Introduction of New Products in the Supply Chain : Optimization and Management of Risks

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Dedicated to my parents

Abstract

The contribution of this thesis is in providing a tractable framework for solving the product rollover problem subject to an uncertain approval date and in determining the optimal strategy to remove a product from the market and introduce a new one. We present our research work in the form of three papers evolving around the product rollover problem from optimization under risk for a constant demand case, to the optimization of the expected net loss for a time dependent product demand rate, and finally optimization under an unknown probability distribution using the data-driven approach.

Shorter product life cycles and rapid product obsolescence provide increasing incentives to introduce new products to markets more quickly. As a consequence of these rapidly changing market conditions, firms focus on improving their new product development processes to reap the advantages of early market entry. Researchers have analyzed market entry, but have rarely provided quantitative approaches for the product rollover problem. This research builds upon the literature by using established optimization methods like the Conditional Value at Risk and the data- driven optimization approach to examine how firms can minimize their net losses during the rollover process. Specifically, our work explicitly optimizes the timing of removal and introduction of old and new products, respectively, the optimal strategy, and the magnitude of net losses.

In the first paper of the thesis, we use the conditional value at risk to optimize the net loss *and investigate the effect of risk perception of the manager on the rollover process. We apply CVaR minimization to a product rollover problem with uncertain regulatory approval date and compare it to the minimization of the classical expected net loss. Results show that the optimal strategy is dependent on the parameters (costs and prices) and/or probability distribution of the approval date and risk. We derive conditions for optimality and unique closed-form solutions for single and dual rollover cases. Furthermore, we present the variation of optimal costs and solutions under different probability distribution families. Many possibilities extensions and directions for research exist, such as, optimizing with respect to a distribution free regulatory approval date, or for different products and lifecycles, and rollover for time-dependent demand.*

In the second paper, we investigate our rollover problem, but for a time-dependent demand rate for the second product trying to approximate the Bass Model. This is a more realistic setting than the first paper where we use to examine the effect on product entry timing decisions.

Finally, in the third paper, we apply the data-driven approach to the product rollover problem where the probability distribution of the approval date is unknown; rather we have historical observations of approval dates. We develop the optimal times of rollover and the show superiority of the data-driven method over the conditional value at risk in the case when it is difficult to guess the probability distribution.

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Introduction de Nouveaux Produits dans la Supply Chain : Optimisation et Management des Risques

Intérêt et question de recherche

Aujourd'hui les consommateurs cherchent les produits les plus récents et ayant des goûts très variés. Avec l'accélération technologique, les cycles de vie des produits se sont raccourcis et donc, de nouveaux produits doivent être introduits au marché plus souvent et les anciens doivent être progressivement retirés.

L'introduction d'un nouveau produit est une source de croissance et d'avantage concurrentiel. Les directeurs du *Marketing et Supply Chain* se sont confrontés à la question de savoir comment gérer avec succès le remplacement de leurs produits et d'optimiser les coûts de la chaîne d' approvisionnement.

Dans une situation idéale, la procédure de *rollover* est efficace et claire: l'ancien produit est vendu jusqu'à une date prévue où un nouveau produit est introduit. Dans la vie réelle, la situation est moins favorable. Une étude de sociétés américaines sur les biens de consommation durable a montré que, pour plusieurs raisons, plus de cinquante pourcent de nouveaux produits n'ont pas réussis après avoir été introduis sur le marché car les lancements des produits se sont confrontés à de nombreuses perturbations de type potentiel aléatoire, comme des retards inattendus ou de logistique industrielle, les problèmes de qualité, les mauvais prévisions de demande, la réaction inattendue des marchés à l'annonce de ces nouveaux produits, etc. ..

La façon d'introduire de nouveaux produits en retirant progressivement les anciens est devenue un problème reconnu dans la gestion. Si la production de l'ancien produit est arrêtée trop tôt, c'est à dire avant que le nouveau produit ne soit suffisamment disponible sur le marché, l'entreprise perd des profits et d'écart d'acquisition. D'autre part, si la production du produit existant est arrêtée trop tard, l'entreprise connaîtra un coût d'obsolescence pour le produit existant, parce que la demande et/ou le prix aurait diminué et ce produit sera considéré par les clients comme "ancienne génération". En outre, si la production du nouveau produit est lancée trop tôt, l'entreprise connaîtra un coût de mise à disposition des stocks jusqu'à ce que le marché se tourne vers ce produit. Le processus de lancement ou d'introduction d'un nouveau produit dans le marché et la suppression d'un ancien est dénoté par *product rollover*.

Une question importante dans la gestion du lancement d'un nouveau produit est de savoir si les deux générations du produit doivent coexister sur le marché pour un temps donné et de savoir s'il y a un chevauchement de quelque sorte dans l'inventaire des produits successifs. Dans ce travail, nous nous concentrons sur trois stratégies fondamentales du product rollover:

- *Planned Stockout Rollover*

- *Single Rollover*

- *Dual Rollover*

Pour la stratégie du *Planned Stockout Rollover* (Voir Figure 1), l'introduction du nouveau produit est prévue de telle sorte qu'il y a une rupture de stock durant la transition au nouveau produit. Au cours de cette période de rupture de stock, aucun produit n'est disponible pour le marché (ce qui aboutit à une sorte de coût de rupture de stock).

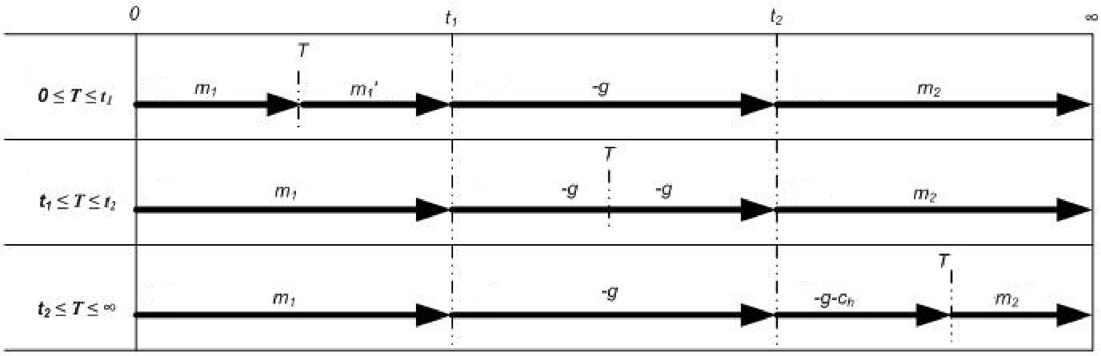

Figure 1: Politique du Planned Stockout Rollover

Suivant la stratégie du *Single Rollover*, le nouveau produit est introduit et simultanément l'ancien produit est retiré du marché, alors nous avons en tout moment un seul produit disponible dans le marché.

D'autre part, la stratégie du *Dual Rollover* (Voir Figure 2) consiste à ce que le nouveau produit soit introduit d'abord, puis l'ancien produit est retiré. Ainsi, dans ce contexte, deux générations du produit coexistent sur le marché pour une certaine durée.

L'avantage de la stratégie du *Dual Rollover*, comparée à la celle du *Planned Stock-*

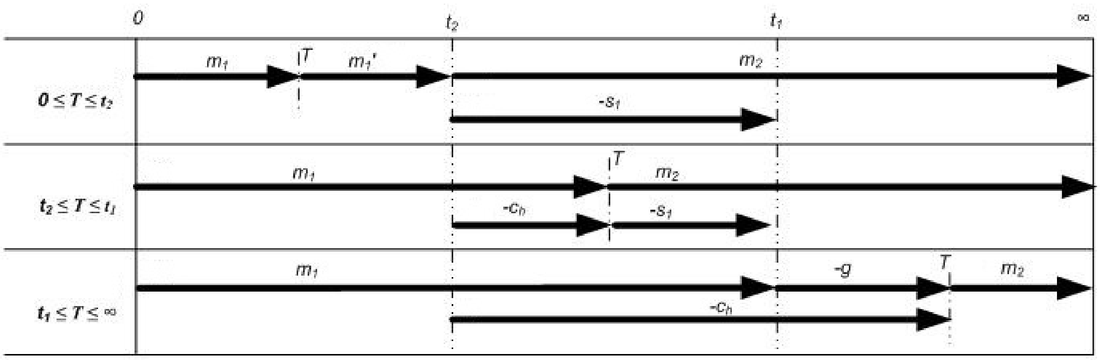

Figure 2: Politique du Dual Rollover

out Rollover est que la première assure une certaine 'protection' contre des événements aléatoires (délais, qualité, niveau de la demande du marché) qui affectent l'élimination prévue de l'ancien produit. L'inconvenient de la politique du *Dual Rollover* est le coût additionnel correspondant à l'inventaire supplémentaire de la chaîne d' approvisionnement.

Le but de notre travail est d'analyser et de caractériser l'optimalité de chaque type de stratégie avec une date de disponibilité stochastique pour l'introduction du nouveau produit sur le marché. Nous considérons une approche quantitative: une telle analyse nécessite un modèle d'évaluation de performance.

Notre modèle de départ est inspiré du modèle de Hill et Sawaya (2004). En résolvant le problème d'optimisation associé, nous présentons les conditions d'optimalité pour les trois politiques : *Planned Stockout, Single, et Dual Rollover*.

Pour résoudre le problème d'optimisation, nous utilisons dans notre première article deux mesures de minimisation: le coût moyen et le coût du *Conditional Value at Risk* (CVaR). Le CVaR est une mesure du risque efficace largement pris en compte dans la littérature de finance. C'est un critère de risque assez récent qui a émergé comme présentant tout à fait des propriétés théoriques intéressantes. On obtient des solutions en forme explicite pour les politiques optimales. En outre, nous caractérisons l'influence des paramètres de coûts sur la structure de la politique optimale. Dans cet esprit, nous analysons aussi le comportement de la politique de rollover optimale dans des contextes différents (plus grande variance etc..).

Dans notre deuxième article, nous examinons le même problème mais avec une demande constante pour le premier produit et une demande linéaire au début puis constante pour le deuxième. Ce modèle est inspiré par la demande de Bass. Dans notre troisième article, la date de disponibilité du nouveau produit existe mais elle est inconnue. La seule information disponible est un ensemble historique d'échantillons qui sont tirées de la vraie distribution. Nous résoudrons le problème avec l'approche *data-driven* est nous obtenons des formulations tractables. Nous développons aussi des bornes sur le nombre d'échantillons nécessaires pour garantir qu'avec une forte probabilité, le coût n'est pas très loin du vrai coût optimal.

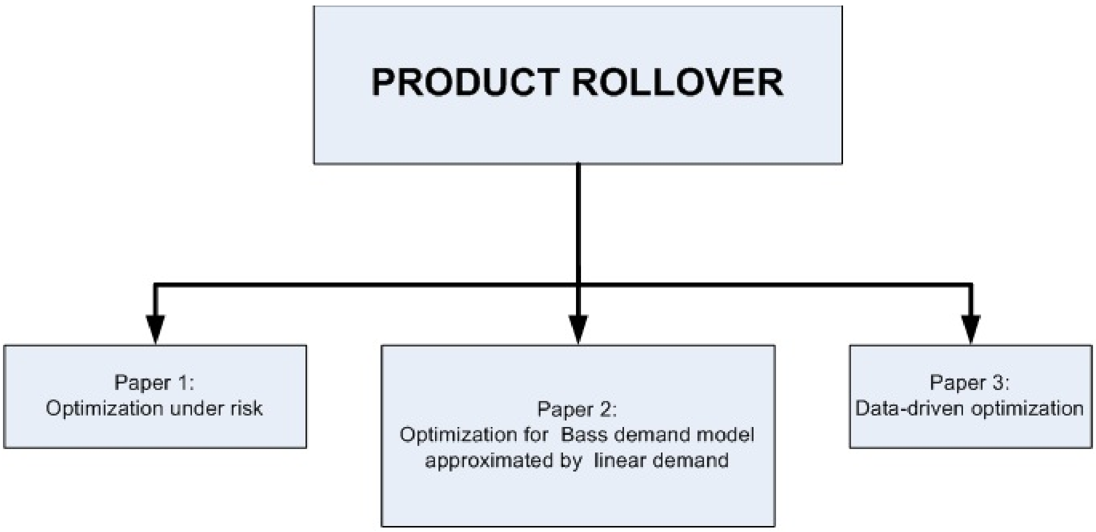

Figure 3: Les Trois Articles de la Recherche

Plusieurs articles ont traité la question de la gestion efficace du lancement de nouveaux produits, et le retirement de l'ancien. Une première tendance de recherche sur le développement de nouveaux produits est principalement de nature qualitative et descriptive (voir Krishnan et Ulrich(2002)) dans les domaines de marketing et de gestion des opérations.

Chryssochoidis et Wong (1998) ont étudié d'un point de vue empirique l'ensemble du processus dans un grand nombre d'entreprises. Cette recherche présente les causes essentielles de retard dans le rollover des produits dans un environnement international. Saunders et Jobber (1994) identifient les différents types de stratégies et de chevauchement dans une application de rollover. Certains facteurs clés sont exposés et associés à l'efficacité de chaque stratégie.

Plusieurs articles ont porté sur l'analyse de l'introduction de nouveaux produits et selon différentes hypothèses et de différents points de vue. Erhun et al (2007) ont menu une étude qualitative sur les différents pilotes touchant les transitions de produits chez Intel Corp. Ils présentent un cadre qui guide les gestionnaires à concevoir et à mettre en oeuvre des politiques appropriées en tenant compte des risques de la transition liée au produit, les processus de fabrication, les caractéristiques chaîne d'approvisionnement, et les politiques de gestion dans un environnement concurrentiel. Les auteurs suggèrent que les entreprises doivent élaborer des stratégies claires pour le lancement de produits, pour ne pas risquer son échec. Ils comparent les stratégies du planned stockout et du dual rollover. La politique du planned stockout rollover peut être considérée d'un risque élevé très sensible au potentiel des événements aléatoires. Au contraire, la politique du dual rollover est moins risquée, mais aboutit à des coûts plus élevés de stocks.

Hendricks et Singhal (1997) ont démontré par une recherche empirique que le retard dans le lancement de nouveaux produits diminue la valeur marchande de l'entreprise. Certains articles portent sur la modélisation quantitative et l'optimisation des processus de renversement.

Lim et Tang (2006) ont développé un modèle déterministe qui permet la détermination des prix de produits nouveaux et anciens ainsi que les dates d'introduction et d'élimination de ces produits. En outre, ils mettent au point des conditions du coût marginal pour déterminer dans quels cas la politique du dual rollover est plus favorable que celle du planned stockout rollover.

Hill et Sawaya (2000) ont examiné un problème de planification et d'élimination de l'ancien produit et l'introduction d'un nouveau qui va le remplacer, en vertu d'une date d'approbation réglementaire incertaine pour le nouveau produit. Ils présentent la structure de la politique optimale.

Un problème très simple a été analysé dans le travail de Ronen et Trietsch (1993) où il examine la question de trouver la date de départ pour une activité dans un environnement aléatoire. Les modèles de risque sensibles à l'inventaire, la modélisation et la gestion de la chaîne d'approvisionnement ont été proposés dans quelques papiers. Tang (2006) présente des modèles quantitatifs divers pour la gestion des risques de la chaîne d'approvisionnement. La plupart d'articles de recherche essayent de maximiser un profit cible prédéterminé, mais ca peut introduire un risque haut.

Design de recherche et méthode CVaR et résultats

En général, la modélisation des risques a constitué un domaine de recherche important dans la finance. Une façon moderne de prendre en compte le risque consiste à se concentrer sur le déficit, grâce à une absolue liée à la perte tolérable ou en définissant une borne sur la valeur à risque conditionnel. Les propriétés théoriques de la mesure de la valeur à risque ont été largement étudié (voir Rockafellar et Uryasev 2000,2002).

Ozler et al. (2009) utilisent la Value at Risk (VaR) comme mesure de risque dans un cadre de newsboy avec multi-produits sous une contrainte de VaR. Le Value at Risk (VaR) est la perte maximale sur un horizon donné qui ne devrait être dépassée qu'avec une probabilité donnée. La VaR est toujours accompagnée du degré de confiance (en général 95 % ou 99 %) et de l'horizon (en général 1, 3, 5, 10 ou 30 jours). Contrairement

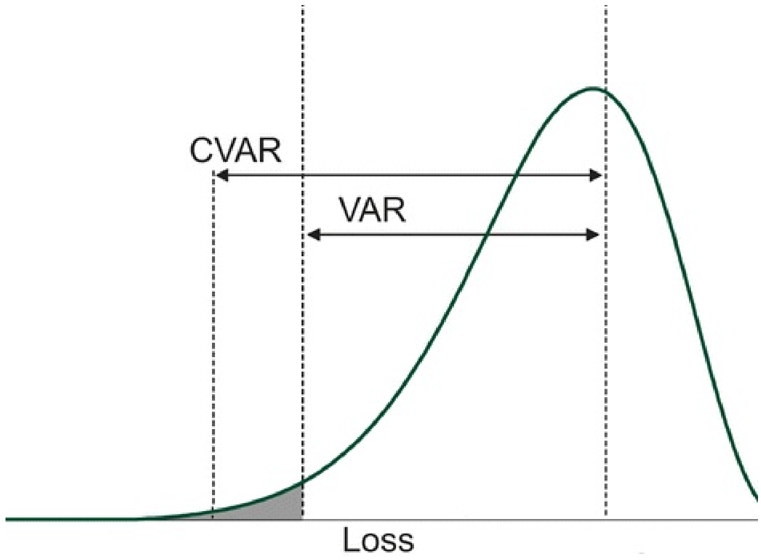

Figure 4: VaR vs CVaR

à la VaR, la " Conditional Value at Risk " (CVaR) nommée aussi " Expected shortfall " est une mesure de risque cohérente.

La Value at Risk est sans doute l'outil le plus utilisé pour mesurer et contrôler les risques financiers, mais cette méthode connait des limites dans les situations de risque extrême. D'autre part, la VaR ne signale rien sur les pertes effectives en cas de dépassement. Ces pertes peuvent être bien plus élevées que le prévoit une VaR normale, en raison de queues de distribution épaisses des rendements. D'autres mesures de risque ont donc été proposées, notamment la VaR conditionnelle ou la CVaR.

La CVaR mesure justement les pertes en dépassement de la VaR. C'est une mesure cohérente du risque. En outre, l'optimisation de portefeuille sous contrainte de CVaR se résous facilement par des méthodes de programmation linéaire, ce qui n'est pas le cas de la VaR (en l'absence de propriété de convexité).

La VaR n'est pas la mesure de risque parfaite. Comme toute mesure, la VaR n'a pas de précision absolue et même parfois, elle n'est pas pratique. Il faut néanmoins préciser que la VaR a ses inconvénients. Il peut être judicieux de déterminer d'autres mesures de risque telle la Conditional Value-at-Risk (CVaR), appelée aussi Expected Shortfall. L'alternative à la VaR est le CVaR.

Le CVaR est en effet une mesure cohérente du risque, contrairement à la VaR, car elle respecte le critère de sous-additivité. Le principe de diversification est donc satisfait, ce qui n'est pas le cas de la VaR : en effet, la VaR globale d'un portefeuille peut être supérieure à la somme des VaR des sous-portefeuilles qui le composent.

Les conditions sur les paramètres et la politique optimale pour le coût moyen sont présentées dans Tableau 1 avec des exemples dans Tableau 2. Par contre, pour le CVaR il y a plusieurs tableaux et conditions pour les politiques optimales du problème qui dépend sur la géométrie de la perte et des coûts.

Demande de Bass

Nous procédons notre travail à résoudre une autre version du problème. Le Modèle de diffusion Basse pour les ventes de nouveaux produits a été présenté par Bass (1969). Depuis sa publication en Management Science, ce modèle a été cité plus de 600 fois et il est considéré comme l'un des modèles les plus remarquables pour les nouveaux produits de prévision. En fait, la majorité des recherches sur les nouveaux produits durables ont porté sur le processus de diffusion.

Ce modèle a été initialement développé pour les biens durables. Cependant, le modèle se révèle applicable à une catégorie plus large de produits et services tels que les produits B2B, les services de télécommunications, les équipements, les semi-conducteurs,

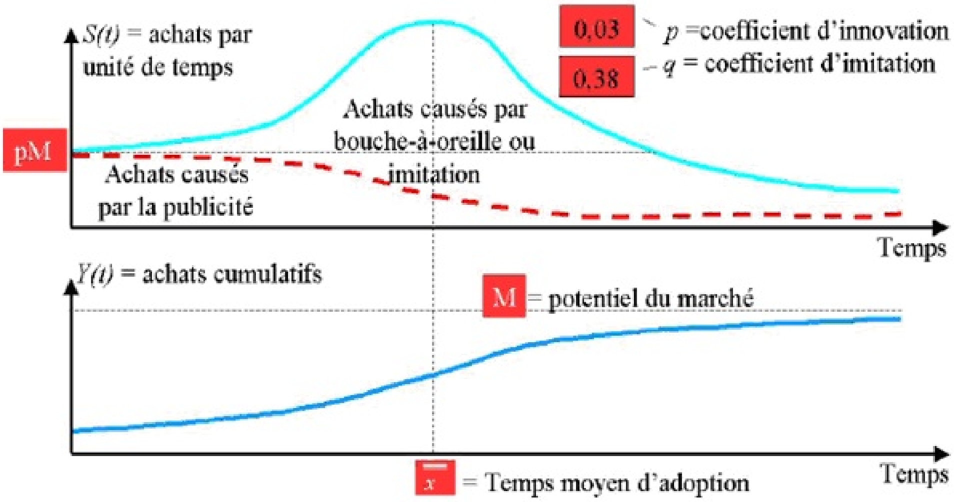

Figure 5: Modéle de Bass

les produits médicaux, les autres produits technologiques et les services.

La communication est un facteur d'influence essentiel du décollage des ventes. La théorie de la diffusion représente l'adoption d'un nouveau produit à partir de la diffusion de l'information. Deux types d'informations sont représentés d'après Bass (1969): l'information dans les médias de masse et le bouche à oreilles. Ces deux types d'informations limitent l'incertitude des consommateurs associée au nouveau produit. Le bouche à oreilles ne pouvant être contrôlé et ne représentant pas une variable d'action pour les firmes, nous nous intéressons uniquement à la politique publi-promotionnelle. De plus, la représentation du bouche à oreille et de la communication impersonnelle est liée à la notion de pénétration du marché déjà abordée plus haut.

Les actions publi-promotionnelle apportent une certaine visibilité au produit et informent les consommateurs sur le nouveau produit. En effet, la publicité dans les médias de masse (télévision, affichage, Ě.) permet une visibilité importante et touche la grande majorité de la population cible. Elle permet de dévoiler les attributs objectifs du nouveau produit comme la couleur, le poids, la taille, la puissanceĚ (Kalish 1985).

Ainsi, l'ensemble des actions de communication a un effet accélérateur du décollage des ventes.

De nombreuses études présentent les paramètres dans diverses industries, des valeurs moyennes de q et p pour les biens durables se sont avérés comme suit : p = 0.03, q = 0.38. Christophe Van den Bulte (Lilien et al.,2000) a construit une base de données de 1586 ensembles de paramètres p et q. Les nouveaux produits qui ont connus une croissance sont ceux de divertissement par exemple, ceux qui ont généralement une vie courte, qui sont très saisonniers, et dont le lancement sur le marché est parfois précédé par la publicité excessive et de la communication marketing.

Nous voulons examiner le problème du rollover avec une demande de Bass pour le nouveau produit, mais ce problème était trop complexe et c'était impossible de trouver des solutions explicites ou des conditions d'optimalités. Nous avons décidé d'utiliser une demande linéaire par morceaux, où la demande du nouveau produit augmente linéairement puis devient constante.

Design de recherche et méthode d'Optimisation Data-Driven et résultats

Le plus souvent, les modèles et algorithmes d'aide à la décision supposent implicitement que les données d'entrée soient connues de manière exacte. Pourtant, la plupart des problèmes rencontrés en pratique peuvent difficilement être traitées dans ce cadre. Le désir de tenir compte d'incertitudes sur les données d'entrée d'un problème d'optimisation n'est pas nouveau mais ce thème de recherche reste l'un des plus actifs jusqu'aujourd'hui, et a connu récemment un fort intérêt pour une grande variété d'applications. Il est clair que quand la distribution de probabilité n'est pas connue, il est impossible d'utiliser la méthode CVaR. Divers modèles ont été proposés pour rendre compte de manière aussi satisfaisante que possible d'incertitudes sur la distribution.

Nous proposons une approche data-driven au problème du rollover avec une date d'admission incertaine.Nos résultats numériques sont très encourageants. Notre méthode est caractérise par

- le travail direct avec les données historiques
- les solutions robustes qui intègrent les préférences de risque en utilisant un paramètre scalaire, plutôt que des fonctions d'utilité
- les solutions explicites sous closed-form

La question de maximiser le profit en présence d'une date d'admissibilité incertaine pour la gestion des rollovers n'a pas reçu assez d'attention dans la littérature quantitative, et a été traité sous l'hypothèse que la distribution de la date d'admissibilité est connue et que le décideur est neutre au risque.

La volatilité de la date d'admissibilité de la plupart des produits n'est pas connue et il est difficile d'obtenir des distributions précises. Scarf (1958) suggère que nous pouvons avoir des raisons de soupçonner que la future demande dans le problème de newsboy proviendra d'une distribution différente des observations historiques. Cette imprévisibilité constitue une incitation forte pour le décideur de mettre en oeuvre des solutions robustes qui donneront de bons résultats pour un large éventail de résultats de la demande réelle, ou dans notre cas la date de l'admissibilité.

La question de l'imperfection de l'information a été traitée dans le passé en supposant que seulement les deux premiers moments sont connus. En 1958, Scarf a dérivé la quantité optimale de commande pour le problème de newsboy classique avec une moyenne et une variance données, et son travail a été poursuivi par Gallego et. al. (2001). Toutefois, une telle méthode est fondée sur l'estimation correcte des deux moments mais manque le lien fort aux préférences des risques, qui dans la pratique joue un rôle clé dans le choix de la solution du problème.

Les préférences du risque ont été considéré par Lau (1980) qui tient compte de deux critères alternatifs: l'espérance d'utilité et la probabilité de parvenir à un certain profit. Plus récemment, Eeckhoudt et al.(2007) ont revisité le cadre fondé sur l'utilité espérée. Cependant, il est difficile en pratique d'articuler l'utilité.

L'approche que nous utilisons ici s'écarte de ces cadres dans deux grands points:

- entièrement piloté par les données, en ce sens que nous construisons directement sur l'échantillon de données disponibles au lieu d'estimer les distributions de probabilité.

- repose sur un paramètre scalaire intégré dans le modèle de robustesse. Ce paramètre correspond ici à un quantile pré-spécifiée. Dans ce cadre, la valeur de la variable aléatoire est déterminée par le calcul de la perte attendue de moins que le quantile. Le décideur se concentre sur une évaluation plus prudente de son perte que celui fourni par une approche devrait-valeur, mais est capable d'adapter le degré de prudence en choisissant le facteur de correction approprié.

L'approche garde la convexité des problèmes lorsque la fonction de perte est convexe. Nous développons notre modèle dans un cadre d'un seul rollover sous horizon infini. Cette méthode est bien adapté pour les problèmes statiques et dynamiques et elle a un lien fort avec l'attitude du décideur à l'égard des risques. De même, elle peut être appliquée dans de nombreux domaines, y compris la gestion de stock et l'optimisation de portefeuille.

Conclusion et piste de future recherches

L'objet de ce travail est d'étudier le problème du product rollover avec une date incertaine d'admission pour le nouveau produit dans plusieurs contextes. On présente trois articles avec une synthèse de la littérature qui illustre ce problème. Malgré l'intérêt des entreprises au product rollover, il n'y a pas encore de recherches qualitatives approfondies dans ce domaine. Amélioration du marketing-coordination des opérations est largement considérée comme une occasion pour améliorer les performances des entreprises. Un besoin important pour le marketing-opérations de coordination est la planification et l'introduction de nouveaux produits où un produit existant est supprimé et un produit de remplacement est progressivement introduit. Ce problème est particulièrement important dans le contexte de la fabrication ou il y a une date d'admission incertaine pour les nouveaux produits. La thèse a formulé cette classe de product rollover comme un problème d'optimisation stochastique. Les articles développés présentent des solutions optimales et uniques ainsi que les politiques optimales du rollover. Des expressions en closed-form ont été développées afin de rendre les résultats faciles à mettre en oeuvre. Notre étude a plusieurs limitations. Le modèle ignore les mesures concurrentielles qui pourraient influer la demande pour le nouveau produit si le nouveau produit est introduit plus tard que le produit du compétiteur. Toutefois, les demandes pour l'admission de produits sont parfois des informations publiques. Il est donc favorable que l'entreprise anticipe la réaction du marché et de la concurrence et prend cela en considération dans le modèle. Dans certaines situations, les anciens et nouveaux produits partagent certaines machines et la capacité de production peut être limitée. Nous pouvons encore utiliser notre modèle, cependant, l'affirmation des capacités devrait être considérée dans le processus de back-ordering. Dans cette thèse, la demande est supposée constante ou linéaire par morceaux pour chaque produit.

Les délais de livraison des produits, les rendements d'approvisionnement, les délais et les rendements de fabrication sont également supposés être déterministe. Un modèle de simulation stochastique peut être mis en oeuvre pour explorer ces questions. Nous avons proposé une approche robuste au problème du rollover qui construit directement sur les données historiques, sans exiger aucune estimation de la distribution de probabilité. Cette approche intègre la robustesse grâce à un paramètre scalaire unique qui peut être ajusté pour atteindre un niveau approprié de protection contre l'incertitude. Par ailleurs, le cadre data-driven présenté dans cette thèse est relié à la théorie des préférences de risque. Nous avons pu dériver les propriétés structurelles des solutions optimales, et ces expressions ont fourni des informations précieuses sur les politiques optimales.

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CHAPTER 1

Introduction

1.1 Overview

The thesis addresses the product rollover problem under an uncertain approval date. The results presented here are based on the research performed during my five years of doctoral studies at HEC. This chapter will provide background and motivation for solving this problem and discuss the objectives, methodology, and scope of the work presented in the remainder of the thesis.

1.1.1 Background and Motivation

Today, product development and introduction to the market are strategic issues for companies. Product life cycles have become short due to technological advancement, and thus, new products have to be introduced and old products phased out frequently. This relatively rapid new product development process can be viewed by a company as a competitive weapon with the underlying cost trade-off. Consider a company that must plan the phase-out of an existing product and the phase-in of a replacement product. If production of the existing product is stopped too early, i.e., before the new product is available for the market, the firm will lose profit and customer goodwill. On the other hand, if production of the existing product is stopped too late, the firm will experience an obsolescence cost for the existing product, because demand and/or price would have decreased as this product can be considered "old generation". Furthermore, if the production of the new product is started too early, the firm will experience an inventory carrying cost until the market will turn to this product since it needs to fill in the pipeline to launch sales (Hill and sawaya (2004)).

As new products appear in the market, many old products could become obsolete, and hence, they should be phased out. The process of launching or introducing a new product in the market place and the removal of an old one is known as *product rollover*.

Classically, there are two rollover strategies: single-product and dual-product rollover. In the single-product rollover strategy, there is a simultaneous introduction of the new product and elimination of the old product, i.e., at any time there is a unique product generation available in the market. On the contrary, in the dual-product rollover strategy, the new product is introduced first and then the old product is phased out. Thus, in this setting, two product generations coexist in the market, for a given time length.

Several papers have addressed the analysis of new product introduction and product rollover processes, under different assumptions and from various viewpoints. Krishnan and Ulrich (2002) present an excellent review of product development decisions encompassing work in marketing, operations management, and engineering design. Erhun et al (2007) conduct a qualitative study on different drivers affecting product transitions at Intel Corp., and they design a framework that guides managers to design and implement appropriate policies taking into consideration transition risks related to the product, manufacturing process, supply chain features, and managerial policies in a competitive environment.

Cohen et al (1996) analyzed performance trade-off for new product development processes. In particular, they proved that it is more favorable to use the faster speed of improvement to develop a better product rather than to develop a product faster, contradicting conventional practice concerning the dominance of incremental over significant improvements in product enhancements.

Saunders and Jobber (1994) study different strategies for the simultaneous deletion and introduction of new products and claim that launch and deletion strategies should be synchronized and that rapid launch strategies should be accompanied by rapid deletion, whereas low price launches should be accompanied by even lower priced deletions.

Billington et al (1998) argue that there has been a low success rate for product rollovers and present many cases of companies that have failed in product rollovers due to technical problems leading to delay in introduction of the new product to the market, excess old product inventory, bad timing of new product announcement, and overly optimistic sales. Furthermore, the authors suggest that companies should have a clear strategy for product rollover in addition to contingency plans in case their strategy fails. They compare and contrast single and dual product rollover strategies. They argue that a single product rollover strategy can be viewed as a high-risk, high return strategy, sensitive to potential random events. On the contrary, the dual product rollover strategy is less risky, but induces higher inventory costs. For complex situations, the authors argue that in addition to the choice of the best strategy, planners should develop contingency plans in anticipation of certain events such as competitors introducing new products, technical problems with the new products, stock-out of old products, and too much inventory of the new or old product.

Hendricks and Singhal (1997) go further and investigate the effect of delays in new product introductions on the market value of the firm. The results of their study indicate that the stock market reacts negatively to delayed product introduction, and that on average, delayed introductions decrease the market value of the firm. Some papers develop quantitative models for the product rollover analysis. Lim and Tang (2006) developed a deterministic model that allows for the determination of prices of old and new products and the times of phase-in and phase-out of the products. Moreover, they developed marginal cost based conditions to determine when a dual product rollover strategy is more favorable than a single rollover one.

Hill and Sawaya (2004) examine the problem of simultaneously planning the phase-out of the old product and the phase-in of a new one that will replace the old product, under an uncertain regulatory approval date for the new product. Furthermore, they exhibit the structure of the optimal policy for an expected profit objective function. In their setting, the manufacturing and procurement lead-times for these products are significant, making it necessary to commit to the planning date before the earliest approval date. The new product is not available for sale until the distribution channel is filled with a minimal number of units. The old product is sold until the firm runs out of inventory or until it is replaced by an approved new product. The firm's policy is to scrap all old product units immediately when an approved new product is available for sale. The fundamental structure of the problem, namely planning a starting date for an activity in a random setting, can be linked to the well known newsboy problem. A very simple setting has been analyzed in the paper of Ronen and Trietsch (1993).

We examine the product rollover under risk. In general, risk modeling has constituted an important research stream for years. In particular, a modern way to take into account the risk consists of focusing on shortfall as in Scaillet (2000). This can be done through an absolute bound on the tolerable loss or by setting a bound on the conditional value at risk as in Artzner et al (1997, 1999). The latter has become a very popular tool in finance.

The conditional value at risk, denoted as CVaR, was introduced in Artzner et al (1997) to remedy several shortcomings of the more familiar value at risk approach. The CVaR measure of risk has very interesting theoretical properties and possesses the attractive feature of being computationally tractable (see Rockafellar and Uryasev (2000, 2002)), in particular, in the framework of stochastic programming. Setting an upper bound constraint on CVaR is often imposed by financial institutions and is thus very relevant in the supply chain context. Risk-sensitivity models in inventory modeling and supply chain management have been proposed in a few papers.

In the real world, managers and planners are not satisfied by maximizing profit only, and rather they may be concerned with other objectives such as trying to attain a predetermined target profit as much as possible. Yet, such a criterion may result in inadequately large losses. To reduce such a loss, Lau (1980),inspired by Markowitz (1952), proposed to minimize the standard deviation of the profit. Yet, profit above some target level is not regarded as a risk to be hedged, but rather additional gain. To minimize a downside risk measure capturing a risk of the profit going down to some target level is more interesting than the other risk measures such as the standard deviation, and in the newsboy framework literature, many researchers consider minimizing such downside risk measures as alternatives to the expected profit maximization.

Tang (2006) provides a review of various quantitative models for managing supply chain risks, yet presents no literature that directly discusses product development and in particular product rollover under an uncertain regulatory approval date. Most inventory related papers try to maximize a predetermined target profit, and that may lead to an increased risk.

Supply chain literature has shown the importance of incorporating a risk measure in inventory management problems, such as Gallego and Moon (1993) who use the maxmin approach to determine the optimal order quantity in a newsboy problem. The maxmin approach is considered as an extremely conservative or pessimistic approach to taking decisions in which one evaluates all the minimum possible returns associated with different decisions and selects the decision yielding maximum value of minimum returns. In their research, they derive the maximin order quantity when only the mean and the variance of the demand variable are known. Thus, in this method, the ordering decisions are based on the worst case within the considered family of demand variables, which often may not reflect real-life demand situations as mention Bertsimas and Thiele (2005).

Parlar and Weng (2003) consider the expected profit in place of the fixed target. These objectives are very intuitive, but the related optimization problems have no convex structure, and accordingly, they are very tough to handle for general distribution functions. Besides, these models seek higher profit, whereas a possibility of suffering great loss is not considered.

On the other hand, Bogataj and Hvalica (2003) use a tradeoff between the expected value criterion and maxmin, others focus on discounted cash flow methodologies such as Luciano and Pecatti (1999), Grubbstro and Thorstenson (1986), and Koltai (2006).Cash flow-oriented models are useful when a time lag exists in the model, i.e., a time lag between starting production of a product and the transportation of the product, or in other words the delay,(Dyckhoff H. et al. (2003)). Chen et al (2005) analyze risk aversion inventory problems comparing risk measures and expected utility optimization.

Ahmed et al (2007) derive the structure of the solution of coherent risk measure optimization for the newsboy loss function.This method is a unified treatment of risk averse and minimax inventory models, the latter objective dealing with minimising worst-case consequences. Borgonovo and Peccati (2006) discuss sensitivity analysis of inventory management models when uncertainty in the input parameters is given full consideration.

Ozler et al (2008) utilize Value at Risk (VaR) as a risk measure in a newsboy framework and investigate the multi-product newsboy problem under a VaR constraint.This formulation does not take into account the risk of earning less than a desired target profit or losing more than an acceptable level due to the randomness of demand. VaR is a popular measure of risk representing the percentile of the loss distribution with a specified confidence level.Furthermore, when analyzed with scenarios, VaR is non-convex as well as non-differentiable, and hence, it is difficult to find a global minimum via conventional optimization techniques. Alternatively, Conditional VaR (CVaR), introduced by Rockafellar and Uryasev (2000) allows the determination of optimal solutions and conditions in a relatively easier way.

Gotoh and Takano (2007) solve the newsboy problem by considering the minimization of the conditional value at risk (CVaR), a preferred risk measure in financial risk management and develop unique closed form solutions due to the convexity of the CVaR. In the supply chain context, van Delft et al (2004) used a CVaR criterion approach in a stochastic programming model aimed at evaluating option purchasing contracts in a risk management perspective. Chen et al (2009) study the newsvendor model under the CVaR criterion for additive and multiplicative demand models, and provide sufficient conditions for the existence and uniqueness of the optimal policy.

The CVaR is a downside risk used in financial risk management, in the single-period newsboy situation. The CVaR enjoys preferable properties that are induced from some axiomatization of rational investors' behavior under uncertainty and, thus, they are meaningful also to a manager who faces uncertain profit/loss situation as in the newsboy problem. In particular, the consistency with the stochastic dominance implies that minimizing the CVaR never conflicts with maximizing the expectation of any risk-averse utility function (Ogryczak and Ruszczynski, 2002). On the other hand, some researchers directly treat the risk aversion through the newsboy's utility function (Eeckhoudt et al., 1995). In practice, utility function is, however, too conceptual to identify and, thus, the use of risk measures has advantage over that of utility functions.

Moreover, the lower partiality of the CVaR plays an important role in preserving the concavity of the profit or, equivalently, the convexity of the cost. In financial portfolio management as in Rockafellar and Uryasev (2002), the return from an asset portfolio is often represented as a linear function of the portfolio, which is to be determined. This is why the standard deviation results in a convex quadratic function. On the contrary, the profit in the newsboy problem is a non-linearly concave function of the order quantities. Consequently, minimizing the standard deviation of the profit may turn into a non-convex optimization, though many researchers introduce it in order to capture the profit variation (Lau, 1980) and develop a CAPM by following the modern portfolio theory (Anvari, 1987).

CVaR preserves the concavity of the profit function by virtue of its lower partiality, and the resulting risk minimization becomes a convex program. By making use of such a nice structure, we use the CVaR measure and achieve analytical results of the CVaR minimizations and thus we are able to find interesting properties of the solutions. The tractability makes the CVaR minimization into a basic tool for more advanced analysis of the risk-averse newsboy problems.

In this thesis, we develop a risk-sensitivity optimization approach for product rollover in a stochastic setting. Namely, we consider a Conditional Value at Risk (CVaR)-type objective function in a product rollover problem under an uncertain approval and where both products have a constant demand rate. Such an uncertain date can correspond to the situation where the replacement product requires an external or an internal approval decision before being sold in the market, as in Hill and Sawaya (2004).We believe our study to be the first that applies a risk-sensitiv optimization model in a stochastic product rollover problem. This approach is important for product rollover situations concerning key products for a company, and for which the risk issue has to be explicitly considered (see Billington et al (1998) for practical examples). In our methodology, we use analytical models which are tractable, yet capture important factors influencing decision making. In particular, we give explicit closed-form expressions for the optimal policies.

Intuition may lead to the hypothesis that, in product rollover stochastic settings, higher regulatory approval date variability result in larger variances and in higher costs. This intuition is correct for many distributions that are commonly used in practice, such as for the normal distribution function. However, we show that in some cases stochastically larger or more variable regulatory approval dates may not necessarily result in a higher optimal cost, because sometimes the variability effects may dominate. On the other hand, a more variable regulatory approval date always leads to a higher optimal average cost. To characterize these regulatory approval date distributions we use stochastic dominance relations.

We proceed with our work and solve another version of the problem where the demand is no longer constant but follows a Bass demand rate. The Bass Diffusion Model for sales of new products was presented by Bass (1969). Since its publication in *Management Science*, it has been cited over 600 times and is one of the most notable models for new-product forecasting. It was originally developed for application only to durable goods. However, the model has proven applicable to a wider class of products and services such as B2B products, telecom services, equipment, semiconductor chips, medical products, and other technology-based products and services.

The Bass model assumes that a population of potential adopters for a new product is subject to two means of communication: mass- media communication and word-of-mouth communication. The former affects potential adopters directly, while the later influences the interaction between customers who already adopted the product as well as the future potential adopters.

For the Bass demand rate, we can no longer consider a constant inventory and have to develop a time varying inventory policy that follows the diffusion of the product. The optimization problem turns into a very complex one where it is not possible to obtain closed form solutions, therefore we perform different numerical simulations for different types of products with different life-cycles from durable goods to intertwinement products to services. To simplify our problem, we try to model the demand with a linearly increasing demand for the second product that becomes constant after a certain period of time. We develop conditions of convexity and give optimal timing decisions for product rollover while comparing our model to a constant demand.

We now introduce our third paper. The purpose of this paper is to analyze and characterize the optimality of each type of strategy (single or dual) for a setting with a stochastic approval date for the new product. Hill and Sawaya (2004) examine the problem of simultaneously planning the phase-out of the old product and the phase-in of a new one that will replace the old product, under an uncertain approval date for the new product. Our problem setting is inspired from their model. In El Khoury et al. (2011), we assumed that the approval date follows a known probability distribution; in practice, however, the volatility of the approval approval date makes it difficult to obtain accurate forecasts of the probability distribution. The assumption that the approval date distribution is known is unrealistic especially since only partial information about the approval is available for the manager.

Thus, we adopt a non-parametric data-driven approach where we build directly upon available historic data samples instead of estimating the probability distributions relying on a scalar parameter to incorporate robustness in the model which corresponds to a pre-specified quantile of the cost. The random variable is determined by computing the expected cost above that quantile, that is, by removing (trimming) the instances of the cost below the quantile and taking the average over the remaining ones. The fraction of data points removed will be referred to as the trimming factor that determines the degree of conservatism. This is a one-sided trimming approach studied by Bertsimas et. al. (2004) and Thiele (2004).The only information available is a set of independent samples drawn independently from the true approval date distribution, but the true distribution is unknown to the manager.This approach was first proposed by Thiele (2004) where she applied it to different variances of the newsboy problem. The importance of this method is the tractability and the possibility of formulating unique closed-form solutions for problems that are convex and piecewise linear.

To our knowledge, this is the first work that addresses the product rollover problem under uncertainty using a data-driven optimization approach. In fact, approval date distributions are very hard to model and often the manager has only historical observations. We derive theoretical insights into the optimal strategies depending on the cost parameters and the degree of conservatism chosen by the decision-maker. We also compare our solutions to the CVaR solutions obtained in our previous work when the probability distribution is known. The structure of this work is as follows: in Chapters 2, 3, and 4 we present our three papers along with their appendicies and in Chapter 5 we present our findings, conclusions, and propose future research directions.

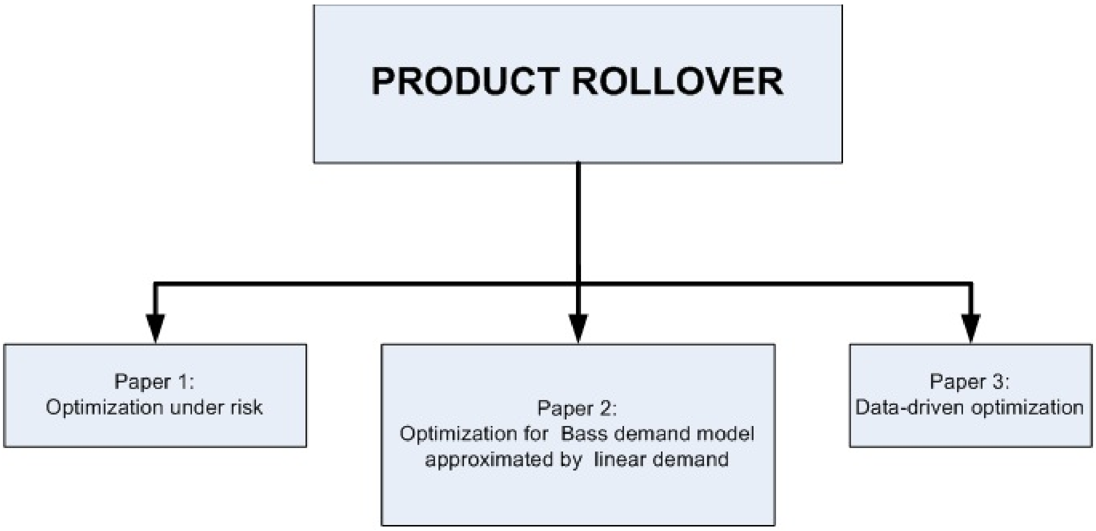

Figure 1.1: Product Rollover under Uncertainty in Three Different Settings

1.1.2 Research Objectives

Previous work on product rollover has mainly focused on the new product development process, such as technology selection and market uncertainty. Researchers usually try to maximize profits. More recent works have expanded the scope that involve pricing.

One of the key challenges for managing product rollovers successfully is determining times of introduction and removal of products. Our main research question is trying to develop an optimal rollover strategy so we try to answer timing issues:

When shall we phase out the old product and introduce the new product? Should we phase out the old product and introduce the new product simultaneously or should we introduce the new product first and then phase out the old product later on?

Whatever strategy the firm decides to adopt, it has to decide how much of the new product to stock and how much is the acceptable lost sales to minimize its losses.

We try to answer our research questions in three papers:

Paper 1: Optimal Strategy for Stochastic Product Rollover under risk using CVaR analysis

We consider an inventory/production rollover process between an old and a new product, with a random approval date for the new product. First, we derive closed forms for the structure of the optimal rollover strategy. Then, we study the impact of uncertainty and of the decision maker's risk position on the optimal strategy structure and on the corresponding cost. We analyze theoretically and partially confirm/infirm some conjectures obtained via empirical research. We illustrate all these results via numerical examples.

Paper 2: Product Rollover Optimization with an Uncertain Approval Date and Piecewise Linear Demand

Consider a company that must plan the phase-out of an existing product and the phase-in of a replacement product. If production of the existing product is stopped too early, i.e., before the new product is available for the market, the firm will lose profit and customer goodwill. On the other hand, if production of the existing product is stopped too late, the firm will experience an obsolescence cost for the existing product. In our paper, we consider a product rollover process with an uncertain approval date for the new product, and develop the optimal rollover strategies by minimizing the expected loss. We derive the optimal strategy and dates to remove an old product and to introduce a new one into the market.

Paper 3: Data-Driven Optimization for Stochastic the Product Rollover Problem

We consider an inventory/production rollover process between an old and a new product, with a random approval date for the new product. Unlike our previous work, the approval date distribution, although exists, is not known. Instead the only information available is a set of independent random samples that are drawn from the true approval date distribution. The analysis we present characterizes the properties of the approval date distribution as a function of the number of historic samples and optimization in a single framework. We present data-driven solutions and incorporate risk preferences using a scalar parameter and tractable formulations leading to closed-form solutions based on the ranking of the historical dates, which provide key insights into the role of the cost parameters and optimal rollover policy. Moreover, we establish bounds on the number of samples required to guarantee that with high probability, the expected cost of the sampling-based policies is arbitrarily close (i.e., with arbitrarily small relative error) compared to the expected cost of the optimal policies which have full access to the approval date distributions. The bounds that we develop are general, easy to compute and do not depend at all on the specific approval date distributions. We finally test the 'robustness' of our solutions though numerical computations.

1.2 Research Methodology

In this section, we discuss the different methodologies and models used in the three different papers.

1.2.1 RISK ANALYSIS: Description of the CVaR CRITERION

In order to introduce the impact of risk in the decision process, we consider our problem in a CVaR-minimization context, an approach mainly developed by Rockafellar and Uryasev (2000, 2002) who propose the following definition to minimize the loss with respect to the decision variables t_1 and t_2

$$l_\beta(t_1, t_2) = \min_{\{\alpha \in \mathbf{R}\}} \left\{ \alpha + \frac{1}{1-\beta} E[L(t_1, t_2, T) - \alpha]^+ \right\}. \qquad (1.2.1)$$

where $z^+ = max\{0, z\}$ and β reflects the degree of risk aversion for the planner (the larger β is, the more risk averse the planner is).

Hsieh and Lu (2010) study the manufacturer's return policy and the retailers' decisions in a supply chain consisting of one manufacturer and two risk-averse retailers under a single-period setting with price-sensitive random demand. They characterize each retailer's risk-embedded objective via conditional value-at-risk, and construct manufacturer-Stackelberg games with and without horizontal price competition between the retailers.

Sawik (2010) studies the optimal selection of supply portfolio in a make-to-order environment in the presence of supply chain disruption risks. Given a set of customer orders for products, the decision maker needs to decide from which supplier to purchase custom parts required for each customer order to minimize total cost and mitigate the impact of disruption risks.

Chen et al. (2009) present some convex stochastic programming models for single and multi-period inventory control problems where the market demand is random and order quantities need to be decided before demand is realized. Both models minimize the expected losses subject to risk aversion constraints expressed through VaR and CVaR measures.

Gotoh and Takano (2007) solve the newsboy problem by considering the minimization of the conditional value at risk (CVaR), a preferred risk measure in financial risk management and develop unique closed form solutions due to the convexity of the CVaR. In the supply chain context, van Delft et al (2004) used a CVaR criterion approach in a stochastic programming model aimed at evaluating option purchasing contracts in a risk management perspective. Chen et al (2009) study the newsvendor model under the CVaR criterion for additive and multiplicative demand models, and provide sufficient conditions for the existence and uniqueness of the optimal policy.

Let $L(t_1, t_2, T)$ be the loss associated with the decision variables t_1, t_2, and the random variable T. Let us denote the distribution function of $L(t_1, t_2, T)$ by

$$\Phi(\eta|t_1, t_2) = Pr\{L(t_1, t_2, T) \leq \eta\}. \tag{1.2.2}$$

For $\beta \in [0, 1)$, we define the β-VaR of this distribution by

$$\alpha_\beta(t_1, t_2) = min\{\alpha|\Phi(\alpha|t_1, t_2) \geq \beta\}, \tag{1.2.3}$$

By definition, it can be expected that the loss $L(t_1, t_2, T)$ exceeds α_β only by $(1 - \beta) \times 100\%$. Rockafellar and Uryasev (2000, 2002) introduce the β-tail distribution function to focus on the upper tail part of the loss distribution as

$$\Phi_\beta(\eta|t_1,t_2) = \begin{cases} 0 & \text{for} \quad \eta < \alpha_\beta(t_1,t_2), \\ \frac{\Phi_\beta(\eta|t_1,t_2)-\beta}{1-\beta} & \text{for} \quad \eta \geq \alpha_\beta(t_1,t_2). \end{cases} \tag{1.2.4}$$

Using the expectation operator $E_\beta[T]$ under the β-tail distribution $\Phi_\beta(\cdot|\cdot,\cdot)$, we define the β-conditional value-at-risk of the loss $L(t_1,t_2,T)$ by

$$E_\beta[L(t_1,t_2,T)]. \tag{1.2.5}$$

To minimize $E_\beta[L(t_1,t_2,T)]$ with respect to the decision variables (here t_1 and t_2), according to Rockafellar and Uryasev (2002) , one can introduce an auxiliary function $l_\beta(\cdot,\cdot,\cdot)$ defined by

$$l_\beta(t_1,t_2,\alpha) := \alpha + \frac{1}{1-\beta} E\big[[L(t_1,t_2,T)-\alpha]^+\big], \tag{1.2.6}$$

where $[Y]^+ := max(Y,0)$. It is known that $l_\beta(t_1,t_2,\alpha)$ is convex with respect to α (see Rockafellar and Uryasev (2002)). Also, the minimal value $\Phi_\beta(t_1,t_2)$ can be achieved by minimizing the function $l_\beta(t_1,t_2,\alpha)$ with respect to t_1, t_2, and α (see Rockafellar and Uryasev (2002)).

In our setting, this minimization problem is represented by the following convex program:

$$\min \qquad l_\beta(t_1,t_2,\alpha) := \alpha + \frac{1}{(1-\beta)} \int_0^\infty [L(t_1,t_2,T)-\alpha]^+ f(T)dT, \tag{1.2.7}$$

$$\text{s.t.} \qquad 0 \leq t_1,t_2 \leq \infty, \quad -\infty \leq \alpha \leq \infty. \tag{1.2.8}$$

1.3 What is the Bass Model and Could we Use it?

Traditionally, diffusion models as Bass (1969) were designed to describe the diffusion of single-purchase durable goods. However, the driving forces of diffusion, namely the combination of external influences and the interaction between customers, are relevant for other product types as well. In this section, we present some interesting examples for the market growth of new products in several industries. We will start by durable goods for which the Bass Model was initially intended, and then we discuss other non-durable goods with different product life-cycles.

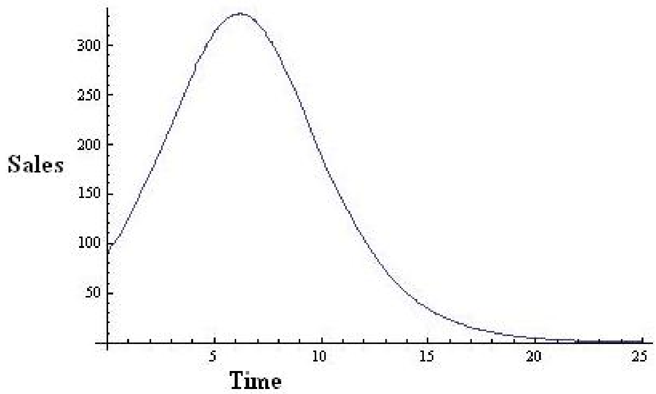

Figure 1.2: Classical Bass Diffusion Model

Though numerous studies have estimated the parameters in various industries, the average values of q and p for durable goods were found to be p_i = 0.03, q_i = 0.38. Estimation issues are discussed in Bass (1969). Christopher Van den Bulte (Lilien et al., 2000) has constructed a database of 1586 sets of p_i and q_i parameters. Figure (1.2) presents the classical Bass model.

The growth of new entertainment products (films, books, and music) usually have a short life, are highly seasonal, and their market launch is sometimes preceded by massive advertising and marketing communication, so that customers can decide to adopt them even prior to launch (Moe and Fader 2002); their distribution is through relatively powerful channels such as exhibitors (theater owners), record stores, and media broadcasting, which share the revenues with the producers and determine product approval to consumers (Ainslie, Dreze, and Zufryden, 2005).

Telecommunication products and services are occupying an increasing share of the world's economies. Many telecom markets are growing markets; the penetration processes of telecom products are interrelated (e.g., penetration of unified messaging services depends on penetration of mobile phones); they depend on the existence of infrastructures (e.g., Skype depends on broadband penetration); in some products and services there are network externalities; some services such as mobile services suffer from fierce competition and high churn rates. Furthermore, the distribution structure of the telecom industry is complex. A single 3G telephony end-user application depends on hardware and software manufacturers, service providers, compatibility issues, and global infrastructures. Despite the market richness and complexity, telecom products are mostly treated as if they were durable goods (Krishnan, Bass, and Kumar, 2000; Jain, Mahajan, and Muller, 1991).

For the Bass demand rate, we can no longer consider a constant inventory and have to develop a time varying inventory policy that follows the diffusion of the product. The optimization problem turns into a very complex one.

In our model, the manufacturing and procurement lead-times for our products are significant, making it necessary to commit to the planning date before the earliest approval date. The new product is not available for sale until the distribution channel is filled with a minimal number of units proportional to demand. The old product is sold until the firm runs out of inventory or until it is replaced by an approved new product. The firm's policy is to scrap all old product units immediately when an approved new product is available for sale. The fundamental structure of the problem, namely planning a starting date for an activity in a random setting, can be linked to the well known newsboy problem. The demand for the old product is constant, whereas the demand of the new product is initially linearly increasing then constant. In our main model, when the new product is delayed, all demand for this product is lost and there is inventory buildup. In another special case, when the new product is delayed, a portion of the demand is lost whereas another portion is maintained (See Figure 1.3). The portion of the demand that was not met but maintained is sold immediately after the approval is granted.

Druehl et Al (2009) argue that delaying a product too long may fail to capitalize on customer willingness-to pay for more advanced technology in addition to the possibility that competitors may (further) infiltrate the market, furthermore, sales of existing product may decline due to market saturation. If a firm introduces too early, it may cannibalize the previous generation too quickly, not taking advantage of market growth. If it waits too long, sales may have slowed considerably as the product has already diffused through the market. If there is not a sufficient base of customers of the innovator type, then the pace will be slow. But once this base of innovators exists, the pace will be increased by either innovators or imitators.

In our problem, the market knows the time at which the new product will be introduced. The customer purchases the product if it has been approved by the regulatory authority where the demand rate is linear and dependent on time. If the product has not been approved, the demand is lost until the date the approval is given, and the demand of the new product remains linearly dependent on time.

Hill and Sawaya (2004) examine the problem of simultaneously planning the phase-

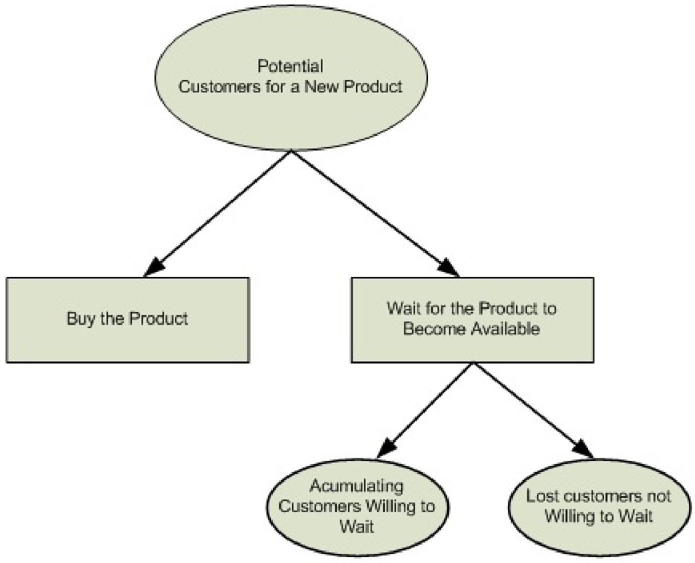

Figure 1.3: New Product Adoption

out of the old product and the phase-in of a new one that will replace the old product, under an uncertain approval date for the new product. Furthermore, they characterize the structure of the optimal policy for an expected profit objective function. Their setting is similar to ours, when the demand of the new and old products are constant and when the new product is not available all of the demand is lost. In this paper, we examine if considering a linearly increasing demand for the new product changes the optimal strategy or the timing decisions compared to the constant demand examined in Hill and Sawaya (2004).

In our first paper, we have considered a problem similar to that of Hill and Sawaya (2004), where demand of both new and old products was constant and where we examined the rollover problem under expected loss and risk minimization. This was important to gain insight on the problem at hand, yet, a constant demand does not apply in real settings where products are subject to a diffusion rate and usually modeled through the Bass Model. Demand usually increases to reach a peak and then decreases after the product achieves maturity. In addition, in our first paper, if neither product was available in the market we assumed that all the demand was lost. Bass model literature contradicts this assumption where, when customers ask for a product and it is not available, not all demand is lost: some customers may be willing to wait at a certain waiting cost and will later purchase the product when it is available. On the other hand, some of the customers will choose not to wait and go on to purchase another product. This decreases lost profit as discussed by Norton and Wilson (1989).

Furthermore, in the first paper we consider the old product demand is equal to the new product demand and that the demand of the new product is not affected if it is delayed. Both of those assumptions are violated in real life settings where there is accumulated inventory and there is a potential market loss when a product is delayed (Druehl et Al (2009)).

In this paper, we model a more realistic setting where demand increases linearly and another special case where not all demand is lost in case of delay. While we can model the old product demand as constant since at the end of the lifecycle of a product, its demand after decreasing becomes constant (See Figure 1.2). The demand of a new product usually increases incrementally over time and this has an effect on product entry timing decisions.

We believe this study to be the first that examines this kind of setting of the product rollover problem. This approach is important for product rollover situations concerning key products for a company. We try to prove the uniqueness of the optimal solutions and approximate the optimal solutions through Mathematica as it is not possible to provide analytical closed-form solutions.

1.4 Unknown Probability Distributions and Data-Driven Optimization

In the data-driven approach, the random variable is determined by computing the expected cost above a certain quantile, that is, by removing (trimming) the instances of the cost below the quantile and taking the average over the remaining ones. The fraction of data points removed will be referred to as the trimming factor which is in fact the same as β used in the CVaR method. We are thus able to compare our solutions using the data-driven approach to the solutions obtained through the CVaR method. In this paper, we replace our original CVaR objective function with an average-based on the drawn samples (Thiele 2006). The sampling-based approximated objective is then minimized.

Suppose that there are N independent samples drawn from the true distribution, labeled as $T_1, ..., T_N$. The data-driven approach approximates the true distribution with the empirical distribution that puts a weight of $1, ...N$ on each of the samples and the expected cost evaluated under this empirical distribution. We denote the $a - quantile$ of the approval date T by $q_a(T)$ where

$$q_a(T) = inf\{t|F(T \leq t) \geq a\}, \tag{1.4.1}$$

for any $a\epsilon(0, 1)$ as have done Levy and Kroll (1978) to describe investor preferences. We adapt their approach to a cost objective as follows:

Theorem 1: $E[U(T_1)] \leq E[U(T_2)]$ for all U decreasing and convex if and only if $E[T_1|T_1 \leq q_a(T_1)] \leq E[T_2|T_2 \leq q_a(T_2)]$ for any $a\epsilon(0, 1)$, and we have strict inequality for some a.

Therefore, a strategy chosen to minimize the tail conditional expectation $E[T_1|T_1 \leq q_a(T_1)]$ is non-dominated. Equivalently, minimizing $E[T_1|T_1 \leq q_a(T_1)]$ for a specific a

guarantees that no other strategy can worsen the value (expected utility) of the random variable for all risk-averse planners. Furthermore, this method does not require any assumptions for the probability distribution of the approval date.

Let N be the total number of observations of T where $(T_{(1)}, ...T_{(N)})$ be those observations ranked in increasing order $(T_{(1)} \leq, ... \leq T_{(N)})$.

Let the trimming factor be the fraction of scenarios that are removed, as $\beta = 1 - a$, and the number of scenarios left after trimming as $N_\beta = \lfloor N(1-\beta) + \beta \rfloor$ so that there is no trimming at $\beta = 0$ $(N_\beta = N)$ and that only the worst scenario remains at $\beta = 1$ $(N_\beta = 1)$.

It follows that the value associated with the random $L_i(t_1, t_2, T)$ is computed by:

$$\frac{1}{N_\beta} \sum_{k=1}^{N_\beta} L_i(t_1, t_2, T)_{(k)} \tag{1.4.2}$$

where $L(t_1, t_2, T)_{(k)}$ is the k^{th} smallest $L(t_1, t_2, T_j)$. From Thiele (2004), problem (1.4.2) becomes

$$\begin{aligned} Min \quad & \frac{1}{N_\beta} \sum_{k=1}^{N} t_k y_k \\ \text{s.t} \quad & \sum_{k=1}^{N} y_k = N_\beta \\ & 0 \leq y_k \leq 1 \forall k \end{aligned} \tag{1.4.3}$$

The feasible set of Eq. 1.4.3 is nonempty and bounded, therefore by strong duality, Eq. 1.4.3 is equivalent to:

$$\begin{aligned} min \quad & N_\beta \phi + \sum_{k=1}^{N} \psi_k \\ & \text{s.t } \phi + \psi_k \geq t_k, \forall k \\ & \psi_k \geq 0 \forall k \end{aligned} \tag{1.4.4}$$

Problem (1.4.3) is a convex problem if $L_i(t_1, t_2, T)$ is convex in t_1 and t_2 , and a linear programming problem since is piecewise linear $L_i(t_1, t_2, T)$ and ς is a polyhedron.

As the cost functions in our product rollover problem are piecewise linear with linear ordering constraints, we will be able to derive tractable, linear programming formulations of the data-driven model.

The conditional value at risk (CVaR) is at the core of the data-driven approach, as the method's objective is to minimize its sample value over the historical realizations of the approval date. CVaR at level β refers to the conditional expectation of losses in the top $100(1-\beta)\%$ and refers to the risk perception of the manager. According to the data-driven approach, the fundamental optimization problem considered here consists of finding the phase-in and phase-out dates which minimize the worst expected cost, the associated optimization problem is

$$\min_{t_1, t_2 \in R^+} \qquad \frac{1}{N_\beta} \sum_{k=1}^{N_\beta} L(t, T)_{(k)}. \tag{1.4.5}$$

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CHAPTER 2

First Paper: Optimal Strategy for Stochastic Product Rollover under Risk using CVAR Analysis

Abstract

We consider an inventory/production rollover process between an old and a new product, with a random approval date for the new product. In absence of risk, this optimization problem consists in finding the phase-in and phase-out dates to minimize the expected loss. In addition, in this paper, we characterize, under risk, the rollover decision making and provide explicit closed-form expressions for the optimal policies. We illustrate all these results via numerical examples and we provide managerial insights for different cases.

KEYWORDS: Product rollover; Uncertain approval date; Planned stockout rollover (PSR); Single product rollover (SPR); Dual product rollover(DPR); Risk sensitive optimization criterion; Conditional value at risk (CVaR); Stochastic dominance; Stochastic comparisons

2.1 INTRODUCTION

Due to rapid technological development and increased variety demanded by consumers, product life cycles have shortened. Therefore, new productshave to be introduced and old products phased out more and more frequently. As new product introduction is a source of growth, renewal and competitive advantage, decision makers are facing the issue of how to successfully manage product replacement and optimize the associated supply chain cost trade-offs. In an ideal setting, the optimal rollover strategy is clear: the old product is phased out at the planned introduction date of the new product, and the new product is readily available. Unfortunately, real-life is quite less favorable.

A study of U.S. durable goods companies showed that, for various reasons, more than 50 percent of new products failed after being introduced to the market. These poor product launch performances are due to numerous potential random disruption in the process (unexpected logistic or industrial delays, quality problems, bad demand forecasts, unexpected market reaction to the new product announcement, etc...). How to phase in new products while phasing out old ones has become a challenging managerial problem in companies. Obviously, when a company is planning the phase-out of an existing product and the phase-in of a replacement product, classical stochastic production/inventory trade-offs have to be considered. If the production of the existing product is stopped too early, i.e., before the new product is available for the market, the firm will lose sales and customer goodwill. On the other hand, if the production of the existing product is stopped too late, the firm will experience an obsolescence cost for the existing product, because demand and/or price would have decreased as this product will be considered "old generation" by the customers. Furthermore, if the production of the new product is started too early, the firm will experience an inventory carrying cost until the market will turn to this product. The process of launching or introducing a new product in the market place and the removal of an old one is known as *product rollover*. In this paper, we focus on three fundamental strategies: rollover with planned stockout, single-product rollover and dual-product rollover. An important issue in new product launch management is whether two product generations coexist in the market for a given length of time; in other words, whether there is an overlapping of some sort in successive product inventory/production/supply chain. In the planned stockout rollover (PSR) strategy, the introduction of the new product is planned in such a way that a stockout phenomenon occurs during the product transition. During this stockout period, no product of any type is available for the market (which induces some kind of backorder cost). In the single-product rollover (SPR) strategy, there is a simultaneous introduction of the new product and elimination of the old product, i.e., at any time there is a unique product generation available in the market. On the contrary, in the dual-product rollover (DPR) strategy, the new product is introduced first and then the old product is phased out. Thus, in this setting, two product generations coexist in the market, for a given length of time. The advantage of the DPR strategy, with respect to the SPR policy, is to allow some protection against potential random events (delays, quality, market demand level) affecting the planned phasing.However, its drawback is the cost corresponding to the additional supply chain inventory. The purpose of this paper is to analyze and characterize the optimality of each type of strategy (PSR, SPR and/or DPR) for a setting with a stochastic approval date for the new product. As we consider a quantitative approach, such an analysis requires a performance evaluation model for the supply chain rollover process between two successive products. We provide a newsboy type planning/inventory model for the rollover process between two successive products. Our starting model is inspired from Hill and Sawaya (2004). By solving the associated optimization problem, we exhibit the optimality conditions for PSR, SPR or DPR. As product rollovers occur in stochastic settings, different rollover strategies exhibit different properties w.r.t. the risk (i.e., cost or profit variability). Efficient risk measure and optimization is a complex issue, extensively considered in finance research literature. A somewhat recent risk criterion, called conditional value at risk and usually denoted as CVaR, has emerged as exhibiting quite interesting theoretical properties and the attractive feature of being computationally tractable (see for example Rockafellar and Uryasev ([30, 31])). Our CVaR model captures the risk issue in the rollover decision making and provides explicit closed-form expressions for the optimal policies. Furthermore, we characterize the influence of the parameters of the setting on the optimal policy structure (namely the different type of costs, the magnitude of the randomness, the manager appetite w.r.t. the risk). We are able to formally prove several conjectures or observations concerning optimal structures reported in other papers, obtained by empirical research. Along this line, we also analyze the behavior of the optimal rollover policy in response to stochastically larger approval process.

2.2 Literature Review

Several papers have addressed the question of efficient management of new product launch, old product destruction/salvage/scrap/liquidation and/or combination of the two processes. A first trend of papers about new product development and launch is mainly of qualitative and descriptive nature (see [23] for a review, encompassing work in marketing, operations management, and engineering design). Chryssochoidis [10, 11] has studied from an empirical point of view the whole process in a large number of companies. This research exhibits critical causes of delay in international product rollover implementation. Saunders and Jobber [33] identify the different types of strategies and overlapping when implementing a phase-in phase-out process. Several papers have addressed the analysis of new product introduction and product rollover processes, under different assumptions and from various viewpoints. Erhun et al ([12]) conduct a qualitative study on different drivers affecting product transitions at Intel Corp., and they develop a framework that guides managers to design and implement appropriate policies taking into consideration transition risks related to the product, manufacturing process, supply chain features, and managerial policies in a competitive environment. The authors suggest that companies should develop clear strategies for product rollover, in addition to contingency plans in case of failure. They compare and contrast single and dual product rollover strategies. They argue that a single product rollover strategy can be viewed as a high-risk, high return strategy, sensitive to potential random events. On the contrary, the dual product rollover strategy is less risky, but it induces higher inventory costs. Hendricks and Singhal ([18]) have shown by empirical research that delays in new product introductions decrease the market value of the firm.

Some papers address quantitative modeling and optimization of rollover processes. Lim and Tang ([25]) developed a deterministic model that allows for the determination of prices of old and new products and the times of phase-in and phase-out of the products. Moreover, they developed marginal cost based conditions to determine when a dual product rollover strategy is more favorable than a single rollover one. Hill and Sawaya ([19]) examine the problem of simultaneously planning the phase-out of the old product and the phase-in of a new one to replace the old product, under an uncertain regulatory approval date for the new product. Furthermore, under a usual expected profit criterion, they exhibit the structure of the optimal policy. The fundamental structure of the problem, namely planning a starting date for an activity in a random setting, can be linked to the well known newsvendor problem. A very simple setting has been analyzed in the paper of Ronen and Trietsch ([32]). In our paper, we develop explicit closed-form expressions for the optimal policies.

Risk-sensitivity models in inventory modeling and supply chain management have been proposed in a few papers. Tang ([38]) provides a review of various quantitative models for managing supply chain risks. Most inventory-related papers maximize a predetermined target profit; yet that may lead to an increased risk. In general, risk modeling has constituted an important research stream in finance. A way to take into account the risk consists of focusing on *shortfall*, through an absolute bound on the tolerable loss or by setting a bound on the *conditional value at risk*. Theoretical properties of the CVaR measure of risk has been extensively studied (see for example [30, 31]). In inventory theory, some papers have adapted standard results to such risk criterion. Ozler et al ([29]) utilize Value at Risk (VaR) as a risk measure in a newsboy framework and investigate the multi-product newsboy problem under a VaR constraint. Some papers ([8, 17]) developed closed form solutions due for a CVaR newsboy problem.

2.3 The product rollover evaluation model

In this section, we will define the product rollover problem and introduce the different notation and assumptions.

2.3.1 Stochastic rollover process and profit/cost rates

The problem context requires a production plan for the phase-out of an existing product (hereafter called *old product*, or *product* 1) and phase-in of a replacement product (called *new product* or *product* 2) under an uncertain (internal or external) approval date, denoted T, for the new product delivery. A typical example for such approval decisions are those of medical devices and pharmaceutical products which cannot be sold until an approval body grants permission. Two decision variables have to be fixed in such a rollover process: t_1, the date the firm plans to phase out the old product and t_2, the date the new product is planned to be ready and available for the market. The existing product is sold until the firm runs out of inventory or until it is replaced by the approved new product. The manufacturing and procurement lead times are assumed to be large, thus making it necessary to commit to the planning dates before the random approval date is revealed. Therefore, the decision process relies exclusively on the probability distribution of this date T. Such large procurement/manufacturing/distribution lead-times are frequent in practice: for instance, the regulatory affairs department in a medical device firm uses a forecast interval for the approval date that is longer than 6 months. During their regular commercial life span, each product has a specific constant demand rate, namely d_1 and d_2. A channel inventory is needed to support each product in the market, which induces carrying inventory cost rates $c_{h,1}$ and $c_{h,2}$. During the commercial life, the contribution-to-profit rate for product i, is defined as

$$m_i = d_i(p_i - c_i) - c_{h,i}, \qquad (i = 1, 2), \tag{2.3.1}$$

with p_i the selling price and c_i the production cost.

Furthermore, in the considered random setting, the profit/cost structure, defined over an infinite time horizon, depends on the relative values of t_1, t_2 and T. Indeed, if the strategy $t_1 \leq t_2$ is chosen, the structure of the profit/cost rates is given in Figure 2.1,

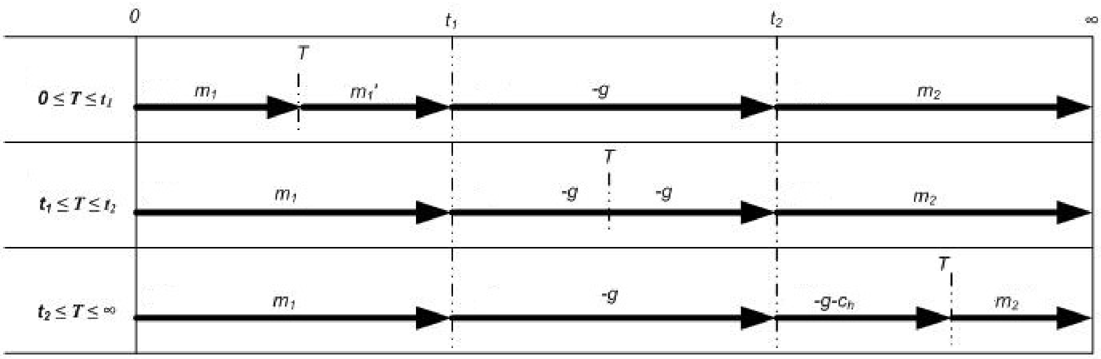

Figure 2.1: the profit rates when $t_1 \leq t_2$

As shown in Figure 2.1, there are three main cases to be considered.

Case 1: $T \leq t_1$**,** Here, the profit rate is m_1 over the time interval $[0, T[$. Then, if $t_1 \leq T \leq t_2$, the new product is approved, but not yet physically available in the supply chain. As the market is assumed to be informed that the new product 2 will soon substitute product 1, the product 1 profit rate changes from m_1 to m_1' as long as product 1 is available, i.e., over $[T, t_1[$. This contribution rate m'_1 is formally given by

$$m'_1 = d'_1(p'_1 - c_1) - c_{h,1}. \tag{2.3.2}$$

Then, over the interval $[t_1, t_2[$, when the old product is sold out, shortages occur until new product 2 delivery date t_2, at a corresponding shortage cost rate g. Once the new product is available, at t_2, the profit rate becomes m_2 over the remaining time horizon $[t_2, \infty[$. Then, if $t_2 \leq T$, the profit/cost rates are similar to the previous situation, except over the interval $[t_2, T[$, where the new product is physically available in the supply chain, but still not approved. A shortage cost rate g occurs until new product 2 is approved. In addition, an inventory cost rate $c_{h,2}$ associated with the product 2 physical inventory is incurred.

If the strategy $t_2 \leq t_1$ is chosen, the structure of the costs and profit rates is given in Figure 2.2. First, let us consider the instance where $T < t_2$. The profit rate is m_1 over the time interval $[0, T[$ and m'_1 over $[T, t_2[$. Then, over the time interval $[t_2, t_1[$, as the new product is approved and physically available, it is sold with a profit rate m_2. However, in the current setting, it is assumed that the firm immediately scraps, at a cost rate s_1, all the remaining inventory of product 1 when an approved product 2 is available for sale, i.e., over the time interval $[T, t_1]$). This is justified by the higher margins for product 2 and by the need to maintain brand equity as a leading-edge provider. Finally, over the remaining time horizon $[t_1, \infty[$, the profit rate becomes to m_2.

Case 2: $t_2 \leq T \leq t_1$ Here, the profit rate is m_1 over $[0, t_2[$. Then over the interval $[t_2, T[$, the profit rate is still m_1, but as the new product is physically available in the supply chain, but not approved for sale, an inventory cost rate $c_{h,2}$ is incurred. Over the remaining horizon starting at T, the new product is sold with a profit rate m_2. In the time interval $[T, t_1[$, the old product is scrapped at a cost rate s_1 while over the remaining time horizon $[t_1, \infty[$, the profit rate becomes to m_2.

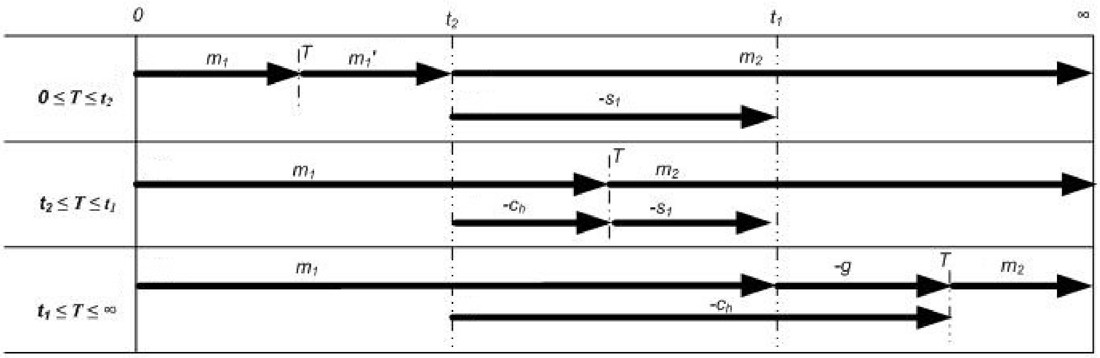

Figure 2.2: the profit rates when $t_2 \leq t_1$

Case 3: $t_1 \leq T$ The profit rate is m_1 over $[0, t_2[$. Then, over the interval $[t_2, t_1[$ the profit rate is still m_1, but an inventory cost rate $c_{h,2}$ has to be incurred. Over $[t_1, T[$, the old product is phased out and the new product is not yet approved. Thus, this creates shortages and a shortage cost rate g is incurred. Finally, over the remaining time horizon $[T, \infty[$, the profit rate reverts to m_2.

2.3.2 Model Notation

For this rollover optimization model, we adopt the following notation. For each product type $i \in \{1, 2\}$, we define

c_i : the unit cost for product i,

p_i : the unit price for product i,

$p_i - c_i$: the gross margin for product i,

d_i : the demand rate for product i,

m_i : the contribution-to-profit rate for product i, defined as $m_i = d_i(p_i - c_i) - c_{h,i}$,

g : the shortage cost rate when the firm has neither of the products to sell,

m_1' : the new contribution-to-profit rate of product 1 after the admissability of product

2 is granted; this value is externally given,

$c_{h,i}$: the carrying cost rate for product i,

s_1 : the scrap cost rate for product 1.

Furthermore, we denote

T : the random approval date for the new product (i.e., for product 2). This random variable has a density probability function $f(\cdot)$ and an associated probability distribution function $F(\cdot)$, both defined over $[0, \infty[$.

The decision variables are,

t_1 : the planned run-out date for inventory of the existing product (i.e., product 1),

t_2 : the planned approval date for the new product (i.e., product 2).

2.3.3 The Global Optimization Criterion

Hill and Sawaya ([19]) solve this problem by maximizing, over the maximal approval date horizon, the contribution to profit, which is the sum of the contribution to profit for products 1 and 2 minus the scrap loss for product 1, the carrying cost for both products, and lost goodwill during the time the firm cannot sell either product. The approval date is the unique random variable of the problem. We consider a performance criterion defined as the difference between the profit, under complete information about approval date, and the profit when the approval date is random and known exclusively through its probability distribution. This performance criterion is defined as follows:

In order to set up the optimization model, let us consider two cases:

Case 1: Availability of perfect information about the approval date

In this case, the regulatory date is known before the decisions t_1 and t_2 are made. This situation is depicted in Figure 2.3. In this ideal setting, the optimal strategy is clear : $t_1 = t_2 = T$, i.e., the old product is phased out at the planned introduction date of the

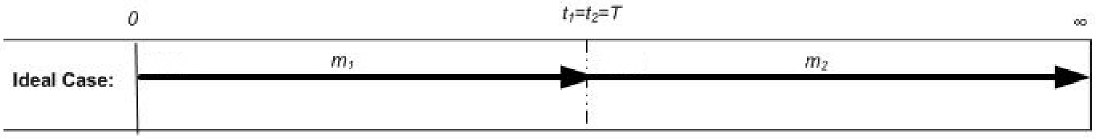

Figure 2.3: Full information case

new product, corresponding to the regulatory date. Over the time interval $[0, T[$, the profit rate is m_1, while on the remaining horizon $[T, \infty]$, the profit rate is m_2.

Case 2: The approval date is random and only known through its probability distribution

In order to characterize the impact of randomness on the rollover process, we consider an objective function defined as the difference between the perfect information cost rate function (Figure 2.3) and the cost rates functions with imperfect information (Figures 2.1 and 2.2). This difference can be interpreted as the loss caused by the randomness of the approval date T. Formally, according to the description given above, these loss functions are piecewise linear and exhibit different structures, depending on the relative values of the decision variables t_1 and t_2. If the planned stock-out strategy $t_1 \leq t_2$ is chosen, the loss rate function is denoted as $L_1(t_1, t_2, T)$ and amounts to

$$\begin{aligned} L_1(t_1, t_2, T) &= (m_1' - m_2)(t_1 - T) + (-m_2 - g)(t_2 - t_1) \text{ if } 0 \leq T \leq t_1, \\ & \quad (-g - m_1)(T - t_1) + (-g - m_2)(t_2 - T) \text{ if } t_1 \leq T \leq t_2, \\ & \quad (-g - m_1)(t_2 - t_1) + (-g - m_1 - c_{h,2})(T - t_2) \text{ if } t_2 \leq T, \\ &= (m_1 + g)[T - t_1]^+ - (g + m_1')[t_1 - T]^+ \\ &+ c_{h,2}[T - t_2]^+ + (m_2 + g)[t_2 - T]^+, \end{aligned} \tag{2.3.3}$$

where $[Y]^+ := \max(Y, 0)$. If the dual rollover strategy $t_2 \leq t_1$ is chosen, the loss function is denoted as $L_2(t_1, t_2, T)$ and is given by

$$\begin{aligned} L_2(t_1, t_2, T) &= (m_1' - m_2)(t_2 - T) - s_1(t_2 - t_1) \text{ if } 0 \le T \le t_1, \\ &\quad -c_{h,2}(T - t_2) - s_1(t_1 - T) \text{ if } t_1 \le T \le t_2, \\ &\quad -c_{h,2}(t_2 - t_1) - (g + m_1)(T - t_1) \text{ if } t_2 \le T, \\ &= (m_2 - m_1' - s_1)[t_2 - T]^+ + c_{h,2}[T - t_2]^+ \\ &+ (m_1 + g)[T - t_1]^+ + s_1[t_1 - T]^+. \end{aligned} \tag{2.3.4}$$

If we formally introduce the two regions, $\mathbf{R_1} = \{(t_1, t_2) \in \mathbb{R}^+ \times \mathbb{R}^+ : t_1 \le t_2\}$ and $\mathbf{R_2} = \{(t_1, t_2) \in \mathbb{R}^+ \times \mathbb{R}^+ : t_1 \ge t_2\}$, the piecewise loss rate functions can be rewritten as

$$L(t_1, t_2, T) = L_i(t_1, t_2, T) \quad \text{if } (t_1, t_2) \in \mathbf{R_i} \quad (i = 1, 2). \tag{2.3.5}$$

with

$$\begin{aligned} L_1(t_1, t_2, T) &= (m_1 + g)[T - t_1]^+ - (g + m_1')[t_1 - T]^+ \\ &\quad + c_{h,2}[T - t_2]^+ + (m_2 + g)[t_2 - T]^+ \end{aligned} \tag{2.3.6}$$

$$\begin{aligned} L_2(t_1, t_2, T) = &\quad (m_2 - m_1' - s_1)[t_2 - T]^+ + c_{h,2}[T - t_2]^+ \\ &\quad + (m_1 + g)[T - t_1]^+ + s_1[t_1 - T]^+, \end{aligned} \tag{2.3.7}$$

On the boundary between regions $\mathbf{R_1}$ and $\mathbf{R_2}$, i.e., for $\mathbf{R_b} = \{(t_1, t_2) \in \mathbb{R}^+ \times \mathbb{R}^+ : t_1 = t_2\}$, the expression of the objective function is obtained from (2.3.6) and/or (2.3.7) as

$$L_b(t, T) = (m_2 - m_1')[t - T]^+ + (m_1 + g + c_{h,2})[T - t]^+. \tag{2.3.8}$$

2.3.4 Parameter Assumptions

We introduce some assumptions for the different parameters. These assumptions are as follows. First the contribution-to-profit rate for the products under regular sales is positive, i.e.,

$$m_1, m_2 > 0. \tag{2.3.9}$$

Furthermore, for product 1, the contribution-to-profit rate under regular sales is greater than the contribution to the profit per period after the new product 2 is available, i.e.,

$$m_1 \geq m_1'. \tag{2.3.10}$$

In order to avoid cases for which it would be optimal to delay infinitely the new product launch, we assume

$$m_2 \geq m_1'. \tag{2.3.11}$$

Finally, as for any classical inventory problem, we assume,

$$g, c_{h,2}, s_1 > 0. \tag{2.3.12}$$

2.4 Formulation of the stochastic product rollover problem

In absence of risk, the classical optimization problem considered here consists of finding the phase-in and phase-out dates which minimize the expected loss. This formulation will be developed in the first subsection. However, the main objective of this paper is also to characterize the rollover decision making, under risk, and provide explicit closed-form expressions for the optimal policies. This formulation is given in the second subsection.

2.4.1 Minimization of the expected loss function problem

The associated optimization problem is

$$\min_{(t_1,t_2)\in \mathbb{R}^+\times\mathbb{R}^+} \qquad l(t_1,t_2) = E_F[L(t_1,t_2,T)], \tag{2.4.1}$$

where $E_F[.]$ is the expectation operator w.r.t. the probability distribution $F(\cdot)$ for the random approval date T. Due to the structure of the cost function given in (2.3.5), we introduce the following auxiliary subproblems, for $i = 1,2$,

$$\min_{(t_1,t_2)\in \mathbf{R_i}} \qquad l_i(t_1,t_2) = E_F[L_i(t_1,t_2,T)], \tag{2.4.2}$$

these function given as

$$\begin{aligned} l_1(t_1,t_2) &= (m_1+g)E_F[T-t_1]^+ - (g+m_1')E_F[t_1-T]^+ \\ &+ c_{h,2}E_F[T-t_2]^+ + (m_2+g)E_F[t_2-T]^+, \end{aligned} \tag{2.4.3}$$

$$\begin{aligned} l_2(t_1,t_2) &= (m_2-m_1'-s_1)E_F[t_2-T]^+ + c_{h,2}E_F[T-t_2]^+ \\ &+ (m_1+g)E_F[T-t_1]^+ + s_1E_F[t_1-T]^+, \end{aligned} \tag{2.4.4}$$

and the boundary problem,

$$\min_{t\in\mathbb{R}^+} \quad l_b(t) = E_F[L_b(t,T)], \tag{2.4.5}$$

with

$$l_b(t) = (m_2-m_1')E_F[t-T]^+ + (m_1+g+c_{h,2})E_F[T-t]^+. \tag{2.4.6}$$

Structural properties of this problem

Solving problem (2.4.1) is not straightforward: indeed, it can be seen that for some parameter values the objective function is not convex over the definition set $\mathbb{R}^+ \times \mathbb{R}^+$. However, we show here that the objective function of each subproblem (2.4.2) is unimodal (or convex) and differentiable when defined over $\mathbb{R}^+ \times \mathbb{R}^+$ (or over $\mathbb{R}^+$ for the boundary function $l_b(\cdot)$). We show how these properties can be used to develop optimality conditions for the solution of the initial problem (2.4.1). The following properties characterize these unimodality/convexity properties and associated optimality conditions.

PROPERTY 1: Under assumption (2.3.10), the loss functions $L_1(\cdot,\cdot,T)$ and $l_1(\cdot,\cdot)$ are strictly jointly convex over $\mathbf{R}^+ \times \mathbf{R}^+$.

Proof. See Appendix A-1.

PROPERTY 2: *If*

$$m_1' < -g \quad \textit{and} \quad (m_2+g)(m_1+g) < -c_{h,2}(g+m_1'), \tag{2.4.7}$$

then the minimum of the loss function $l_1(t_1, t_2)$ over region $\mathbf{R_1}$ is in the interior of region $\mathbf{R_1}$ and the optimal solution of problem (2.4.1) is given by

$$t_1^* = F^{-1}\left(\frac{m_1 + g}{m_1 - m_1'}\right), \quad t_2^* = F^{-1}\left(\frac{c_{h,2}}{m_2 + c_{h,2} + g}\right). \tag{2.4.8}$$

Otherwise, the minimum of the loss function $l_1(t_1, t_2)$ over region $\mathbf{R_1}$ is on the boundary of $\mathbf{R_1}$ and the optimal solution of problem (2.4.1) is given by

$$t_1^* = t_2^* = F^{-1}\left(\frac{m_1 + c_{h,2} + g}{m_2 - m_1' + m_1 + c_{h,2} + g}\right). \tag{2.4.9}$$

The solutions given in Property 2 are unique since $l_1(t_1, t_2)$ is strictly jointly convex over $\mathbf{R}^+ \times \mathbf{R}^+$, as given in Property 1.

2.4.2 Dual Product Rollover

For an expected value minimization objective and dual rollover strategy, the associated optimization problem is

$$l_2(t_1, t_2) = \min_{t_1, t_2 \in \mathbf{R}^+} \left\{ E[L_2(t_1, t_2, T)] \right\}, \tag{2.4.10}$$

$$\text{s.t. } t_1 \geq t_2. \tag{2.4.11}$$

To solve the problem given in (2.4.10), we derive the following properties:

PROPERTY 3: *The loss function $L_2(\cdot, \cdot, T)$ is strictly jointly convex over $\mathbf{R}^+ \times \mathbf{R}^+$ under the assumption $m_2 - m_1' - s_1 + c_{h,2} > 0$.*

COROLLARY 3.1: *The loss function $l_2(t_1, t_2)$ is strictly jointly convex over $\mathbf{R}^+ \times \mathbf{R}^+$ under the assumption $m_2 - m_1' - s_1 + c_{h,2} > 0$.*

PROPERTY 4: *If*

$$m_2 - m_1' - s_1 + c_{h,2} > 0 \text{ and } \quad s_1(m_1 + g + c_{h,2}) < (m_1 + g)(m_2 - m_1'), \tag{2.4.12}$$

then the minimum of the loss function $l_2(t_1, t_2)$ over region $\mathbf{R_2}$ is in the interior of the region $\mathbf{R_2}$ and the optimal solution of problem (2.4.10) is given by

$$t_1^* = F^{-1}\left(\frac{m_1 + g}{m_1 + g + s_1}\right), t_2^* = F^{-1}\left(\frac{c_{h,2}}{m_2 - m_1' + c_{h,2} - s_1}\right). \tag{2.4.13}$$

Otherwise, the minimum of the loss function $l_2(t_1, t_2)$ over region $\mathbf{R_2}$ is on the boundary of $\mathbf{R_2}$ and the optimal solution of problem (2.4.10) is given by

$$t_1^* = t_2^* = F^{-1}\left(\frac{m_1 + c_{h,2} + g}{m_2 - m_1' + m_1 + c_{h,2} + g}\right). \tag{2.4.14}$$

The solutions given in Property 4 are unique since $l_2(t_1, t_2)$ is strictly jointly convex over $\mathbf{R}^+ \times \mathbf{R}^+$, as given in Corollary 3.1.

2.4.3 CVaR Reformulation of the Optimal rollover problem

In order to introduce the impact of risk aversion in the decision process, we consider our problem in a CVaR-minimization context (see [30, 31]). In this section, we exhibit the CVaR reformulation of the rollover optimization problem, we give the corresponding analytical expression for the optimal solutions, and we analyze the impact of risk-aversion on the selected rollover policy.

Conditional Value at Risk formulation

For a given probability distribution $F(\cdot)$ associated with the random approval date T, let us denote the probability distribution function of the loss function $L(t_1, t_2, T)$ by

$$\mathcal{L}_F(\eta | t_1, t_2) = Pr\{L(t_1, t_2, T) \leq \eta\}. \tag{2.4.15}$$

For $\beta \in [0, 1)$, we define the β-VaR of this distribution by

$$\alpha_\beta(t_1, t_2) = min\{\alpha | \mathcal{L}_F(\alpha | t_1, t_2) \geq \beta\}. \tag{2.4.16}$$

It is now possible to introduce the β-tail distribution function to focus on the upper tail part of the loss distribution as

$$\mathcal{L}_{F,\beta}(\eta | t_1, t_2) = \begin{cases} 0 & \text{for} \quad \eta < \alpha_\beta(t_1, t_2), \\ \frac{\mathcal{L}_\beta(\eta | t_1, t_2) - \beta}{1 - \beta} & \text{for} \quad \eta \geq \alpha_\beta(t_1, t_2). \end{cases} \tag{2.4.17}$$

Using the expectation operator $E_\beta[\cdot]$ under the β-tail distribution $\mathcal{L}_{F,\beta}(\cdot|\cdot,\cdot)$, we define the β-conditional value-at-risk of the loss $L(t_1, t_2, T)$ by

$$\tilde{l}_\beta(t_1, t_2) = E_\beta[L(t_1, t_2, T)]. \tag{2.4.18}$$

Finding the optimal rollover structure and the corresponding values for the phase-in and phase-out dates, which minimize the CVaR cost criterion amounts to the optimization problem

$$\min_{(t_1,t_2)\in \mathbb{R}^+ \times \mathbb{R}^+} \tilde{l}_\beta(t_1, t_2) \quad = \quad \Big\{ E_\beta[L(t_1, t_2, T)] \Big\}. \tag{2.4.19}$$

According to [30, 31], it is known that the minimization of $E_\beta[L(t_1, t_2, T)]$ with respect to the decision variables t_1 and t_2, amounts to the minimization of the auxiliary function

$$l_\beta(t_1, t_2, \alpha) := \alpha + \frac{1}{1-\beta} E_F\left[[L(t_1, t_2, T) - \alpha]^+\right]. \tag{2.4.20}$$

It is known that $l_\beta(t_1, t_2, \alpha)$ is convex with respect to α (see [30, 31]). According to the specific structure of the loss function (2.3.3)-(2.3.4), it is natural to associate to (2.4.20) a pair of auxiliary functions

$$l_{\beta,i}(t_1, t_2, \alpha) = \quad \Big\{ \alpha + \frac{1}{1-\beta} E_F[L_i(t_1, t_2, T) - \alpha]^+ \Big\}, \tag{2.4.21}$$

and an auxiliary function on the boundary,

$$l_{\beta,b}(t_1, t_2, \alpha) \quad = \quad \Big\{ \alpha + \frac{1}{1-\beta} E_F[L_b(t, T) - \alpha]^+ \Big\}. \tag{2.4.22}$$

Structural properties of the CVaR Problem

The optimal solution structure is essentially determined by concavity/convexity characteristics of these functions (2.4.21)-(2.4.22) in the regions $\mathbf{R_1}$ and $\mathbf{R_2}$.

PROPERTY 5: The CVaR loss functions $l_{\beta,1}(\cdot,\cdot,\cdot)$ and $l_{\beta,2}(\cdot,\cdot,\cdot)$ are differentiable over $\mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+$. The CVaR loss function $l_{\beta,b}(\cdot,\cdot)$ is differentiable over $\mathbf{R}^+ \times \mathbf{R}^+$.

(see Appendix D).

PROPERTY 6: Under assumption (2.3.10), the CVaR loss function $l_{\beta,1}(\cdot,\cdot,\cdot)$ is strictly jointly convex on $\mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+$.

PROPERTY 7 : Under the assumption $m_2 \geq m_1 \geq m_1'$, for $0 \leq \beta < 1$ values satisfying

$$m_1' < \beta m_1 - g(1-\beta), \text{ and} \tag{2.4.23}$$

$$m_1' < -g \tag{2.4.24}$$

the CVaR-loss function $l_{\beta,1}(\cdot,\cdot,\cdot)$ has a unique finite minimum over $\mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+$ corresponding to

$$t^{*,r}_{\beta,1,1} = F^{-1}\left(\frac{(m_1+g)(1-\beta)}{m_1-m_1'}\right), \tag{2.4.25}$$

$$t^{*,r}_{\beta,2,1} = \left(\frac{m_1+c_{h,2}+g}{m_2+c_{h,2}+g}\right)F^{-1}\left(\frac{c_{h,2}+\beta(m_2+g)}{m_2+c_{h,2}+g}\right) + \left(\frac{m_2-m_1}{m_2+c_{h,2}+g}\right)F^{-1}\left(\frac{c_{h,2}(1-\beta)}{c_{h,2}+m_2+g}\right), \tag{2.4.26}$$

Proof. (see Appendix B-1).

PROPERTY 8 : Under the assumptions $m_1 > m_2 \geq m_1'$, $m_1' < -g$, and $m_2 - m_1 + c_{h,2} > 0$ for $0 \leq \beta < 1$ values satisfying

$$m_1' < \beta m_1 - g(1-\beta) \text{ and} \tag{2.4.27}$$

$$m_2 < m_1 - c_{h,2}(1-\beta), \tag{2.4.28}$$

the CVaR-loss function $l_{\beta,1}(\cdot,\cdot,\cdot)$ has a unique finite minimum over $\mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+$ corresponding to,

$$t^{*,r}_{\beta,1,1} = \left(\frac{m_2-m_1'}{m_1-m_1'}\right)F^{-1}\left(\frac{(m_1+g)(1-\beta)}{m_1-m_1'}\right) + \left(\frac{m_1-m_2}{m_1-m_1'}\right)F^{-1}\left(\frac{m_1-\beta m_1'+g(1-\beta)}{m_1-m_1'}\right), \tag{2.4.29}$$

$$t^{*,r}_{\beta,2,1} = F^{-1}\left(\frac{m_1+g+c_{h,2}\beta}{m_2+g+c_{h,2}}\right). \tag{2.4.30}$$

Proof.(see Appendix B-2).

PROPERTY 9.: Under the assumption $m_2 - m_1' - s_1 + c_{h,2} > 0$, the CVaR-loss function $l_{\beta,2}(\cdot,\cdot,\cdot)$ is strictly jointly convex over $\mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+$.

Proof. It is a known result that if $L_2(\cdot,\cdot,T)$ is convex for any fixed value T, then the CVaR minimization leads to a convex problem (see [30, 31]). Convexity of $L_2(\cdot,\cdot,T)$ over $\mathbf{R}^+ \times \mathbf{R}^+$ was previously proved in Property 3 under assumption $m_2 - m_1' - s_1 > -c_{h,2}$.

Property 10: Under the assumptions $c_{h,2} \geq s_1$ and $m_2 - m_1' - s_1 > 0$, the CVaR-loss function $l_{\beta,2}(\cdot,\cdot,\cdot)$ has a unique minimum over $\mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+$ corresponding

$$t^{*,r}_{\beta,1,2} = F^{-1}\left(\frac{m_1 + g + s_1\beta}{m_1 + g + s_1}\right), \tag{2.4.31}$$

$$\begin{aligned} t^{*,r}_{\beta,2,2} &= \left(\frac{m_2 - m_1'}{m_2 - m_1' - s_1 + c_{h,2}}\right) F^{-1}\left(\frac{c_{h,2}(1-\beta)}{m_2 - m_1' - s_1 + c_{h,2}}\right) \\ &+ \left(\frac{c_{h,2} - s_1}{m_2 - m_1' - s_1 + c_{h,2}}\right) F^{-1}\left(\frac{c_{h,2} + \beta(m_2 - m_1' - s_1)}{m_2 - m_1' - s_1 + c_{h,2}}\right). \end{aligned} \tag{2.4.32}$$

Proof. (see Appendix C-1).

PROPERTY 11: Under the assumptions $c_{h,2} < s_1$ and $m_2 - m_1' - s_1 + c_{h,2} > 0$ for $0 \leq \beta < 1$ values the CVaR-loss function $l_{\beta,2}(\cdot,\cdot,\cdot)$ has a unique finite minimum over $\mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+$ corresponding to

$$\begin{aligned} t^{*,r}_{\beta,1,2} &= \left(\frac{m_1 + g + c_{h,2}}{m_1 + g + s_1}\right) F^{-1}\left(\frac{m_1 + g + s_1\beta}{m_1 + g + s_1}\right) \\ &+ \left(\frac{s_1 - c_{h,2}}{m_1 + g + s_1}\right) F^{-1}\left(\frac{(m_1 + g)(1-\beta)}{m_1 + g + s_1}\right), \end{aligned} \tag{2.4.33}$$

$$t^{*,r}_{\beta,2,2} = F^{-1}\left(\frac{c_{h,2}(1-\beta)}{m_2 - m_1' - s_1 + c_{h,2}}\right) \tag{2.4.34}$$

Proof. The result is direct via first order conditions (see Appendix C-2).

PROPERTY 12: Under assumption (2.3.11), the boundary loss function $l_b(\cdot)$ is strictly convex $\mathbf{R}^+$ and the minimum is given by

$$\begin{aligned} t_b^* &= \left(\frac{m_2 - m_1'}{m_2 - m_1' + m_1 + c_{h,2} + g}\right) F^{-1}\left(\frac{(m_1 + c_{h,2} + g)(1-\beta)}{m_2 - m_1' + m_1 + c_{h,2} + g}\right) \\ &+ \left(\frac{m_1 + c_{h,2} + g}{m_2 - m_1' + m_1 + c_{h,2} + g}\right) F^{-1}\left(\frac{m_1 + c_{h,2} + g + \beta(m_2 - m_1')}{m_2 - m_1' + m_1 + c_{h,2} + g}\right), \end{aligned} \tag{2.4.35}$$

Proof. The result is direct (see Appendix B-2).

Clearly, the structure of the optimal policy depends on the cost parameters and values, and we observe three types of policies: *planned rollover stock-outs*, *single rollover*, and *dual rollover*. The optimal policy structure is displayed in the following table:

	$l_{\beta,1}(t_1,t_2)$ properties: $(t^{r,*}_{\beta,1,1}, t^{r,*}_{\beta,2,1}) \in R_1$	
$l_{\beta,2}(t_1,t_2)$ properties	Case 1 Strictly decreasing w.r.t. t_2 or convex $(t^{r,*}_{\beta,1,2}, t^{r,*}_{\beta,2,2}) \in R_1$	Case 2 Convex $(t^{r,*}_{\beta,1,2}, t^{r,*}_{\beta,2,2}) \in R_2$
Global Optimal Solution Optimal Policy Structure	$(t^{r,*}_{\beta,1,1}, t^{r,*}_{\beta,2,1})$ ↓ *Planned Stockout*	? ↓ ?
	$l_{\beta,1}(t_1,t_2)$ properties: $(t^{r,*}_{\beta,1,1}, t^{r,*}_{\beta,2,1}) \in R_2$	
$l_{\beta,2}(t_1,t_2)$ properties	Case 3 Strictly decreasing w.r.t. t_2 or convex $(t^{r,*}_{\beta,1,2}, t^{r,*}_{\beta,2,2}) \in R_1$	Case 4 Convex $(t^{r,*}_{\beta,1,2}, t^{r,*}_{\beta,2,2}) \in R_2$
Global Optimal Solution Optimal Policy Structure	On the boundary $t_1 = t_2 : t^{*}_{\beta,b}$ ↓ *Single Product Rollover*	$(t^{r,*}_{\beta,1,2}, t^{r,*}_{\beta,2,2})$ ↓ *Dual Product Rollover*

Table 2.1: Convexity properties and structure of the optimal rollover policy

Analysis of results: Impact of Risk Perception on optimal product rollover policies

The optimal policy structure simultaneously depends on the different parameters of the problem, on the probability distribution $F(\cdot)$ and on the risk aversion defined through β. While it is tedious to find explicit necessary and sufficient optimality conditions for each type of rollover policy w.r.t. these different factors, the specific impact of risk aversion over the optimal policy structure can be analyzed. As discussed previously, β reflects the degree of risk aversion for the planner (the larger β is, the more risk averse the planner is). By using above properties of the CVaR-loss functions, the following tables can be developed for two significantly different situations : low risk aversion (i.e., β value near zero) and high risk aversion (i.e., β value near 1).

		$m_2 \ge m_1 \ge m_1'$		$m_1 \ge m_2 \ge m_1'$	
		$\frac{m_1+g}{m_1-m_1'} \le \frac{c_{h,2}}{m_2+g+c_{h,2}}$ $m_1' < -g$	otherwise	$m_1 < m_2 + c_{h,2}$ $m_1' < -g$ $m_1 - m_1' > m_2 + g + c_{h,2}$	otherwise
$c_{h,2} \ge s_1$	$m_2 - m_1' - s_1 > 0$ $\frac{m_1+g}{m_1+g+s_1} > \frac{c_{h,2}}{m_2-m_1'-s_1+c_{h,2}}$	$l_{\beta,1}$: PSO $l_{\beta,2}$: DR $\rightarrow$ ***PSO or DR***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: DR $\rightarrow$ ***DR***	$l_{\beta,1}$: PSO $l_{\beta,2}$: DR $\rightarrow$ ***PSO or DR***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: DR $\rightarrow$ ***DR***
	otherwise	$l_{\beta,1}$: PSO $l_{\beta,2}$: not DR $\rightarrow$ ***PSO***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: not DR $\rightarrow$ ***SR***	$l_{\beta,1}$: PSO $l_{\beta,2}$: not DR $\rightarrow$ ***PSO***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: not DR $\rightarrow$ ***SDR***
$c_{h,2} < s_1$	$m_2 - m_1' - s_1 > 0$ $\frac{m_1+g}{m_1+g+s_1} \le \frac{c_{h,2}}{m_2-m_1'-s_1+c_{h,2}}$	$l_{\beta,1}$: PSO $l_{\beta,2}$: DR $\rightarrow$ ***PSO or DR***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: DR $\rightarrow$ ***DR***	$l_{\beta,1}$: PSO $l_{\beta,2}$: DR $\rightarrow$ ***PSO or DR***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: DR $\rightarrow$ ***DR***
	otherwise	$l_{\beta,1}$: PSO $l_{\beta,2}$: not DR $\rightarrow$ ***PSO***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: not DR $\rightarrow$ ***SR***	$l_{\beta,1}$: PSO $l_{\beta,2}$: not DR $\rightarrow$ ***PSO***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: not DR $\rightarrow$ ***SR***

Table 2.2: Optimal Rollover Policy under low risk averse assumption

	$m_2 \geq m_1 \geq m'_1$	$m_1 \geq m_2 \geq m'_1$			
		$\beta > 1 - \frac{m_1 - m_2}{c_{h,2}}$	$\beta \leq 1 - \frac{m_1 - m_2}{c_{h,2}}$	Otherwise $\beta \leq 1 - \frac{m_1 - m_2}{c_{h,2}}$	$\beta \leq 1 - \frac{m_1 - m_2}{c_{h,2}}$
			$m_1 < m_2 + c_{h,2}$ $m'_1 < -g$ $m_1 - m'_1 > m_2 + g + c_{h,2}$ $-\frac{m'_1 + g}{m_1 - m'_1} < \frac{c_{h,2}}{m_2 + g + c_{h,2}}$	$m_1 \geq m_2 + c_{h,2}$, or $m_1 \geq m_2 + c_{h,2}$, or $m_1 - m'_1 \leq m_2 + g + c_{h,2}$	
$c_{h,2} \geq s_1$; $m_2 - m'_1 - s_1 < 0$	$l_{\beta,1}$: PSO $l_{\beta,2}$: not DR → ***PSO***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: not DR → ***SR***	$l_{\beta,1}$: PSO $l_{\beta,2}$: not DR → ***PSO***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: not DR → ***SR***	$l_{\beta,1}$: PSO(?) $l_{\beta,2}$: not DR → ***PSO or SR***
$c_{h,2} \geq s_1$; $m_2 - m'_1 - s_1 \geq 0$ $\frac{m_1 + g}{m_1 + g + s_1} > \frac{c_{h,2}}{m_2 - m'_1 - s_1 + c_{h,2}}$	$l_{\beta,1}$: PSO $l_{\beta,2}$: DR → ***PSO or DR***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: DR → ***DR***	$l_{\beta,1}$:PSO $l_{\beta,2}$: DR → ***PSO or DR***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: DR → ***DR***	$l_{\beta,1}$: PSO(?) $l_{\beta,2}$: DR → ***PSO or DR***
$c_{h,2} \geq s_1$; $m_2 - m'_1 - s_1 \geq 0$ $\frac{m_1 + g}{m_1 + g + s_1} \leq \frac{c_{h,2}}{m_2 - m'_1 - s_1 + c_{h,2}}$	$l_{\beta,1}$: PSO $l_{\beta,2}$: DR(?) → ***PSO or DR***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: $l_{\beta,2}$: DR(?) → ***SR or DR***	$l_{\beta,1}$: PSO $l_{\beta,2}$: DR(?) → ***PSO or DR***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: DR(?) → ***SR or DR***	$l_{\beta,1}$: PSO(?) $l_{\beta,2}$: DR(?) → ***PSO or DR***
$c_{h,2} < s_1$; $m_2 - m'_1 - s_1 + c_{h,2} \leq 0$	$l_{\beta,1}$: PSO $l_{\beta,2}$: not DR → ***PSO***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: not DR → ***SR***	$l_{\beta,1}$: PSO $l_{\beta,2}$: not DR → ***PSO***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: not DR → ***SR***	$l_{\beta,1}$: PSO(?) $l_{\beta,1}$: PSO(?) → ***PSO or SR***
$c_{h,2} < s_1$; $m_2 - m'_1 - s_1 + c_{h,2} > 0$	$l_{\beta,1}$: PSO $l_{\beta,2}$: DR → ***PSO or DR***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: DR → ***DR***	$l_{\beta,1}$: PSO $l_{\beta,2}$: DR → ***PSO or DR***	$l_{\beta,1}$: not PSO $l_{\beta,2}$: DR → ***DR***	$l_{\beta,1}$: PSO(?) $l_{\beta,2}$: DR → ***PSO or DR***

Table 2.3: Optimal Rollover Policy under high risk averse assumption

It is worth noting that the optimal policy structure is highly dependent on the decision maker risk aversion. A main paper about rollover efficient strategies (Billington et al (1998)) presents the SR rollover strategy as a high risk strategy, suited to situations with low uncertainty and the DR rollover strategy as a low risk strategy, suited to situations with a higher uncertainty. Our theoretical analysis complements (Billington et al (1998)) and rigorously show how each strategy (PSO, SR and DR) can be optimally associated to the risk aversion and uncertainty level. In particular, it can be seen that risk aversion and uncertainty level have a completely different impact on the structure of the optimal policy. Increasing uncertainty level reinforces the rollover policy type (i.e., increase the overlap (positive or negative)), while the decision maker risk aversion can change the optimal policy structure.

2.5 Impact of Uncertainty

In this section, we study the variation of the optimal solution structure, and the associated optimal cost, when increasing stochasticity of the random approval date T. The global motivation of this section consists of theoretically analyzing a conjecture by Billington et al. (1998) claiming that when the variability of the new product approval date increases, then basically the overlap, i.e., the positive/negative gap between t_1 and t_2, in the optimal solution has to increase too. This property is theoretically known as a dispersive ordering property.

2.5.1 A motivating example.

In order to illustrate this mecanism from a heuristic and intuitive point of view, i.e., how significant larger variance (with equal means) nearly induces increasing values for $t_1 - t_2$, we consider a numerical example. If we consider a long administrative agreement procedure involving several technical quality control procedures, in practice the distribution of T can be expected to be a complex combination of a (possibly random) number of general random variable. As a typical illustration, we assume the regulatory approval date to be the sum of a random number of i.i.d. Gaussian variables, i.e., one has $T \sim \sum_{k=1}^{N} T_k$, with T_k i.i.d. Gaussian random variables, with mean μ and standard-deviation σ, and N randomly distributed as a geometric random variable with parameter p. The numerical illustration proceeds as follows. We have considered a numerical example with nominal values $\mu = 319$, $\sigma = 137$ and $p = 0.5$, corresponding to the nominal probability distribution for T, denoted as $F(\cdot)$. Then, we have considered a sequence of alternative distributions $\widetilde{F}_i(\cdot)$ with increasing standard-deviations (but equal means). Recall that the optimal decisions of the considered rollover problem are defined as quantiles of the probability distribution characterizing the regulatory date. We are interested in the evolution of the overlap when problem variability increases.

We thus numerically estimate for which fraction of pairs (a, b) (with $0 < a < b < 1$) the following property holds,

$$(F^{-1}(a) - F^{-1}(b)) - (\widetilde{F}_i^{-1}(a) - \widetilde{F}_i^{-1}(b)) \leq 0, \text{ whenever } 0 < a < b < 1. \tag{2.5.1}$$

Figure 2.4 displays the differences $(F^{-1}(a) - F^{-1}(b)) - (\widetilde{F}_i^{-1}(a) - \widetilde{F}_i^{-1}(b))$, for every pair (a, b) (with $0 < a < b < 1$).

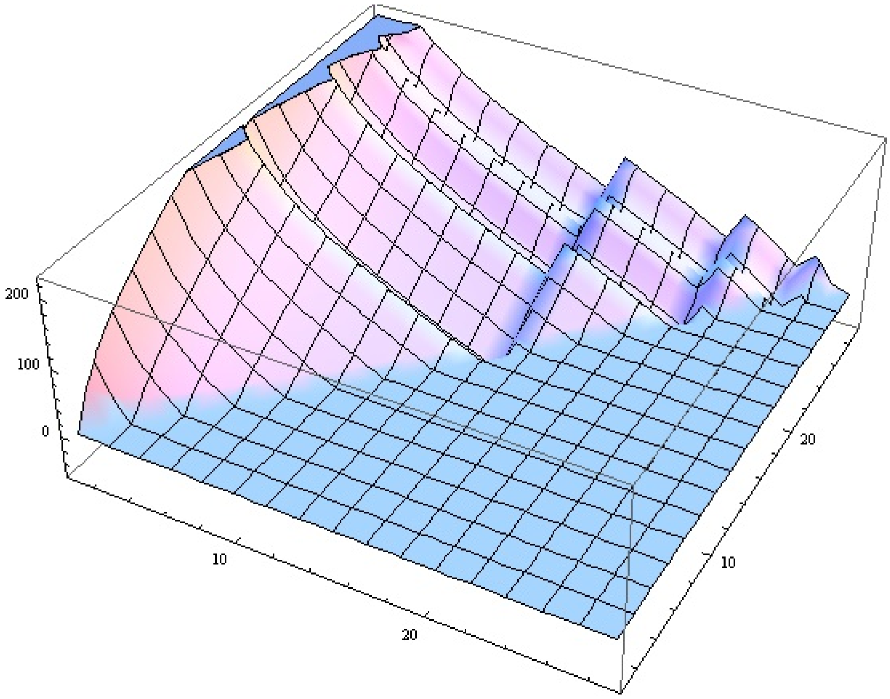

Figure 2.4: Numerical simulations of the differences (2.5.1)

We try to increase the variance until we nearly double it and we give our results in Table 2.4 where we notice that it is enough to increase the variance by 50% to have around 95% of the cases where the gap increases between the rollover dates with increased variance.

Here we formally give conditions guaranteeing this conjecture. It can be seen that the impact of variability on the approval date is threefold : impact on the optimal global cost, impact on the optimal value of each of the two decision variables t_1^* and t_2^*, and impact on the structure of the optimal policy (basically on the size of the overlap, i.e., the difference between the two optimal decision variables, if any). The key element in

Variance Increase %	Valid Cases %
10	90
20	91
30	92
40	93
50	95
60	96
70	98
80	99
90	99
99	99

Table 2.4: Numerical simulations : synthesis

such an analysis is the definition of variability or stochasticity increase between a pair of probability distribution functions.

In order to assess the variability effects on the considered model, we conduct a stochastic comparison between two rollover processes. We consider two rollover processes $i = 1, 2$, with approval dates T_i, known through their cumulative probability distribution functions F_i. We focus here on the variability effects of T_i and thus we assume that the approval dates have equal means, $E[T_1] = E[T_2]$. In order to compare the variabilities of the pair of random variables T_1 and T_2, we will have to define criteria, known as stochastic ordering criteria.

First, we focus on the change, when the problem variability increases, of the optimal solution values (increase or decrease) and on the change of the optimal cost (increase or decrease). This change can be theoretically characterized along the lines of ([37, 40]).

In order to define the concept of variability increase, we consider a stochastic ordering based on a comparison of the spread of the probability density functions.

Second, we focus on the change of the optimal strategy structure, namely the change of the overlap size associated to the optimal policies. We recall that in case of a positive overlap, the pair of products are simultaneously available for the market during some time period, while in case of a negative overlap, no product is available for the market over some time horizon. To do so, we need to use a more restrictive stochastic ordering assumption, known as dispersive ordering condition [20, 21, 36, 40].

2.5.2 Impact of Uncertainty on the cost and on the optimal decisions

The Considered Stochastic Ordering

We consider in this first part the usual stochastic ordering, based on the shapes of the density functions (or the distribution functions), and defined as follows. Let $u(t)$ be a real function defined on an ordered set U of the real line and let $S(u)$ be the number of sign changes of $u(t)$ when t ranges over the entire set U.

Definition. Consider two random variables T_1 and T_2 with same mean, i.e., $E[T_1] = E[T_2]$, having probability distributions $F_1(\cdot)$ and $F_2(\cdot)$ with densities $f_1(\cdot)$ and $f_2(\cdot)$. We say T_1 is more variable than T_2, denoted $T_1 \geq_{var} T_2$, if

$$S(f_1 - f_2) = 2 \text{ with sign sequence +, -, +.} \tag{2.5.2}$$

That is, $f_1(\cdot)$ crosses $f_2(\cdot)$ exactly twice, first from above and then from below. It is known (see [40]), that when $E[T_1] = E[T_2]$, condition (2.5.2) implies that

$$F_1(x) \leq F_2(x) \quad \text{for all } x \quad \text{and} \quad E[h(T_1)] \geq E[h(T_2)] \tag{2.5.3}$$

for all nondecreasing functions $h(\cdot)$. Observe that condition (2.5.2) also implies that

$$S(F_1 - F_2) = 1 \tag{2.5.4}$$

with sign sequence $+,-,+$, in other words, $F_1(\cdot)$ crosses $F_2(\cdot)$ exactly once, and the crossing is from above. Furthermore, it is also known (see [40]) that equation (2.5.4) implies

$$\int_{-\infty}^{t} (F_1(x) - F_2(x))dx \leq 0. \tag{2.5.5}$$

Examples of pairs of distributions satisfying condition (2.5.4) are given in [37] and include a large number of important standard unimodal densities arising in statistical applications, as seen from the following pairs $(i = 1, 2)$:

- $f_i(\cdot)$ are Gamma (Weibull) with shape parameter η_1, η_2, with $\eta_2 < \eta_1$;
- $f_i(\cdot)$ are Uniform (a_i, b_i), with $a_1 < a_2$, $b_1 > b_2$, but $a_1 + b_1 = a_2 + b_2$;
- $F_i(\cdot)$ are Gaussian with parameters μ_i and σ_i, with $\mu_1 = \mu_2$ and $\sigma_2 < \sigma_1$;
- $f_i(\cdot)$ are truncated Gaussian with parameters μ_i and σ_i, with $\mu_1 = \mu_2 \gg 0$ and $\sigma_2 < \sigma_1$;
- $f_1(\cdot)$ is decreasing (e.g., exponential) and $f_2(\cdot)$ is Uniform.

Impact of variability on the decision variables

We now present our results regarding the effect of approval date variability on the optimal times.

Property 13. If $T_1 \geq_{var} T_2$, then there exists a critical number θ_{F_1,F_2} such that

$$\begin{cases} F_1^{-1}(r) \leq F_2^{-1}(r) & \text{if} \quad 0 \leq r \leq \theta_{F_1,F_2}, \\ F_1^{-1}(r) \geq F_2^{-1}(r) & \text{if} \quad \theta_{F_1,F_2} \leq r \leq 1. \end{cases}$$

Proof : the proof follows [37]. Condition $T_1 \geq_{var} T_2$ implies that $F_1(\cdot)$ crosses $F_2(\cdot)$ exactly once for $x = x^*$ (i.e., one has $F_1(x^*) = F_2(x^*)$), and the crossing is from above. That means, there exists x^* such that for $0 < x < x^*$, $F_1(x)$ is at least as large as $F_2(x)$ and for $x < x*$, $F_1(x)$ is at most as large as $F_2(x)$. Setting $\theta_{F_1,F_2} = F_1(x*) = F_2(x*)$, the results regarding the order of $F_i^{-1}(r)$ are immediate.

A direct application of above proposition is the following corollary.

Corollary 13.1 Let us formally denote the distribution dependence of the optimal solutions as $t^{r,*}_{i,j}(F)$ and $t^{b,*}(F)$, with

$$t^{r,*}_{i,j}(F) = F^{-1}(r_{i,j}) \text{ and } t^{b,*}(F) = F^{-1}(r_b) \qquad (i,j = 1,2) \tag{2.5.6}$$

with

$$r_{1,1} = \frac{m_1 + g}{m_1 - m_1'},\ r_{2,1} = \frac{c_{h,2}}{m_2 + c_{h,2} + g}, \tag{2.5.7}$$

$$r_{1,2} = \frac{m_1 + g}{m_1 + g + s_1},\ r_{2,2} = \frac{c_{h,2}}{m_2 - m_1' + c_{h,2} - s_1}, \tag{2.5.8}$$

and

$$r_b = \frac{m_1 + c_{h,2} + g}{m_2 - m_1' + m_1 + c_{h,2} + g}. \tag{2.5.9}$$

Then, if $T_1 \geq_{var} T_2$, then there exists a critical number θ_{F_1,F_2} such that

$$\begin{cases} t^{*,r}_{i,j}(F_1) > t^{*,r}_{i,j}(F_2) & \text{if} \quad F_1^{-1}(r_{i,j}) > \theta_{F_1,F_2}, \\ t^{*,r}_{i,j}(F_1) \leq t^{*,r}_{i,j}(F_2) & \text{if} \quad F_1^{-1}(r_{i,j}) \leq \theta_{F_1,F_2}. \end{cases}$$

This shows that for increasingly variable distributions, the sign of the change of the optimal solutions (i.e., decreasing or increasing with variability) is not straightforward and depends on the order relationship between the threshold θ_{F_1,F_2} and the different ratios (2.5.7)-(2.5.9) defining the optimal solution values.

Impact of variability on the average loss

The following proposition establishes the intuitive result that increasing variability increases the expected loss.

Property 14. If $T_1 \geq_{var} T_2$, then

$$\min_{(t_1,t_2)\in \mathbb{R}^+ \times \mathbb{R}^+} E_{F_1}[L(t_1,t_2,T)] \leq \min_{(t_1,t_2)\in \mathbb{R}^+ \times \mathbb{R}^+} E_{F_2}[L(t_1,t_2,T)]. \tag{2.5.10}$$

Proof. See Appendix E.

2.5.3 Impact of Uncertainty on structure of the optimal rollover policy

Stochastic Ordering Definitions

This subsection analyzes the impact of uncertainty on the structure of the optimal rollover policy, e.g. on the size of the overlap between the planning of the new and the old product. The analysis, focused on the difference between t_1^* and t_2^*, and not on their individual values, relies on another class of stochastic ordering, called dispersive ordering, as defined below.

Definition. Consider two random variables T_1 and T_2 with same mean $E[T_1] = E[T_2]$, having distributions $F_1(\cdot)$ and $F_2(\cdot)$ with densities $f_1(\cdot)$ and $f_2(\cdot)$. T_1 is said to be less dispersed than T_2, denoted by $T_1 <_{disp} T_2$, if

$$F_2^{-1}(a) - F_2^{-1}(b) < F_1^{-1}(a) - F_1^{-1}(b), \text{ whenever } 0 < a < b < 1. \tag{2.5.11}$$

This means that the difference between any pair of quantiles of $F_2(\cdot)$ is smaller than the difference between the corresponding quantiles of $F_1(\cdot)$. It is well known that this condition is more restrictive than (2.5.2) or (2.5.4). Examples include a large number of important standard unimodal densities (see [20]) as pairs of Gamma densities, Uniforms, Gaussians, truncated Gaussians, and others.

Impact of variability on the overlap of the optimal rollover structure

Property 15. If $T_1 >_{disp} T_2$,

then if the optimal policy is dual rollover, the overlap increases, i.e., one has

$$t_{1,F_1}^* - t_{2,F_1}^* > t_{1,F_2}^* - t_{2,F_2}^*, \tag{2.5.12}$$

if the policy is planned stockout, the stockout period increases, i.e.,

$$t_{2,F_1}^* - t_{1,F_1}^* > t_{2,F_2}^* - t_{1,F_2}^*. \tag{2.5.13}$$

Proof This is a direct application of the stochastic ordering to the optimality conditions.

This proposition establishes the general conditions guaranteeing that when the regulatory date process is more random (in some sense), then the optimal policies are reinforced: in case of planned stockout, the stockout period is increased, and in case of dual rollover, the "dual product pipe-line inventory period" is increased. This formally establishes the conjecture empirically given in Billington et al (1998). These authors argue single rollover to be a high-risk, high-return strategy while dual rollover to be less risky. In the next section, a rollover model including the decision maker risk attitude is developed and analyzed.

2.6 Closed-form solutions, numerical experiments, and useful managerial insights

This section presents some closed form solutions, some numerical experiments, and some useful managerial insights.

2.6.1 Examples of Optimal Cost Closed Forms solutions

According to classical inventory theory models (see Gallego [16]), closed forms can be given for the optimal cost for certain probability distributions. Indeed, the loss functions (2.3.3), (2.3.4) and (2.3.8) have a piecewise linear structure which can be exploited.

2.6.2 Numerical experiments

In this section, first, we solve the problem using an exponential distribution and assume that we are in a high risk environment, with $\beta = 0.95$. From the conditions given in Table 2.5, we obtain the optimal rollover strategy to be a dual one with the following

	Optimal loss (The Gaussian case :mean μ and standard deviation σ)
Optimal Planned Stockout	$((m_1 - m_1')F(\frac{t_1^* - \mu}{\sigma}) + (m_2 + g + c_h)F(\frac{t_2^* - \mu}{\sigma}))\,\sigma.$
Optimal Single Product Rollover	$((m_1 + g + c_h + m_2 - m_1')F(\frac{t_b^* - \mu}{\sigma}))\,\sigma.$
Optimal Dual Product Rollover	$((m_1 + g + s_1)F(\frac{t_1^* - \mu}{\sigma}) + (m_2 - m_1' - s_1 + c_h)F(\frac{t_2^* - \mu}{\sigma}))\,\sigma.$

	Optimal loss (The LogNormal case with parameters v and τ)
Optimal Planned Stockout	$(m_1 - m_1')\mu F(\tau - \frac{(Ln(t_1^*) - v)}{\tau}) + (m_1' - m_2)\mu + (m_2 + c_h + g)\mu F(\tau - \frac{(Ln(t_2^*) - v)}{\tau})$
Optimal Single Product Rollover	$(m_2 - m_1' + m_1 + g + c_h)\mu F(\tau - \frac{(Ln(t_1^*) - v)}{\tau}) + (m_1' - m_2)\mu$
Optimal Dual Product Rollover	$(m_1 + g + s_1)\mu F(\tau - \frac{(Ln(t_1^*) - v)}{\tau}) + (m_2 - m_1' - s_1 + c_h)\mu F(\tau - \frac{(Ln(t_2^*) - v)}{\tau}) - (m_2 - m_1')\mu$

Table 2.5: Examples : Closed form for the optimal loss

closed form optimal solutions:

$$t_1^* = -Ln\left(\frac{s_1(1-\beta)}{m_1 + g + s_1}\right)\frac{1}{\lambda} \tag{2.6.1}$$

$$t_2^* = -\left(\frac{m_2 - m_1'}{m_2 - m_1' - s_1 + c_{h,2}}\right)Ln\left(\frac{m_2 - m_1' - s_1 + c_{h,2}\beta}{m_2 - m_1' - s_1 + c_{h,2}}\right)\frac{1}{\lambda} \tag{2.6.2}$$

with $t_1^* = 127.94$, $t_2^* = 0.07$, and optimal expected net loss \$26, 596.

Second, we assume that the approval date T follows a gamma distribution with shape parameter of 80 and scale parameter equal of 0.75. For an expected net loss minimization criterion, we obtain the optimal rollover strategy to be a dual one with $t_1^* = 81.60$, $t_2^* = 43.51$, and optimal expected net loss \$4216.7.

2.6.3 Managerial insights, summary, and future research

In this paper, we apply CVaR minimization to a product rollover problem with uncertain regulatory approval date and compare it to the minimization of the classical expected net loss. Results show that the optimal strategy is dependent on the parameters (costs and prices) and/or probability distribution and risk. We derive conditions for the optimality and the uniqueness of the closed-form solutions for single and dual rollover cases. Furthermore, we present the variation of optimal costs and solutions under different probability distribution families. Many possible extensions and directions for research exist, such as optimizing with respect to a distribution free regulatory approval date, or for different products and lifecycles, and rollover for time-dependent demand. We are currently working on the expected value criterion under a Bass diffusion rate demand and present part of our work in Chapter 3.

Appendix

APPENDIX A: Classical Analysis: Optimal Solutions with Respect to Expected Loss

***APPENDIX A-1: The region* $\mathbf{R_1}$.**

In $\mathbf{R_1}$, the objective loss function is given by

$$
\begin{aligned}
l_1(t_1,t_2) := E[L_1(t_1,t_2,T)] &= \int_0^{t_1}[(m_2-m_1')(t_1-T)+(m_2+g)(t_2-t_1)]f(T)dT \\
&+ \int_{t_1}^{t_2}[(m_1+g)(T-t_1)+(m_2+g)(t_2-T)]f(T)dT \\
&+ \int_{t_2}^{\infty}[(m_1+g)(T-t_1)+c_h(T-t_2)]f(T)dT \\
&= (m_2t_2-m_1't_1)F(t_1)+(m_1'-m_2)G(t_1) \\
&+ m_1(\mu-G(t_1)-t_1(1-F(t_1)))+m_2t_2(F(t_2)-F(t_1)) \\
&+ (c_h+g)(\mu-G(t_2)-t_2(1-F(t_2)))+g(t_2-t_1) \qquad (2.6.3)
\end{aligned}
$$

The associated optimization problem is

$$
\begin{aligned}
\min_{\{(t_1,t_2)\in\mathbf{R_1}\}} \quad l_1(t_1,t_2) &= (m_2t_2-m_1't_1)F(t_1)+(m_1'-m_2)G(t_1) \\
&+ m_1(\mu-G(t_1)-t_1(1-F(t_1)))+m_2t_2(F(t_2)-F(t_1)) \\
&+ (c_h+g)(\mu-G(t_2)-t_2(1-F(t_2)))+g(t_2-t_1). \qquad (2.6.4)
\end{aligned}
$$

Proof of Property 1

The second order derivatives of expression (2.6.3) are given by:

$$\frac{dl_1^2(t_1,t_2)}{dt_1^2} = (m_1-m_1')f(t_1), \qquad (2.6.5)$$

$$\frac{dl_1^2(t_1,t_2)}{dt_2^2} = (m_2+c_h+g)f(t_2). \qquad (2.6.6)$$

It is thus direct to see that the objective function is jointly convex in $\mathbf{R_1}$ since $m_1 > m_1'$.

***APPENDIX A-2: The region* $\mathbf{R_2}$.**

In region $\mathbf{R_2}$, the objective function is given by

$$\begin{aligned} l_2(t_1,t_2) := E[L_2(t_1,t_2,T)] &= \int_0^{t_2}[(m_2-m_1')(t_2-T)+s_1(t_1-t_2)]f(T)dT \\ &+ \int_{t_2}^{t_1}[s_1(t_1-T)+c_{h,2}(T-t_2)]f(T)dT \\ &+ \int_{t_1}^{\infty}[(m_1+g)(T-t_1)+c_{h,2}(T-t_2)]f(T)dT \\ &= ((m_2-m_1')t_2+s_1(t_1-t_2))F(t_2)+(m_1'-m_2)G(t_2) \\ &+ (s_1t_1-c_{h,2}t_2)(F(t_1)-F(t_2))+(c_{h,2}-s_1)(G(t_1)-G(t_2)) \\ &- ((m_1+g)t_1+c_{h,2}t_2)(1-F(t_1)) \\ &+ (m_1+c_{h,2}+g)(\mu-G(t_1)). \end{aligned} \tag{2.6.7}$$

The optimization problem becomes in this case

$$\begin{aligned} \min_{\{(t_1,t_2)\in\mathbf{R_2}\}} l_2(t_1,t_2) &= ((m_2-m_1')t_2+s_1(t_1-t_2))F(t_2)+(m_1'-m_2)G(t_2) \\ &+ (s_1t_1-c_{h,2}t_2)(F(t_1)-F(t_2))+(c_{h,2}-s_1)(G(t_1)-G(t_2)) \\ &- ((m_1+g)t_1+c_{h,2}t_2)(1-F(t_1)) \\ &+ (m_1+c_{h,2}+g)(\mu-G(t_1)). \end{aligned} \tag{2.6.8}$$

Proof of Property 2

The first order derivatives of expression (2.6.8) are given by

$$\frac{dl_2(t_1,t_2)}{dt_1} = -(m_1+g)+(m_1+g+s_1)F(t_1), \tag{2.6.9}$$

$$\frac{dl_2(t_1,t_2)}{dt_2} = -c_{h,2}+(m_2-m_1'+c_{h,2}-s_1)F(t_2). \tag{2.6.10}$$

If $m_2 - m_1' - s_1 < 0$, expression (2.6.10) is negative and $l_2(t_1, t_2)$ is strictly decreasing.

Proof of Property 3

The second order derivatives of expression (2.6.8) are given by

$$\frac{dl_2^2(t_1,t_2)}{dt_1^2} = (m_1 + g + s_1)f(t_1), \tag{2.6.11}$$

$$\frac{dl_2^2(t_1,t_2)}{dt_2^2} = (m_2 - m_1' + c_{h,2} - s_1)f(t_2), \tag{2.6.12}$$

$$\frac{dl_2^2(t_1,t_2)}{dt_1 dt_2} = \frac{dl_2^2(t_1,t_2)}{dt_1 dt_2} = 0. \tag{2.6.13}$$

It is thus direct to see that the objective function is jointly convex in $\mathbf{R_2}$ when $m_2 - m_1' - s_1 + c_{h,2} > 0$, else it is strictly jointly concave.

APPENDIX B: CVaR Minimization

In our setting, the minimization problem is represented by the following convex program:

$$\min \quad l_\beta(t_1, t_2, \alpha) := \alpha + \frac{1}{(1-\beta)} \int_0^\infty [L(t_1, t_2, T) - \alpha]^+ f(T) dT, \qquad (2.6.14)$$

$$\text{s.t.} \quad 0 \le t_1, t_2 \le \infty, \quad -\infty \le \alpha \le \infty. \qquad (2.6.15)$$

This optimization problem and the associated optimality conditions are not straightforward and the optimal solution cannot be expected to be given by classical first order conditions. First, because the $L(\cdot,\cdot,\cdot)$ function is not differentiable for $t_1 = t_2$. Second, due to the $[\cdot]^+$ function in equation (2.6.14), which again is not differentiable around the 0 value. It is the reason why the state space is first divided into complementary regions chosen in such a way that the objective function is differentiable in each region. The standard optimization methods will then be applied over each region. Clearly, the boundaries of the regions, which correspond to states for which the objective function (2.6.14) are not differentiable, and thus, will be carefully analyzed. Depending on the parameters numerical values, it will be shown that while solving the first order optimization conditions two types of situations occur:

- the sign of the derivative of the objective function (2.6.14) with respect to a variable is strictly positive, or strictly negative over the considered region, and in this case, the optimal solution exists on the boundary of the considered region
- the objective function (2.6.14) is strictly convex on the considered region and the optimal solution exists in the interior of the region and is computed via the classical first order optimality conditions.

The solution approach will consist in combining these two ideas in order to analyze the whole state space region. Nevertheless, the whole analysis is tedious, due to the fact that the number of different cases associated to the regions and/or to the boundaries to consider is significative.

First, we will first start by defining the regions. The (t_1, t_2) state space is divided in the regions $(t_1, t_2) : t_1 \leq t_2$ and $(t_1, t_2) : t_1 \geq t_2$, as the expression of the objective function depends on the relative value of t_1 w.r.t. t_2. Second, the α values are decomposed in complementary intervals, such that on each interval the fundamental structure of the term $\Big(L(t_1, t_2, T) - \alpha\Big)$ is fixed.

The optimization problem is given as follows:

$$\begin{aligned} \min\, l_\beta(t_1, t_2, \alpha) &= \alpha + \frac{1}{1-\beta} E_T[L(t_1, t_2, T) - \alpha]^+. \\ \text{s.t.} \quad & 0 \leq t_1, t_2 \leq \infty, \quad -\infty \leq \alpha \leq \infty. \end{aligned} \tag{2.6.16}$$

As the state space has to be divided in two regions, $\mathbf{R_i}$ with $i = 1, 2$, one defines thus

$$l_{\beta,i}(t_1, t_2, \alpha) = \alpha + \frac{1}{1-\beta} E_T[L_i(t_1, t_2, T) - \alpha]^+, \quad \text{with} \quad i = 1, 2. \tag{2.6.17}$$

The region $\mathbf{R_1}$.

In $\mathbf{R_1}$, the associated optimization problem is thus given by

$$\begin{aligned} \min\, l_{\beta,1}(t_1, t_2, \alpha) &= \alpha + \frac{1}{1-\beta} \Big(\int_0^{t_1} [m_1'(T - t_1) + g(t_2 - t_1) + m_2(t_2 - T) - \alpha]^+ f(T) dT \\ &+ \int_{t_1}^{t_2} [m_1(T - t_1) + g(t_2 - t_1) + m_2(t_2 - T) - \alpha]^+ f(T) dT \\ &+ \int_{t_2}^{\infty} [m_1(T - t_1) + g(t_2 - t_1) + (c_{h,2} + g)(T - t_2) - \alpha]^+ f(T) dT \Big) \end{aligned} \tag{2.6.18}$$

$$\text{s.t.} \quad (t_1, t_2) \in \mathbf{R_1}, \quad -\infty \leq \alpha \leq \infty. \tag{2.6.19}$$

We have to consider two cases:

- $m_2 \geq m_1 \geq m_1'$,
- $m_1 \geq m_2 \geq m_1'$.

Appendix B-1 Case 1 : $m_2 \geq m_1 \geq m_1'$

It can be seen that the critical values for the α parameters corresponding to the slope discontinuities for the piecewise linear function (2.6.18), as functions of t_1 and t_2, are given by

$$\tilde{\alpha}_{1,1}(t_1, t_2) = m_1(t_2 - t_1) + g(t_2 - t_1), \quad (2.6.20)$$

$$\tilde{\alpha}_{1,2}(t_1, t_2) = m_2(t_2 - t_1) + g(t_2 - t_1), \quad (2.6.21)$$

$$\tilde{\alpha}_{1,3}(t_1, t_2) = m_2 t_2 - m_1' t_1 + g(t_2 - t_1), \quad (2.6.22)$$

with $\tilde{\alpha}_{1,1}(t_1, t_2) \leq \tilde{\alpha}_{1,2}(t_1, t_2) \leq \tilde{\alpha}_{1,3}(t_1, t_2)$ (see Figure (2.5)),

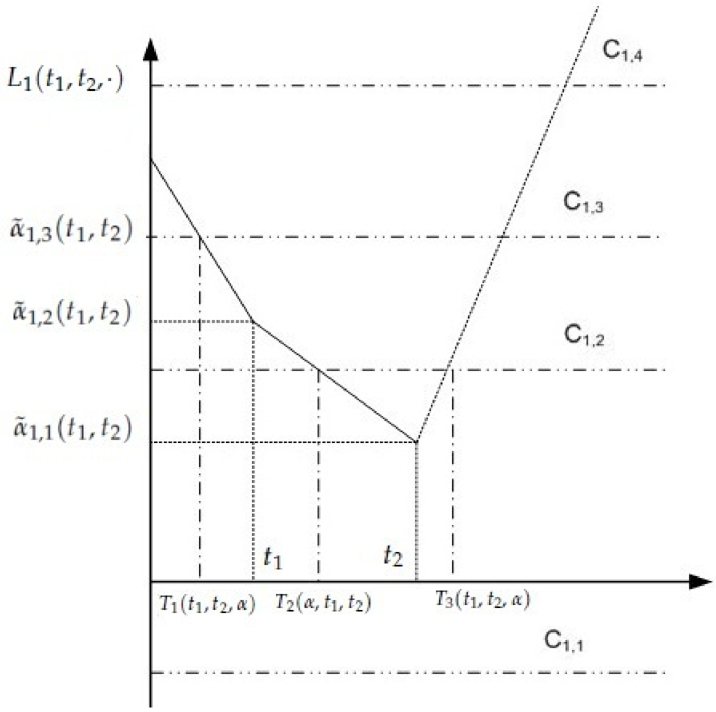

Figure 2.5: Four cases in minimization of CVaR in the region $\mathbf{R_1}$ for Case 1.

In order to characterize the first order conditions, we define the regions $\mathbf{C_{1,1}}$, $\mathbf{C_{1,2}}$, $\mathbf{C_{1,3}}$

and $\mathbf{C_{1,4}}$, as

$$\mathbf{C_{1,1}} = \{(t_1,t_2,\alpha) \text{ with } (t_1,t_2)\in\mathbf{R_1} \text{ and } \alpha\in]\infty,\tilde{\alpha}_{1,1}(t_1,t_2)[, \tag{2.6.23}$$

$$\mathbf{C_{1,2}} = \{(t_1,t_2,\alpha) \text{ with } (t_1,t_2)\in\mathbf{R_1} \text{ and } \alpha\in]\tilde{\alpha}_{1,1}(t_1,t_2),\tilde{\alpha}_{1,2}(t_1,t_2)[, \tag{2.6.24}$$

$$\mathbf{C_{1,3}} = \{(t_1,t_2,\alpha) \text{ with } (t_1,t_2)\in\mathbf{R_1} \text{ and } \alpha\in]\tilde{\alpha}_{1,2}(t_1,t_2),\tilde{\alpha}_{1,3}(t_1,t_2)[, \tag{2.6.25}$$

$$\mathbf{C_{1,4}} = \{(t_1,t_2,\alpha) \text{ with } (t_1,t_2)\in\mathbf{R_1} \text{ and } \alpha\in]\tilde{\alpha}_{1,3}(t_1,t_2),\infty[. \tag{2.6.26}$$

Determination of the optimal policies.

First step : expression of the first order conditions.

The region $\mathbf{C_{1,1}}$.

In this region, the objective function given in expression (2.6.18) is

$$\begin{aligned} l_{\beta,1}(t_1,t_2,\alpha) = \alpha &+ \frac{1}{1-\beta}\Big[(m_2t_2 - m_1't_1)F(t_1) + (m_1' - m_2)G(t_1) \\ &+ m_1(\mu - G(t_1) - t_1(1-F(t_1))) + m_2t_2(F(t_2) - F(t_1)) \\ &+ (c_{h,2}+g)(\mu - G(t_2) - t_2(1-F(t_2))) \\ &+ g(t_2 - t_1) - \alpha\Big]. \end{aligned} \tag{2.6.27}$$

The optimization problem can be rewritten

$$\begin{aligned} \min \quad l_{\beta,1}(t_1,t_2,\alpha) = \alpha &+ \frac{1}{1-\beta}\Big[(m_2t_2 - m_1't_1)F(t_1) + (m_1' - m_2)G(t_1) \\ &+ m_1(\mu - G(t_1) - t_1(1-F(t_1))) + m_2t_2(F(t_2) - F(t_1)) \\ &+ (c_{h,2}+g)(\mu - G(t_2) - t_2(1-F(t_2))) \\ &+ g(t_2 - t_1) - \alpha\Big], \end{aligned} \tag{2.6.28}$$

$$\text{s.t.} \quad (t_1,t_2,\alpha)\in\mathbf{C_{1,1}}. \tag{2.6.29}$$

The first order derivatives of (2.6.28) are given by

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} = \frac{-\beta}{1-\beta} < 0, \tag{2.6.30}$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} = \frac{(m_1 - m_1')}{1-\beta}F(t_1) - \frac{(m_1+g)}{1-\beta}, \tag{2.6.31}$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2 + c_{h,2} + g)}{1-\beta}F(t_2) - \frac{c_{h,2}}{1-\beta}. \tag{2.6.32}$$

The region $\mathbf{C_{1,2}}$.

According to Figure 2.5, let's define $T_2(\alpha, t_1, t_2)$ as the T value corresponding to:

$$\alpha = m_1(T - t_1) + g(t_2 - t_1) + m_2(t_2 - T) \tag{2.6.33}$$

and $T_3(\alpha, t_1, t_2)$ as the T value corresponding to:

$$\alpha = m_1(T - t_1) + g(t_2 - t_1) + (c_{h,2} + g)(T - t_2) \tag{2.6.34}$$

The optimization problem can be rewritten

$$\begin{aligned}
\min \quad l_{\beta,1}(t_1, t_2, \alpha) = \alpha \quad & + \quad \frac{1}{1-\beta}\Bigg((m_1 - m_1')t_1F(t_1) + (m_1' - m_1)G(t_1) \\
& + \quad (m_1 - m_2)G(T_2(t_1, t_2, \alpha)) \\
& + \quad (-m_1t_1 + g(t_2 - t_1) + m_2t_2 - \alpha)F(T_2(t_1, t_2, \alpha)) \\
& + \quad (m_1 + c_{h,2} + g)(\mu - G(T_3(t_1, t_2, \alpha))) \\
& + \quad (-m_1t_1 - gt_1 - c_{h,2}t_2 \\
& - \quad \alpha)(1 - F(T_3(t_1, t_2, \alpha)))\Bigg), \qquad (2.6.35) \\
\text{s.t.} \quad & \quad (t_1, t_2, \alpha) \in \mathbf{C_{1,2}}. \qquad (2.6.36)
\end{aligned}$$

The first order derivatives of (2.6.35) are given by:

$$\begin{aligned}
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} &= \frac{(m_1 - m_1')}{1-\beta}F(t_1) - \frac{(m_1 + g)}{1-\beta}F(T_2(\alpha, t_1, t_2)) \\
&- \frac{(m_1 + g)}{1-\beta}(1 - F(T_3(\alpha, t_1, t_2))), \qquad (2.6.37) \\
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} &= \frac{(m_2 + g)}{1-\beta}F(T_2(\alpha, t_1, t_2)) - \frac{c_{h,2}}{1-\beta}(1 - F(T_3(\alpha, t_1, t_2))), \qquad (2.6.38) \\
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} &= \frac{F(T_3(t_1, t_2, \alpha)) - F(T_2(t_1, t_2, \alpha)) - \beta}{1-\beta}. \qquad (2.6.39)
\end{aligned}$$

The region $\mathbf{C_{1,3}}$.

According to Figure 2.5, let's define $T_1(\alpha, t_1, t_2)$ as the T value corresponding to:

$$\alpha = -m_1'(t_1 - T) + g(t_2 - t_1) + m_2(t_2 - T). \tag{2.6.40}$$

The optimization problem becomes

$$\begin{aligned} \min \quad l_{\beta,1}(t_1,t_2,\alpha) &= \alpha + \frac{1}{1-\beta}\Big[(m_1-m_2)G(T_1(t_1,t_2,\alpha)) + (m_1+c_{h,2}+g)(\mu - G(T_3(t_1,t_2,\alpha))) \\ &+ (-m_1' t_1 + g(t_2-t_1) + m_2 t_2 - \alpha)F(T_1(t_1,t_2,\alpha)) \\ &+ (-m_1t_1 - gt_1 - c_{h,2}t_2 - \alpha)(1-F(T_3(t_1,t_2,\alpha)))\Big], \qquad (2.6.41) \\ \text{s.t.} \quad & (t_1,t_2,\alpha) \in \mathbf{C_{1,3}}. \qquad (2.6.42) \end{aligned}$$

The first order derivatives of (2.6.41) are given by:

$$\begin{aligned} \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} &= -\frac{(m_1'+g)}{1-\beta}F(T_1(\alpha,t_1,t_2)) \\ &- \frac{(m_1+g)(1-F(T_3(\alpha,t_1,t_2)))}{1-\beta}, \qquad (2.6.43) \\ \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} &= \frac{(m_2+g)}{1-\beta}F(T_1(\alpha,t_1,t_2)) - \frac{c_{h,2}}{1-\beta}(1-F(T_3(\alpha,t_1,t_2))), \qquad (2.6.44) \\ \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} &= \frac{F(T_3(t_1,t_2,\alpha)) - F(T_1(t_1,t_2,\alpha)) - \beta}{1-\beta}. \qquad (2.6.45) \end{aligned}$$

The region $\mathbf{C_{1,4}}$.

The optimization problem becomes

$$\begin{aligned} \min \quad l_{\beta,1}(t_1,t_2,\alpha) &= \alpha + \frac{1}{1-\beta}\Big[(m_1+c_{h,2}+g)(\mu - G(T_3(t_1,t_2,\alpha))) \\ &+ (-m_1t_1 - gt_1 - c_{h,2}t_2 - \alpha)(1-F(T_3(t_1,t_2,\alpha)))\Big], \qquad (2.6.46) \\ \text{s.t.} \quad & (t_1,t_2,\alpha) \in \mathbf{C_{1,4}}. \qquad (2.6.47) \end{aligned}$$

The first order derivatives of (2.6.46) are given by:

$$\begin{aligned} \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} &= -\left(\frac{m_1+g}{1-\beta}\right)\Big(1-F(T_3(t_1,t_2,\alpha))\Big), \qquad (2.6.48) \\ \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} &= -\left(\frac{c_{h,2}}{1-\beta}\right)\Big(1-F(T_3(t_1,t_2,\alpha))\Big), \qquad (2.6.49) \\ \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} &= \frac{F(T_3(t_1,t_2,\alpha) - \beta}{1-\beta}. \qquad (2.6.50) \end{aligned}$$

Corollary. In the interior of region $\mathbf{R_1}$, the CVaR loss function $l_{\beta,1}(t_1,t_2,\alpha)$ is differentiable w.r.t. α, t_1 and t_2.

Corollary. By convexity, for fixed t_1 and t_2 in region $\mathbf{R_1}$, the optimal α value can always be found as the solution of the first order conditions.

Corollary. By convexity and derivability, if the optimal solution lies in the interior of the region $\mathbf{R_1}$, then it is given by the solution of the first order condition.

Second step : optimal solution in the interior of a region.

It is direct to see that the only case where the first order conditions possibly have a solution is the region $\mathbf{C_{1,2}}$. Under adequate assumptions, the first order conditions (2.6.37)-(2.6.39) have the solution

$$\begin{aligned} t_1^* &= F^{-1}\left(\frac{(m_1+g)(1-\beta)}{m_1-m_1'}\right), & (2.6.51)\\ t_2^* &= \left(\frac{m_1+c_{h,2}+g}{m_2+c_{h,2}+g}\right)F^{-1}\left(\frac{c_{h,2}+\beta(m_2+g)}{m_2+c_{h,2}+g}\right)\\ &+ \left(\frac{m_2-m_1}{m_2+c_{h,2}+g}\right)F^{-1}\left(\frac{c_{h,2}(1-\beta)}{m_2+c_{h,2}+g}\right), & (2.6.52)\\ \alpha^* &= \left(m_1+c_{h,2}+g\right)F^{-1}\left(\frac{c_{h,2}+\beta(m_2+g)}{m_2+c_{h,2}+g}\right) - (m_1+g)t_1^* - c_{h,2}t_2^*. & (2.6.53) \end{aligned}$$

We also find the following parameter values

$$T_3(t_1^*, t_2^*, \alpha^*) = F^{-1}\left(\frac{c_{h,2}+\beta(m_2+g)}{m_2+c_{h,2}+g}\right), \quad (2.6.54)$$

$$T_2(t_1^*, t_2^*, \alpha^*) = F^{-1}\left(\frac{c_{h,2}(1-\beta)}{m_2+c_{h,2}+g}\right). \quad (2.6.55)$$

Now, several assumptions are required in order to guarantee that this solution belongs to the interior of $\mathbf{C_{1,2}}$. Basically these assumptions are the following

$$\begin{aligned} \frac{(m_1+g)(1-\beta)}{m_1-m_1'} &< 1, & (2.6.56)\\ t_1^* &< t_2^*, & (2.6.57) \end{aligned}$$

$$\begin{aligned} \tilde{\alpha}_{1,1}(t_1^*, t_2^*) = m_1(t_2^*-t_1^*) + g(t_2^*-t_1^*) &< \alpha^*\\ &< \tilde{\alpha}_{1,2}(t_1^*, t_2^*)\\ &= m_2(t_2^*-t_1^*) + g(t_2^*-t_1^*). & (2.6.58) \end{aligned}$$

From Figure (2.5) it can be seen that the last condition is equivalent to

$$t_1^* \leq T_2(t_1^*, t_2^*, \alpha^*), T_2(t_1^*, t_2^*, \alpha^*) \leq t_2^*, \tag{2.6.59}$$

$$t_2^* \leq T_3(t_1^*, t_2^*, \alpha^*). \tag{2.6.60}$$

First condition analysis. The first assumptions is independent of the probability distribution and amounts to the condition on the parameters:

$$m_1' < -g(1-\beta) + \beta m_1. \tag{2.6.61}$$

If $m_1' < -g$, then condition 1 holds for any probability distribution and for all β values one has existence of t_1^*.

If $m_1 \geq m_1' \geq -g$, then for any probability distribution F, there exists a lower bound β_F such that for any β values with $\beta \geq \beta_F$, one has no existence of t_1^*.

Second condition analysis. The second condition is not easy and in general, for arbitrary values of the parameters and of β, it can depend on the probability distribution. However, it can be seen that under parameter conditions corresponding to Case 1, the expression of t_2^* corresponds to a convex combination of $F^{-1}\left(\frac{c_{h,2}+\beta(m_2+g)}{m_2+c_{h,2}+g}\right)$ and of $F^{-1}\left(\frac{c_{h,2}(1-\beta)}{m_2+c_{h,2}+g}\right)$. As a consequence some properties can be found, depending on the order associated with $\frac{c_{h,2}+\beta(m_2+g)}{m_2+c_{h,2}+g}$, $\frac{c_{h,2}(1-\beta)}{m_2+c_{h,2}+g}$ and $\frac{(m_1+g)(1-\beta)}{m_1-m_1'}$.

If $\frac{m_1+g}{m_1-m_1'} < \frac{c_{h,2}}{m_2+c_{h,2}+g}$ then for any probability distribution and for all β values one has $t_1^* < t_2^*$.

If $\frac{m_1+g}{m_1-m_1'} \geq \frac{c_{h,2}}{m_2+c_{h,2}+g}$ then for any probability distribution F, there exists an upper bound β_F such that for any β values with $\beta \leq \beta_F$, one has $t_1^* > t_2^*$.

Third condition analysis. It is direct to see that conditions (2.6.59) and (2.6.60) hold for any distribution and any parameters. In fact, condition (2.6.59) amounts again to $\frac{m_1+g}{m_1-m_1'} < \frac{c_{h,2}}{m_2+c_{h,2}+g}$.

Third step : optimal solution on a boundary of a region. If the optimal solution is not in the interior of a region (i.e., if above conditions do not hold) then the optimal solution has to be found in the boundary region between two regions. The potential boundaries are defined as

- $t_1 = t_2$
- $\alpha(t_1, t_2) = \tilde{\alpha}_{1,i}(t_1, t_2)$ for $i = 1, .., 3$.

The boundary $\mathbf{t_1 = t_2}$.

In this case the associated optimization problem is thus given by (See Figure 2.6.62

$$\begin{aligned} \min l_{\beta,1}(t,t,\alpha) &= \alpha + \frac{1}{1-\beta}\Bigg(\int_0^t [m_1'(T-t) + m_2(t-T) - \alpha]^+ f(T)dT \\ &+ \int_t^\infty [m_1(T-t) + (c_{h,2}+g)(T-t) - \alpha]^+ f(T)dT\Bigg) \\ \text{s.t.} \quad & -\infty \leq \alpha \leq \infty. \end{aligned} \tag{2.6.62}$$

It can be seen that the critical values for the α parameters corresponding to the slope discontinuities for the piecewise linear function (2.6.18), as functions of t are given by

$$\tilde{\alpha}_{1,1}(t,t) = 0, \tag{2.6.63}$$

$$\tilde{\alpha}_{1,2}(t,t) = (m_2 - m_1')t. \tag{2.6.64}$$

with $\tilde{\alpha}_{1,1}(t,t) \leq \tilde{\alpha}_{1,2}(t,t)$ (see Figure (2.5)),

We define the regions $\mathbf{C_{b,1,1}}$, $\mathbf{C_{b,1,2}}$, $\mathbf{C_{b,1,3}}$ and $\mathbf{C_{b,1,4}}$, defined as

$$\mathbf{C_{b,1,1}} = \{(t,\alpha) \quad \text{with} \quad (t,t) \in \mathbf{R_1} \quad \text{and} \quad \alpha \in]\infty, \tilde{\alpha}_{1,1}(t,t)[, \tag{2.6.65}$$

$$\mathbf{C_{b,1,2}} = \{(t,\alpha) \quad \text{with} \quad (t,t_) \in \mathbf{R_1} \quad \text{and} \quad \alpha \in]\tilde{\alpha}_{1,1}(t,t), \tilde{\alpha}_{1,2}(t,t)[, \tag{2.6.66}$$

$$\mathbf{C_{b,1,3}} = \{(t,\alpha) \quad \text{with} \quad (t,t) \in \mathbf{R_1} \quad \text{and} \quad \alpha \in]\tilde{\alpha}_{1,2}(t,t), \infty[. \tag{2.6.67}$$

First step : expression of the first order conditions.

The region $\mathbf{C_{b,1,1}}$.

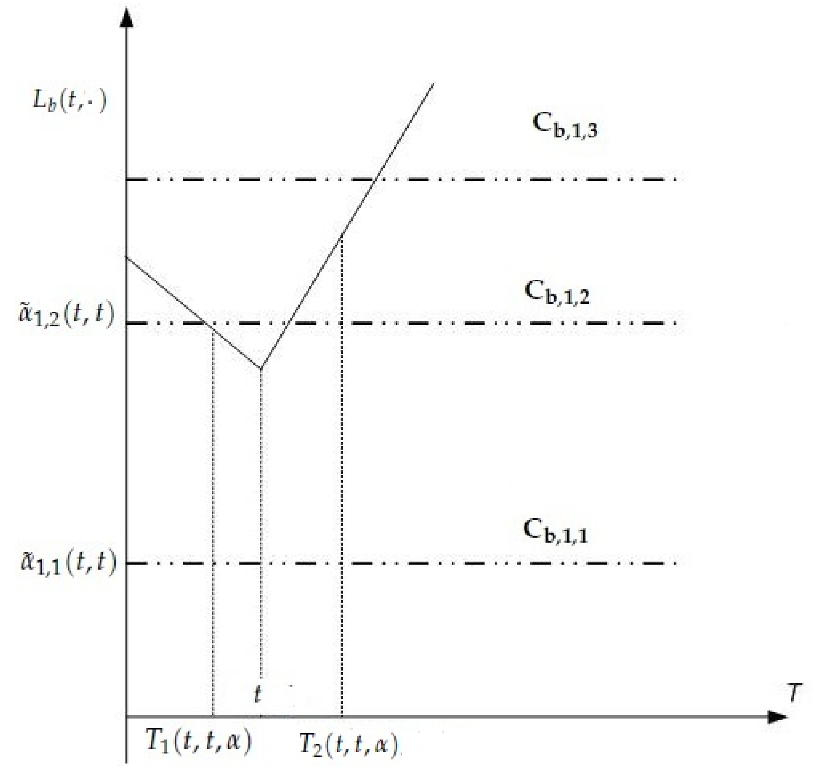

Figure 2.6: Three cases in minimization of CVaR on the boundary

In this region, the objective function given in expression (2.6.18) is

$$\begin{aligned} l_{\beta,1}(t,t,\alpha) = \alpha &+ \frac{1}{1-\beta}\Big[(m_2 t - m_1' t)F(t) + (m_1' - m_2)G(t) \\ &+ m_1(\mu - G(t) - t(1-F(t))) \\ &+ (c_{h,2}+g)(\mu - G(t) - t(1-F(t))) - \alpha\Big]. \end{aligned} \tag{2.6.68}$$

The first order order derivatives of expression (2.6.68) are given by:

$$\frac{dl_{\beta,1}(t,t,\alpha)}{dt} = \frac{(m_1 - m_1' + m_2 + c_{h,2} + g)}{1-\beta}F(t) - \frac{(m_1 + g + c_{h,2})}{1-\beta}, \tag{2.6.69}$$

$$\frac{dl_{\beta,1}(t,t,\alpha)}{d\alpha} = \frac{-\beta}{1-\beta} < 0. \tag{2.6.70}$$

The region $\mathbf{C_{b,1,2}}$.

According to Figure 2.5, let's define $T_1(\alpha,t,t)$ as the T value corresponding to:

$$\alpha = -m_1'(t-T) + g(t-t) + m_2(t-T). \tag{2.6.71}$$

The optimization problem becomes

$$\begin{aligned} \min \quad l_{\beta,1}(t,t,\alpha) &= \alpha + \frac{1}{1-\beta}\Big[(m_1 - m_2)G(T_1(t,t,\alpha)) + (m_1 + c_{h,2} + g)(\mu - G(T_2(t,t,\alpha))) \\ &+ (-m_1' t + m_2 t - \alpha)F(T_1(t,t,\alpha)) \\ &+ (-m_1 t - gt - c_{h,2}t - \alpha)(1 - F(T_2(t,t,\alpha)))\Big], \end{aligned} \tag{2.6.72}$$

$$\text{s.t.} \quad (t,t,\alpha) \in \mathbf{C_{b,1,2}}. \tag{2.6.73}$$

The first order derivatives of (2.6.72) are given by:

$$\begin{aligned} \frac{dl_{\beta,1}(t,t,\alpha)}{dt} &= \frac{(m_2 - m_1')}{1-\beta}F(T_1(\alpha,t,t)) \\ &- \frac{(m_1 + g + c_{h,2})}{1-\beta}(1 - F(T_2(\alpha,t,t))), \end{aligned} \tag{2.6.74}$$

$$\frac{dl_{\beta,1}(t,t,\alpha)}{d\alpha} = \frac{F(T_2(t,t,\alpha)) - F(T_1(t,t,\alpha)) - \beta}{1-\beta}. \tag{2.6.75}$$

The region $\mathbf{C_{b,1,3}}$.

The optimization problem becomes

$$\begin{aligned} \min \quad l_{\beta,1}(t,t,\alpha) &= \alpha + \frac{1}{1-\beta}\Big[(m_1 + c_{h,2} + g)(\mu - G(T_2(t,t,\alpha))) \\ &- ((m_1 + g + c_{h,2})t + \alpha)(1 - F(T_2(t,t,\alpha)))\Big], \end{aligned} \tag{2.6.76}$$

$$\text{s.t.} \qquad (t,t,\alpha) \in \mathbf{C_{b,1,3}}. \tag{2.6.77}$$

The first order derivatives of (2.6.76) are given by:

$$\frac{dl_{\beta,1}(t,t,\alpha)}{dt} = -\left(\frac{m_1 + g + c_{h,2}}{1-\beta}\right)\Big(1 - F(T_2(t,t,\alpha))\Big) < 0, \tag{2.6.78}$$

$$\frac{dl_{\beta,1}(t,t,\alpha)}{d\alpha} = \frac{F(T_2(t,t,\alpha)) - \beta}{1-\beta}. \tag{2.6.79}$$

Second step : optimal solution on the boundary.

It is direct to see that the only case where the first order conditions possibly have a solution is the region $\mathbf{C_{b,1,2}}$. Under adequate assumptions, the first order conditions (2.6.37)-(2.6.39) have the solution

$$\begin{aligned} t^* &= \frac{\Big(m_2 - m_1'\Big)F^{-1}\left(\frac{(m_1+c_{h,2}+g)(1-\beta)}{m_2-m_1'+m_1+c_{h,2}+g}\right)}{m_2 - m_1' + m_1 + c_{h,2} + g} \\ &+ \frac{\Big(m_1 + c_{h,2} + g\Big)F^{-1}\left(\frac{m_1+c_{h,2}+g+\beta(m_2-m_1')}{m_2-m_1'+m_1+c_{h,2}+g}\right)}{m_2 - m_1' + m_1 + c_{h,2} + g}, \end{aligned} \tag{2.6.80}$$

$$\alpha^* = \Big(m_2 - m_1'\Big)t^*. \tag{2.6.81}$$

The assumptions which imply optimality of a solution on the boundary are the complementary conditions which guarantee optimality of interior optimal solution (see (2.6.56)-(2.6.57)).

Appendix B-2 Case 2 : $m_1 \geq m_2 \geq m_1'$

It can be seen that the critical values for the α parameters corresponding to the slope discontinuities for the piecewise linear function (2.6.18), as functions of t_1 and t_2, are given by

$$\tilde{\alpha}_{1,1}(t_1, t_2) = m_2(t_2 - t_1) + g(t_2 - t_1), \tag{2.6.82}$$

$$\tilde{\alpha}_{1,2}(t_1, t_2) = m_1(t_2 - t_1) + g(t_2 - t_1), \tag{2.6.83}$$

$$\tilde{\alpha}_{1,3}(t_1, t_2) = m_2 t_2 - m_1' t_1 + g(t_2 - t_1), \tag{2.6.84}$$

In order to characterize the first order conditions, we define the regions for

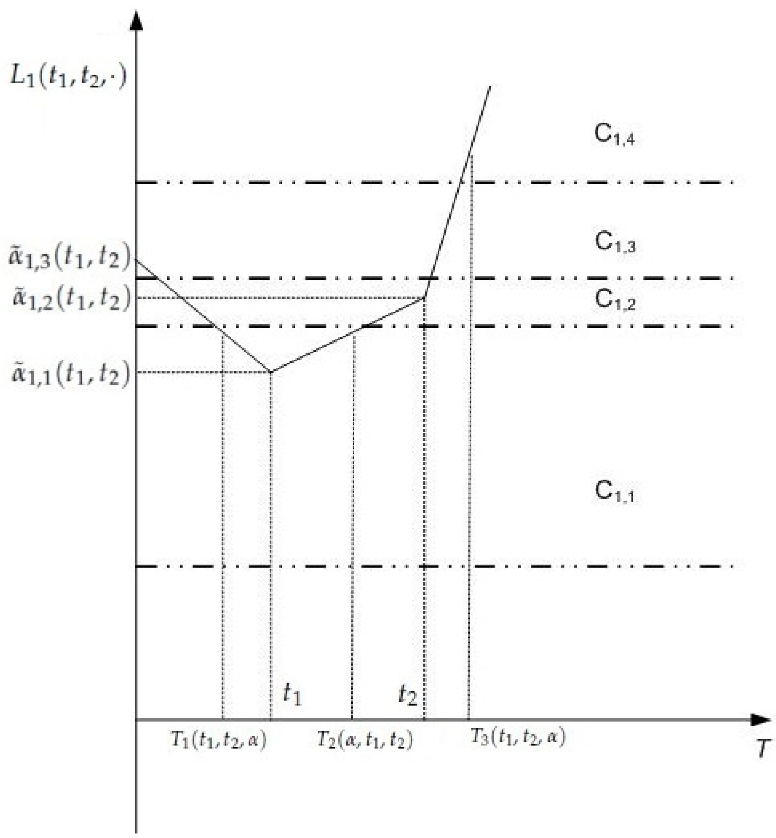

Figure 2.7: Four cases in minimization of CVaR in the region $\mathbf{R_1}$ for $m_1 \geq m_2 \geq m_1'$

- $\tilde{\alpha}_{1,1}(t_1,t_2) \leq \tilde{\alpha}_{1,2}(t_1,t_2) \leq \tilde{\alpha}_{1,3}(t_1,t_2)$ $\mathbf{C_{1,1}}$, $\mathbf{C_{1,2}}$, $\mathbf{C_{1,3}}$ and $\mathbf{C_{1,4}}$, as

$$\mathbf{C_{1,1}} = \{(t_1,t_2,\alpha) \text{ with } (t_1,t_2) \in \mathbf{R_1} \text{ and } \alpha \in]\infty, \tilde{\alpha}_{1,1}(t_1,t_2)[, \tag{2.6.85}$$

$$\mathbf{C_{1,2}} = \{(t_1,t_2,\alpha) \text{ with } (t_1,t_2) \in \mathbf{R_1} \text{ and } \alpha \in]\tilde{\alpha}_{1,1}(t_1,t_2), \tilde{\alpha}_{1,2}(t_1,t_2)[, \tag{2.6.86}$$

$$\mathbf{C_{1,3}} = \{(t_1,t_2,\alpha) \text{ with } (t_1,t_2) \in \mathbf{R_1} \text{ and } \alpha \in]\tilde{\alpha}_{1,2}(t_1,t_2), \tilde{\alpha}_{1,3}(t_1,t_2)[, \tag{2.6.87}$$

$$\mathbf{C_{1,4}} = \{(t_1,t_2,\alpha) \text{ with } (t_1,t_2) \in \mathbf{R_1} \text{ and } \alpha \in]\tilde{\alpha}_{1,3}(t_1,t_2), \infty[. \tag{2.6.88}$$

The main result for CVAR in region $\mathbf{R_1}$

Determination of the optimal policies.

First step : expression of the first order conditions.

The region $\mathbf{C_{1,1}}$.

In this region, the objective function given in expression (2.6.18) is

$$\begin{aligned} l_{\beta,1}(t_1,t_2,\alpha) = \alpha \quad &+ \quad \frac{1}{1-\beta}\Big[(m_2t_2 - m_1't_1)F(t_1) + (m_1' - m_2)G(t_1) \\ &+ \quad m_1(\mu - G(t_1) - t_1(1-F(t_1))) + m_2t_2(F(t_2) - F(t_1)) \\ &+ \quad (c_{h,2} + g)(\mu - G(t_2) - t_2(1-F(t_2))) \\ &+ \quad g(t_2 - t_1) - \alpha\Big]. \end{aligned} \tag{2.6.89}$$

The optimization problem can be rewritten

$$\begin{aligned} \min \quad l_{\beta,1}(t_1,t_2,\alpha) = \alpha \quad &+ \quad \frac{1}{1-\beta}\Big[(m_2t_2 - m_1't_1)F(t_1) + (m_1' - m_2)G(t_1) \\ &+ \quad m_1(\mu - G(t_1) - t_1(1-F(t_1))) + m_2t_2(F(t_2) - F(t_1)) \\ &+ \quad (c_{h,2} + g)(\mu - G(t_2) - t_2(1-F(t_2))) \\ &+ \quad g(t_2 - t_1) - \alpha\Big], \end{aligned} \tag{2.6.90}$$

$$\text{s.t.} \quad (t_1,t_2,\alpha) \in \mathbf{C_{1,1}}. \tag{2.6.91}$$

The first order derivatives of (2.6.90) are given by

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} = \frac{-\beta}{1-\beta} < 0, \tag{2.6.92}$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} = \frac{(m_1-m_1')}{1-\beta}F(t_1) - \frac{(m_1+g)}{1-\beta}, \tag{2.6.93}$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2+c_{h,2}+g)}{1-\beta}F(t_2) - \frac{c_{h,2}}{1-\beta}. \tag{2.6.94}$$

The region $\mathbf{C_{1,2}}$.

According to Figure 2.7, let's define $T_1(\alpha,t_1,t_2)$ as the T value corresponding to:

$$\alpha = -m_1'(t_1-T) + g(t_2-t_1) + m_2(t_2-T). \tag{2.6.95}$$

and $T_2(\alpha,t_1,t_2)$ as the T value corresponding to:

$$\alpha = m_1(T-t_1) + g(t_2-t_1) + m_2(t_2-T) \tag{2.6.96}$$

The optimization problem can be rewritten

$$\begin{aligned}
\min \quad l_{\beta,1}(t_1,t_2,\alpha) = \alpha \quad &+ \quad \frac{1}{1-\beta}\Big(-(m_1'+g)t_1 + (m_2+g)t_2 - \alpha)F(T_1(t_1,t_2,\alpha)) + (m_1'-m_2)G(T_1(t_1,t_2,\alpha)) \\
&+ \quad (m_1-m_2)(G(t_2) - G(T_2(t_1,t_2,\alpha))) \\
&+ \quad (-m_1t_1 + g(t_2-t_1) + m_2t_2 - \alpha)(F(t_2) - F(T_2(t_1,t_2,\alpha))) \\
&+ \quad (m_1+c_{h,2}+g)(\mu - G(t_2)) \\
&+ \quad (-m_1t_1 - gt_1 - c_{h,2}t_2 - \alpha)(1-F(t_2))\Big), \qquad (2.6.97) \\
\text{s.t.} \quad & (t_1,t_2,\alpha) \in \mathbf{C_{1,2}}. \qquad (2.6.98)
\end{aligned}$$

The first order derivatives of (2.6.97) are given by:

$$\begin{aligned}
\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} &= -\frac{(m_1'+g)}{1-\beta}F(T_1(\alpha,t_1,t_2)) + \frac{(m_1+g)}{1-\beta}F(T_2(\alpha,t_1,t_2)) \\
&- \frac{(m_1+g)}{1-\beta}, \qquad (2.6.99) \\
\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} &= \frac{(m_2+g)}{1-\beta}(F(T_1(\alpha,t_1,t_2)) - F(T_2(\alpha,t_1,t_2))) - \frac{c_{h,2}}{1-\beta}(1-F(t_2)) + (\frac{m_1+g)}{1-\beta}F(t_2), \qquad (2.6.100) \\
\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} &= \frac{F(T_2(t_1,t_2,\alpha)) - F(T_1(t_1,t_2,\alpha)) - \beta}{1-\beta}. \qquad (2.6.101)
\end{aligned}$$

The region $\mathbf{C_{1,3}}$.

According to Figure 2.7, let's define $T_3(\alpha,t_1,t_2)$ as the T value corresponding to:

$$\alpha = m_1(T-t_1) + g(t_2-t_1) + (c_{h,2}+g)(T-t_2) \tag{2.6.102}$$

The optimization problem becomes

$$\begin{aligned}
\min \quad l_{\beta,1}(t_1,t_2,\alpha) &= \alpha + \frac{1}{1-\beta}\Big[(m_1-m_2)G(T_1(t_1,t_2,\alpha)) + (m_1+c_{h,2}+g)(\mu - G(T_3(t_1,t_2,\alpha))) \\
&+ (-m_1't_1 + g(t_2-t_1) + m_2t_2 - \alpha)F(T_1(t_1,t_2,\alpha)) \\
&+ (-m_1t_1 - gt_1 - c_{h,2}t_2 - \alpha)(1 - F(T_3(t_1,t_2,\alpha)))\Big], \qquad (2.6.103) \\
\text{s.t.} \quad & (t_1,t_2,\alpha) \in \mathbf{C_{1,3}}. \qquad (2.6.104)
\end{aligned}$$

The first order derivatives of (2.6.103) are given by:

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} = -\frac{(m_1'+g)}{1-\beta}F(T_1(\alpha,t_1,t_2)) - \frac{(m_1+g)}{1-\beta}(1-F(T_3(\alpha,t_1,t_2))), \qquad (2.6.105)$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2+g)}{1-\beta}F(T_1(\alpha,t_1,t_2)) - \frac{c_{h,2}}{1-\beta}(1-F(T_3(\alpha,t_1,t_2))), \qquad (2.6.106)$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_3(t_1,t_2,\alpha)) - F(T_1(t_1,t_2,\alpha)) - \beta}{1-\beta}. \qquad (2.6.107)$$

The region $\mathbf{C_{1,4}}$.

The optimization problem becomes

$$\begin{aligned}
\min \quad l_{\beta,1}(t_1,t_2,\alpha) &= \alpha + \frac{1}{1-\beta}\Big[(m_1+c_{h,2}+g)(\mu - G(T_3(t_1,t_2,\alpha))) \\
&+ (-m_1t_1 - gt_1 - c_{h,2}t_2 - \alpha)(1 - F(T_3(t_1,t_2,\alpha)))\Big], \qquad (2.6.108) \\
\text{s.t.} \quad & (t_1,t_2,\alpha) \in \mathbf{C_{1,4}}. \qquad (2.6.109)
\end{aligned}$$

The first order derivatives of (2.6.108) are given by:

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} = -\left(\frac{m_1+g}{1-\beta}\right)\Big(1 - F(T_3(t_1,t_2,\alpha))\Big), \qquad (2.6.110)$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} = -\left(\frac{c_{h,2}}{1-\beta}\right)\Big(1 - F(T_3(t_1,t_2,\alpha))\Big), \qquad (2.6.111)$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_3(t_1,t_2,\alpha) - \beta}{1-\beta}. \qquad (2.6.112)$$

In the interior of region $\mathbf{R_1}$, the CVaR loss function $l_{\beta,1}(t_1,t_2,\alpha)$ is differentiable w.r.t. α, t_1 and t_2.

By convexity, for fixed t_1 and t_2 in region $\mathbf{R_1}$, the optimal α value can always be found as the solution of the first order condition. By convexity and derivability, if the optimal solution lies in the interior of the region $\mathbf{R_1}$ it is given by the solution of the first order condition.

Second step : optimal solution in the interior of a region.

It is direct to see that the only case where the first order conditions possibly have a solution is the region $\mathbf{C_{1,2}}$. Under adequate assumptions, the first order conditions (2.6.99)-(2.6.101) have the solution

$$\begin{aligned} t_1^* &= \frac{\left(m_2 - m_1'\right) F^{-1}\left(\frac{(m_1+g)(1-\beta)}{m_1-m_1'}\right)}{m_1 - m_1'} \\ &+ \frac{\left(m_1 - m_2\right) F^{-1}\left(\frac{m_1-\beta m_1'+g(1-\beta)}{m_1-m_1'}\right)}{m_1 - m_1'}, \end{aligned} \tag{2.6.113}$$

$$t_2^* = F^{-1}\left(\frac{m_1 + g + c_{h,2}\beta}{m_2 + c_{h,2} + g}\right) \tag{2.6.114}$$

$$\begin{aligned} \alpha^* &= \left(m_1 + c_{h,2} + g\right) F^{-1}\left(\frac{m_1 - \beta m_1' + g(1-\beta)}{m_1 - m_1'}\right) \\ &- (m_1 + g)t_1^* - c_{h,2}t_2^*. \end{aligned} \tag{2.6.115}$$

We also find the following parameter values

$$T_1(t_1^*, t_2^*, \alpha^*) = F^{-1}\left(\frac{(m_1 + g)(1-\beta)}{m_1 - m_1'}\right), \tag{2.6.116}$$

$$T_2(t_1^*, t_2^*, \alpha^*) = F^{-1}\left(\frac{m_1 - \beta m_1' + g(1-\beta)}{m_1 - m_1'}\right). \tag{2.6.117}$$

Now, several assumptions are required in order to guarantee that this solution belongs to the interior of $\mathbf{C_{1,2}}$. Basically these assumptions are the following

$$\frac{(m_1 + g)(1-\beta)}{m_1 - m_1'} < 1, \tag{2.6.118}$$

$$t_1^* < t_2^*, \tag{2.6.119}$$

$$\begin{aligned} \tilde{\alpha}_{3,1}(t_1^*, t_2^*) = m_2(t_2^* - t_1^*) + g(t_2^* - t_1^*) &< \\ \alpha^* < \tilde{\alpha}_{3,2}(t_1^*, t_2^*) &= m_1(t_2^* - t_1^*) + g(t_2^* - t_1^*). \end{aligned} \tag{2.6.120}$$

From Figure (2.7) it can be seen that the last condition is equivalent to

$$T_1(t_1^*, t_2^*, \alpha^*) \leq t_1^*, \tag{2.6.121}$$

$$t_1^* \leq T_2(t_1^*, t_2^*, \alpha^*), \tag{2.6.122}$$

$$T_2(t_1^*, t_2^*, \alpha^*) \leq t_2^*. \tag{2.6.123}$$

First condition analysis. The first assumption is independent of the probability distribution and amounts to the condition on the parameters:

$$m_1' < -g(1-\beta) + \beta m_1. \tag{2.6.124}$$

If $m_1' < -g$ then condition 1 holds for any probability distribution and for all β values one has existence of t_1^*.

If $m_1 \geq m_1' \geq -g$, then for any probability distribution F, there exists a lower bound β_F such that for any β values with $\beta \geq \beta_F$, one has no existence of t_1^*.

Second condition analysis. The second condition is not easy and in general, for arbitrary values of the parameters and of β, can depend on the probability distribution. However, it can be seen that under parameters conditions corresponding to Case 2, expression of t_1^* corresponds to a convex combination of $F^{-1}\left(\frac{(m_1+g)(1-\beta)}{m_1-m_1'}\right)$ and of $F^{-1}\left(\frac{m_1-\beta m_1'+g(1-\beta)}{m_1-m_1'}\right)$. As a consequence some properties can be found, depending on the order associated with $\frac{m_1-\beta m_1'+g(1-\beta)}{m_1-m_1'}$, $\frac{m_2+g+c_{h,2}\beta}{m_2+c_{h,2}+g}$ and $\frac{(m_1+g)(1-\beta)}{m_1-m_1'}$.

If $\frac{m_1-\beta m_1'+g(1-\beta)}{m_1-m_1'} < \frac{m_2+g+c_{h,2}\beta}{m_2+c_{h,2}+g}$ then for any probability distribution and for all β values one has $t_1^* < t_2^*$.

If $\frac{m_1-\beta m_1'+g(1-\beta)}{m_1-m_1'} \geq \frac{m_2+g+c_{h,2}\beta}{m_2+c_{h,2}+g}$ then for any probability distribution F, there exists an upper bound β_F such that for any β values with $\beta \leq \beta_F$, one has $t_1^* \geq t_2^*$.

Third condition analysis. It is direct to see that conditions (2.6.121) and (2.6.122) hold for any distribution and any parameters. In fact, condition (2.6.123) amounts again to $\frac{m_1-\beta m_1'+g(1-\beta)}{m_1-m_1'} < \frac{m_2+g+c_{h,2}\beta}{m_2+c_{h,2}+g}$.

Third step : optimal solution on a boundary of a region. If the optimal solution is not in the interior of a region (i.e., if above conditions do not hold) then the optimal solution has to be found on the boundary of region between two regions (Solved Previously).

- $\tilde{\alpha}_{1,1}(t_1,t_2) \leq \tilde{\alpha}_{1,3}(t_1,t_2) \leq \tilde{\alpha}_{1,2}(t_1,t_2)$ $\mathbf{C_{1,1}}$, $\mathbf{C_{1,2}}$, $\mathbf{C_{1,3}}$ and $\mathbf{C_{1,4}}$, as

$$\mathbf{C_{1,1}} = \{(t_1,t_2,\alpha) \text{ with } (t_1,t_2) \in \mathbf{R_1} \text{ and } \alpha \in]\infty, \tilde{\alpha}_{1,1}(t_1,t_2)[, \tag{2.6.125}$$

$$\mathbf{C_{1,2}} = \{(t_1,t_2,\alpha) \text{ with } (t_1,t_2) \in \mathbf{R_1} \text{ and } \alpha \in]\tilde{\alpha}_{1,1}(t_1,t_2), \tilde{\alpha}_{1,3}(t_1,t_2)[, \tag{2.6.126}$$

$$\mathbf{C_{1,3}} = \{(t_1,t_2,\alpha) \text{ with } (t_1,t_2) \in \mathbf{R_1} \text{ and } \alpha \in]\tilde{\alpha}_{1,3}(t_1,t_2), \tilde{\alpha}_{1,2}(t_1,t_2)[, \tag{2.6.127}$$

$$\mathbf{C_{1,4}} = \{(t_1,t_2,\alpha) \text{ with } (t_1,t_2) \in \mathbf{R_1} \text{ and } \alpha \in]\tilde{\alpha}_{1,2}(t_1,t_2), \infty[. \tag{2.6.128}$$

Proof: *Determination of the optimal policies.*

First step : expression of the first order conditions.

The region $\mathbf{C_{1,1}}$. In this region, the objective function is given by

$$\begin{aligned} l_{\beta,1}(t_1,t_2,\alpha) = \alpha \;&+\; \frac{1}{1-\beta}\Big[(m_2t_2 - m_1't_1)F(t_1) + (m_1' - m_2)G(t_1) \\ &+\; m_1(\mu - G(t_1) - t_1(1-F(t_1))) + m_2t_2(F(t_2) - F(t_1)) \\ &+\; (c_h + g)(\mu - G(t_2) - t_2(1-F(t_2))) + g(t_2 - t_1) \\ &-\; \alpha\Big]. \end{aligned} \tag{2.6.129}$$

The optimization problem can be rewritten

$$\begin{aligned} \min \quad l_{\beta,1}(t_1,t_2,\alpha) = \alpha \;&+\; \frac{1}{1-\beta}\Big[(m_2t_2 - m_1't_1)F(t_1) + (m_1' - m_2)G(t_1) \\ &+\; m_1(\mu - G(t_1) - t_1(1-F(t_1))) + m_2t_2(F(t_2) - F(t_1)) \\ &+\; (c_h + g)(\mu - G(t_2) - t_2(1-F(t_2))) + g(t_2 - t_1) \\ &-\; \alpha\Big], \end{aligned} \tag{2.6.130}$$

$$\text{s.t.} \quad (t_1,t_2,\alpha) \in \mathbf{C_{1,1}}. \tag{2.6.131}$$

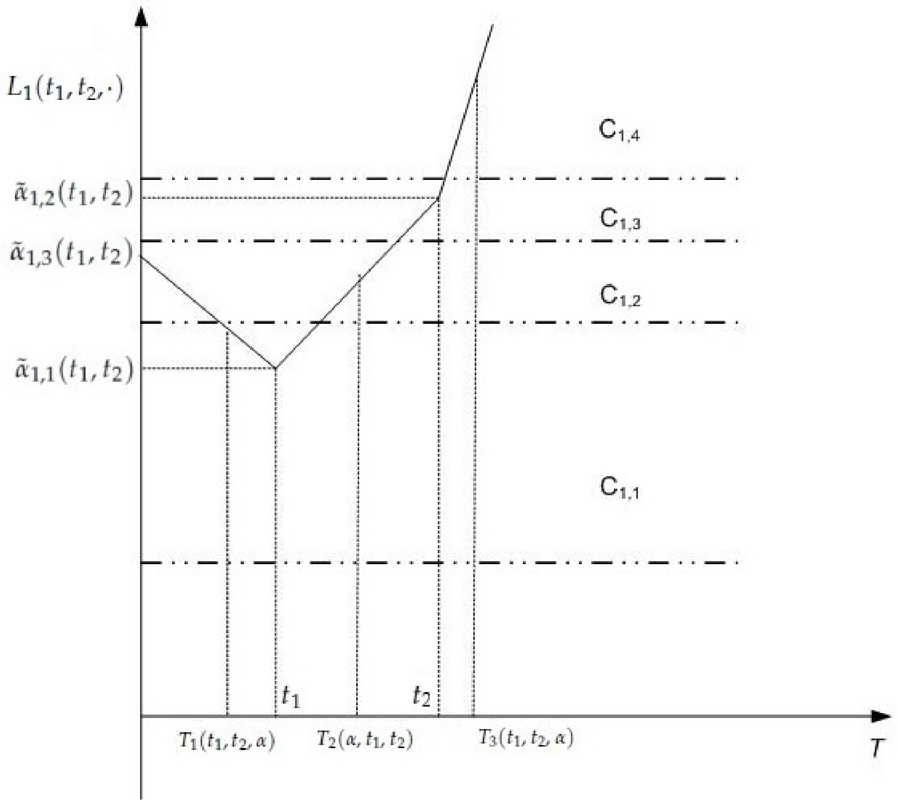

Figure 2.8: Four cases in minimization of CVaR in the region $\mathbf{R_1}$

The first order derivatives of (2.6.130) are given by

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} = \frac{-\beta}{1-\beta} < 0, \tag{2.6.132}$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} = \frac{(m_1-m_1')}{1-\beta}F(t_1) - \frac{(m_1+g)}{1-\beta}, \tag{2.6.133}$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2+c_h+g)}{1-\beta}F(t_2) - \frac{c_h}{1-\beta}. \tag{2.6.134}$$

The region $\mathbf{C_{1,2}}$.

According to Figure 2.8, let's define $T_1(\alpha, t_1, t_2)$ as the T value corresponding to:

$$\alpha = -m_1'(t_1 - T) + g(t_2 - t_1) + m_2(t_2 - T). \tag{2.6.135}$$

and $T_2(\alpha, t_1, t_2)$ as the T value corresponding to:

$$\alpha = m_1(T - t_1) + g(t_2 - t_1) + m_2(t_2 - T) \tag{2.6.136}$$

The optimization problem can be rewritten

$$\begin{aligned} \min \quad l_{\beta,1}(t_1,t_2,\alpha) = \alpha \quad &+ \quad \frac{1}{1-\beta}\Big(-(m_1'+g)t_1 + (m_2+g)t_2 - \alpha)F(T_1(t_1,t_2,\alpha)) + (m_1'-m_2)G(T_1(t_1,t_2,\alpha)) \\ &+ \quad (m_1-m_2)(G(t_2) - G(T_2(t_1,t_2,\alpha))) \\ &+ \quad (-m_1t_1 + g(t_2-t_1) + m_2t_2 - \alpha)(F(t_2) - F(T_2(t_1,t_2,\alpha))) \\ &+ \quad (m_1+c_h+g)(\mu - G(t_2)) \\ &+ \quad (-m_1t_1 - gt_1 - c_ht_2 - \alpha)(1-F(t_2))\Big), \end{aligned} \tag{2.6.137}$$

$$\text{s.t.} \qquad (t_1,t_2,\alpha) \in \mathbf{C_{1,2}}. \tag{2.6.138}$$

The first order derivatives of (2.6.137) are given by:

$$\begin{aligned} \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} &= -\frac{(m_1'+g)}{1-\beta}F(T_1(\alpha,t_1,t_2)) + \frac{(m_1+g)}{1-\beta}F(T_2(\alpha,t_1,t_2)) \\ &- \frac{(m_1+g)}{1-\beta}, \end{aligned} \tag{2.6.139}$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2+g)}{1-\beta}(F(T_1(\alpha,t_1,t_2)) - F(T_2(\alpha,t_1,t_2))) - \frac{c_h}{1-\beta}(1-F(t_2)) + (\frac{m_1+g)}{1-\beta}F(t_2), \tag{2.6.140}$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_2(t_1,t_2,\alpha)) - F(T_1(t_1,t_2,\alpha)) - \beta}{1-\beta}. \tag{2.6.141}$$

The region $\mathbf{C_{1,3}}$.

According to Figure 2.8, let's define $T_3(\alpha, t_1, t_2)$ as the T value corresponding to:

$$\alpha = m_1(T - t_1) + g(t_2 - t_1) + (c_h + g)(T - t_2) \tag{2.6.142}$$

The optimization problem becomes

$$\begin{aligned} \min \quad l_{\beta,1}(t_1, t_2, \alpha) &= \alpha + \frac{1}{1-\beta}\Big[(m_1 - m_2)(G(t_2) - G(T_2(t_1, t_2, \alpha))) + (m_1 + c_h + g)(\mu - G(t_2)) \\ &+ (-m_1 t_1 + g(t_2 - t_1) + m_2 t_2 - \alpha)(F(t_2) - F(T_2(t_1, t_2, \alpha))) \\ &+ (-m_1 t_1 - g t_1 - c_h t_2 - \alpha)(1 - F(t_2))\Big], \qquad (2.6.143) \\ \text{s.t.} \quad & (t_1, t_2, \alpha) \in \mathbf{C_{1,3}}. \qquad (2.6.144) \end{aligned}$$

The first order derivatives of (2.6.143) are given by:

$$\begin{aligned} \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} &= -\frac{(m_1 + g)}{1-\beta}(F(t_2) - F(T_2(\alpha, t_1, t_2))) - \frac{(m_1 + g)}{1-\beta}(1 - F(t_2)), \qquad (2.6.145) \\ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} &= \frac{(m_2 + g)}{1-\beta}(F(t_2) - F(T_2(\alpha, t_1, t_2))) - \frac{c_h}{1-\beta}(1 - F(t_2)), \qquad (2.6.146) \\ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} &= \frac{F(T_2(t_1, t_2, \alpha)) - \beta}{1-\beta}. \qquad (2.6.147) \end{aligned}$$

The region $\mathbf{C_{1,4}}$.
The optimization problem becomes

$$\begin{aligned} \min \quad l_{\beta,1}(t_1, t_2, \alpha) &= \alpha + \frac{1}{1-\beta}\Big[(m_1 + c_h + g)(\mu - G(T_3(t_1, t_2, \alpha))) \\ &+ (-m_1 t_1 - g t_1 - c_h t_2 - \alpha)(1 - F(T_3(t_1, t_2, \alpha)))\Big], \qquad (2.6.148) \\ \text{s.t.} \quad & (t_1, t_2, \alpha) \in \mathbf{C_{1,4}}. \qquad (2.6.149) \end{aligned}$$

The first order derivatives of (2.6.148) are given by:

$$\begin{aligned} \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} &= -\left(\frac{m_1 + g}{1-\beta}\right)\Big(1 - F(T_3(t_1, t_2, \alpha))\Big), \qquad (2.6.150) \\ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} &= -\left(\frac{c_h}{1-\beta}\right)\Big(1 - F(T_3(t_1, t_2, \alpha))\Big), \qquad (2.6.151) \\ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} &= \frac{F(T_3(t_1, t_2, \alpha) - \beta}{1-\beta}. \qquad (2.6.152) \end{aligned}$$

Corollary. In the interior of region $\mathbf{R_1}$, the CVaR loss function $l_{\beta,1}(t_1, t_2, \alpha)$ is differentiable w.r.t. α, t_1 and t_2.

Corollary. By convexity, for fixed t_1 and t_2 in region $\mathbf{R_1}$, the optimal α value can always be found as the solution of the first order condition.

Corollary. By convexity and derivability, if the optimal solution lies in the interior of the region $\mathbf{R_1}$ it is given by the solution of the first order condition.

Second step : optimal solution in the interior of a region. It is direct to see that the only case where the first order conditions possibly have a solution is the region $\mathbf{C_{1,2}}$. Under adequate assumptions, the first order conditions (2.6.139)-(2.6.141) have the solution

$$t_1^* = \frac{\left(m_2 - m_1'\right)F^{-1}\left(\frac{(m_1+g)(1-\beta)}{m_1-m_1'}\right) + \left(m_1 - m_2\right)F^{-1}\left(\frac{m_1-\beta m_1'+g(1-\beta)}{m_1-m_1'}\right)}{m_1 - m_1'}, \tag{2.6.153}$$

$$t_2^* = F^{-1}\left(\frac{m_1 + g + c_h\beta}{m_2 + c_h + g}\right) \tag{2.6.154}$$

$$\alpha^* = \left(m_1 + c_h + g\right)F^{-1}\left(\frac{m_1 - \beta m_1' + g(1-\beta)}{m_1 - m_1'}\right) - (m_1+g)t_1^* - c_h t_2^*. \tag{2.6.155}$$

We also find the following parameter values

$$T_1(t_1^*, t_2^*, \alpha^*) = F^{-1}\left(\frac{(m_1+g)(1-\beta)}{m_1 - m_1'}\right), \tag{2.6.156}$$

$$T_2(t_1^*, t_2^*, \alpha^*) = F^{-1}\left(\frac{m_1 - \beta m_1' + g(1-\beta)}{m_1 - m_1'}\right). \tag{2.6.157}$$

Now, several assumptions are required in order to guarantee that this solution belongs to the interior of $\mathbf{C_{1,2}}$. Basically these assumptions are the following

$$\frac{(m_1+g)(1-\beta)}{m_1 - m_1'} < 1, \tag{2.6.158}$$

$$t_1^* < t_2^*, \tag{2.6.159}$$

$$\begin{aligned} \tilde{\alpha}_{3,1}(t_1^*, t_2^*) = m_1(t_2^* - t_1^*) + g(t_2^* - t_1^*) &< \alpha^* < \tilde{\alpha}_{3,3}(t_1^*, t_2^*) \\ &= (m_2+g)t_2^* - (m_1'+g)t_1^*. \end{aligned} \tag{2.6.160}$$

From Figure (2.8) it can be seen that the last condition is equivalent to

$$T_1(t_1^*, t_2^*, \alpha^*) \le t_1^*, \tag{2.6.161}$$

$$t_1^* \le T_2(t_1^*, t_2^*, \alpha^*), \tag{2.6.162}$$

$$T_2(t_1^*, t_2^*, \alpha^*) \le t_2^*. \tag{2.6.163}$$

First condition analysis. The first assumption is independent of the probability distribution and amounts to the condition on the parameters:

$$m_1' < -g(1-\beta) + \beta m_1. \tag{2.6.164}$$

Corollary. If $m'_1 < -g$ then condition 1 holds for any probability distribution and for all β values one has existence of t_1^*.

Corollary. If $m_1 \geq m'_1 \geq -g$, then for any probability distribution F, there exists a lower bound β_F such that for any β values with $\beta \geq \beta_F$, one has no existence of t_1^*.

Second condition analysis. The second condition is not easy and in general, for arbitrary values of the parameters and of β, can depend on the probability distribution. However, it can be seen that under parameters conditions corresponding to Case 2, expression of t_1^* corresponds to a convex combination of $F^{-1}\left(\frac{(m_1+g)(1-\beta)}{m_1-m'_1}\right)$ and of $F^{-1}\left(\frac{m_1-\beta m'_1+g(1-\beta)}{m_1-m'_1}\right)$. As a consequence some properties can be found, depending on the order associated with $\frac{m_1-\beta m'_1+g(1-\beta)}{m_1-m'_1}$, $\frac{m_2+g+c_h\beta}{m_2+c_h+g}$ and $\frac{(m_1+g)(1-\beta)}{m_1-m'_1}$.

Corollary. If $\frac{m_1-\beta m'_1+g(1-\beta)}{m_1-m'_1} < \frac{m_2+g+c_h\beta}{m_2+c_h+g}$ then for any probability distribution and for all β values one has $t_1^* < t_2^*$.

Corollary. If $\frac{m_1-\beta m'_1+g(1-\beta)}{m_1-m'_1} \geq \frac{m_2+g+c_h\beta}{m_2+c_h+g}$ then for any probability distribution F, there exists an upper bound β_F such that for any β values with $\beta \leq \beta_F$, one has $t_1^* \geq t_2^*$.

Third condition analysis. It is direct to see that conditions (2.6.161) and (2.6.162) hold for any distribution and any parameters. In fact, condition (2.6.163) amounts again to $\frac{m_1-\beta m'_1+g(1-\beta)}{m_1-m'_1} < \frac{m_2+g+c_h\beta}{m_2+c_h+g}$.

Third step : optimal solution on a boundary of a region. If the optimal solution is not in the interior of a region (i.e., if above conditions do not hold) then the optimal solution has to be found on the boundary of region between two regions.

Appendix C: The region $\mathbf{R_2}$.

In $\mathbf{R_2}$, the associated optimization problem is thus given by

$$\begin{aligned} \min\ l_{\beta,2}(t_1,t_2,\alpha) = \alpha \quad &+ \quad \frac{1}{1-\beta}\Bigg(\int_0^{t_2}[-m_1'(t_2-T)+m_2(t_2-T)+s_1(t_1-t_2)-\alpha]^+ f(T)dT \\ &+ \quad \int_{t_2}^{t_1}[+s_1(t_1-T)+c_{h,2}(T-t_2)-\alpha]^+ f(T)dT \\ &+ \quad \int_{t_1}^{\infty}[m_1(T-t_1)+g(T-t_1)+c_{h,2}(T-t_2)-\alpha]^+ f(T)dT\Bigg) \quad (2.6.165) \\ \text{s.t.} \quad & \quad (t_1,t_2)\in \mathbf{R_2}, \quad -\infty \le \alpha \le \infty. \quad (2.6.166) \end{aligned}$$

Depending on the convexity/concavity of $l_{\beta,2}(t_1,t_2,\alpha)$, we divide our analysis into two cases: If $m_2 - m_1' - s_1 + c_{h,2} > 0$ then $l_{\beta,2}(t_1,t_2,\alpha)$ is strictly convex, else if $m_2 - m_1' - s_1 + c_{h,2} < 0$ then $l_{\beta,2}(t_1,t_2,\alpha)$ is strictly concave.

Furthermore, there there are two cases have to be considered, these values characterizing the structure of the solution :

- $c_{h,2} \ge s_1$,
- $c_{h,2} \le s_1$.

Appendix C-1 $c_{h,2} \ge s_1$:

It can be seen that the critical values for the α parameters corresponding to the slope discontinuities for the piecewise linear function (2.6.165), as functions of t_1 and t_2, are given by

$$\tilde{\alpha}_{2,1}(t_1,t_2) = s_1(t_1-t_2), \quad (2.6.167)$$

$$\tilde{\alpha}_{2,2}(t_1,t_2) = (m_2-m_1')t_2 + s_1(t_1-t_2), \quad (2.6.168)$$

$$\tilde{\alpha}_{2,3}(t_1,t_2) = c_{h,2}(t_1-t_2). \quad (2.6.169)$$

with $\tilde{\alpha}_{2,1}(t_1, t_2) \le \tilde{\alpha}_{2,2}(t_1, t_2) \le \tilde{\alpha}_{2,3}(t_1, t_2)$ (see Figure (2.9)).

It can be seen that the critical values for the α parameters corresponding to the slope discontinuities for the piecewise linear function (2.6.165), as functions of t_1 and t_2, are given by

$$\tilde{\alpha}_{2,1}(t_1, t_2) = s_1(t_1 - t_2), \quad (2.6.170)$$

$$\tilde{\alpha}_{2,2}(t_1, t_2) = (m_2 - m_1')t_2 + s_1(t_1 - t_2), \quad (2.6.171)$$

$$\tilde{\alpha}_{2,3}(t_1, t_2) = c_{h,2}(t_1 - t_2). \quad (2.6.172)$$

with $\tilde{\alpha}_{2,1}(t_1, t_2) \le \tilde{\alpha}_{2,2}(t_1, t_2) \le \tilde{\alpha}_{2,3}(t_1, t_2)$ (see Figure (2.9)). In order to characterize the

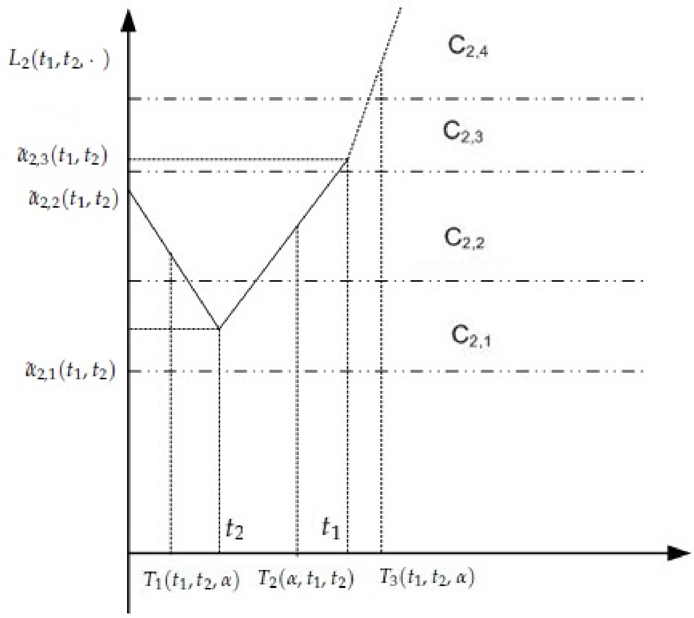

Figure 2.9: Four cases in minimization of CVaR in the region $\mathbf{R_2}$ for $c_{h,2} \ge s_1$

first order conditions, we define the regions $\mathbf{C_{2,1}}$, $\mathbf{C_{2,2}}$, $\mathbf{C_{2,3}}$ and $\mathbf{C_{2,4}}$, as

$$\mathbf{C_{2,1}} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbf{R_2} \text{ and } \alpha \in]\infty, \tilde{\alpha}_{2,1}(t_1, t_2)[, \quad (2.6.173)$$

$$\mathbf{C_{2,2}} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbf{R_2} \text{ and } \alpha \in]\tilde{\alpha}_{2,1}(t_1, t_2), \tilde{\alpha}_{2,2}(t_1, t_2)[, \quad (2.6.174)$$

$$\mathbf{C_{2,3}} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbf{R_2} \text{ and } \alpha \in]\tilde{\alpha}_{2,2}(t_1, t_2), \tilde{\alpha}_{2,3}(t_1, t_2)[, \quad (2.6.175)$$

$$\mathbf{C_{2,4}} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbf{R_2} \text{ and } \alpha \in]\tilde{\alpha}_{2,3}(t_1, t_2), \infty[. \quad (2.6.176)$$

The main result for CVAR in region $\mathbf{R_2}$

Proof: *Determination of the optimal policies.*

First step : expression of the first order conditions.

The region $\mathbf{C_{2,1}}$. In this region, the objective function given in expression (2.6.165) is

$$\begin{aligned} l_{\beta,2} &= \alpha + \frac{1}{1-\beta}\Big[((m_2 - m_1')t_2 + s_1(t_1 - t_2))F(t_2) + (m_1' - m_2)G(t_2) \\ &+(s_1t_1 - c_{h,2}t_2)(F(t_1) - F(t_2)) + (c_{h,2} - s_1)(G(t_1) - G(t_2)) \\ &-((m_1 + g)t_1 + c_{h,2}t_2)(1 - F(t_1)) + (m_1 + c_{h,2} + g)(\mu - G(t_1)) - \alpha\Big] \end{aligned} \tag{2.6.177}$$

The optimization problem can be rewritten

$$\begin{aligned} l_{\beta,2} &= \alpha + \frac{1}{1-\beta}\Big[((m_2 - m_1')t_2 + s_1(t_1 - t_2))F(t_2) + (m_1' - m_2)G(t_2) \\ &+(s_1t_1 - c_{h,2}t_2)(F(t_1) - F(t_2)) + (c_{h,2} - s_1)(G(t_1) - G(t_2)) \\ &-((m_1 + g)t_1 + c_{h,2}t_2)(1 - F(t_1)) + (m_1 + c_{h,2} + g)(\mu - G(t_1)) - \alpha\Big] \end{aligned} \tag{2.6.178}$$

$$\text{s.t.} \quad (t_1, t_2, \alpha) \in \mathbf{C_{2,1}}. \tag{2.6.179}$$

The region $\mathbf{C_{2,1}}$. The first order derivatives of (2.6.178) are given by

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = -\frac{\beta}{(1-\beta)}, \tag{2.6.180}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{(s_1 + m_1 + g)F(t_1) - (m_1 + g)}{(1-\beta)}, \tag{2.6.181}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2 - m_1' - s_1 + c_{h,2})F(t_2) - c_{h,2}}{(1-\beta)}. \tag{2.6.182}$$

The region $\mathbf{C_{2,2}}$.

According to Figure 2.9, let's define $T_1(\alpha, t_1, t_2)$ as the T value corresponding to:

$$\alpha = -m_1'(t_2 - T) + m_2(t_2 - T) + s_1(t_1 - t_2) \tag{2.6.183}$$

and $T_2(\alpha, t_1, t_2)$ as the T value corresponding to:

$$\alpha = +s_1(t_1 - T) + c_{h,2}(T - t_2) \tag{2.6.184}$$

The optimization problem can be rewritten

$$\begin{aligned} l_{\beta,2} &= \alpha + \frac{1}{1-\beta}\Big[(m_2 - m_1')t_2 + s_1(t_1 - t_2) - \alpha\Big]F(T_1(\alpha,t_1,t_2)) \\ &+ \frac{1}{1-\beta}[m_1' - m_2]G(T_1(\alpha,t_1,t_2)) \\ &+ \frac{1}{1-\beta}\Big[(s_1t_1 - c_{h,2}t_2 - \alpha)(F(t_1) - F(T_2(\alpha,t_1,t_2))) \\ &+ (c_{h,2} - s_1)(G(t_1) - G(T_2(\alpha,t_1,t_2)))\Big] \\ &+ \frac{1}{1-\beta}\Big[-((m_1 + g)t_1 + c_{h,2}t_2 + \alpha)(1 - F(t_1)) + (m_1 + c_{h,2} + g)(\mu - G(t_1))\Big], \end{aligned} \tag{2.6.185}$$

$$\text{s.t.} \quad (t_1, t_2, \alpha) \in \mathbf{C_{2,2}}. \tag{2.6.186}$$

The first order derivatives of (2.6.185) are given by:

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{s_1(F(t_1)+F(T_1(\alpha,t_1,t_2))-F(T_2(\alpha,t_1,t_2))+(m_1+g)(F(t_1)-1)}{1-\beta}, \quad (2.6.187)$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2-s_1-m_1')F(T_1(\alpha,t_1,t_2))-c_{h,2}(1-F(T_2(\alpha,t_1,t_2)))}{1-\beta}, \quad (2.6.188)$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_2(\alpha,t_1,t_2))-F(T_1(\alpha,t_1,t_2))-\beta}{1-\beta}. \quad (2.6.189)$$

The region $\mathbf{C_{2,3}}$.

According to Figure 2.9, let's define $T_3(\alpha,t_1,t_2)$ as the T value corresponding to:

$$\alpha = m_1(T-t_1)+g(T-t_1)+c_{h,2}(T-t_2). \quad (2.6.190)$$

The optimization problem becomes

$$\begin{aligned} \min \quad l_{\beta,2}(t_1,t_2,\alpha) &= \alpha+\frac{1}{1-\beta}\Big[(s_1t_1-c_{h,2}t_2-\alpha)(F(t_1)-F(T_2(\alpha,t_1,t_2))) \\ &+ (c_{h,2}-s_1)(G(t_1)-G(T_2(\alpha,t_1,t_2)))\Big] \\ &+ \frac{1}{1-\beta}\Big[-((m_1+g)t_1+c_{h,2}t_2+\alpha)(1-F(t_1))+(m_1+c_{h,2}+g)(\mu-G(t_1))\Big], \quad (2.6.191) \\ \text{s.t.} \quad & (t_1,t_2,\alpha)\in \mathbf{C_{2,3}}. \quad (2.6.192) \end{aligned}$$

The first order derivatives of (2.6.191) are given by:

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{s_1}{1-\beta}(F(t_1)-F(T_2(t_1,t_2,\alpha)))-\frac{1}{1-\beta}(m_1+g)(1-F(t_1))), \quad (2.6.193)$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{-c_{h,2}}{1-\beta}(1-F(T_2(t_1,t_2,\alpha))), \quad (2.6.194)$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_2(t_1,t_2,\alpha))-\beta}{1-\beta}. \quad (2.6.195)$$

The region $\mathbf{C_{2,4}}$.

The optimization problem becomes

$$\begin{aligned} \min \quad l_{\beta,2}(t_1,t_2,\alpha) &= \alpha+\frac{1}{1-\beta}\Big[-((m_1+g)t_1+c_{h,2}t_2+\alpha)(1-F(T_3(t_1,t_2,\alpha)) \\ &+ (m_1+c_{h,2}+g)(\mu-G(T_3(t_1,t_2,\alpha)))\Big] \quad (2.6.196) \\ \text{s.t.} \quad & (t_1,t_2,\alpha)\in \mathbf{C_{2,4}}. \quad (2.6.197) \end{aligned}$$

The first order derivatives of (2.6.196) are given by:

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{-1}{1-\beta}(m_1+g)(1-F(T_3(\alpha,t_1,t_2))), \quad (2.6.198)$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{-c_{h,2}}{1-\beta}(1-F(T_3(\alpha,t_1,t_2))), \quad (2.6.199)$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_3(\alpha,t_1,t_2))-\beta}{1-\beta}. \quad (2.6.200)$$

In the interior of region $\mathbf{R_2}$, the CVaR loss function $l_{\beta,2}(t_1,t_2,\alpha)$ is differentiable w.r.t. α, t_1 and t_2.

By convexity, for fixed t_1 and t_2 in region $\mathbf{R_2}$, the optimal α value can always be found as the solution of the first order condition.

By convexity and derivability, if the optimal solution lies in the interior of the region $\mathbf{R_2}$ it is given by the solution of the first order condition.

Second step : optimal solution in the interior of a region. It is direct to see that the only case where the first order conditions possibly have a solution is the region $\mathbf{C_{2,2}}$. Under adequate assumptions, the first order conditions (2.6.324)-(2.6.326) have the solution

$$t_1^* = F^{-1}\left(\frac{m_1 + g + s_1\beta}{m_1 + g + s_1}\right), \tag{2.6.201}$$

$$\begin{aligned} t_2^* &= \left(\frac{m_2 - m_1'}{m_2 - m_1' - s_1 + c_{h,2}}\right) F^{-1}\left(\frac{c_{h,2}(1-\beta)}{m_2 - m_1' - s_1 + c_{h,2}}\right) \\ &- \left(\frac{s_1 - c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}}\right) F^{-1}\left(\frac{c_{h,2} + \beta(m_2 - m_1' - s_1)}{m_2 - m_1' - s_1 + c_{h,2}}\right), \end{aligned} \tag{2.6.202}$$

$$\alpha^* = \left(c_{h,2} - s_1\right) F^{-1}\left(\frac{c_{h,2} + \beta(m_2 - m_1' - s_1)}{m_2 - m_1' - s_1 + c_{h,2}}\right) + s_1 t_1^* - c_{h,2} t_2^*. \tag{2.6.203}$$

We also find the following parameter values

$$T_1(t_1^*, t_2^*, \alpha^*) = F^{-1}\left(\frac{c_{h,2}(1-\beta)}{m_2 - m_1' - s_1 + c_{h,2}}\right), \tag{2.6.204}$$

$$T_2(t_1^*, t_2^*, \alpha^*) = F^{-1}\left(\frac{c_{h,2} + \beta(m_2 - m_1' - s_1)}{m_2 - m_1' - s_1 + c_{h,2}}\right). \tag{2.6.205}$$

Now, several assumptions are required in order to guarantee that this solution belongs to the interior of $\mathbf{C_{2,2}}$. Basically these assumptions are the following

$$F\left(\frac{c_{h,2} + \beta(m_2 - m_1' - s_1)}{m_2 - m_1' - s_1 + c_{h,2}}\right) < 1, \tag{2.6.206}$$

$$t_2^* < t_1^*, \tag{2.6.207}$$

$$\begin{aligned} \tilde{\alpha}_{2,1}(t_1^*, t_2^*) = s_1(t_1^* - t_2^*) &< \alpha^* < \tilde{\alpha}_{2,2}(t_1^*, t_2^*) \\ &= (m_2 - m_1')t_2^* + s_1(t_1^* - t_2^*). \end{aligned} \tag{2.6.208}$$

From Figure (2.9) it can be seen that the last condition is equivalent to

$$T_1(t_1^*, t_2^*, \alpha^*) \leq t_2^*, \tag{2.6.209}$$

$$t_2^* \leq T_2(t_1^*, t_2^*, \alpha^*), \tag{2.6.210}$$

$$T_2(t_1^*, t_2^*, \alpha^*) \leq t_1^*. \tag{2.6.211}$$

First condition analysis. The first assumption is independent of the probability distribution and amounts to the condition on the parameters:

$$c_{h,2} + \beta(m_2 - m_1' - s_1) < m_2 - m_1' - s_1 + c_{h,2} \tag{2.6.212}$$

If $m_2 - m_1' - s_1 \geq 0$ then condition 1 holds for any probability distribution and for all β values one has existence of t_2^*.

Second condition analysis. The second condition is not easy and in general, for arbitrary values of the parameters and of β, can depend on the probability distribution. However, it can be seen that under parameters conditions corresponding to Case 1, expression of t_2^* corresponds to a convex combination of $F^{-1}\left(\frac{c_{h,2}(1-\beta)}{m_2-m_1'-s_1+c_{h,2}}\right)$ and of $F^{-1}\left(\frac{c_{h,2}+\beta(m_2-m_1'-s_1)}{m_2-m_1'-s_1+c_{h,2}}\right)$. As a consequence some properties can be found, depending on the order associated with $\frac{m_1+g+s_1\beta}{m_1+g+s_1}$, $\frac{c_{h,2}(1-\beta)}{m_2-m_1'-s_1+c_{h,2}}$ and $\frac{c_{h,2}+\beta(m_2-m_1'-s_1)}{m_2-m_1'-s_1+c_{h,2}}$.

If $\frac{c_{h,2}+\beta(m_2-m_1'-s_1)}{m_2-m_1'-s_1+c_{h,2}} < \frac{m_1+g+s_1\beta}{m_1+g+s_1}$ then for any probability distribution one has $t_2^* < t_1^*$.

If $\frac{c_{h,2}+\beta(m_2-m_1'-s_1)}{m_2-m_1'-s_1+c_{h,2}} \geq \frac{m_1+g+s_1\beta}{m_1+g+s_1}$ then for any probability distribution F, there exists an upper bound β_F such that for any β values with $\beta \leq \beta_F$, one has $t_2^* \geq t_1^*$.

If $m_2 - m_1' - s_1 < 0$, then for any probability distribution F, there is no finite minimum inside the region.

Third condition analysis. It is direct to see that conditions (2.6.209) and (2.6.210) hold for any distribution and any parameters. In fact, condition (2.6.211) amounts again to $\frac{c_{h,2}+\beta(m_2-m_1'-s_1)}{m_2-m_1'-s_1+c_{h,2}} < \frac{m_1+g+s_1\beta}{m_1+g+s_1}$.

Third step : optimal solution on a boundary of a region. If the optimal solution is not in the interior of a region (i.e., if above conditions do not hold) then the optimal solution has to be found on the boundary of region between two regions. The potential boundaries are defined as

- $t_1 = t_2$

- $\alpha(t_1, t_2) = \tilde{\alpha}_{2,i}(t_1, t_2)$ for $i = 1, .., 3$.

Solved previously for case $t_1 \leq t_2$.

Now, for the case

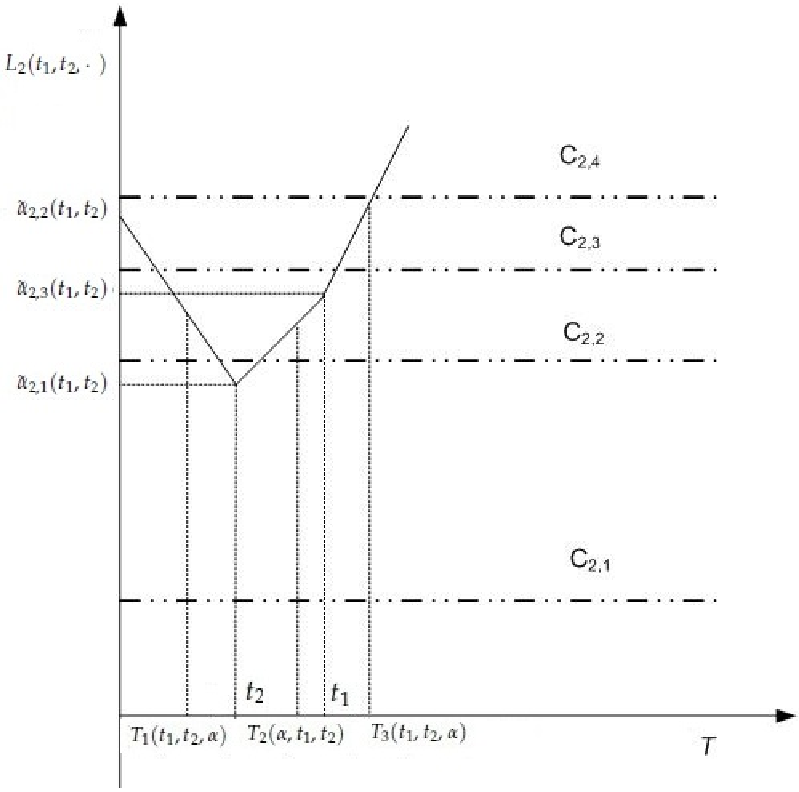

Figure 2.10: Four cases in minimization of CVaR in the region $\mathbf{R_2}$

In order to characterize the first order conditions, we define the regions $\mathbf{C_{2,1}}$, $\mathbf{C_{2,2}}$, $\mathbf{C_{2,3}}$ and $\mathbf{C_{2,4}}$, as

$$\mathbf{C_{2,1}} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbf{R_2} \text{ and } \alpha \in]\infty, \tilde{\alpha}_{2,1}(t_1, t_2)[, \tag{2.6.213}$$

$$\mathbf{C_{2,2}} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbf{R_2} \text{ and } \alpha \in]\tilde{\alpha}_{2,1}(t_1, t_2), \tilde{\alpha}_{2,3}(t_1, t_2)[, \tag{2.6.214}$$

$$\mathbf{C_{2,3}} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbf{R_2} \text{ and } \alpha \in]\tilde{\alpha}_{2,3}(t_1, t_2), \tilde{\alpha}_{2,2}(t_1, t_2)[, \tag{2.6.215}$$

$$\mathbf{C_{2,4}} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbf{R_2} \text{ and } \alpha \in]\tilde{\alpha}_{2,2}(t_1, t_2), \infty[. \tag{2.6.216}$$

Proof:*Determination of the optimal policies.*

First step : expression of the first order conditions.

The region $\mathbf{C_{2,1}}$. In this region, the objective function given in expression is

$$\begin{aligned} l_{\beta,2} &= \alpha + \frac{1}{1-\beta}\Big[((m_2 - m_1')t_2 + s_1(t_1 - t_2))F(t_2) + (m_1' - m_2)G(t_2) \\ &\quad +(s_1t_1 - c_ht_2)(F(t_1) - F(t_2)) + (c_h - s_1)(G(t_1) - G(t_2)) \\ &\quad -((m_1 + g)t_1 + c_ht_2)(1 - F(t_1)) + (m_1 + c_h + g)(\mu - G(t_1)) \\ &- \alpha\Big] \end{aligned} \tag{2.6.217}$$

The optimization problem can be rewritten

$$\begin{aligned} l_{\beta,2} &= \alpha + \frac{1}{1-\beta}\Big[((m_2 - m_1')t_2 + s_1(t_1 - t_2))F(t_2) + (m_1' - m_2)G(t_2) \\ &\quad +(s_1t_1 - c_ht_2)(F(t_1) - F(t_2)) + (c_h - s_1)(G(t_1) - G(t_2)) \\ &\quad -((m_1 + g)t_1 + c_ht_2)(1 - F(t_1)) + (m_1 + c_h + g)(\mu - G(t_1)) \\ &- \alpha\Big] \end{aligned} \tag{2.6.218}$$

$$\text{s.t.} \qquad (t_1, t_2, \alpha) \in \mathbf{C_{2,1}}. \tag{2.6.219}$$

The first order derivatives of (2.6.218) are given by

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{d\alpha} = -\frac{\beta}{(1-\beta)}, \tag{2.6.220}$$

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_1} = \frac{(s_1 + m_1 + g)F(t_1) - (m_1 + g)}{(1-\beta)}, \tag{2.6.221}$$

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 - m_1' - s_1 + c_h)F(t_2) - c_h}{(1-\beta)}. \tag{2.6.222}$$

The region $\mathbf{C_{2,2}}$.

According to Figure 2.10, let's define $T_1(\alpha, t_1, t_2)$ as the T value corresponding to:

$$\alpha = -m_1'(t_2 - T) + m_2(t_2 - T) + s_1(t_1 - t_2) \tag{2.6.223}$$

and $T_2(\alpha, t_1, t_2)$ as the T value corresponding to:

$$\alpha = +s_1(t_1 - T) + c_h(T - t_2) \tag{2.6.224}$$

The optimization problem can be rewritten

$$\begin{aligned} l_{\beta,2} &= \alpha + \frac{1}{1-\beta}\Big[(m_2 - m_1')t_2 + s_1(t_1 - t_2) - \alpha\Big]F(T_1(\alpha, t_1, t_2)) \\ &+ \frac{1}{1-\beta}[m_1' - m_2]G(T_1(\alpha, t_1, t_2)) \\ &+ \frac{1}{1-\beta}\Big[(s_1 t_1 - c_h t_2 - \alpha)(F(t_1) - F(T_2(\alpha, t_1, t_2))) \\ &+ (c_h - s_1)(G(t_1) - G(T_2(\alpha, t_1, t_2)))\Big] \\ &+ \frac{1}{1-\beta}\Big[-((m_1 + g)t_1 + c_h t_2 + \alpha)(1 - F(t_1)) \\ &+ (m_1 + c_h + g)(\mu - G(t_1))\Big], \qquad (2.6.225) \\ \text{s.t.} \quad & (t_1, t_2, \alpha) \in \mathbf{C_{2,2}}. \qquad (2.6.226) \end{aligned}$$

The first order derivatives of (2.6.225) are given by:

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{s_1(F(t_1) + F(T_1(\alpha,t_1,t_2)) - F(T_2(\alpha,t_1,t_2)) + (m_1+g)(F(t_1)-1)}{1-\beta}, \qquad (2.6.227)$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2 - s_1 - m_1')F(T_1(\alpha,t_1,t_2)) - c_h(1 - F(T_2(\alpha,t_1,t_2)))}{1-\beta}, \qquad (2.6.228)$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_2(\alpha,t_1,t_2)) - F(T_1(\alpha,t_1,t_2)) - \beta}{1-\beta}. \qquad (2.6.229)$$

The region $\mathbf{C_{2,3}}$.

Let's define $T_3(\alpha, t_1, t_2)$ as the T value corresponding to:

$$\alpha = m_1(T - t_1) + g(T - t_1) + c_h(T - t_2). \qquad (2.6.230)$$

The optimization problem becomes

$$\begin{aligned} \min \quad l_{\beta,2}(t_1,t_2,\alpha) &= \alpha + \frac{1}{1-\beta}\Big[(m_1' - m_2)G(T_1(t_1,t_2,\alpha)) + ((m_2 - m_1' - s_1)t_2 + s_1 t_1 - \alpha)F(T_1(t_1,t_2,\alpha))\Big] \\ &+ \frac{1}{1-\beta}\Big[-((m_1+g)t_1 + c_h t_2 + \alpha)(1 - F(T_3(t_1,t_2,\alpha)) \\ &+ (m_1 + c_h + g)(\mu - G(T_3(t_1,t_2,\alpha)))\Big], \qquad (2.6.231) \\ \text{s.t.} \quad & (t_1,t_2,\alpha) \in \mathbf{C_{2,3}}. \qquad (2.6.232) \end{aligned}$$

The first order derivatives of (2.6.231) are given by:

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{s_1 F(T_1(\alpha,t_1,t_2)) - (m_1+g)(1 - F(T_3(\alpha,t_1,t_2)))}{1-\beta}, \qquad (2.6.233)$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2 - m_1' - s_1)F(T_1(\alpha,t_1,t_2)) - c_h(1 - F(T_3(\alpha,t_1,t_2)))}{1-\beta}, \qquad (2.6.234)$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_3(\alpha,t_1,t_2)) - F(T_1(\alpha,t_1,t_2)) - \beta}{1-\beta}. \qquad (2.6.235)$$

The region $\mathbf{C_{2,4}}$.
The optimization problem becomes

$$\begin{aligned} \min \quad l_{\beta,2}(t_1,t_2,\alpha) &= \alpha + \frac{1}{1-\beta}\Big[-((m_1+g)t_1 + c_h t_2 + \alpha)(1 - F(T_3(t_1,t_2,\alpha)) \\ &+ (m_1 + c_h + g)(\mu - G(T_3(t_1,t_2,\alpha)))\Big] \quad (2.6.236) \\ \text{s.t.} \quad & (t_1,t_2,\alpha) \in \mathbf{C_{2,4}}. \quad (2.6.237) \end{aligned}$$

The first order derivatives of (2.6.236) are given by:

$$\begin{aligned} \frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} &= \frac{-1}{1-\beta}(m_1+g)(1 - F(T_3(\alpha,t_1,t_2))), \quad (2.6.238) \\ \frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} &= \frac{-c_h}{1-\beta}(1 - F(T_3(\alpha,t_1,t_2))), \quad (2.6.239) \\ \frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} &= \frac{F(T_3(\alpha,t_1,t_2)) - \beta}{1-\beta}. \quad (2.6.240) \end{aligned}$$

Corollary. In the interior of region $\mathbf{R_2}$, the CVaR loss function $l_{\beta,2}(t_1,t_2,\alpha)$ is differentiable w.r.t. α, t_1 and t_2.

Corollary. By convexity, for fixed t_1 and t_2 in region $\mathbf{R_2}$, the optimal α value can always be found as the solution of the first order condition.

Corollary. By convexity and derivability, if the optimal solution lies in the interior of the region $\mathbf{R_2}$ it is given by the solution of the first order condition.

Second step : optimal solution in the interior of a region. It is direct to see that the only case where the first order conditions possibly have a solution is the region $\mathbf{C_{2,2}}$. Under adequate assumptions, the first order conditions (2.6.227)-(2.6.229) have the solution

$$\begin{aligned} t_1^* &= F^{-1}\left(\frac{m_1+g+s_1\beta}{m_1+g+s_1}\right), && (2.6.241)\\ t_2^* &= \left(\frac{m_2-m_1'}{m_2-m_1'-s_1+c_h}\right)F^{-1}\left(\frac{c_h(1-\beta)}{m_2-m_1'-s_1+c_h}\right)\\ &- \left(\frac{s_1-c_h}{m_2-m_1'-s_1+c_h}\right)F^{-1}\left(\frac{c_h+\beta(m_2-m_1'-s_1)}{m_2-m_1'-s_1+c_h}\right), && (2.6.242)\\ \alpha^* &= \left(c_h-s_1\right)F^{-1}\left(\frac{c_h+\beta(m_2-m_1'-s_1)}{m_2-m_1'-s_1+c_h}\right)+s_1t_1^*-c_ht_2^*. && (2.6.243) \end{aligned}$$

We also find the following parameter values

$$T_1(t_1^*,t_2^*,\alpha^*) = F^{-1}\left(\frac{c_h(1-\beta)}{m_2-m_1'-s_1+c_h}\right), \tag{2.6.244}$$

$$T_2(t_1^*,t_2^*,\alpha^*) = F^{-1}\left(\frac{c_h+\beta(m_2-m_1'-s_1)}{m_2-m_1'-s_1+c_h}\right). \tag{2.6.245}$$

Now, several assumptions are required in order to guarantee that this solution belongs to the interior of $\mathbf{C_{2,2}}$. Basically these assumptions are the following

$$\begin{aligned} F\left(\frac{c_h+\beta(m_2-m_1'-s_1)}{m_2-m_1'-s_1+c_h}\right) &< 1, && (2.6.246)\\ t_2^* &< t_1^*, && (2.6.247)\\ \tilde{\alpha}_{2,1}(t_1^*,t_2^*) = s_1(t_1^*-t_2^*) &< \alpha^* < \tilde{\alpha}_{2,3}(t_1^*,t_2^*) = c_h(t_1^*-t_2^*). && (2.6.248) \end{aligned}$$

From Figure (2.10) it can be seen that the last condition is equivalent to

$$T_1(t_1^*,t_2^*,\alpha^*) \le t_2^*, \tag{2.6.249}$$

$$t_2^* \le T_2(t_1^*,t_2^*,\alpha^*), \tag{2.6.250}$$

$$T_2(t_1^*,t_2^*,\alpha^*) \le t_1^*. \tag{2.6.251}$$

First condition analysis. The first assumption is independent of the probability distribution and amounts to the condition on the parameters:

$$c_h+\beta(m_2-m_1'-s_1) < m_2-m_1'-s_1+c_h \tag{2.6.252}$$

Corollary. If $m_2 - m'_1 - s_1 \geq 0$ then condition 1 holds for any probability distribution and for all β values one has existence of t^*_2.

Corollary. If $m_2 - m'_1 - s_1 < 0$, then for any probability distribution F, there exists a lower bound β_F such that for any β values with $\beta \geq \beta_F$, one has no existence of t^*_2.

Second condition analysis. The second condition is not easy and in general, for arbitrary values of the parameters and of β, can depend on the probability distribution. However, it can be seen that under parameters conditions corresponding to Case 1, expression of t^*_2 corresponds to a convex combination of $F^{-1}\left(\frac{c_h(1-\beta)}{m_2-m'_1-s_1+c_h}\right)$ and of $F^{-1}\left(\frac{c_h+\beta(m_2-m'_1-s_1)}{m_2-m'_1-s_1+c_h}\right)$. As a consequence some properties can be found, depending on the order associated with $\frac{m_1+g+s_1\beta}{m_1+g+s_1}$, $\frac{c_h(1-\beta)}{m_2-m'_1-s_1+c_h}$ and $\frac{c_h+\beta(m_2-m'_1-s_1)}{m_2-m'_1-s_1+c_h}$.

Corollary. If $\frac{c_h+\beta(m_2-m'_1-s_1)}{m_2-m'_1-s_1+c_h} < \frac{m_1+g+s_1\beta}{m_1+g+s_1}$ then for any probability distribution one has $t^*_2 < t^*_1$.

Corollary. If $\frac{c_h+\beta(m_2-m'_1-s_1)}{m_2-m'_1-s_1+c_h} \geq \frac{m_1+g+s_1\beta}{m_1+g+s_1}$ then for any probability distribution F, there exists an upper bound β_F such that for any β values with $\beta \leq \beta_F$, one has $t^*_2 \geq t^*_1$.

Third condition analysis. It is direct to see that conditions (2.6.249) and (2.6.250) hold for any distribution and any parameters. In fact, condition (2.6.251) amounts again to $\frac{c_h+\beta(m_2-m'_1-s_1)}{m_2-m'_1-s_1+c_h} < \frac{m_1+g+s_1\beta}{m_1+g+s_1}$.

Third step : optimal solution on a boundary of a region. If the optimal solution is not in the interior of a region (i.e., if above conditions do not hold) then the optimal solution has to be found on the boundary of region between two regions. The potential boundaries are defined as

- $t_1 = t_2$
- $\alpha(t_1, t_2) = \tilde{\alpha}_{2,i}(t_1, t_2)$ for $i = 1, .., 3$.

Solved previously for case $t_1 \leq t_2$.

Appendix C-2 $c_{h,2} \leq s_1$:

It can be seen that the critical values for the α parameters corresponding to the slope discontinuities for the piecewise linear function (2.6.165), as functions of t_1 and t_2, are given by

$$\tilde{\alpha}_{2,1}(t_1, t_2) = s_1(t_1 - t_2), \tag{2.6.253}$$

$$\tilde{\alpha}_{2,2}(t_1, t_2) = (m_2 - m_1')t_2 + s_1(t_1 - t_2), \tag{2.6.254}$$

$$\tilde{\alpha}_{2,3}(t_1, t_2) = c_{h,2}(t_1 - t_2). \tag{2.6.255}$$

with $\tilde{\alpha}_{2,3}(t_1, t_2) \leq \tilde{\alpha}_{2,1}(t_1, t_2) \leq \tilde{\alpha}_{2,2}(t_1, t_2)$ (see Figure (2.11)).

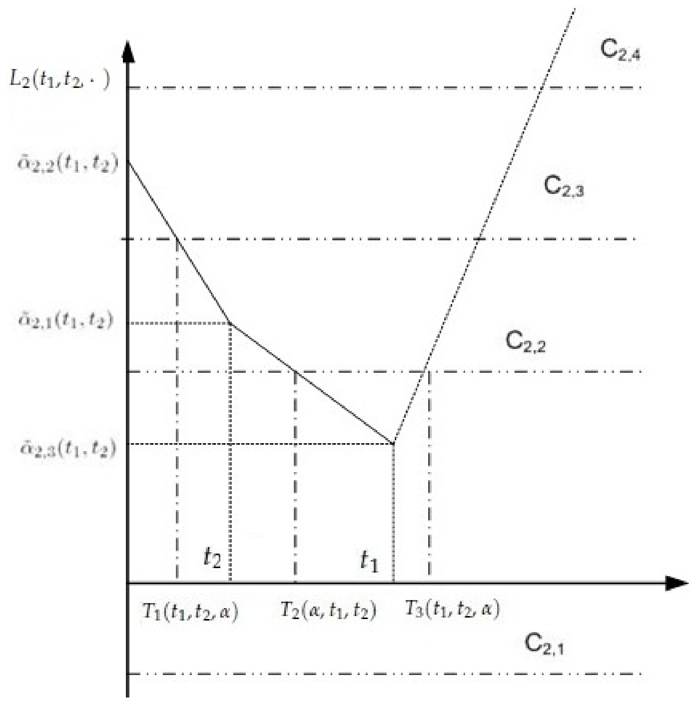

Figure 2.11: Four cases in minimization of CVaR in the region $\mathbf{R_2}$ for $c_{h,2} \leq s_1$

In order to characterize the first order conditions, we define the regions $\mathbf{C_{2,1}}$, $\mathbf{C_{2,2}}$, $\mathbf{C_{2,3}}$

and $\mathbf{C_{2,4}}$, as

$$\mathbf{C_{2,1}} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbf{R_2} \text{ and } \alpha \in]\infty, \tilde{\alpha}_{2,3}(t_1, t_2)[, \tag{2.6.256}$$

$$\mathbf{C_{2,2}} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbf{R_2} \text{ and } \alpha \in]\tilde{\alpha}_{2,3}(t_1, t_2), \tilde{\alpha}_{2,1}(t_1, t_2)[, \tag{2.6.257}$$

$$\mathbf{C_{2,3}} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbf{R_2} \text{ and } \alpha \in]\tilde{\alpha}_{2,1}(t_1, t_2), \tilde{\alpha}_{2,2}(t_1, t_2)[, \tag{2.6.258}$$

$$\mathbf{C_{2,4}} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbf{R_2} \text{ and } \alpha \in]\tilde{\alpha}_{2,2}(t_1, t_2), \infty[. \tag{2.6.259}$$

The main result for CVAR in region $\mathbf{R_2}$ for $c_{h,2} \leq s_1$

Proof:*Determination of the optimal policies.*

First step : expression of the first order conditions.

The region $\mathbf{C_{2,1}}$. In this region, the objective function given in expression (2.6.165) is

$$\begin{aligned} l_{\beta,2} &= \alpha + \frac{1}{1-\beta}\Big[((m_2 - m_1')t_2 + s_1(t_1 - t_2))F(t_2) + (m_1' - m_2)G(t_2) \\ &\quad +(s_1 t_1 - c_{h,2}t_2)(F(t_1) - F(t_2)) + (c_{h,2} - s_1)(G(t_1) - G(t_2)) \\ &\quad -((m_1 + g)t_1 + c_{h,2}t_2)(1 - F(t_1)) + (m_1 + c_{h,2} + g)(\mu - G(t_1)) \\ &- \alpha\Big] \end{aligned} \tag{2.6.260}$$

The optimization problem can be rewritten

$$\begin{aligned} l_{\beta,2} &= \alpha + \frac{1}{1-\beta}\Big[((m_2 - m_1')t_2 + s_1(t_1 - t_2))F(t_2) + (m_1' - m_2)G(t_2) \\ &\quad +(s_1 t_1 - c_{h,2}t_2)(F(t_1) - F(t_2)) + (c_{h,2} - s_1)(G(t_1) - G(t_2)) \\ &\quad -((m_1 + g)t_1 + c_{h,2}t_2)(1 - F(t_1)) + (m_1 + c_{h,2} + g)(\mu - G(t_1)) \\ &- \alpha\Big] \end{aligned} \tag{2.6.261}$$

$$\text{s.t.} \quad (t_1, t_2, \alpha) \in \mathbf{C_{2,1}}. \tag{2.6.262}$$

The first order derivatives of (2.6.261) are given by

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{d\alpha} = -\frac{\beta}{(1-\beta)}, \tag{2.6.263}$$

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_1} = \frac{(s_1 + m_1 + g)F(t_1) - (m_1 + g)}{(1-\beta)}, \tag{2.6.264}$$

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 - m_1' - s_1 + c_{h,2})F(t_2) - c_{h,2}}{(1-\beta)}. \tag{2.6.265}$$

The region $\mathbf{C_{2,2}}$.

According to Figure 2.11, let's define $T_2(\alpha, t_1, t_2)$ as the T value corresponding to:

$$\alpha = +s_1(t_1 - T) + c_{h,2}(T - t_2) \tag{2.6.266}$$

and $T_3(\alpha, t_1, t_2)$ as the T value corresponding to:

$$\alpha = m_1(T - t_1) + g(T - t_1) + c_{h,2}(T - t_2). \tag{2.6.267}$$

The optimization problem can be rewritten

$$\begin{aligned} l_{\beta,2} &= \alpha + \frac{1}{1-\beta}\Big[(m_2 - m_1' - s_1 + c_{h,2})t_2 F(t_2) + (c_{h,2} - s_1)G(T_2(t_1, t_2, \alpha)) \\ &- (m_2 - m_1' - s_1 + c_{h,2})G(t_2) + (s_1 t_1 - c_{h,2} t_2 - \alpha)F(T_2(t_1, t_2, \alpha)) \\ &- ((m_1 + g)t_1 + c_{h,2} t_2 + \alpha)(1 - F(T_3(t_1, t_2, \alpha))) \\ &+ (m_1 + c_{h,2} + g)(\mu - G(T_3(t_1, t_2, \alpha)))\Big], \end{aligned} \tag{2.6.268}$$

$$\text{s.t.} \quad (t_1, t_2, \alpha) \in \mathbf{C_{2,2}}. \tag{2.6.269}$$

The first order derivatives of (2.6.268) are given by:

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_1} = \frac{s_1 F(T_2(\alpha, t_1, t_2)) - (m_1 + g)(1 - F(T_3(\alpha, t_1, t_2)))}{1-\beta}, \tag{2.6.270}$$

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 - m_1' - s_1 + c_{h,2})F(t_2) - c_{h,2} F(T_2(\alpha, t_1, t_2)) - c_{h,2}(1 - F(T_3(\alpha, t_1, t_2)))}{1-\beta}, \tag{2.6.271}$$

$$, \tag{2.6.272}$$

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_3(\alpha, t_1, t_2)) - F(T_2(\alpha, t_1, t_2)) - \beta}{1-\beta}. \tag{2.6.273}$$

The region $\mathbf{C_{2,3}}$.

According to Figure 2.11, let's define $T_1(\alpha, t_1, t_2)$ as the T value corresponding to:

$$\alpha = -m_1'(t_2 - T) + m_2(t_2 - T) + s_1(t_1 - t_2) \tag{2.6.274}$$

The optimization problem becomes

$$\begin{aligned} \min \quad l_{\beta,2}(t_1, t_2, \alpha) &= \alpha + \frac{1}{1-\beta}\Big[(m_1' - m_2)G(T_1(t_1, t_2, \alpha)) + ((m_2 - m_1' - s_1)t_2 + s_1 t_1 - \alpha)F(T_1(t_1, t_2, \alpha))\Big] \\ &+ \frac{1}{1-\beta}\Big[-((m_1 + g)t_1 + c_{h,2} t_2 + \alpha)(1 - F(T_3(t_1, t_2, \alpha)) \\ &+ (m_1 + c_{h,2} + g)(\mu - G(T_3(t_1, t_2, \alpha)))\Big], \end{aligned} \tag{2.6.275}$$

$$\text{s.t.} \quad (t_1, t_2, \alpha) \in \mathbf{C_{2,3}}. \tag{2.6.276}$$

The first order derivatives of (2.6.275) are given by:

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_1} = \frac{s_1 F(T_1(\alpha, t_1, t_2)) - (m_1 + g)(1 - F(T_3(\alpha, t_1, t_2)))}{1-\beta}, \tag{2.6.277}$$

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 - m_1' - s_1)F(T_1(\alpha, t_1, t_2)) - c_{h,2}(1 - F(T_3(\alpha, t_1, t_2)))}{1-\beta}, \tag{2.6.278}$$

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_3(\alpha, t_1, t_2)) - F(T_1(\alpha, t_1, t_2)) - \beta}{1-\beta}. \tag{2.6.279}$$

The region $\mathbf{C}_{2,4}$.
The optimization problem becomes

$$\begin{aligned} \min \quad l_{\beta,2}(t_1,t_2,\alpha) &= \alpha + \frac{1}{1-\beta}\Big[-((m_1+g)t_1 + c_{h,2}t_2 + \alpha)(1 - F(T_3(t_1,t_2,\alpha)) \\ &+ (m_1 + c_{h,2} + g)(\mu - G(T_3(t_1,t_2,\alpha)))\Big] \quad (2.6.280) \\ \text{s.t.} \quad & (t_1,t_2,\alpha) \in \mathbf{C}_{2,4}. \quad (2.6.281) \end{aligned}$$

The first order derivatives of (2.6.280) are given by:

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{-1}{1-\beta}(m_1+g)(1 - F(T_3(\alpha,t_1,t_2))), \quad (2.6.282)$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{-c_{h,2}}{1-\beta}(1 - F(T_3(\alpha,t_1,t_2))), \quad (2.6.283)$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_3(\alpha,t_1,t_2)) - \beta}{1-\beta}. \quad (2.6.284)$$

In the interior of region $\mathbf{R_2}$, the CVaR loss function $l_{\beta,2}(t_1,t_2,\alpha)$ is differentiable w.r.t. α, t_1 and t_2.

By convexity, for fixed t_1 and t_2 in region $\mathbf{R_2}$, the optimal α value can always be found as the solution of the first order condition.

By convexity and derivability, if the optimal solution lies in the interior of the region $\mathbf{R_2}$ it is given by the solution of the first order condition.

Second step : optimal solution in the interior of a region. It is direct to see that the only case where the first order conditions possibly have a solution is the region $\mathbf{C}_{2,2}$. Under adequate assumptions, the first order conditions (2.6.270)-(2.6.273) have the solution

$$\begin{aligned} t_1^* &= F^{-1}\Bigg(\frac{\Big(m_1+g+c_{h,2}\Big)F^{-1}\Big(\frac{m_1+g+s_1\beta}{m_1+g+s_1}\Big)}{m_1+g+s_1} \\ &+ \frac{\Big(s_1-c_{h,2}\Big)F^{-1}\Big(\frac{(m_1+g)(1-\beta)}{m_1+g+s_1}\Big)}{m_1+g+s_1}\Bigg) \quad (2.6.285) \end{aligned}$$

$$t_2^* = F^{-1}\Bigg(\frac{c_{h,2}(1-\beta)}{m_2 - m_1' - s_1 + c_{h,2}}\Bigg), \quad (2.6.286)$$

$$\alpha^* = \Big(c_{h,2} - s_1\Big) F^{-1}\left(\frac{(m_1+g)(1-\beta)}{m_1+g+s_1}\right) + s_1 t_1^* - c_{h,2} t_2^*. \tag{2.6.287}$$

We also find the following parameter values

$$T_2(t_1^*, t_2^*, \alpha^*) = F^{-1}\left(\frac{(m_1+g)(1-\beta)}{m_1+g+s_1}\right), \tag{2.6.288}$$

$$T_3(t_1^*, t_2^*, \alpha^*) = F^{-1}\left(\frac{m_1+g+s_1\beta}{m_1+g+s_1}\right). \tag{2.6.289}$$

Now, several assumptions are required in order to guarantee that this solution belongs to the interior of $\mathbf{C_{2,2}}$. Basically these assumptions are the following

$$F\left(\frac{c_{h,2}(1-\beta)}{m_2 - m_1' - s_1 + c_{h,2}}\right) < 1, \tag{2.6.290}$$

$$t_2^* < t_1^*, \tag{2.6.291}$$

$$\tilde{\alpha}_{2,1}(t_1^*, t_2^*) = c_{h,2}(t_1^* - t_2^*) < \alpha^* < \tilde{\alpha}_{2,3}(t_1^*, t_2^*) = s_1(t_1^* - t_2^*). \tag{2.6.292}$$

From Figure 2.11 it can be seen that the last condition is equivalent to

$$t_2^* \leq T_2(t_1^*, t_2^*, \alpha^*), \tag{2.6.293}$$

$$T_2(t_1^*, t_2^*, \alpha^*) \leq t_1^*, \tag{2.6.294}$$

$$t_1^* \leq T_3(t_1^*, t_2^*, \alpha^*). \tag{2.6.295}$$

First condition analysis. The first assumption is independent of the probability distribution and amounts to the condition on the parameters:

$$c_{h,2}(1-\beta) < m_2 - m_1' - s_1 + c_{h,2} \tag{2.6.296}$$

Second condition analysis. The second condition is not easy and in general, for arbitrary values of the parameters and of β, can depend on the probability distribution. However, it can be seen that under parameters conditions corresponding to Case 1,

expression of t_2^* corresponds to a convex combination of $F^{-1}\left(\frac{(m_1+g)(1-\beta)}{m_1+g+s_1}\right)$ and of $F^{-1}\left(\frac{m_1+g+s_1\beta}{m_1+g+s_1}\right)$. As a consequence some properties can be found, depending on the order associated with $\frac{m_1+g+s_1\beta}{m_1+g+s_1}$, $\frac{c_{h,2}(1-\beta)}{m_2-m_1'-s_1+c_{h,2}}$ and $\frac{(m_1+g)(1-\beta)}{m_1+g+s_1}$.
If $m_2 - m_1' - s_1 + c_{h,2} \geq 0$ then condition 1 holds for any probability distribution and for all β values one has existence of t_2^*.

If $\frac{c_{h,2}(1-\beta)}{m_2-m_1'-s_1+c_{h,2}} \geq \frac{(m_1+g)(1-\beta)}{m_1+g+s_1}$ then for any probability distribution F, there exists an upper bound β_F such that for any β values with $\beta \leq \beta_F$, one has $t_2^* \geq t_1^*$.

If $m_2 - m_1' - s_1 < 0$, then for any probability distribution F, there exists a lower bound β_F such that for any β values with $\beta \geq \beta_F$, one has no existence of t_2^*.

Third condition analysis. It is direct to see the conditions (2.6.295) and (2.6.294) hold for any distribution and any parameters. In fact, condition (2.6.293) amounts again to $\frac{c_{h,2}(1-\beta)}{m_2-m_1'-s_1+c_{h,2}} < \frac{(m_1+g)(1-\beta)}{m_1+g+s_1}$.

Third step : optimal solution on a boundary of a region. If the optimal solution is not in the interior of a region (i.e., if above conditions do not hold) then the optimal solution has to be found on the boundary of region between two regions. The potential boundaries are defined as

- $t_1 = t_2$
- $\alpha(t_1, t_2) = \tilde{\alpha}_{2,i}(t_1, t_2)$ for $i = 1, .., 3$.

Solved previously for case $t_1 \leq t_2$.

APPENDIX D: Differentiability of the CVaR Function

Differentiability of the CVaR loss function $l_{\beta,1}(t_1, t_2, \alpha)$ inside $\mathbf{R_1} \times \mathbf{R}$

Case 1 : $m_2 \geq m_1 \geq m_1'$

According to the 4 regions, the expressions of the the first order conditions of $l_{\beta,1}(t_1, t_2, \alpha)$ are as follows:

The region $\mathbf{C_{1,1}}$.

The first order derivatives of (2.6.28) are given by

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} = \frac{-\beta}{1-\beta}, \tag{2.6.297}$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} = \frac{(m_1 - m_1')}{1-\beta}F(t_1) - \frac{(m_1+g)}{1-\beta}, \tag{2.6.298}$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2 + c_{h,2} + g)}{1-\beta}F(t_2) - \frac{c_{h,2}}{1-\beta}. \tag{2.6.299}$$

The region $\mathbf{C_{1,2}}$.

The first order derivatives of (2.6.35) are given by:

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} = \frac{(m_1 - m_1')}{1-\beta}F(t_1) - \frac{(m_1+g)}{1-\beta}F(T_2(\alpha,t_1,t_2)) - \frac{(m_1+g)}{1-\beta}(1 - F(T_3(\alpha,t_1,t_2))), \tag{2.6.300}$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2+g)}{1-\beta}F(T_2(\alpha,t_1,t_2)) - \frac{c_{h,2}}{1-\beta}(1 - F(T_3(\alpha,t_1,t_2))), \tag{2.6.301}$$

$$\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_3(t_1,t_2,\alpha)) - F(T_2(t_1,t_2,\alpha)) - \beta}{1-\beta}. \tag{2.6.302}$$

The region $\mathbf{C_{1,3}}$.

The first order derivatives of (2.6.41) are given by:

$$\begin{aligned}\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} &= -\frac{(m_1'+g)}{1-\beta}F(T_1(\alpha,t_1,t_2)) \\ &- \frac{(m_1+g)}{1-\beta}(1-F(T_3(\alpha,t_1,t_2))), \quad (2.6.303)\\ \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} &= \frac{(m_2+g)}{1-\beta}F(T_1(\alpha,t_1,t_2)) \\ &- \frac{c_{h,2}}{1-\beta}(1-F(T_3(\alpha,t_1,t_2))), \quad (2.6.304)\\ \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} &= \frac{F(T_3(t_1,t_2,\alpha))-F(T_1(t_1,t_2,\alpha))-\beta}{1-\beta}. \quad (2.6.305)\end{aligned}$$

The region $\mathbf{C_{1,4}}$.

The first order derivatives of (2.6.46) are given by:

$$\begin{aligned}\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} &= -\left(\frac{m_1+g}{1-\beta}\right)\Big(1-F(T_3(t_1,t_2,\alpha))\Big), \quad (2.6.306)\\ \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} &= -\left(\frac{c_{h,2}}{1-\beta}\right)\Big(1-F(T_3(t_1,t_2,\alpha))\Big), \quad (2.6.307)\\ \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} &= \frac{F(T_3(t_1,t_2,\alpha)-\beta}{1-\beta}. \quad (2.6.308)\end{aligned}$$

From the expression of the first order derivatives in the 4 regions, it is clear that $l_{\beta,1}(t_1,t_2,\alpha)$ is differentiable within each region, yet we have to examine the differentiability at the critical points of $\tilde{\alpha}_{1,1}(t_1,t_2)$, $\tilde{\alpha}_{1,2}(t_1,t_2)$, and $\tilde{\alpha}_{1,3}(t_1,t_2)$ as follows:

Differentiability at $\tilde{\alpha}_{1,1}(t_1,t_2)$

If $\alpha \to \tilde{\alpha}^-_{1,1}(t_1,t_2)$, we are in region $\mathbf{C_{1,1}}$ and the first order derivatives are given by the expressions (2.6.297-2.6.299).

If $\alpha \to \tilde{\alpha}^+_{1,1}(t_1,t_2)$, we are in region $\mathbf{C_{1,2}}$ and and the first order derivatives are given by the expressions (2.6.300-2.6.302).

Yet for $\alpha = \tilde{\alpha}^+_{1,1}(t_1,t_2)$, one gets $T_2(t_1,t_2,\alpha) = T_3(t_1,t_2,\alpha) = t_2$, and thus the first order derivatives of region $\mathbf{C_{1,1}}$ become equal to that of region $\mathbf{C_{1,2}}$ and the function is differentiable at $\tilde{\alpha}_{1,1}(t_1,t_2)$.

Now for the differentiability with respect to t_1 and t_2, if F is continuous, then the objective function is derivable with respect to t_1 and t_2.

Differentiability at $\tilde{\alpha}_{1,2}(t_1, t_2)$

If $\alpha \rightarrow \tilde{\alpha}_{1,2}^{-}(t_1, t_2)$, we are in region $\mathbf{C_{1,2}}$ and the first order derivatives are given by the expressions (2.6.300-2.6.302).

If $\alpha \rightarrow \tilde{\alpha}_{1,2}^{+}(t_1, t_2)$, we are in region $\mathbf{C_{1,3}}$ and and the first order derivatives are given by the expressions (2.6.303-2.6.305).

Yet for $\alpha = \tilde{\alpha}_{1,2}^{+}(t_1, t_2)$, one gets $T_1(t_1, t_2, \alpha) = T_2(t_1, t_2, \alpha) = t_1$, and thus the first order derivatives of region $\mathbf{C_{1,2}}$ become equal to that of region $\mathbf{C_{1,3}}$ and the function is differentiable at $\tilde{\alpha}_{1,2}(t_1, t_2)$.

Now for the differentiability with respect to t_1 and t_2, if F is continuous, then the objective function is derivable with respect to t_1 and t_2.

Differentiability at $\tilde{\alpha}_{1,3}(t_1, t_2)$

If $\alpha \rightarrow \tilde{\alpha}_{1,3}^{-}(t_1, t_2)$, we are in region $\mathbf{C_{1,3}}$ and the first order derivatives are given by the expressions (2.6.303-2.6.305).

If $\alpha \rightarrow \tilde{\alpha}_{1,3}^{+}(t_1, t_2)$, we are in region $\mathbf{C_{1,4}}$ and and the first order derivatives are given by the expressions (2.6.306-2.6.308).

Yet for $\alpha = \tilde{\alpha}_{1,3}^{+}(t_1, t_2)$, one gets $T_1(t_1, t_2, \alpha) = 0$, and thus the first order derivatives of region $\mathbf{C_{1,3}}$ become equal to that of region $\mathbf{C_{1,4}}$ and the function is differentiable at $\tilde{\alpha}_{1,3}(t_1, t_2)$.

Now for the differentiability with respect to t_1 and t_2, if F is continuous, then the objective function is derivable with respect to t_1 and t_2.

Case 2 : $m_1 \geq m_2 \geq m_1'$

From the expression of the first order derivatives in the 4 regions, it is clear that $l_{\beta,1}(t_1, t_2, \alpha)$ is differentiable within each region, yet we have to examine the differentiability at the critical points of $\tilde{\alpha}_{1,1}(t_1, t_2)$, $\tilde{\alpha}_{1,2}(t_1, t_2)$, and $\tilde{\alpha}_{1,3}(t_1, t_2)$ as follows:

According to the 4 regions, the expressions of the the first order conditions of $l_{\beta,1}(t_1, t_2, \alpha)$ are as follows:

The region $\mathbf{C_{1,1}}$. The first order derivatives of (2.6.90) are given by

$$\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} = \frac{-\beta}{1-\beta}, \tag{2.6.309}$$

$$\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} = \frac{(m_1 - m_1')}{1-\beta} F(t_1) - \frac{(m_1 + g)}{1-\beta}, \tag{2.6.310}$$

$$\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 + c_{h,2} + g)}{1-\beta} F(t_2) - \frac{c_{h,2}}{1-\beta}. \tag{2.6.311}$$

The region $\mathbf{C_{1,2}}$.

The first order derivatives of (2.6.97) are given by:

$$\begin{aligned}\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} &= -\frac{(m_1' + g)}{1-\beta} F(T_1(\alpha, t_1, t_2)) + \frac{(m_1 + g)}{1-\beta} F(T_2(\alpha, t_1, t_2)) \\ &- \frac{(m_1 + g)}{1-\beta},\end{aligned} \tag{2.6.312}$$

$$\begin{aligned}\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} &= \frac{(m_2 + g)}{1-\beta} (F(T_1(\alpha, t_1, t_2)) - F(T_2(\alpha, t_1, t_2))) \\ &- \frac{c_{h,2}}{1-\beta}(1 - F(t_2)) + (\frac{m_1 + g)}{1-\beta} F(t_2),\end{aligned} \tag{2.6.313}$$

$$\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_2(t_1, t_2, \alpha)) - F(T_1(t_1, t_2, \alpha)) - \beta}{1-\beta}. \tag{2.6.314}$$

The region $\mathbf{C_{1,3}}$.

The first order derivatives of (2.6.103) are given by:

$$\begin{aligned}\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} &= -\frac{(m_1'+g)}{1-\beta}F(T_1(\alpha,t_1,t_2)) \\ &- \frac{(m_1+g)}{1-\beta}(1-F(T_3(\alpha,t_1,t_2))), \qquad (2.6.315)\\ \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} &= \frac{(m_2+g)}{1-\beta}F(T_1(\alpha,t_1,t_2)) \\ &- \frac{c_{h,2}}{1-\beta}(1-F(T_3(\alpha,t_1,t_2))), \qquad (2.6.316)\\ \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} &= \frac{F(T_3(t_1,t_2,\alpha))-F(T_1(t_1,t_2,\alpha))-\beta}{1-\beta}. \qquad (2.6.317)\end{aligned}$$

The region $\mathbf{C_{1,4}}$.

The first order derivatives of (2.6.108) are given by:

$$\begin{aligned}\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} &= -\left(\frac{m_1+g}{1-\beta}\right)\left(1-F(T_3(t_1,t_2,\alpha))\right), \qquad (2.6.318)\\ \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} &= -\left(\frac{c_{h,2}}{1-\beta}\right)\left(1-F(T_3(t_1,t_2,\alpha))\right), \qquad (2.6.319)\\ \frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} &= \frac{F(T_3(t_1,t_2,\alpha)-\beta}{1-\beta}. \qquad (2.6.320)\end{aligned}$$

Differentiability at $\tilde{\alpha}_{1,1}(t_1,t_2)$

If $\alpha \to \tilde{\alpha}^-_{1,1}(t_1,t_2)$, we are in region $\mathbf{C_{1,1}}$ and the first order derivatives are given by the expressions (2.6.309-2.6.311).

If $\alpha \to \tilde{\alpha}^+_{1,1}(t_1,t_2)$, we are in region $\mathbf{C_{1,2}}$ and and the first order derivatives are given by the expressions (2.6.312-2.6.314).

Yet for $\alpha = \tilde{\alpha}^+_{1,1}(t_1,t_2)$, one gets $T_1(t_1,t_2,\alpha) = T_2(t_1,t_2,\alpha) = t_1$, and thus the first order derivatives of region $\mathbf{C_{1,1}}$ become equal to that of region $\mathbf{C_{1,2}}$ and the function is differentiable at $\tilde{\alpha}_{1,1}(t_1,t_2)$.

Now for the differentiability with respect to t_1 and t_2, if F is continuous, then the objective function is derivable with respect to t_1 and t_2.

Differentiability at $\tilde{\alpha}_{1,2}(t_1, t_2)$

If $\alpha \to \tilde{\alpha}_{1,2}^{-}(t_1, t_2)$, we are in region $\mathbf{C_{1,2}}$ and the first order derivatives are given by the expressions (2.6.312-2.6.314).

If $\alpha \to \tilde{\alpha}_{1,2}^{+}(t_1, t_2)$, we are in region $\mathbf{C_{1,3}}$ and and the first order derivatives are given by the expressions (2.6.315-2.6.317).

Yet for $\alpha = \tilde{\alpha}_{1,2}^{+}(t_1, t_2)$, one gets $T_2(t_1, t_2, \alpha) = T_3(t_1, t_2, \alpha) = t_2$, and thus the first order derivatives of region $\mathbf{C_{1,2}}$ become equal to that of region $\mathbf{C_{1,3}}$ and the function is differentiable at $\tilde{\alpha}_{1,2}(t_1, t_2)$.

Now for the differentiability with respect to t_1 and t_2, if F is continuous, then the objective function is derivable with respect to t_1 and t_2.

Differentiability at $\tilde{\alpha}_{1,3}(t_1, t_2)$

If $\alpha \to \tilde{\alpha}_{1,3}^{-}(t_1, t_2)$, we are in region $\mathbf{C_{1,3}}$ and the first order derivatives are given by the expressions (2.6.315-2.6.317).

If $\alpha \to \tilde{\alpha}_{1,3}^{+}(t_1, t_2)$, we are in region $\mathbf{C_{1,4}}$ and and the first order derivatives are given by the expressions (2.6.108-2.6.320).

Yet for $\alpha = \tilde{\alpha}_{1,3}^{+}(t_1, t_2)$, one gets $T_1(t_1, t_2, \alpha) = 0$, and thus the first order derivatives of region $\mathbf{C_{1,3}}$ become equal to that of region $\mathbf{C_{1,4}}$ and the function is differentiable at $\tilde{\alpha}_{1,3}(t_1, t_2)$. Now for the differentiability with respect to t_1 and t_2, if F is continuous, then the objective function is derivable with respect to t_1 and t_2.

Differentiability of the CVaR loss function $l_{\beta,2}(t_1, t_2, \alpha)$ inside $\mathbf{R_2} \times \mathbf{R}$

Case 1: $c_{h,2} \geq s_1$

The region $\mathbf{C_{2,1}}$.

The first order derivatives of (2.6.178) are given by

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = -\frac{\beta}{(1-\beta)}, \tag{2.6.321}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{(s_1+m_1+g)F(t_1)-(m_1+g)}{(1-\beta)}, \tag{2.6.322}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2-m_1'-s_1+c_{h,2})F(t_2)-c_{h,2}}{(1-\beta)}. \tag{2.6.323}$$

The region $\mathbf{C_{2,2}}$.

The first order derivatives of (2.6.185) are given by:

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{s_1(F(t_1)+F(T_1(\alpha,t_1,t_2))-F(T_2(\alpha,t_1,t_2))+(m_1+g)(F(t_1)-1)}{1-\beta}, \tag{2.6.324}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2-s_1-m_1')F(T_1(\alpha,t_1,t_2))-c_{h,2}(1-F(T_2(\alpha,t_1,t_2)))}{1-\beta}, \tag{2.6.325}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_2(\alpha,t_1,t_2))-F(T_1(\alpha,t_1,t_2))-\beta}{1-\beta}. \tag{2.6.326}$$

The region $\mathbf{C_{2,3}}$.

The first order derivatives of (2.6.191) are given by:

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{s_1}{1-\beta}(F(t_1)-F(T_2(t_1,t_2,\alpha)))-\frac{1}{1-\beta}(m_1+g)(1-F(t_1))), \tag{2.6.327}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{-c_{h,2}}{1-\beta}(1-F(T_2(t_1,t_2,\alpha))), \tag{2.6.328}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_2(t_1,t_2,\alpha))-\beta}{1-\beta}. \tag{2.6.329}$$

The region $\mathbf{C_{2,4}}$.

The first order derivatives of (2.6.196) are given by:

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{-1}{1-\beta}(m_1+g)(1-F(T_3(\alpha,t_1,t_2))), \tag{2.6.330}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{-c_{h,2}}{1-\beta}(1-F(T_3(\alpha,t_1,t_2))), \tag{2.6.331}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_3(\alpha,t_1,t_2))-\beta}{1-\beta}. \tag{2.6.332}$$

Differentiability at $\tilde{\alpha}_{2,1}(t_1, t_2)$

If $\alpha \to \tilde{\alpha}^-_{2,1}(t_1, t_2)$, we are in region $\mathbf{C_{2,1}}$ and the first order derivatives are given by the expressions (2.6.321-2.6.323).

If $\alpha \to \tilde{\alpha}^+_{2,1}(t_1, t_2)$, we are in region $\mathbf{C_{2,2}}$ and and the first order derivatives are given by the expressions (2.6.324-2.6.326).

Yet for $\alpha = \tilde{\alpha}^+_{2,1}(t_1, t_2)$, one gets $T_1(t_1, t_2, \alpha) = T_2(t_1, t_2, \alpha) = t_1$, and thus the first order derivatives of region $\mathbf{C_{2,1}}$ become equal to that of region $\mathbf{C_{2,2}}$ and the function is differentiable at $\tilde{\alpha}_{2,1}(t_1, t_2)$.

Now for the differentiability with respect to t_1 and t_2, if F is continuous, then the objective function is derivable with respect to t_1 and t_2.

Differentiability at $\tilde{\alpha}_{2,2}(t_1, t_2)$

If $\alpha \to \tilde{\alpha}^-_{2,2}(t_1, t_2)$, we are in region $\mathbf{C_{1,2}}$ and the first order derivatives are given by the expressions (2.6.324-2.6.326).

If $\alpha \to \tilde{\alpha}^+_{2,2}(t_1, t_2)$, we are in region $\mathbf{C_{2,3}}$ and and the first order derivatives are given by the expressions (2.6.327-2.6.329).

Yet for $\alpha = \tilde{\alpha}^+_{2,2}(t_1, t_2)$, one gets $T_1(t_1, t_2, \alpha) = 0$ or disappears, and thus the first order derivatives of region $\mathbf{C_{2,2}}$ become equal to that of region $\mathbf{C_{2,3}}$ and the function is differentiable at $\tilde{\alpha}_{2,2}(t_1, t_2)$.

Now for the differentiability with respect to t_1 and t_2, if F is continuous, then the objective function is derivable with respect to t_1 and t_2.

Differentiability at $\tilde{\alpha}_{2,3}(t_1, t_2)$

If $\alpha \to \tilde{\alpha}^{-}_{2,3}(t_1, t_2)$, we are in region $\mathbf{C_{2,3}}$ and the first order derivatives are given by the expressions (2.6.327-2.6.329).

If $\alpha \to \tilde{\alpha}^{+}_{2,3}(t_1, t_2)$, we are in region $\mathbf{C_{2,4}}$ and and the first order derivatives are given by the expressions (2.6.330-2.6.332).

Yet for $\alpha = \tilde{\alpha}^{+}_{2,3}(t_1, t_2)$, one gets $T_2(t_1, t_2, \alpha) = T_3(t_1, t_2, \alpha) = t_1$, and thus the first order derivatives of region $\mathbf{C_{2,3}}$ become equal to that of region $\mathbf{C_{2,4}}$ and the function is differentiable at $\tilde{\alpha}_{2,3}(t_1, t_2)$.

Now for the differentiability with respect to t_1 and t_2, if F is continuous, then the objective function is derivable with respect to t_1 and t_2.

Case 2:$c_{h,2} \leq s_1$

The region $\mathbf{C_{2,1}}$.

The first order derivatives of (2.6.261) are given by

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = -\frac{\beta}{(1-\beta)}, \tag{2.6.333}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{(s_1+m_1+g)F(t_1)-(m_1+g)}{(1-\beta)}, \tag{2.6.334}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2-m_1'-s_1+c_{h,2})F(t_2)-c_{h,2}}{(1-\beta)}. \tag{2.6.335}$$

The region $\mathbf{C_{2,2}}$.

The first order derivatives of (2.6.268) are given by:

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{s_1F(T_2(\alpha,t_1,t_2))-(m_1+g)(1-F(T_3(\alpha,t_1,t_2)))}{1-\beta}, \tag{2.6.336}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2-m_1'-s_1+c_{h,2})F(t_2)-c_{h,2}F(T_2(\alpha,t_1,t_2))-c_{h,2}(1-F(T_3(\alpha,t_1,t_2)))}{1-\beta}, \tag{2.6.337}$$

$$, \tag{2.6.338}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_3(\alpha,t_1,t_2))-F(T_2(\alpha,t_1,t_2))-\beta}{1-\beta}. \tag{2.6.339}$$

The region $\mathbf{C_{2,3}}$.

The first order derivatives of (2.6.275) are given by:

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{s_1F(T_1(\alpha,t_1,t_2)) - (m_1+g)(1-F(T_3(\alpha,t_1,t_2)))}{1-\beta}, \tag{2.6.340}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{(m_2-m_1'-s_1)F(T_1(\alpha,t_1,t_2)) - c_{h,2}(1-F(T_3(\alpha,t_1,t_2)))}{1-\beta}, \tag{2.6.341}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_3(\alpha,t_1,t_2)) - F(T_1(\alpha,t_1,t_2)) - \beta}{1-\beta}. \tag{2.6.342}$$

The region $\mathbf{C_{2,4}}$.

The first order derivatives of (2.6.280) are given by:

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_1} = \frac{-1}{1-\beta}(m_1+g)(1-F(T_3(\alpha,t_1,t_2))), \tag{2.6.343}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{dt_2} = \frac{-c_{h,2}}{1-\beta}(1-F(T_3(\alpha,t_1,t_2))), \tag{2.6.344}$$

$$\frac{dl_{\beta,2}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_3(\alpha,t_1,t_2)) - \beta}{1-\beta}. \tag{2.6.345}$$

Differentiability at $\tilde{\alpha}_{2,1}(t_1,t_2)$

If $\alpha \to \tilde{\alpha}_{2,1}^{-}(t_1,t_2)$, we are in region $\mathbf{C_{2,1}}$ and the first order derivatives are given by the expressions (2.6.333-2.6.335).

If $\alpha \to \tilde{\alpha}_{2,1}^{+}(t_1,t_2)$, we are in region $\mathbf{C_{2,2}}$ and and the first order derivatives are given by the expressions (2.6.336-2.6.339).

Yet for $\alpha = \tilde{\alpha}_{2,1}^{+}(t_1,t_2)$, one gets $T_2(t_1,t_2,\alpha) = T_3(t_1,t_2,\alpha) = t_1$, and thus the first order derivatives of region $\mathbf{C_{2,1}}$ become equal to that of region $\mathbf{C_{2,2}}$ and the function is differentiable at $\tilde{\alpha}_{2,1}(t_1,t_2)$.

Now for the differentiability with respect to t_1 and t_2, if F is continuous, then the objective function is derivable with respect to t_1 and t_2.

Differentiability at $\tilde{\alpha}_{2,2}(t_1,t_2)$

If $\alpha \to \tilde{\alpha}_{2,2}^{-}(t_1,t_2)$, we are in region $\mathbf{C_{1,2}}$ and the first order derivatives are given by the expressions (2.6.336-2.6.339).

If $\alpha \to \tilde{\alpha}^{+}_{2,2}(t_1, t_2)$, we are in region $\mathbf{C_{2,3}}$ and and the first order derivatives are given by the expressions (2.6.340-2.6.342).

Yet for $\alpha = \tilde{\alpha}^{+}_{2,2}(t_1, t_2)$, one gets $T_1(t_1, t_2, \alpha) = T_2(t_1, t_2, \alpha) = t_2$ or disappears, and thus the first order derivatives of region $\mathbf{C_{2,2}}$ become equal to that of region $\mathbf{C_{2,3}}$ and the function is differentiable at $\tilde{\alpha}_{2,2}(t_1, t_2)$.

Now for the differentiability with respect to t_1 and t_2, if F is continuous, then the objective function is derivable with respect to t_1 and t_2.

Differentiability at $\tilde{\alpha}_{2,3}(t_1, t_2)$

If $\alpha \to \tilde{\alpha}^{-}_{2,3}(t_1, t_2)$, we are in region $\mathbf{C_{2,3}}$ and the first order derivatives are given by the expressions (2.6.327-2.6.329).

If $\alpha \to \tilde{\alpha}^{+}_{2,3}(t_1, t_2)$, we are in region $\mathbf{C_{2,4}}$ and and the first order derivatives are given by the expressions (2.6.343-2.6.345).

Yet for $\alpha = \tilde{\alpha}^{+}_{2,3}(t_1, t_2)$, one gets $T_1(t_1, t_2, \alpha) = 0$, and thus the first order derivatives of region $\mathbf{C_{2,3}}$ become equal to that of region $\mathbf{C_{2,4}}$ and the function is differentiable at $\tilde{\alpha}_{2,3}(t_1, t_2)$.

Now for the differentiability with respect to t_1 and t_2, if F is continuous, then the objective function is derivable with respect to t_1 and t_2.

APPENDIX E

One finds in the literature on stochastic dominance relations a family of rules to compare variability between the two demands (Fishburn and Vickson 1978).

Definition 1. Approval date T_2 is more n-variable than approval date T_1, denoted by If

$$T_1 \geq_n T_2, \tag{2.6.346}$$

$$H_n(x) \geq 0 \text{ for all } x \geq 0, \tag{2.6.347}$$

Where

$H_1(x) = F_2(x) - F_1(x)$,

$H_n(x) = \int_0^x H_{n-1}(t)dt \ (n = 2, 3, ...)$

In Lemma 1 (below) we show that 2-variability implies higher approval date variances in with higher costs. The proof is based on the following theorem.

Theorem 1 (Fishburn, 1980). If $T_1 \geq_n T_2$ for some $n \geq 2$ then

$$\begin{array}{l} \mu_1 = \mu_2 \\ \sigma_1^2 \neq \sigma_2^2 \end{array} \Rightarrow \sigma_1^2 < \sigma_2^2.$$

Property 14 If $T_1 \geq_{var} T_2$, then

$$\min_{(t_1,t_2)\in\mathbb{R}^+\times\mathbb{R}^+} E_{F_1}[L(t_1,t_2,T)] \leq \min_{(t_1,t_2)\in\mathbb{R}^+\times\mathbb{R}^+} E_{F_2}[L(t_1,t_2,T)]. \tag{2.6.348}$$

Proof. We successively consider the three strategies and the associated cost functions. We show that if the variability increases, each cost function increases. Apply Theorem 1, the definition of $t^*_{j,i}$ and the definition of optimal costs for regions $\mathbf{R_1}$ and $\mathbf{R_2}$ and the boundary in between the regions to see what happens when $T_1 \geq_{var} T_2$

We know that

$$E([t_i - T]^+) = \int_0^{t_1} F(T)dT \quad (2.6.349)$$

and

$$E([T - t_i]^+) = \int_0^{t_1} F(T)dT + \mu - t_1. \quad (2.6.350)$$

For the minimum of $\mathbf{L_1}$ The optimal expected costs associated to probability distributions $F_1(\cdot)$ and $F_2(\cdot)$ can be rewritten as

$$\begin{aligned} E_{F_i}[L_1(t_1^*(F_i), t_2^*(F_i))] &= -gt_1^*(F_i) - m_1 t_1^*(F_i) - c_{h,2} t_2^*(F_i) + (m_1 - m_1') \int_0^{t_1^*(F_i)} F_i(T)dT \\ &+ (m_2 + c_{h,2} + g) \int_0^{t_2^*(F_i)} F_i(T)dT, \quad (2.6.351) \end{aligned}$$

and we thus have the following difference expression

$$\begin{aligned} E_{F_2}[L_1(t_1^*(F_2), t_2^*(F_2))] &- E_{F_1}[L_1(t_1^*(F_2), t_2^*(F_2)] = (m_1 - m_1') \int_0^{t_1^*(F_2)} (F_2(T) - F_1(T))dT \\ &+ (m_2 + c_{h,2} + g) \int_0^{t_2^*(F_2)} (F_2(T) - F_1(T))dT. \quad (2.6.352) \end{aligned}$$

As by optimality, one has

$$\begin{aligned} E_{F_2}[L_1(t_1^*(F_2), t_2^*(F_2))] &- E_{F_1}[L_1(t_1^*(F_1), t_2^*(F_1))] \\ &\geq \\ E_{F_2}[L_1(t_1^*(F_2), t_2^*(F_2))] &- E_{F_1}[L_1(t_1^*(F_2), t_2^*(F_2))], \quad (2.6.353) \end{aligned}$$

by (2.5.5), we conclude

$$E_{F_2}[L_1(t_1^*(F_2), t_2^*(F_2))] - E_{F_1}[L_1(t_1^*(F_1), t_2^*(F_1))] \geq 0. \quad (2.6.354)$$

For the minimum of L_b

The optimal expected costs associated to probability distributions $F_1(\cdot)$ and $F_2(\cdot)$ can be rewritten as

$$\begin{aligned} E_{F_i}[L_b(t_i^*)] &= -(g + m_1 + c_{h,2})t_1^*(F_i) \\ &+ (m_1 - m_1' + m_2 + c_{h,2} + g) \int_0^{t_1^*(F_i)} F_i(T)dT, \quad (2.6.355) \end{aligned}$$

and we thus have the following difference expression

$$\begin{aligned} E_{F_2}[L_b(t_2^*)] \quad - \quad & E_{F_1}[L_b(t_2^*)] = \\ + \quad & (m_2 - m_1' + m_1 + c_{h,2} + g) \int_0^{t_1^*(F_2)} (F_2(T) - F_1(T))dT. \end{aligned} \tag{2.6.356}$$

As by optimality, one has

$$E_{F_2}[L_b(t_2^*)] - E_{F_1}[L_b(t_1^*)] \geq E_{F_2}[L_b(t_2^*)] - E_{F_1}[L_b(t_2^*)], \tag{2.6.357}$$

by (2.5.5), we conclude

$$E_{F_2}[L_b(t_2^*)] - E_{F_1}[L_b(t_2^*)] \geq 0 \tag{2.6.358}$$

$$E_{F_2}[L_b(t_2^*)] - E_{F_1}[L_b(t_2^*)] \geq 0. \tag{2.6.359}$$

For the minimum of $\mathbf{L_2}$ (under assumption $m_2 - m_1' - s_1 + c_{h,2} > 0$). The optimal expected costs associated to probability distributions $F_1(\cdot)$ and $F_2(\cdot)$ can be rewritten as

$$\begin{aligned} E_{F_i}[L_2(t_1^*(F_i), t_2^*(F_i))] \quad = \quad & -c_{h,2}t_2^*(F_i) - (m_1 + g)t_1^*(F_i) + (m_2 - m_1' - s_1 + c_{h,2}) \int_0^{t_2^*(F_i)} F_i(T)dT \\ + \quad & (m_1 + g + s_1) \int_0^{t_1^*(F_i)} F_i(T)dT, \end{aligned} \tag{2.6.360}$$

and we thus have the following difference expression

$$\begin{aligned} E_{F_2}[L_2(t_1^*(F_2), t_2^*(F_2))] \quad - \quad & E_{F_1}[L_2(t_1^*(F_2), t_1^*(F_2))] = +(m_2 - m_1' - s_1 + c_{h,2}) \int_0^{t_2^*(F_2)} (F_2(T) - F_1(T))dT \\ + \quad & \Big(m_1 + g + s_1\Big) \int_0^{t_1^*(F_2)} (F_2(T) - F_1(T))dT. \end{aligned} \tag{2.6.361}$$

As by optimality, one has

$$\begin{aligned} E_{F_2}[L_2(t_1^*(F_2), t_2^*(F_2))] \quad & - \quad E_{F_1}[L_2(t_{1,1}^*, t_2^*(F_1))] \\ & \geq \\ E_{F_2}[L_2(t_1^*(F_2), t_2^*(F_2))] \quad & - \quad E_{F_1}[L_2(t_1^*(F_2), t_1^*(F_2))], \end{aligned} \tag{2.6.362}$$

by (2.5.5), one concludes

$$E_{F_2}[L_2(t_1^*(F_2), t_2^*(F_2))] - E_{F_1}[L_2(t_1^*(F_2), t_1^*(F_2))] \geq 0. \tag{2.6.363}$$

Therefore, as the minimum in each region decreases with $T_1 \geq_2 T_2$, then the global minimum which is the minimum of the minimum found in each region also decreases.

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CHAPTER 3

Second Paper: Product Rollover Optimization with an Uncertain Approval Date and Piecewise Linear Demand

Abstract

Consider a company that must plan the phase-out of an existing product and the phase-in of a replacement product. If production of the existing product is stopped too early, i.e., before the new product is available for the market, the firm will lose profit and customer goodwill. On the other hand, if production of the existing product is stopped too late, the firm will experience an obsolescence cost for the existing product. In our paper, we consider a product rollover process with an uncertain approval date for the new product, and develop the optimal rollover strategies by minimizing the expected loss. The new product demand is piecewise linear, initially it increases linearly until it reaches a certain demand level where it becomes constant. This demand dynamics can be viewed as an approximation for the classical Bass demand dynamics for new products. We derive the optimal strategy and dates to remove an old product and to

introduce a new one into the market.

KEYWORDS: Product rollover; Uncertain approval date, Solo Product Rollover, Dual Product Rollover, Risk management, Bass demand, Product demand diffusion.

3.1 Introduction and Literature Review

Several papers have addressed the analysis of new product introduction and product rollover processes under different assumptions and from various viewpoints such as marketing, operations management, and engineering design. Some researchers such as Erhun et al (2007) have performed qualitative studies on different drivers affecting product transitions and designed a framework that guide managers to design and implement appropriate policies taking into consideration transition risks related to the product, manufacturing process, supply chain features, and managerial policies in a competitive environment. The stock market reacts negatively to delays in product introduction, and that on average, delayed announcements decrease the market value of the firm, as Hendricks and Singhal (1997) claim.

Some papers develop quantitative models for the product rollover analysis. Lim and Tang (2006) developed a deterministic model that allows for the determination of prices of old and new products and the times of phase-in and phase-out of the products. Moreover, they developed marginal cost based conditions to determine when a dual product rollover strategy is more favorable than a single rollover one.

Classically, there are two rollover strategies: planned stockout and dual rollover. In the planned stockout-product rollover strategy, there is a simultaneous introduction of the new product and elimination of the old product, i.e., at any time there is a unique product generation available in the market. On the contrary, in the dual-product rollover strategy, the new product is introduced first and then the old product is phased out.

Thus, in this setting, two product generations coexist in the market, for a given time length. A planned stockout product rollover strategy can be viewed as a high-risk, high return strategy, sensitive to potential random events. On the contrary, the dual product rollover strategy is less risky, but induces higher inventory costs. For complex situations, Billington et al (1998) argue that in addition to the choice of the best strategy, planners should develop contingency plans in anticipation of certain events such as competitors introducing new products, technical problems with the new products, stock-out of old products, and too much inventory of the new or old product.

Few researchers examined different strategies for the simultaneous deletion of old products and the introduction of new products. In general, literature argues that there has been a low success rate for product rollovers and presents many cases of companies that have failed in product rollovers due to technical problems leading to delay in introduction of the new product to the market, excessive old product inventory, bad timing of new product announcement, and overly optimistic sales. It is suggested that companies should have a clear strategy for product rollover in addition to contingency plans in case their strategy fails.

A review of the literature reveals that the timing of market entry is a strategic qualitative decision as well as a tactical quantitative decision. The strategic choice between pioneering and following is a problem of balancing different costs and profits. Furthermore, the tactical decision of entry time is a problem of balancing the risks of premature entry and the missed opportunity of late entry. Usually, firms who enter earlier expect higher returns especially if they are successful, but bear the risk of lower likelihood of success than later entrants.

We examine the problem of simultaneously planning the phase-out of the old product and the phase-in of a new one that will replace the old product, under an uncertain approval date for the new product whose demand is piecewise linear. Initially, demand increases linearly until it reaches a certain level after which it becomes constant. These demand dynamics can be viewed as an approximation for the classical Bass demand dynamics for new products.

The Bass Diffusion Model for sales of new products was presented by Bass (1969). Since its publication in *Management Science*, it has been cited over 600 times and is one of the most notable models for new-product forecasting. It was originally developed for application only to durable goods. However, the model has proven applicable to a wider class of products and services such as B2B products, telecom services, equipment, semiconductor chips, medical products, and other technology-based products and services.

The Bass model assumes that a population of potential adopters for a new product is subject to two means of communication: mass-media communication and word-of-mouth communication. The former affects potential adopters directly, while the later influences the interaction between customers who already adopted the product, as well as the future potential adopters.

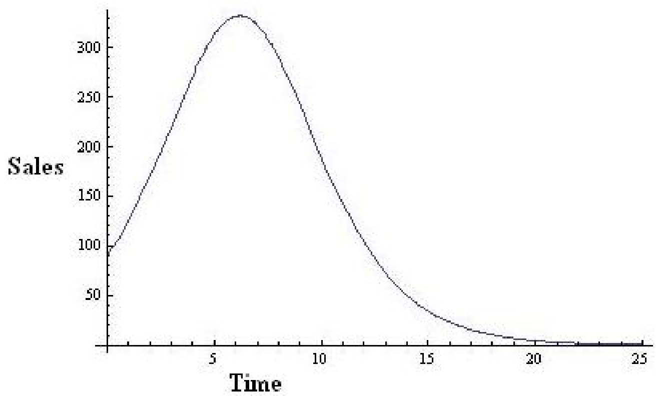

Figure 3.1: Classical Bass Diffusion Model

In our model, the manufacturing and procurement lead-times for our products are significant, making it necessary to commit to the planning date before the earliest approval date. The new product is not available for sale until the distribution channel is filled with a minimal number of units proportional to demand. The old product is sold until the firm runs out of inventory or until it is replaced by an approved new product. The firm's policy is to scrap all old product units immediately when an approved new product is available for sale. The fundamental structure of the problem, namely planning a starting date for an activity in a random setting, can be linked to the well known newsboy problem. The demand for the old product is constant, whereas the demand of the new product is initially linearly increasing then constant. In our main model, when the new product is delayed, all demand for this product is lost and there is inventory buildup. We also study another case, where when the new product is delayed, a portion of the demand is lost whereas another portion is maintained (See Figure 3.2). The portion of the demand that was not met but maintained is sold immediately after the approval is granted.

Druehl et Al (2009) argue that delaying a product too long may fail to capitalize on customer willingness-to pay for more advanced technology in addition to the possibility that competitors may (further) infiltrate the market, furthermore, sales of existing product may decline due to market saturation. If a firm introduces the new product too early, it may cannibalize the previous generation too quickly, not taking advantage of market growth. If it waits too long, sales may have slowed considerably as the product would have already diffused through the market. If there is not a sufficient base of customers of the innovator type, then the pace will be slow. But once this base of innovators exists, the pace will be increased by either innovators or imitators. In our problem, the market knows the time at which the new product will be introduced. The

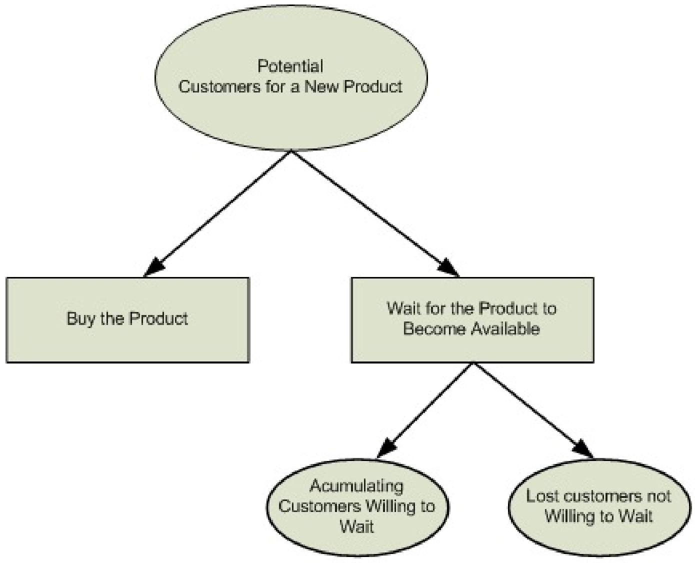

Figure 3.2: New Product Adoption

customer purchases the product if it has been approved by the regulatory authority where the demand rate is piecewise and dependent on time. If the product has not been approved, the demand is lost until approval is given. The demand of the new product remains piecewise linear dependent on time.

Hill and Sawaya (2004) examine the problem of simultaneously planning the phase-out of the old product and the phase-in of a new one that will replace the old product, under an uncertain approval date for the new product. Furthermore, they exhibit the structure of the optimal policy for an expected profit objective function. Their setting is similar to ours, when demands of the new and old products is constant and when the new product is not available all of the demand is lost. In this paper, we examine if considering a piecewise linearly increasing demand for the new product changes the optimal strategy or the timing decisions compared to the constant demand examined in Hill and Sawaya (2004).

In our first paper, we have considered a problem similar to Hill and Sawaya (2004), where demands of both new and old products were constant and we examined the rollover problem under expected loss and risk minimization. This was important to gain insight on the problem at hand, yet, a constant demand does not apply in real settings where products are subject to a diffusion rate and usually modeled through the Bass Model. Demand usually increases to reach a peak and then decreases after the product achieves maturity. In addition, in our first paper, if neither product was available in the market we assumed that all the demand was lost. Bass model literature contradicts this assumption where, when customers ask for a product and it is not available, not all demand is lost: some customers may be willing to wait at a certain waiting cost and will later purchase the product when it is available. On the other hand, some of the customers will choose not to wait and go on to purchase another product. This decreases lost profit as discussed by Norton and Wilson (1989). Due to the difficulty of obtaining closed-form solutions in case when not all demand is lost, we can only produce numerical simulations in this case that we present in our appendix.

Furthermore, in the first paper we consider the old product demand is equal to the new product demand and that the demand of the new product is not affected if it is delayed. Both those assumptions are violated in real life settings where there is accumulated inventory and there is a potential market loss when a product is delayed Druehl et Al (2009).

In this paper, we model a more realistic setting where demand is piecewise linear and another special case where not all demand is lost in case of delay, yet part of the market demand is lost. While we can model the old product demand as constant since at the end of the lifecycle of a product, its demand after decreasing becomes constant (See Figure 3.1). The demand of a new product usually increases incrementally over time and this has effect on product entry timing decisions.

We believe this study to be the first that examines this kind of setting of the product rollover problem. This approach is important for product rollover situations concerning key products for a company. We prove the uniqueness of the optimal solutions and approximate the optimal solutions through Mathematica as it is not possible to provide analytical closed-form solutions in case not all demand is lost.

This paper is organized as follows: we start by presenting the main product rollover evaluation model, then we discuss the optimal conditions and convexity. We then present a numerical study and conclude the paper with a summary of our findings and reflections for future research directions. We provide in an appendix the problem when not all demand is lost and which we can only solve numerically.

3.2 The Product Rollover Evaluation Model

In this section, we will define the product rollover problem and introduce the different notation and assumptions.

3.2.1 Stochastic rollover process and profit/cost rates

The problem context requires a production plan for the phase-out of an existing product (hereafter called *old product*, or *product* 1) and phase-in of a replacement product (called *new product* or *product* 2) under an uncertain (internal or external) approval date, denoted T, for the new product delivery. A typical example for such approval decisions are those of medical devices and pharmaceutical products which cannot be sold until an approval body grants permission. Two decision variables have to be determined in such a rollover process: t_1, the date the firm plans to run-out of the old product and t_2, the date the new product is planned to be ready and available for the market. The existing product is sold until the firm runs out of inventory or until it is replaced by the approved new product. The manufacturing and procurement lead times are assumed to be large, thus making it necessary to commit to the planning dates before the random approval date is revealed. The decision process relies thus exclusively on the probability distribution of this date T. Such large procurement/manufacturing/distribution lead-times are frequent in practice: for instance, the regulatory affairs department in a medical device firm uses a forecast interval for the approval date that is more than 6 months long.

At the end of the lifetime of a product, its demand decreases to become constant, here denoted by d_1. On the other hand, the demand of a new product is piecewise linear, initially increases with respect to time then becomes constant and is denoted by $d_{2,a}(t)$. A channel inventory is needed to support each product in the market, which induces per unit carrying inventory cost rate h. During the commercial life, the contribution-to-profit rate per unit for product i, is defined as

$$m_i = p_i - c_i, \qquad (i = 1, 2), \tag{3.2.1}$$

with p_i the selling price and c_i the production cost per unit.

In the considered random setting, the profit/cost structure, defined over an infinite time horizon, depends furthermore on the relative values of t_1, t_2 and T. Indeed, if the planned stock-out strategy $t_1 \leq t_2$ is chosen, the structure of the profit/cost rates is given in Figure 3.6,

Two main cases have to be considered. First, if $T \leq t_1$, the profit rate is $m_1 - h$ per unit sold of the first product over the time interval $[0, T[$, therefore the total contribution to profit is given by $(m_1 - h)d_1$ per unit time. Then, if $t_1 \leq T \leq t_2$, the new product is approved, but not physically available in the supply chain. The market is assumed to be informed that the new product will substitute the old product only at time t_2. Then, over the interval $[t_1, t_2[$, when the old product is sold out, shortages occur until new product delivery date t_2, at a corresponding shortage cost rate g per unit and the total shortage cost would be gd_1 per unit time. Once the new product is available, at t_2, the profit rate over the remaining time horizon $[t_2, \infty[$:

$$\left(m_2 - h\right) \begin{cases} \int_{t_2}^{\frac{d_2 - b_{2,a}}{a_{2,a}}} \left(a_{2,a}(t-t_2) + b_{2,a}\right) dt & \text{if } 0 \leq t - t_2 \leq \frac{d_2 - b_{2,a}}{a_{2,a}}, \\ \int_{\frac{d_2 - b_{2,a}}{a_{2,a}}}^{\infty} d_2 dt & \text{if } t - t_2 > \frac{d_2 - b_{2,a}}{a_{2,a}}. \end{cases} \tag{3.2.2}$$

Then, for the second case, one has $t_2 \leq T$. The profit/cost rates are similar to the previous situation, except over the interval $[t_2, T[$, where the new product is physically available in the supply chain, but still not approved. Then, over the interval $[t_1, t_2[$, when the old product is sold out, the shortage cost rate is given by gd_1 per unit time. All of the demand $d_{2,a}(t)$ of the new product is lost at a shortage cost rate of g per unit until new product 2 is approved at time T and the shortage cost incurred would be:

$$-g \begin{cases} \int_{t_2}^{\frac{d_2 - b_{2,a}}{a_{2,a}}} \left(a_{2,a}(t-t_2) + b_{2,a}\right) dt & \text{if } 0 \leq t - t_2 \leq \frac{d_2 - b_{2,a}}{a_{2,a}}, \\ \int_{\frac{d_2 - b_{2,a}}{a_{2,a}}}^{T} d_2 dt & \text{if } \frac{d_2 - b_{2,a}}{a_{2,a}} < t - t_2 < T. \end{cases} \tag{3.2.3}$$

On the other hand an inventory proportional to $d_{2,a}(t - t_2)$ at holding cost rate per unit of h and the cost rate is given by:

$$-h \begin{cases} \int_{t_2}^{\frac{d_2 - b_{2,a}}{a_{2,a}}} \left(a_{2,a}(t-t_2) + b_{2,a}\right) dt & \text{if } 0 \leq t - t_2 \leq \frac{d_2 - b_{2,a}}{a_{2,a}}, \\ \int_{\frac{d_2 - b_{2,a}}{a_{2,a}}}^{T} d_2 dt & \text{if } \frac{d_2 - b_{2,a}}{a_{2,a}} < t - t_2 < T. \end{cases} \tag{3.2.4}$$

At T once the approval is given, the contribution to profit is given by

$$(m_2 - h) \begin{cases} \int_{T}^{\frac{d_2 - b_{2,a}}{a_{2,a}}} \left(a_{2,a}(t-T) + b_{2,a}\right) dt & \text{if } 0 \leq t - T \leq \frac{d_2 - b_{2,a}}{a_{2,a}}, \\ \int_{\frac{d_2 - b_{2,a}}{a_{2,a}}}^{\infty} d_2 dt & \text{if } t - T > \frac{d_2 - b_{2,a}}{a_{2,a}}. \end{cases} \tag{3.2.5}$$

In case the dual rollover strategy $t_2 \leq t_1$ is chosen, the structure of the costs and profit rates is given in Figure 3.7. Let us consider first the case $T < t_2$. The profit rate is $m_1 - h$ per unit over the time interval $[0, t_2[$ when $T < t_2$ so the total contribution to profit would be $(m_1 - h)d_1$ per unit time. Then, over the time interval $[t_2, t_1[$, the new product is approved and physically available, it is sold with a profit rate per unit $m_2 - h$ and the contribution to profit would be the same as that given in equation 3.2.2. In the current setting, it is however assumed that the firm scraps, at a cost rate s_1 per unit, all the remaining inventory of product 1 immediately when an approved product 2 is available for sale, i.e., over the time interval $[t_2, t_1]$ if $T < t_2$ and $[T, t_1]$ if $t_2 < T < t_1$) giving a scrap cost of $s_1 d_1$ per unit time. This can be linked to several typical market forces that can be observed in some sectors. First, in some situations, it is considered as important (if not necessary) to provide customers with the latest technology, i.e., with the newest product type. Second, higher demand, higher prices, and higher commissions drive sales organizations to shift to the new product. Third, marketing organizations want products that accentuate the leading edge nature of the firm's brand and do not want to lose the opportunity to sell the best and latest product. This is justified by the higher margins for product 2 and by the need to maintain brand equity as a leading-edge provider.

In the second situation, one has $t_2 \leq T \leq t_1$. The total profit is $(m_1 - h)d_1$ per unit over $[0, t_2[$. Then over the interval $[t_2, T[$, the profit rate is still $(m_1 - h)d_1$ per unit time; however, as the new product is physically available in the supply chain, but the demand $d_{2,a}(t)$ of the new product is lost at a shortage cost rate of g per unit until new product 2 is approved at time T giving a shortage cost given as shown in equation 3.2.3. On the other hand a channel inventory proportional to $d_{2,a}(t)$ is kept at a holding cost rate per unit of h with the cost given in equation 3.2.4. At T once the approval is given, the demand of the new product becomes $d_{2,a}(t - T)$ giving a profit as depicted in equation 3.2.5. In the time interval $[T, t_1[$, the old product is scrapped at a cost rate s_1 per unit or $s_1 d_1$ per unit time.

In the last case, $t_1 \leq T$, the profit/cost rates are similar to the previous situation, over all the time intervals, except that there is no longer any scraping for product 1 as $t_1 < T$.

3.2.2 Notation for the Model

For this rollover optimization model, we adopt the following notation. As we have explained in the previous section, all profits/costs depend on time since the demand of the new product is a piecewise linear function of time.

Deterministic Parameters:

c_i is the per unit cost for product i,

p_i is the per unit unit price for product i,

$p_i - c_i$ is the gross margin per unit for product i,

m_i is the contribution to profit per unit for product i and is defined as

$$m_i = p_i - c_i, \tag{3.2.6}$$

g is the shortage cost per unit when the firm has neither of the products to sell,

h is the carrying cost per unit of product 1 or 2,

s_1 is the per unit scrap cost for product 1 (note that if there is some positive margin when getting rid of product 1 inventory, then one has $s_1 < 0$ and one can speak of "scrap margin". Clearly in this case one has $|s_1| < m_1$,

d_1 is the rate of demand of product 1 per unit time and it is constant,

$d_{2,a}(t)$ is the rate of demand of product 2 when product 2 is granted approval on time and is given by

$$d_{2,a}(t) = \begin{cases} a_{2,a}t + b_{2,a} & \text{if } 0 \leq t \leq \frac{d_2 - b_{2,a}}{a_{2,a}}, \\ d_2 & \text{if } t > \frac{d_2 - b_{2,a}}{a_{2,a}}. \end{cases} \tag{3.2.7}$$

Based on our discussion on late product diffusion in the previous section, $b_{2,a} > d_1$, and $d_2 > 0$ and constant.

Random Parameters:

T is the random approval date for the new product (i.e., for product 2). This random variable has a probability density function $f(\cdot)$ and a probability distribution function $F(\cdot)$ defined on the range $[0, \infty[$, i.e., one has

$$Prob[0 \leq T \leq u] = \int_0^u f(T)dT = F(u). \tag{3.2.8}$$

We denote $G(\cdot)$, the partial distribution function defined as

$$G(t) = \int_0^t Tf(T)dT. \tag{3.2.9}$$

Let μ be mean of the approval date distribution, where $\mu = G(\infty)$.

Decision Variables:

t_1 is the planned run-out date for inventory of the existing product (i.e., product 1),

t_2 is the planned approval date for the new product (i.e., product 2).

t_b is when $t_1 = t_2 = t_b$ which is the case of the single rollover strategy .

The following constraint for the decision variables

$$0 \leq t_1, t_2 \leq \infty. \tag{3.2.10}$$

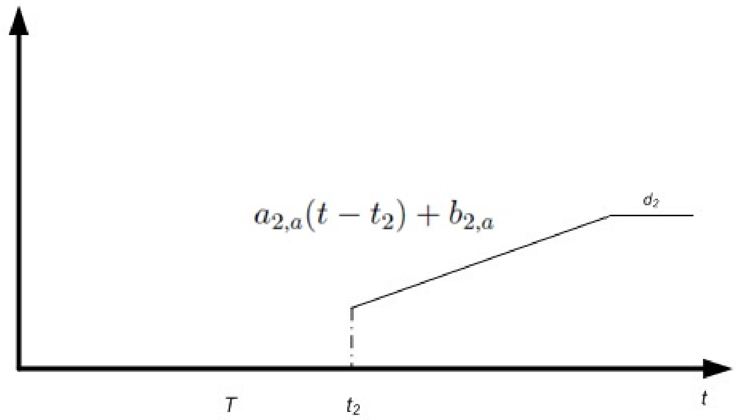

Figure 3.3: Approval granted before t_2

Inventory Policy:

We suppose that the production capacity is unlimited and the firm chooses to produces as much as the cumulative demand at time t.

3.2.3 Demand Process

As we have previously mentioned, the demand of the old product is constant and denoted by d_1. The market knows that a new product will be introduced at time t_2 (Figure 3.3). If $T < t_2$, the customer purchases the product as of time t_2 and the demand of the new product is piecewise linear increasing with time given by

$$d_{2,a}(t) = \begin{cases} a_{2,a}t + b_{2,a} & \text{if } 0 \leq t \leq \frac{d_2 - b_{2,a}}{a_{2,a}}, \\ d_2 & \text{if } t > \frac{d_2 - b_{2,a}}{a_{2,a}}. \end{cases} \tag{3.2.11}$$

On the other hand, if $T > t_2$, all of the demand between t_2 and T is lost (See Figure 3.4).

3.2.4 Net Loss Function

Due to the structure of the problem, the state space has to be divided in two regions, $R_1 = \{t_1, t_2 \in R^+ \text{ with } t_1 \leq t_2\}$ and $\mathbf{R_2} = \{t_1, t_2 \in R^+ \text{ with } t_1 \geq t_2\}$. Over region

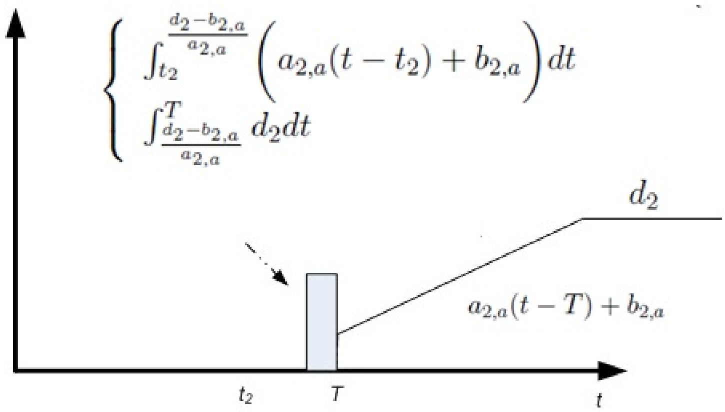

Figure 3.4: Approval granted after t_2

$\mathbf{R_1}$ the objective function is denoted as $L_1(t_1, t_2, T)$ (and $L_2(t_1, t_2, T)$ for region $\mathbf{R_2}$) and both functions are continuous throughout the space and at boundary $t_1 = t_2$. We define the objective function as the net loss incurred between the ideal case or the case of full information and the cases where the approval date is uncertain. Formally, according to the description previously given, the net loss functions $L_1(t_1, t_2, T)$ and $L_2(t_1, t_2, T)$ are continuous and can be decomposed into functions defined on bounded intervals. This decomposition can be expressed as

$$L_j(t_1, t_2, T) = L_{j,i}(t_1, t_2, T) \quad \text{if } T \in I_i, \quad \text{for } j = 1, 2;\ i = 1, 2, ..., k \tag{3.2.12}$$

with k, the functions $L_{j,i}(t_1, t_2, \cdot)$ and the intervals I_i to be defined in the following sections.

Let $L_b(t_b, T)$ be the net loss functions at the boundary $t_1 = t_2 = t_b$ defined as follows:

$$L_b(t_b, T) = L_{b,i}(t_b, T) \quad \text{if } T \in I_i, \quad \text{for } i = 1, 2 \tag{3.2.13}$$

3.2.5 Ideal Case

In this ideal setting, the optimal solution is clear : $t_1 = t_2 = T$, i.e., the old product is sold out at the planned introduction date of the new product, corresponding to the approval date. Over the time interval $[0, T[$, the profit rate is $m_1 - h$ per unit, while on

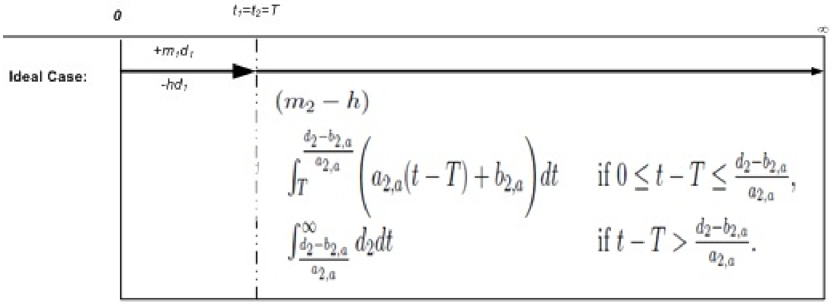

Figure 3.5: Perfect Information Case

the remaining horizon $[T, \infty[$, the profit rate is $m_2 - h$ per unit. In order to characterize the impact of randomness on the rollover process, we consider an objective function defined as the difference between the perfect information cost rate function (Figure 3.5) and the cost rates functions with imperfect information (Figures 3.6 and 3.7). This difference can be interpreted as the loss caused by the randomness of the approval date T. Formally, according to the description given above, these loss functions are piecewise linear and exhibit different structures, depending on the relative values of the decision variables t_1 and t_2.

3.2.6 Planned Stockout Rollover $t_1 \leq t_2$:

For a planned stockout rollover strategy; the company plans to run out of the old product before introducing product 2 into the market. The random approval date T falls into one of these two cases: $0 \leq T < t_2$ and $t_2 \leq T < \infty$. The firm sells product 1 during $(0, t_1)$ at a demand rate of d_1 and a net profit of $m_2 - h$ per unit demand. Between (t_1, t_2) there are no products to sell in the market, incurring a shortage cost of g. The market knows that at time t_2 the firm plans to introduce a new product into the market. If $T \leq t_2$, product 2 will be available in the market and sold at the rate of $d_{2,a}(t)$ with a contribution to profit of $m_2 - h$ per unit. On the other hand, if $t_2 \leq T$, product 2 is not available in the market, no customer is willing to wait and all demand is lost

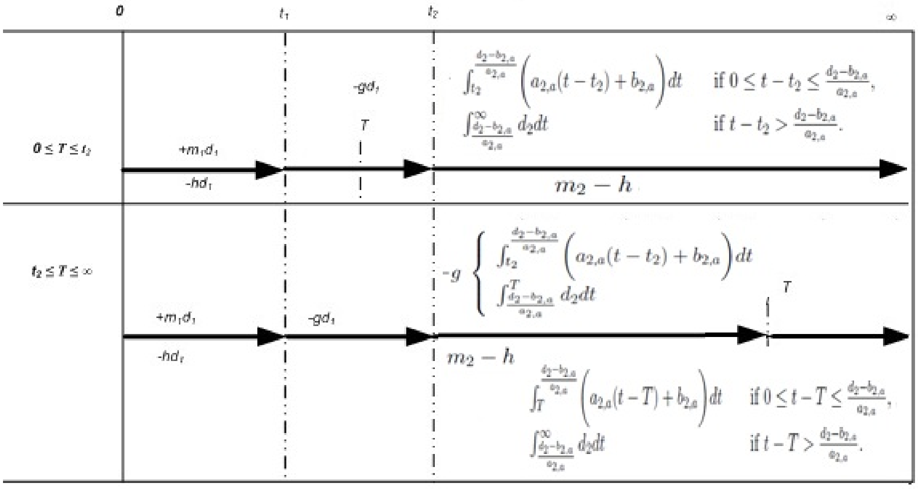

Figure 3.6: Planned Stockout Rollover Strategy

incurring a shortage cost of g per unit. When the product is available in the market, the new product will then be sold at a demand rate of $d_{2,a}(t)$ and a contribution to profit of $m_2 - h$ per unit.

Over region $\mathbf{R_1}$, the decomposition (3.2.12) is given in Figure 3.6. The net loss in this case is given by:

$$L_1(t_1, t_2, T) = \begin{cases} (m_1 - h)(T - t_1)d_1 + g(t_2 - t_1)d_1 + (m_2 - h)d_2(t_2 - T) \\ \text{if } 0 \le T \le t_2, \\ (m_1 - h)(T - t_1)d_1 + g(t_2 - t_1)d_1 + (g + h)d_2(T - t_2) \\ \text{if } t_2 \le T, \end{cases} \tag{3.2.14}$$

and $I_1 = [0, t_2]$ and $I_2 = [t_2, \infty]$.

It is clear from Figure (3.6) and expression (3.2.14) that for any given value of t_2, the firm can always increase the contribution to profit and reduce lost goodwill by increasing t_1. This means that the optimal policy can either be $t_1 = t_2$ or $t_1 > t_2$, which is the same result presented by Hill and Sawaya (2004).

3.2.7 Single Rollover Strategy

For the single rollover strategy, the net the loss function becomes

$$L_b(t_b, T) = \begin{cases} (m_1 - h)(T - t_b)d_1 + (m_2 - h)d_2(t_b - T) \\ \text{if } 0 \le T \le t_b, \\ (m_1 - h)(T - t_b)d_1 + (g + h)d_2(T - t_b)\Big) \\ \text{if } t_b \le T, \end{cases} \quad (3.2.15)$$

and $I_1 = [0, t_b]$ and $I_2 = [t_b, \infty]$.

3.2.8 Dual Rollover $t_2 \le t_1$

Over region $\mathbf{R_2}$, the decomposition is given in Figure 3.7 and the net loss becomes:

$$L_2(t_1, t_2, T) = \begin{cases} (m_1 - h)(T - t_2)d_1 + s_1(t_1 - t_2)d_1 + (m_2 - h)d_2(t_2 - T) \\ \text{if } 0 \le T \le t_2, \\ s_1(t_1 - T)d_1 + (g + h)d_2(T - t_2)\Big) \\ \text{if } t_2 \le T \le t_1, \\ (m_1 - h)(T - t_1)d_1 + (g + h)d_2(T - t_2) \\ \text{if } t_1 \le T, \end{cases} \quad (3.2.16)$$

and $I_1 = [0, t_2]$, $I_2 = [t_2, t_1]$, and $I_3 = [t_1, \infty]$

3.2.9 Parameter Assumptions

As usual in a stochastic production/inventory model, it is necessary to introduce some assumptions for the different parameters. These assumptions are as follows. First the contribution-to-profit rate per unit for the products under regular sales is positive, i.e.,

$$m_1, m_2 > 0. \quad (3.2.17)$$

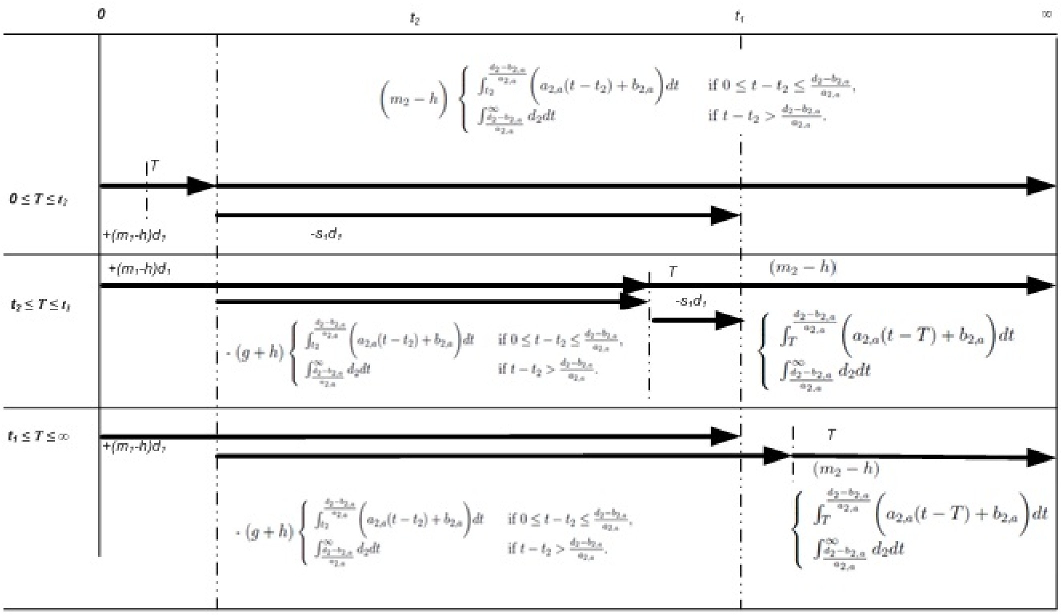

Figure 3.7: Dual Rollover Strategy

We also have the standard assumptions from classical inventory theory, namely

$$g, h, s_1 > 0. \tag{3.2.18}$$

For the demand process we have the following assumptions:

$$a_{2,a} > 0, b_{2,a} > d_1 > 0, d_2 > d_1 \tag{3.2.19}$$

3.3 Optimality Conditions and Convexity

In this section, we present the optimality conditions through the first-order derivatives and try to obtain closed form solutions.

3.3.1 Planned Stock-out

$$\begin{aligned} l_1(t_1,t_2) &= \int_0^{t_2}\Big((m_1-h)(T-t_1)d_1+g(t_2-t_1)d_1 \\ &+ (m_2-h)d_2(t_2-T)\Big)f(T)dT \\ &+ \int_{t_2}^{\infty}\Big((m_1-h)(T-t_1)d_1+g(t_2-t_1)d_1 \\ &+ (g+h)d_2(T-t_2)\Big)f(T)dT \end{aligned} \tag{3.3.1}$$

The first order derivative of $l_1(t_1,t_2)$ with respect to t_1 is given by

$$\frac{dl_1(t_1,t_2)}{dt_1} = -d_1(m_1+g-h) \tag{3.3.2}$$

Expression (3.3.2) is strictly decreasing with respect to t_1, therefore the optimal value of t_1 occurs at the maximum possible value of t_1, which is, in our case, t_2, and therefore the optimal solution occurs on the boundary $t_1 = t_2$ or $t_1 > t_2$, as was proven in Hill and Sawaya (2004).

3.3.2 Single Rollover

$$\begin{aligned} l_b(t_b) &= \int_0^{t_b}\Big((m_1-h)(T-t_b)d_1+(m_2-h)d_2(t_b-T)\Big)f(T)dT \\ &+ \int_{t_b}^{\infty}\Big((m_1-h)(T-t_b)d_1+(g+h)d_2(T-t_b)\Big)f(T)dT \end{aligned} \tag{3.3.3}$$

The first order derivative of $l_b(t_b)$ with respect to t_b is given by

$$\frac{dl_b(t_b)}{dt_b} = d_1(h-m_1)-d_2(g+h)+\Big(m_2+g\Big)d_2F(t_b) \tag{3.3.4}$$

The optimal value of t_b^* occurs when expression (3.3.4) is zero as follows:

$$\frac{dl_b(t_b^*)}{dt_b} = d_1(h-m_1)-d_2(g+h)+\Big(m_2+g\Big)d_2F(t_b^*) = 0 \tag{3.3.5}$$

Therefore, we have

$$t_b^* = F^{-1}\left(\frac{(m_1 - h)d_1 + (g+h)d_2}{(m_2+g)d_2}\right) \tag{3.3.6}$$

Since $m_2 > m_1$ and $d_2 > d_1$, then t_b^* always exists.

3.3.3 Convexity

The second order derivative of $l_b(t_b)$ with respect to t_b is given by

$$\frac{dl_b^2(t_b)}{dt_b^2} = \Big((m_2 - h)d_2 + (g+h)d_2\Big) f(t_b^*) \tag{3.3.7}$$

The second order derivative given in (3.3.7) is convex since $m_2 > h$ (See Figure 3.8 as an example).

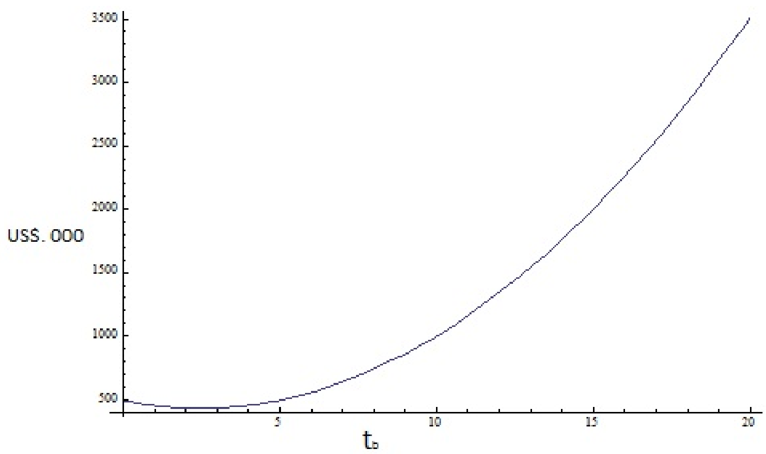

Figure 3.8: Single Rollover with Piecewise Linear Demand for a convex $l_b(t_b)$ where $m_1 = 49$, $m_2 = 50$, $g = 0$, $h = 0$, $a_{2,a} = 105$, $b_{2,a} = 6$, $d_1 = 1$, and $d_2 = 8$, $F(T)$ is uniform $[0, 20]$.

3.3.4 Dual Rollover

$$\begin{aligned} l_2(t_1,t_2) &= \int_0^{t_2}\Big((m_1-h)(T-t_2)d_1+s_1(t_1-t_2)d_1 \\ &+ (m_2-h)d_2(t_2-T)\Big)f(T)dT \\ &+ \int_{t_2}^{t_1}\Big(s_1(t_1-T)d_1+(g+h)d_2(T-t_2)\Big)f(T)dT \\ &+ \int_{t_1}^{\infty}\Big((m_1-h)(T-t_1)d_1+(g+h)d_2(T-t_2)\Big)f(T)dT \end{aligned} \tag{3.3.8}$$

The first order derivative of $l_2(t_1,t_1)$ with respect to t_1 is given by

$$\frac{dl_2(t_1,t_2)}{dt_1} = -d_1(m_1-h)+d_1(m_1-h+s_1)F(t_1) \tag{3.3.9}$$

Setting expression (3.3.9) to zero, we get the optimal value of t_1 to be:

$$t_1^* = F^{-1}\Big(\frac{m_1-h}{m_1-h+s_1}\Big) \tag{3.3.10}$$

The first order derivative of $l_2(t_1,t_1)$ with respect to t_2 is given by

$$\frac{dl_2(t_1,t_2)}{dt_2} = -\Big(g+h\Big)d_2+\Big((m_2+g)d_2-(m_1-h+s_1)d_1\Big)F(t_2) \tag{3.3.11}$$

The optimal value of t_2^* occurs when expression (3.3.11) is zero as follows:

$$t_2^* = F^{-1}\Big(\frac{(g+h)d_2}{(m_2+g)d_2-(m_1-h+s_1)d_1}\Big) \tag{3.3.12}$$

For t_2^* to exist the following condition should be satisfied

$$(m_2+g)d_2-(m_1-h+s_1)d_1 > 0 \tag{3.3.13}$$

Furthermore $t_2^* < t_1^*$, or the following condition should be satisfied

$$(m_2-h)d_2-(m_1-h+s_1)d_1 > 0 \tag{3.3.14}$$

3.3.5 Convexity

The second order derivative of $l_2(t_1,t_1)$ with respect to t_2 is given by

$$\frac{d^2l_2(t_1,t_2)}{dt_2^2} = \Big((m_2+g)d_2-(m_1-h+s_1)d_1\Big)f(t_2) \tag{3.3.15}$$

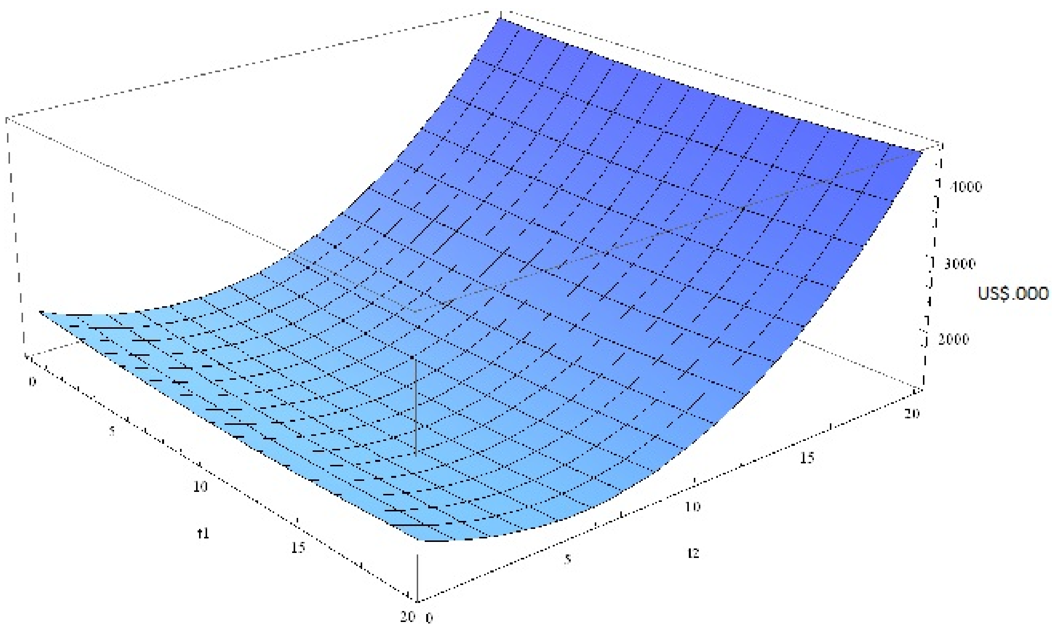

Figure 3.9: Optimal Solution Dual Rollover $t_1^* = 11.11$ and $t_2 = 5.79$ with optimal cost 1.264 million US Dollars for $m_1 = 6$, $m_2 = 22$, $g = 3$, $h = 1$, $d_1 = 5$, $s_1 = 4$, $a_{2,a} = 0.15$, and $b_{2,a} = 20$, $d_2 = 50$ $F(T)$ is uniform $[0, 20]$.

The objective function $l_2(t_1, t_1)$ is convex with respect to t_2 if

$$(m_2 + g)d_2 - (m_1 - h + s_1)d_1 > 0 \tag{3.3.16}$$

3.4 Numerical Example

A hypothetical example is presented here to illustrate our model. The product is selling d1 = 5 units a month of medical device 1 and has submitted product 2 for FDA approval on January 1, 2011. Product 2 is expected to be approved sometime between January 1, 2011 and August 1, 2012, following a uniform distribution. We consider the following parameters for costs in thousands US$ and demand: $m_1 = 20$, $m_2 = 22$, $g = 3$, $h = 10$, $a_{2,a} = 0.15$, $b_{2,a} = 20$, $a_{2,l} = 0.10$, $b_{2,a} = 15$, $w = 8$, $d_1 = 5$, and $d_2 = 50$. Let the probability distribution of time T be a uniform one given by $f(T) = \frac{1}{20}$. For this setting, we can see that the optimal strategy is a dual rollover one. We get the optimal removal date $t_1^* = 11.11$, or November 3, 2011 and the introduction date $t_2^* = 5.79$ or May 24, 2011 (Refer to Figure 3.9).

3.5 Conclusion and Limitations

In this paper, we tried to approximate a bass demand model in a product rollover setting under an uncertain regulatory approval date. There are many limitations to this research as we have not taken into consideration the effect of the customer's willingness to wait or buy another product in case of delay. For the time being, we have presented a model along with closed form solutions and conditions for convexity. We have tried solve the more complex model presented in our appendix, but could only obtain solutions through numerical simulations. It would be worth it to approximate that complex model to get a closed form if possible, but that is left for future research.

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Appendix

The Product Rollover Evaluation Model

In this section, we will define the product rollover problem and introduce the different notation and assumptions.

Stochastic rollover process and profit/cost rates

The problem context requires a production plan for the phase-out of an existing product (hereafter called *old product*, or *product* 1) and phase-in of a replacement product (called *new product* or *product* 2) under an uncertain (internal or external) approval date, denoted T, for the new product delivery. A typical example for such approval decisions are those of medical devices and pharmaceutical products which cannot be sold until an approval body grants permission. Two decision variables have to be determined in such a rollover process: t_1, the date the firm plans to run-out of the old product and t_2, the date the new product is planned to be ready and available for the market. The existing product is sold until the firm runs out of inventory or until it is replaced by the approved new product. The manufacturing and procurement lead times are assumed to be large, thus making it necessary to commit to the planning dates before the random approval date is revealed. The decision process relies thus exclusively on the probability distribution of this date T. Such large procurement/manufacturing/distribution lead-times are frequent in practice: for instance, the regulatory affairs department in a medical device firm uses a forecast interval for the approval date that is more than 6 months long.

At the end of the lifetime of a product, its demand decreases to become constant, here denoted by d_1. On the other hand, the demand of a new product increases with respect to time and is denoted by $d_{2,a}(t)$. A channel inventory is needed to support each product in the market, which induces per unit inventory carrying cost rate h. During the commercial life, the contribution-to-profit rate per unit for product i, is defined as

$$m_i = p_i - c_i, \qquad (i = 1, 2), \tag{3.5.1}$$

with p_i the selling price and c_i the production cost per unit.

In the considered random setting, the profit/cost structure, defined over an infinite time horizon furthermore depends on the relative values of t_1, t_2 and T. Indeed, if the planned stock-out strategy $t_1 \leq t_2$ is chosen, the structure of the profit/cost rates is given in Figure 3.13,

Two main cases have to be considered. First, if $T \leq t_1$, the profit rate is $m_1 - h$ per unit sold of the first product over the time interval $[0, T[$, therefore the total contribution to profit is given by $(m_1 - h)d_1$ per unit time. Then, if $t_1 \leq T \leq t_2$, the new product is approved, but not physically available in the supply chain. The market is assumed to be informed that the new product will substitute the old product only at time t_2. Then, over the interval $[t_1, t_2[$, when the old product is sold out, shortages occur until new product delivery date t_2, at a corresponding shortage cost rate g per unit and the total shortage cost would be gd_1 per unit time. Once the new product is available, at t_2, the profit rate per unit becomes $m_2 - h$ over the remaining time horizon $[t_2, \infty[$ where the demand of the new product is linearly time dependent and defined as $d_{2,a}(t)$ and the total contribution to profit would be $(m_2 - h)d_{2,a}(t - t_2)$ per unit time. Then, for the second case, one has $t_2 \leq T$. The profit/cost rates are similar to the previous situation, except over the interval $[t_2, T[$, where the new product is physically available in the supply chain, but still not approved. A portion ξ of the demand $d_{2,a}(t)$ of the new product is lost at a shortage cost rate of g per unit until new product 2 is approved at time T and the shortage cost incurred would be $g\xi d_{2,a}(t - t_2)$ per unit time. On the other hand a portion $1-\xi$ of the demand $d_{2,a}(t)$ is accumulated at a waiting and holding cost rate per unit of $h+w$ and the invetory and waiting cost would be given by $(h+w)(1-\xi)d_{2,a}(t-t_2)$ per unit time. At T once the approval is given, all of the accumulated demand between t_2 and T is sold at profit $m_2(1-\xi)d_{2,a}(t-t_2)$ per unit time and the demand of the new product becomes $d_{2,l}(t)$ where $a_{2,a} > a_{2,l}$ and $b_{2,a} > b_{2,l}$ and the contribution to profit per unit time is given by $(m_2-h)d_{2,l}(T-t_2)$.

On the other hand, if the dual rollover strategy $t_2 \leq t_1$ is chosen, the structure of the costs and profit rates is given in Figure 3.14. Let us consider first the case $T < t_2$. The profit rate is $m_1 - h$ per unit over the time interval $[0, t_2[$ when $T < t_2$ so the total contribution to profit would be $(m_1-h)d_1$ per unit time. Then, over the time interval $[t_2, t_1[$, the new product is approved and physically available, it is sold with a profit rate per unit $m_2 - h$ and the contribution to profit would be $(m_2-h)d_{2,a}(t-t_2)$ per unit time. In the current setting, it is however assumed that the firm scraps, at a cost rate s_1 per unit, all the remaining inventory of product 1 immediately when an approved product 2 is available for sale, i.e., over the time interval $[t_2, t_1]$ if $T < t_2$ and $[T, t_1]$ if $t_2 < T < t_1$) giving a scrap cost of $s_1 d_1$ per unit time. This can be linked to several typical market forces that can be observed in some sectors. First, in some situations, it is considered as important (if not necessary) to provide customers with the latest technology, i.e., with the newest product type. Second, higher demand, higher prices, and higher commissions drive sales organizations to shift to the new product. Third, marketing organizations want products that accentuate the leading edge nature of the firm's brand and do not want to lose the opportunity to sell the best and latest product. This is justified by the higher margins for product 2 and by the need to maintain brand equity as a leading-edge provider. Finally, over the remaining time horizon $[t_1, \infty[$, the profit rate becomes to $(m_2-h)d_{2,a}(t-t_2)$ per unit time of product.

In the second situation, one has $t_2 \leq T \leq t_1$. The total profit is $(m_1 - h)d_1$ per unit over $[0, t_2[$. Then over the interval $[t_2, T[$, the profit rate is still $(m_1 - h)d_1$ per unit time, but as the new product is physically available in the supply chain, but a portion ξ of the demand $d_{2,a}(t)$ of the new product is lost at a shortage cost rate of g per unit until new product 2 is approved at time T giving a shortage cost of $g\xi d_{2,a}(t - t_2)$ per unit time. On the other hand a portion $1 - \xi$ of the demand $d_{2,a}(t)$ is accumulated at a waiting and holding cost rate per unit of $h + w$ giving a total $(h + w)(1 - \xi)d_{2,a}(t - t_2)$ per unit time. At T once the approval is given, all of the accumulated demand between t_2 and T is sold at profit m_2 for a profit $m_2(1 - \xi)d_{2,a}(t - t_2)$ per unit time and the demand of the new product becomes $d_{2,l}(t)$ where $a_{2,a} > a_{2,l}$ and $b_{2,a} > b_{2,l}$ giving a profit per unit time of $(m_2 - h)d_{2,l}(T - t_2)$. From T until the whole remaining horizon, the new product is sold with a profit rate $(m_2 - h)d_{2,l}(T - t_2)$ per unit time. In the time interval $[T, t_1[$, the old product is scrapped at a cost rate s_1 per unit or $s_1 d_1$ per unit time.

In the last case, $t_1 \leq T$, the profit/cost rates are similar to the previous situation, except over all the time intervals, except that there is no longer any scraping for product 1 as $t_1 < T$.

Notation for the Model

For this rollover optimization model, we adopt the following notation. As we have explained in the previous section, all profit/cost depend on time since the demand of the new product is a initially a linearly increasing function with respect to time.

Deterministic Parameters:

c_i is the per unit cost for product i,

p_i is the per unit unit price for product i,

$p_i - c_i$ is the gross margin per unit for product i,

m_i is the contribution to profit per unit for product i and is defined as

$$m_i = p_i - c_i, \tag{3.5.2}$$

g is the shortage cost per unit when the firm has neither of the products to sell,

h is the carrying cost per unit of product 1 or 2,

s_1 is the per unit scrap cost for product 1 (note that if there is some positive margin when getting rid of product 1 inventory, then one has $s_1 < 0$ and one can speak of "scrap margin". Clearly in this case one has $|s_1| < m_1$,

w is the waiting cost per unit for product 2 if the product is not available,

ξ is the portion of demand of product 2 that is lost when product 2 is not available, and $1 - \xi$ is the portion of demand willing to wait till product 2 is available,

d_1 is the rate of demand of product 1 per unit time and it is constant,

$d_{2,a}(t)$ is the rate of demand of product 2 when product 2 is available on time and is given by $d_{2,a}(t) = a_{2,a}t + b_{2,a}$ where $a_{2,a} > 0$, $b_{2,a} > 0$, and $d_2 > 0$ and constant.

$$d_{2,a}(t) \quad = \quad \begin{cases} a_{2,a}t + b_{2,a} & \text{if } 0 \leq t \leq \frac{d_2 - b_{2,a}}{a_{2,a}}, \\ d_2 & \text{if } t > \frac{d_2 - b_{2,a}}{a_{2,a}}. \end{cases} \tag{3.5.3}$$

$d_{2,l}(t)$ is the rate of demand of product 2 when product 2 is late and is give by

$$d_{2,l}(t) \quad = \quad \begin{cases} a_{2,l}t + b_{2,l} & \text{if } 0 \leq t \leq \frac{d_2 - b_{2,l}}{a_{2,l}}, \\ d_2 & \text{if } t > \frac{d_2 - b_{2,l}}{a_{2,l}}. \end{cases} \tag{3.5.4}$$

where $a_{2,l} > 0$ and $b_{2,l} > 0$.

Based on our discussion on late product diffusion in the previous section, we note that $a_{2,a} > a_{2,l}$, $b_{2,a} > b_{2,l} > d_1$, and $d_2 > 0$ and constant.

Random Parameters:

T is the random approval date for the new product (i.e., for product 2). This random variable has a density probability function $f(\cdot)$ and a probability distribution function $F(\cdot)$ defined on the range $[0, \infty[$, i.e., one has

$$Prob[0 \leq T \leq u] = \int_0^u f(T)dT = F(u). \tag{3.5.5}$$

We denote $G(\cdot)$, the partial distribution function defined as

$$G(t) = \int_0^t Tf(T)dT. \tag{3.5.6}$$

Let μ be mean of the approval date distribution, where $\mu = G(\infty)$.

Decision Variables:

t_1 is the planned run-out date for inventory of the existing product (i.e., product 1),

t_2 is the planned approval date for the new product (i.e., product 2).

t_b is the date when the inventory of the existing product is equal to the approval date of the new product, or $t_1 = t_2 = t_b$ which is the case of the single rollover strategy .

Clearly it is necessary to consider the following constraint for the decision variables

$$0 \leq t_1, t_2 \leq \infty. \tag{3.5.7}$$

Inventory Policy:

We suppose that the production capacity is unlimited and the firm chooses to produces as such as the cumulative demand at time t.

Demand Process

As we have previously mentioned, the demand of the old product is constant and denoted by d_1. The market knows that a new product will be introduced at time t_2. If $T < t_2$, the customer purchases the product if it has been approved by the regulatory authority (Figure 3.10) and the demand of the new product is initially linearly increasing with time given by $d_{2,a}(t) = a_{2,a}t + b_{2,a}$ where $a_{2,a} > 0$ and $b_{2,a} > 0$ until it reaches a time $\frac{d_2 - b_{2,a}}{a_{2,a}}$ when it becomes constant. On the other hand, if $T > t_2$, part of the demand is accumulated between t_2 and T and sold at T or as soon as the approval is given. The demand of the new product then becomes $d_{2,l}(t) = a_{2,l}t + b_{2,l}$ where $a_{2,a} > a_{2,l} > 0$ and $b_{2,a} > b_{2,l} > 0$ (Figure 3.11) until it reaches a time $\frac{d_2 - b_{2,l}}{a_{2,l}}$ when it becomes constant.

Net Loss Function:

Due to the structure of the problem, the state space has to be divided in two regions, $R_1 = \{t_1, t_2 \in R^+ \text{ with } t_1 \leq t_2\}$ and $\mathbf{R_2} = \{t_1, t_2 \in R^+ \text{ with } t_1 \geq t_2\}$. Over region $\mathbf{R_1}$ the objective function is denoted as $L_1(t_1, t_2, T)$ (and $L_2(t_1, t_2, T)$ for region $\mathbf{R_2}$) and are continuous throughout the space and at boundary $t_1 = t_2$ (See Appendix A).

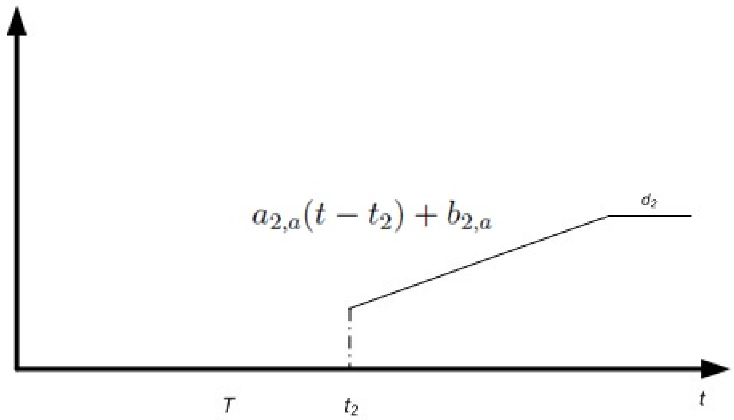

Figure 3.10: Approval granted before t_2

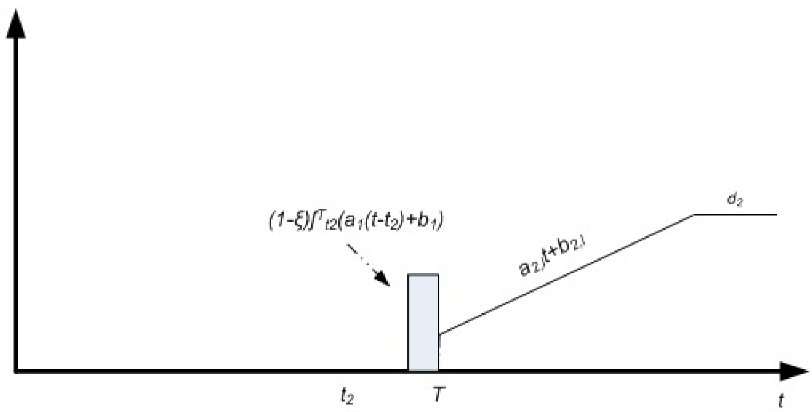

Figure 3.11: Approval granted after t_2

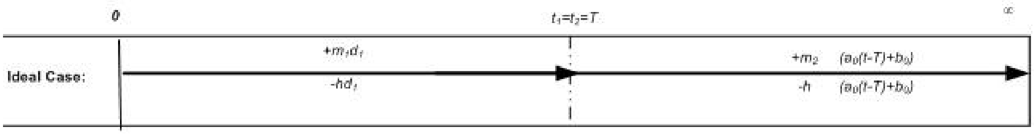

Figure 3.12: Perfect Information Case

We define the objective function as the net loss incurred between the ideal case or the case of full information and the cases where the approval date is uncertain. Formally, according to the description given previously, the net loss functions $L_1(t_1, t_2, T)$ and $L_2(t_1, t_2, T)$ are continuous and can be decomposed into functions defined on bounded intervals. This decomposition can be expressed as

$$L_j(t_1, t_2, T) = L_{j,i}(t_1, t_2, T) \quad \text{if } T \in I_i, \quad \text{for } j = 1, 2;\ i = 1, 2, ..., k \tag{3.5.8}$$

with k, the functions $L_{j,i}(t_1, t_2, \cdot)$ and the intervals I_i to be defined in the following sections.

Let $L_b(t_b, T)$ be the net loss functions at the boundary $t_1 = t_2 = t_b$ defined as follows:

$$L_b(t_b, T) = L_{b,i}(t_b, T) \quad \text{if } T \in I_i, \quad \text{for } i = 1, 2 \tag{3.5.9}$$

Ideal Case

In this ideal setting, the optimal solution is clear : $t_1 = t_2 = T$, i.e., the old product is sold out at the planned introduction date of the new product, corresponding to the approval date. Over the time interval $[0, T[$, the profit rate is $m_1 - h$ per unit, while on the remaining horizon $[T, \infty]$, the profit rate is $m_2 - h$ per unit. In order to characterize the impact of randomness on the rollover process, we consider an objective function defined as the difference between the perfect information cost rate function (Figure 3.12) and the cost rates functions with imperfect information (Figures 3.13 and 3.14). This difference can be interpreted as the loss caused by the randomness of the approval date T. Formally, according to the description given above, these loss functions are piecewise linear and exhibit different structures, depending on the relative values of the decision variables t_1 and t_2.

Planned Stockout Rollover $t_1 \leq t_2$:

For a planned stockout rollover strategy; the company plans to run out of the old product before introducing product 2 into the market. The random approval date T falls into one of these cases, $0 \leq T < t_2$ and $t_2 \leq T < \infty$. The firm sells product 1 during $(0, t_1)$ at a demand rate of d_1 and a net profit of $m_2 - h$ per unit demand. Between (t_1, t_2) there are no products to sell in the market, incurring a shortage cost of g. The market knows that at time t_2 the firm plans to introduce a new product into the market. If $T \leq t_2$, product 2 will be available in the market and sold at the rate of $d_{2,a}(t)$ with a contribution to profit of $m_2 - h$ per unit. On the other hand, if $t_2 \leq T$, product 2 is not available in the market, a proportion of customers $1 - \xi$ decides to wait at a certain waiting and inventory cost $w + h$ per unit and another proportion ξ of the customers decides to give up on purchasing the product incurring a shortage cost of g per unit. When the product is available in the market, the customers who have waited for the product will immediately purchase it and introduce a profit of m_2 per unit to the firm. The new product will then be sold at a demand rate of $d_{2,l}(t)$ and a contribution to profit of $m_2 - h$ per unit.

Over region $\mathbf{R_1}$, the decomposition (3.5.8) is given in Figure 3.13. The net loss in this

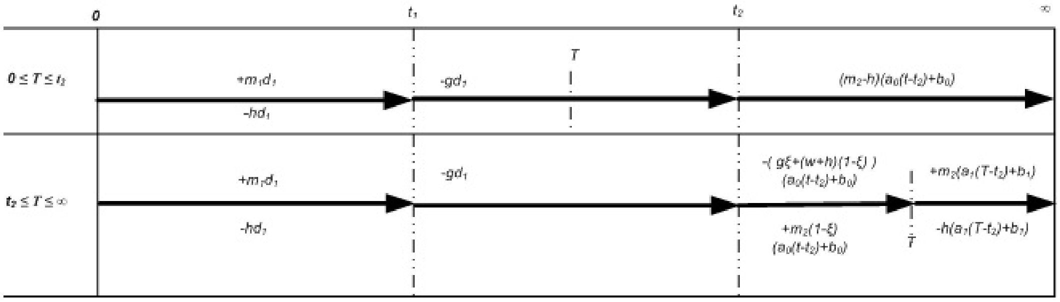

Figure 3.13: Planned Stockout Rollover Strategy

case is given by:

$$L_1(t_1,t_2,T) = \begin{cases} (m_1-h)(T-t_1)d_1 + g(t_2-t_1)d_1 + \Big(m_2-h\Big)\Big(T-t_2\Big)\Big(d_2-b_{2,a}\Big) \\ \text{if } 0 \leq T \leq t_2, \\ (m_1-h)(T-t_1)d_1 + g(t_2-t_1)d_1 + \Big(m_2-h\Big)\Big((a_{2,a}-a_{2,l})\frac{T^2}{2} \\ +(b_{2,l}-b_{2,a})T\Big) \\ +\Big((m_2-h-w)(1-\xi)-g\xi\Big)\Big(a_{2,a}(\frac{T^2-t_2^2}{2}-t_2T)+b_{2,a}(T-t_2)\Big) \\ \text{if } t_2 \leq T, \end{cases} \tag{3.5.10}$$

and $I_1 = [0, t_2]$ and $I_2 = [t_2, \infty]$.

It is clear from Figure (3.13) and expression (3.5.10) that for any given value of t_2, the firm can always increase the contribution to profit and reduce lost goodwill by increasing t_1. This means that the optimal policy can either be $t_1 = t_2$ or $t_1 > t_2$, which is the same result presented by Hill and Sawaya (2004).

Single Rollover Strategy

On the boundary for the single rollover strategy, the net the loss function becomes

$$L_b(t_b, T) = \begin{cases} (m_1 - h)(T - t_b)d_1 + \Big(m_2 - h\Big)\Big(T - t_b\Big)\Big(d_2 - b_{2,a}\Big) \\ \text{if } 0 \le T \le t_b, \\ (m_1 - h)(T - t_b)d_1 + \Big(m_2 - h\Big)\Big((a_{2,a} - a_{2,l})\frac{T^2}{2} + (b_{2,l} - b_{2,a})T\Big)\Big) \\ +\Big((m_2 - h - w)(1 - \xi) - g\xi\Big)\Big(a_{2,a}(\frac{T^2 - t_b^2}{2} - t_b T) + b_{2,a}(T - t_b)\Big) \\ \text{if } t_b \le T, \end{cases} \tag{3.5.11}$$

and $I_1 = [0, t_b]$ and $I_2 = [t_b, \infty]$.

Dual Rollover $t_2 \le t_1$

Over region $\mathbf{R_2}$, the decomposition is given in Figure 3.14) and the net loss becomes:

$$L_2(t_1, t_2, T) = \begin{cases} (m_1 - h)(T - t_2)d_1 + s_1(t_1 - t_2)d_1 + \Big(m_2 - h\Big)\Big(T - t_2\Big)\Big(d_2 - b_{2,a}\Big) \\ \text{if } 0 \le T \le t_2, \\ s_1(t_1 - T)d_1 + \Big(m_2 - h\Big)\Big((a_{2,a} - a_{2,l})\frac{T^2}{2} + (b_{2,l} - b_{2,a})T\Big) \\ +\Big((m_2 - h - w)(1 - \xi) - g\xi\Big)\Big(a_{2,a}(\frac{T^2 - t_2^2}{2} - t_2 T) + b_{2,a}(T - t_2)\Big) \\ \text{if } t_2 \le T \le t_1, \\ (m_1 - h)(T - t_1)d_1 + \Big(m_2 - h\Big)\Big(m_2 - h\Big)\Big((a_{2,a} - a_{2,l})\frac{T^2}{2} + (b_{2,l} - b_{2,a})T\Big) \\ +\Big((m_2 - h - w)(1 - \xi) - g\xi\Big)\Big(a_{2,a}(\frac{T^2 - t_2^2}{2} - t_2 T) + b_{2,a}(T - t_2)\Big) \\ \text{if } t_1 \le T, \end{cases} \tag{3.5.12}$$

and $I_1 = [0, t_2]$, $I_2 = [t_2, t_1]$, and $I_3 = [t_1, \infty]$

Parameter Assumptions

As usually in stochastic production/inventory model, in order to guarantee the significance of the model, it is necessary to introduce some assumptions for the different parameters. These assumptions are as follows. First the contribution-to-profit rate per unit for the products under regular sales is positive, i.e.,

$$m_1, m_2 > 0. \tag{3.5.13}$$

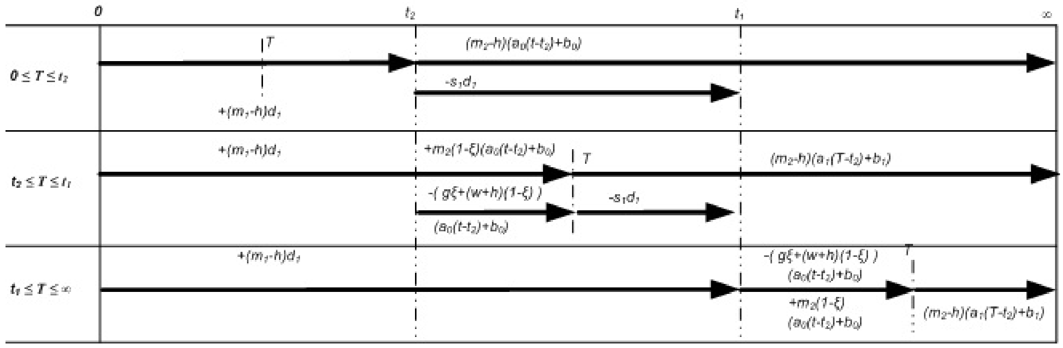

Figure 3.14: Dual Rollover Strategy

We also have the standard assumptions from classical inventory theory,

$$g, c_h, s_1 > 0. \tag{3.5.14}$$

For the demand process we have the following assumptions:

$$a_{2,a} > a_{2,l} > 0, b_{2,a} > b_{2,l} > d_1 > 0, d_2 > d_1 \tag{3.5.15}$$

Optimal Conditions and Convexity

Optimal Conditions

In this section, we present the optimal conditions through the first order derivatives and try to obtain closed form solutions.

Planned Stockout

$$\begin{aligned} l_1(t_1,t_2) &= \int_0^{t_2}\Big((m_1-h)(T-t_1)d_1+g(t_2-t_1)d_1 \\ &+ \Big(m_2-h\Big)\Big(\Big(T-t_2\Big)\Big(d_2-b_{2,a}\Big)\Big)\Big)f(T)dT \\ &+ \int_{t_2}^{\infty}\Big((m_1-h)(T-t_1)d_1+g(t_2-t_1)d_1 \\ &+ \Big(m_2-h\Big)\Big((a_{2,a}-a_{2,l})\frac{T^2}{2}+(b_{2,l}-b_{2,a})T\Big) \\ &+ \Big((m_2-h-w)(1-\xi)-g\xi\Big)\Big(a_{2,a}(\frac{T^2-t_2^2}{2}-t_2T)+b_{2,a}(T-t_2)\Big)\Big)f(T)dT \end{aligned} \tag{3.5.16}$$

The first order derivative of $l_1(t_1, t_2)$ with respect to t_1 is given by

$$\frac{dl_1(t_1, t_2)}{dt_1} = -d_1(m_1 + g - h) \tag{3.5.17}$$

Expression (3.5.17) is strictly decreasing with respect to t_1, therefore the optimal value of t_1 occurs at the maximum possible value of t_1 which is in our case t_2, and therefore the optimal solution occurs on the boundary $t_1 = t_2$ or $t_1 > t_2$, as was reached in Hill and Sawaya (2004).

Single Rollover

$$\begin{aligned}
l_b(t_b) \quad = \quad & \int_0^{t_b} \Big((m_1 - h)(T - t_b)d_1 + \\
+ \quad & \Big(m_2 - h\Big)\Big(T - t_b\Big)\Big(d_2 - b_{2,a}\Big)\Big) f(T)dT \\
+ \quad & \int_{t_b}^{\infty} \Big((m_1 - h)(T - t_b)d_1 \\
+ \quad & \Big(m_2 - h\Big)\Big((a_{2,a} - a_{2,l})\frac{T^2}{2} + (b_{2,l} - b_{2,a})T\Big) \\
+ \quad & \Big((m_2 - h - w)(1 - \xi) - g\xi\Big)\Big(a_{2,a}(\frac{T^2 - t_b^2}{2} - t_2 T) + b_{2,a}(T - t_b)\Big)\Big) f(T)dT
\end{aligned} \tag{3.5.18}$$

The first order derivative of $l_b(t_b)$ with respect to t_b is given by

$$\begin{aligned}
\frac{dl_b(t_b)}{dt_b} \quad = \quad & d_1(h - m_1) + \Big(b_{2,a} - a_{2,a}t_b\Big)\Big((m_2 - h - w)(1 - \xi) - g\xi\Big)\Big(1 - F(t_b)\Big) \\
- \quad & \Big(m_2 - h\Big)\Big(a_{2,a}t_b - b_{2,a}\Big)F(t_b) \\
+ \quad & a_{2,a}\Big((m_2 - h - w)(1 - \xi) - g\xi\Big)\Big(\mu - G(t_b)\Big) \\
+ \quad & \Big(m_2 - h\Big)\Big(\frac{(a_{2,a} - a_{2,l})}{2}t_b^2 + (b_{2,a} - b_{2,l})t_b\Big) f(t_b)
\end{aligned} \tag{3.5.19}$$

The optimal value of t_b^* occurs when expression (3.5.19) is zero as follows:

$$\begin{aligned}
\frac{dl_b(t_b^*)}{dt_b} \quad = \quad & d_1(h - m_1) + \Big(b_{2,a} - a_{2,a}t_b^*\Big)\Big((m_2 - h - w)(1 - \xi) - g\xi\Big)\Big(1 - F(t_b^*)\Big) \\
- \quad & \Big(m_2 - h\Big)\Big(a_{2,a}t_b^* - b_{2,a}\Big)F(t_b^*) \\
+ \quad & a_{2,a}\Big((m_2 - h - w)(1 - \xi) - g\xi\Big)\Big(\mu - G(t_b^*)\Big) \\
+ \quad & \Big(m_2 - h\Big)\Big(\frac{(a_{2,a} - a_{2,l})}{2}t_b^{*2} + (b_{2,a} - b_{2,l})t_b^*\Big) f(t_b^*) = 0
\end{aligned} \tag{3.5.20}$$

Therefore, we have

$$\begin{aligned}
F(t_b^*) &= \frac{d_1(m_1-h) - \Big(b_{2,a} - a_{2,a}t_b^*\Big)\Big((m_2-h-w)(1-\xi) - g\xi\Big)}{((m_2-h)\xi + w(1-\xi) + g\xi)(a_{2,a}t_b^* - b_{2,a})} \\
&- a_{2,a}\frac{\Big((m_2-h-w)(1-\xi) - g\xi\Big)\Big(\mu - G(t_b^*)\Big)}{((m_2-h)\xi + w(1-\xi) + g\xi)(a_{2,a}t_b^* - b_{2,a})} \\
&- \frac{\Big(m_2-h\Big)\Big(\frac{(a_{2,a}-a_{2,l})}{2}t_b^{*2} + (b_{2,a}-b_{2,l})t_b^*\Big)}{((m_2-h)\xi + w(1-\xi) + g\xi)(a_{2,a}t_b^* - b_{2,a})}f(t_b^*)
\end{aligned} \tag{3.5.21}$$

The second order derivative of $l_b(t_b)$ with respect to t_b is given by

$$\begin{aligned}
\frac{dl_b^2(t_b)}{dt_b^2} &= -a_{2,a}\Big((m_2-h-w)(1-\xi) - g\xi\Big)\Big(1 - F(t_b)\Big) \\
&- b_{2,a}\Big((m_2-h-w)(1-\xi) - g\xi\Big)f(t_b) \\
&- \Big(m_2-h\Big)a_{2,a}F(t_b) - \Big(m_2-h\Big)\Big(a_{2,a}t_b - b_{2,a}\Big)f(t_b) \\
&+ \Big(m_2-h\Big)\Big((a_{2,a}-a_{2,l})t_b + (b_{2,a}-b_{2,l})\Big)f(t_b) \\
&+ \Big(m_2-h\Big)\Big(\frac{(a_{2,a}-a_{2,l})}{2}t_b^2 + (b_{2,a}-b_{2,l})t_b\Big)f'(t_b)
\end{aligned} \tag{3.5.22}$$

The second order derivative given in (3.5.22) cannot be guaranteed to be convex. We produce several plots to prove our point, despite that it is not convex, but we can see that there is a unique global minimum. If we study expression (3.5.22), we can say that if f is strictly increasing i.e., $f' > 0$ and $b_{2,a} >>> a_{2,a}$, then (3.5.22) is convex and the minimum in this case is unique.

Another case where we can guarantee convexity is when the demand of product 2 is constant and is not affected by a delay, i.e., $a_{2,a} = a_{2,l} = 0$ and $b_{2,a} = b_{2,l} = d_2$. The second order derivative of $l_b(t_b)$ with respect to t_b is

$$\frac{dl_b^2(t_b)}{d^2t_b} = \Big((m_2-h)\xi + w(1-\xi) + g\xi\Big)d_2f(t_b) \tag{3.5.23}$$

We have $((m_2-h)\xi + w(1-\xi) + g\xi)d_2 > 0$ for all cost parameters, therefore $l_b(t_b)$ is convex.

In the case of constant demand, the optimal value of t_b^* is given by

$$t_b^* = F^{-1}\Big(\frac{d_1(m_1-h) - d_2\Big((m_2-h-w)(1-\xi) - g\xi\Big)}{((m_2-h)\xi + w(1-\xi) + g\xi)d_2}\Big) \tag{3.5.24}$$

Furthermore, for t_b to exist, the following condition should be satisfied:

$$0 < \frac{d_1(m_1 - h) - d_2\Big((m_2 - h - w)(1 - \xi) - g\xi\Big)}{((m_2 - h)\xi + w(1 - \xi) + g\xi)d_2} < 1 \tag{3.5.25}$$

For $\frac{d_1(m_1-h)-d_2\Big((m_2-h-w)(1-\xi)-g\xi\Big)}{((m_2-h)\xi+w(1-\xi)+g\xi)d_2} < 1$, we should have

$$d_1(m_1 - h) < d_2(m_2 - h) \tag{3.5.26}$$

Knowing that $d_1 < d_2$ and $m_1 < m_2$, we have this condition always satisfied.

For $0 < \frac{d_1(m_1-h)-d_2((m_2-h-w)(1-\xi)-g\xi)}{((m_2-h)\xi+w(1-\xi)+g\xi)d_2}$, knowing that $((m_2 - h)\xi + w(1 - \xi) + g\xi)d_2$, we should have

$$0 < d_1(m_1 - h) - d_2((m_2 - h - w)(1 - \xi) - g\xi) \tag{3.5.27}$$

If $d_1(m_1 - h) - d_2((m_2 - h - w)(1 - \xi) - g\xi) < 0$, then expression (3.5.19) is positive for a constant demand and $l_b(t_b)$ is strictly increasing with respect to t_b and the optimal value of t_b occurs at the minimum possible value of t_b, i.e., $t_b = 0$.

Economic Analysis: Examining the condition $d_1(m_1 - h) - d_2((m_2 - h - w)(1 - \xi) - g\xi)$, this condition can be negative when $d_1 m_1 << d_2 m_2$ combined with a low portion of lost demand ξ in case product 2 is late, and a low waiting cost. In other words, product 2 is much valuable for teh customer or in demand than product 1 and the waiting cost is very low compared to the high contribution to profit of product 2.

Now if the inventory cost h is too high then the condition in expression (3.5.25) is violated and $t_b^* \to \infty$.

Convexity when all demand is lost: We examine the convexity in case $\xi = 1$ or when all the demand is lost when $T > t_b$, in this case the second order derivative is

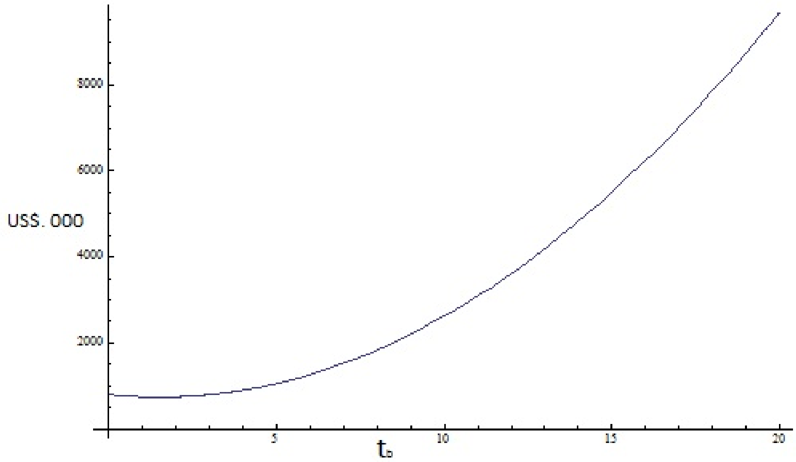

Figure 3.15: Case of All Demand Lost: Single Rollover with Linear Demand

given by:

$$\begin{aligned}\frac{dl_b^2(t_b)}{dt_b^2} &= a_{2,a}g\Big(1-F(t_b)\Big)+b_{2,a}gf(t_b)-\Big(m_2-h\Big)a_{2,a}F(t_b)-\Big(m_2-h\Big)\Big(a_{2,a}t_b-b_{2,a}\Big)f(t_b)\\ &+ \Big(m_2-h\Big)\Big((a_{2,a}-a_{2,l})t_b+(b_{2,a}-b_{2,l})\Big)f(t_b)\\ &+ \Big(m_2-h\Big)\Big(\frac{(a_{2,a}-a_{2,l})}{2}t_b^2+(b_{2,a}-b_{2,l})t_b\Big)f'(t_b) \qquad (3.5.28)\end{aligned}$$

Expression (ref) is positive if $a_{2,a} \to 0$ or in other words,the new product diffuses very slowly, therefore the objective function is convex with respect to t_b if $a_{2,a} \to 0$.

Figure (3.15) presents an example where all demand is lost for the following cost parameters in thousands $m_1 = 20$, $m_2 = 22$, $\xi = 1$, $g = 3$, $h = 10$, and $w = 8$, and the following demand parameters $a_{2,a} = 0.15$, $a_{2,l} = 0.01$, $b_{2,a} = 20$, $a_{2,l} = 0.1$, $b_{2,l} = 15$, $d_1 = 5$, and $d_2 = 50$. The probability distribution is a uniform one where $0 \leq T \leq 20$. The optimal solution in this case is $t_b^* = 9.7$ and $l_b^* = 9626$ U.S.$. On the other hand, Figure (3.16) represents the case of $\xi = 1$ when the demand of the new product is constant and equal to d_2. The optimal solution in this case is $t_b^* = 5.33$ and $l_b^* = 1466670$ U.S.$.

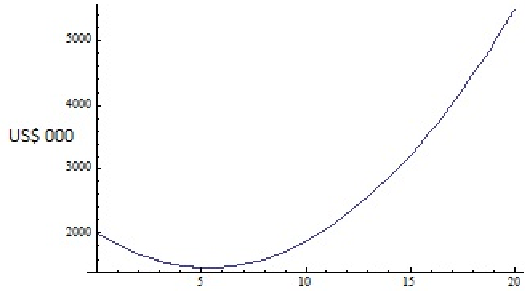

Figure 3.16: Case of All Demand Lost: Single Rollover with Constant Demand

Dual Rollover

$$\begin{aligned}
l_2(t_1,t_2) &= \int_0^{t_2}\Big((m_1-h)(T-t_2)d_1+s_1(t_1-t_2)d_1\\
&+ \Big(m_2-h\Big)\Big(m_2-h\Big)\Big(T-t_2\Big)\Big(d_2-b_{2,a}\Big)\Big)f(T)dT\\
&+ \int_{t_2}^{t_1}\Big(s_1(t_1-T)d_1+\Big(m_2-h\Big)\Big(m_2-h\Big)\Big((a_{2,a}-a_{2,l})\frac{T^2}{2}+(b_{2,l}-b_{2,a})T\Big)\\
&+ \Big((m_2-h-w)(1-\xi)-g\xi\Big)\Big(a_{2,a}(\frac{T^2-t_2^2}{2}-t_2T)+b_{2,a}(T-t_2)\Big)f(T)dT\\
&+ \int_{t_1}^{\infty}\Big((m_1-h)(T-t_1)d_1+\Big(m_2-h\Big)\Big(m_2-h\Big)\Big((a_{2,a}-a_{2,l})\frac{T^2}{2}+(b_{2,l}-b_{2,a})T\Big)\\
&+ \Big((m_2-h-w)(1-\xi)-g\xi\Big)\Big(a_{2,a}(\frac{T^2-t_2^2}{2}-t_2T)+b_{2,a}(T-t_2)\Big)\Big)f(T)dT
\end{aligned} \tag{3.5.29}$$

The first order derivative of $l_2(t_1,t_1)$ with respect to t_1 is given by

$$\frac{dl_2(t_1,t_2)}{dt_1} = -d_1(m_1-h)+d_1((m_1-h)+s_1)F(t_1) \tag{3.5.30}$$

Setting expression (3.5.30) to zero, we get the optimal value of t_1 to be:

$$t_1^* = F^{-1}\Big(\frac{m_1-h}{m_1-h+s_1}\Big) \tag{3.5.31}$$

The first order derivative of $l_2(t_1,t_1)$ with respect to t_2 is given by

$$\begin{aligned}
\frac{dl_2(t_1,t_2)}{dt_2} &= \Big(d_1(h-m_1-s_1)+(m_2-h)(b_{2,a}-a_{2,a}t_2)\Big)F(t_2)\\
&+ a_{2,a}\Big((m_2-h-w)(1-\xi)-g\xi\Big)\Big(\mu-G(t_2)\Big)\\
&+ \Big(b_{2,a}-a_{2,a}t_2\Big)\Big((m_2-h-w)(1-\xi)-g\xi\Big)\Big(1-F(t_2)\Big)\\
&+ \Big(m_2-h\Big)\Big((a_{2,l}-a_{2,a})\frac{t_2^2}{2}+(b_{2,a}-b_{2,l})t_2\Big)f(t_2)
\end{aligned} \tag{3.5.32}$$

The optimal value of t_2^* occurs when expression (3.5.32) is zero as follows:

$$\begin{aligned} F(t_2^*) &= a_{2,a}\frac{\Big((m_2-h-w)(1-\xi)-g\xi\Big)\Big(\mu-G(t_2^*)\Big)}{d_1(m_1-h+s_1)+(a_{2,a}t_2^*-b_{2,a})((m_2-h+g)\xi+w(1-\xi))} \\ &+ \frac{\Big(b_{2,a}-a_{2,a}t_2^*\Big)\Big((m_2-h-w)(1-\xi)-g\xi\Big)}{d_1(m_1-h+s_1)+(a_{2,a}t_2^*-b_{2,a})((m_2-h+g)\xi+w(1-\xi))} \\ &+ \frac{\Big(m_2-h\Big)\Big((a_{2,l}-a_{2,a})\frac{t_2^{*2}}{2}+(b_{2,a}-b_{2,l})t_2^*\Big)}{d_1(m_1-h+s_1)+(a_{2,a}t_2^*-b_{2,a})((m_2-h+g)\xi+w(1-\xi))}f(t_2^*) \end{aligned} \tag{3.5.33}$$

The second order derivative of $l_2(t_1, t_1)$ with respect to t_2 is given by

$$\begin{aligned} \frac{d^2l_2(t_1,t_2)}{dt_2^2} &= \Big(d_2((m_2-h+g)\xi+w(1-\xi))-d_1(m_1-h+s_1)\Big)f(t_2) \\ &- a_{2,a}t_2\Big((m_2-h-w)(1-\xi)-g\xi\Big)\Big(1-F(t_2)\Big) \\ &+ \Big(m_2-h\Big)\Big((a_{2,l}-a_{2,a})\frac{t_2^2}{2}+(b_{2,a}-b_{2,l})t_2\Big)f'(t_2) \\ &+ \Big(m_2-h\Big)\Big((a_{2,l}-a_{2,a})t_2+(b_{2,a}-b_{2,l})\Big)f(t_2) \end{aligned} \tag{3.5.34}$$

The second order derivative given in (3.5.34) cannot be guaranteed to be convex. We produce several plots to prove our point, despite that it is not convex, but we can see that there is a unique global minimum. If we study expression (3.5.34), we can say that if f is strictly increasing i.e., $f' > 0$ and $b_{2,a} >>> a_{2,a}$, then (3.5.34) is convex and the minimum in this case is unique.

Now we consider the special case when the demand of product 2 is constant. Another case where we can guarantee convexity is when the demand of product 2 is constant and is not affected by a delay, i.e., $a_{2,a} = a_{2,l} = 0$ and $b_{2,a} = b_{2,l} = d_2$. In this case, the second order derivative of $l_2(t_1, t_1)$ with respect to t_1 is given by

$$\frac{dl_2^2(t_1,t_2)}{d^2t_1} = d_1((m_1-h)+s_1)f(t_1) \tag{3.5.35}$$

Since $m_1 > h$, expression (3.5.35) is always positive and $l_2(t_1, t_1)$ is convex with respect to t_1.

The second order derivative of $l_2(t_1, t_1)$ with respect to t_2 is given by

$$\frac{dl_2^2(t_1,t_2)}{dt_2^2} = (d_2((m_2-h+g)\xi+w(1-\xi))-d_1(m_1-h+s_1))f(t_2) \tag{3.5.36}$$

We distinguish between two cases:

CASE A

In this case, $(d_2((m_2 - h + g)\xi + w(1 - \xi)) - d_1(m_1 - h + s_1)) > 0$, and therefore $l_2(t_1, t_1)$ is convex with respect to t_2. Now for t_2^* to exist, the following condition has to be satisfied,

$$0 < \frac{-d_2\Big((m_2 - h - w)(1 - \xi) - g\xi\Big)}{d_2((m_2 - h + g)\xi + w(1 - \xi)) - d_1(m_1 - h + s_1)} < 1 \tag{3.5.37}$$

This simplifies to the following two conditions:

$$\frac{-d_2\Big((m_2 - h - w)(1 - \xi) - g\xi\Big)}{d_2((m_2 - h + g)\xi + w(1 - \xi)) - d_1(m_1 - h + s_1)} < 1 \tag{3.5.38}$$

Then

$$d_2(m_2 - h) > d_1(m_1 - h + s_1) \tag{3.5.39}$$

If $d_2(m_2 - h) < d_1(m_1 - h + s_1)$, then the first order derivative of $l_2(t_1, t_2)$ with respect to t_2 is strictly decreasing with respect to t_2, and the optimal value occurs at the maximum possible value of t_2, i.e., $t_2 = t_1 = t_b$.

Economic Analysis: Examining the condition $d_2(m_2 - h) < d_1(m_1 - h + s_1)$, this may occur when the salvage cost s_1 is very high, therefore it makes more economic sense to introduce the new product and remove the old at the same time.

For

$$0 < \frac{-d_2\Big((m_2 - h - w)(1 - \xi) - g\xi\Big)}{d_2((m_2 - h + g)\xi + w(1 - \xi)) - d_1(m_1 - h + s_1)} \tag{3.5.40}$$

then $-d_2\Big((m_2 - h - w)(1 - \xi) - g\xi\Big) > 0$ since $d_2((m_2 - h + g)\xi + w(1 - \xi)) - d_1(m_1 - h + s_1) > 0$. Now, if $-d_2\Big((m_2 - h - w)(1 - \xi) - g\xi\Big) < 0$, then then the first order derivative of $l_2(t_1, t_2)$ with respect to t_2 is strictly increasing with respect to t_2, and the optimal value occurs at the minimum possible value of t_2, i.e., $t_2 = 0$.

Economic Analysis: Examining the condition $-d_2((m_2 - h - w)(1 - \xi) - g\xi)$, this condition can be negative when there is a low portion of lost demand ξ in case product 2 is late and a low waiting cost. In other words, product 2 is very valuable for the customer and the waiting cost is very low compared to the high contribution to profit of product 2.

Now, in the case when t_2 exists, we have to further satisfy a condition $t_2^* < t_1^*$ given by

$$\frac{-d_2\Big((m_2 - h - w)(1 - \xi) - g\xi\Big)}{d_2((m_2 - h + g)\xi + w(1 - \xi)) - d_1(m_1 - h + s_1)} < \frac{m_1 - h}{m_1 - h + s_1} \tag{3.5.41}$$

In case condition (3.5.41) is violated, then the optimal value occurs at $t_1 = t_2 = t_b$.

If the inventory cost h is too high, then the first order derivative of $l_2(t_1, t_2)$ with respect to t_2 is decreasing and the optimal value of t_2^* occurs at $t_2 \to t_1$.

CASE B

- In this case, $(d_2((m_2 - h + g)\xi + w(1 - \xi)) - d_1(m_1 - h + s_1)) < 0$, then $l_2(t_1, t_1)$ is concave with respect to t_2 and the minimum will occur at a boundary depending on $-d_2\Big((m_2 - h - w)(1 - \xi) - g\xi\Big)$.

- If

$$0 < -d_2\Big((m_2 - h - w)(1 - \xi) - g\xi\Big) \tag{3.5.42}$$

Then expression the first order derivative of $l_2(t_1, t_2)$ with respect to t_2 is strictly decreasing with respect to t_2, and the optimal value occurs at the maximum possible value of t_2, i.e., $t_2 = t_1 = t_b$.

Now if

$$-d_2\Big((m_2 - h - w)(1-\xi) - g\xi\Big) < 0 \tag{3.5.43}$$

we distinguish two cases:

- If $\frac{-d_2\Big((m_2-h-w)(1-\xi)-g\xi\Big)}{d_2((m_2-h+g)\xi+w(1-\xi))-d_1(m_1-h+s_1)} < 1$, knowing that $d_2((m_2-h+g)\xi + w(1-\xi)) - d_1(m_1 - h + s_1) < 0$, if $-d_2\Big((m_2 - h - w)(1-\xi) - g\xi\Big) > d_2((m_2 - h + g)\xi + w(1-\xi)) - d_1(m_1 - h + s_1)$, then the first order derivative of $l_2(t_1, t_2)$ with respect to t_2 is decreasing with respect to t_2 and the minimum occurs at $t_1 = t_2 = t_b$.

- If $-d_2\Big((m_2 - h - w)(1-\xi) - g\xi\Big) < d_2((m_2 - h + g)\xi + w(1-\xi)) - d_1(m_1 - h + s_1)$, then the first order derivative of $l_2(t_1, t_2)$ with respect to t_2 is increasing with respect to t_2 and the minimum occurs at $t_2 = 0$.

If the inventory cost h is too high, then the first order derivative of $l_2(t_1, t_2)$ with respect to t_2 is decreasing and the optimal value of t_2^* occurs at $t_2 \to t_1$.

Convexity when all demand is lost: We examine the convexity in case $\xi = 1$ or when all the demand is lost when $T > t_2$, in this case the second order derivative is given by:

$$\begin{aligned}
\frac{d^2 l_2(t_1, t_2)}{dt_2^2} &= \Big(d_2(m_2 - h + g) - d_1(m_1 - h + s_1)\Big) f(t_2) + a_{2,a} t_2 g \Big(1 - F(t_2)\Big) \\
&+ \Big(m_2 - h\Big)\Big((a_{2,l} - a_{2,a})\frac{t_2^2}{2} + (b_{2,a} - b_{2,l})t_2\Big) f'(t_2) \\
&+ \Big(m_2 - h\Big)\Big((a_{2,l} - a_{2,a})t_2 + (b_{2,a} - b_{2,l})\Big) f(t_2)
\end{aligned} \tag{3.5.44}$$

Knowing that $a_{2,l} > a_{2,a}$, expression (3.5.44) is positive and $l_2(t_1, t_2)$ is convex with respect to t_2 if $\Big(d_2(m_2 - h + g) - d_1(m_1 - h + s_1)\Big) > 0$.

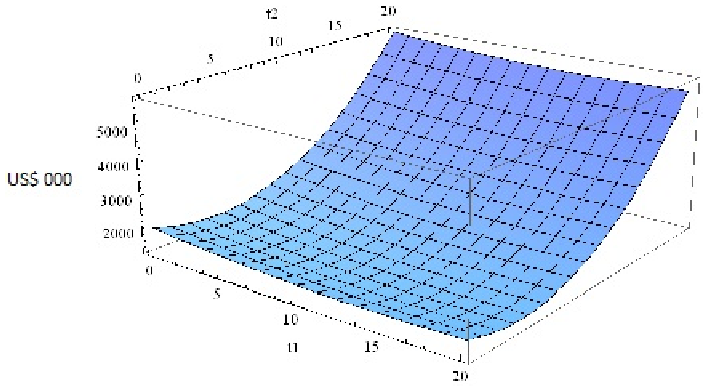

Figure 3.17: Case of All Demand Lost: Dual Rollover with Linear Demand and $\Big(d_2(m_2 - h + g) - d_1(m_1 - h + s_1)\Big) > 0$

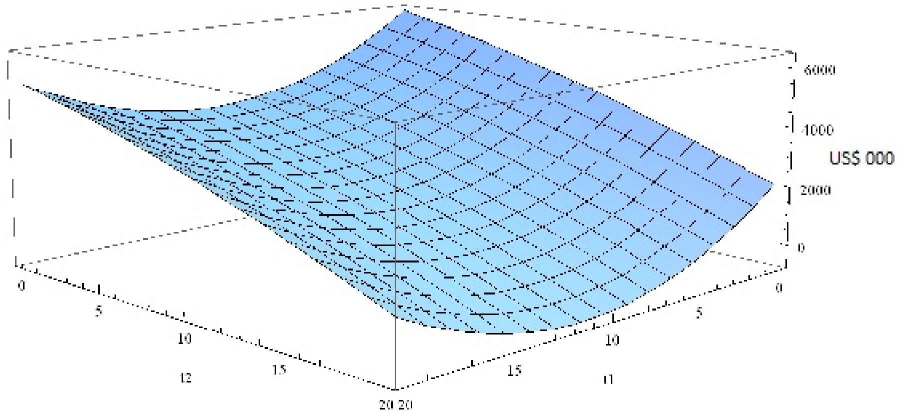

Figure 3.18: Case of All Demand Lost: Dual Rollover with Linear Demand and $\Big(d_2(m_2 - h + g) - d_1(m_1 - h + s_1)\Big) < 0$

Figure (3.17) presents an example where all demand is lost for the following cost parameters in thousands $m_1 = 20$, $m_2 = 22$, $\xi = 1$, $g = 3$, $h = 10$, s_1, and $w = 8$, and the following demand parameters $d_1 = 5$, and $d_2 = 50$. The probability distribution is a uniform one where $0 \le T \le 20$. The optimal solution in this case is $t_1^* = 14.3$, $t_2^* = 4.8$ and $l_b^* = 1180360$ U.S.\$. In another example presented in Figure (3.18), $s_1 = 8$ and $d_1 = 49$, the other parameters are unchanged. The optimal solution in this case is $t_1^* = 14.3$, $t_2^* = 4.8$ and $l_b^* = 1180360$ U.S.\$.

CHAPTER 4

Third Paper: Data-Driven Optimization for the Stochastic Product Rollover Problem

Abstract

We consider an inventory/production rollover process between an old and a new product, with a random approval date for the new product. Unlike our previous work, where the approval date distribution was known, here it is not known. Instead the only information available is a set of independent random samples that are drawn from the true approval date distribution. The analysis we present characterizes the properties of the approval date distribution as a function of the number of historic samples and optimization in a single framework. We present data-driven solutions and incorporate risk preferences using a scalar parameter and tractable formulations leading to closed-form solutions based on the ranking of the historical dates, which provide key insights into the role of the cost parameters and optimal rollover policy. Moreover, we establish bounds on the number of samples required to guarantee that with high probability, the expected cost of the sampling-based policies is arbitrarily close (i.e., with arbitrarily small relative error) compared to the expected cost of the optimal policies which have full access to the approval date *distributions. The bounds that we develop are general, easy to compute and do not depend at all on the specific approval date distributions. We finally test the 'robustness' of our solutions though numerical computations.*

Keywords: Product rollover; Uncertain approval date; Planned stockout rollover (PSR); Single product rollover (SPR); Dual product rollover(DPR); Data-Driven

4.1 Introduction

Frequent introduction of new products and phasing out of old ones creates enormous challenges to managing product rollovers in today's market. It is essential that companies develop clear strategies for product rollover, in addition to contingency plans in case of failure to minimize their loss in the presence of uncertainty. Several papers have addressed the question of efficient management of new product launch, old product destruction/salvage/scrap/sold and/or combination of the two processes.

In an ideal setting, efficient rollover is clear : the old product is sold out at the planned introduction date of the new product, and the new product is readily available.Clearly, when a company is planning the phase-out of an existing product and the phase-in of a replacement product, classical stochastic production/inventory trade-offs have to be considered. If production of the existing product is stopped too early, i.e., before the new product is available for the market, the firm will lose profit and customer goodwill. On the other hand, if production of the existing product is stopped too late, the firm will experience an obsolescence cost for the existing product, because demand and/or price would have decreased as this product will be considered "old generation" by the customers. Furthermore, if the production of the new product is started too early, the firm will experience an inventory carrying cost until the market will turn to this product. Real-life is usually less favorable.

In this paper, we focus on three fundamental strategies: planned stockout rollover (PSR), single-product rollover (SPR) ,and dual-product rollover (DPR). An important issue in new product launch management is whether two product generations should coexist in the market for some time; in other words, whether there should be an overlapping of some sort in successive product inventory/production/supply chain. In the PSR strategy, the introduction of the new product is planned in such a way that a stock-out phenomenon occurs during the product transition. During this stock-out period, no product of any type is available for the market (which introduces some kind of back-order cost). In the SPR strategy, there is a simultaneous introduction of the new product and elimination of the old product, i.e., at any time there is a unique product generation available in the market. On the contrary, in the DPR strategy, the new product is introduced first and then the old product is phased out. Thus, in this setting, two product generations coexist in the market for some time. The advantage of the DPR strategy, with respect to the SPR policy, is to allow some protection against potential random events (delays, quality, market demand level) affecting the planned phasing-out process, but its drawback is the cost corresponding to the additional supply chain inventory.

The purpose of this paper is to analyze and characterize the optimality of each type of strategy (PSR, SPR and/or DPR) for a setting with a stochastic approval date for the new product. Hill and Sawaya (2004) examine the problem of simultaneously planning the phase-out of the old product and the phase-in of a new one that will replace the old product, under an uncertain approval date for the new product. Our problem setting is inspired from their model. In El Khoury et al. (2011), we assumed that the approval date follows a known probability distribution, in practice however, the volatility of the approval approval date makes it difficult to obtain accurate forecasts of the probability distribution. The assumption that the approval date distribution is known is unrealistic especially that only partial information about the approval is available for the manager.

Thus, we adopt a non-parametric data-driven approach where we build directly upon available historic data samples instead of estimating the probability distributions relying on a scalar parameter to incorporate robustness in the model which corresponds to a pre-specified quantile of the cost. The random variable is determined by computing the expected cost above that quantile, that is, by removing (trimming) the instances of the cost below the quantile and taking the average over the remaining ones. The fraction of data points removed will be referred to as the trimming factor that determines the degree of conservatism. This is a one-sided trimming approach studied by Bertsimas et. al. (2004) and Thiele (2004).The only information available is a set of independent samples drawn independently from the true approval date distribution, but the true distribution is unknown to the manager.This approach was first proposed by Thiele (2004) where she applied it to different variances of the newsboy problem. The importance of this method is the tractability and the possibility of formulating unique closed-form solutions for problems that are convex and piecewise linear.

To our knowledge, this is the first work that addresses the product rollover problem under uncertainty using a data-driven optimization approach. In fact, approval date distributions are very hard to model and often the manager has only historical observations. We derive theoretical insights into the optimal strategies depending on the cost parameters and the degree of conservatism chosen by the decision-maker. We also compare our solutions to the Conditional Value at Risk (CVaR) solutions obtained in our previous work when the probability distribution is known.

The structure of this paper is as follows: in section 5.2, we review product rollover and data-driven literature. In section 5.3, we present the stochastic product rollover problem under consideration and in section 5.4 we discuss the data-driven cost approach and compare it to the conditional value at risk. In section 5.5, we give the structural properties and solutions to our problem. In section 5.6, we present the numerical convergence through bounds and finally in section 5.7, we test our solutions through numerical simulations and show that the data-driven approach may give better solutions than the conditional value of risk in case of guessing wrongly the probability distribution. We finally conclude the paper in section 5.8 reporting our findings and proposing future research directions.

4.2 Literature Review

A first trend of papers about new product development and launch is mainly of qualitative and descriptive nature (Chryssochoidis and Wong (1998), Saunders and Jobber (1994), Erhun et al (2007), Hendricks and Singhal (1997)) that guide managers to design and implement appropriate policies taking into consideration transition risks related to the product, manufacturing processes, supply chain features, and managerial policies in a competitive environment.

Choosing the optimal strategy - planned stockout, single, or dual - is central in the product rollover problem. Literature has reported that a planned stock-out rollover (PSR) and single product rollover (SPR) can be viewed as high-risk, high return strategies, sensitive to potential random events. On the contrary, the dual product rollover

(DPR) strategy is less risky, but induces higher inventory costs.

Despite the importance of optimizing revenues of product rollover, very few papers address the problem quantitatively such as Billington et al (1998). Lim and Tang (2006) developed a deterministic model that allows the determination of prices of old and new products and the times of phase-in and phase-out of the products. A very simple setting has been analyzed in the paper of Ronen and Trietsch (1993).

Risk-sensitivity models in inventory modeling and supply chain management have been proposed in many papers. Most inventory-related papers try to maximize a predetermined target profit such as Lau (1980), who first modeled risk. This criterion may result in an unacceptably large loss and researchers like Markowitz (1952) propose to minimize the standard deviation of the profit. Tang (2006) provides a review of various quantitative models for managing supply chain risks.

In general, risk modeling has constituted an important research stream in finance. A way to take into account the risk consists of focusing on shortfall, through an absolute bound on the tolerable loss or by setting a bound on the conditional value at risk. Theoretical properties of the CVaR measure of risk has been extensively studied by Rockafellar and Uryasev (2000). Gotoh and Takano (2007) and Chen et al. (2009) developed closed form solutions using Conditional Value at Risk (CVaR) for the newsboy problem. Others like Ozler et al (2009) utilize Value at Risk (VaR) as risk measure in a newsboy framework and investigate the multi-product newsboy problem under a VaR constraint.

All methods discussed above require the knowledge of the probability distribution of the stochastic variable. In case the variance of the distribution is unknown, the min-max approach is a way to address this situation. Several researchers have chosen this method to solve the newsboy problem when the exact demand distribution is not known like Bienstock and Ozbay (2006) and Gallego et al (2001). The min-max approach knowing only the mean and the variance was first introduced by Scarf (1958). In this method, smaller where smaller profits are preferred if they exhibit less variability. Kasugai and Kasegai (1960) applied dynamic programming and the min-max regret ordering policy to the distribution-free multi-period newsboy problem. Scarf (1959) and Liyanage and Shanthikumar (2005) assume that the 'unknown' distribution belongs to a parametric family of distributions, but the values of the parameters are unknown. Gallego and Moon (1993,1994) extended Scarf's method to the single-period newsboy model with a fixed order quantity and under periodic review.

Moon and Silver (2000) develop distribution-free models and heuristics for a multi-item newsboy problem with a budget constraint and fixed ordering costs. A comprehensive literature reviews and suggestions for future research on the newsboy problem are complied by Khouja (1999)

Supply chain literature has explored sampling-based optimization in the form of a data-driven approach to solve stochastic optimization problems with unknown distributions. In this approach, historical data or sample evaluations are generated from the true distribution. This method was pioneered by van Ryzin and McGill (2000). Bertsimas and de Boer (2005) developed a stochastic gradient algorithm to solve a revenue management problem using the scenario samples. Levi et al. (2007) apply the data-driven framework to the newsvendor problem and establish bounds on the number of samples required to guarantee with some probability that the real expected cost of the sample based policies approximates the expected optimal cost. Ben-Tal and Nemirovski (1998, 1999, 2000), Goldfarb and Iyengar (2003) and Bertsimas and Sim (2004) developed the data-driven approach and applied it to various settings. The data-driven approach is usually appropriate for risk-averse managers, but it can give quite conservative solutions.

The approach proposed in this paper is entirely data-driven building directly upon the sample of available data instead of estimating the probability distributions. It does not rely on utilities but rather on a scalar parameter to incorporate robustness to the model. This scalar parameter corresponds to a pre-specfied quantile of the loss. The random variable is determined by computing the expected revenue below that quantile, that is, by removing (trimming) the instances of the profit above the quantile and taking the average over the remaining ones. The fraction of data points removed will be referred to as the trimming factor. By this, the planner focuses on a more conservative valuation of his revenue/loss and is able to adjust the degree of conservatism by selecting the trimming factor appropriately. Two-sided trimming has been studied by Rousseeuw and Leroy (1987), Ryan (1996), Wilcox (1997) One-sided trimming has been studied by Bertsimas et. al. in (2004) and Levy and Kroll (1978). The importance of this approach lies in the uniquness of the strategy that will minimize losses in the case of convex utilities and allowing for nonparametric estimators and tractable formulations.

4.3 The product rollover evaluation model

In this section, we recall the product rollover problem that we introduced in El Khoury et al. (2011) and introduce the different notation and assumptions.

4.3.1 Stochastic rollover process and profit/cost rates

The problem context requires a production plan for the phase-out of an existing product (here called *old product*, or *product* 1) and phase-in of a replacement product (called *new product* or *product* 2) under an uncertain (internal or external) approval date, denoted T, for the new product delivery. A typical example for such approval decisions are those of medical devices and pharmaceutical products which cannot be sold until an approval body grants permission. Two decision variables have to be fixed in such a rollover process: t_1, the date the firm plans to run-out of the old product and t_2, the date the new product is planned to be ready and available for the market. The existing product is sold until the firm runs out of inventory or until it is replaced by the approved new product. The manufacturing and procurement lead times are assumed to be large, thus making it necessary to commit to the planning dates before the random approval date is revealed. The decision process relies thus exclusively on the probability distribution of this date T, which, in our case, is not known. Such large procurement/manufacturing/distribution lead-times are frequent in practice: for instance, the regulatory affairs department in a medical device firm uses a forecast interval for the approval date that is more than 6 months long. During their regular commercial life, each product has a specific constant demand rate, namely d_1 and d_2. A channel inventory is needed to support each product in the market, which induces inventory cost rates $c_{h,1}$ and $c_{h,2}$. During the commercial life, the contribution-to-profit rate for product i, is defined as

$$m_i = d_i(p_i - c_i) - c_{h,i}, \qquad (i = 1, 2), \tag{4.3.1}$$

with p_i the selling price and c_i the production cost.

In the considered random setting, the profit/cost structure, defined over an infinite time horizon, depends furthermore on the relative values of t_1, t_2 and T. Indeed, if the planned stock-out strategy ($t_1 \leq t_2$) is chosen, the structure of the profit/cost rates is given in Figure 4.2,

Three main cases have to be considered. First, if $T \leq t_1$, the profit rate is m_1 over the time interval $[0, T[$. Then, if $t_1 \leq T \leq t_2$, the new product is approved, but not physically available in the supply chain. As the market is assumed to be informed that the new product 2 will substitute product 1 in a supposedly short delay, the product 1 profit rate changes from m_1 to m_1' as long as product 1 is available, i.e., over $[T, t_1[$. This contribution rate m_1' is formally given by

$$m_1' = d_1'(p_1' - c_1) - c_{h,i}. \tag{4.3.2}$$

Then, over the interval $[t_1, t_2[$, when the old product is sold out, shortages occur until new product 2 delivery date t_2, at a corresponding shortage cost rate g. Once the new product is available, at t_2, the profit rate becomes m_2 over the remaining time horizon $[t_2, \infty[$. Then, for the third case, one has $t_2 \leq T$. The profit/cost rates are similar to the previous situation, except over the interval $[t_2, T[$, where the new product is physically available in the supply chain, but still not approved. A shortage cost rate g occurs until new product 2 is approved. In addition, an inventory cost rate $c_{h,2}$ associated to the product 2 physical inventory is to be incurred.

If the dual rollover strategy ($t_2 \leq t_1$) is chosen, the structure of the costs and profit rates is given in Figure 4.3.

Let us consider first the case $T < t_2$. The profit rate is m_1 over the time interval $[0, T[$ and m_1' over $[T, t_2[$. Then, over the time interval $[t_2, t_1[$, as the new product is approved and physically available, it is sold with a profit rate m_2. In the current setting, it is however assumed that the firm scraps, at a cost rate s_1, all the remaining inventory of product 1 immediately when an approved product 2 is available for sale, i.e., over the time interval $[T, t_1]$). This can be linked to several typical market forces that can be observed in some sectors. First, in some situations, it is considered as important (if not necessary) to provide customers with the latest technology, i.e., with the newest product type. Second, higher demand, higher prices, and higher commissions drive sales organizations to shift to the new product. Third, marketing organizations want products that accentuate the leading edge nature of the firm's brand and do not want to lose the opportunity to sell the best and latest product. This is justified by the higher margins for product 2 and by the need to maintain brand equity as a leading-edge provider. Finally, over the remaining time horizon $[t_1, \infty[$, the profit rate becomes to m_2.

In the second situation, one has $t_2 \leq T \leq t_1$. The profit rate is m_1 over $[0, t_2[$. Then over the interval $[t_2, T[$, the profit rate is still m_1, but as the new product is physically available in the supply chain, but not approved for sale, an inventory cost rate $c_{h,2}$ to be incurred. From T until the whole remaining horizon, the new product is sold with a profit rate m_2. In the time interval $[T, t_1[$, the old product is scrapped at a cost rate s_1. Over the remaining time horizon $[t_1, \infty[$, the profit rate becomes to m_2. In the last case, $t_1 \leq T$. The profit rate is m_1 over $[0, t_2[$. Then, over the interval $[t_2, t_1[$ the profit rate is still m_1, but an inventory cost rate $c_{h,2}$ has to be incurred. Over $[t_1, T[$ old product is sold out and new product is not approved, shortages induce thus a shortage cost rate g. Finally, over the remaining time horizon $[T, \infty[$, the profit rate becomes to m_2.

4.3.2 Model Notation

For this rollover optimization model, we adopt the following notation. For each product type $i \in \{1, 2\}$, we define

c_i : the unit cost for product i,

p_i : the unit price for product i,

$p_i - c_i$: the gross margin for product i,

d_i : the demand rate for product i,

m_i : the contribution-to-profit rate for product i, defined as $m_i = d_i(p_i - c_i) - c_{h,i}$,

g : the loss of goodwill rate when the firm has neither of the products to sell,

m_1' : the new contribution-to-profit rate of product 1 after the admissability of product 2 is granted; this value is externally given,

$c_{h,i}$: the carrying cost rate for product i,

s_1 : the scrap cost rate for product 1.

Furthermore, we denote

T : the random approval date for the new product (i.e., for product 2).

The decision variables are,

t_1 : the planned run-out date for inventory of the existing product (i.e., product 1).

t_2 : the planned availability date for the new product (i.e., product 2),

t_b : is the planned availability date for the new product and the removal of the old product when $t_1 = t_2$ for a single rollover strategy.

4.3.3 The Global Optimization Criterion

We consider a performance criterion defined as the difference between the cost when the approval date is random and known exclusively through observation and the cost under complete information about approval date. This performance criterion is defined as follows.

Let's first consider the perfect information case for which the value of the regulatory date is known before the decisions t_1 and t_2 are made. This situation is depicted in Figure 4.1.

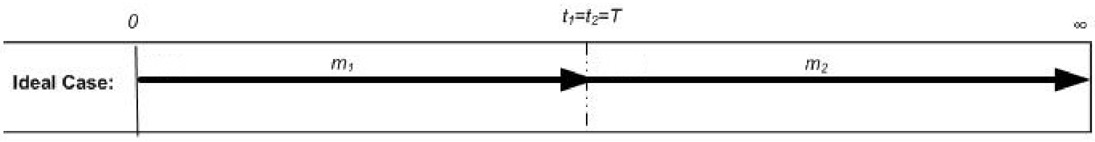

Figure 4.1: Full information case

In this ideal setting, the optimal solution is clear : $t_1 = t_2 = T$, i.e., the old product is sold out at the planned introduction date of the new product, corresponding to the approval date. Over the time interval $[0, T[$, the profit rate is m_1, while on the remaining horizon $[T, \infty]$, the profit rate is m_2.

In order to characterize the impact of randomness on the rollover process, we consider an objective function defined as the difference between the cost rates functions with imperfect information (Figures 4.2 and 4.3) and the perfect information cost rate function (Figure 4.1). This difference can be interpreted as the loss caused by not knowing the approval date T. Formally, according to the description given above, these cost functions are piecewise linear and exhibit different structures, depending on the relative values of the decision variables t_1 and t_2.

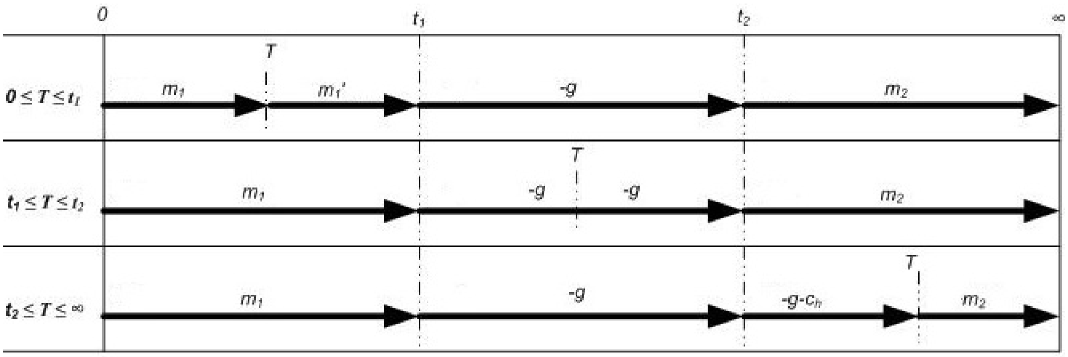

Figure 4.2: The profit rates when $t_1 \leq t_2$

If the planned stock-out strategy ($t_1 \leq t_2$) is chosen, the cost rate function is denoted as $L_1(t_1, t_2, T)$ and amounts to

$$L_1(t_1,t_2,T) = \begin{cases} (m_2 - m_1')(T-t_1) - (m_2+g)(t_2-t_1) & \text{if } 0 \leq T \leq t_1, \\ -(g+m_1)(T-t_1) - (g+m_2)(t_2-T) & \text{if } t_1 \leq T \leq t_2, \\ -(g+m_1)(t_2-t_1) - (g+m_1+c_{h,2})(T-t_2) & \text{if } t_2 \leq T, \end{cases}$$

$$= (m_1+g)[T-t_1]^+ - (g+m_1')[t_1-T]^+$$
$$+ \quad c_{h,2}[T-t_2]^+ + (m_2+g)[t_2-T]^+. \tag{4.3.3}$$

where $[Y]^+ := \max(Y, 0)$.

If the dual rollover strategy ($t_2 \leq t_1$) is chosen, the cost function is denoted as $L_2(t_1, t_2, T)$ and is given by

$$
\begin{aligned}
L_2(t_1, t_2, T) &= \begin{cases} (m_2 - m_1')(T - t_2) - s_1(t_2 - t_1) & \text{if } 0 \leq T \leq t_2, \\ -c_{h,2}(T - t_2) - s_1(t_1 - T) & \text{if } t_2 \leq T \leq t_1, \\ -c_{h,2}(t_2 - t_1) - (g + m_1)(T - t_1) & \text{if } t_1 \leq T, \end{cases} \\
&= (m_2 - m_1' - s_1)[t_2 - T]^+ + c_{h,2}[T - t_2]^+ \\
&+ (m_1 + g)[T - t_1]^+ + s_1[t_1 - T]^+. \qquad (4.3.4)
\end{aligned}
$$

If we formally introduce the two regions, $R_1 = \{(t_1, t_2) \in R^+ \times R^+ \ : \ t_1 \leq t_2\}$ and

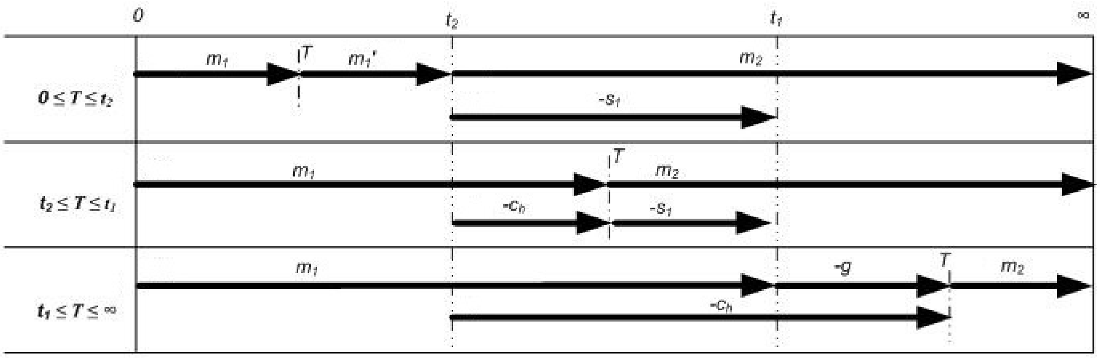

Figure 4.3: The profit rates when $t_2 \leq t_1$

$R_2 = \{(t_1, t_2) \in R^+ \times R^+ \ : \ t_1 \geq t_2\}$, the piecewise loss rate functions can be rewritten as

$$L(t_1, t_2, T) = L_i(t_1, t_2, T) \qquad \text{if } (t_1, t_2) \in R_i \qquad (i = 1, 2). \qquad (4.3.5)$$

On the boundary between regions R_1 and R_2, i.e., for $R_b = \{(t_1, t_2) \in R^+ \times R^+ : t_1 = t_2\}$, the expression of the objective function is obtained from (4.3.3) and/or (4.3.4) as

$$L_b(t, T) \quad = \quad (m_2 - m_1')[t - T]^+ + (m_1 + g + c_{h,2})[T - t]^+. \tag{4.3.6}$$

4.3.4 Parameter Assumptions

As in all stochastic production/inventory models, it is necessary to introduce some assumptions for the different parameters. These assumptions are as follows. First, the contribution-to-profit rate for the products under regular sales is positive, i.e.,

$$m_1, m_2 > 0. \tag{4.3.7}$$

Furthermore, for product 1 (the old product), the contribution-to-profit rate under regular sales is greater than contribution to the profit per period after the new product 2 is available, i.e.,

$$m_1 \geq m_1'. \tag{4.3.8}$$

In order to avoid cases for which it would be optimal to infinitely delay the new product launch, it is assumed that

$$m_2 \geq m_1'. \tag{4.3.9}$$

We also have the standard assumptions from classical inventory theory,

$$g, c_{h,2}, s_1 > 0. \tag{4.3.10}$$

4.4 Data-Driven Cost Approach vs Conditional Value at Risk

In our rollover problem, the manager has to determine the optimal rollover dates and strategy of introduction and removal of two products from the market. The manager has to plan his resources prior to observing the approval date T to satisfy the market while minimizing the net loss given by:

$$\min_{(t_1,t_2)\in R^+} l(t_1,t_2,T) = E[L_i(t_1,t_2,T)] \qquad \text{if } (t_1,t_2) \in R_i \qquad (i = 1,2). \tag{4.4.1}$$

The expectation is taken with respect to the stochastic approval date T, which has a cumulative distribution function (cdf) F.

We have thoroughly studied the net loss function objective function and the optimal solutions and strategies (see El Khoury (2011)). In particular, $l_1(t_1,t_2)$ and $l_b(t_1,t_2)$ are continuous and convex functions (Properties 1 and 2 in El Khoury (2011)). On the other hand, $l_2(t_1,t_2)$ is continuous and convex for $m_2 - m_1' - s_1 + c_{h,2} > 0$ (Property 3 in El Khoury et al. (2011)). Therefore, the optimal solution can be characterized through first-order conditions.

Given the convexity of our objective functions and knowing the probability distribution of T, we could also apply the Conditional Value at Risk (CVaR) approach. For $\beta \in [0,1)$, we define the β-VaR of this distribution by

$$\alpha_\beta(t_1,t_2) = min\{\alpha | \mathcal{L}_F(\alpha|t_1,t_2) \geq \beta\}. \tag{4.4.2}$$

It is now possible to introduce the β-tail distribution function to focus on the upper tail of the loss distribution as

$$\mathcal{L}_{F,\beta}(\eta|t_1,t_2) = \begin{cases} 0 & \text{for} \quad \eta < \alpha_\beta(t_1,t_2), \\ \frac{\mathcal{L}_\beta(\eta|t_1,t_2)-\beta}{1-\beta} & \text{for} \quad \eta \geq \alpha_\beta(t_1,t_2). \end{cases} \tag{4.4.3}$$

Using the expectation operator $E_\beta[\cdot]$ under the β-tail distribution $\mathcal{L}_{F,\beta}(\cdot|\cdot,\cdot)$, we define the β-conditional value-at-risk of the loss $L(t_1,t_2,T)$ by

$$\tilde{l}_{\beta,i}(t_1,t_2) = E_\beta[L_i(t_1,t_2,T)]. \tag{4.4.4}$$

Finding the optimal rollover strategy and the corresponding values of the phase-in and phase-out dates, which minimize the CVaR cost criterion amounts to the optimization problem

$$\min_{(t_1,t_2)\in R^+\times R^+} \tilde{l}_{\beta,i}(t_1,t_2) \quad = \quad \Big\{ E_\beta[L_i(t_1,t_2,T)] \Big\}. \tag{4.4.5}$$

The CVaR is a viable risk measure when the probability distribution of the approval date is known, and this was the case in El Khoury et al. (2011). In real life applications, the probability distribution is rarely known and we have to revert to data-driven optimization methods to calculate optimal solutions.

In the data-driven approach, the random variable is determined by computing the expected cost above a certain quantile, that is, by removing (trimming) the instances of the cost below the quantile and taking the average over the remaining ones. The fraction of data points removed will be referred to as the trimming factor which is in fact the same as β used in the CVaR method. We are therefore able to compare our solutions using the data-driven approach to the solutions obtained through the CVaR method. In this paper, we replace our original CVaR objective function with an average based on the drawn samples (Thiele 2006). The sampling-based approximated objective is then minimized.

Suppose that there are N independent samples drawn from the true distribution, labeled as $T_1, ..., T_N$. The data-driven approach approximates the true distribution with the empirical distribution that puts a weight of $\frac{1}{N}$ on each of the N samples and the expected cost evaluated under this empirical distribution. We denote the $a - quantile$ of the approval date T by $q_a(T)$ where

$$q_a(T) = inf\{t|F(T \leq t) \geq a\}, \tag{4.4.6}$$

for any $a\epsilon(0,1)$ as have done Levy and Kroll (1978) to describe investor preferences. We adapt their appraoch to a cost objective as follows:

Theorem 1: $E[U(T_1)] \leq E[U(T_2)]$ for all U decreasing and convex if and only if $E[T_1|T_1 \leq q_a(T_1)] \leq E[T_2|T_2 \leq q_a(T_2)]$ for any $a\epsilon(0,1)$, and we have strict inequality for some a.

Therefore, a strategy chosen to minimize the tail conditional expectation $E[T_1|T_1 \leq q_a(T_1)]$ is non-dominated. Equivalently, minimizing $E[T_1|T_1 \leq q_a(T_1)]$ for a specific a guarantees that no other strategy can worsen the value (expected utility) of the random variable for all risk-averse planners. Furthermore, this method does not require any assumptions for the probability distribution of the approval date.

Let N be the total number of observations of T where $(T_{(1)}, ...T_{(N)})$ be those observations ranked in increasing order $(T_{(1)} \leq, ... \leq T_{(N)})$.

Let the trimming factor be the fraction of scenarios that are removed, as $\beta = 1 - a$, and the number of scenarios left after trimming as $N_\beta = \lfloor N(1-\beta) + \beta \rfloor$ so that there is no trimming at $\beta = 0$ $(N_\beta = N)$ and that the worst scenario is at $\beta = 1$ $(N_\beta = 1)$.

It follows that the value associated with the random $L_i(t_1, t_2, T)$ is computed by:

$$\frac{1}{N_\beta} \sum_{k=1}^{N_\beta} L_i(t_1, t_2, T)_{(k)} \tag{4.4.7}$$

so we generate random realizations of T based on $T_{(1)}, ...T_{(k)}, T_{(k+1)}, ..., T_{(N)}$, each with equal probability, where $L(t_1, t_2, T)_{(k)}$ is the k^{th} smallest $L(t_1, t_2, T_j)$. From Thiele (2004), problem (4.4.7) becomes

$$\begin{aligned} Min \quad & \frac{1}{N_\beta} \sum_{k=1}^{N} t_k y_k \\ \text{s.t} \quad & \sum_{k=1}^{N} y_k = N_\beta \\ & 0 \leq y_k \leq 1 \forall k \end{aligned} \tag{4.4.8}$$

The feasible set of Eq. 4.4.8 is nonempty and bounded, therefore by strong duality, Eq. 4.4.8 is equivalent to:

$$\begin{aligned} min \quad & N_\beta \phi + \sum_{k=1}^{N} \psi_k && (4.4.9) \\ & \text{s.t } \phi + \psi_k \geq t_k, \forall k \\ & \psi_k \geq 0 \forall k \\ & t_1, t_2 \geq 0 && (4.4.10) \end{aligned}$$

Problem (4.4.8) is a convex problem if $L_i(t_1, t_2, T)$ is convex in t_1 and t_2 , and a linear programming problem since is piecewise linear $L_i(t_1, t_2, T)$.

As the cost functions in our product rollover problem are piecewise linear with linear ordering constraints, we will be able to derive tractable, linear programming formulations of the data-driven model.

The conditional value at risk (CVaR) is at the core of the data-driven approach, as the method's objective is to minimize its sample value over the historical realizations of the approval date. CVaR at level β refers to the conditional expectation of losses in the top $100(1-\beta)\%$ and refers to the risk perception of the manager. According to the data-driven approach, the fundamental optimization problem considered here consists of finding the phase-in and phase-out dates which minimize the maximum (worst) expected cost objective, the associated optimization problem is

$$\min_{t_1, t_2 \in R^+} \quad \frac{1}{N_\beta} \sum_{k=1}^{N_\beta} L(t, T)_{(k)}. \tag{4.4.11}$$

Due to the structure of the cost function given in (4.3.5), we introduce the following auxiliary subproblems, for $i = 1, 2$, (See El Khoury et al.(2011))

$$\min_{(t_1, t_2) \in R_i} \quad \frac{1}{N_\beta} \sum_{k=1}^{N_\beta} L_i(t_1, t_2, T)_{(k)}, \tag{4.4.12}$$

and the boundary problem,

$$\min_{(t_b)\in R^+} \qquad \frac{1}{N_\beta}\sum_{k=1}^{N_\beta} L_b(t_b,T)_{(k)}, \tag{4.4.13}$$

4.5 Structural Properties and Optimal Solutions

The optimal solution structure is essentially determined by convexity characteristics of these functions (4.3.3)-(4.3.4) in the regions R_1 and R_2.

Property 1: Under assumption (4.3.8), the loss function $L_1(t_1,t_2,T)$ is strictly jointly convex on $R^+ \times R^+$.

Proof. See El Khoury et al. (2011)

We distinguish two cases, $m_2 \geq m_1$ and $m_1 \geq m_2$ to solve the following problem (See El Khoury et al. 2011)

$$\min_{(t_1,t_2)\in R_1} \qquad \frac{1}{N_\beta}\sum_{k=1}^{N_\beta} L_1(t_1,t_2,T)_{(k)}. \tag{4.5.1}$$

We know that $L_1(t_1,t_2,T)$ is jointly convex with respect to t_1 and t_2, therefore we are able to apply Theorem 1 and solve for tractable solutions by distinguishing two cases:

- $m_2 \geq m_1$ where $L_1(t_1,t_2,T)$ is strictly decreasing with respect to $T_k < t_2$ and strictly increasing with respect to $T_k > t_2$
- $m_1 \geq m_2$ where $L_1(t_1,t_2,T)$ is strictly decreasing with respect to $T_k < t_1$ and strictly increasing with respect to $T_k > t_1$.

PROPOSITION 1 : Under the assumption $m_2 \geq m_1 \geq m_1'$, if $m_1' < -g$, and $\frac{m_1+g}{m_1-m_1'} < \frac{c_{h,2}}{m_2+c_{h,2}+g}$, problem (4.4.8) has a unique finite minimum over $R_1 \times R_1$ corresponding to,

$$t_1^* = min\left\{T_{(j)}|T_{(j)} \geq T_{(M_\beta^1)}\right\}, \tag{4.5.2}$$

$$t_2^* = min\left\{T_{(h)}|T_{(h)} \geq \left(\frac{m_2 - m_1}{m_2 + c_{h,2} + g}\right)T_{(M_\beta^2)} + \left(\frac{m_1 + c_{h,2} + g}{m_2 + c_{h,2} + g}\right)T_{(N-N_\beta+M_\beta^2)}\right\}. \tag{4.5.3}$$

where $M_\beta^1 = \lceil \frac{m_1+g}{m_1-m_1'} N_\beta \rceil$ and $M_\beta^2 = \lceil \frac{c_{h,2}}{m_2+c_{h,2}+g} N_\beta \rceil$, otherwise, there exists no finite minimum for problem (4.4.8) in R^1 and the optimal rollover strategy will either be single or dual rollover.

Proof. See Appendix A, Proposition 1.

PROPOSITION 2 : Under the assumption $m_1 \geq m_2 \geq m_1'$, if $m_1' < -g$ and $\frac{m_1+g}{m_1-m_1'} < \frac{c_{h,2}}{m_2+c_{h,2}+g}$ problem (4.4.8) has a unique finite minimum over $R_1 \times R_1$ corresponding to,

$$t_1^* = min\left\{T_{(j)}|T_{(j)} \geq \left(\frac{m_2 - m_1'}{m_1 - m_1'}\right)T_{(M_\beta^1)} + \left(\frac{m_1 - m_2}{m_1 - m_1'}\right)T_{(N-N_\beta+M_\beta^1)}\right\}, \tag{4.5.4}$$

$$t_2^* = min\left\{T_{(h)}|T_{(h)} \geq T_{(N-N_\beta+M_\beta^2)}\right\}. \tag{4.5.5}$$

where $M_\beta^1 = \lceil \frac{m_1+g}{m_1-m_1'} N_\beta \rceil$ and $M_\beta^2 = \lceil \frac{c_{h,2}}{m_2+c_{h,2}+g} N_\beta \rceil$, otherwise, there exists no finite minimum for problem (4.4.8) in R^1 and the optimal rollover strategy will either be single or dual rollover.

Proof. See Appendix A, Proposition 2.

Property 2: Under the assumption $m_2 - m_1' - s_1 + c_{h,2} > 0$,the loss function $L_2(t_1, t_2, T)$ is strictly jointly convex over $\mathbf{R}^+ \times \mathbf{R}^+$, else it is strictly concave and the optimal strategy will be a planned stockout or single rollover.

We distinguish two cases, $c_{h,2} \geq s_1$ and $s_1 \geq c_{h,2}$ to solve the following problem (See El Khoury et al. 2011)

$$\min_{(t_1,t_2)\in R_2} \qquad \frac{1}{N_\beta} \sum_{k=1}^{N_\beta} L_2(t_1, t_2, T)_{(k)}. \tag{4.5.6}$$

We know that $L_2(t_1, t_2, T)$ is jointly convex with respect to t_1 and t_2 for $m_2 - m_1' - s_1 + c_{h,2}$, therefore we are able to apply Theorem 1 and solve for tractable solutions by distinguishing two cases:

- $c_{h,2} \geq s_1$ where $L_2(t_1, t_2, T)$ is strictly decreasing with respect to $T_k < t_2$ and strictly increasing with respect to $T_k > t_2$
- $s_1 \geq c_{h,2}$ where $L_2(t_1, t_2, T)$ is strictly decreasing with respect to $T_k < t_1$ and strictly increasing with respect to $T_k > t_1$.

PROPOSITION 3 : Under the assumption $c_{h,2} \geq s_1$, if $m_2 - m_1' - s_1 > 0$ and $(m_1 + g + ch_h)s_1 < (m_1 + g)(m_2 - m_1')$ problem (4.4.8) has a unique finite minimum over $R_2 \times R_2$ corresponding to,

$$t_1^* = min\left\{ T_{(j)} | T_{(j)} \geq T_{(N-N_\beta+M_\beta^1)} \right\}, \tag{4.5.7}$$

$$t_2^* = min\left\{ T_{(h)} | T_{(h)} \geq \left(\frac{m_2 - m_1'}{m_2 - m_1' - s_1 + c_{h,2}} \right) T_{(M_\beta^2)} + \left(\frac{c_{h,2} - s_1}{m_2 - m_1' - s_1 + c_{h,2}} \right) T_{(N-N_\beta+M_\beta^2)} \right\} \tag{4.5.8}$$

where $M_\beta^1 = \lceil \frac{m_1+g}{m_1+g+s_1} N_\beta \rceil$ and $M_\beta^2 = \lceil \frac{c_{h,2}}{m_2-m_1'-s_1+c_{h,2}} N_\beta \rceil$, otherwise, there exists no finite minimum for problem (4.4.8) in R^2 and the optimal rollover strategy will either be single or planned stockout.

Proof. See Appendix B, Proposition 3.

PROPOSITION 4 : Under the assumption $c_{h,2} < s_1$, if $m_2 - m_1' - s_1 > 0$ and $(m_1 + g + ch_h)s_1 < (m_1 + g)(m_2 - m_1')$ problem (4.4.8) has a unique finite minimum over $R_2 \times R_2$ corresponding to,

$$\begin{aligned} t_1^* &= min\left\{T_{(j)}|T_{(j)} \geq \left(\frac{m_1 + g + c_{h,2}}{m_1 + g + s_1}\right)T_{(N-N_\beta+M_\beta^1)} \right. \\ &+ \left. \left(\frac{s_1 - c_{h,2}}{m_1 + g + s_1}\right)T_{(M_\beta^1)}\right\}, \qquad (4.5.9) \\ t_2^* &= min\left\{T_{(h)}|T_{(h)} \geq T_{(M_\beta^2)}\right\} \qquad (4.5.10) \end{aligned}$$

where $M_\beta^1 = \lceil \frac{m_1+g}{m_1+g+s_1} N_\beta \rceil$ and $M_\beta^2 = \lceil \frac{c_{h,2}}{m_2-m_1'-s_1+c_{h,2}} N_\beta \rceil$, otherwise, there exists no finite minimum for problem (4.4.8) in R^2 and the optimal rollover strategy will either be single or planned stockout.

Proof. See Appendix B, Proposition 4.

Property 3: Under the assumption (4.3.9),the loss function $L_b(t_b, T)$ is strictly jointly convex on $R^+ \times R^+$ over $\mathbf{R}^+$. *Proof.* See El Khoury et al. (2011)

Proposition 5 : A unique finite minimum on the boundary exists over R^+ corresponding to,

$$\begin{aligned} t_b^* &= min\left\{T_{(j)}|T_{(j)} \geq \left(\frac{m_2 - m_1'}{m_1 - m_1' + m_2 + c_{h,2} + g}\right)T_{(M_\beta)} \right. \\ &+ \left. \left(\frac{m_1 + g + c_{h,2}}{m_1 - m_1' + m_2 + c_{h,2} + g}\right)T_{(N-N_\beta+M_\beta)}\right\} \qquad (4.5.11) \end{aligned}$$

where $M_\beta = \lceil \frac{m_1+g+c_{h,2}}{m_1-m_1'+m_2+c_{h,2}+g} N_\beta \rceil$.

Proof. See Appendix C, Proposition 5.

4.6 Numerical Convergence: Bound Analysis

In this paper, we consider the rollover problem under the assumption that the explicit approval date distribution is not know, but the only information available is a set of independent samples drawn from the true distribution. We have already studied the model in El Khoury et al. (2011) where the date distribution was given explicitly and we developed closed-form solutions. However, in most real-life situations, the true distributions are not available or may be too complex to work with. Thus, a data-driven algorithmic framework is recommended and gives very reasonable solutions. In this section, we establish bounds on the number of samples required to guarantee that with a high pre-specified confidence probability the expected cost of sampling-based policies is close, with a relative small error, compared to the expected cost of the optimal policies which have full access to the date distributions. The bounds that we develop are general, easy to compute and do not depend at all on the specific demand distributions. They depend on the specified confidence probability and the relative error, as well as on the ratio between the cost parameters. This approach was suggested by Levy et al. (2007) who discuss the robust optimization solution with respect to the original problem as a function of N_β. For a specified *accuracy level* $\epsilon > 0$ and a *confidence level* $1 - \delta$ (where $0 < \delta < 1$), there exists a number of samples N_β such that, with probability at least $1 - \delta$, the optimal solution has an expected cost $l_i(\hat{t_1}, \hat{t_2})$ that is at most $(1 + \epsilon)l_i(t_1^*, t_2^*)$. They also define "closeness" between $l_i(\hat{t_1}, \hat{t_2})$ and (t_1^*, t_2^*) by how close are $F(\hat{t_1})$ and $F(\hat{t_2})$ to $F(t_1^*)$ and $F(t_2^*)$ respectively.

Our results are valid for negative values of T and for any date distribution T.

For each strategy, planned stock-out, single, and dual rollovers, the worst-case bound is different. Therefore we propose the following theorems :

THEOREM A Consider a planned stock-out rollover problem with a random variable T and $E[T] < \infty$. Let $0 < \epsilon \leq 1$ be a specified accuracy level and $1 - \epsilon$ (for $0 < \delta < 1$) be a specified confidence level. Suppose that

$$N_\beta \geq Max\left(\frac{9}{2\epsilon^2}\left(\frac{min(-(m_1'+g),m_1+g)}{m_1-m_1'}\right)^{-2} log\left(\frac{2}{\delta}\right), \frac{9}{2\epsilon^2}\left(\frac{c_{h,2}}{m_2+g+c_{h,2}}\right)^{-2} log\left(\frac{2}{\delta}\right)\right)$$

and the data-driven counterpart is solved with respect to N_β i.i.d samples of T. Let $\hat{T_1}$ be the optimal solution to the data-driven counterpart and $\hat{t_1}$ denote its realization. Then, with probability at least $1 - \delta$, the expected cost of $\hat{t_1}$ is at most $1 + \epsilon$ times the expected cost of an optimal solution t_1^* to the rollover problem. In other words, $l_1(\hat{T}_1, t_2) \leq (1 + \epsilon) l_1(t_1^*, t_2)$ with probability at least $1 - \delta$.

Proof. See Appendix D.

THEOREM B Consider a dual rollover problem with random variable T and $E[T] < \infty$. Let $0 < \epsilon \leq 1$ be a specified accuracy level and $1 - \epsilon$ (for $0 < \delta < 1$) be a specified confidence level. Suppose that

$$N_\beta \geq Max\left(\frac{9}{2\epsilon^2}\left(\frac{s_1}{m_1+g+s_1}\right)^{-2} log\left(\frac{2}{\delta}\right), \frac{9}{2\epsilon^2}\left(\frac{min(m_2-m_1'-s_1,c_{h,2})}{m_2-m_1'+c_{h,2}-s_1}\right)^{-2} log\left(\frac{2}{\delta}\right)\right)$$

and the data-driven counterpart is solved with respect to N_β i.i.d samples of T. Let $\hat{T_1}$ be the optimal solution to the data-driven counterpart and $\hat{t_1}$ denote its realization. Then, with probability at least $1 - \delta$, the expected cost of $\hat{t_1}$ is at most $1 + \epsilon$ times the expected cost of an optimal solution t_1^* to the rollover problem. In other words, $l_1(\hat{T}_1, t_2) \leq (1 + \epsilon) l_1(t_1^*, t_2)$ with probability at least $1 - \delta$.

Proof. See Appendix E.

THEOREM C Consider a single rollover problem with random variable T and $E[T] < \infty$. Let $0 < \epsilon \leq 1$ be a specified accuracy level and $1 - \epsilon$ (for $0 < \delta < 1$) be a specified confidence level. Suppose that

$$N_\beta \geq \frac{9}{2\epsilon^2}\left(\frac{min(m_2-m_1',m_1+g+c_{h,2})}{m_2-m_1'+m_1+g+c_{h,2}}\right)^{-2} log\left(\frac{2}{\delta}\right)$$

and the data-driven problem is solved with respect to N_β i.i.d samples of T. Let $\hat{T}_1$ be the optimal solution to the data-driven counterpart and $\hat{t_1}$ denote its realization. Then, with probability at least $1-\delta$, the expected cost of $\hat{t_1}$ is at most $1+\epsilon$ times the expected cost of an optimal solution t_1^* to the rollover problem. In other words, $l_b(\hat{T}_1) \leq (1+\epsilon)l_b(t_1^*)$ with probability at least $1-\delta$.

Proof. See Appendix F.

THEOREM D: For the expected cost to be at most $1+\epsilon$ times the expected cost of an optimal solution t_1^* to the rollover problem with probability at least $1-\delta$, then the upper bound N should satisfy

$$\prod_{w=N-N_\beta+1}^{N} w \geq (1-\delta)N_\beta\left(\frac{\beta}{1-\beta}\right)^{N_\beta} \tag{4.6.1}$$

where N_β is one of the bounds calculated in the Theorems A, B, or C depending on the rollover strategy.

Proof: By definition, the trimming factor or the fraction of scenarios that are removed is β, and the number of scenarios left after trimming as $N_\beta = \lfloor N(1-\beta)+\beta \rfloor$. We recall the definition of a binomial probability distribution where for having a probability of N_β successes/observations left as follows:

$$\binom{N}{N_\beta}(1-\beta)^{N_\beta}\beta^{N-N_\beta} \tag{4.6.2}$$

We know that we have $1-\beta$ of the scenarios taken, therefore we have

$$1-\delta \leq \binom{N}{N_\beta}(1-\beta)^{N_\beta}\beta^{N-N_\beta} \tag{4.6.3}$$

	Planned Stock-out Rollover			Dual Rollover			Planned Stock-out Rollover			Dual Rollover		
N_β	t_1	t_2	Data-driven Cost	t_1	t_2	Data-driven Cost	t_1	t_2	CVAR Optimal Cost	t_1	t_2	CVAR Optimal Cost
5	21.50	44.21	2106.95	60.00	18.01	966.67	6.00	32.22	1462.69	58.71	5.21	461.57
10	11.66	38.55	1624.82	60.00	5.38	463.59	6.00	32.22	1462.69	58.71	5.21	461.57
50	8.08	36.28	1521.64	56.53	5.60	467.62	6.00	32.22	1462.69	58.71	5.21	461.57
100	6.51	33.38	1471.40	59.62	7.45	478.41	6.00	32.22	1462.69	58.71	5.21	461.57
500	6.47	33.13	1468.59	58.94	5.25	461.64	6.00	32.22	1462.69	58.71	5.21	461.57
1,000	5.89	32.42	1465.37	58.99	5.12	461.69	6.00	32.22	1462.69	58.71	5.21	461.57
5,000	6.16	32.20	1461.13	58.94	5.26	461.64	6.00	32.22	1462.69	58.71	5.21	461.57
10,000	6.13	32.38	1463.02	58.82	5.20	461.59	6.00	32.22	1462.69	58.71	5.21	461.57
100,000	6.04	32.21	1462.24	58.70	5.18	461.58	6.00	32.22	1462.69	58.71	5.21	461.57
1,000,000	6.00	32.23	1462.76	58.72	5.20	461.57	6.00	32.22	1462.69	58.71	5.21	461.57

Table 4.1: Optimal Costs for the two Strategies for Different Sample Values

Simplifying, we get

$$\prod_{w=N-N_\beta+1}^{N} w \geq (1-\delta)N_\beta \left(\frac{\beta}{1-\beta}\right)^{N_\beta} \tag{4.6.4}$$

4.7 Numerical Experiments

4.7.1 Bound Analysis and Convergence

In our numerical convergence section, we have computed the worst-case upper bounds on the number of samples required, and we see in this section that we need a significantly fewer number of samples to achieve *close* optimal costs. We start by simulating for different numbers of samples the data-driven solution attained and the optimal cost and we compare it to the optimal solution when the distribution of T is known. We simulate a case where we have the following parameters: $m_1 = 20$, $m_1' = -30$, $m_2 = 40$, $g = 5$, $s_1 = 3$, $c_{h,2} = 9$, and for a uniform distribution $[0, 60]$ for T where $\beta = 0.8$. We see that for a total number of samples $N = 500$, we can get an error of less than 1% in both planned stockout and dual rollover strategies (See Figure 4.4 and Table 1) and we have the optimal strategy as dual both in the CVaR and the data-driven.

We present another example with Gaussian distribution for T with mean being 50 and

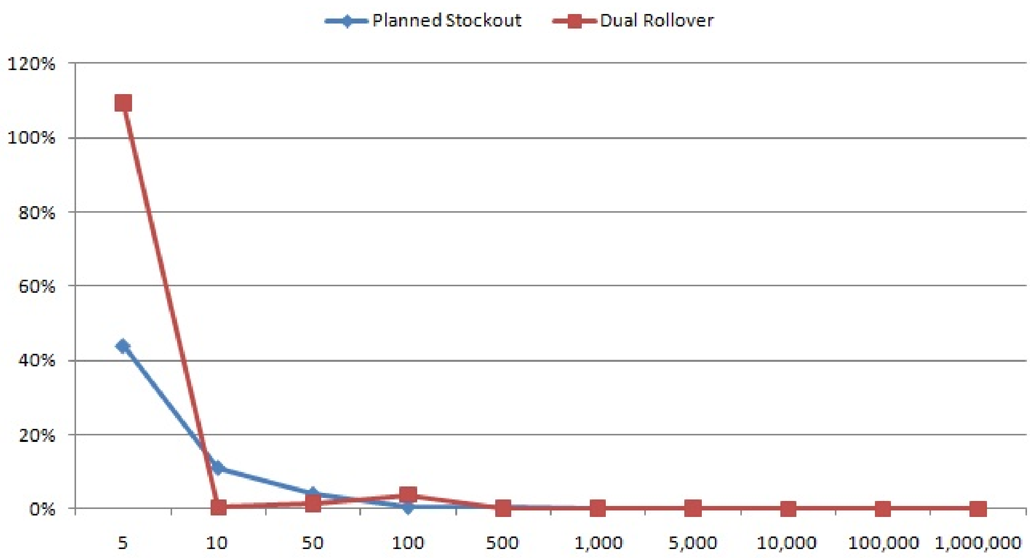

Figure 4.4: Percentage Error vs. N_β

variance 2 and the cost parameters $m_1 = 20$, $m_1' = -30$, $m_2 = 40$, $g = 5$, $s_1 = 3$, $c_{h,2} = 9$ in thousands, and $\beta = 0.9$. The optimal solution with the CVAR approach is $t_1 = 54.6$, $t_2 = 46.05$ and optimal cost 72813 US\$. We simulate for different values of N and get an error of around 2% for $N \geq 100$. We plot the errors with respect to N in Figure 4.5.

4.7.2 Effect of 'Wrongly' Guessing the Probability Distribution

In this section, we try to prove the superiority of the data-driven approach to the CVAR when we wrongly estimate the probability distribution. In other words, suppose that we have a set of historic date samples, we estimate the mean and the variance from this set and the only information available to us is this mean and variance. We try to guess the probability distribution and apply the CVAR method. We will see through different examples that as N increases, the data-driven approach gives better solutions than the CVAR one in case we wrongly guess the probability distribution family.

We examine two cases, when we estimate correctly the probability distribution method and another case when we estimate the mean or standard deviation of the probability

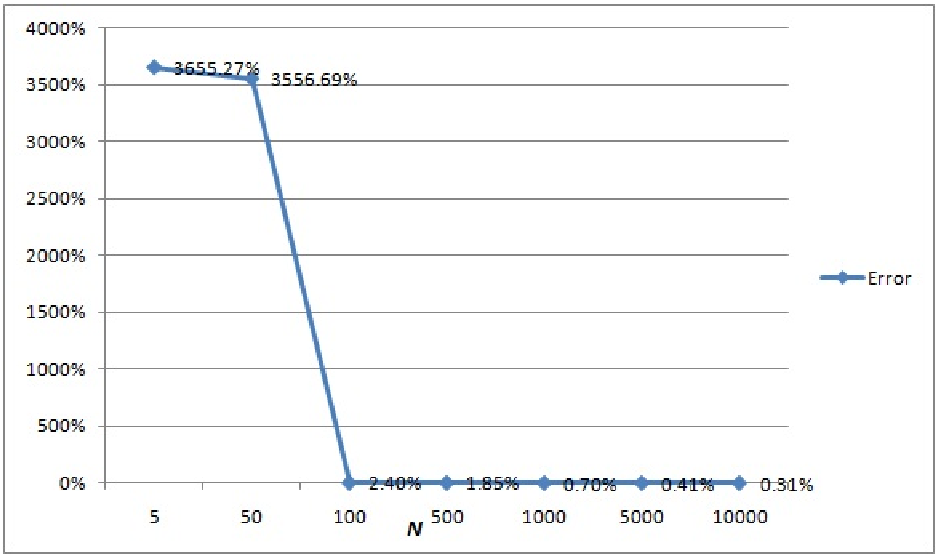

Figure 4.5: Percentage Error vs. N for the Dual Rollover Strategy

distribution correctly. Consider the following case:

$m_1 = 20$, $m'_1 = -30$, $m_2 = 40$, $g = 5$, $s_1 = 3$, $c_{h,2} = 9$ with the data samples generated through a uniform probability distribution$[0, 100]$ for T. The optimal strategy is the dual rollover one and if we correctly use the uniform distribution to calculate the CVAR optimal cost we get an average error of 1.7% for $N \geq 500$. Now we examine the case where we wrongly estimate the probability distribution to be a normal distribution mean 50 and variance 29, in other words the same mean and variance as the correct probability distribution. We get the optimal strategy to be planned stockout one, unlike the real one that we should have (dual) with the optimal dates $t_1 = 2.29$ and $t_2 = 52.39$. The optimal cost is \$2901 compared to a an optimal cost of 299 for a uniform distribution (See Figure 4.6). We can conclude, in this case, that incorrectly estimating the probability distribution can lead to an incorrect rollover strategy and around ten times greater costs.

N	t_1	t_2	Data Driven Cost	t_1	t_2	CVAR Cost
5	60.00	39.42	3850.57	59.36	4.97	299.89
50	60.00	7.70	3199.93	59.36	4.97	299.89
100	59.34	6.40	321.19	59.36	4.97	299.89
500	59.90	5.08	310.13	59.36	4.97	299.89
1000	59.67	5.25	303.51	59.36	4.97	299.89
5000	59.48	5.15	303.16	59.36	4.97	299.89
10000	59.39	4.81	303.49	59.36	4.97	299.89

Table 4.2: Dual Rollover Optimal Dates and Costs for data-driven and CVAR for Correct Guessing Case

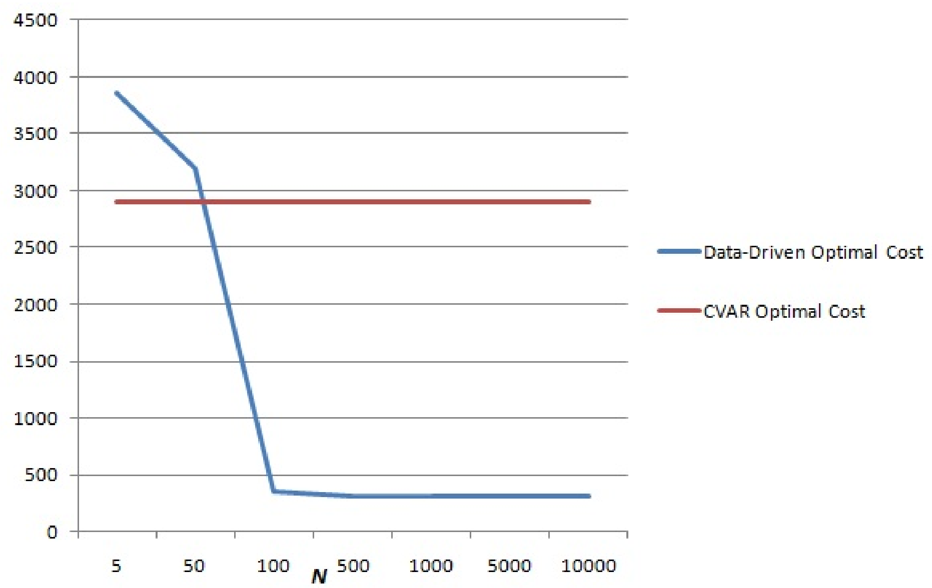

Figure 4.6: Optimal Costs in Case of Wrong Probability Distribution Guessing

4.8 Conclusion and Future Research Directions

In our third paper, we have proposed a data-driven approach to the rollover problem that builds directly upon the historical data without requiring any probability distribution. We have compared our approach to the CVAR one, and we showed that the data-driven approach can give the correct rollover strategy and a very close optimal cost with a relatively *low* number of observations. We have also showed that, in case a probability distribution has been wrongly estimated, the data-driven approach is far superior and can provide more valuable insights into the rollover strategy.

We have also established bounds on the number of samples required to guarantee that with high pre-specified confidence probability the expected cost of sampling-based policies is close, with a relative small error, compared to the expected cost of the optimal policies which have full access to the date distributions.

Having obtained these results, we believe that the data-driven approach can be used for other extensions of the rollover problem. It would be worth it to solve for optimal pricing, inventory and other extensions of the rollover problem using the data-driven approach.

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Appendices

APPENDIX A

For a planned stock-out product rollover strategy the net cost is given by:

$$\begin{aligned} L_1(t_1,t_2,T) &= (m_1+g)[T-t_1]^+ - (g+m_1')[t_1-T]^+ \\ &\quad + c_{h,2}[T-t_2]^+ + (m_2+g)[t_2-T]^+. \end{aligned} \tag{4.8.1}$$

We can rewrite (4.8.1) as follows:

$$\begin{aligned} L_1(t_1,t_2,T) &= -(m_1+g)t_1 + c_{h,2}t_2 - (m_1'-m_1)[t_1-T]^+ \\ &+ (m_2+c_{h,2}+g)[t_2-T]^+ + (m_1+c_{h,2}+g)T \end{aligned} \tag{4.8.2}$$

Our goal is to minimize the trimmed mean of the cost:

$$\begin{aligned} \min_{0\le t_1\le t_2} \quad & -(m_1+g)t_1 - c_{h,2}t_2 + \frac{1}{N_\beta}\sum_{k=1}^{N_\beta}\Big(-(m_1'-m_1)[t_1-T]^+ \\ + \quad & (m_2+c_{h,2}+g)[t_2-T]^+ + (m_1+c_{h,2}+g)T\Big)_k. \end{aligned} \tag{4.8.3}$$

where for any $y \epsilon R^n$, $y_{(k)}$ is the k^{th} smallest component of y.

We know that $L_1(t_1,t_2,T)$ is jointly convex with respect to t_1 and t_2, therefore we are able to apply Theorem 1 and solve for tractable solutions by distinguishing two cases:

- $m_2 \geq m_1$ where $L_1(t_1,t_2,T)$ is strictly decreasing with respect to $T_k < t_2$ and strictly increasing with respect to $T_k > t_2$
- $m_1 \geq m_2$ where $L_1(t_1,t_2,T)$ is strictly decreasing with respect to $T_k < t_1$ and strictly increasing with respect to $T_k > t_1$.

Case 1: $m_2 \geq m_1$

Proposition 1:

(a) The optimal times t_1 and t_2 in (4.8.3) are the solution of the linear programming problem:

$$\min_{0 \le t_1 \le t_2} \quad -(m_1+g)t_1 - c_{h,2}t_2 + \frac{1}{N_\beta}\sum_{k=1}^{N}\psi_k^2 + \frac{1}{N_\beta}\sum_{k=1}^{N}\psi_k^1, \tag{4.8.4}$$

$$\text{s.t} \quad \phi + \psi_k^1 - \Big((m_1' - m_1)Z_k^1\Big) \ge 0, \forall k$$

$$\phi + \psi_k^2 + \Big((m_2 + c_{h,2} + g)Z_k^2\Big) \ge (m_1 + c_{h,2} + g)T_k, \forall k \tag{4.8.5}$$

$$Z_k^1 + t_1 \ge T_k \forall k,$$

$$Z_k^2 + t_2 \ge T_k \forall k,$$

$$Z_k^1 \ge 0, Z_k^2 \ge 0, \psi_k \ge 0 \forall k$$

Moreover, $t_1^* = T_{(j)}$ for some j and $t_2^* = T_{(h)}$ for some h.

(b) Let $M_\beta^1 = \lceil \frac{m_1+g}{m_1-m_1'} N_\beta \rceil . t_1^*$ satisfies

$$t_1^* = min\Big\{ T_{(j)} | T_{(j)} \ge T_{(M_\beta^1)} \Big\} \tag{4.8.6}$$

(c) Let S_β be the set of the N_β worst-case scenarios at optimality, that is $\sum_{k=1}^{N_\beta} L_1(t_1, t_2, T)_{(i)} = \sum_{i \epsilon S_\beta} L_1(t_1, t_2, T_i)$, and let $T_{(j)}^{S_\beta}$ the $j-th$ highest approval date within that set. We have:

$$t_1^* = T_{(M_\beta^1)}^{S_\beta} \tag{4.8.7}$$

where M_β^1 is defined in (b).

(d) Let $M_\beta^2 = \lceil \frac{c_{h,2}}{m_2+c_{h,2}+g} N_\beta \rceil . t_2^*$ satisfies

$$t_2^* = min\Big\{ T_{(h)} | T_{(h)} \ge \Big(\frac{m_2 - m_1}{m_2 + c_{h,2} + g}\Big) T_{(M_\beta^2)} + \Big(\frac{m_1 + c_{h,2} + g}{m_2 + c_{h,2} + g}\Big) T_{(N - N_\beta + M_\beta^2)} \Big\} \tag{4.8.8}$$

(e) Let S_β be the set of the N_β worst-case scenarios at optimality, that is $\sum_{k=1}^{N_\beta} L_2(t_1, t_2, T)_{(i)} = \sum_{i \epsilon S_\beta} L_2(t_1, t_2, T_i)$, and let $T_{(h)}^{S_\beta}$ the $h-th$ highest approval date within that set. We have:

$$t_2^* = T_{(M_\beta^2)}^{S_\beta} \tag{4.8.9}$$

where M^2_β is defined in (d).

(f)If $t_1^* < t_2^*$, then the optimal strategy may be planned stock-out, else it is a single or dual rollover one.

Proof

(a) Let $L_1(t_1, t_2, T) = L_1(t_1, T) + L_2(t_2, T)$ where $L_1(t_1, T) = -(m_1' - m_1)[t_1 - T]^+$ and $L_2(t_2, T) = +(m_2 + c_{h,2} + g)[t_2 - T]^+ + (m_1 + c_{h,2} + g)T$. We know that $L_1(t_1, t_2, T)$ is continuous and piecewise linear.

We consider $L_1(t_1, T)$ which is non-decreasing in T, and the k^{th} smallest $[t_1 - T]^+$ at t_1 is equal to $[t_1 - T_{(k)}]^+$.

Applying Theorem 1 to Problem (4.8.3), at optimality, $t_2^* = T_{(h)}$ for some h because the function to minimize in $L_2(t_2, T)$ is convex piecewise linear with breakpoints in the set $(T_{(i)})$.

Therefore, the worst case scenarios of $L_1(t_1, T)$ and $L_2(t_2, T)$ would give the N_β worst case scenarios of $L_1(t_1, t_2, T)$ and Problem (4.8.3) is equivalent to:

$$
\begin{aligned}
Min \quad & \phi_1 + \phi_2 + \frac{1}{N_\beta}\sum_{k=1}^{N} \psi_k^1 + \frac{1}{N_\beta}\sum_{k=1}^{N} \psi_k^2 && (4.8.10)\\
\text{s.t} \quad & \phi_1 + \psi_k^1 \geq 0, \forall k \\
& \phi_2 + \psi_k^2 \geq +(m_1 + c_{h,2} + g)T_k, \forall k \\
& \psi_k^1, \psi_k^2 \geq 0 \forall k, \\
& t \epsilon \varsigma.
\end{aligned}
$$

Problem (4.8.10) is a convex problem since $L_1(t_1, T)$ and $L_2(t_2, T)$ are decreasing in t_1 and increasing t_2 respectively and a linear programming problem since $L_1(t_1, T)$ and $L_2(t_2, T)$are piecewise linear and ς is a polyhedron.

For any vector t with ranked components $T_{(1)} \leq ... \leq T_{(N)}$, is the optimal solution of:

$$\begin{aligned} Max \quad & \frac{1}{N_\beta} \sum_{k=1}^{N} t_k y_k \\ \text{s.t} \quad & \sum_{k=1}^{N} y_k = N_\beta \\ & 0 \leq y_k \leq 1 \forall k \end{aligned} \tag{4.8.11}$$

The feasible set of Eq. 4.8.11 is nonempty and bounded, therefore by strong duality, Eq. 4.8.11 is equivalent to:

$$\begin{aligned} Min \quad & N_\beta \phi_1 + N_\beta \phi_2 + \sum_{k=1}^{N} \left(\psi_k^1 + \psi_k^2 \right) \\ \text{s.t} \quad & \phi_1 + \psi_k^1 \geq t_k^1, \forall k \\ & \phi_2 + \psi_k^2 \geq t_k^2, \forall k \\ & \psi_k^1 \geq 0, \psi_k^2 \geq 0 \forall k \end{aligned} \tag{4.8.12}$$

Reinjecting Eq. (4.8.12) into Eq. (4.8.3) with $t_k^1 = 0$ and $t_k^2 = +(m_1 + c_{h,2} + g)T_k$ for all k yields Eq. (4.8.10).

It follows immediately that Eq. (4.8.10) is a convex problem since $L_1(t_1, T)$ and $L_2(t_2, T)$ are convex in t_1 and t_2. Moreover, since $L_1(t_1, T)$ and $L_2(t_2, T)$ are (convex) piecewise linear in t_1 and t_2 and ς is a polyhedron, then Eq. (4.8.10) is a linear programming problem.

As the cost functions in our product rollover problem are piecewise linear with linear ordering constraints, Theorem 1 will allow us to derive tractable, linear programming formulations of the data-driven models.

(b) The slope of the cost function with respect to t_1 is : $-(m_1+g)-\frac{1}{N_\beta}(m'_1-m_1).\{i\epsilon S(t_1), T_i \leq t_1\}$ where $S(t_1)$ is the set of indices of the N_β smallest $(m_1-m'_1)[t_1-T]^+$ at t_1 given. It is easy to show that for any $i\epsilon S(t_1)$ and for any k such that $T_k \leq T_i \leq t_1$, $k\epsilon S(t_1)$ as well. Similarly, for any $i\epsilon S(t_1)$ and any k such that $T_k \geq T_i \geq t_1$, $k\epsilon S(t_1)$. Hence, $S(t_1)$ consists of the indices of $T_{(1)}, ..., T_{(M^1_\beta)}$ and $T_{(N-N_\beta+M^1_\beta+1)}, ...T_N$ for some $0 \leq M^1_\beta \leq N$, with $T_{M^1_\beta} \leq t_1 \leq T_{(N-N_\beta+M^1_\beta+1)}$. The slope of the trimmed cost function is then proportional to $-\frac{m_1+g}{m_1-m'_1}N_\beta + M^1_\beta$, and at optimality M^1_β is equal to $\lceil \frac{m_1+g}{m_1-m'_1}N_\beta \rceil$. We now have to determine the optimal value of t_1.

Let $f^j_i = (m_1-m'_1)[T_{(j)}-T_{(i)}]^+$ be the cost realized when $t_1 = T_{(j)}$ and $T = T_{(i)}$, for all i and j. The optimal M^1_β is the largest integer less than or equal to N_β such that $f^j_{M^1_\beta} \geq f^j_{N-N_\beta+M^1_\beta}$. (Otherwise, we would remove M^1_β from $S(t_1)$ and add $N-N_\beta+M^1_\beta$ instead.) Plugging the expression of $f^j_{M^1_\beta}$ and $f^j_{N-N_\beta+M^1_\beta}$ yields:

$$-(m_1-m'_1)T_{(M^1_\beta)} + (m_1-m'_1)T_{(j)} \geq 0 \tag{4.8.13}$$

Combining the previous results, Equation (4.8.6) follows immediately.

(c) Considering only the scenarios in $S_{1\beta}$, we inject $N = N_\beta$ into Equation (4.8.6).

(d) The slope of the cost function with respect to t_2 is : $-c_{h,2}+\frac{1}{N_\beta}(m_2+c_{h,2}+g).\{i\epsilon S(t_2), T_i \leq t_2\}$ where $S(t_2)$ is the set of indices of the N_β smallest $(m_2+c_{h,2}+g)[t_2-T]^+ + (m_1 + c_{h,2}+g)T$ at t_2 given. It is easy to show that for any $i\epsilon S(t_2)$ and for any k such that $T_k \leq T_i \leq t_2$, $k\epsilon S(t_2)$ as well. Similarly, for any $i\epsilon S(t_2)$ and any k such that $T_k \geq T_i \geq t_2$, $k\epsilon S(t_2)$. Hence, $S(t_2)$ consists of the indices of $T_{(1)}, ..., T_{(M^2_\beta)}$ and $T_{(N-N_\beta+M^2_\beta+1)}, ...T_N$

for some $0 \leq M_\beta^2 \leq N$, with $T_{M_\beta^2} \leq t_2 \leq T_{(N-N_\beta+M_\beta^2+1)}$. The slope of the trimmed cost function is then proportional to $-\frac{c_{h,2}}{m_2+c_{h,2}+g}N_\beta + M_\beta^2$, and at optimality M_β^2 is equal to $\lceil \frac{c_{h,2}}{m_2+c_{h,2}+g}N_\beta \rceil$. We now have to determine the optimal value of t_2.

Let $f_i^h = (m_2 + c_{h,2} + g)[T_{(h)} - T_{(i)}]^+ + (m_1 + c_{h,2} + g)T_{(i)}$ be the cost realized when $t_2 = T_{(h)}$ and $T = T_{(i)}$, for all i and h. The optimal M_β^2 is the largest integer less than or equal to N_β such that $f_{M_\beta^2}^h \geq f_{N-N_\beta+M_\beta^2}^h$. (Otherwise, we would remove M_β^2 from $S(t_2)$ and add $N - N_\beta + M_\beta^2$ instead.) Plugging the expression of $f_{M_\beta^2}^h$ and $f_{N-N_\beta+M_\beta^2}^h$ yields:

$$-(m_2 - m_1)T_{(M_\beta^2)} + (m_2 + c_{h,2} + g)T_{(h)} \geq +(m_1 + c_{h,2} + g)T_{(N-N_\beta+M_\beta^2)} \quad (4.8.14)$$

Combining the previous results, Equation (4.8.8) follows immediately.

(e) Considering only the scenarios in S_β, we inject $N = N_\beta$ into Equation (4.8.8).

(f) For the planned stock-out solutions to exist, the condition $t_1^* < t_2^*$ or $\frac{m_1+g}{m_1-m_1'} < \frac{c_{h,2}}{m_2+c_{h,2}+g}$ should be satisfied, furthermore, for t_1^* to exist, $g < -m_1'$.

Remark:

When $N \to \infty$, $N_\beta \to N$, therefore expression (4.8.8) becomes:

$$t_2^* = min\left\{ T_{(h)} | T_{(h)} \geq T_{(M_\beta^2)} \right\} \quad (4.8.15)$$

where $M_\beta^2 = \lceil \frac{c_{h,2}}{m_2+c_{h,2}+g}N_\beta \rceil$ and in this case we go back to having the same solution as when the probability distribution of T is known.

Case 2: $m_2 \leq m_1$

Proposition 2:

We can re-write expression (4.8.3) as follows:

$$\begin{aligned}\min_{0\le t_1\le t_2} \quad & -(m_1+g)t_1 - c_{h,2}t_2 \\ + \quad & \frac{1}{N_\beta}\sum_{k=1}^{N_\beta}\Big(-(m_1'-m_1)[t_1-T]^+ - (m_2-m_1)T + (m_2+c_{h,2}+g)[t_2-T]^+ \\ + \quad & (m_2+c_{h,2}+g)T\Big)_k . \end{aligned} \tag{4.8.16}$$

(a) The optimal times t_1 and t_2 in (4.8.3) are the solution of the linear programming problem:

$$\min_{0\le t_1\le t_2} \quad -(m_1+g)t_1 - c_{h,2}t_2 + \frac{1}{N_\beta}\sum_{k=1}^{N}\psi_k^2 + \frac{1}{N_\beta}\sum_{k=1}^{N}\psi_k^1, \tag{4.8.17}$$

$$\tag{4.8.18}$$

$$\begin{aligned}\text{s.t} \quad & \phi + \psi_k^1 + \Big((m_1-m_1')Z_k^1\Big) \ge -(m_2-m_1)T_k, \forall k \\ & \phi + \psi_k^2 + \Big((m_2+c_{h,2}+g)Z_k^2\Big) \ge +(m_2+c_{h,2}+g)T_k, \forall k \qquad (4.8.19) \\ & Z_k^1 + t_1 \ge T_k \forall k, \\ & Z_k^2 + t_2 \ge T_k \forall k, \\ & Z_k^1 \ge 0, Z_k^2 \ge 0, \psi_k \ge 0 \forall k \end{aligned}$$

Moreover, $t_1^* = T_{(j)}$ for some j and $t_2^* = T_{(h)}$ for some h.

(b) Let $M_\beta^1 = \lceil \frac{m_1+g}{m_1-m_1'} N_\beta \rceil$.$t_1^*$ satisfies

$$t_1^* = min\Big\{T_{(j)} | T_{(j)} \ge \Big(\frac{m_2-m_1'}{m_1-m_1'}\Big)T_{(M_\beta^1)} + \Big(\frac{m_1-m_2}{m_1-m_1'}\Big)T_{(N-N_\beta+M_\beta^1)}\Big\} \tag{4.8.20}$$

(c) Let S_β be the set of the N_β worst-case scenarios at optimality, that is $\sum_{k=1}^{N_{1\beta}} L_1(t_1,t_2,T)_{(i)} = \sum_{i\epsilon S_\beta} L_1(t_1,t_2,T_i)$, and let $T_{(j)}^{S_\beta}$ the $j-th$ highest approval date within that set. We have:

$$t_1^* = T_{(M_\beta^1)}^{S_\beta} \tag{4.8.21}$$

where M^1_β is defined in (b).

(d) Let $M^2_\beta = \lceil \frac{c_{h,2}}{m_2+c_{h,2}+g} N_\beta \rceil . t^*_2$ satisfies

$$t^*_2 = min\left\{ T_{(h)} | T_{(h)} \geq T_{(N-N_\beta+M^2_\beta)} \right\} \tag{4.8.22}$$

(e) Let S_β be the set of the N_β worst-case scenarios at optimality, that is $\sum_{k=1}^{N_\beta} L_2(t_1, t_2, T)_{(i)} = \sum_{i\epsilon S_\beta} L_2(t_1, t_2, T_i)$, and let $T^{S_\beta}_{(h)}$ the $h-th$ highest approval date within that set. We have:

$$t^*_2 = T^{S_\beta}_{(M^2_\beta)} \tag{4.8.23}$$

where M^2_β is defined in (d).

(f)If $t^*_1 < t^*_2$, then the optimal strategy may be planned stock-out, else it is a single or dual rollover one.

Proof (a) (a) Let $L_1(t_1, t_2, T) = L_1(t_1, T) + L_2(t_2, T)$ where $L_1(t_1, T) = -(m'_1 - m_1)[t_1 - T]^+ - (m_2 - m_1)T$ and $L_2(t_2, T) = +(m_2 + c_{h,2} + g)[t_2 - T]^+ + (m_2 + c_{h,2} + g)T$. We know that $L_1(t_1, t_2, T)$ is continuous and piecewise linear.

We consider $L_1(t_1, T)$ which is nonincreasing in T, and the k^{th} smallest $[t_1 - T]^+$ at t_1 is equal to $[t_1 - T_{(k)}]^+$.

Applying Theorem 1 to Problem (4.8.16), at optimality, $t^*_2 = T_{(h)}$ for some h because the function to minimize in $L_2(t_2, T)$ is convex piecewise linear with breakpoints in the set $(T_{(i)})$.

Therefore, the worst case scenarios of $L_1(t_1, T)$ and $L_2(t_2, T)$ would give the N_β worst

case scenarios of $L_1(t_1, t_2, T)$. and Problem (4.8.16) is equivalent to:

$$\begin{aligned} Min \quad & \phi_1 + \phi_2 + \frac{1}{N_\beta} \sum_{k=1}^{N} \psi_k^1 + \frac{1}{N_\beta} \sum_{k=1}^{N} \psi_k^2 \\ \text{s.t} \quad & \phi_1 + \psi_k^1 \geq -(m_2 - m_1)T_k, \forall k \\ & \phi_2 + \psi_k^2 \geq +(m_2 + c_{h,2} + g)T_k, \forall k \\ & \psi_k^1, \psi_k^2 \geq 0 \forall k, \\ & t \epsilon \varsigma. \end{aligned} \tag{4.8.24}$$

Problem (4.8.24) is a convex problem since $L_1(t_1, T)$ and $L_2(t_2, T)$ are decreasing in t_1 and increasing t_2 respectively and a linear programming problem since $L_1(t_1, T)$ and $L_2(t_2, T)$are piecewise linear and ς is a polyhedron.

It follows immediately that Eq. (4.8.24) is a convex problem since $L_1(t_1, T)$ and $L_2(t_2, T)$ are convex in t_1 and t_2. Moreover, since $L_1(t_1, T)$ and $L_2(t_2, T)$ are (convex) piecewise linear in t_1 and t_2 and ς is a polyhedron, then Eq. (4.8.24) is a linear programming problem.

As the cost functions in our product rollover problem are piecewise linear with linear ordering constraints, Theorem 1 will allow us to derive tractable, linear programming formulations of the data-driven models.

(b) The slope of the cost function with respect to t_1 is : $-(m_1 + g) - \frac{1}{N_\beta}(m_1' - m_1).\{i \epsilon S(t_1), T_i \leq t_1\}$ where $S(t_1)$ is the set of indices of the N_β smallest $(m_1 - m_1')[t_1 - T]^+$ at t_1 given. It is easy to show that for any $i \epsilon S(t_1)$ and for any k such that $T_k \leq T_i \leq t_1$, $k \epsilon S(t_1)$ as well. Similarly, for any $i \epsilon S(t_1)$ and any k such that $T_k \geq T_i \geq t_1$, $k \epsilon S(t_1)$. Hence, $S(t_1)$ consists of the indices of $T_{(1)}, ..., T_{(M_\beta^1)}$ and $T_{(N - N_\beta + M_\beta^1 + 1)}, ...T_N$ for some $0 \leq M_\beta^1 \leq N$, with $T_{M_\beta^1} \leq t_1 \leq T_{(N - N_\beta + M_\beta^1 + 1)}$. The slope of the trimmed cost function is then proportional to $-\frac{m_1 + g}{m_1 - m_1'} N_\beta + M_\beta^1$, and at optimality M_β^1 is equal to $\lceil \frac{m_1 + g}{m_1 - m_1'} N_\beta \rceil$. We now have to determine the optimal value of t_1.

Let $f_i^j = (m_1 - m_1')[T_{(j)} - T_{(i)}]^+ - (m_2 - m_1)T$ be the cost realized when $t_1 = T_{(j)}$ and $T = T_{(i)}$, for all i and j. The optimal M_β^1 is the largest integer less than or equal to N_β such that $f^j_{M_\beta^1} \geq f^j_{N-N_\beta+M_\beta^1}$. (Otherwise, we would remove M_β^1 from $S(t_1)$ and add $N - N_\beta + M_\beta^1$ instead.) Plugging the expression of $f^j_{M_\beta^1}$ and $f^j_{N-N_\beta+M_\beta^1}$ yields:

$$-(m_2 - m_1')T_{(M_\beta^1)} + (m_1 - m_1')T_{(j)} \geq -(m_2 - m_1)T_{(N-N_\beta+M_\beta^1)} \quad (4.8.25)$$

Combining the previous results, Equation (4.8.20) follows immediately.

(c) Considering only the scenarios in $S_{1\beta}$, we inject $N = N_\beta$ into Equation (4.8.20).

(d) The slope of the cost function with respect to t_2 is : $-c_{h,2} + \frac{1}{N_\beta}(m_2 + c_{h,2} + g).\{i\epsilon S(t_2), T_i \leq t_2\}$ where $S(t_2)$ is the set of indices of the N_β smallest $(m_2 + c_{h,2} + g)[t_2 - T]^+ + (m_1 + c_{h,2} + g)T$ at t_2 given. It is easy to show that for any $i\epsilon S(t_2)$ and for any k such that $T_k \leq T_i \leq t_2$, $k\epsilon S(t_2)$ as well. Similarly, for any $i\epsilon S(t_2)$ and any k such that $T_k \geq T_i \geq t_2$, $k\epsilon S(t_2)$. Hence, $S(t_2)$ consists of the indices of $T_{(1)}, ..., T_{(M_\beta^2)}$ and $T_{(N-N_\beta+M_\beta^2+1)}, ...T_N$ for some $0 \leq M_\beta^2 \leq N$, with $T_{M_\beta^2} \leq t_2 \leq T_{(N-N_\beta+M_\beta^2+1)}$. The slope of the trimmed cost function is then proportional to $-\frac{c_{h,2}}{m_2+c_{h,2}+g}N_\beta + M_\beta^2$, and at optimality M_β^2 is equal to $\lceil \frac{c_{h,2}}{m_2+c_{h,2}+g}N_\beta \rceil$. We now have to determine the optimal value of t_2.

Let $f_i^h = (m_2 + c_{h,2} + g)[T_{(h)} - T_{(i)}]^+ + (m_2 + c_{h,2} + g)T_{(i)}$ be the cost realized when $t_2 = T_{(h)}$ and $T = T_{(i)}$, for all i and h. The optimal M_β^2 is the largest integer less than or equal to N_β such that $f^h_{M_\beta^2} \geq f^h_{N-N_\beta+M_\beta^2}$. (Otherwise, we would remove M_β^2 from $S(t_2)$ and add $N - N_\beta + M_\beta^2$ instead.) Plugging the expression of $f^h_{M_\beta^2}$ and $f^h_{N-N_\beta+M_\beta^2}$ yields:

$$(m_2 + c_{h,2} + g)T_{(h)} \geq (m_2 + c_{h,2} + g)T_{(N-N_\beta+M_\beta^2)} \quad (4.8.26)$$

Combining the previous results, Equation (4.8.22) follows immediately.

(e) Considering only the scenarios in S_β, we inject $N = N_\beta$ into Equation (4.8.22).

(f) For the planned stock-out solutions to exist, the condition $t_1^* < t_2^*$ or $\frac{m_1+g}{m_1-m_1'} < \frac{c_{h,2}}{m_2+c_{h,2}+g}$ should be satisfied, furthermore, for t_1^* to exist, $g < -m_1'$.

Remark:

When $N \to \infty$, $N_\beta \to N$, therefore expression (4.8.22) becomes:

$$t_2^* = min\left\{T_{(h)}|T_{(h)} \geq T_{(M_\beta^2)}\right\} \tag{4.8.27}$$

where $M_\beta^2 = \lceil \frac{c_{h,2}}{m_2+c_{h,2}+g} N_\beta \rceil$ and in this case we go back to having the same solution as when the probability distribution of T is known.

Appendix B

For a dual product rollover strategy the net cost is given by:

$$\begin{aligned} L_2(t_1,t_2,T) &= (m_1+g)[T-t_1]^+ + s_1[t_1-T]^+ \\ &\quad +c_{h,2}[T-t_2]^+ - (m_2-m_1'-s_1)[t_2-T]^+. \end{aligned} \tag{4.8.28}$$

We can rewrite (4.8.28) as follows:

$$\begin{aligned} L_2(t_1,t_2,T) &= -(m_1+g)t_1 - c_{h,2}t_2 + (m_1+g+s_1)[t_1-T]^+ \\ &+ (m_2-m_1'-s_1+c_{h,2})[t_2-T]^+ + (m_1+c_{h,2}+g)T \end{aligned} \tag{4.8.29}$$

Our goal is to minimize the trimmed mean of the cost:

$$\begin{aligned} \min_{0 \leq t_2 \leq t_1} \quad & -(m_1+g)t_1 - c_{h,2}t_2 + \frac{1}{N_\beta}\sum_{k=1}^{N_\beta}\Big(+(m_1+g+s_1)[t_1-T]^+ \\ + \quad & (m_2-m_1'-s_1+c_{h,2})[t_2-T]^+ + (m_1+c_{h,2}+g)T\Big)_k . \end{aligned} \tag{4.8.30}$$

where for any $y \epsilon R^n$, $y_{(k)}$ is the k^{th} smallest component of y.

We distinguish two major cases:

Case A: $m_2 - m_1' - s_1 + c_{h,2} > 0$

and **Case B:** $m_2 - m_1' - s_1 + c_{h,2} < 0$.

Case A: $m_2 - m_1' - s_1 + c_{h,2} > 0$

For this case,problem (4.8.30) is convex with respect to t_2 (and t_1) and we distinguish two subcases: $c_{h,2} \geq s_1$ and $c_{h,2} < s_1$. We know that $L_2(t_1, t_2, T)$ is jointly convex with respect to t_1 and t_2 when $m_2 - m_1' - s_1 + c_{h,2}$, therefore we are able to apply Theorem 1 and solve for tractable solutions by distinguishing two cases:

- $c_{h,2} \geq s_1$ where $L_2(t_1, t_2, T)$ is strictly decreasing with respect to $T_k < t_2$ and strictly increasing with respect to $T_k > t_2$
- $s_1 \geq c_{h,2}$ where $L_2(t_1, t_2, T)$ is strictly decreasing with respect to $T_k < t_1$ and strictly increasing with respect to $T_k > t_1$.

Case 1: $c_{h,2} \geq s_1$

Proposition 3:

(a) The optimal times t_1 and t_2 in (4.8.30) are the solution of the linear programming problem:

$$\begin{aligned}
\min_{0\le t_2\le t_1} \quad & -(m_1+g)t_1 - c_{h,2}t_2 + \frac{1}{N_\beta}\sum_{k=1}^{N}\psi_k^2 + \frac{1}{N_\beta}\sum_{k=1}^{N}\psi_k^1, && (4.8.31)\\
\text{s.t} \quad & \phi + \psi_k^1 + \left((m_1+g+s_1)Z_k^1\right) \ge (m_1+g+s_1)T_k, \forall k \\
& \phi + \psi_k^2 + \left((m_2-m_1'-s_1+c_{h,2})Z_k^2\right) \ge (c_{h,2}-s_1)T_k, \forall k && (4.8.32)\\
& Z_k^1 + t_1 \ge T_k \forall k, \\
& Z_k^2 + t_2 \ge T_k \forall k, \\
& Z_k^1 \ge 0, Z_k^2 \ge 0, \psi_k \ge 0 \forall k
\end{aligned}$$

Moreover, $t_1^* = T_{(j)}$ for some j and $t_2^* = T_{(h)}$ for some h.

(b) Let $M_\beta^1 = \lceil \frac{m_1+g}{m_1+g+s_1} N_\beta \rceil$.$t_1^*$ satisfies

$$t_1^* = min\left\{ T_{(j)} | T_{(j)} \ge T_{(N-N_\beta+M_\beta^1)} \right\} \quad (4.8.33)$$

(c) Let S_β be the set of the N_β worst-case scenarios at optimality, that is $\sum_{k=1}^{N_{1\beta}} L_1(t_1,t_2,T)_{(i)} = \sum_{i\epsilon S_\beta} L_1(t_1,t_2,T_i)$, and let $T_{(j)}^{S_\beta}$ the $j-th$ highest approval date within that set. We have:

$$t_1^* = T_{(M_\beta^1)}^{S_\beta} \quad (4.8.34)$$

where M_β^1 is defined in (b).

(d) Let $M_\beta^2 = \lceil \frac{c_{h,2}}{m_2-m_1'-s_1+c_{h,2}} N_\beta \rceil$. t_2^* satisfies

$$\begin{aligned}
t_2^* &= min\left\{ T_{(h)} | T_{(h)} \ge \left(\frac{m_2-m_1'}{m_2-m_1'-s_1+c_{h,2}}\right) T_{(M_\beta^2)} \right. \\
&+ \left. \left(\frac{c_{h,2}-s_1}{m_2-m_1'-s_1+c_{h,2}}\right) T_{(N-N_\beta+M_\beta^2)} \right\} \quad (4.8.35)
\end{aligned}$$

(e) Let S_β be the set of the N_β worst-case scenarios at optimality, that is $\sum_{k=1}^{N_\beta} L_2(t_1,t_2,T)_{(i)} = \sum_{i\epsilon S_\beta} L_2(t_1,t_2,T_i)$, and let $T_{(h)}^{S_\beta}$ the $h-th$ highest approval date within that set. We have:

$$t_2^* = T_{(M_\beta^2)}^{S_\beta} \quad (4.8.36)$$

where M_β^2 is defined in (d).

(f)If $t_1^* > t_2^*$, then the optimal strategy may be dual, else it is a single or dual rollover one.

Proof

(a) Let $L_2(t_1, t_2, T) = L_1(t_1, T) + L_2(t_2, T)$ where $L_1(t_1, T) = (m_1 + g + s_1)[t_1 - T]^+ + (m_1 + g + s_1)T$ and $L_2(t_2, T) = +(m_2 - m_1' - s_1 + c_{h,2})[t_2 - T]^+ + (c_{h,2} - s_1)T$. We know that $L(t_1, t_2, T)$ is continuous and piecewise linear.

Applying Theorem 1 to Problem (4.8.30), at optimality, $t_1^* = T_{(j)}$ for some j because the function to minimize in $L_1(t_1, T)$ is convex piecewise linear with breakpoints in the set $(T_{(i)})$.

Applying Theorem 1 to Problem (4.8.30), at optimality, $t_2^* = T_{(h)}$ for some h because the function to minimize in $L_2(t_2, T)$ is convex piecewise linear with breakpoints in the set $(T_{(i)})$.

Therefore, the worst case scenarios of $L_1(t_1, T)$ and $L_2(t_2, T)$ would give the N_β worst

case scenarios of $L_2(t_1, t_2, T)$ and Problem (4.8.28) is equivalent to:

$$\begin{aligned} Min \quad & \phi_1 + \phi_2 + \frac{1}{N_\beta}\sum_{k=1}^{N}\psi_k^1 + \frac{1}{N_\beta}\sum_{k=1}^{N}\psi_k^2 \\ \text{s.t} \quad & \phi_1 + \psi_k^1 \geq +(m_1 + g + s_1)T_k, \forall k \\ & \phi_2 + \psi_k^2 \geq +(c_{h,2} - s_1)T_k, \forall k \\ & \psi_k^1, \psi_k^2 \geq 0 \forall k, \\ & t \epsilon \varsigma. \end{aligned} \tag{4.8.37}$$

Problem (4.8.37) is a convex problem since $L_1(t_1, T)$ and $L_2(t_2, T)$ are increasing in t_1 and increasing t_2 respectively and a linear programming problem since $L_1(t_1, T)$ and $L_2(t_2, T)$are piecewise linear and ς is a polyhedron.

It follows immediately that Eq. (4.8.37) is a convex problem since $L_1(t_1, T)$ and $L_2(t_2, T)$ are convex in t_1 and t_2. Moreover, since $L_1(t_1, T)$ and $L_2(t_2, T)$ are (convex) piecewise linear in t_1 and t_2 and ς is a polyhedron, then Eq. (4.8.37) is a linear programming problem.

As the cost functions in our product rollover problem are piecewise linear with linear ordering constraints, Theorem 1 will allow us to derive tractable, linear programming formulations of the data-driven models.

(b) The slope of the cost function with respect to t_1 is : $-(m_1 + g) + \frac{1}{N_\beta}(m_1 + g + s_1).\{i\epsilon S(t_1), T_i \leq t_1\}$ where $S(t_1)$ is the set of indices of the N_β smallest $(m_1 + g + s_1)[t_1 - T]^+ + (m_1 + g + s_1)T$ at t_1 given. It is easy to show that for any $i\epsilon S(t_1)$ and for any k such that $T_k \leq T_i \leq t_1$, $k\epsilon S(t_1)$ as well. Similarly, for any $i\epsilon S(t_1)$ and any k such that $T_k \geq T_i \geq t_1$, $k\epsilon S(t_1)$. Hence, $S(t_1)$ consists of the indices of $T_{(1)}, ..., T_{(M_\beta^1)}$ and $T_{(N-N_\beta+M_\beta^1+1)}, ...T_N$ for some $0 \leq M_\beta^1 \leq N$, with $T_{M_\beta^1} \leq t_1 \leq T_{(N-N_\beta+M_\beta^1+1)}$. The slope of the trimmed cost function is then proportional to $-\frac{m_1+g}{m_1+g+s_1}N_\beta + M_\beta^1$, and at optimality M_β^1 is equal to $\lceil \frac{m_1+g}{m_1+g+s_1}N_\beta \rceil$. We now have to determine the optimal value of t_1.

Let $f_i^j = (m_1 + g + s_1)[T_{(j)} - T_{(i)}]^+ + (m_1 + g + s_1)T_{(i)}$ be the cost realized when $t_1 = T_{(j)}$ and $T = T_{(i)}$, for all i and j. The optimal M_β^1 is the largest integer less than or equal to N_β such that $f_{M_\beta^1}^j \geq f_{N-N_\beta+M_\beta^1}^j$. (Otherwise, we would remove M_β^1 from $S(t_1)$ and add $N - N_\beta + M_\beta^1$ instead.) Plugging the expression of $f_{M_\beta^1}^j$ and $f_{N-N_\beta+M_\beta^1}^j$ yields:

$$(m_1 + g + s_1)T_{(j)} \geq (m_1 + g + s_1)T_{(N-N_\beta^1+M_\beta^1)} \tag{4.8.38}$$

Combining the previous results, Equation (4.8.33) follows immediately.

(c) Considering only the scenarios in $S_{1\beta}$, we inject $N = N_\beta$ into Equation (4.8.33).

(d) The slope of the cost function with respect to t_2 is : $-c_{h,2} + \frac{1}{N_\beta}(m_2 - m_1' - s_1 + c_{h,2}).\{i\epsilon S(t_2), T_i \leq t_2\}$ where $S(t_2)$ is the set of indices of the N_β smallest $+(m_2 - m_1' - s_1 + c_{h,2})[t_2 - T]^+ + (c_{h,2} - s_1)T_i$ at t_2 given. It is easy to show that for any $i\epsilon S(t_2)$ and for any k such that $T_k \leq T_i \leq t_2$, $k\epsilon S(t_2)$ as well. Similarly, for any $i\epsilon S(t_2)$ and any k such that $T_k \geq T_i \geq t_2$, $k\epsilon S(t_2)$. Hence, $S(t_2)$ consists of the indices of $T_{(1)}, ..., T_{(M_\beta^2)}$ and $T_{(N-N_\beta+M_\beta^2+1)}, ...T_N$ for some $0 \leq M_\beta^2 \leq N$, with $T_{M_\beta^2} \leq t_2 \leq T_{(N-N_\beta+M_\beta^2+1)}$. The slope of the trimmed cost function is then proportional to $-\frac{c_{h,2}}{m_2-m_1'-s_1+c_{h,2}}N_\beta + M_\beta^2$, and at optimality M_β^2 is equal to $\lceil \frac{c_{h,2}}{m_2-m_1'-s_1+c_{h,2}}N_\beta \rceil$. We now have to determine the optimal value of t_2.

Let $f_i^h = (m_2 - m_1' - s_1 + c_{h,2})[T_{(h)} - T_{(i)}]^+ + (c_{h,2} - s_1)T_{(i)}$ be the cost realized when $t_2 = T_{(h)}$ and $T = T_{(i)}$, for all i and h. The optimal M_β^2 is the largest integer less than or equal to N_β such that $f_{M_\beta^2}^h \geq f_{N-N_\beta+M_\beta^2}^h$. (Otherwise, we would remove M_β^2 from $S(t_2)$ and add $N - N_\beta + M_\beta^2$ instead.) Plugging the expression of $f_{M_\beta^2}^h$ and $f_{N-N_\beta+M_\beta^2}^h$ yields:

$$-(m_2 - m_1')T_{(M_\beta^2)} + (m_2 - m_1' - s_1 + c_{h,2})T_{(h)} \geq (c_{h,2} - s_1)T_{(N-N_\beta+M_\beta^2)} \tag{4.8.39}$$

Combining the previous results, Equation (4.8.35) follows immediately.

(e) Considering only the scenarios in S_β, we inject $N = N_\beta$ into Equation (4.8.35).

Case 2: $c_{h,2} < s_1$

Proposition 4:

(a) The optimal times t_1 and t_2 in (4.8.30) are the solution of the linear programming problem:

$$\min_{0 \le t_2 \le t_1} \quad -(m_1+g)t_1 - c_{h,2}t_2 + \frac{1}{N_\beta}\sum_{k=1}^{N}\psi_k^2 + \frac{1}{N_\beta}\sum_{k=1}^{N}\psi_k^1, \tag{4.8.40}$$

$$\begin{aligned}
\text{s.t} \quad & \phi + \psi_k^1 + \left((m_1+g+s_1)Z_k^1\right) \ge +(m_1+g+c_{h,2})T_k, \forall k \\
& \phi + \psi_k^2 + \left((m_2 - m_1' - s_1 + c_{h,2})Z_k^2\right) \ge 0T_k, \forall k \qquad (4.8.41)\\
& Z_k^1 + t_1 \ge T_k \forall k, \\
& Z_k^2 + t_2 \ge T_k \forall k, \\
& Z_k^1 \ge 0, Z_k^2 \ge 0, \psi_k \ge 0 \forall k
\end{aligned}$$

Moreover, $t_1^* = T_{(j)}$ for some j and $t_2^* = T_{(h)}$ for some h.

(b) Let $M_\beta^1 = \lceil \frac{m_1+g}{m_1+g+s_1} N_\beta \rceil . t_1^*$ satisfies

$$t_1^* = min\left\{ T_{(j)} | T_{(j)} \ge \left(\frac{m_1+g+c_{h,2}}{m_1+g+s_1}\right) T_{(N-N_\beta+M_\beta^1)} + \left(\frac{s_1 - c_{h,2}}{m_1+g+s_1}\right) T_{(M_\beta^1)} \right\} \tag{4.8.42}$$

(c) Let S_β be the set of the N_β worst-case scenarios at optimality, that is $\sum_{k=1}^{N_{1\beta}} L_1(t_1,t_2,T)_{(i)} = \sum_{i \epsilon S_\beta} L_1(t_1,t_2,T_i)$, and let $T_{(j)}^{S_\beta}$ the $j-th$ highest approval date within that set. We have:

$$t_1^* = T_{(M_\beta^1)}^{S_\beta} \tag{4.8.43}$$

where M_β^1 is defined in (b).

(d) Let $M_\beta^2 = \lceil \frac{c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}} N_\beta \rceil . t_2^*$ satisfies

$$t_2^* = min\left\{ T_{(h)} | T_{(h)} \geq T_{(M_\beta^2)} \right\} \tag{4.8.44}$$

(e) Let S_β be the set of the N_β worst-case scenarios at optimality, that is $\sum_{k=1}^{N_\beta} L_2(t_1, t_2, T)_{(i)} = \sum_{i \epsilon S_\beta} L_2(t_1, t_2, T_i)$, and let $T_{(h)}^{S_\beta}$ the $h - th$ highest approval date within that set. We have:

$$t_2^* = T_{(M_\beta^2)}^{S_\beta} \tag{4.8.45}$$

where M_β^2 is defined in (d).

(f)If $t_1^* > t_2^*$, then the optimal strategy may be dual, else it is a single or dual rollover one.

Proof

(a) Let $L_2(t_1, t_2, T) = L_1(t_1, T) + L_2(t_2, T)$ where $L_1(t_1, T) = (m_1 + g + s_1)[t_1 - T]^+ + (m_1 + g + c_{h,2})T$ and $L_2(t_2, T) = +(m_2 - m_1' - s_1 + c_{h,2})[t_2 - T]^+$. We know that $L(t_1, t_2, T)$ is continuous and piecewise linear.

Applying Theorem 1 to Problem (4.8.30), at optimality, $t_1^* = T_{(j)}$ for some j because the function to minimize in $L_1(t_1, T)$ is convex piecewise linear with breakpoints in the set $(T_{(i)})$.

Applying Theorem 1 to Problem (4.8.30), at optimality, $t_2^* = T_{(h)}$ for some h because the function to minimize in $L_2(t_2, T)$ is convex piecewise linear with breakpoints in the set $(T_{(i)})$.

Therefore, the worst case scenarios of $L_1(t_1, T)$ and $L_2(t_2, T)$ would give the N_β worst

case scenarios of $L_2(t_1, t_2, T)$ and Problem (4.8.30) is equivalent to:

$$\begin{aligned}
Max \quad & \phi_1 + \phi_2 + \frac{1}{N_\beta}\sum_{k=1}^{N}\psi_k^1 + \frac{1}{N_\beta}\sum_{k=1}^{N}\psi_k^2 \\
\text{s.t} \quad & \phi_1 + \psi_k^1 \geq +(m_1 + g + c_{h,2})T_k, \forall k \\
& \phi_2 + \psi_k^2 \geq 0, \forall k \\
& \psi_k^1, \psi_k^2 \geq 0 \forall k, \\
& t \epsilon \varsigma.
\end{aligned} \tag{4.8.46}$$

Problem (4.8.46) is a convex problem since $L_1(t_1, T)$ and $L_2(t_2, T)$ are increasing in t_1 and increasing t_2 respectively and a linear programming problem since $L_1(t_1, T)$ and $L_2(t_2, T)$are piecewise linear and ς is a polyhedron.

It follows immediately that Eq. (4.8.46) is a convex problem since $L_1(t_1, T)$ and $L_2(t_2, T)$ are convex in t_1 and t_2. Moreover, since $L_1(t_1, T)$ and $L_2(t_2, T)$ are (convex) piecewise linear in t_1 and t_2 and ς is a polyhedron, then Eq. (4.8.46) is a linear programming problem.

As the cost functions in our product rollover problem are piecewise linear with linear ordering constraints, Theorem 1 will allow us to derive tractable, linear programming formulations of the data-driven models.

(b) The slope of the cost function with respect to t_1 is : $-(m_1 + g) + \frac{1}{N_\beta}(m_1 + g + s_1).\{i \epsilon S(t_1), T_i \leq t_1\}$ where $S(t_1)$ is the set of indices of the N_β smallest $(m_1 + g + s_1)[t_1 - T]^+ + (m_1 + g + s_1)T$ at t_1 given. It is easy to show that for any $i \epsilon S(t_1)$ and for any k such that $T_k \leq T_i \leq t_1$, $k \epsilon S(t_1)$ as well. Similarly, for any $i \epsilon S(t_1)$ and any k such that $T_k \geq T_i \geq t_1$, $k \epsilon S(t_1)$. Hence, $S(t_1)$ consists of the indices of $T_{(1)}, ..., T_{(M_\beta^1)}$ and $T_{(N - N_\beta + M_\beta^1 + 1)}, ...T_N$ for some $0 \leq M_\beta^1 \leq N$, with $T_{M_\beta^1} \leq t_1 \leq T_{(N - N_\beta + M_\beta^1 + 1)}$. The slope of the trimmed cost function is then proportional to $-\frac{m_1 + g}{m_1 + g + s_1}N_\beta + M_\beta^1$, and at optimality M_β^1 is equal to $\lceil \frac{m_1 + g}{m_1 + g + s_1}N_\beta \rceil$. We now have to determine the optimal value of t_1.

Let $f_i^j = (m_1 + g + s_1)[T_{(j)} - T_{(i)}]^+ + (m_1 + g + c_{h,2})T_{(i)}$ be the cost realized when $t_1 = T_{(j)}$ and $T = T_{(i)}$, for all i and j. The optimal M_β^1 is the largest integer less than or equal to N_β such that $f_{M_\beta^1}^j \geq f_{N-N_\beta+M_\beta^1}^j$. (Otherwise, we would remove M_β^1 from $S(t_1)$ and add $N - N_\beta + M_\beta^1$ instead.) Plugging the expression of $f_{M_\beta^1}^j$ and $f_{N-N_\beta+M_\beta^1}^j$ yields:

$$-(s_1 - c_{h,2})T_{(M_\beta^1)} + (m_1 + g + s_1)T_{(j)} \geq (m_1 + g + c_{h,2})T_{(N-N_\beta^1+M_\beta^1)} \tag{4.8.47}$$

Combining the previous results, Equation (4.8.42) follows immediately.

(c) Considering only the scenarios in $S_{1\beta}$, we inject $N = N_\beta$ into Equation (4.8.42).

(d) The slope of the cost function with respect to t_2 is : $-c_{h,2} + \frac{1}{N_\beta}(m_2 - m_1' - s_1 + c_{h,2}).\{i\epsilon S(t_2), T_i \leq t_2\}$ where $S(t_2)$ is the set of indices of the N_β smallest $+(m_2 - m_1' - s_1 + c_{h,2})[t_2 - T]^+$ at t_2 given. It is easy to show that for any $i\epsilon S(t_2)$ and for any k such that $T_k \leq T_i \leq t_2$, $k\epsilon S(t_2)$ as well. Similarly, for any $i\epsilon S(t_2)$ and any k such that $T_k \geq T_i \geq t_2$, $k\epsilon S(t_2)$. Hence, $S(t_2)$ consists of the indices of $T_{(1)}, ..., T_{(M_\beta^2)}$ and $T_{(N-N_\beta+M_\beta^2+1)}, ...T_N$ for some $0 \leq M_\beta^2 \leq N$, with $T_{M_\beta^2} \leq t_2 \leq T_{(N-N_\beta+M_\beta^2+1)}$. The slope of the trimmed cost function is then proportional to $-\frac{c_{h,2}}{m_2-m_1'-s_1+c_{h,2}}N_\beta + M_\beta^2$, and at optimality M_β^2 is equal to $\lceil \frac{c_{h,2}}{m_2-m_1'-s_1+c_{h,2}}N_\beta \rceil$. We now have to determine the optimal value of t_2.

Let $f_i^h = (m_2 - m_1' - s_1 + c_{h,2})[T_{(h)} - T_{(i)}]^+ + (c_{h,2} - s_1)T_{(i)}$ be the cost realized when $t_2 = T_{(h)}$ and $T = T_{(i)}$, for all i and h. The optimal M_β^2 is the largest integer less than or equal to N_β such that $f_{M_\beta^2}^h \geq f_{N-N_\beta+M_\beta^2}^h$. (Otherwise, we would remove M_β^2 from $S(t_2)$ and add $N - N_\beta + M_\beta^2$ instead.) Plugging the expression of $f_{M_\beta^2}^h$ and $f_{N-N_\beta+M_\beta^2}^h$ yields:

$$(m_2 - m_1' - s_1 + c_{h,2})T_{(h)} \geq (m_2 - m_1' - s_1 + c_{h,2})T_{M_\beta^2} \tag{4.8.48}$$

Combining the previous results, Equation (4.8.44) follows immediately.

(e) Considering only the scenarios in S_β, we inject $N = N_\beta$ into Equation (4.8.44).

(f) For the dual rollover solutions to exist, the conditions $t_2^* < t_1^*$ or $s_1(m_1 + g + c_{h,2}) < (m_1 + g)(m_2 - m_1')$, $m_2 - m_1' - s_1 > 0$, and $m_2 - m_1' - s_1 + c_{h,2} > 0$ should be satisfied.

Case B: $m_2 - m_1' - s_1 + c_{h,2} < 0$

For this case,problem (4.8.30) is concave with respect to t_1, but strictly decreasing with respect tot_2, and therefore disregarding the distribution of T, we know that the optimal solution should always be the greatest possible value of t_2, i.e., $t_2 = t_1$, and by this we go back to the single rollover strategy.

APPENDIX C

For a single rollover strategy the net cost is given by:

$$L_b(t_b, T) = (m_1 + g + c_{h,2})[T - t_b]^+ - (m_1' - m_2)[t_b - T]^+. \tag{4.8.49}$$

We can rewrite (4.8.49) as follows:

$$\begin{aligned} L_b(t_b, T) &= -(m_1 + g + c_{h,2})t_b - (m_1' - m_1 - m_2 - c_{h,2} - g)[t_b - T]^+ \\ &+ (m_1 + c_{h,2} + g)T. \end{aligned} \tag{4.8.50}$$

Our goal is to minimize the trimmed mean of the cost:

$$\begin{aligned} \min_{0 \le t_b} -(m_1 + g + c_{h,2})t_b &+ \frac{1}{N_\beta} \sum_{k=1}^{N_\beta} \Bigg((m_1 - m_1' + m_2 + c_{h,2} + g)[t_b - T]^+ \\ &+ (m_1 + g + c_{h,2})T \Bigg)_k . \end{aligned} \tag{4.8.51}$$

where for any $y \epsilon R^n$, $y_{(k)}$ is the k^{th} smallest component of y.

Proposition 5:

(a) The optimal time t_b in (4.8.51) are the solution of the linear programming problem:

$$\min_{0\leq t_b\leq t_2} \quad -(m_1+g+c_{h,2})t_b+\frac{1}{N_\beta}\sum_{k=1}^{N}\psi_k, \tag{4.8.52}$$

$$\text{s.t} \quad \phi+\psi_k+\Big((m_1-m_1'+m_2+c_{h,2}+g)Z_k\Big)\geq +(m_1+g+c_{h,2})T_k, \forall k$$

$$Z_k+t_b\geq T_k\forall k,$$

$$Z_k\geq 0, \psi_k\geq 0, \forall k$$

$$0\leq t_b. \tag{4.8.53}$$

Moreover, $t_b^*=T_{(j)}$ for some j.

(b) Let $M_\beta=\lceil\frac{m_1+g+c_{h,2}}{m_1-m_1'+m_2+c_{h,2}+g}N_\beta\rceil$.$t_b^*$ satisfies

$$\begin{aligned} t_b^* &= min\Bigg\{T_{(j)}|T_{(j)}\geq\left(\frac{m_2-m_1'}{m_1-m_1'+m_2+c_{h,2}+g}\right)T_{(M_\beta)} \\ &+ \left(\frac{m_1+g+c_{h,2}}{m_1-m_1'+m_2+c_{h,2}+g}\right)T_{(N-N_\beta+M_\beta)}\Bigg\} \end{aligned} \tag{4.8.54}$$

(c) Let S_β be the set of the N_β worst-case scenarios at optimality, that is $\sum_{k=1}^{N_{1\beta}} L_b(t_b,T)_{(i)}=\sum_{i\epsilon S_\beta} L_b(t_b,T_i)$, and let $T_{(j)}^{S_\beta}$ the $j-th$ highest approval date within that set. We have:

$$t_b^*=T_{(M_\beta)}^{S_\beta} \tag{4.8.55}$$

where M_β is defined in (b).

Proof (a) This follows from applying Theorem 1 to Problem (4.8.51). At optimality, $t_b^*=T_{(j)}$ for some j because the function to minimize in (4.8.51) is convex piecewise linear with breakpoints in the set $(T_{(i)})$.

(b) The slope of the cost function with respect to t_b is : $-(m_1+g+c_{h,2})+\frac{1}{N_\beta}(m_1-$

$m_1' + m_2 + c_{h,2} + g).\{i\epsilon S(t_b), T_i \leq t_b\}$ where $S(t_b)$ is the set of indices of the N_β smallest $(m_1 - m_1' + m_2 + c_{h,2} + g)[t_b - T]^+ + (m_1 + g + c_{h,2})T_i$ at t_b given. It is easy to show that for any $i\epsilon S(t_b)$ and for any k such that $T_k \leq T_i \leq t_b$, $k\epsilon S(t_b)$ as well. Similarly, for any $i\epsilon S(t_b)$ and any k such that $T_k \geq T_i \geq t_b$, $k\epsilon S(t_b)$. Hence, $S(t_b)$ consists of the indices of $T_{(1)}, ..., T_{(M_\beta)}$ and $T_{(N-N_\beta+M_\beta+1)}, ...T_N$ for some $0 \leq M_\beta \leq N$, with $T_{M_\beta} \leq t_b \leq T_{(N-N_\beta+M_\beta+1)}$. The slope of the trimmed cost function is then proportional to $-\frac{m_1+g+c_{h,2}}{m_1-m_1'+m_2+c_{h,2}+g}N_\beta + M_\beta$, and at optimality M_β is equal to $\lceil \frac{m_1+g+c_{h,2}}{m_1-m_1'+m_2+c_{h,2}+g}N_\beta \rceil$. We now have to determine the optimal value of t_b.

Let $f_i^j = (m_1 - m_1' + m_2 + c_{h,2} + g)[T_{(j)} - T_{(i)}]^+ + (m_1 + g + c_{h,2})T_{(i)}$ be the cost realized when $t_b = T_{(j)}$ and $T = T_{(i)}$, for all i and j. The optimal M_β is the largest integer less than or equal to N_β such that $f_{M_\beta}^j \geq f_{N-N_\beta+M_\beta}^j$. (Otherwise, we would remove M_β from $S(t_b)$ and add $N - N_\beta + M_\beta$ instead.) Plugging the expression of $f_{M_\beta}^j$ and $f_{N-N_\beta+M_\beta}^j$ yields:

$$
\begin{aligned}
-(m_2 - m_1')T_{(M_\beta)} + (m_1 - m_1' \quad + \quad & m_2 + c_{h,2} + g)T_{(j)} \geq \\
& +(m_1 + g + c_{h,2})T_{(N-N_\beta+M_\beta)} \qquad (4.8.56)
\end{aligned}
$$

Combining the previous results, Equation (4.8.54) follows immediately.

Remark:

When $N \to \infty$, $N_\beta \to N$, therefore expression (4.8.54) becomes:

$$
t_b^* = min\left\{ T_{(j)} | T_{(j)} \geq T_{(M_\beta)} \right\} \qquad (4.8.57)
$$

where $M_\beta = \lceil \frac{m_1+g+c_{h,2}}{m_1-m_1'+m_2+c_{h,2}+g}N_\beta \rceil$ and in this case we go back to having the same solution as when the probability distribution of T is known.

(c) Considering only the scenarios in $S_{1\beta}$, we inject $N = N_\beta$ into Equation (4.8.54).

Appendix D

Since we have two variables, we need to calculate the worst bound for each variable, then take the maximum of the bounds to obtain the worst bound required. The expected net cost function was previously defined in our first work and proved to be convex is given by:

$$\begin{aligned} l_1(t_1,t_2) &= (m_1+g)E[T-t_1]^+ - (g+m_1')E[t_1-T]^+ \\ &+ c_{h,2}E[T-t_2]^+ + (m_2+g)E[t_2-T]^+. \end{aligned} \quad (4.8.58)$$

It is important to start this section by recalling from our previous work that the existence of t_1^* and t_2^* is possible only if the following conditions are satisfied:

$$m_1' < -g \quad (4.8.59)$$

and

$$m_1' > -g\left(\frac{m_2+g+c_{h,2}}{c_{h,2}}\right) - m_1\left(\frac{m_2+g}{c_{h,2}}\right), \quad (4.8.60)$$

We denote the right-hand and left-hand derivatives of $l_1(t_1,t_2)$ with respect to t_1 by $l_1^r(t_1,t_2)$ and $l_1^l(t_1,t_2)$, respectively and express them as follows:

$$l_1^r(t_1,t_2) = -(m_1+g) + (m_1-m_1')F(t_1), \quad (4.8.61)$$

and

$$l_1^l(t_1,t_2) = -(m_1+g) + (m_1-m_1')F(T<t_1). \quad (4.8.62)$$

Since F is assumed continuous, then $l_1(t_1,t_2)$ is continuously differentiable with

$$l_1'(t_1,t_2) = -(m_1+g) + (m_1-m_1')F(t_1). \quad (4.8.63)$$

From the classical optimization theory, t_1^* zeros the derivative and we have $l_1^r(t_1^*,t_2) \geq 0$

and $l_1^l(t_1^*, t_2) \leq 0$ and 0 is a sub-gradient at t_1^*.

Definition 1.1 Let $\hat{t_1}$ be a realization of $\hat{T_1}$ with $\psi > 0$. $\hat{t_1}$ is ψ-accurate if $F(\hat{t_1}) \geq \frac{m_1+g}{m_1-m_1'} - \psi$ and $\bar{F}(\hat{t_1}) \geq -\frac{m_1'+g}{m_1-m_1'} - \psi$.

This definition can be translated to bounds on the right-hand and left-hand derivatives of l_1 at $\hat{t_1}$. Observe that $F(T < t_1) = 1 - \bar{F}(t_1)$. It is straightforward to verify that we could equivalently define $\hat{t_1}$ to be ψ-accurate exactly when $l_1^r(\hat{t_1}, t_2) \geq -\psi(m_1 - m_1')$ and $l_1^l(\hat{t_1}, t_2) \leq \psi(m_1 - m_1')$. This implies that there exists a sub-gradient $r \in \Delta l_1(\hat{t_1}, t_2)$ such that $|r| \leq \psi(m_1 - m_1')$. Intuitively, this implies that, for ψ sufficiently small, 0 is 'almost' a sub-gradient at $\hat{t_1}$, and hence $\hat{t_1}$ is "close" to being optimal.

LEMMA 1.1 Let $\psi > 0$ and assume that $\hat{t_1}$ is $\psi - accurate$. Then:

(i)

$$l_1(\hat{t_1}, t_2) - l_1(t_1^*, t_2) \leq \psi(m_1 - m_1')|\hat{t_1} - t_1^*|. \tag{4.8.64}$$

(ii)

$$l_1(t_1^*, t_2) \geq \left(\frac{(m_1 - m_1')(m_1 + g)}{m_1 - m_1'} - \psi max(m_1 - m_1', m_1 + g)\right)|\hat{t_1} - t_1^*|. \tag{4.8.65}$$

Proof. Suppose $\hat{t_1}$ is ψ-accurate. Clearly, either $\hat{t_1} \geq t_1^*$ or $\hat{t_1} < t_1^*$. Suppose first that $\hat{t_1} \geq t_1^*$. We will obtain an upper bound on the difference $l_1(\hat{t_1}, t_2) - l_1(t_1^*, t_2)$. If $T \in (-\infty, \hat{t_1})$, then the difference between the costs incurred by $\hat{t_1}$ and t_1^* is at most $-(m_1' + g)(\hat{t_1} - t_1^*)$. On the other hand, if $T \in [\hat{t_1}, \infty)$, then t_1^* has higher cost than $\hat{t_1}$, by exactly$(m_1 + g)(\hat{t_1} - t_1^*)$. Since $\hat{t_1}$ is ψ-accurate, we have the following

$$F([T \in [\hat{t_1}, \infty)]) = F(T \geq \hat{t_1}) = \bar{F}(\hat{t_1}) \geq -\frac{m_1' + g}{m_1 - m_1'} - \psi \tag{4.8.66}$$

and

$$P([T \in [0, \hat{t_1})]) = F(T < \hat{t_1}) = 1 - \bar{F}(\hat{t_1}) \leq \frac{m_1 + g}{m_1 - m_1'} + \psi \tag{4.8.67}$$

Therefore,

$$\begin{aligned} l_1(\hat{t}_1, t_2) - l_1(t_1^*, t_2) \quad & \leq -\left(m_1' + g\right)\left(\frac{m_1 + g}{m_1 - m_1'} + \psi\right)\left(\hat{t}_1 - t_1^*\right) \\ & -\left(m_1 + g\right)\left(-\frac{m_1' + g}{m_1 - m_1'} - \psi\right)\left(\hat{t}_1 - t_1^*\right) \\ & = \psi(m_1 - m_1')(\hat{t}_1 - t_1^*) \end{aligned} \tag{4.8.68}$$

Similarly, if $\hat{t}_1 < t_1^*$, then for each realization $T \in (\hat{t}_1, \infty)$ the difference between the costs of $\hat{t}_1$ and t_1^*, respectively, is at most $(m_1 + g)(t_1^* - \hat{t}_1)$, and if $T \in (-\infty, \hat{t}_1]$, then the cost of $\hat{t}_1$ exceeds the cost of t_1^* by exactly $-(m_1' + g)(t_1^* - \hat{t}_1)$. Given that $\hat{t}_1$ is assumed to be ψ-accurate, we have

$$F(T \leq \hat{t}_1) = \bar{F}(\hat{t}_1) \geq \frac{m_1 + g}{m_1 - m_1'} - \psi \tag{4.8.69}$$

and

$$F(T > \hat{t}_1) = 1 - \bar{F}(\hat{t}_1) \leq -\frac{m_1' + g}{m_1 - m_1'} + \psi \tag{4.8.70}$$

Therefore

$$\begin{aligned} l_1(\hat{t}_1, t_2) - l_1(t_1^*, t_2) \quad & \leq \left(m_1 + g\right)\left(-\frac{m_1' + g}{m_1 - m_1'} + \psi\right)\left(t_1^* - \hat{t}_1\right) \\ & +\left(m_1' + g\right)\left(\frac{m_1 + g}{m_1 - m_1'} - \psi\right)\left(t_1^* - \hat{t}_1\right) \\ & = \psi(m_1 - m_1')(t_1^* - \hat{t}_1) \end{aligned} \tag{4.8.71}$$

The proof of part *(i)* then follows.

The above arguments also imply that if $\hat{t}_1 \geq t_1^*$ then

$$l_1(\hat{t}_1, t_2) \geq E[(T \geq \hat{t}_1)(m_1 + g)(\hat{t}_1 - t_1^*)] = (m_1 + g)\bar{F}(\hat{t}_1)(\hat{t}_1 - t_1^*). \tag{4.8.72}$$

We conclude that $l_1(t_1^*, t_2)$ is at least $\left(m_1 + g\right)\left(-\frac{m_1'+g}{m_1-m_1'} - \psi\right)\left(\hat{t}_1 - t_1^*\right)$. Similarly, in the case $\hat{t}_1 < t_1^*$, we conclude that $l_1(t_1^*, t_2)$ is at least

$$E[(T \leq \hat{t}_1)(-m_1' - g)(t_1^* - \hat{t}_1)] \geq -\left(m_1' + g\right)\left(\frac{m_1 + g}{m_1 - m_1'} - \psi\right)\left(t_1^* - \hat{t}_1\right). \tag{4.8.73}$$

In other words,

$$l_1(t_1^*, t_2) \geq \left(\frac{-(m_1' + g)(m_1 + g)}{m_1 - m_1'} - \psi max(-m_1' - g, m_1 + g) \right) |\hat{t_1} - t_1^*|. \tag{4.8.74}$$

COROLLARY 1.1 *For a given accuracy level $\epsilon \in (0, \leq 1]$, if $\hat{t_1}$ is ψ-accurate for*

$$\psi = \frac{\epsilon}{3} \frac{min(-m_1' - g, m_1 + g)}{m_1 - m_1'}, \tag{4.8.75}$$

then the cost of $\hat{t_1}$ is at most $(1 + \epsilon)$ dates the optimal cost, i.e., $l_1(\hat{t_1}, t_2) \leq (1 + \epsilon) l_1(t_1^*, t_2)$.

Proof.

Let $\psi = \frac{\epsilon}{3} \frac{min(-m_1' - g, m_1 + g)}{m_1 - m_1'}$ By Lemma 1.1, we know that in this case

$$l_1(\hat{t_1}, t_2) - l_1(t_1^*, t_2) \leq \psi(m_1 - m_1')|\hat{t_1} - t_1^*|. \tag{4.8.76}$$

and that

$$l_1(t_1^*, t_2) \geq \left(\frac{-(m_1' + g)(m_1 + g)}{m_1 - m_1'} - \psi max(-m_1' - g, m_1 + g) \right) |\hat{t_1} - t_1^*|. \tag{4.8.77}$$

It is then sufficient to show that

$$\psi\left(m_1 - m_1'\right) \leq \epsilon\left(\frac{-(m_1' + g)(m_1 + g)}{m_1 - m_1'} - \psi max(-m_1' - g, m_1 + g) \right) \tag{4.8.78}$$

Indeed,

$$\begin{aligned} \psi(m_1 - m_1') &\leq (2+\epsilon)\psi max(-m_1' - g, m_1 + g) - \epsilon\psi max(-m_1' - g, m_1 + g) \\ &= \frac{(2+\epsilon)\epsilon}{3} \frac{max(-m_1' - g, m_1 + g) min(-m_1' - g, m_1 + g)}{m_1 - m_1'} - \epsilon\psi max(-m_1' - g, m_1 + g) \\ &\leq \epsilon\left(\frac{-(m_1' + g)(m_1 + g)}{m_1 - m_1'} - \psi max(-m_1' - g, m_1 + g) \right) \end{aligned} \tag{4.8.79}$$

We substitute $\psi = \frac{\epsilon}{3} \frac{min(-m_1' - g, m_1 + g)}{m_1 - m_1'}$ in the first inequality and the second inequality follows as $\epsilon \leq 1$. We conclude that $l_1(\hat{T}_1, t_2) - l_1(t_1^*, t_2) \leq \epsilon l_1(t_1^*, t_2)$, from which the corollary follows.

We now establish upper bounds on the number of samples N_β required in order to guarantee that $\hat{t_1}$ is ψ-accurate with high probability (for each specified $\psi > 0$ and confidence probability $1-\delta$). Since $\hat{T}_1$ is the sample $\frac{m_1+g}{m_1-m_1'}$- quantile and t_1^* is the true $\frac{m_1+g}{m_1-m_1'}$-quantile, we can use known results regarding the convergence of sample quantiles to the true quantiles or more generally, the convergence of the empirical CDF $F_{N_\beta}(t_1)$ to the true CDF $F(t_1)$. (For N_β independent random samples all distributed according to T, we define $F_{N_\beta}(t_1) := \frac{1}{N_\beta}\sum_{k=1}^{N_\beta} T_1^k$, where for each $k = 1, ..., N_\beta$, $T_1^k = (T_1^k \leq T)$, and $T_1^1, ..., T_1^{N_\beta}$ are i.i.d. according to T.)

Lemma 1.2 *For each $\psi > 0$ and $0 < \delta < 1$, if the number of samples is $= N_\beta > N_\beta(\psi, \delta) = \frac{1}{2}\frac{1}{\psi^2}log(\frac{2}{\delta})$, then $\hat{T}_1$, the $\frac{m_1+g}{m_1-m_1}$-quantile of the sample, is ψ-accurate with probability at least $1-\delta$.*

Lemma 1.2 is a direct consequence of the fact that the empirical CDF converges uniformly and exponentially fast to the true CDF.

Combining Lemma 1.1, Corollary 1.1 and Lemma 1.2 above, we can obtain the following theorem.

THEOREM A.1 Consider a rollover problem specified by a date distribution T with $E[T] < \infty$. Let $0 < \epsilon \leq 1$ be a specified accuracy level and $1-\epsilon$ (for $0 < \delta < 1$) be a specified confidence level. Suppose that $N_\beta \geq \frac{9}{2\epsilon^2}\left(\frac{min(-(m_1'+g), m_1+g)}{m_1-m_1'}\right)^{-2} log\left(\frac{2}{\delta}\right)$ and the data-drive counterpart is solved with respect to N_β i.i.d samples of T. Let $\hat{T}_1$ be the optimal solution to the data-drive counterpart and $\hat{t_1}$ denote its realization. Then, with probability at least $1-\delta$, the expected cost of $\hat{t_1}$ is at most $1+\epsilon$ times the expected cost of an optimal solution t_1^* to the rollover problem. In other words,

$l_1(\hat{T}_1, t_2) \leq (1+\epsilon) l_1(t_1^*, t_2)$ with probability at least $1-\delta$.

N_β does not depend on the date distribution T, but on the square of the reciprocal of $\frac{min(-(m_1'+g), m_1+g)}{m_1-m_1'}$. This means that large samples are required when might be large when $\frac{m_1+g+c_{h,2}}{m_2-m_1'+m_1+g+c_{h,2}}$ is very close to either 0 or 1. Since the optimal solution t_1^* is the $\frac{m_1+g}{m_1-m_1'}$-quantile of T, this is consistent with the well-known fact that in order to approximate an extreme quantile one needs many samples. N_β is a worst-case upper bound and it is likely that in many cases a significantly fewer number of samples will suffice.

We now continue our analysis with respect to t_2. We already proved that $l_1(t_1, t_2)$ is convex in t_2. We denote right-hand and left-hand derivatives of $l_1(t_1, t_2)$ with respect to t_2, denoted by $l_1^r(t_1, t_2)$ and $l_1^l(t_1, t_2)$, respectively and are given by:

$$l_1^r(t_1, t_2) = -c_{h,2} + (m_2 + g + c_{h,2}) F(t_2) \tag{4.8.80}$$

and

$$l_1^l(t_1, t_2) = -c_{h,2} + (m_2 + g + c_{h,2}) F(T < t_2). \tag{4.8.81}$$

Since F is assumen continuous, then $l_1(t_1, t_2)$ is continuously differentiable with

$$l_1'(t_1, t_2) = -c_{h,2} + (m_2 + g + c_{h,2}) F(t_2). \tag{4.8.82}$$

Using the explicit expressions of the derivatives, one can characterize the optimal solution t_2^*. Specifically, t_2^*=inf$\{t_2 : F(t_2) \geq \frac{c_{h,2}}{m_2+g+c_{h,2}}\}$. That is, t_2^* is the $\frac{c_{h,2}}{m_2+g+c_{h,2}}$-quantile of the distribution of T. From the classical optimization thereof, we have $l_1'(t_1, t_2^*) = 0$ and $l_1^r(t_1, t_2^*) \geq 0$ and $l_1^l(t_1, t_2^*) \leq 0$ and 0 is a sub-gradient at t_2^*.

Definition 1.2 Let $\hat{t}_2$ be a realization of $\hat{T}_2$ and let $\psi > 0$. $\hat{t}_2$ is ψ-accurate if $F(\hat{t}_2) \geq \frac{c_{h,2}}{m_2+g+c_{h,2}} - \psi$ and $\bar{F}(\hat{t}_2) \geq \frac{m_2+g}{m_2+g+c_{h,2}} - \psi$.

This definition can be translated to bounds on the right-hand and left-hand derivatives of l_1 at $\hat{t}_2$. Observe that $F(T < t_2) = 1 - \bar{F}(t_2)$ and equivalently $\hat{t}_2$ is ψ-accurate exactly when $l_1^r(t_1, \hat{t}_2) \geq -\psi(m_2 + g + c_{h,2})$ and $l_1^l(t_1, \hat{t}_2) \leq \psi(m_2 + g + c_{h,2})$. This implies that there exists a sub-gradient $r \in \Delta l_1(t_1, \hat{t}_2)$ such that $|r| \leq \psi(m_2 + g + c_{h,2})$. This implies that, for ψ sufficiently small, 0 is "almost" a sub-gradient at $\hat{t}_2$, and hence $\hat{t}_2$ is "close" to being optimal.

LEMMA 1.3 Let $\psi > 0$ and assume that $\hat{t}_2$ is $\psi - accurate$. Then:

(i)

$$l_1(t_1, \hat{t}_2) - l_1(t_1, t_2^*) \leq \psi(m_2 + g + c_{h,2})|\hat{t}_2 - t_2^*|. \tag{4.8.83}$$

(ii)

$$l_1(t_1, t_2^*) \geq \left(\frac{c_{h,2}(m_2 + g)}{m_2 + g + c_{h,2}} - \psi(m_2 + g) \right) |\hat{t}_2 - t_2^*|. \tag{4.8.84}$$

Proof. Suppose $\hat{t}_2$ is ψ-accurate, we have either $\hat{t}_2 \geq t_2^*$ or $\hat{t}_2 < t_2^*$. Suppose first that $\hat{t}_2 \geq t_2^*$. We will obtain an upper bound on the difference $l_1(t_1, \hat{t}_2) - l_1(t_1, t_2^*)$. If the realized date $T \in (-\infty, \hat{t}_2)$, then the difference between the costs incurred by $\hat{t}_2$ and t_2^* is at most $(m_2 + g)(\hat{t}_2 - t_2^*)$. On the other hand, if $T \in [\hat{t}_2, \infty)$, then t_2^* has higher cost than $\hat{t}_2$, by exactly $c_{h,2}(\hat{t}_2 - t_2^*)$. $\hat{t}_2$ is assumed to be ψ-accurate, we have

$$P([T \in [\hat{t}_2, \infty)]) = F(T \geq \hat{t}_2) = \bar{F}(\hat{t}_2) \geq \frac{m_2 + g}{m_2 + g + c_{h,2}} - \psi \tag{4.8.85}$$

and

$$P([T \in [0, \hat{t}_2)]) = F(T < \hat{t}_2) = 1 - \bar{F}(\hat{t}_2) \leq \frac{c_{h,2}}{m_2 + g + c_{h,2}} + \psi \tag{4.8.86}$$

Therefore

$$\begin{aligned} l_1(t_1,\hat{t_2}) - l_1(t_1,t_2^*) \quad & \leq \Big(m_2+g\Big)\Big(\frac{c_{h,2}}{m_2+g+c_{h,2}}+\psi\Big)\Big(\hat{t_2}-t_2^*\Big) \\ & -c_{h,2}\Big(\frac{m_2+g}{m_2+g+c_{h,2}}-\psi\Big)\Big(\hat{t_2}-t_2^*\Big) \\ & = \psi(m_2+g+c_{h,2})(\hat{t_2}-t_2^*) \end{aligned} \tag{4.8.87}$$

Similarly, if $\hat{t_2} < t_2^*$, then for each realization $T \in (\hat{t_2}, \infty)$ the difference between the costs of $\hat{t_2}$ and t_2^*, respectively, is at most $c_{h,2}(t_2^* - \hat{t_2})$, and if $T \in (-\infty, \hat{t_2}]$, then the cost of t_2^* exceeds the cost of $\hat{t_2}$ by exactly $(m_2+g)(t_2^*-\hat{t_2})$. $\hat{t_2}$ is assumed to be ψ-accurate, we have

$$F(T \leq \hat{t_2}) = \bar{F}(\hat{t_2}) \geq \frac{c_{h,2}}{m_2+g+c_{h,2}} - \psi \tag{4.8.88}$$

and

$$F(T > \hat{t_2}) = 1 - \bar{F}(\hat{t_2}) \leq \frac{m_2+g}{m_2+g+c_{h,2}} + \psi \tag{4.8.89}$$

Therefore

$$\begin{aligned} l_1(t_1,\hat{t_2}) - l_1(t_1,t_2^*) \quad & \leq c_{h,2}\Big(\frac{m_2+g}{m_2+g+c_{h,2}}+\psi\Big)\Big(t_2^*-\hat{t_2}\Big) \\ & -\Big(m_2+g\Big)\Big(\frac{c_{h,2}}{m_2+g+c_{h,2}}-\psi\Big)\Big(t_2^*-\hat{t_2}\Big) \\ & = \psi(m_2+g+c_{h,2})(t_2^*-\hat{t_1}) \end{aligned} \tag{4.8.90}$$

The proof of part *(i)* then follows.

The above arguments also imply that if $\hat{t_2} \geq t_2^*$ then

$$l_1(t_1,t_2^*) \geq E[(T \geq \hat{t_2})(\hat{t_2}-t_2^*)c_{h,2}] = c_{h,2}\bar{F}(\hat{t_2})(\hat{t_2}-t_2^*). \tag{4.8.91}$$

We conclude that $l_1(t_1,t_2^*)$ is at least

$$c_{h,2}\Big(\frac{m_2+g}{m_2+g+c_{h,2}}-\psi\Big)\Big(\hat{t_2}-t_2^*\Big). \tag{4.8.92}$$

Similarly, in the case $\hat{t_2} < t_2^*$, we conclude that $l_1(t_1, t_2^*)$ is at least

$$E[(T \leq \hat{t_2})(m_2 + g) \qquad (t_2^* - \hat{t_2})] \geq \left(m_2 + g\right)\left(\frac{c_{h,2}}{m_2 + g + c_{h,2}} - \psi\right)\left(t_2^* - \hat{t_2}\right). \tag{4.8.93}$$

In other words,

$$l_1(t_1, t_2^*) \geq \left(\frac{c_{h,2}(m_2 + g)}{m_2 + g + c_{h,2}} - \psi max(m_2 + g, c_{h,2})\right)|\hat{t_2} - t_2^*|. \tag{4.8.94}$$

Since we know that $c_{h,2} < m_2$, then expression (4.8.94) becomes

$$l_1(t_1, t_2^*) \geq \left(\frac{c_{h,2}(m_2 + g)}{m_2 + g + c_{h,2}} - \psi(m_2 + g)\right)|\hat{t_2} - t_2^*|. \tag{4.8.95}$$

COROLLARY 1.2 For a given accuracy level $\epsilon \in (0, \leq 1]$, if $\hat{t_2}$ is ψ-accurate for

$$\psi = \frac{\epsilon}{3}\frac{c_{h,2}}{m_2 + g + c_{h,2}}, \tag{4.8.96}$$

then the cost of $\hat{t_2}$ is at most $(1 + \epsilon)$ dates the optimal cost, i.e., $l_1(t_1, \hat{t_2}) \leq (1 + \epsilon)l_1(t_1, t_2^*)$.

PROOF.

Let $\psi = \frac{\epsilon}{3}\frac{c_{h,2}}{m_2+g+c_{h,2}}$ By Lemma 1.3, we know that in this case

$$l_1(t_1, \hat{t_2}) - l_1(t_1, t_2^*) \leq \psi(m_2 + g + c_{h,2})|\hat{t_2} - t_2^*|. \tag{4.8.97}$$

and that

$$l_1(t_1, t_2^*) \geq \left(\frac{c_{h,2}(m_2 + g)}{m_2 + g + c_{h,2}} - \psi(m_2 + g)\right)|\hat{t_2} - t_2^*|. \tag{4.8.98}$$

It is then sufficient to show that

$$\psi\left(m_2 + g + c_{h,2}\right) \leq \epsilon\left(\frac{c_{h,2}(m_2 + g)}{m_2 + g + c_{h,2}} - \psi(m_2 + g)\right) \tag{4.8.99}$$

Indeed,

$$\begin{aligned} \psi(m_2 + g + c_{h,2}) &\leq (2 + \epsilon)\psi(m_2 + g) - \epsilon\psi(m_2 + g) && (4.8.100) \\ &= \frac{(2 + \epsilon)\epsilon}{3}\frac{(m_2 + g)c_{h,2}}{m_2 + g + c_{h,2}} - \epsilon\psi(m_2 + g) \\ &\leq \epsilon\left(\frac{c_{h,2}(m_2 + g)}{m_2 + g + c_{h,2}} - \psi(m_2 + g)\right) && (4.8.101) \end{aligned}$$

We substitute $\psi = \frac{\epsilon}{3}\frac{c_{h,2}}{m_2+g+c_{h,2}}$ in the first inequality and the second inequality follows since $\epsilon \leq 1$. We conclude that $l_1(t_1, \hat{T}_2) - l_1(t_1, t_2^*) \leq l_1(t_1, t_2^*)$, from which the corollary follows.

We now establish upper bounds on N_β to guarantee that $\hat{t_2}$ is ψ-accurate with high probability (for each specified $\psi > 0$ and confidence probability $1-\delta$). Since $\hat{T}_2$ is the sample $\frac{c_{h,2}}{m_2+g+c_{h,2}}$- quantile and t_2^* is the true $\frac{c_{h,2}}{m_2+g+c_{h,2}}$-quantile, we can use known results regarding the convergence of sample quantiles to the true quantiles or more generally, the convergence of the empirical CDF $F_{N_\beta}(t_2)$ to the true CDF $F(t_2)$. (For N_β independent random samples all distributed according to T, we define $F_{N_\beta}(t_2) := \frac{1}{N_\beta}\sum_{k=1}^{N_\beta} T^k$, where for each $k = 1, ..., N_\beta$, $T_2^k = (T_2^k \leq T)$, and $T_2^1, ..., T_2^{N_\beta}$ are i.i.d. according to T.)

Lemma 1.4 *For each $\psi > 0$ and $0 < \delta < 1$, if the number of samples is $= N_\beta > N_\beta(\psi, \delta) = \frac{1}{2}\frac{1}{\psi^2}log(\frac{2}{\delta})$, then $\hat{T}_2$, the $\frac{c_{h,2}}{m_2+g+c_{h,2}}$-quantile of the sample, is ψ-accurate with probability at least $1-\delta$.*

Lemma 1.4 is a direct consequence of the fact that the empirical CDF converges uniformly and exponentially fast to the true CDF.

Combining Lemma 1.3, Corollary 1.2 and Lemma 1.4 above, we can obtain the following theorem.

THEOREM A.2 Consider a rollover problem specified by a date distribution T with $E[T] < \infty$. Let $0 < \epsilon \leq 1$ be a specified accuracy level and $1-\epsilon$ (for $0 < \delta < 1$) be a specified confidence level. Suppose that $N_\beta \geq \frac{9}{2\epsilon^2}\left(\frac{c_{h,2}}{m_2+g+c_{h,2}}\right)^{-2} log\left(\frac{2}{\delta}\right)$ and the data-drive counterpart is solved with respect to N_β i.i.d samples of T. Let $\hat{T}_1$ be the optimal solution to the data-drive counterpart and $\hat{t_2}$ denote its realization. Then, with probability at least $1-\delta$, the expected cost of $\hat{t_2}$ is at most $1+\epsilon$ times the expected cost of an optimal solution t_2^* to the rollover problem. In other words, $l_1(t_1, \hat{T_2}) \leq (1+\epsilon) l_1(t_1, t_2^*)$ with probability at least $1-\delta$.

N_β does not depend on the date distribution T, but on the square of the reciprocal of $\frac{c_{h,2}}{m_2+g+c_{h,2}}$. This means that large samples are required when might be large when $\frac{c_{h,2}}{m_2+g+c_{h,2}}$ is very close to either 0 or 1. Since the optimal solution t_2^* is the $\frac{c_{h,2}}{m_2+g+c_{h,2}}$-quantile of T, this is consistent with the well-known fact that in order to approximate an extreme quantile one needs many samples. N_β is a worst-case upper bound and it is likely that in many cases a significantly fewer number of samples will suffice.

Appendix E

Since we have two variables, we need to calculate the worst bound for each variable, then take the maximum of each. We have defined the expected loss function to be convex and given by

$$\begin{aligned} l_2(t_1, t_2) &= (m_2 - m_1' - s_1)E[t_2 - T]^+ + c_{h,2}E[T - t_2]^+ \\ &+ (m_1 + g)E[T - t_1]^+ + s_1 E[t_1 - T]^+. \end{aligned} \tag{4.8.102}$$

It is important to start this section by recalling from our previous work that the existence of t_1^* and t_2^* is possible only if the following conditions are satisfied:

$$m_2 - m_1' - s_1 > 0 \tag{4.8.103}$$

and

$$s_1 < \frac{(m_1+g)(m_2-m_1')}{m_1+g+c_{h,2}}. \tag{4.8.104}$$

We denote the right-hand and left-hand derivatives of $l_2(t_1, t_2)$ by $l_2^r(t_1, t_2)$ and $l_2^l(t_1, t_2)$, respectively and are given by:

$$l_2^r(t_1, t_2) = -(m_1 + g) + (m_1 + g + s_1)F(t_1), \tag{4.8.105}$$

and

$$l_2^l(t_1, t_2) = -(m_1 + g) + (m_1 + g + s_1)(T < t_1). \tag{4.8.106}$$

Since F is continuous, then $l_2(t_1, t_2)$ is continuously differentiable with

$$l_2'(t_1, t_2) = -(m_1 + g) + (m_1 + g + s_1)F(t_1). \tag{4.8.107}$$

The optimal solution t_1^* is given by:

$$t_1^* = inf\{t_1 : F(t_1) \geq \frac{m_1 + g}{m_1 + g + s_1}\}. \tag{4.8.108}$$

t_1^* is the $\frac{m_1+g}{m_1+g+s_1}$-quantile of the distribution of T. From the classical optimization theory, $l_2'(t_1^*, t_2) = 0$, and $l_2^r(t_1^*, t_2) \geq 0$ and $l_2^l(t_1^*, t_2) \leq 0$ and 0 is a sub-gradient at t_1^*.

Definition 2.1 Let $\hat{t_1}$ be some realization of $\hat{T_1}$ and let $\psi > 0$. We will say that $\hat{t_1}$ is ψ-accurate if $F(\hat{t_1}) \geq \frac{m_1+g}{m_1+g+s_1} - \psi$ and $\bar{F}(\hat{t_1}) \geq \frac{s_1}{m_1+g+s_1} - \psi$.

This definition can be translated to bounds on the right-hand and left-hand derivatives of l_2 at $\hat{t_1}$. Observe that $F(T < t_1) = 1 - \bar{F}(\hat{t_1})$ and equivalently $\hat{t_1}$ is ψ-accurate exactly when $l_2^r(\hat{t_1}, t_2) \geq -\psi(m_1 + g + s_1)$ and $l_2^l(\hat{t_1}, t_2) \leq \psi(m_1 + g + s_1)$. This implies that there exists a sub-gradient $r \in \Delta l_2(\hat{t_1}, t_2)$ such that $|r| \leq \psi(m_1 + g + s_1)$. Intuitively, this implies that, for ψ sufficiently small, 0 is 'almost' a sub-gradient at $\hat{t_1}$, and hence $\hat{t_1}$ is 'close' to being optimal.

LEMMA 2.1 Let $\psi > 0$ and assume that $\hat{t_1}$ is $\psi - accurate$. Then:

(i)

$$l_2(\hat{t_1}, t_2) - l_2(t_1^*, t_2) \leq \psi(m_1 + g + s_1)|\hat{t_1} - t_1^*|. \tag{4.8.109}$$

(ii)

$$l_2(t_1^*, t_2) \geq \left(\frac{s_1(m_1 + g)}{m_1 + g + s_1} - \psi(m_1 + g)\right)|\hat{t_1} - t_1^*|. \tag{4.8.110}$$

Proof. Suppose $\hat{t_1}$ is ψ-accurate. We have either $\hat{t_1} \geq t_1^*$ or $\hat{t_1} < t_1^*$. Suppose first that $\hat{t_1} \geq t_1^*$. We will obtain an upper bound on the difference $l_2(\hat{t_1}, t_2) - l_2(t_1^*, t_2)$. Clearly, if the realized date $T \in (-\infty, \hat{t_1})$, then the difference between the costs incurred by $\hat{t_1}$ and t_1^* is at most $s_1(\hat{t_1} - t_1^*)$. On the other hand, if $T \in [\hat{t_1}, \infty)$, then t_1^* has higher cost than $\hat{t_1}$, by exactly$(m_1 + g)(\hat{t_1} - t_1^*)$. Now since $\hat{t_1}$ is assumed to be ψ-accurate, we have

$$P([T \in [\hat{t_1}, \infty)]) = F(T \geq \hat{t_1}) = \bar{F}(\hat{t_1}) \geq \frac{s_1}{m_1 + g + s_1} - \psi \tag{4.8.111}$$

and

$$P([T \in [0, \hat{t_1})]) = F(T < \hat{t_1}) = 1 - \bar{F}(\hat{t_1}) \leq \frac{m_1 + g}{m_1 + g + s_1} + \psi \tag{4.8.112}$$

Therefore

$$\begin{aligned} l_2(\hat{t_1}, t_2) - l_2(t_1^*, t_2) \quad & \leq s_1\left(\frac{m_1 + g}{m_1 + g + s_1} + \psi\right)\left(\hat{t_1} - t_1^*\right) \\ & -\left(m_1 + g\right)\left(\frac{s_1}{m_1 + g + s_1} - \psi\right)\left(\hat{t_1} - t_1^*\right) \\ & = \psi(m_1 + g + s_1)(\hat{t_1} - t_1^*) \end{aligned} \tag{4.8.113}$$

Similarly, if $\hat{t_1} < t_1^*$, then for each realization $T \in (\hat{t_1}, \infty)$ the difference between the costs of $\hat{t_1}$ and t_1^*, respectively, is at most $(m_1 + g)(t_1^* - \hat{t_1})$, and if $T \in (-\infty, \hat{t_1}]$, then the cost of $\hat{t_1}$ exceeds the cost of t_1^* by exactly $s_1(t_1^* - \hat{t_1})$. Given that $\hat{t_1}$ is ψ-accurate, we know that

$$F(T \leq \hat{t_1}) = \bar{F}(\hat{t_1}) \geq \frac{m_1 + g}{m_1 + g + s_1} - \psi \tag{4.8.114}$$

and

$$F(T > \hat{t_1}) = 1 - \bar{F}(\hat{t_1}) \leq \frac{s_1}{m_1 + g + s_1} + \psi \tag{4.8.115}$$

Therefore

$$\begin{aligned} l_2(\hat{t_1}, t_2) - l_2(t_1^*, t_2) \quad & \leq \Big(m_1 + g\Big)\Big(\frac{s_1}{m_1 + g + s_1} + \psi\Big)\Big(t_1^* - \hat{t_1}\Big) \\ & - s_1\Big(\frac{m_1 + g}{m_1 + g + s_1} - \psi\Big)\Big(t_1^* - \hat{t_1}\Big) \\ & = \psi(m_1 + g + s_1)(t_1^* - \hat{t_1}) \end{aligned} \tag{4.8.116}$$

The proof of part *(i)* then follows.

The above arguments also imply that if $\hat{t_1} \geq t_1^*$ then

$$l_2(\hat{t_1}, t_2) \geq E[(T \geq \hat{t_1})(m_1 + g)(\hat{t_1} - t_1^*)] = (m_1 + g)\bar{F}(\hat{t_1})(\hat{t_1} - t_1^*). \tag{4.8.117}$$

We conclude that $l_2(t_1^*, t_2)$ is at least $\Big(m_1 + g\Big)\Big(\frac{s_1}{m_1+g+s_1} - \psi\Big)\Big(\hat{t_1} - t_1^*\Big)$. Similarly, in the case $\hat{t_1} < t_1^*$, we conclude that $l_2(t_1^*, t_2)$ is at least

$$\begin{aligned} E[(T \leq \hat{t_1}) \quad & s_1(t_1^* - \hat{t_1})] \\ & \geq s_1\Big(\frac{m_1 + g}{m_1 + g + s_1} - \psi\Big)\Big(t_1^* - \hat{t_1}\Big). \end{aligned} \tag{4.8.118}$$

In other words,

$$l_1(\hat{t_1}, t_2) \geq \Big(\frac{s_1(m_1 + g)}{m_1 + g + s_1} - \psi(m_1 + g)\Big)|\hat{t_1} - t_1^*|. \tag{4.8.119}$$

COROLLARY 2.1 For a given accuracy level $\epsilon \in (0, \leq 1]$, if $\hat{t_1}$ is ψ-accurate for $\psi = \frac{\epsilon}{3}\frac{s_1}{m_1+g+s_1}$, then the cost of $\hat{t_1}$ is at most $(1 + \epsilon)$ dates the optimal cost, i.e., $l_2(\hat{t_1}, t_2) \leq (1 + \epsilon)l_2(t_1^*, t_2)$.

PROOF.

Let $\psi = \frac{\epsilon}{3}\frac{s_1}{m_1+g+s_1}$ By Lemma 2.1, we know that in this case

$$l_2(\hat{t_1}, t_2) - l_2(t_1^*, t_2) \leq \psi(m_1 + g + s_1)|\hat{t_1} - t_1^*|. \tag{4.8.120}$$

and that

$$l_2(t_1^*, t_2) \geq \left(\frac{s_1(m_1+g)}{m_1+g+s_1} - \psi(m_1+g) \right) |\hat{t_1} - t_1^*|. \tag{4.8.121}$$

It is then sufficient to show that

$$\psi\left(m_1+g+s_1 \right) \leq \epsilon \left(\frac{s_1(m_1+g)}{m_1+g+s_1} - \psi(m_1+g) \right) \tag{4.8.122}$$

Indeed,

$$\begin{aligned} \psi(m_1+g+s_1) &\leq (2+\epsilon)\psi(m_1+g) - \epsilon\psi(m_1+g) & (4.8.123)\\ &= \frac{(2+\epsilon)\epsilon}{3} \frac{(m_1+g)s_1}{m_1+g+s_1} - \epsilon\psi(m_1+g) \\ &\leq \epsilon \left(\frac{s_1(m_1+g)}{m_1+g+s_1} - \psi(m_1+g) \right) & (4.8.124) \end{aligned}$$

We substitute $\psi = \frac{\epsilon}{3} \frac{s_1}{m_1+g+s_1}$ in the first inequality and the second inequality follows as $\epsilon \leq 1$. We conclude that $l_2(\hat{T}_1, t_2) - l_2(t_1^*, t_2) \leq l_2(t_1^*, t_2)$, from which the corollary follows.

We now establish upper bounds on N_β required in order to guarantee that $\hat{t_1}$ is ψ-accurate with high probability (for each specified $\psi > 0$ and confidence probability $1 - \delta$). Since $\hat{T}_1$ is the sample $\frac{m_1+g}{m_1+g+s_1}$- quantile and t_1^* is the true $\frac{m_1+g}{m_1+g+s_1}$-quantile, we can use known results regarding the convergence of sample quantiles to the true quantiles or more generally, the convergence of the empirical CDF $F_{N_\beta}(t_1)$ to the true CDF $F(t_1)$. (For N_β independent random samples all distributed according to T, we define $F_{N_\beta}(t_1) := \frac{1}{N_\beta} \sum_{k=1}^{N_\beta} T^k$, where for each $i = 1, ..., N_\beta$, $T_1^k = (T_1^k \leq T)$, and $T_1^1, ..., T_1^{N_\beta}$ are i.i.d. according to T.)

Lemma 2.2 *For each $\psi > 0$ and $0 < \delta < 1$, if the number of samples is $= N_\beta > N_\beta(\psi, \delta) = \frac{1}{2} \frac{1}{\psi^2} log(\frac{2}{\delta})$, then $\hat{T}_1$, the $\frac{m_1+g}{m_1+g+s_1}$-quantile of the sample, is ψ-accurate with probability at least* $1 - \delta$. Lemma 2.2 is a direct consequence of the fact that the empirical CDF converges uniformly and exponentially fast to the true CDF.

Combining Lemma 2.1, Corollary 2.1 and Lemma 2.2 above, we can obtain the following theorem.

THEOREM B.1 Consider a rollover problem specified by a date distribution T with $E[T] < \infty$. Let $0 < \epsilon \leq 1$ be a specified accuracy level and $1 - \epsilon$ (for $0 < \delta < 1$) be a specified confidence level. Suppose that $N_\beta \geq \frac{9}{2\epsilon^2}\left(\frac{s_1}{m_1+g+s_1}\right)^{-2} log\left(\frac{2}{\delta}\right)$ and the data-drive counterpart is solved with respect to N_β i.i.d samples of T. Let $\hat{T_1}$ be the optimal solution to the data-drive counterpart and $\hat{t_1}$ denote its realization. Then, with probability at least $1 - \delta$, the expected cost of $\hat{t_1}$ is at most $1 + \epsilon$ times the expected cost of an optimal solution t_1^* to the rollover problem. In other words, $l_1(\hat{T}_1, t_2) \leq (1+\epsilon)l_1(t_1^*, t_2)$ with probability at least $1 - \delta$.

N_β does not depend on the date distribution T, but on the square of the reciprocal of $\frac{s_1}{m_1+g+s_1}$. This means that large samples are required when might be large when $\frac{m_1+g}{m_1+g+s_1}$ is very close to either 0 or 1. Since the optimal solution t_1^* is the $\frac{m_1+g}{m_1+g+s_1}$-quantile of T, this is consistent with the well-known fact that in order to approximate an extreme quantile one needs many samples. N_β is a worst-case upper bound and it is likely that in many cases a significantly fewer number of samples will suffice.

We now continue our analysis with respect to t_2. We already proved that $l_2(t_1, t_2)$ is convex in t_2. We denote the right-hand and left-hand derivatives of $l_2(t_1, t_2)$ by $l_2^r(t_1, t_2)$ and $l_2^l(t_1, t_2)$, respectively and are given below

$$l_2^r(t_1, t_2) = -c_{h,2} + (m_2 - m_1' - s_1 + c_{h,2})F(t_2), \tag{4.8.125}$$

and

$$l_2^l(t_1, t_2) = -c_{h,2} + (m_2 - m_1' - s_1 + c_{h,2})F(T < t_2). \tag{4.8.126}$$

Since F is continuous $l_2(t_1, t_2)$ is continuously differentiable with

$$l_2'(t_1, t_2) = -c_{h,2} + (m_2 - m_1' - s_1 + c_{h,2})F(t_2). \tag{4.8.127}$$

We characterize the optimal solution t_2^* by

$$t_2^* = inf\{t_2 : F(t_2) \geq \frac{c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}}\} \tag{4.8.128}$$

and t_2^* is the $\frac{c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}}$-quantile of the distribution of T. From the classical optimization theory, we have $l_2'(t_1, t_2^*) = 0$ and $l_2^r(t_1, t_2^*) \geq 0$ and $l_2^l(t_1, t_2^*) \leq 0$ and 0 is a sub-gradient at t_2^*

Definition 2.2 Let $\hat{t_2}$ be a realization of $\hat{T}_2$ and let $\psi > 0$. $\hat{t_2}$ is ψ-accurate if $F(\hat{t_2}) \geq \frac{c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}} - \psi$ and $\bar{F}(\hat{t_2}) \geq \frac{m_2 - m_1' - s_1}{m_2 - m_1' - s_1 + c_{h,2}} - \psi$.

This definition can be translated to bounds on the right-hand and left-hand derivatives of l_2 at $\hat{t_2}$. Observe that $F(T < t_2) = 1 - \bar{F}(t_2)$ and equivalently we define $\hat{t_2}$ to be ψ-accurate exactly when $l_2^r(t_1, \hat{t_2}) \geq -\psi(m_2 - m_1' - s_1 + c_{h,2})$ and $l_2^l(t_1, \hat{t_2}) \leq \psi(m_2 - m_1' - s_1 + c_{h,2})$. This implies that there exists a sub-gradient $r \in \Delta l_2(t_1, \hat{t_2})$ such that $|r| \leq \psi(m_2 - m_1' - s_1 + c_{h,2})$. Intuitively, this implies that, for ψ sufficiently small, 0 is "almost" a sub-gradient at $\hat{t_2}$, and hence $\hat{t_2}$ is "close" to being optimal.

LEMMA 2.3 Let $\psi > 0$ and assume that $\hat{t_2}$ is $\psi - accurate$. Then:

(i)

$$l_2(t_1, \hat{t_2}) - l_2(t_2^*) \leq \psi(m_2 - m_1' - s_1 + c_{h,2})|\hat{t_2} - t_2^*|. \tag{4.8.129}$$

(ii)

$$l_2(t_1, t_2^*) \geq \left(\frac{(m_2 - m_1' - s_1)c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}} - \psi max(m_2 - m_1' - s_1, c_{h,2})\right)|\hat{t_2} - t_2^*|. \tag{4.8.130}$$

Proof. Suppose $\hat{t_2}$ is ψ-accurate. We have $\hat{t_2} > t_2^*$ or $\hat{t_2} < t_2^*$. Suppose first that $\hat{t_2} > t_2^*$. We will obtain an upper bound on the difference $l_2(t_1, \hat{t_2}) - l_2(t_1, t_2^*)$. If the realized date $T \in (-\infty, \hat{t_2})$, then the difference between the costs incurred by $\hat{t_2}$ and t_2^* is at most $(m_2 - m_1' - s_1)(\hat{t_2} - t_2^*)$. On the other hand, if $T \in [\hat{t_2}, \infty)$, then t_2^* has higher cost than $\hat{t_2}$, by exactly$c_{h,2}(\hat{t_2} - t_2^*)$. Given that $\hat{t_2}$ is ψ-accurate, we have the following

$$F([T \in [\hat{t_2}, \infty)]) = F(T \geq \hat{t_2}) = \bar{F}(\hat{t_2}) \geq \frac{m_2 - m_1' - s_1}{m_2 - m_1' - s_1 + c_{h,2}} - \psi \tag{4.8.131}$$

and

$$P([T \in [0, \hat{t_2})]) = F(T < \hat{t_2}) = 1 - \bar{F}(\hat{t_2}) \leq \frac{c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}} + \psi \tag{4.8.132}$$

Therefore

$$\begin{aligned} l_2(t_1, \hat{t_2}) - l_2(t_1, t_2^*) \quad & \leq \left(m_2 - m_1' - s_1\right)\left(\frac{c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}} + \psi\right)\left(\hat{t_2} - t_2^*\right) \\ & -c_{h,2}\left(\frac{m_2 - m_1' - s_1}{m_2 - m_1' - s_1 + c_{h,2}} - \psi\right)\left(\hat{t_2} - t_2^*\right) \\ & = \psi(m_2 - m_1' - s_1 + c_{h,2})(\hat{t_2} - t_2^*) \end{aligned} \tag{4.8.133}$$

Similarly, if $\hat{t_2} < t_2^*$, then for each realization $T \in (\hat{t_2}, \infty)$ the difference between the costs of $\hat{t_2}$ and t_2^*, respectively, is at most $c_{h,2}(t_2^* - \hat{t_2})$, and if $T \in (-\infty, \hat{t_2}]$, then the cost of $\hat{t_2}$ exceeds the cost of t_2^* by exactly $(m_2 - m_1' - s_1)(t_2^* - \hat{t_2})$. As $\hat{t_2}$ is assumed to be ψ-accurate, we know that

$$F(T \leq \hat{t_2}) = \bar{F}(\hat{t_2}) \geq \frac{c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}} - \psi \tag{4.8.134}$$

and

$$F(T > \hat{t_2}) = 1 - \bar{F}(\hat{t_2}) \leq \frac{m_2 - m_1' - s_1}{m_2 - m_1' - s_1 + c_{h,2}} + \psi \tag{4.8.135}$$

Therefore

$$\begin{aligned} l_2(t_1, \hat{t_2}) - l_2(t_1, t_2^*) \quad & \leq c_{h,2}\left(\frac{m_2 - m_1' - s_1}{m_2 - m_1' - s_1 + c_{h,2}} + \psi\right)\left(t_2^* - \hat{t_2}\right) \\ & -\left(m_2 - m_1' - s_1\right)\left(\frac{c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}} - \psi\right)\left(t_2^* - \hat{t_2}\right) \\ & = \psi(m_2 - m_1' - s_1 + c_{h,2})(t_2^* - \hat{t_2}) \end{aligned} \tag{4.8.136}$$

The proof of part *(i)* then follows.

The above arguments also imply that if $\hat{t_2} \geq t_2^*$ then

$$l_2(t_1, t_2^*) \geq E[(T \geq \hat{t_2})c_{h,2}(\hat{t_2} - t_2^*)] = c_{h,2}\bar{F}(\hat{t_2})(\hat{t_2} - t_2^*). \tag{4.8.137}$$

We conclude that $l_2(t_1, t_2^*)$ is at least $c_{h,2}\left(\frac{m_2 - m_1' - s_1}{m_2 - m_1' - s_1 + c_{h,2}} - \psi\right)\left(\hat{t_2} - t_2^*\right)$. Similarly, in the case $\hat{t_2} < t_2^*$, we conclude that $l_2(t_1, t_2^*)$ is at least

$$\begin{aligned} E[(T \leq \hat{t_2})(m_2 - m_1' - s_1) \quad & (t_2^* - \hat{t_2})] \geq \\ & \left(m_2 - m_1' - s_1\right)\left(\frac{c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}} - \psi\right)\left(t_2^* - \hat{t_2}\right). \end{aligned} \tag{4.8.138}$$

In other words,

$$l_2(t_1, t_2^*) \geq \left(\frac{c_{h,2}(m_2 - m_1' - s_1)}{m_2 - m_1' - s_1 + c_{h,2}} - \psi max(m_2 - m_1' - s_1, c_{h,2})\right)|\hat{t_2} - t_2^*|. \tag{4.8.139}$$

COROLLARY 2.2 For a given accuracy level $\epsilon \in (0, \leq 1)$, if $\hat{t_2}$ is ψ-accurate for $\psi = \frac{\epsilon}{3}\frac{min(m_2 - m_1' - s_1, c_{h,2})}{m_2 - m_1' - s_1 + c_{h,2}}$, then the cost of $\hat{t_2}$ is at most $(1 + \epsilon)$ dates the optimal cost, i.e., $l_2(t_1, \hat{t_2}) \leq (1 + \epsilon)l_2(t_1, t_2^*)$.

PROOF.

Let $\psi = \frac{\epsilon}{3}\frac{min(m_2 - m_1' - s_1, c_{h,2})}{m_2 - m_1' - s_1 + c_{h,2}}$ By Lemma 2.3, we know that in this case

$$l_2(t_1, \hat{t_2}) - l_2(t_1, t_2^*) \leq \psi(m_2 - m_1' - s_1 + c_{h,2})|\hat{t_2} - t_2^*|. \tag{4.8.140}$$

and that

$$l_2(t_1, t_2^*) \geq \left(\frac{(m_2 - m_1' - s_1)c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}} - \psi max(m_2 - m_1' - s_1, c_{h,2})\right)|\hat{t_2} - t_2^*|. \tag{4.8.141}$$

It is then sufficient to show that

$$\begin{aligned} & \psi\left(m_2 - m_1' - s_1 + c_{h,2}\right) \leq \\ \epsilon\left(\frac{(m_2 - m_1' - s_1)c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}}\right. & \left. - \psi max(m_2 - m_1' - s_1 +, c_{h,2})\right) \end{aligned} \tag{4.8.142}$$

Indeed,

$$\begin{aligned}
\psi(m_2 - m_1' + m_1 + g + c_{h,2}) &\leq (2+\epsilon)\psi max(m_2 - m_1' - s_1, c_{h,2}) - \epsilon\psi max(m_2 - m_1' - s_1, c_{h,2}) && (4.8.143)\\
&= \frac{(2+\epsilon)\epsilon}{3}\frac{max(m_2 - m_1' - s_1, c_{h,2}) min(m_2 - m_1' - s_1, c_{h,2})}{m_2 - m_1' - s_1 + c_{h,2}} - \epsilon\psi max(m_2 - m_1' - s_1, c_{h,2})\\
&\leq \epsilon\left(\frac{(m_2 - m_1' - s_1)c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}} - \psi max(m_2 - m_1' - s_1, c_{h,2})\right) && (4.8.144)
\end{aligned}$$

We substitute $\psi = \frac{\epsilon}{3}\frac{min(m_2 - m_1' - s_1, c_{h,2})}{m_2 - m_1' - s_1 + c_{h,2}}$ in the first inequality and The second inequality follows as $\epsilon \leq 1$. We conclude that $l_2(t_1, \hat{T}_2) - l_2(t_1, t_2^*) \leq l_2(t_1, t_2^*)$, from which the corollary follows.

We now establish upper bounds on N_β required to guarantee that $\hat{t_2}$, the realization of $\hat{T}_2$, is ψ-accurate with high probability (for each specified $\psi > 0$ and confidence probability $1 - \delta$). Since $\hat{T}_2$ is the sample $\frac{c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}}$- quantile and t_2^* is the true $\frac{c_{h,2}}{m_2 - m_1' - s_1 + c_{h,2}}$-quantile, we can use known results regarding the convergence of sample quantiles to the true quantiles or more generally, the convergence of the empirical CDF $F_{N_\beta}(t_2)$ to the true CDF $F(t_2)$. (For N_β independent random samples all distributed according to T, we define $F_{N_\beta}(t_2) := \frac{1}{N_\beta}\sum_{k=1}^{N_\beta} T^k$, where for each $k = 1, ..., N_\beta$, $T^k = (T_2^k \leq T)$, and $T_2^1, ..., T_2^{N_\beta}$ are i.i.d. according to T.)

Lemma 2.4 *For each $\psi > 0$ and $0 < \delta < 1$, if the number of samples is $= N_\beta > N_\beta(\psi, \delta) = \frac{1}{2}\frac{1}{\psi^2}log(\frac{2}{\delta})$, then $\hat{T}_2$, the $\frac{c_{h,2}}{m_2 - m_1' + c_{h,2} - s_1}$-quantile of the sample, is ψ-accurate with probability at least $1 - \delta$.*

Lemma 2.4 is a direct consequence of the fact that the empirical CDF converges uniformly and exponentially fast to the true CDF.

Combining Lemma 2.3, Corollary 2.2 and Lemma 2.4 above, we can obtain the following theorem.

THEOREM B.2 Consider a rollover problem specified by a date distribution T with $E[T] < \infty$. Let $0 < \epsilon \leq 1$ be a specified accuracy level and $1 - \epsilon$ (for $0 < \delta < 1$) be a specified confidence level. Suppose that $N_\beta \geq \frac{9}{2\epsilon^2}\left(\frac{min(m_2 - m_1' - s_1, c_{h,2})}{m_2 - m_1' + c_{h,2} - s_1}\right)^{-2} log\left(\frac{2}{\delta}\right)$ and the data-drive counterpart is solved with respect to N_β i.i.d samples of T. Let $\hat{T}_1$ be the optimal solution to the data-driven counterpart and $\hat{t_2}$ denote its realization. Then, with probability at least $1 - \delta$, the expected cost of $\hat{t_2}$ is at most $1 + \epsilon$ times the expected cost of an optimal solution t_2^* to the rollover problem. In other words, $l_2(t_1, \hat{T}_2) \leq (1 + \epsilon) l_2(t_1, t_2^*)$ with probability at least $1 - \delta$.

N_β does not depend on the date distribution T, but on the square of the reciprocal of $\frac{min(c_{h,2}, m_2 - m_1' - s_1)}{m_2 - m_1' + c_{h,2} - s_1}$. This means that large samples are required when might be large when $\frac{c_{h,2}}{m_2 - m_1' + c_{h,2} - s_1}$ is very close to either 0 or 1. Since the optimal solution t_2^* is the $\frac{c_{h,2}}{m_2 - m_1' + c_{h,2} - s_1}$-quantile of T, this is consistent with the well-known fact that in order to approximate an extreme quantile one needs many samples. N_β is a worst-case upper bound and it is likely that in many cases a significantly fewer number of samples will suffice.

Appendix F

We have defined the expected loss function at the boundary to be convex and given by

$$l_b(t_b) \quad = \quad (m_2 - m_1') E[t_b - T]^+ + (m_1 + g + c_{h,2}) E[T - t_b]^+. \tag{4.8.145}$$

We denote the right-hand and left-hand derivatives of $l_b(t_b)$, denoted by $l_b^r(t_b)$ and $l_b^l(t_b)$, the one-sided derivatives of $l_b(t_b)$ as follows:

$$l_b^r(t_b) = -(m_1 + g + c_{h,2}) + (m_2 - m_1' + m_1 + g + c_{h,2}) F(t_b), \tag{4.8.146}$$

$$where F(t_b) := F(T \leq t_b)$$

and

$$l_b^l(t_b) = -(m_1 + g + c_{h,2}) + (m_2 - m_1' + m_1 + g + c_{h,2})F(T < t_b). \tag{4.8.147}$$

We assume F to be continuous and therefore $l_b(t_b)$ is continuously differentiable with

$$l_b'(t_b) = -(m_1 + g + c_{h,2}) + (m_2 - m_1' + m_1 + g + c_{h,2})F(t_b). \tag{4.8.148}$$

We can characterize the optimal solution t_b^* by

$$t_b^* = inf\{t_b : F(t_b) > \frac{m_1 + g + c_{h,2}}{m_2 - m_1' + m_1 + g + c_{h,2}}\}, \tag{4.8.149}$$

and t_b^* is the $\frac{m_1+g+c_{h,2}}{m_2-m_1'+m_1+g+c_{h,2}}$-quantile of the distribution of T. From classical optimization theory, $l_b'(t_b^*) = 0$, $l_b^r(t_b^*) \geq 0$, and $l_b^l(t_b^*) \leq 0$ with 0 being a sub-gradient at t_b^*, and hence optimality conditions for $l_b(t_b)$ are satisfied (see Scarf (1959) for details).

Definition 3.1 Let $\hat{t_b}$ be a realization of $\hat{T}_1$ and $\psi > 0$. We define $\hat{t_b}$ to be ψ-accurate if $F(\hat{t_b}) \geq \frac{m_1+g+c_{h,2}}{m_2-m_1'+m_1+g+c_{h,2}} - \psi$ and $\bar{F}(\hat{t_b}) \geq \frac{m_2-m_1'}{m_2-m_1'+m_1+g+c_{h,2}} - \psi$.

This definition can be translated to bounds on the right-hand and left-hand derivatives of l_b at $\hat{t_b}$. We know that$F(T < t_b) = 1 - \bar{F}(t_b)$ and equivalently $\hat{t_b}$ is ψ-accurate when $l_b^r(\hat{t_b}) \geq -\psi(m_2 - m_1' + m_1 + g + c_{h,2})$ and $l_b^l(\hat{t_b}) \leq \psi(m_2 - m_1' + m_1 + g + c_{h,2})$. This implies that there exists a sub-gradient $r \in \Delta l_b(\hat{t_b})$ such that $|r| \leq \psi(m_2 - m_1' + m_1 + g + c_{h,2})$. Therefore, for ψ sufficiently small, 0 is 'almost' a sub-gradient at $\hat{t_b}$, and hence $\hat{t_b}$ is "close" to being optimal.

LEMMA 3.1 Let $\psi > 0$ and assume that $\hat{t_b}$ is $\psi - accurate$. Then:

(i)

$$l_b(\hat{t_b}) - l_b(t_b^*) \leq \psi(m_2 - m_1' + m_1 + g + c_{h,2})|\hat{t_b} - t_b^*|. \tag{4.8.150}$$

(ii)

$$l_b(t_b^*) \geq \left(\frac{(m_2 - m_1')(m_1 + g + c_{h,2})}{m_2 - m_1' + m_1 + g + c_{h,2}} - \psi max(m_2 - m_1', m_1 + g + c_{h,2})\right)|\hat{t_b} - t_b^*|. \tag{4.8.151}$$

Proof. Suppose $\hat{t_b}$ is ψ-accurate. We have $\hat{t_b} \geq t_b^*$ or $\hat{t_b} < t_b^*$. Suppose first that $\hat{t_b} > t_b^*$. We will obtain an upper bound on the difference $l_b(\hat{t_b}) - l_b(t_b^*)$.If $T \in (-\infty, \hat{t_b})$, then the difference between the costs incurred by $\hat{t_b}$ and t_b^* is at most $(m_2 - m_1')(\hat{t_b} - t_b^*)$. On the other hand, if $T \in [\hat{t_b}, \infty)$, then t_b^* has higher cost than $\hat{t_b}$, by exactly$(m_1 + c_{h,2} + g)(\hat{t_b} - t_b^*)$. Given that $\hat{t_b}$ is assumed ψ-accurate, we have the following:

$$P([T \in [\hat{t_b}, \infty)]) = F(T \geq \hat{t_b}) = \bar{F}(\hat{t_b}) \geq \frac{m_2 - m_1'}{m_2 - m_1' + m_1 + g + c_{h,2}} - \psi \tag{4.8.152}$$

and

$$F([T \in [0, \hat{t_b})]) = F(T < \hat{t_b}) = 1 - \bar{F}(\hat{t_b}) \leq \frac{m_1 + g + c_{h,2}}{m_2 - m_1' + m_1 + g + c_{h,2}} + \psi \tag{4.8.153}$$

Therefore

$$\begin{aligned} l_b(\hat{t_b}) - l_b(t_b^*) \quad & \leq \left(m_2 - m_1'\right)\left(\frac{m_1 + g + c_{h,2}}{m_2 - m_1' + m_1 + g + c_{h,2}} + \psi\right)\left(\hat{t_b} - t_b^*\right) \\ & -\left(m_1 + g + c_{h,2}\right)\left(\frac{m_2 - m_1'}{m_2 - m_1' + m_1 + g + c_{h,2}} - \psi\right)\left(\hat{t_b} - t_b^*\right) \\ & = \psi(m_2 - m_1' + m_1 + g + c_{h,2})(\hat{t_b} - t_b^*) \end{aligned} \tag{4.8.154}$$

Similarly, if $\hat{t_b} < t_b^*$, then for each realization $T \in (\hat{t_b}, \infty)$ the difference between the costs of $\hat{t_b}$ and t_b^*, respectively, is at most $(m_1 + g + c_{h,2})(t_b^* - \hat{t_b})$, and if $T \in (-\infty, \hat{t_b}]$, then the cost of $\hat{t_b}$ exceeds the cost of t_b^* by exactly $(m_2 - m_1')(t_b^* - \hat{t_b})$. Given that $\hat{t_b}$ is assumed ψ-accurate, we have

$$F(T \leq \hat{t_b}) = \bar{F}(\hat{t_b}) \geq \frac{m_1 + g + c_{h,2}}{m_2 - m_1' + m_1 + g + c_{h,2}} - \psi \tag{4.8.155}$$

and

$$F(T > \hat{t_b}) = 1 - \bar{F}(\hat{t_b}) \leq \frac{m_2 - m_1'}{m_2 - m_1' + m_1 + g + c_{h,2}} + \psi \tag{4.8.156}$$

We conclude that

$$\begin{aligned} l_b(\hat{t_b}) - l_b(t_b^*) \quad &\leq \Big(m_1 + g + c_{h,2}\Big)\Big(\frac{m_2 - m_1'}{m_2 - m_1' + m_1 + g + c_{h,2}} + \psi\Big)\Big(t_b^* - \hat{t_b}\Big) \\ &-\Big(m_2 - m_1'\Big)\Big(\frac{m_1 + g + c_{h,2}}{m_2 - m_1' + m_1 + g + c_{h,2}} - \psi\Big)\Big(t_b^* - \hat{t_b}\Big) \\ &= \psi(m_2 - m_1' + m_1 + g + c_{h,2})(t_b^* - \hat{t_b}) \end{aligned} \quad (4.8.157)$$

The proof of part *(i)* then follows.

The above arguments also imply that if $\hat{t_b} \geq t_b^*$ then

$$l_b(t_b^*) \geq E[(T \geq \hat{t_b})(m_1 + g + c_{h,2})(\hat{t_b} - t_b^*)] = (m_1 + g + c_{h,2})\bar{F}(\hat{t_b})(\hat{t_b} - t_b^*). \quad (4.8.158)$$

We conclude that $l_b(t_b^*)$ is at least $\Big(m_1 + g + c_{h,2}\Big)\Big(\frac{m_2 - m_1'}{m_2 - m_1' + m_1 + g + c_{h,2}} - \psi\Big)\Big(\hat{t_b} - t_b^*\Big)$.

Similarly, in the case $\hat{t_b} < t_b^*$, we conclude that $l_b(t_b^*)$ is at least

$$E[(T \leq \hat{t_b})(m_2 - m_1')(t_b^* - \hat{t_b})] \geq \quad (4.8.159)$$
$$\Big(m_2 - m_1'\Big)\Big(\frac{m_1 + g + c_{h,2}}{m_2 - m_1' + m_1 + g + c_{h,2}} - \psi\Big)\Big(t_b^* - \hat{t_b}\Big).$$

In other words,

$$l_b(t_b^*) \geq \quad (4.8.160)$$
$$\Big(\frac{(m_2 - m_1')(m_1 + g + c_{h,2})}{m_2 - m_1' + m_1 + g + c_{h,2}} - \psi max(m_2 - m_1', m_1 + g + c_{h,2})\Big)|\hat{t_b} - t_b^*|.$$

COROLLARY 3.1 *For a given accuracy level $\epsilon \in (0, \leq 1]$, if $\hat{t_b}$ is ψ-accurate for*

$$\psi = \frac{\epsilon}{3}\frac{min(m_2 - m_1', m_1 + g + c_{h,2})}{m_2 - m_1' + m_1 + g + c_{h,2}}, \quad (4.8.161)$$

then the cost of $\hat{t_b}$ is at most $(1 + \epsilon)$ dates the optimal cost, i.e.,

$$l_b(\hat{t_b}) \leq (1 + \epsilon)l_b(t_b^*). \quad (4.8.162)$$

PROOF.

Let $\psi = \frac{\epsilon}{3}\frac{min(m_2-m_1',m_1+g+c_{h,2})}{m_2-m_1'+m_1+g+c_{h,2}}$ By Lemma 3.1, we know that in this case

$$l_b(\hat{t_b}) - l_b(t_b^*) \le \psi(m_2 - m_1' + m_1 + g + c_{h,2})|\hat{t_b} - t_b^*|. \tag{4.8.163}$$

and that

$$l_b(t_b^*) \ge \left(\frac{(m_2-m_1')(m_1+g+c_{h,2})}{m_2-m_1'+m_1+g+c_{h,2}} - \psi max(m_2-m_1', m_1+g+c_{h,2})\right)|\hat{t_b}-t_b^*|. \tag{4.8.164}$$

It is then sufficient to show that

$$\psi\left(m_2-m_1'+m_1+g+c_{h,2}\right) \le \epsilon\left(\frac{(m_2-m_1')(m_1+g+c_{h,2})}{m_2-m_1'+m_1+g+c_{h,2}} - \psi max(m_2-m_1', m_1+g+c_{h,2})\right) \tag{4.8.165}$$

Indeed,

$$\begin{aligned}
& \psi(m_2 - m_1' + m_1 + g + c_{h,2}) \le (2+\epsilon)\psi max(m_2 - m_1', m_1 + g + c_{h,2}) \\
& -\epsilon\psi max(m_2 - m_1', m_1 + g + c_{h,2}) \\
= \;& \frac{(2+\epsilon)\epsilon}{3}\frac{max(m_2 - m_1', m_1 + g + c_{h,2})min(m_2 - m_1', m_1 + g + c_{h,2})}{m_2 - m_1' + m_1 + g + c_{h,2}} \\
& -\epsilon\psi max(m_2 - m_1', m_1 + g + c_{h,2}) \\
\le \;& \epsilon\left(\frac{(m_2 - m_1')(m_1 + g + c_{h,2})}{m_2 - m_1' + m_1 + g + c_{h,2}} - \psi max(m_2 - m_1', m_1 + g + c_{h,2})\right)
\end{aligned} \tag{4.8.166}$$

We substitute $\psi = \frac{\epsilon}{3}\frac{min(m_2-m_1',m_1+g+c_{h,2})}{m_2-m_1'+m_1+g+c_{h,2}}$ in the first equality and the second inequality follows as $\epsilon \le 1$. We conclude that $l_b(\hat{T}_1) - l_b(t_b^*) \le \epsilon l_b(t_b^*)$, from which the corollary follows.

We now establish upper bounds on the number of samples N_β required in order to guarantee that $\hat{t_b}$ is ψ-accurate with high probability (for each specified $\psi > 0$ and confidence probability $1-\delta$). Since $\hat{T}_1$ is the sample $\frac{m_1+g+c_{h,2}}{m_2-m_1'+m_1+g+c_{h,2}}$- quantile and t_b^* is the true $\frac{m_1+g+c_{h,2}}{m_2-m_1'+m_1+g+c_{h,2}}$-quantile, we can use known results regarding the convergence of sample quantiles to the true quantiles or more generally, the convergence of the empirical CDF $F_{N_\beta}(t_b)$ to the true CDF $F(t_b)$. (For N_β independent random samples all distributed according to T, we define $F_{N_\beta}(t_b) := \frac{1}{N_\beta}\sum_{k=1}^{N} T^k$, where for each

$k = 1, ..., N_\beta$, $T^k = (T^k \leq T)$, and $T^1, ..., T_\beta^N$ are i.i.d. according to T.)

Lemma 3.2 *For each $\psi > 0$ and $0 < \delta < 1$, if the number of samples is $= N_\beta > N_\beta(\psi, \delta) = \frac{1}{2}\frac{1}{\psi^2}log(\frac{2}{\delta})$, then $\hat{T}_b$, the $\frac{m_1+g+c_{h,2}}{m_2-m_1'+m_1+g+c_{h,2}}$-quantile of the sample, is ψ-accurate with probability at least $1-\delta$.*

Lemma 3.2 is a direct consequence of the fact that the empirical CDF converges uniformly and exponentially fast to the true CDF (Hoeffding Inequality (1963)).

Combining Lemma 3.1, Corollary 3.1 and Lemma 3.2 above, we can obtain the following theorem.

THEOREM C Consider a rollover problem specified by a date distribution T with $E[T] < \infty$. Let $0 < \epsilon \leq 1$ be a specified accuracy level and $1-\epsilon$ (for $0 < \delta < 1$) be a specified confidence level. Suppose that $N_\beta \geq \frac{9}{2\epsilon^2}\left(\frac{min(m_2-m_1', m_1+g+c_{h,2})}{m_2-m_1'+m_1+g+c_{h,2}}\right)^{-2} log\left(\frac{2}{\delta}\right)$ and the data-driven problem is solved with respect to N_β i.i.d samples of T. Let $\hat{T}_1$ be the optimal solution to the data-driven counterpart and $\hat{t_b}$ denote its realization. Then, with probability at least $1-\delta$, the expected cost of $\hat{t_b}$ is at most $1+\epsilon$ times the expected cost of an optimal solution t_b^* to the rollover problem. In other words, $l_b(\hat{T}_1) \leq (1+\epsilon)l_b(t_b^*)$ with probability at least $1-\delta$.

N_β does not depend on the date distribution T, but on the square of the reciprocal of $\frac{min(m_2-m_1', m_1+g+c_{h,2})}{m_2-m_1'+m_1+g+c_{h,2}}$. This means that large samples are required when might be large when $\frac{m_1+g+c_{h,2}}{m_2-m_1'+m_1+g+c_{h,2}}$ is very close to either 0 or 1. Since the optimal solution t_b^* is the $\frac{m_1+g+c_{h,2}}{m_2-m_1'+m_1+g+c_{h,2}}$-quantile of T, this is consistent with the well-known fact that in order to approximate an extreme quantile one needs many samples. N_β is a worst-case upper bound and it is likely that in many cases a significantly fewer number of samples will suffice.

CHAPTER 5

Limitations and Future Research Directions

This PhD thesis is based on three papers. We have presented different variations of the product rollover problem. My research work has mainly been concerned with solving the problem and providing managerial insights. We have used tractable methodologies to provide closed form solutions when possible.

Each of the three papers answers questions, but also suggests new ones for future research. Some specific follow-up questions are discussed in the respective papers. In all three papers, we develop the problem over a single period for one product. Real-world supply chain structures are often more complex, and may have multiple rollover processes over several cycles. In the context of product rollover, it is extremely interesting to extend the results to more complex systems and extend it to include capacity constraints. One difficulty is how to establish valid order bounds for stages that serve multiple products or demand markets. Supply chains with both capacity constraints and multiple products are left for future investigations.

This speaks to a general theme: it is often challenging to analytically address various challenges one at the time. However, when multiple complications arise it is often very difficult, if not impossible, to mathematically characterize the necessary optimal conditions and determine uniqueness of solutions. The resolution may be higher reliance on simulation and numerical solutions.

In our three papers, we assume that discounting cash flows is not necessary, and this may be a limitation. All parameters should be adjusted to reflect the effect of interest and tax also.

Furthermore, our model ignores competitive action that might impact the demand for the new product if the new product is introduced later than the competition's product. However, planners usually know that their product are subject to an approval date in our case thus giving management some forewarning about competitive actions. Therefore, managers most likely can anticipate the market reaction to new product introduction and estimate reflective price and demand parameters for the model.

In some situations, the old and new products share some machine and labor capacity. We can still use our model to optimize rollover procedure, however, capacity contention should be considered in the inventory build-up.

We have assumed in our model that procurement leadtimes, procurement yields, manufacturing leadtimes and manufacturing yields are deterministic. A stochastic simulation model could be implemented to explore these issues.

From both a theoretical and a practical perspective, it would be desirable to explore further the points listed above. It would be valuable to investigate in depth how different these issues affect optimality criteria and conditions and lead to different strategies, and if there are specific situations for which our assumption framework is not particularly well suited.

Introduction de Nouveaux Produits dans la Supply Chain : Optimisation et Management des Risques

Résumé

Les consommateurs d'aujourd'hui ont des goûts très variés et cherchent les produits les plus récents. Avec l'accélération technologique, les cycles de vie des produits se sont raccourcis et donc, de nouveaux produits doivent être introduits au marché plus souvent et progressivement, les anciens doivent y être retirés. L'introduction d'un nouveau produit est une source de croissance et d'avantage concurrentiel. Les directeurs du Marketing et Supply Chain se sont confrontés à la question de savoir comment gérer avec succès le remplacement de leurs produits et d'optimiser les coûts de la chaîne d'approvisionnement associée. Dans une situation idéale, la procédure de rollover est efficace et claire: l'ancien produit est vendu jusqu'à une date prévue où un nouveau produit est introduit. Dans la vie réelle, la situation est moins favorable. Le but de notre travail est d'analyser et de caractériser la politique optimale du rollover avec une date de disponibilité stochastique pour l'introduction du nouveau produit sur le marché. Pour résoudre le problème d'optimisation, nous utilisons dans notre premier article deux mesures de minimisation: le coût moyen et le coût de la valeur conditionnelle à risque. On obtient des solutions en forme explicite pour les politiques optimales. En outre, nous caractérisons l'influence des paramètres de coûts sur la structure de la politique optimale. Dans cet esprit, nous analysons aussi le comportement de la politique de rollover optimale dans des contextes différents. Dans notre deuxième article, nous examinons le même problème mais avec une demande constante pour le premier produit et une demande linéaire au début puis constante pour le deuxième. Ce modèle est inspiré par la demande de Bass. Dans notre troisième article, la date de disponibilité du nouveau produit existe mais elle est inconnue. La seule information disponible est un ensemble historique d'échantillons qui sont tirés de la vraie distribution. Nous résoudrons le problème avec l'approche data driven et nous obtenons des formulations tractables. Nous développons aussi des bornes sur le nombre d'échantillons nécessaires pour garantir qu'avec une forte probabilité, le coût n'est pas très loin du vrai coût optimal.

Mots-clés: Product rollover, date de disponibilité, planned stock-out rollover, single product rollover, dual product rollover, critère d'optimisation des risques, valeur conditionnelle à risque, dominance stochastique; comparaison stochastique, modèle de Bass, théorie de diffusion, optimisation data-driven.

Introduction of New Products in the Supply Chain: Optimization and Management of Risks

Abstract

Shorter product life cycles and rapid product obsolescence provide increasing incentives to introduce new products to markets more quickly. As a consequence of rapidly changing market conditions, firms focus on improving their new product development processes to reap the benefits of early market entry. Researchers have analyzed market entry, but have seldom provided quantitative approaches for the product rollover problem. This research builds upon the literature by using established optimization methods to examine how firms can minimize their net loss during the rollover process. Specifically, our work explicitly optimizes the timing of removal of old products and introduction of new products, the optimal strategy, and the magnitude of net losses when the market entry approval date of a new product is unknown. In the first paper, we use the conditional value at risk to optimize the net loss and investigate the effect of risk perception of the manager on the rollover process. We compare it to the minimization of the classical expected net loss. We derive conditions for optimality and unique closed-form solutions for single and dual rollover cases. In the second paper, we investigate the rollover problem, but for a time-dependent demand rate for the second product trying to approximate the Bass Model. Finally, in the third paper, we apply the data-driven optimization approach to the product rollover problem where the probability distribution of the approval date is unknown. We rather have historical observations of approval dates. We develop the optimal times of rollover and show the superiority of the data-driven method over the conditional value at risk in case where it is difficult to guess the real probability distribution.

Keywords: Product rollover, uncertain approval date, planned stock-out rollover, single product rollover, dual product rollover, risk sensitive optimization criterion, conditional value at risk, stochastic dominance; stochastic comparisons, bass demand, product demand diffusion, data-driven optimization.