Some Applications of Differential-Difference Algebra to Creative Telescoping

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Summary

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Summary

What is creative telescoping?

Example: for every $x \in [0, 1/4)$, our goal is to prove the identity

$$F(x) \triangleq \sum_{n=0}^{+\infty} f(x,n) = \frac{1}{\sqrt{1-4x}}, \quad \text{where } f(x,n) = \binom{2n}{n} x^n.$$

Creative telescoping: find $L(x, D_x)(f) = \Delta_n(g)$, where

$$\underbrace{L(x, D_x)}_{\textit{Telescoper}} = 2 - (1-4x)D_x \quad \text{and} \quad \underbrace{g(x,n)}_{\textit{Certificate}} = \frac{n}{x} \cdot f.$$

$$\sum_n \text{ and } L \text{ commute } + \text{ "Nice" cond.: } \sum_{n=0}^{+\infty} \Delta_n(g) = 0$$

$$\Rightarrow \quad L(F(x)) = L\left(\frac{1}{\sqrt{1-4x}}\right) = 0$$

$$+ \qquad \Rightarrow \quad \text{The identity holds.}$$

$$F\,|_{x=0} = \frac{1}{\sqrt{1-4x}}\,|_{x=0} = 1$$

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Algorithms for creative telecoping

CT for sols of first-order D-Δ systems

hypergeom.	Zeilberger1990, Petkovšek-Wilf-Z.1996, Gessel1995, Abramov-Le2002
hyperexp.	Almkvist-Z.1990
q-hypergeom.	Wilf-Z.1992, Koornwinder1993, Paule-Riese1997
hyper.-hyper.	W-Z.1992, A-Z.1990, Koepf1998

CT for sols of high-order D-Δ systems

holonomic	Zeilberger1990, Takayama1992, Chyzak-Salvy1998 Chyzak2000
non-holonomic	Majewicz1996, Kauers2007, Chen-Sun2008, Chyzak-Kauers-Salvy2009

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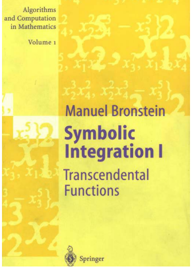
Summary

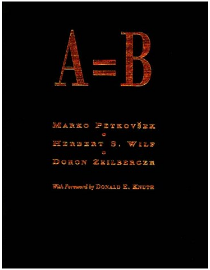
Motivation and approach

Complexity analysis $\Rightarrow$ fast algorithms $\Rightarrow$ fast implementation.

Risch's algorithm $\longrightarrow$ Zeilberger-style algorithms.



$\longrightarrow$



Hermite reduction

Creative telescoping

Notation

Rational functions:

- k : a field of characteristic zero;
- $k(x, y)$: the rational-function field in x and y over k;

Linear differential operators:

- $D_x = \frac{\partial}{\partial x}$, $D_y = \frac{\partial}{\partial y}$;
- $k(x, y)\langle D_x, D_y\rangle$: the ring of linear differential operators.

Rational case

Specialized telescoping problem for rational functions:

Given $f \in k(x, y)$, construct the minimal telescoper $L \in k(x)\langle D_x \rangle$ and $g \in k(x, y)$ s.t.

$$L(x, D_x)(f) = D_y(g).$$

Classical tool: Hermite reduction for rational-function integration

$$\int f\, dy = g + \sum_{i=1}^{n} c_i \log(h_i).$$

Applications:

Differential annihilators for diagonals and algebraic functions.

Main results on rational-function telescoping

Algorithms:

- Hermite: a new method based on Hermite reduction;
- RatAZ: improvements over Almkvist-Zeilberger's algorithm.

Arithmetic complexity:

CT for bivariate rational functions has polynomial-time complexity.

$$\text{Hermite: } \tilde{\mathcal{O}}(d^7), \qquad \text{RatAZ: } \tilde{\mathcal{O}}(d^9)$$

Implementation:

MAPLE function rational_creative_telescoping in MGFUN (Algolib 14.0.)

Hermite reduction for creative telescoping

Let $f = P/Q \in k(x, y)$, Q^* be the sqfr. part of Q, and $d_y^* = \deg_y(Q^*)$.

Hermite reduction w.r.t. y:

$$f = D_y(g) + \frac{a}{Q^*}, \quad \deg_y a < d_y^*.$$

Idea:

For $i = 0, 1, 2, \ldots,$ reduce

$$D_x^i(f) = D_y(g_i) + a_i/Q^*, \quad \deg_y(a_i) < d_y^*,$$

until $\exists\ \eta_0, \ldots, \eta_i \in k(x)$ with $\eta_i \neq 0$ s.t. $\sum_{j=0}^{i} \eta_j a_j = 0$.

$$\sum_{j=0}^{i} \eta_j a_j = 0 \quad \Longleftrightarrow \quad \underbrace{\sum_{j=0}^{i} \eta_j D_x^j}_{\text{telescoper}}(f) = D_y \underbrace{\left(\sum_{j=0}^{i} \eta_j g_j\right)}_{\text{certificate}}.$$

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Observations

- Lemma on order bound: Given $f = P/Q \in k(x,y)$,

  its minimal telescoper has order at most d_y^* ($\le \deg_y Q$).

- Certificates are optional:

$$a_j \in k(x)[y] \quad \text{and} \quad \deg_y(a_j) < d_y^*$$

$$\sum_{j=0}^{i} \eta_j(x) \cdot a_j = 0 \quad \Longrightarrow \quad \begin{cases} \text{Telescoper:} & \sum_{j=0}^{i} \eta_j D_x^j; \\ \text{Certificate:} & \sum_{j=0}^{i} \eta_j g_j. \end{cases}$$

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Improved Almkvist-Zeilberger's algorithm

AZ's algorithm: Given $f \in k(x, y)$, set $L = \sum_{i=0}^{\rho} \eta_i D_x^i$ and $g = r\, f$ in

$$L(x, D_x)(f) = D_y(g),$$

$$\Downarrow$$

$$D_y(r) + qr = \sum_{i=0}^{\rho} \eta_i p_i, \quad \text{for } \rho = 0, 1, \ldots,$$

where $q = D_y(f)/f$ and $p_i = D_x^i(f)/f$.

Improvements:

- Better denominator bounds (extending an idea in GeddesLe2002);
- Tighter degree bound (specializing a formula in Gerhard2001).

Complexity estimates for minimal telescopers

- For $f = P/Q \in k(x, y)$,

$$d_x = \max(\deg_x P, \deg_x Q), \quad d_y = \max(\deg_y P, \deg_y Q);$$

- L: minimal telescoper for f; g: certificate;

$$L(x, D_x)(f) = D_y(g).$$

- $2 \leq \omega \leq 3$: exponent of matrix multiplication over k.

Method	ord(L)	$\deg_x(L)$	$\deg_x(g)$	$\deg_y(g)$	Complexity
Hermite	d_y	$\mathcal{O}(d_x d_y^2)$	$\mathcal{O}(d_x d_y^2)$	$\mathcal{O}(d_y^2)$	$\tilde{\mathcal{O}}(d_x d_y^{\omega+3})$
RatAZ	d_y	$\mathcal{O}(d_x d_y^2)$	$\mathcal{O}(d_x d_y^2)$	$\mathcal{O}(d_y^2)$	$\tilde{\mathcal{O}}(d_x d_y^{2\omega+2})$

(Complexity is in terms of arithmetic operations in k.)

Implementations and timings (dense data)

- `AZ`: Function `Zeilberger` in DEtools;
- `RatAZ`: Improved `AZ` for rational functions;
- `Hermite`: Hermite reduction based method.

d	1	2	3	4	5	6	7
`AZ`	0.054	0.158	2.731	64.75	619.0	$>$ hr	$>$ hr
`RatAZ`	0.019	0.059	0.402	4.461	34.13	220.5	792.1
`Hermite`	0.016	0.057	0.398	2.664	18.80	106.2	422.5

(Timing in seconds.)

Data set: P and Q are in $\mathbb{Z}[x, y]$ generated by `randpoly()`,

$$f = \frac{P}{Q}, \quad d = d_x = d_y \in \{1, 2, \ldots, 7\}.$$

Application to diagonals

Definition. For $f = P/Q \in k(x, y)$ with $Q(0,0) \neq 0$, expand

$$f = \sum_{i,j \geq 0} f_{i,j} x^i y^j.$$

Define

$$\mathrm{diag}(f) := \sum_{i=0}^{\infty} f_{i,i} x^i.$$

Lemma (Lipshitz, 1988). If L is a telescoper for $\frac{f(y, x/y)}{y}$ w.r.t. y, then

$$L(\mathrm{diag}(f)) = 0.$$

Remark: Certificates are not needed at all.

Lattice path in the plane

Let $S_d = \{(i,j) \in \mathbb{N}^2 | i + j = d\}$,

$$f(x,y,d) = \frac{1}{1 - \sum_{(i,j)\in S_d} x^i y^j}, \qquad \text{for } 11 \leq d \leq 20.$$

d	11	12	13	14	15	16	17	18	19	20
AZ	48.7	5.72	144.	12.4	400.	23.9	1016.	46.7	$>$ hr.	81.2
RatAZ	43.8	5.61	129.	11.8	269.	27.9	663.4	45.8	2976.	88.4
Hermite	11.7	2.55	31.9	5.71	91.3	12.8	227.8	21.1	617.9	40.3
Order	11	6	13	7	15	8	17	9	19	10

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Motivation

Wilf and Zeilberger's conjecture: Let h be a hyper-hyper function. Then

$$h \text{ is holonomic} \quad \Leftrightarrow \quad h \text{ is proper.}$$

Multiplicative structure distinguishes

proper hyper-hyper functions from arbitrary ones.

- Differential case: multivariate Christopher's theorem;
- Difference case: the Ore-Sato theorem
- Mixed case: ?

Multivariate hyperexponential-hypergeometric functions

- k: an algebraically closed field of characteristic zero;
- $\mathbf{x} = (x_1, \ldots, x_m), \quad \mathbf{y} = (y_1, \ldots, y_n)$;
- D_i: derivation $\partial/\partial x_i$, $\quad E_j$: shift operator $y_j \to y_j + 1$.

Hyper-Hyper: $h(\mathbf{x}, \mathbf{y})$ is hyperexponential-hypergeometric over $k(\mathbf{x}, \mathbf{y})$ if

$$\text{all } \frac{D_i(h)}{h} \text{ and } \frac{E_j(h)}{h} \text{ are rational functions in } k(\mathbf{x}, \mathbf{y}).$$

Examples:

$$\frac{1}{x_1 x_2 + y_1 + y_2}, \exp\left(\frac{1}{x_1 + x_2}\right), (x_1 x_2 + 1)^{\sqrt{2}}, x_1^{y_1} x_2^{y_2}, (y_1 + y_2)!, \text{ etc.}$$

Classical structure theorems

Ore-Sato theorem (Ore1930, Sato1960's). A hypergeometric term $h(\mathbf{y})$ can be written as

$$f(\mathbf{y}) \cdot \prod_{j=1}^{n} u_j^{y_j} \cdot \prod_{p=1}^{P} (\phi_{p,1} y_1 + \cdots + \phi_{p,n} y_n + \varphi_p)!^{e_p},$$

where $f \in k(\mathbf{y})$, $u_j, \varphi_p \in k$, and $\phi_{p,j}, e_p \in \mathbb{Z}$.

Christopher's theorem (C. 1999). A hyperexponential function $h(\mathbf{x})$ can be written as

$$\exp(f) \prod_{\ell=1}^{L} g_\ell^{c_\ell},$$

where $f, g_\ell \in k(\mathbf{x})$, and $c_\ell \in k$.

Compatible rational functions

A first-order system

$$D_1(z) = a_1 z, \ldots, D_m(z) = a_m z, \quad E_1(z) = b_1 z, \ldots, E_n(z) = b_n z$$

has a nonzero solution iff $b_1 \cdots b_n \neq 0$ and the following compatibility conditions (CCs) hold:

$$D_i(a_j) = D_j(a_i), \quad \text{for } 1 \leq i < j \leq m,$$

$$\frac{E_i(b_j)}{b_j} = \frac{E_j(b_i)}{b_i}, \quad \text{for } 1 \leq i < j \leq n,$$

$$\frac{D_i(b_j)}{b_j} = E_j(a_i) - a_i, \quad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

Definition. $a_1, \ldots, a_m, b_1, \ldots, b_n$ are compatible if $b_1 \cdots b_n \neq 0$ and the above CCs hold.

Bivariate mixed case: Feng-Singer-Wu's theorem

Theorem (FSW, 2010). Let $a, b \in k(x, y)$ with $b \neq 0$. Then

$$E_y(a) - a = \frac{D_x(b)}{b},$$

if and only if there exist $f \in k(x, y)$, $\beta, \gamma \in k(x)$ and $\alpha \in k(y)$ s.t.

$$a = \frac{D_x(f)}{f} + y\frac{D_x(\beta)}{\beta} + \gamma \quad \text{and} \quad b = \frac{E_y(f)}{f}\beta\alpha.$$

Multivariate extension of FSW's lemma

Theorem 1 (new). Let $a_1, \ldots, a_m, b_1, \ldots, b_n \in k(\mathbf{x}, \mathbf{y})$ with $b_1 \cdots b_n \neq 0$. Then they satisfy the compatibility conditions:

$$D_i(a_j) = D_j(a_i), \quad \frac{E_i(b_j)}{b_j} = \frac{E_j(b_i)}{b_i}, \quad \text{and} \quad \frac{D_i(b_j)}{b_j} = E_j(a_i) - a_i,$$

if and only if there exist $f \in k(\mathbf{x}, \mathbf{y})$, $\beta_j, \gamma_i \in k(\mathbf{x})$, $\alpha_j \in k(\mathbf{y})$, s.t.

$$a_i = \frac{D_i(f)}{f} + \sum_{j=1}^{n} y_j \frac{D_i(\beta_j)}{\beta_j} + \gamma_i(\mathbf{x}), \quad 1 \leq i \leq m,$$

$$b_j = \frac{E_j(f)}{f} \cdot \beta_j \cdot \alpha_j(\mathbf{y}), \quad 1 \leq j \leq n,$$

and γ_i, α_j are compatible.

Multiplicative structure

Theorem 2 (new). A multivariate hyper-hyper function can be written as

$$f \cdot \prod_{j=1}^{n} \beta_j^{y_j} \cdot H_1(\mathbf{x}) \cdot H_2(\mathbf{y}),$$

where $f \in k(\mathbf{x}, \mathbf{y})$, $\beta_j \in k(\mathbf{x})$, H_1 is hyperexponential over $k(\mathbf{x})$, and H_2 is hypergeometric over $k(\mathbf{y})$.

Properness

Cor. A multivariate hyper-hyper function can be written as

$$\underbrace{f \cdot \prod_{j=1}^{n} \beta_j^{y_j}}_{Mixed} \cdot \underbrace{\exp(g_0) \prod_{\ell=1}^{L} g_\ell^{c_\ell}}_{Christopher} \cdot \underbrace{\prod_{p=1}^{P} (\phi_{p,1} y_1 + \cdots + \phi_{p,t} y_n + \varphi_p)!^{e_p}}_{Ore-Sato},$$

where $f \in k(\mathbf{x}, \mathbf{y})$, $g_0, g_\ell, \beta_j \in k(\mathbf{x})$, $c_\ell, \varphi_p \in k$, and $\phi_{p,j}, e_p \in \mathbb{Z}$.

Definition (Wilf–Zeilberger, 1992). A hyper-hyper function is proper if it admits the above form, in which f is a polynomial in $k[\mathbf{x}, \mathbf{y}]$.

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Telescoping problems: the bivariate case

Let $\Delta_m = E_m - 1$.

Discrete case:

$$\sum_m h(n, m) \quad \rightsquigarrow \quad L(n, E_n)(h) = \Delta_m(g)$$

Continuous case:

$$\int_a^b h(x, y)\, dy \quad \rightsquigarrow \quad L(x, D_x)(h) = D_y(g)$$

Mixed case:

$$\sum_n h(x, n) \quad \rightsquigarrow \quad L(x, D_x)(h) = \Delta_n(g)$$

$$\int_a^b h(x, n)\, dx \quad \rightsquigarrow \quad L(n, E_n)(h) = D_x(g)$$

Zeilberger-style algorithms and their termination

Zeilberger's algorithm: (Discrete case, 1990)

0. Initialize $\rho := 0$;
1. Set $L_\rho := \sum_{i=0}^{\rho} \ell_i(n) E_n^i$;
2. Solve $L_\rho(h) = \Delta_m(g_\rho)$ via ParaGosper;
3. If find a nontrivial solution in Step 2, return; otherwise increase ρ by 1 and go to Step 1.

Problem: When does Zeilberger's algorithm terminate?

$$\text{Termination} \quad \Leftrightarrow \quad \text{Existence of telescopers}$$

Sufficient condition: h is proper $\Rightarrow$ $\mathcal{Z}$ terminates on h

Existence criteria: the bivariate hyper-hyper case

- Differential case (Bernstein 1971, Lipshitz 1988):

  Hyperexponential $\Rightarrow$ Holonomic $\Rightarrow$ Telescoper exists

- Shift case (Abramov 2002), q-Shift case (Chen–Hou–Mu 2005):

  $$L(n, E_n)(h) = \Delta_m(g) \quad \Leftrightarrow \quad h = \Delta_m(h_1) + \text{ proper term}$$

- Mixed case:

  Problem: Given a hyper-hyper function $h(x, n)$, decide:

  - $\exists\, L \in k(x)\langle D_x\rangle \setminus \{0\}$ s.t. $L(x, D_x)(h) = \Delta_n(g)$?
  - $\exists\, L \in k(n)\langle E_n\rangle \setminus \{0\}$ s.t. $L(n, E_n)(h) = D_x(g)$?

  where g is hyper-hyper over $k(x, n)$.

Ingredient 1: Splitness

Definition. A polynomial $p \in k[x, n]$ is split if

$$p = p_1(x) \cdot p_2(n) \text{ with } p_1 \in k[x] \text{ and } p_2 \in k[n].$$

Lemma. Let L be either in $k(x, n)\langle D_x \rangle$ or in $k(x, n)\langle E_n \rangle$. Assume that

$$L(f) = p \quad \text{for some } f \in k(x, n) \text{ and } p \in k[x, n].$$

Then $\mathrm{den}(f)$ is split if the leading coeff of L is split.

Ingredient 2: Standard representations

Notation.

$$\mathcal{H}(a, b) := \{h \mid D_x(h) = a\,h,\ E_n(h) = b\,h\}\,.$$

Definition. $(f, \alpha, \beta, \gamma) \in k(x, n)^4$ is called a standard form of $h(x, n)$ if

$$h \in f\beta^n\mathcal{H}(\gamma, \alpha),$$

where

- any factor of $\mathrm{den}(f)$ is not split,
- $\alpha \in k(n)$ has monic numerator and denominator,
- $\beta, \gamma \in k(x)$.

Lemma. A hyper-hyper function is proper iff the first component of its standard form is a polynomial in $k[x, n]$.

Ingredient 3: Additive decompositions

(Abramov-Petkovšek 2002, Geddes–Le–Li 2004)

AP finds h_1 with $h/h_1 \in k(x,n)$, s.t.,

$$h - \Delta_n(h_1) \in \frac{v}{u} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \tilde{\alpha}(n)),$$

where $(u, v, \beta\tilde{\alpha})$ satisfies technical cond.'s (AP-triple).

Lemma. $h = \Delta_n(g) \;\Rightarrow\; u \in k(x)$.

GLL finds h_1 with $h/h_1 \in k(x,n)$, s.t.,

$$h - D_x(h_1) \in \frac{v}{u} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \alpha(n)),$$

where $\left(u,\ v,\ n\frac{D_x(\beta)}{\beta} + \tilde{\gamma}\right)$ satisfies technical cond.'s (GLL-triple).

Lemma. $h = D_x(g) \;\Rightarrow\; u \in k(n)$.

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Lemma. $h = D_x(g) \;\Rightarrow\; u \in k(n)$.

Ingredient 4: Applying L to additive decompositions

Differential:

$$h = \Delta_n(h_1) + h_2 \quad \text{AP decomp.}$$

$$\Downarrow$$

$$\forall\ L \in k(x)\langle D_x\rangle, \quad L(h) = \Delta_n(L(h_1)) + L(h_2) \quad \text{AP decomp.}$$

Recurrence:

$$h = D_x(h_1) + h_2 \quad \text{GLL decomp.}$$

$$\Downarrow$$

$$\forall\ L \in k(n)\langle E_n\rangle, \quad L(h) = D_x(L(h_1)) + L(h_2) \quad \text{GLL decomp.}$$

Two new criteria for existence

Theorem 1 (difference case): Let $h = \Delta_n(h_1) + h_2$ be an AP decomposition of h w.r.t. n. Then

$$h \text{ has a telescoper w.r.t. } n \quad \Leftrightarrow \quad h_2 \text{ is either zero or proper.}$$

Theorem 2 (differential case): Let $h = D_x(h_1) + h_2$ be a GLL decomposition of h w.r.t. x. Then

$$h \text{ has a telescoper w.r.t. } x \quad \Leftrightarrow \quad h_2 \text{ is either zero or proper.}$$

Telescopers for rational functions

Example:

$$h(x, n) = \frac{1}{(x+n)^s},$$

- $s = 1$: h has no telescoper w.r.t. n or x;
- $s > 1$:
  - h has no telescoper w.r.t. n,
  - h has a telescoper w.r.t. x,

$$h = D_x\left(\frac{-1}{(s-1)(x+n)^{s-1}}\right).$$

Telescopers for non-proper functions

Example:

$$h(x, n) = \frac{-x + 2\,nx + 2\,n^2}{(x+n)^2\,x} \cdot x^n \cdot e^{-x}.$$

Though h is not proper, it still has a telescoper w.r.t. x since

$$h = D_x\left(\frac{1}{x+n} x^n e^{-x}\right) + \frac{1}{x} \cdot x^n \cdot e^{-x}.$$

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Main contributions

- Hermite reduction based algorithm:
  - a lower complexity than RatAZ;
  - separating the computation of L and that of g;
  - a MAPLE implementation integrated into Mgfun(Algolib 14.0).

  (ISSAC2010 with Bostan, Chyzak, and Li)

- Structure theorem for hyper-hyper functions;

$$\text{Hyper-Hyper} = \text{Mixed} \cdot \text{Christopher} \cdot \text{Ore-Sato}$$

- Two criteria for the existence of telescopers:
  - h has a telescoper w.r.t. $n \quad \Leftrightarrow \quad h = \Delta_n(h_1) + \text{proper term}.$
  - h has a telescoper w.r.t. $x \quad \Leftrightarrow \quad h = D_x(h_1) + \text{proper term}.$

Ongoing work and Perspectives

- Extending the Hermite-reduction based method to

  hypergeometric, hyperexponential, or multivariate rational cases

- Existence criteria for telescopers in the q-setting:

(L, g)	D_y	$S_n - 1$	$E_n - 1$
$L(x, D_x)$	✓	✓	?
$L(n, E_n)$	✓	✓	?
$L(n, Q_n)$	?	?	✓

- Wilf and Zeilberger's conjecture in the general mixed case.

Thank you!

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