Test of homogeneity of variance in wavelet domain.

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EMS 2009 TOULOUSE

July 29, 2009

Stationarity and Generalized Spectral Density

- $X \stackrel{\text{def}}{=} \{X_k\}_{k\in\mathbb{Z}}$: real-valued process, not necessarily stationary.
- Define $[\boldsymbol{\Delta} X]_n \stackrel{\text{def}}{=} X_n - X_{n-1}$ and $\boldsymbol{\Delta}^K = \boldsymbol{\Delta} \circ \boldsymbol{\Delta}^{K-1}$ for $K \geq 1$.

Definition 1

X is K-th order difference stationary if $\boldsymbol{\Delta}^K X$ covariance stationary.

- f a non-negative 2π-periodic symmetric function.

Definition 2

X admits *generalized spectral density* f if $\boldsymbol{\Delta}^K X$ weakly stationary with spectral density function

$$f_K(\lambda) = |1 - \mathrm{e}^{-\mathrm{i}\lambda}|^{2K} f(\lambda) . \quad (1)$$

Wavelet decomposition

Let ϕ a scale function and ψ a wavelet. Associate to $\{x_k, k \in \mathbb{Z}\}$ the function :

$$\mathbf{x}(t) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} x_k \, \phi(t-k), \quad t \in \mathbb{R} \, .$$

The wavelet coefficient $W_{j,k}$ at scale $j \geq 0$ and location $k \in \mathbb{Z}$ is :

$$W_{j,k} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \mathbf{x}(t) \psi_{j,k}(t) \, \mathrm{d}t \, .$$

It only depends on $x_1, \ldots, x_n$ for all $(j,k) : j \geq 0, 0 \leq k < n_j$ with $n_j = 2^{-j} n + O(1)$.

Computation

For all $j \geq 0$, $(W_{j,k})_k$ can be expressed as a linear filtering of $(x_k)_k$ followed by 2^{-j}-downsampling :

$$W_{j,k} = \sum_{l \in \mathbb{Z}} x_l \, h_{j,2^j k - l},$$

where, the impulse response $h_{j,\cdot}$ has length $O(2^j)$.
Standard assumptions on (ϕ, ψ) imply that

$$h_{j,\cdot} = (1 - B)^M (\tilde{h}_{j,\cdot}),$$

where $(\tilde{h}_{j,k})_k$ is a finite sequence, B the lag operator and M is called the number of vanishing moments of the wavelet.

The between-scale process

If $M \geq K$

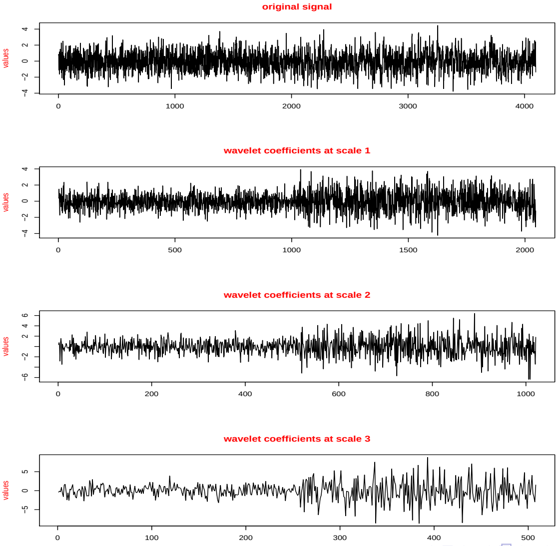
- $\{W_{j,k}\}_{k\in\mathbb{Z}}$ $j \geq 0$ is stationary but the two–dimensional process $\{[W_{j,k},\ W_{j',k}]^T\}_{k\in\mathbb{Z}}$ with $j \neq j'$, is not stationary.
- Consider instead the between-scale process

$$\{[W_{j,k},\ \mathbf{W}_{j,k}(j-j')]^T\}_{k\in\mathbb{Z}}\ ,\ \text{ for } j > j' = j - u, \text{ where}$$

$$\mathbf{W}_{j,k}(u) \stackrel{\text{def}}{=} [W_{j-u,2^u k},\ W_{j-u,2^u k+1},\ \dots,\ W_{j-u,2^u k+2^u-1}]^T\ .$$

are non overlapping blocks of wavelet coefficients at the finer scale j'.

Change-points detection in the wavelet domain : AR(1)-ARMA(1,1)



Test hypotheses

Let $X_1, \ldots, X_n$, n observations of a process and $W_{j,k}$ the associated wavelet coefficients. Define $\sigma^2_{j,k} = \mathrm{Var}\big(W_{j,k}\big)$

$$\begin{cases} \mathcal{H}_0 : \sigma^2_{j,1} = \cdots = \sigma^2_{j,n_j}, & \text{for all } j \in \{J_1, \ldots, J_2\} \\ \mathcal{H}_1 : \sigma^2_{j,1} = \cdots = \sigma^2_{j,k_j} \neq \sigma^2_{j,k_j+1} = \ldots = \sigma^2_{j,n_j} & \text{for at least one } j \in \{J_1, \ldots, J_2\} \end{cases}$$

where J_1 is the finest scale and J_2 is the coarsest scale.
Under $\mathcal{H}_0$, we suppose that X is a Gaussian process with generalized spectral density given by $f(\lambda)$ + assumptions on f and on the wavelet (e.g: $M \geq K$).

Functional CLT

Define the partial sums: $S_{J_1,J_2}(t) = \frac{1}{\sqrt{n_{J_2}}}\left[\sum_{i=1}^{\lfloor n_j t \rfloor} W_{j,i}^2\right]_{j=J_1,\dots,J_2}$

Theorem 1 (Under $\mathcal{H}_0$)

$$\left(S_{J_1,J_2}(t) - \mathbb{E}\left[S_{J_1,J_2}(t)\right]\right) \xrightarrow{\mathcal{L}} \Gamma_{J_1,J_2}^{1/2}\left(B_{J_1}(t),\dots,B_{J_2}(t)\right)^T,$$

in $D^{J_2-J_1+1}[0,1]$, where $\{B_j(t)\}_{j=J_1,\dots,J_2}$ are independent Brownian motions and Γ_{J_1,J_2} is a covariance matrix depending on f, ϕ, ψ, J_1, J_2.

Bartlett estimator of the covariance

Consider the following process

$$Y_{J_1,J_2,k} = \left(W^2_{J_2,k}, \ldots, \sum_{l=0}^{2^{J_2-j}-1} W^2_{j,2^{J_2-j}k+l}, \cdots \sum_{l=0}^{2^{J_2-J_1}-1} W^2_{J_1,2^{J_2-J_1}k+l} \right).$$

The Bartlett estimator of the covariance matrix of the square wavelet's coefficients between scales $J_1, \ldots, J_2$ is the $(J_2 - J_1 + 1) \times (J_2 - J_1 + 1)$ symmetric definite positive matrix $\hat{\Gamma}_{J_1,J_2}$ where :

$$\hat{\gamma}_{J_1,J_2}(\ell) = n_{J_2}^{-1} \sum_{k=1}^{n_{J_2}-|\ell|} \left(Y_{J_1,J_2,k} - \bar{Y}_{J_1,J_2} \right) \left(Y_{J_1,J_2,k+\ell} - \bar{Y}_{J_1,J_2} \right)^T ,$$

$$\hat{\Gamma}_{J_1,J_2} = \sum_{\ell=-q(n_{j_2})}^{q(n_{J_2})} w_\ell[q(n_{J_2})] \hat{\gamma}_{J_1,J_2}(\ell) ,$$

Consistency of the Bartlett estimator.

and $w_l(q(n_j)) = 1 - \frac{|l|}{1+q(n_j)}$ are the Bartlett weights.

Theorem 2 (Under $\mathcal{H}_0$)

Assume that $q(n_{J_2}) \to \infty$ *and* $\frac{q(n_{J_2})}{n_{J_2}} \to 0$ *as* $n_{J_2} \to \infty$, *then*

$$\hat{\Gamma}_{J_1,J_2} \xrightarrow{P} \Gamma_{J_1,J_2},$$

where $\Gamma_{J_1,J_2}(i,j) = \sum_{\tau \in \mathbb{Z}} \mathrm{Cov}(Y_{j,0}, Y_{i,\tau})$ *with* $1 \leq i \leq j \leq J_2 - J_1 + 1$.

Test statistic (1)

$$T_{J_1,J_2}(t) \stackrel{\text{def}}{=} \left(S_{J_1,J_2}(t) - tS_{J_1,J_2}(1)\right)^T \hat{\Gamma}_{J_1,J_2}^{-1} \left(S_{J_1,J_2}(t) - tS_{J_1,J_2}(1)\right) \quad (2)$$

converges in weakly in the Skorokhod space $D([0,1])$

$$T_{J_1,J_2}(t) \xrightarrow{\mathcal{L}} \sum_{\ell=1}^{J_2-J_1+1} \left[B_\ell^0(t)\right]^2$$

where $t \mapsto (B_1^0(t), \ldots, B_{J_2-J_1+1}^0(t))$ is a vector of $J_2 - J_1 + 1$ independent Brownian bridges.

Test statistic (2)

$$\mathrm{CVM}(J_1, J_2) \stackrel{\mathrm{def}}{=} \int_0^1 T_{J_1,J_2}(t)\mathrm{d}t\ ,$$

which converges to $$\int_0^1 \sum_{\ell=1}^{J_2-J_1+1} \left[B_\ell^0(t)\right]^2 \mathrm{d}t\ .$$

It is also possible to use the max. functional leading to an analogue of the Kolmogorov-Smirnov statistics,

$$\mathrm{KSM}(J_1, J_2) \stackrel{\mathrm{def}}{=} \sup_{0\leq t\leq 1} T_{J_1,J_2}(t)$$

which converges to $$\sup_{0\leq t\leq 1} \sum_{\ell=1}^{J_2-J_1+1} \left[B_\ell^0(t)\right]^2\ .$$

Asymptotic estimator of the Covariance matrix

Suppose that $f(\lambda) = |1 - \mathrm{e}^{\mathrm{i}\lambda}|^{-2d} f^*(\lambda)$,

- $d \in \mathbb{R}$: memory parameter
- $f^*(0) > 0$, $f^* \in C^2$.

From Theorem 1 in [Moulines et al 2008], for two scales $i \leq j$

$$\sigma_j^2(d, f^*) \approx f^*(0)\,\mathrm{K}\,2^{2jd} \text{ and}$$

$$\mathrm{Cov}\left(W_{j,k}^{(X)}, \mathbf{W}_{j,k'}^{(X)}(u)\right) \approx f^*(0)\mathrm{Cov}\left(W_{j,k}^{(d)}, \mathbf{W}_{j,k'}^{(d)}(u)\right),$$

where K depends on ψ and d and $\left\{W_{j,k}^{(d)}\right\}_k$ are the wavelet coefficients at scale j of the Generalized Fractionary Brownian Motion (GFBM) with Hurst index $H = d + 1/2$.

In this case, as $J_1 \to \infty$, with $J_2 - J_1 = L$,

$$\Gamma_{J_1,J_2} \to A^2 \mathbf{M}_L(d),$$

where $\mathbf{M}_L(d)$ is the covariance matrix of the square of the GFBM wavelet's coefficients between $J_1, \ldots, J_2$.
A and d can be estimated using the **Abry and Veitch** log-regression estimator in the wavelet domain which relies on the approximation

$$\log(\sigma_j^2) \approx \log(A) + 2d(j - J_1) \log 2. \quad (3)$$

For each j, σ_j^2 is estimated with the scalogram

$$\hat{\sigma}_j^2 = \frac{1}{n_j} \sum_{i=1}^{n_j} \left(W_{j,i}^{(X)} \right)^2 .$$

Level of the test based on Bartlett estimator of the Covariance Matrix.

$n = 8192$		WN	$MA(1)$	$AR(1)$	$ARFIMA(1, d, 1)$	
					$d = 0.2$	$d = 0.3$
$J_2 = 3$	KSM	0.02	0.02	0.062	0.089	0.1
$J_2 = 3$	CVM	0.027	0.012	0.041	0.076	0.082
$J_2 = 4$	KSM	0.04	0.04	0.18	0.2	0.29
$J_2 = 4$	CVM	0.021	0.032	0.091	0.116	0.192
$J_2 = 5$	KSM	0.014	0.017	0.31	0.302	0.341
$J_2 = 5$	CVM	0.02	0.024	0.17	0.159	0.256

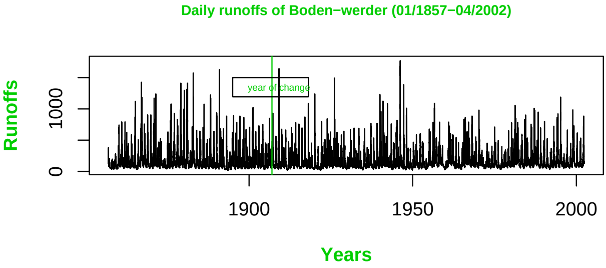
Table: Empirical level of KSM − CVM on 8192 observations of four different classes of Gaussian processes using the Bartlett estimator when the asymptotic level is set to 0.05 and the finest scale to $J_1 = 1$.

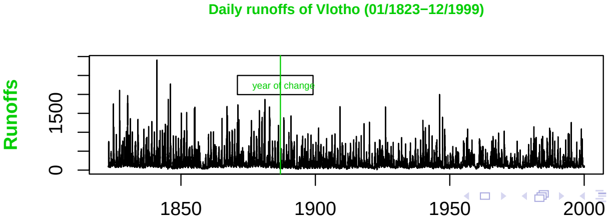
Level of the test based on the Asymptotic Covariance Matrix.

$n = 8192$		WN	$MA(1)$	$AR(1)$	$ARFIMA(1, d, 1)$	
					$d = 0.2$	$d = 0.3$
$J_2 = 7$	KSM	0.052	0.042	0.082	0.109	0.21
$J_2 = 7$	CVM	0.029	0.025	0.035	0.086	0.092
$J_2 = 8$	KSM	0.071	0.070	0.08	0.074	0.11
$J_2 = 8$	CVM	0.032	0.036	0.041	0.046	0.09
$J_2 = 9$	KSM	0.074	0.07	0.1	0.151	0.181
$J_2 = 9$	CVM	0.041	0.04	0.096	0.129	0.159

Table: Empirical level of $\texttt{KSM} - \texttt{CVM}$ on 8192 observations of four different classes of Gaussian processes using the asymptotic estimator when the asymptotic level is set to 0.05 and the finest scale $J_1 = 5$.

Change point detection on real data





Change points on Weser

	Boden-werder	Vlotho
$pvalue_{bartlett}$	$2e^{-9}$	$3e^{-11}$
$pvalue_{asymptotic}$	$1e^{-5}$	$1e^{-10}$
year of change	1909	1887

Table: Change-point in the river Weser

Conclusion

- These tests are able to detect the presence of non-stationarity for both long-range and short-range processes.
- The theorical results are rigously justified.
- The estimation of the Asymptotic Covariance matrix though the long memory parameter yields better results than the Bartlett estimator.