A journey through second order BSDEs and other contemporary issues in mathematical finance.

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A Journey through Second-Order BSDEs and other Contemporary Issues in Mathematical Finance

Voyage au Cœur des EDSRs du Second Ordre et autres Problèmes Contemporains de Mathématiques Financières

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[1]Le, ou plutôt les, problèmes en question se sont bien évidemment produits deux jours après avoir écrit ces lignes.
[2]bon...presque immédiatement.

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A journey through second-order BSDEs and contemporary issues in mathematical finance

Abstract: This PhD dissertation presents two independent research topics dealing with contemporary issues in mathematical finance, the second one being divided into into two distinct problems.

Throughout the first part of the dissertation, we study the notion of second order backward stochastic differential equations (2BSDE in the following), first introduced by Cheredito, Soner, Touzi and Victoir [25], then reformulated by Soner, Touzi and Zhang [107]. We start by proving an extension of their existence and uniqueness results to the case of a continuous generator with linear growth.

Then, we pursue our study with another extension to the case of a quadratic generator. The theoretical results obtained in that chapter allow us to solve a problem of utility maximization for an investor in an incomplete market, the source of incompleteness being on one hand the restrictions on the class of admissible trading strategies, and on the other hand the fact that the volatility of the market is uncertain. We prove the existence of optimal strategies, we characterize the value function of the problem thanks to a 2BSDE and solve explicetely several examples which give further insight into the main modifications introduced by the uncertain volatility framework.

We conclude the first part of the dissertation by introducing the notion of 2BSDEs reflected on an obstacle. We prove existence and uniqueness of the solutions of those equations and propose an application to the pricing problem of American options under volatility uncertainty.

The first chapter of the second part of the dissertation deals with a problem of option pricing in an illiquidity model. We provide asymptotic expansions of those prices in the infinite liquidity limit and highlight a transition phase effect depending on the regularity of the payoff considered. We also give numerical results.

Finally, the last chapter of this thesis is devoted to a Principal/Agent problem with moral hazard. A bank (the agent) has a certain number of defaultable loans and is ready to exchange their interests with the promess of payments. The bank can influence the default probabilities by choosing whether it monitors the loans or not, this monitoring being costly for the bank. Those choices are only known by the bank itself. Investors (the principal) want to design contracts which maximize their utility while implicitely giving incentives to the bank to monitor all the loans at all times. We solve explicitely this optimal control problem, we describe the associated optimal contract and its economic implications and provide some numerical simulations.

Keywords: Second order backward stochastic differential equations, singular probability measures, quasi-sure stochastic analysis, non-linear Feynman-Kac formula, fully non-linear PDEs, linear growth generator, quadratic growth generator, robust utility maximization, volatility uncertainty, obstacle problem, American options, super-replication, viscosity solutions, liquidity, asymptotic expansions, bank monitoring, CDSs, principal/agent problem, HJB equation.

Voyage au cœur des EDSRs du second ordre et autres problèmes contemporains de mathématiques financières

Résumé: Cette thèse présente deux principaux sujets de recherche indépendants, le dernier étant décliné sous la forme de deux problèmes distincts.

Dans toute la première partie de la thèse, nous nous intéressons à la notion d'équations différentielles stochastiques rétrogrades du second ordre (dans la suite 2EDSR), introduite tout d'abord par Cheredito, Soner, Touzi et Victoir [25] puis reformulée récemment par Soner, Touzi et Zhang [107]. Nous prouvons dans un premier temps une extension de leurs résultats d'existence et d'unicité lorsque le générateur considéré est seulement continu et à croissance linéaire.

Puis, nous poursuivons notre étude par une nouvelle extension au cas d'un générateur quadratique. Ces résultats théoriques nous permettent alors de résoudre un problème de maximisation d'utilité pour un investisseur dans un marché incomplet, à la fois car des contraintes sont imposées sur ses stratégies d'investissement, et aussi parce que la volatilité du marché est supposée être inconnue. Nous prouvons dans notre cadre l'existence de stratégies optimales, caractérisons la fonction valeur du problème grâce à une EDSR du second ordre et résolvons explicitement certains exemples qui nous permettent de mettre en exergue les modifications induites par l'ajout de l'incertitude de volatilité par rapport au cadre habituel.

Nous terminons cette première partie en introduisant la notion d'EDSR du second ordre avec réflexion sur un obstacle. Nous prouvons l'existence et l'unicité des solutions de telles équations, et fournissons une application possible au problème de courverture d'options Américaines dans un marché à volatilité incertaine.

Le premier chapitre de la seconde partie de cette thèse traite d'un problème de pricing d'options dans un modèle où la liquidité du marché est prise en compte. Nous fournissons des développements asymptotiques de ces prix au voisinage de liquidité infinie et mettons en lumière un phénomène de transition de phase dépendant de la régularité du payoff des options considérées. Quelques résultats numériques sont également proposés.

Enfin, nous terminons cette thèse par l'étude d'un problème Principal/Agent dans un cadre d'aléa moral. Une banque (qui joue le rôle de l'agent) possèdant un certain nombre de prêts, souhaite échanger leurs intérêts contre des flux de capitaux. La banque peut influencer les probabilités de défaut de ces emprunts en exerçant ou non une activité de surveillance coûteuse. Ces choix de la banque ne sont connus que d'elle seule. Des investisseurs (qui jouent le rôle de principal) souhaitent mettre en place des contrats qui maximisent leur utilité tout en incitant implicitement la banque à exercer une activité de surveillance constante. Nous résolvons ce problème de contrôle optimal explicitement, décrivons le contrat optimal associé ainsi que ses implications économiques et fournissons quelques simulations numériques.

Mots-clés: équations différentielles stochastiques rétrogrades du second ordre, mesures de probabilité singulières, analyse stochastique quasi-sûre, formule de Feynman-Kac non-linéaire, EDP complètement non-linéaires, générateur à croissance linéaire, générateur à croissance quadratique, maximisation d'utilité robuste, incertitude de volatilité, problème d'obstacle, options Américaines, surréplication, solutions de viscosité, liquidité, développements asymptotiques, surveillance des banques, CDS, problème principal/agent, équation de HJB.

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Chapitre 1

Introduction

Always be wary of any helpful item that weighs less than its operating manual.

Terry Pratchett

1.1 Une étude des EDSRs du second ordre

1.1.1 Des EDSRs classiques aux EDSRs du second ordre

1.1.1.1 Quelques préliminaires sur les EDSRs classiques

L'objectif principal de la première partie de cette thèse est l'étude des Equations Différentielles Stochastiques Rétrogrades du Second Ordre (notées dans la suite 2EDSRs) ainsi que de quelques unes de leurs possibles applications à des problèmes liés aux mathématiques financières. Commençons néanmoins par rappeler la notion classique d'EDSR (dans le cas réel pour simplifier la présentation). Plaçons-nous sur un espace probabilisé $(\Omega, \mathbb{F}, \mathbb{P})$ sur lequel est construit un mouvement Brownien W (d-dimensionel) dont la filtration naturelle et augmentée est notée $(\mathcal{F}_t)_{t\geq 0}$. Une EDSR à horizon déterministe T est alors une équation du type

$$y_t = \xi + \int_t^T f_s(y_s, z_s)ds - \int_t^T z_s dW_s,\ 0 \leq t \leq T,\ \mathbb{P} - p.s. \tag{1.1.1}$$

Les données d'une telle équation sont

- La condition terminale ξ, qui est une variable aléatoire $\mathcal{F}_T$-mesurable à valeurs dans $\mathbb{R}$.
- Le générateur $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, supposé mesurable par rapport à $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$, où $\mathcal{P}$ désigne la tribu des évènements prévisibles.

Résoudre une telle équation consiste alors à trouver un couple de processus $(y_t, z_t)_{0\leq t\leq T}$, $\mathcal{F}_t$-adaptés et vérifiant (1.1.1). Le vocable *rétrogrades* vient ici du fait que seule la condition terminale de l'équation est connue, au sens où $y_T = \xi$, et c'est de là que vient en grande partie la complexité de ce problème. En effet, puisque nous recherchons une solution adaptée, un simple retournement du temps est ici inenvisageable. Ceci explique la nécéssité de chercher la solution sous la forme non pas d'un, mais de deux processus, le processus z ayant pour but justement de garantir l'adaptabilité de la solution.

Les EDSRs ont été introduites pour la première fois par Bismut dans le cas d'un générateur linéaire [10], mais le véritable point de départ de la théorie telle qu'elle est connue aujourd'hui reste l'article de Pardoux et Peng [83], dans lequel est prouvé le théorème suivant.

Theorem 1.1.1 (Pardoux et Peng [83]). *Supposons que le générateur f est Lipschitz en (y,z), uniformément en (s,ω) et*

$$\mathbb{E}^{\mathbb{P}}\left[|\xi|^2+\int_0^T |f_s(0,0)|^2\,ds\right]<+\infty.$$

Alors l'EDSR (1.1.1) a une unique solution (y,z) telle que z soit un processus de carré intégrable.

Depuis ce premier résultat général d'existence, une littérature toujours plus vaste, à laquelle nous nous intéresserons par la suite, s'est attachée à affaiblir de plus en plus les hypothèses du théorème précédent. Cet engouement s'explique en partie par le très grand nombre de champs d'applications de la théorie des EDSRs, qu'il s'agisse de problèmes de contrôle stochastique, de jeux stochastiques, ou des problèmes de gestion de portefeuille... Nous renvoyons le lecteur à l'article [43] pour une revue détaillée des applications en finance. Néanmoins, c'est le lien extrêmement étroit qui existe entre la théorie des EDSRs et la théorie des Equations aux Dérivées Partielles (EDPs dans la suite) qui demeure la raison principale de cet intérêt marqué de la communauté mathématique. Revenons maintenant sur cette connexion.

Considérons une classe d'EDSRs particulières, dites Markoviennes. Pour ces équations, l'aléatoire de la condition terminale et du générateur est supposé être entièrement généré par une certaine diffusion. Plus précisément, (y,z) est solution de

$$y_t=g(X_T)+\int_t^T f(s,X_s,y_s,z_s)ds-\int_t^T z_s dW_s,\ 0\leq t\leq T,\ \mathbb{P}-p.s., \tag{1.1.2}$$

où f et g sont des fonctions déterministes et où $(x_t)_{0\leq t\leq T}$ est solution de l'EDS

$$X_t=x+\int_0^t b(s,X_s)ds+\int_0^t \sigma(s,X_s)dW_s,\ 0\leq t\leq T,\ \mathbb{P}-p.s.$$

Soit alors l'EDP

$$\begin{aligned}&\frac{\partial u}{\partial t}(t,x)+\mathcal{L}u(t,x)+f\left(t,x,u(t,x),\nabla u(t,x)\sigma(t,x)\right)=0,\ (t,x)\in[0,T)\times\mathbb{R}\\ &u(T,.)=g(.),\end{aligned} \tag{1.1.3}$$

où $\mathcal{L}$ est le générateur infinitésimal associé à la diffusion dont est solution X, donné par

$$\mathcal{L}v(t,x):=\frac{1}{2}\mathrm{Tr}\left[a(t,x)\nabla^2 v(t,x)\right]+b(t,x).\nabla v(t,x),$$

où $a(t,x):=\sigma(t,x)'\sigma(t,x)$.

Si nous supposons que cette EDP possède une solution régulière, une simple application de la formule d'Itô montre que $(u(t,X_t),\nabla u(t,X_t)\sigma(t,X_t))$ est solution de l'EDSR (1.1.2). Ce résultat, qui n'est rien d'autre qu'une généralisation de la formule de Feynman-Kac, nous confère ainsi une interprétation probabiliste de l'EDP (1.1.3) et ouvre la voie de la simulation numérique de solutions d'EDPs par des méthodes probabilistes, qui ont comme grand avantage de ne pas (ou peu) souffrir de problèmes liés à la dimension. De telles méthodes ont fait l'objet de nombreux travaux, parmi lesquels nous pouvons citer Zhang [118], [119] et Bouchard et Touzi [17]. Notons cependant que la théorie des EDSRs ne fournit une telle interprétation probabiliste que pour des EDP dites quasi-linéaires, au sens où la dépendance en la Hessienne dans (1.1.3) ne peut être que linéaire. En effet, les termes faisant intervenir la Hessienne ne proviennent que de la variation quadratique de X dans la formule d'Itô. Dès lors, le monde des EDPs complètement non-linéaires nous demeure fermé. Etant donné l'extrême importance que de telles équations peuvent revêtir dans de nombreux domaines des mathématiques, de la physique ou encore de l'ingénierie, il est on ne peut plus naturel et désirable d'étendre les résultats ci-dessus à une classe plus grande d'EDPs. C'est exactement cette volonté qui a été à l'origine de la première définition de la notion de 2EDSRs.

1.1.1.2 Une première formulation des 2EDSRs

Comme nous l'avons vu précédemment, si nous voulons étendre l'interprétation probabiliste aux EDPs complètement non-linéaires, il est nécessaire d'avoir une extension qui permet de considérer des processus avec des coefficients de diffusion différents. C'est exactement l'approche suivie dans l'article [25] par Cheredito, Soner, Touzi et Victoir, qui introduisent une généralisation naturelle de la notion d'EDSRs markoviennes sous la forme suivante.

Trouver un quadruplet (Y, Z, Γ, A) de processus $\mathcal{F}_t^T$-progressivement mesurable (i.e. progressivement mesurable par rapport à la filtration naturelle complétée de $W_s^t := W_s - W_t$), tel que

$$\begin{aligned}
dY_s &= h(s, X_s^{t,x}, Y_s, Z_s, \Gamma_s)ds - Z_s \circ dX_s^{t,x},\ t \leq s \leq T,\ \mathbb{P}-p.s. \\
dZ_s &= A_s ds + \Gamma_s dX_s^{t,x},\ t \leq s \leq T,\ \mathbb{P}-p.s. \\
Y_T &= g(X_T^{t,x}),
\end{aligned} \tag{1.1.4}$$

où $Z_s \circ dX_s^{t,x}$ désigne l'intégrale de Stratonovich, qui est reliée à l'intégrale d'Itô par

$$Z_s \circ dX_s^{t,x} := Z_s dX_s^{t,x} + \frac{1}{2} d\left\langle Z, X^{t,x}\right\rangle_s,$$

où la dynamique de $X^{t,x}$ est cette fois homogène en temps (par souci de simplicité)

$$X_s^{t,x} = x + \int_t^s b(X_u^{t,x})du + \int_t^s \sigma(X_u^{t,x})dW_u,\ t \leq s \leq T,\ \mathbb{P}-p.s.,$$

et où la fonction déterministe h est supposée uniformément Lipschitz en y, à croissance polynomiale en x, z et γ et décroissante en γ.

L'idée derrière cette formulation est d'ajouter une équation permettant d'avoir un contrôle sur la variation quadratique du processus Z avec le Brownien W, qui représente, au moins moralement, la partie qui fera intervenir la Hessienne dans l'EDP associée.

Considérons l'EDP

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) - h\left(t, x, u(t,x), \nabla u(t,x), \nabla^2 u(t,x)\right) = 0,\ (t,x) \in [0,T) \times \mathbb{R} \\
&u(T,.) = g(.).
\end{aligned} \tag{1.1.5}$$

Si cette EDP (1.1.5) admet une solution régulière, alors la formule d'Itô nous permet à nouveau d'obtenir que (Y, Z, Γ, A) défini ci-dessous est solution de (1.1.4).

$$\begin{aligned}
Y_s &= u(s, X_s^{t,x}),\ t \leq s \leq T,\ \mathbb{P}-p.s. \\
Z_s &= \nabla u(s, X_s^{t,x}),\ t \leq s \leq T,\ \mathbb{P}-p.s. \\
\Gamma_s &= \nabla^2 u(s, X_s^{t,x}),\ t \leq s \leq T,\ \mathbb{P}-p.s. \\
A_s &= \mathcal{L}\nabla u(s, X_s^{t,x}),\ t \leq s \leq T,\ \mathbb{P}-p.s.,
\end{aligned}$$

où

$$\mathcal{L}v(t,x) := \frac{\partial v}{\partial t}(t,x) + \frac{1}{2}\mathrm{Tr}\left[\nabla^2 v(t,x) a(x)\right],$$

avec $a(x) := \sigma(x)\sigma(x)'$.

Ce résultat, somme toute relativement simple, nous conduit naturellement aux questions suivantes. Qu'en est-il de l'unicité d'une telle solution, et surtout dans quel espace cette hypothétique unicité a-t-elle lieu ? Est-il possible de définir une telle 2EDSR dans le cas d'un générateur et d'une condition terminale non Markoviens ? L'article [25] fournit des réponses partielles à ces questions. Commençons par l'unicité. Pour cela, il est utile d'introduire deux problèmes, qui appartiennent à la famille des problèmes de cible stochastique du second ordre (voir [105] pour de plus amples détails). Fixons des constantes positives p, q et m, et considérons $\mathcal{A}_m^{t,x}$ la famille des processus de la forme

$$Z_s = z + \int_t^s A_u du + \int_t^s \Gamma_u dX_u^{t,x},\ t \leq s \leq T,\ \mathbb{P}-p.s.,$$

où $z \in \mathbb{R}^d$, $(A_s)_{t\leq s\leq T}$ et $(\Gamma_s)_{t\leq s\leq T}$ sont des processus $\mathcal{F}_t^T$-progressivement mesurables à valeurs respectivement dans $\mathbb{R}^d$ et $\mathbb{S}^{d,>0}$ tels que

$$\max\left\{|Z_s|, |A_s|, |\Gamma_s|\right\} \leq m\left(1 + \left|X_s^{t,x}\right|^p\right),\ t \leq s \leq T,$$

et

$$|\Gamma_s - \Gamma_r| \leq m\left(1 + \left|X_s^{t,x}\right|^q + \left|X_r^{t,x}\right|^q\right)\left(|s-r| + \left|X_s^{t,x} - X_r^{t,x}\right|\right),\ \text{pour tout } (r,s) \in [t,T].$$

Définissons alors $\mathcal{A}^{t,x} := \cup_{m\geq 0}\mathcal{A}_m^{t,x}$ et considérons pour un $Z \in \mathcal{A}^{t,x}$ la solution de l'EDS suivante

$$Y_s^{t,x,y,Z} = y + \int_t^s h(u, X_u^{t,x}, Y_u^{t,x,y,Z}, Z_u, \Gamma_u)du + \int_t^s Z_u \circ dX_u^{t,x},\ t \leq s \leq T,\ \mathbb{P}-p.s.$$

Le résultat principal de [25] (voir Théorème 4.9) nous indique alors en substance que

$$\begin{aligned} V(t,x) &:= \inf\left\{y,\ Y_T^{t,x,y,Z} \geq g(X_T) \text{ pour un } Z \in \mathcal{A}^{t,x}\right\} \\ U(t,x) &:= \sup\left\{y,\ Y_T^{t,x,y,Z} \leq g(X_T) \text{ pour un } Z \in \mathcal{A}^{t,x}\right\}, \end{aligned}$$

sont respectivement des sur et sous-solutions de viscosité de l'EDP (1.1.5), et par conséquent que si cette EDP vérifie un principe de comparaison au sens de la viscosité, alors (1.1.4) a une unique solution dans la classe $\mathcal{A}^{t,x}$.

Partant de ce résultat d'unicité, il est naturel de se demander si la classe $\mathcal{A}^{t,x}$ peut être étendue à une classe plus familière et moins technique, comme cela pouvait être le cas avec les EDSRs classiques, pour lesquelles toute la théorie est faite dans $L^2(\mathbb{P})$. Malheureusement, un contre-exemple en dimension $d = 1$ est donné dans [107] (voir exemple 7.1). En choisissant une condition terminale nulle et un générateur $f(t,y,z,\gamma) := \frac{c}{2}\gamma$ pour une constante $c \neq 1$, ils prouvent l'existence d'une solution non nulle à (1.1.4). En outre, comme nous le verrons ultérieurement dans le second chapitre de cette thèse, au moment de traiter le problème de sur-réplication dans le modèle de Çetin-Jarrow-Protter (qui est un cas particulier des problèmes de cible stochastique du second ordre mentionné ci-dessus), il est presque impossible de modifier la définition de $\mathcal{A}^{t,x}$ sans trivialiser la dépendence en Γ du problème.

Enfin, il n'existe à ce jour aucun résultat d'existence pour les 2EDSR définies de cette façon, en dehors du cas trivial traité ci-dessus. Ainsi, si cette définition permet effectivement d'avoir une interprétation probabiliste pour des EDP complètement non-linéaires (grâce à laquelle des algorithmes numériques ont été élaborés par Fahim et al. [50] puis utilisés en pratique par Guyon et Labordère [58] dans le cadre de modèles à volatilité incertaine), elle ne fournit pas une théorie d'existence et d'unicité satisfaisante, puisque l'existence n'est prouvée que dans le cas Markovien trivial, et l'unicité n'a lieu que dans un espace extrêmement technique et peu naturel, en comparaison des espaces habituels dans la théorie classique des EDSRs. Une reformulation du problème était donc nécessaire.

1.1.1.3 Formulation quasi-sûre des 2EDSRs

Cette nouvelle formulation des 2EDSRs, introduite par Soner, Touzi et Zhang dans [107] est, au premier abord, beaucoup moins naturelle que la précédente. De ce fait, attachons-nous dans un premier temps à fournir des intuitions venant du lien avec les EDPs mentionné précédemment. Considérons donc à nouveau l'EDP

$$\begin{aligned}&\frac{\partial u}{\partial t}(t,x) - h\left(t,x,u(t,x),\nabla u(t,x),\nabla^2 u(t,x))\right) = 0,\ (t,x)\in[0,T)\times\mathbb{R}\\ &u(T,.) = g(.),\end{aligned} \tag{1.1.6}$$

et supposons que h est convexe et décroissante en γ. Par des résultats classiques d'analyse convexe (voir Rockafeller [94]), nous savons que

$$h(t,x,y,z,\gamma) = \sup_{a\in\mathbb{S}^{d,>0}}\left\{\frac{1}{2}\mathrm{Tr}\left[a\gamma\right] - f(t,x,y,z,a)\right\},$$

où f est la transformée de Fenchel-Legendre de h. Il est donc naturel de s'attendre à ce que, dans un certain sens à préciser, nous ayons

$$u(t,x) = \sup_{a\in\mathbb{S}^{d,>0}} u^a(t,x),$$

où u^a est solution de l'EDP

$$\begin{aligned}&\frac{\partial u^a}{\partial t}(t,x) - \frac{1}{2}\mathrm{Tr}\left[a\nabla^2 u^a(t,x)\right] - f\left(t,x,u^a(t,x),\nabla u^a(t,x),a)\right) = 0,\ (t,x)\in[0,T)\times\mathbb{R}\\ &u^a(T,.) = g(.).\end{aligned} \tag{1.1.7}$$

L'EDP (1.1.7) étant semi-linéaire, nous savons qu'elle peut être associée à une EDSR classique. Ce raisonnement nous suggère donc qu'une définition cohérente pour une 2EDSR qui serait associée à (1.1.7) est d'introduire un processus Y qui vérifierait formellement

$$Y_t = \sup_{a\in\mathbb{S}^{d,>0}} y_t^a,$$

où

$$y_t^a = g(X_T^a) + \int_t^T f(s,X_s^a,y_s^a,z_s^a,a_s)ds - \int_t^T z_s^a dX_s^a,$$

avec

$$X_t^a = \int_0^t a_s^{1/2} dW_s.$$

L'idée principale de [107] est alors d'adapter à ce contexte l'analyse stochastique quasi-sûre développée par Denis et Martini [38] (et qui partage un lien plus qu'étroit avec la théorie des G-expectation de Peng, voir [88] et [40]), qui correspond à une extension de l'analyse stochastique habituelle dans un cadre où l'on ne travaille plus sous une seule mesure de probabilité fixée, mais sous une famille non-dominée de mesures de probabilité. Sans rentrer dans les détails techniques qui seront traités dans le corps de cette thèse, voyons dans notre exemple illustratif comment cette théorie peut nous être utile. Partons de notre mesure de probabilité de référence et définissons

$$\mathbb{P}^a := \mathbb{P} \circ (X^a_\cdot)^{-1},$$

qui n'est autre que la loi sous $\mathbb{P}$ du processus X^a. Modulo certaines technicalités (hautement non triviales comme nous le verrons), notre Y_t pourrait être considéré comme le supremum de solutions d'EDSRs ayant la même condition terminale et le même générateur, mais définies sous ces différentes mesures de probabilité $\mathbb{P}^a$. Ces dernières s'avérant en général être singulières, l'analyse stochastique quasi-sûre peut alors être utilisée.

Ces observations ont donc mené Soner, Touzi et Zhang [107] à considérer la fomulation suivante (simplifiée pour une présentation plus claire, une définition rigoureuse sera donnée dans le chapitre suivant, voir Section 2.2.4) pour une 2EDSR.

Trouver un triplet de processus (Y, Z, K), K étant un processus croissant nul en 0, tel que

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s, \widehat{a}_s)ds - \int_t^T Z_s dB_s + K_T - K_t, \ \mathbb{P} - p.s. \text{ pour tout } \mathbb{P}, \tag{1.1.8}$$

où B est le processus canonique et où la famille de mesure de probabilité, sous laquelle l'équation (1.1.8) est considérée, consiste en les lois sous la mesure de Wiener de processus du type des X^a ci-dessus. Le processus $\widehat{a}$ est quant à lui une version de la densité de la variation quadratique de B définie pour tout ω, et qui donc a la propriété très intéressante que pour tout processus $(a_t)_{t\geq 0}$, $\widehat{a}$ coïncide $\mathbb{P}^a$-p.s. avec a . Le processus K a également une interprétation heuristique. Comme nous l'avons vu, si cette définition est cohérente, Y doit formellement être supérieur à toutes les solutions d'EDSRs classiques avec la même condition terminale et le même générateur, sous les différentes mesures de probabilité considérées. Le rôle du processus K est alors de "pousser" le processus Y pour s'assurer que cette propriété soit tout le temps vérifiée.

Soner, Touzi et Zhang démontrent alors l'existence et l'unicité des solutions des équations du type (1.1.8) lorsque le générateur f est uniformément Lipschitz en (y, z), sous des conditions d'intégrabilité sur ξ et f et sous un certain nombre d'autres conditions techniques qui seront détaillées plus tard. En outre, comme dans le cadre des EDSRs classiques, une formule du type Feynman-Kac est prouvée et elle fournit le lien entre EDPs complètement non-linéaires et 2EDSRs. Notons néanmoins que cette formulation reposant de manière cruciale sur la convexité du générateur, elle ne permet d'obtenir une représentation probabiliste pour des EDPs du type (1.1.6) que lorsque la fonction h est convexe et décroissante en γ.

Revenons succintement sur les idées principales de leur preuve. Tout d'abord, l'unicité est obtenue par le biais d'une représentation de la solution Y comme un supremum en un certain sens des solutions y^a d'EDSRs classiques définies sous les différentes mesures de probabilité considérées. Cette représentation fournit alors un candidat naturel pour la solution. Néanmoins, les mesures de probabilité considérées étant singulières, il est alors extrêmement difficile de gérer les problémes liés aux ensembles négligeables. De ce fait, Soner, Touzi et Zhang commencent par construire ω par ω une solution dans le cas où la condition terminale est régulière (plus précisément lorsqu'elle est uniformément continue et bornée en ω), puis étendent leurs résultats à la fermeture de cet espace pour une norme appropriée. Ce type de preuve s'éloigne donc considérablement des preuves d'existence dans le cas des EDSRs classiques, qui reposent essentiellement sur des arguments de type point fixe et sur des estimations a priori des solutions.

Notre objectif dans la première partie de cette thèse a d'abord été d'étendre ce résultat d'existence et d'unicité de [107] à des 2EDSRs possédant des générateurs non nécessairement Lipschitz, avec comme but d'appliquer cette nouvelle théorie à des problématiques concrètes de mathématiques financières. Ainsi, les trois premiers chapitres s'inscrivent dans cette optique, et nous y démontrons des extensions de la théorie des 2EDSRs à des générateurs à croissance linéaire puis quadratique, avant d'utiliser ces résultats pour résoudre un problème de maximisation d'utilité pour un investisseur dans un marché incomplet où la volatilité est incertaine. Enfin, nous définissons une notion de 2EDSR dont la solution est contrainte à demeurer au-dessus d'un certain obstacle fixé, et proposons une application au pricing d'options Américaines en volatilité incertaine. Au-delà de ces extensions, nous nous sommes attachés, autant que faire se peut, à dégager une première ébauche d'un socle commun de techniques pouvant être utilisées dans cette théorie. En effet, nous l'avons déjà mentionné, la preuve d'existence dans [107] n'utilise pratiquement pas les techniques habituelles de la littérature des EDSRs, et, nous le verrons, elle semble assez difficilement généralisable à des situations plus complexes. Pour autant, nous verrons que contrairement au cas habituel, le fait de travailler sous plusieurs mesures de probabilité singulières engendre des problèmes techniques nombreux qui nous ont empêchés d'utiliser des techniques issues de la théorie classique des EDSRs dans la plupart des cas. Nous revenons un peu plus en détails sur ces problèmes dans le dernier chapitre de cette première partie.

1.1.2 2EDSRs à croissance linéaire

1.1.2.1 Articles sources

A la suite de l'article de Pardoux et Peng [83], de nombreux auteurs ont cherché à affaiblir les hypothèses sur le générateur permettant de prouver l'existence de solutions pour une EDSR. Un premier pas en ce sens a d'abord été effectué par Hamadène [60], qui prouve, dans le cas unidimensionel, l'existence d'une solution lorsque la condition terminale est bornée et que le générateur est localement Lipschitzien et à croissance sous-linéaire en (y, z). Par la suite, Lepeltier et San Martin [72] vont prouver l'existence de solutions maximales et minimales pour une EDSR dont le générateur est continu et à croissance linéaire en (y, z), avec une condition terminale de carré intégrable.

Notons que dans ces articles, la question de l'unicité n'est pas envisagée, et il a d'ailleurs été prouvé par Pardoux et Peng dans [84] que sans hypothèses supplémentaires, cette dernière n'avait pas nécessairement lieu. Cette question sera soulevée dans un article de Pardoux [85], dans lequel il obtient l'existence et l'unicité pour des EDSRs dont le générateur est uniformément Lipschitz en z, continu en y avec une croissance quelconque, et vérifie la condition suivante, dite de monotonie

$$(y - y')(f_t(y, z) - f_t(y', z)) \leq \mu(y - y')^2, \text{ pour tout } (t, y, y', z), \tag{1.1.9}$$

qui remplace le caractère Lipschitz en y.

Ces trois articles ont en commun l'approche utilisée pour construire une solution. Il s'agit essentiellement d'approximer le générateur f par une une suite de fonctions $(f_n)_{n\geq 0}$ qui converge de manière monotone en un certain sens vers f et qui sont Lipschitziennes. Tout le problème est alors d'arriver à prouver la convergence des solutions des EDSRs associées aux f_n. Ce type de preuve est couramment appelé preuve par approximation monotone, et constitue probablement l'une des techniques les plus usitées dans la théorie classique des EDSRs.

1.1.2.2 Motivation et nouveaux résultats

La principale motivation de ce premier chapitre de la thèse réside dans le fait qu'elle constitue une sorte de banc d'essai nous permettant de constater jusqu'où les techniques classiques de la théorie des EDSRs peuvent être utilisées dans le cadre des 2EDSRs. Nous nous sommes donc basés sur les trois articles cités précédemment pour obtenir un résultat d'existence et d'unicité pour des 2EDSRs dont le générateur est continu, à croissance linéaire en y, vérifiant la condition de monotonie (1.1.9), et uniformément Lipschitz en z. Il s'avère, contrairement au cas classique, que bien que ces hypothèses soient suffisantes pour prouver l'unicité d'une solution, elles ne nous ont pas permis de conclure quant au problème de l'existence. Pour cela, il nous a fallu considérer un générateur uniformément continu en y. La raison est fondamentale et inhérente au cadre des 2EDSRs. Dans toutes les preuves par approximation monotone, il est nécessaire à un moment ou un autre d'utiliser le théorème de convergence monotone, afin de déduire d'une convergence presque-sûre une convergence dans $L^2(\mathbb{P})$ par exemple. Et là où le bât blesse, c'est que ce théorème n'est plus forcément valable dans un cadre quasi-sûr. En effet, supposons donnée une suite de variables aléatoires $(X_n)_{n\geq 0}$ qui décroît vers 0, $\mathbb{P}-p.s.$ pour toutes les mesures de probabilité $\mathbb{P}$ considérées. Alors, nous ne pouvons pas affirmer de manière générale que

$$\sup_{\mathbb{P}} \mathbb{E}^{\mathbb{P}}[X_n] \downarrow 0,$$

un contre-exemple étant fourni à la fin de la première partie de la thèse (voir Section 6).

Ce problème avait déjà été abordé dans [40] (voir théorème 31). Par la suite, nous montrons que la condition ajoutée sur notre générateur permet d'appliquer ce théorème aux quantités qui nous intéressent. Plus précisément, suivant la technique introduite par Lepeltier et San Martin [72], nous approximons le générateur de notre 2EDSR par inf-convolution et montrons sous nos hypothèses que cette approximation converge uniformément en (y, z) vers le générateur initial. Dès lors, nous parvenons à n'utiliser le théorème de convergence montone que pour des quantités faisant intervenir le supremum en y et z du générateur et de son approximation. Ces quantités sont, sous nos hypothèses, définies pour tout ω et régulières en ω. N'ayant alors pas à gérer les problématiques liées aux ensembles négligeables, nous pouvons appliquer le théorème de convergence monotone de [40]. Notons qu'il s'agit de l'unique cas dans cette thèse où nous pouvons prouver l'existence pour l'2EDSR par une technique d'approximation monotone. Encore une fois, nous examinons plus précisément ce problème dans le court chapitre 6 qui clôt la première partie de la thèse.

Nous développons ensuite une stratégie de preuve qui diffère sensiblement des preuves classiques. Nous prenons ici le parti de ne pas rentrer plus dans les détails, les notations à introduire pour un exposé rigoureux et parlant étant trop lourdes, et nous renvoyons aux remarques qui émaillent le corps de la thèse.

1.1.3 2EDSRs à croissance quadratique

1.1.3.1 Articles sources

Depuis la fin des années 1990, l'intérêt pour les EDSRs dites à croissance quadratique (au sens où le générateur est à croissance quadratique en z) ont reçu une attention toute particulière, du fait de leur intérêt dans des problèmes liés aux mesures de risque dynamiques ou à la gestion de portefeuille avec contraintes (voir [46]). Ainsi, la question d'existence et d'unicité d'une solution dans le cas où le générateur vérifie

$$|f_t(y,z)| \leq |l_t| + c_t |y| + \frac{\delta}{2} |z|^2, \tag{1.1.10}$$

δ étant une constante positive et c et l des processus adaptés suffisamment intégrables, a d'abord été résolue par Kobylanski [68] dans le cas d'une condition terminale bornée. Rappellons brièvement les différentes étapes de la preuve. Dans un premier temps, à l'aide d'un changement de variable exponentiel, l'auteur parvient à établir des estimations a priori pour les solutions de l'EDSR, puis approche le problème quadratique par une suite de problèmes Lipschitziens et parvient à prouver des convergences suffisamment fortes pour pouvoir passer à la limite dans les EDSRs. La question de l'unicité est ensuite traitée par des techniques héritées des EDPs, et sera grandement simplifiée dans [62] par l'utilisation du caractère BMO du processus z.

Une autre preuve intéressante et très différente de celle de Kobylanski est due à Tevzadze [112] pour un générateur qui est en plus localement Lipschitz en z. Il utilise un argument de point fixe lorsque la condition terminale de l'EDSR est suffisament petite, puis coupe son EDSR en morceaux qui rentrent dans ce cadre et parvient enfin à les réunir pour obtenir une solution de l'EDSR initiale.

Plus récemment, Briand et Hu dans [14] et [15] ont étendu ces résultats au cas de conditions finales non bornées. Plus précisément, ils imposent que la condition terminale ξ ait suffisamment de moments exponentiels. Leur preuve consiste alors à se ramener au cas d'une condition terminale bornée puis à localiser astucieusement pour pouvoir obtenir de bonnes convergences. Ils obtiennent aussi un résultat d'unicité lorsque le générateur est convexe en z.

Enfin, une dernière avancée très récente dans ce domaine a été éffectuée par Barrieu et El Karoui [7] qui traitent de manière plus générale le problème d'existence pour une condition terminale ayant des moments exponentiels, mais en adoptant un point de vue forward, au sens où elles utilisent essentiellement des résultats de stabilité pour certaines familles de semi-martingales. L'intérêt de cette nouvelle approche est que des résultats d'existence plus puissants sont obtenus plus facilement que dans la littérature existante. Notons que cette approche avait déjà été envisagée par Barrieu, El Karoui et Cazanave dans un cadre moins général, voir [6].

1.1.3.2 Motivation et nouveaux résultats

Motivés par le problème de maximisation d'utilité en volatilité incertaine que nous traitons dans le troisième chapitre de la thèse, nous avons essayé de généraliser la littérature existante pour prouver des résultats d'existence et d'unicité pour des 2EDSRs quadratiques. Nous considérons, suivant Tevzadze [112], un générateur uniformément Lipschitz en y, localement Lipschitz en z et vérifiant une condition de croissance similaire à (1.1.10). Nous obtenons alors l'unicité par les mêmes techniques que dans [107]. Notons que dans cette preuve, le caractère BMO du processus Z joue un rôle tout à fait fondamental, beaucoup plus que dans le cas des EDSRs classiques. En ce qui concerne l'existence, nous proposons une preuve utilisant le même genre d'argument que la preuve initiale dans le cas Lipschitzien de [107], et permet d'obtenir l'existence en considérant le même ensemble de mesures de probabilité que dans [107] (contrairement au cas linéaire du Chapitre 2, où cet ensemble doit être restreint).

Enfin, nous étendons à ce cadre quadratique les liens avec les EDPs complètement non-linéaires prouvés dans [107] dans le cas Lipschitzien.

1.1.4 Maximisation d'utilité en volatilité incertaine

1.1.4.1 Littérature existante

Un problème aujourd'hui classique en mathématiques financières est celui auquel fait face un agent économique qui a la possibilité d'investir son argent dans un marché financier, répartissant sa richesse entre un actif non-risqué rémunéré à un taux fixe r (tel qu'un compte en banque) et pour simplifier, un unique actif risqué noté S, tel une action. Etant donné un horizon d'investissement fixe et fini T, le but de l'agent est de trouver une répartition optimale lui permettant de maximiser l'espérance de son utilité en T. Partant d'une richesse initiale x, nous notons π le portefeuille de l'agent (aussi appelé stratégie d'investissement), processus adapté représentant la richesse investie à chaque instant dans l'actif risqué. Puis, quitte à actualiser toutes les quantités, nous pouvons sans perte de généralité considérer le cas $r = 0$. Le processus de richesse de l'agent est alors donné à chaque instant par

$$X_t^{x,\pi} := x + \int_0^t \pi_s \frac{dS_s}{S_s}.$$

L'approche standard et aujourd'hui classique pour ce type de problème a d'abord été introduite par Von Neumann et Morgenstern dans [113], où ils supposent que l'ensemble des préférences de l'agent peut être réprésenté par une fonction d'utilité U et une mesure de probabilité fixée $\mathbb{P}$. La fonction U est supposée croissante (l'agent est de plus en plus heureux lorsque sa richesse s'accroît) et strictement concave (cette propriété traduit un phénomène de satiété). Dans ce cadre, l'agent cherche à résoudre le problème de maximisation suivant

$$V(x) := \sup_{\pi \in \mathcal{A}} \mathbb{E}^{\mathbb{P}} \left[U(X_T^{x,\pi}) \right],$$

où $\mathcal{A}$ est l'ensemble des portefeuilles admissibles, c'est-à-dire l'ensemble des stratégies que l'agent est autorisé à employer.

Ce problème, dit d'investissement optimal, fut résolu pour la première fois dans les années 60 par Merton [79] dans un cas particulier. Il suppose notamment que l'actif S suit une dynamique de Black-Scholes, c'est-à-dire qu'il s'agit d'un Brownien géométrique, qu'aucune restriction n'est appliquée sur les stratégies de portefeuille (et donc que le marché considéré est complet) et que la fonction d'utilité est donnée par

$$U(x) = \frac{x^\gamma}{\gamma},\ 0 < \gamma < 1.$$

Sa preuve repose sur des techniques classiques de contrôle optimal, puisqu'il parvient à résoudre explicitement l'équation d'Hamilton-Jacobi-Bellman associée au problème puis à appliquer un théorème de vérification. Néanmoins, il faudra attendre le milieu des années 80 pour que le problème en marché complet avec des fonctions d'utilité générales soit résolu par Pliska [90]. Dans cette preuve, beaucoup plus d'inspiration probabiliste, les outils majeurs deviennent l'utilisation de théorie de la dualité convexe, ainsi que l'existence d'une unique probabilité martingale pour S (le marché étant complet).

Par la suite, de nombreux auteurs se sont attachés à se débarasser des limites de la formulation de Merton, en particulier de l'hypothèse de marché complet et parfait, qui s'avère beaucoup trop restrictive et peu réaliste du point de vue des applications. Ainsi, dès 1976, Constantinides et Magill [27] se sont intéressés au problème posé par l'ajout de coûts de transaction, donnant naissance à une littérature assez prolifique. Parmi toutes les autres généralisations envisagées, une s'intéresse à l'introduction de contraintes sur les stratégies de portefeuille admissibles. Les premiers pas dans cette direction ont été réalisés par Cvitanić et Karatzas [30] et Zariphopoulou [117], qui imposent que les stratégies π demeurent dans un certain ensemble fixé. Là encore, les techniques utilisées relèvent de la dualité convexe ou de techniques de contrôle. Similairement, des résultats dans des modèles généraux de semi-martingales ont été obtenus par Kramkov et Schachermayer [67]. Il faudra ensuite attendre le début des années 2000 pour qu'un lien soit établi entre ce problème d'investissement optimal et la théorie des EDSRs. Ainsi, El Karoui et Rouge [46] considèrent le problème de prix d'indifférence pour une utilité exponentielle (qui est intiment lié au problème d'investissement optimal) dans le cas où les stratégies d'investissement sont contraintes à demeurer dans un ensemble convexe et fermé. Ils parviennent alors à prouver que la fonction valeur de leur problème est reliée à une EDSR dont le générateur est quadratique en z. Puis, poursuivant dans la même veine, Hu, Imkeller et Müller [62] généralisent les résultats de [46] aux cas des utilités puissance et logarithme, lorsque les stratégies de portefeuille sont contraintes dans un ensemble fermé. Une fois encore, ils parviennent à établir un lien avec des EDSRs quadratiques. Plus récemment encore, ces résultats ont été étendus au cas de marchés à sauts par El Karoui, Jeanblanc, Matoussi et Ngoupeyou [48].

Une toute autre direction de généralisation concerne la problématique sur l'incertitude liée au modèle. En effet, dans l'approche originale de Merton, une mesure de probabilité $\mathbb{P}$ est fixée. Cela signifie que nous supposons que l'agent connaît parfaitement la mesure de probabilité historique qui décrit la dynamique de l'actif risqué S. En réalité, il est beaucoup plus raisonnable de penser que l'agent fait face à une certaine incertitude sur le modèle à considérer, et qu'il envisage plusieurs mesures de probabilité plausibles. Cette généralisation porte le nom d'investissement optimal robuste, et consiste alors à résoudre le problème

$$V^{rob}(x) := \sup_{\pi\in\mathcal{A}} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[U(X_T^{x,\pi})\right],$$

où $\mathcal{P}$ est l'ensemble des mesures de probabilité plausibles envisagées par l'agent.

Dans ce cadre, l'agent cherche donc à maximiser l'espérance de l'utilité de sa richesse finale en considérant que la "Nature" le met dans le pire état possible pour lui. Dans ce cadre, les propriétés de l'ensemble $\mathcal{P}$ sont déterminantes pour la résolution du problème. Les premiers articles sur ce problème se sont intéressés à des ensembles $\mathcal{P}$ dominés, permettant par exemple de modéliser une incertitude sur le drift de la diffusion de S. Ainsi Gilboa et Schmeidler [55], puis Bordigoni, Matoussi et Schweizer [16] le résolvent par des techniques de contrôle. De même, des résultats ont été obtenus par Gundel [57], Quenez [91], Schied [97], Schied et Wu [98] et Skiadas [99] dans le cas d'une filtration continue. Globalement, ces différentes approches reposent essentiellement sur la dualité convexe.

La situation se complique beaucoup lorsque l'ensemble $\mathcal{P}$ n'est plus dominé, cas permettant de modéliser une incertitude sur la volatilité de la diffusion de S. Le problème de pricing d'options dans ce cadre avait déjà été étudié par Avellaneda, Lévy et Paras [2] et Lyons [76], et plus récemment par Denis et Martini [38] grâce à l'analyse stochastique quasi-sûre. Néanmoins, le problème d'investissement optimal n'a été traité que très récemment par Denis et Kervarec [39], qui parviennent à développer une théorie de la dualité dans ce cadre et prouvent ensuite l'existence d'une stratégie et d'une mesure de probabilité optimale, lorsque la fonction d'utilité obtenue est bornée. Notons que leur approche prend également en compte une incertitude par rapport au drift de S. Enfin, Epstein et Ji [49] ont formulé un modèle d'utilité dynamique dans ce cadre de volatilité incertaine.

1.1.4.2 Motivation et nouveaux résultats

Nous l'avons vu, El Karoui et Rouge [46] puis Hu, Imkeller et Müller [62] ont établi un lien entre le problème d'investissement optimal en marché incomplet (avec des contraintes sur les stratégies d'investissement) et les EDSRs quadratiques pour des fonctions d'utilité particulières. En outre, nous avons déjà mentionné le lien étroit qui existe entre le problème de pricing sous volatilité incertaine, résolu par l'analyse stochastique quasi-sûre dans [38], et la définition même des 2EDSRs. Dès lors, il est naturel de s'attendre à ce que le problème d'investissement optimal robuste avec des contraintes sur les stratégies d'investissement soit relié à des 2EDSRs quadratiques.

Cette intuition, qui nous a guidés tout au long de ce chapitre de la thèse, s'est avérée exacte. Ainsi, nous avons pu, en suivant les grandes idées de [46] et [62], résoudre le problème d'investissement optimal robuste gâce à une 2EDSR, dans le cas particulier d'une fonction d'utilité exponentielle, puissance et logarithme. Plus précisément, nous avons montré que

$$V^{rob}(x) = U(x - Y_0),$$

où Y_0 est la valeur initiale de la solution d'une certaine 2EDSR quadratique.

De plus, nous prouvons l'existence d'une stratégie optimale. Par rapport aux résultats de Denis et Kervarec, notre approche nous permet de résoudre explicitement le problème dans certains cas particuliers qui mettent en lumière les différences principales entre le problème classique et le problème robuste. Ainsi, si nous supposons que le processus de volatilité de S reste dans l'intervalle $[\underline{\sigma}, \overline{\sigma}]$, alors nous démontrons que le problème d'utilité robuste, sans aucune contrainte sur les stratégies, avec une utilité puissance est en fait équivalent au problème classique de Merton lorsque la volatilité de S est égale à la constante $\overline{\sigma}$. Ce résultat quelque peu surprenant doit être analysé à la lumière des résultats connus sur le problème de pricing en volatilité incertaine. En effet, nous savons que dans ce cas la probabilité optimale ne fait intervenir que les deux bornes $\underline{\sigma}$ et $\overline{\sigma}$, le passage de l'une à l'autre étant gouverné par la convexité du payoff g de l'option considérée. Cette propriété se déduit directement de l'EDP dite de Black-Scholes-Barrenblatt

$$-\frac{\partial v}{\partial t} - \frac{1}{2} \sup_{\underline{\sigma} \leq \sigma \leq \overline{\sigma}} \left(\sigma \frac{\partial^2 v}{\partial x^2} \right) = 0, \;\; v(T, .) = g(.),$$

dont est solution le prix de l'option en volatilité incertaine.

Nous intuitons alors une EDP similaire pour le problème d'investissement optimal robuste qui permet de comprendre le résultat de notre exemple. En outre, nous fournissons d'autres exemples explicites pour lesquels la probabilité optimale ne fait pas intervenir que les deux bornes $\underline{\sigma}$ et $\overline{\sigma}$, mais toutes les valeurs possibles dans l'intervalle. Etant donné que le problème d'investissement optimal robuste permet de calculer des prix d'indifférence robustes d'options (voir [46] dans le cas classique), cela met en évidence une différence fondamentale entre le pricing classique et le pricing par indifférence dans le cadre de la volatilité incertaine.

1.1.5 2EDSRs réfléchies sur un obstacle

1.1.5.1 Articles sources

Les EDSRs réfléchies sur un obstacle fixé ont été introduites par El Karoui, Kapoudjian, Pardoux, Peng et Quenez [44]. Il s'agit du premier cas d'EDSR avec contraintes, pour lesquelles on impose que la solution y_t reste systématiquement au-dessus d'un obstacle S_t. Un processus croissant dont le but est de "pousser" la solution de l'EDSR vers le haut est introduit. Plus précisément, nous disons que le triplet de processus adaptés (y_t, z_t, k_t), où k est un processus croissant, est solution de l'EDSR réfléchie sur l'obstacle S avec condition terminale ξ et générateur f si

$$\begin{aligned}
&y_t = \xi + \int_t^T f_s(y_s, z_s)ds - \int_t^T z_s dW_s + k_T - k_t,\ 0 \leq t \leq T,\ \mathbb{P}-p.s.\\
&y_t \geq S_t,\ 0 \leq t \leq T,\ \mathbb{P}-p.s.\\
&\int_0^T (y_s - S_s)dk_s = 0,\ \mathbb{P}-p.s.
\end{aligned} \tag{1.1.11}$$

La dernière condition dans (1.1.11) stipule que le processus croissant k est minimal au sens où il n'agit que lorsque y touche l'obstacle. Elle permet d'obtenir l'unicité de la solution d'une telle équation. Dans [44], une théorie d'existence et d'unicité avec des hypothèses similaires à celle de [83] est développée, et il est prouvé que les EDSRs réfléchies fournissent une représentation probabiliste pour des EDP quasi-linéaires avec un obstacle. Un lien étroit avec les problèmes d'arrêt optimal est également mis en évidence. Ce lien trouve alors son application dans le problème de pricing des options Américaines, résolu par El Karoui et Quenez grâce aux EDSRs réfléchies dans [43].

Avant de poursuivre, revenons rapidement sur les preuves d'existence pour les EDSRs réfléchies données dans [44]. La première utilise les propriétés de l'enveloppe de Snell et des arguments de point fixe, tandis que la seconde utilise le principe dit de pénalisation. Ce dernier consiste à considérer la suite d'EDSRs suivantes

$$y_t^n = \xi + \int_t^T f_s(y_s^n, z_s^n)ds - \int_t^T z_s^n dW_s + k_T^n - k_t^n,$$

où

$$k_t^n := n\int_0^t (y_s^n - S_s)^- ds.$$

S'inspirant en partie des techniques de preuve par approximation monotone, ils parviennent à montrer que (y^n, z^n, k^n) converge en un sens suffisamment fort vers une solution de (1.1.11).

Par la suite, ces problèmes d'EDSRs pour lesquels la solution (y, z) est contrainte ont été généralisés par Peng [86] puis Peng et Xu [87], et utilisés pour résoudre des problèmes de pricing d'options en marchés incomplets.

1.1.5.2 Nouveaux résultats

Avec comme motivation la résolution du problème de pricing d'options Américaines dans un marché à volatilité incertaine, nous nous sommes attachés dans ce pénultième chapitre de la première partie de la thèse à définir une notion d'EDSR du second ordre réfléchie sur une barrière inférieure (c'est-à-dire que la solution doit demeurer au-dessus d'un obstacle). Dans ce cas, et encore une fois sans rentrer dans les détails techniques, le problème prend la forme suivante.

Trouver un triplet de processus (Y, Z, K), K étant un processus croissant nul en 0, tel que

$$\begin{aligned}
&Y_t = \xi + \int_t^T f_s(Y_s, Z_s, \widehat{a}_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ \mathbb{P}-p.s. \text{ pour tout } \mathbb{P}\\
&Y_t \geq S_t,\ \mathbb{P}-p.s. \text{ pour tout } \mathbb{P}.
\end{aligned} \tag{1.1.12}$$

Intuitivement, le processus K a cette fois-ci une interprétation un peu différente que dans le cas des 2EDSRs sans réflexion. En effet, il permet d'une part de maintenir le processus Y au-dessus des solutions d'EDSRs réfléchies avec les mêmes paramètres sous les différentes mesures de probabilité. D'autre part, il assure que le processus Y reste au-dessus de l'obstacle S. Cette interprétation heuristique est importante, car elle met en lumière le fait que, contrairement aux EDSRs réfléchies classiques, il y a dans notre cadre une différence fondamentale entre la réflexion sur un obstacle inférieur et sur un obstacle supérieur. En effet, dans le cas classique, le problème avec obstacle supérieur se résoud en ajoutant un processus k décroissant. Ainsi, dans notre cas, nous nous retrouverions avec un processus décroissant lié à l'obstacle, et un processus croissant lié au cadre des 2EDSRs. Il faudrait alors considérer dans (1.1.12) un processus à variation finie, ce qui rendrait le problème beaucoup plus complexe.

Cela étant dit, nous proposons, comme pour les 2EDSRs à générateur quadratique, une preuve d'existence s'inspirant des techniques introduites dans [107], mais demandant plus de travail préalable, des résultats nécessaires n'existant pas à notre connaissance dans la littérature.

Enfin, nous utilisons nos résultats pour obtenir une caractérisation d'un prix de courverture pour des options Américaines en volatilité incertaine, qui sont naturellement reliées aux 2EDSRs réfléchies.

1.1.6 Perspectives

1.1.6.1 Du point de vue théorique

Le sujet des EDSRs du second ordre est un sujet très récent, la formulation que nous utilisons datant seulement de 2010. En ce sens, le recul que nous pouvons avoir sur ces notions n'est pas total, et l'un des buts de cette partie de la thèse est de fournir des outils et des bases permettant de mieux appréhender cette théorie dans de futures recherches. Dans cette optique, nous avons prouvé des généralisations des résultats d'existence et d'unicité de [107], en proposant quand cela nous a été possible des preuves utilisant des outils similaires à ceux de la théorie classique des EDSRs. Ces résultats sont certes un premier pas, mais ne sont pas encore totalement satisfaisants. Ainsi, dans les cas quadratiques et réfléchis, nous n'avons pas réussi à prouver l'existence grâce à des techniques d'approximation monotone. Cette absence est dommageable, car les preuves que nous proposons semblent difficilement généralisables à des cas plus complexes. En effet, elles utilisent de manière cruciale le fait que nos résultats d'unicité fournissent un candidat naturel pour l'existence. Or, il existe dans la littérature classique des EDSRs des cas pour lesquels l'unicité n'a pas lieu, mais pour lesquels il est possible de prouver l'existence de solutions minimales et maximales (voir par exemple [68] dans le cas quadratique). De tels résultats ne nous semblent accessibles que par des preuves par approximation monotone. Dans le même ordre d'idées, nous n'avons pas ici fournit de lien entre les 2EDSRs réflćhies sur un obstacle et les EDPs complètement non-linéaires avec obstacle. Ce lien, auquel nous pouvons naturellement nous attendre étant donnés les résultats de [44] pour les EDSRs réfléchies, est à notre sens un axe important pour de futures recherches. Or, dans le cas classique, la preuve du lien repose essentiellement sur le fait que la méthode par pénalisation permet de construire la solution d'une EDSR réfléchie comme limite d'une suite d'EDSRs, pour lesquelles le lien avec les EDPs quasi-linéaires est déjà connu. Ainsi, une preuve par pénalisation de l'existence pour les 2EDSRs réfléchies serait un premier pas dans cette direction.

Ces restrictions sont dues de manière plus générale au fait que l'analyse stochastique quasi-sûre, et plus généralement la théorie des capacités dont elle est issue, ne permettent pas d'utiliser à notre guise des résultats indispensables comme le théorème de convergence monotone ou même le théorème de convergence dominée. Un travail important, et qui reste aujourd'hui à faire, sur ces notions profondes devrait nous permettre de disposer d'outils très certainement plus adaptés à l'étude des EDSRs du second ordre.

Dans un autre ordre d'idées, nous avions remarqué que la théorie actuelle des EDSRs du second ordre ne fournit une représentation probabiliste que pour des EDPs complètement non-linéaires avec une dépendance convexe (ou concave) par rapport à la matrice Hessienne. Cette restriction est intimement liée à l'interprétation en terme de contrôle stochastique des 2EDSRs, et parvenir à la généraliser constitue un problème très intéressant. Des premiers résultats dans ce sens ont d'ailleurs été obtenus par Jianfeng Zhang et son étudiant Triet Pham dans des travaux non publiés à l'heure actuelle (voir [89] pour un premier aperçu). Ils s'intéressent à une notion de 2EDSR permettant de représenter des solutions d'EDP de la forme

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) - h\left(t,x,u(t,x),\nabla u(t,x),\nabla^2 u(t,x)\right)\right) = 0,\ \ (t,x)\in[0,T)\times\mathbb{R}\\
&u(T,.) = g(.),
\end{aligned} \tag{1.1.13}$$

où h s'écrit comme la différence de deux fonctions convexes en γ.

Ainsi, le générateur de l'EDP s'écrit alors comme la somme d'une fonction convexe et d'une fonction concave en γ. Dans, ce cas, au moins formellement, il faudrait considérer une 2EDSR dans laquelle le processus K ajouté n'est plus croissant, mais seulement à variation finie (et donc pouvant s'écrire comme différence de deux processus croissants). En ce sens, ce problème se rapprocherait de celui des 2EDSRs réfléchies sur un obstacle supérieur que nous avons déjà mentionné.

1.1.6.2 Du point de vue des applications

Les applications des 2EDSRs que nous avons proposées dans ce chapitre concernent uniquement des extensions de problèmes classiques de mathématiques financières à un cadre de volatilité incertaine. Il est certain que de très nombreuses autres applications à des problèmes dans ce cadre sont possibles (par exemple des problèmes de jeux stochastiques qui peuvent dans le cas classique être reliés à des EDSRs, voir [59] par exemple), la volatilité incertaine et les 2EDSRs étant extrêmement interconnectées. Néanmoins, cela ne doit pas nous faire oublier que plus généralement, les 2EDSRs ont un lien avec certaines EDPs non linéaires. Encore plus que cela, il a été prouvé par Soner, Touzi et Zhang dans [108] que les 2EDSRs jouaient un rôle prépondérant dans la formulation duale des problèmes de cible stochastique du second ordre. Un exemple de tels problèmes est celui de sur-réplication sous contrainte Gamma, étudié par Soner et Touzi dans [103]. Or, ce problème tel qu'il est défini, ne fait intervenir qu'une seule mesure de probabilité. Le lien qu'il partage alors avec les 2EDSRs, mentionné dans le paragraphe 4.3 de [108], n'est pas totalement clair. Comment justifier que la version duale de ce problème fasse intervenir une famille de mesures de probabilité singulières ? Cette compréhension nécessaire pour de plus larges applications de la théorie reste aujourd'hui encore partielle, et devrait constituer, à notre sens, une des pistes primordiales pour des recherches ultérieures.

1.2 Deux problèmes récents de mathématiques financières

1.2.1 Développements asymptotiques du coût de sur-réplication au voisinage de grande liquidité

1.2.1.1 Articles sources

Le modèle aujourd'hui classique de Black-Scholes, qui fournit un paradigme pour le pricing et la couverture d'options, repose de manière fondamentale sur le fait que le marché considéré doit être sans friction et compétitif. Autrement dit, les investisseurs peuvent acheter ou vendre à chaque instant n'importe quelle quantité d'actions sans aucun impact sur leur prix, et ces transactions ne sont soumises à aucun coût supplémentaire ou restriction. Ces hypothèses ne sont, bien entendu, pas réalistes, et la littérature en mathématiques financières regorge de travaux cherchant à les relaxer. Un des axes suivis par ces travaux concerne la prise en compte de ce que l'on appelle communément le risque de liquidité, qui représente les risques additionels supportés par un investisseur dus à la temporalité et à la taille de la transaction qu'il effectue à un instant donné. Récemment, de nombreux auteurs ont proposé des méthodes permettant d'incorporer le risque de liquidité dans le cadre du pricing et de la couverture des produits dérivés (voir par exemple [3], [18], [19] et [20]). La caractéristique commune de ces travaux réside en ce qu'ils modélisent le risque de liquidité comme un coût de transaction non-linéaire, dont l'origine est à chercher dans la perturbation de l'offre et la demande sur le marché lorsqu'un investisseur veut échanger d'importantes quantités sur une courte période de temps. Notre premier travail de cette seconde partie s'intéresse plus particulièrement au modèle introduit par Çetin, Jarrow et Protter dans [18], que nous allons maintenant détailler.

Le marché financier considéré est constitué de deux actifs, l'un est non risqué et son prix est normalisé à 1. Le deuxième actif est risqué et sensible au risque de liquidité. Plus précisément, les auteurs introduisent une courbe de demande qui modélise le prix de cet actif

$$\mathbf{S}(t, S(t), \nu),$$

où ν correspond au volume de la transaction et où $S(t) = \mathbf{S}(t, S(t), 0)$ est le processus de prix marginal défini comme solution d'une certaine EDS

$$\frac{dS(t)}{S(t)} = \mu(t, S(t))dt + \sigma(t, S(t))dW(t).$$

Ainsi, la fonction $\mathbf{S}$, supposée régulière, indique le prix à l'instant t d'une action pour un volume de transaction fixé ν. Partant de cette modélisation, Çetin, Jarrow et Protter prouvent que l'existence de ce coût de liquidité rend impossible pour un investisseur d'utiliser des stratégies d'investissement à variation quadratique infinie, ces dernières engendrant un coût de liquidité infini. En outre, les stratégies d'investissement continues et à variation finie n'entraînent quant à elles aucun coût de liquidité. Intuitivement, ce résultat est dû au fait qu'une transaction d'un certain volume peut alors être décomposée en une somme de petites transactions permettant de rendre le coût de liquidité négligeable. Ainsi, ils prouvent qu'un tel marché pour lequel il existe une unique probabilité martingale est approximativement complet, au sens où n'importe quel actif peut être répliqué par une suite de stratégies convergeant dans L^2. Ainsi, d'après leurs résultats, l'introduction du risque de liquidité ne modifie pas les prix des options. Notons que ce phénomêne ne se produit qu'en temps continu. En effet, [20] traite une version en temps discret de ce problème pour lequel un coût de liquidité non nul existe, ce dernier disparaissant dans la limite continue.

A la suite de ces résultats, Çetin, Soner et Touzi [21] se sont aussi intéressés à ce problème. Leur intuition est que cette absence de coût de liquidité en temps continu est un paradoxe du à la modélisation adoptée dans [18]. Ils parviennent alors à démontrer en imposant des restrictions sur les stratégies d'investissement admissibles (essentiellement sur leur dynamique et sur leur gamma) que le coût de sur-réplication d'une option de payoff g était donné par l'unique solution au sens de la viscosité de l'EDP suivante

$$-\frac{\partial v}{\partial t}(t,s) - \frac{s^2\sigma^2}{4l}\left[-l^2 + \left(\left(\frac{\partial^2 v}{\partial s^2} + l\right)^+\right)^2\right] = 0, \tag{1.2.1}$$

avec comme condition terminale $v(T,.) = g(.)$ et où $l(t,s) := \left[4\frac{\partial \mathbf{S}}{\partial \nu}(t,s,0)\right]^{-1}$ est l'indice de liquidité du marché.

Notons que l'ensemble des stratégies d'investissement admissibles est assez semblable à celui que nous avons déjà décrit dans la première formulation des 2EDSRs, et de ce fait assez technique, et surtout, sa définition ne peut pratiquement pas être modifiée, sous peine de trivialiser le problème. Il est donc a priori difficile de savoir si l'approche de [21] est plus pertinente. Heureusement, Soner et Gökay ont prouvé dans [56] que la limite en temps continu de ce modèle dans un cadre binomial donnait exactement la même EDP que (1.2.1). Comme les stratégies d'investissement dans le modèle binomial ne sont pas restreintes, le résultat de [56] confirme la pertinence de celui de [21]. Cette différence entre le temps discret et le temps continu, ce dernier nécessitant des restrictions supplémentaires des stratégies admissibles, n'est pas une surprise en soi. En effet, même dans des marchés sans friction, le choix des stratégies admissibles est fondamental en temps continu. Ainsi, il est nécessaire pour éviter l'apparition d'opportunités d'arbitrage d'imposer certaines contraintes d'intégrabilité ou de borner inférieurement les processus de richesse. Les restrictions dans ce cadre de marché illiquide doivent donc être considérées comme des généralisations de ces dernières.

Enfin, notons que l'EDP (1.2.1) est complètement non-linéaire, et, comme le problème de sur-réplication sous contrainte gamma évoqué plus haut, ce problème admet une version duale qui peut être reliée à une 2EDSR.

1.2.1.2 Nouveaux résultats

Lorsque l'on observe l'EDP (1.2.1), on constate aisément que lorsque l tend vers l'infini, c'est-à-dire lorsque la liquidité du marché devient infinie, elle tend formellement vers l'EDP classique de Black-Scholes. Nous pouvons donc nous attendre à ce que les prix d'options en fassent de même. Afin d'étudier ce problème, nous introduisons un nouvel indice de liquidité

$$l^\varepsilon(t,s) := \frac{l(t,s)}{\varepsilon},$$

de telle sorte que le paramètre ε gouverne la distance entre ce modèle et le modèle de Black-Scholes. Notre but dans ce chapitre est de prouver des développements asymptotiques du prix de sur-réplication donné par (1.2.1) quand ε tend vers 0.

Comme cela est généralement le cas avec de tels problèmes de développements asymptotiques, nous commençons par effectuer des calculs formels et parvenons ensuite, en utilisant des techniques de la théorie des solutions de viscosité, à prouver, lorsque le payoff g est suffisamment régulier, un développement du type

$$v^\varepsilon = v^{BS} + \varepsilon v^{(1)} + \ldots + \varepsilon^n v^{(n)} + \; + o(\varepsilon^n), \tag{1.2.2}$$

où v^ε est le prix de sur-réplication et v^{BS} le prix Black-Scholes correspondant.

Ensuite, nous prouvons le même résultat pour des payoffs convexes et continus vérifiant certaines propriétés dont les Call et les Put font partie. Dans ce cas, la preuve fait usage de techniques de régularisation des payoffs par convolution.

Nous remarquons ensuite que dans le cas d'une option digitale, dont le payoff est discontinu, le terme $v^{(1)}$ qui apparaît dans (1.2.2) devient infini. Cela signifie que ce développement ne peut plus être valable dans ce cas. Nous intuitons alors, ceci étant confirmé par nos simulations numériques, que le terme de premier ordre dans ce cas est donné par ε^{α} pour un certain $0 < \alpha < 1$. Nous adaptons alors nos méthodes précédentes, mais ne parvenons qu'à prouver que si cet exposant α existe, alors il est nécessairement compris entre 2/5 et 1.

1.2.1.3 Perspectives

Clairement, une piste possible d'amélioration de nos résultats serait d'obtenir complètement le développement dans le cas d'une option digitale. Néanmoins, il faudrait pour cela très certainement utiliser des techniques différentes de celles que nous avons employées, celles-ci ne nous semblant pas pouvoir véritablement être poussées plus loin. De plus, le caractère non-linéaire de notre EDP et le fait que nous travaillons uniquement avec des solutions au sens de la viscosité ne nous permettent pas d'utiliser les approches classiques de Fouque, Papanicolaou et Sircar [54]. Un autre axe de recherche possible serait de traiter non plus seulement des options Européennes dans ce cadre, mais aussi des options dites "path-dependent" comme les options Américaines.

1.2.2 Incitations optimales dans un problème Principal/Agent

Dans ce dernier chapitre de la thèse, nous traitons d'un problème dit de Principal-Agent avec aléa moral, introduit pour la première fois dans [82]. Nous considérons ainsi une banque (jouant ici le rôle de l'agent) qui a la possibilité d'établir un ensemble d'emprunts identiques rémunérés à un taux μ fixé. Cette dernière a des capacités financières limitées, mais a face à elle des investisseurs (le principal) qui sont prêts à mettre en place des contrats avec la banque leur permettant de récupérer les intérêts des emprunts, contre lesquels ils versent une rente, fixée par contrat, à la banque. A chaque instant, la banque a la possibilité d'exercer ou non une surveillance sur les emprunts, la surveillance permettant de réduire la probabilité de défaut de l'emprunt, mais occasionnant un surcoût opérationnel pour la banque. Ces actions ne sont pas observables pour les investisseurs, d'où le terme d'aléa moral, la banque pouvant prendre des décisions défavorables pour les investisseurs. Notons que dans notre modélisation, nous supposons qu'à chaque instant de défaut l'ensemble de tous les emprunts restant peut être liquidé avec une certaine probabilité, qui est elle aussi fixée par le contrat.

Nous partons alors du principe que les investisseurs souhaitent que la banque surveille à chaque instant l'ensemble des emprunts n'ayant pas encore fait défaut. Sachant que la banque choisit son action en maximisant son utilité, qui dépend de sa compensation stipulée dans le contrat, tout le problème des investisseurs est alors de concevoir un contrat leur permettant de maximiser leur utilité et qui oblige implicitement la banque à effectuer une surveillance constante. Nous avons alors devant nous un problème de maximisation sous contrainte pour lequel l'ensemble d'admissibilité est assez complexe et difficile à décrire. L'étape cruciale dans la résolution consiste alors à introduire de nouveaux contrôles par le biais d'un théorème de représentation de martingales, permettant d'avoir une description mathématique simple de notre ensemble d'admissibilité. Ce type de raisonnement a été introduit dans le cadre de problèmes Principal/Agent en temps continu par Sannikov [96], puis repris par Biais et al. dans [9]. Une fois ramenés à un problème de contrôle optimal classique, nous considérons le système récursif d'équations d'Hamilton-Jacobi-Bellman associé. Nous parvenons, sous certaines conditions sur les paramètres du modèle, à résoudre explicitement ce système, puis nous prouvons un théorème de vérification nous assurant que nous avons ainsi bien obtenu la fonction valeur du problème.

Nous obtenons également le contrat optimal qui met bien en lumière le rapport de force entre les investisseurs et la banque dans ce problème. La banque est supposée plus impatiente que les investisseurs (au sens où elle actualise sa richesse à un taux supérieur) et souhaite donc toucher une rente. Le contrat optimal que nous obtenons stipule que des paiements ne sont faits à la banque que quand son utilité a atteint un niveau suffisant, signe pour les investisseurs que les emprunts sont bien gérés par la banque. Sachant qu'à chaque instant de défaut l'utilité de la banque subit un saut négatif, cette dernière est alors incitée à surveiller les emprunts pour diminuer d'autant plus la probabilité d'un défaut. Quant à la décision de liquidation de l'ensemble des emprunts après un défaut (qui signifie l'arrêt définitif des paiements reçus par la banque), tant que l'utilité de la banque est suffisamment importante, les investisseurs n'ont pas besoin de cette menace supplémentaire et la probabilité de liquidation est nulle. Par contre, en dessous d'un certain seuil, celle-ci devient strictement positive, incitant une fois encore la banque à exercer une surveillance accrue.

Puis, nous examinons le cas particulier où la banque et les investisseurs sont aussi patients. Nous prouvons alors que dans ce cas l'aléa moral ne fait plus perdre d'utilité à l'ensemble constitué de la banque et des investisseurs, et tout se passe comme si nous étions dans un cadre de coopération totale. Nous fournissons également des simulations numériques qui éclairent nos résultats.

Enfin, notons que dans [82], les résultats concernant le contrat optimal que nous obtenons, permettent de décrire l'implémentation d'une titrisation complète de l'ensemble des emprunts considérés, par le biais de l'émission par la banque d'Asset Backed Credit Default Swaps (ABCDS). Contrairement à des CDS traditionels, pour lesquels le vendeur (ici la banque) ne doit verser une compensation à l'acheteur qu'à l'unique instant où un défaut intervient, en échange de quoi il reçoit périodiquement des paiements, les ABCDS sont des contrats dans lequels les paiments sont effectués au jour le jour. Le contrat optimal permet alors pour les investisseurs de savoir quand et comment ils doivent payer la banque pour que celle-ci ait une politique responsable vis-à-vis de la surveillance des emprunts titrisés.

Partie I

Second order backward stochastic differential equations

Chapitre 2

2BSDEs with linear growth

Contents

2.1 Introduction

Backward stochastic differential equations (BSDEs for short) appeared in Bismuth [10] in the linear case, and then have been widely studied since the seminal paper of Pardoux and Peng [83]. Their range of applications includes notably probabilistic numerical methods for partial differential equations, stochastic control, stochastic differential games, theoretical economics and financial mathematics.

On a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ generated by an $\mathbb{R}^d$-valued Brownian motion B, a solution to a BSDE consists on finding a pair of progressively measurable processes (Y, Z) such that

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s,\ t \in [0,T],\ \mathbb{P} - a.s.$$

where f (also called the driver) is a progressively measurable function and ξ is an $\mathcal{F}_T$-measurable random variable.

Pardoux and Peng proved existence and uniqueness of the above BSDE provided that the function f is uniformly Lipschitz in y and z and that ξ and $f_s(0,0)$ are square integrable, but also that if the randomness in f and ξ is induced by the current value of a state process defined by a forward stochastic differential equation, then the solution to the BSDE could be related to the solution of a semilinear PDE by means of a generalized Feynman-Kac formula. Since their pioneering work, many efforts have been made to relax the assumptions on the driver f. For instance, Lepeltier and San Martin [72] have proved the existence of a solution when f is only continuous in (y, z) with linear growth, and Kobylanski [68] obtained the existence and uniqueness of a solution when f is continuous and has quadratic growth in z and the terminal condition ξ is bounded.

More recently, motivated by applications in financial mathematics and probabilistic numerical methods for PDEs (see [21], [50] and [105]), Cheredito, Soner, Touzi and Victoir [25] introduced the notion of Second order BSDEs (2BSDEs), which are connected to the larger class of fully nonlinear PDEs. Then, Soner, Touzi and Zhang [107] provided a complete theory of existence and uniqueness for 2BSDEs under uniform Lipschitz conditions similar to those of Pardoux and Peng. Their key idea was to reinforce the condition that the 2BSDE must hold $\mathbb{P}-a.s.$ for every probability measure $\mathbb{P}$ in a non-dominated class of mutally singular measures (see Section 2.2 for precise definitions).

Our aim in this chapter is to relax the Lipschitz-type hypotheses of [107] on the driver of the 2BSDE to prove an existence and uniqueness result. In Section 2.3, inspired by Pardoux [85], we study 2BSDEs with a driver which is Lipschitz in some sense in z, uniformly continuous with linear growth in y and satisfies a monotonicity condition. We then prove existence and uniqueness and highlight one of the main difficulties when dealing with 2BSDEs. Indeed, the main tool in the proof of existence is to use monotonic approximations (as in [68], [72] or [81] among many others). However, since we are working under a family of non-dominated probability measures, the monotone or dominated convergence theorem may fail (a question we further study in Chapter 6), which in turn raises subtle technical difficulties in the proofs.

2.2 Preliminaries

Let $\Omega := \left\{\omega \in C([0,T],\mathbb{R}^d) : \omega_0 = 0\right\}$ be the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{0\leq t\leq T}|\omega_t|$, B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0\leq t\leq T}$ the filtration generated by B, and $\mathbb{F}^+ := \left\{\mathcal{F}_t^+\right\}_{0\leq t\leq T}$ the right limit of $\mathbb{F}$. We first recall the notations introduced in [107].

2.2.1 The local martingale measures

We say a probability measure $\mathbb{P}$ is a local martingale measure if the canonical process B is a local martingale under $\mathbb{P}$. By Karandikar [64], we know that we can give pathwise definitions of the quadratic variation $\langle B\rangle_t$ and its density $\widehat{a}_t$.

Let $\overline{\mathcal{P}}_W$ denote the set of all local martingale measures $\mathbb{P}$ such that

$$\langle B\rangle_t \text{ is absolutely continuous in } t \text{ and } \widehat{a} \text{ takes values in } \mathbb{S}_d^{>0},\ \mathbb{P}-a.s. \tag{2.2.1}$$

where $\mathbb{S}_d^{>0}$ denotes the space of all $d\times d$ real valued positive definite matrices.

We recall from [107], the class $\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$ consisting of all probability measures

$$\mathbb{P}^\alpha := \mathbb{P}_0\circ (X^\alpha)^{-1} \text{ where } X_t^\alpha := \int_0^t \alpha_s^{1/2}dB_s,\ t\in[0,1],\ \mathbb{P}_0-a.s., \tag{2.2.2}$$

and we concentrate on the subclass $\widetilde{\mathcal{P}}_S \subset \overline{\mathcal{P}}_S$ defined by

$$\widetilde{\mathcal{P}}_S := \left\{\mathbb{P}^\alpha \in \overline{\mathcal{P}}_S,\ \underline{a}\leq \alpha\leq \bar{a},\ \mathbb{P}_0-a.s.\right\}, \tag{2.2.3}$$

for fixed matrices $\underline{a}$ and $\bar{a}$ in $\mathbb{S}_d^{>0}$. We recall from [108] that every $\mathbb{P}\in\overline{\mathcal{P}}_S$ (and thus in $\widetilde{\mathcal{P}}_S$) satisfies the Blumenthal zero-one law and the martingale representation property.

Definition 2.2.1. *We say that a property holds $\widetilde{\mathcal{P}}_S$-quasi surely ($\widetilde{\mathcal{P}}_S-q.s.$ for short) if it holds $\mathbb{P}-a.s.$ for all $\mathbb{P}\in\widetilde{\mathcal{P}}_S$.*

2.2.2 The non-linear generator

We consider a map $H_t(\omega, y, z, \gamma) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$, where $D_H \subset \mathbb{R}^{d\times d}$ is a given subset containing 0.

Define the corresponding conjugate of H w.r.t. γ by

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2}\mathrm{Tr}(a\gamma) - H_t(\omega, y, z, \gamma) \right\} \text{ for } a \in S_d^{>0}$$
$$\widehat{F}_t(y, z) := F_t(y, z, \widehat{a}_t) \text{ and } \widehat{F}_t^0 := \widehat{F}_t(0, 0).$$

We denote by $D_{F_t(y,z)} := \{a,\ F_t(\omega, y, z, a) < +\infty\}$ the domain of F in a for a fixed (t, ω, y, z).

As in [107] we fix a constant $\kappa \in (1, 2]$ and restrict the probability measures in

Definition 2.2.2. *$\mathcal{P}_H^\kappa$ consists of all $\mathbb{P} \in \widetilde{\mathcal{P}}_S$ such that*

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{F}_t^0\right|^\kappa dt\right)^{\frac{2}{\kappa}}\right] < +\infty.$$

It is clear that $\mathcal{P}_H^\kappa$ is decreasing in κ, and $\widehat{a}_t \in D_{F_t}$, $dt \times d\mathbb{P} - a.s.$ for all $\mathbb{P} \in \mathcal{P}_H^\kappa$. We will also note $\overline{\mathcal{P}}_H^\kappa$ the closure for the weak topology of $\mathcal{P}_H^\kappa$.

Remark 2.2.1. *Unlike in [107], we assume that the bounds on the density of the quadratic variation $\widehat{a}$ are uniform with respect to the underlying probability measure. In particular, this ensures that the family $\mathcal{P}_H^\kappa$ is weakly relatively compact and that $\overline{\mathcal{P}}_H^\kappa$ is weakly compact.*

We now state our main assumptions on the function F which will be our main interest in the sequel

Assumption 2.2.1. (i) *The domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z).*

(ii) *For fixed (y, z, a), F is $\mathbb{F}$-progressively measurable.*

(iii) *We have the following uniform Lipschitz-type property*

$$\forall (y, z, z', a, t, \omega),\ \left|F_t(\omega, y, z, a) - F_t(\omega, y, z', a)\right| \leq C\left|a^{1/2}(z - z')\right|.$$

(iv) *F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.*

(v) *F is uniformly continuous in y, uniformly in (z, t, ω, a), and has the following growth property*

$$\exists C > 0 \ s.t.\ \forall (t, y, a, \omega),\ |F_t(\omega, y, 0, a)| \leq |F_t(\omega, 0, 0, a)| + C(1 + |y|).$$

(vi) *We have the following monotonicity condition. There exists $\mu > 0$ such that*

$$(y_1 - y_2)(F_t(\omega, y_1, z, a) - F_t(\omega, y_2, z, a)) \leq \mu |y_1 - y_2|^2,\ \text{for all } (t, \omega, y_1, y_2, z, a)$$

(vii) *F is continuous in a.*

Remark 2.2.2. *Let us comment on the above assumptions. Assumptions 2.2.1* (i) *and* (iv) *are taken from [107] and are needed to deal with the technicalities induced by the quasi-sure framework. Assumptions 2.2.1* (ii) *and* (iii) *are quite standard in the classical BSDE litterature. Then, Assumptions 2.2.1* (v) *and* (vi) *where introduced by Pardoux in [85] in a more general setting (namely with a general growth condition in y, and only a continuity assumption on y) and are also quite common in the litterature (see [12], [13] and [73]). Let us immediately point out that as explained in Remark 2.3.4 below, we must restrict ourselves to linear growth in y, because of the technical difficulties due to the 2BSDE framework. Moreover, we need to assume uniform conitnuity in y to ensure that we have a strong convergence result for the approximation we will consider (see also Remark 2.3.4). Finally, Assumption 2.2.1* (vii) *is needed in our framework to obtain technical results concerning monotone convergence in a quasi-sure setting.*

2.2.3 The spaces and norms

We now recall from [107] the spaces and norms which will be needed for the formulation of the second order BSDEs. Notice that all subsequent notations extend to the case $\kappa = 1$.

For $p \geq 1$, $L_H^{p,\kappa}$ denotes the space of all $\mathcal{F}_T$-measurable scalar r.v. ξ with

$$\|\xi\|^p_{L_H^{p,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[|\xi|^p\right] < +\infty.$$

$\mathbb{H}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}^d$-valued processes Z with

$$\|Z\|^p_{\mathbb{H}_H^{p,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T |\widehat{a}_t^{1/2} Z_t|^2 dt\right)^{\frac{p}{2}}\right] < +\infty.$$

$\mathbb{D}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes Y with

$$\mathcal{P}_H^\kappa - q.s. \text{ càdlàg paths, and } \|Y\|^p_{\mathbb{D}_H^{p,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} |Y_t|^p\right] < +\infty.$$

For each $\xi \in L_H^{1,\kappa}$, $\mathbb{P} \in \mathcal{P}_H^\kappa$ and $t \in [0,T]$ denote

$$\mathbb{E}_t^{H,\mathbb{P}}[\xi] := \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \ \mathbb{E}_t^{\mathbb{P}'}[\xi] \text{ where } \mathcal{P}_H^\kappa(t^+,\mathbb{P}) := \left\{\mathbb{P}' \in \mathcal{P}_H^\kappa : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t^+\right\}.$$

Here $\mathbb{E}_t^{\mathbb{P}}[\xi] := E^{\mathbb{P}}[\xi|\mathcal{F}_t]$. Then we define for each $p \geq \kappa$,

$$\mathbb{L}_H^{p,\kappa} := \left\{\xi \in L_H^{p,\kappa} : \|\xi\|_{\mathbb{L}_H^{p,\kappa}} < +\infty\right\} \text{ where } \|\xi\|^p_{\mathbb{L}_H^{p,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\underset{0\leq t\leq T}{\operatorname{ess\,sup}^{\mathbb{P}}} \left(\mathbb{E}_t^{H,\mathbb{P}}[|\xi|^\kappa]\right)^{\frac{p}{\kappa}}\right].$$

Finally, we denote by $\mathrm{UC}_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi : \Omega \to \mathbb{R}$ with respect to the $\|\cdot\|_\infty$-norm, and we let

$$\mathcal{L}_H^{p,\kappa} := \text{the closure of } \mathrm{UC}_b(\Omega) \text{ under the norm } \|\cdot\|_{\mathbb{L}_H^{p,\kappa}}, \text{ for every } 1 \leq \kappa \leq p.$$

2.2.4 Formulation

We shall consider the following second order BSDE (2BSDE for short), which was first defined in [107]

$$Y_t = \xi + \int_t^T \widehat{F}_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t, \ 0 \leq t \leq T, \ \mathcal{P}_H^\kappa - q.s. \tag{2.2.4}$$

Definition 2.2.3. *For $\xi \in L_H^{2,\kappa}$, we say $(Y,Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a solution to 2BSDE (2.2.4) if :*

- $Y_T = \xi$, $\mathcal{P}_H^\kappa - q.s.$
- $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$, *the process $K^\mathbb{P}$ defined below has non-decreasing paths $\mathbb{P}-a.s.$*

$$K_t^\mathbb{P} := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ 0 \le t \le T,\ \mathbb{P}-a.s. \tag{2.2.5}$$

- *The family $\left\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$ satisfies the minimum condition*

$$K_t^\mathbb{P} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^\mathbb{P}} \mathbb{E}_t^{\mathbb{P}'}\left[K_T^{\mathbb{P}'}\right],\ 0 \le t \le T,\ \mathbb{P}-a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \tag{2.2.6}$$

Moreover if the family $\left\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$ can be aggregated into a universal process K, we call (Y,Z,K) a solution of 2BSDE (2.2.4).

Following [107], in addition to Assumption 2.2.1, we will always assume

Assumption 2.2.2. (i) *$\mathcal{P}_H^\kappa$ is not empty.*

(ii) *The process $\widehat{F}^0$ satisfies the following integrability condition*

$$\phi_H^{2,\kappa} := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\left[\underset{0 \le t \le T}{\operatorname{ess\,sup}^\mathbb{P}} \left(\mathbb{E}_t^{H,\mathbb{P}}\left[\int_0^T |\hat{F}_s^0|^\kappa ds\right]\right)^{\frac{2}{\kappa}}\right] < +\infty. \tag{2.2.7}$$

Before going on, let us recall one of the main results of [107]. For this, we first recall their assumptions on the generator F

Assumption 2.2.3. (i) *The domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z).*

(ii) *For fixed (y,z,a), F is $\mathbb{F}$-progressively measurable.*

(iii) *We have the following uniform Lipschitz-type property*

$$\forall (y, y', z, z', a, t, \omega),\ \left|F_t(\omega, y, z, a) - F_t(\omega, y', z', a)\right| \le C\left(\left|y - y'\right| + \left|a^{1/2}(z - z')\right|\right).$$

(iv) *F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.*

Theorem 2.2.1 (Soner, Touzi, Zhang [107])**.** *Let Assumptions 2.2.2 and 2.2.3 hold. Then, for any $\xi \in \mathcal{L}_H^{2,\kappa}$, the 2BSDE (2.2.4) has a unique solution $(Y,Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$.*

2.2.5 Representation and uniqueness of the solution

We follow once more Soner, Touzi and Zhang [107]. For any $\mathbb{P} \in \mathcal{P}_H^\kappa$, $\mathbb{F}$-stopping time τ, and $\mathcal{F}_\tau$-measurable random variable $\xi \in \mathbb{L}^2(\mathbb{P})$, let $(y^\mathbb{P}, z^\mathbb{P}) := (y^\mathbb{P}(\tau,\xi), z^\mathbb{P}(\tau,\xi))$ denote the unique solution to the following standard BSDE (existence and uniqueness under our assumptions follow from Pardoux [85])

$$y_t^\mathbb{P} = \xi + \int_t^\tau \widehat{F}_s(y_s^\mathbb{P}, z_s^\mathbb{P})ds - \int_t^\tau z_s^\mathbb{P} dB_s,\ 0 \le t \le \tau,\ \mathbb{P}-a.s. \tag{2.2.8}$$

We then have similarly as in Theorem 4.4 of [107]

Theorem 2.2.2. *Let Assumptions 2.2.1 and 2.2.2 hold. Assume* $\xi \in \mathbb{L}^{2,\kappa}_H$ *and that* $(Y,Z) \in \mathbb{D}^{2,\kappa}_H \times \mathbb{H}^{2,\kappa}_H$ *is a solution of the 2BSDE (2.2.4). Then, for any* $\mathbb{P} \in \mathcal{P}^\kappa_H$ *and* $0 \leq t_1 < t_2 \leq T$,

$$Y_{t_1} = \underset{\mathbb{P}^{'} \in \mathcal{P}^\kappa_H(t_1,\mathbb{P})}{\text{ess sup}^{\mathbb{P}}} \; y^{\mathbb{P}^{'}}_{t_1}(t_2, Y_{t_2}), \;\; \mathbb{P}-a.s. \tag{2.2.9}$$

Consequentlty, the 2BSDE (2.2.4) has at most one solution in $\mathbb{D}^{2,\kappa}_H \times \mathbb{H}^{2,\kappa}_H$.

Proof. The proof is almost identical to the proof of Theorem 4.4 in [107], and we reproduce it here for the convenience of the reader. First, if (2.2.9) holds, then

$$Y_t = \underset{\mathbb{P}^{'} \in \mathcal{P}^\kappa_H(t^+,\mathbb{P})}{\text{ess sup}^{\mathbb{P}}} \; y^{\mathbb{P}^{'}}_t(T,\xi), \;\; t \in [0,T], \;\; \mathbb{P}-a.s. \text{ for all } \mathbb{P} \in \mathcal{P}^\kappa_H,$$

and thus is unique. Then, since we have that $d\langle Y, B\rangle_t = Z_t dB_t$, $\mathcal{P}^\kappa_H - q.s.$, Z is also unique. We shall now prove (2.2.9).

(i) Fix $0 \leq t_1 < t_2 \leq T$ and $\mathbb{P} \in \mathcal{P}^\kappa_H$. For any $\mathbb{P}^{'} \in \mathcal{P}^\kappa_H(t_1^+,\mathbb{P})$, we have

$$Y_t = Y_{t_2} + \int_t^{t_2} \widehat{F}_s(Y_s,Z_s)ds - \int_t^{t_2} Z_s dB_s + K^{\mathbb{P}^{'}}_{t_2} - K^{\mathbb{P}^{'}}_t, \;\; 0 \leq t \leq t_2, \;\; \mathbb{P}^{'}-a.s.$$

and that $K^{\mathbb{P}^{'}}$ is nondecreasing, $\mathbb{P}^{'}-a.s.$ Then, we can apply a generalized comparison theorem proved by Lepeltier, Matoussi and Xu (see Theorem 4.1 in [73]) under $\mathbb{P}^{'}$ to obtain $Y_{t_1} \geq y^{\mathbb{P}^{'}}_{t_1}(t_2,Y_{t_2})$, $\mathbb{P}^{'}-a.s.$ Since $\mathbb{P}^{'} = \mathbb{P}$ on $\mathcal{F}^+_t$, we get $Y_{t_1} \geq y^{\mathbb{P}^{'}}_{t_1}(t_2,Y_{t_2})$, $\mathbb{P}-a.s.$ and thus

$$Y_{t_1} \geq \underset{\mathbb{P}^{'} \in \mathcal{P}^\kappa_H(t_1^+,\mathbb{P})}{\text{ess sup}^{\mathbb{P}}} \; y^{\mathbb{P}^{'}}_{t_1}(t_2, Y_{t_2}), \;\; \mathbb{P}-a.s.$$

(ii) We now prove the reverse inequality. Fix $\mathbb{P} \in \mathcal{P}^\kappa_H$. We will show in (iii) below that

$$C^{\mathbb{P}}_{t_1} := \underset{\mathbb{P}^{'} \in \mathcal{P}^\kappa_H(t_1^+,\mathbb{P})}{\text{ess sup}^{\mathbb{P}}} \; \mathbb{E}^{\mathbb{P}^{'}}_{t_1}\left[\left(K^{\mathbb{P}^{'}}_{t_2} - K^{\mathbb{P}^{'}}_{t_1}\right)^2\right] < +\infty, \;\; \mathbb{P}-a.s.$$

For every $\mathbb{P}^{'} \in \mathcal{P}^\kappa_H(t^+,\mathbb{P})$, denote

$$\delta Y := Y - y^{\mathbb{P}^{'}}(t_2,Y_{t_2}) \text{ and } \delta Z := Z - z^{\mathbb{P}^{'}}(t_2,Y_{t_2}).$$

By the Lipschitz Assumption 2.2.1(iii) and the monotonicity Assumption 2.2.1(vi), there exist a bounded process λ and a process η which is bounded from above such that

$$\delta Y_t = \int_t^{t_2} \left(\eta_s \delta Y_s + \lambda_s \widehat{a}_s^{1/2} \delta Z_s\right) ds - \int_t^{t_2} \delta Z_s dB_s + K^{\mathbb{P}^{'}}_{t_2} - K^{\mathbb{P}^{'}}_t, \;\; t \leq t_2, \;\; \mathbb{P}^{'}-a.s.$$

Define for $t_1 \leq t \leq t_2$

$$M_t := \exp\left(\int_{t_1}^t \left(\eta_s - \frac{1}{2}|\lambda_s|^2\right) ds - \int_{t_1}^t \lambda_s \widehat{a}_s^{-1/2} dB_s\right), \;\; \mathbb{P}^{'}-a.s.$$

By Itô's formula, we obtain, as in [107], that

$$\delta Y_{t_1} = \mathbb{E}^{\mathbb{P}^{'}}_{t_1}\left[\int_{t_1}^{t_2} M_t dK^{\mathbb{P}^{'}}_t\right] \leq \mathbb{E}^{\mathbb{P}^{'}}_{t_1}\left[\sup_{t_1 \leq t \leq t_2} (M_t)(K^{\mathbb{P}^{'}}_{t_2} - K^{\mathbb{P}^{'}}_{t_1})\right],$$

since $K^{\mathbb{P}'}$ is non-decreasing. Then, because λ is bounded and η is bounded from above, we have for every $p \geq 1$

$$\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\sup_{t_1 \leq t \leq t_2} (M_t)^p\right] \leq C_p,\ \mathbb{P}' - a.s.$$

Then it follows from the Hölder inequality that

$$\delta Y_{t_1} \leq C(C_{t_1}^{\mathbb{P}'})^{1/3}\left(\mathbb{E}_{t_1}^{\mathbb{P}'}\left[K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\right]\right)^{1/3},\ \mathbb{P}' - a.s.$$

By the minimum condition (2.2.6) and since $\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})$ is arbitrary, this ends the proof.

(iii) It remains to show that the estimate for $C_{t_1}^{\mathbb{P}'}$ holds. By definition of the family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\}$, the linear growth condition in y of the generator and the Lipschitz condition in z, we have

$$\sup_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\right)^2\right] \leq C\left(1 + \|Y\|_{\mathbb{D}_H^{2,\kappa}}^2 + \|Z\|_{\mathbb{H}_H^{2,\kappa}}^2 + \phi_H^{2,\kappa}\right) < +\infty.$$

Then we proceed exactly as in the proof of Theorem 4.4 in [107].

□

2.3 Proof of existence

2.3.1 Preliminary results

In order to prove existence, we need an approximation of continuous functions by Lipschitz functions proved by Lepeltier and San Martin in [72]. In the present framework, we also use the well-known approximation of Moreau and Yosida (see [80] and [115])

Define

$$F_t^n(y, z, a) := \inf_{(u,v) \in \mathbb{Q}^{d+1}} \left\{F_t(u, v, a) + n\,|y - u| + n\left|a^{1/2}(z - v)\right|^2\right\},$$

where the minimum over rationals $(u, v) \in \mathbb{Q}^{d+1}$ coincides with the minimum over real parameters by our continuity assumptions, and ensures the measurability of F^n.

Lemma 2.3.1. *Let C be the constant in Assumption 2.2.1. We have*

(i) *F^n is well defined for $n \geq C$ and we have*

$$|F_t^n(y, z, a)| \leq |F_t(0, 0, a)| + C(1 + |y| + |a^{1/2}z|),\ \textit{for all } (y, z, a, n, t, \omega).$$

(ii) $|F_t^n(y, z_1, a) - F_t^n(y, z_2, a)| \leq C|a^{1/2}(z_1 - z_2)|$, *for all* $(y, z_1, z_2, a, t, \omega)$.

(iii) $|F_t^n(y_1, z) - F_t^n(y_2, z)| \leq n\,|y_1 - y_2|$, *for all* $(y_1, y_2, z, a, t, \omega)$.

(iv) *The sequence $(F_t^n(y, z, a))_n$ is increasing, for all (t, y, z, a).*

(v) *If F is decreasing in y, then so is F^n.*

Proof. (i). Clearly $F^n \leq F$. By the linear growth assumption on F, this provides one side of the inequality. Then, using the Lipschitz property Assumption 2.2.1(iii)

$$\begin{aligned}
&F_t^n(y,z,a)\\
&\geq \inf_{(u,v)\in\mathbb{Q}^{d+1}} \left\{-C(1+|u|+|a^{\frac{1}{2}}z|+|a^{\frac{1}{2}}(z-v)|)-|F_t(0,0,a)|+n(|y-u|+|a^{\frac{1}{2}}(z-v)|^2)\right\}\\
&=-|F_t(0,0,a)|-C(1+|y|+|a^{1/2}z|)-\frac{C^2}{4n}\\
&\geq -|F_t(0,0,a)|-C^{'}(1+|y|+|a^{1/2}z|).
\end{aligned}$$

Hence the result.

(ii) Denote

$$G_t^n(u,z,a) := \inf_{v\in\mathbb{Q}^d}\left\{F_t(u,v,a)+n\left|a^{1/2}(z-v)\right|^2\right\},$$

so that

$$F_t^n(y,z,a) := \inf_{u\in\mathbb{Q}}\left\{G_t^n(u,z,a)+n\,|y-u|\right\}.$$

We first prove that G^n is uniformly Lipschitz in z in the sense of (ii) above with a constant independent of n. This is actually a simple consequence of Theorems 5.1 and 7.3 in Chapter 1 in [26]. Indeed, thanks to Theorem 5.1(d) of [26], we know that the Fréchet derivative of G^n exists and that its proximal subgradient is actually included in the proximal subgradient of F. Then thanks to the characterization of Lipschitz functions in terms of proximal subgradients of Theorem 7.3 of [26], we get the result.

We next prove that F^n remains uniformly Lipschitz in z in the sense of (ii) above. Let $(z_1,z_2)\in\mathbb{R}^{2d}$ and y_ε be such that

$$F_t^n(y,z_2,a)\geq G_t^n(y_\varepsilon,z_2,a)+n\,|y_\varepsilon-y|-\varepsilon.$$

Then we have

$$\begin{aligned}
F_t^n(y,z_1,a)-F_t^n(y,z_2,a) &\leq G_t^n(y_\varepsilon,z_1,a)+n\,|y-y_\varepsilon|-G_t^n(y_\varepsilon,z_2,a)-n\,|y-y_\varepsilon|+\varepsilon\\
&\leq C|a^{1/2}(z_1-z_2)|+\varepsilon.
\end{aligned}$$

By arbitrariness of ε and by interchanging the roles of z_1 and z_2 we get the desired result.

For (iii), let $\varepsilon>0$ and let y_ε be such that

$$F_t^n(y_1,z,a)\geq \inf_{v\in\mathbb{Q}^d}\left\{F_t(y_\varepsilon,z,a)+n\left|a^{1/2}(z-v)\right|^2\right\}+n\,|y_1-y_\varepsilon|-\varepsilon.$$

Then we have

$$\begin{aligned}
F_t^n(y_1,z,a) &\geq \inf_{v\in\mathbb{Q}^d}\left\{F_t(y_\varepsilon,z,a)+n\left|a^{\frac{1}{2}}(z-v)\right|^2\right\}+n(|y_2-y_\varepsilon|+|y_1-y_\varepsilon|-|y_2-y_\varepsilon|)-\varepsilon\\
&\geq F_t^n(y_2,z,a)-n\,|y_1-y_2|-\varepsilon.
\end{aligned}$$

By exchanging the roles of y_1 and y_2 and by the arbitrariness of ε we obtain (iii).

(iv) is trivial by definition.

(v) We now assume that F is decreasing in y. In particular, F is C^1 in y for *a.e.* y. Define for all y, $h_{n,y,v,t,a}(u) := F_t(u,v,a) + n|y-u|$. For $u \leq y$, $h_{n,y,v,t,a}$ is clearly decreasing in u. Then, its minimum in u can only be attained at y or at a point strictly greater than y.

Therefore we can write

$$F_t^n(y,z,a) = \inf_{v\in\mathbb{Q}^d}\left\{\min\left\{F_t(y,v,a), \inf_{u\in\mathbb{Q},\, u>y}\{F_t(u,v)+n(u-y)\}\right\} + n\left|a^{1/2}(z-v)\right|^2\right\}$$
$$= \inf_{v\in\mathbb{Q}^d}\left\{\min\left\{F_t(y,v,a), \inf_{u\in\mathbb{Q},\, u>0}\{F_t(u+y,v,a)+nu\}\right\} + n\left|a^{1/2}(z-v)\right|^2\right\},$$

and under this form it is clear that F^n is decreasing in y. □

Remark 2.3.1. *In Theorem* 5.1 *in Chapter* 1 *of [26], the function considered is supposed to be bounded from below. However, a careful reading of the proof shows that this hypothesis is only necessary to prove that the inf-convolution is well defined, which we already know to hold true is our case for n large enough, thanks to our linear growth hypothesis.*

Remark 2.3.2. *Unlike [72] or [78], we do not use linear inf-convolution for our approximation, but a mix of linear and quadratic inf-convolution. This is due to our crucial need that our approximation remains uniformly Lipschitz in z in the sense of Assumption 2.2.1*(iii). *It is then more convenient to use quadratic inf-convolution in the variable z to be able to use the results of non-smooth analysis of [26].*

We note that in above lemmas, and in all subsequent results, we shall denote by C a generic constant which may vary from line to line and depends only on the dimension d, the maturity T and the constants in Assumptions 2.2.1 and 2.2.2. We shall also denote by C_κ a constant which may depend on κ as well.

Let us now note that we can always consider without loss of generality that the constant μ in Assumption 2.2.1(vi) is equal to 0. Indeed, we have the following Lemma

Lemma 2.3.2. *Let $\lambda > 0$, then $(Y_t, Z_t, \{K_t^{\mathbb{P}}, \mathbb{P}\in\mathcal{P}_H^\kappa\})$ solve the 2BSDE* (2.2.4) *if and only if $\left(e^{\lambda t}Y_t, e^{\lambda t}Z_t, \left\{\int_0^t e^{\lambda s}dK_s^{\mathbb{P}}, \mathbb{P}\in\mathcal{P}_H^\kappa\right\}\right)$ solve the 2BSDE with terminal condition $\overline{\xi} := e^{\lambda T}\xi$ and driver $\widetilde{F}_t^{(\lambda)}(y,z) := F_t^{(\lambda)}(y,z,\widehat{a}_t)$, where $F_t^{(\lambda)}(y,z,a) := e^{\lambda t}F_t(e^{-\lambda t}y, e^{-\lambda t}z, a) - \lambda y$.*

Proof. The fact that the two solutions solve the corresponding equations is a simple consequence of Itô's formula. The only thing that we have to check is that the family $\{K_t^{\mathbb{P}}, \mathbb{P}\in\mathcal{P}_H^\kappa\}$ satisfies the minimum condition (2.2.6) if and only if it is verified by the family $\left\{\int_0^t e^{\lambda s}dK_s^{\mathbb{P}}, \mathbb{P}\in\mathcal{P}_H^\kappa\right\}$. First of all, it is clear that

$$\int_0^t e^{\lambda s}dK_s^{\mathbb{P}} = \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\, \mathbb{E}_t^{\mathbb{P}'}\left[\int_0^T e^{\lambda s}dK_s^{\mathbb{P}'}\right] \Leftrightarrow \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\, \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T e^{\lambda s}dK_s^{\mathbb{P}'}\right] = 0.$$

Now for every $t\in[0,T]$, $\mathbb{P}\in\mathcal{P}_H^\kappa$ and $\mathbb{P}'\in\mathcal{P}_H^\kappa(t,\mathbb{P})$, the result follows from

$$\mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T e^{\lambda(s-T)}dK_s^{\mathbb{P}'}\right] \leq \mathbb{E}_t^{\mathbb{P}'}\left[K_T^{\mathbb{P}'} - K_t^{\mathbb{P}}\right] \leq \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T e^{\lambda s}dK_s^{\mathbb{P}'}\right],\ \mathbb{P}-a.s.$$

□

Thus, if we choose $\lambda = \mu$ then $F^{(\mu)}$ satisfies

$$(y_1 - y_2)(F_t^{(\mu)}(\omega, y_1, z, a) - F_t^{(\mu)}(\omega, y_2, z, a)) \leq 0, \text{ for all } (t, \omega, y_1, y_2, z, a).$$

As a consequence of Lemma 2.3.2, we can assume without loss of generality that our driver is decreasing in y. Therefore, frow now on this assumption will replace Assumption 2.2.1(vi).

As explained in Remark 2.3.4, we will actually need a strong convergence result for the sequence $\widehat{F}_t^n(y,z) := F_t^n(y, z, \widehat{a}_t^{1/2})$. Let us define the following quantity

$$\widetilde{F}_t^n := \sup_{(y,z,a)\in\mathbb{R}^{d+1}\times[\underline{a},\bar{a}]} \left\{ F_t(y,z,a) - F_t^n(y,z,a) \right\}.$$

We then have the following result

Lemma 2.3.3. *Let Assumption 2.2.1 hold. Then the sequence $\widehat{F}^n$ converges uniformly globally in (y, z) and for all $0 \leq t \leq T$ and all $\varepsilon > 0$*

$$\sup_{\mathbb{P}\in\overline{\mathcal{P}}_H^\kappa} \mathbb{E}^{\mathbb{P}} \left[\left| \widetilde{F}_t^n \right|^{2+\varepsilon} \right] \leq C,$$

for some C independent of n.

The proof is relegated to the Appendix.

Remark 2.3.3. *Notice that in the above Lemma, we consider a supremum over $\overline{\mathcal{P}}_H^\kappa$ and not over $\mathcal{P}_H^\kappa$. This is important because we are going to apply the monotone convergence Theorem of [40] to the quantity $\left|\widetilde{F}_t^n\right|$ in the sequel, and this theorem can only be used under a weakly compact family of probability measures.*

We are now in a position to state the main result of this section

Theorem 2.3.1. *Suppose there exists $\varepsilon > 0$ such that $\xi \in \mathcal{L}_H^{2,\kappa} \cap L_H^{2+\varepsilon,\kappa}$ and*

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}} \left[\int_0^T \left| \widehat{F}_t^0 \right|^{2+\varepsilon} dt \right] < +\infty. \tag{2.3.1}$$

Then, under Assumptions 2.2.1, 2.2.2 and 2.3.3, there exists a unique solution $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ of the 2BSDE (2.2.4).

2.3.2 Proof of the main result

For a fixed n, consider the following 2BSDE

$$Y_t^n = \xi + \int_t^T \widehat{F}_s^n(Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dB_s + K_T^n - K_t^n, \ 0 \leq t \leq T, \ \mathcal{P}_H^\kappa - q.s. \tag{2.3.2}$$

By Lemma 2.3.1 and our Assumptions 2.2.1 and 2.2.2 we know that all the requirements of Theorem 4.7 of [107] are fullfilled. Thus, we know that for all n the above 2BSDE has a unique solution $(Y^n, Z^n) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$. Moreover, if we introduce the following standard BSDEs for all $\mathbb{P} \in \mathcal{P}_H^\kappa$

$$y_t^{\mathbb{P},n} = \xi + \int_t^T \widehat{F}_s^n(y_s^{\mathbb{P},n}, z_s^{\mathbb{P},n})ds - \int_t^T z_s^{\mathbb{P},n} dB_s,\ 0 \leq t \leq T,\ \mathbb{P}-a.s., \tag{2.3.3}$$

we have the already mentioned representation (see Theorem 4.4 in [107])

$$Y_t^n = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ y_t^{\mathbb{P}',n},\ 0 \leq t \leq T,\ \mathbb{P}-a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \tag{2.3.4}$$

The idea of the proof of existence is to prove that the limit in a certain sense of the sequence (Y^n, Z^n) is a solution of the 2BSDE (2.2.4). We first provide a priori estimates which are uniform in n on the solutions of (2.3.2) and (2.3.3).

Lemma 2.3.4. *There exists a constant $C_\kappa > 0$ such that for all n large enough*

$$\|Y^n\|^2_{\mathbb{D}_H^{2,\kappa}} + \|Z^n\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\left|K_T^{\mathbb{P},n}\right|^2 + \sup_{0\leq t\leq T}|y_t^{\mathbb{P},n}|^2 + \int_0^T |\widehat{a}_s^{1/2} z_s^{\mathbb{P},n}|^2 ds\right] \leq C_\kappa\left(1 + \|\xi\|^2_{\mathcal{L}_H^{2,\kappa}} + \phi_H^{2,\kappa}\right).$$

Proof. Let us consider the following BSDE

$$u_t^{\mathbb{P}} = |\xi| + \int_t^T |\widehat{F}_s^0| + C\left(1 + |u_s^{\mathbb{P}}| + |\widehat{a}_s^{1/2} v_s^{\mathbb{P}}|\right) ds - \int_t^T v_s^{\mathbb{P}} dB_s,\ 0 \leq t \leq T,\ \mathbb{P}-a.s. \tag{2.3.5}$$

Since its generator is clearly Lipschitz it has a unique solution. Moreover, we can apply the comparison theorem of El Karoui, Peng and Quenez [42] to obtain that, due to our uniform growth assumption and (i), (ii), (iii) of Lemma 2.3.1

$$\forall\ m \leq n \text{ large enough},\ \forall\ \mathbb{P} \in \mathcal{P}_H^\kappa,\ y^{\mathbb{P},m} \leq y^{\mathbb{P},n} \leq u^{\mathbb{P}},\ \mathbb{P}-a.s.$$

Now, following line-by-line the proof of Lemma 4.3 in [107] we obtain that for all $\mathbb{P} \in \mathcal{P}_H^\kappa$ and all $0 \leq t \leq T$

$$|u_t^{\mathbb{P}}| \leq C_\kappa\left(1 + \mathbb{E}_t^{\mathbb{P}}\left[|\xi|^\kappa + \int_t^T |\widehat{F}_s^0|^\kappa ds\right]^{1/\kappa}\right).$$

Therefore by definition of the norms and the representation (2.3.4) we have

$$\|Y^n\|^2_{\mathbb{D}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|y_t^{\mathbb{P},n}|^2\right] \leq C_\kappa\left(1 + \|\xi\|^2_{\mathcal{L}_H^{2,\kappa}} + \phi_H^{2,\kappa}\right).$$

We next apply Itô's formula to $(Y_t^n)^2$ under each $\mathbb{P} \in \mathcal{P}_H^\kappa$, we have $\mathbb{P}-a.s.$

$$(Y_t^n)^2 + \int_t^T |\widehat{a}_s^{1/2} Z_s^n|^2 ds \leq \xi^2 + 2\int_t^T Y_s^n \widehat{F}_s^n(Y_s^n, Z_s^n)ds - 2\int_t^T Y_{s^-}^n Z_s^n dB_s + 2\int_t^T Y_{s^-}^n dK_s^{\mathbb{P},n}.$$

Thus, since $(Y^n, Z^n) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$, by taking expectation we obtain that for all $\mathbb{P} \in \mathcal{P}_H^\kappa$,

$$\mathbb{E}^{\mathbb{P}}\left[\int_t^T |\widehat{a}_s^{1/2} Z_s^n|^2 ds\right] \leq \|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + 2\mathbb{E}^{\mathbb{P}}\left[\int_t^T Y_s^n \widehat{F}_s^n(Y_s^n, Z_s^n)ds + \sup_{0\leq t\leq T}|Y_t^n| K_T^{\mathbb{P},n}\right].$$

Now the uniform growth condition (i) of Lemma 2.3.1 and the elementary inequality $2ab \leq a^2/\varepsilon + \varepsilon b^2$, $\forall\ \varepsilon > 0$ yield

$$\mathbb{E}^{\mathbb{P}}\left[\int_t^T |\widehat{a}_s^{1/2} Z_s^n|^2 ds\right] \leq \|\xi\|^2_{\mathbb{L}^{2,\kappa}_H} + C\left(1 + \phi_H^{2,\kappa} + \mathbb{E}^{\mathbb{P}}\left[\int_t^T |Y_s^n|^2 ds\right]\right)$$
$$+ \frac{1}{3}\mathbb{E}^{\mathbb{P}}\left[\int_t^T |\widehat{a}_s^{1/2} Z_s^n|^2 ds\right] + 2\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} |Y_t^n| K_T^{\mathbb{P},n}\right]$$
$$i.e.\ \frac{2}{3}\mathbb{E}^{\mathbb{P}}\left[\int_0^T |\widehat{a}_s^{1/2} Z_s^n|^2 ds\right] \leq \|\xi\|^2_{\mathbb{L}^{2,\kappa}_H} + C\left(1 + (1+\varepsilon^{-1})\|Y^n\|^2_{\mathbb{D}^{2,\kappa}_H}\right) + \varepsilon\mathbb{E}^{\mathbb{P}}\left[(K_T^{\mathbb{P},n})^2\right].$$

But by definition of $K_T^{\mathbb{P},n}$, it is clear that

$$\mathbb{E}^{\mathbb{P}}\left[(K_T^{\mathbb{P},n})^2\right] \leq C_0\left(1 + \|\xi\|^2_{\mathbb{L}^{2,\kappa}_H} + \phi_H^{2,\kappa} + \|Y^n\|^2_{\mathbb{D}^{2,\kappa}_H} + \int_0^T |\widehat{a}_s^{1/2} Z_s^n|^2 ds\right). \tag{2.3.6}$$

Choosing $\varepsilon = \frac{1}{3C_0}$, reporting (2.3.6) in the previous inequality and taking supremum over $\mathbb{P}$ yields

$$\|Z^n\|^2_{\mathbb{H}^{2,\kappa}_H} \leq C_\kappa\left(1 + \|\xi\|^2_{\mathbb{L}^{2,\kappa}_H} + \phi_H^{2,\kappa}\right),$$

from which we can then deduce the result for $K_T^{\mathbb{P},n}$.

Finally, we can show similarly, by applying Itô's formula to $y_t^{\mathbb{P},n}$ instead, that

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\int_0^T |\widehat{a}_s^{1/2} z_s^{\mathbb{P},n}|^2 ds\right] \leq C_\kappa\left(1 + \|\xi\|^2_{\mathbb{L}^{2,\kappa}_H} + \phi_H^{2,\kappa}\right).$$

□

Now from the comparison theorems for 2BSDEs (see Corolary 4.5 in [107]) and BSDEs and (iv) of Lemma 2.3.1, we recall that we have for all $0 \leq t \leq T$

$$y_t^{\mathbb{P},n} \leq y_t^{\mathbb{P},n+1}\ \mathbb{P}-a.s. \text{ for all } \mathbb{P}\in\mathcal{P}_H^\kappa, \text{ and } Y_t^n \leq Y_t^{n+1},\ \mathcal{P}_H^\kappa - q.s.$$

Remark 2.3.4. *If we were in the classical framework, this $\mathbb{P}-$almost sure convergence of $y^{\mathbb{P},n}$ together with the estimates of Lemma 2.3.4 would be sufficient to prove the convergence in the usual $\mathbb{H}^2$ space, thanks to dominated convergence theorem. However, in our case, since the norms involve the supremum over a family of probability measures, this theorem can fail. This is exactly the major difficulty when considering the 2BSDE framework, since most of the techniques used in the standard BSDE litterature to prove existence results involve approximations. In order to solve this problem, we need more regularity to be able apply the monotone convergence Theorem* 31 *of [40]. This is exactly why we had to add the assumption of uniform conitnuity in y for our proof to work. Moreover, this also explains why, as already mentioned in Remark 2.2.2, we cannot generalize completely the results of Pardoux [85]. Indeed, restricting ourselves to linear growth in y allows us to use the approximation by inf-convolution which has some very nice properties. If we had considered general growth in y, then it would have been extremely difficult to find reasonable conditions on the driver $\widehat{F}$ in order to have uniform convergence of the approximation.*

Next, we prove that

Lemma 2.3.5. *Under the hypotheses of Theorem 2.3.1, we have*

$$\sup_{0\leq t\leq T}\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[|y_t^{\mathbb{P}}-y_t^{\mathbb{P},n}|^{2+\varepsilon'}\right]\underset{n\to+\infty}{\to}0,$$

for any $0<\varepsilon'<\varepsilon$, *with the same* ε *as in* (2.3.1)*.*

Proof. Thanks to Lemma 2.3.1, we know that we can apply the same proof as that of Theorem 2.3 of [85] in order to get for each $\mathbb{P}\in\mathcal{P}_H^\kappa$

$$\mathbb{E}^{\mathbb{P}}\left[|y_t^{\mathbb{P}}-y_t^{\mathbb{P},n}|^{2+\varepsilon'}\right]\leq C\int_0^T\mathbb{E}^{\mathbb{P}}\left[|\widehat{F}_s(y_s^{\mathbb{P}},z_s^{\mathbb{P}})-\widehat{F}_s^n(y_s^{\mathbb{P}},z_s^{\mathbb{P}})|^{2+\varepsilon'}\right]ds\leq C\int_0^T\mathbb{E}^{\mathbb{P}}\left[|\widetilde{F}_s^n|^{2+\varepsilon'}\right]ds.$$

By Lemma 2.3.3, we know that F^n converges uniformly in (y,z). Since F^n and F are also continuous in a by Assumption 2.2.1(vii), the convergence is also uniform in $a\in[\underline{a},\bar{a}]$ by Dini's lemma. Thus, $|\widetilde{F}_s^n|^{2+\varepsilon'}$ decreases to 0 for every $\omega\in\Omega$.

Then, since F and F^n are uniformly continuous in ω on the whole space Ω, then $\left|\widetilde{F}_t^n\right|^{2+\varepsilon'}$ is continuous in ω on Ω, and therefore quasi-continuous in the sense of [40]. Moreover, we have again by Lemma 2.3.3 and the fact that $\varepsilon'<\varepsilon$

$$\sup_{\mathbb{P}\in\overline{\mathcal{P}}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\left(\left|\widetilde{F}_t^n\right|^{2+\varepsilon'}\right)^{1+\varepsilon''}\right],$$

for some $\varepsilon''>0$.

Hence, we have by classical arguments (namely Hölder and Markov inequalities) that

$$\lim_{N\to+\infty}\sup_{\mathbb{P}\in\overline{\mathcal{P}}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\left|\widetilde{F}_t^n\right|^{2+\varepsilon'}1_{|\widetilde{F}_t^n|^{2+\varepsilon'}>N}\right]=0.$$

Therefore, we can apply the monotone convergence theorem of [40] under the family $\overline{\mathcal{P}}_H^\kappa$ to obtain that

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\left|\widetilde{F}_t^n\right|^{2+\varepsilon'}\right]\leq\sup_{\mathbb{P}\in\overline{\mathcal{P}}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\left|\widetilde{F}_t^n\right|^{2+\varepsilon'}\right]\underset{n\to+\infty}{\longrightarrow}0,\ \text{for all } 0\leq t\leq T.$$

Finally, the required result follows from the standard dominated convergence Theorem for the integral with respect to the Lebesgue measure.

□

We continue with the following result

Lemma 2.3.6. *Assume moreover that there exists an* $\varepsilon>0$ *such that* $\xi\in L_H^{2+\varepsilon,\kappa}$ *and*

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\int_0^T\left|\widehat{F}_t^0\right|^{2+\varepsilon}dt\right]<+\infty.$$

Then, we have

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|Y_t^n-Y_t^p|^2\right]\longrightarrow 0,\ \text{as } n,p\longrightarrow+\infty.$$

Proof. By the representation (2.3.4), we have for all n,p large enough

$$\sup_{0\leq t\leq T}|Y_t^n-Y_t^p|\leq \sup_{0\leq t\leq T}\ \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ |y_t^{\mathbb{P}',n}-y_t^{\mathbb{P}',p}|,\ \mathbb{P}-a.s. \text{ for all } \mathbb{P}\in\mathcal{P}_H^\kappa.$$

Then, we easily get

$$\sup_{0\leq t\leq T}|Y_t^n-Y_t^p|^2\leq \sup_{0\leq t\leq T}\left(\underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[\sup_{0\leq s\leq T}|y_s^{\mathbb{P}',n}-y_s^{\mathbb{P}',p}|\right]\right)^2,\ \mathbb{P}-a.s. \text{ for all } \mathbb{P}\in\mathcal{P}_H^\kappa.$$

Taking expectations yields for all $\mathbb{P}\in\mathcal{P}_H^\kappa$

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|Y_t^n-Y_t^p|^2\right]\leq \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left(\underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[\sup_{0\leq s\leq T}|y_s^{\mathbb{P}',n}-y_s^{\mathbb{P}',p}|\right]\right)^2\right].$$

We next use the generalization of Doob maximal inequality Proposition 2.4.1 of the Appendix (see also Lemma 6.2 in [106]), to obtain that for all $\varepsilon'<\varepsilon$

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|Y_t^n-Y_t^p|^2\right]\leq C\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq s\leq T}|y_s^{\mathbb{P},n}-y_s^{\mathbb{P},p}|^{2+\varepsilon'}\right]^{\frac{2}{2+\varepsilon'}}.$$

Thus it suffices to show that the right-hand side tends to 0 as $n,p\to+\infty$. We start by stating some new a priori estimates with our new integrability assumptions for ξ and $\widehat{F}^0$ (see the Appendix for the proof)

$$\sup_n\left\{\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|y_t^{\mathbb{P},n}\right|^{2+\varepsilon'}\right]+\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T\left|\widehat{a}_t^{1/2}z_t^{\mathbb{P},n}\right|^2dt\right)^{\frac{2+\varepsilon'}{2}}\right]\right\}$$
$$\leq C\left(1+\|\xi\|^{2+\varepsilon'}_{L_H^{2+\varepsilon',\kappa}}+\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\int_0^T\left|\widehat{F}_s^0\right|^{2+\varepsilon'}ds\right]\right).\tag{2.3.7}$$

Let us note $\delta\widehat{F}_t^{n,p}:=\widehat{F}^n(t,y_t^{\mathbb{P},n},z_t^{\mathbb{P},n})-\widehat{F}^p(t,y_t^{\mathbb{P},p},z_t^{\mathbb{P},p})$. Applying Itô's formula to $|y_t^{\mathbb{P},n}-y_t^{\mathbb{P},p}|^2$ and taking conditional expectations yields, for all $\mathbb{P}\in\mathcal{P}_H^\kappa$

$$|y_t^{\mathbb{P},n}-y_t^{\mathbb{P},p}|^2+\mathbb{E}_t^{\mathbb{P}}\left[\int_t^T|\widehat{a}_s^{1/2}(z_s^{\mathbb{P},n}-z_s^{\mathbb{P},p})|^2ds\right]\leq 2\mathbb{E}_t^{\mathbb{P}}\left[\int_t^T\left|y_s^{\mathbb{P},n}-y_s^{\mathbb{P},p}\right|\left|\delta\widehat{F}_s^{n,p}\right|ds\right].\tag{2.3.8}$$

Now since $\varepsilon'>0$, it follows from Doob maximal inequality (in the classical form under a single measure) and the Cauchy-Schwarz inequality that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left(\mathbb{E}_t^{\mathbb{P}}\left[\int_t^T|\widehat{a}_s^{1/2}(z_s^{\mathbb{P},n}-z_s^{\mathbb{P},p})|^2ds\right]\right)^{\frac{2+\varepsilon'}{2}}\right]&\leq C\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T\left|y_s^{\mathbb{P},n}-y_s^{\mathbb{P},p}\right|\left|\delta\widehat{F}_s^{n,p}\right|ds\right)^{\frac{2+\varepsilon'}{2}}\right]\\&\leq C\left(\mathbb{E}^{\mathbb{P}}\left[\int_0^T\left|y_s^{\mathbb{P},n}-y_s^{\mathbb{P},p}\right|^{2+\varepsilon'}ds\right]\right)^{1/2}\\&\quad\times\left(\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T\left|\delta\widehat{F}_s^{n,p}\right|^2ds\right)^{\frac{2+\varepsilon'}{2}}\right]\right)^{1/2}\end{aligned}\tag{2.3.9}$$

By the uniform growth property (i) of Lemma 2.3.1, we have for all $t \in [0,T]$

$$|\delta\widehat{F}_t^{n,p}|^2 \leq C\left(1+|\widehat{F}_t^0|^2+|y_t^{\mathbb{P},n}|^2+|y_t^{\mathbb{P},p}|^2+|\widehat{a}_t^{1/2}z_t^{\mathbb{P},n}|^2+|\widehat{a}_t^{1/2}z_t^{\mathbb{P},p}|^2\right),\ \mathbb{P}-a.s.,\ \forall\,\mathbb{P}\in\mathcal{P}_H^\kappa.$$

Hence using the uniform a priori estimates of (2.3.7), we have

$$\begin{aligned}\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T|\delta\widehat{F}_t^{n,p}|^2dt\right)^{\frac{2+\varepsilon'}{2}}\right] &\leq C\left(1+\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T\left|\widehat{F}_s^0\right|^2ds\right)^{\frac{2+\varepsilon'}{2}}\right]\right)\\ &\quad+C\left(\sup_n\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|y_t^{\mathbb{P},n}|^{2+\varepsilon'}\right]\right)\\ &\quad+C\left(\sup_n\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T|\widehat{a}_s^{1/2}z_s^{\mathbb{P},n}|^2ds\right)^{\frac{2+\varepsilon'}{2}}\right]\right)\\ &\leq C\left(1+\|\xi\|^{2+\varepsilon}_{\mathbb{L}_H^{2+\varepsilon',\kappa}}+\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\int_0^T\left|\widehat{F}_s^0\right|^{2+\varepsilon'}ds\right]\right).\end{aligned}\qquad(2.3.10)$$

We then have

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\int_0^T\left|y_t^{\mathbb{P},n}-y_t^{\mathbb{P},p}\right|^{2+\varepsilon'}dt\right]\leq\int_0^T\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\left|y_t^{\mathbb{P},n}-y_t^{\mathbb{P},p}\right|^{2+\varepsilon'}\right]dt\underset{n,p\to+\infty}{\longrightarrow}0,\qquad(2.3.11)$$

where we used Lemma 2.3.5 and the standard Lebesgue dominated convergence Theorem.

Therefore, plugging the estimate (2.3.10) in (2.3.9), using (2.3.11), and sending n,p to $+\infty$, we see that

$$\begin{aligned}\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T|\widehat{a}_s^{1/2}(z_s^{\mathbb{P},n}-z_s^{\mathbb{P},p})|^2ds\right)^{\frac{2+\varepsilon'}{2}}\right]&\leq\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left(\mathbb{E}_t^{\mathbb{P}}\left[\int_t^T|\widehat{a}_s^{1/2}(z_s^{\mathbb{P},n}-z_s^{\mathbb{P},p})|^2ds\right]\right)^{\frac{2+\varepsilon'}{2}}\right]\\ &\underset{n,p\to+\infty}{\longrightarrow}0.\end{aligned}\qquad(2.3.12)$$

Then, by Itô's formula we similarly obtain

$$\begin{aligned}\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|y_t^{\mathbb{P},n}-y_t^{\mathbb{P},p}|^{2+\varepsilon'}\right]&\leq C\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|\int_t^T(y_s^{\mathbb{P},n}-y_s^{\mathbb{P},p})(z_s^{\mathbb{P},n}-z_s^{\mathbb{P},p})dB_s\right|^{\frac{2+\varepsilon'}{2}}\right]\\ &\quad+2\left(\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\int_0^T|y_t^{\mathbb{P},n}-y_t^{\mathbb{P},p}|^{2+\varepsilon'}dt\right]\right)^{1/2}\\ &\quad\times\left(\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T|\delta\widehat{F}_t^{n,p}|^2dt\right)^{\frac{2+\varepsilon'}{2}}\right]\right)^{1/2}.\end{aligned}\qquad(2.3.13)$$

By previous calculations we know that the second term tends to 0 as $n,p\to+\infty$. For the first one, we have by the Burkholder-Davis-Gundy inequality (where we recall that the constants involved are universal and thus do not depend on $\mathbb{P}$)

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|\int_t^T (y_s^{\mathbb{P},n}-y_s^{\mathbb{P},p})(z_s^{\mathbb{P},n}-z_s^{\mathbb{P},p})dB_s\right|^{\frac{2+\varepsilon'}{2}}\right] &\leq C_0\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T |y_s^{\mathbb{P},n}-y_s^{\mathbb{P},p}|^2|\widehat{a}_s^{\frac{1}{2}}(z_s^{\mathbb{P},n}-z_s^{\mathbb{P},p})|^2ds\right)^{\frac{2+\varepsilon'}{4}}\right]\\ &\leq C_0\mathbb{E}^{\mathbb{P}}\left[\left(\sup_{0\leq t\leq T}|y_t^{\mathbb{P},n}-y_t^{\mathbb{P},p}|^2\int_0^T|\widehat{a}_s^{\frac{1}{2}}(z_s^{\mathbb{P},n}-z_s^{\mathbb{P},p})|^2ds\right)^{\frac{2+\varepsilon'}{4}}\right]\\ &\leq \frac{1}{2C}\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|y_t^{\mathbb{P},n}-y_t^{\mathbb{P},p}|^{2+\varepsilon'}\right]\\ &\quad+\frac{CC_0^2}{4}\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T|\widehat{a}_s^{\frac{1}{2}}(z_s^{\mathbb{P},n}-z_s^{\mathbb{P},p})|^2ds\right)^{\frac{2+\varepsilon'}{2}}\right],\end{aligned}$$

where C is the constant in (2.3.13).

Reporting this in the above inequality (2.3.13) and letting n,p go to $+\infty$ then finishes the proof. □

Remark 2.3.5. *In contrast with the classical case, we proved here the convergence of Y^n in $\mathbb{D}_H^{2,\kappa}$ before proving any convergence for Z^n. Proceeding in this order is crucial because of the process $K^{\mathbb{P},n}$, which prevents us from using the usual techniques. Then, it is natural to use the representation formula for Y^n to control the $\mathbb{D}_H^{2,\kappa}$ norm of Y^n-Y^p by a certain norm of $y^{\mathbb{P},n}-y^{\mathbb{P},p}$. It turns out that we end up with a norm which is closely related to the $\mathbb{L}_H^{2,\kappa}$ norm. However, this norm for the process $y^{\mathbb{P},n}$ is not tractable for classical BSDEs, therefore, we have to use the generalized Doob inequality (which is currently conjectured to be the best possible, see Remark* 2.9 *in [107]) to return to the usual norm for $y^{\mathbb{P},n}$. This in turn forces us to assume stronger integrability assumptions on ξ and $\widehat{F}^0$.*

We just have proved that the sequence $(Y^n)_n$ is Cauchy in the Banach $\mathbb{D}_H^{2,\kappa}$. Thus it converges to Y in $\mathbb{D}_H^{2,\kappa}$. Let us now focus on Z^n and $K^{\mathbb{P},n}$.

Lemma 2.3.7. *There exist a process $Z\in\mathbb{H}_H^{2,\kappa}$ and a non-decreasing process $K^{\mathbb{P}}\in\mathbb{D}^2(\mathbb{P})$ such that*

$$\|Z^n-Z\|^2_{\mathbb{H}_H^{2,\kappa}}+\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|K_t^{\mathbb{P},n}-K_t^{\mathbb{P}}|^2\right]\longrightarrow 0,\ \text{as } n\to+\infty.$$

Proof. Denote $\delta\widehat{F}_t^{n,p}:=\widehat{F}_t^n(Y_t^n,Z_t^n)-\widehat{F}_t^p(Y_t^p,Z_t^p)$. Applying Itô's formula to $|Y_t^n-Y_t^p|^2$ and taking expectations yields, for all $\mathbb{P}\in\mathcal{P}_H^\kappa$

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[|Y_0^n-Y_0^p|^2+\int_0^T|\widehat{a}_t^{1/2}(Z_t^n-Z_t^p)|^2dt\right]&\leq 2\mathbb{E}^{\mathbb{P}}\left[\int_0^T(Y_t^n-Y_t^p)\delta\widehat{F}_t^{n,p}dt\right]\\ &\quad+2\mathbb{E}^{\mathbb{P}}\left[\int_0^T(Y_t^n-Y_t^p)d(K_t^{\mathbb{P},n}-K_t^{\mathbb{P},n})\right].\end{aligned}$$

Then

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T |\widehat{a}_t^{1/2}(Z_t^n - Z_t^p)|^2 dt\right] \leq 2\left(\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\int_0^T |Y_t^n - Y_t^p|^2 dt\right]\right)^{1/2}$$
$$\times\left(\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\int_0^T |\delta\widehat{F}_t^{n,p}|^2 dt\right]\right)^{1/2}$$
$$+2\left(\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|Y_t^n - Y_t^p|^2\right]\right)^{1/2}$$
$$\times\left(\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[|K_T^{\mathbb{P},n}|^2 + |K_T^{\mathbb{P},p}|^2\right]\right)^{1/2}.$$

Notice that the right-hand side tends to 0 uniformly in $\mathbb{P}$ as $n,p\to+\infty$ due to Lemmas 2.3.4 and 2.3.6 and (2.3.10). Thus (Z^n) is a Cauchy sequence in $\mathbb{H}_H^{2,\kappa}$ and therefore converges to a process $Z\in\mathbb{H}_H^{2,\kappa}$.

Now by (2.2.5), we have

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|K_t^{\mathbb{P},n} - K_t^{\mathbb{P},p}|^2\right] \leq C\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[|Y_0^n - Y_0^p|^2 + \sup_{0\leq t\leq T}|Y_t^n - Y_t^p|^2\right]$$
$$+C\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\int_0^T|\delta\widehat{F}_t^{n,p}|^2dt + \sup_{0\leq t\leq T}\left|\int_t^T(Z_s^n - Z_s^p)dB_s\right|^2\right].$$

The first two terms on the right-hand side tend to 0 as $n,p\to+\infty$ thanks to Lemma 2.3.6. For the last one, using BDG inequality and the result we just proved on the sequence (Z^n), we see that it also tends to 0. Thus, in order to finish the proof, we need to show that the term involving $\delta\widehat{F}^{n,p}$ converges to 0. This is deduced from the following facts

- $Y^n \nearrow Y$ in $\mathbb{D}_H^{2,\kappa}$ and $dt\times\mathcal{P}_H^\kappa - q.s.$
- By Lemma 2.3.3, $\widehat{F}^n$ converges uniformly to $\widehat{F}$ in (y,z). Moreover, we have from the Lipschitz property of F in z and its uniform continuity in y

  $$\left|\widehat{F}_t(Y_t,Z_t) - \widehat{F}_t^n(Y_t^n,Z_t^n)\right| \leq \rho(|Y_t^n - Y_t|) + \left|\widehat{F}_t(Y_t^n,Z_t) - \widehat{F}_t^n(Y_t^n,Z_t)\right| + C|\widehat{a}_t^{1/2}(Z_t - Z_t^n)|,$$

  for some modulus of continuity ρ.

  When taking expectation under $\mathbb{P}$ and supremum over all $\mathbb{P}\in\mathcal{P}_H^\kappa$, the convergence to 0 for the term involving $Z-Z^n$ is clear by our previous result . For the second one in the right-hand side above, we can use the same arguments as in the proof of Lemma 2.3.5 to obtain that it also converges to 0. Finally, we recall that since the space Ω is convex, it is a classical result that we can choose the modulus of continuity ρ to be concave, non-decreasing and sub-linear. We then have by Jensen inequality

  $$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\rho(|Y_t^n - Y_t|)\right] \leq \rho\left(\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[|Y_t^n - Y_t|\right]\right) \longrightarrow 0,$$

  by Lemma 2.3.6.

Consequently, for all $\mathbb{P} \in \mathcal{P}_H^\kappa$, there exists a non-decreasing and progressively measurable process $K^{\mathbb{P}}$ such that

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} |K_t^{\mathbb{P},n} - K_t^{\mathbb{P}}|^2\right] \underset{n\to+\infty}{\to} 0.$$

Moreover, since the $K_t^{\mathbb{P},n}$ are càdlàg, so is $K_t^{\mathbb{P}}$. □

Proof. [Proof of Theorem 2.3.1] Taking limits in the 2BSDE (2.3.2), we obtain that $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a solution of the 2BSDE (2.2.4). To conclude the proof of existence, it remains to check the minimum condition (2.2.6). But for all $\mathbb{P} \in \mathcal{P}_H^\kappa$, we know that $K^{\mathbb{P},n}$ verifies (2.2.6). Then we can pass to the limit in the minimum condition verified by $K^{\mathbb{P},n}$. Indeed, we have for all $\mathbb{P}$

$$K_t^{\mathbb{P},n} = \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\left[K_T^{\mathbb{P}',n}\right], \ \mathbb{P}-a.s.$$

Extracting a subsequence if necessary, the left-hand side converges to $K_t^{\mathbb{P}}$, $\mathcal{P}_H^\kappa - q.s.$ Then as in the beginning of the proof of Lemma 2.3.6, we can write

$$\underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\left[K_T^{\mathbb{P}',n}\right] = \lim_{m\to+\infty} \downarrow \mathbb{E}^{\mathbb{P}_m}\left[K_T^{\mathbb{P}_m,n}\right],$$

where $(\mathbb{P}_m)_{m\geq 0}$ is a sequence of probability measures belonging to $\mathcal{P}_H^\kappa(t^+, \mathbb{P})$.

Then by Lemma 2.3.7, we know that $\mathbb{E}^{\mathbb{P}}[K_T^{\mathbb{P},n}]$ converges to $\mathbb{E}^{\mathbb{P}}[K_T^{\mathbb{P}}]$ uniformly in $\mathbb{P} \in \mathcal{P}_H^\kappa$. Thus we can take the limit in n in the above expression and switch the limits in n and m. This shows that

$$K_t^{\mathbb{P}} = \lim_{m\to+\infty} \downarrow \mathbb{E}^{\mathbb{P}_m}\left[K_T^{\mathbb{P}_m}\right] \geq \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\left[K_T^{\mathbb{P}'}\right].$$

The converse inequality is trivial since the process $K^{\mathbb{P}}$ is non-decreasing. Thus, the minimum condition is satisfied, and the proof of Theorem 2.3.1 is complete. □

Remark 2.3.6. *In comparison with the classical BSDE framework, we had to add some assumptions here to prove existence of a solution. The question is whether these assumptions can be weakened by using another construction for the solution. For instance, we may use the so called regular conditional probability distributions as in Chapters 3 and 5 of the present document. However, as mentioned in Remark 3.4.1, even though we could construct a candidate solution when the terminal condition is in $UC_b(\Omega)$, when trying to check that the family $\left\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$ obtained verifies the minimum condition, our monotonicity assumption is not sufficient. Thus, regardless of the solution construction method, we have to add some assumptions to prove existence.*

2.4 Appendix

Proposition 2.4.1. *Let ξ be some $\mathcal{F}_T$ random variable, and let $n \geq 1$. Then, for all $p > n$*

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left(\underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}[|\xi|]\right)^n\right] \leq C \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\left(\mathbb{E}^{\mathbb{P}}[|\xi|^p]\right)^{\frac{n}{p}}.$$

Proof. The proof follows the ideas of the proof of Lemma 6.2 in [106], which closely follows the classical proof of the Doob maximal inequality (see also [110] for related results).

Fix some $\mathbb{P}$, and let us note $X_t^{\mathbb{P}} := \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}[|\xi|]$. Then, it can be shown that $X_t^{\mathbb{P}}$ is a $\mathbb{P}$-supermartingale, and thus admits a càdlàg version. For all $\lambda > 0$, let us define the following $\mathbb{F}^+$ stopping times

$$\tau_\lambda^{\mathbb{P}} = \inf\left\{t \geq 0,\ X_t^{\mathbb{P}} \geq \lambda,\ \mathbb{P} - a.s.\right\}.$$

Define $X^{\mathbb{P},*} := \sup_{0\leq t\leq T} X_t^{\mathbb{P}}$. Then, we have

$$\mathbb{P}(X^{\mathbb{P},*} \geq \lambda) = \mathbb{P}(\tau_\lambda^{\mathbb{P}} \leq T) \leq \frac{1}{\lambda}\mathbb{E}^{\mathbb{P}}\left[X_{\tau_\lambda^{\mathbb{P}}}^{\mathbb{P}} 1_{\tau_\lambda^{\mathbb{P}}\leq T}\right].$$

As previously in this chapter, we know that there exists a sequence $(\mathbb{P}_n)_{n\geq 0} \subset \mathcal{P}_H^\kappa\left((\tau_\lambda^{\mathbb{P}})^+, \mathbb{P}\right)$ such that

$$X_{\tau_\lambda^{\mathbb{P}}}^{\mathbb{P}} = \lim_{n\to+\infty} \uparrow \mathbb{E}_{\tau_\lambda^{\mathbb{P}}}^{\mathbb{P}_n}[|\xi|].$$

Hence, using in this order the monotone convergence theorem and the fact that all the $\mathbb{P}_n$ coincide with $\mathbb{P}$ on $\mathcal{F}_{\tau_\lambda^{\mathbb{P}}}$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[X_{\tau_\lambda^{\mathbb{P}}}^{\mathbb{P}} 1_{\tau_\lambda^{\mathbb{P}}\leq T}\right] &= \lim_{n\to+\infty} \uparrow \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}_{\tau_\lambda^{\mathbb{P}}}^{\mathbb{P}_n}[|\xi|]\, 1_{\tau_\lambda^{\mathbb{P}}\leq T}\right] \\
&= \lim_{n\to+\infty} \uparrow \mathbb{E}^{\mathbb{P}_n}\left[\mathbb{E}_{\tau_\lambda^{\mathbb{P}}}^{\mathbb{P}_n}[|\xi|]\, 1_{\tau_\lambda^{\mathbb{P}}\leq T}\right] \\
&= \lim_{n\to+\infty} \uparrow \mathbb{E}^{\mathbb{P}_n}\left[|\xi|\, 1_{\tau_\lambda^{\mathbb{P}}\leq T}\right] \\
&\leq \lim_{n\to+\infty} \uparrow \left(\mathbb{E}^{\mathbb{P}_n}[|\xi|^p]\right)^{1/p} \mathbb{P}_n(\tau_\lambda^{\mathbb{P}} \leq T)^{1-\frac{1}{p}} \\
&\leq \mathbb{P}(\tau_\lambda^{\mathbb{P}} \leq T)^{1-\frac{1}{p}} \lim_{n\to+\infty} \uparrow \left(\mathbb{E}^{\mathbb{P}_n}[|\xi|^p]\right)^{1/p} \\
&\leq \mathbb{P}(\tau_\lambda^{\mathbb{P}} \leq T)^{1-\frac{1}{p}} \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \left(\mathbb{E}^{\mathbb{P}}[|\xi|^p]\right)^{1/p}.
\end{aligned}$$

Using this estimate, we finally get

$$\mathbb{P}(X^{\mathbb{P},*} \geq \lambda) \leq \frac{1}{\lambda^p} \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}[|\xi|^p].$$

Now, for every $\lambda_0 > 0$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[(X^{\mathbb{P},*})^n\right] &= n\int_0^{+\infty} \lambda^{n-1}\mathbb{P}(X^{\mathbb{P},*} \geq \lambda)d\lambda \\
&= n\int_0^{\lambda_0} \lambda^{n-1}\mathbb{P}(X^{\mathbb{P},*} \geq \lambda)d\lambda + n\int_{\lambda_0}^{+\infty} \lambda^{n-1}\mathbb{P}(X^{\mathbb{P},*} \geq \lambda)d\lambda \\
&\leq \lambda_0^n + n \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}[|\xi|^p] \int_{\lambda_0}^{+\infty} \frac{d\lambda}{\lambda^{p-n+1}} \\
&= \lambda_0^n + n \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}[|\xi|^p] \frac{\lambda_0^{n-p}}{p-n}.
\end{aligned}$$

Now choosing $\lambda_0 = \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \left(\mathbb{E}^{\mathbb{P}}[|\xi|^p]\right)^{1/p}$, this yields

$$\mathbb{E}^{\mathbb{P}}\left[(X^{\mathbb{P},*})^n\right] \leq \left(1 + \frac{n}{p-n}\right) \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \left(\mathbb{E}^{\mathbb{P}}[|\xi|^p]\right)^{n/p}.$$

Hence the result. □

Proof. [Proof of Lemma 2.3.3] The uniform convergence result is a simple consequence of a result already proved by Lasry and Lyons in [70] (see the Theorem on page 4 and Remark (iv) on page 5). For the inequality, since $\widehat{F}$ is uniformly continuous in y, there exists a modulus of continuity ρ with linear growth. Then, it follows that

$$\begin{aligned} F_t - F_t^n &= \sup_{(u,v)\in\mathbb{Q}^{d+1}, u\geq y} \left\{ F_t(y,z,a) - F_t(u,v,a) - n\,|y-u| - n\left|a^{\frac{1}{2}}(z-v)\right|^2 \right\} \\ &\leq \sup_{(u,v)\in\mathbb{Q}^{d+1}, u\geq y} \left\{ C\rho(|y-u|) + C\left|a^{1/2}(z-v)\right| - n\left|a^{1/2}(z-v)\right|^2 - n\,|y-u| \right\} \\ &= \frac{C^2}{4n} + \sup_{u\geq 0}\left\{C\rho(u) - nu\right\}. \end{aligned}$$

Since ρ has linear growth in y, the function on the right-hand side above is clearly decreasing in n and thus is dominated by a constant, which gives us the result. □

Proposition 2.4.2. *Assume that there exists an $\varepsilon > 0$ such that $\xi \in L_H^{2+\varepsilon,\kappa}$ and*

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{F}_t^0\right|^{2+\varepsilon} dt\right] < +\infty.$$

Then we have

$$\begin{aligned} &\sup_n \left\{ \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|y_t^{\mathbb{P},n}\right|^{2+\varepsilon}\right] + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{a}_t^{1/2} z_t^{\mathbb{P},n}\right|^2 dt\right)^{\frac{2+\varepsilon}{2}}\right]\right\} \\ &\leq C\left(1 + \|\xi\|^{2+\varepsilon}_{L_H^{2+\varepsilon,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\int_0^T\left|\widehat{F}_s^0\right|^{2+\varepsilon} ds\right]\right). \end{aligned}$$

Proof. Fix a $\mathbb{P}$, and consider the BSDE (2.3.5). As in the proof of Lemma 2.3.4, we have that for all n, $y^{\mathbb{P},n} \leq u^{\mathbb{P}}$, $\mathbb{P}$-a.s. Now let α be some positive constant which will be fixed later and let $\eta \in (0,1)$. By Itô's formula we have

$$\begin{aligned} e^{\alpha t}\left|u_t^{\mathbb{P}}\right|^2 + \int_t^T e^{\alpha s}\left|\widehat{a}_s^{1/2} v_s^{\mathbb{P}}\right|^2 ds &= e^{\alpha T}|\xi|^2 + 2\int_t^T e^{\alpha s} u_s^{\mathbb{P}}\left(\left|\widehat{F}_s^0\right| + C\right) ds \\ &\quad + 2C\int_t^T u_s^{\mathbb{P}}\left(e^{\alpha s}\left|u_s^{\mathbb{P}}\right| + \left|\widehat{a}_s^{1/2} v_s^{\mathbb{P}}\right|\right) ds - \alpha\int_t^T e^{\alpha s}\left|u_s^{\mathbb{P}}\right|^2 ds \\ &\quad - 2\int_t^T e^{\alpha s} u_s^{\mathbb{P}} v_s^{\mathbb{P}} dB_s \\ &\leq e^{\alpha T}|\xi|^2 + 2\int_t^T e^{\alpha s}\left|u_s^{\mathbb{P}}\right|\left(\left|\widehat{F}_s^0\right| + C\right) ds \\ &\quad + \left(2C + \frac{C^2}{\eta} - \alpha\right)\int_t^T e^{\alpha s}\left|u_s^{\mathbb{P}}\right|^2 ds + \eta\int_t^T e^{\alpha s}\left|\widehat{a}_s^{1/2} v_s^{\mathbb{P}}\right|^2 ds \\ &\quad - 2\int_t^T e^{\alpha s} u_s^{\mathbb{P}} v_s^{\mathbb{P}} dB_s. \end{aligned}$$

Now choose α such that $\nu := \alpha - 2C - \frac{C^2}{\eta} \geq 0$. We obtain

$$e^{\alpha t}\left|u_t^{\mathbb{P}}\right|^2 + (1-\eta)\int_t^T e^{\alpha s}\left|\widehat{a}_s^{1/2}v_s^{\mathbb{P}}\right|^2 ds + \nu\int_t^T e^{\alpha s}\left|u_t^{\mathbb{P}}\right|^2 ds \leq e^{\alpha T}\left|\xi\right|^2 + 2\int_t^T e^{\alpha s}\left|u_s^{\mathbb{P}}\widehat{F}_s^0\right| ds$$
$$+ 2C\int_t^T e^{\alpha s}\left|u_s^{\mathbb{P}}\right| ds$$
$$- 2\int_t^T e^{\alpha s}u_s^{\mathbb{P}}v_s^{\mathbb{P}} dB_s. \tag{2.4.1}$$

Taking conditional expectation in (2.4.1) yields

$$e^{\alpha t}\left|u_s^{\mathbb{P}}\right|^2 \leq \mathbb{E}_t^{\mathbb{P}}\left[e^{\alpha T}\left|\xi\right|^2 + 2\int_t^T e^{\alpha s}\left|u_s^{\mathbb{P}}\right|\left(\left|\widehat{F}_s^0\right| + C\right) ds\right].$$

By Doob's maximal inequality, we then get for all $\beta \in (0,1)$, since $\varepsilon > 0$

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} e^{\alpha t\frac{2+\varepsilon}{2}}\left|u_t^{\mathbb{P}}\right|^{2+\varepsilon}\right] \leq C\mathbb{E}^{\mathbb{P}}\left[e^{\alpha T\frac{2+\varepsilon}{2}}\left|\xi\right|^{2+\varepsilon}\right]$$
$$+ C\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left(e^{\alpha t\frac{2+\varepsilon}{4}}\left|u_s^{\mathbb{P}}\right|^{\frac{2+\varepsilon}{2}}\right)\left(\int_0^T C + \left|\widehat{F}_s^0\right| ds\right)^{\frac{2+\varepsilon}{2}}\right]$$
$$\leq C\left(1 + \left\|\xi\right\|_{L_H^{2+\varepsilon,\kappa}}^{2+\varepsilon}\right) + \beta\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} e^{\alpha t\frac{2+\varepsilon}{2}}\left|u_t^{\mathbb{P}}\right|^{2+\varepsilon}\right]$$
$$+ \beta\frac{C^2}{4}\mathbb{E}^{\mathbb{P}}\left[\int_0^T\left|\widehat{F}_s^0\right|^{2+\varepsilon} ds\right].$$

Since α is positive and $y^{\mathbb{P},n} \leq u^{\mathbb{P}}$, we get finally for all n

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|y_t^{\mathbb{P},n}\right|^{2+\varepsilon}\right] \leq C\left(1 + \left\|\xi\right\|_{L_H^{2+\varepsilon,\kappa}}^{2+\varepsilon} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\int_0^T\left|\widehat{F}_s^0\right|^{2+\varepsilon} ds\right]\right). \tag{2.4.2}$$

Now apply Itô's formula to $\left|y^{\mathbb{P},n}\right|^2$ and put each side to the power $\frac{2+\varepsilon}{2}$, we have easily

$$\left(\int_0^T\left|\widehat{a}_t^{1/2}z_t^{\mathbb{P},n}\right|^2 dt\right)^{\frac{2+\varepsilon}{2}} \leq C\left(\left|\xi\right|^{2+\varepsilon} + \left(\int_0^T\left|y_t^{\mathbb{P},n}\widehat{F}_t^n(y_t^{\mathbb{P},n}, z_t^{\mathbb{P},n})\right| dt\right)^{\frac{2+\varepsilon}{2}}\right)$$
$$+ C\left(\int_0^T y_t^{\mathbb{P},n}z_t^{\mathbb{P},n} dB_t\right)^{\frac{2+\varepsilon}{2}}$$

Since $\widehat{F}^n$ satisfies a uniform linear growth property, we get

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T\left|\widehat{a}_t^{1/2}z_t^{\mathbb{P},n}\right|^2 dt\right)^{\frac{2+\varepsilon}{2}}\right] \leq C\left\|\xi\right\|_{L_H^{2+\varepsilon,\kappa}}^{2+\varepsilon} + C\left(1 + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\int_0^T\left|\widehat{F}_t^0\right|^{2+\varepsilon} dt\right]\right)$$
$$+ C\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|y_t^{\mathbb{P},n}\right|^{2+\varepsilon}\right]$$
$$+ \frac{1}{3}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T\left|\widehat{a}_t^{1/2}z_t^{\mathbb{P},n}\right|^2 dt\right)^{\frac{2+\varepsilon}{2}}\right]$$
$$+ C\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T y_t^{\mathbb{P},n}z_t^{\mathbb{P},n} dB_t\right)^{\frac{2+\varepsilon}{2}}\right]$$

Hence, we get for all $\gamma \in (0,1)$ by BDG inequality and (2.4.2)

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{a}_t^{1/2} z_t^{\mathbb{P},n}\right|^2 dt\right)^{\frac{2+\varepsilon}{2}}\right] &\leq C\left(1+\|\xi\|^{2+\varepsilon}_{L_H^{2+\varepsilon,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{F}_t^0\right|^{2+\varepsilon} dt\right]\right) \\
&\quad + C\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|y_t^{\mathbb{P},n}\right|^2 \left|\widehat{a}_t^{1/2} z_t^{\mathbb{P},n}\right|^2 dt\right)^{\frac{2+\varepsilon}{4}}\right] \\
&\leq C\left(1+\|\xi\|^{2+\varepsilon}_{L_H^{2+\varepsilon,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{F}_t^0\right|^{2+\varepsilon} dt\right]\right) \\
&\quad + \gamma\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{a}_t^{1/2} z_t^{\mathbb{P},n}\right|^2 dt\right)^{\frac{2+\varepsilon}{2}}\right] \\
&\quad + \frac{C^2}{4\gamma}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|y_t^{\mathbb{P},n}\right|^{2+\varepsilon}\right],
\end{aligned}$$

which ends the proof. □

Chapitre 3

2BSDEs with quadratic growth

Contents

3.1 Introduction

Motivated by a robust utility maximization problem under volatility uncertainty (see the following Chapter), our aim here is to go beyond the results of Chapter 2 to prove an existence and uniqueness result for 2BSDEs whose generator has quadratic growth in z. The question of existence and uniqueness of solutions to these quadratic equations in the classical case was first examined by Kobylanski [68], who proved existence and uniqueness of a solution by means of approximation techniques borrowed from the PDE litterature, when the generator is continuous and has quadratic growth in z and the terminal condition ξ is bounded. Then, Tevzadze [112] has given a direct proof for the existence and uniqueness of a bounded solution in the Lipschitz-quadratic case, proving the convergence of the usual Picard iteration. Following those works, Briand and Hu [14] have extended the existence result to unbounded terminal condition with exponential moments and proved uniqueness for a convex coefficient [15]. Finally, Barrieu and El Karoui [7] recently adopted a completely different approach, embracing a forward point of view to prove existence under conditions similar to those of Briand and Hu.

In this Chapter, we propose to prove existence and uniqueness in the 2BSDE case. First in Section 3.4, we use the method introduced by Soner, Touzi and Zhang [107] to construct the solution to the quadratic 2BSDE path by path. We did not however managed to use the classical techniques of the BSDE litterature to prove existence. Indeed, we emphasize that since we are working under a family of singular probability measures, the monotone convergence theorem is known to hold only when the random variables considered are regular in ω. This induces some major technicalities which are difficult to deal with, and we refer to Chapter 6 for a more detailed discussion about this problem. Finally, in Section 3.5, we extend the results of Soner, Touzi and Zhang on the connections between fully non-linear PDEs and 2BSDEs to the quadratic case.

3.2 Preliminaries

The set up and notations are the same as the ones introduced in Chapter 2, we will therefore limit ourselves here to introduce the specific notations corresponding to our quadratic framework.

3.2.1 The local martingale measures

As in [107], we concentrate on the subclass $\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$ consisting of all probability measures

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1} \text{ where } X^\alpha_t := \int_0^t \alpha_s^{1/2} dB_s,\ t \in [0,1],\ \mathbb{P}_0 - a.s. \tag{3.2.1}$$

for some $\mathbb{F}$-progressively measurable process α satisfying $\int_0^T |\alpha_s|\, ds < +\infty$. We recall from [108] that every $\mathbb{P} \in \overline{\mathcal{P}}_S$ satisfies the Blumenthal zero-one law and the martingale representation property.

Notice that the set $\overline{\mathcal{P}}_S$ is bigger that the set $\widetilde{\mathcal{P}}_S$ introduced in Chapter 2.

3.2.2 The non-linear generator

We consider a map $H_t(\omega, y, z, \gamma) : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$, where $D_H \subset \mathbb{R}^{d\times d}$ is a given subset containing 0.

Define the corresponding conjugate of H w.r.t. γ by

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2}\mathrm{Tr}(a\gamma) - H_t(\omega, y, z, \gamma) \right\} \text{ for } a \in S_d^{>0}$$

$$\widehat{F}_t(y,z) := F_t(y, z, \widehat{a}_t) \text{ and } \widehat{F}^0_t := \widehat{F}_t(0,0).$$

We denote by $D_{F_t(y,z)}$ the domain of F in a for a fixed (t, ω, y, z).

As in [107] we restrict the probability measures in $\mathcal{P}_H \subset \overline{\mathcal{P}}_S$

Definition 3.2.1. *$\mathcal{P}_H$ consists of all $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that*

$$\underline{a}_{\mathbb{P}} \leq \widehat{a} \leq \bar{a}_{\mathbb{P}},\ dt \times d\mathbb{P} - a.s. \text{ for some } \underline{a}_{\mathbb{P}}, \bar{a}_{\mathbb{P}} \in S_d^{>0}, \text{ and } \widehat{a} \in D_{F_t(y,z)}.$$

We now state our main assumptions on the function F which will be our main interest in the sequel

Assumption 3.2.1. (i) *The domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z).*

(ii) *For fixed (y, z, γ), F is $\mathbb{F}$-progressively measurable.*

(iii) *F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.*

(iv) *F is continuous in z and has the following growth property. There exists $(\alpha, \beta, \gamma) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*$ such that*

$$|F_t(\omega, y, z, a)| \leq \alpha + \beta |y| + \frac{\gamma}{2}\left|a^{1/2}z\right|^2,\ \text{for all } (t, y, z, \omega, a).$$

(v) *F is C^1 in y and C^2 in z, and there are constants r and θ such that for all (t,ω,y,z,a),*

$$|D_yF_t(\omega,y,z,a)| \leq r, \;\; |D_zF_t(\omega,y,z,a)| \leq r + \theta \left|a^{1/2}z\right|,$$

$$|D^2_{zz}F_t(\omega,y,z,a)| \leq \theta.$$

Remark 3.2.1. *Let us comment on the above assumptions. Assumptions 3.2.1* (i) *and* (iii) *are taken from [107] and are needed to deal with the technicalities induced by the quasi-sure framework. Assumptions 3.2.1* (ii) *and* (iv) *are quite standard in the classical BSDE litterature. Finally, Assumption 3.2.1* (v) *was introduced by Tevzadze in [112] for quadratic BSDEs and is essential in our framework to prove existence of the solution to 2BSDEs in Section 3.4.*

Remark 3.2.2. *When the terminal condition is small enough, Assumption 3.2.1* (v) *can be replaced by a weaker one, as shown by Tevzadze in [112]. We will therefore sometimes consider*

Assumption 3.2.2. (i) *The domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω,y,z).*

(ii) *For fixed (y,z,γ), F is $\mathbb{F}$-progressively measurable.*

(iii) *F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.*

(iv) *F is continuous in z and has the following growth property. There exists $(\alpha,\beta,\gamma) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*$ such that*

$$|F_t(\omega,y,z,a)| \leq \alpha + \beta\,|y| + \frac{\gamma}{2}\left|a^{1/2}z\right|^2, \text{ for all } (t,y,z,\omega,a).$$

(v) *We have the following "local Lipschitz" assumption in z, $\exists \mu > 0$ and a progressively measurable process $\phi \in \mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$ such that for all (t,y,z,z',ω,a),*

$$\left|F_t(\omega,y,z,a) - F_t(\omega,y,z',a) - \phi_t.a^{1/2}(z-z')\right| \leq \mu a^{1/2}\left|z-z'\right|\left(\left|a^{1/2}z\right| + \left|a^{1/2}z'\right|\right).$$

(vi) *We have the following uniform Lipschitz-type property in y*

$$\left|F_t(\omega,y,z,a) - F_t(\omega,y',z,a)\right| \leq C\left|y-y'\right|, \text{ for all } (y,y',z,t,\omega,a).$$

Furthermore, we observe that our subsequent proof of uniqueness of a solution of our quadratic 2BSDE only use Assumption (3.2.2).

Remark 3.2.3. *By Assumption 3.2.1*(v)*, we have that $\widehat{F}^0_t$ is actually bounded, so the strong integrability condition*

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{F}^0_t\right|^\kappa dt\right)^{\frac{2}{\kappa}}\right] < +\infty,$$

with a constant $\kappa \in (1,2]$ introduced in [107] is not needed here. Moreover, this implies that $\widehat{a}$ always belongs to D_{F_t} and thus that $\mathcal{P}_H$ is not empty.

3.2.3 The spaces and norms

We only add here some particular spaces which are particularly linked to our quadratic growth framework.

L^∞_H denotes the space of random variables which are bounded quasi-surely and take as a norm

$$\|\xi\|_{L^\infty_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} \|\xi\|_{L^\infty(\mathbb{P})}\,.$$

$\mathbb{BMO}(\mathcal{P}_{\rm H})$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}^d$-valued processes Z with

$$\|Z\|_{\mathbb{BMO}(\mathcal{P}_{\rm H})} := \sup_{\mathbb{P}\in\mathcal{P}_H} \left\|\int_0^\cdot Z_s dB_s\right\|_{\mathrm{BMO}(\mathbb{P})} < +\infty,$$

where $\|\cdot\|_{\mathrm{BMO}(\mathbb{P})}$ is the usual $\mathrm{BMO}(\mathbb{P})$ norm under $\mathbb{P}$.

$\mathbb{D}^\infty_H$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes Y with

$$\mathcal{P}_H - q.s. \text{ càdlàg paths, and } \|Y\|_{\mathbb{D}^\infty_H} := \sup_{0\leq t\leq T} \|Y_t\|_{L^\infty_H} < +\infty.$$

In the case $p=+\infty$ the natural generalization of the norm $\mathbb{L}^p_H$ is the norm L^∞_H introduced above. Therefore, we will use the latter in order to be consistent with the notations of Chapter 2.

Finally, we denote by $\mathrm{UC}_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi:\Omega\to\mathbb{R}$ with respect to the $\|\cdot\|_\infty$-norm, and we let

$$\mathcal{L}^p_H := \text{the closure of } \mathrm{UC}_b(\Omega) \text{ under the norm } \|\cdot\|_{\mathbb{L}^p_H}, \text{ for every } p\geq 1.$$

3.2.4 Some properties of the space $\mathbb{BMO}(\mathcal{P}_{\rm H})$

We present here some results on the space $\mathbb{BMO}(\mathcal{P}_{\rm H})$ which generalize the already known results on the spaces $\mathrm{BMO}(\mathbb{P})$ for a given $\mathbb{P}$.

Lemma 3.2.1. *Let M be a continuous $\mathbb{P}$-local martingale for all $\mathbb{P}\in\mathcal{P}_H$ such that $M\in\mathbb{BMO}(\mathcal{P}_{\rm H})$. Then there exists $r>1$, such that $\mathcal{E}(M)\in L^r_H$.*

Proof. By Theorem 3.1 in [66], we know that if $\|M\|_{\mathrm{BMO}(\mathbb{P})}\leq\Phi(r)$ for some one-to-one function Φ from $(1,+\infty)$ to $\mathbb{R}^*_+$, then $\mathcal{E}(M)$ is in $L^r(\mathbb{P})$. Here, since $M\in\mathbb{BMO}(\mathcal{P}_{\rm H})$, the same r can be used for all the probability measures. □

Lemma 3.2.2. *M be a continuous $\mathbb{P}$-local martingale for all $\mathbb{P}\in\mathcal{P}_H$ such that $M\in\mathbb{BMO}(\mathcal{P}_{\rm H})$. Then there exists $r>1$, such that for all $t\in[0,T]$*

$$\sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}_t\left[\left(\frac{\mathcal{E}(M)_t}{\mathcal{E}(M)_T}\right)^{\frac{1}{r-1}}\right] < +\infty.$$

Proof. This is a direct application of Theorem 2.4 in [66] for all $\mathbb{P}\in\mathcal{P}_H$. □

We emphasize that the two previous Lemmas are absolutely crucial to our proof of uniqueness and existence. Besides, they will also play a major role in the following chapter.

3.2.5 Formulation

We shall consider the following second order BSDE (2BSDE for short), which was first defined in [107]

$$Y_t = \xi + \int_t^T \widehat{F}_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ 0 \le t \le T,\ \mathcal{P}_H - q.s. \tag{3.2.2}$$

Definition 3.2.2. *We say* $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ *is a solution to 2BSDE (3.2.2) if :*

- $Y_T = \xi,\ \mathcal{P}_H - q.s.$
- *For all* $\mathbb{P} \in \mathcal{P}_H$*, the process* $K^{\mathbb{P}}$ *defined below has non-decreasing paths* $\mathbb{P} - a.s.$

$$K_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ 0 \le t \le T,\ \mathbb{P} - a.s. \tag{3.2.3}$$

- *The family* $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\}$ *satisfies the minimum condition*

$$K_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}\left[K_T^{\mathbb{P}'}\right],\ 0 \le t \le T,\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H. \tag{3.2.4}$$

Moreover if the family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\}$ *can be aggregated into a universal process* K*, we call* (Y, Z, K) *a solution of 2BSDE (3.2.2).*

3.3 Representation and uniqueness of the solution

We follow once more Soner, Touzi and Zhang [107]. For any $\mathbb{P} \in \mathcal{P}_H$, $\mathbb{F}$-stopping time τ, and $\mathcal{F}_\tau$-measurable random variable $\xi \in \mathbb{L}^\infty(\mathbb{P})$, we define $(y^{\mathbb{P}}, z^{\mathbb{P}}) := (y^{\mathbb{P}}(\tau, \xi), z^{\mathbb{P}}(\tau, \xi))$ as the unique solution of the following standard BSDE (existence and uniqueness have been proved under our assumptions by Tevzadze in [112])

$$y_t^{\mathbb{P}} = \xi + \int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} dB_s,\ 0 \le t \le \tau,\ \mathbb{P} - a.s. \tag{3.3.1}$$

First, we introduce the following simple generalization of the comparison Theorem proved in [112] (see Theorem 2).

Proposition 3.3.1. *Let Assumptions 3.2.1 hold true. Let* ξ_1 *and* $\xi_2 \in L^\infty(\mathbb{P})$ *for some probability measure* $\mathbb{P}$*, and* V^i*,* $i = 1, 2$ *be two adapted, càdlàg non-decreasing processes null at 0. Let* $(Y^i, Z^i) \in \mathbb{D}^\infty(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P})$*,* $i = 1, 2$ *be the solutions of the following BSDEs*

$$Y_t^i = \xi^i + \int_t^T \widehat{F}_s(Y_s^i, Z_s^i)ds - \int_t^T Z_s^i dB_s + V_T^i - V_t^i,\ \mathbb{P} - a.s.,\ i = 1, 2,$$

respectively. If $\xi_1 \ge \xi_2$*,* $\mathbb{P} - a.s.$ *and* $V^1 - V^2$ *is non-decreasing, then it holds* $\mathbb{P} - a.s.$ *that for all* $t \in [0, T]$

$$Y_t^1 \ge Y_t^2.$$

Proof. First of all, we need to justify the existence of the solutions to those BSDEs. Actually, this is a simple consequence of the existence results of Tevzadze [112] and for instance Proposition 3.1 in [77]. Then, the above comparison is a mere generalization of Theorem 2 in [112]. □

We then have similarly as in Theorem 4.4 of [107] or Theorem 2.2.2

Theorem 3.3.1. *Let Assumptions 3.2.2 hold. Assume $\xi \in \mathbb{L}^\infty_H$ and that $(Y,Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ is a solution to 2BSDE (3.2.2). Then, for any $\mathbb{P} \in \mathcal{P}_H$ and $0 \le t_1 < t_2 \le T$,*

$$Y_{t_1} = \underset{\mathbb{P}' \in \mathcal{P}_H(t_1,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \; y^{\mathbb{P}'}_{t_1}(t_2, Y_{t_2}), \;\; \mathbb{P} - a.s. \tag{3.3.2}$$

Consequentlty, the 2BSDE (3.2.2) has at most one solution in $\mathbb{D}^\infty_H \times \mathbb{H}^2_H$.

Before proceeding with the proof, we will need the following Lemma which shows that in our 2BSDE framework, we still have a deep link between quadratic growth and the BMO spaces.

Lemma 3.3.1. *Let Assumption 3.2.2 hold. Assume $\xi \in \mathbb{L}^\infty_H$ and that $(Y,Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ is a solution to 2BSDE (3.2.2). Then $Z \in \mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$.*

Proof. Denote $\mathcal{T}_0^T$ the collection of stopping times taking values in $[0,T]$ and for each $\mathbb{P} \in \mathcal{P}_H$, let $(\tau_n^{\mathbb{P}})_{n \ge 1}$ be a localizing sequence for the $\mathbb{P}$-local martingale $\int_0^\cdot Z_s dB_s$. By Itô's formula under $\mathbb{P}$ applied to $e^{-\nu Y_t}$, which is a càdlàg process, for some $\nu > 0$, we have for every $\tau \in \mathcal{T}_0^T$

$$\begin{aligned}\frac{\nu^2}{2}\int_\tau^{\tau_n^{\mathbb{P}}} e^{-\nu Y_t}\left|\widehat{a}_t^{1/2} Z_t\right|^2 dt &= e^{-\nu Y_{\tau_n^{\mathbb{P}}}} - e^{-\nu Y_\tau} - \nu \int_\tau^{\tau_n^{\mathbb{P}}} e^{-\nu Y_{t^-}} dK_t^{\mathbb{P}} - \nu \int_\tau^{\tau_n^{\mathbb{P}}} e^{-\nu Y_t}\widehat{F}_t(Y_t, Z_t)dt \\ &\quad + \nu \int_\tau^{\tau_n^{\mathbb{P}}} e^{-\nu Y_{t^-}} Z_t dB_t - \sum_{\tau \le s \le \tau_n^{\mathbb{P}}} e^{-\nu Y_s} - e^{-\nu Y_{s^-}} + \nu \Delta Y_s e^{-\nu Y_{s^-}}.\end{aligned}$$

Since $Y \in \mathbb{D}^\infty_H$, $K^{\mathbb{P}}$ is non-decreasing and since the contribution of the jumps is negative because of the convexity of the function $x \to e^{-\nu x}$, we obtain with Assumption 3.2.1(iv)

$$\begin{aligned}\frac{\nu^2}{2}\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^{\tau_n^{\mathbb{P}}} e^{-\nu Y_t}\left|\widehat{a}_t^{1/2} Z_t\right|^2 dt\right] &\le e^{\nu \|Y\|_{\mathbb{D}^\infty_H}}\left(1 + \nu T\left(\alpha + \beta \|Y\|_{\mathbb{D}^\infty_H}\right)\right) \\ &\quad + \frac{\nu\gamma}{2}\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^{\tau_n^{\mathbb{P}}} e^{-\nu Y_t}\left|\widehat{a}_t^{1/2} Z_t\right|^2 dt\right].\end{aligned}$$

By choosing $\nu = 2\gamma$, we then have

$$\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^{\tau_n^{\mathbb{P}}} e^{-2\gamma Y_t}\left|\widehat{a}_t^{1/2} Z_t\right|^2 dt\right] \le \frac{1}{\gamma} e^{2\gamma \|Y\|_{\mathbb{D}^\infty_H}}\left(1 + 2\gamma T\left(\alpha + \beta \|Y\|_{\mathbb{D}^\infty_H}\right)\right).$$

Finally, by monotone convergence and Fatou's lemma we get that

$$\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^{T} \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt\right] \le \frac{1}{\gamma} e^{4\gamma \|Y\|_{\mathbb{D}^\infty_H}}\left(1 + 2\gamma T\left(\alpha + \beta \|Y\|_{\mathbb{D}^\infty_H}\right)\right),$$

which provides the result by arbitrariness of $\mathbb{P}$ and τ. □

Proof. [Proof of Theorem 3.3.1] The proof follows the lines of the proof of Theorem 4.4 in [107], but we have to deal with some specific difficulties due to our quadratic growth assumption. First (3.3.2) implies that

$$Y_t = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \; y_t^{\mathbb{P}'}(T,\xi), \;\; t \in [0,T], \;\; \mathbb{P} - a.s. \text{ for all } \mathbb{P} \in \mathcal{P}_H,$$

and thus is unique. Then, since we have that $d\langle Y, B\rangle_t = Z_t d\langle B\rangle_t$, $\mathcal{P}_H - q.s.$, Z is also unique. We now prove (3.3.2) in three steps.

(i) Fix $0 \leq t_1 < t_2 \leq T$ and $\mathbb{P} \in \mathcal{P}_H$. For any $\mathbb{P}^{'} \in \mathcal{P}_H(t_1^+, \mathbb{P})$, we have

$$Y_t = Y_{t_2} + \int_t^{t_2} \widehat{F}_s(Y_s, Z_s)ds - \int_t^{t_2} Z_s dB_s + K_{t_2}^{\mathbb{P}^{'}} - K_t^{\mathbb{P}^{'}},\ t_1 \leq t \leq t_2,\ \mathbb{P}^{'} - a.s.$$

and that $K^{\mathbb{P}^{'}}$ is nondecreasing, $\mathbb{P}^{'} - a.s.$ Then, we can apply the comparison Theorem 3.3.1 under $\mathbb{P}^{'}$ to obtain $Y_{t_1} \geq y_{t_1}^{\mathbb{P}^{'}}(t_2, Y_{t_2})$, $\mathbb{P}^{'} - a.s.$ Since $\mathbb{P}^{'} = \mathbb{P}$ on $\mathcal{F}_t^+$, we get $Y_{t_1} \geq y_{t_1}^{\mathbb{P}^{'}}(t_2, Y_{t_2})$, $\mathbb{P} - a.s.$ and thus

$$Y_{t_1} \geq \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}^{'} \in \mathcal{P}_H(t_1^+, \mathbb{P})} y_{t_1}^{\mathbb{P}^{'}}(t_2, Y_{t_2}),\ \mathbb{P} - a.s.$$

(ii) We now prove the reverse inequality. Fix $\mathbb{P} \in \mathcal{P}_H$. Let us assume for the time being that

$$C_{t_1}^{\mathbb{P},p} := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}^{'} \in \mathcal{P}_H(t_1^+, \mathbb{P})} \mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\left(K_{t_2}^{\mathbb{P}^{'}} - K_{t_1}^{\mathbb{P}^{'}}\right)^p\right] < +\infty,\ \mathbb{P} - a.s.,\ \text{for all } p \geq 1.$$

For every $\mathbb{P}^{'} \in \mathcal{P}_H(t^+, \mathbb{P})$, denote

$$\delta Y := Y - y^{\mathbb{P}^{'}}(t_2, Y_{t_2}) \text{ and } \delta Z := Z - z^{\mathbb{P}^{'}}(t_2, Y_{t_2}).$$

By the Lipschitz Assumption 3.2.2(vi) and the local Lipschitz Assumption 3.2.2(v), there exist a bounded process λ and a process η with

$$|\eta_t| \leq \mu\left(\left|\widehat{a}_t^{1/2} Z_t\right| + \left|\widehat{a}_t^{1/2} z_t^{\mathbb{P}^{'}}\right|\right),\ \mathbb{P}^{'} - a.s.$$

such that

$$\delta Y_t = \int_t^{t_2}\left(\lambda_s \delta Y_s + (\eta_s + \phi_s)\widehat{a}_s^{1/2}\delta Z_s\right)ds - \int_t^{t_2} \delta Z_s dB_s + K_{t_2}^{\mathbb{P}^{'}} - K_t^{\mathbb{P}^{'}},\ t \leq t_2,\ \mathbb{P}^{'} - a.s.$$

Define for $t_1 \leq t \leq t_2$

$$M_t := \exp\left(\int_{t_1}^t \lambda_s ds\right),\ \mathbb{P}^{'} - a.s.$$

Now, since $\phi \in \mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$, by Lemma 3.3.1, we know that the $\mathbb{P}^{'}$-exponential martingale $\mathcal{E}\left(\int_0^{\cdot}(\phi_s + \eta_s)\widehat{a}_s^{-1/2} dB_s\right)$ is a $\mathbb{P}^{'}$-uniformly integrable martingale (see Theorem 2.3 in the book by Kazamaki [66]). Therefore we can define a probability measure $\mathbb{Q}^{'}$, which is equivalent to $\mathbb{P}^{'}$, by its Radon-Nykodym derivative

$$\frac{d\mathbb{Q}^{'}}{d\mathbb{P}^{'}} = \mathcal{E}\left(\int_0^T (\phi_s + \eta_s))\widehat{a}_s^{-1/2} dB_s\right).$$

Then, by Itô's formula, we obtain, as in [107], that

$$\delta Y_{t_1} = \mathbb{E}_{t_1}^{\mathbb{Q}^{'}}\left[\int_{t_1}^{t_2} M_t dK_t^{\mathbb{P}^{'}}\right] \leq \mathbb{E}_{t_1}^{\mathbb{Q}^{'}}\left[\sup_{t_1 \leq t \leq t_2}(M_t)(K_{t_2}^{\mathbb{P}^{'}} - K_{t_1}^{\mathbb{P}^{'}})\right],$$

since $K^{\mathbb{P}^{'}}$ is non-decreasing. Then, since λ is bounded, we have that M is also bounded and thus for every $p \geq 1$

$$\mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\sup_{t_1 \leq t \leq t_2}(M_t)^p\right] \leq C_p,\ \mathbb{P}^{'} - a.s. \tag{3.3.3}$$

Since $(\eta+\phi)\widehat{a}_s^{-1/2}$ is in $\mathbb{BMO}(\mathcal{P}_{\rm H})$, we know by Lemma 3.2.1 that there exists $r>1$, independent of the probability measure considered, such that $\mathcal{E}\left(\int_0^T(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)\in L^r_H$. Then it follows from the Hölder inequality and Bayes Theorem that

$$\delta Y_{t_1}\leq\frac{\left(\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\mathcal{E}\left(\int_0^T(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)^r\right]\right)^{\frac{1}{r}}}{\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\mathcal{E}\left(\int_0^T(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)\right]}\left(\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\left(\sup_{t_1\leq t\leq t_2}M_t\right)^q\left(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'}\right)^q\right]\right)^{\frac{1}{q}}$$

$$\leq C\left(C_{t_1}^{\mathbb{P},4q-1}\right)^{\frac{1}{4q}}\left(\mathbb{E}_{t_1}^{\mathbb{P}'}\left[K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'}\right]\right)^{\frac{1}{4q}}.$$

By the minimum condition (3.2.4) and since $\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})$ is arbitrary, this ends the proof.

(iii) It remains to show that the estimate for $C_{t_1}^{\mathbb{P},p}$ holds for $p\geq 1$. By definition of the family $\left\{K^{\mathbb{P}},\mathbb{P}\in\mathcal{P}_H\right\}$, we have

$$\mathbb{E}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'}\right)^p\right]\leq C\left(1+\|Y\|^p_{\mathbb{D}^\infty_H}+\|\xi\|^p_{\mathbb{L}^\infty_H}+\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\left(\int_{t_1}^{t_2}\left|\widehat{a}_t^{1/2}Z_t\right|^2dt\right)^p\right]\right)$$
$$+C\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\left(\int_{t_1}^{t_2}Z_tdB_t\right)^p\right].$$

Thus by the energy inequalities for BMO martingales and by Burkholder-Davis-Gundy inequality, we get that

$$\mathbb{E}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'}\right)^p\right]\leq C\left(1+\|Y\|^p_{\mathbb{D}^\infty_H}+\|\xi\|^p_{\mathbb{L}^\infty_H}+\|Z\|^{2p}_{\mathbb{BMO}_{\rm H}}+\|Z\|^p_{\mathbb{BMO}_{\rm H}}\right).$$

Therefore, we have proved that

$$\sup_{\mathbb{P}'\in\mathcal{P}_H(t_1^+,\mathbb{P})}\mathbb{E}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'}\right)^p\right]<+\infty.$$

Then we proceed exactly as in the proof of Theorem 4.4 in [107].

□

We conclude this section by showing some a priori estimates which will be useful in the sequel.

Theorem 3.3.2. *Let Assumption 3.2.2 hold.*

(i) *Assume that $\xi\in\mathbb{L}^\infty_H$ and that $(Y,Z)\in\mathbb{D}^\infty_H\times\mathbb{H}^2_H$ is a solution to 2BSDE (3.2.2). Then, there exists a constant C such that*

$$\|Y\|_{\mathbb{D}^\infty_H}+\|Z\|^2_{\mathbb{BMO}(\mathcal{P}_{\rm H})}\leq C\left(1+\|\xi\|_{\mathbb{L}^\infty_H}\right)$$
$$\forall p\geq 1,\quad\sup_{\mathbb{P}\in\mathcal{P}_H,\ \tau\in\mathcal{T}_0^T}\mathbb{E}_\tau^{\mathbb{P}}\left[(K_T^{\mathbb{P}}-K_\tau^{\mathbb{P}})^p\right]\leq C\left(1+\|\xi\|^p_{\mathbb{L}^\infty_H}\right).$$

(ii) *Assume that $\xi^i\in\mathbb{L}^\infty_H$ and that $(Y^i,Z^i)\in\mathbb{D}^\infty_H\times\mathbb{H}^2_H$ is a corresponding solution to 2BSDE (3.2.2), $i=1,2$. Denote $\delta\xi:=\xi^1-\xi^2$, $\delta Y:=Y^1-Y^2$, $\delta Z:=Z^1-Z^2$ and $\delta K^{\mathbb{P}}:=K^{\mathbb{P},1}-K^{\mathbb{P},2}$. Then, there exists a constant C such that*

$$\|\delta Y\|_{\mathbb{D}^\infty_H}\leq C\|\delta\xi\|_{\mathbb{L}^\infty_H}$$
$$\|\delta Z\|^2_{\mathbb{BMO}(\mathcal{P}_{\rm H})}\leq C\|\delta\xi\|_{\mathbb{L}^\infty_H}\left(1+\left\|\xi^1\right\|_{\mathbb{L}^\infty_H}+\left\|\xi^2\right\|_{\mathbb{L}^\infty_H}\right)$$
$$\forall p\geq 1,\ \sup_{\mathbb{P}\in\mathcal{P}_H}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|\delta K_t^{\mathbb{P}}\right|^p\right]\leq C\|\xi\|^{\frac{p}{2}}_{\mathbb{L}^\infty_H}\left(1+\left\|\xi^1\right\|^{\frac{p}{2}}_{\mathbb{L}^\infty_H}+\left\|\xi^2\right\|^{\frac{p}{2}}_{\mathbb{L}^\infty_H}\right).$$

Proof.

(i) By Theorem 3.3.1 we know that for all $\mathbb{P}\in\mathcal{P}_H$ and for all $t\in[0,T]$ we have

$$Y_t = \underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}}\ y_t^{\mathbb{P}'}.$$

Then by Lemma 1 in [14], we know that for all $\mathbb{P}\in\mathcal{P}_H$

$$\left|y_t^{\mathbb{P}}\right| \leq \frac{1}{\gamma}\log\left(\mathbb{E}_t^{\mathbb{P}}\left[\psi(|\xi|)\right]\right),\ \text{where } \psi(x):=\exp\left(\gamma\alpha\frac{e^{\beta T}-1}{\beta}+\gamma e^{\beta T}x\right).$$

Thus, we obtain

$$\left|y_t^{\mathbb{P}}\right| \leq \alpha\frac{e^{\beta T}-1}{\beta}+e^{\beta T}\left\|\xi\right\|_{\mathbb{L}_H^\infty},$$

and by the representation recalled above, the estimate of $\|Y\|_{\mathbb{D}_H^\infty}$ is obvious.

By the proof of Lemma 3.3.1, we have now

$$\|Z\|^2_{\mathbb{BMO}(\mathcal{P}_\mathrm{H})} \leq Ce^{C\|Y\|_{\mathbb{D}_H^\infty}}\left(1+\|Y\|_{\mathbb{D}_H^\infty}\right)\leq C\left(1+\|\xi\|_{\mathbb{L}_H^\infty}\right).$$

Finally, we have for all $\tau\in\mathcal{T}_0^T$, for all $\mathbb{P}\in\mathcal{P}_H$ and for all $p\geq 1$, by definition

$$(K_T^{\mathbb{P}}-K_\tau^{\mathbb{P}})^p = \left(Y_\tau-\xi-\int_\tau^T \widehat{F}_t(Y_y,Z_t)dt+\int_\tau^T Z_tdB_t\right)^p.$$

Therefore, by our growth Assumption 3.2.1(iv)

$$\begin{aligned}
\mathbb{E}_\tau^{\mathbb{P}}\left[(K_T^{\mathbb{P}}-K_\tau^{\mathbb{P}})^p\right] &\leq C\left(1+\|\xi\|^p_{\mathbb{L}_H^\infty}+\|Y\|^p_{\mathbb{D}_H^\infty}+\mathbb{E}_\tau^{\mathbb{P}}\left[\left(\int_\tau^T\left|\widehat{a}_t^{1/2}Z_t\right|^2dt\right)^p\right]\right)\\
&\quad+C\mathbb{E}_\tau^{\mathbb{P}}\left[\left(\int_\tau^T Z_tdB_t\right)^p\right]\\
&\leq C\left(1+\|\xi\|^p_{\mathbb{L}_H^\infty}+\|Z\|^{2p}_{\mathbb{BMO}(\mathcal{P}_\mathrm{H})}+\|Z\|^{p}_{\mathbb{BMO}(\mathcal{P}_\mathrm{H})}\right)\\
&\leq C\left(1+\|\xi\|^p_{\mathbb{L}_H^\infty}\right),
\end{aligned}$$

where we used again the energy inequalities and the BDG inequality. This provides the estimate for $K^{\mathbb{P}}$ by arbitrariness of τ and $\mathbb{P}$.

(ii) With the same notations and calculations as in step (ii) of the proof of Theorem 3.3.1, it is easy to see that for all $\mathbb{P}\in\mathcal{P}_H$ and for all $t\in[0,T]$, we have

$$\delta y_t^{\mathbb{P}} = \mathbb{E}_t^{\mathbb{Q}}\left[M_T\delta\xi\right]\leq C\left\|\delta\xi\right\|_{\mathbb{L}_H^\infty},$$

since M is bounded and we have (3.3.3). By Theorem 3.3.1, the estimate for δY follows.

Now apply Itô's formula under a fixed $\mathbb{P}\in\mathcal{P}_H$ to $|\delta Y|^2$ between a given stopping time $\tau\in\mathcal{T}_0^T$ and T

$$\begin{aligned}
\mathbb{E}_\tau^{\mathbb{P}}\left[|\delta Y_\tau|^2+\int_\tau^T\left|\widehat{a}_t^{1/2}\delta Z_t\right|^2dt\right] &\leq \mathbb{E}_\tau^{\mathbb{P}}\left[|\delta\xi|^2+2\int_\tau^T\delta Y_t\left(\widehat{F}_t(Y_t^1,Z_t^1)-\widehat{F}_t(Y_t^2,Z_t^2)\right)dt\right]\\
&\quad-2\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T\delta Y_{t^-}d(\delta K_t^{\mathbb{P}})\right].
\end{aligned}$$

Then, we have by Assumption 3.2.1(iv) and the estimates proved in (i) above

$$\begin{aligned}\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T \left|\widehat{a}_t^{1/2}\delta Z_t\right|^2 dt\right] &\leq C\left\|\delta Y\right\|_{\mathbb{D}_H^\infty}\left(1+\sum_{i=1}^2 \left\|Y^i\right\|_{\mathbb{D}_H^\infty}+\left\|Z^i\right\|_{\mathbb{BMO}(\mathcal{P}_H)}\right)\\ &\quad+\left\|\delta\xi\right\|_{\mathbb{L}_H^\infty}^2+2\left\|\delta Y\right\|_{\mathbb{D}_H^\infty}\mathbb{E}_\tau^{\mathbb{P}}\left[\left|K_T^{\mathbb{P},1}-K_\tau^{\mathbb{P},1}\right|+\left|K_T^{\mathbb{P},2}-K_\tau^{\mathbb{P},2}\right|\right]\\ &\leq C\left\|\delta\xi\right\|_{\mathbb{L}_H^\infty}\left(1+\left\|\xi^1\right\|_{\mathbb{L}_H^\infty}+\left\|\xi^2\right\|_{\mathbb{L}_H^\infty}\right),\end{aligned}$$

which implies the required estimate for δZ.

Finally, by definition, we have for all $\mathbb{P}\in\mathcal{P}_H$ and for all $t\in[0,T]$

$$\delta K_t^{\mathbb{P}} = \delta Y_0 - \delta Y_t - \int_0^t \widehat{F}_s(Y_s^1, Z_s^1) - \widehat{F}_s(Y_s^2, Z_s^2)ds + \int_0^t \delta Z_s dB_s.$$

By Assumptions 3.2.2(v) and (vi), it follows that

$$\begin{aligned}\sup_{0\leq t\leq T}\left|\delta K_t^{\mathbb{P}}\right| &\leq C\left(\left\|\delta Y\right\|_{\mathbb{D}_H^\infty}+\int_0^T\left|\widehat{a}_s^{1/2}\delta Z_s\right|\left(1+\left|\widehat{a}_s^{1/2}Z_s^1\right|+\left|\widehat{a}_s^{1/2}Z_s^2\right|\right)ds\right)\\ &\quad+\sup_{0\leq t\leq T}\left|\int_0^t \delta Z_s dB_s\right|,\end{aligned}$$

and by Cauchy-Schwarz, BDG and energy inequalities, we see that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|\delta K_t^{\mathbb{P}}\right|^p\right] &\leq C\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T\left(1+\left|\widehat{a}_s^{1/2}Z_s^1\right|^2+\left|\widehat{a}_s^{1/2}Z_s^2\right|^2\right)ds\right)^p\right]^{\frac{1}{2}}\\ &\quad\times\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T\left|\widehat{a}_s^{1/2}\delta Z_s\right|^2 ds\right)^p\right]^{\frac{1}{2}}\\ &\quad+C\left(\left\|\delta\xi\right\|_{\mathbb{L}_H^\infty}^p+\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T\left|\widehat{a}_s^{1/2}\delta Z_s\right|^2 ds\right)^{p/2}\right]\right)\\ &\leq C\left\|\delta\xi\right\|_{\mathbb{L}_H^\infty}^{p/2}\left(1+\left\|\xi^1\right\|_{\mathbb{L}_H^\infty}^{p/2}+\left\|\xi^2\right\|_{\mathbb{L}_H^\infty}^{p/2}\right).\end{aligned}$$

□

Remark 3.3.1. *Let us note that the proof of* (i) *only requires that Assumption 3.2.2*(iv) *holds true, whereas* (ii) *also requires Assumption 3.2.2*(v) *and* (vi).

3.4 Proof of existence

In the article [107], the main tool to prove existence of a solution is the so-called regular conditional probability distributions of Stroock and Varadhan [111], which allows to construct a solution to the 2BSDE when the terminal condition belongs to the space $\mathrm{UC_b}(\Omega)$. In this section we will generalize their approach to the quadratic case.

3.4.1 Notations

For the convenience of the reader, we recall below some of the notations introduced in [107].

- For $0 \leq t \leq T$, denote by $\Omega^t := \left\{\omega \in C\left([t,1],\mathbb{R}^d\right),\ w(t)=0\right\}$ the shifted canonical space, B^t the shifted canonical process, $\mathbb{P}_0^t$ the shifted Wiener measure and $\mathbb{F}^t$ the filtration generated by B^t.

- For $0 \leq s \leq t \leq T$ and $\omega \in \Omega^s$, define the shifted path $\omega^t \in \Omega^t$

$$\omega_r^t := \omega_r - \omega_t,\ \forall r \in [t,T].$$

- For $0 \leq s \leq t \leq T$ and $\omega \in \Omega^s$, $\widetilde{\omega} \in \Omega^t$ define the concatenation path $\omega \otimes_t \widetilde{\omega} \in \Omega^s$ by

$$(\omega \otimes_t \widetilde{\omega})(r) := \omega_r 1_{[s,t)}(r) + (\omega_t + \widetilde{\omega}_r) 1_{[t,T]}(r),\ \forall r \in [s,T].$$

- For $0 \leq s \leq t \leq T$ and a $\mathcal{F}_T^s$-measurable random variable ξ on Ω^s, for each $\omega \in \Omega^s$, define the shifted $\mathcal{F}_T^t$-measurable random variable $\xi^{t,\omega}$ on Ω^t by

$$\xi^{t,\omega}(\widetilde{\omega}) := \xi(\omega \otimes_t \widetilde{\omega}),\ \forall \widetilde{\omega} \in \Omega^t.$$

  Similarly, for an $\mathbb{F}^s$-progressively measurable process X on $[s,T]$ and $(t,\omega) \in [s,T] \times \Omega^s$, the shifted process $\left\{X_r^{t,\omega}, r \in [t,T]\right\}$ is $\mathbb{F}^t$-progressively measurable.

- For a $\mathbb{F}$-stopping time τ, the r.c.p.d. of $\mathbb{P}$ (noted $\mathbb{P}_\tau^\omega$) induces naturally a probability measure $\mathbb{P}^{\tau,\omega}$ (that we also call the r.c.p.d. of $\mathbb{P}$) on $\mathcal{F}_T^{\tau(\omega)}$ which in particular satisfies that for every bounded and $\mathcal{F}_T$-measurable random variable ξ

$$\mathbb{E}^{\mathbb{P}_\tau^\omega}\left[\xi\right] = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}\left[\xi^{\tau,\omega}\right].$$

- We define similarly as in Section 3.4 the set $\bar{\mathcal{P}}_S^t$, by restricting to the shifted canonical space Ω^t, and its subset $\mathcal{P}_H^t$.

- Finally, we define our "shifted" generator

$$\widehat{F}_s^{t,\omega}(\widetilde{\omega},y,z) := F_s(\omega \otimes_t \widetilde{\omega}, y, z, \widehat{a}_s^t(\widetilde{\omega})),\ \forall (s,\widetilde{\omega}) \in [t,T] \times \Omega^t.$$

  Then note that since F is assumed to be uniformly continuous in ω under the $\mathbb{L}^\infty$ norm, then so is $\widehat{F}^{t,\omega}$.

3.4.2 Existence when ξ is in $\mathrm{UC_b}(\Omega)$

When ξ is in $\mathrm{UC_b}(\Omega)$, by definition there exists a modulus of continuity function ρ for ξ and F in ω. Then, for any $0 \leq t \leq s \leq T$, $(y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$ and $\omega, \omega' \in \Omega$, $\tilde{\omega} \in \Omega^t$,

$$\left|\xi^{t,\omega}\left(\tilde{\omega}\right) - \xi^{t,\omega'}\left(\tilde{\omega}\right)\right| \leq \rho\left(\left\|\omega - \omega'\right\|_t\right) \text{ and } \left|\widehat{F}_s^{t,\omega}\left(\tilde{\omega},y,z\right) - \widehat{F}_s^{t,\omega'}\left(\tilde{\omega},y,z\right)\right| \leq \rho\left(\left\|\omega - \omega'\right\|_t\right),$$

where $\|\omega\|_t := \sup_{0 \leq s \leq t} |\omega_s|$, $0 \leq t \leq T$.

To prove existence, as in [107], we define the following value process V_t pathwise:

$$V_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}_H^t} \mathcal{Y}_t^{\mathbb{P},t,\omega}(T,\xi),\ \text{for all } (t,\omega) \in [0,T] \times \Omega, \tag{3.4.1}$$

where, for any $(t_1,\omega)\in[0,T]\times\Omega$, $\mathbb{P}\in\mathcal{P}_H^{t_1}$, $t_2\in[t_1,T]$, and any $\mathcal{F}_{t_2}$-measurable $\eta\in\mathbb{L}^\infty(\mathbb{P})$, we denote $\mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2,\eta):=y_{t_1}^{\mathbb{P},t_1,\omega}$, where $\left(y^{\mathbb{P},t_1,\omega},z^{\mathbb{P},t_1,\omega}\right)$ is the solution of the following BSDE on the shifted space Ω^{t_1} under $\mathbb{P}$

$$y_s^{\mathbb{P},t_1,\omega}=\eta^{t_1,\omega}+\int_s^{t_2}\widehat{F}_r^{t_1,\omega}\left(y_r^{\mathbb{P},t_1,\omega},z_r^{\mathbb{P},t_1,\omega}\right)dr-\int_s^{t_2}z_r^{\mathbb{P},t_1,\omega}dB_r^{t_1},\ s\in[t_1,t_2],\ \mathbb{P}-\text{a.s.} \tag{3.4.2}$$

We recall that since the Blumenthal zero-one law holds for all our probability measures, $\mathcal{Y}_t^{\mathbb{P},t,\omega}(1,\xi)$ is constant for any given (t,ω) and $\mathbb{P}\in\mathcal{P}_H^t$. Therefore, the process V is well defined. However, we still do not know anything about its measurability. The following Lemma answers this question.

Lemma 3.4.1. *Let Assumptions 3.2.1 hold true and let ξ be in $\mathrm{UC_b}(\Omega)$. Then*

$$|V_t(\omega)|\leq C\left(1+\|\xi\|_{\mathbb{L}_H^\infty}\right),\ \textit{for all}\ (t,\omega)\in[0,T]\times\Omega.$$

Furthermore

$$\left|V_t(\omega)-V_t(\omega')\right|\leq C\rho\left(\left\|\omega-\omega'\right\|_t\right),\ \textit{for all}\ \left(t,\omega,\omega'\right)\in[0,T]\times\Omega^2.$$

In particular, V_t is $\mathcal{F}_t$-measurable for every $t\in[0,T]$.

Proof. (i) For each $(t,\omega)\in[0,T]\times\Omega$ and $\mathbb{P}\in\mathcal{P}_H^t$, note that

$$\begin{aligned}y_s^{\mathbb{P},t,\omega}=\xi^{t,\omega}&-\int_s^T\left[\widehat{F}_r^{t,\omega}(0)+\lambda_r y_r^{\mathbb{P},t,\omega}+\eta_r\left(\widehat{a}_r^t\right)^{1/2}z_r^{\mathbb{P},t,\omega}+\phi_r\left(\widehat{a}_r^t\right)^{1/2}z_r^{\mathbb{P},t,\omega}\right]dr\\&-\int_s^T z_r^{\mathbb{P},t,\omega}dB_r^t,\ s\in[t,T],\ \mathbb{P}-a.s.\end{aligned}$$

where λ is bounded and η satisfies

$$|\eta_r|\leq\mu\left|\widehat{a}_t^{1/2}z_r^{\mathbb{P},t,\omega}\right|,\ \mathbb{P}-a.s.$$

Then proceeding exactly as in the second step of the proof of Theorem 3.3.1, we can define a bounded process M and a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that

$$\left|y_t^{\mathbb{P},t,\omega}\right|\leq\mathbb{E}_t^{\mathbb{Q}}\left[M_T\left|\xi^{t,\omega}\right|\right]\leq C\left(1+\|\xi\|_{\mathbb{L}_H^\infty}\right).$$

By arbitrariness of $\mathbb{P}$, we get $|V_t(\omega)|\leq C(1+\|\xi\|_{\mathbb{L}_H^\infty})$.

(ii) The proof is exactly the same as above, except that we need to use uniform continuity in ω of $\xi^{t,\omega}$ and $\widehat{F}^{t,\omega}$. In fact, if we define for $(t,\omega,\omega')\in[0,T]\times\Omega^2$

$$\delta y:=y^{\mathbb{P},t,\omega}-y^{\mathbb{P},t,\omega'},\ \delta z:=z^{\mathbb{P},t,\omega}-z^{\mathbb{P},t,\omega'},\ \delta\xi:=\xi^{t,\omega}-\xi^{t,\omega'},\ \delta\widehat{F}:=\widehat{F}^{t,\omega}-\widehat{F}^{t,\omega'},$$

then we get with the same notations

$$|\delta y_t|=\mathbb{E}^{\mathbb{Q}}\left[M_T\delta\xi+\int_t^T M_s\delta\widehat{F}_s ds\right]\leq C\rho(\left\|\omega-\omega'\right\|_t).$$

We get the result by arbitrariness of $\mathbb{P}$. □

Then, we show the same dynamic programming principle as Proposition 4.7 in [108]

Proposition 3.4.1. *Under Assumption 3.2.1 and for $\xi \in \mathrm{UC_b}(\Omega)$, we have for all $0 \le t_1 < t_2 \le T$ and for all $\omega \in \Omega$*

$$V_{t_1}(\omega) = \sup_{\mathbb{P}\in\mathcal{P}_H^{t_1}} \mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2, V_{t_2}^{t_1,\omega}).$$

The proof is almost the same as the proof in [108], but we give it for the convenience of the reader.

Proof. Without loss of generality, we can assume that $t_1 = 0$ and $t_2 = t$. Thus, we have to prove

$$V_0(\omega) = \sup_{\mathbb{P}\in\mathcal{P}_H} \mathcal{Y}_0^{\mathbb{P}}(t, V_t).$$

Denote $(y^{\mathbb{P}}, z^{\mathbb{P}}) := (\mathcal{Y}^{\mathbb{P}}(T,\xi), \mathcal{Z}^{\mathbb{P}}(T,\xi))$

(i) For any $\mathbb{P} \in \mathcal{P}_H$, it follows from Lemma 4.3 in [108], that for $\mathbb{P} - a.e.\ \omega \in \Omega$, the r.c.p.d. $\mathbb{P}^{t,\omega} \in \mathcal{P}_H^t$. By Tevzadze [112], we know that when the terminal condition is small, quadratic BSDEs whose generator satisfies Assumption (3.2.2) (v) can be constructed via Picard iteration. Thus, it means that at each step of the iteration, the solution can be formulated as a conditional expectation under $\mathbb{P}$. Then, for general terminal conditions, Tevzadze showed that if the generator satisfies Assumption (2.2.1) (v), the solution of the quadratic BSDE can be written as a sum of quadratic BSDEs with small terminal condtion. By the properties of the r.p.c.d., this implies that

$$y_t^{\mathbb{P}}(\omega) = \mathcal{Y}_t^{\mathbb{P}^{t,\omega},t,\omega}(T,\xi), \text{ for } \mathbb{P} - a.e.\ \omega \in \Omega.$$

By definition of V_t and the comparison principle for quadratic BSDEs, we deduce that $y_0^{\mathbb{P}} \le \mathcal{Y}_0^{\mathbb{P}}(t, V_t)$ and it follows from the arbitrariness of $\mathbb{P}$ that

$$V_0(\omega) \le \sup_{\mathbb{P}\in\mathcal{P}_H} \mathcal{Y}_0^{\mathbb{P}}(t, V_t).$$

(ii) For the other inequality, we proceed as in [108]. Let $\mathbb{P} \in \mathcal{P}_H$ and $\varepsilon > 0$. The idea is to use the definition of V as a supremum to obtain an ε-optimizer. However, since V depends obviously on ω, we have to find a way to control its dependence in ω by restricting it in a small ball. But, since the canonical space is separable, this is easy. Indeed, there exists a partition $(E_t^i)_{i\ge1} \subset \mathcal{F}_t$ such that $\|\omega - \omega'\|_t \le \varepsilon$ for any i and any $\omega, \omega' \in E_t^i$.

Now for each i, fix an $\widehat{\omega}_i \in E_t^i$ and let, as advocated above, $\mathbb{P}_t^i$ be an $\varepsilon-$optimizer of $V_t(\widehat{\omega}_i)$. If we define for each $n \ge 1$, $\mathbb{P}^n := \mathbb{P}^{n,\varepsilon}$ by

$$\mathbb{P}^n(E) := \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^n \mathbb{E}^{\mathbb{P}_t^i}\left[1_E^{t,\omega}\right] 1_{E_t^i}\right] + \mathbb{P}(E \cap \widehat{E}_t^n), \text{ where } \widehat{E}_t^n := \cup_{i>n} E_t^i,$$

then, by the proof of Proposition 4.7 in [108], we know that $\mathbb{P}^n \in \mathcal{P}_H$ and that

$$V_t \le y_t^{\mathbb{P}^n} + \varepsilon + C\rho(\varepsilon),\ \mathbb{P}^n - a.s. \text{ on } \cup_{i=1}^n E_t^i.$$

Let now $(y^n, z^n) := (y^{n,\varepsilon}, z^{n,\varepsilon})$ be the solution of the following BSDE on $[0,t]$

$$y_s^n = \left[y_t^{\mathbb{P}^n} + \varepsilon + C\rho(\varepsilon)\right] 1_{\cup_{i=1}^n E_t^i} + V_t 1_{\widehat{E}_t^n} + \int_s^t \widehat{F}_r(y_r^n, z_r^n)dr - \int_s^t z_r^n dB_r,\ \mathbb{P}^n - a.s. \tag{3.4.3}$$

Note that since $\mathbb{P}^n = \mathbb{P}$ on $\mathcal{F}_t$, the equality (3.4.3) also holds $\mathbb{P} - a.s.$ By the comparison theorem, we know that $\mathcal{Y}_0^{\mathbb{P}}(t, V_t) \le y_0^n$. Using the same arguments and notations as in the proof of Lemma 3.4.1, we obtain

$$\left|y_0^n - y_0^{\mathbb{P}^n}\right| \le C\mathbb{E}^{\mathbb{Q}}\left[\varepsilon + \rho(\varepsilon) + \left|V_t - y_t^{\mathbb{P}^n}\right| 1_{\widehat{E}_t^n}\right].$$

Then, by Lemma 3.4.1, we have

$$\mathcal{Y}_0^{\mathbb{P}}(t,V_t) \leq y_0^n \leq V_0(\omega) + C\left(\varepsilon + \rho(\varepsilon) + \mathbb{E}^{\mathbb{Q}}\left[\Lambda 1_{\widehat{E}_t^n}\right]\right).$$

The result follows from letting n go to $+\infty$ and ε to 0. □

Define now for all (t,ω), the $\mathbb{F}^+$-progressively measurable process

$$V_t^+ := \overline{\lim_{r\in\mathbb{Q}\cap(t,T],r\downarrow t}} V_r.$$

Lemma 3.4.2. *Under the conditions of the previous Proposition, we have*

$$V_t^+ = \lim_{r\in\mathbb{Q}\cap(t,T],r\downarrow t} V_r,\ \mathcal{P}_H - q.s.$$

and thus V^+ is càdlàg $\mathcal{P}_H - q.s.$

Proof. Actually, we can proceed exactly as in the proof of Lemma 4.8 in [108], since the theory of g-expectations of Peng has been extended by Ma and Song in [77] to the quadratic case (see in particular their Corollary 5.6 for our purpose). □

Finally, proceeding exactly as in Steps 1 and 2 of the proof of Theorem 4.5 in [108], and in particular using the Doob-Meyer decomposition proved in [77] (Theorem 5.2), we can get the existence of a universal process Z and a family of non-decreasing processes $\left\{K^{\mathbb{P}}, \mathbb{P}\in\mathcal{P}_H\right\}$ such that

$$V_t^+ = V_0^+ - \int_0^t \widehat{F}_s(V_s^+, Z_s)ds + \int_0^t Z_s dB_s - K_t^{\mathbb{P}},\ \mathbb{P}-a.s.\ \forall \mathbb{P}\in\mathcal{P}_H.$$

For the sake of completeness, we provide the representation (3.3.2) for V and V^+, and that, as shown in Proposition 4.11 of [108], we actually have $V = V^+$, $\mathcal{P}_H - q.s.$, which shows that in the case of a terminal condition in $UC_b(\Omega)$, the solution of the 2BSDE is actually $\mathbb{F}$-progressively measurable. This will be important in Section 5.4.

Proposition 3.4.2. *Assume that $\xi\in UC_b(\Omega)$ and that Assumption 3.2.1 hold. Then we have*

$$V_t = \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})}{}^{\mathbb{P}}\ \mathcal{Y}_t^{\mathbb{P}'}(T,\xi)\ \text{and}\ V_t^+ = \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{}^{\mathbb{P}}\ \mathcal{Y}_t^{\mathbb{P}'}(T,\xi),\ \mathbb{P}-a.s.,\ \forall\mathbb{P}\in\mathcal{P}_H.$$

Besides, we also have for all t

$$V_t = V_t^+,\ \mathcal{P}_H - q.s.$$

Proof. The proof for the representations is the same as the proof of proposition 4.10 in [108], since we also have a stability result for quadratic BSDEs under our assumptions. For the equality between V and V^+, we also refer to the proof of Proposition 4.11 in [108]. □

To be sure that we have found a solution to our 2BSDE, it remains to check that the family of non-decreasing processes above satisfies the minimum condition. Let $\mathbb{P}\in\mathcal{P}_H$, $t\in[0,T]$ and $\mathbb{P}\in\mathcal{P}_H(t^+,\mathbb{P})$. From the proof of Theorem 3.3.1, we have with the same notations

$$\begin{aligned}\delta V_t = \mathbb{E}_t^{\mathbb{Q}'}\left[\int_t^T M_t dK_t^{\mathbb{P}'}\right] &\geq \mathbb{E}_t^{\mathbb{Q}'}\left[\inf_{t\leq s\leq T}(M_s)(K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'})\right] \\ &= \frac{\mathbb{E}_t^{\mathbb{P}'}\left[\mathcal{E}\left(\int_0^T(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)\inf_{t\leq s\leq T}(M_s)(K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'})\right]}{\mathbb{E}_t^{\mathbb{P}'}\left[\mathcal{E}\left(\int_0^T(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)\right]}\end{aligned}$$

For notational convenience, denote $\mathcal{E}_t := \mathcal{E}\left(\int_0^t (\phi_s + \eta_s)\widehat{a}_s^{-1/2} dB_s\right)$. Let r be the number given by Lemma 3.2.2 applied to $\mathcal{E}$. Then we estimate

$$\begin{aligned}
&\mathbb{E}_t\left[K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'}\right] \\
&\leq \mathbb{E}_t^{\mathbb{P}'}\left[\frac{\mathcal{E}_T}{\mathcal{E}_t}\inf_{t\leq s\leq T}(M_s)(K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'})\right]^{\frac{1}{2r-1}} \mathbb{E}_t^{\mathbb{P}'}\left[\left(\frac{\mathcal{E}_T}{\inf_{t\leq s\leq T}(M_s)\mathcal{E}_t}\right)^{\frac{1}{2(r-1)}}(K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'})\right]^{\frac{2(r-1)}{2r-1}} \\
&\leq (\delta V_t)^{\frac{1}{2r-1}}\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(\frac{\mathcal{E}_T}{\mathcal{E}_t}\right)^{\frac{1}{r-1}}\right]\right)^{\frac{r-1}{2r-1}}\left(\mathbb{E}_t^{\mathbb{P}'}\left[\inf_{t\leq s\leq T}(M_s)^{-\frac{2}{r-1}}\right]\mathbb{E}_t^{\mathbb{P}'}\left[(K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'})^4\right]\right)^{\frac{r-1}{2(2r-1)}} \\
&\leq C\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}\right)^4\right]\right)^{\frac{r-1}{2(2r-1)}}(\delta V_t)^{\frac{1}{2r-1}}.
\end{aligned}$$

By following the arguments of the proof of Theorem 3.3.1 (ii) and (iii), we then deduce the minimum condition.

Remark 3.4.1. *In order to prove the minimum condition it is fundamental that the process M above is bounded from below. For instance, it would not be the case if we had replaced the Lipschitz assumption on y by a monotonicity condition as in Chapter 2.*

3.4.3 Main result

We are now in position to state the main result of this section

Theorem 3.4.1. *Let $\xi \in \mathcal{L}_H^\infty$. Under Assumption 3.2.1, there exists a unique solution $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ of the* 2BSDE (3.2.2).

Proof. For $\xi \in \mathcal{L}_H^\infty$, there exists $\xi_n \in \mathrm{UC_b}(\Omega)$ such that $\|\xi - \xi_n\| \underset{n\to+\infty}{\to} 0$. Then, thanks to the a priori estimates obtained in Proposition 3.3.2, we can proceed exactly as in the proof of Theorem 4.6 (ii) in [107] to obtain the solution as a limit of the solution of the 2BSDE (3.2.2) with terminal condition ξ_n (see also the proof of Theorem 5.4.1 in Chapter 5) □

3.5 Connection with fully nonlinear PDEs

In this section, we place ourselves in the general case of Section 3.4, and we assume moreover that all the nonlinearity in H only depends on the current value of the canonical process B (the so-called Markov property)

$$H_t(\omega, y, z, \gamma) = h(t, B_t(\omega), y, z, \gamma),$$

where $h : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times D_h \to \mathbb{R}$ is a deterministic map. Then, we define as in Section 3.4 the corresponding conjugate and bi-conjugate functions

$$f(t, x, y, z, a) := \sup_{\gamma\in D_h}\left\{\frac{1}{2}\mathrm{Tr}\,[a\gamma] - h(t, x, y, z, \gamma)\right\} \tag{3.5.1}$$

$$\widehat{h}(t, x, y, z, \gamma) := \sup_{a\in\mathbb{S}_d^{>0}}\left\{\frac{1}{2}\mathrm{Tr}\,[a\gamma] - f(t, x, y, z, a)\right\} \tag{3.5.2}$$

We denote $\mathcal{P}_h := \mathcal{P}_H$, and following [107], we strengthen Assumption 3.2.1

Assumption 3.5.1. (i) *The domain D_{f_t} of the map $a \to f(t,x,y,z,a)$ is independent of (x,y,z).*

(ii) *On D_{f_t}, f is uniformly continuous in t, uniformly in a.*

(iii) *f is continuous in z and has the following growth property. There exists (α,β,γ) such that*

$$|f(t,x,y,z,a)| \leq \alpha + \beta|y| + \frac{\gamma}{2}\left|a^{1/2}z\right|^2, \text{ for all } (t,x,y,z,a).$$

(iv) *f is C^1 in y and C^2 in z, and there are constants r and θ such that for all (t,x,y,z,a)*

$$|D_y f(t,x,y,z,a)| \leq r, \; |D_z f(t,x,y,z,a)| \leq r + \theta\left|a^{1/2}z\right|$$

$$|D^2_{zz} f(t,x,y,z,a)| \leq \theta.$$

(v) *f is uniformly continuous in x, uniformly in (t,y,z,a), with a modulus of continuity ρ which has polynomial growth.*

Remark 3.5.1. *As mentioned in Remark 3.2.2, when the terminal condition is small enough, Assumption 3.5.1* (iv) *can be replaced by the following weaker assumptions.*

(iv')[a] *There exists $\mu > 0$ and a bounded $\mathbb{R}^d$-valued function ϕ such that for all $(t,y,z,z^{'},a)$*

$$\left|f(t,x,y,z,a) - f(t,x,y,z^{'},a) - \phi(t).a^{\frac{1}{2}}(z-z^{'})\right| \leq \mu a^{\frac{1}{2}}\left|z-z^{'}\right|\left(\left|a^{\frac{1}{2}}z\right| + \left|a^{\frac{1}{2}}z^{'}\right|\right).$$

(iv')[b] *f is Lipschitz in y, uniformly in (t,x,z,a).*

Let now $g : \mathbb{R}^d \to \mathbb{R}$ be a Lebesgue measurable and bounded function. Our object of interest here is the following 2BSDE with terminal condition $\xi = g(B_T)$

$$Y_t = g(B_T) + \int_t^T f(s, B_s, Y_s, Z_s, \widehat{a}_s^{1/2})ds - \int_t^T Z_s dB_s + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}, \; \mathcal{P}_h - q.s. \tag{3.5.3}$$

The aim of this section is to generalize the results of [107] and establish the connection $Y_t = v(t, B_t)$, $\mathcal{P}_h - q.s.$, where v is the solution in some sense of the following fully nonlinear PDE

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) + \widehat{h}\left(t,x,v(t,x),Dv(t,x),D^2v(t,x)\right) = 0, \; t \in [0,T) \\ v(T,x) = g(x). \end{cases} \tag{3.5.4}$$

Following the classical terminology in the BSDE litterature, we say that the solution of the 2BSDE is Markovian if it can be represented by a deterministic function of t and B_t. In this subsection, we will construct such a function following the same spirit as in the construction in the previous section.

With the same notations for shifted spaces, we define for any $(t,x) \in [0,T] \times \mathbb{R}^d$

$$B_s^{t,x} := x + B_s^t, \text{ for all } s \in [t,T].$$

Let now τ be an $\mathbb{F}^t$-stopping time, $\mathbb{P} \in \mathcal{P}_h^t$ and η a $\mathbb{P}$-bounded $\mathcal{F}_\tau^t$-measurable random variable. Similarly as in (3.4.2), we denote $(y^{\mathbb{P},t,x}, z^{\mathbb{P},t,x}) := (\mathcal{Y}^{\mathbb{P},t,x}(\tau,\eta), \mathcal{Z}^{\mathbb{P},t,x}(\tau,\eta))$ the unique solution of the following BSDE

$$y_s^{\mathbb{P},t,x} = \eta + \int_s^\tau f(u, B_u^{t,x}, y_u^{\mathbb{P},t,x}, z_u^{\mathbb{P},t,x}, \widehat{a}_u^t)du - \int_s^\tau z_u^{\mathbb{P},t,x} dB_u^{t,x}, \; t \leq s \leq \tau, \; \mathbb{P} - a.s. \tag{3.5.5}$$

Then we define the following deterministic function (by virtue of the Blumenthal $0-1$ law)

$$u(t,x) := \sup_{\mathbb{P}\in\mathcal{P}_h^t} \mathcal{Y}_t^{\mathbb{P},t,x}(T, g(B_T^{t,x})), \text{ for } (t,x)\in[0,T]\times\mathbb{R}^d. \tag{3.5.6}$$

We then have the following Theorem, which is actually Theorem 5.9 of [107] in our framework

Theorem 3.5.1. *Let Assumption 3.5.1 hold, and assume that g is bounded and uniformly continuous. Then the 2BSDE (3.5.3) has a unique solution $(Y,Z)\in\mathbb{D}_H^\infty\times\mathbb{H}_H^2$ and we have $Y_t = u(t,B_t)$. Moreover, u is uniformly continuous in x, uniformly in t and right-continuous in t.*

Proof. The existence and uniqueness for the 2BSDE follows directly from Theorem 3.4.1. Since $\xi\in\mathrm{UC_b}(\Omega)$, we have with the notations of the previous section $V_t = u(t,B_t)$. But, by Proposition 3.4.2, we know that $Y_t = V_t$, hence the first result.

Then the uniform continuity of u is a simple consequence of Lemma 3.4.1. Finally, the right-continuity of u in t can be obtained exactly as in the proof of Theorem 5.9 in [107]. □

3.5.1 Non-linear Feynman-Kac formula in the quadratic case

Exactly as in the classical case and as in Theorem 5.3 in [107], we have a non-linear version of the Feynaman-Kac formula. Notice however that the proof is more involved than in the classical case, mainly due to the technicalities introduced by the quasi-sure framework.

Theorem 3.5.2. *Let Assumption 3.5.1 hold true. Assume further that $\widehat{h}$ is continuous in its domain, that D_f is independent of t and is bounded both from above and away from 0. Let $v\in C^{1,2}([0,T),\mathbb{R}^d)$ be a classical solution of (3.5.4) with $\{(v,Dv)(t,B_t)\}_{0\leq t\leq T}\in\mathbb{D}_H^\infty\times\mathbb{H}_H^2$. Then*

$$Y_t := v(t,B_t),\ Z_t := Dv(t,B_t),\ K_t := \int_0^t k_s ds,$$

is the unique solution of the quadratic 2BSDE (3.5.3), where

$$k_t := \widehat{h}(t,B_t,Y_t,Z_t,\Gamma_t) - \frac{1}{2}\mathrm{Tr}\left[\widehat{a}_t^{1/2}\Gamma_t\right] + f(t,B_t,Y_t,Z_t,\widehat{a}_t) \text{ and } \Gamma_t := D^2 v(t,B_t).$$

Proof. The proof follows line-by-line the proof of Theorem 5.3 in [107], so we omit it. □

3.5.2 The viscosity solution property

As usual when dealing with possibly discontinuous viscosity solutions, we introduce the following upper and lower-semicontinuous envelopes

$$u_*(t,x) := \underline{\lim}_{(t',x')\to(t,x)} u(t',x'),\ u^*(t,x) := \overline{\lim}_{(t',x')\to(t,x)} u(t',x')$$
$$\widehat{h}_*(\vartheta) := \underline{\lim}_{(\vartheta')\to(\vartheta)} \widehat{h}(\vartheta'),\ \widehat{h}^*(\vartheta) := \overline{\lim}_{(\vartheta')\to(\vartheta)} \widehat{h}(\vartheta')$$

In order to prove the main Theorem of this subsection, we will need the following Proposition, whose proof (which is rather technical) is omitted, since it is exactly the same as the proof of Propositions 5.10 and 5.14 and Lemma 6.2 in [107].

Proposition 3.5.1. *Let Assumption 3.5.1 hold. Then for any bounded function g*

(i) *For any (t,x) and arbitrary* $\mathbb{F}^t$*-stopping times* $\{\tau^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_h^t\}$, *we have*

$$u(t,x) \leq \sup_{\mathbb{P}\in\mathcal{P}_h^t} \mathcal{Y}_t^{\mathbb{P},t,x}(\tau^{\mathbb{P}}, u^*(\tau^{\mathbb{P}}, B_{\tau^{\mathbb{P}}}^{t,x})).$$

(ii) *If in addition g is lower-semicontinuous, then*

$$u(t,x) = \sup_{\mathbb{P}\in\mathcal{P}_h^t} \mathcal{Y}_t^{\mathbb{P},t,x}(\tau^{\mathbb{P}}, u(\tau^{\mathbb{P}}, B_{\tau^{\mathbb{P}}}^{t,x})).$$

Now we can state the main Theorem of this section

Theorem 3.5.3. *Let Assumption 3.5.1 hold true. Then*

(i) *u is a viscosity subsolution of*

$$-\partial_t u^* - \widehat{h}^*(\cdot, u^*, Du^*, D^2u^*) \leq 0, \text{ on } [0,T)\times\mathbb{R}^d.$$

(ii) *If in addition g is lower-semicontinuous and* D_f *is independent of t, then u is a viscosity supersolution of*

$$-\partial_t u_* - \widehat{h}_*(\cdot, u_*, Du_*, D^2u_*) \geq 0, \text{ on } [0,T)\times\mathbb{R}^d.$$

Proof. The proof follows closely the proof of Theorem 5.11 in [107], with some minor modifications (notably when we prove (3.5.10)). We provide it for the convenience of the reader.

(i) Assume to the contrary that

$$0 = (u^*-\phi)(t_0,x_0) > (u^*-\phi)(t,x) \text{ for all } (t,x)\in[0,T)\times\mathbb{R}^d\backslash\{(t_0,x_0)\}, \tag{3.5.7}$$

for some $(t_0,x_0)\in[0,T)\times\mathbb{R}^d$ and

$$\left(-\partial_t\phi - \widehat{h}^*(\cdot,\phi,D\phi,D^2\phi)\right)(t_0,x_0) > 0, \tag{3.5.8}$$

for some smooth and bounded function ϕ (we can assume w.l.o.g. that ϕ is bounded since we are working with bounded solutions of 2BSDEs).

Now since ϕ is smooth and since by definition $\widehat{h}^*$ is upper-semicontinuous, there exists an open ball $\mathcal{O}(r,(t_0,x_0))$ centered at (t_0,x_0) with radius r, which can be chosen less than $T-t_0$, such that

$$-\partial_t\phi - \widehat{h}(\cdot,\phi,D\phi,D^2\phi) \geq 0, \text{ on } \mathcal{O}(r,(t_0,x_0)).$$

By definition of $\widehat{h}$, this implies that for any $\alpha\in\mathbb{S}_d^{>0}$

$$-\partial_t\phi - \frac{1}{2}\text{Tr}\left[\alpha D^2\phi\right] + f(\cdot,\phi,D\phi,\alpha) \geq 0, \text{ on } \mathcal{O}(r,(t_0,x_0)). \tag{3.5.9}$$

Let us now denote

$$\mu := -\max_{\partial\mathcal{O}(r,(t_0,x_0))}(u^*-\phi).$$

By (3.5.7), this quantity is strictly positive.

Let now (t_n,x_n) be a sequence in $\mathcal{O}(r,(t_0,x_0))$ such that $(t_n,x_n)\to(t_0,x_0)$ and $u(t_n,x_n)\to u^*(t_0,x_0)$. Denote the following stopping time

$$\tau_n := \inf\left\{s>t_n,\ (s,B_s^{t_n,x_n}\notin\mathcal{O}(r,(t_0,x_0))\right\}.$$

Since $r < T - t_0$, we have $\tau_n < T$ and therefore $(\tau_n, B^{t_n,x_n}_{\tau_n}) \in \partial\mathcal{O}(r,(t_0,x_0))$. Hence, we have

$$c_n := (\phi - u)(t_n, x_n) \to 0 \text{ and } u^*(\tau_n, B^{t_n,x_n}_{\tau_n}) \le \phi(\tau_n, B^{t_n,x_n}_{\tau_n}) - \mu.$$

Fix now some $\mathbb{P}^n \in \mathcal{P}^{t_n}_h$. By the comparison Theorem for quadratic BSDEs, we have

$$\mathcal{Y}^{\mathbb{P}_n,t_n,x_n}_{t_n}(\tau_n, u^*(\tau_n, B^{t_n,x_n}_{\tau_n})) \le \mathcal{Y}^{\mathbb{P}_n,t_n,x_n}_{t_n}(\tau_n, \phi(\tau_n, B^{t_n,x_n}_{\tau_n}) - \mu).$$

Then proceeding exactly as in the second step of the proof of Theorem 3.3.1, we can define a bounded process M_n, whose bounds only depend on T and the Lipchitz constant of f in y, and a probability measure $\mathbb{Q}_n$ equivalent to $\mathbb{P}_n$ such that

$$\mathcal{Y}^{\mathbb{P}_n,t_n,x_n}_{t_n}(\tau_n, \phi(\tau_n, B^{t_n,x_n}_{\tau_n}) - \mu) - \mathcal{Y}^{\mathbb{P}_n,t_n,x_n}_{t_n}(\tau_n, \phi(\tau_n, B^{t_n,x_n}_{\tau_n})) = -\mathbb{E}^{\mathbb{Q}_n}_{t_n}[M_{\tau_n}\mu] \le -\mu',$$

for some strictly positive constant μ' which is independent of n.

Hence, we obtain by definition of c_n

$$\mathcal{Y}^{\mathbb{P}_n,t_n,x_n}_{t_n}(\tau_n, u^*(\tau_n, B^{t_n,x_n}_{\tau_n})) - u(t_n,x_n) \le \mathcal{Y}^{\mathbb{P}_n,t_n,x_n}_{t_n}(\tau_n, \phi(\tau_n, B^{t_n,x_n}_{\tau_n})) - \phi(t_n,x_n) + c_n - \mu'. \tag{3.5.10}$$

With the same arguments as above, it is then easy to show with Itô's formula that

$$\mathcal{Y}^{\mathbb{P}_n,t_n,x_n}_{t_n}(\tau_n, \phi(\tau_n, B^{t_n,x_n}_{\tau_n})) - \phi(t_n,x_n) = \mathbb{E}^{\mathbb{Q}_n}_{t_n}\left[-\int_{t_n}^{\tau_n} M^n_s \psi^n_s ds\right],$$

where

$$\psi^n_s := (-\partial_t\phi - \frac{1}{2}\mathrm{Tr}\left[\widehat{a}^t_s D^2\phi\right] + f(\cdot, D\phi, \widehat{a}^t_s))(s, B^{t_n,x_n}_s).$$

But by (3.5.9) and the definition of τ_n, we know that for $t_n \le s \le \tau_n$, $\psi^n_s \ge 0$. Recalling (3.5.10), we then get

$$\mathcal{Y}^{\mathbb{P}_n,t_n,x_n}_{t_n}(\tau_n, u^*(\tau_n, B^{t_n,x_n}_{\tau_n})) - u(t_n,x_n) \le c_n - \mu'.$$

Since c_n does not depend on $\mathbb{P}_n$, we immediately get

$$\sup_{\mathbb{P}\in\mathcal{P}^{t_n}_h} \mathcal{Y}^{\mathbb{P}_n,t_n,x_n}_{t_n}(\tau_n, u^*(\tau_n, B^{t_n,x_n}_{\tau_n})) - u(t_n,x_n) \le c_n - \mu'.$$

The right-hand side is strictly negative for n large enough, which contradicts Proposition 3.5.1(i).

(ii) We also proceed by contradiction. Assuming to the contrary that

$$0 = (u_* - \phi)(t_0,x_0) < (u_* - \phi)(t,x) \text{ for all } (t,x) \in [0,T) \times \mathbb{R}^d \setminus \{(t_0,x_0)\}, \tag{3.5.11}$$

for some $(t_0,x_0) \in [0,T) \times \mathbb{R}^d$ and

$$\left(-\partial_t\phi - \widehat{h}_*(\cdot, \phi, D\phi, D^2\phi)\right)(t_0,x_0) < 0, \tag{3.5.12}$$

for some smooth and bounded function ϕ (we can assume w.l.o.g. that ϕ is bounded since we are working with bounded solutions of 2BSDEs).

Now we have by definition $\widehat{h}_* \le \widehat{h}$, hence

$$\left(-\partial_t\phi - \widehat{h}(\cdot, \phi, D\phi, D^2\phi)\right)(t_0,x_0) < 0, \tag{3.5.13}$$

Unlike with the subsolution property, we do not know whether $D^2\phi(t_0,x_0)\in D_{\widehat{h}}$ or not. If it is the case, then by the definition of $\widehat{h}$, there exists some $\bar{\alpha}\in\mathbb{S}_d^{>0}$ such that

$$\left(-\partial_t\phi-\frac{1}{2}\mathrm{Tr}\left[\bar{\alpha}D^2\phi\right]+f(\cdot,\phi,D\phi,\bar{\alpha})\right)(t_0,x_0)<0, \tag{3.5.14}$$

which implies in particular that $\bar{\alpha}\in D_f$.

If $D^2\phi(t_0,x_0)\notin D_{\widehat{h}}$, we still have that $\partial_t\phi(t_0,x_0)$ is finite, and thus $\bar{\alpha}\in D_f$ and (3.5.13) holds.

Now since ϕ is smooth and since D_f does not depend on t, there exists an open ball $\mathcal{O}(r,(t_0,x_0))$ centered at (t_0,x_0) with radius r, which can be chosen less than $T-t_0$, such that

$$-\partial_t\phi-\frac{1}{2}\mathrm{Tr}\left[\bar{\alpha}D^2\phi\right]+f(\cdot,\phi,D\phi,\bar{\alpha})\leq 0, \text{ on } \mathcal{O}(r,(t_0,x_0)).$$

Let us now denote

$$\mu:=\min_{\partial\mathcal{O}(r,(t_0,x_0))}(u_*-\phi).$$

By (3.5.11), this quantity is strictly positive.

Let now (t_n,x_n) be a sequence in $\mathcal{O}(r,(t_0,x_0))$ such that $(t_n,x_n)\to(t_0,x_0)$ and $u(t_n,x_n)\to u_*(t_0,x_0)$. Denote the following stopping time

$$\tau_n:=\inf\left\{s>t_n,\ (s,B_s^{t_n,x_n}\notin\mathcal{O}(r,(t_0,x_0))\right\}.$$

Since $r<T-t_0$, we have $\tau_n<T$ and therefore $(\tau_n,B_{\tau_n}^{t_n,x_n})\in\partial\mathcal{O}(r,(t_0,x_0))$. Hence, we have

$$c_n:=(\phi-u)(t_n,x_n)\to 0 \text{ and } u_*(\tau_n,B_{\tau_n}^{t_n,x_n})\geq\phi(\tau_n,B_{\tau_n}^{t_n,x_n})+\mu.$$

Now for each n consider the probability measure $\bar{\mathbb{P}}^n:=\mathbb{P}^{\bar{\alpha}}$ induced by the constant diffusion $\bar{\alpha}$ from time t_n onwards. It is clearly in $\mathcal{P}_h^{t_n}$. Then, arguing exactly as in (i), we prove that

$$u(t_n,x_n)-\mathcal{Y}_{t_n}^{\bar{\mathbb{P}}^n,t_n,x_n}(\tau_n,u_*(\tau_n,B_{\tau_n}^{t_n,x_n}))\leq c_n-\mu',\ \bar{\mathbb{P}}^n-a.s.$$

For n large enough, the right-hand side becomes strictly negative, which contradicts Proposition 3.5.1(ii).

□

Chapitre 4

Robust utility maximization

Contents

4.1 Introduction

The BSDE theory finds one of its application in the problem of utility maximization which can be formulated as follows

$$V^{\xi}(x) := \sup_{\pi\in\tilde{\mathcal{A}}} \inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}^{\mathbb{Q}}[U(X_T^{\pi}-\xi)],$$

where $\tilde{\mathcal{A}}$ is a given set of admissible trading strategies, $\mathcal{P}$ is the set of all considered possible probability measures, U is a utility function, X_T^{π} is the liquidation value of a trading strategy π with positive initial capital $X_0^{\pi}=x$ and ξ is a terminal liability, equal to 0 if U is only defined on $\mathbb{R}^+$.

In the standard problem of utility maximization, $\mathcal{P}$ contains only one probability measure $\mathbb{P}$. This means that the investor knows the "historical" probability $\mathbb{P}$ that describes the dynamics of the state process. In reality, the investor may have some uncertainty on this probability, which means that there can be several objective probability measures in $\mathcal{P}$. In this case, we call the problem robust utility maximization. Many authors introduce a set of probability measures which is absolutely continuous with respect to a reference probability measure $\mathbb{P}$. This is going to be the case if we take into account drift uncertainty, which is not significant in determining options prices, but is important in studying utility maximization. However, if we want to work in the framework of the uncertain volatility model introduced by Avellaneda, Levy and Paras. [2] and Lyons [76], the set of probability measures becomes non-dominated.

The usual approach for the standard utility maximization problem is due to Von Neumann and Morgenstern [113]. In the seminal paper [79], Merton was the first to study the problem of portfolio selection with utility maximization by stochastic optimal control techniques. Then in [67], Kramkov and Schachermayer studied the problem of utility maximization in a general semimartingale model by means of duality theory. Later, El Karoui and Rouge [46] considered the indifference pricing problem with exponential utility. They assumed that the admissible trading strategies set was closed and convex, and showed that the solution is related to a standard BSDE with quadratic growth. Following their ideas, Hu, Imkeller and Müller, in [62], used a similar approach to extend their results to power and logarithmic utility functions. Moreover, they considered a set of admissible strategies which is only closed. In that case, the maximization problem was also found to be related to BSDEs with quadratic generator. In a more recent paper [48], El Karoui, Jeanblanc, Matoussi and Ngoupeyou studied the indifference price of an unbounded claim in an incomplete jump-diffiusion model by considering the risk aversion represented by an exponential utility function. Using the dynamic programming equation, they found that the price of an unbounded credit derivatives was solution of a quadratic BSDE with jumps.

The problem of robust utility maximization with dominated models has been introduced by Gilboa and Schmeidler [55]. In [16], Bordigoni, Matoussi and Schweizer solved the robust problem by stochastic control techniques and proved that the solution was also related to a BSDE. Some results in the robust maximization problem have also been obtained in Gundel [57] , Quenez [91], Schied [97], Schied and Wu [98], Skiadas [99] in the case of continuous filtrations. The overall approach relies essentially on convex duality ideas.

Robust utility maximization with non-dominated models, encompassing the case of the UVM model, has been studied for the first time by Denis and Kervarec [39]. In this article, they first establish a duality theory for robust utility maximization and then show that there exists a least favorable probability. They also take into account uncertainty about the drift. The utility funtion U in their framwork is supposed to be bounded and to satisfy Inada conditions. More recently, in [49], Epstein and Ji formulate a model of utility for a continuous-time framework that captures the decision-maker's concern with ambiguity or model uncertainty, even though they do not study the maximization problem of robust utility *per se*.

In the present framework, we study robust utility maximization with non-dominated models via 2BSDEs techniques. For this purpose, we recall the 2BSDEs framework in Section 4.2. Largely inspired by [46] and [62], in Sections 4.3, 4.4, 4.5 and 4.6 we solve the problem for robust exponential utility, robust power utility and robust logarithmic utility, which, unlike in [39], are not bounded. In particular, we prove the existence of optimal strategy. In Section 4.7, we give some examples where we can explicitly solve the robust utility maximization problems by finding the solution of the associated 2BSDEs, and we provide some intuitions and comparisons with the classical dynamic programming approach adopted in the seminal work of Merton [79].

4.2 Preliminaries

We start by recalling some notations and notions related to the theory of 2BSDEs, which are the main tools in our approach to the robust utility maximization problem. The framework here in the same as in Chapter 3.

4.2.1 The quadratic generator

We consider a map $H_t(\omega, z, \gamma) : [0, T] \times \Omega \times \mathbb{R}^d \times D_H \to \mathbb{R}$, where $D_H \subset \mathbb{R}^{d\times d}$ is a given subset containing 0.

Define the corresponding conjugate of H w.r.t. γ by

$$F_t(\omega, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2}\mathrm{Tr}(a\gamma) - \mathrm{H_t}(\omega, z, \gamma) \right\} \text{ for } a \in S_d^{>0}$$

$$\widehat{F}_t(z) := F_t(z, \widehat{a}_t) \text{ and } \widehat{F}_t^0 := \widehat{F}_t(0).$$

We denote by $D_{F_t(z)}$ the domain of F in a for a fixed (t, ω, z). As in Chapter 3, the generator F is supposed to verify either Assumption 3.2.1 or Assumption 3.2.2.

We recall that Assumption 3.2.2 is weaker than Assumption 3.2.1, but is sufficient to have existence of the quadratic 2BSDE defined below only if the norm of the terminal condition ξ is small enough. Notice that this will always be the case with power and logarithmic utilities for which the terminal condition of the 2BSDE will be 0.

4.2.2 Quadratic 2BSDE

In the sequel we will have to deal with the following type of 2BSDEs

$$Y_t = \xi - \int_t^T \widehat{F}(Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ 0 \le t \le T,\ \mathcal{P}_H - q.s. \tag{4.2.1}$$

We recall here one of the results proved in Chapter 3

Theorem 4.2.1. *Let $\xi \in \mathcal{L}_H^\infty$. Under Assumption 2.2.2 or Assumption 2.2.2 with the addition that the norm of ξ is small enough, there exists a unique solution $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ of the 2BSDE (4.2.1).*

4.3 Robust utility maximization

We will now present the main problem of this paper and introduce a financial market with volatility uncertainty.

The financial market consists of one bond with interest rate zero and d stocks. The price process is given by

$$dS_t = \mathrm{diag}\,[S_t]\,(b_t dt + dB_t),\ \mathcal{P}_H - q.s.$$

where b is an $\mathbb{R}^d$-valued uniformly bounded stochastic process which is uniformly continuous in ω for the $||\cdot||_\infty$ norm.

Remark 4.3.1. *The volatility is implicitly embedded in the model. Indeed, under each $\mathbb{P} \in \mathcal{P}_H$, we have $dB_s \equiv \widehat{a}_t^{1/2} dW_t^{\mathbb{P}}$ where $W^{\mathbb{P}}$ is a Brownian motion under $\mathbb{P}$. Therefore, $\widehat{a}^{1/2}$ plays the role of volatility under each $\mathbb{P}$ and thus allows us to model the volatility uncertainty. We also note that we make the uniform continuity assumption for b to ensure that the 2BSDE obtained later satisfies Assumptions 2.2.1.*

We then denote $\pi = (\pi_t)_{0\leq t\leq T}$ a trading strategy, which is a d-dimensional $\mathbb{F}$-progressively measurable process, supposed to take its value in some closed set $\tilde{A}$. In the sequel, we denote $\tilde{\mathcal{A}}$ the set of admissible trading strategies, which will be defined for each of the three utility functions in the following sections.

The process π_t^i describes the amount of money invested in stock i at time t, with $1 \leq i \leq d$. The number of shares is $\frac{\pi_t^i}{S_t^i}$. So the liquidation value of a trading strategy π with positive initial capital x is given by the following wealth process:

$$X_t^\pi = x + \int_0^t \pi_s(dB_s + b_s ds),\ 0 \leq t \leq T,\ \mathcal{P}_H - q.s.$$

The problem of the investor in this financial market is to maximize his expected utility under model uncertainty from his total wealth $X_T^\pi - \xi$ where ξ is a liability at time T which is a random variable assumed to be $\mathcal{F}_T$-measurable and in $\mathcal{L}_H^\infty$. Then the value function V of the maximization problem can be written as

$$V^\xi(x) := \sup_{\pi\in\tilde{\mathcal{A}}} \inf_{\mathbb{Q}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{Q}}[U(X_T^\pi - \xi)]. \tag{4.3.1}$$

In the case where $\mathcal{P}_H$ contains only one probability measure, the problem reduces to the classical utility maximization problem.

Remark 4.3.2. *Due to the construction of 2BSDE, we need the liability ξ to be in the class $\mathcal{L}_H^\infty$. It is easy to see that ξ can be constant, deterministic or in the form of $g(B_T)$ where g is a Lipschitz bounded function, such as a Put or a Call spread payoff function. However, we notice that vanilla options payoffs with underlying S may not be in $\mathcal{L}_H^\infty$. Indeed, we have in the one-dimensioal framework*

$$S_T = S_0 \exp\left(\int_0^T b_t dt - \frac{1}{2}\langle B\rangle_T + B_T\right),\ \mathcal{P}_H - q.s.$$

Since the quadratic variation of the canonical process can be written as follows

$$\overline{\lim_{n\to+\infty}} \sum_{i\leq 2^n t} \left(B_{\frac{i+1}{2^n}}(\omega) - B_{\frac{i}{2^n}}(\omega)\right)^2,$$

it is not too difficult to see that S can be approximated by a sequence of random variables in $\mathrm{UC_b}(\Omega)$. *Besides, this sequence converges in $\mathcal{L}_H^2$. However, we cannot be sure that it also converges in $\mathcal{L}_H^\infty$, which is our space of interest here.*

Of course, in an uncertain volatility framework, this seems to be a major drawback. Nevertheless, to deal with these options, it suffices to redo the whole 2BSDE construction from scratch but taking exponential of the Brownian motion under the Wiener measure as the canonical process instead of the Brownian motion itself. This would amount to restrict ourselves to the subset $\mathcal{P}_H^+$ of $\mathcal{P}_H$, containing the local martingale measure which make the canonical process a positive continuous martingale.

To find the value funtion V and an optimal trading strategy π^*, we follow the ideas of the general *martingale optimality principle* approach as in [46] and [62], but adapt it here to a nonlinear framework.

Let R^π be a family of processes which satisfy the following properties

Properties 4.3.1. (i) $R_T^\pi = U(X_T^\pi - \xi)$ *for all* $\pi \in \tilde{\mathcal{A}}$.

(ii) $R_0^\pi = R_0$ *is constant for all* $\pi \in \tilde{\mathcal{A}}$.

(iii) *We have*

$$\underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}[U(X_T^\pi - \xi)] \leq R_t^\pi, \ \forall \pi \in \tilde{\mathcal{A}}$$

$$R_t^{\pi^*} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}[U(X_T^{(\pi^*)} - \xi)] \text{ for some } \pi^* \in \tilde{\mathcal{A}}, \ \mathbb{P}-a.s. \text{ for all } \mathbb{P} \in \mathcal{P}_H.$$

Then it follows

$$\inf_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[U(X_T^\pi - \xi)] \leq R_0 = \inf_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[U(X_T^{(\pi^*)} - \xi)] = V^\xi(x). \tag{4.3.2}$$

In the following sections we will follow the ideas of Hu, Imkeller and Müller [62] to construct such a family for our three utility functions.

4.4 Robust exponential utility

In this section, we will consider the exponential utility function which is defined as

$$U(x) = -\exp(-\beta x), \ x \in \mathbb{R} \text{ for } \beta > 0.$$

In our context, the set of admissible trading strategies is defined as follows

Definition 4.4.1. *Let* $\tilde{A}$ *be a closed set in* $\mathbb{R}^{1\times d}$. *The set of admissible trading strategies* $\tilde{\mathcal{A}}$ *consists of all d-dimensional progressively measurable processes,* $\pi = (\pi_t)_{0\leq t\leq T}$ *satisfying*

$$\pi \in \mathbb{BMO}(\mathcal{P}_\mathrm{H}) \text{ and } \pi_\mathrm{t} \in \tilde{\mathrm{A}}, \ \mathrm{dt} \otimes \mathcal{P}_\mathrm{H} - \mathrm{a.e.}$$

Remark 4.4.1. *Many authors shed light on the natural link between BMO class, exponential uniformly integrable class and BSDEs with quadratic growth. See [14], [7] and [62] among others. In the standard utility maximization problem studied in [62], their trading strategies satisfy a uniform integrability assumption on the family* $(\exp X_\tau^\pi)_\tau$. *Since the optimal strategy is a BMO martingale, it is easy to see that the utility maximization problem can also be solved if the uniform integrability assumption is replaced by a BMO assumption. However, at the end of the day, those two assumptions are deeply linked, as shown in the context of quadratic semimartingales in [7]. Nonetheless, in our framework, as explained below in Remark 4.4.3, we need to generalize the BMO martingale assumption instead of the uniform integrability assumption.*

4.4.1 Characterization of the value function and existence of an optimal strategy

The investor wants to solve the maximization problem

$$V^\xi(x) := \sup_{\pi \in \tilde{\mathcal{A}}} \inf_{\mathbb{Q} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{Q}}\left[-\exp\left(X_T^\pi - \xi\right)\right] \tag{4.4.1}$$

To construct R^π, we set

$$R_t^\pi = -\exp(-\beta(X_t^\pi - Y_t)), \ t \in [0,T], \ \pi \in \tilde{\mathcal{A}}.$$

where $(Y,Z)\in\mathbb{D}^\infty_H\times\mathbb{H}^2_H$ the unique solution of the following 2BSDE with quadratic generator:

$$Y_t=\xi-\int_t^T Z_s dB_s-\int_t^T \widehat{F}(s,Z_s)ds+K_T^{\mathbb{P}}-K_t^{\mathbb{P}},\ \mathbb{P}-a.s.,\ \forall\mathbb{P}\in\mathcal{P}_H.$$

The generator $\widehat{F}$ is chosen so that R^π satisfies the Properties 4.3.1.

Remark 4.4.2. *From Theorem 3.3.1, we have the following representation*

$$Y_t=\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})} y_t^{\mathbb{P}'}(T,\xi).$$

So that, in general, Y_0 is only $\mathcal{F}_{0^+}$-measurable and therefore not a constant. But by Proposition 3.4.2, we know that we actually have $\mathbb{P}-a.s.$ for all $\mathbb{P}$

$$Y_0=\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(0^+,\mathbb{P})} y_0^{\mathbb{P}'}(T,\xi)=\sup_{\mathbb{P}'\in\mathcal{P}_H} y_0^{\mathbb{P}'}(T,\xi).$$

So Y_0 is a constant by the Zero-One Blumenthal law.

Let us now define for all $a\in\mathbb{S}_d^{>0}$ such that $\underline{a}\leq a\leq\overline{a}$ the set A_a by

$$A_a:=a^{1/2}\widetilde{A}=\left\{a^{1/2}b\ :\ b\in\widetilde{A}\right\}.$$

The set A_a is still closed. Moreover, since $\widetilde{A}\neq\varnothing$ and $a\in[\underline{a},\ \overline{a}]$, we have

$$\min\left\{|r|\,,\ r\in A_a\right\}\leq k, \tag{4.4.2}$$

for some constant k independent of a.

We can now state the main result of this section

Theorem 4.4.1. *Let Assumption 3.2.2, with the addition that the norm of ξ is small enough, or Assumption 3.2.1, with the addition that the closed domain $\widetilde{A}$ is C^2, hold. Then, the value function of the optimization problem (4.4.1) is given by*

$$V^\xi(x)=-\exp\left(-\beta\left(x-Y_0\right)\right),$$

where Y_0 is defined as the initial value of the unique solution $(Y,Z)\in\mathbb{D}^\infty_H\times\mathbb{H}^2_H$ of the following 2BSDE

$$Y_t=\xi-\int_t^T Z_s dB_s-\int_t^T \widehat{F}_s(Z_s)ds+K_T^{\mathbb{P}}-K_t^{\mathbb{P}},\ \mathbb{P}-a.s.,\ \forall\mathbb{P}\in\mathcal{P}_H. \tag{4.4.3}$$

The generator is defined as follows

$$\widehat{F}_t(\omega,z,a):=F_t(\omega,z,\widehat{a}_t), \tag{4.4.4}$$

where for all $t\in[0,T]$, $z\in\mathbb{R}^d$ and $a\in\mathbb{S}_d^{>0}$

$$F_t(\omega,z,a)=-\frac{\beta}{2}\mathrm{dist}^2\left(a^{1/2}z+\frac{1}{\beta}\theta_t(\omega),A_a\right)+z'a^{1/2}\theta_t(\omega)+\frac{1}{2\beta}\left|\theta_t(\omega)\right|^2,\ \textit{for } a\in S_d^{>0},$$

were $\theta_t(\omega)=a^{-1/2}b_t(\omega)$.

Moreover, there exists an optimal trading strategy π^* satisfying

$$\widehat{a}_t^{1/2}\pi_t^*\in\Pi_{A_{\widehat{a}_t}}\left(\widehat{a}_t^{1/2}Z_t+\frac{1}{\beta}\widehat{\theta}_t\right),\quad t\in[0,T],\ \mathcal{P}_H-q.s. \tag{4.4.5}$$

where $\widehat{\theta}_t:=\widehat{a}_t^{-1/2}b_t$.

Proof.

Step 1: We first show that the 2BSDE (4.4.3) has an unique solution. We need to verify that the generator $\widehat{F}$ satisfies the conditions of Assumption 3.2.2 or 3.2.1.

First of all, F defined above is a convex function of a, and thus F can be written as the Fenchel transform of a function

$$H_t(\omega, z, \gamma) := \sup_{a \in D_F} \left\{ \frac{1}{2}\mathrm{Tr}(a\gamma) - \mathrm{F_t}(\omega, \mathrm{z}, \mathrm{a}) \right\} \text{ for } \gamma \in \mathbb{R}^{d \times d}.$$

That F satisfies the first two conditions of either Assumption 3.2.2 or 3.2.1 is obvious. For Assumptions 3.2.2(iii) and 3.2.1(iii), the assumption of boundedness and uniform continuity in ω on b implies that b^2 is uniformly continuous in ω. Since b and b^2 are the only non-deterministic terms in F, then F is also uniformly continuous in ω.

Then, since we consider the distance function to a closed set, we know that it is attained for some element. From this, it is clear that the generator of this 2BSDE is purely quadratic. Besides, as recalled earlier in (4.4.2), there exists a constant $k \geq 0$ such that

$$\min\left\{|d| : d \in A_{\widehat{a}_t}\right\} \leq k \quad \text{for } dt \otimes \mathbb{P} - a.e., \text{ for all } \mathbb{P} \in \mathcal{P}_H.$$

Then we get, for all $z \in \mathbb{R}^d$, $t \in [0,T]$,

$$\mathrm{dist}^2\left(\widehat{a}_t^{1/2} z + \frac{1}{\beta}\widehat{\theta}_t, A_{\widehat{a}_t}\right) \leq 2\left|\widehat{a}_t^{1/2} z\right|^2 + 2\left(\frac{1}{\beta}\left|\widehat{\theta}_t\right| + k\right)^2.$$

Thus, we get from the boundedness of $\widehat{\theta}$

$$\left|\widehat{F}_t(z)\right| \leq c_0 + c_1 \left|\widehat{a}_t^{1/2} z\right|^2,$$

that is to say that Assumptions 3.2.2(iv) and 3.2.1(iv) are satisfied.

Finally, Assumption 3.2.2(v) is clear from the Lipschitz property of the distance function, and Assumption 3.2.1(v) is also clear by our regularity assumption on $\widetilde{A}$ in that case.

The terminal condition ξ is in $\mathcal{L}_H^\infty$ and we have proved that the generator $\widehat{F}$ satisfies Assumption 3.2.2 or Assumption 3.2.1, therefore Theorem 4.2.1 states that the 2BSDE (4.4.3) has a unique solution in $\mathbb{D}_H^\infty \times \mathbb{H}_H^2$.

Step 2: We first decompose R^π as the product of a process M^π and a non-decreasing process A^π that is constant for some $\pi^* \in \tilde{\mathcal{A}}$.

Define for all $\mathbb{P} \in \mathcal{P}_H$

$$M_t^\pi = e^{-\beta(x - Y_0)} \exp\left(-\int_0^t \beta(\pi_s - Z_s)dB_s - \frac{1}{2}\int_0^t \beta^2 \left|\widehat{a}_s^{1/2}(\pi_s - Z_s)\right|^2 ds - \beta K_t^{\mathbb{P}}\right), \ \mathbb{P} - a.s.$$

We can then write

$$R^\pi = M^\pi A^\pi,$$

with

$$A_t^\pi = -\exp\left(\int_0^t v(s, p_s, Z_s)ds\right),$$

and

$$v(t, \pi, z) \ := -\beta\pi b_t + \beta\widehat{F}_t(z) + \frac{1}{2}\beta^2\left|\widehat{a}_t^{1/2}(\pi - z)\right|^2.$$

Clearly, we may rewrite $v(t,\pi_t,Z_t)$ in the following form

$$\begin{aligned}\frac{1}{\beta}v(t,\pi_t,Z_t) &= \frac{\beta}{2}\left|\widehat{a}_t^{1/2}\pi_t\right|^2 - \beta\pi_t'\widehat{a}_t^{1/2}\left(\widehat{a}_t^{1/2}Z_t+\frac{1}{\beta}\widehat{\theta}_t\right)+\frac{\beta}{2}\left|\widehat{a}_t^{1/2}Z_t\right|^2+\widehat{F}_t(Z_t)\\ &= \frac{\beta}{2}\left|\widehat{a}_t^{1/2}\pi_t-\left(\widehat{a}_t^{1/2}Z_t+\frac{1}{\beta}\widehat{\theta}_t\right)\right|^2 - Z_t'\widehat{a}_t^{1/2}\widehat{\theta}_t-\frac{1}{2\beta}\left|\widehat{\theta}_t\right|^2+\widehat{F}_t(Z_t).\end{aligned}$$

By a classical measurable selection theorem (see [33] (chapitre III) or [41] or Lemma 3.1 in [42]), we can define a progressively measurable process π^* satisfying (4.4.5). Then, it follows from the definition of $\widehat{F}$ that $\mathcal{P}_H - q.s.$

- $v(t,\pi_t,Z_t)\geq 0$ for all $\pi\in\tilde{\mathcal{A}}$.
- $v(t,\pi_t^*,Z_t)=0$,

which implies that A^π is always non-increasing for all π and is equal to -1 for π^*.

Step 3: In this step, we show that the processes

$$\int_0^\cdot Z_s dB_s,\quad \int_0^\cdot \pi_s^* dB_s,$$

are $\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$ martingales.

First of all, by Lemma 3.3.1, we know that $\int_0^\cdot Z_s dB_s$ is a $\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$ martingale.

By the triangle inequality and the definition of π^* together with (4.4.2), we have

$$\begin{aligned}\left|\widehat{a}_t^{1/2}\pi_t^*\right| &\leq \left|\widehat{a}_t^{1/2}Z_t+\frac{1}{\beta}\widehat{\theta}_t\right|+\left|\widehat{a}_t^{1/2}\pi_t^*-\left(\widehat{a}_t^{1/2}Z_t+\frac{1}{\beta}\widehat{\theta}_t\right)\right|\\ &\leq 2\left|\widehat{a}_t^{1/2}Z_t\right|+\frac{2}{\beta}\left|\widehat{\theta}_t\right|+k\leq 2\left|\widehat{a}_t^{1/2}Z_t\right|+k_1,\end{aligned}$$

where k_1 is a bound on $\widehat{\theta}$.

Then, for every probability $\mathbb{P}\in\mathcal{P}_H$ and every stopping time $\tau\leq T$,

$$\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T\left|\widehat{\theta}_t\pi_t^*\right|^2dt\right]\leq \mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T 8\left|\widehat{a}_t^{1/2}Z_t\right|^2dt+2Tk_1^2\right],$$

and therefore

$$\left\|\int_0^\cdot \pi_s^*dB_s\right\|_{\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})}\leq 8\left\|\int_0^\cdot Z_sdB_s\right\|_{\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})}+2Tk_1^2.$$

This implies the $\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$ martingale property of $\int_0^\cdot \pi_s^* dB_s$ as desired.

Step 4: We then prove that $\pi^*\in\tilde{\mathcal{A}}$ and $R^{\pi^*}\equiv -M^{\pi^*}$ satisfies Property (iii) of 4.3.1, that is to say

$$\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}\ \mathbb{E}_t^{\mathbb{P}'}\left[M_T^{\pi^*}\right]=M_t^{\pi^*}\quad \mathbb{P}-a.s.\ \forall\mathbb{P}\in\mathcal{P}_H.$$

For a fixed $\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})$, we denote

$$L_t:=\int_0^t\beta(\pi_s^*-Z_s)dB_s+\frac{1}{2}\int_0^t\beta^2\left|\widehat{a}_s^{1/2}(\pi_s^*-Z_s)\right|^2ds+\beta K_t^{\mathbb{P}'},$$

then with Itô's formula, we obtain, thanks to the $\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$ property proved in Step 3

$$\begin{aligned}\mathbb{E}_t^{\mathbb{P}'}\left[M_T^{\pi^*}\right]-M_t^{\pi^*}&=-\beta\mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s^-}^{\pi^*}dK_s^{\mathbb{P}'}\right]\\&+\mathbb{E}_t^{\mathbb{P}'}\left[\sum_{t\leq s\leq T}e^{-L_s}-e^{-L_{s^-}}+e^{-L_{s^-}}(L_s-L_{s^-})\right].\end{aligned}\tag{4.4.6}$$

First, we prove

$$\underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s^-}^{\pi^*}dK_s^{\mathbb{P}'}\right]=0,\ \mathbb{P}-\text{a.s.}$$

For every t and every $\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})$, we have

$$0\leq\mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s^-}^{\pi^*}dK_s^{\mathbb{P}'}\right]\leq\mathbb{E}_t^{\mathbb{P}'}\left[\left(\sup_{t\leq s\leq T}M_s^{\pi^*}\right)\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)\right].$$

Besides, since $K^{\mathbb{P}'}$ is non-decreasing, we obtain for all $s\geq t$

$$M_s^{\pi^*}\leq e^{-\beta(x-Y_0)}\mathcal{E}\left(\beta\int_t^s(Z_u-\pi_u^*)\,dB_u\right)$$

Then, again thanks to Step 3, we know that

$$\int_0^t(Z_s-\pi_s^*)\,dB_s\in\mathbb{BMO}(\mathcal{P}_{\mathrm{H}}),$$

and thus the exponential martingale above is a uniformly integrable martingale for all $\mathbb{P}$ and is in L_H^r for some $r>1$ (see Lemma 3.2.1). Thus, by Hölder inequality,

$$\mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s^-}^{\pi^*}dK_s^{\mathbb{P}'}\right]\leq e^{\beta(Y_0-x)}\mathbb{E}_t^{\mathbb{P}'}\left[\sup_{t\leq s\leq T}\mathcal{E}^r\left(\beta\int_t^s(Z_u-\pi_u^*)\,dB_u\right)\right]^{\frac{1}{r}}\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)^q\right]^{\frac{1}{q}}.$$

With Doob's maximal inequality, we have

$$\mathbb{E}_t^{\mathbb{P}'}\left[\sup_{t\leq s\leq T}\mathcal{E}^r\left(\beta\int_t^s(Z_u-\pi_u^*)\,dB_u\right)\right]^{1/r}\quad\leq\quad C\mathbb{E}_t^{\mathbb{P}'}\left[\mathcal{E}^r\left(\beta\int_t^T(Z_u-\pi_u^*)\,dB_u\right)\right]^{1/r}<+\infty.$$

where C is an universal constant that can change value from line to line.

Then by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)^q\right]^{1/q}&\leq C\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)\right]\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)^{2q-1}\right]\right)^{\frac{1}{2q}}\\&\leq C\left(\underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)^{2q-1}\right]\right)^{\frac{1}{2q}}\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)\right]\right)^{\frac{1}{2q}}.\end{aligned}$$

Arguing as in the proof of Theorem 3.3.1 we know that

$$\left(\underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)^{2q-1}\right]\right)^{\frac{1}{2q}}<+\infty.$$

Hence, we obtain

$$0\leq \underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s^-}^{\pi^*}dK_s^{\mathbb{P}'}\right]\leq C\underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\left(\mathbb{E}_t^{\mathbb{P}'}\left[\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)\right]\right)^{\frac{1}{2q}}=0,$$

which means

$$\underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s^-}^{\pi^*}dK_s^{\mathbb{P}'}\right]=0.$$

Finally, we have

$$\begin{aligned}
&\underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s^-}^{\pi^*}dK_s^{\mathbb{P}'}-\sum_{t\leq s\leq T}\exp(-\beta L_s)-\exp(-\beta L_{s^-})+\beta\exp(-\beta L_{s^-})(L_s-L_{s^-})\right]\\
&\leq \underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_{s^-}^{\pi^*}dK_s^{\mathbb{P}'}\right]\\
&\quad-\underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[\sum_{t\leq s\leq T}\exp(-\beta L_s)-\exp(-\beta L_{s^-})+\beta\exp(-\beta L_{s^-})(L_s-L_{s^-})\right]\\
&\leq 0,
\end{aligned}$$

because the function $x\to\exp(-x)$ is convex and the jumps of L are positive.

Hence, using (4.4.6), we have

$$\underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[M_T^{\pi^*}-M_t^{\pi^*}\right]\geq 0.$$

But by definition M^{π^*} is the product of a martingale and a positive decreasing process and is therefore a supermartingale. This implies that

$$\underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[M_T^{\pi^*}-M_t^{\pi^*}\right]=0.$$

Finally, π^* is an admissible strategy, R^{π^*} satisfies Property 4.3.1(iii) and

$$\begin{aligned}
R_0^{\pi^*}&=\inf_{\mathbb{P}\in\mathcal{P}_H}\mathbb{E}^{\mathbb{P}}\left[-\exp\left(-\beta\left(x+\int_0^T\pi_s^*\left(dB_s+\theta_s ds\right)-\xi\right)\right)\right]\\
&=-\exp\left(-\beta\left(x-Y_0\right)\right).
\end{aligned}$$

Step 5: Next we will show that for all $\pi\in\tilde{\mathcal{A}}$, R^π satisfies Property (iii) of 4.3.1, that is,

$$\underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}[-\exp(-\beta(X_T^\pi-\xi))]\leq R_t^\pi,\ \mathbb{P}-a.s.$$

Since $\pi\in\tilde{\mathcal{A}}$, the process

$$\int_0^t\left(Z_s-\pi_s\right)dB_s,$$

is in $\mathbb{BMO}(\mathcal{P}_{\rm H})$. Then the process

$$N^\pi = \exp\left(-\beta(x - Y_0)\right)\mathcal{E}\left(-\beta\int_0^\cdot (\pi_s - Z_s)\,dB_s\right),$$

is a uniformly integrable martingale under each $\mathbb{P} \in \mathcal{P}_H$.

As in the previous steps, we write R^π as $R^\pi_t = M^\pi_t A^\pi_t$, where A^π is a negative non-increasing process. We then have

$$\begin{aligned}\underset{\mathbb{P}'\in\mathcal{P}_H(s^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}_s\left[M^\pi_t A^\pi_t\right] &\leq \underset{\mathbb{P}'\in\mathcal{P}_H(s^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}_s\left[M^\pi_t A^\pi_s\right],\ \mathbb{P}-a.s.\\ &= \underset{\mathbb{P}'\in\mathcal{P}_H(s^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}_s\left[M^\pi_t\right] A^\pi_s,\ \mathbb{P}-a.s.\end{aligned}$$

because A^π is negative. By the same arguments as in Step 3 for $M^{(\pi^*)}$, we have

$$\underset{\mathbb{P}'\in\mathcal{P}_H(s^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}_s\left[M^\pi_t\right] = M^\pi_s,\ \mathbb{P}-a.s.$$

Therefore the following inequality holds

$$\underset{\mathbb{P}'\in\mathcal{P}_H(s^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}_s\left[R^\pi_t\right] \leq R^\pi_s,\ \mathbb{P}-a.s.$$

which ends the proof. □

Remark 4.4.3. *We see here why it is essential in our context to have strong integrability assumptions on the trading strategies. Indeed, in the proof of the above property for M^{π^*}, the fact that the stochastic integral*

$$\int_0^\cdot \pi^*_s dB_s,$$

is in $\mathbb{BMO}(\mathcal{P}_{\rm H})$ allowed us to control the moments of its stochastic exponential, which in turn allowed us to deduce from the minimal property for $K^{\mathbb{P}}$ a similar minimal property for

$$\int_0^\cdot M^{\pi^*}_s dK^{\mathbb{P}}_s.$$

This term is new when compared with the context of [62]. To deal with it, we have to impose the $\mathbb{BMO}(\mathcal{P}_{\rm H})$ property. Le us note however that since the optimal strategy already has that property, we do not lose a lot by restricting the strategies.

Remark 4.4.4. *We note that our approach still works when there are no constraints on trading strategies. In this case, the 2BSDE related to the maximization problem has an uniformly Lipschitz generator, and we are in the context of complete markets.*

4.4.2 A min-max property

By comparing the value function of our robust utility maximization problem and the one presented in [62] for standard utility maximization problem, we are able to prove a min-max property similar to the one proved by Denis and Kervarec in [39]. We observe that we were only able to prove this property after having solved the initial problem, unlike in the approach of [39].

Theorem 4.4.2. *Under the previous assumptions on the probability measures set $\mathcal{P}_H$ and the admissible strategies set $\tilde{\mathcal{A}}$, the following min-max property holds.*

$$\sup_{\pi\in\tilde{\mathcal{A}}}\inf_{\mathbb{P}\in\mathcal{P}_H}\mathbb{E}^{\mathbb{P}}\left[R_T^\pi\right]=\inf_{\mathbb{P}\in\mathcal{P}_H}\sup_{\pi\in\tilde{\mathcal{A}}}\mathbb{E}^{\mathbb{P}}\left[R_T^\pi\right]=\inf_{\mathbb{P}\in\mathcal{P}_H}\sup_{\pi^{\mathbb{P}}\in\tilde{\mathcal{A}}^{\mathbb{P}}}\mathbb{E}^{\mathbb{P}}\left[R_T^\pi\right],$$

where $\tilde{\mathcal{A}}^{\mathbb{P}}$ is the set consisting of trading strategies which are in $\widetilde{A}$ and such that the process $\left(\int_0^t\pi_s dB_s\right)_{0\leq t\leq T}$ is in BMO($\mathbb{P}$).

Proof. First note that we have

$$D:=\sup_{\pi\in\tilde{\mathcal{A}}}\inf_{\mathbb{P}\in\mathcal{P}_H}\mathbb{E}^{\mathbb{P}}\left[R_T^\pi\right]\leq\inf_{\mathbb{P}\in\mathcal{P}_H}\sup_{\pi\in\tilde{\mathcal{A}}}\mathbb{E}^{\mathbb{P}}\left[R_T^\pi\right]\leq\inf_{\mathbb{P}\in\mathcal{P}_H}\sup_{\pi^{\mathbb{P}}\in\tilde{\mathcal{A}}^{\mathbb{P}}}\mathbb{E}^{\mathbb{P}}\left[R_T^\pi\right]=:C.$$

Indeed, the first inequality is obvious and the second one follows from the fact that for all $\mathbb{P}$, $\tilde{\mathcal{A}}\subset\tilde{\mathcal{A}}^{\mathbb{P}}$.

It remains to prove that $C\leq D$. By the previous sections, we know that

$$D=-\exp\left(-\beta\left(x-Y_0\right)\right).$$

Moreover, we know from Chapter 3 that we have a representation for Y_0,

$$Y_0=\sup_{\mathbb{P}\in\mathcal{P}_H}y_0^{\mathbb{P}},$$

where $y_0^{\mathbb{P}}$ is the solution of the standard BSDE with the same generator $\widehat{F}$.

On the other hand, we observe from [62] that

$$C=\inf_{\mathbb{P}\in\mathcal{P}_H}\left[-\exp\left(-\beta\left(x-y_0^{\mathbb{P}}\right)\right)\right],$$

implying that $C=D$. □

4.4.3 Indifference pricing via robust utility maximization

It has been shown in [46] that in a market model with constraints on the portfolios, if we define the indifference price for a claim Φ as the smallest number p such that

$$\sup_{\pi}\mathbb{E}\left[-\exp\left(-\beta\left(X^{x+p,\pi}-\Phi\right)\right)\right]\geq\sup_{\pi}\mathbb{E}\left[-\exp\left(-\beta X^{x,\pi}\right)\right],$$

where $X^{x,\pi}$ is the wealth associated with the portfolio π and initial value x, then this problem turns into the resolution of a BSDE with quadratic growth generator.

In our framework of uncertain volatility, the problem of indifference pricing of a contingent claim ϕ boils down to solve the following equation in p

$$V^0(x) = V^\Phi(x+p).$$

Thanks to our results, we know that if $\psi \in \mathcal{L}_H^\infty$ then the two sides of the above equality can be calculated by solving 2BSDEs. The price p can therefore be calculated as soon as we are able to solve the 2BSDEs (explicitely or numerically). We provide two examples in Section 4.7.

4.5 Robust power utility

In this section, we will consider the power utility function.

$$U(x) = -\frac{1}{\gamma}x^{-\gamma},\ x>0\ \gamma>0.$$

Here we shall use a different notion of trading strategy: $\rho = (\rho^i)_{i=1,\dots,d}$ denotes the proportion of wealth invested in stock i. The number of shares of stock i is given by $\frac{\rho_t^i X_t}{S_t^i}$.

Then the wealth process is defined as

$$X_t^\rho = x + \int_0^t \sum_{i=1}^d \frac{X_s^\rho \rho_{i,s}}{S_{i,s}} dS_{i,s} = x + \int_0^t X_s^\rho \rho_s \left(dB_s + b_s ds\right),\ \mathcal{P}_H - q.s. \tag{4.5.1}$$

and the initial capital x is positive.

The wealth process X^ρ can be written as

$$X_t^\rho = x\mathcal{E}\left(\int_0^t \rho_s(dB_s + b_s ds)\right),\ t\in[0,T],\ \mathcal{P}_H - q.s.$$

Then for every $\rho \in \tilde{\mathcal{A}}$, the wealth process X^ρ is a local $\mathbb{P}$-martingale bounded from below, hence, a $\mathbb{P}$-supermartingale, for all $\mathbb{P}\in\mathcal{P}_H$.

In the present setting, the set of admissible strategies is defined as follows

Definition 4.5.1. *The set of admissible trading strategies $\tilde{\mathcal{A}}$ consists of all $\mathbb{R}^d$-valued progressively measurable processes $\rho = (\rho_t)_{0\leq t\leq T}$ satisfying*

$$\rho \in \mathbb{BMO}(\mathcal{P}_H) \text{ and } \rho \in \tilde{A},\ dt\otimes\mathcal{P}_H - a.e.$$

We suppose that there is no liability ($\xi = 0$). Then the investor faces the maximization problem

$$V(x) = \sup_{\rho\in\tilde{\mathcal{A}}} \inf_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^\mathbb{P}\left[U(X_T^\rho)\right]. \tag{4.5.2}$$

In order to find the value function and an optimal strategy, we apply the same method as in the exponential utility case. We therefore have to construct a stochastic process $\tilde{R}^\rho$ with terminal value

$$\tilde{R}_T^\rho = U\left(x + \int_0^T X_s \rho \frac{dS_s}{S_s}\right).$$

satisfying Properties 4.3.1.

Then the value function will be given by $V(x) = \tilde{R}_0^x$. Applying the utility function to the wealth process yields

$$-\frac{1}{\gamma}\left(X_t^{\rho,x}\right)^{-\gamma} = -\frac{1}{\gamma}x^{-\gamma}\exp\left(-\int_0^t \gamma\rho_s dB_s - \int_0^t \gamma\rho_s b_s ds + \frac{1}{2}\int_0^t \gamma\left|\widehat{a}_s^{1/2}\rho_s\right|^2 ds\right). \tag{4.5.3}$$

This equation suggests the following choice

$$\tilde{R}_t^\rho = -\frac{1}{\gamma}x^{-\gamma}\exp\left(-\int_0^t \gamma\rho_s dB_s - \int_0^t \gamma\rho_s b_s ds + \frac{1}{2}\int_0^t \gamma\left|\widehat{a}_s^{1/2}\rho_s\right|^2 ds + Y_t\right),$$

where $(Y, Z) \in \mathbb{D}_H^{\infty,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is the unique solution of the following 2BSDE

$$Y_t = 0 - \int_t^T Z_s dBs - \int_t^T \widehat{F}_s(Z_s)ds + K_T - K_t,\ t \in [0,T]\ \mathcal{P}_H - q.s. \tag{4.5.4}$$

In order to get Property 4.3.1 (iii) for $\tilde{R}^\rho$, we have to construct $\widehat{F}_t(z)$ such that, for $t \in [0,T]$

$$\gamma\rho_t b_t - \frac{1}{2}\gamma\left|\widehat{a}_t^{1/2}\rho_t\right|^2 - \widehat{F}_t(Z_t) \le -\frac{1}{2}\left|\widehat{a}_t^{1/2}(\gamma\rho_t - Z_t)\right|^2 \text{ for all } \rho \in \tilde{\mathcal{A}}, \tag{4.5.5}$$

with equality for some $\rho^* \in \tilde{\mathcal{A}}$. This is equivalent to

$$\widehat{F}_t(Z_t) \ge -\frac{1}{2}\gamma(1+\gamma)\left|\widehat{a}_t^{1/2}\rho_t - \frac{1}{1+\gamma}\left(-\widehat{a}_t^{1/2}Z_t + \widehat{\theta}_t\right)\right|^2 - \frac{1}{2}\frac{\gamma\left|-\widehat{a}_t^{1/2}Z_t + \widehat{\theta}_t\right|^2}{1+\gamma} + \frac{1}{2}\left|\widehat{a}_t^{1/2}Z_t\right|^2.$$

Hence, the appropriate choice for $\widehat{F}$ is

$$\widehat{F}_t(z) = -\frac{\gamma(1+\gamma)}{2}\mathrm{dist}^2\left(\frac{-\widehat{a}_t^{1/2}z + \widehat{\theta}_t}{1+\gamma}, A_{\widehat{a}_t}\right) + \frac{\gamma\left|-\widehat{a}_t^{1/2}z + \widehat{\theta}_t\right|^2}{2(1+\gamma)} + \frac{1}{2}\left|\widehat{a}_t^{1/2}z\right|^2, \tag{4.5.6}$$

and a candidate for the optimal strategy must satisfy

$$\rho_t^* \in \Pi_{A_{\widehat{a}_t}}\left(\frac{1}{1+\gamma}(-\widehat{a}_t^{1/2}Z_t + \widehat{\theta}_t)\right),\ t \in [0,T].$$

We summarize this in the following Theorem.

Theorem 4.5.1. *Let Assumption 3.2.2 or Assumption 3.2.1 with the addition that the closed domain $\widetilde{A}$ is C^2 hold. Then, the value function of the optimization problem (4.5.2) is given by*

$$V(x) = -\frac{1}{\gamma}x^{-\gamma}\exp(Y_0) \text{ for } x > 0$$

where Y_0 is defined as the unique solution $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ of the quadratic 2BSDE

$$Y_t = 0 - \int_t^T Z_s dBs - \int_t^T \widehat{F}_s(Z_s)ds + K_T - K_t,\ t \in [0,T]\ \mathcal{P}_H - q.s. \tag{4.5.7}$$

with

$$\widehat{F}_t(z) = -\frac{\gamma(1+\gamma)}{2}dist^2\left(\frac{1}{1+\gamma}(-\widehat{a}_t^{1/2}z + \widehat{\theta}_t), A_{\widehat{a}_t}\right) + \frac{\gamma\left|-\widehat{a}_t^{1/2}z + \widehat{\theta}_t\right|^2}{2(1+\gamma)} + \frac{1}{2}\left|\widehat{a}_t^{1/2}z\right|^2.$$

Moreover, there exists an optimal trading strategy $\rho^* \in \tilde{\mathcal{A}}$ with the property

$$\widehat{a}_t^{1/2}\rho_t^* \in \Pi_{A_{\widehat{a}_t}}\left(\frac{1}{1+\gamma}(-\widehat{a}_t^{1/2}Z_t + \widehat{\theta}_t)\right),\ t\in[0,T]. \tag{4.5.8}$$

Proof. The proof is very similar to the case of robust exponential utility. First we can show, with the same arguments, that the generator $\widehat{F}$ satisfies the conditions of Assumption 2.2.1 or Assumption 2.2.2, hence there exists a unique solution to the 2BSDE (4.5.7).

Let then ρ^* denote the progressively measurable process, constructed with a measurable selection theorem, which realizes the distance in the definition of $\widehat{F}$. The same arguments as in the case of robust exponential utility show that $\rho^* \in \tilde{\mathcal{A}}$.

Then with the choice we made for $\widehat{F}$, we have the following multiplicative decomposition

$$\tilde{R}_t^\rho = -\frac{1}{\gamma}x^{-\gamma}\mathcal{E}\left(-\int_0^t (\gamma\rho_s - Z_s)\,dB_s\right)e^{-\gamma K_t^{\mathbb{P}}}\exp\left(-\int_0^t v_s ds\right),$$

where

$$v_t = \gamma\rho_t b_t - \frac{1}{2}\gamma\left|\widehat{a}_t^{1/2}\rho_t\right|^2 - \widehat{F}_t(Z_t) + \frac{1}{2}\left|\widehat{a}_t^{1/2}(\gamma\rho_t - Z_t)\right|^2 \leq 0,\ dt\otimes d\mathbb{P}-\text{a.s.}$$

Then since the stochastic integral $\int_0^t(\rho_s - Z_s)dB_s$ is in $\mathbb{BMO}(\mathcal{P}_{\rm H})$, the stochastic exponential above is a uniformly integrable martingale. By exactly the same arguments as before, we have

$$\underset{\mathbb{P}'\in\mathcal{P}_H(s^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_s^{\mathbb{P}'}\left[\tilde{R}_t^\rho\right] \leq \tilde{R}_s^\rho,\quad s\leq t,\ \mathbb{P}-a.s.$$

with equality for ρ^*.

Hence, the terminal value $\tilde{R}_T^\rho$ is the utility of the terminal wealth of the trading strategy ρ. Consequently,

$$\underset{\mathbb{P}'\in\mathcal{P}_H(0^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}}\left[U\left(X_T^{(\rho,x)}\right)\right] \leq \tilde{R}_0^{(x)} = -\frac{1}{\gamma}x^{-\gamma}\exp(Y_0)\ \text{ for all } \rho\in\tilde{\mathcal{A}}.$$

□

Remark 4.5.1. *Of course, the min-max property of Theorem 4.4.2 still holds.*

4.6 Robust logarithmic utility

In this section, we consider another important utility function

$$U(x) = \log(x),\ x>0.$$

Here we use a the same notion of trading strategies as in the power utility case: $\rho = (\rho^i)_{i=1,\ldots,d}$ denotes the part of the wealth invested in stock i. The number of shares of stock i is given by $\frac{\rho_t^i X_t}{S_t^i}$. Then the wealth process is defined as:

$$X_t^\rho = x + \int_0^t \sum_{i=1}^d \frac{X_s^\rho \rho_{i,s}}{S_{i,s}} dS_{i,s} = x + \int_0^t X_s^\rho\rho_s\left(dB_s + b_s ds\right),\ \mathcal{P}_H - q.s. \tag{4.6.1}$$

and the initial capital x is positive.

The wealth process X^ρ can be written as

$$X_t^\rho = x\mathcal{E}\left(\int_0^t \rho_s(dB_s + b_s ds)\right),\ t\in[0,T],\ \mathcal{P}_H - q.s.$$

In this case, the set of admissible strategies is defined as follows

Definition 4.6.1. *The set of admissible trading strategies $\tilde{\mathcal{A}}$ consists of all $\mathbb{R}^d$-valued progressively measurable processes ρ satisfying*

$$\sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\int_0^T |\widehat{a}_t^{1/2}\rho_t|^2 dt\right] < \infty,$$

and $\rho \in \tilde{A}$, $dt\otimes d\mathbb{P} - a.s.$, $\forall \mathbb{P}\in\mathcal{P}_H$.

For the logarithmic utility, the agent has no liability at time T ($\xi = 0$). Then the optimization problem is given by

$$\begin{aligned} V(x) &= \sup_{\rho\in\tilde{\mathcal{A}}} \inf_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}[\log(X_T^\rho)] \\ &= \log(x) + \sup_{\rho\in\tilde{\mathcal{A}}} \inf_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\int_0^T \rho_s dB_s + \int_0^T (\rho_s b_s - \frac{1}{2}|\widehat{a}_s^{1/2}\rho_s|^2)ds\right]. \end{aligned} \tag{4.6.2}$$

We have the following theorem.

Theorem 4.6.1. *Let Assumption 3.2.2 or Assumption 3.2.1 with the addition that the closed domain $\widetilde{A}$ is C^2 hold. Then, the value function of the optimization problem (4.6.2) is given by*

$$V(x) = \log(x) - Y_0 \quad \text{for } x > 0,$$

where Y_0 is defined as the unique solution $(Y,Z)\in\mathbb{D}_H^\infty\times\mathbb{H}_H^2$ of the quadratic 2BSDE

$$Y_t = 0 - \int_t^T Z_s dB_s - \int_t^T \widehat{F}_s ds + K_T^{\mathbb{P}} - K_t^{\mathbb{P}},\ t\in[0,T],\ \mathbb{P}-a.s.,\ \forall\mathbb{P}\in\mathcal{P}_H. \tag{4.6.3}$$

The generator is defined by

$$\widehat{F}_s = F_s(\widehat{a}_s),$$

where

$$F_s(a) = -\frac{1}{2}dist^2(\theta_s, A_a) + \frac{1}{2}|\theta_s|^2,\ for\ a\in S_d^{>0}.$$

Moreover, there exists an optimal trading strategy $\rho^*\in\tilde{\mathcal{A}}$ with the property

$$\widehat{a}_t^{1/2}\rho_t^* \in \Pi_{A_{\widehat{a}_t}}\left(\widehat{\theta}_t\right). \tag{4.6.4}$$

Proof. The proof is very similar to the case of exponential and power utility. First we show that there exists an unique solution to the 2BSDE (4.6.3). We then write, for $t\in[0,T]$

$$R_t^\rho = M_t^\rho + A_t^\rho,$$

where

$$\begin{aligned} M^\rho &= \log(x) - Y_0 + \int_0^t (\rho_s - Z_s)\, dB_s + K_t^{\mathbb{P}}, \\ A^\rho &= \int_0^t \left(-\frac{1}{2}\left|\widehat{a}_s^{1/2}\rho_s - \widehat{\theta}_s\right|^2 + \frac{1}{2}\left|\widehat{\theta}_s\right|^2 - \widehat{F}_s\right) ds. \end{aligned}$$

Then, we similarly prove that ρ^*, which can be consructed by means of a classical measurable selection argument, is in $\tilde{\mathcal{A}}$. Note in particular that ρ^* only depends on $\widehat{\theta}$, $\widehat{a}^{1/2}$ and the closed set $\tilde{A}$ describing the constraints on the trading strategies.

Next, due to Definition (4.6.1), the stochastic integral in R^ρ is a martingale under each $\mathbb{P}$ for all $\rho \in \tilde{\mathcal{A}}$. Moreover, $\widehat{F}$ is chosen to make A non-increasing for all ρ and a constant for ρ^*. Thus, the minimum condition of $K^{\mathbb{P}}$ implies that R^ρ satisfies the Property (iii) of 4.3.1.

Furthermore, the initial value Y_0 of the simple 2BSDE (4.6.3) satisfies

$$Y_0 = -\sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\int_0^T \widehat{F}_s ds\right].$$

Hence,

$$V(x) = R_0^{\rho^*}(x) = \log(x) + \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\int_0^T \widehat{F}_s ds\right].$$

□

Remark 4.6.1. *Of course, the min-max property of Theorem 4.4.2 still holds.*

4.7 Examples

In general, it is difficult to solve BSDEs and 2BSDEs explicitely. In this section, we will give some examples where we have an explicit solution. In particular, we show how the optimal probability measure is chosen. In all our examples, we will work in dimension one, $d = 1$.

First, we deal with robust exponential utility. We consider the case where there are no constraints on trading strategies, that is $\widetilde{A} = \mathbb{R}$. Then the associated 2BSDE has a generator which is linear in z. In the first example, we consider a deterministic terminal liability ξ and show that we can compare our result with the one obtained by solving the HJB equation in the standard Merton's approach, working with the probability measure associated to the constant process $\bar{a}$. In the second example, we show that with a random payoff $\xi = -B_T^2$, where B is the canonical process, we end up with an optimal probability measure which is not of Bang-Bang type (Bang-Bang type means that, under this probability measure, the density of the quadratic variation $\widehat{a}$ takes only the two extreme values, $\underline{a}$ and $\overline{a}$). We emphasize that this example does not have real financial significance, but shows nonetheless that one cannot expect the optimal probability measure to depend only on the two bounds for the volatility unlike with option pricing in the UVM model.

4.7.1 Example 1: Deterministic payoff

In this example, we suppose that b is a constant in $\mathbb{R}$. From Theorem 4.4.1, we know that the value function of the robust maximization problem is given by

$$V^\xi(x) = -\exp\left(-\beta\left(x - Y_0\right)\right),$$

where Y is the solution of a 2BSDE with quadratic generator. When there are no constraints, the 2BSDE can be written as follows

$$Y_t = \xi - \int_t^T Z_s dB_s - \int_t^T \widehat{F}_s(Z_s)ds + K_T^{\mathbb{P}} - K_t^{\mathbb{P}},\ \mathbb{P}-a.s.,\ \forall \mathbb{P}\in\mathcal{P}_H.$$

and the generator is given by

$$F_t(\omega, z, a) = bz + \frac{b^2}{2\beta a}, \text{ for } a \in S_d^{>0}.$$

Then we can solve explicitly the correpondent BSDEs with the same generator under each $\mathbb{P}$. Let

$$M_t = e^{-\int_0^t \frac{1}{2} b^2 \widehat{a}_s^{-1} ds - \int_0^t b \widehat{a}_s^{-1} dB_s}.$$

By applying Itô's formula to $y_t^{\mathbb{P}} M_t$, we have

$$y_0^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}} \left[\xi M_T - \frac{b^2}{2\beta} \int_0^T \widehat{a}_s^{-1} M_s ds \right].$$

Since $\underline{a} \leq \widehat{a} \leq \bar{a}$, we derive that

$$y_0^{\mathbb{P}} \leq \xi - \frac{1}{2\beta} \frac{b^2}{\bar{a}} T.$$

Therefore, by the representation of Y, we have

$$Y_0 \leq \xi - \frac{1}{2\beta} \frac{b^2}{\bar{a}} T.$$

Moreover, under the specific probability measure $\mathbb{P}^{\bar{a}} \in \mathcal{P}_H$, we have

$$y_0^{\mathbb{P}^{\bar{a}}} = \xi - \frac{1}{2\beta} \frac{b^2}{\bar{a}} T.$$

This implies that $Y_0 = y_0^{\mathbb{P}^{\bar{a}}}$, which means that the robust utility maximization problem is degenerated and is equivalent to a standard utility maximization problem under the probability measure $\mathbb{P}^{\bar{a}}$. We give more details and intuitions about this result in Example 4.7.3 below.

4.7.2 Example 2 : Non-deterministic payoff

In this subsection, we consider a non-deterministic payoff $\xi = -B_T^2$. As in the first example, there are no constraints on trading strategies. Then, the 2BSDE has a linear generator. We can verify that $-B_T^2$ can be written as the limit under the norm $\|\cdot\|_{\mathbb{L}_H^2}$ of a sequence which is in $\mathrm{UC_b}(\Omega)$, and thus is in $\mathcal{L}_H^{2,\kappa}$, which is the terminal condition set for 2BSDE with Lipschitz generator. Here, we suppose that b is a deterministic continuous function of time t.

By the same method as in the previous example, let

$$M_t = e^{-\int_0^t \frac{1}{2} b_s^2 \widehat{a}_s^{-1} ds - \int_0^t b_s \widehat{a}_s^{-1} dB_s},$$

then we obtain

$$y_0^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}} \left[-M_T B_T^2 - \int_0^T \frac{b_s^2}{2\beta} \widehat{a}_s^{-1} M_s ds \right].$$

By applying Itô's formula to $M_t B_t$, we have

$$dM_t B_t = M_t dB_t + B_t dM_t - b_t M_t dt.$$

Since b is deterministic, by taking expectation under $\mathbb{P}$ and localizing if necessary, we obtain

$$\mathbb{E}^{\mathbb{P}} [M_T B_T] = \mathbb{E}^{\mathbb{P}} \left[-\int_0^T b_t M_t dt \right] = -\int_0^T b_t dt.$$

Again, by applying Itô's formula to $-M_t B_t^2$, we have

$$-dM_t B_t^2 = -2M_t B_t dB_t - B_t^2 dM_t - \widehat{a}_t M_t dt + 2b_t M_t B_t dt.$$

Therefore $y_0^{\mathbb{P}}$ can be rewritten as

$$y_0^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}\left[\int_0^T -M_t\left(\widehat{a}_t + \frac{b_s^2}{2\beta\widehat{a}_t}\right)dt\right] - \int_0^T 2b_t\left(\int_0^t b_s ds\right)dt.$$

By analysing the map $g : x \in \mathbb{R}^+ \longmapsto x - \frac{b_s^2}{2\beta x}$, we know that $g' = 1 - \frac{b_s^2}{2\beta x^2}$, then g is non-decreasing when $x^2 \geq \frac{b^2}{2\beta}$. Consider b a deterministic positive continuous non-decreasing function of time t such that

$$\frac{b_0^2}{2\beta} \leq \underline{a}^2 \leq \overline{a}^2 \leq \frac{b_T^2}{2\beta}.$$

Let $\underline{t}$ such that $\frac{b_{\underline{t}}^2}{2\beta} = \underline{a}$ and $\bar{t}$ such that $\frac{b_{\bar{t}}^2}{2\beta} = \bar{a}$, and $a^* := \underline{a}\mathbf{1}_{0\leq t\leq \underline{t}} + \frac{b_t}{\sqrt{2\beta}}\mathbf{1}_{\underline{t}\leq t\leq \bar{t}} + \overline{a}\mathbf{1}_{\bar{t}\leq t\leq T}$, then as in Example 4.7.1, we can show that $\mathbb{P}^{a^*}$ is an optimal probability measure, which is not of Bang-Bang type.

4.7.3 Example 3 : Merton's approach for robust power utility

Here, we deal with robust power utility. As in Example 4.7.1, we suppose that b is a constant in $\mathbb{R}$ and $\xi = 0$. First, we consider the case where $\widetilde{A} = \mathbb{R}$. From Theorem 4.5.1, $\widehat{F}_t(z)$ can be rewritten as

$$\widehat{F}_t(z) = \frac{\gamma\left|-\widehat{a}_t^{1/2}z + b\widehat{a}_t^{-1}\right|^2}{2(1+\gamma)} + \frac{1}{2}\left|\widehat{a}_t^{1/2}z\right|^2,$$

which is quadratic and linear in z. According to BSDEs theory, we can solve explicitly the corresponding BSDEs with this generator under each probability measure $\mathbb{P}$. We use an exponential transformation and let

$$\alpha := 1 + \frac{\gamma}{1+\gamma},\ y'^{\mathbb{P}} := e^{-\alpha y_t^{\mathbb{P}}},\ z'^{\mathbb{P}} := e^{-\alpha y_t^{\mathbb{P}}} z_t^{\mathbb{P}}.$$

By applying Itô's formula, we know that $(y'^{\mathbb{P}}, z'^{\mathbb{P}})$ is the solution of the following linear BSDE

$$dy_t'^{\mathbb{P}} = -\alpha y'^{\mathbb{P}}\left[\frac{\gamma}{2(1+\gamma)}\left(b^2\widehat{a}^{-1} - 2bz^{\mathbb{P}}\right)dt + z'^{\mathbb{P}}dB_t\right]$$

with the terminal condition $y_T'^{\mathbb{P}} = 1$.

Let

$$\lambda_t := \frac{\alpha\gamma}{2(1+\gamma)}b^2\widehat{a}^{-1},\ \eta_t := -\frac{\gamma}{2(1+\gamma)}2b\widehat{a}^{-1/2},\ \text{and } M_t := e^{\int_0^t \lambda_s - \frac{\eta_s^2}{2}ds + \int_0^t \widehat{a}_s^{-1/2}\eta_s dB_s}.$$

By applying Itô's formula to $y_t'^{\mathbb{P}} M_t$, we obtain

$$y_t'^{\mathbb{P}} = \mathbb{E}_t^{\mathbb{P}}\left[M_T/M_t\right],\ \text{so } y_0^{\mathbb{P}} = -\frac{1}{\alpha}\ln\left(\mathbb{E}^{\mathbb{P}}\left[M_T\right]\right).$$

Since $\underline{a} \leq \widehat{a} \leq \bar{a}$, we derive that

$$y_0^{\mathbb{P}} \leq -\frac{\gamma}{2(1+\gamma)}\frac{b^2}{\bar{a}}T.$$

Thus by the representation of Y, we have

$$Y_0 \leq -\frac{\gamma}{2(1+\gamma)}\frac{b^2}{\bar{a}}T.$$

Moreover, under the specific probability measure $\mathbb{P}^{\bar{a}} \in \mathcal{P}_H$, we have

$$y_0^{\mathbb{P}^{\bar{a}}} = -\frac{\gamma}{2(1+\gamma)}\frac{b^2}{\bar{a}}T.$$

This implies that $Y_0 = y_0^{\mathbb{P}^{\bar{a}}}$. Finally, the value of the robust power utility maximization problem is

$$V(x) = -\frac{1}{\gamma}x^{-\gamma}\exp\left(Y_0\right).$$

As in Example 4.7.1, the robust utility maximization problem is degenerate, and becomes a standard utility maximization problem under the probability measure $\mathbb{P}^{\overline{a}}$. In order to shed more light on this somehow surprising result, we first recall the HJB equation obtained by Merton [79] in the standard utility maximization problem

$$-\frac{\partial v}{\partial t} - \sup_{\delta\in\tilde{A}}\left[\mathcal{L}^{\delta}v(t,x)\right] = 0,$$

together with the terminal condition

$$v(T,x) = U(x) := -\frac{x^{-\gamma}}{\gamma},\ x\in\mathbb{R}_+,\ \gamma>0,$$

where $\mathcal{L}^{\delta}v(t,x) = x\delta b\frac{\partial v}{\partial x} + \frac{1}{2}x^2\delta^2\sigma^2\frac{\partial^2 v}{\partial x^2}$, with a constant volatility σ.

It turns out that, when $\tilde{A} = \mathbb{R}$, the value function is given by

$$v(t,x) = \exp\left(\frac{b^2}{2\sigma^2}\frac{-\gamma}{(1+\gamma)}(T-t)\right)U(x),\ (t,x)\in[0,T]\times\mathbb{R}_+.$$

Let $\sigma^2 = \overline{a}$, we have $v(0,x) = V(x)$, the result given by our 2BSDE method. Intuitively and formally speaking, the HJB equation for the robust maximization problem should then be

$$-\frac{\partial v}{\partial t} - \sup_{\delta\in\tilde{A}}\inf_{\alpha\in[\underline{a},\bar{a}]}\left[\mathcal{L}^{\delta,\alpha}v(t,x)\right] = 0$$

together with the terminal condition $v(T,x) = U(x),\ x\in\mathbb{R}_+$, where

$$\mathcal{L}^{\delta,\alpha}v(t,x) = x\delta b\frac{\partial v}{\partial x} + \frac{1}{2}x^2\delta^2\alpha\frac{\partial^2 v}{\partial x^2}.$$

Note that the value function we obtained from our 2BSDE approach solves the above PDE.

Now consider the case $\tilde{A} = \mathbb{R}$, if the second derivative of v is positive, then the term

$$\inf_{\delta\in\tilde{A}}\left[\mathcal{L}^{\delta,\underline{a},\bar{a}}v(t,x)\right]$$

becomes infinite, so the above PDE has no meaning. This implies that v is concave. Then $\overline{a}$ is the minimizer. This explains why the robust utility maximization problem degenerates in the case $\tilde{A} = \mathbb{R}$. However, it is clear that when, for instance, we impose no short-sale and no large sales conditions (that is to say $\tilde{A}$ is a segment), then the problem should not degenerate and the optimal probability measure switches between the two bounds $\underline{a}$ and $\overline{a}$.

Finally, notice that using the language of G-expectation introduced by Peng in [88], if we let

$$G(\Gamma) = \frac{1}{2} \sup_{\underline{a} \leq \alpha \leq \overline{a}} \alpha\Gamma = \frac{1}{2} \left(\overline{a} \left(\Gamma \right)^{+} - \underline{a} \left(\Gamma \right)^{-} \right),$$

then the above PDE can be rewritten as follows

$$-\frac{\partial v}{\partial t} + \inf_{\delta \in \tilde{A}} \left[\mathcal{L}^{\delta, \underline{a}, \bar{a}} v(t,x) \right] = 0,$$

where

$$\mathcal{L}^{\delta, \underline{a}, \bar{a}} v(t,x) = x^2 \delta^2 G \left(-\frac{\partial^2 v}{\partial x^2} \right).$$

Then, our PDE plays the same role for Merton's PDE as the Black-Scholes-Barrenblatt PDE plays for the usual Black-Scholes PDE, by replacing the second derivative terms by their non-linear versions.

Chapitre 5

Reflected 2BSDEs

Contents

5.1 Introduction

Reflected backward stochastic differential equations (RBSDEs for short) were introduced by El Karoui et al. [44] to study related obstacle problems for PDE's and American options pricing. In this case, the solution Y of the BSDE is constrained to stay above a given obstacle process S. In order to achieve this, a non-decreasing process K is added to the solution

$$
\begin{aligned}
&Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ t \in [0,T],\ \mathbb{P}-a.s.\\
&Y_t \geq S_t,\ t \in [0,T],\ \mathbb{P}-a.s.\\
&\int_0^T (Y_s - S_s)dK_s = 0,\ \mathbb{P}-a.s.,
\end{aligned}
$$

where the last condition, also known as the Skorohod condition means that the process K is minimal in the sense that it only acts when Y reaches the obstacle S. This condition is crucial to obtain the wellposedness of the classical RBSDEs.

Our aim in this chapter is to provide a complete theory of existence and uniqueness of Second order RBSDEs (2RBSDEs) under the Lipschitz-type hypotheses of [107] on the driver. We will show that in this context, the definition of a 2RBSDE with a lower obstacle S is very similar to that of a 2BSDE. We do not need to add another increasin process, unlike in the classical case, and we do not need to impose a condition similar to the Skorohod condition. The only change necessary is in the minimal condition that the increasing process K of the 2RBSDE must satisfy.

The rest of this chapter is organised as follows. In Section 5.2, we recall briefly some notations, provide the precise definition of 2RBSDEs and show how they are connected to classical RBSDEs. Then, in Section 5.3, we show a representation formula for the solution of a 2RBSDEs which in turn implies uniqueness. We then provide some links between 2RBSDEs and optimal stopping problems. In Section 5.4, we give a proof of existence by means of r.c.p.d. techniques, as in Chapter 3 for quadratic 2BDSEs. Let us mention that this proof required to extend existing results on the theory of g-martingales of Peng (see [86]) to the reflected case. Since to the best of our knowledge, those results do not exist in the litterature, we prove them in the Appendix in Section 5.6. Finally, we use these new objects in Section 5.5 to study the pricing problem of American options in a market with volatility uncertainty.

5.2 Preliminaries

We place ourselves in the same framework as in Chapter 3.

5.2.1 The non-linear generator

We consider a map $H_t(\omega, y, z, \gamma) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$, where $D_H \subset \mathbb{R}^{d\times d}$ is a given subset containing 0.

Define the corresponding conjugate of H w.r.t. γ by

$$F_t(\omega, y, z, a) := \sup_{\gamma\in D_H} \left\{\frac{1}{2}\mathrm{Tr}(a\gamma) - H_t(\omega, y, z, \gamma)\right\} \text{ for } a \in S_d^{>0}$$
$$\widehat{F}_t(y, z) := F_t(y, z, \widehat{a}_t) \text{ and } \widehat{F}_t^0 := \widehat{F}_t(0, 0).$$

We denote by $D_{F_t(y,z)}$ the domain of F in a for a fixed (t, ω, y, z).

We now state our main assumptions on the function F which will be our main interest in the sequel

Assumption 5.2.1. (i) *The domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z).*

(ii) *For fixed (y, z, a), F is $\mathbb{F}$-progressively measurable.*

(iii) *We have the following uniform Lipschitz-type property in y and z*

$$\forall (y, y', z, z', t, a, \omega),\ |F_t(\omega, y, z, a) - F_t(\omega, y', z', a) \leq C\left(|y - y'| + |a^{1/2}(z - z')|\right).$$

(iv) *F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.*

5.2.2 Formulation

First, we consider a process S which will play the role of our lower obstacle. We will always assume that S verifies the following properties

(i) S is $\mathbb{F}$-progressively measurable and càdlàg.

(ii) S is uniformy continuous in ω in the sense that for all t

$$|S_t(\omega) - S_t(\widetilde{\omega})| \leq \rho\left(\|\omega - \widetilde{\omega}\|_t\right),\ \forall\ (\omega, \widetilde{\omega}) \in \Omega^2,$$

for some modulus of continuity ρ and where we define $\|\omega\|_t := \sup_{0\leq s\leq t} |\omega(s)|$.

Then, we shall consider the following second order RBSDE (2RBSDE for short) with lower obstacle S

$$Y_t = \xi + \int_t^T \widehat{F}_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ 0 \leq t \leq T,\ \mathcal{P}_H^\kappa - q.s. \tag{5.2.1}$$

We follow Soner, Touzi and Zhang [107]. For any $\mathbb{P} \in \mathcal{P}_H^\kappa$, $\mathbb{F}$-stopping time τ, and $\mathcal{F}_\tau$-measurable random variable $\xi \in \mathbb{L}^2(\mathbb{P})$, let $(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P}}) := (y^{\mathbb{P}}(\tau, \xi), z^{\mathbb{P}}(\tau, \xi), k^{\mathbb{P}}(\tau, \xi))$ denote the unique solution to the following standard RBSDE with obstacle S (existence and uniqueness have been proved under our assumptions by Lepeltier and Xu in [74])

$$\begin{cases} y_t^{\mathbb{P}} = \xi + \int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} dB_s + k_\tau^{\mathbb{P}} - k_t^{\mathbb{P}},\ 0 \leq t \leq \tau,\ \mathbb{P} - a.s. \\ y_t^{\mathbb{P}} \geq S_t,\ \mathbb{P} - a.s. \\ \int_0^t \left(y_{s^-}^{\mathbb{P}} - S_{s^-}\right) dk_s^{\mathbb{P}} = 0,\ \mathbb{P} - a.s.,\ \forall t \in [0, T]. \end{cases}$$

Definition 5.2.1. *For $\xi \in \mathbb{L}_H^{2,\kappa}$, we say $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a solution to the 2RBSDE (5.2.1) if*

- $Y_T = \xi$, $\mathcal{P}_H^\kappa - q.s.$
- *$\forall \mathbb{P} \in \mathcal{P}_H^\kappa$, the process $K^{\mathbb{P}}$ defined below has non-decreasing paths $\mathbb{P} - a.s.$*

$$K_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ 0 \leq t \leq T,\ \mathbb{P} - a.s. \tag{5.2.2}$$

- *We have the following minimum condition*

$$K_t^{\mathbb{P}} - k_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}\left[K_T^{\mathbb{P}'} - k_T^{\mathbb{P}'}\right],\ 0 \leq t \leq T,\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \tag{5.2.3}$$

- $Y_t \geq S_t$, $\mathcal{P}_H^\kappa - q.s.$

Following [107], in addition to Assumption 5.2.1, we will always assume

Assumption 5.2.2. (i) *$\mathcal{P}_H^\kappa$ is not empty.*

(ii) *The processes $\widehat{F}^0$ and S satisfy the following integrability conditions*

$$\phi_H^{2,\kappa} := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\underset{0\leq t\leq T}{\operatorname{ess\,sup}^{\mathbb{P}}} \left(\mathbb{E}_t^{H,\mathbb{P}}\left[\int_0^T |\hat{F}_s^0|^\kappa ds\right]\right)^{\frac{2}{\kappa}}\right] < +\infty \tag{5.2.4}$$

$$\psi_H^{2,\kappa} := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\underset{0\leq t\leq T}{\operatorname{ess\,sup}^{\mathbb{P}}} \left(\mathbb{E}_t^{H,\mathbb{P}}\left[\left(\sup_{0\leq s\leq T} (S_s)^+\right)^\kappa\right]\right)^{\frac{2}{\kappa}}\right] < +\infty. \tag{5.2.5}$$

5.2.3 Connection with standard RBSDEs

If H is linear in γ, that is to say

$$H_t(y,z,\gamma) := \frac{1}{2}\mathrm{Tr}\left[a_t^0\gamma\right] - f_t(y,z),$$

where $a^0 : [0,T]\times\Omega \to S_d^{>0}$ is $\mathbb{F}$-progressively measurable and has uniform upper and lower bounds. As in [107], we no longer need to assume any uniform continuity in ω in this case. Besides, the domain of F is restricted to a^0 and we have

$$\widehat{F}_t(y,z) = f_t(y,z).$$

If we further assume that there exists some $\mathbb{P}\in\overline{\mathcal{P}}_S$ such that $\widehat{a}$ and a^0 coincide $\mathbb{P}-a.s.$ and $\mathbb{E}^{\mathbb{P}}\left[\int_0^T |f_t(0,0)|^2\,dt\right] < +\infty$, then $\mathcal{P}_H^\kappa = \{\mathbb{P}\}$.

Then, unlike with 2BSDEs, it is not immediate from the minimum condition (5.2.3) that the process $K^{\mathbb{P}} - k^{\mathbb{P}}$ is actually null. However, we know that $K^{\mathbb{P}} - k^{\mathbb{P}}$ is a martingale with finite variation. Since $\mathbb{P}$ satisfy the martingale representation property, this martingale is also continuous, and therefore it is null. Thus we have

$$0 = k^{\mathbb{P}} - K^{\mathbb{P}},\ \mathbb{P}-a.s.,$$

and the 2RBSDE is equivalent to a standard RBSDE. In paticular, we see that the part of $K^{\mathbb{P}}$ which increases only when $Y_{t^-} > S_{t^-}$ is null, which means that $K^{\mathbb{P}}$ satisfies the usual Skorohod condition with respect to the obstacle.

5.3 Uniqueness of the solution and other properties

5.3.1 Representation and uniqueness of the solution

We have similarly as in Theorem 4.4 of [107]

Theorem 5.3.1. *Let Assumptions 5.2.1 and 5.2.2 hold. Assume $\xi\in\mathbb{L}_H^{2,\kappa}$ and that (Y,Z) is a solution to 2RBSDE (5.2.1). Then, for any $\mathbb{P}\in\mathcal{P}_H^\kappa$ and $0\le t_1<t_2\le T$,*

$$Y_{t_1} = \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}),\ \mathbb{P}-a.s. \tag{5.3.1}$$

$$\tag{5.3.2}$$

Consequently, the 2RBSDE (5.2.1) has at most one solution in $\mathbb{D}_H^{2,\kappa}\times\mathbb{H}_H^{2,\kappa}$.

Proof. The proof follows the lines of the proof of Theorem 4.4 in [107]. First,

$$Y_t = \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ y_t^{\mathbb{P}'}(T,\xi),\ t\in[0,T],\ \mathbb{P}-a.s.,\ \text{for all } \mathbb{P}\in\mathcal{P}_H^\kappa,$$

and thus is unique. Then, since we have that $d\langle Y,B\rangle_t = Z_t d\langle B\rangle_t,\ \mathcal{P}_H^\kappa - q.s.$, Z is unique. Finally, the process $K^{\mathbb{P}}$ is uniquely determined. We shall now prove (5.3.1).

(i) Fix $0 \leq t_1 < t_2 \leq T$ and $\mathbb{P} \in \mathcal{P}_H^\kappa$. For any $\mathbb{P}^{'} \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})$, we have

$$Y_t = Y_{t_2} + \int_t^{t_2} \widehat{F}_s(Y_s, Z_s)ds - \int_t^{t_2} Z_s dB_s + K_{t_2}^{\mathbb{P}^{'}} - K_t^{\mathbb{P}^{'}},\ t_1 \leq t \leq t_2,\ \mathbb{P}^{'} - a.s.$$

Now, it is clear that we can always decompose the non-decreasing process $K^{\mathbb{P}}$ into

$$K_t^{\mathbb{P}^{'}} = A_t^{\mathbb{P}^{'}} + B_t^{\mathbb{P}^{'}},\ \mathbb{P}^{'} - a.s.,$$

were $A^{\mathbb{P}^{'}}$ and $B^{\mathbb{P}^{'}}$ are two non-decreasing processes such that $A^{\mathbb{P}^{'}}$ only increases when $Y_{t^-} = S_{t^-}$ and $B^{\mathbb{P}^{'}}$ only increases when $Y_{t^-} > S_{t^-}$. With that decomposition, we can apply a generalisation of the usual comparison theorem proved by El Karoui et al. (see Theorem 5.2 in [47]), whose proof is postponed to the appendix, under $\mathbb{P}^{'}$ to obtain $Y_{t_1} \geq y_{t_1}^{\mathbb{P}^{'}}(t_2, Y_{t_2})$ and $A_{t_2}^{\mathbb{P}^{'}} - A_{t_1}^{\mathbb{P}^{'}} \leq k_{t_2}^{\mathbb{P}^{'}} - k_{t_1}^{\mathbb{P}^{'}},\ \mathbb{P}^{'} - a.s.$ Since $\mathbb{P}^{'} = \mathbb{P}$ on $\mathcal{F}_t^+$, we get $Y_{t_1} \geq y_{t_1}^{\mathbb{P}^{'}}(t_2, Y_{t_2})$, $\mathbb{P} - a.s.$ and thus

$$Y_{t_1} \geq \underset{\mathbb{P}^{'} \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ y_{t_1}^{\mathbb{P}^{'}}(t_2, Y_{t_2}),\ \mathbb{P} - a.s.$$

(ii) We now prove the reverse inequality. Fix $\mathbb{P} \in \mathcal{P}_H^\kappa$. We will show in (iii) below that

$$C_{t_1}^{\mathbb{P}} := \underset{\mathbb{P}^{'} \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\left(K_{t_2}^{\mathbb{P}^{'}} - k_{t_2}^{\mathbb{P}^{'}} - K_{t_1}^{\mathbb{P}^{'}} + k_{t_1}^{\mathbb{P}^{'}}\right)^2\right] < +\infty,\ \mathbb{P} - a.s.$$

For every $\mathbb{P}^{'} \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})$, denote

$$\delta Y := Y - y^{\mathbb{P}^{'}}(t_2, Y_{t_2}),\ \delta Z := Z - z^{\mathbb{P}^{'}}(t_2, Y_{t_2}) \text{ and } \delta K^{\mathbb{P}^{'}} := K^{\mathbb{P}^{'}} - k^{\mathbb{P}^{'}}(t_2, Y_{t_2}).$$

By the Lipschitz Assumption 5.2.1(iii), there exist two bounded processes λ and η such that for all $t_1 \leq t \leq T_2$

$$\delta Y_t = \int_t^{t_2} \left(\lambda_s \delta Y_s + \eta_s \widehat{a}_s^{1/2} \delta Z_s\right) ds - \int_t^{t_2} \delta Z_s dB_s + \delta K_{t_2}^{\mathbb{P}^{'}} - \delta K_{t_1}^{\mathbb{P}^{'}},\ \mathbb{P}^{'} - a.s.$$

Define for $t_1 \leq t \leq t_2$ the following continuous process

$$M_t := \exp\left(\int_{t_1}^t \left(\lambda_s - \frac{1}{2}|\eta_s|^2\right) ds - \int_{t_1}^t \eta_s \widehat{a}_s^{-1/2} dB_s\right),\ \mathbb{P}^{'} - a.s.$$

Note that since λ and η are bounded, we have for all $p \geq 1$

$$\mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\sup_{t_1 \leq t \leq t_2} (M_t)^p + \sup_{t_1 \leq t \leq t_2} (M_t^{-1})^p\right] \leq C_p,\ \mathbb{P}^{'} - a.s. \tag{5.3.3}$$

Then, by Itô's formula, we obtain

$$\delta Y_{t_1} = \mathbb{E}_{t_1}^{\mathbb{P}^{'}}\left[\int_{t_1}^{t_2} M_t d\delta K_t^{\mathbb{P}^{'}}\right]. \tag{5.3.4}$$

Let us now prove that the process $K^{\mathbb{P}'}-k^{\mathbb{P}'}$ is non-decreasing. By the minimum condtion (5.2.3), it is clear that it is acually a $\mathbb{P}'$-submartingale. Let us apply the Doob-Meyer decomposition under $\mathbb{P}'$, we get the existence of a $\mathbb{P}'$-martingale $N^{\mathbb{P}'}$ and a non-decreasing process $P^{\mathbb{P}'}$, both null at 0, such that

$$K_t^{\mathbb{P}'}-k_t^{\mathbb{P}'}=N_t^{\mathbb{P}'}+P_t^{\mathbb{P}'},\ \mathbb{P}'-a.s.$$

Then, since we know that all the probability measures in $\mathcal{P}_H^\kappa$ satisfy the martingale representation property, the martingale $N^{\mathbb{P}'}$ is continuous. Besides, by the above equation, it also has finite variation. Hence, we have $N^{\mathbb{P}'}=0$, and the result follows.

Returning back to (5.3.4), we can now write

$$\begin{aligned}
\delta Y_{t_1} &\leq \mathbb{E}_{t_1}^{\mathbb{P}'}\left[\sup_{t_1\leq t\leq t_2}(M_t)\left(\delta K_{t_2}^{\mathbb{P}'}-\delta K_{t_1}^{\mathbb{P}'}\right)\right]\\
&\leq \left(\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\sup_{t_1\leq t\leq t_2}(M_t)^3\right]\right)^{1/3}\left(\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\left(\delta K_{t_2}^{\mathbb{P}'}-\delta K_{t_1}^{\mathbb{P}'}\right)^{3/2}\right]\right)^{2/3}\\
&\leq C(C_{t_1}^{\mathbb{P}})^{1/3}\left(\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\delta K_{t_2}^{\mathbb{P}'}-\delta K_{t_1}^{\mathbb{P}'}\right]\right)^{1/3},\ \mathbb{P}-a.s.
\end{aligned}$$

Taking the essential infimum on both sides finishes the proof.

(iii) It remains to show that the estimate for $C_{t_1}^{\mathbb{P}}$ holds. But by definition, we clearly have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'}-k_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'}+k_{t_1}^{\mathbb{P}'}\right)^2\right] &\leq C\left(\|Y\|^2_{\mathbb{D}_H^{2,\kappa}}+\|Z\|^2_{\mathbb{H}_H^{2,\kappa}}+\phi_H^{2,\kappa}\right)\\
&\quad + C\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|y_t^{\mathbb{P}}\right|^2+\int_0^T\left|\widehat{a}_t^{1/2}z_s^{\mathbb{P}}\right|^2ds\right]\\
&< +\infty,
\end{aligned}$$

since the last term on the right-hand side is finite thanks to the integrability assumed on ξ and $\widehat{F}^0$.

Then we can proceed exactly as in the proof of Theorem 4.4 in [107].

□

Remark 5.3.1. *Let us now justify the minimum condition (5.2.3). Assume for the sake of clarity that the generator $\widehat{F}$ is equal to 0. By the above Theorem, we know that if there exists a solution to the 2RBSDE (5.2.1), then the process Y has to satisfy the representation (5.3.1). Therefore, we have a natural candidate for a possible solution of the 2RBSDE. Now, assume that we could construct such a process Y satisfying the representation (5.3.1) and which has the decomposition (5.2.1). Then, taking conditional expectations in $Y-y^{\mathbb{P}}$, we end up with exactly the minimum condition (5.2.3).*

Finally, the following comparison Theorem follows easily from the classical one for RBSDEs (see for instance Theorem 5.2 in [47] and Theorem 3.4 in [74]) and the representation (5.3.1).

Theorem 5.3.2. *Let (Y,Z) and (Y',Z') be the solutions of 2RBSDEs with terminal conditions ξ and $\xi^{'}$, lower obstacles S and $S^{'}$ and generators $\widehat{F}$ and $\widehat{F}^{'}$ respectively, and let $(y^{\mathbb{P}},z^{\mathbb{P}},k^{\mathbb{P}})$ and $(y'^{\mathbb{P}},z'^{\mathbb{P}},k'^{\mathbb{P}})$ the solutions of the associated RBSDEs. Assume that they both verify our Assumptions 5.2.1 and 5.2.2 and that we have*

- $\xi \leq \xi^{'}$, $\mathcal{P}_H^\kappa - q.s.$
- $\widehat{F}_t(y_t'^{\mathbb{P}}, z_t'^{\mathbb{P}}) \leq \widehat{F}_t^{'}(y_t'^{\mathbb{P}}, z_t'^{\mathbb{P}})$, $\mathbb{P}-a.s.$, *for all* $\mathbb{P} \in \mathcal{P}_H^\kappa$.
- $S_t \leq S_t^{'}$, $\mathcal{P}_H^\kappa - q.s.$

Then $Y \leq Y'$, $\mathcal{P}_H^\kappa - q.s.$

Remark 5.3.2. *Note that in our context, in the above comparison Theorem, even if the obstacles S and $S^{'}$ are identical, we cannot compare the increasing processes $K^{\mathbb{P}}$ and $K'^{\mathbb{P}}$. This is due to the fact that the processes $K^{\mathbb{P}}$ do not satisfy the Skorohod condition.*

5.3.2 Some properties of the solution

Now that we have proved the representation (5.3.1), we can show, as in the classical framework, that the solution Y of the 2RBSDE is linked to an optimal stopping problem

Proposition 5.3.1. *Let (Y,Z) be the solution to the above 2RBSDE (5.2.1). Then for each $t \in [0,T]$ and for all $\mathbb{P} \in \mathcal{P}_H^\kappa$*

$$Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}^{'} \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}^{'}}\left[\int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}^{'}}, z_s^{\mathbb{P}^{'}})ds + S_\tau 1_{\tau<T} + \xi 1_{\tau=T}\right], \ \mathbb{P}-a.s. \tag{5.3.5}$$

$$= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}}\left[\int_t^\tau \widehat{F}_s(Y_s,Z_s)ds + A_\tau^{\mathbb{P}} - A_t^{\mathbb{P}} + S_\tau 1_{\tau<T} + \xi 1_{\tau=T}\right], \ \mathbb{P}-a.s. \tag{5.3.6}$$

where $\mathcal{T}_{t,T}$ is the set of all stopping times valued in $[t,T]$ and where $A_t^{\mathbb{P}} := \int_0^t 1_{Y_{s^-}>S_{s^-}} dK_s^{\mathbb{P}}$ is the part of $K^{\mathbb{P}}$ which only increases when $Y_{s^-} > S_{s^-}$.

Proof. By Proposition 3.1 in [74], we know that for all $\mathbb{P} \in \mathcal{P}_H^\kappa$

$$y_t^{\mathbb{P}} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}}\left[\int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}})ds + S_\tau 1_{\tau<T} + \xi 1_{\tau=T}\right], \ \mathbb{P}-a.s.$$

Then the first equality is a simple consequence of the representation formula (5.3.1). For the second one, we proceed exactly as in the proof of Proposition 3.1 in [74]. Fix some $\mathbb{P} \in \mathcal{P}_H^\kappa$ and some $t \in [0,T]$. Let $\tau \in \mathcal{T}_{t,T}$. We obtain by taking conditional expectation in (5.2.1)

$$\begin{aligned} Y_t &= \mathbb{E}_t^{\mathbb{P}}\left[Y_\tau + \int_t^\tau \widehat{F}_s(Y_s,Z_s)ds + K_\tau^{\mathbb{P}} - K_t^{\mathbb{P}}\right] \\ &\geq \mathbb{E}_t^{\mathbb{P}}\left[\int_t^\tau \widehat{F}_s(Y_s,Z_s)ds + S_\tau 1_{\tau<T} + \xi 1_{\tau=T} + A_\tau^{\mathbb{P}} - A_t^{\mathbb{P}}\right]. \end{aligned}$$

This implies that

$$Y_t \geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}}\left[\int_t^\tau \widehat{F}_s(Y_s,Z_s)ds + A_\tau^{\mathbb{P}} - A_t^{\mathbb{P}} + S_\tau 1_{\tau<T} + \xi 1_{\tau=T}\right], \ \mathbb{P}-a.s.$$

Fix some $\varepsilon > 0$ and define the stopping time $D_t^{\mathbb{P},\varepsilon} := \inf\{u \geq t,\ Y_u \leq S_u + \varepsilon,\ \mathbb{P} - a.s.\} \wedge T$. It is clear by definition that on the set $\left\{D_t^{\mathbb{P},\varepsilon} < T\right\}$, we have $Y_{D_t^{\mathbb{P},\varepsilon}} \leq S_{D_t^{\mathbb{P},\varepsilon}} + \varepsilon$. Similarly, on the set $\left\{D_t^{\mathbb{P},\varepsilon} = T\right\}$, we have $Y_s > S_s + \varepsilon$, for all $t \leq s \leq T$. Hence, for all $s \in [t, D_t^{\mathbb{P},\varepsilon}]$, we have $Y_{s^-} > S_{s^-}$. This implies that $K_{D_t^{\mathbb{P},\varepsilon}} - K_t = A_{D_t^{\mathbb{P},\varepsilon}} - A_t$, and therefore

$$Y_t \leq \mathbb{E}_t^{\mathbb{P}}\left[\int_t^{D_t^{\mathbb{P},\varepsilon}} \widehat{F}_s(Y_s, Z_s)ds + A^{\mathbb{P}}_{D_t^{\mathbb{P},\varepsilon}} - A_t^{\mathbb{P}} + S_{D_t^{\mathbb{P},\varepsilon}} 1_{D_t^{\mathbb{P},\varepsilon} < T} + \xi 1_{D_t^{\mathbb{P},\varepsilon} = T}\right] + \varepsilon,$$

which ends the proof by arbitrariness of ε. □

Then, if we have more information on the obstacle S, we can give a more explicit representation for the processes $K^{\mathbb{P}}$, just as in the classical case (see Proposition 4.2 in [43]).

Assumption 5.3.1. *S is a semi-martingale of the form*

$$S_t = S_0 + \int_0^t U_s ds + \int_0^t V_s dB_s + C_t,\ \mathcal{P}_H^\kappa - q.s.$$

where C is càdlàg process of integrable variation such that the measure dC_t is singular with respect to the Lebesgue measure dt and which admits the following decomposition

$$C_t = C_t^+ - C_t^-,$$

where C^+ and C^- are non-decreasing processes. Besides, U and V are respectively $\mathbb{R}$ and $\mathbb{R}^d$-valued $\mathcal{F}_t$ progressively measurable processes such that

$$\int_0^T (|U_t| + |V_t|^2)dt + C_T^+ + C_T^- \leq +\infty,\ \mathcal{P}_H^\kappa - q.s.$$

Proposition 5.3.2. *Let Assumptions 5.2.1, 5.2.2 and 5.3.1 hold. Let (Y, Z) be the solution to the 2RBSDE (5.2.1), then*

$$Z_t = V_t,\ dt \times \mathcal{P}_H^\kappa - q.s. \text{ on the set } \{Y_{t^-} = S_{t^-}\}, \tag{5.3.7}$$

and there exists a progressively measurable process $(\alpha_t^{\mathbb{P}})_{0\leq t\leq T}$ such that $0 \leq \alpha \leq 1$ and

$$1_{Y_{t^-} = S_{t^-}} dK_t^{\mathbb{P}} = \alpha_t^{\mathbb{P}} 1_{Y_{t^-} = S_{t^-}} \left(\left[\widehat{F}_t(S_t, V_t) + U_t\right]^- dt + dC_t^-\right).$$

Proof. First, for all $\mathbb{P} \in \mathcal{P}_H^\kappa$, the following holds $\mathbb{P} - a.s.$

$$Y_t - S_t = Y_0 - S_0 - \int_0^t \left(\widehat{F}_s(Y_s, Z_s) + U_s\right) ds + \int_0^t (Z_s - V_s)dB_s - K_t^{\mathbb{P}} - C_t^+ + C_t^-.$$

Now if we denote L_t the local time at 0 of $Y_t - S_t$, then by Itô-Tanaka formula under each $\mathbb{P}$

$$\begin{aligned}(Y_t - S_t)^+ = (Y_0 - S_0)^+ &- \int_0^t 1_{Y_{s^-} > S_{s^-}} \left(\widehat{F}_s(Y_s, Z_s) + U_s\right) ds + \int_0^t 1_{Y_{s^-} > S_{s^-}} (Z_s - V_s) dB_s \\ &- \int_0^t 1_{Y_{s^-} > S_{s^-}} d(K_t^{\mathbb{P}} + C_t^+ - C_t^-) + \frac{1}{2} L_t \\ &+ \sum_{0 \le s \le t} (Y_s - S_s)^+ - (Y_{s^-} - S_{s^-})^+ - 1_{Y_{s^-} > S_{s^-}} \Delta(Y_s - S_s) \\ = (Y_0 - S_0)^+ &- \int_0^t 1_{Y_{s^-} > S_{s^-}} \left(\widehat{F}_s(Y_s, Z_s) + U_s\right) ds + \int_0^t 1_{Y_{s^-} > S_{s^-}} (Z_s - V_s) dB_s \\ &- \int_0^t 1_{Y_{s^-} > S_{s^-}} d(K_t^{\mathbb{P}} + C_t^+ - C_t^-) + \frac{1}{2} L_t \\ &+ \sum_{0 \le s \le t} (Y_s - S_s)^+ - (Y_{s^-} - S_{s^-})^+ - 1_{Y_{s^-} > S_{s^-}} \Delta(Y_s - S_s).\end{aligned}$$

However, we have $(Y_t - S_t)^+ = Y_t - S_t$, hence by identification of the martingale part

$$1_{Y_{t^-} = S_{t^-}} (Z_t - V_t) dB_t = 0, \ \mathcal{P}_H^\kappa - q.s.$$

from which the first statement is clear.

Since the jump part is obviously positive and $L$and C^+ are non-decreasing processes, we also have

$$1_{Y_{t^-} = S_{t^-}} dK_t^{\mathbb{P}} \le -1_{Y_{t^-} = S_{t^-}} \left(\left(\widehat{F}_t(Y_t, Z_t) + U_t\right) dt - dC_t^- \right).$$

The second statement follows then easily. □

5.3.3 A priori estimates

We conclude this section by showing some a priori estimates which will be useful in the sequel.

Theorem 5.3.3. *Let Assumptions 5.2.1 and 5.2.2 hold. Assume $\xi \in \mathbb{L}_H^{2,\kappa}$ and $(Y, Z, K) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{I}_H^{2,\kappa}$ is a solution to the 2RBSDE (5.2.1). Let $\{(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P}})\}_{\mathbb{P} \in \mathcal{P}_H^\kappa}$ be the solutions of the corresponding BSDEs (5.2.2). Then, there exists a constant C_κ depending only on κ, T and the Lipschitz constant of $\widehat{F}$ such that*

$$\|Y\|^2_{\mathbb{D}_H^{2,\kappa}} + \|Z\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}} \left[(K_T^{\mathbb{P}})^2 \right] \le C \left(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + \phi_H^{2,\kappa} + \psi_H^{2,\kappa} \right),$$

and

$$\sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \left\{ \left\| y^{\mathbb{P}} \right\|^2_{\mathbb{D}^2(\mathbb{P})} + \left\| z^{\mathbb{P}} \right\|^2_{\mathbb{H}^2(\mathbb{P})} + \left\| k^{\mathbb{P}} \right\|^2_{\mathbb{I}^2(\mathbb{P})} \right\} \le C \left(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + \phi_H^{2,\kappa} + \psi_H^{2,\kappa} \right).$$

Proof. By Lemma 2 in [61], we know that there exists a constant C_κ depending only on κ, T and the Lipschitz constant of $\widehat{F}$, such that for all $\mathbb{P}$

$$\left| y_t^{\mathbb{P}} \right| \le C_\kappa \mathbb{E}_t^{\mathbb{P}} \left[|\xi|^\kappa + \int_t^T \left| \widehat{F}_s^0 \right|^\kappa ds + \sup_{t \le s \le T} (S_s^+)^\kappa \right]. \tag{5.3.8}$$

Let us note immediately, that in [61], the result is given with an expectation and not a conditional expectation, and more importantly that the process considered are continuous. However, the generalization is easy for the conditional expectation. As far as the jumps are concerned, their proof only uses Itô's formula for smooth convex functions, for which the jump part can been taken care of easily in the estimates. Then, one can follow exactly their proof to get our result.

This immediately provides the estimate for $y^{\mathbb{P}}$. Now by definition of our norms, we get from (5.3.8) and the representation formula (5.3.1) that

$$\|Y\|^2_{\mathbb{D}^{2,\kappa}_H} \leq C_\kappa \left(\|\xi\|^2_{\mathbb{L}^{2,\kappa}_H} + \phi^{2,\kappa}_H + \psi^{2,\kappa}_H\right). \tag{5.3.9}$$

Now apply Itô's formula to $|Y|^2$ under each $\mathbb{P} \in \mathcal{P}^\kappa_H$. We get as usual for every $\varepsilon > 0$

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt\right] &\leq C\mathbb{E}^{\mathbb{P}}\left[|\xi|^2 + \int_0^T |Y_t|\left(\left|\widehat{F}^0_t\right| + |Y_t| + \left|\widehat{a}_t^{1/2} Z_t\right|\right) dt\right] \\
&\quad + \mathbb{E}^{\mathbb{P}}\left[\int_0^T |Y_t|\, dK^{\mathbb{P}}_t\right] \\
&\leq C\left(\|\xi\|_{\mathbb{L}^{2,\kappa}_H} + \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} |Y_t|^2 + \left(\int_0^T \left|\widehat{F}^0_t\right| dt\right)^2\right]\right) \\
&\quad + \varepsilon\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt + \left|K^{\mathbb{P}}_T\right|^2\right] + \frac{C^2}{\varepsilon}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} |Y_t|^2\right].
\end{aligned} \tag{5.3.10}$$

Then by definition of our 2RBSDE, we easily have

$$\mathbb{E}^{\mathbb{P}}\left[\left|K^{\mathbb{P}}_T\right|^2\right] \leq C_0\mathbb{E}^{\mathbb{P}}\left[|\xi|^2 + \sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt + \left(\int_0^T \left|\widehat{F}^0_t\right| dt\right)^2\right], \tag{5.3.11}$$

for some constant C_0, independent of ε.

Now set $\varepsilon := (2(1+C_0))^{-1}$ and plug (5.3.11) in (5.3.10). One then gets

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt\right] \leq C\mathbb{E}^{\mathbb{P}}\left[|\xi|^2 + \sup_{0\leq t\leq T} |Y_t|^2 + \left(\int_0^T \left|\widehat{F}^0_t\right| dt\right)^2\right].$$

From this and the estimate for Y, we immediately obtain

$$\|Z\|_{\mathbb{H}^{2,\kappa}_H} \leq C\left(\|\xi\|^2_{\mathbb{L}^{2,\kappa}_H} + \phi^{2,\kappa}_H + \psi^{2,\kappa}_H\right).$$

Then the estimate for $K^{\mathbb{P}}$ comes from (5.3.11). The estimates for $z^{\mathbb{P}}$ and $k^{\mathbb{P}}$ can be proved similarly.

□

Theorem 5.3.4. *Let Assumptions 5.2.1 and 5.2.2 hold. For $i = 1, 2$, let $(Y^i, Z^i, \{K^{\mathbb{P},i}, \mathbb{P} \in \mathcal{P}^\kappa_H\})$ be the solutions to the 2RBSDE (5.2.1) with terminal condition ξ^i and lower obstacle S. Then, there exists a constant C_κ depending only on κ, T and the Lipschitz constant of F such that*

$$\left\|Y^1 - Y^2\right\|_{\mathbb{D}^{2,\kappa}_H} \leq C\left\|\xi^1 - \xi^2\right\|_{\mathbb{L}^{2,\kappa}_H},$$

and

$$\begin{aligned}
&\left\|Z^1 - Z^2\right\|^2_{\mathbb{H}^{2,\kappa}_H} + \sup_{\mathbb{P}\in\mathcal{P}^\kappa_H} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|K^{\mathbb{P},1}_t - K^{\mathbb{P},2}_t\right|^2\right] \\
&\leq C\left\|\xi^1 - \xi^2\right\|_{\mathbb{L}^{2,\kappa}_H}\left(\left\|\xi^1\right\|_{\mathbb{L}^{2,\kappa}_H} + \left\|\xi^1\right\|_{\mathbb{L}^{2,\kappa}_H} + (\phi^{2,\kappa}_H)^{1/2} + (\psi^{2,\kappa}_H)^{1/2}\right).
\end{aligned}$$

Proof. As in the previous Proposition, we can follow the proof of Lemma 3 in [61], to obtain that there exists a constant C_κ depending only on κ, T and the Lipschitz constant of $\widehat{F}$, such that for all $\mathbb{P}$

$$\left|y_t^{\mathbb{P},1}-y_t^{\mathbb{P},2}\right|\leq C_\kappa\mathbb{E}_t^{\mathbb{P}}\left[\left|\xi^1-\xi^2\right|^\kappa\right]. \tag{5.3.12}$$

Now by definition of our norms, we get from (5.3.12) and the representation formula (5.3.1) that

$$\left\|Y^1-Y^2\right\|^2_{\mathbb{D}^{2,\kappa}_H}\leq C_\kappa\left\|\xi^1-\xi^2\right\|^2_{\mathbb{L}^{2,\kappa}_H}. \tag{5.3.13}$$

Applying Itô's formula to $\left|Y^1-Y^2\right|^2$, under each $\mathbb{P}\in\mathcal{P}_H^\kappa$, leads to

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\int_0^T\left|\widehat{a}_t^{\frac12}(Z_t^1-Z_t^2)\right|^2dt\right]&\leq C\mathbb{E}^{\mathbb{P}}\left[\left|\xi^1-\xi^2\right|^2\right]+\mathbb{E}^{\mathbb{P}}\left[\int_0^T\left|Y_t^1-Y_t^2\right|d(K_t^{\mathbb{P},1}-K_t^{\mathbb{P},2})\right]\\&\quad+C\mathbb{E}^{\mathbb{P}}\left[\int_0^T\left|Y_t^1-Y_t^2\right|\left(\left|Y_t^1-Y_t^2\right|+|\widehat{a}_t^{\frac12}(Z_t^1-Z_t^2)|\right)dt\right]\\&\leq C\left(\left\|\xi^1-\xi^2\right\|^2_{\mathbb{L}^{2,\kappa}_H}+\left\|Y^1-Y^2\right\|^2_{\mathbb{D}^{2,\kappa}_H}\right)\\&\quad+\frac12\mathbb{E}^{\mathbb{P}}\left[\int_0^T\left|\widehat{a}_t^{1/2}(Z_t^1-Z_t^2)\right|^2dt\right]\\&\quad+C\left\|Y^1-Y^2\right\|_{\mathbb{D}^{2,\kappa}_H}\left(\mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^2\left(K_T^i\right)^2\right]\right)^{\frac12}.\end{aligned}$$

The estimate for (Z^1-Z^2) is now obvious from the above inequality and the estimates of Proposition 5.3.3.

Finally the estimate for the difference of the increasing processes is obvious by definition. □

5.4 A direct existence argument

In the articles [107], the main tool to prove existence of a solution is the so called regular conditional probability distributions of Stroock and Varadhan [111]. Indeed, it allows to construct a solution to the 2BSDE when the terminal condition belongs to the space $\mathrm{UC_b}(\Omega)$. In this section we will generalize their approach to the reflected case. We refer to Chapter 3 for the notations.

5.4.1 Existence when ξ is in $\mathrm{UC_b}(\Omega)$

When ξ is in $\mathrm{UC_b}(\Omega)$, we know that there exists a modulus of continuity function ρ for ξ, F and S in ω. Then, for any $0\leq t\leq s\leq T$, $(y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$ and $\omega,\omega'\in\Omega$, $\tilde\omega\in\Omega^t$,

$$\left|\xi^{t,\omega}(\tilde\omega)-\xi^{t,\omega'}(\tilde\omega)\right|\leq\rho\left(\left\|\omega-\omega'\right\|_t\right),\ \left|\widehat{F}_s^{t,\omega}(\tilde\omega,y,z)-\widehat{F}_s^{t,\omega'}(\tilde\omega,y,z)\right|\leq\rho\left(\left\|\omega-\omega'\right\|_t\right)$$

$$\left|S_s^{t,\omega}(\tilde\omega)-S_s^{t,\omega'}(\tilde\omega)\right|\leq\rho\left(\left\|\omega-\omega'\right\|_t\right).$$

We then define for all $\omega\in\Omega$

$$\Lambda(\omega):=\sup_{0\leq s\leq t}\Lambda_t(\omega), \tag{5.4.1}$$

where

$$\Lambda_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}_H^t}\left(\mathbb{E}^{\mathbb{P}}\left[\left|\xi^{t,\omega}\right|^2 + \int_t^T |\widehat{F}_s^{t,\omega}(0,0)|^2 ds + \left(\sup_{t\le s\le T}(S_s^{t,\omega})^+\right)^2\right]\right)^{1/2}.$$

Now since $\widehat{F}^{t,\omega}$ is also uniformly continuous in ω, we have

$$\Lambda(\omega) < \infty \text{ for some } \omega\in\Omega \text{ iff it holds for all } \omega\in\Omega. \tag{5.4.2}$$

Moreover, when Λ is finite, it is uniformly continuous in ω under the $\mathbb{L}^\infty$-norm and is therefore $\mathcal{F}_T$-measurable.

Now, by Assumption 5.2.2, we have

$$\Lambda_t(\omega) < \infty \text{ for all } (t,\omega)\in[0,T]\times\Omega. \tag{5.4.3}$$

To prove existence, we define the following value process V_t pathwise

$$V_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}_H^t}\mathcal{Y}_t^{\mathbb{P},t,\omega}(T,\xi), \text{ for all } (t,\omega)\in[0,T]\times\Omega, \tag{5.4.4}$$

where, for any $(t_1,\omega)\in[0,T]\times\Omega$, $\mathbb{P}\in\mathcal{P}_H^{t_1}$, $t_2\in[t_1,T]$, and any $\mathcal{F}_{t_2}$-measurable $\eta\in\mathbb{L}^\infty(\mathbb{P})$, we denote $\mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2,\eta) := y_{t_1}^{\mathbb{P},t_1,\omega}$, where $\left(y^{\mathbb{P},t_1,\omega}, z^{\mathbb{P},t_1,\omega}, k^{\mathbb{P},t_1,\omega}\right)$ is the solution of the following RBSDE with lower obstacle $S^{t_1,\omega}$ on the shifted space Ω^{t_1} under $\mathbb{P}$

$$y_s^{\mathbb{P},t_1,\omega} = \eta^{t_1,\omega} + \int_s^{t_2}\widehat{F}_r^{t_1,\omega}\left(y_r^{\mathbb{P},t_1,\omega}, z_r^{\mathbb{P},t_1,\omega}\right)dr - \int_s^{t_2} z_r^{\mathbb{P},t_1,\omega}dB_r^{t_1} + k_{t_2}^{\mathbb{P},t_1,\omega} - k_{t_1}^{\mathbb{P},t_1,\omega} \tag{5.4.5}$$

$$y_t^{\mathbb{P},t_1,\omega} \ge S_t^{t_1,\omega},\ \mathbb{P}-a.s.$$

$$\int_{t_1}^{t_2}\left(y_{s^-}^{\mathbb{P},t_1,\omega} - S_{s^-}^{t_1,\omega}\right)dk_s^{\mathbb{P},t_1,\omega} = 0,\ \mathbb{P}-a.s. \tag{5.4.6}$$

In view of the Blumenthal zero-one law, $\mathcal{Y}_t^{\mathbb{P},t,\omega}(T,\xi)$ is constant for any given (t,ω) and $\mathbb{P}\in\mathcal{P}_H^t$. Moreover, since $\omega_0 = 0$ for all $\omega\in\Omega$, it is clear that, for the $y^{\mathbb{P}}$ defined in (5.2.2),

$$\mathcal{Y}^{\mathbb{P},0,\omega}(t,\eta) = y^{\mathbb{P}}(t,\eta) \text{ for all } \omega\in\Omega.$$

Lemma 5.4.1. *Let Assumptions 5.2.1 and 5.2.2 hold and consider some ξ in $\mathrm{UC_b}(\Omega)$. Then for all $(t,\omega)\in[0,T]\times\Omega$ we have $|V_t(\omega)|\le C(1+\Lambda_t(\omega))$. Moreover, for all $(t,\omega,\omega')\in[0,T]\times\Omega^2$, $|V_t(\omega)-V_t(\omega')|\le C\rho(\|\omega-\omega'\|_t)$. Consequently, V_t is $\mathcal{F}_t$-measurable for every $t\in[0,T]$.*

Proof. (i) For each $(t,\omega)\in[0,T]\times\Omega$ and $\mathbb{P}\in\mathcal{P}_H^t$, let α be some positive constant which will be fixed later and let $\eta\in(0,1)$. By Itô's formula we have, since $\widehat{F}$ is uniformly Lipschitz and since by (5.4.6) $\int_t^T e^{\alpha s}\left(y_{s^-}^{\mathbb{P},t,\omega} - S_{s^-}^{t,\omega}\right)dk_s^{\mathbb{P},t,\omega} = 0$

$$
\begin{aligned}
e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}\right|^2 &+ \int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},t,\omega}\right|^2 ds \leq e^{\alpha T}\left|\xi^{t,\omega}\right|^2 + 2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}\right|\left|\widehat{F}_s^{t,\omega}(0)\right| ds \\
&+ 2C\int_t^T \left|y_s^{\mathbb{P},t,\omega}\right|\left(\left|y_s^{\mathbb{P},t,\omega}\right| + \left|(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},t,\omega}\right|\right) ds - 2\int_t^T e^{\alpha s} y_{s^-}^{\mathbb{P},t,\omega} z_s^{\mathbb{P},t,\omega} dB_s^t \\
&+ 2\int_t^T e^{\alpha s} S_{s^-}^{t,\omega} dk_s^{\mathbb{P},t,\omega} - \alpha\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}\right|^2 ds \\
&\leq e^{\alpha T}\left|\xi^{t,\omega}\right|^2 + \int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(0)\right|^2 ds - 2\int_t^T e^{\alpha s} y_{s^-}^{\mathbb{P},t,\omega} z_s^{\mathbb{P},t,\omega} dB_s^t + \eta\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},n}\right|^2 ds \\
&+ \left(2C + C^2 + \frac{C^2}{\eta} - \alpha\right)\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}\right|^2 ds + 2\sup_{t\leq s\leq T} e^{\alpha s}(S_s^{t,\omega})^+(k_T^{\mathbb{P},t,\omega} - k_t^{\mathbb{P},t,\omega}).
\end{aligned}
$$

Now choose α such that $\nu := \alpha - 2C - C^2 - \frac{C^2}{\eta} \geq 0$. We obtain for all $\varepsilon > 0$

$$
\begin{aligned}
e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}\right|^2 + (1-\eta)\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},t,\omega}\right|^2 ds \leq\ & e^{\alpha T}\left|\xi^{t,\omega}\right|^2 + \int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(0,0)\right|^2 ds \\
&+ \frac{1}{\varepsilon}\left(\sup_{t\leq s\leq T} e^{\alpha s}(S_s^{t,\omega})^+\right)^2 + \varepsilon(k_T^{\mathbb{P},t,\omega} - k_t^{\mathbb{P},t\omega})^2 \\
&- 2\int_t^T e^{\alpha s} y_{s^-}^{\mathbb{P},t,\omega} z_s^{\mathbb{P},t,\omega} dB_s^t. \qquad (5.4.7)
\end{aligned}
$$

Taking expectation in (5.4.7) yields

$$
\left|y_t^{\mathbb{P},t,\omega}\right|^2 + (1-\eta)\mathbb{E}^{\mathbb{P}}\left[\int_t^T \left|(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},t,\omega}\right|^2 ds\right] \leq C\Lambda_t(\omega)^2 + \varepsilon\mathbb{E}^{\mathbb{P}}\left[(k_T^{\mathbb{P},t,\omega} - k_t^{\mathbb{P},t,\omega})^2\right].
$$

Now by definition, we also have for some constant C_0 independent of ε

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[(k_T^{\mathbb{P},t,\omega} - k_t^{\mathbb{P},t,\omega})^2\right] &\leq C_0\mathbb{E}^{\mathbb{P}}\left[\left|\xi^{t,\omega}\right|^2 + \int_t^T \left|\widehat{F}_s^{t,\omega}(0,0)\right|^2 ds + \int_t^T \left|y_s^{\mathbb{P},t,\omega}\right|^2 ds\right] \\
&\quad + \mathbb{E}^{\mathbb{P}}\left[\int_t^T \left|(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},t,\omega}\right|^2 ds\right] \\
&\leq C_0\left(\Lambda_t(\omega) + \mathbb{E}^{\mathbb{P}}\left[\int_t^T \left|y_s^{\mathbb{P},t,\omega}\right|^2 ds + \int_t^T \left|(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},t,\omega}\right|^2 ds\right]\right).
\end{aligned}
$$

Choosing η small enough and $\varepsilon = \frac{1}{2C_0}$, Gronwall inequality then implies

$$
\left|y_t^{\mathbb{P},t,\omega}\right|^2 \leq C(1 + \Lambda_t(\omega)).
$$

The result then follows from arbitrainess of $\mathbb{P}$.

(ii) The proof is exactly the same as above, except that one has to use uniform continuity in ω of $\xi^{t,\omega}$, $\widehat{F}^{t,\omega}$ and $S^{t,\omega}$. Indeed, for each $(t,\omega) \in [0,T] \times \Omega$ and $\mathbb{P} \in \mathcal{P}_H^t$, let α be some positive constant which will be fixed later and let $\eta \in (0,1)$. By Itô's formula we have, since $\widehat{F}$ is uniformly Lipschitz

$$
\begin{aligned}
&e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}-y_t^{\mathbb{P},t,\omega'}\right|^2+\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|^2 ds\leq e^{\alpha T}\left|\xi^{t,\omega}-\xi^{t,\omega'}\right|^2\\
&\quad+2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|\left(\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|+\left|(\widehat{a}_s^t)^{\frac{1}{2}}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|\right)ds\\
&\quad+2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|\left|\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})-\widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})\right|ds\\
&\quad+2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})d(k_s^{\mathbb{P},t,\omega}-k_s^{\mathbb{P},t,\omega'})-\alpha\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|^2 ds\\
&\quad-2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})dB_s^t\\
&\leq e^{\alpha T}\left|\xi^{t,\omega}-\xi^{t,\omega'}\right|^2+\int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})-\widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})\right|^2 ds\\
&\quad+\left(2C+C^2+\frac{C^2}{\eta}-\alpha\right)\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|^2 ds\\
&\quad+\eta\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|^2 ds\\
&\quad-2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})dB_s^t\\
&\quad+2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})d(k_s^{\mathbb{P},t,\omega}-k_s^{\mathbb{P},t,\omega'}).
\end{aligned}
$$

By the Skorohod condition (5.4.6), we also have

$$
\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})d(k_s^{\mathbb{P},t,\omega}-k_s^{\mathbb{P},t,\omega'})\leq\int_t^T e^{\alpha s}(S_{s^-}^{t,\omega}-S_{s^-}^{t,\omega'})d(k_s^{\mathbb{P},t,\omega}-k_s^{\mathbb{P},t,\omega'}).
$$

Now choose α such that $\nu:=\alpha-2C-C^2-\frac{C^2}{\eta}\geq 0$. We obtain for all $\varepsilon>0$

$$
\begin{aligned}
&e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}-y_t^{\mathbb{P},t,\omega'}\right|^2+(1-\eta)\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|^2 ds\\
&\leq e^{\alpha T}\left|\xi^{t,\omega}-\xi^{t,\omega'}\right|^2+\int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})-\widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})\right|^2 ds\\
&\quad+\frac{1}{\varepsilon}\left(\sup_{t\leq s\leq T}e^{\alpha s}(S_s^{t,\omega}-S_s^{t,\omega'})^+\right)^2+\varepsilon(k_T^{\mathbb{P},t,\omega}-k_T^{\mathbb{P},t,\omega'}-k_t^{\mathbb{P},t\omega}+k_t^{\mathbb{P},t,\omega'})^2\\
&\quad-2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})dB_s^t.
\end{aligned}\tag{5.4.8}
$$

The end of the proof is then similar to the previous step, using the uniform continuity in ω of ξ, F and S. □

Then, we show the same dynamic programming principle as Proposition 4.7 in [108]

Proposition 5.4.1. *Under Assumptions 5.2.1, 5.2.2 and for $\xi\in\mathrm{UC_b}(\Omega)$, we have for all $0\leq t_1<t_2\leq T$ and for all $\omega\in\Omega$*

$$
V_{t_1}(\omega)=\sup_{\mathbb{P}\in\mathcal{P}_H^{t_1}}\mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2,V_{t_2}^{t_1,\omega}).
$$

The proof is almost the same as the proof in [108], but we give it for the convenience of the reader.

Proof. Without loss of generality, we can assume that $t_1 = 0$ and $t_2 = t$. Thus, we have to prove

$$V_0(\omega) = \sup_{\mathbb{P}\in\mathcal{P}_H} \mathcal{Y}_0^{\mathbb{P}}(t, V_t).$$

Denote $(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P}}) := (\mathcal{Y}^{\mathbb{P}}(T,\xi), \mathcal{Z}^{\mathbb{P}}(T,\xi), \mathcal{K}^{\mathbb{P}}(T,\xi))$

(i) For any $\mathbb{P} \in \mathcal{P}_H$, we know by Lemma 4.3 in [108], that for $\mathbb{P}-a.e.\ \omega \in \Omega$, the r.c.p.d. $\mathbb{P}^{t,\omega} \in \mathcal{P}_H^t$. Now thanks to the paper of Xu and Qian [92], we know that the solution of reflected BSDEs with Lipschitz generator can be constructed via Picard iteration. Thus, it means that at each step of the iteration, the solution can be formulated as a conditional expectation under $\mathbb{P}$. By the properties of the r.p.c.d., this entails that

$$y_t^{\mathbb{P}}(\omega) = \mathcal{Y}_t^{\mathbb{P}^{t,\omega},t,\omega}(T,\xi), \text{ for } \mathbb{P}-a.e.\ \omega \in \Omega. \tag{5.4.9}$$

Hence, by definition of V_t and the comparison principle for RBSDEs, we get that $y_0^{\mathbb{P}} \leq \mathcal{Y}_0^{\mathbb{P}}(t, V_t)$. By arbitrariness of $\mathbb{P}$, this leads to

$$V_0(\omega) \leq \sup_{\mathbb{P}\in\mathcal{P}_H} \mathcal{Y}_0^{\mathbb{P}}(t, V_t).$$

(ii) For the other inequality, we proceed as in [108]. Let $\mathbb{P} \in \mathcal{P}_H$ and $\varepsilon > 0$. By separability of Ω, there exists a partition $(E_t^i)_{i\geq 1} \subset \mathcal{F}_t$ such that $\|\omega - \omega'\|_t \leq \varepsilon$ for any i and any $\omega, \omega' \in E_t^i$. Now for each i, fix an $\widehat{\omega}_i \in E_t^i$ and let $\mathbb{P}_t^i$ be an $\varepsilon-$optimizer of $V_t(\widehat{\omega}_i)$.

Now if we define for each $n \geq 1$, $\mathbb{P}^n := \mathbb{P}^{n,\varepsilon}$ by

$$\mathbb{P}^n(E) := \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^n \mathbb{E}^{\mathbb{P}_t^i}\left[1_E^{t,\omega}\right] 1_{E_t^i}\right] + \mathbb{P}(E \cap \widehat{E}_t^n), \text{ where } \widehat{E}_t^n := \cup_{i>n} E_t^i.$$

Then, by the proof of Proposition 4.7 in [108], we know that $\mathbb{P}^n \in \mathcal{P}_H$. Besides, by Lemma 5.4.1 and its proof, we know that V and $\mathcal{Y}^{\mathbb{P},t,\omega}$ are uniformly continuous in ω and thus

$$\begin{aligned} V_t &\leq V_t(\widehat{\omega}_i) + C\rho(\varepsilon) \leq \mathcal{Y}_t^{\mathbb{P}_t^i,t,\widehat{\omega}_i}(T,\xi) + \varepsilon + C\rho(\varepsilon) \\ &\leq \mathcal{Y}_t^{\mathbb{P}_t^i,t,\omega}(T,\xi) + \varepsilon + C\rho(\varepsilon) = \mathcal{Y}_t^{(\mathbb{P}^n)^{t,\omega},t,\omega}(T,\xi) + \varepsilon + C\rho(\varepsilon). \end{aligned}$$

Then, it follows from (5.4.9) that

$$V_t \leq y_t^{\mathbb{P}^n} + \varepsilon + C\rho(\varepsilon), \ \mathbb{P}^n - a.s. \text{ on } \cup_{i=1}^n E_t^i. \tag{5.4.10}$$

Let now $(y^n, z^n, k^n) := (y^{n,\varepsilon}, z^{n,\varepsilon}, k^{n,\varepsilon})$ be the solution of the following RBSDE with lower obstacle S on $[0,t]$

$$y_s^n = \left[y_t^{\mathbb{P}^n} + \varepsilon + C\rho(\varepsilon)\right] 1_{\cup_{i=1}^n E_t^i} + V_t 1_{\widehat{E}_t^n} + \int_s^t \widehat{F}_r(y_r^n, z_r^n)dr - \int_s^t z_r^n dB_r + k_t^n - k_s^n, \ \mathbb{P}-a.s. \tag{5.4.11}$$

By the comparison principle for RBSDEs, we know that $\mathcal{Y}_0^{\mathbb{P}}(t, V_t) \leq y_0^n$. Then since $\mathbb{P}^n = \mathbb{P}$ on $\mathcal{F}_t$, the equality (5.4.11) also holds $\mathbb{P}-a.s.$ Using the same arguments and notations as in the proof of Lemma 5.4.1, we obtain

$$\left|y_0^n - y_0^{\mathbb{P}^n}\right|^2 \leq C\mathbb{E}^{\mathbb{P}}\left[\varepsilon^2 + \rho(\varepsilon)^2 + \left|V_t - y_t^{\mathbb{P}^n}\right|^2 1_{\widehat{E}_t^n}\right].$$

Then, by Lemma 5.4.1, we have

$$\mathcal{Y}_0^{\mathbb{P}}(t,V_t) \leq y_0^n \leq y_0^{\mathbb{P}^n} + C\left(\varepsilon + \rho(\varepsilon) + \left(\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^2 1_{\widehat{E}_t^n}\right]\right)^{1/2}\right)$$
$$\leq V_0(\omega) + C\left(\varepsilon + \rho(\varepsilon) + \left(\mathbb{E}^{\mathbb{P}}\left[\Lambda_t^2 1_{\widehat{E}_t^n}\right]\right)^{1/2}\right).$$

Then it suffices to let n go to $+\infty$ and ε to 0. □

Define now for all (t,ω), the $\mathbb{F}^+$-progressively measurable process

$$V_t^+ := \overline{\lim_{r\in\mathbb{Q}\cap(t,T],r\downarrow t}} V_r.$$

Lemma 5.4.2. *Under the conditions of the previous Proposition, we have*

$$V_t^+ = \lim_{r\in\mathbb{Q}\cap(t,T],r\downarrow t} V_r,\ \mathcal{P}_H - q.s.$$

and thus V^+ is càdlàg $P_H - q.s.$.

Proof. For each $\mathbb{P}$, let $(\bar{\mathcal{Y}}^{\mathbb{P}}, \bar{\mathcal{Z}}^{\mathbb{P}})$ be the solution of the BSDE with generator $\widehat{F}$ and terminal condition ξ at time T. We define

$$\widetilde{V}^{\mathbb{P}} := V - \bar{\mathcal{Y}}^{\mathbb{P}}.$$

Then, $\widetilde{V}^{\mathbb{P}} \geq 0$, $\mathbb{P} - a.s.$

For any $0 \leq t_1 < t_2 \leq T$, let $(y^{\mathbb{P},t_2}, z^{\mathbb{P},t_2}, k^{\mathbb{P},t_2}) := (\mathcal{Y}^{\mathbb{P}}(t_2,V_{t_2}), \mathcal{Z}^{\mathbb{P}}(t_2,V_{t_2}), \mathcal{K}^{\mathbb{P}}(t_2,V_{t_2}))$. Since we have for $\mathbb{P} - a.e.\ \omega$, $\mathcal{Y}_{t_1}^{\mathbb{P}}(t_2,V_{t_2})(\omega) = \mathcal{Y}^{\mathbb{P},t_1,\omega}(t_2,V_{t_2}^{t_1,\omega})$, we get from Proposition 5.4.1

$$V_{t_1} \geq y_{t_1}^{\mathbb{P},t_2},\ \mathbb{P} - a.s.$$

Denote

$$\widetilde{y}_t^{\mathbb{P},t_2} := y_t^{\mathbb{P},t_2} - \bar{\mathcal{Y}}_t^{\mathbb{P}},\ \widetilde{z}_t^{\mathbb{P},t_2} := \widehat{a}_t^{-1/2}(z_t^{\mathbb{P},t_2} - \bar{\mathcal{Z}}_t^{\mathbb{P}}).$$

Then $\widetilde{V}_{t_1}^{\mathbb{P}} \geq \widetilde{y}_{t_1}^{\mathbb{P},t_2}$ and $(\widetilde{y}^{\mathbb{P},t_2}, \widetilde{z}^{\mathbb{P},t_2})$ satisfies the following RBSDE with lower obstacle $S - \bar{\mathcal{Y}}^{\mathbb{P}}$ on $[0,t_2]$

$$\widetilde{y}_t^{\mathbb{P},t_2} = \widetilde{V}_{t_2}^{\mathbb{P}} + \int_t^{t_2} f_s^{\mathbb{P}}(\widetilde{y}_s^{\mathbb{P},t_2}, \widetilde{z}_s^{\mathbb{P},t_2})ds - \int_t^{t_2} \widetilde{z}_s^{\mathbb{P},t_2} dW_s^{\mathbb{P}} + k_{t_2}^{\mathbb{P},t_2} - k_t^{\mathbb{P},t_2},$$

where

$$f_t^{\mathbb{P}}(\omega,y,z) := \widehat{F}_t(\omega, y + \bar{\mathcal{Y}}_t^{\mathbb{P}}(\omega), \widehat{a}_t^{-1/2}(\omega)(z + \bar{\mathcal{Z}}_t^{\mathbb{P}}(\omega))) - \widehat{F}_t(\omega, \bar{\mathcal{Y}}_t^{\mathbb{P}}(\omega), \bar{\mathcal{Z}}_t^{\mathbb{P}}(\omega)).$$

By the definition given in the Appendix, $\widetilde{V}^{\mathbb{P}}$ is a positive weak reflected $f^{\mathbb{P}}$-supermartingale under $\mathbb{P}$. Since $f^{\mathbb{P}}(0,0) = 0$, we can apply the downcrossing inequality proved in the Appendix in Theorem 5.6.3 to obtain classically that for $\mathbb{P} - a.e.\ \omega$, the limit

$$\lim_{r\in\mathbb{Q}\cup(t,T],r\downarrow t} \widetilde{V}_r^{\mathbb{P}}(\omega)$$

exists for all t.

Finally, since $\bar{\mathcal{Y}}^{\mathbb{P}}$ is continuous, we get the result. □

Proceeding exactly as in Steps 1 et 2 of the proof of Theorem 4.5 in [108], we can then prove that V^+ is a strong reflected $\widehat{F}$-supermartingale. Then, using the Doob-Meyer decomposition proved in the Appendix in Theorem 5.6.2 for all $\mathbb{P}$, we know that there exists a unique ($\mathbb{P} - a.s.$) process $\overline{Z}^{\mathbb{P}} \in \mathbb{H}^2(\mathbb{P})$ and unique non-decreasing càdlàg square integrable processes $A^{\mathbb{P}}$ and $B^{\mathbb{P}}$ such that

- $V_t^+ = V_0^+ - \int_0^t \widehat{F}_s(V_s^+, \overline{Z}_s^{\mathbb{P}})ds + \int_0^t \overline{Z}_s^{\mathbb{P}} dB_s - A_t^{\mathbb{P}} - B_t^{\mathbb{P}},\ \mathbb{P}-a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H.$
- $V_t^+ \geq S_t,\ \mathbb{P}-a.s.\ \forall \mathbb{P} \in \mathcal{P}_H.$
- $\int_0^T (V_{t^-} - S_{t^-})\, dA_t^{\mathbb{P}},\ \mathbb{P}-a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H.$
- $A^{\mathbb{P}}$ and $B^{\mathbb{P}}$ never act at the same time.

We then define $K^{\mathbb{P}} := A^{\mathbb{P}} + B^{\mathbb{P}}$. By Karandikar [64], since V^+ is a càdlàg semimartingale, we can define a universal process $\overline{Z}$ which aggregates the family $\left\{\overline{Z}^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$.

We next prove the representation (5.3.1) for V and V^+, and that, as shown in Proposition 4.11 of [108], we actually have $V = V^+$, $\mathcal{P}_H - q.s.$, which shows that in the case of a terminal condition in $UC_b(\Omega)$, the solution of the 2RBSDE is actually $\mathbb{F}$-progressively measurable.

Proposition 5.4.2. *Assume that $\xi \in UC_b(\Omega)$ and that Assumptions 5.2.1 and 5.2.2 hold. Then we have*

$$V_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})} \mathcal{Y}_t^{\mathbb{P}'}(T,\xi) \text{ and } V_t^+ = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})} \mathcal{Y}_t^{\mathbb{P}'}(T,\xi),\ \mathbb{P}-a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H.$$

Besides, we also have for all t

$$V_t = V_t^+,\ \mathcal{P}_H - q.s.$$

Proof. The proof for the representations is the same as the proof of proposition 4.10 in [108], since we also have a stability result for RBSDEs under our assumptions. For the equality between V and V^+, we also refer to the proof of Proposition 4.11 in [108]. □

Therefore, in the sequel we will use V instead of V^+.

Finally, we have to check that the minimum condition (5.2.3) holds. Fix $\mathbb{P}$ in $\mathcal{P}_H^\kappa$ and $\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})$. By the Lipschitz property of F, we know that there exists bounded processes λ and η such that

$$V_t - y_t^{\mathbb{P}'} = \int_t^T \lambda_s (V_s - y_s^{\mathbb{P}'}) ds - \int_t^T \widehat{a}_s^{1/2} (\overline{Z}_s - z_s^{\mathbb{P}'})(\widehat{a}_s^{-1/2} dB_s - \eta_s ds) + K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'} - k_T^{\mathbb{P}'} + k_t^{\mathbb{P}'}. \tag{5.4.12}$$

Then, one can define a probability measure $\mathbb{Q}'$ equivalent to $\mathbb{P}'$ such that

$$V_t - y_t^{\mathbb{P}'} = e^{-\int_0^t \lambda_u du} \mathbb{E}_t^{\mathbb{Q}'} \left[\int_t^T e^{\int_0^s \lambda_u du} d(K_s^{\mathbb{P}'} - k_s^{\mathbb{P}'}) \right].$$

Now define the following càdlàg non-decreasing processes

$$\overline{K}_s^{\mathbb{P}'} := \int_0^s e^{\int_0^u \lambda_r dr} dK_u^{\mathbb{P}'},\ \overline{k}_s^{\mathbb{P}'} := \int_0^s e^{\int_0^u \lambda_r dr} dk_u^{\mathbb{P}'}.$$

By the representation (5.3.1), we deduce that the process

$$\overline{K}^{\mathbb{P}'} - \overline{k}^{\mathbb{P}'}$$

is a $\mathbb{Q}'$-submartingale. Using Doob-Meyer decomposition and the fact that all the probability measures we consider satisfy the martingale representation property, we deduce as in Step (ii) of the proof of Theorem 5.3.1 that this process is actually non-decreasing. Then by definition, this entails that the process $K^{\mathbb{P}'} - k^{\mathbb{P}'}$ is also non-decreasing.

Let us denote

$$P_t^{\mathbb{P}'} := K^{\mathbb{P}'} - k^{\mathbb{P}'}.$$

Returning to (5.4.12) and defining a process M as in Step (ii) of the proof of Theorem 5.3.1, we obtain that

$$V_t - y_t^{\mathbb{P}'} = \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T M_s dP_s^{\mathbb{P}'}\right] \geq \mathbb{E}_t^{\mathbb{P}'}\left[\inf_{t\leq s\leq T} M_s\left(P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}\right)\right].$$

Then, we have

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}'}\left[P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}\right] \\
&= \mathbb{E}_t^{\mathbb{P}'}\left[\left(\inf_{t\leq s\leq T} M_s\right)^{1/3}\left(P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}\right)\left(\inf_{t\leq s\leq T} M_s\right)^{-1/3}\right] \\
&\leq \left(\mathbb{E}_t^{\mathbb{P}'}\left[\inf_{t\leq s\leq T} M_s\left(P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}\right)\right]\mathbb{E}_t^{\mathbb{P}'}\left[\sup_{t\leq s\leq T} M_s^{-1}\right]\mathbb{E}_t^{\mathbb{P}'}\left[\left(P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}\right)^2\right]\right)^{1/3} \\
&\leq C\left(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}\left[\left(P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}\right)^2\right]\right)^{1/3}\left(V_t - y_t^{\mathbb{P}'}\right)^{1/3}.
\end{aligned}$$

Arguing as in Step (iii) of the proof of Theorem 5.3.1, the above inequality shows that we have

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}\left[P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}\right] = 0,$$

that is to say that the minimum condition (5.2.3) is satisfied.

5.4.2 Main result

We are now in position to state the main result of this section

Theorem 5.4.1. *Let $\xi \in \mathcal{L}_H^{2,\kappa}$. Under Assumptions 5.2.1 and 5.2.2, there exists a unique solution $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ of the* 2BSDE (5.2.1).

Proof. The proof follow the lines of the proof of Theorem 4.7 in [107]. In general for a terminal condition $\xi \in \mathcal{L}_H^{2,\kappa}$, there exists by definition a sequence $(\xi_n)_{n\geq 0} \subset \mathrm{UC_b}(\Omega)$ such that

$$\lim_{n\to+\infty} \|\xi_n - \xi\|_{\mathbb{L}_H^{2,\kappa}} = 0 \text{ and } \sup_{n\geq 0} \|\xi_n\|_{\mathbb{L}_H^{2,\kappa}} < +\infty.$$

Let (Y^n, Z^n) be the solution to the 2RBSDE (5.2.1) with terminal condition ξ_n and

$$K_t^{\mathbb{P},n} := Y_0^n - Y_t^n - \int_0^t \widehat{F}_s(Y_s^n, Z_s^n)ds + \int_0^t Z_s^n dB_s,\ \mathbb{P} - a.s.$$

By the estimates of Proposition 5.3.4, we have as $n, m \to +\infty$

$$\|Y^n - Y^m\|^2_{\mathbb{D}_H^{2,\kappa}} + \|Z^n - Z^m\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} |K_t^{\mathbb{P},n} - K_t^{\mathbb{P},m}|\right] \leq C_\kappa \|\xi_n - \xi_m\|_{\mathbb{L}_H^{2,\kappa}} \to 0.$$

Extracting a subsequence if necessary, we may assume that

$$\|Y^n - Y^m\|^2_{\mathbb{D}^{2,\kappa}_H} + \|Z^n - Z^m\|^2_{\mathbb{H}^{2,\kappa}_H} + \sup_{\mathbb{P}\in\mathcal{P}^\kappa_H} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} |K^{\mathbb{P},n}_t - K^{\mathbb{P},m}_t|\right] \leq \frac{1}{2^n}. \quad (5.4.13)$$

This implies by Markov inequality that for all $\mathbb{P}$ and all $m \geq n \geq 0$

$$\mathbb{P}\left[\sup_{0\leq t\leq T}\left\{|Y^n_t - Y^m_t|^2 + |K^{\mathbb{P},n}_t - K^{\mathbb{P},m}_t|^2\right\} + \int_0^T |\widehat{a}^{1/2}_t (Z^n_s - Z^m_s)|^2 dt > n^{-1}\right] \leq Cn2^{-n}. \quad (5.4.14)$$

Define

$$Y := \overline{\lim_{n\to+\infty}} Y^n,\ Z := \overline{\lim_{n\to+\infty}} Z^n,\ K^{\mathbb{P}} := \overline{\lim_{n\to+\infty}} K^{\mathbb{P},n},$$

where the $\overline{\lim}$ for Z is taken componentwise. All those processes are clearly $\mathbb{F}^+$-progressively measurable.

By (5.4.14), it follows from Borel-Cantelli Lemma that for all $\mathbb{P}$ we have $\mathbb{P} - a.s.$

$$\lim_{n\to+\infty}\left[\sup_{0\leq t\leq T}\left\{|Y^n_t - Y_t|^2 + |K^{\mathbb{P},n}_t - K^{\mathbb{P}}_t|^2\right\} + \int_0^T |\widehat{a}^{1/2}_t (Z^n_s - Z_s)|^2 dt\right] = 0.$$

It follows that Y is càdlàg, $\mathcal{P}^\kappa_H - q.s.$, and that $K^{\mathbb{P}}$ is a càdlàg non-decreasing processe, $\mathbb{P} - a.s.$ Furthermore, for all $\mathbb{P}$, sending m to infinity in (5.4.13) and applying Fatou's lemma under $\mathbb{P}$ gives us that $(Y, Z) \in \mathbb{D}^{2,\kappa}_H \times \mathbb{H}^{2,\kappa}_H$.

Finally, we can proceed exactly as in the regular case ($\xi \in \mathrm{UC_b}(\Omega)$) to show that the minimum condition (5.2.3) holds.

□

5.5 American Options under volatility uncertainty

First let us recall the link between American options and RBSDEs in the classical framework (see [43] for more details). Let $\mathcal{M}$ be a standard financial complete market (1 risky asset S and a bond). It is well known that in some constrained cases the pair wealth-portfolio $(X^{\mathbb{P}}, \pi^{\mathbb{P}})$ satisfies:

$$X^{\mathbb{P}}_t = \xi + \int_t^T b(s, X^{\mathbb{P}}_s, \pi^{\mathbb{P}}_s)ds - \int_t^T \pi_s\sigma_s dW_s$$

where W is a Brownian motion under the underlying probability measure $\mathbb{P}$, b is convex and Lipschitz with respect to (x, π). In addition we assume that the process $(b(t, 0, 0))_{t\leq T}$ is square-integrable and $(\sigma_t)_{t\leq T}$, the volatility matrix of the n risky assets, is invertible and its inverse $(\sigma_t)^{-1}$ is bounded. The classical case corresponds to $b(t, x, \pi) = -r_t x - \pi.\sigma_t\theta_t$, where θ_t is the risk premium vector.

When the American option is exercised at a stopping time $\nu \geq t$ the yield is given by

$$\tilde{S}_\nu = S_\nu \mathbf{1}_{[\nu<T]} + \xi_T \mathbf{1}_{[\nu=T]}.$$

Let t be fixed and let $\nu \geq t$ be the exercising time of the contingent claim. Then, since the market is complete, there exists a unique pair $(X^{\mathbb{P}}_s(\nu, \tilde{S}_\nu), \pi^{\mathbb{P}}_s(\nu, \tilde{S}_\nu)) = (X^{\mathbb{P},\nu}_s, \pi^{\mathbb{P},\nu}_s)$ which replicates $\tilde{S}_\nu$, *i.e.*,

$$-dX^{\mathbb{P},\nu}_s = b(s, X^{\mathbb{P},\nu}_s, \pi^{\mathbb{P},\nu}_s)dt - \pi^{\mathbb{P},\nu}_s \sigma_s dW_s,\ s \leq \nu;\ X^{\mathbb{P},\nu}_\nu = \tilde{S}_\nu.$$

Therefore the price of the contingent claim is given by:

$$X_t^{\mathbb{P}} = \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{t,T}} X_t^{\mathbb{P}}(\nu, \tilde{S}_\nu).$$

Then, the link with RBSDE is given by the following Theorem of [43]

Theorem 5.5.1. *There exist $\pi^{\mathbb{P}} \in \mathbb{H}^2(\mathbb{P})$ and a non-decreasing continuous process $k^{\mathbb{P}}$ such that for all $t \in [0,T]$*

$$\begin{cases} X_t^{\mathbb{P}} = \xi + \int_t^T b(s, X_s^{\mathbb{P}}, \pi_s^{\mathbb{P}})ds - \int_t^T \pi_s^{\mathbb{P}} \sigma_s dW_s + k_T^{\mathbb{P}} - k_t^{\mathbb{P}} \\ X_t^{\mathbb{P}} \geq S_t \\ \int_0^T (X_t^{\mathbb{P}} - S_t) dk_t^{\mathbb{P}} = 0. \end{cases}$$

Furthermore, the stopping time $D_t^{\mathbb{P}} = \inf\{s \geq t, X_s^{\mathbb{P}} = S_s\} \wedge T$ is optimal after t.

Let us now go back to our uncertain volatility framework. The pricing of European contingent claims has already been treated in that context by Avellaneda, Lévy and Paras in [2], Denis and Martini in[38] with capacity theory and more recently by Vorbrink in [114] using the G-expectation framework.

We still consider a financial market with one risky asset S, whose dynamics are given by

$$dS_t = r_t dt + dB_t, \ \mathcal{P}_H^\kappa - q.s.$$

and we assume as above that our wealth process has the following dynamic

$$X_t = \xi + \int_t^T b(s, X_s, \pi_s)ds - \int_t^T \pi_s dB_s, \ \mathcal{P}_H^\kappa - q.s.$$

In order to be in our 2RBSDE framework, we have to assume that b satisfies Assumptions 5.2.1 and 5.2.2. The main difference is that now b must satisfy stronger integrability conditions and also that it has to be uniformly continuous in ω (when we assume that $\widehat{a}$ in the expression of b is constant). For instance, in the classical case recalled above, it means that r must be uniformly continuous in ω, which is the case if for example it is deterministic. Finally, we must assume that $\xi \in \mathcal{L}_H^{2,\kappa}$. This is going to be the case for all Lipschitz functions of S, if we assume that r is uniformly continuous in ω, which includes Call and Put options. Finally, since S is going to be the obstacle, it has to be uniformly continuous in ω. This is why we consider an asset which is given by a Brownian motion with drift and not a geometric Brownian motion. Indeed, the geometric Brownian motion may not be uniformly continuous in ω.

Remark 5.5.1. *Of course, from a financial point of view, assets driven by a Brownian motion instead of a geometric Brownian motion, especially in a market with volatility uncertainty, have much less interest. Nonetheless, notice that we could get rid of this restriction by doing the same construction as in [107] for 2BSDEs when considering a canonical process which is an exponential Brownian motion under the Wiener measure.*

Following the intuitions in the papers mentioned above, it is natural in our now incomplete market to consider as a superhedging price for our contingent claim

$$X_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})} X_t^{\mathbb{P}'}, \ \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}_H^\kappa,$$

where $X_t^{\mathbb{P}}$ is the price at time t of the contingent claim in the complete market mentioned at the beginning, with underlying probability measure $\mathbb{P}$. Notice immediately that we do not claim that this price is the superreplicating price in our context, in the sense that it would be the smallest one for which there exists a strategy which superreplicates the American option quasi-surely.

The following Theorem is then a simple consequence of the previous one

Theorem 5.5.2. *There exist $\pi \in \mathbb{H}_H^{2,\kappa}$, a family of non-decreasing càdlàg processes $K^{\mathbb{P}}$ such that for all $t \in [0,T]$ and for all $\mathbb{P} \in \mathcal{P}_H^\kappa$*

$$\begin{cases} X_t = \xi + \int_t^T b(s, X_s, \pi_s)ds - \int_t^T \pi_s \sigma_s dW_s + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}, \ \mathbb{P}-a.s. \\ X_t \geq S_t, \ \mathbb{P}-a.s. \\ K_t^{\mathbb{P}} - k_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}\left[K_T^{\mathbb{P}'} - k_T^{\mathbb{P}'}\right], \ \mathbb{P}-a.s. \end{cases}$$

Furthermore, for all ε, the stopping time $D_t^\varepsilon = \inf\{s \geq t, X_s \leq S_s + \varepsilon, \ \mathcal{P}_H^\kappa - q.s.\} \wedge T$ is ε-optimal after t. Besides, for all $\mathbb{P}$, if we consider the stopping times $D_t^{\mathbb{P},\varepsilon} = \inf\left\{s \geq t, X_s^{\mathbb{P}} \leq S_s + \varepsilon, \ \mathbb{P}-a.s.\right\} \wedge T$, which are ε-optimal for the American options under each $\mathbb{P}$, then for all $\mathbb{P}$

$$D_t^\varepsilon \geq D_t^{\varepsilon,\mathbb{P}}, \ \mathbb{P}-a.s. \tag{5.5.1}$$

Proof. The existence of the processes is a simple consequence of Theorem 5.4.1 and the fact that X is the superhedging price of the contingent claim comes from the representation formula (5.3.1). Then, the ε-optimality of D_t^ε and the inequality (5.5.1) are clear by definition.

□

Remark 5.5.2. *The formula (5.5.1) confirms the natural intuition that the smallest optimal time to exercise the American option when the volatility is uncertain is the supremum, in some sense, of all the optimal stopping times for the classical American options for each volatility scenarii.*

5.6 Appendix : Supersolutions of reflected BSDEs

In this section, we extend some of the results of Peng [86] concerning g-supersolution of BSDEs to the case of RBSDEs. Let us note that the majority of the following proofs follows straightforwardly from the original proofs of Peng, with some minor modifications due to the added reflection. However, we still provide most of them since, to the best of our knowledge, they do not appear anywhere else in the litterature.

In the following, we fix a probability measure $\mathbb{P}$

5.6.1 Definitions and first properties

Let us be given the following objects

- A function $g_s(\omega, y, z)$, $\mathbb{F}$-progressively measurable for fixed y and z, uniformly Lipschitz in (y,z) and such that
$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T |g_s(0,0)|^2 \, ds\right] < +\infty.$$
- A terminal condition ξ which is $\mathcal{F}_T$-measurable and in $L^2(\mathbb{P})$.

- A càdlàg process V with $\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|V_t|^2\right] < +\infty$.
- A càdlàg non-decreasing process S such that $\mathbb{E}^{\mathbb{P}}\left[\left(\sup_{0\leq t\leq T}(S_t)^+\right)^2\right] < +\infty$.

We want to study the following problem. Finding $(y,z,k)\in\mathbb{D}^2(\mathbb{P})\times\mathbb{H}^2(\mathbb{P})\times\mathbb{I}^2(\mathbb{P})$ such that

$$\begin{cases} y_t = \xi + \int_t^T g_s(y_s,z_s)ds - \int_t^T z_s dW_s + k_T - k_t + V_T - V_t,\ 0\leq t\leq T,\ \mathbb{P}-a.s. \\ y_t \geq S_t,\ \mathbb{P}-a.s. \\ \int_0^T (y_{s^-} - S_{s^-})\, dk_s = 0,\ \mathbb{P}-a.s.,\ \forall t\in[0,T]. \end{cases}$$

We first have a result of existence and uniqueness

Proposition 5.6.1. *Under the above hypotheses, there exists a unique solution $(y,z,k)\in\mathbb{D}^2(\mathbb{P})\times\mathbb{H}^2(\mathbb{P})\times\mathbb{I}^2(\mathbb{P})$ to the reflected BSDE (5.6.1).*

Proof. Consider the following penalized BSDE, whose existence and uniqueness are ensured by the results of Peng [86]

$$y_t^n = \xi + \int_t^T g_s(y_s^n, z_s^n)ds - \int_t^T z_s^n dW_s + k_T^n - k_t^n + V_T - V_t,$$

where $k_t^n := n\int_0^t (y_s^n - S_s)^- ds$.

Then, define $\widetilde{y}_t^n := y_t^n + V_t$, $\widetilde{\xi} := \xi + V_T$, $\widetilde{z}_t^n := z_t^n$, $\widetilde{k}_t^n := k_t^n$ and $\widetilde{g}_t(y,z) := g_t(y-V,z)$. We have

$$\widetilde{y}_t^n = \widetilde{\xi} + \int_t^T \widetilde{g}_s(\widetilde{y}_s^n, \widetilde{z}_s^n)ds - \int_t^T \widetilde{z}_s^n dW_s + \widetilde{k}_T^n - \widetilde{k}_t^n,$$

Then, since we know by Lepeltier and Xu [74], that the above penalization procedure converges to a solution of the corresponding RBSDE, existence and uniqueness are then simple generalization of the classical results in RBSDE theory. □

We also have a comparison theorem in this context

Proposition 5.6.2. *Let ξ_1 and $\xi_2\in L^2(\mathbb{P})$, V^i, $i=1,2$ be two adapted, càdlàg processes and $g_s^i(\omega,y,z)$ two functions, which all verify the above assumptions. Let $(y^i,z^i,k^i)\in\mathbb{D}^2(\mathbb{P})\times\mathbb{H}^2(\mathbb{P})\times\mathbb{I}^2(\mathbb{P})$, $i=1,2$ be the solutions of the following RBSDEs with lower obstacle S^i*

$$y_t^i = \xi^i + \int_t^T g_s^i(y_s^i, z_s^i)ds - \int_t^T z_s^i dW_s + k_T^i - k_t^i + V_T^i - V_t^i,\ \mathbb{P}-a.s.,\ i=1,2,$$

respectively. If

- $\xi_1 \geq \xi_2$, $\mathbb{P}-a.s.$
- $V^1 - V^2$ *is non-decreasing,* $\mathbb{P}-a.s.$
- $S^1 \geq S^2$, $\mathbb{P}-a.s.$
- $g_s^1(y_s^1, z_s^1) \geq g_s^2(y_s^1, z_s^1)$, $dt\times d\mathbb{P}-a.s.$

then it holds $\mathbb{P}-a.s.$ *that for all* $t\in[0,T]$

$$Y_t^1 \geq Y_t^2.$$

Besides, if $S^1 = S^2$*, then we also have* $dK^1 \leq dK^2$.

Proof. The first part can be proved exactly as in [44], whereas the second one comes from the fact that the penalization procedure converges in this framework, as seen previously. □

Remark 5.6.1. *If we replace the deterministic time* T *by a stopping time* τ*, then all the above is still valid.*

From now on, we will specialize the discussion to the case where the process V is actually in $\mathbb{I}^2(\mathbb{P})$ and consider the following RBSDE

$$\begin{cases} y_t = \xi + \int_{t\wedge\tau}^{\tau} g_s(y_s,z_s)ds + V_\tau - V_{t\wedge\tau} + k_\tau - k_{t\wedge\tau} - \int_{t\wedge\tau}^{\tau} z_s dW_s,\ 0\leq t\leq\tau,\ \mathbb{P}-a.s. \\ y_t \geq S_t,\ \mathbb{P}-a.s. \\ \int_0^\tau (y_{s^-} - S_{s^-})\, dk_s = 0,\ \mathbb{P}-a.s.,\ \forall t\in[0,\tau]. \end{cases}$$

Definition 5.6.1. *If* y *is a solution of a RBSDE of the form* (5.6.1)*, then we call* y *a reflected* g*-supersolution on* $[0,\tau]$*. If* $V=0$ *on* $[0,\tau]$*, then we call* y *a reflected* g*-solution.*

We now face a first difference from the case of non-reflected supersolution. Since in our case we have two increasing processes, if a g-supersolution is given, there can exist several increasing processes V and k such that (5.6.1) is satisfied. Indeed, we have the following proposition

Proposition 5.6.3. *Given* y *a* g*-supersolution on* $[0,\tau]$*, there is a unique* $z\in\mathbb{H}^2(\mathbb{P})$ *and a unique couple* $(k,V)\in(\mathbb{I}^2(\mathbb{P}))^2$ *(in the sense that the sum* $k+V$ *is unique), such that* (y,z,k,V) *satisfy* (5.6.1)*. Besides, there exists a unique quadruple* (y,z,k',V') *satisfying* (5.6.1) *such that* k' *and* V' *never act at the same time.*

Proof. If both (y,z,k,V) and (y,z^1,k^1,V^1) satisfy (5.6.1), then applying Itô's formula to $(y_t-y_t)^2$ gives immediately that $z=z^1$ and thus $k+V=k^1+V^1$, $\mathbb{P}-a.s.$

Then, if (y,z,k,V) satisfying (5.6.1) is given, then it is easy to construct (k',V') such that

- k' only increases when $y_{t^-} = S_{t^-}$.
- V' only increases when $y_{t^-} > S_{t^-}$.
- $V'_t + k'_t = V_t + k_t$, $dt\times d\mathbb{P}-a.s.$

and such a couple is unique. □

Remark 5.6.2. *We give a counter-example to the general uniqueness in the above Proposition. Let* $T=2$ *and consider the following RBSDE*

$$\begin{cases} y_t = -2 + 2 - t + k_2 - k_t - \int_t^2 z_s dW_s,\ 0\leq t\leq 2,\ \mathbb{P}-a.s. \\ y_t \geq -\frac{t^2}{2},\ \mathbb{P}-a.s. \\ \int_0^2 \left(y_{s^-} + \frac{t^2}{2}\right) dk_s = 0,\ \mathbb{P}-a.s.,\ \forall t\in[0,2]. \end{cases}$$

We then have $z=0$, $y_t = 1_{0\leq t\leq 1}\left(\frac{1}{2}-t\right) - \frac{t^2}{2}1_{1<t\leq 2}$ *and* $k_t = 1_{t\geq 1}\frac{t^2-1}{2}$.

However, we can also take

$$V_t' = t1_{t\leq 1} + \left(\frac{t^2}{4}+\frac{t}{4}+\frac{1}{2}\right)1_{1<t\leq 2} \text{ and } k_t' = 1_{t\geq 1}\left(\frac{t^2}{4}+\frac{3}{4}t-1\right).$$

Following Peng [86], this allows us to define

Definition 5.6.2. *Let y be a supersolution on $[0,\tau]$ and let (y,z,k,V) be the related unique triple in the sense of the RBSDE (5.6.1), where k and V never act at the same time. Then we call (z,k,V) the decomposition of y.*

5.6.2 Monotonic limit theorem

We now study a limit theorem for reflected g-supersolutions, which is very similar to theorems 2.1 and 2.4 of [86].

We consider a sequence of reflected g-supersolutions

$$\begin{cases} y_t^n = \xi^n + \int_t^T g_s(y_s^n, z_s^n)ds + V_T^n - V_t^n + k_T^n - k_t^n - \int_t^T z_s^n dW_s,\ 0\leq t\leq \tau,\ \mathbb{P}-a.s. \\ y_t^n \geq S_t,\ \mathbb{P}-a.s. \\ \int_0^\tau \left(y_{s^-}^n - S_{s^-}\right)dk_s^n = 0,\ \mathbb{P}-a.s.,\ \forall t\in[0,T], \end{cases}$$

where the V^n are in addition supposed to be continuous.

Theorem 5.6.1. *If we assume that (y_t^n) increasingly converges to (y_t) with*

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|y_t|^2\right] < +\infty,$$

and that (k_t^n) decreasingly converges to (k_t), then y is a g-supersolution, that is to say that there exists $(z,V)\in\mathbb{H}^2(\mathbb{P})\times\mathbb{I}^2(\mathbb{P})$ such that

$$\begin{cases} y_t = \xi + \int_t^T g_s(y_s, z_s)ds + V_T - V_t + k_T - k_t - \int_t^T z_s dW_s,\ 0\leq t\leq T,\ \mathbb{P}-a.s. \\ y_t \geq S_t,\ \mathbb{P}-a.s. \\ \int_0^T (y_{s^-} - S_{s^-})\,dk_s = 0,\ \mathbb{P}-a.s.,\ \forall t\in[0,T], \end{cases}$$

Besides, z is the weak (resp. strong) limit of z^n in $\mathbb{H}^2(\mathbb{P})$ (resp. in $\mathbb{H}^p(\mathbb{P})$ for $p<2$) and V_t is the weak limit of V_t^n in $L^2(\mathbb{P})$.

Before proving the Theorem, we will need the following Lemma

Lemma 5.6.1. *Under the hypotheses of Theorem 5.6.1, there exists a constant $C>0$ independent of n such that*

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T |z_s^n|^2 ds + (V_T^n)^2 + (k_T^n)^2\right] \leq C.$$

Proof. We have

$$A_T^n + k_T^n = y_0^n - y_T^n - \int_0^T g_s(y_s^n, z_s^n)ds + \int_0^T z_s^n dW_s$$
$$\leq C\left(\sup_{0\leq t\leq T}|y_t^n| + \int_0^T |z_s^n|\,ds + \int_0^T |g_s(0,0)|\,ds + \left|\int_0^T z_s^n dW_s\right|\right). \tag{5.6.1}$$

Besides, we also have for all $n \geq 1$, $y_t^1 \leq y_t^n \leq y_t$ and thus $|y_t^n| \leq \left|y_t^1\right| + |y_t|$, which in turn implies that

$$\sup_n \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|y_t^n|^2\right] \leq C.$$

Reporting this in (5.6.1) and using BDG inequality, we obtain

$$\mathbb{E}^{\mathbb{P}}\left[(V_T^n)^2 + (k_T^n)^2\right] \leq \mathbb{E}^{\mathbb{P}}\left[(V_T^n + k_T^n)^2\right]$$
$$\leq C_0\left(1 + \mathbb{E}^{\mathbb{P}}\left[\int_0^T |g_s(0,0)|^2\,ds + \int_0^T |z_s^n|^2\,ds\right]\right). \tag{5.6.2}$$

Then, using Itô's formula, we obtain classically for all $\varepsilon > 0$

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T |z_s^n|^2\,ds\right] \leq \mathbb{E}^{\mathbb{P}}\left[(y_T^n)^2 + 2\int_0^T y_s^n g_s(y_s^n, z_s^n)ds + 2\int_0^T y_{s^-}^n d(V_s^n + k_s^n)\right]$$
$$\leq \mathbb{E}^{\mathbb{P}}\left[C\left(1 + \sup_{0\leq t\leq T}|y_t^n|^2\right) + \int_0^T \frac{|z_s^n|^2}{2}ds + \varepsilon\left(|V_T^n|^2 + |k_T^n|^2\right)\right]. \tag{5.6.3}$$

Then, from (5.6.2) and (5.6.3), we obtain by choosing $\varepsilon = \frac{1}{4C_0}$ that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T |z_s^n|^2\,ds\right] \leq C.$$

Reporting this in (5.6.1) ends the proof. □

Proof. [Proof of Theorem 5.6.1] By Lemma 5.6.1 and its proof we first have

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T |g_s(y_s^n, z_s^n)|^2\,ds\right] \leq C\mathbb{E}^{\mathbb{P}}\left[\int_0^T |g_s(0,0)|^2 + |y_s^n|^2 + |z_s^n|^2\,ds\right] \leq C.$$

Thus $g_s(y_s^n, z_s^n)$ and z^n are bounded in $\mathbb{H}^2(\mathbb{P})$, and there exists subsequences which converge respectively to some g_s and z_s. Therefore, for every stopping time τ, we also have the following weak convergences

$$\int_0^\tau z_s^n dW_s \to \int_0^\tau z_s dW_s,\ \int_0^\tau g_s(y_s^n, z_s^n)ds \to \int_0^\tau \bar{g}_s ds,$$
$$V_\tau^n \to -y_\tau + y_0 - k_\tau - \int_0^\tau \bar{g}_s ds + \int_0^\tau z_s dW_s.$$

Then by the section theorem, it is clear that V and k are non-decreasing, and by Lemma 2.2 of [86] we know that y, V and k are càdlàg. We now show the strong convergence of z^n. Following Peng [86], we apply Itô's formula between two stopping times τ and σ. Since V^n is continuous, we obtain

$$\mathbb{E}^{\mathbb{P}}\left[\int_\sigma^\tau |z_s^n - z_s|^2\,ds\right] \leq \mathbb{E}^{\mathbb{P}}\left[|y_\tau^n - y_\tau|^2 + \sum_{\sigma\leq t\leq\tau}(\Delta(V_t + k_t))^2\right]$$
$$+ 2\mathbb{E}^{\mathbb{P}}\left[\int_\sigma^\tau |y_s^n - y_s|\,|g_s(y_s^n, z_s^n) - \bar{g}_s|\,ds + \int_\sigma \tau(y_s^n - y_s)d(V_s + k_s)\right].$$

Then we can finish exactly as in [86] to obtain the desired convergence. Since g is supposed to be Lipschitz, we actually have

$$\bar{g}_s = g_s(y_s, z_s),\ \mathbb{P}-a.s.$$

Finally, since for each n, we have $y_t^n \geq S_t$, we have $y_t \geq S_t$. For the Skorohod condition, we have, since the k^n are decreasing

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\int_0^T y_{t^-} - S_{t^-} dk_t\right] &\leq \mathbb{E}^{\mathbb{P}}\left[\int_0^T y_{t^-} - y_{t^-}^n dk_t + \int_0^T y_{t^-}^n - S_{t^-} dk_t^n\right]\\ &= \mathbb{E}^{\mathbb{P}}\left[\int_0^T y_{t^-} - y_{t^-}^n dk_t\right].\end{aligned}$$

Then, we have

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T y_{t^-} - y_{t^-}^n dk_t\right] \leq \left(\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|y_t^1 - y_t\right|^2\right]\right)^{1/2}\left(\mathbb{E}^{\mathbb{P}}\left[k_T^2\right]\right)^{1/2} < +\infty$$

Therefore by Lebesgue dominated convergence Theorem, we obtain that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T y_{t^-} - y_{t^-}^n dk_t\right] \to 0,$$

and thus

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T y_{t^-} - S_{t^-} dk_t\right] \leq 0,$$

which ends the proof. □

5.6.3 Doob-Meyer decomposition

We now introduce the notion of reflected g-(super)martingales.

Definition 5.6.3. (i) *A reflected g-martingale on $[0,T]$ is a reflected g-solution on $[0,T]$.*

(ii) (Y_t) *is a reflected g-supermartingale in the strong (resp. weak) sense if for all stopping time $\tau \leq T$ (resp. all $t \leq T$), we have $\mathbb{E}^{\mathbb{P}}[|Y_\tau|^2] < +\infty$ (resp. $\mathbb{E}^{\mathbb{P}}[|Y_t|^2] < +\infty$) and if the reflected g-solution (y_s) on $[0,\tau]$ (resp. $[0,t]$) with terminal condition Y_τ (resp. Y_t) verifies $y_\sigma \leq Y_\sigma$ for every stopping time $\sigma \leq \tau$ (resp. $y_s \leq Y_s$ for every $s \leq t$).*

As in the case without reflection, under mild conditions, a reflected g-supermartingale in the weak sense corresponds to a reflected g-supermartingale in the strong sense. Besides, thanks to the comparison Theorem, it is clear that a g-supersolution on $[0,T]$ is also a g-supermartingale in the weak and strong sense on $[0,T]$. The following Theorem adresses the converse property, which gives us a non-linear Doob-Meyer decomposition.

Theorem 5.6.2. *Let (Y_t) be a right-continuous reflected g-supermartingale on $[0,T]$ in the strong sense with*

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|Y_t|^2\right] < +\infty.$$

Then (Y_t) *is a reflected* g*-supersolution on* $[0,T]$*, that is to say that there exists a unique triple* $(z,k,V) \in \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$ *such that*

$$\begin{cases} Y_t = Y_T + \int_t^T g_s(Y_s,z_s)ds + V_T - V_t + k_T - k_t - \int_t^T z_s dW_s,\ 0 \leq t \leq T,\ \mathbb{P}-a.s. \\ Y_t \geq S_t,\ \mathbb{P}-a.s. \\ \int_0^T (Y_{s^-} - S_{s^-})\, dk_s = 0,\ \mathbb{P}-a.s.,\ \forall t \in [0,T]. \\ V \text{ and } k \text{ never act at the same time.} \end{cases}$$

We follow again [86] and consider the following sequence of RBSDEs

$$\begin{cases} y_t^n = Y_T + \int_t^T g_s(y_s^n, z_s^n)ds + n\int_t^T (Y_s - y_s^n)ds + k_T^n - k_t^n - \int_t^T z_s^n dW_s,\ 0 \leq t \leq T \\ y_t^n \geq S_t,\ \mathbb{P}-a.s. \\ \int_0^T \left(y_{s^-}^n - S_{s^-}\right) dk_s^n = 0,\ \mathbb{P}-a.s.,\ \forall t \in [0,T], \end{cases}$$

We then have

Lemma 5.6.2. *For all* n*, we have*

$$Y_t \geq y_t^n.$$

Proof. The proof is exactly the same as the proof of Lemma 3.4 in [86], so we omit it. □

Proof. [Proof of Theorem 5.6.2] The uniqueness is due to the uniqueness for reflected g-supersolutions proved in Proposition 5.6.3. For the existence part, we first notice that since $Y_t \geq y_t^n$ for all n, by the comparison Theorem for RBSDEs, we have $y_t^n \leq y_t^{n+1}$ and $dk_t^n \geq dk_t^{n+1}$. Therefore they converge monotonically to some processes y and k. Besides, y is bounded from above by Y. Therefore, all the conditions of Theorem 5.6.1 are satisfied and y is a reflected g-supersolution on $[0,T]$ of the form

$$y_t = Y_T + \int_t^T g_s(y_s,z_s)ds + V_T - V_t + k_T - k_t - \int_t^T z_s dW_s,$$

where V_t is the weak limit of $V_t^n := n\int_0^t (Y_s - y_s^n)ds$.

From Lemma 5.6.1, we have

$$\mathbb{E}^{\mathbb{P}}[(V_T^n)^2] = n^2 \mathbb{E}^{\mathbb{P}}\left[\int_0^T |Y_s - y_s^n|^2\, ds\right] \leq C.$$

It then follows that $Y_t = y_t$, which ends the proof. □

5.6.4 Downcrossing inequality

In this section we prove a downcrossing inequality for reflected g-supermartingales in the spirit of the one proved in [22]. We use the same notations as in the classical theory of g-martingales (see [22] and [86] for instance).

Theorem 5.6.3. *Assume that* $g(0,0) = 0$*. Let* (Y_t) *be a positive reflected* g*-supermartingale in the weak sense and let* $0 = t_0 < t_1 < ... < t_i = T$ *be a subdivision of* $[0,T]$*. Let* $0 \leq < a < b$*, then there exists* $C > 0$ *such that* $D_a^b[Y,n]$*, the number of downcrossings of* $[a,b]$ *by* $\{Y_{t_j}\}$*, verifies*

$$\mathcal{E}^{-\mu}[D_a^b[Y,n]] \leq \frac{C}{b-a}\mathcal{E}^{\mu}[Y_0 \wedge b],$$

where μ *is the Lipschitz constant of* g*.*

Proof. Consider

$$\begin{cases} y_t^i = Y_{t_i} + \int_t^{t_i} ds + \int_t^T (\mu \left|y_s^i\right| + \mu \left|z_s^i\right| ds + k_T^n - k_{t_j}^n - \int_t^{t_i} z_s^i dW_s,\ 0 \leq t \leq t_i,\ \mathbb{P}-a.s. \\ y_t^i \geq S_t,\ \mathbb{P}-a.s. \\ \int_0^{t_i} \left(y_{s^-}^i - S_{s^-}\right) dk_s^i = 0,\ \mathbb{P}-a.s.,\ \forall t \in [0, t_i]. \end{cases}$$

We define $a_s^i := -\mu \text{sgn}(z_s^i) 1_{t_{j-1} < s \leq t_j}$ and $a_s := \sum_{i=0}^n a_s^i$. Let $\mathbb{Q}^a$ be the probability measure defined by

$$\frac{d\mathbb{Q}^a}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T a_s dW_s\right).$$

We then have easily that $y_t^i \geq 0$ since $Y_{t_i} \geq 0$ and

$$y_t^i = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,t_i}} \mathbb{E}_t^{\mathbb{Q}^a}\left[e^{-\mu(\tau - t)} S_\tau 1_{\tau < t_i} + Y_{t_i} e^{-\mu(t_i - t)} 1_{\tau = t_i}\right].$$

Since Y is reflected g-supermartingale (and thus also a reflected $g^{-\mu}$-supermartingale where $g_s^{-\mu}(y,z) := -\mu(|y| + |z|))$, we therefore obtain

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t_{i-1},t_i}} \mathbb{E}_{t_{i-1}}^{\mathbb{Q}^a}\left[e^{-\mu(\tau - t_{i-1})} S_\tau 1_{\tau < t_i} + Y_{t_i} e^{-\mu(t_i - t_{i-1})} 1_{\tau = t_i}\right] \leq Y_{t_{i-1}}.$$

Hence, by choosing $\tau = t_j$ above, we get

$$\mathbb{E}_{t_{i-1}}^{\mathbb{Q}^a}\left[Y_{t_i} e^{-\mu(t_i - t_{i-1})}\right] \leq Y_{t_{i-1}},$$

which implies that $(e^{-\mu t_i} Y_{t_i})_{0 \leq i \leq n}$ is a $\mathbb{Q}^a$-supermartingale. Then we can finish the proof exactly as in [22]. □

Chapitre 6

Non-dominated Monotone Convergence Theorem

Contents

6.1 Introduction

As we pointed out in the Introduction, one of the most important techniques used in the classical BSDE litterature to prove existence of solutions is the so-called approximation technique. This technique is important, because it allows to prove existence of maximal or minimal solutions of some BSDEs, even though uniqueness may not hold (see for instance [68] in the quadratic case). Similarly, this is the proof of existence of reflected BSDEs by penalization in [44] which allowed them to prove a probabilitic representation of solution of quasi-linear PDEs with obstacles. Therefore, we feel that it would be important to be able to use the same techniques in the 2BSDE framework. However, their use raises subtle technical difficulties in that case, and this short chapter aims at providing further insight into those problems. More precisely, we recall the monotone convergence theorem proved in [40] and show why its assumptions are too restrictive to use it in our context.

6.2 A counter-example to the monotone convergence theorem

Let $\mathcal{P}_0 \subset \overline{\mathcal{P}}_W$. In general in a non-dominated framework, the monotone convergence Theorem may not hold, that is to say that even if we have a sequence of random variables X_n which decreasingly converges $\mathcal{P}_0 - q.s.$ to 0, then we may not have that

$$\sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}[X_n] \downarrow 0.$$

Indeed, let us consider for instance the set

$$\mathcal{P}_1 := \left\{\mathbb{P}^p := \mathbb{P}_0 \circ (\sqrt{p}B),\ p \in \mathbb{N}^*\right\}.$$

Then, define $Y_n := \frac{B_1^2}{n}$. It is clear that the sequence Y_n decreases $\mathbb{P}$-a.s. to 0 for all $\mathbb{P} \in \mathcal{P}_1$. But we have for all p and all n

$$\mathbb{E}^{\mathbb{P}^p}\left[Y_n\right] = \frac{p}{n},$$

which implies that $\sup_{\mathbb{P}\in\mathcal{P}_1} \mathbb{E}^{\mathbb{P}}[Y_n] = +\infty$.

It therefore clear that it is necessary to add more assumptions in order to recover a monotone convergence theorem. Notice that those assumptions will concern both the family $\mathcal{P}_0$ and the random variables considered. For instance, in the above example, this the fact that the set $\mathcal{P}_1$ considered is not weakly compact which implies that the monotone convergence theorem can fail.

6.3 The monotone convergence theorem of [40]

We start this section by recalling some definitions

Definition 6.3.1. (i) *We say that a family of random variables $(X^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ is $\mathcal{P}_0$-uniformly integrable if*

$$\lim_{C\to+\infty} \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left[\left|X^{\mathbb{P}}\right| 1_{|X^{\mathbb{P}}|>C}\right] = 0.$$

(ii) *We say that a family of random variables $(X^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ is $\mathcal{P}_0$-quasi continuous if for all $\mathbb{P}\in\mathcal{P}_0$ and for all $\varepsilon>0$, there exists an open set $\mathcal{O}^{\mathbb{P},\varepsilon}$ such that*

$$\mathbb{P}(\mathcal{O}^{\mathbb{P},\varepsilon}) \leq \varepsilon \text{ and } X^{\mathbb{P}} \text{ is continuous in } \omega \text{ outside } \mathcal{O}^{\mathbb{P},\varepsilon}.$$

If the above definitions, if the family of random variables can be aggregated into one universal random variable X, we keep the same terminology, with the following modification

(iii) *We say that an aggregated random variables X is $\mathcal{P}_0$-quasi continuous if for all $\varepsilon>0$, there exists an open set $\mathcal{O}^{\varepsilon}$ such that*

$$\sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{P}(\mathcal{O}^{\varepsilon}) \leq \varepsilon \text{ and } X \text{ is continuous in } \omega \text{ outside } \mathcal{O}^{\varepsilon}.$$

Remark 6.3.1. *A sufficient condition for a family of random variables $(X^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ to be $\mathcal{P}_0$-uniformly integrable is that*

$$\sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left[\left|X^{\mathbb{P}}\right|^{1+\varepsilon}\right] < +\infty, \text{ for some } \varepsilon>0.$$

We then have the following monotone convergence theorem proved in [40]

Theorem 6.3.1 (Denis, Hu, Peng)**.** *Let $\mathcal{P}_0$ be a weakly compact family and let X_n be a sequence of $\mathcal{P}_0$-uniformly integrable random variables which verifies that*

$$X_n(\omega) \downarrow 0, \text{ for every } \omega \in \Omega\backslash\mathcal{N}, \text{ where } \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{P}(\mathcal{N}) = 0.$$

Then

$$\sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}[X_n] \downarrow 0.$$

It is obvious that this Theorem is particularly suited for a capacity framework, as considered in [38] or [40]. However in our case, we do not work with capacities, and all our properties hold only $\mathbb{P}$-a.s. for all probability measures in $\mathcal{P}_0$, which generally makes this Theorem not general enough for our purpose. The notable exception is of course the first Chapter of this thesis, where we managed to apply Theorem 6.3.1 to quantities which were not only defined for every ω, but were also continuous in ω. In that case, this difference between capacities and probabilities is not important.

We will now look at the different assumptions of Theorem 6.3.1 and show why they cannot always be met in our context.

6.4 The aggregation problem

If we want to prove existence for 2BSDEs with approximation techniques in more general cases, we are bound to have to prove that

$$\sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}[|y_t^{\mathbb{P}} - y_t^{n,\mathbb{P}}|] \downarrow 0,$$

where $y^{\mathbb{P}}$ and $y^{n,\mathbb{P}}$ are respectively the solutions of the BSDEs associated with the 2BSDE with generator F, and the solutions of the BSDEs associated with the 2BSDE with generator F^n.

In the linear case of Chapter 2, we managed to control those quantities by the difference of the generators, and we used the fact that the convergence of F^n to F was uniform in y and z. However in more general cases, this strong convergence may not hold, and one of our only remaining options is to try and use a monotone convergence theorem.

Let us note

$$X_n^{\mathbb{P}} = |y_t^{\mathbb{P}} - y_t^{n,\mathbb{P}}|.$$

The first problem in that case is that we have a family of random variables indexed by the underlying probability measure. Thus, if we want to use Theorem 6.3.1, we need to aggregate this family. However, this is generally not possible. Indeed, let us consider for instance the Wiener measure $\mathbb{P}_0$ and the measure $\mathbb{P}^2$ which is the law under the Wiener measure of $\sqrt{2}B_.$. Then, if we consider BSDEs with generator equal to 0 and terminal condition B_T^2, we can compute that

$$\begin{aligned} y_t^{\mathbb{P}_0} &= T - t + B_t^2,\ \mathbb{P}_0 - a.s. \\ y_t^{\mathbb{P}_2} &= 2(T-t) + B_t^2,\ \mathbb{P}_2 - a.s. \end{aligned}$$

Hence, in general, solutions of BSDEs with the same generator and the same terminal condition but considered under singular probability measures cannot be aggregated.

6.5 Application in the case of the family $\widetilde{\mathcal{P}}_S$

Since we are working with the set $\widetilde{\mathcal{P}}_S$ defined in (2.2.3), it is naturel to wonder whether it verifies the assumptions of Theorem 6.3.1 or not. First of all, it has been proved in [38] and [40] that with our definition (2.2.2), $\widetilde{\mathcal{P}}_S$ is weakly relatively compact. However, it is not closed for the weak topology. Hence, if we want to apply Theorem 6.3.1, it is necessary to restrict ourselves to a subset of $\widetilde{\mathcal{P}}_S$ which is closed for the weak topology or to be able to work with the closure of $\widetilde{\mathcal{P}}_S$ instead. In Chapter 2, we use the second possibility, since the quantity to which we apply the theorem is defined for all ω, and thus can easily be defined on the support of all probability measures in the closure of $\widetilde{\mathcal{P}}_S$. However, as we already emphasized above, we will need in general to use a monotone convergence theorem with families of solutions of BSDEs indexed by the underlying probability measure. The main problem in that case is that a probability measure in the closure of $\widetilde{\mathcal{P}}_S$ do not necessarily satisfy the predictable representation property. Since the filtration of the canonical process is quasi-left continuous, we know that in that case, it is necessary to add a martingale orthogonal to B in the definition of a BSDE (see El Karoui and Huang [45] for more details). This therefore would further complicate an already complicated situation.

Let us now provide a simple result related to weak compactness assumption of Theorem 6.3.1.

Lemma 6.5.1. *Let $\mathcal{A}$ be a family of $\mathcal{F}$-progressively measurable processes $(\alpha_t)_{0\leq t\leq T}$ taking values between $\underline{a}$ and $\bar{a}$ and defined $\mathbb{P}_0 - a.s.$, which is compact, in the sense that from any sequence in $\mathcal{A}$, we can extract a subsequence which converges $dt \times d\mathbb{P}_0 - a.s.$ Then the restriction of $\widetilde{\mathcal{P}}_S$ to the processes in $\mathcal{A}$ is weakly compact.*

Proof. Let $\mathbb{P}^{\alpha^n}$ be a sequence of probbility measures in this restriction which converges weakly to some $\mathbb{P}$. Since $\mathcal{A}$ is compact, extracting a subsequence if necessary, we may assume that

$$\alpha^n \to \alpha, \ dt \times d\mathbb{P}_0 - a.s.,$$

for some process $\alpha \in \mathcal{A}$.

Then, by the dominated convergence theorem for stochastic integrals, we know that in probability

$$X_t^{\alpha^n} \to X_t^{\alpha}.$$

We can then extract a further subsequence such that this convergence holds $dt \times d\mathbb{P}_0 - a.s.$ Consequently, for any continuous bounded function g, we have

$$\mathbb{E}^{\mathbb{P}^{\alpha^n}}\left[g(B_\cdot)\right] = \mathbb{E}^{\mathbb{P}_0}\left[g(X_\cdot^{\alpha^n})\right] \to \mathbb{E}^{\mathbb{P}_0}\left[g(X_\cdot^{\alpha})\right] = \mathbb{E}^{\mathbb{P}^{\alpha}}\left[g(B_\cdot)\right],$$

which means that $\mathbb{P}^{\alpha^n}$ converges weakly to $\mathbb{P}^{\alpha}$, and therefore by uniqueness of the limit $\mathbb{P} = \mathbb{P}^{\alpha}$. Hence, we have a closed subset of a weakly relatively compact set, which is therefore weakly compact. □

Therefore, by restricting the set of probability measures, we can always recover the weak compactness assumption.

6.6 Conclusion

During our research, we tried to generalize the monotone convergence Theorem 6.3.1, but our results (which are not reported here since they are not useful) only improved it by allowing to consider non-aggregated family of random-variables which satisfied a certain consistency condition (which is closely related to the consistency condition of [109]). But families of solutions of BSDEs indexed by the underlying probability measure do not satisfy this consistency condition. This is the main reason why we did not manage to prove existence of 2BSDEs by approximation techniques. However, we want to emphasize that if one were to prove a monotone convergence theorem which could be applied to our quantities of interest, then we already have those proofs of existence, using the general approach we introduced in Chapter 2. Nonetheless, the technical difficulties behind such a monotone convergence theorem are far from trivial, and we will hopefully manage to address them in future research.

Partie II

Contemporary issues in mathematical finance

Chapitre 7

Large liquidity expansion of the superhedging costs

Contents

7.1 Introduction

The classical option pricing equation of Black & Scholes is derived under several simplifying assumptions. The "infinite" liquidity of the underlying stock process is one of them. In an attempt to understand the impact of liquidity, Çetin, Jarrow, Protter and collaborators [18, 19, 20] postulated the existence of a supply curve $\mathbf{S}(t,s,\nu)$ which is the price of a share of the stock when one wants to buy ν shares at time t. In the Black & Scholes setting, this price function is taken to be independent of ν corresponding to infinite amount of supply, hence infinite liquidity. In a recent paper, Çetin, Soner and Touzi [21] used this model and studied the liquidity premium in the price of an option written on such a stock with less than infinite liquidity. They characterized the option price by a nonlinear Black & Scholes equation, given in (7.2.3) below. In this pricing equation the liquidity manifests itself by means of a *liquidity function* ℓ, which is given by

$$\ell(t,s) := \left[4\,\frac{\partial \mathbf{S}}{\partial \nu}(t,s,0)\right]^{-1}, \quad (t,s) \in [0,T] \times \mathbb{R}_+.$$

The liquidity function ℓ measures the level of liquidity of the market. Namely, the larger ℓ is, the more liquid the market is.

The main result of [21] is the characterization of the liquidity premium as the unique viscosity solution of a nonlinear Black-Scholes equation (7.2.3), which is very similar to the one derived by Barles and Soner [5]. This nonlinear equation can only be solved numerically as no explicit solutions are available. Motivated by this fact, in this paper we obtain rigorous asymptotic expansions for the liquidity premium. For vanilla options with sufficiently regular payoff, this expansion can be calculated explicitly giving further insight into the liquidity effects.

As stated the chief objective of this paper is to analyze the large liquidity effect. Thus, we assume that the supply function depends on a small parameter ε

$$\mathbf{S}^\varepsilon(t,s,\nu) := \mathbf{S}(t,s,\varepsilon\nu), \qquad (t,s) \in [0,T] \times \mathbb{R}_+.$$

Then, the corresponding liquidity function is given by

$$\ell^\varepsilon(t,s) := \frac{1}{\varepsilon}\ell(t,s), (t,s) \in [0,T] \times \mathbb{R}_+ .$$

Hence, as ε tends to zero, the market becomes completely liquid. So we expect the price of an option V^ε to converge to the classical Black-Scholes price, v^{BS}, and we are interested in expansions of the form

$$V^\varepsilon = v^{BS} + \varepsilon v^{(1)} + \ldots + \varepsilon^n v^{(n)} + \ +o(\varepsilon^n).$$

Indeed, we prove this type of results and identify the functions $v^{(n)}$ in some cases. In particular, we show that

$$v^{(1)}(t,s) = \int_t^T \ \mathbb{E}_{t,s}\left[\frac{S_u^2\sigma^2(u,S_u)}{4\ell(u,S_u)}\left(v_{ss}^{BS}(u,S_u)\right)^2\right] du. \tag{7.1.1}$$

This is exactly the liquidity premium of the standard Black-Scholes hedge.

The chapter is organized as follows.The problem is introduced in the next section and the approach is formally introduced in Section 7.3. Under a strong smoothness assumption, full expansion is obtained in Section 7.4. A quick convergence result is proved in Section 7.5. The Call option is studied in Section 7.6 and the Digital option in the final section.

7.2 The general setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a Brownian motion W with completed canonical filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$, where $T > 0$ is fixed maturity. The marginal price process S_t is defined by the stochastic differential equation

$$\frac{dS_t}{S_t} = \sigma(t,S_t)dW_t,$$

where σ is assumed to be bounded, Lipschitz-continuous and uniformly elliptic.

Given a continuous portfolio strategy Y with finite quadratic variation process $\langle Y \rangle$, the *small time liquidation value of the portfolio* is given by

$$dZ_t^{\varepsilon,Y} = Y_t dS_t - [4\ell^\varepsilon(t,S_t)]^{-1}\, d\langle Y\rangle_t = Y_t dS_t - \varepsilon\, [4\ell(t,S_t)]^{-1}\, d\langle Y\rangle_t.$$

The dependence of the process Z on its initial condition is suppressed for simplicity. Given a function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfying

$$g \quad \text{is bounded from below and} \quad \sup_{s>0} \frac{g(s)}{1+s} \ < \ \infty, \tag{7.2.1}$$

the super-hedging cost is defined by

$$V^\varepsilon(t,s) \;:=\; \inf\left\{z \;:\; Z_t^{\varepsilon,Y} = z \text{ and } Z_T^{\varepsilon,Y} \geq g\left(S_T\right)\ \mathbb{P}\text{-}as \text{ for some } Y \in \mathcal{A}_{t,s}\right\}, \qquad (7.2.2)$$

where the time origin is removed to t and the initial condition for the price process is $S_t = s$. We refer to [21] for the precise definition of the set of admissible strategies $\mathcal{A}_{t,s}$.

This problem is similar to the super-replication problem studied extensively in [23, 24, 25, 102, 103, 104]. In the above setting, it is shown in Çetin, Soner and Touzi [21] that the value function of the super-hedging problem is the unique viscosity solution of the following nonlinear equation,

$$-V_t^\varepsilon + \hat{H}^\varepsilon\left(t,s,V_{ss}^\varepsilon\right) \;=\; 0, \quad \text{on} \quad [0,T)\times(0,\infty), \qquad (7.2.3)$$

satisfying the terminal condition $V^\varepsilon(T,.) = g$ and the growth condition

$$-C \;\leq\; V^\varepsilon(t,s) \;\leq\; C(1+s),\ (t,s)\in[0,T]\times\mathbb{R}_+,\ \text{for some constant } C>0\,. \qquad (7.2.4)$$

Here, $\hat{H}^\varepsilon$ denotes the elliptic majorant of the first guess operator H^ε:

$$\hat{H}^\varepsilon(t,s,\gamma) := \sup_{\beta\geq 0}\ H^\varepsilon(t,s,\gamma+\beta)$$
$$H^\varepsilon(t,s,\gamma) := -\frac{1}{2}s^2\sigma^2(t,s)\gamma - \varepsilon[4\ell(t,s)]^{-1}s^2\sigma^2(t,s)\gamma^2.$$

By direct calculation, it follows that

$$\hat{H}^\varepsilon(t,s,\gamma) = -\frac{1}{2}s^2\sigma^2(t,s)\left[\gamma + \left(\gamma+\frac{\ell(t,s)}{\varepsilon}\right)^- + \frac{\varepsilon}{2\ell(t,s)}\left(\gamma+\left(\gamma+\frac{\ell(t,s)}{\varepsilon}\right)^-\right)^2\right].$$

For $\varepsilon = 0$, both $\hat{H}^\varepsilon$, H^ε coincides with the following standard elliptic operator,

$$\hat{H}^0(t,s,\gamma) = H^0(t,s,\gamma) = -\frac{1}{2}s^2\sigma^2(t,s)\gamma, \qquad (t,s,\gamma)\in[0,T]\times\mathbb{R}_+\times\mathbb{R}.$$

Hence, the equation (7.2.3) reduces to the linear Black-Scholes equation

$$-\frac{\partial v^{BS}}{\partial t} - \frac{1}{2}s^2\sigma^2(t,s)v_{ss}^{BS} = 0. \qquad (7.2.5)$$

We recall the well-known fact that its unique solution, v^{BS}, is the Black-Scholes price,

$$v^{BS}(t,s) = \mathbb{E}_{t,s}\left[g(S_T)\right],\ (t,s)\in[0,T]\times\mathbb{R}_+,$$

where we used the notation $\mathbb{E}_{t,s} = \mathbb{E}[\,\cdot \mid S_t = s]$.

7.3 Formal calculations and assumptions

It is formally clear that as the market becomes more liquid, V^ε should converge to the Black-Scholes price v^{BS}. Indeed, this is proved in Section 7.5. We are also interested in a Taylor expansion of V^ε in the parameter ε, i.e.,

$$V^\varepsilon(t,s) = v^{BS}(t,s) + \varepsilon v^{(1)}(t,s) + \varepsilon^{(2)}v^2(t,s) + \ldots + \varepsilon^n v^{(n)}(t,s) + o(\varepsilon^n), \qquad (7.3.1)$$

where $o(\varepsilon^n)$ is the standard notation, indicating that $o(\varepsilon^n)/\varepsilon^n$ converges to zero as ε tends to zero.

Indeed, under sufficient regularity

$$v^{(n)}(t,s) = \frac{1}{n!} \left.\frac{\partial^n V^\varepsilon(t,s)}{\partial \varepsilon^n}\right|_{\varepsilon=0}.$$

Thus, formally differentiate the equation (7.2.3) n-times with respect to ε and then set ε to zero. Using the above formal definition of $v^{(n)}$, we arrive at,

$$0 = -v_t^{(n)} - \frac{1}{2}s^2\sigma^2(t,s)v_{ss}^{(n)} - F_n(t,s) \tag{7.3.2}$$

$$F_n(t,s) = \frac{s^2\sigma^2(t,s)}{4\ell(t,s)} \sum_{k=0}^{n-1} \left[v_{ss}^{(k)}(t,s)\ v_{ss}^{(n-1-k)}(t,s)\right], \tag{7.3.3}$$

where we set $v^{(0)} := v^{BS}$. For all $n \geq 1$, the terminal data is $v^{(n)}(T,\cdot) \equiv 0$, so that the Feymann-Kac formula yields

$$v^{(n)}(t,s) = \sum_{k=0}^{n-1} \mathbb{E}_{t,s}\left[\int_t^T \left(\frac{S_u^2\sigma^2}{4\ell} v_{ss}^{(k)} v_{ss}^{(n-1-k)}\right)(u,S_u)du\right]. \tag{7.3.4}$$

In particular, $v^{(1)}$ is given as in (7.1.1).

The above calculations yield a rigorous proof when the pay-off is sufficiently regular. We will prove this in Section 7.4. On the other hand, for some discontinuous pay-offs the above functions may not be finite. For instance, for a digital option, $v^{(1)} \equiv \infty$. Indeed, if we take

$$g(s) := 1_{s\geq K},\ \sigma(t,s) \equiv \sigma \text{ and } \ell(t,s) \equiv \ell,$$

we compute that

$$\begin{aligned} v^{(1)}(t,s) = {} & \frac{1}{8\pi\ell\sigma^2}\int_t^T \frac{(u-t)e^{-\left(\frac{1}{\sigma\sqrt{T+u-2t}}\ln\left(\frac{s}{K}\right)+\frac{\sigma}{2}\frac{T-2u+t}{\sqrt{T+u-2t}}\right)^2}}{(T-u)^{\frac{3}{2}}(T+u-2t)^{\frac{3}{2}}} \\ & + \frac{1}{8\pi\ell\sigma^2}\int_t^T \frac{e^{-\left(\frac{1}{\sigma\sqrt{T+u-2t}}\ln\left(\frac{s}{K}\right)+\frac{\sigma}{2}\frac{T-2u+t}{\sqrt{T+u-2t}}\right)^2}}{\sqrt{T-u}(T+u-2t)^{\frac{3}{2}}} \left(\frac{\ln\left(\frac{s}{K}\right)}{\sigma\sqrt{T+u-2t}} + \frac{\sigma}{2}\frac{T-2u+t}{\sqrt{T+u-2t}}\right)^2. \end{aligned}$$

The first term above is actually $+\infty$ because of the non-integrability of $(T-u)^{-3/2}$ near T.

In such cases, the expansion is not valid and a careful study of the behavior of V^ε near the terminal data is needed. This will be done in Section 7.7. However, we first prove the full expansion in the "smooth" case. Then, in Section 7.6, we consider the Call option proving the expansion up to $n = 2$. Clearly, this later result extends to all Put options. Also, remarks on other payoffs and higher expansions are given in Remarks 7.6.2 and 7.6.1.

7.4 Expansion for smooth pay-offs

In this section, we prove the expansion under the assumption that there is a constant $\hat{C}$ so that

$$-\hat{C} \leq v^{(n)}(t,s) \leq \hat{C}(1+s), \left|(s^2+1)v_{ss}^{(n)}(t,s)\right| \leq \hat{C}, \tag{7.4.1}$$
$$|F_n(t,s)| \leq \hat{C},\ \forall\ (t,s) \in [0,T]\times\mathbb{R}_+,\ n = 1,2,\ldots.$$

Clearly, this is an implicit assumption on the pay-off g. Essentially, it holds for all smooth pay-offs growing at most linearly. In particular, (7.4.1) holds if $\sigma(t,s) \equiv \sigma$, $\ell(t,s) \equiv \ell$ and if there exists a constant C so that

$$-C \le g(s) \le C(1+s), \quad \left|(s^2+1)\frac{\partial^n}{\partial s^n} g(s)\right| \le C, \ \forall\ s \in \mathbb{R}_+,\ n = 2, 3, \ldots.$$

This is proved by using the homogenity of the Black-Scholes equation and differentiating it repeatedly.

Following the techniques developed in the papers [53, 51, 71, 100, 101], for an integer $n \ge 0$ we define,

$$V^{\varepsilon,n}(t,s) := \frac{V^\varepsilon(t,s) - \sum_{k=0}^{n-1} \varepsilon^k v^{(k)}(t,s)}{\varepsilon^n}\ , \tag{7.4.2}$$

where as before we set $v^{(0)} = v^{BS}$.

Theorem 7.4.1. *Assume (7.4.1). Then, for every $n = 1, 2, \ldots$, there are constants C_n and $\varepsilon_0 > 0$ so that for every $\varepsilon \in (0, \varepsilon_0]$, and $n = 1, 2, \ldots$,*

$$v^{BS}(t,s) \le V^\varepsilon(t,s) \le v^{\varepsilon,n}(t,s) := \sum_{k=0}^{n-1} [\varepsilon^k v^{(k)}(t,s)] + \varepsilon^n C_n (T-t). \tag{7.4.3}$$

In particular, as $\varepsilon \downarrow 0$, V^ε converges to the Black-Scholes price v^{BS} uniformly on compact sets. Moreover, for every $n \ge 1$, $V^{\varepsilon,n}$ converges to $v^{(n)}$, again uniformly on compact sets.

Proof. Clearly, $v^{BS} \le V^\varepsilon$. We continue by proving the upper bound. Let $v^{\varepsilon,n}$ be as in (7.4.3) with a constant C_n to be determined below. Using (7.3.2), we calculate that

$$\begin{aligned} -v_t^{\varepsilon,n}(t,s) + \hat{H}^\varepsilon(t,s,v_{ss}^{\varepsilon,n}(t,s)) &\ge -v_t^{\varepsilon,n}(t,s) + H^\varepsilon(t,s,v_{ss}^{\varepsilon,n}(t,s)) \\ &= -v_t^{\varepsilon,n} - \frac{1}{2}s^2\sigma^2 v_{ss}^{\varepsilon,n} - \frac{\varepsilon s^2\sigma^2}{4\ell(t,s)}(v_{ss}^{\varepsilon,n})^2 \\ &= \varepsilon^n C_n + \sum_{k=1}^{n-1}[\varepsilon^k\ F_k(t,s)] - \frac{\varepsilon s^2\sigma^2}{4\ell(t,s)}(v_{ss}^{\varepsilon,n})^2. \end{aligned}$$

In view of (7.3.3),

$$\frac{\varepsilon s^2\sigma^2}{4\ell(t,s)}(v_{ss}^{\varepsilon,n})^2 - \sum_{k=1}^{n-1}[\varepsilon^k\ F_k(t,s)] = \varepsilon^n F_n(t,s) + \varepsilon^{n+1}\frac{s^2\sigma^2}{4\ell(t,s)}\ g^\varepsilon(t,s),$$

where $g^\varepsilon(t,s)$ is a quadratic function $v_{ss}^{(k)}(t,s)$ for $k \le n$ and possibly powers of ε. Hence by (7.4.1), there is a constant C_n,

$$\left|\sum_{k=1}^{n-1}[\varepsilon^k\ F_k(t,s)] - \frac{\varepsilon s^2\sigma^2}{4\ell(t,s)}(v_{ss}^{\varepsilon,n})^2\right| \le \varepsilon^n C_n.$$

Hence, we conclude that $v^{\varepsilon,n}$ is a supersolution of (7.2.3). Moreover, by (7.4.1), $-C \le v^{\varepsilon,n}(t,s) \le C(1+s)$. Then, by the comparison theorem for (7.2.3) (Theorem 6.1 of [21]), we conclude that $V^\varepsilon(t,s) \le v^{\varepsilon,n}(t,s)$.

In particular, this estimate implies the convergence of V^ε to v^{BS}. To prove the convergence of $V^{\varepsilon,n}$, we first observe that

$$V^\varepsilon = \sum_{k=0}^{n} [\varepsilon^k v^{(n)}(t,s)] + \varepsilon^n V^{\varepsilon,n}.$$

Using the equations (7.2.3) and (7.3.2), we conclude that $V^{\varepsilon,n}$ is a viscosity solution of

$$-V_t^{\varepsilon,n} - \frac{1}{2}s^2\sigma^2(t,s)V_{ss}^{\varepsilon,n} + F^{\varepsilon,n}\left(t,s,V_{ss}^{\varepsilon,n}\right) = 0, \qquad (t,s) \in [0,T) \times \mathbb{R}_+,$$

where

$$F^{\varepsilon,n}(t,s,\gamma) := \frac{1}{\varepsilon^n}\left[\hat{H}^\varepsilon(t,s,v_{ss}^{\varepsilon,n}(t,s) + \varepsilon^n\gamma) + \frac{1}{2}s^2\sigma^2 v_{ss}^{\varepsilon,n} + \sum_{k=1}^{n-1} \varepsilon^k F^k(t,s)\right].$$

Tedious but a straightforward calculation shows that

$$\lim_{(t',s',\gamma',\varepsilon)\to(t,s,\gamma,0)} F^{\varepsilon,n}(t',s',\gamma') \quad = \quad F^n(t,s),$$

where F_n is as in (7.3.3).

Then, by the classical stability results of viscosity solutions [4, 29, 52], the Barles-Perthame semi-relaxed limits

$$\underline{v}^{(n)}(t,s) := \liminf_{(t',s',\varepsilon)\to(t,s,0)} V^{\varepsilon,n}(t',s') \text{ and } \overline{v}^{(n)}(t,s) := \limsup_{(t',s',\varepsilon)\to(t,s,0)} V^{\varepsilon,n}(t',s'),$$

are, respectively, a viscosity supersolution and a subsolution of the equation (7.3.2) satisfied by $v^{(n)}$. Moreover it follows from (7.4.3) that

$$\underline{v}^{(n)}(T,\cdot) = \overline{v}^{(n)}(T,\cdot) = 0 = v^{(n)}(T,\cdot).$$

We now use the comparison result for the linear partial differential equation (7.3.2), and conclude that $\underline{v}^{(n)} \geq \overline{v}^{(n)}$. Since

$$\underline{v}^{(n)}(t,s) \leq \liminf_{\varepsilon\to 0} V^{\varepsilon,n}(t,s) \leq \limsup_{\varepsilon\to 0} V^{\varepsilon,n}(t,s) \leq \overline{v}^{(n)}(t,s),$$

on $[0,T]\times\mathbb{R}_+$, this proves that $\underline{v}^{(n)} = \overline{v}^{(n)} = v^{(n)}$.

Hence, $V^{\varepsilon,n}$ converges to the unique solution $v^{(n)}$, uniformly on compact sets.

□

7.5 A general convergence result

In this section, we prove an easy convergence result under the following general assumption. We assume that

$$cs^2 \leq \ell(t,s), \tag{7.5.1}$$

for some constant c and that

Assumption 7.5.1. *There is a decreasing sequence of smooth approximation* $g_m \geq g$ *of the payoff* g *satisfying* (7.4.1) *with* $n = 1, 2$. *Let* $v_m^{(n)}$, F_m^n *be the previously defined functions with payoff* g_m. *Then,* $F_m^1(t,s) \leq c_m$ *for some constant* c_m.

This assumption is satisfied by all Lipschitz or for all bounded pay-offs.

Theorem 7.5.1. *Assume* (7.2.1), (7.5.1) *and that Assumption 7.5.1 holds true. Then, as the liquidity parameter goes to infinity, or equivalently as* $\varepsilon \downarrow 0$, V^ε *converges to the Black-Scholes price* v^{BS}.

Proof. Let c_m be as above and set

$$u^\varepsilon(t,s) := v_m^{BS}(t,s) + \varepsilon c_m (T-t).$$

As in the proof of Theorem 7.4.1, we can show that u^ε is a super-solution of (7.2.3). Hence, $V^\varepsilon \leq u^\varepsilon$. Therefore,

$$\limsup_{\varepsilon \downarrow 0} \; V^\varepsilon(t,s) \leq v_m^{BS}(t,s).$$

By (7.2.1), $v_m^{BS}(t,s)$ converges to $v^{BS}(t,s)$. Since $V^\varepsilon \geq v^{BS}$, this proves the convergence of V^ε to v^{BS}.

□

7.6 First order expansion for convex payoffs

One major limitation of our previous result is that the Call pay-off does not satisfy the Assumption (7.4.1). Therefore, in this section, we prove the first term in the Taylor expansion (7.3.1), i.e.,

$$V^\varepsilon(t,s) = v^{BS}(t,s) + \varepsilon v^{(1)}(t,s) + o(\varepsilon), \tag{7.6.1}$$

for convex payoffs satisfying weaker assumptions than (7.4.1). In particular, we will show that call options verify those assumptions.

7.6.1 The general result

In order to capitalize on the results we have already obtained for smooth payoffs, we will also consider a regularized version of our problem

$$\begin{aligned} -V_t^{\varepsilon,\alpha} + \widehat{H}^\varepsilon(t,s,V_{ss}^{\varepsilon,\alpha}) = 0, \;\; \text{for } (t,s) \in [0,T) \times \mathbb{R}_+ \\ V^{\varepsilon,\alpha}(T,s) = \widehat{g}_\alpha(s), \end{aligned} \tag{7.6.2}$$

where $\widehat{g}_\alpha(s) = \phi_\alpha * g(s)$ with $\phi_\alpha(\cdot) := \frac{1}{\alpha}\phi(\frac{\cdot}{\alpha})$ and ϕ is a positive, symmetric bump function on $\mathbb{R}$, compactly supported in $[-1,1]$ and satisfying

$$\int_{-1}^{1} \phi(u)du = 1.$$

By convexity of g, for all $\alpha > 0$ we have $\widehat{g}_\alpha \geq g$, so that by monotony of our problem

$$V^\varepsilon \leq V^{\varepsilon,\alpha}.$$

Thus, since the main idea of our proof is to find a super-solution of (7.2.3), we see that it is enough to find a super-solution of (7.6.2). Let $v^{BS,\alpha}$ and $v^{(1),\alpha}$, respectively, be the Black-Scholes price and the first-order expansion term for the regularized option. We now state our assumptions

Assumption 7.6.1. (i) $v^{BS} + v^{BS,\alpha} + v^{(1)} + v^{(1),\alpha} < +\infty$.

(ii) *As α tends to 0 we have*

$$v^{BS,\alpha}(t,s) = v^{BS}(t,s) + O(\alpha^2)$$
$$v^{(1),\alpha}(t,s) = v^{(1)}(t,s) + o(1).$$

(iii) *There exists a constant c_* independent of s, $T-t$ and α and $(\nu,\beta) \in [0,1]\times[1/2,1]$ such that $1 < 2\beta+\nu < 2$ and*

$$\frac{s^2\sigma^2}{4\ell}(v_{ss}^{(1),\alpha}(t,s))^2 \leq \frac{c_*}{(T-t)^{1-\nu}\alpha^{2+2\nu}}, \qquad s\left|v_{ss}^{BS,\alpha}(t,s)\right| \leq \frac{c_*}{(T-t)^{1-\beta}\alpha^{2\beta-1}}.$$

This assumption will be proved to be verified by Call options payoffs in subsection 7.6.2.

Let $V^{\varepsilon,1}$ be as (7.4.2), i.e.

$$V^{\varepsilon,1}(t,s) := \frac{V^\varepsilon(t,s) - v^{BS}(t,s)}{\varepsilon}.$$

Theorem 7.6.1. *Let Assumption 7.6.1 hold true and let $a \in (\frac{1}{2}, \frac{1}{2\beta+\nu})$. Then for every $(t,s) \in [0,T]\times\mathbb{R}_+$ we have,*

$$v^{BS} \leq V^\varepsilon \leq v^{BS,\varepsilon^a} + \varepsilon v^{(1),\varepsilon^a} + c_*(T-t)^{\beta+\frac{\nu-1}{2}}\varepsilon^{2-a(\nu+2\beta)} + c_*(T-t)^\nu\varepsilon^{3-2a(1+\nu)}.$$

Moreover, $V^\varepsilon \to v^{BS}$, $V^{\varepsilon,1}\to v^{(1)}$ uniformly on compact sets, and (7.6.1) holds true.

Proof. It is clear that $V^\varepsilon \geq v^{BS}$. To prove the reverse inequality, we start by following a technique similar to the one used in the proof of Theorem 7.4.1. Set

$$v^{\varepsilon,2} := v^{BS,\varepsilon^a} + \varepsilon v^{(1),\varepsilon^a} + c_*(T-t)^{\beta+\frac{\nu-1}{2}}\varepsilon^{2-a(\nu+2\beta)} + c_*(T-t)^\nu\varepsilon^{3-2a(1+\nu)}.$$

We calculate that for $(t,s)\in[0,T)\times\mathbb{R}_+$

$$\begin{aligned}
&-v_t^{\varepsilon,2} + \hat{H}^\varepsilon(t,s,v_{ss}^{\varepsilon,2}) \geq -v_t^{\varepsilon,2} + H^\varepsilon(t,s,v_{ss}^{\varepsilon,2})\\
&= \frac{c_*\varepsilon^{2-a(\nu+2\beta)}}{(T-t)^{1-\beta-\frac{\nu-1}{2}}} + \frac{c_*\varepsilon^{3-2a(1+\nu)}}{(T-t)^{1-\nu}} - v_t^{BS,\varepsilon^a} - \varepsilon v_t^{(1),\varepsilon^a} - \frac{1}{2}s^2\sigma^2 v_{ss}^{\varepsilon,2} - \frac{\varepsilon s^2\sigma^2}{4\ell}\left(v_{ss}^{\varepsilon,2}\right)^2\\
&= \frac{c_*\varepsilon^{2-a(\nu+2\beta)}}{(T-t)^{1-\beta-\frac{\nu-1}{2}}} + \frac{c_*\varepsilon^{3-2a(1+\nu)}}{(T-t)^{1-\nu}} - \frac{s^2\sigma^2}{4\ell}(v_{ss}^{(1),\varepsilon^a})^2\varepsilon^3 - \frac{s^2\sigma^2}{2\ell}v_{ss}^{BS,\varepsilon^a}v_{ss}^{(1),\varepsilon^a}\varepsilon^2.
\end{aligned}$$

In view of Assumption 7.6.1(iii), this quantity is always positive. We now analyze the terminal condition. In view of the conditions imposed on a,β and ν

$$v^{\varepsilon,2}(T,s) = v^{BS,\varepsilon^a}(T,s) = \widehat{g}_{\varepsilon^a}(s).$$

Hence, $v^{\varepsilon,2}$ is a super-solution of (7.6.2) and therefore of (7.2.3). Then, by the comparison theorem for (7.2.3) (proved in [21]), we conclude that $V^{\varepsilon}(t,s) \le v^{\varepsilon,2}(t,s)$.

We now let ε go to 0 in the above inequalities. This proves that V^{ε} converges to v^{BS} uniformly on compact sets.

Finally, by Assumption 7.6.1(ii)

$$0 \le V^{\varepsilon,1}(t,s) \le v^{(1)}(t,s) + o\left(\varepsilon^{\min\{1-a(2\beta+\nu),2-2a(1+\nu)\}}\right) + O\left(\varepsilon^{2a-1}\right),$$

where it is clear with our conditions on a, β and ν that the $o(\cdot)$ and $O(\cdot)$ above go to 0 as ε tends to 0.

Using this estimate, we then prove the convergence of $V^{\varepsilon,1}$ exactly as in Theorem 7.4.1. □

Remark 7.6.1. Higher expansions can be proved similarly, provided that we extend Assumption 7.6.1 for $n \ge 2$.

7.6.2 Expansion for the Call option

In this section, we take

$$g(s) = (s-K)^{+}, \ \sigma(t,s) \equiv \sigma, \ \ell(t,s) \equiv \ell,$$

and we verify that Assumptions 7.6.1(ii) and 7.6.1(iii) are satisfied, since Assumption 7.6.1(i) is trivial.

Straightforward but tedious calculations using the Feynman-Kac formula yield

$$v_{ss}^{BS,\alpha}(t,s) = \frac{1}{\sigma s\sqrt{2\pi\tau}} \int_{-1}^{1} \phi(u) \exp\left(-\frac{1}{2} d_1(s, K+\alpha u, \tau)^2\right) du$$

$$v^{(1),\alpha}(t,s) = \frac{1}{8\ell\pi} \int_0^{\tau} \int_{-1}^{1} \int_{-1}^{1} \frac{\phi(x)\phi(y) h_\alpha(\tau, v, s, K, x, y)}{\sqrt{v(2\tau - v)}} dx dy dv,$$

where

$$\tau = T - t,$$

$$d_1(s,k,t) = \frac{1}{\sigma\sqrt{t}} \ln(s/k) + \frac{1}{2}\sigma\sqrt{t}$$

$$\delta(\tau,v,s,k) = \frac{1}{\sigma\sqrt{2\tau - v}} \ln(s/k) - \frac{\sigma}{2} \frac{\tau - 2v}{\sqrt{2\tau - v}}$$

$$\begin{aligned} h_\alpha(\tau,v,s,k,x,y) = \exp&\left(-\delta(\tau,v,s,k)^2 + \frac{\delta(\tau,v,s,k)}{\sigma\sqrt{2\tau-v}}\left(\log\left(1+\frac{\alpha x}{k}\right) + \log\left(1+\frac{\alpha y}{k}\right)\right)\right) \\ \times \exp&\left(-\frac{\tau}{2\sigma^2 v(2\tau - v)}\left(\log\left(1+\frac{\alpha x}{k}\right) - \log\left(1+\frac{\alpha y}{k}\right)\right)^2\right) \\ \times \exp&\left(-\frac{1}{\sigma^2(2\tau - v)} \log\left(1+\frac{\alpha x}{k}\right) \log\left(1+\frac{\alpha y}{k}\right)\right). \end{aligned}$$

The following two propositions, whose proof is relagated to the Appendix, ensure that Assumptions 7.6.1(ii) and 7.6.1(iii) are satisfied.

Proposition 7.6.1. *There exists a constant c_*, independent of s, τ and α so that for all $(\nu,\beta) \in [0,1] \times [1/2,1]$:*

$$s\left|v_{ss}^{BS,\alpha}(t,s)\right| \leq \frac{c_*}{\tau^{1-\beta}\alpha^{2\beta-1}}, \qquad \frac{s^2\sigma^2}{4\ell}(v_{ss}^{(1),\alpha}(t,s))^2 \leq \frac{c_*}{\tau^{1-\nu}\alpha^{2+2\nu}}.$$

Proposition 7.6.2. *As α tends to 0 we have the following expansions*

$$v^{BS,\alpha}(t,s) = v^{BS}(t,s) + \alpha^2 \frac{e^{-\frac{1}{2}d_0(s,K,\tau)^2}}{2K\sigma\sqrt{2\pi\tau}} \int_{-1}^{1} \phi(v)v^2 dv + O(\alpha^4)$$

$$v^{(1),\alpha}(t,s) = v^{(1)}(t,s) - \alpha \frac{e^{-\frac{1}{2}d_0(s,K,\tau)^2}}{8K\sigma\ell\sqrt{2\pi\tau}} \int_{-1}^{1}\int_{-1}^{1} \phi(x)\phi(y)|x-y|dxdy + o(\alpha),$$

where $d_0(s,k,\tau) = \frac{1}{\sigma\sqrt{\tau}}\ln(s/k) - \frac{1}{2}\sigma\sqrt{\tau}$.

Remark 7.6.2. It is not hard to show that the results of Propositions 7.6.1 and 7.6.2 hold for all convex linear combination of call or put options. However, we cannot use the above proof for, say, a call spread option whose payoff is neither convex nor concave.

7.6.3 Numerical experiments

In order to have a better grasp of the liquidity effects, we also solved numerically (with simple finite difference methods) the PDE (7.2.3). We represent below the behaviour of the liquidity premium (that is to say $V^{\varepsilon} - v^{BS}$) when the time to maturity t and the spot price vary

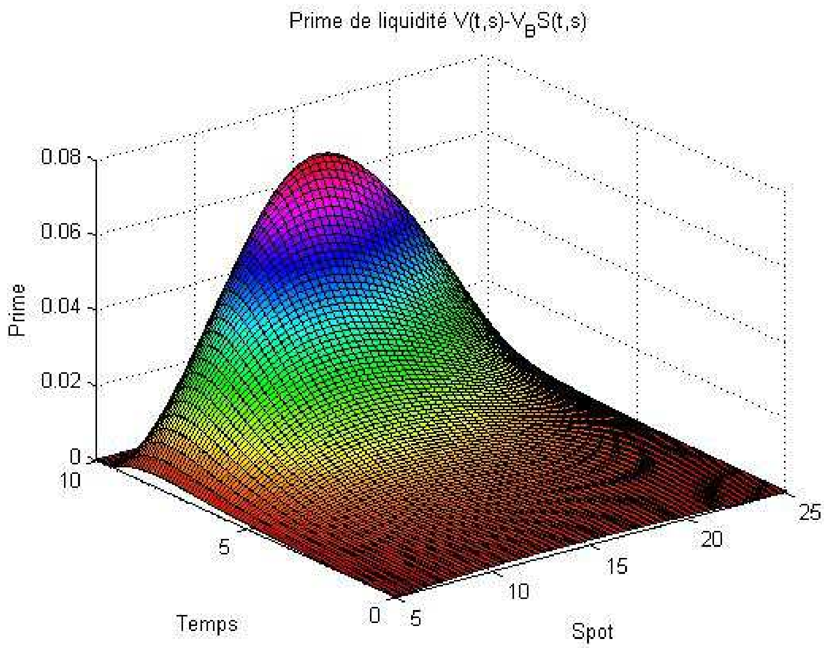

Figure 7.1: Call liquidity premium - $T = 10$, $K = 15$, $\sigma = 0.5$, $\varepsilon = 0.1$, $\ell = 1$

In the above figure, the liquidity effect is strongly marked for ATM options and disapears quickly for ITM and OTM options. This was to be expected. Indeed, our calculations showed that the liquidity effect is, for the first order, driven by the Γ of the call option (see (7.8.1)), which explodes for ATM options near maturity. Moreover, with our set of parameters, the first order correction is at most 0.06 for a BS price of 8.56, which means that the hedge against liquidity risk is not that expensive when the illiquidity is not too strong. We now compare the real liquidity premium with its first-order expansion term.

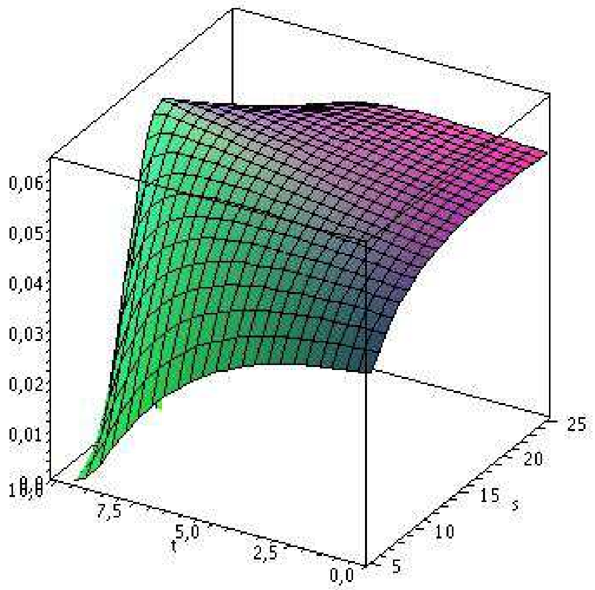

Figure 7.2: Call first order liquidity premium - $T = 10$, $K = 15$, $\sigma = 0.5$, $\varepsilon = 0.1$, $\ell = 1$

A rapid examination of the above figure shows that the first order approximation remains excellent as long as we do not go too far from the maturity time T and we stay close to the money $s = K$. Otherwise, the first order overvalues the liquidity premium.

7.7 Digital option

In this section, we analyze the specific example of a Digital option in the context of Black-Scholes model with constant liquidity parameter

$$g(s) := 1_{s \geq K}, \text{ and } \sigma(t,s) \equiv \sigma, \ \ell(t,s) \equiv \ell.$$

7.7.1 Theoretical bounds

As pointed out earlier, for the Digital option, the first-order term that we obtained formally is equal to $+\infty$. Thus, the expansion (7.3.1) is no longer valid and our aim in this section is to find bounds for the first-order of the expansion. We start by approximating the option by a sequence of regularized call spreads. Then the original problem (7.2.3) is replaced by

$$\begin{aligned} &-V_t^{\varepsilon,\alpha} + \widehat{H}^{\varepsilon}(t,s,V_{ss}^{\varepsilon,\alpha}) = 0, \text{ for } (t,s) \in [0,T) \times \mathbb{R}_+ \\ &V^{\varepsilon,\alpha}(T,s) = \widehat{g}_{\alpha}(s), \end{aligned} \tag{7.7.1}$$

where $\widehat{g}_\alpha(s) = \phi_\alpha * g_\alpha(s)$ with $g_\alpha(s) = \frac{(s-K+2\alpha)^+ - (s-K+\alpha)^+}{\alpha}$.

Since ϕ_α has compact support in $[-\alpha, \alpha]$, notice that $\widehat{g}_\alpha \geq g$. Then, since the terminal condition is smooth, it follows from the comparison principle that

$$V^\varepsilon(t,s) \leq V^{\varepsilon,\alpha}(t,s), \text{ for } (t,s,\alpha) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+^*. \tag{7.7.2}$$

With the same notations as in the previous section, we directly calculate using again the Feynman-Kac formula that

$$v_{ss}^{BS,\alpha}(t,s) = \frac{1}{\sigma s \alpha \sqrt{2\pi\tau}} \int_{-1}^{1} \phi(u) \left(e^{-\frac{1}{2} d_1(s,K+\alpha u - 2\alpha,\tau)^2} - e^{-\frac{1}{2} d_1(s,K+\alpha u-\alpha,\tau)^2} \right) du$$

$$v^{(1),\alpha}(t,s) = \frac{1}{8\ell\pi\alpha^2} \int_0^\tau \int_{-1}^1 \int_{-1}^1 \frac{\phi(x)\phi(y)\widehat{h}_\alpha(\tau,v,s,K,x,y)}{\sqrt{v(2\tau - v)}} dxdydv,$$

where

$$\widehat{h}_\alpha(\tau,v,s,K,x,y) = \sum_{1\leq i,j \leq 2} h_\alpha(\tau,v,s,K,x-i,y-j).$$

Then, we have the two following propositions which are proved exactly as in the call option case (since the functions involved here are essentially the same)

Proposition 7.7.1. *There exists a constant c_*, independent of s, τ and α so that for all $(\nu,\beta) \in [0,1] \times [1/2,1]$*

$$s\left|v_{ss}^{BS,\alpha}(t,s)\right| \leq \frac{c_*}{\tau^{1-\beta}\alpha^{2\beta}}, \qquad \frac{s^2\sigma^2}{4\ell}(v_{ss}^{(1),\alpha}(t,s))^2 \leq \frac{c_*}{\tau^{1-\nu}\alpha^{6+2\nu}}.$$

Proposition 7.7.2. *As α tends to 0 we have the following expansions*

$$v^{BS,\alpha}(t,s) = v^{BS}(t,s) + \frac{3}{2}\alpha \frac{e^{-\frac{1}{2}d_0(s,K,\tau)^2}}{K\sigma\sqrt{2\pi\tau}} + O(\alpha^2)$$

$$v^{(1),\alpha}(t,s) = \alpha^{-1}\frac{e^{-\frac{1}{2}d_0(s,K,\tau)^2}}{8K\sigma\ell\sqrt{2\pi\tau}} \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y)(|x-y-1|+|x-y+1|-2|x-y|)dxdy + o(\alpha^{-1}).$$

Define $V^{\varepsilon,1,c}$ by

$$V^{\varepsilon,1,c}(t,s) := \frac{V^\varepsilon(t,s) - v^{BS}(t,s)}{\varepsilon^c}.$$

Theorem 7.7.1. *Let $(\beta,\nu) \in [1/2,1] \times [0,1]$ be such that $\gamma := \frac{2\beta+\nu-1}{2\beta+\nu+4} \in (0,1)$ and set $a := \frac{2}{5}(1-\gamma)$. Then for all $(t,s) \in [0,T] \times \mathbb{R}_+$,*

$$v^{BS} \leq V^\varepsilon \leq v^{BS,\varepsilon^a} + \varepsilon v^{(1),\varepsilon^a} + c_*(T-t)^{\beta+\frac{\nu-1}{2}}\varepsilon^{2-3a-a(\nu+2\beta)} + c_*(T-t)^\nu \varepsilon^{3-2a(3+\nu)}.$$

In particular, V^ε converges to v^{BS}, uniformly on compact sets and

$$0 \leq \liminf_{(t',s',\varepsilon)\to(t,s,0)} V^{\varepsilon,1,a}(t',s',a) \leq \limsup_{(t',s',\varepsilon)\to(t,s,0)} V^{\varepsilon,1,a}(t',s') \leq \frac{3}{2}\frac{e^{-\frac{1}{2}d_0(s,K,\tau)^2}}{K\sigma\sqrt{2\pi\tau}} + c_*(T-t)^{\frac{5\gamma}{2(1-\gamma)}},$$

i.e. the order of the expansion is at least $2/5$.

Proof. It is clear that $V^\varepsilon \geq v^{BS}$. To prove the reverse inequality, we start by following a technique similar to the one used in the proof of Theorem 7.6.1. Set

$$v^{\varepsilon,2} := v^{BS,\varepsilon^a} + \varepsilon v^{(1),\varepsilon^a} + c_*(T-t)^{\beta+\frac{\nu-1}{2}}\varepsilon^{2-3a-a(\nu+2\beta)} + c_*(T-t)^{\nu}\varepsilon^{3-2a(3+\nu)}.$$

We proceed exactly as in Theorem 7.6.1 using Proposition 7.7.1. The result is

$$-v_t^{\varepsilon,2}(t,s) + \hat{H}^\varepsilon(t,s,v_{ss}^{\varepsilon,2}(t,s)) \geq 0, \text{ for } (t,s) \in [0,T) \times \mathbb{R}_+.$$

We now analyze the terminal condition. Since $2\beta + \nu > 1$, we have

$$v^{\varepsilon,2}(T,s) = v^{BS,\varepsilon^a}(T,s).$$

Hence, $v^{\varepsilon,2}$ is a super-solution of (7.6.2) and therefore of (7.2.3). Then, by the comparison theorem for (7.2.3) (proved in [21]), we conclude that $V^\varepsilon(t,s) \leq v^{\varepsilon,2}(t,s)$.

Then by Proposition 7.7.2 and the conditions imposed on a, β and ν, we obtain easily the uniform convergence on compact sets of V^ε to v^{BS} by letting ε go to 0.

Now for the first order term, we would like to use our expansions and obtain a finite majorant for $V^{\varepsilon,1,c}$ with the largest possible c. It is easy to argue that $c = a$ is the best choice possible. This, in turn, imposes the following condition

$$a \leq \min\left\{\frac{1}{2}, \frac{2}{4+2\beta+\nu}, \frac{3}{7+2\nu}\right\} = \frac{2}{4+2\beta+\nu}.$$

Now it follows that, for all $\gamma > 0$ small enough, there are β and ν satisfying our conditions so that $\frac{2}{4+2\beta+\nu} = \frac{2}{5}(1-\gamma)$. It suffices then to take the lim inf and lim sup in the inequality to prove the result. □

7.7.2 Numerical results

7.7.2.1 The digital option liquidity premium

In this section, we provide numerical results for the case of the Digital option. As in the section 7.6.3 the PDE (7.2.3) is solved with finite difference method. We represent below the behaviour of the liquidity premium when the time to maturity t and the spot price vary. Qualitatively, the liquidity premium behaves as in the Call case. However, as expected the effects of illiquidity are even stronger for ATM options near maturity, since the Γ of a digital option explodes faster. Moreover, with our set of parameters, the first order correction to the price is at most 0.04 for a BS price of 0.21, which means that the hedge against liquidity risk is much more expensive in the case of a digital option, for a same level of liquidity in the market.

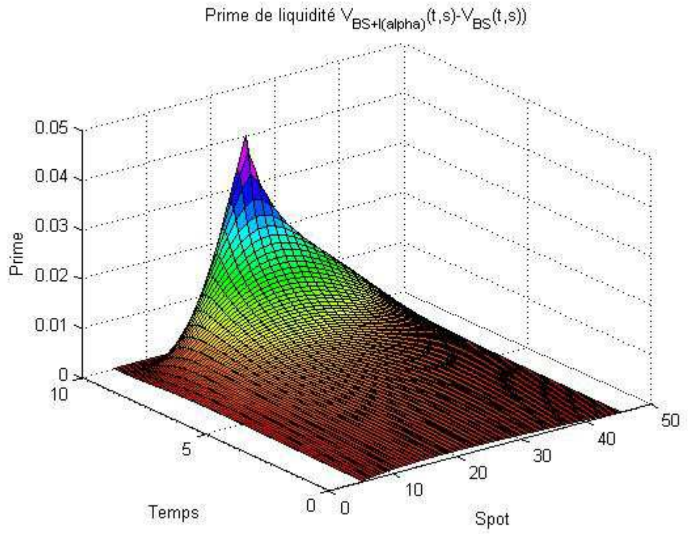

Figure 7.3: Digital liquidity premium - $T = 10$, $K = 25$, $\sigma = 0.5$, $\varepsilon = 0.1$, $\ell = 1$

7.7.2.2 Numerical confirmation of the expansion order

We represent below the liquidity premium for a fixed value of the spot when the parameter ε varies with a logarithmic scale.

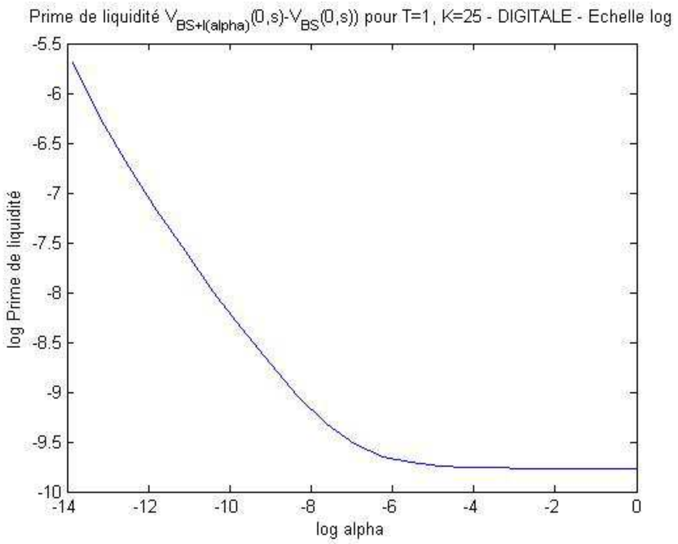

Figure 7.4: $\log\left(V^\varepsilon - v^{BS}\right)$ - $T = 1$, $K = 25$, $s = 15$, $\sigma = 0.5$, $\varepsilon = 0.1$, $\ell = 1$

For small values of ε we observe the expected linear behaviour of $\log\left(V^\varepsilon - v^{BS}\right)$. The slope of the above curve is roughly equal to $1/2$ (the exact value here is 0.54), which is close to our minimal value of $2/5$. The numerical results suggest that the true expansion order lies in the interval $[2/5, 1/2]$.

It is also important to realize the financial implications of our results. We just have highlighted the fact that the first order effect exhibits a phase transition for discontinuous payoff, in the sense that derivative securities of the type of digital options induce a cost of illiquidity which vanishes at a significantly slower rate than the continuous payoff case. This means that derivative with discontinuous payoff are more rapidly affected by the illiquidity cost.

7.8 Appendix

Proof. [Proof of Proposition 7.6.1] We start by proving the inequality for $v_{ss}^{BS,\alpha}$. By dominated convergence, it is clear that $sv_{ss}^{BS,\alpha}$ goes to 0 when s approaches 0 or $+\infty$. Hence for $\alpha \neq 0$, it also converges to 0 when τ tends to 0. Thus $sv_{ss}^{BS,\alpha}$ is less than a constant C_α independent of s and τ. However, when α tends to zero, we obtain the classical expression of the Γ of a call option

$$v_{ss}^{BS}(t,s) = \frac{e^{-\frac{1}{2}d_1(s,K,\tau)^2}}{s\sigma\sqrt{2\pi\tau}}, \tag{7.8.1}$$

which is known to explode only when $s = K$ and $\tau \to 0$. Therefore, to understand the dependence in α of C_α, we only have to study the behaviour of $sv_{ss}^{BS,\alpha}$ when $s = K$ and when both α and τ go to 0.

Let us therefore take $\alpha = \varepsilon^a$ and $\tau = \varepsilon^b$ with a and b strictly positive numbers. For all $\beta \in [1/2, 1]$ we have

$$\tau^{1-\beta}\alpha^{2\beta-1}sv_{ss}^{BS,\varepsilon^a} = \frac{\varepsilon^{(b/2-a)(1-2\beta)}}{\sigma\sqrt{2\pi}}\int_{-1}^{1}\phi(u)e^{-\frac{1}{2}\left(\frac{\sigma\varepsilon^{b/2}}{2}-\frac{\varepsilon^{-b/2}}{\sigma}\log\left(1+\frac{\varepsilon^a u}{K}\right)\right)^2}du.$$

Therefore, if $a < b/2$ (i.e. if τ goes to 0 faster than α) the quantity above always goes to 0 when $\varepsilon \to 0$ due to the exponential term. If $a \geq b/2$, the exponential term goes to 1, but since $\beta \in [1/2, 1]$ the above expression has always a finite limit. Hence the inequality for $sv_{ss}^{BS,\alpha}$.

A change of variable and direct calculations imply that, for all $\nu \in [0, 1]$, we have

$$\tau^{\frac{1-\nu}{2}}\alpha^{1+\nu}sv_{ss}^{(1),\alpha}(t,s) = \frac{\alpha^{1+\nu}\tau^{-\frac{1+\nu}{2}}}{8\ell\pi s}\int_0^1\int_{(-1,1)^2}\frac{\phi(x)\phi(y)\widetilde{h}_\alpha(\tau,\tau v,s,K,x,y)}{\sqrt{v}(2-v)^{3/2}}dxdydv, \tag{7.8.2}$$

where

$$\begin{aligned}\frac{\widetilde{h}_\alpha(\tau,v,s,K,x,y)}{h_\alpha(\tau,v,s,K,x,y)} &= 2 + \left(2\delta(\tau,\tau v,s,K) - \frac{\log\left(1+\frac{\alpha x}{K}\right)+\log\left(1+\frac{\alpha y}{K}\right)}{\sigma\sqrt{\tau(2-v)}}\right)^2 \\ &+ \left(2\delta(\tau,\tau v,s,K) - \frac{\log\left(1+\frac{\alpha x}{K}\right)+\log\left(1+\frac{\alpha y}{K}\right)}{\sigma\sqrt{\tau(2-v)}}\right)\sigma\sqrt{\tau(2-v)}.\end{aligned}$$

Using the same arguments as in the proof of the previous inequality, we can show again that the only problem corresponds to the case where $s = K$ and α and τ go to 0. Using the same notations, we have

$$h_{\varepsilon^a}(\varepsilon^b, \varepsilon^b v, s, s, x, y) = \exp\left(-\frac{\sigma^2\varepsilon^b(1-2v)^2}{4(2-v)} + \frac{(1-2v)\left(\log\left(1+\frac{\varepsilon^a x}{K}\right) + \log\left(1+\frac{\varepsilon^a y}{K}\right)\right)}{2(2-v)}\right)$$
$$\times \exp\left(-\frac{\varepsilon^{-b}}{\sigma^2(2-v)}\log\left(1+\frac{\varepsilon^a x}{K}\right)\log\left(1+\frac{\varepsilon^a y}{K}\right)\right)$$
$$\times \exp\left(-\frac{\varepsilon^{-b}\left(\log\left(1+\frac{\alpha x}{K}\right) - \log\left(1+\frac{\alpha y}{K}\right)\right)^2}{2\sigma^2 v(2-v)}\right)$$

$$\frac{\widetilde{h}_{\varepsilon^a}(\varepsilon^b, v, s, s, x, y)}{h_{\varepsilon^a}(\varepsilon^b, v, s, s, x, y)} = 2 + \left(\frac{\sigma\varepsilon^{\frac{b}{2}}(1-2v)}{\sqrt{2-v}} + \varepsilon^{-b}\frac{\log\left(1+\frac{\varepsilon^a x}{K}\right) + \log\left(1+\frac{\varepsilon^a y}{K}\right)}{\sigma\sqrt{2-v}}\right)^2$$
$$- \left(\frac{\sigma\varepsilon^{\frac{b}{2}}(1-2v)}{\sqrt{2-v}} + \varepsilon^{-b}\frac{\log\left(1+\frac{\varepsilon^a x}{K}\right) + \log\left(1+\frac{\varepsilon^a y}{K}\right)}{\sigma\sqrt{2-v}}\right)\sigma\sqrt{2-v}\varepsilon^{\frac{b}{2}}.$$

Therefore, if $a < b/2$, $\widetilde{h}_{\varepsilon^a}$ always goes to 0. Otherwise, the integral has a finite limite but since $\nu \in [0,1]$ and $a \geq b/2$, the expression in (7.8.2) has a finite limit. This proves the second inequality. □

Proof. [Proof of Proposition 7.6.2] The first result is straightforward and only uses the fact that the function ϕ is symmetric, which allows us to get rid off the odd terms in the expansion. For the second one, we directly calculate that

$$v^{(1),\alpha} = \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y)\frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2 v(1-\frac{v}{2\tau})} + o(\alpha^2)}}{8\pi\ell\sqrt{v(2\tau-v)}}dxdydv$$
$$+ \alpha \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y)\frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2 v(1-\frac{v}{2\tau})} + o(\alpha^2)}\delta}{8\pi\ell K\sigma\sqrt{v}(2\tau-v)}(x+y)dxdydv$$
$$+ \alpha^2 \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y)\frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2 v(1-\frac{v}{2\tau})} + o(\alpha^2)}\left(2(x+y)^2\delta^2 + \sigma\sqrt{2\tau-v}(x^2+y^2)\delta - 2xy\right)}{16\pi\ell K^2\sigma^2\sqrt{v}(2\tau-v)^{3/2}}dxdydv$$
$$+ o\left(\alpha^2 \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y)\frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2 v(1-\frac{v}{2\tau})} + o(\alpha^2)}}{8\pi\ell\sqrt{v(2\tau-v)}}dxdydv\right),$$

where we suppressed the arguments of the functions $v^{(1),\alpha}$ and δ for notational simplicity.

Note that all the above integrals are well-defined and finite. Then using dominated convergence and the fact that ϕ is symmetric, it is easy to show that

$$
\begin{aligned}
v^{(1),\alpha} &= \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2 v(1-\frac{v}{2\tau})} + o(\alpha^2)}}{\sqrt{8\pi\ell v(2\tau - v)}} dxdydv \\
&+ \alpha \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2}\delta}{8\pi\ell K\sigma\sqrt{v}(2\tau - v)} (x+y) dxdydv \\
&+ \alpha^2 \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2}\left(2(x+y)^2\delta^2 + \sigma\sqrt{2\tau - v}(x^2+y^2)\delta - 2xy\right)}{16\pi\ell K^2\sigma^2\sqrt{v}(2\tau - v)^{3/2}} dxdydv + o\left(\alpha^2\right) \\
&= \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2 v(1-\frac{v}{2\tau})} + o(\alpha^2)}}{8\pi\ell\sqrt{v(2\tau - v)}} dxdydv + o(\alpha).
\end{aligned}
$$

Now the first term in the expansion above goes clearly to $v^{(1)}$ as α tends to 0. Then we have

$$
v^{(1),\alpha} - v^{(1)} = \int_0^\tau \int_{-1}^1 \int_{-1}^1 \frac{e^{-\delta(\tau,v,s,K)^2}\phi(x)\phi(y)}{8\pi\ell\sqrt{v(2\tau - v)}} \left(e^{-\frac{\alpha^2(x-y)^2}{4K^2\sigma^2 v(1-\frac{v}{2\tau})} + o(\alpha^2)} - 1 \right) dxdydv + o(\alpha).
$$

Using the change of variable $u = \frac{\alpha|x-y|}{2K\sigma\sqrt{v}}$, the first term above can be rewritten as

$$
\frac{\alpha}{8\pi\ell K\sigma} \int_{\frac{\alpha|x-y|}{2K\sigma\sqrt{\tau}}}^{+\infty} \int_{-1}^1 \int_{-1}^1 \frac{e^{-\delta(\tau, \frac{\alpha^2(x-y)^2}{4K^2\sigma^2u^2}, s, K)^2}\phi(x)\phi(y)|x-y|}{\sqrt{2\tau - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2u^2}}} \frac{e^{-\frac{u^2}{1-\frac{\alpha^2(x-y)^2}{8\tau K^2\sigma^2u^2}} + o(\alpha^2)} - 1}{u^2} dxdydu.
$$

A simple application of the dominated convergence and Fubini theorems shows that the above integral (without the α factor) has a finite limit as α approaches 0 and is given by

$$
\frac{e^{-\frac{1}{2}d_0(s,K,\tau)^2}}{8\pi\ell K\sigma\sqrt{2\tau}} \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y)|x-y| dxdy \int_0^{+\infty} \frac{e^{-u^2} - 1}{u^2} du.
$$

Since the last integral is equal to $\sqrt{\pi}$, we obtain the second expansion. □

Chapitre 8

Bank monitoring incentives

Contents

8.1 Introduction

In the recent years, there has been a rather significant increase in the interest in continuous time Principal-Agent models and their applications. Notwithstanding the fact that a general resolution of these problems is intrinsically technical, they often lead to elegant solutions with precise and clear economic predictions. In general, a Principal-Agent problem is a problem of optimal contracting between two parties, one of which, namely the agent, may affect the value of the outcome process with his actions. The classical literrature usually study three main types of such problems

(i) The *first-best case, also called risk sharing, where the Principal and the Agent have exactly the same information. They have to agree how to share the risk between themselves, and usually, the principal is assumed to have the upper hand in the sense that he offers the contract and also dictates the agent's actions.*

(ii) *The second-best case, where the actions of the agent are hidden to the principal or cannot be contracted upon. Because of this, there is generally a loss in expected utility for the principal, who can only attain the* second-best *reward. Besides, since the agent can in this case choose actions which are not in the best interest of the principal, we sometimes talk about* moral hazard.

(iii) *The third-best case, sometimes also called adverse selection, where the principal does not know some key characteristics of the agent, and also does not know the actions he undertakes. In that case, the principal can only attain his* third-best *reward.*

The model we will discuss in this paper, and which will be decribed in more details in the following sections, belongs to the second class. Indeed, we consider a bank (the Agent) which has the opportunity to set up a pool of defaultable loans, and investors (the Principal), who want to invest in it. The bank has the choice to either monitor a given loan or not, these actions being unobservable for the investors. Of course, the investors would like the bank to monitor all the loans, in order to decrease the risk they face. However, since they cannot choose directly the actions they would like the bank to perform, they have to be aware, given a contract c, which action $a(c)$ will be optimal for the bank to choose. Thus we have to deal with a problem of incentives, where the investors indirectly influence the bank to monitor the loans, by offering an appropriate contract.

In its whole generality, the mathematical treatment of those problems is as follows. We first have to solve the agent's problem for a given fixed contract c

$$V_A(c) := \sup_a \mathbb{E}\left[U_A(c,a)\right],$$

where U_A is the utility function of the agent.

Then, if we assume for simplicity that there exists a unique optimal action $a(c)$ which solves this problem, we then have to solve the principal's problem

$$V_P := \sup_c \left\{\mathbb{E}\left[U_P(c,a(c))\right] + \lambda \mathbb{E}\left[U_A(c,a(c))\right]\right\},$$

where U_P is the utility function of the principal and where λ is a Lagrange multiplier associated to the the constraint that the agent has to accept the contract.

Because of the almost limitless choices for c, this problem is usually simplified to be more tractable. Thus, it is generally assumed that the agent does not directly act on the outcome, but that he instead chooses the distribution of the outcome by choosing specific actions. This actually means that he chooses the probability measure $\mathbb{P}^a$ under which the above expectation are taken. This setting, which will be described more rigorously in the following section corresponds to a weak formulation of the stochastic control problem of the agent.

As shown in [31], a general theory can be used to solve these problems, by means of Forward-Backward stochastic differential equations. However, we will show, using a different approach proposed by Sannikov [96], which leads to explicit solutions that we will then be able to analyze. This paper is deeply related to the recent and fertile literature on dynamic moral hazard, as illustrated by DeMarzo and Sannikov [35], DeMarzo and Fishman [36], [37], Biais et al. [8] or Sannikov [96]. Many papers deal with frequent and infinitesimal risk, but Sannikov [95] also has Poisson risk. A difference is that jumps are associated with upside cash-flow shocks, which leads to predictable downsizing and qualitatively different results. In Biais et al. [9], moral hazard is about large and unfrequent risks. As in our model and unlike in the Brownian case, investors inflict sharp reductions in the agent's continuation utility when losses occur and unpredictable downsizing when performance is poor. Firm size dynamics is different because the agent can expand through investment and follow asymptotically a positive growth trend. Our analysis offers a first description of unpredictable downsizing in a non-stationary context. Let us note that the modelization we adopt was first introduced by Pagès in a paper in preparation [82].

The rest of the paper is organised as follows. In section 8.2, we present the model itself, describe the contracts and give our main assumptions. Then, in Section 8.3, we formaly derive a candidate optimal-contract by solving the HJB equation associated to our control problem and then use a standard verification argument to show that it is indeed the optimal contract. Finally, in Section 8.6 we present and comment some numerical results.

8.2 The model

8.2.1 Notations and preliminaries

We consider a model with universal risk neutrality in which time is continuous and indexed by $t \in [0, \infty)$. The risk-free interest rate is supposed, without loss of generality, to be equal to 0. As mentioned in the Introduction, a bank has the opportunity to set up a pool of I unit loans indexed by $i = 1, \ldots, I$ which are ex ante identical. Each loan is a defaultable perpetuity yielding cash flow μ per unit time until it defaults. Once a loan defaults it gives no further payments. The infinite maturity assumption is made for tractability, and we may think of corporate or mortgage loans which, once repaid, are replaced by loans with identical characteristics.

Denote

$$N_t = \sum_{i=1}^{I} 1_{\{\tau^i \leq t\}},$$

the sum of individual loan default indicators, where τ^i denotes the default time of loan i. The current size of the pool is $I - N_t$. Since all loans are a priori identical, they can be reindexed in any order after defaults.

The action of the bank consists on deciding at each time t whether it monitors the different loans. These actions are summarized by the functions e_t^i defined by

$$\text{For } 1 \leq i \leq I - N_t,\ e_t^i = 1 \text{ if loan } i \text{ is monitored at time } t, \text{ and } e_t^i = 0 \text{ otherwise.}$$

Usually, the bank prefers not to monitor, as it enjoys a flow of private benefits B per non-defaulted loan it does not monitor. These private benefits capture the opportunity cost of various wasteful activities the bank can indulge in when it shirks. It is natural to assume that these benefits depend on the number of loans which have not defaulted, since a larger pool requires more monitoring.

The rate at which loan i defaults is controlled by the hazard rate α_t^i specifying its instantaneous probability of default conditional on history up to time t. Individual hazard rates are assumed to depend both on the monitoring choice of the bank and on the size of the pool. Specifically, we choose to model the hazard rate of a non-defaulted loan i at time t by

$$\alpha_t^i = \alpha_{I-N_t}\left(1 + (1 - e_t^i)\varepsilon\right), \tag{8.2.1}$$

where the parameters $\{\alpha_j\}_{1 \leq j \leq I}$ represent individual "baseline" risk under monitoring when the number of loans is j and ε is the proportional impact of shirking on default risk. This modelization comes from a relatively large strand of litterature. Indeed there is compelling evidence about the importance of servicers in securitization. Ashcraft and Shuermann [1] discuss the frictions that arise in the atomized setting of securitization and show that the servicer's role is not confined to the collection and remittance of loan payments. These activities have consequences for the performance of loans, with an impact of plus or minus 10 percent on loss according to a Moody's estimate.

Remark 8.2.1. *Note that according to Equation* (8.2.1)*, we assume that monitoring affects risk only at the time it is exerted. Moreover, letting α_t^i depend on the size $I - N_t$ is just a way to model imperfect correlation across default times. As in models recently introduced in the credit field (see for instance Davis and Lo [32], Jarrow and Yu [63] or Yu [116]), default correlation is induced by contagion effects in the pool, i.e., individual defaults may be informative about the default risk of surviving loans. Uncorrelated default risk would correspond to a constant $\{\alpha_j\}_{1\leq j\leq I}$.*

We define a shirking process k by

$$k_t = \sum_{i=1}^{I-N_t} \left(1 - e_t^i\right),$$

which represents the number of loans that the bank fails to monitor at time t.

Then, according to (8.2.1), and given the independence of the loans, we may define an aggregate default intensity by

$$\lambda_t^k = \alpha_{I-N_t} \left(I - N_t + \varepsilon k_t\right). \tag{8.2.2}$$

The bank can fund the pool internally with capital K at a cost $r \geq 0$. Positive internal funding costs reflect bank's limited access to capital or deposits and may include any regulatory or agency costs associated with this source of financing. The bank can also raise funds from competitive investors who value income streams at the prevailing riskless interest rate of zero. We assume that both the bank and investors observe the history of defaults and liquidations.

8.2.2 Description of the contracts

We will now describe more precisely the terms of the contracts which can be signed between the investors and the bank. The contracts are agreed upon at time 0 and determine how cash flows are shared and how loans are liquidated, conditionally on past defaults and liquidations. Without loss of generality, they specify that an investor receives cash flows from the pool and makes transfers to the bank. We denote by $D = \{D_t\}_{t\geq 0}$ the positive and increasing process describing cumulative transfers from the investor to the bank. For simplicity, we assume that the measure dD_t is absolutely continuous with respect to the Lebesgue measure, that is to say

$$D_t = \int_0^t \delta_s ds,$$

for some predictable process δ which is positive, and such that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^\tau |\delta_s|\, ds\right] < +\infty, \tag{8.2.3}$$

where τ is the liquidation time of the pool.

Let then $H_t := 1_{\{t\geq\tau\}}$ be the liquidation indicator of the whole pool. The contract specifies the probability θ_t with which the pool is maintained given default ($dN_t = 1$), so that at each point in time

$$dH_t = \begin{cases} 0 & \text{with probability } \theta_t, \\ dN_t & \text{with probability } 1 - \theta_t. \end{cases}$$

With our notations,the hazard rates associated with the default and liquidation processes N_t and H_t are λ_t^k and $(1 - \theta_t)\,\lambda_t^k$, respectively.

The contract also specifies when liquidation occurs. We assume that liquidations can only take the form of the stochastic liquidation of all loans following immediately default. The above properties translate into

$$\mathbb{P}\left(\tau \in \left\{\tau^1, ..., \tau^I\right\}\right) = 1, \text{ and } \mathbb{P}(\tau = \tau^i | \mathcal{F}_{\tau^i}, \tau > \tau^{i-1}) = 1 - \theta_{\tau^i}.$$

Remark 8.2.2. *Of course, the type of liquidation that we choose is far from being the most general. Indeed, liquidations could be partial, involving the removal of a state-dependent number of loans from the pool, or stochastic, implying that part or all of the loans in the pool are liquidated with state-dependent probability. This choice is first motivated by the fact that it leads to a stochastic control problem which can be solved explicitly and for which we therefore have further insight in the economic implications. However, we will show in 8.4.3, heuristically, that this choice is, under some conditions, not so arbitrary.*

We summarize the above details of the contracts, which are completely specified by the choice of (δ, θ). Each infinitesimal time interval $(t, t + dt)$ unfolds as follows:

- $I - N_t$ loans are performing at time t.
- The bank chooses to leave $k_t \leq I - N_t$ loans unmonitored and monitors the $I - N_t - k_t$ other loans, enjoying private benefits $k_t B \, dt$.
- The investor receives $(I - N_t) \, \mu \, dt$ from the cash flows generated by the pool and pays $\delta_t dt \geq 0$ as fees to the bank.
- With probability $\lambda_t^k \, dt$ defined by (8.2.2) there is a default $(dN_t = 1)$.
- Given default the pool is maintained $(dH_t = 0)$ with probability θ_t or liquidated $(dH_t = 1)$ with probability $1 - \theta_t$.

As recalled in the introduction, we assume that the bank's monitoring decision is not observable. This leads to a dynamic moral hazard problem, where the contract (δ, θ) uses observations on defaults to give the bank incentives to monitor. We assume that investors can fully commit to such contracts.

8.2.3 Economic assumptions

In this section we give some Assumptions on our parameters, arising from economic considerations.

Assumption 8.2.1. *Individual default risk is non-decreasing with past default*

$$\alpha_j \leq \alpha_{j-1}, \qquad \textit{for all } j \leq I. \tag{8.2.4}$$

Estimates from dynamic models (See for exemple Laurent et al. [69] or Lopatin and Misirpashaev [75]) of default risk show that aggregate intensity tends to increase with the number of defaults. Assumption 8.2.1 is weaker and allows for periods during which the underlying individual hazard rates are constant, implying that aggregate intensity is locally decreasing as new defaults occur.

Now, we recall that we ultimately want to design contracts for which it is optimal for the bank to monitor all the loans at all times. Therefore, it is necessary that we impose a condition which tells us that the bank can benefit, on average, more from monitoring than from doing nothing. Let us then define, for $1 \leq j \leq I$, by $\overline{\alpha}_j$ the harmonic mean of $(\alpha_i)_{1\leq i\leq j}$

$$\frac{1}{\overline{\alpha}_j} := \frac{1}{j}\sum_{i=1}^{j}\frac{1}{\alpha_i}.$$

The quantity $j/\overline{\alpha}_j$ is therefore the expected time before liquidation when we start with j loans and the bank always monitors all the loans (see (8.2.1)). Notice then that everything happens as if each loan had an expected duration of $1/\overline{\alpha}_j$. Indeed, if we assume that at time 0 the pool contains j loans and that they are monitored at all times, then their expected total value is given by

$$\begin{aligned}
\mathbb{E}\left[\int_0^{\tau}(j-N_t)\mu dt\right] &= \sum_{i=1}^{j}\mathbb{E}\left[\int_{\tau^{i-1}}^{\tau^i}(j-i+1)\mu dt\right] \\
&= \mu\sum_{i=1}^{j}(j-i+1)\mathbb{E}\left[\tau^i-\tau^{i-1}\right] \\
&= \mu\sum_{i=1}^{j}\frac{j-i+1}{\lambda_{i-j+1}} \\
&= \mu\sum_{i=1}^{j}\frac{1}{\alpha i} \\
&= \frac{j\mu}{\overline{\alpha}_j}.
\end{aligned}$$

Similarly, the expected total loss for the bank due to monitoring is given by

$$\mathbb{E}\left[\int_0^{\tau}(j-N_t)\frac{rB}{\varepsilon}dt\right] = r\frac{jB}{\varepsilon\overline{\alpha}_j},$$

since the loss per unit time for the bank when it monitors is $\frac{rB}{\varepsilon}$.

Therefore, from the economic viewpoint, the expected benefits from monitoring are greater than its costs if we have for all j

$$\frac{j\mu - r\frac{jB}{\varepsilon\overline{\alpha}_j}}{\lambda_j} \geq \frac{j\mu + jB}{\lambda_j(1+\varepsilon)}. \tag{8.2.5}$$

Indeed, with monitoring, the instantaneous flow of income is equal to the yield of all the loans $j\mu$ minus the cost of monitoring. Then, when there is no monitoring, the income of the bank are increased by its private benefits B per loan, and are thus equal to $j\mu + jB$, but the probability of default increases.

We summarize this in the following assumption.

Assumption 8.2.2. *We have for all j*

$$\frac{r}{\overline{\alpha}_j} \leq \frac{\mu\varepsilon - B}{B}\frac{\varepsilon}{1+\varepsilon}.$$

We will give later in 8.4.3 an heuristic justification of the above assumption. Also note that the above assumptions will be in force throughout the paper and will always be implicitly assumed.

Finally, we assume

Assumption 8.2.3.

$$\mu \geq \overline{\alpha}_I. \tag{8.2.6}$$

This Assumption will be justified in Section 8.5.

8.3 Optimal contracting

Before going on, let us now describe the stochastic basis on which we are working. We will always place ourselves on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ on which N is a Poisson process whith intensity λ_t^0 (which is defined by (8.2.2)) and where $\mathbb{P}$ is the reference probability measure. We denote $(\mathcal{F}_t^N)_{t\geq 0})$ the completed natural filtration of N and by $(\mathcal{G}_t)_{t\geq 0}$ the minimal filtration containing $(\mathcal{F}_t^N)_{t\geq 0})$ and that makes the liquidation time of the pool τ a $\mathcal{G}$-stopping time. We note that this filtration satisfies the usual hypotheses, and therefore we will always consider super or submartingales in their càdlàg version.

8.3.1 Incentive compatibility and limited liability

As recalled in the introduction, in order to make the problem tractable, we consider that the monitoring choices of the bank affect only the distribution of the size of the pool. To formalize this, recall that by definition, the shirking process k is $\mathcal{G}-$predictable and bounded. Then, gy Girsanov Theorem, we can define a probability measure $\mathbb{P}^k$ equivalent to $\mathbb{P}$ such that

$$N_t - \int_0^t \lambda_t^k ds,$$

is a $\mathbb{P}^k$-martingale.

More precisely, we have from Brémaud [11] (Chapter VI, Theorem T3), that on $\mathcal{G}_t$

$$\frac{d\mathbb{P}^k}{d\mathbb{P}} = Z_t^k,$$

where Z^k is the unique solution of the following SDE

$$Z_t^k = 1 + \int_0^t Z_{s^-}^k \left(\frac{\lambda_s^k}{\lambda_s^0} - 1\right)\left(dN_s - \lambda_s^0 ds\right),\ 0 \leq t \leq T,\ \mathbb{P} - a.s.$$

Then, given a contract (δ, θ) and a shirking process k, the bank's expected utility at $t = 0$ is given by

$$u_0^k(\delta, \theta) := \mathbb{E}^{\mathbb{P}^k}\left[\int_0^\tau e^{-rt}(\delta_t dt + Bk_t\, dt)\right], \tag{8.3.1}$$

while that of the investor is

$$v_0^k(\delta, \theta) := \mathbb{E}^{\mathbb{P}^k}\left[\int_0^\tau (I - N_t)\,\mu\, dt - \delta_t dt\right]. \tag{8.3.2}$$

Following, Sannikov [96], we give now the definition of an incentive-compatible shirking process.

Definition 8.3.1. *A shirking decision k is incentive-compatible with respect to the contract (δ, θ) if it maximizes* (8.3.1).

Then, the problem faced by the investors is to design a contract (δ,θ) and an incentive-compatible advice on the monitoring k that maximize their expected discounted payoff, subject to a given reservation utility for the bank

$$\begin{aligned} v_I(u) \quad &:= \quad \sup_{(\delta,\theta,k)} \mathbb{E}^{\mathbb{P}^k}\left[\int_0^\tau (I-N_t)\,\mu dt - \delta_t dt_t\right] \\ \text{subject to} \quad & \qquad \mathbb{E}^{\mathbb{P}^k}\left[\int_0^\tau e^{-rt}(\delta_t dt + Bk_t\,dt)\right] \geq u_0 \\ & \qquad k \text{ incentive-compatible with respect to } (\delta,\theta)\,. \end{aligned} \tag{8.3.3}$$

This allows us to define a first set of admissible contracts for a given monitoring advice k

$$\begin{aligned} \mathcal{A}^k(x) := \{(\delta,\theta),\ &\theta \text{ is a predictable process with values in } [0,1], \\ &\delta \text{ is a positive predictable process which satisfies (8.2.3)}, \\ &k \text{ is incentive-compatible with } (\delta,\theta) \text{ and } u_0^k(\delta,\theta) \geq x\}. \end{aligned} \tag{8.3.4}$$

Notice that we will put more restrictions on this set at the end of the section.

Now, our aim is to find a practical condition which is equivalent to the property that a shirking process k is incentive-compatible with a given contract. This will be achieved thanks to martingale arguments. Indeed, consider the bank's expected lifetime utility, conditional on $\mathcal{G}_t$

$$\begin{aligned} U_t^k(\delta,\theta) \quad &:= \quad \mathbb{E}^{\mathbb{P}^k}\left[\int_0^\tau e^{-rs}\,(\delta_s + Bk_s)\,ds \;\Big|\; \mathcal{G}_t\right] \\ &= \quad \int_0^{t\wedge\tau} e^{-rs}\,(\delta_s + Bk_s)\;ds + e^{-rt}u_t^k(\delta,\theta), \end{aligned} \tag{8.3.5}$$

where u_t^k is the dynamic version of the bank's continuation utility defined as

$$u_t^k(\delta,\theta) := 1_{\{t\leq\tau\}}\mathbb{E}^{\mathbb{P}^k}\left[\int_t^\tau e^{-r(s-t)}\,(\delta_s + Bk_s)\;ds \;\Big|\; \mathcal{G}_t\right], \tag{8.3.6}$$

Since we are working with the completed natural filtration of a Poisson process, since U_t^k is a $\mathcal{G}_t$-martingale under $\mathbb{P}^k$ and is in L^1 because of the integrability assumptions we made, the martingale representation theorem for point processes (see [11], Chapter III, Theorems $T9$ and $T17$, and Chapter VI, Theorems $T2$ and $T3$) implies that there are predictable processes h^1 and h^2 such that the bank's continuation utility u^k satisfies the following "promise-keeping" equation until liquidation occurs

$$du_t^k + (\delta_t + Bk_t)\;dt = ru_t^k\,dt - h_t^1\left(dN_t - \lambda_t^k\,dt\right) - h_t^2\left(dH_t - (1-\theta_t)\lambda_t^k\,dt\right), \tag{8.3.7}$$

where the dependence of h^1 and h^2 on k has been suppressed for notational convenience.

Remark 8.3.1. *The processes h^1 and h^2 have the following financial meaning. Upon default ($dN_t = 1$), the bank's continuation utility jumps from u_t^k to $u_t^k - h_t^1$ if the pool is maintained ($dH_t = 0$) and from u_t^k to $u_t^k - h_t^1 - h_t^2$ when it is liquidated ($dH_t = 1$). Therefore, h^1 and h^2 can be interpreted as the penalties faced by the bank given default. The penalty h^1 arises if a loan defaults, while the penalty h^2 reflects the risk of liquidation incurred with probability $1-\theta_t$.*

For the rest of the dynamic, if there is no default ($dN_t = 0$), the bank's dividend has two components. One is $\delta_t + Bk_t$, the contractual fee complemented by any private benefits as a result of not monitoring. The other corresponds to the expected cost of the penalties, $\lambda_t^k\left(h_t^1 + (1-\theta_t)h_t^2\right)$. The promise-keeping equation (8.3.7) states that in the absence of jumps the expected rate of change in the bank's continuation payoff plus the rate of dividends must be equal to the bank's discount rate r.

The introduction of the processes h^1 and h^2 provides exactly what we wanted, that is to say a practical way of characterizing contracts for which a given k is incentive-compatible. This is the object of the following proposition, inspired by Sannikov [96]

Proposition 8.3.1. *Given a contract (δ,θ) and a shirking process k, it is incentive-compatible if and only if for all $t\in[0,\tau]$ and for all $i=1...I$, the following holds almost-surely,*

$$\left(\frac{B}{\varepsilon\alpha_{I-N_t}}-h_t^1-(1-\theta_t)h_t^2\right)(k_t-i)\geq 0. \tag{8.3.8}$$

Proof. Consider an arbitrary strategy $\widehat{k}$ specifying the number of unmonitored loans at any point in time until liquidation. Let u_t^k denote the continuation utility in (8.3.6) resulting from the decision to monitor all loans at all times. Define by

$$\widehat{U}_t=\int_0^{t\wedge\tau}e^{-rs}\left(\delta_s+B\widehat{k}_s\right)ds+e^{-rt}u_t^k \tag{8.3.9}$$

the lifetime utility of the bank viewed as of time t if it follows the strategy $\widehat{k}$ before time t, and plans to switch to k afterwards.

We have for all $t\in[0,\tau]$

$$\begin{aligned}
d\widehat{U}_t&=e^{-rt}\left(\delta_t+B\widehat{k}_t\right)dt+e^{-rt}\left(du_t^k-ru_t^k\,dt\right)\\
&=e^{-rt}B(\widehat{k}_t-k_t)\,dt-e^{-rt}\left(h_t^1(dN_t-\lambda_t^k\,dt)+h_t^2(dH_t-(1-\theta_t)\lambda_t^k\,dt)\right)\\
&=e^{-rt}\left(B-\alpha_{I-N_t}\varepsilon(h_t^1+(1-\theta_t)h_t^2)\right)(\widehat{k}_t-k_t)dt\\
&\quad-e^{-rt}\left(h_t^1(dN_t-\lambda_t^{\widehat{k}}\,dt)+h_t^2(dH_t-(1-\theta_t)\lambda_t^{\widehat{k}}\,dt)\right),
\end{aligned}$$

where we have used the promise-keeping equation (8.3.7) for u^k.

Therefore, the first term on the right-hand side

$$e^{-rt}\left(B-\alpha_{I-N_t}\varepsilon(h_t^1+(1-\theta_t)h_t^2)\right)(\widehat{k}_t-k_t),$$

is the drift of $\widehat{U}$ under $\mathbb{P}^{\widehat{k}}$.

Note also that by definition, h^1 and h^2 are integrable, and therefore the martingale part of $\widehat{U}$ is a true $\mathbb{P}^{\widehat{k}}$-martingale.

(i) Now assume that (8.3.8) does not hold on a set of positive measure, and choose $\widehat{k}$ such that it maximizes the quantity

$$\left(B-\alpha_{I-N_t}\varepsilon(h_t^1+(1-\theta_t)h_t^2)\right)\widehat{k}_t,$$

for all t.

Then, the drift of $\widehat{U}$ under $\mathbb{P}^{\widehat{k}}$ is non-negative and strictly positive on a set of positive measure. Therefore $\widehat{U}$ is a $\mathbb{P}^{\widehat{k}}$-submartingale. This implies the existence of a time $t^*>0$ such that

$$\mathbb{E}^{\mathbb{P}^{\widehat{k}}}[\widehat{U}_{t^*}]>\widehat{U}_0=u_0^k.$$

Therefore, if the agent follows this strategy $\widehat{k}$ until the time t^* and then switches to the strategy k, his utility is strictly greater than the utility obtained from following the strategy k all the time. This contradicts the fact that the strategy k is incentive-compatible.

(ii) With the same notations as above, assume that (8.3.8) holds for the strategy k. Then this means that $\widehat{U}$ is a $\mathbb{P}^{\widehat{k}}$-supermartingale, regardless of the choice of strategy $\widehat{k}$. Moreover, since $\widehat{U}$ is positive (because δ is assumed to be positive), it has a last element (see Problem 3.16 in [65] for instance). Then, we have by the optional sampling Theorem

$$u_0^k = \widehat{U}_0 \geq \mathbb{E}^{\mathbb{P}^{\widehat{k}}}\left[\widehat{U}_\tau\right] = u_0^{\widehat{k}},$$

where we used (8.3.9) and the fact that $u_\tau^k = 0$ for the last inequality.

This means that the strategy k maximizes the expected utility of the agent and is therefore incentive-compatible. □

Since we are mainly interested in designing optimal contracts for which the bank is detered from shirking, we will actually only consider contracts for which $k = 0$ is incentive-compatible. In that particular case, the above Proposition can be simplified as follows.

Corollary 8.3.1. *Given a contract (δ, θ), $k = 0$ is incentive-compatible if and only if*

$$h_t^1 + (1-\theta_t)h_t^2 \geq \frac{B}{\varepsilon \alpha_{I-N_t}}, \ t \in [0,\tau], \ \mathbb{P} - a.s. \tag{8.3.10}$$

Remark 8.3.2. *Corollary 8.3.1 states that, given that the pool has i remaining loans, in order to induce the bank to monitor all loans, the continuation payoff must drop in expectation by at least the quantity*

$$b_i := \frac{B}{\varepsilon \alpha_i},$$

following default.

In order to specify further our admissible strategies, we have to put some restrictions on h^1 and h^2. First, we assume that the bank has limited liability. This means that the bank's continuation utility must exceed the lower bound b_{i-1}, because otherwise it would not be possible for the investors to apply the required penalties following default. This implies that the pool can be maintained only if the following condition is not violated

$$\text{For all } 1 \leq i \leq I, \ u_{t^-}^0 - h_t^1 \geq b_{i-1}, \ \text{on } \{N_t = I - i\}. \tag{8.3.11}$$

For the second condition, we assume that the bank forfeits any rights to cash flows once the pool is liquidated. The constraint $u_\tau^0 = 0$ implies in turn that at all times

$$u_t^0 = h_t^1 + h_t^2. \tag{8.3.12}$$

Indeed, the utility of the bank must jump to 0 just after the liquidation of the pool. Since the penalty after liquidation is exactly $h^1 + h^2$, (8.3.12) must hold at each time.

Notice that the introduction of the processes h^1 and h^2 is crucial in this problem, since it allows us to obtain a set of admissible contract which is greatly simplified, the implicit incentive-compatibility condition being replaced by an explicit one. Our set of admissible strategies is therefore

$$\widetilde{\mathcal{A}}^0(x) := \{(\delta,\theta,h^1,h^2),\ \theta \text{ is a predictable process with values in } [0,1],$$
$$\delta \text{ is a positive predictable process which satisfies (8.2.3)},$$
$$h^1 \text{ and } h^2 \text{ are predictable processes, integrable, and satisfy } u^0_{t^-} - h^1_t \geq b_{I-N_t-1},$$
$$u^0_t = h^1_t + h^2_t,\ x \leq u^0_0(\delta,\theta).\}. \tag{8.3.13}$$

8.3.2 Reduction to a stochastic control problem and HJB equation

Under condition (8.3.10), $k = 0$ is incentive-compatible. That being taken care of, solving for the optimal contract involves maximizing an investor's expected utility and is therefore a classical stochastic control problem. Let $v_j(u)$ denote the investor's value function, i.e., the maximum expected utility an investor can achieve given a pool of size j and a reservation utility for the bank u. Then, we expect the investor's value function to solve the following system of Hamilton-Jacobi-Bellman equations with initial condition $v_0(u) = 0$

$$-\sup_{(\delta,\theta,h^1,h^2)\in\mathcal{C}^j} \Big\{\left(ru + \lambda_j\left(h^1 + (1-\theta)h^2\right) - \delta\right) v_j'(u) + j\mu - \delta$$
$$-\theta\lambda_j\left(v_j(u) - v_{j-1}(u-h^1)\right) - (1-\theta)\lambda_j v_j(u)\Big\} = 0,\ u > b_j, \tag{8.3.14}$$

where the $\mathcal{C}^j$ are our admissible strategies sets defined by

$$\mathcal{C}^j := \left\{(\delta,\theta,h^1,h^2),\ \delta \geq 0,\ \theta \in [0,1],\ h^1 + (1-\theta)h^2 \geq b_j,\ u - h^1 \geq b_{j-1},\ u = h^1 + h^2\right\}.$$

Remark 8.3.3. *The above equations have an economic interpretation.*

- *The first term is the change in the investor's utility due to the drift of the utility of the bank, as can be seen in (8.3.7)).*
- *The second term is the revenue the investor gets from the loans after payment to the bank, which is exactly equal to $j\mu - \delta$.*
- *The last two terms correspond to the predictable loss in investor utility following default, depending on whether or not the pool is maintained. The bank's continuation utility is reduced from u to $u - h^1$ with probability $\theta\lambda_j$ (and the number of loans remaining in the pool decreases to $j-1$) and brought down to zero with probability $(1-\theta)\lambda_j$.*

Remark 8.3.4. *We will see in the next section that our control problem is singular. Therefore the above HJB equation (8.3.14) is not exactly the correct one, and we will consider instead a variational inequality.*

Given the constraints in the definition of $\mathcal{C}^i$, it is more convenient to reparametrize the problem in terms of the variable $z := \theta(u - h^1)$. This leads to the simpler system of HJB equations

$$\sup_{(\delta,\theta,z)\in\widetilde{\mathcal{C}}^j} \left\{(ru + \lambda_j(u-z) - \delta)\, v_j'(u) + j\mu - \delta - \lambda_j\left(v_j(u) - \theta v_{j-1}\left(\frac{z}{\theta}\right)\right)\right\} = 0,\ u > b_j, \tag{8.3.15}$$

where the constraints become

$$\widetilde{\mathcal{C}}^j := \left\{(\delta,\theta,z),\ \delta\geq 0,\ \theta\in\left[0, 1\wedge\frac{u-b_j}{b_{j-1}}\right],\ \text{and } z\in[b_{j-1}\theta, u-b_j]\right\}.$$

Our strategy now is to guess a candidate optimal contract by solving the above system of HJB equations, and to prove that the conjectured contract is indeed optimal by means of a verification argument. However, since $j=1$ is a degenerate special case, it is convenient to treat monitoring with a single loan first before turning to the general case.

8.3.3 Single loan: Constant utility

We will always assume that $r\leq\lambda_1$ (this condition will be explained when we will treat the general case). We provide below a solution of the HJB equation which is compatible with our problem, in the sense that the initial conditions for v_1 are obtained from our formulation of the Principal/Agent problem.

Since there is only one loan, when it defaults the pool is automatically liquidated, which means that θ is always equal to 1. Since $v_0=0$ and $b_0=0$, optimizing first with respect to δ yields the following variational inequality for $u>b_1$

$$\min\left\{-\sup_{b_1\leq h^1\leq u}\left\{\left(ru+\lambda_1 h^1\right)v_1'(u)+\mu-\lambda_1 v_1(u)\right\}, v_1'(u)+1\right\}=0.$$

Moreover, it appears that $\delta=0$ as long as $v_1'(u)+1>0$.

It is clear from the above equation that the right-derivative of v_1 at b_1 is equal to $\frac{\lambda_1 v_1(b_1)-\mu}{b_1(r+\lambda_1)}$. We therefore have to consider three cases.

- If $\lambda_1 v_1(b_1)>\mu$, then at least on a small interval on the right of b_1, we have $v_1'\geq 0$. Thus on this interval the equation becomes

$$(r+\lambda_1)\,uv_1'(u)+\mu-\lambda_1 v_1(u)=0,$$

  whose solution is given by

$$\widetilde{v}_1(u) := \left(v_1(b_1)-\frac{\mu}{\lambda_1}\right)\left(\frac{u}{b_1}\right)^{\frac{\lambda_1}{\lambda_1+r}}+\frac{\mu}{\lambda_1}.$$

  Since this function has a derivative which is always positive, this means that in that case $v_1'+1>0$ and therefore δ is always equal to 0. Thus it follows that the investor's utility is equal to (see (8.3.2) when $k=0$)

$$v_1(b_1)=\mathbb{E}\left[\int_0^{\tau^1}\mu ds\right]=\frac{\mu}{\lambda_1},$$

  contradicting the fact that $\lambda_1 v_1(b_1)>\mu$. Hence this case is not possible.

- If $\lambda_1 v_1(b_1)=\mu$, then using (8.3.2) with $k=0$, we obtain that $\delta=0$ (since we assumed that $\delta\geq 0$). Plugging this in (8.3.1), we get that the bank utility is equal to 0 even if the loan has not defaulted, which contradicts the fact that it should remain above its minimum level b_1. Hence this case is not possible either.

- Finally, if $\lambda_1 v_1(b_1) < \mu$, then at least on a small interval on the right of b_1, we have $v_1' \leq 0$. Thus on this interval the variational inequality becomes

$$\min\left\{-\left(ru + \lambda_1 b_1\right) v_1'(u) - \mu + \lambda_1 v_1(u), v_1'(u) + 1\right\} = 0.$$

The solution of the first ODE in the system is given by

$$\widehat{v}_1(u) := \left(v_1(b_1) - \frac{\mu}{\lambda_1}\right)\left(\frac{ru + \lambda_1 b_1}{rb_1 + \lambda_1 b_1}\right)^{\frac{\lambda_1}{r}} + \frac{\mu}{\lambda_1}.$$

It is clear that if $r \leq \lambda_1$ then in that case the above function is concave for $u > b_1$, its derivative decreases to $-\infty$ and is therefore always negative. We will also verify next that we have

$$v_1'(b_1) \geq -1. \tag{8.3.16}$$

This implies that the solution v_1 is equal to $\widehat{v}_1$ until its derivative reaches the value -1 at some uniquely defined point γ_1. Thus, we have a solution on the interval $[b_1, \gamma_1]$. In that case we know that $\delta = 0$ for $u < \gamma_1$. In order to obtain the value of δ when $u = \gamma_1$, we return to the bank's utility dynamics given by (8.3.7)

$$du_t^0 = (ru_t^0 - \delta_t + \lambda_1(h_t^1 + (1 - \theta_t)h_t^2)dt, \text{ for } t < \tau^1.$$

Since $h^1 = b_1$, $h_2 = u - b_1$ and $\theta = 1$, we obtain

$$du_t^0 = (ru_t^0 - \delta_t + \lambda_1 b_1)dt, \text{ for } t < \tau^1.$$

Hence, if $u_0^0 < \gamma_1$ then $\delta = 0$ and thus the utility of the bank keeps on increasing until the default occurs or until the time t^* for which $u_{t^*}^0 = \gamma_1$. Then, δ_t should be chosen so that u_t^0 stays constant after that time t^*, that is to say

$$\delta_t = 1_{t=t^*}(r\gamma_1 + \lambda_1 b_1).$$

Indeed, if $\delta_{t^*} < r\gamma_1 + \lambda_1 b_1$ then u^0 keeps on increasing after t^* and therefore δ_t is equal to 0 except at t^*, and thus the utility of the bank given by (8.3.1) is 0, which contradicts the fact that it should stay above b_1. We obtain similarly a contradiction when $\delta_{t^*} > r\gamma_1 + \lambda_1 b_1$.

Now we want to calculate $v_1(b_1)$. First, in this case $u_0^0 = b_1$, and we therefore have after some calculations

$$t^* = \frac{1}{r}\ln\left(\frac{r\gamma_1 + \lambda_1 b_1}{rb_1 + \lambda_1 b_1}\right),$$

and thus by definition

$$\begin{aligned} v_1(b_1) &= \frac{\mu}{\lambda_1} - (r\gamma_1 + \lambda_1 b_1)\mathbb{E}^{\mathbb{P}}\left[1_{t^*<\tau^1}(\tau^1 - t^*)\right] \\ &= \frac{\mu - (r\gamma_1 + \lambda_1 b_1)e^{-\lambda_1 t^*}}{\lambda_1} \\ &= \frac{\mu}{\lambda_1} - \frac{r\gamma_1 + \lambda_1 b_1}{\lambda_1}\left(\frac{rb_1 + \lambda_1 b_1}{r\gamma_1 + \lambda_1 b_1}\right)^{\frac{\lambda_1}{r}}. \end{aligned}$$

Now recall that we have to verify that (8.3.16) holds. With the above value of $v_1(b_1)$, we obtain

$$v_1'(b_1) = -\left(\frac{rb_1+\lambda_1 b_1}{r\gamma_1+\lambda_1 b_1}\right)^{\frac{\lambda_1}{r}-1} \geq -1.$$

Hence, it remains to verify that we indeed have that u_0^0 calculated with (8.3.1) is equal to b_1. We have

$$\begin{aligned} u_0^0 &= (r\gamma_1+\lambda_1 b_1)\mathbb{E}^{\mathbb{P}}\left[1_{t^*<\tau^1}\int_{t^*}^{\tau} e^{-rs}ds\right] \\ &= \frac{r\gamma_1+\lambda_1 b_1}{r}\mathbb{E}^{\mathbb{P}}\left[1_{t^*<\tau^1}\left(e^{-rt^*}-e^{-r\tau^1}\right)\right] \\ &= \frac{r\gamma_1+\lambda_1 b_1}{r+\lambda_1}e^{-(\lambda_1+r)t^*} \\ &= b_1\left(\frac{rb_1+\lambda_1 b_1}{r\gamma_1+\lambda_1 b_1}\right)^{\frac{\lambda_1}{r}}. \end{aligned}$$

Thus, $u_0^0 = b_1$ if and only if we actually have $\gamma_1 = b_1$, which means that v_1 should be linear above b_1

$$v_1(u) = v_1(b_1) - u + b_1,\ u \geq b_1.$$

We now need to verify that

$$-(ru+\lambda_1 b_1)\,v_1'(u) - \mu + \lambda_1 v_1(u) \geq 0,\ u \geq b_1.$$

We have

$$-(ru+\lambda_1 b_1)\,v_1'(u) - \mu + \lambda_1 v_1(u) = r(u-b_1) \geq 0,$$

which shows that we indeed have found a solution of the variational inequality.

Finally, we compute that

$$v_1'(b_1^-) - v_1(b_1^+) = \frac{\mu - b_1(r+\lambda_1)}{\lambda_1 b_1} + 1 = \frac{\mu - rb_1}{\lambda_1 b_1} \geq \frac{\mu+B}{1+\varepsilon} > 0,$$

by Assumption 8.2.2, we have $\mu - rb_1 \geq \frac{\mu+B}{1+\varepsilon} > 0$, which implies that v_1 is concave.

Summarizing all the above, we have shown the following proposition

Proposition 8.3.2. *Under the assumption $r \leq \lambda_1$, then the function v_1 defined by*

$$v_1(u) := b_1 - u + \frac{\mu - b_1(r+\lambda_1)}{\lambda_1},\ u > b_1,$$

which we extend to $[0, b_1]$ by

$$v_1(u) = \frac{v_1(b_1)}{b_1}u,\ u \leq b_,.$$

is a solution of (8.3.15).

Moreover in that case v_1 is concave.

Remark 8.3.5. *In the case $j = 1$ the utility of the bank is always b_1 and the bank receives constant payments $\delta_t = rb_1 + \lambda_1 b_1$ until the loan defaults. We refer to Section 8.3.6 for the proof that the contract described above is indeed the optimal one when there is only one loan in the pool.*

8.3.4 Formal derivation of a candidate optimal contract

In this section we will derive formally a simple system of ordinary differential equations from the HJB equations (8.3.15), and we will show next that their solutions are actually regular solutions of (8.3.15).

Step (i) Optimizing first with respect to δ yields the following variational inequality for $u > b_j$

$$\min\left\{-\sup_{(\theta,z)\in\widetilde{\mathcal{B}}^j}\left\{(ru+\lambda_j(u-z))\,v_j'(u)+j\mu-\lambda_j\left(v_j(u)-\theta v_{j-1}\left(\frac{z}{\theta}\right)\right)\right\},v_j'(u)+1,\right\}=0. \tag{8.3.17}$$

where

$$\widetilde{\mathcal{B}}^j:=\left\{(\theta,z),\ \theta\in\left[0,1\wedge\frac{u-b_j}{b_{j-1}}\right],\ \text{and } z\in[b_{j-1}\theta,u-b_j]\right\}.$$

We continue our guess of the value function assuming that all the functions v_j are concave (a property which needs to be verified by our candidate). Then the first derivative of v_j is decreasing. Let us also assume that there exists a level $\gamma_j > b_j$ (which is going to be our free boundary) such that

$$v_j'(\gamma_j)=-1,\ v_j'(u)>-1,\ \text{for } u<\gamma_j,$$

Then as long as $u < \gamma_j$, v_j satisfies the first equation in (8.3.17). Therefore, equation (8.3.17) tells us that the bank receives cash from the investors only when its utility attains the level γ_j (since $\delta = 0$ is optimal before that). We also assume (and we will verify) that our candidate satisfy

$$-\sup_{(\theta,z)\in\widetilde{\mathcal{B}}^j}\left\{(ru+\lambda_j(u-z))\,v_j'(u)+j\mu-\lambda_j\left(v_j(u)-\theta v_{j-1}\left(\frac{z}{\theta}\right)\right)\right\}\geq 0,\ u\geq\gamma_j.$$

This means that v_j becomes linear above γ_j, and that the variational inequality (8.3.17) takes the simpler form

$$\begin{aligned}&-\sup_{(\theta,z)\in\widetilde{\mathcal{B}}^j}\left\{(ru+\lambda_j(u-z))\,v_j'(u)+j\mu-\lambda_j\left(v_j(u)-\theta v_{j-1}\left(\frac{z}{\theta}\right)\right)\right\}=0,\ b_j<u\leq\gamma_j\\&v_j'(u)+1=0,\ u>\gamma_j.\end{aligned}$$

Now in order to know which level γ_j should be chosen, it is natural to require our solution to be maximal in the sense that for each $u > b_j$

$$\gamma_j\longrightarrow v_j(u),$$

is maximal at the chosen value of γ_j. Of course, it is not clear at all whether such a value exists. Nonetheless, we will prove that this heuristic approach can be proved rigorously, and that our maximality assumption has a clear economic meaning.

Step (ii) We next turn to the liquidation decision, one finds as first-order condition with respect to θ

$$v_{j-1}\left(\frac{z}{\theta}\right)-\frac{z}{\theta}v_{j-1}'\left(\frac{z}{\theta}\right)\geq 0. \tag{8.3.18}$$

Once again, if v_{j-1} is concave, the above inequality (8.3.18) is always verified. This means that the function

$$\theta\longrightarrow\theta v_{j-1}\left(\frac{z}{\theta}\right),$$

is non-decreasing, which implies that the optimal θ corresponds to its upper bound.

There are then two cases

(i) $u \in [b_j, b_j + b_{j-1})$ and $\theta = \frac{u-b_j}{b_{j-1}}$.

(ii) $u \in [b_j + b_{j-1}, \gamma_j]$ and $\theta = 1$.

In the first interval, the pool is liquidated with strictly positive probability following default. The continuation decision θ reflects the position of u in $[b_j, b_j + b_{j-1})$. Thus if a default occurs in that interval, either the bank's continuation utility drops to the minimum threshold b_j or the pool is liquidated. In contrast there is no liquidation in the interval $[b_j + b_{j-1}, \gamma_j)$, which we refer to as "probation." It will be verified that $\gamma_j \geq b_j + b_{j-1}$, implying that payments are made to the bank when its continuation utility is in $[b_j, b_j + b_{j-1})$.

Step (iii) Finally consider the decision regarding z. First, if $u \in [b_j, b_j + b_{j-1})$, then z has to be equal to $u - b_j$. Then, in the probation interval $\theta = 1$ and z is constrained in the range $[b_{j-1}, u - b_j]$. We continue our guess of a candidate solution assuming that

$$v'_{j-1}(u - b_j) - v'_j(u) \geq 0, \tag{8.3.19}$$

a condition which needs to be verified by the resulting candidate.

Then, since v_{j-1} is supposed to be concave, we have for all $z \in [b_{j-1}, u - b_j]$

$$v'_{j-1}(z) - v'_j(u) \geq 0.$$

From this, we obtain that the function $z \longrightarrow -zv'_j + v_{j-1}(z)$ is non-decreasing, which in turn implies that the supremum over z is also attained at $u - b_j$ in the probation interval.

Summarizing all the above formal calculations, we end up with the following system of ODEs, which should lead us to a solution of the HJB equation on the interval $[b_j, \gamma_j]$

$$\begin{aligned} &(ru + \lambda_j b_j)\, v'_j(u) + j\mu - \lambda_j \left(v_j(u) - v_{j-1}(u - b_j)\right) = 0, \qquad u \in (b_j + b_{j-1}, \gamma_j] \\ &(ru + \lambda_j b_j)\, v'_j(u) + j\mu - \lambda_j \left(v_j(u) - \frac{u - b_j}{b_{j-1}} v_{j-1}(b_{j-1})\right) = 0, \qquad u \in (b_j, b_j + b_{j-1}]\,. \end{aligned}$$

We next extend the value function v_j to the interval $[0, b_j]$ by setting

$$v_j(u) := \frac{u}{b_j} v_j(b_j), \qquad u \in [0, b_j]\,, \tag{8.3.20}$$

and to the interval $(\gamma_j, +\infty)$ by

$$v_j(u) := v_j(\gamma_j) - u + \gamma_j.$$

Then the above system of ODEs simplifies to

$$\begin{aligned} &(ru + \lambda_j b_j)\, v'_j(u) + j\mu - \lambda_j \left(v_j(u) - v_{j-1}(u - b_j)\right) = 0, \qquad u \in (b_j, \gamma_j] \\ &v'_j(u) = -1,\ \ u \geq \gamma_j. \end{aligned} \tag{8.3.21}$$

Recall that we will need to verify that the solution obtained from (8.3.21) satisfies all the properties we used to derive our candidate.

8.3.5 Solving the HJB equation

We now provide conditions under which the heuristic derivation of the previous section indeed corresponds to a solution of the original system of HJB equations (8.3.15). Since we already solved the problem for $j = 1$, we assume here that $j \geq 2$.

Let us define

$$\overline{v}_j := v_j(b_j),$$

and for $x > 0$ and $0 < \beta \leq 1$the functions

$$\phi_\beta(x) := \left(\frac{1+x}{1+(1+\beta)x}\right)^{\frac{1}{x}-1}, \ \psi_\beta(x) := \frac{\phi_\beta(x) - x}{(1-x)\phi\beta(x)}.$$

Remark 8.3.6. *Then, it is easy to prove that the functions ψ_β can be extended to continuous functions on $\mathbb{R}_+$ which decrease from 1 to $\frac{1}{2}$ and that for all $x \geq 0$*

$$\psi_1(x) = \inf_{0<\beta\leq 1} \psi_\beta(x).$$

We have the following results.

Proposition 8.3.3. *Assume that*

$$\frac{r}{\lambda_j} - 1 \leq \frac{\overline{v}_{j-1}}{b_{j-1}}. \tag{8.3.22}$$

(i) *The ordinary differential equations (8.3.21), along with (8.3.20), have unique maximal solutions v_j for $j \geq 2$. The functions v_j are globally concave, differentiable everywhere except at b_j and twice differentiable everywhere except at b_j and $b_j + b_{j-1}$. The endogenous thresholds $\gamma_j \geq b_j + b_{j-1}$ are uniquely determined by*

$$\frac{r}{\lambda_j} - 1 \in \partial v_{j-1}(\gamma_j - b_j), \tag{8.3.23}$$

where $\partial v_j(u)$ is the subdifferential of v_j at u and verify

$$\gamma_j \leq b_j + \gamma_{j-1}. \tag{8.3.24}$$

(ii) *The λ_j can be chosen recursively so that*

$$\left(v'_{j-1}(b^+_{j-1})\right)^+ \frac{b_{j-1}}{\overline{v}_{j-1}} \leq \psi_1\left(\frac{r}{\lambda_j}\right), \tag{8.3.25}$$

In that case, the functions v_j also verify

$$v'_j(u) - v'_{j-1}(u - b_j) \leq 0, \ \textit{for all } u \geq b_j. \tag{8.3.26}$$

The proof is rather tedious and is relegated to the Appendix.

Remark 8.3.7. *Notice that in the proof, the Assumption 8.2.1 is only used to prove that the b_j are increasing and thus that*

$$\psi_1\left(\frac{r}{\lambda_j}\right) \leq \psi_{\frac{b_{j-1}}{b_j}}\left(\frac{r}{\lambda_j}\right).$$

If one were satisfied, instead of condition (8.3.25), by the condtion

$$\left(v'_{j-1}(b^+_{j-1})\right)^+ \frac{b_{j-1}}{\overline{v}_{j-1}} \leq \psi_{\frac{b_{j-1}}{b_j}}\left(\frac{r}{\lambda_j}\right),$$

which is implicit, then we no longer need Assumption 8.2.1.

Now since the functions v_j constructed in Proposition 8.3.3 are globally concave, have a derivative which is greater than -1 for $u < \gamma_j$ and equal to -1 for $u \geq \gamma_j$ and satisfy (8.3.26), we can apply the heuristic arguments of Section 8.3.4 to obtain the following corollary.

Corollary 8.3.2. *Under the assumptions of Proposition 8.3.3, the functions v_j constructed in the same Proposition solve the HJB equation (8.3.14).*

Proof. The only remaining property to prove is that for $u \geq \gamma_j$, we have

$$-\left(ru + \lambda_j b_j\right) v'_j(u) - j\mu + \lambda_j \left(v_j(u) - v_{j-1}(u - b_j)\right) \geq 0.$$

We compute

$$\begin{aligned}
&-\left(ru + \lambda_j b_j\right) v'_j(u) - j\mu + \lambda_j \left(v_j(u) - v_{j-1}(u - b_j)\right)\\
&= ru + \lambda_j b_j - j\mu + \lambda_j \left(v_j(\gamma_j) - u + \gamma_j - v_{j-1}(u - b_j)\right)\\
&= r(u - \gamma_j) + \lambda_j \left(v_{j-1}(\gamma_j - b_j) + \gamma_j - b_j - v_{j-1}(u - b_j) - u + b_j\right)\\
&\geq r(u - \gamma_j) - \lambda_j (u - \gamma_j) \left(1 + \sup_{\gamma_j - b_j \leq x \leq u - b_j} v'_{j-1}(x)\right)\\
&\geq r(u - \gamma_j) - \lambda_j (u - \gamma_j) \frac{r}{\lambda_j}\\
&= 0,
\end{aligned}$$

where we used the fact that v_{j-1} is concave, that $u \to v_{j-1}(u) + u$ is increasing and that $v'_{j-1}(\gamma_j - b_j) \leq \frac{r}{\lambda_j} - 1$.

In particular, this shows that

$$\begin{aligned}
-\sup_{(\delta,\theta,h^1,h^2)\in\mathcal{C}^j} &\left\{\left(ru + \lambda_j \left(h^1 + (1-\theta)h^2\right) - \delta\right) v'_j(u) + j\mu - \delta\right.\\
&\left. -\lambda_j \left(v_j(u) - \theta v_{j-1}(u - h^1)\right)\right\} \geq 0, \; u \geq \gamma_j.
\end{aligned}$$

□

Let us finally describe the contract (δ, θ) which can be deduced from the above results. Starting from a reservation utility $x \leq \gamma_I$ for the bank, the following contract unfolds.

Contract 8.3.1. (i) *Given size j, the pool remains in operation (i.e. there is no liquidation) with one less unit at any time there is a default in the range $[b_j + b_{j-1}, \gamma_j]$.*

(ii) *The flow of fees paid to the bank given j is $\delta^j_t = \lambda_j b_j + r\gamma_j$ as long as $u_t = \gamma_j$ and no default occurs. Otherwise $\delta^j_t = 0$.*

(iii) *Liquidation of the whole pool occurs with probability* $\theta_t^j = (u_t - b_j)/b_{j-1}$ *in the range* $[b_j, b_j + b_{j-1})$.

To summarize, we have for j *given and with the original notations of* (8.3.14)

$$\begin{aligned} \delta^j(u) &:= 1_{u=\gamma_j}(\lambda_j b_j + r\gamma_j) \\ \theta^j(u) &:= 1_{b_j \le u < b_j + b_{j-1}} \frac{u - b_j}{b_{j-1}} + 1_{b_j + b_{j-1} \le u \le \gamma_j} \\ h^{1,j}(u) &:= (u - b_{j-1}) 1_{b_j \le u < b_j + b_{j-1}} + b_j 1_{b_j + b_{j-1} \le u \le \gamma_j} \\ h^{2,j}(u) &:= u - h^{1,j}(u). \end{aligned} \tag{8.3.27}$$

Remark 8.3.8. *If the reservation utility for the bank* x *is greater than* γ_I *then the contract should specify in addition that a transfer is immediately made to the bank so that its utility returns to the level* γ_I. *This means that instead of considering transfers* $(D_t)_{t\ge 0}$ *which are only absolutely continuous with respect to the Lebesgue measure, we have to add a Dirac mass at* 0. *Our proofs can then be easily adjusted to that case, therefore we will not treat it. Moreover, notice that the contract 8.3.1 is clearly in* $\widetilde{\mathcal{A}}^0(x)$.

8.3.6 The verification theorem

In this section, we prove our main result

Theorem 8.3.1. *Let* $u_0 \le \gamma_I$ *be the reservation utility for the bank. Then, the optimal contract in* $\widetilde{\mathcal{A}}^0(x)$ *for the problem* (8.3.3) *is the contract 8.3.1.*

We decompose the proof in two parts. First, we show that the bank can obtain a level of utility u_0 and the investors $v_I(u_0)$, for any $u_0 \ge b_I$, with this contract. The second part, reported in Proposition 8.3.5, shows that for any contract (δ, θ) which makes the shirking decision $k = 0$ incentive-compatible, the utility the investors can obtain is bounded from above by $v_I(u_0)$, where u_0 is the utility obtained by the bank.

Proposition 8.3.4. *Let the assumptions of Proposition 8.3.3 hold true. For any starting condition* $u_0 > b_I$, *we define the process* u_t *as the solution of the following SDE for* $j = 0..I-1$

$$\begin{aligned} du_t = &(ru_t - \delta^{I-N_t}(u_t))dt - h^{1,I-N_t}(u_t)(dN_t - \lambda_{I-N_t}dt) \\ &- h^{2,I-N_t}(u_t)(dH_t - \lambda_{I-N_t}(1 - \theta^{I-N_t}(u_t))dt), \ t < \tau. \end{aligned} \tag{8.3.28}$$

Then, the contract defined by $\left(\delta^{I-N_t}(u_t), \theta^{I-N_t}(u_t)\right)$ *is incentive compatible, has value* u_0 *for the bank and value* $v_I(u_0)$ *for the investors.*

Proof. First, the drift and volatility in the SDE (8.3.28) are clearly Lipschitz. This guarantees the existence and uniqueness of the solution for all t. Moreover, it is also clear by definitions of δ^{I-N_t}, θ^{I-N_t}, $h^{1,I-N_t}$ and $h^{2,I-N_t}$ that

$$ru_t - \delta^{I-N_t} + \lambda_{I-N_t}\left(h^{1,I-j}(u_t) + (1 - \theta^{I-N_t}(u_t))h^{2,I-N_t}(u_t)\right) \ge 0.$$

Hence u_t remains below γ_{I-N_t}. Moreover, when N jumps, we have at the time of the jump

$$\begin{aligned} u_t &= u_{t^-} - h_t^{1,I-N_{t^-}} \\ &= b_{I-N_t} 1_{b_{I-N_{t^-}} \leq u_{t^-} < b_{I-N_t} + b_{I-N_{t^-}}} + (u_{t^-} - b_{N_{t^-}}) 1_{b_{I-N_t} + b_{I-N_{t^-}} \leq u_{t^-} \leq \gamma_{I-N_{t^-}}} \\ &\geq b_{I-N_t}. \end{aligned}$$

Therefore, we always have $u_t \geq b_{I-N_t}$ for $t < \tau$. Hence, the process u is bounded.

Moreover, it is clear by construction that this contract makes the shirking decision $k = 0$ incentive-compatible. Indeed, we have after some calculations for all j

$$h^{1,I-N_t}(u_t) + (1 - \theta^{I-N_t}(u_t)) h^{2,I-N_t}(u_t) = b_{I-N_t}, \ t < \tau,$$

which is exactly (8.3.10).

Then, using the equation (8.3.7) for the continuation utility of the bank obtained with the contract $(\delta^{I-N_t}(u_t), \theta^{I-N_t}(u_t))$, we obtain

$$\begin{aligned} d\left(e^{-rt}(u_t^0 - u_t)\right) &= e^{-rt}\left((h_t^1 - h^{1,I-N_t}(u_t))(dN_t - \lambda_{I-N_t} dt)\right) \\ &\quad + e^{-rt}\left((h_t^2 - h^{2,I-N_t}(u_t))(dH_t - \lambda_{I-N_t}(1 - \theta^{I-N_t}(u_t)) dt)\right). \end{aligned}$$

Since $h^{1,N_t}(u_t)$ and $h^{2,I-N_t}(u_t)$ are bounded because u_t is bounded and since h_t^1 and h_t^2 are in the space $L^1(\mathbb{P})$ by construction, we can take the conditionnal expectation above to obtain

$$\mathbb{E}_t\left[u_{t+s}^0 - u_{t+s}\right] = e^{rs}(u_t^0 - u_t).$$

u^0 remains bounded, because the δ^j are bounded for all j (recall (8.3.6)) and u is bounded, thus the left-hand side above must remain bounded. Since $r > 0$, letting s go to $+\infty$ implies that necessarily $u_t = u_t^0$, $\mathbb{P} - a.s.$ and in particular that the bank overall utility is

$$u_0^0 = u_0.$$

Let us now turn our attention to the investors. Define

$$G_t := \int_0^t ((I - N_s)\mu - \delta(u_s)) ds + v_{I-N_t}(u_t), \tag{8.3.29}$$

where the v_j are those defined in Proposition 8.3.3.

Let us place ourselves on the interval $[\tau_j \wedge \tau, \tau_{j+1} \wedge \tau)$. We have shown before that u_t remains above b_{I-j}. But we know by construction that v_{I-j} is continuous on $[b_{I-j}, +\infty)$ and has a derivative which can be continuously extended on $[b_{I-j}, +\infty)$. Hence we can apply the change of variable formula for locally bounded processes (see [34], Chapter VI, Section 92) to obtain for all $t \geq 0$

$$\begin{aligned} G_t = v_I(u_0) &+ \sum_{j=0}^{I-1} \int_{\tau_j \wedge t}^{\tau_{j+1} \wedge t} (I - j)\mu - \delta^{I-j}(u_s) + v'_{I-j}(u_s)\left(r u_s - \delta^{I-j}(u_s)\right) ds \\ &+ \sum_{j=0}^{I-1} \int_{\tau_j \wedge t}^{\tau_{j+1} \wedge t} \lambda_{I-j} v'_{I-j}(u_s)\left(h^{1,I-j}(u_s) + (1 - \theta^{I-j}(u_s)) h^{2,I-j}(u_s)\right) ds \\ &+ \sum_{j=0}^{I-1} \sum_{\tau_j \wedge t \leq s \leq \tau_{j+1} \wedge t} v_{I-j}(u_s) - v_{I-j}(u_{s^-}). \end{aligned} \tag{8.3.30}$$

Let us decompose the jumps of v_j. We have

$$\begin{aligned} v_j(u_s) - v_j(u_{s^-}) &= \Delta N_s \left((1-\Delta H_s)\, v_{j-1}\left(u_{s^-} - h^{1,j}(u_{s^-})\right) - v_j\left(u_{s^-}\right)\right) \\ &= \Delta N_s \left(v_{j-1}\left(u_{s^-} - h^{1,j}(u_{s^-})\right) - v_j\left(u_{s^-}\right)\right) - \Delta H_s v_{j-1}\left(u_{s^-} - h^{1,j}(u_{s^-})\right), \end{aligned}$$

which implies that

$$\begin{aligned} \sum_{\tau_j\wedge t \le s \le \tau_{j+1}\wedge t} v_{I-j}(u_s) - v_{I-j}(u_{s^-}) = &\int_{\tau_j\wedge t}^{\tau_{j+1}\wedge t} \left(v_{I-j-1}\left(u_{s^-} - h^{1,I-j}(u_{s^-})\right) - v_{I-j}\left(u_{s^-}\right)\right) dN_s \\ &- \int_{\tau_j\wedge t}^{\tau_{j+1}\wedge t} v_{I-j-1}\left(u_{s^-} - h^{1,I-j}(u_{s^-})\right) dH_s. \end{aligned}$$

From this, we obtain

$$\begin{aligned} G_t = v_I(u_0) &+ \sum_{j=0}^{I-1} \int_{\tau_j\wedge t}^{\tau_{j+1}\wedge t} (I-j)\mu - \delta^{I-j}(u_s) + v'_{I-j}(u_s)\left(ru_s - \delta^{I-j}(u_s)\right) ds \\ &+ \sum_{j=0}^{I-1} \int_{\tau_j\wedge t}^{\tau_{j+1}\wedge t} \lambda_{I-j} v'_{I-j}(u_s)\left(h^{1,I-j}(u_s) + (1-\theta^{I-j}(u_s)) h^{2,I-j}(u_s)\right) ds \\ &+ \sum_{j=0}^{I-1} \int_{\tau_j\wedge t}^{\tau_{j+1}\wedge t} \lambda_{I-j}\left(v_{I-j-1}\left(u_s - h^{1,I-j}(u_s)\right) - v_{I-j}\left(u_s\right)\right) ds \\ &- \sum_{j=0}^{I-1} \int_{\tau_j\wedge t}^{\tau_{j+1}\wedge t} \lambda_{I-j}(1-\theta^{I-j}) v_{I-j-1}\left(u_s - h^{1,I-j}(u_s)\right) ds \\ &+ \sum_{j=0}^{I-1} \int_{\tau_j\wedge t}^{\tau_{j+1}\wedge t} \left(v_{I-j-1}\left(u_{s^-} - h^{1,I-j}(u_{s^-})\right) - v_{I-j}\left(u_{s^-}\right)\right)\left(dN_s - \lambda_{I-j} ds\right) \\ &- \sum_{j=0}^{I-1} \int_{\tau_j\wedge t}^{\tau_{j+1}\wedge t} v_{I-j-1}\left(u_{s^-} - h^{1,I-j}(u_{s^-})\right)\left(dH_s - \lambda_{I-j}(1-\theta^{I-j}(u_{s^-})) ds\right). \end{aligned}$$

Using the fact that the v_j solve the HJB equation 8.3.21, we deduce

$$\begin{aligned} G_t = v_I(u_0) &+ \sum_{j=0}^{I-1} \int_{\tau_j\wedge t}^{\tau_{j+1}\wedge t} \left(v_{I-j-1}\left(u_{s^-} - h^{1,I-j}(u_{s^-})\right) - v_{I-j}\left(u_{s^-}\right)\right)\left(dN_s - \lambda_{I-j} ds\right) \\ &- \sum_{j=0}^{I-1} \int_{\tau_j\wedge t}^{\tau_{j+1}\wedge t} v_{I-j-1}\left(u_{s^-} - h^{1,I-j}(u_{s^-})\right)\left(dH_s - \lambda_{I-j}(1-\theta^{I-j}(u_{s^-})) ds\right). \end{aligned} \tag{8.3.31}$$

Hence, G is a bounded martingale until time τ (since δ is bounded by definition and u_t and thus the $v_j(u_t)$ are also bounded) and we have, since $u_\tau = 0$

$$\mathbb{E}\left[\int_0^\tau \left((I-N_t)\mu - \delta_t\right) dt\right] = \mathbb{E}[G_\tau] = G_0 = v_I(u_0),$$

which is the desired result. □

We now show that $v_I(u_0)$ is an upper bound for the utility the investor can obtain from any contract which makes the shirking decision $k = 0$ incentive-compatible.

Proposition 8.3.5. *For any contract* $(\delta, \theta) \in \widetilde{\mathcal{A}}^0(u_0)$*, the utility the investors can obtain is bounded from above by* $v_I(u_0)$*, where* u_0 *is the utility obtained by the bank.*

Proof. We define as in the previous proof the quantity G_t for an arbitrary contract (δ, θ). By applying the change of variable formula and arguing exactly as before we can obtain that the drift of G is actually negative, using again (8.3.14). Indeed, we know that for any $(\delta, \theta, h^1, h^2) \in \widetilde{\mathcal{A}}^0(u_0)$, we have from Corollary 8.3.2 and its proof that for all j

$$\left(ru_t + \lambda_j\left(h_t^1 + (1-\theta_t)h_t^2\right) - \delta_t\right) v_j'(u_t) + j\mu - \lambda_j\left(v_j(u_t) - \theta_t v_{j-1}(u_t - h_t^1)\right) \leq 0.$$

Hence, using again (8.3.31), we have

$$\begin{aligned} G_{t\wedge\tau} \leq v_I(u) &+ \int_0^{\tau\wedge t} \left(v_{I-N_s-1}\left(u_{s^-} - h_s^{1,I-N_s}\right) - v_{I-N_s}\left(u_{s^-}\right)\right)\left(dN_s - \lambda_{I-N_s} ds\right) \\ &- \int_0^{\tau\wedge t} v_{I-N_s-1}\left(u_{s^-} - h_s^{1,I-N_s}\right)\left(dH_s - \lambda_{I-N_s}(1-\theta_s^{I-N_s})ds\right). \end{aligned} \tag{8.3.32}$$

Now we have

$$\begin{aligned} &\mathbb{E}\left[\int_0^{\tau\wedge t} \left|v_{I-N_s-1}\left(u_s - h_s^{1,I-N_s}\right) - v_{I-N_s}\left(u_s\right)\right| ds\right] \\ &\leq \mathbb{E}\left[\int_0^{\tau\wedge t} \left|v_{I-N_s-1}\left(u_s - h_s^{1,I-N_s}\right) - v_{I-N_s-1}\left(u_s - b_{I-N_s}\right)\right| ds\right] \\ &\quad + \mathbb{E}\left[\int_0^{\tau\wedge t} \left|v_{I-N_s-1}\left(u_s - b_{I-N_s}\right) - v_{I-N_s}\left(u_s\right)\right| ds\right] \end{aligned}$$

Then, from (8.3.26), we know that for all j the function $u \longrightarrow v_j(u) - v_{j-1}(u - b_j)$ is decreasing. Moreover, for u large enough (namely $u \geq \gamma_j \vee (\gamma_{j-1} + b_j)$) we have

$$v_j(u) - v_{j-1}(u - b_j) = v_j(\gamma_j) + \gamma_j - v_{j-1}(\gamma_{j-1}) + \gamma_{j-1} - b_j,$$

which implies that for all j the function $u \longrightarrow v_j(u) - v_{j-1}(u - b_j)$ is bounded.

Moreover, we have

$$\begin{aligned} &\mathbb{E}\left[\int_0^{\tau\wedge t} \left|v_{I-N_s-1}\left(u_s - h^{1,I-N_s}\right) - v_{I-N_s-1}\left(u_s - b_{I-N_s}\right)\right| ds\right] \\ &\leq \mathbb{E}\left[\int_0^{\tau\wedge t} \left|h_s^{1,I-N_s} - b_{I-N_s}\right| \sup_{b_{I-N_s} < u \leq \gamma_{I-N_s}} \left|v_{I-N_s}'(u)\right| ds\right] \\ &\leq C\left(1 + \mathbb{E}\left[\int_0^{\tau\wedge t} |u_s|\, ds\right]\right) \\ &\leq C\left(1 + \mathbb{E}\left[\int_0^{\tau\wedge t} ue^{(r+2\lambda)s} ds\right]\right) < +\infty, \end{aligned}$$

where $\lambda := \sup_{1\leq j\leq I} \lambda_j$, and where we used successively the fact that the derivative of the v_j can be extended to a continuous function on $[b_j, \gamma_j]$ which is therefore bounded on that compact, then the fact that by the limited liability condition (8.3.11) we have $h_t^1 \leq u_t$ and finally that conditionnaly on the fact that there are j loans left in the pool, the drift of u_t given by (8.3.7) is equal to,

$$\begin{aligned} ru_t + \lambda_j \left(h_t^1 + (1-\theta_t)h_t^2\right) - \delta_t &\leq ru_t + \lambda_j \left(h_t^1 + (1-\theta_t)(u_t - h_t^1)\right) \\ &\leq ru_t + \lambda_j \left(u_t - b_{j-1} + (1-\theta_t)u_t\right)\right) \\ &\leq u_t(r + 2\lambda_j), \end{aligned}$$

where we used the fact that h^1, b_j and λ_j are positive. Hence, u_t increases at a rate lower than $r + 2\lambda$.

Similarly, we have

$$\begin{aligned} &\mathbb{E}\left[\int_0^{\tau\wedge t} \left|v_{I-N_s-1}\left(u_{s^-} - h_s^{1,I-N_s}\right)\right| ds\right] \\ &= \mathbb{E}\left[\int_0^{\tau\wedge t} \left|v_{I-N_s-1}\left(u_{s^-} - h_s^{1,I-N_s}\right) - v_{I-N_s-1}\left(u_{s^-} - h_s^{1,I-N_s} - h_s^{2,I-N_s}\right)\right| ds\right] \\ &\leq \mathbb{E}\left[\int_0^{\tau\wedge t} \left|h_s^{2,I-N_s}\right| \sup_{b_{I-N_s}<u\leq\gamma_{I-N_s}} \left|v'_{I-N_s-1}(u)\right| ds\right] \\ &\leq \mathbb{E}\left[\int_0^{\tau\wedge t} |u_s| \sup_{b_{I-N_s}<u\leq\gamma_{I-N_s}} \left|v'_{I-N_s-1}(u)\right| ds\right] < +\infty. \end{aligned}$$

Taking expectations in (8.3.32), we therefore obtain

$$\begin{aligned} v_I(u_0) &\geq \mathbb{E}\left[\int_0^\tau \left((I-N_s)\mu - \delta_s\right) ds\right] + \mathbb{E}\left[1_{t<\tau}\left(\int_t^\tau \left(\delta_s - (I-N_s)\mu\right) ds + v_{I-N_t}(u_t)\right)\right] \\ &= \mathbb{E}\left[\int_0^\tau \left((I-N_s)\mu - \delta_s\right) ds\right] + \mathbb{E}\left[1_{t<\tau}\mathbb{E}_t\left[\int_t^\tau \left(\delta_s - (I-N_s)\mu\right) ds + v_{I-N_t}(u_t)\right]\right] \\ &= \mathbb{E}\left[\int_0^\tau \left((I-N_s)\mu - \delta_s\right) ds\right] + \mathbb{E}\left[1_{t<\tau}\left(u_t + v_{I-N_t}(u_t) - \mathbb{E}_t\left[\int_t^\tau (I-N_s)\,\mu ds\right]\right)\right] \\ &\geq \mathbb{E}\left[\int_0^\tau \left((I-N_s)\mu - \delta_s\right) ds\right] + \mathbb{E}\left[1_{t<\tau}\left(-I\mu\tau + u_t + v_{I-N_t}(u_t)\right)\right]. \end{aligned} \tag{8.3.33}$$

Then, we know that for all j the function $u \longrightarrow u + v_j(u)$ is increasing before γ_j and is constant for $u \geq \gamma_j$. It is therefore bounded and we have

$$\left|-I\mu\tau + u_t + v_{I-N_t}(u_t)\right| \leq I\mu\tau + \sup_{1\leq j\leq I} |\gamma_j + v_j(\gamma_j)| \leq C(1+\tau),$$

for some positive constant C. This quantity being integrable, we can apply the dominated convergence theorem in (8.3.33) and let t go to $+\infty$ to obtain

$$v_I(u_0) \geq \mathbb{E}\left[\int_0^\tau \left((I-N_s)\mu - \delta_s\right) ds\right],$$

which is the desired result. □

8.4 Economic interpretations and some heuristic justifications

In this section we provide some remarks and explanations which give more economic meaning to the results we obtained on the investor's value function and the optimal contract. Finally, we provide some heuristic justifications of some of the assumptions we made.

8.4.1 The properties of the investor's value function

We describe below the economic interpretation of the properties of the investor's value function v_I described in Proposition 8.3.3.

- The investor's value functions v_j are concave on $[b_j, \infty]$. This property reflects the inefficiency arising from stochastic liquidation. Investor value reacts all the more strongly to u, considered as indicator of performance, as u is low. Since liquidation arises in the interval $[b_j, b_j + b_{j-1}]$ and is all the more likely as u is low, the highest inefficiency arises when the bank is constrained at the reservation level b_j.

- Equation (8.3.23) is essentially the "smooth pasting" condition $v_j''(\gamma_j) = 0$ ensuring that γ_j is optimal. If the v_j were strictly concave at γ_j, more surplus could be obtained by marginally raising the threshold beyond that level. It looks slightly complicated because v_{j-1} may not be differentiable at $\gamma_j - b_j$. When $\gamma_j > b_j + b_{j-1}$ however, condition (8.3.23) reads more simply

$$\frac{r}{\lambda_j} = 1 + v_{j-1}'(\gamma_j - b_j). \tag{8.4.1}$$

  The interpretation of (8.4.1) is the following. The right-hand side is the social value of performance following immediately default under target given j, i.e., the shadow price of performance should the size become $j-1$. As long as this is greater than the expected cost of performance given j, r/λ_j, the bank's target γ_j can be raised. Note that by construction $\gamma_j - b_j < \gamma_{j-1}$ when $r > 0$, so the bank's utility following default never starts above target given $j-1$. This implies that, whenever a default occurs, payments are suspended for some time.

- The decision to operate a larger pool is ruled by (8.3.22). Indeed, the concavity of the extended value function implies that $1 + v_j'(u)$ is bounded above by $1 + \overline{v}_j/b_j$. If the condition did not hold, the expected cost of performance r/λ_j would be higher than its social value, implying that it would not be efficient to let the bank operate a pool of size j.

8.4.2 The optimal contract

The investor's strategy relies on two instruments: the prospect of future payments if there is no default for some time (the carrot), and the risk of stochastic liquidation if there are poor performances (the stick). The minimum rent consistent with monitoring is b_j when there are still j loans left in the pool. Given track record $u_t \geq b_j$, it makes sense for investors to encourage the bank to improve its credentials before making payments. To keep the bank participating, they let the rent grow at a rate consistent with pool size and rate of impatience. The definition of the contract (8.3.1) determines how far the target γ_j should be away from b_j given j. Once the target is reached, the payment of fees are resumed.

Hence, the bank is encouraged to monitor the loans, because it knows that after each default, it is going to have to wait before she receives payments again. Moreover, it also knows that if its utility becomes too low (which probably means that it is doing a poor job monitoring the loans), then it faces the risk of a premature stochastic liquidatin which would stop the payments forever.

Finally, let us comment on the choice of the level of the reservation utility of the bank u_0. If we refer to the shape of the function v_I (see Figure 8.6 below), if the investors were in a monopolistic position for example, they should choose the level u_0 which corresponds to the maximum of v_I. However, we assumed that the investors were competitive, which means that the bank will only choose contract which gives her the maximum level of utility. Hence at the equilibrium, the initial level of utility for the bank should be γ_I. The competitive behavior of the investors will also play a role in Section 8.4.3(i) below.

8.4.3 Heuristic justification of some of our assumptions

In this section, we show successively that our Assumption 8.2.2 somehow justifies the fact that the shirking decision $k = 0$ under which we solved the problem is optimal from a social point of view, then we justify heuristically under an additionnal assumption that our stochastic liquidation policy is also optimal.

(i) **Justification of Assumption 8.2.2**

The analysis has been carried out assuming that $k_t = 0$ was incentive-compatible. Suppose in contrast that we let the bank completely shirks and thus reap private benefits over the infinitesimal interval $[t, t + dt)$ and to revert to the optimal policy afterwards. Let w and w' be the pool values associated with the optimal and alternative policies, respectively, that is to say the value of the bank plus that of the investors. If we start at time t with j loans letf in the pool, at date $t + dt$, when the optimal policy is resumed, the system is either in state (j, u_t) or in state $(j - 1, u_t - b_j)$, where $u_t \in [b_j, \gamma_j]$. As the two policies are identical starting $t + dt$, we have $w_{t+dt} = w'_{t+dt}$.

Under the optimal policy, the dynamics of w over $[t + dt)$ are given by

$$dw_t + (j\mu - ru_t - \lambda_j \Delta w_t)\, dt = 0,$$

where $\Delta w_t = v_j(u_t) - v_{j-1}(u_t - b_j) + b_j$ is the social loss incurred in case of default. Note that $dw_t \geq 0$ since $\frac{dv_j}{du} \geq -1$ and the drift of u_t is positive in that case.

Then, under the alternative policy, the dynamics of w' are given by

$$dw'_t + (j(\mu + B) - \lambda_j(1 + \varepsilon)\Delta w_t)\ dt = 0.$$

Note that interest charges in favor of the bank have not been reckoned in. The social planner can dispense with the promise-keeping constraint (including interest rate charges in the dynamics of w' would only reinforce the conclusion) as the bank fails to monitor anyway.

Using Assumption 8.2.2, we obtain

$$\begin{aligned} \frac{j\mu - ru_t}{\lambda_j} &\geq \frac{j\mu - r\gamma_j}{\lambda_j} \\ &\geq \frac{j\mu - r\sum_{i \leq J} b_i}{\lambda_j} \\ &= \frac{\mu - rB/\left(\varepsilon\overline{\alpha}_j\right)}{\alpha_j} \\ &\geq \frac{\mu + B}{\alpha_j(1 + \varepsilon)}. \end{aligned}$$

From this we obtain

$$\begin{aligned} 0 &\leq dw_t = \lambda_j \left(\Delta w_t - \frac{j\mu - ru_t}{\lambda_j}\right) dt \\ &\leq \lambda_j(1 + \varepsilon) \left(\Delta w_t - \frac{\mu + B}{\alpha_j(1 + \varepsilon)}\right) dt \\ &= dw'_t. \end{aligned}$$

This implies $w_t > w'_t$, as desired, since the value of w and w' are the same at $t + dt$. This, of course, does not constitue a proof, *per se*, but gives a heuristic link between our Assumption 8.2.2 which was designed thanks to economic arguments, and the fact that the social optimum should be attained when the bank does not shirk at all, as long as the bank is not too impatient (which is the meaning of Assumption 8.2.2, which gives an upper bound for r). Moreover, let us also note that we consider that the investors make their decision to design a contract which is incentive-compatible with the shirking decision $k = 0$, because they want to maximize the global utility, not only their own. Indeed, as already mentioned earlier, since the investors are competitive, those who consider only their utility will be eliminated (i.e. not contracted upon) by the mechanisms of the market. This is why our criterion here is the global utility.

(ii) **Comparison with other liquidation policies** We assume here that the sequence $\left(\frac{\overline{v}_j}{b_j}\right)_{1\leq j\leq I}$ is increasing. This assumption is of course an implicit one on the parameter of the model. Moreover, our numerical experimentations tend to show that this happens when the α_j do not decrease too rapidly.

Under this assumption, we want to show that, using the optimal value function as a reference, no other liquidation policy can improve upon the optimal stochastic liquidation (SL) specified in the paper. Let us fix j, the number of loans left in the pool and let $u \in [b_j, b_j + b_{j-1})$ be the level of utility of the bank. We assume of course that it is in the liquidation interval which is the only one where the stochastic liquidation may occur.

Consider *stochastic* policies leading to the partial removal of loans. The most general policy specifies that, with probability $\theta'_i \geq 0$, where i takes value in a subset of $\{1, \dots, j-1\}$, the residual number of loans is i and with probability $1 - \sum_{i=1}^{j-1} \theta'_i$ the pool is liquidated. For the alternative policy to be incentive-compatible, the expected penalties must be such that

$$\sum_{i=1}^{j-1} \theta'_i(u - b_i) + (1 - \sum_{i=1}^{j-1} \theta'_i)u \quad = \quad u - \sum_{i=1}^{j-1} \theta'_i b_i \geq b_j.$$

The corresponding efficiency loss is

$$\sum_{i=1}^{j-1} \theta'_i(\overline{v}_j - \overline{v}_i) + \left(1 - \sum_{i=1}^{j-1} \theta'_i\right) \overline{v}_j = \overline{v}_j - \sum_{i=1}^{j-1} \theta'_i \overline{v}_i$$
$$\text{subject to } \sum_{i=1}^{j-1} \theta'_i b_i \leq u - b_j.$$

Now by our assumption, we have

$$\overline{v}_j - \sum_{i=1}^{j-1} \theta'_i \overline{v}_i = \overline{v}_j - \sum_{i=1}^{j-1} \theta'_i b_i \frac{\overline{v}_i}{b_i} \geq \overline{v}_j - \frac{\overline{v}_{j-1}}{b_{j-1}} \sum_{i=1}^{j-1} \theta'_i b_i \geq \overline{v}_j - \overline{v}_{j-1} \frac{u - b_j}{b_{j-1}} = \overline{v}_j - \theta \overline{v}_{j-1}.$$

Now we can attain this lower bound by choosing all the θ'_i equal to 0 except $\theta'_{j-1} = \theta$. This is exactly our stochastic liquidation policy as desired.

8.5 What happens when $r = 0$?

In this section we treat our problem in the special case where the bank is as patient as the investors. We will see that in that case, the optimal contract leads to the first-best utility for the investors.

Since most of the proofs follow exactly the same arguments as in the case $r > 0$, we will only sketch them. First, we give the analogue of Proposition 8.3.3 in that case.

Proposition 8.5.1. *Assume that $r = 0$.*

(i) *The ordinary differential equations (8.3.21), along with (8.3.20), have unique maximal solutions v_j for $j \geq 1$. The functions v_j are globally concave, differentiable everywhere except at b_j and twice differentiable everywhere except at b_j and $b_j + b_{j-1}$. The endogenous thresholds γ_j are uniquely determined by*

$$\gamma_j = \sum_{i=1}^{j} b_i. \tag{8.5.1}$$

(ii) *We also have*

$$v_j'(u) - v_{j-1}'(u - b_j) \leq 0, \text{ for all } u \geq b_j. \tag{8.5.2}$$

Proof. (i) When $r = 0$, the solution of (8.3.21) for a given $\gamma \geq b_j$ is

$$\begin{aligned} v_j(u) &= \frac{j\mu}{\lambda_j} + e^{\frac{u-\gamma}{b_j}} \left(v_{j-1}(\gamma - b_j) - b_j\right) + \int_u^\gamma \frac{e^{\frac{u-x}{b_j}}}{b_j} v_{j-1}(x - b_j)dx, \ b_j < u \leq \gamma \\ v_j(u) &= \gamma - u + v_j(\gamma), \ u > \gamma. \end{aligned}$$

Using the same arguments as in the proof of Proposition 8.3.3, it is easily proved that the choice of γ which leads to the maximum solution is

$$\gamma_j = \gamma_{j-1} + b_j.$$

Reasoning by induction, we can then prove similarly that the functions v_j verify all the desired properties. Moreover, since $\gamma_1 = b_1$, we obtain that

$$\gamma_j = \sum_{i=1}^{j} b_i.$$

(ii) We can prove that

$$\begin{aligned} v_j'(u) &= \int_u^{\gamma_j} \frac{e^{\frac{u-x}{b_j}}}{b_j} \frac{dv_{j-1}}{du}(x - b_j)dx - e^{\frac{u-\gamma_j}{b_j}}, \ b_j < u \leq \gamma_j \\ \frac{dv_j}{du}(u) &= -1, \ u > \gamma_j. \end{aligned}$$

By the concavity of v_{j-1}, this implies that for $b_j < u \leq \gamma_j$

$$v_j'(u) - v_{j-1}'(u - b_j) \leq -e^{\frac{u-\gamma_j}{b_j}} \left(v_{j-1}'(u - b_j) + 1\right) \leq 0.$$

Since (8.5.2) is clear when $u > \gamma_j$, this proves (ii). □

Thanks to Proposition 8.5.1, we have a concave solution of the HJB equation, then using the same techniques as in the case $r > 0$, we can verify that the optimal contract is given by

Contract 8.5.1. (i) *Given size j, the pool remains in operation (i.e. there is no liquidation) with one less unit at any time there is a default in the range $[b_j + b_{j-1}, \gamma_j]$.*

(ii) *The flow of fees paid to the bank given j is $\delta_t^j = \lambda_j b_j$ as long as $u_t = \gamma_j$ and no default occurs. Otherwise $\delta_t^j = 0$.*

(iii) *Liquidation of the whole pool occurs with probability $\theta_t^j = (u_t - b_j)/b_{j-1}$ in the range $[b_j, b_j + b_{j-1})$.*

As advocated in Section 8.4.2, we know that the initial reservation utility of the bank should be fixed at γ_I. However in that case, we have

$$v_I(\gamma_I) = \frac{j\mu}{\lambda_j} - b_j + v_{j-1}(\gamma_{j-1}) = \frac{1}{\alpha_i}\left(\mu - \frac{B}{\varepsilon}\right) + v_{j-1}(\gamma_{j-1}) = \frac{I}{\overline{\alpha}_I}\left(\mu - \frac{B}{\varepsilon}\right).$$

Therefore, the social value of the contract is

$$\gamma_I + v_I(\gamma_I) = \frac{I\mu}{\overline{\alpha}_I},$$

which is exactly equal to $\mathbb{E}\left[\int_0^\tau \mu(I - N_t)dt\right]$, that is to say the social value which can be attained in the first-best. Hence, when the bank is no longer impatient, our contract leads to the first-best. This was to be expected, since there is no longer loss in utility due to the fact that the bank has to be penalized because of its impatience.

Finally, let us note a problem we have not addressed until now. Normally, the investors value function should be less than the initial investment (otherwise the bank woul not be to appeal to them), and the total value of the contract should be greater than the initial investment (otherwise there is no gain). This translates into

$$v_I(\gamma_I) \leq I, \text{ and } v_I(\gamma_I) + \gamma_I \geq I, \tag{8.5.3}$$

since the initial investment is equal to I, the loans being unitary.

In the general case, it seems extremely difficult to find practical sufficient conditions for this to happen. Nonetheless, the case $r = 0$ can provide some intuitions. Indeed, the above inequalities are equivalent, in the case $r = 0$ to

$$\mu - \frac{B}{\varepsilon} \leq \overline{\alpha}_I \leq \mu.$$

The right-hand side corresponds to Assumption 8.2.3, while the left-hand side tells us that the cost of shirking should be sufficiently large. Indeed, as B/ε goes to 0 the investors become more and more insensitive to the actions of the bank, which will lead to a non-optimal situation where the bank shirks. In the general case, we can intuitively expect that (8.5.3) is going to hold if we have a lower bound (which is bound to depend on r) on B/ε.

8.6 Numerical results

In this section we present some numerical results to illustrate the properties we already obtained. Following the empirical estimates of [28] concerning the values of the α_j, we choose to work with a pool of $I = 20$ loans with

μ	0.06
r	0.02
B	0.002
ε	0.25
$(\alpha_j)_{1\leq j\leq 14}$	0.055
$(\alpha_j)_{15\leq j\leq 18}$	0.05
$(\alpha_j)_{19\leq j\leq 20}$	0.044

With those values, the Assumption 8.2.2 is clearly satisfied. Let us now comment on those values. As already mentioned, the values for the α_j and for ε are taken from the litterature. We let the α_j be piecewise constant to model the fact that the change in default intensity begins to be non-negligeable after a certain fraction (here 75 %) of the pool has already defaulted. Finally the interest rate of the bank and the yield of the loans are taken close to what is usually the norm in the markets.

Using the fact that the v_j have a semi-explicit form, we used numerical integrations techniques to obtain the functions v_j for $j = 1..20$. They are represented below. With our numerical values, it appears that the v_j are increasing with j, as should be expected, just as the γ_j and the $\overline{v}_j/b_j$, which at least in this case, justifies all the assumption we made in this model (see Section 8.4.3). Besides, the condition (8.3.25) of Proposition 8.3.3 is always verified here. We also represent below the values of γ_j and $\overline{v}_j/b_j$ for $j = 1..20$.

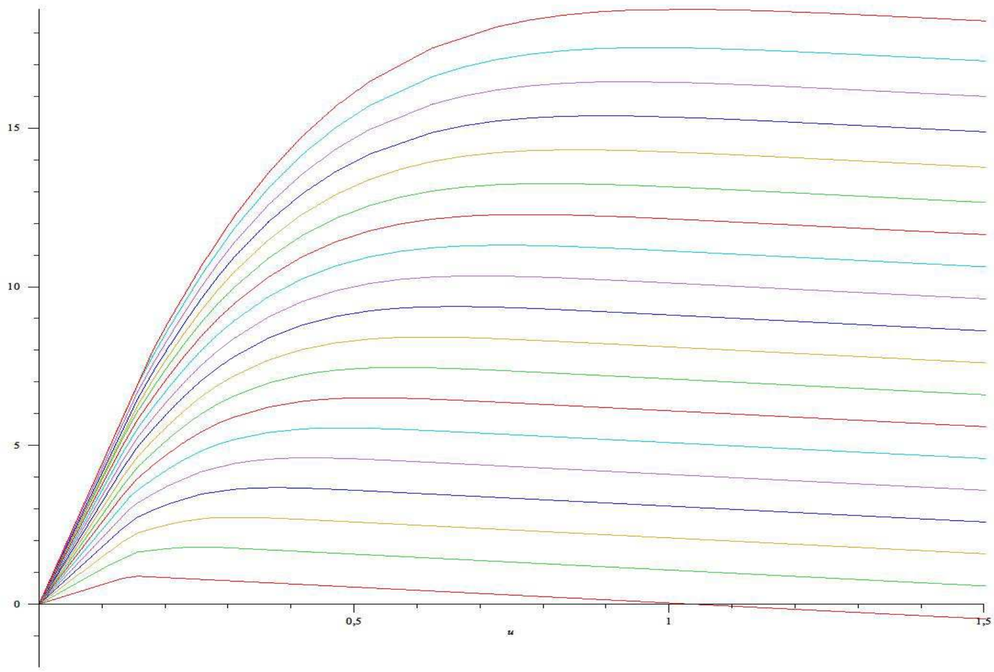

Figure 8.1: Functions $v_j(u)$ for $j = 1..20$.

Finally notice that the contract between the bank and the investors generates a positive social surplus. Indeed, the social suplus is given by

$$v_{20}(\gamma_{20}) + \gamma_{20} - 20 = 1.14.$$

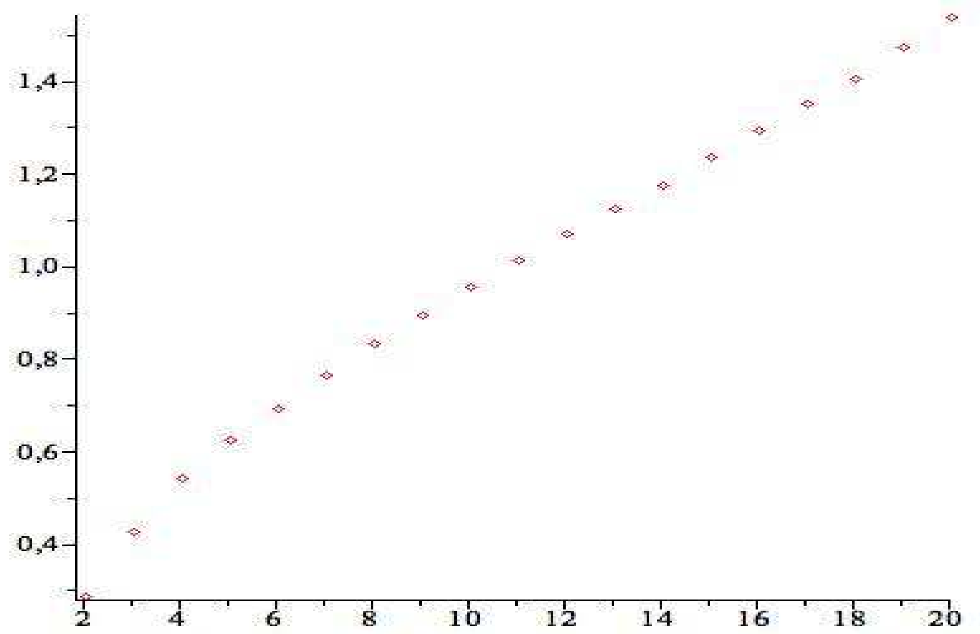

Figure 8.2: Values of γ_j for $j = 1..20$.

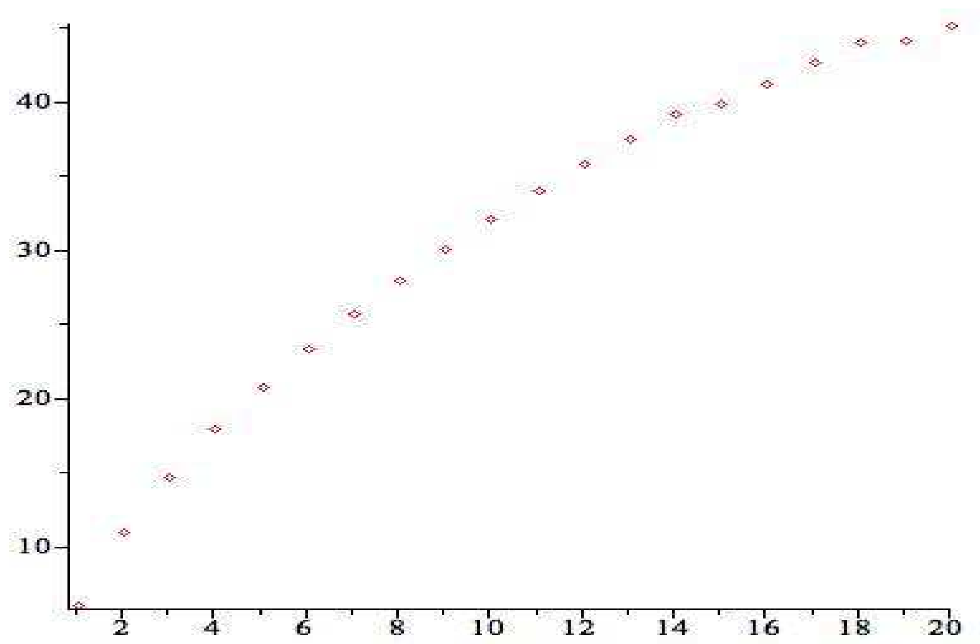

Figure 8.3: Values of $\frac{\overline{v}_j}{b_j}$ for $j = 1..20$.

Moreover, we have $v_{20}(\gamma_{20}) = 19.59$ which means (8.5.3) is satisfied in this numerical example. Moreover, the capital that the bank has to invest corresponds to roughly 3% of the total amount. In other words, it is sufficiently small to not deter the bank from going into that contract.

8.7 Appendix

Proof. [Proof of Proposition 8.3.3(i)] We will show the result by induction.

- Initialization with $j = 2$

The solution of the ODE (8.3.21) for $j = 2$ and a given fixed value of $\gamma \geq b_2$ can be easily calculated and is given by

$$\widetilde{v}_2(u,\gamma) := (ru+\lambda_2 b_2)^{\frac{\lambda_2}{r}} \int_u^\gamma \frac{2\mu + \lambda_2 v_1(x-b_2)}{(rx+\lambda_2 b_2)^{\frac{\lambda_2}{r}+1}} dx$$
$$+ \left(v_1(\gamma - b_2) + \frac{2\mu - (r\gamma + \lambda_2 b_2)}{\lambda_2}\right)\left(\frac{ru+\lambda_2 b_2}{r\gamma+\lambda_2 b_2}\right)^{\frac{\lambda_2}{r}}, \; b_2 < u \leq \gamma,$$

and $\widetilde{v}_2(u,\gamma) = \gamma - u + v_2(\gamma)$ for $u > \gamma$.

Now since we have shown that v_1 is everywhere twice differentiable except at b_1, we have for every $\gamma \neq b_1 + b_2$ and every $b_2 < u \leq \gamma$

$$\frac{\partial \widetilde{v}_2}{\partial \gamma}(u,\gamma) = \left(v_1'(\gamma - b_2) + 1 - \frac{r}{\lambda_2}\right)\left(\left(\frac{ru+\lambda_2 b_2}{r\gamma+\lambda_2 b_2}\right)^{\frac{\lambda_2}{r}} 1_{u\leq\gamma} + 1_{u>\gamma}\right).$$

Thus, the above expression always has the sign of $v_1'(\gamma - b_2) + 1 - \frac{r}{\lambda_2}$, that is to say that it is positive for $\gamma < b_1 + b_2$ and negative for $\gamma > b_1 + b_2$. Hence, we clearly have for all $b_2 < u$

$$\sup_{\gamma \geq b_2} \widetilde{v}_2(u,\gamma) = \widetilde{v}_2(u, b_1+b_2),$$

which means that the maximal solution of (8.3.21) for $j = 2$ corresponds to the choice $\gamma_2 = b_1 + b_2$, which also happens to correspond to the unique solution of

$$\frac{r}{\lambda_2} - 1 \in \partial v_1(\gamma_2 - b_1).$$

Then, after some calculations, we obtain that for all $b_2 < u < b_1 + b_2$

$$v_2''(u) = -\left(\lambda_2 - r + \lambda_2 \frac{\overline{v}_1}{b_1}\right) \frac{(ru+\lambda_2 b_2)^{\frac{\lambda_2}{r}-1}}{(r(b_1+b_2)+\lambda_2 b_2)^{\frac{\lambda_2}{r}}} \leq 0,$$

because of (8.3.22).

Hence, since v_2, is linear on $[b_1 + b_2, +\infty)$ and is differentiable at $b_1 + b_2$, it is concave on $(b_2, +\infty)$. Now if we consider the linear extrapolation of v_2 over $[0, b_1]$ by (8.3.20), we just need to verify that the left-derivative of v_2 at b_2 is less than its right-derivative to obtain the concavity of v_2 over $[0, +\infty]$. Taking the limit for $u \downarrow b_2$ in the equation (8.3.21), we obtain

$$v_2'(b_2^+) = \frac{\lambda_2 \overline{v}_2 - 2\mu}{b_2(r+\lambda_2)}.$$

This implies that

$$v_2'(b_2^-) - v_2'(b_2^+) = \frac{2\mu}{b_2\lambda_2} + v_2'(b_2^+)\frac{r}{\lambda_2} \geq \frac{r}{\lambda_2}\left(\frac{2\mu}{rb_2} - 1\right).$$

Now recall Assumption 8.2.2, which can be expressed, thanks to (8.2.5), as

$$\frac{\mu}{r\bar{b}_2} \geq 1 + \frac{1}{\varepsilon} + \frac{\overline{\alpha}_2}{r},$$

where

$$\bar{b}_2 := \frac{B}{\varepsilon\overline{\alpha}_2} = \frac{b_1 + b_2}{2}.$$

Hence,

$$\frac{2\mu}{rb_2} \geq \frac{b_1 + b_2}{b_2}\left(1 + \frac{1}{\varepsilon} + \frac{\overline{\alpha}_2}{r}\right) \geq 1,$$

which means that $v_2'(b_2^-) - v_2'(b_2^+) \geq 0$.

- Heredity : $j \geq 3$

Let us now suppose that the maximal solution of (8.3.21) v_{j-1} has been constructed for some $j \geq 3$, that it is globally concave on $[0, +\infty)$, everywhere differentiable except at b_{j-1}, everywhere twice differentiable except at b_{j-1} and $b_{j-1} + b_{j-2}$, and that the corresponding $\gamma_{j-1} \geq b_{j-1} + b_{j-2}$. Let us now construct the maximal solution corresponding to j. Exactly as in the case $j = 2$, the solution of the ODE (8.3.21) and a given fixed value of $\gamma \geq b_j$ can be easily calculated and is given by

$$\begin{aligned}\widetilde{v}_j(u,\gamma) :=& (ru + \lambda_j b_j)^{\frac{\lambda_j}{r}} \int_u^\gamma \frac{j\mu + \lambda_j v_{j-1}(x - b_j)}{(rx + \lambda_j b_j)^{\frac{\lambda_j}{r}+1}} dx \\ &+ \left(v_{j-1}(\gamma - b_j) + \frac{j\mu - (r\gamma + \lambda_j b_j)}{\lambda_j}\right)\left(\frac{ru + \lambda_j b_j}{r\gamma + \lambda_j b_j}\right)^{\frac{\lambda_j}{r}}, \; b_j < u \leq \gamma,\end{aligned}$$

and $\widetilde{v}_j(u,\gamma) = \gamma - u + v_j(\gamma)$ for $u > \gamma$.

Note also that from (8.3.21) it is clear that v_j is differentiable everywhere except at b_j, and twice differentiable everywhere except at b_j and $b_j + b_{j-1}$.

Now since we assumed that v_{j-1} is everywhere differentiable except at b_{j-1}, we have for every $\gamma \neq b_{j-1} + b_j$ and every $b_j < u \leq \gamma$

$$\frac{\partial \widetilde{v}_j}{\partial \gamma}(u,\gamma) = \left(v_{j-1}'(\gamma - b_j) + 1 - \frac{r}{\lambda_j}\right)\left(\left(\frac{ru + \lambda_j b_j}{r\gamma + \lambda_j b_j}\right)^{\frac{\lambda_j}{r}} 1_{u\leq\gamma} + 1_{u>\gamma}\right).$$

Thus, since v_{j-1} is concave and its derivative non-increasing, we can conclude as in the case $j = 2$ that the maximal solution is uniquely determined by the choice γ_j which corresponds to the solution of

$$\frac{r}{\lambda_j} - 1 \in \partial v_{j-1}(\gamma_j - b_j).$$

More precisely, using (8.3.22), we have only two cases. Either,

$$v_{j-1}'(b_{j-1}^+) \leq \frac{r}{\lambda_j} - 1 \leq \frac{\overline{v}_{j-1}}{b_{j-1}},$$

and $\gamma_j = b_{j-1} + b_j$, or

$$\frac{r}{\lambda_j} - 1 < v_{j-1}'(b_{j-1}^+),$$

and $b_{j-1} + b_j < \gamma_j \leq \gamma_{j-1} + b_j$.

Let us now study the concavity. We can differentiate twice the equation (8.3.21) on $(b_j, b_j + b_{j-1})$ since $v_{j-1}(u - b_j)$ is linear and thus twice differentiable on this open interval. We then obtain easily

$$v_j''(u) = v_j''((b_j + b_{j-1})^-)\left(\frac{ru + \lambda_j b_j}{r(b_j + b_{j-1}) + \lambda_j b_j}\right)^{\frac{\lambda_j}{r}-2}, \; b_j < u < b_j + b_{j-1}. \tag{8.7.1}$$

There are then two cases. If $\gamma_j = b_j + b_{j-1}$, differentiating once (8.3.21) and then taking the limit $u \uparrow b_j + b_{j-1}$, we get

$$(r(b_j + b_{j-1}) + \lambda_j b_j)v_j''((b_j + b_{j-1})^-) = \lambda_j\left(\frac{r}{\lambda_j} - 1 - \frac{\overline{v}_{j-1}}{b_{j-1}}\right) \leq 0.$$

Since $v_j''(u) = 0$ for $u > b_j + b_{j_1}$, we have proved the concavity on $(b_j, +\infty)$.

Now if $\gamma_j > b_j + b_{j-1}$, differentiating once (8.3.21) and taking limits on both sides of $b_j + b_{j-1}$, we obtain

$$v_j''((b_j + b_{j-1})^+) - v_j''((b_j + b_{j-1})^-) = \frac{\lambda_j}{r(b_j + b_{j-1}) + \lambda_j b_j}\left(\frac{\overline{v}_{j-1}}{b_{j-1}} - v_{j-1}'(b_{j-1}^+)\right), \tag{8.7.2}$$

where the right-hand side is positive by the concavity of v_{j-1}.

Next, we differentiate twice (8.3.21) on $(b_j + b_{j-1}, \gamma_j]$. We obtain easily

$$v_j''(u) = \lambda_j(ru + \lambda_j b_j)^{\frac{\lambda_j}{r}-2}\int_u^{\gamma_j} \frac{v_{j-1}''(x - b_j)}{(ru + \lambda_j b_j)^{\frac{\lambda_j}{r}-1}}dx. \tag{8.7.3}$$

Note that we should normally distinguish between the cases $b_j + b_{j-1} + b_{j-2} \leq \gamma_j$ or not, since v_{j-1} is not twice differentiable at $b_{j-1} + b_{j-2}$. However, since we know that v_j is twice differentiable at $b_j + b_{j-1} + b_{j-2}$, this actually does not change the result. Since v_{j-1} is concave, (8.7.3) implies that v_j is concave on $(b_j + b_{j-1}, +\infty)$. Then with (8.7.2) we obtain that the left second derivative of v_j at $b_j + b_{j-1}$ is negative, which, thanks to (8.7.1) shows finally the concavity on $(b_j, +\infty)$.

Finally, it remains to show that $v_j'(b_j^+) \leq \frac{\overline{v}_j}{b_j}$. We take the limit for $u \downarrow b_j$ in the equation (8.3.21), we obtain

$$v_j'(b_j^+) = \frac{\lambda_j \overline{v}_j - j\mu}{b_j(r + \lambda_j)}.$$

This implies that

$$v_j'(b_j^-) - v_j'(b_j^+) = \frac{j\mu}{b_j\lambda_j} + v_j'(b_j^+)\frac{r}{\lambda_j} \geq \frac{r}{\lambda_j}\left(\frac{j\mu}{rb_j} - 1\right).$$

Now recall Assumption 8.2.2, which can be expressed, thanks to (8.2.5), as

$$\frac{\mu}{r\bar{b}_j} \geq 1 + \frac{1}{\varepsilon} + \frac{\overline{\alpha}_j}{r},$$

where

$$\bar{b}_j := \frac{B}{\varepsilon\overline{\alpha}_j} = \frac{1}{j}\sum_{i=1}^{j} b_i.$$

Hence,

$$\frac{j\mu}{rb_j} \geq \frac{j\bar{b}_j}{b_j}\left(1+\frac{1}{\varepsilon}+\frac{\overline{\alpha}_j}{r}\right) \geq 1,$$

which means that $v_1'(b_j^-) - v_j'(b_j^+) \geq 0$ and that v_j is concave on $[0,+\infty)$ □

Proof. [Proof of Proposition 8.3.3(ii)] First of all, by the properties of the function ψ_1 recalled in Remark 8.3.6, it is clear that we can always find a λ_j such that (8.3.25) is satisfied. Then, if for a fixed $j \geq 2$ we have $v_{j-1}'(b_{j-1}^+) \leq 0$, by differentiating (8.3.21), we immediately have for $u > b_j$ and $u \neq b_j + b_{j-1}$

$$\lambda_j\left(v_j'(u) - v_{j-1}'(u-b_j)\right) = (ru+\lambda_j b_j)v_j''(u) + rv_j'(u). \tag{8.7.4}$$

Since we have proved in (i) that the v_j are concave, it is clear that if $v_{j-1}'(b_{j-1}^+) \leq 0$, the right-hand side above is negative. Then by left and right continuity of v_{j-1}' at b_{j-1}, the result extends to $u = b_j + b_{j-1}$. Hence the desired property (8.3.26). In particular, this proves the result for $j=2$ since $v_1'(b_1^+) = -1$.

Note also that the property (8.3.26) clearly holds for v_j when $u > \gamma_j$. Indeed, we have

$$v_j' = -1$$

and we know that the derivative of v_{j-1} is always greater than -1.

Let us now show the rest of the result by induction. Since (8.3.26) is true for $j=2$, let us fix a $j \geq 3$ and assume that

$$v_{j-1}'(u) - v_{j-2}'(u-b_{j-1}) \leq 0,\ \ u > b_{j-1}. \tag{8.7.5}$$

Now if $v_{j-1}'(b_{j-1}^+) \leq 0$, we already know that the property 8.3.26 is true for v_j, so we will assume that $v_{j-1}'(b_{j-1}^+) > 0$. Moreover, by our remark above, we know that (8.3.26) holds true for v_j when $u > \gamma_j$. Let us then first prove that (8.3.26) for v_j when $u > b_j + b_{j-1}$. If $\gamma_j = b_j + b_{j-1}$, there is nothing to do. Otherwise, we have using successively (8.7.4) and (8.7.3)

$$\begin{aligned}\lambda_j\left(v_j'(u) - v_{j-1}'(u-b_j)\right) &= (ru+\lambda_j b_j)v_j''(u) + rv_j'(u) \\ &= (ru+\lambda_j b_j)^{\frac{\lambda_j}{r}-1}\int_u^{\gamma_j}\frac{\lambda_j v_{j-1}''(x-b_j)}{(rx+\lambda_j b_j)^{\frac{\lambda_j}{r}-1}}dx + rv_j'(u). \end{aligned} \tag{8.7.6}$$

Now if we differentiate (8.3.21) and solve the corresponding ODE for v_j', we obtain

$$v_j'(u) = (ru+\lambda_j b_j)^{\frac{\lambda_j}{r}-1}\int_u^{\gamma_j}\frac{\lambda_j v_{j-1}'(x-b_j)}{(rx+\lambda_j b_j)^{\frac{\lambda_j}{r}}}dv - \left(\frac{ru+\lambda_j b_j}{r\gamma_j+\lambda_j b_j}\right)^{\frac{\lambda_j}{r}-1}. \tag{8.7.7}$$

Using (8.7.7) in (8.7.6), we obtain for $u > b_j + b_{j-1}$

$$\begin{aligned}&\lambda_j\left(\frac{dv_j}{du}(u) - \frac{dv_{j-1}}{du}(u-b_j)\right) \\ &= \lambda_j(ru+\lambda_j b_j)^{\frac{\lambda_j}{r}-1}\int_u^{\gamma_j}\frac{(rx+\lambda_j b_j)v_{j-1}''(x-b_j) + rv_{j-1}'(x-b_j)}{(rx+\lambda_j b_j)^{\frac{\lambda_j}{r}}}dv - r\left(\frac{ru+\lambda_j b_j}{r\gamma_j+\lambda_j b_j}\right)^{\frac{\lambda_j}{r}-1}. \end{aligned} \tag{8.7.8}$$

Then we have for all $x \geq u > b_j + b_{j-1}$ and $x \neq b_j + b_{j-1} + b_{j-2}$

$$\begin{aligned}(rx + \lambda_j b_j)v''_{j-1}(x - b_j) + rv'_{j-1}(x - b_j) &= (r(x - b_j) + \lambda_{j-1}b_{j-1})v''_{j-1}(x - b_j) + rv'_{j-1}(x - b_j) \\ &\quad + (\lambda_j b_j - \lambda_{j-1}b_{j-1} + rb_j)\, v''_{j-1}(x - b_j) \\ &= \lambda_{j-1}\left(v'_{j-1}(x - b_j) - v'_{j-2}(x - b_j - b_{j-1})\right) \\ &\quad + (\lambda_j b_j - \lambda_{j-1}b_{j-1} + rb_j)\, v''_{j-1}(x - b_j) \\ &\leq (\lambda_j b_j - \lambda_{j-1}b_{j-1} + rb_j)\, v''_{j-1}(x - b_j),\end{aligned}$$

where we used the induction hypothesis (8.7.5) in the last inequality.

Since v_{j-1} is concave, the sign of the right-hand side above is given by the sign of

$$\lambda_j b_j - \lambda_{j-1}b_{j-1} + rb_j = \frac{JB}{\varepsilon} - \frac{(J-1)B}{\varepsilon} + rb_j = \frac{B}{\varepsilon} + rb_j \geq 0.$$

Reporting this in (8.7.8) implies

$$v'_j(u) - v'_{j-1}(u - b_j) \leq 0, \;\; u > b_j + b_{j-1}.$$

It remains to prove (8.3.26) when $b_j < u < b_j + b_{j-1}$. In that case, (8.3.26) can be written

$$v'_j(u) - \frac{\overline{v}_{j-1}}{b_{j-1}} \leq 0, \;\; b_j < u < b_j + b_{j-1},$$

which is equivalent by concavity of v_j to

$$v'_j(b_j^+) - \frac{\overline{v}_{j-1}}{b_{j-1}} \leq 0.$$

Now using (8.7.7), we also have

$$\begin{aligned}v'_j(b_j^+) &= (ru + \lambda_j b_j)^{\frac{\lambda_j}{r}-1} \int_{b_j}^{b_j+b_{j-1}} \frac{\lambda_j \frac{\overline{v}_{j-1}}{b_{j-1}}}{(rx + \lambda_j b_j)^{\frac{\lambda_j}{r}}} dv \\ &\quad + v'_{j-1}(b_j + b_{j-1}) \left(\frac{ru + \lambda_j b_j}{r(b_j + b_{j-1}) + \lambda_j b_j}\right)^{\frac{\lambda_j}{r}-1} \\ &= \frac{\overline{v}_{j-1}}{b_{j-1}} \frac{\lambda_j}{\lambda_j - r}\left(1 - \left(\frac{rb_j + \lambda_j b_j}{r(b_j + b_{j-1}) + \lambda_j b_j}\right)^{\frac{\lambda_j}{r}-1}\right) \\ &\quad + v'_{j-1}(b_j + b_{j-1}) \left(\frac{ru + \lambda_j b_j}{r(b_j + b_{j-1}) + \lambda_j b_j}\right)^{\frac{\lambda_j}{r}-1} \\ &\leq \frac{\overline{v}_{j-1}}{b_{j-1}} \frac{\lambda_j}{\lambda_j - r}\left(1 - \left(\frac{rb_j + \lambda_j b_j}{r(b_j + b_{j-1}) + \lambda_j b_j}\right)^{\frac{\lambda_j}{r}-1}\right) \\ &\quad + v'_{j-1}(b_{j-1}^+) \left(\frac{ru + \lambda_j b_j}{r(b_j + b_{j-1}) + \lambda_j b_j}\right)^{\frac{\lambda_j}{r}-1} \\ &= \phi_{\frac{b_{j-1}}{b_j}}\left(\frac{r}{\lambda_j}\right) \frac{\overline{v}_{j-1}}{b_{j-1}} \left(\frac{\phi_{\frac{b_{j-1}}{b_j}}\left(\frac{r}{\lambda_j}\right) - 1}{\phi_{\frac{b_{j-1}}{b_j}}\left(\frac{r}{\lambda_j}\right)(x-1)} + \frac{v'_{j-1}(b_{j-1}^+)}{\frac{\overline{v}_{j-1}}{b_{j-1}}}\right),\end{aligned}$$

which implies

$$v_j'(b_j^+) - \frac{\overline{v}_{j-1}}{b_{j-1}} \leq \phi_{\frac{b_{j-1}}{b_j}}\left(\frac{r}{\lambda_j}\right)\frac{\overline{v}_{j-1}}{b_{j-1}}\left(\frac{v_{j-1}'(b_{j-1}^+)}{\frac{\overline{v}_{j-1}}{b_{j-1}}} - \psi_{\frac{b_{j-1}}{b_j}}\left(\frac{r}{\lambda_j}\right)\right).$$

By Assumption 8.2.1, we know that $b_j \geq b_{j-1}$, hence with (8.3.25) and what we recalled earlier about the functions ψ_β in Remark 8.3.6, we have

$$\frac{\overline{v}_{j-1}}{b_{j-1}} \leq \psi\left(\frac{r}{\lambda_j}\right) \leq \psi_{\frac{b_{j-1}}{b_j}}\left(\frac{r}{\lambda_j}\right),$$

which implies the desired property and ends finally the proof.

□

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