.592e-05
Robust j^*+1	2.624e-05	1.628e-05	2.992e-05	2.051e-05	1.784e-05
Robust Black-Scholes	1.383e-05	1.144e-05	1.526e-05	1.482e-05	1.22e-05

Figure B.31: *Estimation of the integrated volatility with the last traded price for Total, second week.*

2.2.2 Volatility estimation with the bid price

	Day 1	Day 2	Day 3	Day 4	Day 5
ZMA	3.724e-05	6.047e-05	3.445e-05	4.58e-05	3.636e-05
Garman-Klass	5.9e-06	7e-06	7.86e-06	6.254e-05	7.25e-06
j^*	5.9e-07	8.9e-07	8.4e-07	1.23e-06	6.7e-07
j^*+1	7.3e-07	1.27e-06	9.9e-07	1.57e-06	8e-07
Black-Scholes	3.6e-07	6.3e-07	3.5e-07	9.4e-07	4e-07
Robust j^*	1.75e-06	2.6e-06	2.49e-06	3.43e-06	1.97e-06
Robust j^*+1	2.16e-06	3.68e-06	2.91e-06	4.37e-06	2.34e-06
Robust Black-Scholes	1e-06	1.79e-06	9.9e-07	2.69e-06	1.12e-06

Figure B.32: *Estimation of the integrated volatility with the bid price for Total, first week.*

	Day 6	Day 7	Day 8	Day 9	Day 10
ZMA	NA	0	2.057e-05	0	2.094e-05
Garman-Klass	2.418e-05	1.88e-05	1.664e-05	2.758e-05	3.916e-05
j^*	2.41e-06	1.42e-06	3.25e-06	2.7e-06	3.67e-06
j^*+1	2.74e-06	1.66e-06	3.68e-06	3.19e-06	4.4e-06
Black-Scholes	1.06e-06	1.1e-06	1.25e-06	1.93e-06	2.11e-06
Robust j^*	6.95e-06	3.63e-06	8.81e-06	6.95e-06	8.43e-06
Robust j^*+1	7.92e-06	4.2e-06	9.89e-06	8.34e-06	1.03e-05
Robust Black-Scholes	2.97e-06	2.8e-06	3.33e-06	5.07e-06	5.11e-06

Figure B.33: *Estimation of the integrated volatility with the bid price for Total, second week.*

2.3 Gaz de France

- Tick first week=0.01.
- Tick second week=0.01.
- Volume on the two weeks=14309727.

2.3.1 Volatility estimation with the last traded price

	Day 1	Day 2	Day 3	Day 4	Day 5
ZMA	0.00021208	0.00021733	0.00016261	0.00031609	9.892e-05
Garman-Klass	1.988e-05	2.74e-05	2.72e-05	3.457e-05	2.513e-05
j^*	1.326e-05	3.086e-05	1.08e-05	3.06e-05	5.26e-06
j^*+1	1.5e-05	3.563e-05	1.317e-05	3.502e-05	6.56e-06
Black-Scholes	7.55e-06	1.448e-05	7.35e-06	1.912e-05	3.35e-06
Robust j^*	3.849e-05	7.647e-05	2.979e-05	8.446e-05	1.388e-05
Robust j^*+1	4.373e-05	8.958e-05	3.635e-05	9.583e-05	1.699e-05
Robust Black-Scholes	2.17e-05	3.752e-05	2.015e-05	5.206e-05	9.04e-06

Figure B.34: *Estimation of the integrated volatility with the last traded price for Gaz de France, first week.*

	Day 6	Day 7	Day 8	Day 9	Day 10
ZMA	8.587e-05	0.00012742	0.0001237	8.962e-05	0.00013441
Garman-Klass	2.761e-05	4.256e-05	8.085e-05	2.752e-05	2.564e-05
j^*	8.92e-06	1.048e-05	1.203e-05	4.51e-06	6.46e-06
j^*+1	1.082e-05	1.278e-05	1.416e-05	5.41e-06	7.73e-06
Black-Scholes	4.42e-06	6.25e-06	2.73e-06	2.63e-06	4.66e-06
Robust j^*	2.045e-05	2.954e-05	2.285e-05	1.187e-05	1.821e-05
Robust j^*+1	2.473e-05	3.592e-05	3.562e-05	1.444e-05	2.186e-05
Robust Black-Scholes	1.121e-05	1.727e-05	6.33e-06	7.12e-06	1.313e-05

Figure B.35: *Estimation of the integrated volatility with the last traded price for Gaz de France, second week.*

2.3.2 Volatility estimation with the bid price

	Day 1	Day 2	Day 3	Day 4	Day 5
ZMA	9.274e-05	8.379e-05	4.001e-05	0.00011642	3.387e-05
Garman-Klass	1.847e-05	2.347e-05	2.498e-05	3.457e-05	2.504e-05
j^*	4.86e-06	4.14e-06	2.43e-06	6.17e-06	1.54e-06
$j^* + 1$	6.28e-06	4.82e-06	3.31e-06	8.31e-06	1.87e-06
Black-Scholes	2.26e-06	1.71e-06	1.28e-06	3.56e-06	5.2e-07
Robust j^*	1.198e-05	1.235e-05	6.74e-06	1.741e-05	4.56e-06
Robust $j^* + 1$	1.553e-05	1.427e-05	9.13e-06	2.319e-05	5.49e-06
Robust Black-Scholes	5.68e-06	5.01e-06	3.59e-06	9.85e-06	1.54e-06

Figure B.36: *Estimation of the integrated volatility with the bid price for Gaz de France, first week.*

	Day 6	Day 7	Day 8	Day 9	Day 10
ZMA	0	4.117e-05	0.0001161	0	8.744e-05
Garman-Klass	2.845e-05	4e-05	7.656e-05	3.009e-05	2.364e-05
j^*	3.74e-06	2.33e-06	6.43e-06	5e-07	3.77e-06
$j^* + 1$	5.02e-06	3.81e-06	7.71e-06	8.3e-07	5.19e-06
Black-Scholes	1.35e-06	8.8e-07	6.2e-07	2.3e-07	1.85e-06
Robust j^*	1.023e-05	6.82e-06	1.904e-05	1.47e-06	9.99e-06
Robust $j^* + 1$	1.359e-05	1.111e-05	2.283e-05	2.44e-06	1.295e-05
Robust Black-Scholes	3.6e-06	2.56e-06	1.79e-06	6.6e-07	5.19e-06

Figure B.37: *Estimation of the integrated volatility with the bid price for Gaz de France, second week.*

2.4 Renault

- Tick first week=0.05.
- Tick second week=0.01.
- Volume on the two weeks=14309727. .

2.4.1 Volatility estimation with the last traded price

	Day 1	Day 2	Day 3	Day 4	Day 5
ZMA	7.776e-05	6.994e-05	0	6.701e-05	0.00011226
Garman-Klass	7.64e-06	8.77e-06	2.535e-05	2.811e-05	3.612e-05
j^*	3.06e-06	2.78e-06	1.79e-06	3.18e-06	8.72e-06
j^*+1	3.35e-06	3.14e-06	1.94e-06	3.7e-06	1.094e-05
Black-Scholes	2.78e-06	2.44e-06	1.47e-06	2.22e-06	5.59e-06
Robust j^*	8.46e-06	6.53e-06	4.87e-06	8.15e-06	1.85e-05
Robust j^*+1	9.24e-06	7.35e-06	5.3e-06	9.4e-06	2.187e-05
Robust Black-Scholes	7.64e-06	5.79e-06	4.12e-06	5.84e-06	1.344e-05

Figure B.38: *Estimation of the integrated volatility with the last traded price for Renault, first week.*

	Day 6	Day 7	Day 8	Day 9	Day 10
ZMA	6.87e-05	4.593e-05	0	3.082e-05	8.264e-05
Garman-Klass	3.148e-05	1.533e-05	1.402e-05	5.284e-05	2.458e-05
j^*	5.62e-06	2.89e-06	6.5e-07	1.65e-06	7.2e-06
j^*+1	6.71e-06	3.42e-06	8.4e-07	2.13e-06	9.3e-06
Black-Scholes	3.55e-06	1.82e-06	4.3e-07	1.12e-06	5.04e-06
Robust j^*	1.62e-05	7.74e-06	1.82e-06	4.52e-06	2.005e-05
Robust j^*+1	1.919e-05	9.14e-06	2.3e-06	5.75e-06	2.567e-05
Robust Black-Scholes	1.034e-05	4.8e-06	1.16e-06	2.99e-06	1.37e-05

Figure B.39: *Estimation of the integrated volatility with the last traded price for Renault, second week.*

2.4.2 Volatility estimation with the bid price

	Day 1	Day 2	Day 3	Day 4	Day 5
ZMA	1.974e-05	1.726e-05	0	3.635e-05	4.839e-05
Garman-Klass	9.44e-06	8.78e-06	2.15e-05	2.425e-05	3.204e-05
j^*	5.6e-07	3.2e-07	3.5e-07	1.36e-06	2.77e-06
$j^* + 1$	6.3e-07	4.8e-07	4.7e-07	1.54e-06	3.43e-06
Black-Scholes	4.2e-07	1.7e-07	2.4e-07	7e-07	1.82e-06
Robust j^*	1.59e-06	9.6e-07	9.8e-07	3.29e-06	6.49e-06
Robust $j^* + 1$	1.76e-06	1.46e-06	1.32e-06	3.79e-06	7.91e-06
Robust Black-Scholes	1.21e-06	5.2e-07	6.9e-07	1.77e-06	4.62e-06

Figure B.40: *Estimation of the integrated volatility with the bid price for Renault, first week.*

	Day 6	Day 7	Day 8	Day 9	Day 10
ZMA	4.925e-05	1.815e-05	0	2.297e-05	0
Garman-Klass	3.165e-05	1.567e-05	1.402e-05	5.157e-05	2.25e-05
j^*	3.65e-06	1.39e-06	5.1e-07	1.24e-06	3.36e-06
$j^* + 1$	4.68e-06	1.66e-06	7.2e-07	1.59e-06	4.14e-06
Black-Scholes	1.93e-06	7.3e-07	2.4e-07	6.4e-07	2.34e-06
Robust j^*	1.04e-05	3.86e-06	1.16e-06	3.57e-06	9.79e-06
Robust $j^* + 1$	1.33e-05	4.6e-06	1.51e-06	4.62e-06	1.206e-05
Robust Black-Scholes	5.41e-06	2.02e-06	6.1e-07	1.85e-06	6.78e-06

Figure B.41: *Estimation of the integrated volatility with the bid price for Renault, second week.*

Appendix C

Appendix of part III

1 Besov spaces

For the definition of Besov spaces, we refer to Cohen [28] and Triebel [108]. Let Δ_h^n be the operator defined by $\Delta_h^1 f(x) = f(x+h) - f(x)$ and $\Delta_h^n f(x) = \Delta_h^1(\Delta_h^{n-1})f(x)$. The n-th order L^p modulus of smoothness of f on $[0,1]$ is

$$\omega_n(f,t)_p = \sup_{|h|\leq t} \|\Delta_h^n f\|_{\mathrm{L}^p(\Omega_{h,n})},$$

where $\Omega_{h,n} = \{x \in [0,1]; x + kh \in [0,1], k = 0,\dots,n\}$. For $p,q \geq 1$, $s > 0$, the Besov space $\mathcal{B}_{p,q}^s([0,1])$ consists of those functions $f \in \mathrm{L}^p[0,1]$ such that

$$\{2^{sj}\omega_n(f,2^{-j})_p\}_{j\geq 0} \in l^q,$$

where $n \in \mathbb{N}$ and $s < n$. It is a Banach space when equipped with the norm

$$\|f\|_{\mathcal{B}_{p,q}^s([0,1])} = \|f\|_{\mathrm{L}^p} + \|\{2^{sj}\omega_n(f,2^{-j})_p\}_{j\geq 0}\|_{l^q}.$$

For p or q less than 1 and $s > \max\{1/p - 1, 0\}$, the Besov space $\mathcal{B}_{p,q}^s([0,1])$ can be defined the same way but is only a quasi-Banach space.

2 Besov spaces and Schauder basis

For $j \geq 0, k = 1,\dots,2^j$, let $\chi_{jk} = 2^{j/2}(1_{[(k-1)/2^j,(2k-1)/2^{j+1}]} - 1_{[(2k-1)/2^{j+1},k/2^j]})$, where 1. is the indicator function. The Schauder basis is defined as follows:

$$\phi_0(t) = 1_{[0,1]},\ \phi_1(t) = t1_{[0,1]},\ \phi_{jk}(t) = \int_0^t \chi_{jk}(s)\mathrm{d}s,\ s \in [0,1],$$

Any continuous function $f\colon [0,1] \to \mathbb{R}$ can be expressed, for all $t \in [0,1]$, as

$$f(t) = f_0\phi_0(t) + f_1\phi_1(t) + \sum_{j\geq 0}\sum_{k=1}^{2^j} f_{jk}\phi_{jk}(t),$$

where $f_0 = f(0)$, $f_1 = f(1) - f(0)$ and

$$f_{jk} = 2^{j/2+1}[-f(\{2k\}2^{-(j+1)}) + 2f(\{2k-1\}2^{-(j+1)}) - f(\{2k-2\}2^{-(j+1)})],$$

with convergence in uniform norm.

Proposition C.1 *(CKR) Let $0 < s < 1$, $1 \leq p, q \leq \infty$. If $s > 1/p$, then $\mathcal{B}^s_{p,q}([0,1])$ is a space of real continuous functions on $[0,1]$, isomorphic to a space of real sequences, with the following equivalence between the norms:*

$$\|f\|_{\mathcal{B}^s_{p,q}([0,1])} \sim \max\big\{|f_{-1}|, |f_0|, \big(\sum_{j\geq 0} 2^{-jq(1/2-s+1/p)}\big(\sum_{k=1}^{2^j}|f_{jk}|^p\big)^{q/p}\big)^{1/q}\big\},$$

where $f_0 = f(0)$, $f_1 = f(1) - f(0)$ and

$$f_{jk} = 2^{j/2+1}[-f(\{2k\}2^{-(j+1)}) + 2f(\{2k-1\}2^{-(j+1)}) - f(\{2k-2\}2^{-(j+1)})].$$

Appendix D

Appendix of part IV

1 Relative microstructure noise index, last traded prices

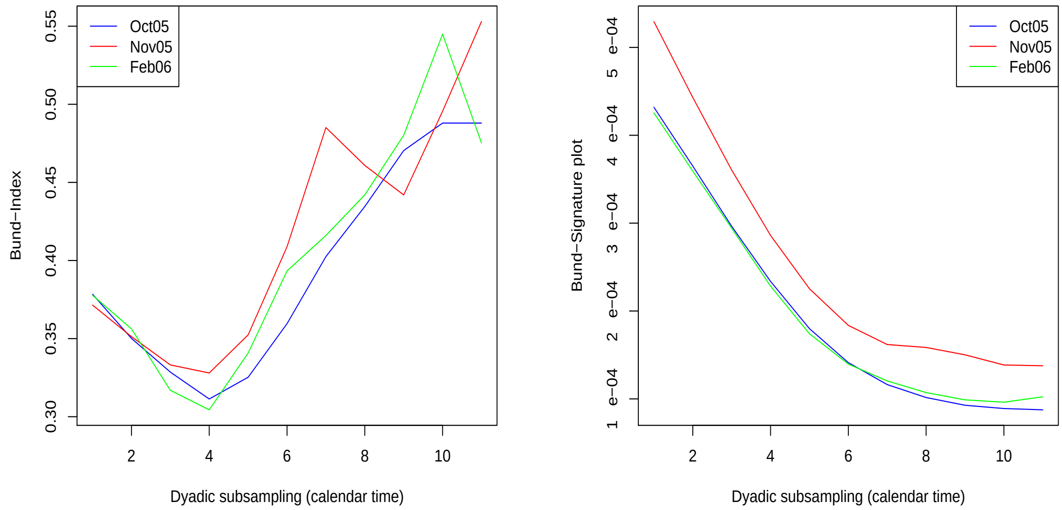

Figure D.1: *Relative microstructure noise index (left) and relative signature plot (right) for the Bund, Oct 05, Nov 05, Feb 06.*

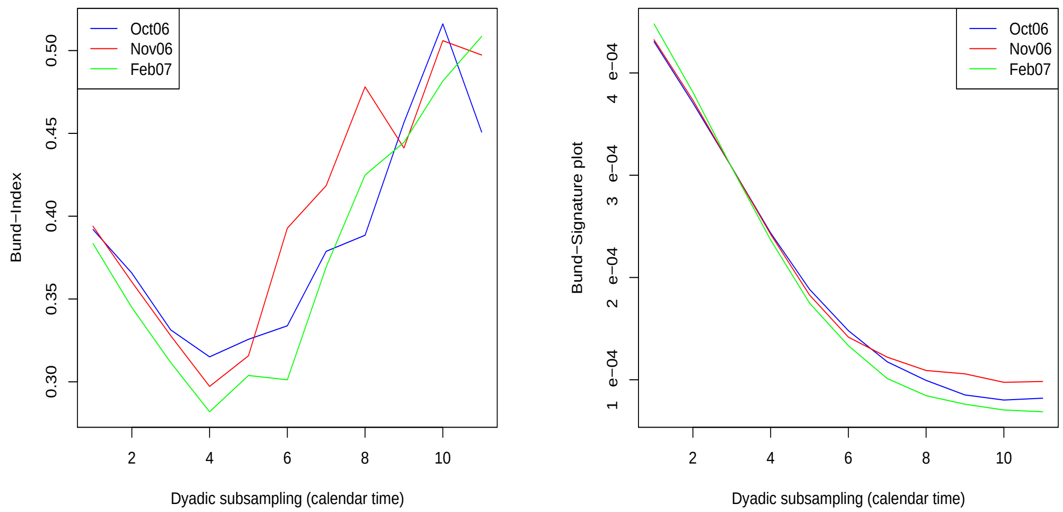

Figure D.2: *Relative microstructure noise index (left) and relative signature plot (right) for the Bund, Oct 06, Nov 06, Feb 07.*

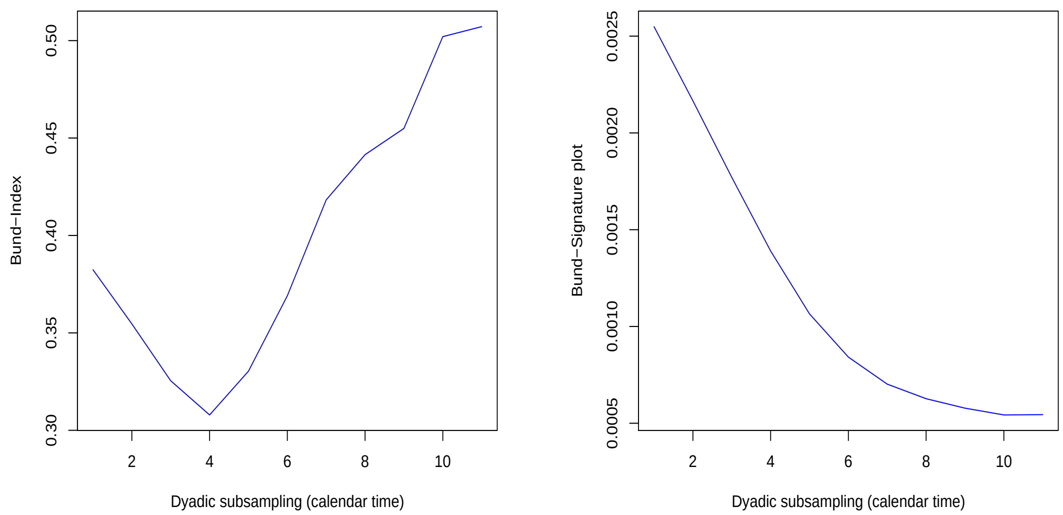

Figure D.3: *Relative microstructure noise index (left) and relative signature plot (right) for the Bund, aggregated data: Oct 05, Nov 05, Feb 06, Oct 06, Nov 06, Feb 07.*

2 Absolute microstructure noise index, bid prices

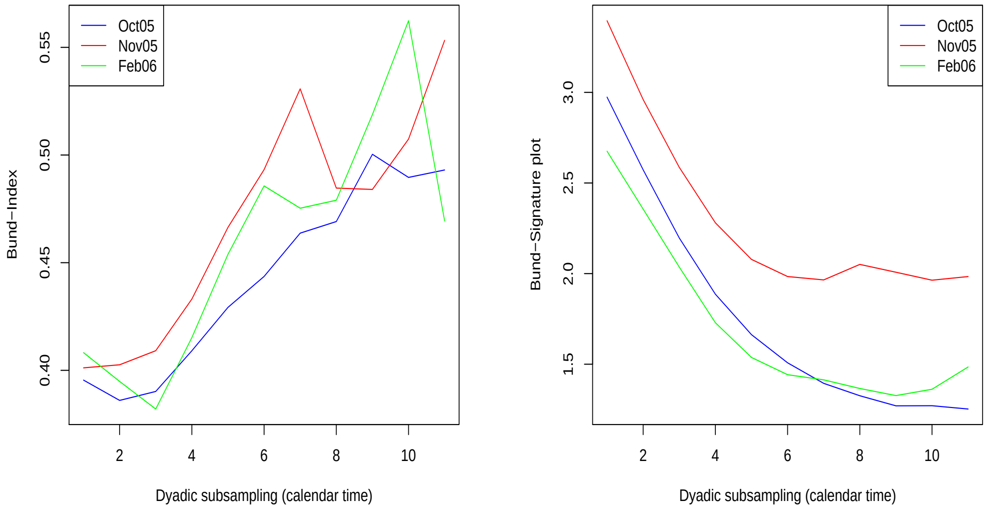

Figure D.4: *Absolute microstructure noise index (left) and absolute signature plot (right), for the Bund, bid price, Oct 05, Nov 05, Feb 06.*

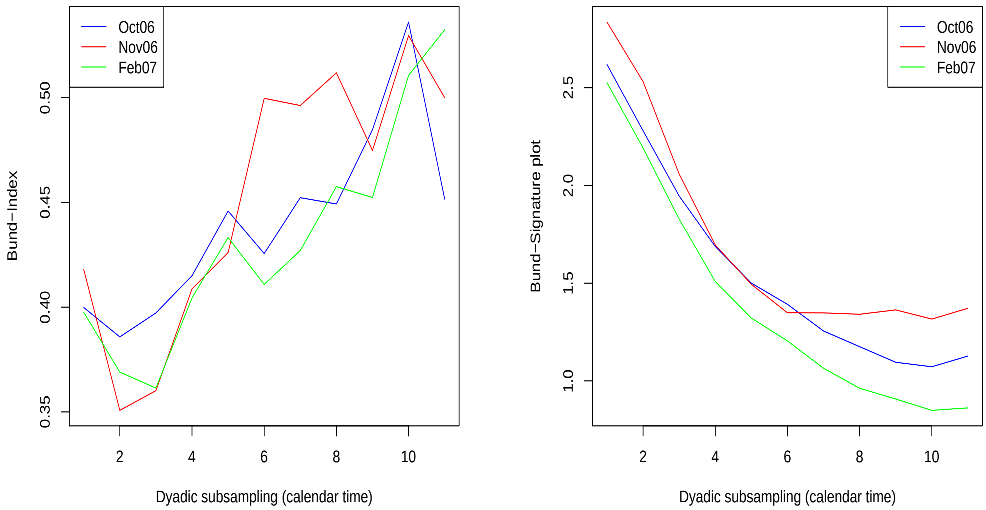

Figure D.5: *Absolute microstructure noise index (left) and absolute signature plot (right), for the Bund, bid price, Oct 06, Nov 06, Feb 07.*

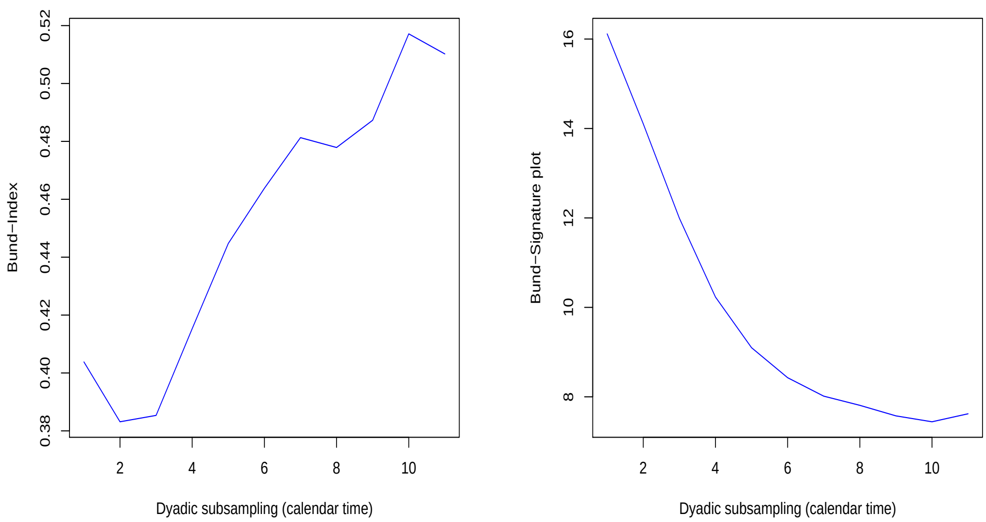

Figure D.6: *Absolute microstructure noise index (left) and absolute signature plot (right), for the Bund, bid price, aggregated data: Oct 05, Nov 05, Feb 06, Oct 06, Nov 06, Feb 07.*

3 Relative microstructure noise index, bid prices

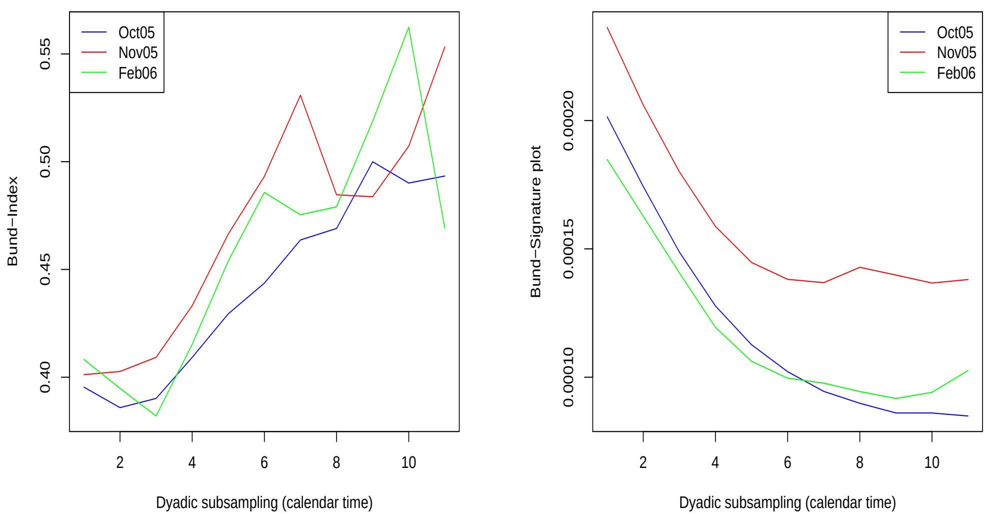

Figure D.7: *Relative microstructure noise index (left) and relative signature plot (right), for the Bund, bid price, Oct 05, Nov 05, Feb 06.*

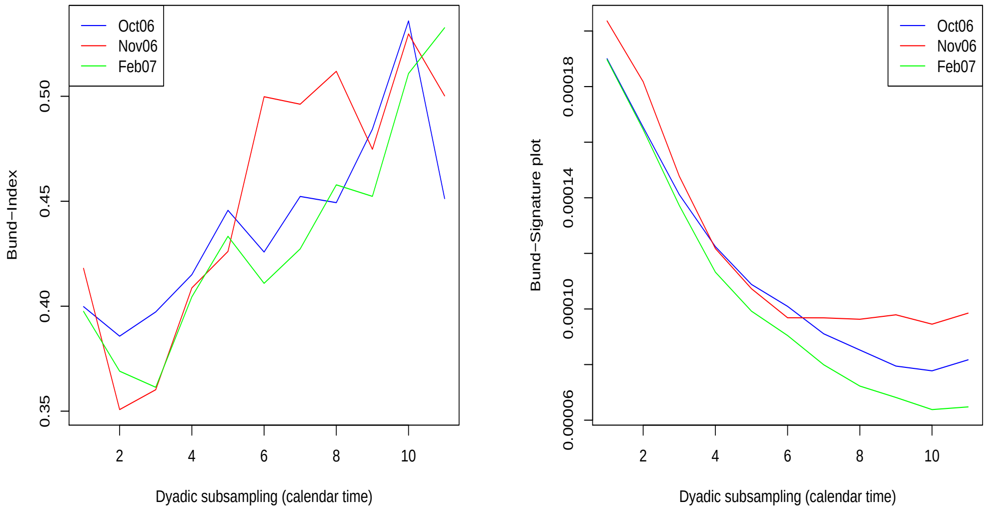

Figure D.8: *Relative microstructure noise index (left) and relative signature plot (right), for the Bund, bid price, Oct 06, Nov 06, Feb 07.*

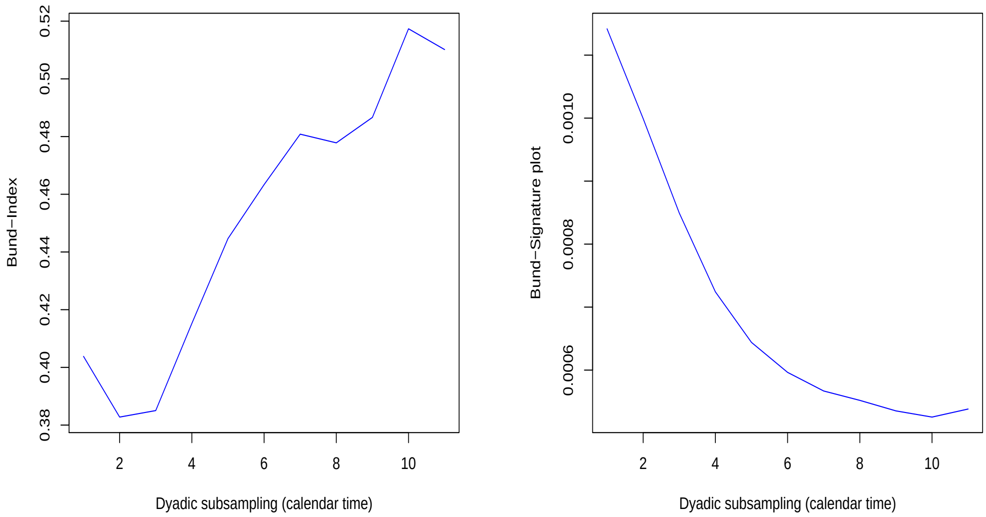

Figure D.9: *Relative microstructure noise index (left) and relative signature plot (right), for the Bund, bid price, aggregated data: Oct 05, Nov 05, Feb 06, Oct 06, Nov 06, Feb 07.*

4 Absolute microstructure noise index, mid prices

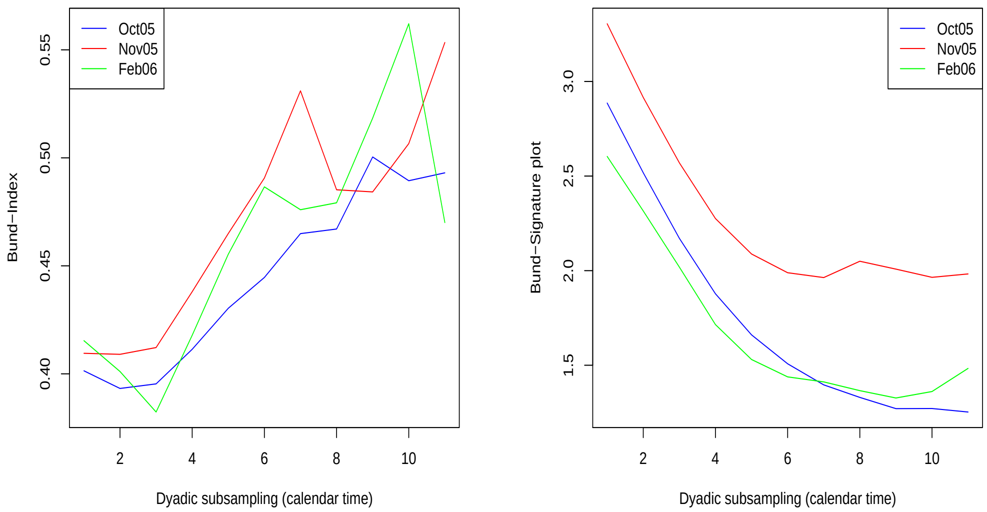

Figure D.10: *Absolute microstructure noise index (left) and absolute signature plot (right), for the Bund, bid price, Oct 05, Nov 05, Feb 06.*

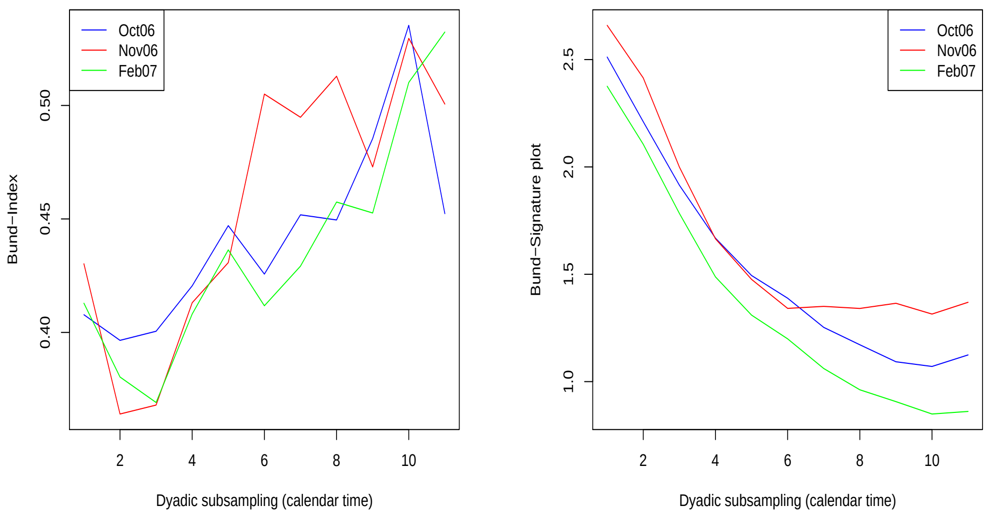

Figure D.11: *Absolute microstructure noise index (left) and absolute signature plot (right), for the Bund, bid price, Oct 06, Nov 06, Feb 07.*

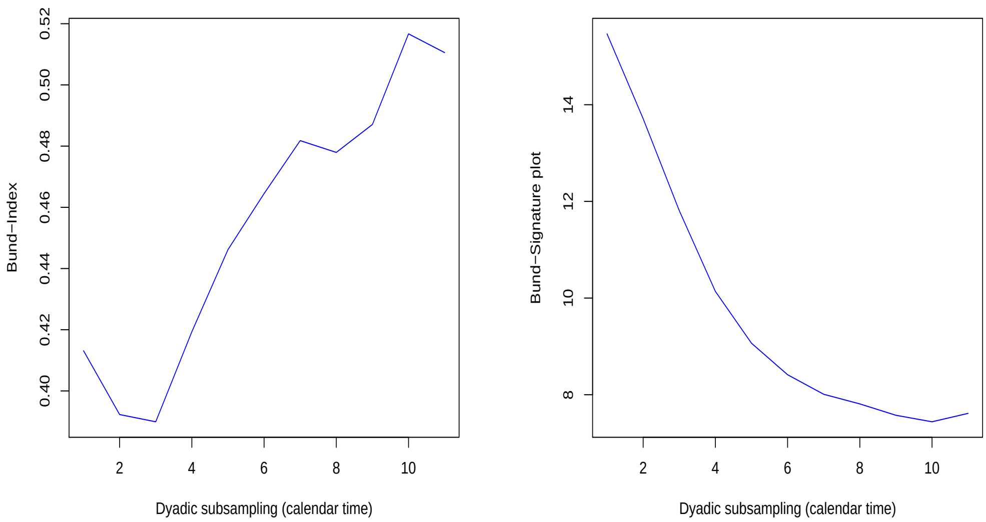

Figure D.12: *Absolute microstructure noise index (left) and absolute signature plot (right), for the Bund, bid price, aggregated data: Oct 05, Nov 05, Feb 06, Oct 06, Nov 06, Feb 07.*

5 Relative microstructure noise index, mid prices

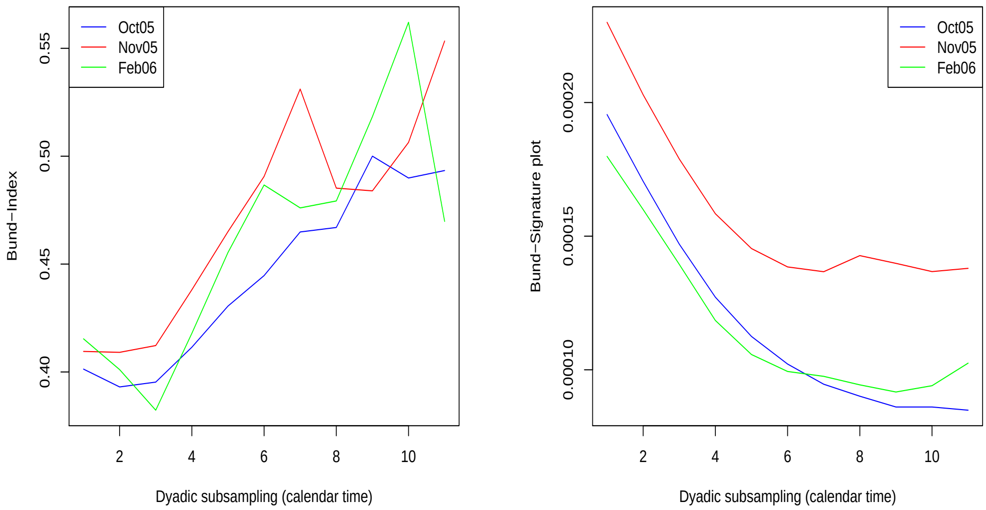

Figure D.13: *Relative microstructure noise index (left) and relative signature plot (right), for the Bund, bid price, Oct 05, Nov 05, Feb 06.*

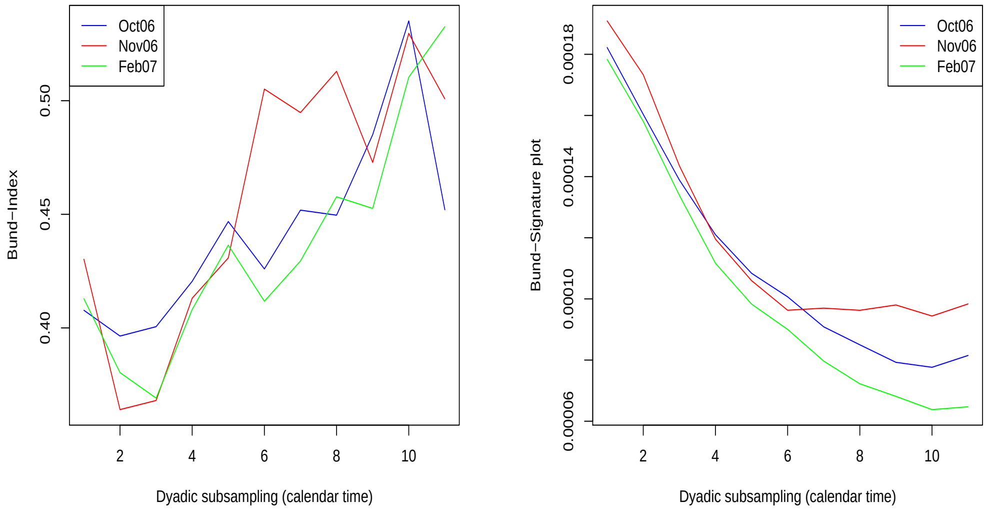

Figure D.14: *Relative microstructure noise index (left) and relative signature plot (right), for the Bund, bid price, Oct 06, Nov 06, Feb 07.*

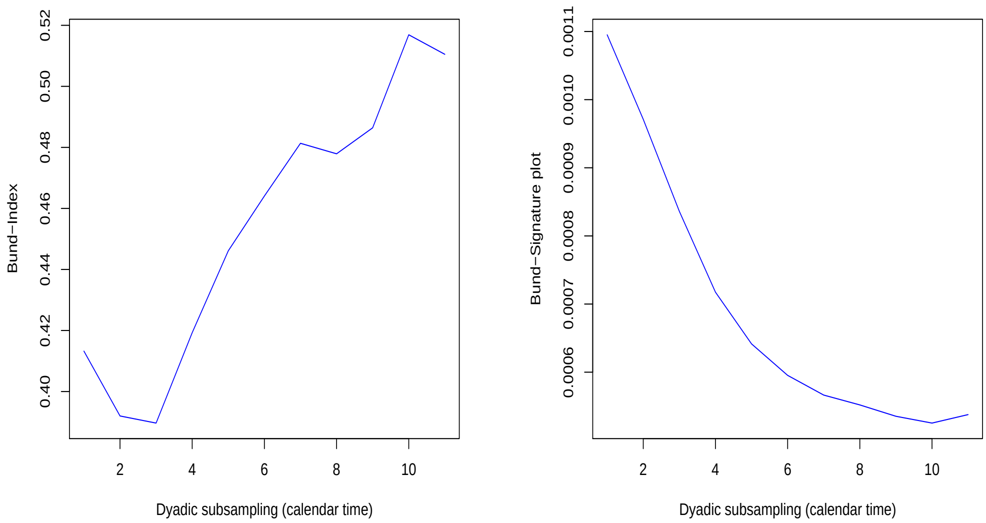

Figure D.15: *Relative microstructure noise index (left) and relative signature plot (right), for the Bund, bid price, aggregated data: Oct 05, Nov 05, Feb 06, Oct 06, Nov 06, Feb 07.*

6 The behavior of the 1-variation, bid prices

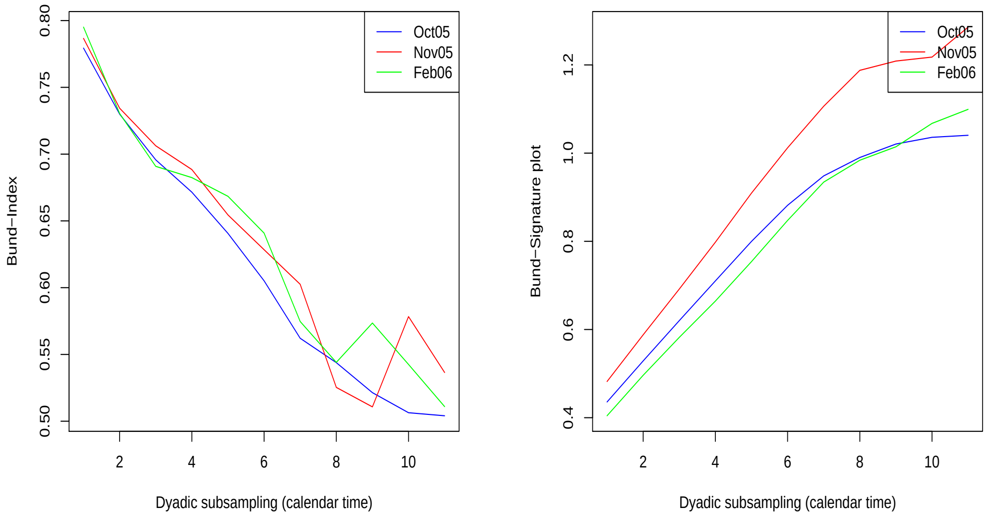

Figure D.16: *Microstructure noise index* S_q^1 *(left) and signature plot for* $p = 1$*(right), for the Bund, bid price, Oct 05, Nov 05, Feb 06.*

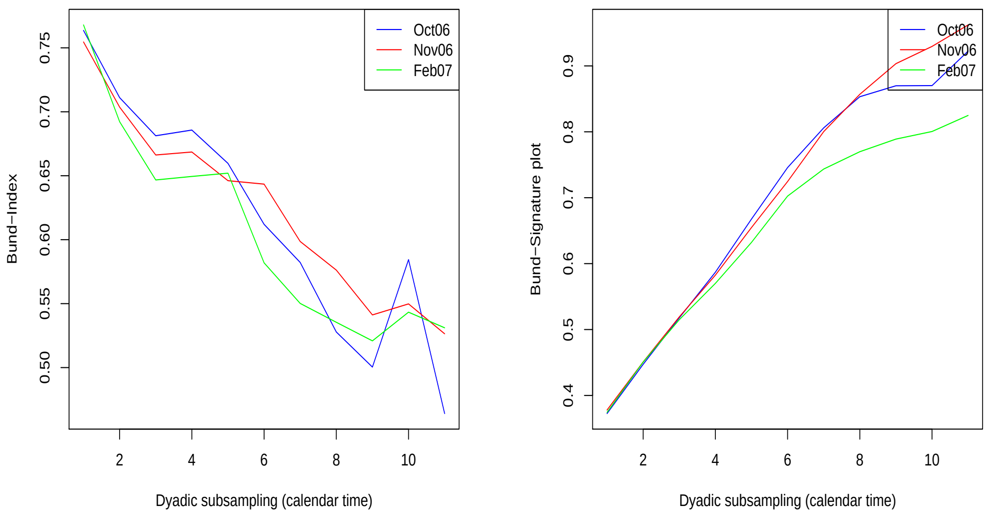

Figure D.17: *Microstructure noise index* S_q^1 *(left) and signature plot for* $p = 1$*(right), for the Bund, bid price, Oct 06, Nov 06, Feb 07.*

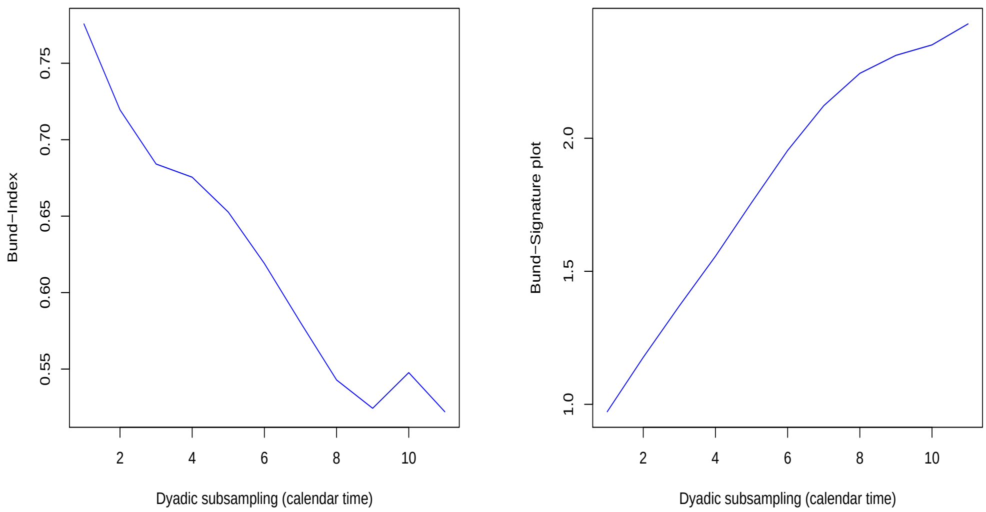

Figure D.18: *Microstructure noise index* S_q^1 *(left) and signature plot for* $p = 1$ *(right), for the Bund, bid price, aggregated data: Oct 05, Nov 05, Feb 06, Oct 06, Nov 06, Feb 07.*

7 Stochastic volatility model, numerical illustrations

We now consider the same specifications as in section 4 and section 5 but in the case where X comes from the GARCH(1,1) diffusion model studied by Andersen and Bollerslev and Meddahi:

$$\mathrm{d}\log X_t = \nu_t \mathrm{d}W_t,$$

$$\mathrm{d}\nu_t^2 = 0.035(0.636 - \nu_t^2)\mathrm{d}t + 0.144\nu_t^2 \mathrm{d}W_t^1,$$

with W_t and W_t^1 two independent Brownian motions. For each model, we compute 50 simulations with $n = 2^{19}$ and give the average absolute microstructure index and signature plot for V equal to

$$0, 2.10^{-4}, 4.10^{-4}, 6.10^{-4}, 8.10^{-4}, 10^{-3}.$$

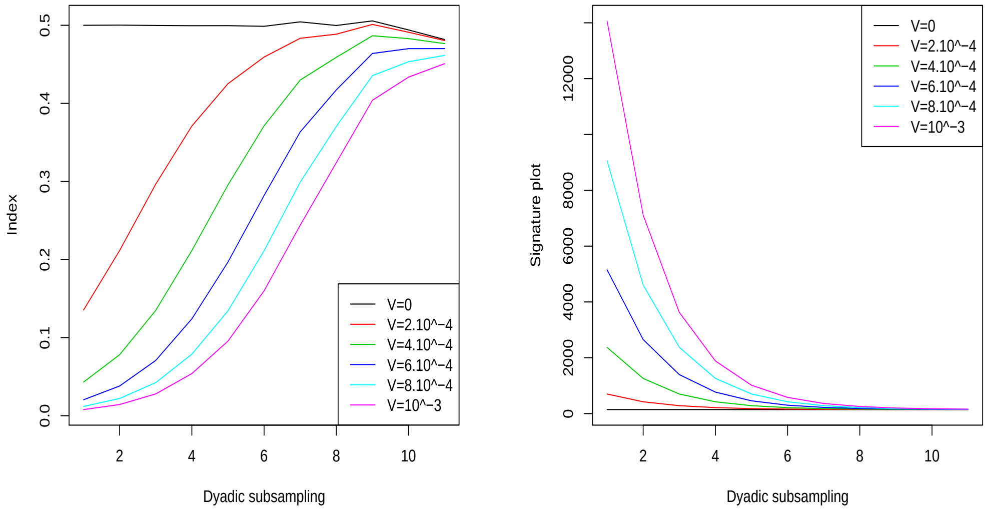

Figure D.19: *Model M1, Microstructure noise index (left) and signature plot (right).*

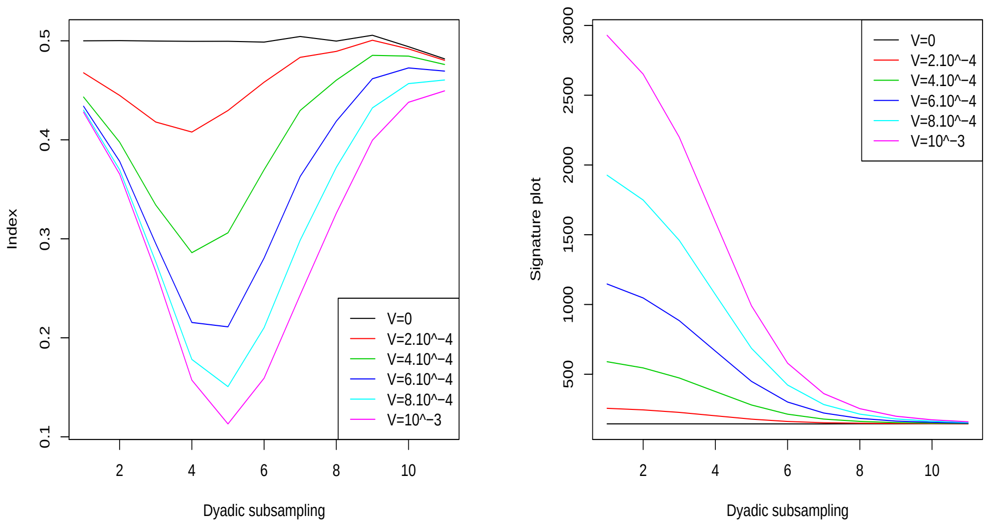

Figure D.20: *Model M2, Microstructure noise index (left) and signature plot (right).*

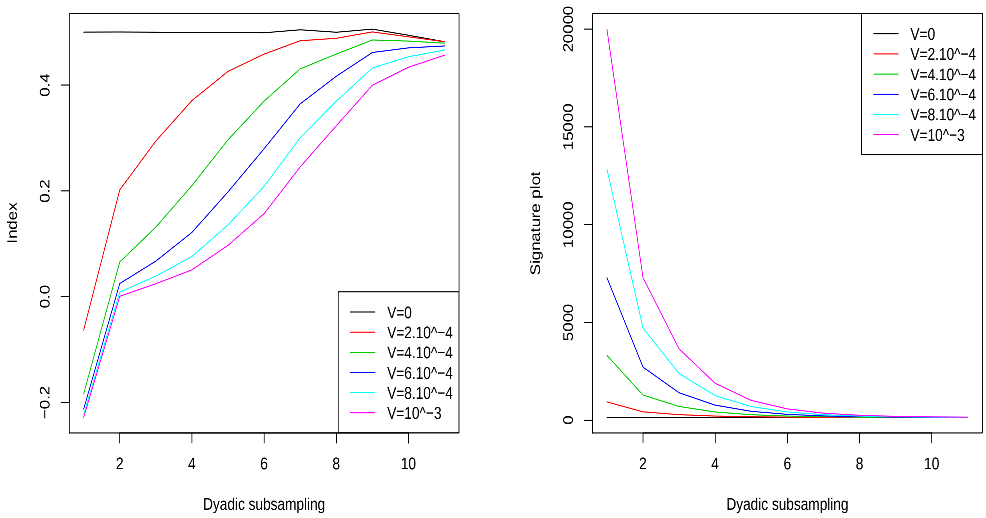

Figure D.21: *Model M3, Microstructure noise index (left) and signature plot (right).*

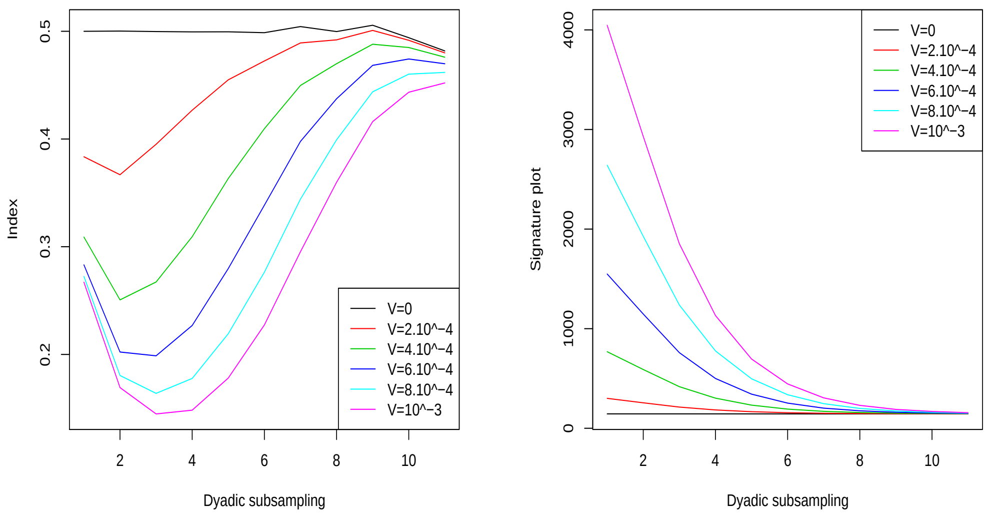

Figure D.22: *Model M4, Microstructure noise index (left) and signature plot (right).*

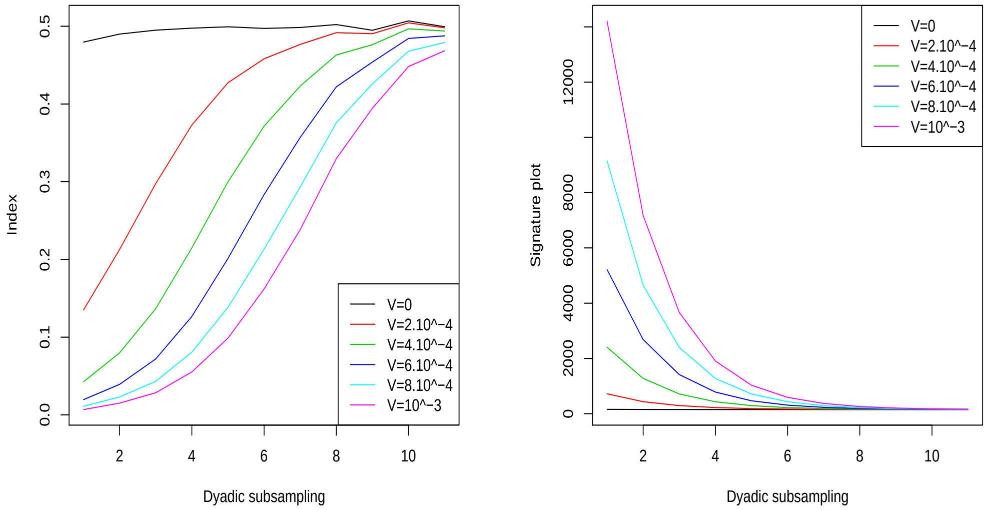

Figure D.23: *Model M'1, Microstructure noise index (left) and signature plot (right).*

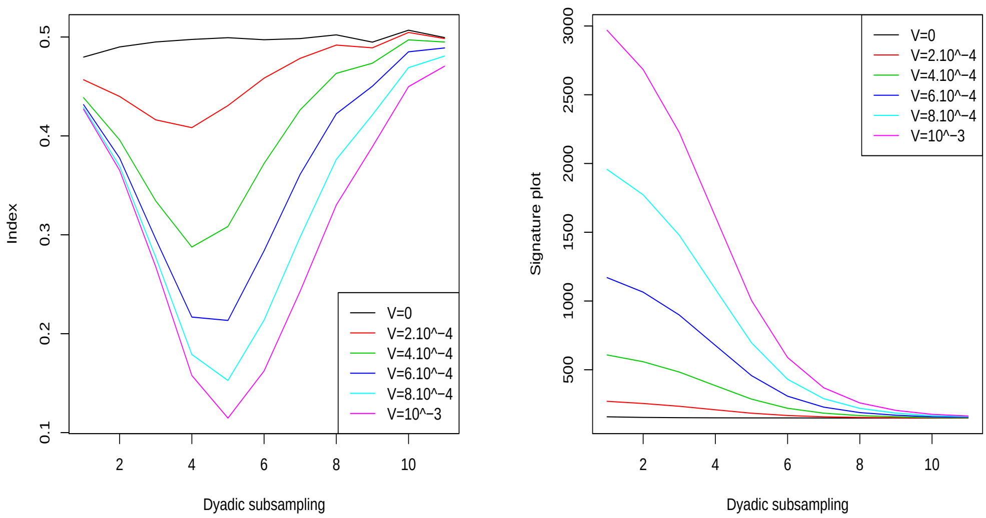

Figure D.24: *Model M'2, Microstructure noise index (left) and signature plot (right).*

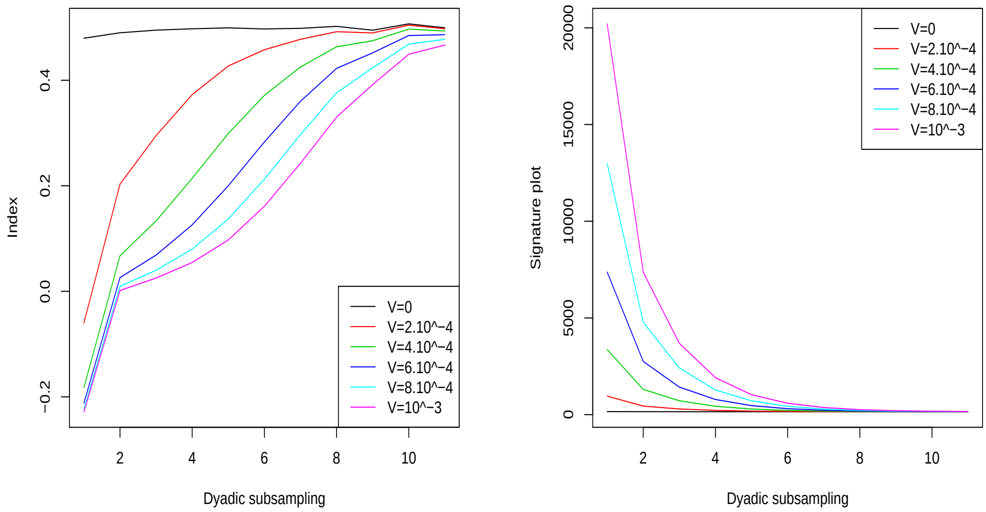

Figure D.25: *Model M'3, Microstructure noise index (left) and signature plot (right).*

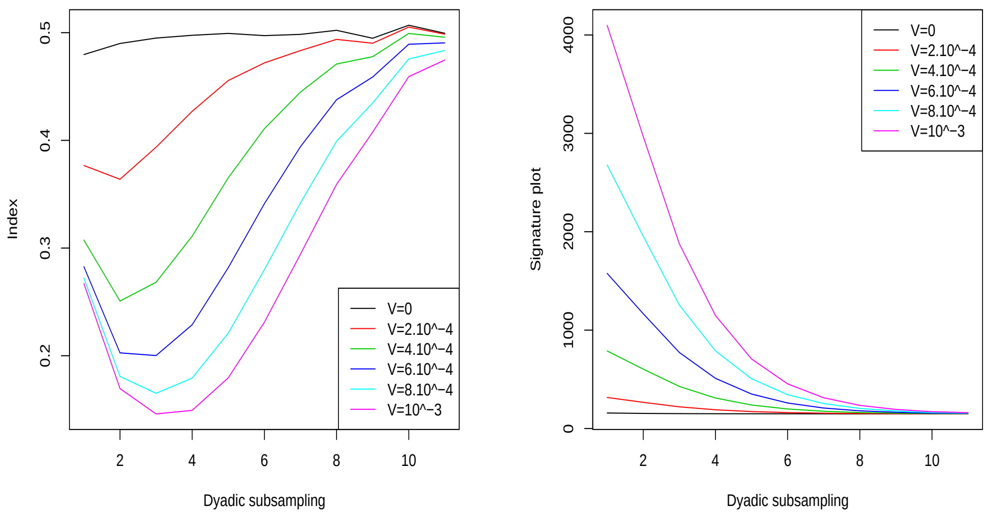

Figure D.26: *Model M'4, Microstructure noise index (left) and signature plot (right).*

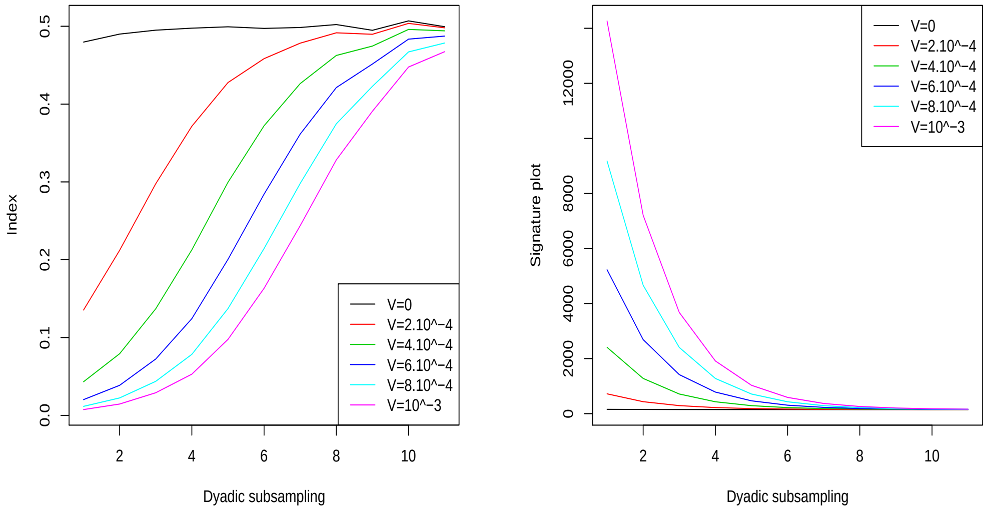

Figure D.27: *Model M'5, Microstructure noise index (left) and signature plot (right).*

Appendix E

Stochastic volatility

We use in this work the framework of stochastic volatility models. Indeed it enables to reproduce several stylized facts of the markets, see Shephard [105]. For example, we can obtain heavy tailed distributions for the returns.

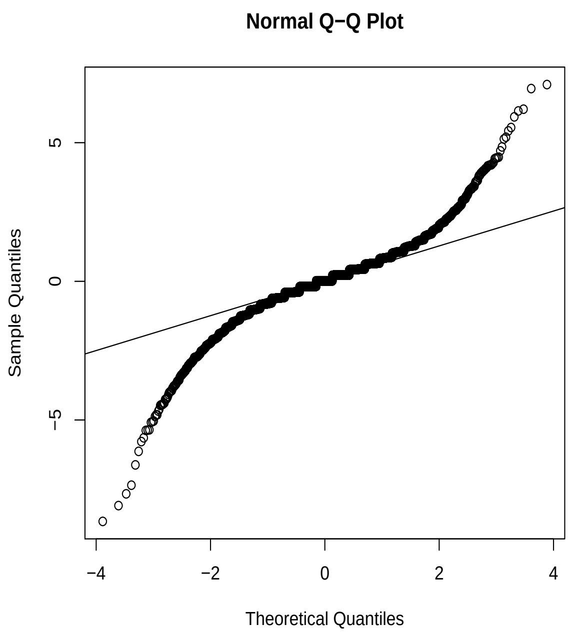

Figure E.1: *QQ-plot of the 20 minutes log returns of the Bund contract, centered and normalized, from 01/09/2005 to 31/01/2007, from 9 am to 18.30, Paris time. The QQ-plot is significantly far from the line, what suggests the presence of heavy tails.*

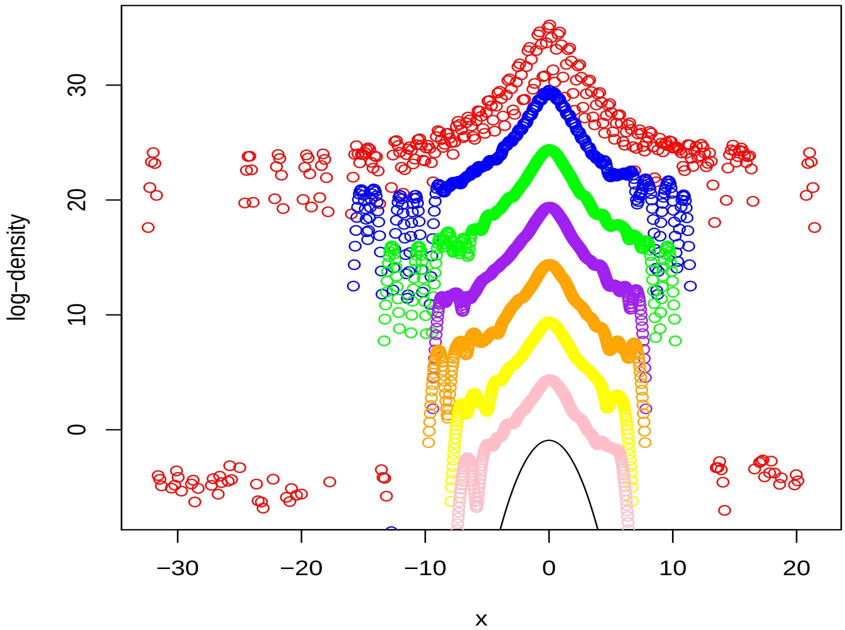

Figure E.2: *Logarithms of the kernel estimations of the densities of the log return at 1 (red), 5 (blue), 10 (green), 20 (purple), 30 (orange), 45 (yellow) et 60 (pink) minutes of the Bund contract, centered and normalized, from 01/09/2005 to 31/01/2007, from 9 am to 18.30, Paris time. The log-density of a standard Gaussian variable is in colored in black. The log-density are shifted up for reading convenience. The tails are fatter than for a Gaussian variable. The smallest the sampling period, the fattest the tails.*

How to study the volatility in practice? Consider a diffusion process (Y_t) (the price of the asset), such that

$$\frac{\mathrm{d}Y_t}{Y_t} = \sigma_t \mathrm{d}B_t + \mu_t \mathrm{d}t.$$

The practitioner can only observe the price at some discrete times. Thus, the coefficient σ_t is not observed and so its analysis is complex. A classical approach is to try to build an approximation of the volatility. Suppose that a day j is made of the time interval $[T_j^o, T_j^c]$ and that we observe the price with subsampling period Δ. From the sample

$$\{Y_{T_j^o + k\Delta},\ k = 0, \ldots, \lfloor \Delta^{-1}(T_j^c - T_j^o) \rfloor\},$$

a daily measure of the volatility can be

$$\frac{1}{T_j^c - T_j^o} \sum_{k=0}^{\lfloor \Delta^{-1}(T_j^c - T_j^o) \rfloor} \big(\log(Y_{T_j^o+(k+1)\Delta}) - \log(Y_{T_j^o+k\Delta}) \big)^2.$$

Indeed, in the continuous semi-martingales context, for big enough $\Delta^{-1}(T_j^c - T_j^o)$, the "proxi" that we have defined of the daily integrated volatility

$$\frac{1}{T_j^c - T_j^o} \int_{T_j^o}^{T_j^c} \sigma_s^2 \mathrm{d}s.$$

Note that in practice, we have to take a big enough sampling period and to avoid the problems of microstructure noise and so to stay in the semi-martingale context. The computation of this "proxi" on financial data leads to the following graphs.

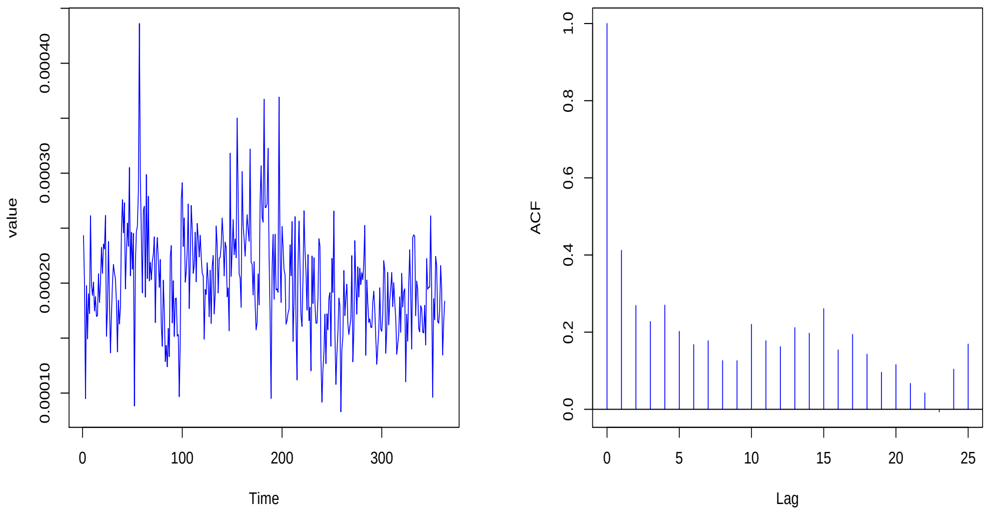

Figure E.3: *Volatility proxy (left) and its autocorrelogram (right) for the Bund contract, from 2005-09-01 to 2007-01-31, from 9 am to 18.30, Paris time, sampling frequency 10 minutes. We see in particular that the volatility seems quite stationary. The slow decay of the autocorrelogram suggests some persistence in the volatility*

Bibliography

[1] Aït-Sahalia, Y. and Jacod, J. (2006) *Testing for jumps in a discretely observed process.* Working paper.

[2] Aït-Sahalia, Y. and Jacod, J. (2007) *Volatility estimators for discretely sampled Lévy processes.* Ann. Statist., **35**, 355-392.

[3] Aït-Sahalia, Y. and Mykland, P.A. (2004) *Estimators of diffusions with randomly spaced observations: a general theory.* Ann. Statist., **32**, 2186-2222.

[4] Aït-Sahalia, Y., Mykland, P.A. and Zhang, L. (2005) *How often to sample a continuous time process in the presence of market microstructure noise.* Review of Financial Studies, **18**, 351-416.

[5] Aït-Sahalia, Y., Mykland, P.A. and Zhang, L. (2005) *Ultra high frequency estimation with dependent microstructure noise.* Working paper.

[6] Aldous, D.J. and Eagleson, G.K. (1978) *On mixing and stability of limit theorems.* Annals of Probability, **6**, 325-331.

[7] Almeida, A. (2005) *Wavelet bases in generalized Besov spaces.* J. Math. Anal. Appl., **304**, 198-211.

[8] Andersen, T.G. and Bollerslev, T. (1997) *Intraday periodicity and volatility persistence in financial markets.* International Economic Review, **39**, 885-905.

[9] Andersen, T.G., Bollerslev, T., Diebold, F.X. and Labys, P. (2000) *Great Realizations.* Risk, **13**, 105-108.

[10] Andersen, T., Bollerslev, T. and Meddahi, N. (2006) *Market microstructure noise and realized volatility forecasting.* Working paper.

[11] Arcones, M.A. (1995) *On the law of the iterated logarithm for Gaussian processes.* Journal of Theoretical Probability, **8**, 877-903.

[12] Bandi, F.M. and Russel, J.R. (2005) *Microstructure noise, realized variance and optimal sampling.* To appear in Review of Economic Studies.

[13] Bandi, F.M., Russel, J.R. and Yang, C. (2006) *Realized volatility and option pricing.* Working paper.

[14] Barndorff-Nielsen, O.E., Nicolato, E. and Shephard, N. (2002) *Some recent advances in stochastic volatility modelling.* Quant. Finance, **2**, 11-23.

[15] Barndorff-Nielsen, O.E. and Shephard, N. (2001) *Non-Gaussian Ornstein-Uhlenbeck based models and some of their uses in financial economics (with discussion).* Journal of the Royal Statistical Society, Series B**63**, 167-241.

[16] Barndorff-Nielsen, O.E. and Shephard, N. (2002) *Econometric analysis of realized volatility and its use in estimating stochastic volatility models.* Journal of the Royal Statistical Society, Ser. B, **64**, 253-280.

[17] Baxter, G. (1956) *A strong limit theorem for Gaussian processes.* Proc. Amer. Math. Soc., **7**, 522-527.

[18] Berzin, C. and León, J.R. (2005) *Convergence in fractional models and applications.* Electronic journal of probability, **10**, 326-370.

[19] Bibby, B.M. and Sørensen, M. (1995) *Martingale estimating functions for discretely observed diffusion processes.* Bernoulli, **1**, 17-39.

[20] Boufoussi, B. et Ouknine, Y. (1999) *Régularité du temps local du processus symétrique stable en norme Besov.* Stochastics and Stoch. Rep., **66**, 167-175.

[21] Boufoussi, B. et Roynette, B. (1993) *Le temps local brownien appartient p.s à l'espace de Besov* $\mathcal{B}^{1/2}_{p,\infty}$. C.R.A.S. Paris, **1316**, 843-848.

[22] Breidt, F., Crato, N. and De Lima, P. (1998) *The detection and estimation of long-memory in stochastic volatility.* Journal of Econometrics, **83**, 325–348.

[23] Brouste, A. (2006) *Simple branching process wavelet series.* Preprint.

[24] Cavalier, L., Golubev, G.K., Lepski, O.V. and Tsybakov, A.B. (2003) *Block thresholding and sharp adaptive estimation in severely ill-posed inverse problems.* Theory of Probability and its Applications, **48(3)**, 534-556.

[25] Cheridito, P., Kawaguchi, H. and Maejima M. (2003) *Fractional Ornstein-Uhlenbeck processes.* Electronic Journal of probability, **8**, 1-14.

[26] Ciesielski, Z., Kerkyacharian, G. et Roynette B. (1993) *Quelques espaces fonctionnels associés à des processus gaussiens.* Stu. Math., **107**, 172-204.

[27] Cœurjolly, J.F. (2001) *Estimating the parameters of a fractional Brownian motion by discrete variations of its sample paths.* Stat. Inf. Stoc. Pro., **4(2)**, 199-227.

[28] Cohen, A. (1999) *Wavelet methods in Numerical Analysis.* Handbook of Numerical Analysis Vol VII, Ed. P.G. Ciarlet and J.L. Lions, Elsevier Science.

[29] Comte, F., Coutin, L. and Renault, E. (2003) *Affine fractional stochastic volatility models with application to option pricing.* Preprint, university of Montreal.

[30] Comte, F., Genon-Catalot, V. and Rozenholc, Y. (2007) *Penalized nonparametric mean square estimation of the coefficients of diffusion processes.* To appear in Bernoulli.

[31] Comte, F. and Renault, E. (1996) *Long memory continuous time models.* Journal. of Econometrics, **73**, 101-150.

[32] Comte, F. and Renault, E. (1998) *Long-memory in continuous-time stochastic volatility models.* Mathematical Finance, **8**, 291-323.

[33] Dacunha-Castelle, D. et Florens, D. (1986) *Estimation of the coefficients of a diffusion from discrete observations.* Stochastics, **19**, 263-284.

[34] Daubechies, I. (1988) *Orthonormal bases of compactly supported wavelets.* Comm. Pure and Appl. Math, **41**, 909-996.

[35] Delattre, S. (1997) *Estimation du coefficient de diffusion d'un processus de diffusion avec erreurs d'arrondi.* Ph.D Thesis, University Paris 6.

[36] Delattre, S. and Jacod, J. (1997) *A central limit theorem for normalized functions of the increments of a diffusion process, in the presence of round-off errors.* Bernoulli, **3**(1), 1-28.

[37] Deo, R., Hurvich, C. and Lu, Y. (2004) *On the log-periodogram regression estimator of the memory parameter in long memory stochastic volatility model.* Econometric Theory, **17(4)**, 686-710.

[38] Donhal, G. (1987). *On estimating the diffusion coefficient.* J. Applied Probab., **34**, 105-114.

[39] Donoho, D., Johnstone, I., Kerkyacharian, G. and Picard, D. (1996) *Density estimation by wavelet thresholding*. Annals of Statistics, **24**, 508–539.

[40] Doukhan, P., Oppenheim, G. and Taqqu, M. (eds) (2003) *Long-range dependence: theory and applications*. Birkhäuser, Boston.

[41] Fitzsimmons, P.J. and Getoor, R.K. (1992) *Limit theorems and variation properties for fractional derivatives of the local time of a stable process*. Ann. of IHP, section B, **28-2**, 311-333.

[42] Florens, D. (1989) *Approximate discrete time schemes for statistics of diffusion processes*. Statistics, **20**, 547-557.

[43] Florens, D. (1993) *On estimating the diffusion coefficient from discrete observations*. J. Appl. Probab., **30**, 790-804.

[44] Fristedt, B., Taylor, S.J. (1973) *Strong variation for the sample path function of a stable process*. Duke math J., **40**, 259-278.

[45] Garman, M.B. and Klass, M.J. (1980) *On the estimation of security price volatilities from historical data*. The Journal of Business, **53(1)**, 67-78.

[46] Gatheral, J. and Oomen, R. (2007) *Zero-intelligence realized variance estimation*. Working paper.

[47] Gayraud, G. and Tribouley, K. (1999) *Wavelet methods to estimate an integrated functional: Adaptivity and asymptotic law*. Statistics and Probability letters, **44**, 109-122.

[48] Genon-Catalot, V. and Jacod, J. (1993) *On estimating the diffusion coefficient for multidimensional processes*. Ann. IHP-Probab., **29**, 119-151.

[49] Genon-Catalot, V. and Jacod, J. (1994). *Estimation of the diffusion coefficient for difffusion processes: random sampling*. Scand. J. Statist., **21**, 193-221.

[50] Genon-Catalot, V., Jeantheau T. and Laredo, C. (1999) *Parameter estimation for discretely observed stochastic volatility models*. Bernoulli, **5 (5)**, 855-872.

[51] Genon-Catalot, V., Jeantheau T. and Laredo, C. (2000) *Stochastic volatility models as hidden Markov models and statistical applications*. Bernoulli, **6 (6)**, 1051-1079.

[52] Genon-Catalot, V., Laredo, C. and Picard, D. (1992). *Nonparametric estimation of the diffusion coefficient by wavelets methods.* Scand. J. Statist., **19**, 319-335.

[53] Gladyshev, E.G. (1961) *A new limit theorem for stochastic processes with Gaussian increments.* Theor. Probability Appl., **6**, 52-61.

[54] Gloter, A. and Hoffmann, M. (2004) *Stochastic volatility and fractional Brownian motion.* Stoch. Proc. and Appl., **113**, 143-172.

[55] Gloter, A. and Hoffmann, M. (2005) *Estimation of the Hurst parameter from discrete noisy data.* To appear, Ann. Statist.

[56] Gloter, A. and Jacod, J. (1997) *Diffusions with measurement errors, I. Local Asymptotic Normality, II. Optimal estimators.* ESAIM PS, **5**, 225-260.

[57] Gonçalves, S. and Meddahi, N. (2005) *Bootstrapping realized volatility.* Working paper.

[58] Hansen, P.R. and Lunde, A. (2006) *Realized variance and market microstructure noise.* Journal of Business and Economics Statistics, **24**(2), 127-161.

[59] Harvey, A.C. (1998) *Long-memory in stochastic volatility.* In J.Knight and S.Satchell, *Forecasting volatility in financial markets*, 307-320, Oxford, Butterworth-Heineman.

[60] Hoffmann, M. (1999) *Adaptative estimation in diffusion processes.* Stoc. Proc. Appl., **79**, 135-163.

[61] Hoffmann, M. (2001) *On estimating the diffusion coefficient: parametric versus nonparametric.* Ann. IHP-Probab., **37 (3)**, 339-372.

[62] Hoffmann, M. (2002) *Rate of convergence for parametric estimation in a stochastic volatility model.* Stoc. Proc. Appl., **97 (1)**, 135-163.

[63] Hull, J. and White, A. (1988) *An analysis of the bias in option pricing caused by a stochastic volatility.* Adv. Futures Options. Res., **3**, 29-61.

[64] Hurvich, C., Moulines, E. and Soulier, P. (2005) *Estimating long-memory in stochastic volatility.* Econometrica, **73(4)**, 1283-1328.

[65] Hurvich, C. and Ray, K. (2003) *The local whittle estimator of long-memory stochastic volatility.* Journal of Financial Econometrics, **1(3)**, 445-470.

[66] Hurvich, C. and Soulier, P. (2002) *Testing for long-memory in volatility.* Econometric Theory, **18(6)**, 1291-1308.

[67] Ibragimov, A. and Hasminski, R.Z. (1980) *Statistical Estimation Theory.* Springer-Verlag, New York.

[68] Istas J. and Lang G. (1997) *Quadratic variations and estimation of the local Hölder index of a Gaussian process.* Ann. Inst. Henri Poincaré, **33(4)**, 407-436.

[69] Jacobsen, J. (2001) *Discretely observed diffusions: classes of estimating functions and small* Δ*-optimality.* Scandinavian Journal of Statistics, **33(4)**, 123-149.

[70] Jacod, J. (1996) *La variation quadratique du brownien en présence d'erreurs d'arrondi.* Asterisque, **236**, 155-162.

[71] Jacod, J. (1997) *On continuous conditional Gaussian martingales and stable convergence in law.* Séminaire de probabilités (Strasbourg), **31**, 232-246.

[72] Jacod, J. (2000) *Non-parametric kernel estimation of the coefficient of a diffusion.* Scandinavian Journal of Statistics, **27(1)**, 83-96.

[73] Jacod, J. (2006) *Asymptotic properties of realized power variations and related functionals of semimartingales.* To appear in Stoc. Proc. Appl.

[74] Jacod, J. and Protter, P. (1998) *Asymptotic error distributions for the Euler method for stochastic differential equations.* The Annals of Probability, **26**, 267-307.

[75] Jacod, J. and Shiryaev, A.N. (2003) *Limit theorems for stochastic processes.* Springer-Verlag, New York, 2nd edition.

[76] Jensen M. (2004) *Semiparametric bayesian inference of long-memory stochastic volatility models.* Journal of Time Series Analysis, **25(6)**, 895-922.

[77] Kerkyacharian G. and Picard D. (2004) *Regression in random design and warped wavelet.* Bernoulli, **10(6)**, 1053-1105.

[78] Kessler, M. (1997) *Estimation of an ergodic diffusion from discrete observations.* Scandinavian Journal of Statistics, **24**, 211-229.

[79] Kessler, M. (2000) *Simple and explicit estimating functions for a discretely observed diffusion process.* Scandinavian Journal of Statistics, **27**, 65-82.

[80] Kessler, M. and Sørensen, M. (1999) *Estimating equations based on eigenfunctions for a discretely observed diffusion process.* Bernoulli, **5(2)**, 299-314.

[81] Kosulajeff, P. (1937) *Sur la répartition de la partie fractionaire d'une variable aléatoire*. Mat. Sb. (N.S.), **2**, 1017-1019.

[82] Lang G. and Roueff F. (2001) *Semi-parametric estimation of the Hölder exponents of stationary Gaussian processes with minimax rates.* Stat. Inf. Stoc. Pro, **4(3)**, 283-306.

[83] Large, J. (2006) *Estimating quadratic variation when quoted prices change by a constant increment*. Working paper.

[84] Lee, J. (2004) *Wavelet transform for estimating the memory parameter in long memory stochastic volatility model.* Preprint, National University of Singapore.

[85] Li, Y. and Mykland, P. (2006) *Determining the volatility of a price process in the presence of rounding errors.* Technical report, **573**, The University of Chicago, Department of Statistics.

[86] Li, Y. and Mykland, P. (2007) *Are volatility estimators robust with respect to modeling assumptions?* To appear in Bernoulli.

[87] Mandelbrot, B. and Van Ness, J. (1968) *Fractional Brownian motions, fractional noises and applications*. SIAM Rev., **10**, 422-437.

[88] Manstavicius, M. (2004) *p-variation of strong Markov processes.* Ann. of Probability, **32(3)**, 2053-2066.

[89] Manstavicius, M. (2005) *A non Markovian process with unbounded p-variation.* Elect. Comm. in Probab., **10**, 17-28.

[90] Marcus, M. and Rosen, J. (1992) *p-variation of the local times of symmetric stable processes and of Gaussian processes with stationary increments.* Ann. of Probability, **20(4)**, 1685-1713.

[91] Melino, A. and Turnbull, S.M. (1990) *Pricing foreign currency options with stochastic volatility*. J. Econom., **45**, 239-265.

[92] Merton, R.C. (1971) *Optimum consumption and portfolio rules in a continuous-time model.* Journal of Economic Theory, **3**, 373-413.

[93] Monroe, I. (1972) *On the γ-variation of processes with stationary independent increments.* Ann. Math. Stat., **43**, 1293-1220.

[94] Meddahi, N. (2002) *A theoretical comparison between integrated and realized volatility.* Journal of Applied Econometrics, **17**, 475-508.

[95] Meyer, Y. (1990) *Ondelettes et opérateurs I-II.* Hermann.

[96] Musiela, M. and Rutkowski, M. (2005) *Martingale methods in financial modelling.* Springer.

[97] Norros, I., Valkeila, E. and Virtamo, Y. (1999) *An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions.* Bernoulli, **5(4)**, 571-587.

[98] Nualart, D. and Ouknine Y. (2003) *Besov regularity of stochastic integrals with respect to the fractional Brownian motion with parameter $H > 1/2$.* Journal of Theoretical Probability, **16**, 451-470.

[99] Ohanissian, A., Russel, J. and Tsay R. (2004) *True or spurious long memory in volatility: does it matter for option pricing?* Preprint.

[100] Rényi, A. (1963) *On stable sequences of events.* Sankyā Series A, **25**, 293-302.

[101] Revuz, D. and Yor, M. (1999) *Continuous martingales and Brownian motion.* Springer, 3rd edition.

[102] Robinson, P.M. (2001) *The memory of stochastic volatility models.* J. of Econometrics, **101**, 195-218.

[103] Roynette B. (1993) *Mouvement brownien et espaces de Besov.* Stochastics and Stochastics Reports, **43, 3-4**, 221-260.

[104] Schilling R.L. (2000) *Function spaces as path spaces of Feller processes.* Math. Nachr., **217**, 147-174.

[105] Shephard, N. (2006) *Stochastic volatility.* New Palgrave Dic. of Econ., 2nd edition.

[106] Taylor, S. (2000) *Consequences for option pricing of a long-memory in volatility.* Preprint.

[107] Teyssière, G. (2003) *Interaction models for common long-range in asset price volatilities*. In Processes with Long Range Correlations: Theory and Applications, Lecture Notes in Physics. G. Rangarajan and M. Ding editors, **621**, 251-269, Springer Verlag.

[108] Triebel H. (1983) *Theory of function spaces*. Birkhäuser, Basel.

[109] Triebel H. (2006) *Theory of function spaces III*. Birkhäuser, Basel.

[110] Tukey, J.W. (1939) *On the distribution of the fractional part of a statistical variable*. Mat. Sb. (N.S.), **4**, 561-562.

[111] Wood, A. and Chan, G. (1994) *Simulation of stationary Gaussian processes in* $[0,1]^d$. Journal of computational and graphical statistics, **3(4)**, 409-432.

[112] Yoshida, N. (1992). Estimation for diffusion processes from discrete observations. *J. Multivariate Anal.*, **41**, 220-242.

[113] Zhang, L. (2006) *Efficient estimation of stochastic volatility using noisy observations: a multi-scale approach*. To appear in Bernoulli.

[114] Zhang, L., Mykland, P.A. and Aït-Sahalia, Y. (2005) *A tale of two time scales: determining integrated volatility with noisy high-frequency data*. Journal of the American Statistical Association, **100(472)**, Theory and Methods, 1394-1411.