Jeux répétés en réseaux et communication

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A mes parents et mon frère
A la mémoire de Jean-Pierre
A Guillaume

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Introduction

1

La théorie des jeux répétés modélise des situations dans lesquelles plusieurs agents interagissent de façon stratégique au cours du temps. Les agents (joueurs) prennent des décisions (actions) à chaque étape, ce qui génère un paiement. Le paiement global d'un joueur est fonction de la suite de ses paiements d'étapes. [1] La répétition d'une interaction au cours du temps peut permettre aux agents de coopérer, alors que ce ne serait pas possible si le jeu n'était joué qu'une seule fois. Par exemple, considérons l'exemple suivant, appelé dilemme du prisonnier :

FIGURE 1.1: Le dilemme du prisonnier

	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$

Le joueur 1 choisit la ligne du tableau, le joueur 2 la colonne, et les nombres dans chaque cellule correspondent aux paiements des joueurs 1 et 2 respectivement. Si l'interaction entre les joueurs 1 et 2 n'a lieu qu'une seule fois, le seul équilibre de Nash (c'est-à-dire profil d'actions stratégiquement rationnel) est (D, D), dont résulte un paiement de 1 pour chaque joueur. Au contraire, si les joueurs interagissent un nombre infini de fois, et si les joueurs observent les actions passées de leur adversaire, alors le résultat $(3, 3)$ devient un paiement d'équilibre. En effet, étudions la stratégie suivante: chaque joueur choisit l'action C aussi longtemps que l'adversaire en a fait autant, et choisit D pour toujours dès que l'adversaire joue D. Si les joueurs pensent que leur adversaire joue cette stratégie, alors ils n'ont pas d'incitations à jouer l'action D, puisque le gain de court terme obtenu en jouant D à une étape est plus faible que les pertes futures qui en découlent.

Ainsi, la répétition d'un jeu au cours du temps augmente l'ensemble des paiements d'équilibres. Dans le cas du dilemme du prisonnier, coopérer est un équilibre

1. Différentes évaluations du paiement global existent et sont présentées dans la partie 1.1.

du jeu répété. Dans le cas d'un oligopole (Cournot ou Bertrand), la collusion entre les entreprises est possible si la concurrence se répète infiniment au cours du temps. Ce type de résultat est connu sous le nom de folk théorème (voir Aumann et Shapley, 1976 ; Rubinstein, 1977 ; Fudenberg et Maskin, 1986 et Sorin, 1986): tous les paiements réalisables et individuellement rationnels sont des paiements d'équilibre du jeu répété.[2] Ce type de résultat repose sur le fait que les actions passées de tous les joueurs sont observables: on parle alors d'*observation parfaite*. Cependant, dans de nombreux contextes, il ne paraît pas réaliste de faire cette hypothèse (par exemple, lorsque la décision d'un agent concerne son niveau d'effort). On parle alors d'*observation imparfaite*.

Dans les démonstrations usuelles, l'hypothèse d'observation parfaite est cruciale pour obtenir un folk théorème et construire les équilibres correspondants. En présence d'observation imparfaite, les phénomènes suivants peuvent apparaître : (1) des déviations peuvent être indétectables, c'est-à-dire qu'un joueur peut choisir une autre action que celle prescrite sans modifier l'information des autres joueurs ; (2) lorsqu'une déviation est détectée, plusieurs joueurs peuvent être soupçonnés, et il n'est pas toujours possible de punir plusieurs joueurs en même temps ; et (3) certaines déviations peuvent être détectées uniquement par certains joueurs. Introduire de la communication entre les joueurs peut alors permettre de transmettre ses propres observations, et d'obtenir de l'information de la part des autres joueurs. De plus, on peut modéliser certaines situations avec observation imparfaite par un réseau : chaque joueur observe uniquement les actions jouées par ses voisins dans le passé (Ben-Porath et Kahneman, 1996 ; Renault et Tomala, 1998 ; Tomala, 2011). Ben-Porath et Kahneman (1996) supposent que tous les joueurs peuvent communiquer entre eux, alors que Renault et Tomala (1998) et Tomala (2011) étudient un modèle dans lequel la communication est locale, c'est-à-dire chaque joueur ne peux communiquer qu'avec ses voisins. La communication et les réseaux vont alors de pair : la communication entre individus crée un réseau parmi eux, modélisant le flux d'information.

De façon indépendante, les réseaux de communication sont largement étudiés en cryptographie (voir par exemple Dolev et al, 1993), notamment en ce qui concerne la transmission d'information dans un réseau. Cependant, les notions d'incitations et d'équilibre sont souvent ignorées dans cette littérature, qui se caractérise plutôt par la recherche certaines propriétés de la transmission d'information ("reliability", "security", etc).

Enfin, de nombreux travaux en économie étudient des réseaux sociaux, dans lesquels les agents n'interagissent qu'avec leurs voisins (voir par exemple Jackson, 2008 ;

2. Différentes concepts d'équilibre sont présentés dans la partie 1.1.6.

Goyal, 2009). Cependant, ces modèles ne considèrent pas d'interactions répétées au cours du temps.

Cette thèse cherche à répondre aux questions suivantes : que se passe-t-il lorsque des interactions locales sont répétées au cours du temps ? Quels sont les équilibres de tels jeux répétés ? Quel est l'influence de la nature de la communication ?

Dans la partie suivante, un modèle général de jeux répétés avec signaux est présenté. La partie 1.2 présente les principaux résultats présents dans les littératures sur les jeux répétés, les réseaux économiques et la cryptographie. Puis, les contributions de la thèse et un résumé des résultats sont présentés (partie 1.3), et des applications économiques sont développées (partie 1.4). Enfin, des questions ouvertes et pistes pour des recherches futures sont présentées dans la partie 1.5.

1.1 Modèle général

1.1.1 Données du jeu

Un jeu répété Γ (à information complète) [3] est décrit par les données suivantes :

- un ensemble de joueurs N, avec $N = \{1, \dots, n\}$;
- pour chaque joueur $i \in N$, un sous-ensemble $\mathcal{I}^i \subseteq N$ représentant les joueurs avec lesquels le joueur i interagit. On fait l'hypothèse suivante: pour tout joueur $j \in N$, $j \in \mathcal{I}^i$ si et seulement si $i \in \mathcal{I}^j$;
- pour chaque joueur $i \in N$, un ensemble fini et non vide d'actions A^i. On note $A^{\mathcal{I}^i} = \prod_{j \in \mathcal{I}^i} A^j$;
- une fonction de paiement $g \, : \, A^{\mathcal{I}^i \cup \{i\}} \to \mathbb{R}^N$;
- une *structure d'observation* : pour chaque joueur i, un ensemble fini et non vide de signaux Y^i, et une probabilité de transition $Q \, : \, A \to \Delta(Y)$, avec $A = \prod_{i \in N} A^i$ et $Y = \prod_{i \in N} Y^i$.

Le jeu se déroule de la manière suivante. A chaque étape $t \in \mathbb{N}^*$, chaque joueur $i \in N$ choisit une action $a_t^i \in A^i$ indépendamment des autres joueurs (les décisions sont simultanées). Si le profil d'actions choisies est $a_t \in A$, alors un profil de signaux $y_t = (y_t^i)_{i \in N}$ est tiré selon $Q(.|a_t)$, et le joueur i observe y_t^i. La suite des paiements du joueur i est alors $(g^i(a_t^{\mathcal{I}^i \cup \{i\}}))_{t \geq 1}$, avec $a^{\mathcal{I}^i \cup \{i\}} = (a^j, a^i)_{j \in \mathcal{I}^i}$.

Différents réseaux peuvent modéliser les structures d'interaction, d'observation ou de communication, que l'on définit dans les parties suivantes. De façon générale, $G = (N, E)$ est un graphe non orienté, avec $E \subseteq N \times N$ est un ensemble d'arêtes.

3. Dans cette thèse, on ne s'intéresse qu'aux jeux répétés à information complète, appelés aussi superjeux. Dans les jeux répétés à information incomplète, un état de la nature est tiré selon une distribution aléatoire au début du jeu. Certains joueurs peuvent être informés de cet état, d'autres non.

Pour chaque joueur $i \in N$, on définit $\mathcal{N}(i) = \{j \neq i \ : \ ij \in E\}$ qui représente l'ensemble des voisins du joueur i dans le graphe G. Comme G est non orienté, on déduit : $i \in \mathcal{N}(j) \Leftrightarrow j \in \mathcal{N}(j)$. Par la suite, on différencie le graphe d'interaction G_I, le graphe d'observation G_M (M pour "monitoring" en Anglais) et le graphe de communication G_C, et les ensembles de voisins du chaque joueur i correspondants, $\mathcal{N}_I(i)$, $\mathcal{N}_M(i)$ et $\mathcal{N}_C(i)$.

1.1.2 Structures d'interaction

Dans cette partie, on définit différentes structures d'interaction d'un jeu, en fonction de l'ensemble des joueurs avec lesquels chaque joueur interagit.

Définition 1.1.1. Interaction globale.
Un jeu est dit à *interaction globale* si tous les joueurs interagissent entre eux: $\mathcal{I}^i = N$ pour tout joueur $i \in N$.

Définition 1.1.2. Interaction locale.
Un jeu est dit à *interaction locale* s'il existe une réseau d'interaction (non-orienté) G_I tel que les joueurs interagissent uniquement avec leurs voisins dans le graphe G_I: pour tout joueur $i \in N$, $\mathcal{I}^i = \mathcal{N}_I(i)$ dans G_I.

Remarque 1.1.3. Si le graphe G_I est complet dans la définition d'interaction locale, alors le jeu est à interaction globale.

1.1.3 Structures d'observation

Les définitions suivantes introduisent différents modèles d'observation dans un jeu répété.

Définition 1.1.4. Observation parfaite.
Le jeu répété est dit à *observation parfaite* si le signal de chaque joueur révèle le profil d'actions de tous les joueurs : pour chaque joueur $i \in N$, $Y^i = A$ et $Q((y^i)_{i \in N}|a) = \mathbb{1}_{\{\forall i,\, y^i = a\}}$.

Définition 1.1.5. Observation locale.
Un jeu répété est dit à *observation locale* s'il existe un réseau d'observation G_M tel que chaque joueur observe uniquement les actions de ses voisins dans G_M : pour chaque joueur $i \in N$, $Y^i = A^{\mathcal{N}_M^i}$ et $Q((y^i)_{i \in N}|a) = \mathbb{1}_{\{\forall i,\, y^i = a^{\mathcal{N}_M(i) \cup \{i\}}\}}$.

Remarque 1.1.6. Si le graphe G_M est complet dans la définition d'observation locale, alors le jeu est à observation parfaite.

Définition 1.1.7. Paiements observables.
Un jeu répété est à *paiements observables* lorsque chaque joueur i peut déduire son propre paiement de son action et de son signal. C'est le cas s'il existe une application $\rho : A^i \times Y^i \to \mathbb{R}$ telle que pour chaque profil d'actions a, $Q((y^i)_{i \in N}|a) = \mathbb{1}_{\{\forall i,\, g^i(a) = \rho(a^i, y^i)\}}$.

Remarque 1.1.8. Si un jeu est à interaction locale et observation locale avec $G_I = G_M$, alors les paiements du jeu sont observables.

1.1.4 Structures de communication

Un jeu répété est dit *avec communication* si les joueurs peuvent communiquer à chaque étape. Pour cela, chaque joueur $i \in N$ est doté d'un ensemble non vide de messages M^i qu'il peut envoyer. Cet ensemble peut être fini ou infini selon le modèle étudié. A chaque étape $t \in \mathbb{N}^*$, chaque joueur $i \in N$ choisit son action a_t^i, et envoie en même temps un message $m_t^i(j)$ à tous les joueurs $j \in \mathcal{C}^i$, avec $\mathcal{C}^i \subseteq N$ l'ensemble des joueurs avec lesquels i peut communiquer. On définit maintenant différentes structures de communication en fonction de l'ensemble des joueurs avec lesquels chaque joueur peut communiquer d'une part, et de la nature de la communication d'autre part.

Définition 1.1.9. Communication globale.
Dans un jeu, on parle de *communication globale* si chaque joueur $i \in N$ peut communiquer avec tous les autres joueurs: $\mathcal{C}^i = N$ pour tout joueur $i \in N$.

Définition 1.1.10. Communication locale.
Dans un jeu répété, on parle de *communication locale* lorsqu'il existe un réseau de communication G_C tel que les joueurs ne peuvent communiquer qu'avec leurs voisins dans G_C: pour chaque joueur $i \in N$, $\mathcal{C}^i = \mathcal{N}_C(i)$ dans G_C.

Remarque 1.1.11. Dans la définition de communication locale, si le réseau G_C est le graphe complet, alors la communication est globale.

Définition 1.1.12. Communication privée versus publique.
On parle de *communication publique* lorsqu'à chaque étape $t \geq 1$, chaque joueur $i \in N$ est contraint d'envoyer le même message à tous les joueurs dans $\mathcal{C}^i$:

$$\forall i \in N,\, \forall j \in \mathcal{C}^i,\, m_t^i(j) = m_t^i.$$

A l'inverse, on parle de *communication privée* lorsque chaque joueur $i \in N$ peut envoyer des messages différents à différents joueurs.

Remarque 1.1.13. En cryptologie et en informatique plus généralement, la communication publique est appelée *broadcast*; la communication privée, *unicast.*

Remarque 1.1.14. Dans la littérature économique, le terme *annonces publiques* fait généralement référence à une communication publique et globale.

Dans la partie suivante, on définit les stratégies du jeu répété.

1.1.5 Stratégies

On suppose que les joueurs ont une *mémoire parfaite*, c'est-à-dire qu'ils se rappellent des toutes les actions qu'ils ont jouées dans le passé, de tous les signaux qu'ils ont observés, et s'il y a de la communication, de tous les messages qu'ils ont reçus et envoyés. La description complète du jeu est connaissance commune. On définit l'ensemble des histoires privées d'un joueur $i \in N$ jusqu'à l'étape t par $H_t^i = (A^i \times Y^i)^t$ s'il n'y a pas de communication, et par $H_t^i = (A^i \times Y^i \times (M^i)^{\mathcal{C}^i} \times (M^j)_{\{j\,:\,i \in \mathcal{C}^j\}})^t$ si la communication est autorisée (H_0^i est un singleton). Un élément h_t^i est une histoire privée du joueur i de longueur t. Une *stratégie d'action* du joueur i est définie par $\sigma^i = (\sigma_t^i)_{t \geq 1}$, où pour chaque étape t, σ_t^i est une application de H_{t-1}^i dans $\Delta(A^i)$ ($\Delta(A^i)$ est l'ensemble des distributions de probabilités sur A^i). Si la communication est autorisée, on définit une *stratégie de communication* du joueur i par $\phi^i = (\phi_t^i)_{t \geq 1}$, où pour chaque étape t, ϕ_t^i est une application de H_{t-1}^i dans $\Delta((M^i)^{\mathcal{C}^i})$. Une *stratégie de comportement* du joueur $i \in N$ est la donnée de la paire (σ^i, ϕ^i) ou de σ^i selon que la communication est autorisée ou non. On appelle Σ^i l'ensemble des stratégies d'action du joueur i et Φ^i l'ensemble de ses stratégies de communication. On note $\sigma = (\sigma^i)_{i \in N} \in \prod_{i \in N} \Sigma^i$ le profil des stratégies d'action des joueurs, et $\phi = (\phi^i)_{i \in N} \in \prod_{i \in N} \Phi^i$ le profil de leurs stratégies de communication. Soit H_t l'ensemble des histoires (totales) de longueur t représentant les suites d'actions, de signaux et éventuellement de messages durant t étapes. On note H_∞ l'ensemble des histoires infinies possibles.

Un profil σ (respectivement (σ, ϕ) avec communication) définit une distribution de probabilité $\mathbb{P}_\sigma$ (respectivement $\mathbb{P}_{\sigma,\phi}$) sur l'ensemble des parties H_∞, et l'on note $\mathbb{E}_\sigma$ (respectivement $\mathbb{E}_{\sigma,\phi}$) l'espérance correspondante.

1.1.6 Paiements et équilibres du jeu répété

On étudie maintenant l'évaluation des paiements du jeu répété et d'équilibres. Plusieurs variantes du jeu répété sont classiquement étudiées, on ne s'intéresse ici qu'aux jeux infiniment répétés. Dans ces derniers, on distingue les *jeux escomptés* et les *jeux uniformes* (appelés aussi *jeux non escomptés*).

1.1.6.1 Jeu escompté

Dans le jeu escompté, noté $\Gamma_\delta(g)$, chaque joueur $i \in N$ maximise l'espérance de la somme (normalisée) des paiements escomptés, c'est-à-dire :

$$\gamma^i_\delta(\sigma, \phi) = \mathbb{E}_{\sigma,\phi}\left[(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}g^i_t(a^i_t, a^{\mathcal{I}^i}_t)\right],$$

où $\delta \in [0,1)$ est un taux d'escompte commun. On considère plusieurs notions d'équilibres.

Définition 1.1.15. Equilibre de Nash du jeu répété.
Un profil (σ, ϕ) est un équilibre de Nash si aucun joueur ne peut augmenter son paiement espéré en jouant unilatéralement une stratégie alternative (τ^i, ψ^i).

Un vecteur de paiements $(\gamma^1_(\sigma, \phi), \ldots, \gamma^n(\sigma, \phi)) \in \mathbb{R}^N$ est un *paiement d'équilibre de Nash du jeu répété* s'il existe un taux d'escompte $\bar{\delta} \in (0,1)$ tel que, pour tout $\delta \in (\bar{\delta}, 1)$, $(\gamma^1(\sigma, \phi), \ldots, \gamma^n(\sigma, \phi))$ est induit par un équilibre de Nash du jeu δ-escompté.

On note $E^N_\delta(g)$ l'ensemble des paiements d'équilibre de Nash du jeu répété. Un inconvénient des équilibres de Nash du jeu répété est l'absence de *rationalité séquentielle* de la part des joueurs. La rationalité séquentielle requiert que les stratégies soient optimales en dehors du chemin d'équilibre., c'est-à-dire que les joueurs maximisent leur paiement après n'importe quelle histoire, y compris après avoir observé une déviation. Avant de définir plusieurs concepts d'équilibre basés sur la rationalité séquentielle, on définit la notion de système de croyances.

Définition 1.1.16. Système de croyances.
Un *système de croyances* est une suite $\mu = (\mu^i_t)_{t\geq 1, i\in N}$ avec $\mu^i_t \;\; H^i_t \to \Delta(H_t)$: : étant donnée l'histoire privée h^i du joueur i, $\mu^i_t(h^i)$ est la distribution de probabilités représentant la croyance du joueur i sur l'histoire totale.

En fonction des propriétés voulues du système de croyances que l'on considère, on peut définir différentes notions d'équilibres basées sur la rationalité séquentielle, en particulier l'*équilibre parfait Bayésien* et l'*équilibre séquentiel* (Kreps et Wilson, 1982).

Définition 1.1.17. Equilibre parfait Bayésien Un *équilibre parfait Bayésien* (PBE par la suite)[4] est un élément $((\sigma, \phi), \mu)$, avec (σ, ϕ) un profil de stratégies et μ un système de croyances tel que :

4. PBE fait référence à la terminologie anglaise, “perfect Bayesian equilibrium”.

(i) *rationalité séquentielle :* pour chaque joueur $i \in N$ et chaque histoire privée h^i, (σ^i, ϕ^i) est une meilleure réponse dans le jeu de continuation après h^i, étant données les stratégies des autres joueurs et la croyance du joueur i sur le passé ;

(ii) *cohérence faible des croyances :* les croyances sont obtenues à partir de (σ, ϕ) par la règle de Bayes lorsqu'elle s'applique, et sont quelconques ailleurs.

Un paiement d'équilibre parfait Bayésien est un élément $(\gamma^1(\sigma, \phi), \ldots, \gamma^n(\sigma, \phi))$ dans $\mathbb{R}^n$, où (σ, ϕ) est un PBE.

Il n'y a pas de définition reconnue sur les restrictions qu'impose le PBE sur les histoires hors équilibre, c'est-à-dire sur ce que signifie "lorsqu'elle s'applique" par rapport à la règle de Bayes. Fudenberg et Tirole (1991) donnent une définition pour les jeux avec observation parfaite des actions. Récemment, González-Díaz et Meléndez-Jiménez (2011) ont introduit une définition pour des jeux généraux, incluant les jeux avec observation imparfaite. Quoi qu'il en soit, dans les constructions d'équilibres parfaits Bayésiens de cette thèse (Chapitres 3 et 4), n'importe quelle définition usuelle de PBE convient (voir discussion à la fin de la Partie 3.2). On note $E_\delta^{PBE}(g)$ l'ensemble des paiements d'équilibre parfait Bayésien du jeu $\Gamma_\delta(g)$.

A la différence des PBE, l'équilibre séquentiel, introduit par Kreps et Wilson (1982), repose sur la *cohérence forte des croyances*, définie comme suit.

Définition 1.1.18. Equilibre séquentiel. [5]
Un *équilibre séquentiel* est un élément $((\sigma, \phi), \mu)$, avec (σ, ϕ) un profil de stratégies et μ un système de croyances tel que :

(i) *rationalité séquentielle :* pour chaque joueur $i \in N$ et chaque histoire privée h^i, (σ^i, ϕ^i) est une meilleure réponse dans le jeu de continuation après h^i, étant données les stratégies des autres joueurs et la croyance du joueur i sur le passé ;

(ii) *cohérence forte des croyances :* les croyances doivent être *consistantes*, c'est-à-dire $((\sigma, \phi), \mu)$ est la limite [6] d'une suite $((\sigma^k, \phi^k), \mu^k)$, où pour chaque k, σ^k et ϕ^k sont complètement mixtes (elles attribuent une probabilité strictement positive à chaque action, ou chaque message respectivement, après n'importe quelle histoire), et μ^k est l'unique croyance dérivée selon la règle de Bayes.

Un paiement d'équilibre séquentiel est un élément $(\gamma^1(\sigma, \phi), \ldots, \gamma^n(\sigma, \phi))$ dans $\mathbb{R}^n$, où (σ, ϕ) est un équilibre séquentiel.

On note $E_\delta^S(g)$ l'ensemble des paiements d'équilibre séquentiel du jeu $\Gamma_\delta(g)$.

5. A l'origine, Kreps et Wilson ont introduit les équilibres séquentiels pour les jeux finiment répétés, mais peuvent être adaptés aux jeux infiniment répétés.

6. On considère généralement la convergence uniforme.

Pour finir, Ely, Hörner et Olszewski (2005) ont défini un raffinement d'équilibre séquentiel, appelé *équilibre belief-free*. Dans cette thèse, on considère une adaptation de la définition d'Ely, Hörner et Olszewski. Pour cela, on note, pour chaque joueur $i \in N$, $(\sigma^i, \phi^i)_{|h_t^i}$ la stratégie de continuation de i après l'histoire privée h_t^i, et $(\sigma^{-i}, \phi^{-i})_{|h_t^{-i}}$ le profil de stratégies de continuation des joueurs $j \neq i$ après leurs histoires privées h_t^j. La notion d'équilibre belief-free est définie comme suit.

Définition 1.1.19. Equilibre belief-free
Un profil de stratégies (σ, ϕ) est un *équilibre belief-free* si $\forall h_t \in H_t$, $\forall i \in N$,

$$(\sigma^i, \phi^i)_{|h_t^i} \in \mathrm{BR}((\sigma^{-i}, \phi^{-i})_{|h_t^{-i}}).$$

où h_t^i est la projection de h_t sur H_t^i, et pour tout joueur $j \neq i$, h_t^j est la projection de h_t sur H_t^j. Un paiement d'équilibre belief-free est un élément $(\gamma^1(\sigma, \phi), \ldots, \gamma^n(\sigma, \phi))$ dans $\mathbb{R}^n$, avec (σ, ϕ) un équilibre belief-free.

On note $E_\delta^B(g)$ l'ensemble des paiements d'équilibre belief-free du jeu $\Gamma_\delta(g)$. La notion d'équilibre belief-free raffine la notion d'équilibre séquentiel, puisque les équilibres belief-free imposent la rationalité séquentielle pour n'importe quelles croyances. Dit autrement, un profil de stratégies (σ, ϕ) est un équilibre belief-free si, pour toute histoire privée $h_t^i \in H_t^i$ du joueur i, $(\sigma^i, \phi^i)_{|h_t^i} \in \mathrm{BR}((\sigma^{-i}, \phi^{-i})_{|h_t^{-i}})$, pour toutes histoires privées h_t^j des joueurs $j \neq i$ qui sont possibles étant donnée la structure d'observation. Par exemple, si deux joueurs i et j observent parfaitement les actions choisies par un joueur k à chaque étape t, alors pour chaque $t \geq 1$, a_t^k doit être la même dans h_t^i et h_t^j.

Remarque 1.1.20. Dans le contexte des jeux répétés à information complète, comme c'est le cas ici, un équilibre belief-free existe toujours. En effet, n'importe quelle suite d'équilibres du jeu en un coup qui est indépendante de l'histoire est un équilibre belief-free.

Proposition 1.1.21. $E_\delta^B(g) \subseteq E_\delta^S(g) \subseteq E_\delta^{PBE}(g)$.

1.1.6.2 Jeu non escompté

Le paiement moyen d'un joueur $i \in N$ jusqu'à l'étape $T \geq 1$ si le profil de stratégies (σ, ϕ) est joué est :[7]

$$\gamma_T^i(\sigma, \phi) = \mathbb{E}_{\sigma,\phi}\left(\frac{1}{T}\sum_{t=1}^{T} g^i(a_t^i, a_t^{\mathcal{I}^i})\right).$$

7. Dans cette partie, on suppose que la communication est autorisée. Dans le cas contraire, la donnée de ϕ est omis dans ce qui suit.

Dans le jeu répété non escompté, les joueurs maximisent $\gamma_T^i(\sigma,\phi)$ quand T tend vers l'infini, et l'on note $\Gamma_\infty(g)$ ce jeu uniforme. On s'intéresse à la notion d'équilibre uniforme introduite par Fudenberg et Levine (1991) et Sorin (1992).

Définition 1.1.22. Equilibre uniforme.
Un profil de stratégies (σ,ϕ) est un équilibre uniforme de $\Gamma_\infty(g)$ si :

(1) Pour tout $\epsilon > 0$, (σ,ϕ) est un ϵ-équilibre de Nash de tout jeu finiment répété assez long, c'est-à-dire :

$$\exists T_0,\ \forall T \geq T_0,\ \forall i \in N,\ \forall \tau^i \in \Sigma^i,\ \forall \psi^i \in \Phi^i,\ \ \gamma_T^i(\tau^i,\sigma^{-i},\psi^i,\phi^{-i}) \leq \gamma_T^i(\sigma,\phi)+\epsilon.$$

(2) Pour chaque joueur $i \in N$, $(\gamma_T^i(\sigma,\phi))$ converge vers un élément $\gamma^i(\sigma,\phi)$ quand T tend vers l'infini.

Un paiement d'équilibre uniforme est un élément $(\gamma^1(\sigma,\phi),\ldots,\gamma^n(\sigma,\phi))$ dans $\mathbb{R}^n$, avec (σ,ϕ) un équilibre uniforme.

On note $E_\infty(g)$ l'ensemble des paiements d'équilibres uniformes du jeu $\Gamma_\infty(g)$.

1.1.7 Paiements réalisables et individuellement rationnels

Pour chaque profil d'actions $a \in A$, soient $g(a) = (g^1(a^1,a^{\mathcal{I}^1}),\ldots,g^n(a^n,a^{\mathcal{I}^n}))$ et $g(A) = \{g(a)\ :\ a \in A\}$. L'enveloppe convexe de $g(A)$, notée $\operatorname{co} g(A)$, représente l'ensemble des *paiements réalisables* du jeu. Par convexité et compacité, $\operatorname{co} g(A)$ contient $E_\infty(g)$, $E_\delta^N(g)$, $E_\delta^{PBE}(g)$, $E_\delta^S(g)$ et $E_\delta^B(g)$. On définit maintenant le niveau de punition d'un joueur $i \in N$.

Définition 1.1.23. Minmax indépendant.
Pour chaque joueur $i \in N$, le niveau *minmax indépendant* du joueur i est défini par :

$$\underline{v}^i = \min_{x^{\mathcal{I}^i} \in \prod_{j\in\mathcal{I}^i}\Delta(A^j)}\ \max_{x^i\in\Delta(A^i)} g^i(x^i,x^{\mathcal{I}^i}).$$

Soient $IR = \{g = (g^1,\ldots,g^n) \in \mathbb{R}^n\ :\ \forall i \in N,\ g^i \geq \underline{v}^i\}$ et $IR^* = \{g = (g^1,\ldots,g^n) \in \mathbb{R}^n\ :\ \forall i \in N,\ g^i > \underline{v}^i\}$ les ensembles des paiements *individuellement rationnels* et *strictement individuellement rationnels* respectivement. Enfin, on note $V^* = \operatorname{co} g(A) \cap IR^*$ l'ensemble des paiements réalisables et strictement individuellement rationnels.

Dans certains jeux répétés avec observation imparfaite, il est possible d'obtenir des paiements d'équilibres inférieurs au niveau minmax indépendant (voir Renault et Tomala, 1998). On peut alors considérer le niveau minmax corrélé, défini comme suit.

Définition 1.1.24. Minmax corrélé.
Pour chaque joueur $i \in N$, le niveau *minmax corrélé* du joueur i est défini par :

$$\underline{w}^i = \min_{x^{\mathcal{I}^i} \in \Delta(A^{\mathcal{I}^i})} \max_{x^i \in \Delta(A^i)} g^i(x^i, x^{\mathcal{I}^i}).$$

Dans un jeu à observation parfaite et sans communication privée (avec ou sans communication publique), les niveaux minmax indépendant et corrélé coïncident: en effet, tous les joueurs possèdent la même information à chaque étape, et il n'est pas possible pour un sous-ensemble de joueurs de se corréler secrètement. Cependant, cela n'est pas toujours le cas lorsque l'observation est imparfaite, ou lorsque les joueurs peuvent communiquer de façon privée. On peut définir comme précédemment les paiements réalisables et (strictement) individuellement rationnels en remplaçant le minmax indépendant par le minmax corrélé. Que le jeu soit répété ou non, il est évident que les paiements d'équilibre dans $E_\infty(g)$, $E_\delta^N(g)$, $E_\delta^{PBE}(g)$, $E_\delta^S(g)$ et $E_\delta^B(g)$ sont supérieurs à $\underline{w}^i$.

Par la suite, lorsque cela n'est pas précisé, on considère le niveau minmax indépendant des joueurs.

Un folk théorème consiste à caractériser l'ensemble des paiements d'équilibre d'un jeu répété escompté ou uniforme. Plus précisément, un folk théorème établit que tous les paiements réalisables et individuellement rationnels sont des paiement d'équilibre du jeu répété, escompté ou uniforme. Dans la partie suivante, on présente les principaux résultats existant.

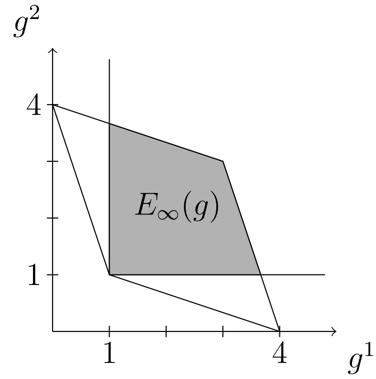
1.2 Revue de littérature

1.2.1 Jeux répétés

A l'origine, le folk théorème a été établi dans les jeux à interaction globale, observation parfaite et sans communication, en considérant les équilibres de Nash du jeu escompté ou les équilibres uniformes du jeu non escompté (Aumann et Shapley, 1976 ; Rubinstein, 1977 ; Fudenberg et Maskin, 1986 et Sorin, 1986).[8] Si l'on reprend l'exemple du dilemme du prisonnier (Figure 1.1), on en déduit que les paiements d'équilibres uniformes du jeu non escompté, notés $E_\infty(g)$, sont les paiements réalisables et individuellement rationnels, dans les jeux avec interaction globale, observation parfaite et sans communication (le niveau minmax, indépendant ou corrélé, de chaque joueur est de 1) :

8. Un folk théorème avec équilibres de Nash du jeu escompté peut être étendu facilement aux équilibres uniformes du jeu non escompté, et vice versa.

Figure 1.2:



Ce résultat est étendu par Fudenberg et Maskin (1986, 1991) aux équilibres sous-jeux parfait,[9] toujours pour des jeux à interaction globale, observation parfaite et sans communication.

Concernant les jeux à observation imparfaite, des résultats existent en ce qui concerne les jeux à observation imparfaite et publique (Fudenberg, Levine et Maskin, 1994), c'est-à-dire qu'un signal public est observé par tous les joueurs. Le modèle de collusion de Green et Porter (1984) concerne également un jeu à observation imparfaite et publique, le prix de marché étant utilisé comme un signal public observable. Pour obtenir un folk théorème, la communication n'est pas nécessaire en présence d'observation publique.

Pour les jeux à observation imparfaite et privée (pas de signal public), Lehrer (1989, 1992 entre autres) a étudié le cas des jeux non escomptés à deux joueurs. Avec plus de deux joueurs, la difficulté réside dans la nécessité pour les joueurs de coordonner leur comportement pour punir d'éventuels déviants. Des travaux récents étudient les équilibres belief-free dans des jeux répétés escomptés à observation privée (voir Mailath et Samuelson, 2006, pour présentation des différents résultats existants). Fudenberg et Levine (1991) établissent un folk théorème pour des jeux escomptés avec observation imparfaite et privée sans communication explicite (ils étudient les équilibres séquentiels). Ces derniers considèrent des signaux aléatoires privés induits par le profil d'actions des joueurs. Avec de la communication globale et publique, Compte (1998), Kandori et Matsushima (1998) et Obara (2009) ont établi des conditions suffisantes pour obtenir un folk théorème du jeu escompté (équilibres séquentiels) : ces quatre auteurs considèrent des signaux aléatoires, mais restreignent leur attention à un sous-ensemble de stratégies autorisées. Tous les modèles précédemment cités ne comportent ni réseau d'interaction, ni d'observation, ni de communication.

9. Un *équilibre sous-jeux parfait* est un profil de stratégies σ tel que, pour toute histoire $h_t \in H_t$, $\sigma_{|h_t}$ est un équilibre de Nash de $\Gamma_\infty(g)$.

Ben-Porath et Kahneman (1996) étudient des jeux répétés avec observation locale et communication publique et globale. Leur principal résultat est le suivant.

Théorème 1.2.1. ***Ben-Porath et Kahneman (1996).***
Considérons un jeu répété escompté avec interaction globale, communication publique et globale, et observation locale. Alors, les assertions suivantes sont équivalentes.

(1) Pour toute fonction de paiement g telle que l'intérieur de V^* est non vide, pour tout paiement $v \in V^*$, il existe un taux d'escompte $\bar{\delta} \in (0,1)$ tel que pour tout $\delta \in (\bar{\delta}, 1)$, v est un paiement d'équilibre séquentiel du jeu δ-escompté.

(2) Chaque joueur i dans N a au moins deux voisins dans le graphe d'observation G_M.

Ben-Porath et Kahneman établissent aussi que la condition (2) du Théorème 1.2.1 est nécessaire et suffisante pour l'obtention d'un folk théorème du jeu non escompté en considérant des équilibres uniformes.

Renault et Tomala (1998) étudient également des jeux répétés à observation locale, mais, à la différence de Ben-Porath et Kahneman, ils introduisent de la communication locale et publique. Ils considèrent les équilibres uniformes du jeu non escompté, et leur résultat est le suivant.

Théorème 1.2.2. ***Renault et Tomala (1998).***
Considérons un jeu répété uniforme avec interaction globale, observation locale, communication publique et locale. Le graphe G modélise à la fois les structures d'observation et de communication : $G = G_M = G_C$. Alors, les assertions suivantes sont équivalentes.

(1) L'ensemble des paiements d'équilibre uniforme est l'ensemble des paiements réalisables et individuellement rationnels, c'est-à-dire :

$$E_\infty(g) = \operatorname{co} g(A) \cap IR.$$

(2) Le graphe G est 2-connexe. [10]

Tomala (2011) a étendu le Théorème 1.2.2 lorsque le réseau G est *partiellement connu* des joueurs, c'est-à-dire que les joueurs ne connaissent que le nombre total de joueurs n ainsi que leurs voisins dans le graphe G.

10. Un graphe G est dit 2-connexe si pour tout i in N, le graphe $G - \{i\}$ est connexe, où $G - \{i\}$ représente le graphe dans lequel le joueur i, ainsi que les arêtes correspondantes, a été supprimé. De façon équivalente par le théorème de Menger, le graphe G est 2-connexe si tous joueurs i et j peuvent être joints par deux chemins indépendants, c'est-à-dire deux chemins n'ayant aucun sommet en commun autre que i et j (voir par exemple, Diestel, 2000, Théorème 3.3.1 page 62).

Récemment, Nava et Piccione (2011) et Cho (2011) ont étudié des jeux dans des réseaux avec interaction locale et observation locale, mais ces auteurs ne prouvent pas de folk théorème.

1.2.2 Réseaux économiques

Il existe également une littérature sur le réseaux économiques (pour un aperçu des principaux résultats dans cette littérature, voir Goyal, 2009, et Jackson, 2008). Certains auteurs (Galeotti et al, 2010 ; Bramoullé et Kranton, 2007) étudient des réseaux d'interactions dans lesquels le paiement d'un joueur ne dépend que de sa propre action et de celles de ses voisins (interaction locale). Cependant, ces travaux ne considèrent pas d'interactions répétées.

1.2.3 Cryptographie

La transmission d'information dans des réseaux de communication fait l'objet de nombreux travaux en cryptographie (voir par exemple Dolev et al, 1993). La communication est alors locale, privée ("unicast") ou publique ("broadcast"). Cependant, mises à part quelques exceptions (Renault et Tomala, 2008), les notions de rationalité ou d'équilibre sont ignorées dans cette littérature : les déviants ne sont pas supposés répondre à des incitations. Ces travaux se concentrent sur la recherche de protocoles de communication satisfaisant certaines propriétés ("reliability", "security", etc). Les techniques de la cryptographie s'avèrent néanmoins très utiles pour la construction de stratégies de communication visant à identifier un éventuel déviant.

1.3 Contributions de la thèse

Cette partie est consacrée à l'exposition des principaux résultats de cette thèse.

1.3.1 Chapitre 2

Dans ce chapitre, j'étudie des jeux répétés avec interaction locale, paiements observables, communication locale et privée. Le graphe G représente à la fois les structures d'interaction et de communication du jeu répété : $G = G_I = G_C$. Les joueurs observent uniquement leur propre paiement à chaque étape, et l'on fait l'hypothèse que les fonctions de paiement des joueurs sont sensibles aux déviations unilatérales (Hypothèse PAYOFFS, P par la suite).

Hypothèse PAYOFFS (P) . *Pour chaque joueur $i \in N$, chaque voisin $j \in \mathcal{N}(i)$ dans le graphe G, chaque actions $b^j, c^j \in A^j$ telles que $b^j \neq c^j$, $a^i \in A^i$, $a^{\mathcal{N}(i)\setminus\{j\}} \in$*

$A^{\mathcal{N}(i)\setminus\{j\}}$:

$$g^i(a^i, a^{\mathcal{N}(i)\setminus\{j\}}, b^j) \neq g^i(a^i, a^{\mathcal{N}(i)\setminus\{j\}}, c^j).$$

Exemple 1.3.1. Les fonctions de paiements suivantes vérifient l'hypothèse P :

- pour tout joueur $i \in N$, soient $A^i \subset \mathbb{N}$ et $g^i(a^i, a^{\mathcal{N}(i)}) = f\left(\sum_{j \in \mathcal{N}(i)\cup\{i\}} a^j\right)$ avec f strictement monotone ;
- profits des entreprises dans un duopole à la Cournot ;
- pour tout joueur $i \in N$, soient $A^i \subset \mathbb{R}$ et $g^i(a^i, a^{\mathcal{N}(i)}) = \sum_{j \in \mathcal{N}(i)} a^j - a^i$ (ce jeu peut s'interpréter comme un dilemme du prisonnier généralisé à n joueurs) ;
- de façon générale, pour tout joueur $i \in N$, soient $A^i \subset \mathbb{R}$ et g^i strictement monotone par rapport à chaque argument.

Le résultat principal est le suivant.

Théorème 1.3.1. *Soit un jeu répété escompté avec interaction locale, communication privée et locale, et paiement observables. Le graphe G modélise à la fois les structures d'interaction et de communication : $G = G_I = G_C$. Alors, les assertions suivantes sont équivalentes.*

(1) Pour toute fonction de paiement qui satisfait l'Hypothèse P, tout paiement réalisable et strictement individuellement rationnel est un paiement d'équilibre de Nash du jeu répété.

(2) Le graphe G satisfait la Condition NETWORKS (Condition N par la suite) : pour chaque joueur $i \in N$, pour tous voisins $j, k \in \mathcal{N}(i)$ tels que $j \neq k$, il existe un joueur $\ell \in \mathcal{N}(j) \setminus \{k\} \triangle \mathcal{N}(k) \setminus \{j\}$ tel qu'il y ait un chemin de ℓ à i qui ne passe ni par j ni par k, avec :

$$\mathcal{N}(j) \setminus \{k\} \triangle \mathcal{N}(k) \setminus \{j\} = (\mathcal{N}(j) \setminus \{k, \mathcal{N}(k)\}) \bigcup (\mathcal{N}(k) \setminus \{j, \mathcal{N}(j)\}).$$

Si la Condition N n'est pas satisfaite, il est possible de construire une fonction de paiement particulière g telle que l'hypothèse P est satisfaite, et telle qu'il existe un paiement réalisable et strictement individuellement rationnel qui n'est pas un équilibre de Nash du jeu répété. Intuitivement, la construction d'un tel exemple repose sur le fait que: (i) il existe un joueur i et deux de ses voisins, j et k, tels que personne ne peut à la fois différencier certaines déviations des joueurs j et k, et transmettre de façon fiable cette information au joueur i; et (ii) g est telle qu'il n'existe pas de profil de stratégies qui permette de punir aux niveaux minmax les joueurs j et k simultanément (le joueur i récompense le joueur j quand il punit le joueur k, et vice versa). En d'autres termes, il n'y a pas de rationalité jointe au sens de Renault et Tomala (2004).

La Condition N est également suffisante pour avoir un folk théorème. Lorsqu'elle est satisfaite, on construit un protocole de communication qui permet d'identifier un joueur qui dévie (de façon unilatérale). On déduit alors le folk théorème.

Ce résultat s'étend si les graphes G_M et G_C sont différents (voir page 55). De plus, la Condition N du Théorème 1.3.1 est également nécessaire et suffisante pour l'obtention d'un folk théorème en prenant en compte les équilibres uniformes du jeu non escompté (voir page 55). Le résultat est également vrai si le réseau G est partiellement connu des joueurs (voir page 54).

Les stratégies de communication construites dans ce chapitre permettent aussi d'étendre le résultat de Renault et Tomala (1998) à la communication privée et locale.

Proposition 1.3.2. *Soit un jeu répété uniforme avec interaction globale, communication privée et locale, et observation locale. Le graphe G modélise à la fois les structures d'observation et de communication et l'on a : $G = G_M = G_C$. Alors, les assertions suivantes sont équivalentes.*

(1) L'ensemble des paiements d'équilibre uniforme est l'ensemble des paiements réalisables et individuellement rationnels, c'est-à-dire :

$$E_\infty(g) = \operatorname{co} g(A) \cap IR.$$

(2) Le graphe G est 2-connexe.

1.3.2 Chapitre 3

Dans ce chapitre, on considère des jeux répétés avec interaction locale, paiements observables et communication globale. Cependant, contrairement au chapitre précédent, on impose de la rationalité séquentielle en étudiant les équilibres parfaits Bayésiens. Le graphe G représente la structure d'interaction : $G = G_I$. On suppose également que le joueurs peuvent communiquer de façon privée ou publique, et que la liste des destinataires d'un message public est *certifiable*, c'est-à-dire tous les joueurs observent la liste des destinataires d'un message public. Les joueurs n'observent que leur propre paiement à chaque étape, et les fonctions de paiement que l'on considèrent satisfont l'Hypothèse P. Le résultat principal est le suivant.

Théorème 1.3.2. *Considérons un jeu répété escompté avec interaction locale, communication globale, privée ou publique, et paiement observables. Le graphe G modélise la structure d'interaction : $G = G_I$. On suppose également que la liste des destinataires d'un message public est certifiable. Alors, les assertions suivantes sont équivalentes.*

(1) *Pour toute fonction de paiement g telle que l'Hypothèse P est satisfaite et l'intérieur de V^* est non vide, pour tout paiement v dans V^*, il existe un taux d'escompte $\bar{\delta} \in (0,1)$ tel que pour tout $\delta \in (\bar{\delta}, 1)$, v est un paiement d'équilibre parfait Bayésien du jeu δ-escompté.*

(2) *Le réseau G est tel que, pour tous joueurs $i, j \in N$, i et j ont au moins un voisin différent (Condition DIFFERENT NEIGHBORS, DN par la suite), c'est-à-dire :*

$$\forall i, j \in N, \quad \mathcal{N}(i) \setminus \{j\} \triangle \mathcal{N}(j) \setminus \{i\} \neq \emptyset. \tag{1.1}$$

Si la Condition DN n'est pas satisfaite, on construit une fonction de paiement particulière telle que l'hypothèse P est satisfaite et $\operatorname{int} V^* \neq \emptyset$, et telle qu'il existe un paiement réalisable et strictement individuellement rationnel qui n'est pas un équilibre de Nash du jeu répété. La preuve repose, comme dans le chapitre précédent, sur une absence de rationalité jointe.

Si la Condition DN est satisfaite, on construit un protocole de communication qui permet d'identifier un joueur qui dévie de façon unilatérale. On construit alors une stratégie d'équilibre parfait Bayésien, qui consiste à punir un joueur qui dévie pendant un nombre fixé d'étapes, puis à récompenser les autres joueurs, afin qu'ils aient une incitation à punir.

La Condition DN du Théorème 1.3.2 est également nécessaire et suffisante pour l'obtention d'un folk théorème du jeu non escompté si l'on considère les équilibres uniformes (voir page 107). De plus, le Théorème 1.3.2 s'étend si l'on considère les niveaux minmax corrélés des joueurs (voir page 104).

1.3.3 Chapitre 4

Dans ce chapitre, on étudie l'influence de la nature de la communication. On considère pour cela des jeux répétés escomptés avec interaction globale et observation locale. On considère deux types de communication : globale et privée d'une part, et locale et publique d'autre part. Le premier résultat étend le Theorème 1.2.1 de Ben-Porath et Kahneman (1996) à la communication privée et aux équilibres belief-free.

Théorème 1.3.3. *Considérons un jeu répété escompté avec interaction globale, observation locale, et communication globale et privée. Le graphe G modélise la structure d'observation : $G = G_M$. Alors, les assertions suivantes sont équivalentes.*

(1) *Pour toute fonction de paiement g telle que l'intérieur de V^* est non vide, et pour tout vecteur $v \in V^*$, il existe un taux d'escompte $\bar{\delta} \in (0,1)$ tel que pour tout $\delta \in (\bar{\delta}, 1)$, v est un paiement d'équilibre belief-free du jeu δ-escompté.*

(2) *Le graphe G vérifie la Condition 2-NEIGHBORS (Condition 2N par la suite): chaque joueur $i \in N$ a au moins deux voisins dans G.*

La nécessité de la Condition 2N pour l'obtention d'un folk théorème (en considérant les équilibres de Nash du jeu répété) est prouvée par Ben-Porath et Kahneman (1996). La preuve de la suffisance de la Condition 2N repose sur la construction d'une stratégie de communication qui permet d'identifier un joueur qui dévie en action, y compris lors de déviations multilatérales.

Les résultats obtenus lorsque l'on considère de la communication locale et publique sont les suivants.

Théorème 1.3.4. *Considérons un jeu répété escompté à $n = 4$ joueurs, avec interaction globale, observation locale, et communication locale et publique. Le graphe G modélise à la fois les structures d'observation et de communication : $G = G_M = G_C$. Alors, les assertions suivantes sont équivalentes.*

(1) Pour toute fonction de paiement g telle que l'intérieur de V^* est non vide, et pour tout vecteur $v \in V^*$, il existe un taux d'escompte $\bar{\delta} \in (0,1)$ tel que pour tout $\delta \in (\bar{\delta}, 1)$, v est un paiement d'équilibre parfait Bayésien du jeu δ-escompté.

(2) Le graphe G est 2-connexe.

Théorème 1.3.5. *Considérons un jeu répété escompté à n joueurs, avec interaction globale, observation locale et communication locale et publique. Le graphe G modélise à la fois les structures d'observation et de communication : $G = G_M = G_C$. Supposons que $N = \{x^1, x^2, \ldots, x^n\}$, avec $n \geq 7$, et que le réseau G est tel que (Condition C) :*

$$\forall i,\ j \in \{1, \ldots, n\}, \quad i - j = -3, -2, -1, 1, 2, 3\ [n] \Rightarrow i \in \mathcal{N}(j). \tag{1.2}$$

Alors, pour toute fonction de paiement g telle que l'intérieur de V^* est non vide, et pour tout vecteur $v \in V^*$, il existe un taux d'escompte $\bar{\delta} \in (0,1)$ tel que pour tout $\delta \in (\bar{\delta}, 1)$, v est un paiement d'équilibre parfait Bayésien du jeu δ-escompté.

1.4 Applications

Une première application des chapitres 2 et 3 est un modèle de concurrence avec interaction locale : seulement quelques entreprises concurrentes (les concurrents les plus directs) ont un impact sur le profit d'une entreprise donnée. De plus, les entreprises n'observent que leur propre profit : en effet, il peut être coûteux voire impossible d'observer les décisions de ses adversaires, y compris des plus proches. Un exemple d'un tel marché est le suivant. Considérons un marché avec différenciation verticale, considérons un marché avec trois entreprises: l'entreprise L vend un produit de luxe, l'entreprise B vend un produit de basse qualité, et l'entreprise

I vend un produit de qualité intermédiaire. L'entreprise L n'est pas en concurrence avec l'entreprise B, cependant toutes les deux sont en concurrence directe avec l'entreprise I. De plus, imaginons que les entreprises font face à deux types de décisions: la capacité de production et la qualité du bien vendu. Si les entreprises n'observent que leur propre profit, l'entreprise I peut être incapable de punir simultanément les entreprises L et B. Par exemple, punir l'entreprise L peut requérir de la part de l'entreprise I d'augmenter la qualité de son produit, et cela peut être avantageux pour l'entreprise B qui subit une concurrence moins importante de la part de l'entreprise I (et réciproquement pour punit l'entreprise B). Cet exemple met en avant l'importance de l'identification d'une entreprise qui dévie. Dans les Chapitres 2 et 3, des conditions nécessaires et suffisantes à la collusion dans un tel modèle sont établies.

Une autre application concerne les jeux de partenariat (Radner, 1986). Supposons que la production est conduite par une équipe dont le but est de maintenir un certain niveau d'effort individuel de la part de chaque membre. Le niveau d'effort de chaque membre est soit non observable (chapitres 2 et 3), soit observable uniquement par ses voisins directs (chapitre 4). On est alors en présence d'aléa moral, l'effort représentant un coût pour l'agent. Selon les différents types d'interaction, d'observation et de communication considérés, les résultats présentés dans cette thèse permettent d'établir des conditions nécessaires et suffisantes à la coordination dans de telles situations.

1.5 Perspectives

1.5.1 Tableaux récapitulatifs

Les tableaux ci-dessous synthétisent les cas dans lesquels les conditions nécessaires et suffisantes pour un folk théorème sont connues à ce jour, selon les différents modèles d'interaction, d'observation et de communication considérés. [11] Le premier tableau (tableau 1.1) concerne les résultats avec équilibres uniformes du jeu non escompté. [12] Le tableau 1.2 concerne les résultats avec équilibres parfaits Bayésiens.

11. Pour les jeux à interaction globale et observation parfaite (sans communication), les folk théorèmes obtenus dans la littérature sont vrais pour toute fonction de paiement. Ainsi, le cas à interaction locale et observation parfaite est un cas particulier des jeux à interaction globale à observation parfaite. Ces résultats sont connus et omis dans les tableaux ci-dessous.

12. On peut considérer de façon équivalente les équilibres de Nash du jeu escompté (voir footnote 8 page 11)

TABLE 1.1: Equilibres uniformes

<table>
<tr><td></td><td></td><td>Observation locale</td><td colspan="2">Observation paiements [a]
(+ Hypothèse P)</td></tr>
<tr><td></td><td></td><td>Interaction globale/locale [b]</td><td>Interaction globale</td><td>Interaction locale</td></tr>
<tr><td rowspan="4">Communication</td><td>globale publique</td><td>Ben-Porath et Kahneman (1996)</td><td rowspan="4">Chapitre 2 : impossible sans hypothèse supplémentaire [c]</td><td>Chapitre 3</td></tr>
<tr><td>globale privée</td><td>Chapitre 4</td><td>Chapitre 2</td></tr>
<tr><td>locale publique</td><td>Renault et Tomala (1998)</td><td>?</td></tr>
<tr><td>locale privée</td><td>Chapitre 2</td><td>Chapitre 2</td></tr>
</table>

a. Les joueurs n'observent que leur propre paiement à chaque étape.
b. Les résultats cités sont obtenus pour des jeux à interaction globale pour toute fonction de paiement. Ainsi, les jeux à interaction locale sont un cas particulier.
c. Voir Exemple 2.2.6 page 34 et Partie 2.3.

TABLE 1.2: Equilibres parfaits Bayésiens

<table>
<tr><td></td><td></td><td>Observation locale</td><td colspan="2">Observation paiements [a]
(+ Hypothèse P)</td></tr>
<tr><td></td><td></td><td>Interaction globale/locale</td><td>Interaction globale</td><td>Interaction locale</td></tr>
<tr><td rowspan="4">Communication</td><td>globale publique</td><td>Ben-Porath et Kahneman (1996) [b]</td><td rowspan="4">Chapitre 3 : impossible sans hypothèse supplémentaire [f]</td><td>Chapitre 3</td></tr>
<tr><td>globale privée</td><td>Chapitre 4 [c]</td><td>? + Chapitre 3 [d]</td></tr>
<tr><td>locale publique</td><td>$n = 4$: Chapitre 4, $n > 4$: ? [e]</td><td>?</td></tr>
<tr><td>locale privée</td><td>?</td><td>?</td></tr>
</table>

a. Les joueurs n'observent que leur propre paiement à chaque étape.
b. En réalité, Ben-Porath et Kahneman étudient les équilibres séquentiels.
c. Le résultat est obtenu pour des équilibres belief-free.
d. Dans le Chapitre 3, on obtient une condition suffisante pour l'obtention d'un folk théorème avec PBE, si l'on considère les niveaux minmax en stratégies pures (voir Corollaire 3.7.1 page 105).
e. Voir cas particuliers, Chapitre 4.
f. Voir Partie 3.6.

1.5.2 Recherches futures

De nombreuses questions restent ouvertes. En particulier, une question importante est de savoir si le Théorème 1.3.2 du Chapitre 3 s'étend si l'on considère de la communication uniquement privée (et globale). En effet, bien que la communication privée soit autorisée dans le Chapitre 3, le folk théorème repose sur le fait que la liste des destinataires est certifiable lorsqu'un joueur envoie un message public. Sans cette hypothèse, la stratégie construite n'est pas un équilibre parfait Bayésien (voir la discussion du Chapitre 3, page 105).

D'autre part, je pense que les résultats du Chapitre 4 concernant des jeux à interaction globale, observation locale, et communication locale et publique sont généralisables. Dans ce contexte, une conjecture est que l'on peut obtenir un folk théorème (en considérant les équilibres parfaits Bayésiens) si et seulement si le graphe représentant les structures d'observation et de communication est 2-connexe.

Enfin, il serait intéressant d'étudier l'influence de la nature de la communication dans les précédents travaux sur les jeux répétés. La majorité des modèles avec communication présents dans la littérature considèrent des jeux avec communication globale et publique (Compte, 1998 ; Kandori et Matsushima, 1998; Obara, 2009). Dans quelle mesure la nature globale et publique de la communication est-elle nécessaire pour l'obtention de ces folk théorèmes ? Répondre à cette question permettrait de mieux cerner le rôle de la communication dans les jeux répétés.

Local interaction, private and local communication 2

This chapter corresponds to the paper entitled A Folk Theorem for Repeated Games Played on a Network *([30]).*

*I consider repeated games with private monitoring played on a network: each player has a set of neighbors with whom he interacts and communicates. Interaction is local: a player's payoff depends on his own and his neighbors' actions only. I introduce local and private communication: at each stage, players can send costless messages to each of their neighbors (*i.e. *communication is local), and they can send different messages to distinct neighbors (*i.e. *communication is private). Monitoring is private in that players observe their stage payoff but not the actions chosen by their neighbors. Payoffs are assumed to be sensitive to unilateral deviations. The main result is to establish a necessary and sufficient condition on the network for a Nash folk theorem to hold, for any such payoff function.*

2.1 Introduction

Consider a model of decentralized competition with local interaction, in which only direct competitors have an impact on a given firm's profit. For instance, in a competition with differentiated products, the intensity of competition might vary with the goods' attributes such as location, quality, etc., and we can think of firms as directly interacting only with competitors which are close in terms of goods. In repeated interactions, a firm's decisions may provoke reactions from its neighbors, and thus propagate to neighbors of neighbors, and finally to the entire market. Differentiation does not only affect profits, but also information, as firms might lack incentives or expertise to monitor distant competitors. Often, a firm gets little more information than its own profit. Does such limited information suffice to sustain collusion when interaction and communication are local?

In this paper, I study a repeated game played on a network, where the nodes are the players (firms) and an edge between two nodes indicates that they are neighbors (direct competitors). Each player interacts and communicates with his neighbors only: (i) a player's payoff depends only on his own action and on his neighbors' actions, and (ii) each player can send private messages to his neighbors at each stage of play. Players observe their stage payoff only. Here, the network specifies both the monitoring and the communication structures.

This paper tackles the following question: for which networks is cooperation achievable? More precisely, I seek necessary and sufficient conditions on the network for a *full* Nash folk theorem to hold, *i.e.* conditions under which *all* feasible, strictly individually rational payoffs are Nash equilibrium payoffs in a repeated game with patient players.

The key to the characterization lies in understanding under which conditions can information be conveyed through a network. More precisely, I address the following question: for which networks can players detect and identify deviators? Assume that, whenever a player deviates, each of his neighbors observes a change in his payoff, thereby detects the deviation. Since monitoring is imperfect, the identity of the deviator might be unknown. I provide a necessary and sufficient condition on the network's topology under which any deviator can be identified by a communication protocol. This condition is the following: for every player i and for every pair $\{j, k\}$ of neighbors of i, there exists another player ℓ such that (a) player ℓ is a neighbor of j or k, but not of both, and (b) there is a path in the network from ℓ to i going neither through j nor k (Condition NETWORKS, henceforth Condition N). This condition has a simple interpretation: assume that player i detects a deviation which he can ascribe to j or to k. Condition N states that there must be another player ℓ who is a neighbor of j but not of k (or of k but not of j). If the deviator is j, and ℓ is

j's neighbor, then ℓ can inform player i that he also detected a unilateral deviation; whereas if player k is the deviator, then player ℓ does not detect the deviation and can report this fact to player i.

From Condition N, I derive a folk theorem for a wide range of payoff functions. Namely, the payoff functions I consider are such that any unilateral changes of action alter the payoffs of the deviator's neighbors. Hence, unilateral deviations are detected by neighbors. However, since payoffs are observed but not actions, deviators may not be identifiable. Such payoff functions arise naturally in Cournot oligopolies with decreasing demand: changes in competitors' quantities have an impact on the market price, and thus on individual profits. The main result of this paper (Theorem 2.2.12) is that a Nash folk theorem holds for any such payoff function if and only if the network satisfies Condition N.

Condition N is indeed necessary for the folk theorem to hold. Given a network for which it is not satisfied, I exhibit a payoff function for which a feasible and strictly individually rational payoff is not a Nash equilibrium outcome (see Section 2.3). The main argument relies on failure of joint rationality in the sense of Renault and Tomala ([38]), that is: (i) there is no strategy profile that delivers mutual minmax payoffs, so that it is not possible to punish all players simultaneously, and (ii) the deviator cannot be identified and several players have to be suspected. The payoff function that I construct satisfies point (i) (see [37] for a similar example). Regarding point (ii), I prove that if Condition N is not satisfied, then there is a player i, and two of his neighbors j and k, such that no player can both distinguish between j and k's deviations and truthfully reports this information to player i. Crucially, the proof relies on the fact that communication is private. Indeed, if public communication among neighbors were allowed, then Condition N would be unnecessary (see discussion in Section 2.5).

The proof of the sufficiency of Condition N (Section 2.4) is inspired by the methods of Renault and Tomala ([37]) and Tomala ([44]). These authors study games with public communication where each player observes the moves of his neighbors in a network, and where communication is constrained by the network. However, their constructions rely on public communication, and with private communication as assumed here, their protocols fail (see discussion in Section 2.4). My (first) protocol departs from [37] and [44], in that players send sets of innocents instead of sets of suspects. The key argument is that whenever player i tells to some neighbor that player j is innocent, this has to be true, whether player i is deviating or not. Indeed, either i is innocent and tells the truth, or i is guilty, but then j is innocent (under unilateral deviations). Moreover, by contrast with [37] and [44], only the deviator's neighbors have to identify him in order to punish. As a matter of fact, it is not always the case that all players in the game know the identity of the deviator (see

Example 2.4.6 in Section 2.4). Basically, the issue is that it is not always possible to differentiate between deviations in action and deviations in communication. Indeed, it may not be possible to know if a neighbor j deviates in communication when he reports a deviation of his own neighbor k, or if he tells the truth and k has deviated in action. Nevertheless, I prove that *deviator identification by neighbors* is sufficient for the folk theorem to hold. In addition, it is still possible to construct a strategy that enables all players to identify the deviator, instead of his neighbors' only. Doing so requires the network to be 2-connected in addition to Condition N. In that case, I construct a (second) communication protocol in order to distinguish between action and communication deviations. Eventually, it enables all the players to identify the deviator when an action deviation occurs. To be clear, my protocols would work in the setup of [37] and [44], which entails that public communication is not necessary for their folk theorem to hold. Hence, a corollary of my result is that the folk theorems in [37] and [44] extend to private communication (see Remark 2.4.13 in Section 2.4).

Part (b) of Condition C is satisfied in many environments: direct and private communication is easy to conduct between firms l and i, provided that l and i know each other. Moreover, it is not too demanding to assume that firms know the direct competitors of their direct competitors (no more is needed), *i.e.* firm i knows firm l. Part (a) is satisfied in many environments with product differentiation. For vertical differentiation, consider a market in which a firm produces a luxury good and is not competing with a firm producing a low quality good. However, both might be competing with a firm producing a product of intermediate quality. For horizontal differentiation, a good example is the Hotelling market model ([24]) in which firms compete with their closest rivals. In this case, it appears relevant to restrict communication among neighbors, particularly when the market considered is geographically large (e.g. gas stations).

It is worth noting that connectivity has an ambiguous effect on information. On one hand, when the connectivity increases, a deviation is detected by more players and it is easier for an informed player (*i.e.* one able to clear a suspect) to transmit truthful information to another player. On the other hand, more links means more suspects for a given deviation, which implies less precise information (see Example 2.2.6 in Section 2.2.2.2). Therefore, more links may make it more difficult to support some payoffs at equilibrium.

Finally, let me stress that my main result is a folk theorem for Nash equilibria. An important open issue is to prove a similar result for sequential or perfect Bayesian equilibria, that is, for an equilibrium concept which requires the strategies to be sequentially rational after every history, including histories off the equilibrium path. The key difficulty is to provide players with incentives to follow the communication protocol constructed in Section 2.4. The issue is that, following an action deviation by some player i, player j may deviate and send false messages. As a consequence, the content of the message "k is innocent" becomes manipulable. How to update sets of innocents becomes challenging, as it is not enough to make conjectures about a single unilateral deviation, but, possibly, about several consecutive deviations. This issue is discussed in Section 2.5.

Related literature. This paper is related to two independent strands of literature on repeated games and networks. On the repeated game side, the folk theorem was originally established for Nash equilibria ([3, 41, 15, 42]), and extended by Fudenberg and Maskin ([15]) to subgame-perfect equilibria. A key assumption is perfect monitoring. A large part of the literature on folk theorems with imperfect monitoring has focused on imperfect public monitoring (see e.g. [14]). Under imperfect private monitoring, as assumed here, Lehrer (see [31, 32]) provides a fairly comprehensive study of the equilibrium payoff set in the two-player undiscounted case. Also, Fudenberg and Levine ([13]) prove a folk theorem with imperfect private monitoring for undiscounted games, or approximate equilibria of discounted games. Regarding discounted games, there is a large recent literature on imperfect private monitoring (see [11, 22, 23] among others, and [33] for a general survey). Part of the literature allows for public communication between game stages. In this setting, Compte ([8]), Kandori and Matsushima ([26]), and Obara ([35]) provide sufficient conditions for a folk theorem to hold. Closer to my setting, Ben-Porath and Kahneman ([5]) establish a folk theorem for games with public communication where each player observes the moves of his neighbors in a network. The closest paper to the present work is Renault and Tomala ([37]). As mentioned before, these authors studied repeated games with the same signals as in [5] (*i.e.* each player observes his neighbors' moves) but communication is constrained by the network. They establish a sufficient and necessary condition on networks for a Nash folk theorem to hold. [1] Finally, the study of communication protocols in order to identify the deviator is introduced by Tomala ([44]), again with observation of the neighbors' moves. Both [37] and [44] consider public communication among neighbors, *i.e.* each player is constrained to send the same message to all his neighbors at each stage.

Like Renault and Tomala ([37]) and Tomala ([44]), I prove a folk theorem for Nash equilibria. However, contrary to [37] and [44] (and to many others, [5, 7, 27, 34, 45]) in which monitoring among neighbors is perfect, I assume here that it is imperfect: payoffs encapsulate all the agent's feedback about his neighbors' play.

1. Actually, Renault and Tomala ([37]) consider the uniform equilibria of the undiscounted repeated game. Their proof easily extends if Nash equilibria of the discounted repeated game are studied.

(For instance, firms infer rivals' likely behavior from their own profits.) Moreover, I do not allow for public communication among neighbors, as opposed to [37] and [44]. Indeed, although a player may send the same message to all his neighbors, they are not sure that they received the same message: no player knows the messages received by others. The main contribution of this paper is to fully characterize the networks for which a folk theorem holds in this setup, that is, under imperfect local monitoring, and private and local communication.

This paper is also related to the literature on social and economic networks (for an overview of the networks literature, see Goyal, [20], and Jackson, [25]). Networks in which the payoff of a player depends on his own action and on the actions of his neighbors have been studied, among others, by Galeotti and al. ([18]), and Bramoullé and Kranton ([6]). However, most of this literature does not consider repeated games.

The paper is organized as follows. In Section 2.2, I introduce the model, the conditions on payoffs and networks, and the main result. In Section 2.3, I prove that Condition N is necessary for a folk theorem to hold. Section 2.4 is devoted to prove that this condition is also sufficient, and introduces communication protocols in order to identify a deviator. Finally, Section 2.5 develops some extensions and raises open questions.

2.2 The model

2.2.1 Preliminaries

Consider a repeated game played on a network, in which the players interact and communicate with their neighbors. This is described by the following data:

- a finite set $N = \{1, \dots, n\}$ of players, with $n \geq 3$. [2]
- For each player $i \in N$, a non-empty finite set A^i of his actions (with $\sharp A^i \geq 2$). I denote $A = \prod_{i \in N} A^i$.
- An undirected graph $G = (N, E)$ in which the vertices are the players N, and $E \subseteq N \times N$ is a set of links. Let $\mathcal{N}(i) = \{j \neq i \,:\, ij \in E\}$ be the set of player i's neighbors. Since G is undirected, the following holds: $i \in \mathcal{N}(j) \Leftrightarrow j \in \mathcal{N}(i)$.
- For each player $i \in N$, a payoff function $g^i \,:\, \prod_{j \in \mathcal{N}(i) \cup \{i\}} A^j \to \mathbb{R}$, *i.e.* the stage payoff of player i depends on his own and his neighbors' actions only.

For each player i in N, I denote by $G - i$ the graph obtained from G by removing i and its links. More precisely, $G - i = (N \setminus \{i\}, E')$ where $E' = \{jk \in (N \setminus \{i\}) \times (N \setminus \{i\}) \,:\, j \in \mathcal{N}(k) \text{ and } k \in \mathcal{N}(j)\}$. Both interaction and communication possibilities are given by the network G. Throughout the paper, I use the following notations:

2. The 2-player case reduces to perfect monitoring, see Footnote 5 in Section 2.2.2.1.

$A^{\mathcal{N}(i)\cup\{i\}} = \prod_{j\in\mathcal{N}(i)\cup\{i\}} A^j$, $a^{\mathcal{N}(i)} = (a^j)_{j\in\mathcal{N}(i)}$ and $g = (g^1, \ldots, g^n)$ denotes the payoff vector. The repeated game unfolds as follows. At every stage $t \in \mathbb{N}^*$:

(i) players first choose (simultaneously) actions in their action sets. Let $a_t = (a_t^i)$ be the action profile at stage t.

(ii) Then, each player $i \in N$ observes his stage payoff $g^i(a_t^i, a_t^{\mathcal{N}(i)})$. A player cannot observe the actions chosen by his opponents, even those chosen by his neighbors.

(iii) Finally, each player sends costless messages to his neighbors. Communication is private (or *unicast*). That is to say, each player can send different messages to different neighbors, and communication between a pair of neighbors is secret (*i.e.* no other player can learn the message, or change it content). Let M^i be the non-empty finite set of messages available to player i, and denote by $m_t^i(j)$ the message sent by player i to his neighbor $j \in \mathcal{N}(i)$ at stage t. The specification of the set M^i is described in Section 2.4.

I assume that players have perfect recall and that the whole description of the game is common knowledge. For each stage t, denote by H_t^i the set of histories of player i up to stage t, that is $(A^i \times (M^i)^{\mathcal{N}(i)} \times (M^j)_{j\in\mathcal{N}(i)} \times \{g^i\})^t$, where $\{g^i\}$ is the range of the payoff function g^i (H_0^i is a singleton). An element of H_t^i is called an i-history of length t. A *behavior strategy* of a player i is a pair (σ^i, ϕ^i) where $\sigma^i = (\sigma_t^i)_{t\geq 1}$, $\phi^i = (\phi_t^i)_{t\geq 1}$, and for each stage t, σ_t^i is a mapping from H_{t-1}^i to $\Delta(A^i)$, with $\Delta(A^i)$ the set of probability distributions over A^i, and ϕ_t^i is a mapping from $H_{t-1}^i \times \{a_t^i\} \times \{g_t^i\}$ to $\Delta((M^i)^{\mathcal{N}(i)})$. I call σ^i the *action strategy* of player i and ϕ^i his *communication strategy*. Each player can deviate from σ^i or from ϕ^i, henceforth I shall distinguish between action deviations and communication deviations accordingly. Let Σ^i be the set of action strategies of player i and Φ^i his set of communication strategies. I denote by $\sigma = (\sigma^i)_{i\in N} \in \prod_{i\in N} \Sigma^i$ the joint action strategy of the players and by $\phi = (\phi^i)_{i\in N} \in \prod_{i\in N} \Phi^i$ their joint communication strategy. Let H_t be the set of histories of length t, *i.e.* the sequences of actions, payoffs and messages up to stage t. A profile (σ, ϕ) defines a probability distribution $\mathbb{P}_{\sigma,\phi}$ over the set of plays, and I denote by $\mathbb{E}_{\sigma,\phi}$ the corresponding expectation. I consider the discounted infinitely repeated game, where the overall payoff function of each player i in N is the expected discounted sum of payoffs. That is, for each player i in N:

$$\gamma_\delta^i(\sigma, \phi) = \mathbb{E}_{\sigma,\phi}\left[(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} g_t^i(a_t^i, a_t^{\mathcal{N}(i)})\right],$$

where $\delta \in [0, 1)$ is a common discount factor. A strategy profile (σ, ϕ) is a Nash equilibrium if no player can increase his discounted payoff by switching unilaterally to an alternative strategy (τ^i, ψ^i). I study equilibrium outcomes of repeated games with high discount factor.

Definition 2.2.1. A payoff vector $x = (x^i)_{i \in N} \in \mathbb{R}^N$ is an equilibrium payoff of the repeated game if there exists a discount factor $\bar{\delta} \in (0,1)$ such that, for any $\delta \in (\bar{\delta}, 1)$, x is induced by a Nash equilibrium of the δ-discounted game.

Let $\Gamma_\delta(G,g)$ be the δ-discounted game described above, and let $E_\delta(G,g)$ be its associated set of equilibrium payoffs. For each $a \in A$, I use the following notations: $g(a) = (g^1(a^1, a^{\mathcal{N}(1)}), \ldots, g^n(a^n, a^{\mathcal{N}(n)}))$ and $g(A) = \{g(a) \,:\, a \in A\}$. Let $\operatorname{co} g(A)$ be the convex hull of $g(A)$, this is the set of feasible payoffs. It is straightforward that $E_\delta(G,g) \subseteq \operatorname{co} g(A)$. The (independent) minmax level of player i is defined by: [3]

$$v^i = \min_{x^{\mathcal{N}(i)} \in \prod_{j \in \mathcal{N}(i)} \Delta(A^j)} \max_{x^i \in \Delta(A^i)} g^i(x^i, x^{\mathcal{N}(i)}).$$

I denote by $IR^*(G,g) = \{x = (x^1, \ldots, x^n) \in \mathbb{R}^N \,:\, x^i > v^i \; \forall i \in N\}$ the set of strictly individually rational payoffs. The aim of this paper is to characterize the networks G for which a folk theorem holds, *i.e.* each feasible and strictly individually rational payoff is an equilibrium payoff of the repeated game for discount factors close enough to one. In the next section, I introduce the conditions on payoff functions and on networks, and the main result.

Now, I recall some definitions of graph theory that are used throughout the paper (the reader is referred to [9]).

Definition 2.2.2. (i) A *graph* is a pair $G = (V, E)$ where V is a set of nodes, and $E \subseteq V \times V$ is a set of links (or edges).

(ii) A *subgraph* G' of G, written as $G' \subseteq G$, is a pair $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$.

(iii) A *walk* in $G = (V, E)$ is a sequence of links $e_1, \ldots, e_K$ such that $e_k e_{k+1} \in E$ for each $k \in \{1, \ldots, K-1\}$. A *path* from node i to node j is a walk such that $e_1 = i$, $e_K = j$, and each node in the sequence $e_1, \ldots, e_K$ is distinct. The number of links of a path is referred as its *length*.

(iv) The *distance* $d_G(i,j)$ in G of two nodes i, j is the length of a shortest path from i to j in G; if no such path exists, $d_G(i,j) := +\infty$.

(v) Two paths from i to j are *independent* if they have no common vertices except i and j.

(vi) G is called *k-connected* (for $k \in \mathbb{N}$) if $\sharp V \geq k$ and $G - X$ is connected for every set $X \subseteq V$ with $\sharp X < k$, where $G - X$ represents the graph where all nodes in X have been removed (and the corresponding links). By Menger theorem, a graph is k-connected if any two of its nodes can be joined by k independent paths (see for instance [9], Theorem 3.3.1 p. 62).

3. It is sometimes possible to drive equilibrium payoffs below this bound, see Renault and Tomala ([37]) for illuminating examples.

(vii) A *connected component* of a graph G is a maximal subgraph in which any two vertices are connected to each other by some path.

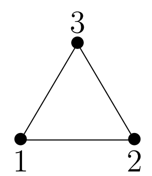
2.2.2 Structures of payoffs and networks

2.2.2.1 Payoff functions

I first show that a necessary condition for a folk theorem to hold is that the payoff functions are sufficiently rich to enable players to at least detect deviations.

Example 2.2.3. Consider the 3-player game played on the following network:

FIGURE 2.1:



and with payoff matrix (Player 1 chooses the row, Player 2 the column, and Player 3 the matrix):

P3 plays A

P1 \ *P2*	A	B
A	0,0,4	0,0,4
B	3,5,0	3,4,3

P3 plays B

P1 \ *P2*	A	B
A	0,0,0	0,0,0
B	4,4,0	4,3,0

The payoff vector $(3, 4, 3)$ is feasible and strictly individually rational, but is not an equilibrium payoff.[4] Indeed, for any discount factor $\delta \in [0, 1)$, Player 2 has an incentive to deviate by playing A at each stage. No matter whether Player 1 chooses A or B, he does not know whether Player 2 plays A or B, and thus does not detect Player 2's deviation.[5] Although Player 3 detects Player 2's deviation (his payoff changes from 3 to 0), he has no way to report truthfully this information to Player 1. Indeed, Player 1 has to play A in order to minmax Player 2, which gives Player 3 a payoff of 4. Player 3 has thus an incentive to lie and tell Player 1 he detected a

4. Notice that the payoff $(3, 4, 3)$ is only achieved by the action profile BBA.

5. Notice that B is dominated (in the stage game) by A for Player 2. Moreover, Player 1 gets as much information on Player 2's action (and on Player 3's action) no matter whether he plays A or B, hence one cannot test him on his knowledge.

deviation from Player 2. Therefore $(3, 4, 3)$ is not an equilibrium payoff and one can show that equilibrium payoffs are bounded away from $(3, 4, 3)$.

Now, assume that the payoff matrix is the following:

P3 plays A

P1 \ *P2*	A	B
A	0,0,4	0,0,4
B	3,5,0	3+ϵ,4,3

P3 plays B

P1 \ *P2*	A	B
A	0,0,0	0,0,0
B	4,4,0	4,3,0

with $\epsilon > 0$. Then, Player 1 detects Player 2's deviation. In addition, all players can detect the deviations of their opponents, and identify the deviator. Therefore, if Player 2 deviates by playing A, Player 1 then starts punishing him by playing A. For a discount factor close enough to one δ, $(3+\epsilon, 4, 3)$ is a Nash equilibrium payoff of the repeated game. ◇

The previous example shows that if a deviation of some player i does not change the payoffs of his neighbors, then this deviation is undetectable and may be profitable. As a consequence, it may possible for some feasible and strictly individually rational payoffs not to be equilibrium payoffs of the repeated game (for a similar phenomenon, see [31]). Hence, it is not possible to obtain a folk theorem for all payoff functions g. I introduce the following assumption. [6]

Assumption PAYOFFS (P) . *For each player* $i \in N$, *each neighbor* $j \in \mathcal{N}(i)$, *every actions* $b^j, c^j \in A^j$ *such that* $b^j \neq c^j$, $a^i \in A^i$, $a^{\mathcal{N}(i)\setminus\{j\}} \in A^{\mathcal{N}(i)\setminus\{j\}}$*:*

$$g^i(a^i, a^{\mathcal{N}(i)\setminus\{j\}}, b^j) \neq g^i(a^i, a^{\mathcal{N}(i)\setminus\{j\}}, c^j).$$

Example 2.2.4. The following payoff functions satisfy Assumption P:

- for each player i in N, let $A^i \subset \mathbb{N}$ and $g^i(a^i, a^{\mathcal{N}(i)}) = f\left(\sum_{j\in\mathcal{N}(i)\cup\{i\}} a^j\right)$ with f strictly monotone;
- firms' profits in Cournot games;
- for each player i in N, let $A^i \subset \mathbb{R}$ and $g^i(a^i, a^{\mathcal{N}(i)}) = \sum_{j\in\mathcal{N}(i)} a^j - a^i$ (this game can be seen as a generalized prisoner's dilemma for n players);
- more generally, for each player i in N, let $A^i \subset \mathbb{R}$ and g^i strictly monotone with respect to each argument.

Example 2.2.3 puts forward the importance of detecting the deviations of neighbors for a folk theorem to hold. Yet, in this example, the particular payoff profile I consider is an equilibrium (in the modified version) since players also identify the

6. This assumption implies that the two-player case reduces to perfect monitoring.

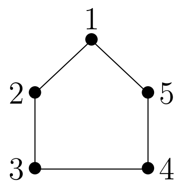
deviator in case of a deviation. I consider more general payoff functions than in Example 2.2.3. Indeed, Assumption P implies that neighbors' deviations are detectable, but the deviator may not be identified. In the next section, I introduce a necessary and sufficient condition on networks for a folk theorem to hold.

2.2.2.2 Condition on networks

The next example puts forward the importance of graph connectivity, and gives the basic intuition behind the proof of necessity in Section 2.3.

Example 2.2.5. Consider the 5-player game played on the following network:

FIGURE 2.2:



in which $\mathcal{N}(i) \equiv \{i-1, i+1\} \mod 5$, for each player i in N. Assume that Player 1 has three actions U, M and D, and that each other player has two actions, C and D. For simplicity, I only define the payoff functions of Players 1, 2 and 3, which are given by the following matrix (Player 2 chooses the row, Player 1 the column, Player 3 the matrix):

P3 plays C

P2 \ *P1*	C	D
U	1,1,1	0,3,1
M	1,0,3	0,0,3
D	1,4,0	0,4,0

P3 plays D

P2 \ *P1*	C	D
U	0,1,4	1,3,4
M	0,0,3	1,0,3
D	0,4,0	1,4,0

In addition, g^4 and g^5 are such that:

- if Player 4 (respectively Player 5) plays D, then each of his neighbors gets an additional payoff of $\epsilon > 0$;
- if Player 1 (respectively Player 3) chooses C, his neighbor 5 (respectively Player 4) receives an additional payoff of $\epsilon > 0$.

It is then easy to check that Assumption P is satisfied. The minmax level of each player 1, 2 and 3 is zero, hence the outcome $(1,1,1)$ is feasible and strictly individually rational. Note that the only way to get $(1,1,1)$ is that Players 1, 2 and 3 choose the action profile (U, C, C). However, Players 1 and 3 both have an incentive to deviate by playing D in order to get a payoff of 3 for Player 1, and a payoff of 4 for Player 3. Both deviations yield a payoff of zero for Player 2. Furthermore, consider a deviation of Player 1 (respectively Player 3) who chooses between actions C and D with probability $\frac{1}{2}$-$\frac{1}{2}$ at each stage. The deviations of Player 1 and 3 induce the same distribution of payoffs for Player 2, and that for every strategy (possibly mixed) of the non-deviating players. [7] Hence, Player 2 cannot infer the identity of the deviator from the observation of his payoffs.

On the contrary, Player 5 (respectively Player 4) can differentiate between the two deviations: if Player 1 deviates (respectively Player 3), then Player 5's payoff (respectively Player 4's) goes down by $\epsilon > 0$. Yet, neither Player 4 nor Player 5 can transmit truthfully this information to Player 2. Indeed, all paths from Player 4 to Player 2 (respectively from Player 5 to Player 2) go either through Player 1 or Player 3, both of whom are suspects. These ideas are explained formally in a general case in Section 2.3. Basically, I prove in Section 2.3 that there exist communication strategies for Players 1 and 3 such that the deviations of Players 1 and 3 induce the same probability distributions over the sequences of messages and payoffs received by Player 2. Therefore, Player 2 cannot differentiate between these deviations of Players 1 and 3. Hence, either "Player 1 always plays C and D with probability $\frac{1}{2}$-$\frac{1}{2}$ and pretends that 3 does" is a profitable deviation for Player 1, or "Player 3 always plays C and D with probability $\frac{1}{2}$-$\frac{1}{2}$ and pretends that 1 does" is a profitable deviation for Player 3. Indeed, the payoffs are such that Player 2 cannot punish both Player 1 and Player 3. Punishing Player 1 requires Player 2 to play M, in which case Player 3 gets a payoff of 3. On the contrary, punishing Player 3 requires Player 2 to play D, in which case Player 1 gets a payoff of 4. Again, this argument is formally developed in Section 2.3. [8] As a consequence, the outcome $(1, 1, 1)$ is not an equilibrium payoff of the repeated game, and the folk theorem fails for this network. $\diamond$

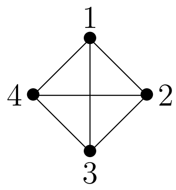
The previous example gives a five-player network for which the folk theorem fails (for some payoff function). That is to say, there exists a payoff function (satisfying Assumption P) such that some feasible and strictly individually rational payoff is not an equilibrium payoff, no matter how patients the players are. Matters would be different if Players 4 and 5 could inform Player 2 directly about a deviation by Player 1 or by Player 3. This suggests that more connected networks are less prone to manipulation. However, the next example illustrates how connectivity may also stand in the way of a folk theorem.

7. Following the terminology of Aumann and Maschler (1966, re-edited in 1995, [4]), this is a jointly controlled lottery over Player 2's payoffs.

8. This is a failure of joint rationality following the terminology of Renault and Tomala ([38]).

Example 2.2.6. Consider the 4-player game played on the following network:

FIGURE 2.3:



in which $\mathcal{N}(i) = N \setminus \{i\}$ for each i in N (complete graph). Assume that $A^i = \{0, 1\}$ for each player i in N. The payoff functions of the players are the following:

$$\begin{aligned} g^1(a) &= 2a^1 - a^2 - a^3 - a^4, \\ g^2(a) = g^3(a) = &= -a^2 - a^3 + \begin{cases} 0 & \text{if } a^1 = a^4, \\ -1 & \text{otherwise,} \end{cases} \\ g^4(a) &= -3a^1 + a^2 + a^3 + 2a^4. \end{aligned}$$

Clearly, Assumption P is satisfied. The minmax level is $v^i \leq -1$ for each player $i \in N$. Consider the payoff vector $(0, 0, 0, 0)$, only obtained when all players choose action 0, and which is feasible and strictly individually rational. It is then not possible for Players 2 and 3 to differentiate between the deviations of Players 1 and 4. Indeed, if Player 1 deviates in action at some stage t by playing 1, then everybody detects the deviation (Assumption P). However, the graph being complete, Players 2 and 3 suspect everybody. As in Example 2.2.5, the deviations of Players 1 and 4 that consist in choosing between 0 and 1 with probability $\frac{1}{2}$-$\frac{1}{2}$ induce the same distributions over the payoffs of Players 2 and 3, for any strategy of the non-deviating players. Moreover, it is not possible for Players 2 and 3 to minmax both Players 1 and 4 simultaneously: punishing Player 1 requires Players 2 and 3 to play 1, in which case Player 4 guarantees a payoff of 1; and punishing Player 4 requires Players 2 and 3 to play 0, in which case Player 1 guarantees a payoff of 1. Therefore, either Player 1 or Player 4 has an incentive to deviate by playing between actions 0 and -1 with probability $\frac{1}{2}$ each, and the folk theorem fails. $\diamond$

A sufficient condition to rule out the last example is to require distinct players to have different neighbors. More formally, the necessary and sufficient condition on the networks for a folk theorem to hold is the following.

Condition NETWORKS (N) . *For each player $i \in N$, for every neighbors $j, k \in \mathcal{N}(i)$ such that $j \neq k$, there exists $\ell \in \mathcal{N}(j) \setminus \{k\} \triangle \mathcal{N}(k) \setminus \{j\}$, such that there is a* *path from ℓ to i which goes neither through j nor k, where:*

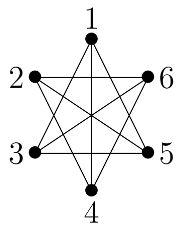
$$\mathcal{N}(j) \setminus \{k\} \triangle \mathcal{N}(k) \setminus \{j\} = \Big(\mathcal{N}(j) \setminus \{k, \mathcal{N}(k)\}\Big) \bigcup \Big(\mathcal{N}(k) \setminus \{j, \mathcal{N}(j)\}\Big).$$

Actually, Condition N captures two requirements. First, the set difference is non-empty, hence players j and k do not have the same neighbors. Second, for each player who is a neighbor of both j and k, there is a player neighbor of only one of them, who can send relevant information to player i on a path that goes neither through j nor k. Example 2.2.6 illustrates the necessity of the first requirement: in my setup, more connectivity might imply less information, since more links may increase the number of suspects when a deviation occurs. As for the second requirement, Example 2.2.5 displays a payoff function for which the punishment of player j (respectively Player 1 in the example) or of player k (respectively Player 3) requires player i (respectively Player 2) to be informed of the deviator's identity.

Intuitively, the consequence of Condition N is the following. Take a player i in N and suppose that there is at some stage an action deviation of one of his neighbors. If the network satisfies Condition N, then there exists a player ℓ who is a neighbor of either j or k, say j, but not of both. On one hand, if the deviator is j, then player ℓ observes a change in his payoff (when Assumption P is satisfied), and concludes that k is innocent (since k is not his neighbor). On the other hand, if the deviator is k, player ℓ concludes that player j is innocent because player ℓ does not observe a change in his payoff. In both situations, player ℓ can transmit his information to player i, as there is path from ℓ to i which goes neither through player j nor player k. Moreover, when a player, say m, informs player i that some other player, say j, is innocent, player i concludes that player j is indeed innocent: either player m is not telling the truth, thus deviates, and player j is indeed innocent under unilateral deviations; or, player m is innocent, thus telling the truth, and player j also is innocent. As a consequence, the truthfulness of the fact "j is innocent" is not manipulable. Eventually, player i learns the identity of the deviator, which is required in order to punish him (at least for some payoff functions, see Examples 2.2.5, 2.2.6, and Section 2.3). A folk theorem then follows. As an example, consider the network below.

Example 2.2.7. Consider the 6-player game played on the following network:

FIGURE 2.4:



and suppose that Assumption P holds. Condition N obviously holds. Consider, without loss of generality, Player 1's point of view (the network is symmetric), and assume that Player 3 deviates in action at some stage $t > 0$. Player 1 then observes a change in his payoff (Assumption P), and suspects his neighbors Players 3, 4 and 5. On the other hand, Player 2 is not a neighbor of Player 3 and does not observe a change in his payoff, hence Player 2 concludes that his neighbors Players 4, 5 and 6 are innocent. Player 2 transmits this information to Players 4 and 5. Then, Player 4 is able to inform Player 1 that Player 5 is innocent, and Player 5 is able to inform Player 1 that Player 4 is innocent. Player 1 eventually knows that Player 3 is guilty, since he is the unique suspect left.

The important facts here are that Player 2 can clear Players 4 and 5, and that there exists a path from Player 2 to Player 1 which goes neither through Player 3 nor 4. This path is used to transmit the information that Player 4 is innocent. In the same way, there is a path from Player 2 to Player 1 which goes neither through 3 nor 5. ◇

I now state a few properties of the networks for which Condition N holds. First, as mentioned, players must have different sets of neighbors. More formally, the following property holds.

Lemma 2.2.8. *If the graph G satisfies Condition N, then for each pair of players i and j who are in the same connected component of at least 3 players, $\mathcal{N}(i) \setminus \{j\} \neq \mathcal{N}(j) \setminus \{i\}$.*

Proof. Take a graph G which satisfies Condition N, and a connected component $C \subseteq G$ with at least three nodes. Assume that there exists a pair of players i and j in C such that $\mathcal{N}(i) \setminus \{j\} = \mathcal{N}(j) \setminus \{i\}$. Since C contains at least three nodes, there exists $k \neq i, j$ such that $k \in \mathcal{N}(i) \cap \mathcal{N}(j)$. Hence, i and j are in $\mathcal{N}(k)$, but then there exists no $\ell \in \mathcal{N}(i) \setminus \{j\} \triangle \mathcal{N}(j) \setminus \{i\}$ such that $\ell \neq i$, j, which contradicts Condition N. □

As shown in Example 2.2.5, having different neighbors is not sufficient for a folk theorem to hold.

Lemma 2.2.9. *If the network G satisfies Condition N, then no player has exactly two neighbors:* $\forall i \in N$, $\sharp\mathcal{N}(i) \neq 2$.

Proof. Take a graph G which satisfies Condition N. Assume that there exists a player i in N who has exactly two neighbors, say j and k: $\mathcal{N}(i) = \{j, k\}$. Condition N implies that there exist a player ℓ in N such that $\ell \neq j$, k, and a path from ℓ to i going neither through j nor k. Therefore, player i has a neighbor different from j and k, a contradiction. □

From the two previous lemmas, I deduce a family of networks satisfying Condition N.

Corollary 2.2.10. *A 3-connected network in which players have different neighbors,* i.e. *for any pair* $(i, j) \in N^2$, $\mathcal{N}(i) \setminus \{j\} \neq \mathcal{N}(j) \setminus \{i\}$, *satisfies Condition N.*

Remark 2.2.11. One can check that there is no connected graph satisfying Condition N with less than six players.

Before presenting the main result, I display some graphs that satisfy Condition N (Figure 2.5) and some that do not (Figure 2.6). Regarding Figure 2.5, notice that G_1 is 3-connected, whereas G_2, G_3 and G_4 are not even 2-connected. Moreover, one can easily check that no circle, complete graph, star network or tree satisfies Condition N (see respectively G_5, G_6, G_7 and G_8).

FIGURE 2.5: Networks satisfying Condition N

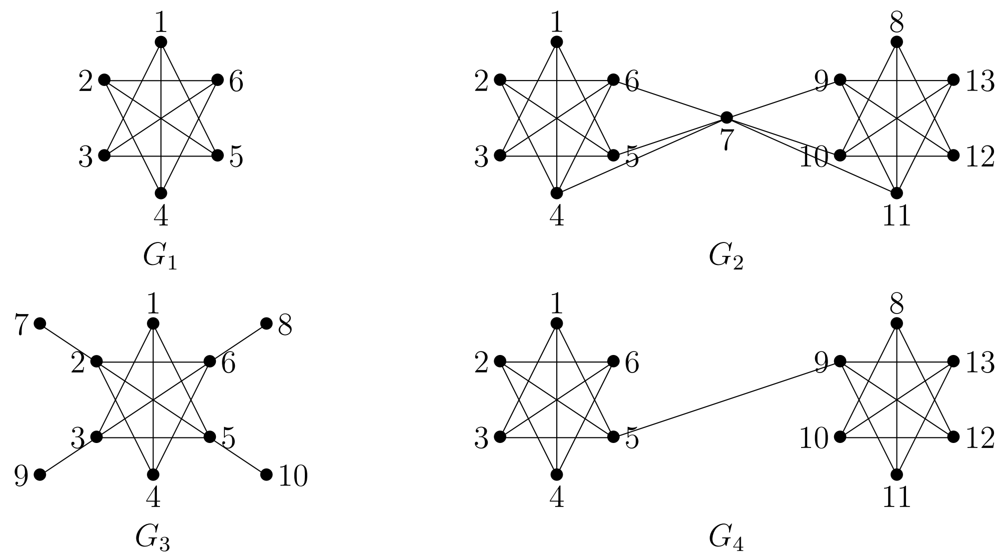
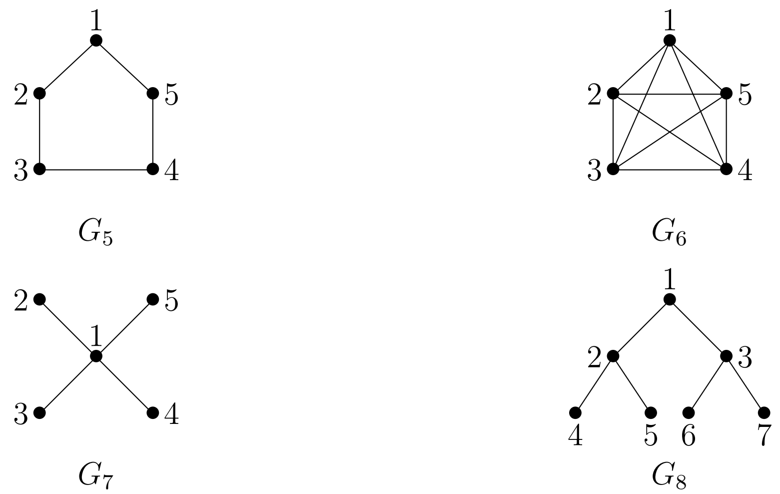


FIGURE 2.6: Networks not satisfying Condition N



2.2.3 Main result

The main result of the paper is the following.

Theorem 2.2.12. *The following statements are equivalent.*

1. *The network G satisfies Condition N.*
2. *For every payoff function that satisfies Assumption P, any feasible and strictly individually rational payoff is an equilibrium payoff of the repeated game.*

In the next section, I prove that Condition N is necessary for the folk theorem to hold, *i.e.* for the family of networks that do not satisfy Condition N, I exhibit a particular payoff function g such that there exists a feasible and strictly individually rational payoff which is not an equilibrium payoff of the repeated game. In Section 2.4, I prove that Condition N is sufficient for the folk theorem to hold.

Remark 2.2.13. Henceforth, I focus on connected networks only. If a network G has several connected components, the payoff function of each player depends only on his own and his neighbors' actions. In addition, players cannot communicate with players who are not in the same connected component. Therefore, I model different connected components as different games.

2.3 Necessary condition

In this section, I prove the following: *for every payoff function for which Assumption P holds, if each feasible and strictly individually rational payoff is an equilibrium payoff of the repeated game, then Condition N must hold.* For this purpose, I take a network G for which Condition N does not hold, and I construct a payoff function

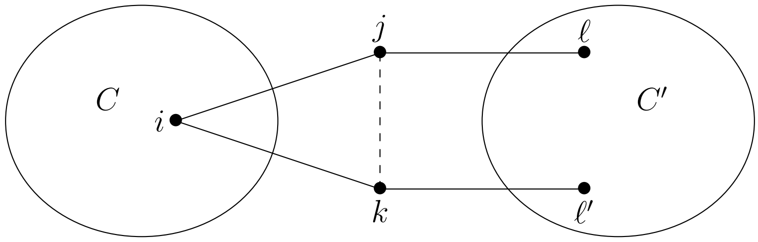
g such that there exists a payoff $u \in \operatorname{co} g(A) \cap IR^*(G, g)$ which is not an equilibrium payoff of the repeated game. Intuitively, when Condition N is violated, there exists a player i, two of his neighbors j and k, and two deviations of j and k such that: for any action profile (possibly mixed), both j's and k's deviations induce the same distribution over player i's payoffs, and the same distribution over the messages received by player i. Following the terminology of Fudenberg, Levine and Maskin ([14]), *pairwise identifiability* fails. The payoffs are constructed so that there exists a feasible and strictly individually rational payoff which is not *jointly rational* (in the sense of Renault and Tomala in [38]; see Example 3.1 therein for a similar phenomenon).

From now on, assume that Condition N is not satisfied. The following holds:

$$\exists i \in N,\ \exists j,k \in \mathcal{N}(i),\ j \neq k,\ \text{s.t. } \forall \ell \in \mathcal{N}(j) \setminus \{k\} \triangle \mathcal{N}(k) \setminus \{j\},$$
$$\text{every path from } \ell \text{ to } i \text{ goes either through } j \text{ or through } k.$$

Therefore, in the graph $G - \{jk\}$ where j and k have been removed, i is not in the same connected component as ℓ, for every $\ell \in \mathcal{N}(j) \setminus \{k\} \triangle \mathcal{N}(k) \setminus \{j\}$, see Figure 2.7.

FIGURE 2.7: Condition N is not satisfied



Here, every player $\ell \in \mathcal{N}(j) \setminus \{k\} \triangle \mathcal{N}(k) \setminus \{j\}$ is in the subgraph C'; each player i in the subgraph C is either a neighbor of both players j and k, *i.e.* $i \in \mathcal{N}(j) \cap \mathcal{N}(k)$, or of none of them, *i.e.* $i \in N \setminus (\mathcal{N}(j) \cup \mathcal{N}(k))$. Moreover, C and C' are disconnected components of $G - \{jk\}$. Finally, players j and k may be neighbors or not, as indicated by the dashed line. For brevity, I assume from now on that players j and k are not neighbors. The proof can be easily extended to the case in which j and k are neighbors.

Consider the payoff functions for players i, j and k represented by the following matrix (where player i chooses the row, player j the column, and player k the matrix):[9]

9. For Assumption P to hold if players j and k are neighbors, I assume that if player j plays D it adds $\epsilon > 0$ to player k's payoff, and symmetrically if player k plays D it adds $\epsilon > 0$ to player j's payoff.

k plays C

$i \setminus j$	C	D
C	1,0,2	0,0,2
D	1,2,0	0,6,0

k plays D

$i \setminus j$	C	D
C	0,0,6	1,0,6
D	0,2,0	1,6,0

I write $u(a^i, a^j, a^k)$ for this payoff vector. Player j's payoff does not depend on k's action, nor does k's payoff depends on j's. Accordingly, I write $u^j(a^j, a^i)$ and $u^k(a^k, a^i)$ in what follows. To complete the description of g, each player $m \neq i, j, k$ has two actions C and D such that:

(i) for each player m such that $m \notin \mathcal{N}(j) \cap \mathcal{N}(k)$, player m's payoff at stage t is:

$$g^m(a_t^m, a_t^{\mathcal{N}(m)}) = \ell_t \frac{\epsilon}{n}$$

for some $\epsilon > 0$, and $\ell_t = \sharp\{\ell : \ell \in \mathcal{N}(m) \cup \{m\} \text{ and } a_t^\ell = C\}$ (ℓ_t is the number of m's neighbors including himself who play C at stage t);

(ii) for each player m such that $m \in \mathcal{N}(j) \cap \mathcal{N}(k)$, player m's payoff at stage t is:

$$g^m(a_t^m, a_t^{\mathcal{N}(m)}) = \begin{cases} 1 + \ell_t^{-jk} \frac{\epsilon}{n} & \text{if } a^j = a^k; \\ \ell_t^{-jk} \frac{\epsilon}{n} & \text{otherwise.} \end{cases}$$

for some $\epsilon > 0$, and $\ell_t^{-jk} = \sharp\{\ell : \ell \in \mathcal{N}(m) \setminus \{j,k\} \cup \{m\} \text{ and } a_t^\ell = C\}$ (ℓ_t is the number of m's neighbors distinct from j and k, including himself, who play C at stage t);

(iii) for players i, j and k:

$$\begin{aligned} g^i(a_t^i, a_t^{\mathcal{N}(i)}) &= u^i(a_t^i, a_t^j, a_t^k) - \ell_t^i \frac{\epsilon}{n}, \\ g^j(a_t^j, a_t^{\mathcal{N}(j)}) &= u^j(a_t^j, a_t^i) + \ell_t^j \frac{\epsilon}{n}, \\ g^k(a_t^k, a_t^{\mathcal{N}(k)}) &= u^k(a_t^k, a_t^i) + \ell_t^k \frac{\epsilon}{n}, \end{aligned}$$

where $\ell_t^i = \sharp\{\ell : \ell \in \mathcal{N}(i) \setminus \{j,k\} \text{ and } a_t^\ell = D\}$, $\ell_t^j = \sharp\{\ell : \ell \in \mathcal{N}(j) \setminus \{i\} \text{ and } a_t^\ell = D\}$, $\ell_t^k = \sharp\{\ell : \ell \in \mathcal{N}(k) \setminus \{i\} \text{ and } a_t^\ell = D\}$ and $u^i(a_t^i, a_t^j, a_t^k)$, $u^j(a_t^j, a_t^i)$, and $u^k(a_t^k, a_t^i)$ are defined by the matrix above.

This payoff function g has the following properties:[10]

(i) g satisfies Assumption P;

(ii) $v^i < 0$, $v^j = 0$, and $v^k = 0$;

(iii) C is a dominant strategy for each player $\ell \neq i, j, k$;

10. It is possible to construct a payoff function g which satisfies these properties when players have more than two actions. This construction is described in Appendix 2.6.1.

(iv) the outcome $(1,1,1)$ (representing the payoffs of players i, j and k) is feasible and strictly individually rational;

(v) if $a^\ell \neq C$ for every $\ell \in \mathcal{N}(i)$, then $g^i(a^i_t, a^{\mathcal{N}(i)}_t) < 1$. Hence, the unique way to obtain the outcome $(1,1,1)$ is that player i chooses between C and D with probability $\frac{1}{2}$- $\frac{1}{2}$, and that all his neighbors (including players j and k) take action C;

(vi) player i cannot punish both players j and k: player i has to play C in order to minmax player j, which enables player k to get a payoff of 6; and player i has to choose action D in order to minmax player k, which enables player j to get 6;

(vii) for each player $m \in \mathcal{N}(j) \cap \mathcal{N}(k)$ (including player i), for every $a^m \in \{C, D\}$, and for every $a^{\mathcal{N}(m)\setminus\{j,k\}}$, the following properties hold:

$$g^m(a^m, a^{\mathcal{N}(m)\setminus\{j,k\}}, a^j = C, a^k = C) = g^m(a^m, a^{\mathcal{N}(m)\setminus\{j,k\}}, a^j = D, a^k = D),$$

$$g^m(a^m, a^{\mathcal{N}(m)\setminus\{j,k\}}, a^j = C, a^k = D) = g^m(a^m, a^{\mathcal{N}(m)\setminus\{j,k\}}, a^j = D, a^k = C).$$

Assume now that $(1,1,1)$ is a Nash equilibrium of the repeated game, and let the profiles $\bar{\sigma} = (\bar{\sigma}^i, \bar{\sigma}^j, \bar{\sigma}^k, (\bar{\sigma}^m)_{m \neq i,j,k})$ and $\bar{\phi} = (\bar{\phi}^i, \bar{\phi}^j, \bar{\phi}^k, (\bar{\phi}^m)_{m \neq i,j,k})$ be an equilibrium yielding a payoff of $\gamma_\delta = (1,1,1)$ for players i, j and k. I construct deviations (τ^j, ψ^j) and (τ^k, ψ^k) for players j and k such that:

(1) both deviations induce the same probability distributions over the sequences of messages and payoffs received by player i (deviations are indistinguishable).

(2) I will deduce from (1) that: $\gamma^j(\tau^j, \bar{\sigma}^{-j}, \psi^j, \bar{\phi}^{-j}) + \gamma^k(\tau^k, \bar{\sigma}^{-k}, \psi^k, \bar{\phi}^{-k}) \geq 3$.

The latter equation contradicts that $(\bar{\sigma}, \bar{\phi})$ is an equilibrium of the repeated game. Indeed, $\gamma^j(\bar{\sigma}, \bar{\phi}) + \gamma^k(\bar{\sigma}, \bar{\phi}) = 2$ and $\gamma^j(\tau^j, \bar{\sigma}^{-j}, \psi^j, \bar{\phi}^{-j}) + \gamma^k(\tau^k, \bar{\sigma}^{-k}, \psi^k, \bar{\phi}^{-k}) \geq 3$. Therefore, either (τ^j, ψ^j) is a profitable deviation for player j, or (τ^k, σ^k) is a profitable deviation for player k. The construction of these deviations is inspired by Renault and Tomala ([39]), and the formal description is in Appendix 2.6.2. The main ideas are the following. Define (τ^j, ψ^j) as follows (the construction of (τ^k, ψ^k) is symmetric):

- at each stage, player j chooses between C and D with probability $\frac{1}{2}$-$\frac{1}{2}$ (instead of C with probability 1);
- at each stage, player j sends the blank message $\varnothing$ to all his neighbors in the subgraph C';
- player j constructs fictitious messages and fictitious actions for players in C' corresponding to the situation "player k plays $\tau^k = D$, and sends the blank message to his neighbors in C' at each stage." Player j then sends messages to players in C, including player i, according to the equilibrium strategy $\bar{\phi}^j$ after having replaced the true history (in the component C') by the fictitious one.

Intuitively, under such a deviation, player i has no way to deduce whether j or k deviated, even when $(\bar{\sigma}, \bar{\phi})$ is a mixed strategy. Indeed, for every pure action profile, both player j's and k's deviations induce the same payoffs for player i, since g satisfies property (vii). In addition, notice that player i's payoff is 1 if j and k choose the same action, and 0 otherwise. Therefore, since τ^j and τ^k prescribe to choose C and D with probability $\frac{1}{2}$-$\frac{1}{2}$ at each stage, $(\tau^j, \psi^j, \bar{\sigma}^{-j}, \bar{\phi}^{-j})$ and $(\tau^k, \psi^k, \bar{\sigma}^{-k}, \bar{\phi}^{-k})$ induce the same distribution over the payoffs of player i, and that even if $\bar{\sigma}^j$ and $\bar{\sigma}^k$ are mixed.

Moreover, both deviations induce the same distribution over the messages received by player i, even when $\bar{\phi}$ is a mixed strategy. [11] The main point is that player i's histories have the same distribution under the profiles $(\tau^j, \bar{\sigma}^{-j}, \psi^j, \bar{\phi}^{-j})$ and $(\tau^k, \bar{\sigma}^{-k}, \psi^k, \bar{\phi}^{-k})$. Recall that I denote player i's private history up to stage t by h^i_t. The following lemma holds.

Lemma 2.3.1. *For every private history h^i_t of player i at any stage t:*

$$\mathbb{P}_{\tau^j, \bar{\sigma}^{-j}, \psi^j, \bar{\phi}^{-j}}(h^i_t) = \mathbb{P}_{\tau^k, \bar{\sigma}^{-k}, \psi^k, \bar{\phi}^{-k}}(h^i_t).$$

The proof is given in Appendix 2.6.3. I conclude that player i cannot distinguish between the deviations (τ^j, ψ^j) and (τ^k, ψ^k). Now, I define the following numbers b_t and c_t:

$$\begin{aligned} b_t &= \mathbb{P}_{\tau^j, \bar{\sigma}^{-j}, \psi^j, \bar{\phi}^{-j}}(a^i_t = C) = \mathbb{P}_{\tau^k, \bar{\sigma}^{-k}, \psi^k, \bar{\phi}^{-k}}(a^i_t = C), \\ c_t &= \mathbb{P}_{\tau^j, \bar{\sigma}^{-j}, \psi^j, \bar{\phi}^{-j}}(a^i_t = D) = \mathbb{P}_{\tau^k, \bar{\sigma}^{-k}, \psi^k, \bar{\phi}^{-k}}(a^i_t = D). \end{aligned}$$

Under $(\tau^j, \psi^j, \bar{\sigma}^{-j}, \bar{\phi}^{-j})$, player j's expected payoff at stage t is then: $g^j_t(\tau^j, \bar{\sigma}^{\mathcal{N}(j)}) \geq 4c_t \geq 4(1 - b_t)$. As a consequence, $\gamma^j_\delta(\tau^j, \psi^j, \bar{\sigma}^{-j}, \bar{\phi}^{-j}) \geq 4(1-\delta)\sum_{t=1}^{+\infty} \delta^{t-1}(1 - b_t)$. Since $(\bar{\sigma}, \bar{\phi})$ is an equilibrium of the repeated game, there must exist $\bar{\delta} \in (0, 1)$ such that for any $\delta \in (\bar{\delta}, 1)$, $4(1-\delta)\sum_{t=1}^{+\infty} \delta^{t-1}(1-b_t) \leq 1$, so that $(1-\delta)\sum_{t=1}^{+\infty} \delta^{t-1} b_t \geq \frac{3}{4}$. In the same way, player k's expected payoff under $(\tau^k, \psi^k, \bar{\sigma}^{-k}, \bar{\phi}^{-k})$ at stage t is: $g^k_t(\tau^k, \bar{\sigma}^{\mathcal{N}(k)}) \geq 4b_t$. Hence, $\gamma^k_\delta(\tau^k, \psi^k, \bar{\sigma}^{-k}, \bar{\phi}^{-k}) \geq 4(1-\delta)\sum_{t=1}^{+\infty} \delta^{t-1} b_t$. However, there exists $\bar{\delta} \in (0, 1)$ such that for any $\delta \in (\bar{\delta}, 1)$, $(1-\delta)\sum_{t=1}^{+\infty} \delta^{t-1} b_t \geq \frac{3}{4}$, so $\gamma^k_\delta(\tau^k, \psi^k, \bar{\sigma}^{-k}, \bar{\phi}^{-k}) \geq 3$. This contradicts the fact that $(\bar{\sigma}, \bar{\phi})$ is an equilibrium of the repeated game. As a conclusion, Condition N is necessary for the folk theorem to hold.

2.4 Construction of the equilibrium strategy

In this section, I assume that the network G satisfies Condition N, and show that any feasible and strictly individually rational payoff is an equilibrium payoff of

11. Indeed, player j draws a fictitious history, using $\bar{\phi}^j$. If $\bar{\phi}^j$ is mixed, so is ψ^j.

the repeated game for every payoff function satisfying Assumption P. From now on, suppose Assumption P holds. I take a point $\bar{\gamma} = (\bar{\gamma}^1, \ldots, \bar{\gamma}^n)$ in $\text{co}\, g(A) \cap IR^*(G, g)$ and construct a Nash equilibrium $(\bar{\sigma}, \bar{\phi})$ of the discounted game with payoff $\bar{\gamma}$ for any discount factor close enough to one. The strategy profile can be decomposed into three phases. First, there is a stream of pure action profiles that yields the desired payoff. This is how the game starts off and how it unfolds as long as no player deviates. Second, there is a communication phase (to be described more precisely) in case of an action deviation, whose purpose is to inform the deviator's neighbors of the deviator's identity. Third, there is a punishment phase.

The communication phase builds on the methods of Renault and Tomala ([37]) and Tomala ([44]). Their constructions rely on two notions: (i) the set of suspected players each player sends in case of a deviation, and (ii) the length of each path of communication (*i.e.* they compare the dates of the messages received to the date of the alleged deviation). Although the messages containing the sets of suspected players are manipulable by those players, the length of communication is not in their setup. This is the key argument for the players to identify the deviator: the length of communication gives the true date of the deviation, and this allows the players to update their sets of suspects (by linking the date of the deviation to the distance from the deviator). This argument crucially relies on public communication among neighbors, as it ensures that the length of communication is not manipulable. With private communication, as assumed here, the length of communication becomes manipulable. To see this, assume that there is no action deviation at stage $t-1$, and that a player k reports at stage t that his neighbor k' told him at stage $t-1$ she detected a deviation. It is not possible to know whether the deviation occurred at stage $t-1$ (the deviator being therefore k') or at stage t (the deviator being k). Hence in my setup, the sets of suspected players as well as the length of communication are manipulable by the players, and the procotols of [37] and [44] fail. My protocol circumvents this issue by relying on sets of innocents, instead of sets of suspects and communication lengths (see Section 2.4.2).

I first provide the details for the three phases of the strategy profile. Section 2.4.2 describes the communication phase more precisely. Sufficiency is then established in Section 2.4.3. Finally, Section 2.4.4 discusses additional properties arising in the case of 2-connected networks.

2.4.1 Description of the equilibrium strategy

The equilibrium path. For each player i in N and each stage $t > 0$, choose $\bar{a}_t^i \in A^i$ such that

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}g_t^i(\bar{a}_t^i, \bar{a}_t^{\mathcal{N}(i)}) = \bar{\gamma}^i.$$

This is possible when $\delta \geq 1 - \frac{1}{n}$ (the existence is proved by Sorin, Proposition 4 p.151 in [42]).

During this phase, player i plays action $\bar{a}_t^i$ at stage t. Player i sends a blank message $\varnothing$ to his neighbors at each stage t: player i does not transmit any information to his neighbors.

Punishment phase. For each player k in N and every neighbor $i \in \mathcal{N}(k)$, fix $\underline{\sigma}^{i,k} \in \Sigma^i$ such that for every ϕ^i and (σ^k, ψ^k),

$$\gamma_\delta^k(\sigma^k, (\underline{\sigma}^{i,k})_{i\in\mathcal{N}(k)}, \psi^k, (\phi^i)_{i\in\mathcal{N}(k)}) \leq \sum_{t=1}^{\infty}(1-\delta)\delta^{t-1}v^k.$$

During this phase, player i plays a minmax strategy against his neighbor k. Notice that only player k's neighbors are able to minmax him, hence the specification of the strategies of the other players is omitted.

Communication phase. Two kinds of deviations might occur, in action or in communication: a deviating player k may stop playing $\bar{a}_t^k$ at some stage t, or may send spurious messages, or both. The deviator is allowed to choose any possible strategy (τ^k, ψ^k). In particular, this deviator may stop playing $\bar{a}_t^k$ and try to avoid other players to identify him by sending false messages. He may also take action $\bar{a}_t^k$ at each stage t but send spurious messages to induce wrong conclusions on the other players (for instance identify an innocent player as guilty). Moreover, he may send different messages to different players to prevent other players' coordination.

The communication phase is such that, if a player deviates in communication but not in action, then all players keep playing they equilibrium actions and payoffs are not affected.

Moreover, when there is an action deviation, only the deviator's neighbors have to identify him in order to punish him. However, all the deviator's neighbors have to identify him: for some payoff functions, it may not be possible to punish several neighbors and it may also be needed that all the deviator's neighbors start playing the punishment strategy to force him to his minmax. During this phase, each player i in N should play action $\bar{a}_t^i$ at stage t. The communication part of the strategy $(\bar{\sigma}, \bar{\phi})$ has the additional property that the deviator is identified by all his neighbors when there is an action deviation. This communication phase is constructed in the next section.

2.4.2 The communication protocol

The communication phase can be described as a *communication protocol*: a specification of how players choose their messages, the number of communication rounds and an output rule for each player. In my context, the communication protocol starts as soon as there is a deviation, and do not stop before the deviator's neighbors find out the identity of the deviator (at least when it is an action deviation).

I first introduce some definitions and then construct a communication protocol which has the desired properties.

2.4.2.1 Definitions

A communication protocol specifies: a finite set of messages, a strategy for each player, a number of rounds of communication and an output rule for each player. Our aim is to construct a communication protocol such that:

(i) if, for every stage $t > 0$ and every player i in N, $g_t^i = g^i(\bar{a}_t^i, \bar{a}_t^{\mathcal{N}(i)})$ and for every neighbor $j \in \mathcal{N}(i)$, player i received the blank message from j, that is $m_t^j(i) = \varnothing$, then the protocol does not start;

(ii) if there is a stage $t > 0$ such that:
- for each stage $s < t$ and each player i in N, $g_s^i = g^i(\bar{a}_s^i, \bar{a}_s^{\mathcal{N}(i)})$ and $m_s^j(i) = \varnothing$ for each neighbor $j \in \mathcal{N}(i)$;
- there is a single player k who starts deviating at stage t, *i.e.* either $a_t^k \neq \bar{a}_t^k$ or there exists a player $j \in \mathcal{N}(k)$ such that $m_t^k(j) \neq \varnothing$;

then, the protocol starts.

(iii) If there is an action deviation at some stage $t' \geq t$, *i.e.* $a_{t'}^k \neq \bar{a}_{t'}^k$, and if all players, except possibly player k, perform the protocol, then after the specified number of rounds, each neighbor of player k outputs the name of k.

In this case, I say that *deviator identification by neighbors* is possible for the network G. Before describing formally this notion, Iet me make the following observation.

Remark 2.4.1. An action deviation of player k at some stage $t > 0$ is not directly observable, because players do not observe actions. However, since Assumption P holds, an action deviation induces a change in payoff for each player $j \in \mathcal{N}(k)$, who should then start the protocol.

I give now formal definitions. I denote by θ the first stage at which some player starts deviating, *ie* θ is the stopping time

$$\theta = \inf\{t \geq 1 \ : \ \exists k \in N \text{ s.t. } \big(a_t^k \neq \bar{a}_t^k \text{ or } \exists j \in \mathcal{N}(k) \text{ s.t. } m_t^k(j) \neq \varnothing\big)\}.$$

Also denote by θ_A the first stage at which some player starts deviating in action, *i.e.* θ_A is the stopping time $\theta_A = \inf\{t \geq 1 \; : \; \exists k \in N \text{ s.t. } a_t^k \neq \bar{a}_t^k\}$. The following definitions are adapted from [44].

Definition 2.4.2. A communication protocol
A communication protocol is a tuple $(T, (M^i)_{i\in N}, \sigma, \phi, (\mathbf{x}^i)_{i\in N})$ with:

- an integer T and message spaces M^i for $i \in N$,
- an action strategy profile $\sigma = (\sigma^i)_{i\in N}$ and a communication strategy profile $\phi = (\phi^i)_{i\in N}$,
- a family of random variables $(\mathbf{x}^i)_{i\in N}$ where $\mathbf{x}^i$ is $\mathcal{H}^i_{\theta_A+T}$-measurable with values in $N \cup \{OK\}$.

Definition 2.4.3. Deviator identification by neighbors
Deviator identification by neighbors is possible for the network G if there exists a communication protocol $(T, (M^i)_{i\in N}, \sigma, \phi, (\mathbf{x}^i)_{i\in N})$ such that, for each player $k \in N$, each behavior strategy (τ^k, ψ^k) and each integer t such that $\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}(\theta_A = t) > 0$, then:

$$\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}\left(\forall i \in \mathcal{N}(k),\; \mathbf{x}^i = k \mid \theta_A = t\right) = 1.$$

The interpretation of deviator identification by neighbors is the following. The action strategy σ^k prescribes player k to choose $a_t^k = \bar{a}_t^k$ at each stage $t > 0$. The communication strategy ϕ^k prescribes player k to send to each neighbor $j \in \mathcal{N}(k)$ the blank message $m_t^k(j) = \varnothing$ at stage t, as long as for every stage $s < t$, $g_s^k = g_s^k(\bar{a}^k, \bar{a}^{\mathcal{N}(k)})$ and $m_s^j(k) = \varnothing$ for each neighbor $j \in \mathcal{N}(k)$. An alternative strategy (τ^k, ψ^k) for player k such that $\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}(\theta_A < +\infty) > 0$ is henceforth called an action deviation. The definition requires that, under an action deviation, the protocol ends in finite time for the deviator's neighbors. That is, T stages of communication after the action deviation, each player $i \in \mathcal{N}(k)$ comes up with a value for $\mathbf{x}^i$. Deviator identification by neighbors is possible if, whenever there is an action deviation of some player $k \in N$, each of his neighbors $i \in \mathcal{N}(k)$ finds out the name of the deviating player with probability one. If no player ever deviates (neither in action nor in communication), each player outputs OK. Recall that the players may not be able to distinguish between action and communication deviations, that is why the protocol starts in both cases. In the next section, I construct a communication protocol to prove that under Assumption P, deviator identification by neighbors is possible for any network that satisfies Condition N.

2.4.2.2 Deviator identification by neighbors

I now prove the following result.

Proposition 2.4.4. *Suppose that Assumption P holds and that G satisfies Condition N. Then, deviator identification by neighbors is possible for the network G*

Remark 2.4.5. In fact, Condition N is also a necessary condition on G to enable deviator identification by neighbors. The proof uses the same ideas as in Section 2.3.

The formal proof of Proposition 2.4.2.2 is given in Section 2.6.4: given a network that satisfies Condition N, I construct a protocol that satisfies the requirements of Definition 2.4.3 for deviator identification by neighbors. The formal description of the protocol is also given in Appendix 2.6.4. The main ideas are as follows. When a player i in N detects a deviation, that is either he observes a change in his payoff or he receives a message different from the blank one, then he starts sending to his neighbors his information about the sets of innocent players computed as follows:

- if the deviation detected is in action, *i.e.* he observed a change in his payoff, then he clears all the players that are not his neighbors since the deviator must be one of his neighbors.
- On the other hand, if he does not observe a change in his payoff, then he clears all his neighbors.
- Player i also updates his set of innocent players with the sets received by his neighbors: for instance, if one of his neighbors $j \in \mathcal{N}(i)$ sends the set of innocents $\{j, \ell, m\}$, he adds players ℓ and m to his own set of innocents. Indeed, either j is the deviator then ℓ and m are cleared (recall that I focus on unilateral deviations); or j is performing the protocol obediently, then ℓ and m are really innocent. However, player i cannot clear player j. In other words, the information "j claims ℓ and m to be innocent" is not manipulable by player j.[12]

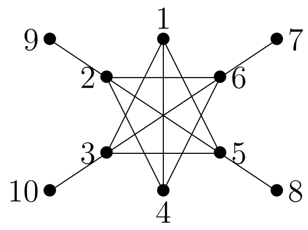
There may be several deviations at different stages, either in communication and/or in action. Actually, each player i sends a list of sets, each set being linked with a delay corresponding to the stage at which the alleged deviation happened. At the end of the protocol, for each player $i \in \mathcal{N}(k)$, if one of player i's neighbors only, say player $k \in \mathcal{N}(i)$, is not cleared, then player i outputs the name of k. Otherwise, player i outputs OK.

Let me now illustrate how the protocol works on an example.

Example 2.4.6. Consider the game played on the following network:

12. As pointed out in the introduction, this is the main difference with the protocols of Renault and Tomala ([37]) and Tomala ([44]). Both protocols rely on sending sets of suspects, which is a manipulable information. Obviously, a given set of suspects enables to compute the corresponding set of innocent players, but their protocols are not based on that idea. Instead, they consider the length of communication. This is not manipulable in their context of public communication among neighbors, whereas it is with private communication.

FIGURE 2.8:



and suppose that Assumption P holds. Assume now that Player 2 deviates from the equilibrium path at some stage t. Players 4, 5, 6 and 9 start the protocol at the end of stage t; Players 1, 3, 7 and 8 at stage $t+1$ and Player 10 at stage $t+2$. The evolution of the sets of innocents (that each player sends at each stage, except Player 2) is described in the following table:

Player	t	$t+1$	$t+2$	$t+3$	t+4
1		$N \setminus \{2,6,8\}$	$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$
3		$N \setminus \{2,4,7,8\}$	$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$
4	$N \setminus \{1,2,6\}$	$N \setminus \{2,6\}$	$N \setminus \{2,6\}$	$N \setminus \{2\}$	$N \setminus \{2\}$
5	$N \setminus \{1,2,3,8\}$	$N \setminus \{1,2,3,8\}$	$N \setminus \{2,8\}$	$N \setminus \{2\}$	$N \setminus \{2\}$
6	$N \setminus \{2,3,4,7\}$	$N \setminus \{2,4\}$	$N \setminus \{2,4\}$	$N \setminus \{2\}$	$N \setminus \{2\}$
7		$N \setminus \{2,3,4\}$	$N \setminus \{2,4\}$	$N \setminus \{2,4\}$	$N \setminus \{2\}$
8		$N \setminus \{1,2,3\}$	$N \setminus \{1,2,3\}$	$N \setminus \{2\}$	$N \setminus \{2\}$
9	$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$
10			$N \setminus \{2,4,7,8\}$	$N \setminus \{2\}$	$N \setminus \{2\}$

At the end of stage $t+2$, every neighbor of Player 2 outputs the name of Player 2 and they start to minmax him at stage $t+10$.

All messages sent at stage $t+4$ seem to clear every player except Player 2. However, regarding the output rule defined previously, it does not imply that each player concludes that Player 2 is guilty. Indeed, Players 7, 8 and 10 never know who the deviator is since all the information they have comes from their unique neighbor. In particular, when Player 5 starts punishing Player 2 at stage $t+10$, Player 8 keeps performing the protocol and outputs the name of Player 5. Then, Player 8 starts punishing Player 5 at stage $t+11$. This is due to the fact that Player 8 cannot differentiate between the two following histories: a communication deviation followed by an action deviation of Player 5 on one hand, and an action deviation of Player 2 on the other hand. As a consequence, there are too many punished players. Therefore, punishing players might find themselves punished later on. But since I consider Nash equilibria, sequential rationality is not required in this case, which is off the equilibrium path. $\diamond$

The latter example shows that it may be impossible for some players to distinguish between communication deviations and action deviations. That is why I append delays to sets of innocent players. Indeed, in the previous example, Player 8 starts a new protocol (with a new delay) at stage $t+10$ at the moment when Player 5 starts to minmax Player 2. [13]

In the next section, I prove that the strategy so constructed is an equilibrium of the repeated game for any discount factor close enough to one.

2.4.3 The equilibrium property

In this section, I prove the sufficiency part of Theorem 2.2.12: *if the network G satisfies Condition N, then for every payoff function satisfying Assumption P, every feasible and strictly individually rational payoff is an equilibrium payoff of the repeated game.*

Take a network G which satisfies Condition N and suppose that Assumption P holds. I prove that the strategy $(\bar{\sigma}, \bar{\phi})$ constructed above is a Nash equilibrium. Recall that $(\bar{\sigma}, \bar{\phi})$ consists in the following:

- each player i in N plays $\bar{a}^i_t$ and sends $m^i_t(j) = \varnothing$ for each $j \in \mathcal{N}(i)$ at each stage t in case of no deviation (equilibrium path);
- when there is a deviation of a player k, each player i, except possibly k, starts the protocol for deviator identification by neighbors (Section 2.4.2.2) and plays $(\tilde{\sigma}, \tilde{\phi})$, with $\tilde{\sigma}^i_t = \bar{a}^i_t$ for each stage t during the protocol. Since some players may output the name of the deviator k before the end of the protocol, each player should not stop communicating before T rounds after θ_A (in fact, $T = n$ the number of players, see Appendix 2.6.4).
- Finally, if there is an action deviation of player k, then each neighbor of player k who outputs the name of k before stage $\theta_A + T$ starts minmaxing him at stage $\theta_A + T + 1$. If a player i outputs OK then he comes back to the equilibrium path by playing $\bar{a}^i_t$ and sending $m^i_t(j) = \varnothing$. [14] Notice that several players might be minmaxed, as in Example 2.4.6: each player i, who outputs the name of a player, starts punishing him.

Now, I check that $\bar{\gamma} \in \operatorname{co} g(A) \cap IR^*(G, g)$ is an equilibrium payoff. If there is no action deviation, the induced payoff vector is indeed $\bar{\gamma}$: if there are communication deviations only, each player i who performs the protocol either outputs OK, since

13. This is another difference with the protocols of [37] and [44]. In their setup, all players identify the deviator no matter whether the deviation is in action or in communication. For that reason, their protocol is executed only once in their model.

14. If for some player i in N, the protocol never ends, then he takes action $\bar{a}^i_t$ and sends messages according to the protocol.

he does not observe a change in his payoff, or keeps following the protocol. Suppose that player k stops playing action $\bar{a}_t^k$ at some stage t; without loss of generality, let $t = 1$. The protocol ends before stage $T = n$. Player k's discounted payoff is at most:

$$\sum_{t=1}^{T}(1-\delta)\delta^{t-1}B + \sum_{t>T}(1-\delta)\delta^{t-1}v^k = (1-\delta^T)B + \delta^T v^k$$

where B is an upper bound on the payoffs in the stage game. Since $\bar{\gamma}^k > v^k$, player k's expected discounted payoff is less than $\bar{\gamma}^k$ for δ close to one.

2.4.4 2-connected networks

In this section, I display some additional properties for 2-connected networks (notice that Condition N does not ensure 2-connectedness, see for instance the networks G_2, G_3 and G_4 in Figure 2.7). I show first that 2-connectedness enables each player to differentiate between action and communication deviations. I derive in Section 2.4.4.2 the existence of a protocol such that if every player i in N performs it (except possibly the deviator), then each player $i \neq k$ outputs the name of the deviating player when there is an action deviation (and not only the deviator's neighbors).

2.4.4.1 Distinguishing action and communication deviations

In this section, I construct a communication protocol such that:

(i) if, for every stage $t > 0$ and every player i in N, $g_t^i = g^i(\bar{a}_t^i, \bar{a}_t^{\mathcal{N}(i)})$ and for every neighbor $j \in \mathcal{N}(i)$, player i receives the blank message from j, $m_t^j(i) = \varnothing$, then the protocol does not start;

(ii) if there is a stage $t > 0$ such that:
- for each stage $s < t$ and each player i in N, $g_s^i = g^i(\bar{a}_s^i, \bar{a}_s^{\mathcal{N}(i)})$ and $m_s^j(i) = \varnothing$ for each neighbor $j \in \mathcal{N}(i)$;
- and there is a single player k who starts deviating at stage t, *i.e.* either $a_t^k \neq \bar{a}_t^k$ or there exists a player $j \in \mathcal{N}(j)$ such that $m_t^k(j) \neq \varnothing$;

then, the protocol starts.

(iii) In addition, for every stage $t' \geq t$ where t is the stage defined by the second property, if there is an action deviation at stage t', *i.e.* $a_{t'}^k \neq \bar{a}_{t'}^k$, and if all players, except possibly player k, perform the protocol, then after the specified number of rounds, each player i outputs A (for Action deviation) regarding the deviation at stage t'. On the other hand, for every stage $t' \geq t$, if there was no action deviation at stage t', each player i in N outputs NA (for No Action deviation).

In other words, for each stage $t' > t$, after a finite number of rounds of communication, each player who performs the protocol is able to tell whether there was an action deviation or not at stage t'. In this case, I say that *deviation differentiation* is possible for the network G. The formal definition of deviation differentiation is given in Appendix 2.6.5. Now, I show that 2-connected networks have the following property.

Proposition 2.4.7. *Suppose that g satisfies Assumption A and that G is 2-connected and satisfies Condition N. Then, deviation differentiation is possible for the network G.*

Remark 2.4.8. As in Proposition 2.4.4, Condition N is also a necessary condition on G to enable deviaton differentiation.

The formal proof of Proposition 2.4.7 is given in Section 2.6.5. Given a 2-connected network satisfying Condition N, I construct a protocol which satisfies the requirements of Definition 2.6.2 for deviation differentiation. The main ideas are as follows. Let t^i be defined as in Section 2.4.2.2. When a player i in N detects a deviation at stage t^i, either he observes a change in his payoff or he receives a message different from the blank one. Then he starts sending to all his neighbors ordered sequences of players linked with dates. If player i observes a change in his payoff at stage t^i, then he sends $(0,(i))$ to his neighbors at the end of stage t^i, which means "player i detects an action deviation 0 stages before." If player i receives $m^i_{t^i}(j) \neq \varnothing$ from a neighbor $j \in \mathcal{N}(i)$, then i transmits this message adding first his name at the end and adding $+1$ to the delay. For instance, if player i receives $(s,(m,\ell))$ from a neighbor $\ell \in \mathcal{N}(i)$, then he transmits to all his neighbors at stage t^i+1 the new message $(s+1,(m,\ell,i))$ which means "i says that ℓ says that m says that he detected a deviation $s+1$ stages before." Other messages are disregarded. The delay represents the date of the alleged action deviation, and the sequence of players represents the path of communication. After T rounds of communication, each player who performs the protocol analyzes all the sequences of players received for each delay. The following properties hold (see Section 2.6.5 for a formal proof).

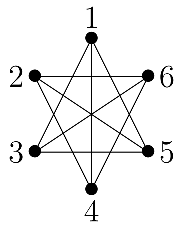
(i) If there is at least one player in all sequences regarding stage t, then there was no action deviation at stage t and each player i outputs NA.

(ii) If there is no single player who appears in all sequences regarding stage t, then there was an action deviation at stage t and each player i outputs A.

The idea is that if there is no action deviation at stage t, then each message received comes from the deviating player, hence his name appears in all sequences of players. On the other hand, if there is an action deviation at stage t, there are at least two neighbors of the deviator who start sending messages since G is 2-connected, and each player receives this information from at least two independent paths. The protocol is described formally in Appendix 2.6.5. Let me now introduce a simple example to illustrate how this protocol works.

Example 2.4.9. Consider a 6-player game played on the following network:

FIGURE 2.9:



Suppose Assumption P holds and that at some stage $t > 0$, Player 1 deviates in communication and sends to Players 4 and 5 the message $m_t^1(j) = (1,(3,1))$, with $j \in \{4,5\}$. This message means: "Player 3 tells me that he detected an action deviation one stage before (therefore at stage t-1)". Player 4 then starts the protocol for deviation differentiation and, for each $j \in \{1,2,6\}$, $m_{t+1}^4(j) = (2,(3,1,4))$. In the same way, for each $j \in \{1,2,3\}$, $m_{t+1}^5(j) = (2,(3,1,5))$. Each message sent after stage $t+1$ starts with the same sequence $(s,(3,1))$, $s > 0$, except if Player 1 sends new spurious messages. But, in each case, Player 1 appears in all sequences of players. And at stage $t+n+1$, each player, except possibly Player 1, outputs NA. However, notice that Player 3 may also appear in all sequences, so it may be impossible to distinguish between a communication deviation at stage $t-1$ of Player 3 on one hand, and a communication deviation of Player 1 at stage t on the other hand. ◇

In the next section, I use the protocol for deviation differentiation to show that 2-connected networks enable all the players to identify the deviator when there is an action deviation.

2.4.4.2 Deviator identification

In this section, I introduce the notion of *deviator identification* by all players, not by neighbors only.

Definition 2.4.10. Deviator identification
Deviator identification is possible for the network G if there exists a communication protocol $(T,(M^i)_{i\in N},\sigma,\phi,(\mathbf{x}^i)_{i\in N})$ such that, for each player $k \in N$, each strategy (τ^k,ψ^k) and each integer t such that $\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}(\theta_A = t) > 0$, then:

$$\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}\left(\forall i \in N \setminus \{k\},\ \mathbf{x}^i = k \mid \theta_A = t\right) = 1.$$

The difference between deviator identification by neighbors and deviator identification is that in the latter notion, every player outputs the name of the deviator when there is an action deviation, while the first notion only requires the deviator's neighbors to output his name. The following property holds.

Proposition 2.4.11. *Suppose Assumption P holds and that G is 2-connected and satisfies Condition N. Then, deviator identification is possible for the network G.*

Remark 2.4.12. As in Proposition 2.4.4, Condition N is also a necessary condition to enable deviator identification for a network G.

The formal proof of Proposition 2.4.11 is given in Appendix 2.6.6. The idea is to combine both protocols constructed for the case of deviator identification by neighbors and for the case of deviation differentiation. As before, each neighbor of the deviator identifies him in finite time. Moreover, when there is an action deviation at some stage t, each player i in N detects that it is an action deviation (Proposition 2.4.7). Then the result follows from the fact that each player who is not a neighbor of the deviator does not suspect any communication deviation of any neighbor anymore. Notice that if deviator identification is possible for a network G, then, when the strategy profile $(\bar{\sigma}, \bar{\phi})$ (as described in Section 2.4.1) is played, and when there is an action deviation at some stage t, thereafter only the deviator is minmaxed since all the players in the game identify him.

Remark 2.4.13. For the case in which all players interact with each other, and observe their neighbors' moves, Renault and Tomala ([37]), and Tomala ([44]), show that 2-connectedness is necessary and sufficient for a Nash folk theorem to hold under the assumption that communication is public among neighbors (or *broadcast*). It is easy to see that my protocols for deviation differentiation and deviator identification work out in their setup too. As a consequence, public communication is not necessary in their setup.

2.5 Extensions and open questions

In this section, I discuss some extensions of the model and state open problems.

Partially known networks. Theorem 2.2.12 is still valid if the network G is *partially known* to the players, *i.e.* they only know their neighbors and the number of players in the game. To prove a folk theorem for partially known networks, the communication protocol constructed in Section 2.4 is modified as follows: instead of sending sets of innocents, players tell whether their neighbors are guilty or innocent.

Hence, each player can easily compute his set of innocents among his neighbors, which is sufficient for the folk theorem to hold.

Uniform equilibrium. Condition N is also a necessary and sufficient condition to obtain a folk theorem for uniform sequential equilibria of the undiscounted repeated game (see [13] and [43]). Namely, every feasible and individually rational payoff is a uniform sequential equilibrium payoff for every payoff function g that satisfies Assumption P if and only if Condition N is satisfied. Following Fudenberg and Levine ([13]), the idea is to construct equilibrium strategies such that a deviator is punished for a long but finite number of periods, then players return to the equilibrium path.

Uncoupling interaction, monitoring and communication networks. In my model, interaction, monitoring and communication structures are captured by the same network. A possible extension of my model is to consider a communication network, G_C, an interaction network G_I and a subnetwork $G_M \subset G_I$ representing the monitoring structure. Each player can communicate with his neighbors in G_C, and his payoff depends on his own action as well as on the actions of his neighbors in G_I. However, each player i only detects the deviations of a subset of his neighbors in G_I, this subset being the set of player i's neighbors in the monitoring graph G_M. Denote by $\mathcal{N}_I(i)$, $\mathcal{N}_M(i)$ and $\mathcal{N}_C(i)$ the set of player i's neighbors in G_C, G_I and G_M respectively. Assumption P is adapted as follows.

Assumption P2 . *For each player $i \in N$, each neighbor $j \in \mathcal{N}_M(i)$, every actions b^j, c^j in A^j, a^i in A^i, $a^{\mathcal{N}_I(i)\setminus\{j\}}$ in $A^{\mathcal{N}_I(i)\setminus\{j\}}$: $g^i(a^i, a^{\mathcal{N}_I(i)\setminus\{j\}}, b^j) \neq g^i(a^i, a^{\mathcal{N}_I(i)\setminus\{j\}}, c^j)$.*

I derive the following Nash folk theorem.

Proposition 2.5.1. *The following statements are equivalent.*

1. *For any payoff function that satisfies Assumption P2, each feasible and strictly individually rational payoff is an equilibrium payoff of the repeated game.*
2. *For each player i in N, each neighbors j, k in $\mathcal{N}_I(i)$, there exists a player ℓ in $\mathcal{N}_M(j) \setminus \{k\} \triangle \mathcal{N}_M(k) \setminus \{j\}$ such that there is a path from player ℓ to i in G_C which goes neither through j nor k.*

The proof is a straightforward adaptation of the proof of Theorem 2.2.12.

Broadcast communication. An alternative communication model is that of public (or *broadcast*) communication: each player is restricted to send the same message to all his neighbors at each stage. In particular, if players must communicate through actions, communication often is broadcast (see [5] and [37]). Broadcast communication is also well studied in computer science (see e.g [12]). Under public communication among neighbors, Condition N remains sufficient for a folk theorem to hold, since the equilibrium strategy constructed in Section 2.4 actually uses broadcast communication only. Yet, necessary conditions remain unknown. In particular, for the case in which players j and k are not neighbors in Figure 2.7, it becomes possible for a player $l \in \mathcal{N}(j) \setminus \{k\} \triangle \mathcal{N}(k) \setminus \{j\}$ to transmit relevant information to player i (see the techniques of [39]). Hence, the necessary and sufficient conditions for a folk theorem must be weaker than Condition N under broadcast communication.

Sequential rationality. As said in the introduction, an important problem is to impose sequential rationality ([28]). First, it is obvious that Condition N is too weak, since the network must be at least 2-connected in addition. Indeed, in Example 2.4.6, Player 5 is minmaxed although he did not deviate. Indeed, since the network is not 2-connected, players cannot distinguish between the deviations of Players 2 and 9, hence both are minmaxed. If sequential rationality were imposed, then Player 2 might have an incentive not to report the deviations of Player 9, and a folk theorem would fail. Therefore, 2-connectedness is necessary for a folk theorem to hold with sequential equilibria. However, even for 2-connected graphs, the problem remains open, as explained in the introduction.

2.6 Proofs

2.6.1 Payoff function of Section 2.3 with more than two actions

In this section, I modify the payoff function constructed in Section 2.3 for games in which players have more than two actions. For this purpose, I duplicate rows, columns, matrices…in the following manner. For each player $p \in N$, identify A^p with $\{1, \dots, k_p\}$, where $k_p = \sharp A^p$. The payoff functions are the following.

- For each player $m \neq i, j, k$ such that $m \notin \mathcal{N}(j) \cap \mathcal{N}(k)$, player m's payoff at stage t is $g^m(a_t^m, a_t^{\mathcal{N}(m)}) = \ell_t \frac{\epsilon}{n} + \frac{1}{\sum_{p \in \mathcal{N}(m) \cup \{m\}} k_p}$.
- For each player $m \neq i, j, k$ such that $m \in \mathcal{N}(j) \cap \mathcal{N}(k)$, player m's payoff at stage t is:

$$g^m(a_t^m, a_t^{\mathcal{N}(m)}) = \begin{cases} 1 + \ell_t^{-jk} \frac{\epsilon}{n} + \frac{1}{\sum_{p \in \mathcal{N}(m) \setminus \{j,k\} \cup \{m\}} k_p} & \text{if } a^j = a^k; \\ \ell_t^{-jk} \frac{\epsilon}{n} + \frac{1}{\sum_{p \in \mathcal{N}(m) \setminus \{j,k\} \cup \{m\}} k_p} & \text{otherwise.} \end{cases}$$

– Player i's payoff at stage t is:

$$g^i(a_t^i, a_t^{\mathcal{N}(i)}) = \begin{cases} u^i(a_t^i, a_t^j, a_t^k) & \text{if } a_t^p \in \{1,2\} \text{ for each } \ p \in \mathcal{N}(i), \\ u^i(a_t^i, a_t^j, a_t^k) - \frac{1}{\sum_{p\in\mathcal{N}(i)\cup\{i\}} k_p} & \text{if } a_t^i,\ a_t^j,\ a_t^k \in \{1,2\},\ a_t^m \geq 3 \\ & \forall m \in \mathcal{N}(i) \setminus \{i,j,k\}, \\ 6 - \frac{1}{\sum_{p\in\mathcal{N}(i)\cup\{i\}} k_p} & \text{otherwise.} \end{cases}$$

– For each player $n \in \{j,k\}$, player n's payoff at stage t is:

$$g^n(a_t^n, a_t^{\mathcal{N}(n)}) = \begin{cases} u^n(a_t^i, a_t^j, a_t^k) & \text{if } a_t^p \in \{1,2\} \text{ for each } \ p \in \mathcal{N}(n), \\ u^n(a_t^i, a_t^j, a_t^k) + \frac{1}{\sum_{p\in\mathcal{N}(n)\cup\{n\}} k_p} & \text{if } a_t^i,\ a_t^j,\ a_t^k \in \{1,2\},\ a_t^m \geq 3 \\ & \forall m \neq i,j,k, \\ 6 + \frac{1}{\sum_{p\in\mathcal{N}(n)\cup\{n\}} k_p} & \text{otherwise.} \end{cases}$$

It is easy to see that this payoff function satisfies Assumption P, and that all the desired properties for the proof of necessity in Section 2.3 hold.

2.6.2 Construction of the deviations (τ^j, ψ^j) and (τ^k, ψ^k)

I define formally (τ^j, ψ^j). Denote by $C(j) = C \cap \mathcal{N}(j)$, $C(k) = C \cap \mathcal{N}(k)$, $C'(j) = C' \cap \mathcal{N}(j)$ and $C'(k) = C' \cap \mathcal{N}(k)$. Notice that $C(j) = C(k)$ and that $C'(j) \cap C'(k) = \emptyset$ by definition. First, I show that the deviations of j and k are indistinguishable according to player i. Assume that:

– each player $\ell \in C'$ uses some pure strategy (σ^ℓ, ϕ^ℓ);

– up to the beginning of stage $T+1$, player j has played a sequence of action $a^j(T+1) = (a_1^j, \ldots, a_{T+1}^j)$ and has sent to each neighbor n in $C'(j)$ a sequence of messages $m^j(n)(T) = (m_1^j(n), \ldots, m_T^j(n))$. I set $m^j(C'(j))(T) = (m^j(n)(T))_{n\in C(j)}$;

– up to the beginning of stage $T+1$, player k has played a sequence of actions $a^k(T+1) = (a_1^k, \ldots, a_{T+1}^k)$ and sent to each neighbor in $C'(k)$ the messages $m^k(\ell)(T) = (m_1^k(\ell), \ldots, m_T^k(\ell))$. Again, set $m^k(C'(k))(T) = (m^k(\ell)(T))_{\ell\in C'(k)}$.

By induction on T, this defines the messages sent by players $\ell \in C'$ at stages $1, \ldots, t+1$. Denote the corresponding sequence of messages received by player j from his neighbors in $C'(j)$ at stages $1, \ldots, t$ by:

$$m^{C'(j)}(t)\ \left[(\sigma^\ell, \phi^\ell)_{\ell\in C'}, a^j(t), m^j(C'(j))(t-1), a^k(t), m^k(C'(k))(t-1) \right].$$

Notice that the message sent by player j does not depend directly on the actions of other players, but on payoffs. Symmetrically I construct (with similar notations):

$$m^{C'(k)}(t)\ \left[(\sigma^\ell, \phi^\ell)_{\ell\in C'}, a^k(t), m^k(C'(k))(t-1), a^j(t), m^j(C'(j))(t-1) \right].$$

In the same way, define the sequences of payoffs of player j at stages $1, \ldots, t$ by

$$g^j(t)\left[a^i(t), m^i(j)(t), m^i(k)(t), (\sigma^\ell, \phi^\ell)_{\ell \in C'}, a^j(t), m^j(C'(j))(t-1), a^k(t), m^k(C'(k))(t-1)\right],$$

where I denote $a^i(t) = (a^i_1, \ldots, a^i_t)$, $m^i(j)(t) = (m^i_1(j), \ldots, m^i_t(j))$, and $m^i(k)(t) = (m^i_1(k), \ldots, m^i_t(k))$. Symmetrically, define the sequences of payoffs of player k at stages $1, \ldots, t$ as follows:

$$g^k(t)\left[a^i(t), m^i(j)(t), m^i(k)(t), (\sigma^\ell, \phi^\ell)_{\ell \in C'}, a^j(t), m^j(C'(j))(t-1), a^k(t), m^k(C'(k))(t-1)\right].$$

The strategy (τ^j, ψ^j) is defined as follows:

- at each stage, player j chooses between C and D with probability $\frac{1}{2}$-$\frac{1}{2}$ each;
- before stage 1, player j selects a fictitious pure strategy (σ^ℓ, ϕ^ℓ) according to the distribution $(\bar{\sigma}^\ell, \bar{\phi}^\ell)$ for each player $\ell \in C'$ and a fictitious pure strategy (σ^j, ϕ^j) according to $(\bar{\sigma}^i, \bar{\phi}^j)$; [15]
- at each stage, player j sends the blank message to all neighbors in $C'(j)$;
- at stage $t = 1$, player j plays according to (σ^j, ϕ^j). At the end of stage $t = 1, \ldots, T-1$, player j knows the sequences of messages $m^j(\ell)(t)$ he actually sent and the messages $m^\ell(j)(t)$ actually received from each neighbor $\ell \in \mathcal{N}(j)$ up to stage t. He also knows at the beginning of stage $t+1$ the sequences of actions he actually played, $\sigma^j(t+1)$, and his payoffs up to stage $t+1$. At stage $t+1$, player j sends to each neighbor in $C(j)$ the following message:

$$\begin{aligned} \phi^j \quad \Bigg[& \sigma^j(t+1), m^j(C(j))(t), m^{C(j)}(j)(t), \\ & g^j(t)\big[\,(\sigma^\ell, \phi^\ell)_{\ell \in C'}, \sigma^j(t+1), \phi^j(t), \tau^k(t+1), \psi^k(C'(k))(t)\,\big], \\ & m^{C'(j)}(t)\big[\,(\sigma^\ell, \phi^\ell)_{\ell \in C'}, \sigma^j(t+1), \phi^j(t), \tau^k(t+1), \psi^k(C'(k))(t)\,\big] \Bigg], \end{aligned}$$

where $\tau^k(t+1) = (\tau^k_1, \ldots, \tau^k_{t+1}) = (D, \ldots, D)$ and

$$\psi^k(C'(k))(t) = (\psi^k_1(\ell), \ldots, \psi^k_t(\ell))_{\ell \in C'(k)} = (\varnothing, \ldots, \varnothing).$$

The interpretation is that player j plays the communication strategy he would have played if he was playing $(\bar{\sigma}^j, \bar{\phi}^j)$ and if player k was playing (τ^k, ψ^k).

Finally, (τ^k, σ^k) is defined by symmetry.

2.6.3 Proof of Lemma 2.3.1.

Proof of Lemma 2.3.1

15. By Kuhn's theorem ([29]), the behavior strategy $(\bar{\sigma}^\ell, \bar{\phi}^\ell)$ is identified with a probability distribution on the set of pure strategies.

First of all, notice that $(\tau^j, \psi^j, \bar{\sigma}^{-j}, \bar{\phi}^{-j})$ and $(\tau^k, \psi^k, \bar{\sigma}^{-k}, \bar{\phi}^{-k})$ induce the same distribution of payoffs for player i, even if $\bar{\sigma}^j$ and $\bar{\sigma}^k$ are mixed: player i gets a payoff of 1 or 0 with probability $\frac{1}{2}$-$\frac{1}{2}$.

Let $N = C' \cup \{j\}$ and $N' = C' \cup \{k\}$. Fix two vectors of pure strategies $(\sigma^N, \phi^N) = (\sigma^j, \phi^j, \sigma^{C'}, \phi^{C'})$ and $(\tilde{\sigma}^{N'}, \tilde{\phi}^{N'}) = (\tilde{\sigma}^k, \tilde{\phi}^k, \tilde{\sigma}^{C'}, \tilde{\phi}^{C'})$ and consider the events:

$$\begin{aligned} H(j)(\sigma^N, \phi^N, \tilde{\sigma}^{N'}, \tilde{\phi}^{N'}) &= \{j \text{ playing } (\tau^j, \psi^k) \text{ first selects } (\sigma^N, \phi^N) \text{ and each} \\ &\quad \text{player } i \text{ in } N' \text{ playing } (\bar{\sigma}^i, \bar{\phi}^i) \text{ selects } (\tilde{\sigma}^i, \tilde{\phi}^i)\}, \\ H(k)(\sigma^N, \phi^N, \tilde{\sigma}^{N'}, \tilde{\phi}^{N'}) &= \{k \text{ playing } (\tau^k, \psi^k) \text{ first selects } (\tilde{\sigma}^{N'}, \tilde{\phi}^{N'}) \text{ and each} \\ &\quad \text{player } i \text{ in } N \text{ playing } (\bar{\sigma}^i, \bar{\phi}^i) \text{ selects } (\sigma^i, \phi^i)\}. \end{aligned}$$

The probability under $(\tau^j, \psi^j, \bar{\sigma}^{-j}, \bar{\phi}^{-j})$ of $H(j)(\sigma^N, \phi^N, \tilde{\sigma}^{N'}, \tilde{\phi}^{N'})$ is the product:

$$\prod_{i \in C'} \bar{\sigma}^i(\sigma^i) \times \bar{\phi}^i(\phi^i) \times \bar{\sigma}^j(\sigma^j) \times \bar{\phi}^j(\phi^j) \times \bar{\sigma^i}(\tilde{\sigma}^i) \times \bar{\phi}^i(\tilde{\phi}^i) \times \bar{\sigma}^k(\tilde{\sigma}^k) \times \bar{\phi}^k(\tilde{\phi}^k),$$

which is also the probability under $(\tau^k, \psi^k, \bar{\sigma}^{-k}, \bar{\phi}^{-k})$ of $H(k)(\sigma^N, \phi^N, \tilde{\sigma}^{N'}, \tilde{\phi}^{N'})$. This holds for each $(\sigma^N, \phi^N, \tilde{\sigma}^{N'}, \tilde{\phi}^{N'})$, so it is sufficient to prove that the following equality holds:

$$\begin{aligned} &\mathbb{P}_{\tau^j, \bar{\sigma}^{-j}, \psi^j, \bar{\phi}^{-j}} \left(h_T^i \mid H(j)(\sigma^N, \phi^N, \tilde{\sigma}^{N'}, \tilde{\phi}^{N'}) \right) = \\ &\quad \mathbb{P}_{\tau^k, \bar{\sigma}^{-k}, \psi^k, \bar{\phi}^{-k}} \left(h_T^i \mid H(k)(\sigma^N, \phi^N, \tilde{\sigma}^{N'}, \tilde{\phi}^{N'}) \right). \end{aligned} \tag{2.1}$$

It is easy to prove by induction that, for each $t \in \{0, \ldots, T-1\}$:

$$\begin{aligned} &\mathbb{P}_{\tau^j, \bar{\sigma}^{-j}, \psi^j, \bar{\phi}^{-j}} \left(h_{t+1}^i \mid H(j)(\sigma^N, \phi^N, \tilde{\sigma}^{N'}, \tilde{\phi}^{N'}), h_t^i \right) = \\ &\quad \mathbb{P}_{\tau^k, \bar{\sigma}^{-k}, \psi^k, \bar{\phi}^{-k}} \left(h_{t+1}^i \mid H(k)(\sigma^N, \phi^N, \tilde{\sigma}^{N'}, \tilde{\phi}^{N'}), h_t^i \right), \end{aligned}$$

which proves (2.1) and Lemma 2.3.1 follows directly. □

2.6.4 Proof of Proposition 2.4.4

In this section, I construct formally the protocol in order to prove Proposition 2.4.4. Take any player i in N. Let t^i be the first stage at which player i detects a deviation, that is $t^i = \inf\{t \geq 1 \,:\, \exists k \in \mathcal{N}(i),\ a_t^k \neq \bar{a}_t^k \text{ or } m_t^k(i) \neq \varnothing\}$. Denote also by $t_A^i = \{t \geq 1 \,:\, \exists k \in \mathcal{N}(i),\ a_t^k \neq \bar{a}_t^k\}$ the first stage at which player i detects an action deviation. Equivalently, since Assumption P is satisfied, t_A^i represents the first stage at which player i observes a change in his payoff: $t_A^i = \inf\{t \geq 1 \,:\, g_t^i \neq g^i(\bar{a}_t^i, \bar{a}_t^{\mathcal{N}(i)})\}$. In the same way, let $t_C^i = \inf\{t \geq 1 \,:\, \exists k \in \mathcal{N}(i),\ m_t^k(i) \neq \varnothing\}$. Obviously, $t^i = \inf\{t_A^i, t_C^i\}$. Let also $\theta = \inf_{i \in N} t^i$. Obviously, $\theta = \inf\{t \geq 1 \,:\, \exists k \in N,\ a_t^k \neq \bar{a}_t^k \text{ or } \exists j \in \mathcal{N}(k) \text{ s.t. } m_t^k(j) \neq \varnothing\}$. In the same way, let $\theta_A = \inf_{i \in N} t_A^i$ and $\theta_C = \inf_{i \in N} t_C^i$. One has $\theta = \inf\{\theta_A, \theta_C\}$. The communication protocol is as follows.

PROTOCOL FOR DEVIATOR IDENTIFICATION BY NEIGHBORS

The message space. All players communicate using the same finite set of messages M with:

$$M = \{(s, I_s)_{s \in S} \ : \ S \subseteq \{0, \dots, n\}, \text{ with } \forall s \in S, \ I_s \subseteq N\}$$

where n is the number of players in the network. A message is a list of pairs, each pair being composed of an integer and a subset of players. The interpretation is as follows. In each pair, the integer s represents a delay and refers to a stage. The subset of players I_s is a set of innocent players corresponding to a deviation that occurred s stages before. The maximal delay is bounded by the number of players n.

The strategy of player i. Player i always takes the action $\bar{a}^i_t$ when he performs the protocol and the message he sends is a function of his observations. At the end of stage t^i, player i starts the protocol. For each stage $t \geq t^i$, let $S^i(t) \subseteq \{0, \dots, n\}$ be the set of delays used in the message sent by player i at stage t. Denote also by $I^i_s(t)$ the set of innocent players according to player i at stage t regarding the deviation that occurred s stages before, with $s \in S^i(t)$: $I^i_s(t)$ thus represents player i's set of innocent players for a deviation that happened at stage $t - s$. At each stage $t \geq t^i$, player i sends to each neighbor $j \in \mathcal{N}(i)$ the message $m^i_j(t) = (s, I^i_s(t))_{s \in S^i(t)}$ computed as follows.

(i) Delay 0: if player i detects an action deviation at stage t, that is $g^i_t \neq g^i(\bar{a}^i_t, \bar{a}^{\mathcal{N}(i)}_t)$, then, at the end of stage t, player i sends to all his neighbors the pair $(0, I^i_0(t))$ where his set of innocent is $I^i_0(t) = N \backslash \mathcal{N}(i)$. This means that for each neighbor $j \in \mathcal{N}(i)$,

$$(0, N \setminus \mathcal{N}(i)) \in m^i_t(j).$$

(ii) Delays $1, \dots, n$: suppose that player i receives the messages $(m^j_t(i))_{j \in \mathcal{N}(i)}$ from his neighbors at stage t, where for every player $j \in \mathcal{N}(i)$, $m^j_t(i) = (s, I^j_s(t))_{s \in S^j(t)}$ (all other messages are disregarded). Then, the message sent by player i at stage $t+1$, $m^i_{t+1}(j) = (s, I^i_s(t+1))_{s \in S^i(t+1)}$, satisfies the following rule.

1. Each delay increases by one: if $s \in \bigcup_{j \in \mathcal{N}(i)} S^j(t) \cup S^i(t)$ and $s \neq n$, then $s + 1 \in S^i(t+1)$.
2. Then, player i updates his sets of innocents linked to each delay $s \in S^i(t+1) \setminus \{0\}$ as follows.

– If $s-1 \in S^i(t)$, then:

$$I^i_s(t+1) = \bigcup_{j \in \mathcal{N}(i)} \left(I^j_{s-1}(t) \setminus \{j\}\right) \cup I^i_{s-1}(t).$$

The new set of innocents of player i at stage $t+1$ for the deviation at stage $t+1-s$ contains player i's set of innocents at stage t and all the players cleared by each neighbor $j \in \mathcal{N}(i)$ at stage t (except player j himself).

– Otherwise, if $s-1 \notin S^i(t)$, then:

$$I^i_s(t+1) = \bigcup_{j \in \mathcal{N}(i)} \left(I^j_{s-1}(t) \setminus \{j\}\right) \cup \mathcal{N}(i).$$

For deviations undetected by player i at stage t, his set of innocents at stage $t+1$ contains all the players cleared by his neighbors at stage t (where the sender is removed from his set of innocents as previously) as well as his neighbors $j \in \mathcal{N}(i)$.

For all other histories, the message is arbitrary (histories which are not consistent with unilateral deviations are disregarded). This ends the definition of the strategies. Denote by $(\tilde{\sigma}, \tilde{\phi})$ this strategy profile.

The output rule. If $t^i_A < +\infty$, then for any $t \geq t^i_A$, let $X^i(t)$ be the set of suspects of player i regarding the deviation at stage t^i_A, that is, for every $t \geq t^i_A$:

$$X^i(t) = \mathcal{N}(i) \cap \left(N \setminus I^i_{t-t^i_A}(t)\right).$$

This set of suspects contains the neighbors of player i that have not been cleared up to stage t. The output rule $\mathbf{x}^i$ of player i is defined as follows. Consider the first stage T^i at which player i identifies the deviating player in case of an action deviation of a neighbor:

$$T^i = \inf\{t \geq t^i_A \,:\, \sharp X^i(t) = 1\}.$$

If $T^i = +\infty$, set $\mathbf{x}^i = OK$. Otherwise, there exists x such that $X^i(T^i) = \{x\}$ and define $\mathbf{x}^i = x$. In other words, when player i's set of suspects concerning an action deviation of one of his neighbors is reduced to x, player i concludes that x is the deviator.

The number of rounds. Let the number of rounds of communication (after an action deviation) be $T = n$, where n is the number of players in the game. Notice that if $\theta_A = +\infty$, the protocol may never stop if the deviator keeps sending spurious messages indefinitely. However, this is not an issue, as communication is costless.

I take up this protocol $(\tilde{\sigma}, \tilde{\phi})$, and prove the following lemma.

Lemma 2.6.1. *Let T the number of rounds be equal to n. Consider a player k, a strategy (τ^k, ψ^k) and an integer t such that $\mathbb{P}_{\tau^k,\tilde{\sigma}^{-k},\psi^k,\tilde{\phi}^{-k}}(\theta_A = t) > 0$. Then, for each player $i \neq k$,*

$$\mathbb{P}_{\tau^k,\tilde{\sigma}^{-k},\psi^k,\tilde{\phi}^{-k}}\big(\forall i \in \mathcal{N}(k),\ X^i(t+n) = \{k\} \mid \theta_A = t\big) = 1.$$

Note that Proposition 2.4.4 directly follows.

Proof of Lemma 2.6.1 Suppose that Assumption P is satisfied. Take a network G that satisfies Condition N and is connected (see Remark 2.2.13). Since the 2-player case is trivial, assume that $n \geq 6$ (see Remark 2.2.11). Fix a player k and assume that player k stops playing action $\bar{a}_t^k$ at stage t, while each player i in N chooses action $\bar{a}_s^i$ for each stage $s \leq t$. One notices that the protocol could have started before, if there were communication deviations only. However, the proof is the same whether one omits them (as I do) or not.[16] For simplicity I omit all the deviations that happens before stage t. I prove that the protocol defined in Section 2.4.2.2 is such that, for $T = n$:

- for each player $i \neq \mathcal{N}(k)$, each player $j \in \mathcal{N}(i)$ such that $j \neq k$, $j \notin X^i(t+T)$;
- $k \in X^i(t+T)$ for every $i \in \mathcal{N}(k)$.

First, each neighbor $i \in \mathcal{N}(k)$ of player k observes a change in his payoff at stage t (because of Assumption P) and thus starts the protocol at the end of stage t by sending the message $m_t^i(j) = (0, N \setminus \mathcal{N}(i))$ to his neighbors $j \in \mathcal{N}(i)$. At the end of stage t, $X^i(t) = \mathcal{N}(i)$. Since Condition N is satisfied, for each player $j \in \mathcal{N}(i)$, $j \neq k$, there exists a player $\ell \in \mathcal{N}(j) \setminus \{k\} \triangle \mathcal{N}(k) \setminus \{j\}$ such that there is a path from ℓ to i which goes neither through j nor k. Consider the two following two cases:

- if $\ell \in \mathcal{N}(k) \setminus \mathcal{N}(j)$, then player ℓ starts the protocol at the end of stage t by sending the message $(0, I_0^\ell(t))$, with $j \in I_0^\ell(t)$. Since G is connected, the distance between ℓ and j is at most $n-2$ (recall that there is a path which goes neither through j nor k). Then, at some stage $s \leq t+n-3$, there exists a player $m \in \mathcal{N}(i)$ such that $m \neq j$, k, and $m_t^m(i) = (s-t, I_{s-t}^m(s))$ with $j \in I_{s-t}^m(s)$ since along the path which goes neither through j nor k, all the players are following the protocol. Finally, at stage $s+1 \leq t+n-2$, the following holds: $j \in I_{s+1-t}^i(s)$, so $j \notin X^i(t+n-1)$.
- On the other hand, if $\ell \in \mathcal{N}(j) \setminus \mathcal{N}(k)$, then the distance between ℓ and k is at most 3, as the sequence k, i, j, ℓ is a path in G. So, player ℓ starts the protocol

16. It adds only some pairs of delays and sets in the messages that are not taken into account to analyze the deviation that occurred at stage t.

at stage $s \leq t+3$. Again, there is a path from player ℓ to i which goes neither through j nor k, its maximal length is $n-3$ and all the players along this path follow the protocol. So, as before, at some stage $s \leq t+3+n-4 = t+n-1$, there exists a player $m \in \mathcal{N}(i)$ such that $m \neq j$, k, and $m_t^m(i) = (s-t, I_{s-t}^m(s))$ with $j \in I_{s-t}^m(s)$. Finally, at stage $s+1 \leq t+n$, one has $j \in I_{s+1-t}^i(s)$, so $j \notin X^i(t+n)$.

I now prove the second point. Each player $i \in \mathcal{N}(k)$ starts the protocol at the end of stage t because of Assumption P. So, at the end of stage t and for each player $i \in \mathcal{N}(k)$, $I_0^i(t) = N \setminus \mathcal{N}(i)$ and $k \notin I_0^i(t)$. On the other hand, every player $j \notin \mathcal{N}(k)$ starts the protocol at stage $s \leq t+n$ as the graph is connected. Since all the players except k perform the protocol, there exists no player $\ell \neq k$ such that $\ell \in \mathcal{N}(j)$ and $k \in I_{s+n-1}^\ell(s-1)$. (Recall that if player k sends a set of innocents to which he belongs, a player who follows the protocol does not clear player k.) Because player j did not observe a change in his payoff at stage t, I conclude that $k \notin I_{s-t}^j(s)$. Then, the unique player that may transmit the name of player k in his set of innocents regarding the deviation at stage t is player k himself. So, for each player $i \in \mathcal{N}(k)$, $k \in X^i(t+n)$.

Finally, I conclude that for each player $i \in \mathcal{N}(k)$, $X^i(t+n) = \{k\}$, which proves Lemma 2.6.1. □

2.6.5 Deviation differentiation

In this part, I define formally deviation differentiation before proving Proposition 2.4.7. For that, I modify the communication protocol's definition given in Section 2.4.2.1. A communication protocol is now a tuple $(T, (M^i)_{i\in N}, \sigma, \phi, (\mathbf{x}^i(t))_{i\in N})$ with:

- an integer T,
- an action strategy $\sigma = (\sigma^i)_{i\in N}$ and a communication strategy $\phi = (\phi^i)_{i\in N}$,
- a family of random variables $(\mathbf{x}^i(t))_{i\in N}$ where $\mathbf{x}^i(t)$ is $\mathcal{H}_{t+T}^i$-mesurable with values in $A \cup NA$.

Note that the output function now depends on the stage of the deviation considered. Let θ and θ_A be the stopping times defined as in Section 2.4.2.1.

Definition 2.6.2. Deviation differentiation

Deviation differentiation is possible for the network G if there exists a communication protocol $(T, (M^i)_{i\in N}, \sigma, \phi, (\mathbf{x}^i(t))_{i\in N})$ such that, for each player $k \in N$, each strategy (τ^k, ψ^k) and each integer t such that $\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}(\theta = t) > 0$, then, for each $t' \geq t$:

$$\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}\left(\forall i \in N,\ \mathbf{x}^i(t') = A \mid a_{t'}^k \neq \bar{a}_{t'}^k\right) = 1$$
$$\text{and} \quad \mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}\left(\forall i \in N,\ \mathbf{x}^i(t') = NA \mid a_{t'}^k = \bar{a}_{t'}^k\right) = 1.$$

The interpretation is the following. The action strategy σ^k prescribes player k to choose $a_t^k = \bar{a}_t^k$ at each stage $t > 0$. The communication strategy ϕ^k prescribes player k to send to every neighbor $j \in \mathcal{N}(k)$ the blank message $m_t^k(j) = \varnothing$ at stage t, as far as for each stage $s < t$, $g_s^k = g_s^k(\bar{a}^k, \bar{a}^{\mathcal{N}(k)})$ and $m_s^j(k) = \varnothing$ for every $j \in \mathcal{N}(k)$. An alternative strategy (τ^k, ψ^k) for player k such that $\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}(\theta = t) > 0$ is henceforth called a deviation. The definition requires that, for each stage $t' \geq t$ and after T stages of communication, each player $i \in N$ comes up with a value for $\mathbf{x}^i(t')$. Deviation differentiation is possible if, whenever there is an action deviation at stage t', each player $i \in N$ outputs A with probability one. On the contrary, whenever there is no action deviation at stage t', each player outputs NA.

Before proving Proposition 2.4.7, I introduce the following communication protocol.

PROTOCOL FOR DEVIATION DIFFERENTIATION

The message space. All players communicate using the same finite set of messages M with:

$$M = \{(s, (x_1, \ldots, x_{s+1}))_{s \in S} \ : \ S \subseteq \{0, \ldots, n\},\ x_1, \ldots, x_{s+1} \in N\}$$

where n is the number of players in the network. Now, $(x_1, \ldots, x_{s+1})$ is an ordered sequence of players and s represents a delay which is bounded by the number of players n.

The strategy of player i. When he performs the protocol, player i always takes the action $\bar{a}_t^i$ and sends some messages as a function of his observations. Let t^i be defined as in Section 2.4.2.2. At the end of stage t^i, player i starts the protocol. For each stage $t \geq t^i$ and each player i in N, let $S^i(t) \subseteq \{0, \ldots, n\}$ be the set of delays used in the message player i sends at stage t. For each stage $t \geq t^i$, player i sends to his neighbors $j \in \mathcal{N}(i)$ the message $m_j^i(t) = (s, (x_1, \ldots, x_{s+1}))$, where $x_1, \ldots, x_{s+1}$ is in N and $s \in S^i(t)$, computed as follows.

- **(i)** Delay 0: if player i detects an action deviation at stage t, that is, if $g_t^i \neq g^i(\bar{a}_t^i, \bar{a}_t^{\mathcal{N}(i)})$, then, at the end of stage t, player i sends to all his neighbors the pair $(0, (x_1 = i))$.
- **(ii)** Delays $1, \ldots, n$: for each player $i \in N$, let $(x_1^{is}(t), \ldots, x_{s+1}^{is}(t))_{s \in S^i(t)}$ be the ordered sequences of players used in the message sent by player i at stage t. Suppose that player i receives at stage t the messages $(m_t^j(i))_{j \in \mathcal{N}(i)}$ from his neighbors, where for every player $j \in \mathcal{N}(i)$, $m_t^j(i) = (s, (x_1^{js}(t), \ldots, x_{s+1}^{js}(t) =$

$j))_{s\in S^j(t)}$ (all other messages are disregarded). Then, the messages sent by player i at stage $t+1$, $m^i_{t+1}(j)$, satisfy the following rule:

$$m^i_{t+1}(j) = \Big((s+1,(x_1^{i(s+1)}(t+1),\dots,j,x_{s+2}^{i(s+1)}(t+1)=i))_{s+1\in S^j(t)\setminus\{n\}}\Big)_{j\in\mathcal{N}(i)}.$$

For each sequence of players sent or received at stage t (with less than n players), player i increments the delay of one and adds his name i to the end of the sequence.

For all other histories, the message is arbitrary (histories which are not consistent with unilateral deviations are disregarded). This ends the definition of the strategies. Denote by (σ^*,ϕ^*) this strategy profile.

The output rule. For each $i\in N$, for each $t'\geq\theta$ (where θ is defined as in Section 2.4.2.1), set $\mathbf{x}^i(t')=A$ if the following two points are satisfied:

1. there exists at least one player who sends before stage $t'+T$ a message containing a sequence of players linked to an eventual deviation at stage t', which implies that stage t' is a potential stage of deviation. Formally:

$$\exists\ell\in N,\ \exists\ell'\in\mathcal{N}(\ell),\ \exists t^\ell\in[t',t'+T],$$
$$\exists m^\ell_{t^\ell}(\ell')=(s,(x_1^{\ell s}(t^\ell),\dots,x_{s+1}^{\ell s}(t^\ell)=\ell))_{s\in S^\ell(t^\ell)},$$
$$\exists s^\ell\in S^\ell(t^\ell)\text{ s.t. } t^\ell-s^\ell=t';$$

2. there exists no player $k\in N$ such that:

$$\forall j\in\mathcal{N}(i),\ \forall t\in[t',t'+T],\ \forall m^j_t(i)=(s,(x_1^{js}(t),\dots,x_{s+1}^{js}(t)=j))_{s\in S^j(t)},$$
$$k\in(x_1^{js}(t),\dots,x_{s+1}^{js}(t)=j)\text{ for }s=t-t'.$$

   Regarding the alleged deviation at stage t', no player appears in all the sequences transmitted before stage $t'+T$.

Otherwise, set $\mathbf{x}^i(t')=NA$.

The number of rounds. The number of rounds of communication is equal to the number of players n after each $t\geq\theta$ such that there exists $k\in N$ such that $a^k_t\neq\bar{a}^k_t$ or $\exists j\in\mathcal{N}(k)$ s.t. $m^k_t(j)\neq\varnothing$. Otherwise, $T=+\infty$. Notice that, as before, the protocol may never stop if the deviator keeps sending spurious messages infinitely. However, this is not an issue as communication is non costly.

I now prove Proposition 2.4.7.

Proof of Proposition 2.4.7 Suppose that Assumption P holds, and take a network G that satisfies Condition N and that is 2-connected. Since the 2-player case is trivial, assume that $n \geq 6$ (see Remark 2.2.11). Fix a player k and assume that player k deviates at some stage t, that is either $a_t^k \neq \bar{a}_t^k$ or there exists $j \in \mathcal{N}(k)$ such that $m_t^k(j) \neq \varnothing$, while for each stage $s \leq t$ and each player $i \in N$, $a_s^i = \bar{a}_s^i$ and $m_s^i(j) = \varnothing$ for every $j \in \mathcal{N}(i)$. I prove that the protocol for deviation differentiation defined above is such that, for every $t' \geq t$:

– if $a_{t'}^k \neq \bar{a}_{t'}^k$, for each player i in N, $\mathbf{x}^i(t') = A$ at stage $t' + T$;

– otherwise, $\mathbf{x}^i(t') = NA$ at stage $t' + T$;

with $T = n$.

First, take $t' \geq t$ such that $a_{t'}^k \neq \bar{a}_{t'}^k$. Since G is connected, each player i in N, except possibly player k, has started the protocol with a delay corresponding to stage t' before stage $t' + n$. Take a neighbor j of player k, $j \in \mathcal{N}(k)$. Since G is 2-connected, for each player i in N, there are at least two independent paths between player j and player i. So, if all the players except possibly k perform the protocol for deviation differentiation, then for each player i in N, the following holds:

1. player j performs the protocol, hence there exists some player $j' \in \mathcal{N}(j)$ such that $m_{t'}^j(j') = (s, (x_1^{js}(t'), \ldots, x_{s+1}^{js}(t') = j))_{s \in S^j(t')}$, $0 \in S^j(t')$ and $t' - 0 = t'$;
2. and:

$$\exists j^1, j^2 \in \mathcal{N}(i) \text{ s.t. } \exists t^1, t^2 \in [t', T + t'],$$
$$\exists m_{t^1}^{j^1}(i) = (s, (x_1^{j^1 s}(t^1), \ldots, x_{s+1}^{j^1 s}(t^1) = j^1))_{s \in S^{j^1}(t^1)},$$
$$\exists m_{t^2}^{j^2}(i) = (s, (x_1^{j^2 s}(t^2), \ldots, x_{s+1}^{j^2 s}(2^1) = j^2))_{s \in S^{j^2}(t^2)},$$
$$\exists (s^1, s^2) \in S^{j^1}(t^1) \times S^{j^2}(t^2) \text{ s.t. } t^1 - s^1 = t^2 - s - 2 = t' \text{ s.t.}$$
$$\forall (p, q) \in [1, \ldots, s^1 + 1] \times [1, \ldots, s^2 + 1], \;\; x_p^{j^1 s^1}(t^1) \neq x_r^{j^2 s^2}(t^2).$$

The interpretation is that there exist two neighbors of player i, named j^1 and j^2, who send messages for the deviation at stage t' to player i such that there are no player in common in the sequences of players. Notice that this property still holds if player i is in $\mathcal{N}(k)$, because $\sharp\mathcal{N}(k) \geq 2$ since G is 2-connected. I deduce then that there exists no player $\ell \in N$ such that for every player $i \in N$:

$$\forall j \in \mathcal{N}(i), \, \forall t \in [t', t' + T], \, \forall m_t^j(i) = (s, (x_1^{js}(t), \ldots, x_{s+1}^{js}(t) = j))_{s \in S^j(t)},$$
$$\ell \in (x_1^{js}(t), \ldots, x_{s+1}^{js}(t) = j) \text{ with } s = t - t'.$$

So for each player i in N, $\mathbf{x}^i(t') = A$ at stage $t' + n$.

For the second point, suppose that $a^k_{t'} = \bar{a}^k_{t'}$. Two cases are possible.

– If there exists a player $\ell \in N$ for which there exist $\ell' \in \mathcal{N}(\ell)$, $t^\ell \in [t', t'+T]$ and a message of the form $m^\ell_{t^\ell}(\ell') = (s, (x_1^{\ell s}(t^\ell), \ldots, x_{s+1}^{\ell s}(t^\ell) = \ell))_{s \in S^\ell(t^\ell)}$ with $s^\ell = t^\ell - t'$, then as all the players except possibly k perform the protocol, it must be the case that $\ell = k$. Moreover, for each $i \in N$:

$$\forall j \in \mathcal{N}(i),\ \forall t \in [t', t'+T],\ \forall m^j_t(i) = (s, (x_1^{js}(t), \ldots, x_{s+1}^{js}(t) = j))_{s \in S^j(t)},$$
$$k \in (x_1^{js}(t), \ldots, x_{s+1}^{js}(t) = j) \text{ with } s = t - t',$$

since player k is the unique player who starts transmitting messages about stage t' (and because no player $i \neq k$ observes a change in his payoff at stage t').

– Otherwise, there exists no player who sends before stage $t' + T$ a message containing a sequence of players linked to an eventual deviation at stage t', which implies that stage t' is not a potential stage of deviation.

So, if $a^k_{t'} = \bar{a}^k_{t'}$, then each player $i \in N$, except possibly player k, outputs NA at stage $t' + n$. This ends the proof of Proposition 2.4.7. □

2.6.6 Proof of Proposition 2.4.11

Proof of Proposition 2.4.11 Suppose that Assumption P holds, and take a network G that satisfies Condition N and that is 2-connected. Again, assume $n \geq 6$. Consider the following protocol, based on the previous two protocols. For each player $i \in N$, player i starts at stage t^i (where t^i is defined as in Section 2.4.2.2) the protocol for deviation differentiation (see Section 2.6.5) until $\theta_A + n + 1$. Player i then starts the protocol for deviator identification by neighbors. Notice that if $\theta_A = +\infty$, player i keeps performing the protocol for deviation differentiation forever. The output rule of player i is just adapted from the protocol for deviator identification by neighbors. For each player i and each $t \geq \theta_A + n + 1$, let $X^i(t)$ be the set of suspects of player i at stage t, that is, for every $t \geq \theta_A + n + 1$: $X^i(t) = N \setminus I^i_{t-\theta_A}(t)$. The output rule $\mathbf{x}^i$ of player i is defined as follows. Consider the first stage T^i at which player i identifies the deviating player at stage θ_A: $T^i = \inf\{t \geq \theta_A + n + 1 \ :\ \sharp X^i(t) = 1\}$. If $T^i = +\infty$, set $\mathbf{x}^i = OK$. Otherwise, there exists x such that $X^i(T^i) = \{x\}$ and define $\mathbf{x}^i = x$. In other words, when player i's set of suspects reduces to x, player i concludes that x is the deviator.

I now prove that this new protocol satisfies the requirements of Definition 2.4.10, with $T = 3n - 1$ as the number of communication rounds. Fix a player k and assume that player k stops playing action $\bar{a}^k_t$ at stage t, while each player i in N chooses action $\bar{a}^i_s$ for each stage $s \leq t$.

First, since G is 2-connected, Proposition 2.4.7 is satisfied, and then, at stage $t+n$, each player i in N outputs A and thus knows that there was a deviation at stage θ_A. Then, each player $i \in N$, except possibly player k, starts the protocol for deviator identification by neighbors at stage $t+n+1$. For each player $j \neq k$, consider the network $G - \{jk\}$ where players j and k have been removed. It only remains to prove the following claim:

Properties 2.6.3. *Fix a player $j \neq k$. In each connected component C of $G - \{jk\}$, there exists a player $\ell \in C$ such that $\ell \in \mathcal{N}(j)$.*

Assume that this claim holds. Take a player $j \neq k$ and a component C of $G - \{jk\}$, then there exists $\ell \in C$ such that $\ell \in \mathcal{N}(j)$. The following two cases only are possible.

- Either $\ell \in \mathcal{N}(j) \cap \mathcal{N}(k)$. Then $j \in I^{\ell}_{2n+1}(t+2n+1)$ (where $I^{\ell}_{2n+1}(t+2n+1)$ is defined as in Section 2.4.2.2) because of Proposition 2.4.4. I conclude that for each player $i \in C$, $j \in I^{i}_{3n-1}(t+3n-1)$, since the distance between ℓ and i is at most $n-2$.
- Or $\ell \in \mathcal{N}(j) \setminus \mathcal{N}(k)$. Then, because player ℓ knows that there was an action deviation at stage t (Proposition 2.4.7), and since player ℓ did not observe a change in his payoff at stage t (Assumption P), player ℓ knows that the deviator is not one of his neighbors (and player ℓ does not believe any of his neighbors might have deviated in communication, as I consider unilateral deviations only). So again $j \in I^{\ell}_{n+1}(t+n+1)$ at stage $t+n+1$, and for each player $i \in C$, $j \in I^{i}_{2n+1}(t+2n-1)$ as before.

I conclude that if Claim 2.6.3 is true, then for each player $j \neq k$ and each player $i \in N$, except possibly player k, $j \in I^{i}_{3n-1}(t+3n-1)$, so each player i outputs the name of k.

Now, I prove Claim 2.6.3. Take a player $j \in N$, such that $j \neq k$ and consider the graph $G - \{jk\}$. Fix a connected component C of $G - \{jk\}$, and take a player $i \in C$. Since the original graph G is 2-connected, there exists a path in G between player i and j which does not go through k. Then there exists a player $\ell \in N$ such that this path can be written as $i, \dots, \ell, j$. In that case, $\ell \in C$ and $\ell \in \mathcal{N}(j)$, which proves Claim 2.6.3 and concludes the proof of Proposition 2.4.11. $\square$

3 Sequential rationality, local interaction, global communication

This chapter corresponds to the paper entitled Communication in Repeated Network Games with Private Monitoring.

I consider repeated games with private monitoring played on a network. Each player has a set of neighbors with whom he interacts: a player's payoff depends on his own and his neighbors' actions only. Monitoring is private and imperfect: each player observes his stage payoff but not the actions of his neighbors. Players can communicate costlessly at each stage: communication can be public, private or a mixture of both. Payoffs are assumed to be sensitive to unilateral deviations. The main result is that a folk theorem holds if and only if no two players have the same set of neighbors.

3.1 Introduction

Consider a repeated game played on a network, where nodes represent players, and edges link neighbors. Interaction is local: a player's payoff depends only on his own and his neighbors' actions. Players observe their stage payoff only, hence monitoring is private, local and imperfect. Hence, both interaction and monitoring structures are given by the network. In addition, players can send costless messages at each stage. Communication can be private or public, that is: players can send different messages to distinct players (e.g. private emails), or they can communicate publicly with a subset of players (e.g. Carbon Copy). In the latter case, the Carbon Copy list of players is certifiable. (Players could also use Blind Carbon Copy, that is a mixture of private and public communication.) This paper circumscribes the networks for which a *full* folk theorem holds in this setup, *i.e.* under which conditions *all* feasible, strictly individually rational payoffs are equilibrium payoffs in the repeated game when players are sufficiently patient. The main result is that a folk theorem holds if and only if no two players have the same set of neighbors (Condition DN). For a wide class of payoff functions, I construct a perfect Bayesian equilibrium (henceforth, PBE) for the family of networks that satisfy Condition DN. Condition DN is also necessary: if it is is not satisfied, then the folk theorem does not hold.

The key to the characterization lies in understanding when communication makes it possible (i) to transmit precise information about players' deviations (detection and identification) and (ii) to coordinate players' behavior. Throughout the paper, I assume that the payoff functions are such that any unilateral deviation affects each neighbor's payoff (Assumption P). Hence, neighbors' deviations are detectable, although deviators may not be identifiable. The condition on the networks' topology for a folk theorem to hold for any payoff function that satisfies Assumption P, is that any two players must have different set of neighbors (Condition DN). It has a simple interpretation. Assume that player i detects a neighbor's deviation, and that, according to player i, the deviator could be either his neighbor j or his neighbor k. The condition states that there exists another player ℓ who (without loss of generality) is a neighbor of j but not of k. If indeed player j is the deviator, then player ℓ can confirm this to player i, since he has also detected a unilateral deviation; whereas if player k is the deviator, then ℓ can report to player i that j is innocent, since ℓ has not detected a deviation. Now, player ℓ could deviate by not reporting his information. This issue is handled by requiring deviating players to confess their deviation afterwards, that is: the equilibrium strategies are such that, each player confesses a deviation after a history in which he did deviate.[1] Under unilateral deviations, if player ℓ does not report his information truthfully, then the player

1. Close arguments can be found in Ashkenazi-Golan ([2]).

who deviates first, either j or k, will confess to player i.

The construction of the equilibrium strategy when Condition DN is satisfied is adapted from Fudenberg and Maskin ([15]). However, since histories are private here, several modifications must be made. In particular, a communication protocol is developed in order to identify the deviator when there is a deviation. Moreover, the punishment and reward phases have to be adapted. Indeed, Condition DN does not rule out networks for which a player, say k, has a single neighbor, say ℓ. In that case, player ℓ might have an incentive not to report player k's deviations for which player ℓ is the unique monitor. However, with the same argument as before which relies on unilateral deviations, if player ℓ does not report player k's deviation, then player k confesses. Nevertheless, it might be impossible for players other than k and ℓ to distinguish between the two following histories: "player k deviates in action at stage t, and his unique monitor ℓ truthfully reports the deviation" and "player k does not deviate at stage t, whereas player ℓ lies when reporting player k's deviation." However, it is then possible to punish both players k and ℓ since no player is a neighbor of both of them: I require player k (respectively ℓ) to minmax player ℓ (respectively k), and all player ℓ's neighbors to minmax player ℓ. [2] Finally, public communication also serves the purpose of coordinating players' actions. In particular, minmax strategies might be mixed, and pure actions are not monitored even by neighbors. This is an obstacle to detect deviations during punishment phases, and to provide incentives (rewards) for the minmaxing players to randomize according to the distribution given by their minmax strategies. To tackle this problem, players announce the pure actions they actually play during the punishment phase (these announcements are made simultaneously with the choice of action). Therefore, players can detect the deviations of their neighbors during the punishment phase, and can reward them accordingly thereafter.

To prove that Condition DN is also necessary for a folk theorem to hold, consider the case for which two players, say j and k, have the same neighbors. I construct a particular payoff function for which there is a feasible and strictly individually rational payoff that is not an equilibrium payoff, no matter how little players discount the future. More precisely, the payoff function is such that there exists a common neighbor of both j and k, say player i, who cannot differentiate between some deviations of players j and k. In addition, player i is unable to punish both: intuitively, player i rewards player k (respectively j) when he punishes player j (respectively player k). This is a failure of joint rationality, following the terminology of Renault

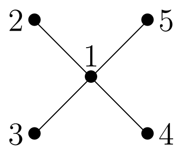
2. This joint punishment of players k and ℓ implies that the reward phase has to be modified from Fudenberg and Maskin ([15]), in a way such that both players k and ℓ have lower continuation payoffs after having been minmaxed than after the punishment phase of another player (see Section 3.5.2.4).

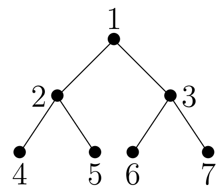
and Tomala ([38]).

Applications. Condition DN is satisfied in many environments with product differentiation. Regarding horizontal differentiation, Hotelling competition ([24]) provides a good example in which firms compete with the closest rivals. The networks G_1 and G_3 displayed in Figure 3.4, which both satisfy Condition DN, depict such environments. As far as vertical differentiation is concerned, consider a market in which a firm sells a luxury product, and may not be competing with a firm producing a low quality product. However, both might be competing with a firm selling a product of intermediate quality. Moreover, consider the situation in which firms make two kinds of decisions: the capacity of production and the quality of the product sold. Then, the firm selling a product of intermediary quality might not be able to punish simultaneously the firm that sells the luxury product on the one hand, and the firm that sells the low quality product on the other hand. Indeed, punishing the firm producing the luxury product may require to intensify the competition with this firm, for instance by increasing the quality of the (intermediary) product sold. Yet, this implies less competition with the firm that sells the low quality product, which can take advantage of this situation. This example puts forward the importance of the identification of the deviator, when a deviation occurs, in order to sustain collusion. My main result shows that, if there are three firms only, the folk theorem does not hold in this setup; on the contrary, collusion is sustainable if there are four firms, whose products are ranked with respect to their quality (identification of the deviator is then possible, since no pair of players have the same neighbors).

An other application of interest is a partnership game (see [36]). Consider a partnership in which production is conducted by a team whose goal is to maintain a certain level of individual effort by its members. Each member's effort is not observable and there is moral hazard (effort is costly). Assume that each member's compensation depends on the output of a subset of members only, referred to as the direct colleagues, and this output depends on the level of effort of these members. For instance, the head of a subteam's compensation may depend on his own, his direct subordinates' and on his own chief's levels of effort: this defines a subteam. In addition, agents may communicate with each other via emails, either privately or publicly. If a member is reported by his direct colleagues, *i.e.* identified as exerting low effort, the group can punish him by reducing his share in the total profit, which raises other members' shares. This paper shows that coordination is sustainable if and only if any two members have a non-common colleague. In particular, the following network structures (G_1 and G_2) illustrate when a folk theorem cannot hold.

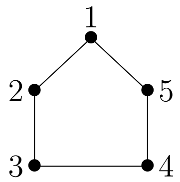
FIGURE 3.1:

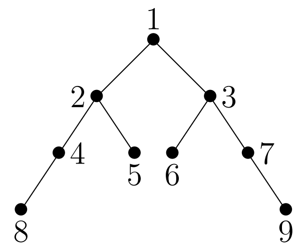




Also, no complete network satisfies Condition DN, which entails that compensation must not depend on the efforts of the whole firm's members in order to enable coordination. However, if the members form a circle (with at least five members, network G_3 in the figure below), then coordination is sustainable. Moreover, the tree structure depicted by the network G_4 also enables a folk theorem to hold.

FIGURE 3.2:





Related literature. This paper lies at the juncture of two independent literatures: repeated games and social networks. Regarding repeated games, the folk theorem was originally established for Nash equilibria ([3, 15, 41, 42]) and extended by Fudenberg and Maskin ([15, 16]) to subgame-perfect equilibria. A key assumption is perfect monitoring. A large fraction of the literature on folk theorems with imperfect monitoring has focused on imperfect public monitoring (see [14]). The model of collusion by Green and Porter ([21]) also considers imperfect public monitoring, where the market price serves as a commonly observed signal. There is a large recent literature on imperfect private monitoring (see [33] for details). Fudenberg and Levine ([13]) establish a folk theorem with imperfect private monitoring without explicit communication. They consider private random signals induced by the action profile of all other players. With public communication, Compte ([8]), Kandori and Matsushima ([26]), and Obara ([35]) provide sufficient conditions for a folk theorem to hold. All three consider random signals, whereas signals are deterministic in my setup. Yet they consider restricted sets of strategies, and they provide sufficient conditions only for a folk theorem to hold, whereas Condition DN is also necessary in my setup.

Closer to my setting, Ben-Porath and Kahneman ([5]) establish a necessary and sufficient condition for a folk theorem for the case in which (i) each player observes his neighbors' moves, and (ii) communication is public. Renault and Tomala ([37]) and Tomala ([44]) study undiscounted repeated games with the same signals as in [5] (*i.e.* each player observes his neighbors' moves) but communication is constrained by the network structure (they also establish a necessary and sufficient condition for a folk theorem to hold). All these papers [5, 37, 44] (and also [7, 27, 34, 45]) assume that monitoring among neighbors is perfect. To the contrary, I assume here that it is imperfect: payoffs encapsulate all an agent's feedback about his neighbors' play (for instance, firms infer rivals' likely behavior from their own profits). Besides, in all these papers, interaction is global, *i.e.* each player interacts with all other players. Here, interactions revolve around a network. Moreover, because of Condition DN, I prove that the folk theorem even fails if the network is complete. Recently, Nava and Piccione ([34]) and Cho ([7]) study games in networks with local interaction. Yet, both assume that each player perfectly observes his neighbors' moves. Finally, repeated games with local interaction and where players only observe their payoffs are studied in Chapter 2. This paper departs from Chapter 2 in several ways. First, communication is assumed to be private and local (*i.e.* players can communicate only with their neighbors) in Chapter 2. On the contrary here, communication is global (*i.e.* players can communicate with all opponents), and can be private or public. This seems less restrictive in terms of applications: a good example is communication via emails. Indeed, it is often the case that players know the identity of their opponents, and they are then able to communicate with all of them via emails. Moreover, emails enable players to certify the list of receivers (Carbon Copy), hence allowing public communication seems natural. Second, contrary Chapter 2 where a Nash folk theorem is established, I impose sequential rationality. The difficulty here is to provide incentives to follow the communication protocol in case of a deviation, and to follow the equilibrium strategy (in particular for the minmaxing players during the punishment phase after a deviation). As opposed to Chapter 2, the communication protocol constructed is robust to unilateral deviations. The main idea follows from the fact that players are required to confess their past deviations. With this strategy, a majority rule prevents a deviator during the communication protocol to block the identification of the deviator. The incentives for following the equilibrium strategy are given by adapting the construction of Fudenberg and Maskin ([15]) as explained previously.

This paper is also related to the literature on social and economic networks (for an overview of the networks literature, see Goyal, [20], and Jackson, [25]). Networks in which a player's payoff depends on his own and his neighbor's actions have been studied among others by Galeotti and al. ([18]) and Bramoullé and Kranton ([6]).

However, this literature does not account for repeated games in general.

The paper is organized as follows. The model is introduced in Section 3.2. In Section 3.3, I discuss the assumption on payoff functions. The main result is stated in Section 3.4. Section 3.5 establishes the sufficiency of Condition DN: for that purpose, I construct a communication protocol which aims at identifying the deviator when a deviation occurs; I then construct a perfect Bayesian equilibrium. Section 3.6 proves that the folk theorem fails if Condition DN is not satisfied. Finally, Section 3.7 develops some extensions and raises open questions.

3.2 The setup

Consider a repeated game played on a fixed network where players interact only with their neighbors. This is described by the following data:

- a finite set $N = \{1, \dots, n\}$ of players ($n \geq 3$);[3]
- for each player $i \in N$, a non-empty finite set A^i of actions (with $\sharp A^i \geq 2$);
- an undirected graph $G = (N, E)$ in which the vertices are the players and $E \subseteq N \times N$ is a set of links. Let $\mathcal{N}(i) = \{j \neq i \,:\, ij \in E\}$ be the set of player i's neighbors. Since G is undirected, the following holds: $i \in \mathcal{N}(j) \Leftrightarrow j \in \mathcal{N}(i)$;
- for each player $i \in N$, a payoff function of the form $g^i \,:\, \prod_{j \in \mathcal{N}(i) \cup \{i\}} A^j \to \mathbb{R}$, *i.e.* player i's stage payoff depends on his own and his neighbors' actions only;
- finally, a non-empty finite set M^i of player i's messages. I assume that the cardinality of M^i is large (see Remark 3.5.12 for more details), the specification of the set M^i is given in Section 3.5.

I use the following notations: $A = \prod_{i \in N} A^i$, $N^{-i} = N \backslash \{i\}$, $A^{\mathcal{N}(i) \cup \{i\}} = \prod_{j \in \mathcal{N}(i) \cup \{i\}} A^j$, $a^{\mathcal{N}(i)} = (a^j)_{j \in \mathcal{N}(i)}$ and $g = (g^1, \dots, g^n)$ denote the payoff vector.

Throughout the paper, the graph G is assumed to be connected. Indeed, since interaction is local, players in different connected components do not interact with each other. Therefore, I model different connected components as different games.

In addition, I introduce costless communication. Players are able to communicate both privately and publicly. First, each player can send different messages to distinct players. Second, players can make public announcements to all players or to a subset of players only. For instance, if a player i makes a public announcement to a subset S of players, then the list S is certifiable, that is: each player s in S knows that all members in S received the same message (this is common knowledge among the players in S), although he does not know the messages received by players who are not in S. Let M^i be a non-empty finite set of player i's messages. Let $m^i_t(j)$ represent the private message sent by player i to player $j \in N$ at stage t, and $m^i_t(S)$

3. The 2-player case reduces to perfect monitoring, see Footnote 6 in Section 3.3.

the public message sent by i at stage t to players in $S \subseteq N$ (hence, $m_t^i(N)$ is a public announcement to all players).

The repeated game unfolds as follows. At every stage $t \in \mathbb{N}^*$:

(i) simultaneously, players choose actions in their action sets and send messages to all players, either publicly or privately as described above.

(ii) Let $a_t = (a_t^i)$ be the action profile at stage t. At the end of stage t, each player $i \in N$ observes his stage payoff $g^i(a_t^i, a_t^{\mathcal{N}(i)})$. A player cannot observe the actions chosen by others, even by his neighbors.

Hence, both interaction and monitoring possibilities are given by the network G. In addition, I assume perfect recall and that the whole description of the game is common knowledge. For each stage t, denote by H_t^i the set of player i's private histories up to stage t, that is $H_t^i = (A^i \times (M^i)^{N^{-i}} \times (M^j)_{j \in N^{-i}} \times \{g^i\})^t$, where $\{g^i\}$ is the range of g^i (H_0^i is a singleton). An element of h_t^i is called an i-history of length t. An *action strategy* for player i is denoted by $\sigma^i = (\sigma_t^i)_{t \geq 1}$ where for each stage t, σ_t^i is a mapping from H_{t-1}^i to $\Delta(A^i)$ (where $\Delta(A^i)$ denotes the set of probability distributions over A^i). A *communication strategy* for player i is denoted by $\phi^i = (\phi_t^i)_{t \geq 1}$ where for each stage t, ϕ_t^i is a mapping from H_{t-1}^i to $\Delta((M^i)^{N^{-i}})$. Note that a player may deviate from σ^i or from ϕ^i, thus I shall distinguish between action and communication deviations accordingly. A *behavior strategy* of a player i is a pair (σ^i, ϕ^i). Let Σ^i be the set of player i's action strategies and Φ^i his set of communication strategies. I denote by $\sigma = (\sigma^i)_{i \in N} \in \prod_{i \in N} \Sigma^i$ the players' joint action strategy and by $\phi = (\phi^i)_{i \in N} \in \prod_{i \in N} \Phi^i$ their joint communication strategy. Let H_t be the set of histories of length t that consists of the sequences of actions, payoffs and messages for t stages. Let H_∞ be the set of all possible infinite histories. A profile (σ, ϕ) defines a probability distribution, $\mathbb{P}_{\sigma,\phi}$, over the set of plays H_∞, and I denote $\mathbb{E}_{\sigma,\phi}$ the corresponding expectation. I consider the discounted infinitely repeated game, in which the overall payoff function of each player i in N is the expected sum of discounted payoffs. That is, for each player i in N:

$$\gamma_\delta^i(\sigma, \phi) = \mathbb{E}_{\sigma,\phi}\left[(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} g_t^i(a_t^i, a_t^{\mathcal{N}(i)})\right],$$

where $\delta \in [0, 1)$ is a common discount factor. Let $\mathcal{G}_\delta(G, g)$ be the δ-discounted game.

The solution concept is perfect Bayesian equilibrium (PBE from now on). While there is no agreed upon definition of what restrictions this involves after histories off the equilibrium path, this plays no role in my construction, and any definition will do. [4] In fact, I only specify strategies after private histories along which only

4. Fudenberg and Tirole ([17]) give a definition for games with observables actions, which my game does not have. A recent definition for general games, including games with imperfect monitoring, is given by González-Díaz and Meléndez-Jiménez ([19]).

unilateral deviations, if any, have taken place. In addition, the construction has the property that, after such histories, the specified play is optimal no matter what beliefs a player holds about his opponents' play, provided that the beliefs are such that: for every player $i \in N$, if player i observes a private history compatible with a history along which no deviation has taken place (respectively along which only unilateral deviations have taken place), then player i believes that no deviation has taken place (respectively only unilateral deviations have taken place). Plainly, this suffices to ensure optimality. Given that play after other histories (*i.e.*, histories that involve simultaneous deviations) is irrelevant, the partial strategy and beliefs that I define can be completed in any arbitrary fashion. Details are given in Section 3.5.2.

Let $E_\delta(G, g)$ its associated set of PBE payoffs. For each $a \in A$, I denote $g(a) = (g^1(a^1, a^{\mathcal{N}(1)}), \ldots, g^n(a^n, a^{\mathcal{N}(n)}))$ and $g(A) = \{g(a) \ : \ a \in A\}$. Let $\operatorname{co} g(A)$ be the convex hull of $g(A)$, which is the set of feasible payoffs. Player i's (independent) minmax level is defined by:[5]

$$\underline{v}^i = \min_{x^{\mathcal{N}(i)} \in \prod_{j \in \mathcal{N}(i)} \Delta(A^j)} \max_{x^i \in \Delta(A^i)} g^i(x^i, x^{\mathcal{N}(i)}).$$

I normalize the payoffs of the game such that $(\underline{v}^1, \ldots, \underline{v}^n) = (0, \ldots, 0)$. I denote by $IR^*(G, g) = \{g = (g^1, \ldots, g^n) \in \mathbb{R}^n \ : \ \forall i \in N, \, g^i > 0\}$ the set of strictly individually rational payoffs. Finally, let $V^* = \operatorname{co} g(A) \cap IR^*(G, g)$ be the set of feasible and strictly individually rational payoffs.

The aim of this paper is to characterize the networks G for which a folk theorem holds, that is: each feasible and strictly individually rational payoff is a PBE payoff for all discount factors close enough to one. In the next section, I display and motivate the class of payoff functions I consider.

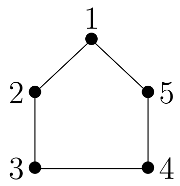
3.3 A class of payoff functions

I first show that a necessary condition for a folk theorem is that the payoff functions are sufficiently rich to enable players to detect deviations.

Example 3.3.1. Consider the 5-player game played on the following network:

5. It is sometimes possible here to drive equilibrium payoffs below this bound, see Section 3.7.

FIGURE 3.3:



G_1

The action sets of the players are $A^1 = A^2 = A^3 = \{C, D\}$ and $A^4 = A^5 = \{0, 1\}$. The payoff functions of players 1, 2 and 3 are given by the following matrices (player 2 chooses the row, player 1 the column, and player 3 the matrix):

3 plays C

$2 \setminus 1$	C	D
C	1,0,2	0,0,2
D	1,2,0	0,6,0

3 plays D

$2 \setminus 1$	C	D
C	0,0,6	1,0,6
D	0,2,0	1,6,0

In addition, the payoff functions of players 4 and 5 are $g^4(a^3, a^4, a^5) = a^4 + a^5 - 1$ and $g^5(a^1, a^4, a^5) = 2a^4 + a^5 - 1$. Notice that $\operatorname{int} V^* \neq \emptyset$. The minmax level of each player $i \in N$ is $\underline{v}^i = 0$. Therefore, the payoff vector $(1, 1, 1, 1, 2)$ is feasible and strictly individually rational. Notice also that the only way the get this payoff vector is that player 1 chooses between C and D with probability $\frac{1}{2}$-$\frac{1}{2}$, that players 2 and 3 choose action D, and that players 4 and 5 choose action 1. Moreover, playing 1 is a dominant strategy for both players 4 and 5.

However, $(1, 1, 1, 1, 2)$ is not a Nash equilibrium payoff of the repeated game: for any discount factor $\delta \in [0, 1)$, either player 1 or player 3 has an incentive to deviate by choosing between C and D with probability $\frac{1}{2}$-$\frac{1}{2}$. The key argument is that both deviations induce the same distribution of signals for player 2. First, player 2's payoff is 1 if players 1 and 3 take the same action, 0 otherwise. Hence, the deviations of players 1 and 3 induce the same distribution of payoffs for player 2, and that for every strategy (possibly mixed) of the non-deviating players.[6] Hence, player 2 cannot infer the identity of the deviator from the observation of his payoffs. Second, neither player 4 nor player 5 has relevant information on the actions chosen by players 1 and 3. As a consequence, player 2 cannot identify the deviator. In addition, player 2 cannot punish both players 1 and 3: punishing player 1 requires

6. Following the terminology of Aumann and Maschler (1966, re-edited in 1995, [4]), this is a jointly controlled lottery over player 2's payoffs.

player 2 to play C, in which case player 3 can get a payoff of 6. [7]. On the contrary, punishing player 3 requires player 2 to play D, in which case player 1 can get a payoff of 6. As a consequence, either "player 1 always plays C and D with probability $\frac{1}{2}$-$\frac{1}{2}$ and pretends that player 3 does" is a profitable deviation for player 1, or "player 3 always plays C and D with probability $\frac{1}{2}$-$\frac{1}{2}$ and pretends that player 1 does" is a profitable deviation for player 3. [8] Therefore, the outcome $(1,1,1,1,2)$ is not a Nash equilibrium payoff of the repeated game and the folk theorem fails.

On the other hand, let g^1, g^2, g^3 and g^4 remain unchanged, and assume that player 5's payoff is the following:

$$g^5(a^1,a^4,a^5) = \begin{cases} 2a^4+a^5-1 & \text{if } a^1=C \\ 2a^4+a^5-1+\epsilon & \text{if } a^1=D \end{cases}$$

with $\epsilon>0$. Player 5 is now able to detect player 1's deviations. Therefore, if player 2 detects either player 1's or player 3's deviation, he can obtain from player 5 the name of the deviator (player 5 clears player 1 if he does not detect any deviation, and clears player 3 otherwise). However, player 5 could lie about the identity of the deviator. I circumvent this issue by requiring a deviating player to confess thereafter. Hence, if player 5 lies, then the initial deviator, say player 1, confesses, and player 3 claims his innocence. A majority rule then gives the name of the deviator. For a large enough discount factor, it is possible to construct a perfect Bayesian equilibrium with payoff vector $(1,1,1,1,2)$ (see Section 3.5 for the construction). ◇

The previous example shows that if a deviation of a player i does not alter all his neighbors' payoffs, then it may possible for some feasible and strictly individually rational payoffs not to be (Nash) equilibrium payoffs of the repeated game. [9] Hence, it is not possible to get a folk theorem for all payoff functions g. I introduce the following assumption. [10]

Assumption PAYOFFS (P) . *For each player $i\in N$, each neighbor $j\in\mathcal{N}(i)$, every actions $b^j,c^j\in A^j$ such that $b^j\neq c^j$, $a^i\in A^i$, $a^{\mathcal{N}(i)\setminus\{j\}}\in A^{\mathcal{N}(i)\setminus\{j\}}$:*

$$g^i(a^i,a^{\mathcal{N}(i)\setminus\{j\}},b^j)\neq g^i(a^i,a^{\mathcal{N}(i)\setminus\{j\}},c^j).$$

Example 3.3.2. The following payoff functions satisfy Assumption P:

- for each player i in N, let $A^i\subset\mathbb{N}$ and $g^i(a^i,a^{\mathcal{N}(i)})=f\left(\sum_{j\in\mathcal{N}(i)\cup\{i\}}a^j\right)$ with f strictly monotone;

7. Formally, this argument does not rule out that there exists a strategy that trades off punishing the two players in a way that drives both of their payoffs below 1; however, it is not hard to show that this is already too much to ask; the details can be found in Section 3.6.
8. These arguments are formally developed in Section 3.6.
9. For a similar phenomenon, see [31].
10. This assumption implies that the two-player case reduces to perfect monitoring.

- for each player i in N, let $A^i \subset \mathbb{R}$ and $g^i(a^i, a^{\mathcal{N}(i)}) = \sum_{j \in \mathcal{N}(i)} a^j - a^i$ (this game can be seen as a generalized prisoner's dilemma for n players);
- firms' profits in Cournot games;
- more generally, for each player i in N, let $A^i \subset \mathbb{R}$ and g^i strictly monotone with respect to each argument.

In the next section, I introduce a necessary and sufficient condition on the networks for a folk theorem to hold.

3.4 The main result

Theorem 3.4.1. *The following statements are equivalent:*

1. *For each payoff function g that satisfies Assumption P and such that the interior of V^* is nonempty, for any payoff v in V^*, there exists $\bar{\delta} \in (0,1)$ such that for all $\delta \in (\bar{\delta}, 1)$, v is a PBE vector payoff of the δ-discounted game.*
2. *The graph G is such that, for every pair of players $i, j \in N$, i and j have different set of neighbors (Condition DN for Different Neighbors), that is:*

$$\forall i, j \in N, \quad \mathcal{N}(i) \setminus \{j\} \neq \mathcal{N}(j) \setminus \{i\}. \tag{DN}$$

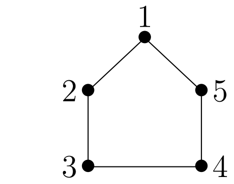
Assume that Condition DN is satisfied, that Assumption P holds, and that V^* is of full dimension. Then, for any $v \in V^*$, it is possible to construct a PBE with payoff v for a discount factor close to one. I construct this strategy in Section 3.5. Intuitively, Condition DN makes it possible to construct a communication protocol that enables players to identify the deviator when a (unilateral) deviation occurs.

In addition, Condition DN prevents the network from having too many links, since it may lead to less information. The idea is that too many links increase the likelihood for two players to have the same neighbors, which violates Condition DN. If two players have the same neighbors, their deviations may not be distinguishable, and I exhibit particular payoff function for which the folk theorem fails (see Section 3.6 for the general proof). In particular, the folk theorem fails for complete networks: for some payoff functions satisfying Assumption P, the issue is that a deviation of any player is detected when then network is complete, but every player suspects everybody, and identification of the deviator may not be possible. This is an obstacle for a folk theorem to hold, see Section 3.6 for details.

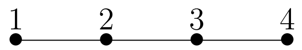
Now, I show some properties of the networks satisfying Condition DN. First, I display some examples of graphs that satisfy Condition DN (Figure 3.4), and some that do not (Figure 3.5). Notice that Condition DN is not monotonic with respect to the number of links, contrary to connectivity. For instance, both G_9 and G_{10} are

2-connected, whereas only G_9 satisfies Condition DN. Also, neither G_3 nor G_4 are 2-connected, whereas only G_3 satisfies Condition DN.

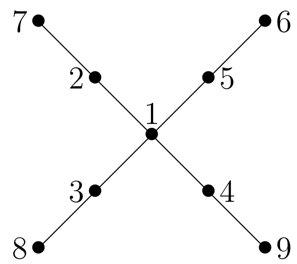
FIGURE 3.4: Networks satisfying DN



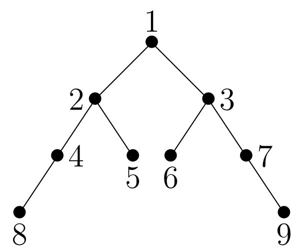
G_1



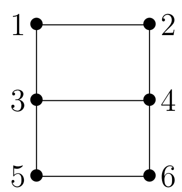
G_3



G_5

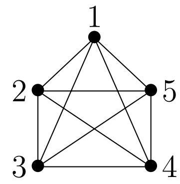


G_7

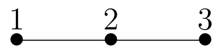


G_9

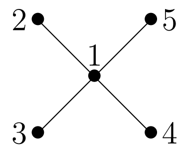
FIGURE 3.5: Networks not satisfying DN



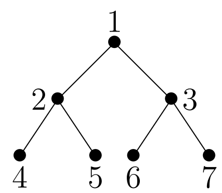
G_2



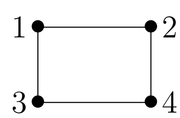
G_4



G_6



G_8



G_{10}

I now exhibit some properties of the networks that satisfy Condition DN.

Proposition 3.4.2. *Assume that the network G is connected and satisfies Condition DN. The following statements hold:*

(i) there are at least $n = 4$ players;
(ii) if $n = 4$, then G is a line (graph G_3 in Figure 3.4).

The proof of Proposition 3.4.2 is given in the appendix. The next proposition shows that, for the family of networks for which Condition DN holds, a player cannot be the unique neighbor of several players.

Proposition 3.4.3. *If a connected network satisfies Condition DN, then for every players j and j^1 such that $\mathcal{N}(j^1) = \{j\}$, and every $\ell \in \mathcal{N}(j)$ so that $\ell \neq j^1$, player ℓ has at least two neighbors.*

Proof. Take any network G, two players j and j^1, and assume that $\mathcal{N}(j^1) = \{j\}$. Take a player $\ell \in \mathcal{N}(j)$ so that $\ell \neq j^1$. If $\sharp\mathcal{N}(\ell) = 1$, then $\mathcal{N}(\ell) = \{j\}$. Therefore, $\mathcal{N}(\ell) = \mathcal{N}(j^1)$, and Condition DN is violated. □

I now study the trees satisfying Condition DN. Recall that a tree is a connected graph without cycles, and that the distance in a graph between two nodes i and j is the length of the shortest path from i to j (the reader is referred to [9] for usual definitions of graph theory).

Corollary 3.4.4. *The following statements hold.*
(i) A tree satisfies Condition DN if and only if the distance between any two terminal nodes is greater than or equal to three.
(ii) No star satisfies Condition DN.

Going back to Figures 3.4 and 3.5, the distance between two terminal nodes in G_7 is at least three, whereas the distance between nodes 4 and 5 in G_8 (or between 6 and 7) equals two.

Proof. Given Proposition 3.4.3, the proof of point (ii) is straightforward. Regarding point (i), take first a tree G for which Condition DN holds. Proposition 3.4.3 directly implies that there are no terminal nodes i and j in G such that the distance between i and j equals 2.

For the converse, take any tree G for which the distance between two terminal nodes is greater than three. Obviously, any two terminal nodes have a non-common neighbor. Take now two nodes i and j in G such that at least one of them is not terminal, say i. Any node that is not terminal has at least two neighbors, let $\mathcal{N}(i) = \{i_1, i_2\}$. The network G being a tree, there is a unique path p between i and j. Therefore, if i_1 (respectively i_2) is in p, then i_2 (respectively i_1) is in $\mathcal{N}(i) \setminus \{j\} \triangle \mathcal{N}(j) \setminus \{i\}$, where $\triangle$ stands for the symmetric difference.[11] Hence, G satisfies Condition DN. □

11. The set $\mathcal{N}(i) \setminus \{j\} \triangle \mathcal{N}(j) \setminus \{i\}$ contains the players $\ell \neq i, j$ such that ℓ is a neighbor of i or j, but not of both.

3.5 Sufficiency of Condition DN

In this section, I assume that the network G satisfies Condition DN of Theorem 3.4.1, namely: for any pair of players i, j in N, i and j have at least one different neighbor. Take a payoff function that satisfies Assumption P and such that $\operatorname{int} V^*$ is non-empty (this is assumed throughout this section). I take a point $v = (v^1, \ldots, v^n)$ in V^* and I construct a PBE of the repeated game (σ^*, ϕ^*) with payoff v for a large enough discount factor.

First of all, notice that each player i detects an action deviation from a pure action profile if and only if he observes a change in his stage payoff (Assumption P). Therefore, there is an action deviation from a pure action profile of some player k at stage t if and only if all the neighbors of player k detect it stage t. Nevertheless, player k's neighbors may not be able to identify the deviator.

I first introduce a communication protocol which identifies the deviator when there is a deviation (Section 3.5.1). Then, I construct the equilibrium strategy (Section 3.5.2), and prove the equilibrium property (Section 3.5.3).

3.5.1 Identifying a deviator

In this section, I construct a communication protocol which identifies the deviator when there is a deviation. Then, I exhibit some properties of this protocol.

3.5.1.1 Construction of the communication protocol

Here, I assume that the players coordinate on playing a sequence on pure actions, $\bar{a}_t = (\bar{a}^i_t)_{i \in N}$ at each stage $t > 0$, and as long as no deviation is detected, on sending the message $\bar{m}_t(N) = (\bar{m}^i_t(N))$, where for each player $i \in N$, $\bar{m}^i_t(N) = \mathcal{N}(i) \cup \{i\}$. This message means that player i did not deviate, and did not detect any action deviation at stage $t-1$. Hence, according to player i, his neighbors and himself are innocent regarding any possible action deviation at stage $t-1$. From now on, I assume that the network G is such that Condition DN holds, and that g satisfies Assumption P.

We aim at identifying the deviator when a deviation occurs. Like in Example 3.3.1, it is not always possible to punish several players, and the identification of the deviator may therefore be needed. Nevertheless, for the family of networks that satisfy Condition DN, it is not always possible to identify the deviator. Take for instance a player k who has a single neighbor ℓ. It may be impossible for players $i \neq k, \ell$ to differentiate between the two following deviations:

- player k deviates in action at stage t and does not confess any deviation regarding stages t and $t+1$;

- player ℓ deviates in communication at stage $t+1$ and claims that player k deviated at stage t.

These deviations still have to be punished, otherwise either player ℓ or player k might have an incentive to deviate. Yet, this is not an obstacle for a folk theorem to hold. Indeed, no player is a neighbor of both players k and ℓ, hence it is possible to minmax both players k and ℓ: while being minmaxed, player ℓ is prescribed to minmax player k. Therefore, if one of these deviations occurs, players only have to know that the deviator is either k or ℓ.

I then construct a *communication protocol*, that is: a profile of communication strategies and an output rule for each player. In my context, the communication protocol starts as soon as there is a (unilateral) deviation (either from $\bar{a}_t$, or from $\bar{m}_t(N)$), and does not stop before players find out the identity of the deviator (at least when it is an action deviation), or at least a subset of two players, k and ℓ, containing the deviator when the situation described above appears.

Each player $i \in N$ starts the protocol every time he detects a deviation from $(\bar{a}, \bar{m}(N))$. Players may start the communication protocol at different stages. Indeed, consider the situation where there is an action deviation of some player j at stage t. Player j's neighbors start the protocol at the end of stage t, whereas other players may not start it before the end of stage $t+1$, when they receive messages from player j's neighbors. I now construct the communication protocol.

The message space. All players communicate using the same finite set of messages $\tilde{M} = N$, with N the set of players.

The strategy of player i. Player i always takes the action $\bar{a}_t^i$ when he performs the protocol.[12] I denote by $\tilde{\phi}^i$ his communication strategy during the communication protocol, which consists in announcing sets of innocents publicly to all players as follows:

- if player i detects an action deviation at stage t, then he announces $\tilde{m}_{t+1}^i(N) = N \setminus \mathcal{N}(i)$ at stage $t+1$: player i claims that all his neighbors are suspects regarding a deviation at stage t; or, in other words, that all other players, including himself, are innocent.
- If player i deviates in action at stage t, then he announces $\tilde{m}_{t+1}^i(N) \supseteq N^{-i}$ at stage $t+1$: player i confesses at stage $t+1$ that he deviated at stage t.

The output rule. Denote by $X_{t+1}^i \subset N$ player i's set of suspected players at stage $t+1$ regarding a possible deviation at stage t. For each player i in N, the set X_{t+1}^i

12. In the construction of the equilibrium strategy in the next section, player i keeps playing the action corresponding to the phase of the game, instead of playing $\bar{a}_t^i$ (see Section 3.5.3).

is computed as follows.

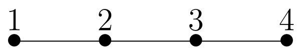
(i) Only announcements of the form $m^j_{t+1}(N) = \mathcal{N}(j) \cup \{j\}$, $m^j_{t+1}(N) = N \setminus \mathcal{N}(j)$ or $m^j_{t+1}(N) = N^{-j}$ for each player $j \in N$ are taken into account. Other announcements are disregarded: in particular, private messages are ignored. Notice also that player i takes into account his own announcement $m^i_{t+1}(N)$.

(ii) For every player j such that $\sharp\mathcal{N}(j^1) \geq 2$ for each $j^1 \in \mathcal{N}(j)$:
- if there exist at least two players j^1 and j^2 such that $j^1 \neq j^2$, $j \in m^{j^1}_{t+1}(N)$ and $j \in m^{j^2}_{t+1}(N)$, then $j \notin X^i_{t+1}$ (*i.e.* j is cleared);
- otherwise, $j \in X^i_{t+1}$ (*i.e.* j is identified as suspect).

(iii) For every pair of players k and ℓ such that $\mathcal{N}(k) = \{\ell\}$:
- if there exist at least two players k^1 and k^2 such that $k^1 \neq k^2$, $k \in m^{k^1}_{t+1}(N)$ and $k \in m^{k^2}_{t+1}(N)$:
  - if there exist at least two players ℓ^1 and ℓ^2 such that $\ell^1 \neq \ell^2$, $\ell \in m^{\ell^1}_{t+1}(N)$ and $\ell \in m^{\ell^2}_{t+1}(N)$, then $\ell \notin X^i_{t+1}$ (*i.e.* ℓ is suspected);
  - otherwise, $\ell \in X^i_{t+1}$ (*i.e.* ℓ is identified as suspect);
- otherwise, $\{k, \ell\} \in X^i_{t+1}$ (*i.e.* both k and ℓ are identified as suspects).

I now introduce an example to show how this protocol works.

Example 3.5.1. Consider the 4-player game played on the following network:



for which Condition DN is satisfied. In addition, take a payoff function g for which Assumption P holds.

1. Assume first that player 2 deviates in action at some stage t and, for simplicity, does not deviate in communication at stage t (but possibly at stage $t+1$). Hence, each player i, except possibly k, announces publicly $\bar{m}^i_t(N) = \mathcal{N}(i) \cup \{i\}$ at stage t. At stage $t+1$, players 1 and 3 should start the protocol, and stick to actions $\bar{a}^1_t$ and $\bar{a}^3_t$ respectively. In addition, strategies $\tilde{\phi}^1_{t+1}$ and $\tilde{\phi}^3_{t+1}$ prescribe to announce publicly $m^1_{t+1}(N) = N \setminus \mathcal{N}(1) = \{1, 3, 4\}$ and $m^3_{t+1}(N) = N \setminus \mathcal{N}(3) = \{1, 3\}$ respectively. Player 4 starts the protocol at the end of stage $t+1$ only, and should announce $m^4_{t+1}(N) = \bar{m}^4_{t+1} = \{3, 4\}$. Finally, player 2 should announce $\{1, 3, 4\}$ publicly to all players at stage $t+1$. Under unilateral deviations, players 1, 3 and 4 appear in the public announcements at stage $t+1$ of at least two different players, hence each player i clears players 1, 3 and 4. Moreover, player 2 appears in at most one public announcement, therefore each player i identifies player 2 as the deviator.

2. Consider now the case in which there is no action deviation at stage t but a communication deviation of player 2 at stage $t+1$ who announces $m^2_{t+1}(N) = N \setminus \mathcal{N}(2) = \{2,4\}$ publicly to all players (recall that private messages are not taken into account in the strategy constructed). Under ϕ^*, players 1, 3 and 4 announce respectively $\{1,2\}$, $\{2,3,4\}$ and $\{3,4\}$ publicly to all players at stage $t+1$ since by assumption there is no action deviation at stage t. Hence, $X^i_t = \{1,2\}$ for each player i in N. The two following cases are then possible.
   - *Player* 2 *also deviates in action at stage* $t+1$. At stage $t+2$, players 1, 2, 3 and 4 should announce respectively $\{1,3,4\}$, $\{1,3,4\}$, $\{1,3\}$ and $\{3,4\}$ publicly to all players. Even if one player deviates at stage $t+2$, each player clears players 1, 3 and 4 and identifies player 2, hence $X^i_{t+2} = \{2\}$ for each player i.
   - *Player* 2 *does not deviate in action at stage* $t+1$. Then there is no action deviation at stage $t+1$ under unilateral deviations.

3. Suppose now that player 1 deviates in action and, for simplicity, does not deviate in communication at stage t. At stage $t+1$, players 1, 2, 3 and 4 should announce respectively $\{2,3,4\}$, $\{2,4\}$, $\{2,3,4\}$ and $\{3,4\}$ publicly to all players. Therefore, the name of player 1 appears in at most one public announcement, whereas the names of players 2, 3 and 4 are in at least two distinct players' public announcements. Hence, $X^i_{t+1} = \{1,2\}$ for each player i.

4. Assume now that there is no action deviation at stage t, and that player 1 deviates in communication at stage $t+1$ by announcing $m^1_{t+1}(N) = N^{-1} = \{2,3,4\}$ publicly to all players: player 1 lies when he confesses his action deviation at stage t. At stage $t+1$, the public announcements of players 2, 3 and 4 are respectively $\{1,2\}$, $\{2,3,4\}$ and $\{3,4\}$, hence $X^i_t = \{1,2\}$ for each player i in N.

5. Finally, suppose that there is no action deviation at stage t, and that player 1 deviates in communication at stage $t+1$ by announcing publicly $m^1_{t+1}(N) = N \setminus \mathcal{N}(1) = \{1,3,4\}$. The public announcements of players 2, 3 and 4 at stage $t+1$ are respectively $\{1,2\}$, $\{2,3,4\}$ and $\{3,4\}$. All players appears in at least two messages, hence $X^i_t = \emptyset$ for each player i. Each player deduces that there was no action deviation at stage t, and that player 1 deviates in communication at stage $t+1$. ◇

3.5.1.2 Properties of the communication protocol

I now exhibit some properties of the communication protocol constructed in the previous section. All the proofs are in the appendix. With a slight abuse of notation, I call $\tilde{\phi}^i$ the communication strategy of player i which consists in sending the message $\bar{m}^i(N)$ when not performing the protocol, and to send messages according to the strategy prescribed by the communication protocol otherwise. The next lemma proves that, if a player who has more than two neighbors deviates in action at some stage t, then he is identified as suspect, and all other players are cleared at stage $t+1$.

Lemma 3.5.2. *Consider a connected network G that satisfies Condition DN, and suppose that Assumption P holds. For each player j such that $\sharp\mathcal{N}(j) \geq 2$, if j deviates in action at some stage t and if all players, except possibly a deviator, follow $\tilde{\phi}$ at stages t and $t+1$, then $X^i_{t+1} = \{j\}$ for each player $i \in N$.*

Notice that there may be a deviation from a player $d \neq j$ at stage $t+1$. Yet, in that case, player j does not deviate at stage $t+1$ under unilateral deviations. The next lemma shows how the protocol works when a player k has a single neighbor ℓ.

Lemma 3.5.3. *Consider a connected network G that satisfies Condition DN, and suppose that Assumption P holds. For every players k and ℓ such that $\mathcal{N}(k) = \{\ell\}$, if:*

(a) either k deviates in action at stage t;

*(b) or, there is no action deviation at stage t, $m^k_{t+1} = m^k_{t+1}(N)$ and $m^k_{t+1}(N) \supseteq N^{-k}$ (*i.e.* player k confesses publicly to all players);*

*(c) or, there is no action deviation at stage t, $m^\ell_{t+1} = m^\ell_{t+1}(N)$ and $m^\ell_{t+1}(N) \supseteq N \setminus \mathcal{N}(\ell)$ (*i.e.* player ℓ claims publicly that he detected a deviation at stage t);*

and if all players, except possibly a deviator, follow $\tilde{\phi}$ at stages t and $t+1$, then $X^i_{t+1} = \{k, \ell\}$ for each player $i \in N$.

The next lemma shows that no communication deviation other than those of Lemma 3.5.3 can induce the identification of an innocent player as a deviator.

Lemma 3.5.4. *Consider a connected network G that satisfies Condition DN, and suppose that Assumption P holds. For each player k in N, if there is no action deviation at stage t, and if player k deviates in communication at stage $t+1$ in such a way that:*

- *either $\mathcal{N}(k) = \{\ell\}$, $m^k_{t+1} \neq m^k_{t+1}(N)$ or $\{N^{-k}, \mathcal{N}(k) \cup \{k\}\} \notin m^k_{t+1}(N)$;*
- *or there exists k' such that $\mathcal{N}(k') = k$, $m^k_{t+1} \neq m^k_{t+1}(N)$ or $\{N \setminus \mathcal{N}(k), \mathcal{N}(k) \cup \{k\}\} \notin m^k_{t+1}(N)$;*
- *or $m^k_{t+1} \neq m^k_{t+1}(N)$ or $\{\mathcal{N}(k) \cup \{k\}\} \notin m^k_{t+1}(N)$;*

and if all players, except possibly a deviator, follow $\tilde{\phi}$ *at stage* $t+1$*, then* $X^i_{t+1} = \emptyset$ *for each player* $i \in N$*.*

3.5.2 The equilibrium strategy

Take a payoff $v \in V^*$. In this section, I construct the PBE strategy, denoted (σ^*, ϕ^*), with payoff v (the proof of the equilibrium property is in Section 3.5.3). More precisely, I construct a restriction of the PBE strategy to a particular class of private histories; namely, the histories along which only unilateral deviations from (σ^*, ϕ^*), if any, have taken place. Formally, I denote by $H^i_t(U|(\sigma^*, \phi^*))$ the set of private histories for player i such that: either no deviation (from (σ^*, ϕ^*)) has occurred, or only unilateral deviations have taken place. That is to say, for any history in $H^i_t(U|(\sigma^*, \phi^*))$, no multilateral deviation has occurred. Similarly, denote by $H_t(U|(\sigma^*, \phi^*))$ the set of total histories along which only unilateral deviations, if any, have taken place.

I define now, for every history in $H_t(U|(\sigma^*, \phi^*))$, a strategy profile which can be decomposed into four phases. First, there is a stream of pure action profiles that yields the desired payoff. This is how the game starts off and how it unfolds as long as no player deviates. Second, there is a communication phase (the communication protocol previously described) in case of a deviation, whose purpose is to inform the deviator's neighbors of the deviator's identity. Third, there is a punishment phase, and finally, a reward phase.

Before constructing the equilibrium strategy, notice that deviations from the equilibrium strategy are of two kinds: action and communication deviations. Since communication is costless, communication deviations have to be punished only if they affect continuation payoffs. In the strategy profile I construct, communication deviations that do not affect continuation payoffs are not punished. More precisely, if a player starts sending spurious messages although no player has deviated in action, and if in addition all players *learn* that there was no action deviation–*i.e.* all players are cleared by the communication protocol–then the deviator is not punished. On the other hand, for some communication deviations without any action deviation, it may not be possible for some players to be aware that there was no action deviation: in this case, punishments of several players take place in my construction. In any case, a player can deviate both in communication and in action, and is then punished (action deviations always lead to the deviator's punishment). Now, I construct the four phases of the strategy, then I specify the beliefs (Section 3.5.2.5).

3.5.2.1 Phase I: equilibrium path

For each player i in N and each stage $t > 0$, choose $\bar{a}^i_t \in A^i$ such that

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}g^i_t(\bar{a}^i_t, \bar{a}^{\mathcal{N}(i)}_t) = v^i.$$

This is possible when $\delta \geq 1 - \frac{1}{n}$ (existence is proved by Sorin, Proposition 4 p.151 in [42]). Moreover, Fudenberg and Maskin (1991) prove that for every $\epsilon > 0$, there exists $\delta_\epsilon < 1$ such that for all $\delta \geq \delta_\epsilon$ and every $v \in V^*$ such that $v^i \geq v$ for all i, the deterministic sequence of pure actions $\bar{a}_t$ can be constructed so that the continuation payoffs at each stage are within ϵ of v (Lemma 2 p. 432 in [16]).[13]

During this phase, player i should play action $\bar{a}^i_t$ at stage t. Moreover, at every period, player i should announce $\bar{m}^i_t(N) = \mathcal{N}(i) \cup \{i\}$ publicly to all players. This message means that player i did not deviate and did not detect any action deviation at stage $t-1$.[14] According to player i, his neighbors and himself are innocent regarding any possible action deviation at stage $t-1$. Player i then announces his *set of innocents* publicly to all players at stage t, denoted I^i_t, made of himself and his neighbors: $I^i_t = \mathcal{N}(i) \cup \{i\}$.

3.5.2.2 Phase II: communication phase

Player $i \in N$ enters phase II each time he detects a unilateral deviation from (σ^*, ϕ^*). For instance, in phase I, player i enters phase II at the end of stage t either when he detects an action deviation at stage t, or when he receives a public message different from $\bar{m}^j_t(N) = \mathcal{N}(j) \cup \{j\}$ from some player $j \in N^{-i}$ at stage t (only public announcements are taken into account here). During this phase, player i performs the communication protocol constructed in Section 3.5.1.

Remark 3.5.5. From Lemmas 3.5.2, 3.5.3 and 3.5.4, it turns out that if a player j deviates in action at some stage t, then he is identified as deviator, and can be punished. Either he is the unique suspect, or he has a single neighbor who also is suspected, in which case both can be punished. Moreover, for any pair of players k and ℓ such that $\mathcal{N}(k) = \{\ell\}$, and if there is no action deviation at stage t, both k and ℓ are identified under communication deviations (b) and (c) of Lemma 3.5.3; for all other communication deviations, everybody is cleared. For the latter case, all players know that there was no action deviation at stage t, thus only a communication deviation at stage $t+1$. That is why no player has to be punished since there is no

13. If it was not the case, some player would prefer to deviate from $\bar{a}$, even if doing so caused his opponents to minmax him thereafter.

14. Recall that, at each stage, messages are sent before observing stage payoffs. This assumption can be relaxed: with a slight modification, the strategy construction is still valid for the case in which messages are sent after the observation of stage payoffs.

impact on continuation payoffs. Only players k's and ℓ's communication deviations that are indistinguishable with player k's action deviation have to be punished.

During phase II, players should stick to the action strategy they were playing in the previous phase. For instance, if players are following the equilibrium path at stage t when they detect some deviation, they should enter the communication phase and keep playing $\bar{a}_{t+1}$ at stage $t+1$. This part of the strategy is thereby purely communicative. Notice that player i may start the protocol in phase I, phase III or phase IV.

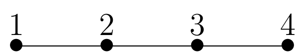
I now describe how the transition is made from phase II to another phase. Denote by plan $\{j\}$ (respectively $\{k, \ell\}$) the punishment phase (phase III) in which player j (respectively players k and ℓ) is minmaxed. The transition rule to another phase is the following:

- if $X^i_{t+1} = \emptyset$, then keep playing according to the current action phase and use the corresponding communication strategy;
- if $X^i_{t+1} = \{j\}$ for some player $j \in N$ such that $\sharp\mathcal{N}(j) \geq 2$ and $\sharp\mathcal{N}(j^1) \geq 2$ for each $j^1 \in \mathcal{N}(j)$, then start plan $\{j\}$;
- if either $X^i_{t+1} = \{\ell\}$ or $X^i_{t+1} = \{k, \ell\}$, with $\mathcal{N}(k) = \{\ell\}$, then start plan $\{k, \ell\}$;
- otherwise, play an arbitrary Nash equilibrium of the one-shot game (history incompatible with unilateral deviations).

Remark 3.5.6. Assume $\mathcal{N}(k) = \{\ell\}$. Player ℓ may prefer plan $\{\ell\}$ to plan $\{k, \ell\}$, or the opposite. That is the reason why the transition rule prevents plan $\{\ell\}$ to happen: if player ℓ is identified as a suspect, then players start plan $\{k, \ell\}$ in order to minmax both, no matter whether player k is also suspected. Moreover, the identification is such that player ℓ is always suspected when player k is, so that plan $\{k\}$ never occurs either.

The next example takes up Example 3.5.1 to show how this transition rule works.

Example 3.5.7. Consider the 4-player game of Example 3.5.1 played on the following network:



for which Condition DN is satisfied. In addition, take a payoff function g for which Assumption P holds.

1. Assume first that player 2 deviates in action at some stage t and, for simplicity, does not deviate in communication at stage t (but possibly at stage $t + 1$). From Example 3.5.1, each player i identifies player 2 as the deviator. The transition rule prescribes players to enter phase III in order to minmax

both players 1 and 2 (plan $\{1,2\}$).

2. Assume now that there is no action deviation at stage t, and that player 2 deviates in communication at stage $t+1$ by announcing $m^2_{t+1}(N) = N\backslash\mathcal{N}(2) = \{2,4\}$ publicly to all players. Then, $X^i_t = \{1,2\}$ for each player i in N. Players then start plan $\{1,2\}$. The reason for this joint punishment is that players 3 and 4 do not differentiate between the histories "player 1 deviates in action at stage t and in communication at stage $t+1$ since he does not confess" and "player 1 deviates neither in action at stage t nor in communication at stage $t+1$ and player 2 deviates in communication at stage $t+1$". The two following cases are then possible.
   - If player 2 also deviates in action at stage $t+1$, then $X^i_{t+2} = \{2\}$ for each player i. Nevertheless, players also start plan $\{1,2\}$.
   - If player 2 does not deviate in action at stage $t+1$, then there is no action deviation at stage $t+1$ under unilateral deviations. Players keep playing plan $\{1,2\}$ until a new possible deviation.

3. Assume now that player 1 deviates in action and, for simplicity, does not deviate in communication at stage t. Then, $X^i_{t+1} = \{1,2\}$ for each player i. Again, players 3 and 4 cannot differentiate between this deviation of player 1 and the deviation of player 2 described in the previous case. Therefore, all players start plan $\{1,2\}$.

4. Assume now that there is no action deviation at stage t, and that player 1 deviates in communication at stage $t+1$ by announcing $m^1_{t+1}(N) = N^{-1} = \{2,3,4\}$ publicly to all players: player 1 lies when he confesses his action deviation at stage t. Then, $X^i_t = \{1,2\}$ for each player i in N. Again, all players start plan $\{1,2\}$.

5. Finally, assume that there is no action deviation at stage t, and that player 1 deviates in communication at stage $t+1$ by announcing publicly $m^1_{t+1}(N) = N \setminus \mathcal{N}(1) = \{1,3,4\}$. Then, $X^i_t = \emptyset$ for each player i: each player deduces that there was no action deviation at stage t, and that player 1 deviates in communication at stage $t+1$. However, communication being costless, my construction does not require to punish player 1. The transition rule prescribes all players to keep playing according to the phase in which the game is and to use the corresponding communication strategy. $\diamond$

This concludes the description of the communication phase.

3.5.2.3 Phase III: punishment phase

First case: plan $\{j\}$. Consider the situation in which each player enters a punishment phase in order to minmax player j. Notice first that only player j's neighbors are able to punish him. Besides, since minmax strategies might be mixed, minmaxing players' deviations might be undetectable: players may not know the sequences of pure actions their neighbors should play. For that reason, announcements are added in the construction. Players' strategies during phase III, denoted $\hat{\sigma}$ and $\hat{\phi}$, are as follows.

During this phase, each player's communication strategy is twofold:

(i) first, at each stage s, each player i in N announces his set of innocents I_s^i publicly to all players (as in phase I). For instance, $I_s^i = \mathcal{N}(i) \cup \{i\}$ belongs to player i's public announcement at stage s if player i does not detect any action deviation at stage $s-1$.

(ii) In addition, each player i reveals his pure action a_s^i publicly to all players at each stage s. With these announcements, all players know the pure actions each player should have played at each stage s. This enables the players to detect deviations and start phase II if needed.
Recall that players choose actions and messages simultaneously at each stage, hence player j has not received his opponents' announcements when he chooses his action and sends his messages. Therefore, player j is indeed punished when minmax strategies are mixed.

A message of any player i during phase III has then the following form: for each stage s in phase III, $m_s^i(N) = (I_s^i, a_s^i)$.

Action strategies of the players are as follows. At each stage $s \geq t+2$ during phase III (the length of phase III, denoted $T(\delta)$, is adapted in Section 3.5.2.4):

- each player $i \in \mathcal{N}(j)$ plays according to his minmax strategy against j, denoted (P_j^i) (recall that P_j^i can be a mixed strategy). Denote by $P(j) = (P_j^i)_{i\in\mathcal{N}(j)}$ the profile of minmax strategies of player j's neighbors against him. For any strategy (σ^j, ψ^j) of player j:

$$\begin{aligned}\gamma_\delta^j(\sigma^j, P(j), \psi^j, (\phi^i)_{i\in\mathcal{N}(j)}) &\leq \sum_{t=1}^{\infty}(1-\delta)\delta^{t-1}\underline{v}^j \\ &\leq 0,\end{aligned}$$

  where ϕ^i is an arbitrary communication strategy of a minmaxing player $i \in \mathcal{N}(j)$.

- Player j commits to play an arbitrary pure action P^j during his punishment, hence his deviations are detectable. In particular, player j is supposed to play P^j even if a player $i \in \mathcal{N}(j)$ deviates by privately reporting to player j the sequence of pure actions he will play during plan $\{j\}$. Hence, player j cannot use this information to get a higher payoff. If he deviates from P^j, then player j only lengthens his punishment (see Section 3.5.3).

- The actions of other players $m \neq i, j$ are arbitrary. Recall that each player m should announce publicly the pure action chosen at each stage, hence his deviations are detectable.

During phase III, each player i in N starts the communication phase at some stage s if:

- either player i detects an action deviation at stage $s-1$, *i.e.* his payoff at stage $s-1$ is not the one he would get if all his neighbors would have played the pure actions they announce at stage $s-1$;
- or, there exists $\tilde{i} \in \mathcal{N}(j) \cap \mathcal{N}(i)$ such that $a^{\tilde{i}}_{s-1}$ is not in the support of $P^{\tilde{i}}(k)$ (this deviation is regarded as an action deviation);[15]
- or, there exists a player $m \in N$ such that $I^m_s \neq \mathcal{N}(s) \cup \{s\}$.

If player i never starts phase II (or if all players are cleared), then player i goes to phase IV at the end of the punishment phase, hence at stage $t + 2 + T(\delta)$.

Second case: plan $\{k, \ell\}$. Consider now the situation in which two players k and ℓ, with $\mathcal{N}(k) = \ell$, are minmaxed from stage $t+2$ on. It means that either player k or player ℓ have been identified as suspect at the end of stage $t+1$.

Remark 3.5.8. The strategy constructed in phase II is such that, whenever the state is $\{k, \ell\}$, it must be the case that $\mathcal{N}(k) = \{\ell\}$ without loss of generality (see Lemmas 3.5.2, 3.5.3 and 3.5.4 in Section 3.5.3). In this case, phase III is such that both k and ℓ are punished. It is possible since each player has to punish only one suspect among k or ℓ.

Each player's communication strategy during phase III is the same as for the first case. However, action strategies differ and are as follows. At each stage $s \geq t+2$ during phase III:

- each player $i \in \mathcal{N}(\ell)$, including player k, plays according to his minmax strategy against ℓ, denoted $(P^i(\ell))$ (recall that $P^i(\ell)$ can be a mixed strategy).
- Player ℓ plays according to his minmax strategy against k, $P^\ell(k)$.
- The actions of other players $m \neq i, k, \ell$ are arbitrary. As explained above, each player m should still announce publicly the pure action chosen at each stage.

Each player i in N starts phase II at some stage s if:

- either player i detects an action deviation at stage $s-1$, *i.e.* his payoff at stage $s-1$ is not the one he would get if all his neighbors would have played the pure action they announced at stage $s-1$;

15. Notice that one can have $i = j$.

- or there exists $\tilde{i} \in \mathcal{N}(\ell) \cap \mathcal{N}(i)$ such that $a_{s-1}^{\tilde{i}}$ is not in the support of $P^{\tilde{i}}(\ell)$;[16]
- or $i \in \mathcal{N}(\ell)$ and a_{s-1}^{ℓ} is not in the support of $P^{\ell}(k)$;
- or there exists a player m such that $I_s^m \neq \mathcal{N}(s) \cup \{s\}$.

If player i never starts phase II (or if all players are cleared), then player i goes to phase IV at the end of the punishment phase, hence at stage $t + 2 + T(\delta)$. This concludes the description of the punishment phase.

In the next section, I define a reward phase such that players k's and ℓ's continuation payoffs after the punishment phase vary according to their realized payoff during phase III (see Section 3.5.2.4). Therefore, they are indifferent between the pure actions in the support of their minmax strategies. In particular, they cannot benefit from a minmaxing player reporting privately his sequence of pure actions.

3.5.2.4 Phase IV: reward phase

The aim of this phase is twofold.

(i) In order to provide each minmaxing player (who is not minmaxed himself) with an incentive to play his minmax strategy in phase III, an additional bonus $\rho > 0$ is added to his average payoff. If the discount factor is large enough, the potential loss during the punishment is compensated by the future bonus.

(ii) Moreover, to induce each minmaxing player to draw his pure actions according to the right distribution of his minmax strategy, I add a phase so that his payoff in the continuation game varies with his realized payoff in a way that makes him indifferent between the pure actions in the support of his minmax strategy. As in Fudenberg and Maskin ([15, 16]), it is convenient to require that any feasible and strictly individually rational continuation payoff must be exactly attained. Otherwise, a minmaxing player might not be exactly indifferent between his pure actions in the support of his minmax strategy.

The possibility of defining such rewards relies on the full dimensionality of the payoff set (recall that $\operatorname{int} V^* \neq \emptyset$). I use the insights of Fudenberg and Maskin (1986). The formal construction of these rewards is now given.

Choose $(v_r^1, \dots, v_r^n) \in \operatorname{int} V^*$ such that for each player $i \in N$, $v^i > v_r^i$. Since V^* has full dimension, there exists $\rho > 0$ such that for each player j such that $\sharp\mathcal{N}(j) \geq 2$ and $\sharp\mathcal{N}(j^1) \geq 2$ for every $j^1 \in \mathcal{N}(j)$, the following holds:

$$v_r(j) = (v_r^1 + \rho, \dots, v_r^{j-1} + \rho, v_r^j, v_r^{j+1} + \rho, \dots, v_r^n + \rho) \in V^*,$$

and for every players k and ℓ such that $\mathcal{N}(k) = \{\ell\}$:

$$v_r(k,\ell) = (v_r^1 + \rho, \dots, v_r^{k-1} + \rho, v_r^k, v_r^{k+1} + \rho, v_r^{\ell-1} + \rho, v_r^\ell + \frac{\rho}{2}, v_r^{\ell+1} + \rho, \dots, v_r^n + \rho) \in V^*.$$ [17]

16. Notice that one can have $i = \ell$ and $\tilde{i} = k$, or the reverse.

17. This element exists since $\operatorname{int} V^* \neq \emptyset$. Notice that the non-equivalent utilities condition introduced by Abreu, Dutta and Smith ([1]) is not sufficient to ensure that $v_r(k,\ell)$ exists and lies in

For each player i in N, let $w^i(d)$ be player i's realized discounted average payoff during phase III when punishing player d in N. Denote by $\bar{g}^i$ player i's greatest one-shot payoff, that is:

$$\bar{g}^i = \max_{a^i,(a^j)_{j\in\mathcal{N}(i)}} g^i(a^i, a^j).$$

Lemma 3.5.9. *For every $\epsilon > 0$ small enough, there exists $\delta^* > \delta_\epsilon$* [18] *such that for all $\delta > \delta^*$, there exists an integer $T(\delta)$* [19] *(if several take the smallest) such that for every pair of players i and d:*

$$(1-\delta^2)\bar{g}^i + \delta^{T(\delta)+3}v_r^i \;<\; v_r^i - \epsilon, \tag{3.1}$$
$$(1-\delta^2)\bar{g}^i + \delta^{T(\delta)+3}v_r^i \;<\; (1-\delta^{T(\delta)+2})w^i(d) + \delta^{T(\delta)+2}(v_r^i + \rho), \tag{3.2}$$
$$(1-\delta^2)\bar{g}^i + \delta^{T(\delta)+3}v_r^i \;<\; (1-\delta^2)w^i(d) + \delta^2(v_r^i + \rho). \tag{3.3}$$

Proof. Choose $\epsilon > 0$ such that, for every players i and d, $\epsilon < \min_i v_r^i$ and

$$-w^i(d) < \frac{v_r^i - \epsilon}{v_r^i}\left(\rho - w^i(d)\right). \tag{3.4}$$

This is possible since $\rho > 0$. Now, I can rewrite Equation (4) as follows:

$$0 < \left(\frac{v_r^i - \epsilon}{v_r^i}\right)\rho - \left(1 - \frac{v_r^i - \epsilon}{v_r^i}\right) w^i(d).$$

Then, there exists $x \in (0,1)$ such that $xv_r^i < v_r^i - \epsilon$ and $xv_r^i < (1-x)w^i(d) + x(v_r^i + \rho)$ (take x close to but lower than $\frac{v_r^i-\epsilon}{v_r^i}$). Then, I choose δ^* close to one and $\delta^{T(\delta)}$ close to x as ϵ tends to zero (notice that x is close to 1 as ϵ tends to zero). The left hand side of Equations (1), (2) and (3) tends to xv_r^i as ϵ tends to zero and Equations (1) and (2) directly follow. The right hand side of Equation (3) tends to $v_r^i + \frac{\rho}{2}$ as ϵ tends to zero and Equation (3) follows directly since $\rho > 0$. As a consequence, δ^* exists. □

Define now, for every pair of players i and d:

$$z^i(d) = \begin{cases} w^i(d)\frac{1-\delta^{T(\delta)}}{\delta^{T(\delta)}}, & \text{if } i \in \mathcal{N}(d), \\ 0, & \text{otherwise.} \end{cases}$$

Finally, let $a_t(j, \delta, (z^i(j))_{i\in\mathcal{N}(j)})$ and $a_t(k, \ell, \delta, (z^\ell(k)), (z^i(\ell))_{i\in\mathcal{N}(\ell)})$ be deterministic sequences of pure actions that result in the following payoffs:

$$(v_r^j, (v_r^i + \rho - z^i(j))_{i\neq j}) \in V^*, \tag{3.5}$$
$$(v_r^k - z^k(\ell), v_r^\ell - z^\ell(k), (v_r^i + \rho - z^i(\ell))_{i\neq k,\ell}) \in V^*, \tag{3.6}$$

V^*.

18. Recall that δ_ϵ is defined in Section 3.5.2.1 such that for every $\epsilon > 0$, there exists $\delta_\epsilon < 1$ such that for all $\delta \geq \delta_\epsilon$ and every $v \in V^*$ such that $v^i \geq v$ for all i, the deterministic sequence of pure actions $\bar{a}_t$ can be constructed so that the continuation payoffs at each stage are within ϵ of v.

19. Recall that $T(\delta)$ is the length of phase III.

and whose continuation payoffs are within ϵ of (5) and (6) respectively.

Lemma 3.5.10. *The sequences* $a_t(j, \delta, (z^i(j))_{i \in \mathcal{N}(j)})$ *and* $a_t(k, \ell, \delta, (z^\ell(k)), (z^i(\ell))_{i \in \mathcal{N}(\ell)})$ *exist for* ϵ *close to zero and* δ *close to one.*

Proof. Consider a sequence (ϵ_n, δ_n) such that ϵ_n tends to zero and δ_n tends to one as n tends to infinity. By construction, $T(\delta)$ is the smallest integer satisfying equations (3), (4) and (5), and $\delta^{T(\delta)}$ is close to $\frac{v_r^i - \epsilon}{v_r^i}$. Hence, $\delta_n^{\mu(\delta_n)}$ is close to one for n sufficiently large, which implies that $z^i(j) = w^i(j) \frac{1 - \delta_n^{\mu(\delta_n)}}{\delta_n^{\mu(\delta_n)}}$ tends to zero as n tends to infinity. As a consequence, for n sufficiently large, $\rho - z^i(j) > 0$ and the payoffs in (8) are in V^* and bounded away from the axes by at least ϵ_n. By Lemma 2 in Fudenberg and Maskin (page 432 in [16]), this implies that for n sufficiently large, there exists $\bar{\delta}$ close to one (and greater than $\delta^* > \delta_\epsilon$) such that, for every $\delta > \bar{\delta}$, there exists a sequence of pure actions $a_t(j, \delta_n, (z^i(j))_{i \in N})$ with payoffs (8) and whose continuation payoffs are within ϵ_n of (8). Similar arguments apply to prove that $a_t(k, \ell, \delta, (z^\ell(k)), (z^i(\ell))_{i \in \mathcal{N}(\ell)})$ exists. □

The strategy σ^{i*} for any player i in phase IV is then the following. On one hand, if only player j is minmaxed in phase III, player i starts playing $a_t(j, \delta, (z^i(j))_{i \in N})$at stage $t + 3 + T(\delta)$ until a new possible deviation. On the other hand, if both players k and ℓ are minmaxed in phase III, then player i starts playing the sequence $a_t(k, \ell, \delta, (z^\ell(k))_{i \in \mathcal{N}(k)}, (z^i(\ell))_{i \in \mathcal{N}(\ell)})$ at stage $t+3+T(\delta)$ until a new possible deviation.

Intuitively, these rewards are such that, if plan $\{j\}$ or plan $\{k, \ell\}$, with $\mathcal{N}(k) = \{\ell\}$, is played, each player $i \neq k, \ell$ has an incentive to play his minmax strategy against j due to the additional bonus of ρ thereafter. In addition, players k and ℓ have no incentive to deviate during plan $\{k, \ell\}$, otherwise they would only lengthen their punishment, postponing positive payoffs (recall that by construction plans $\{k\}$ and $\{\ell\}$ are never played). Finally, when punishing any player d in N, each minmaxing player i has no incentive to draw the sequence of pure actions according to another distribution, whose support is included in the support of $P^i(d)$: any expected advantage that player i gets from playing some pure action in phase III is subsequently removed in phase IV.

For each player $i \in N$, ϕ^* is the same as before: when a player i does not detect any deviation at stage t, then he should send the message $m_{t+1}^i = \mathcal{N}(i) \cup \{i\}$. When a player i detects a deviation, he starts phase II.

Finally, notice that play after histories that involve simultaneous deviations is irrelevant. Hence, the partial strategy that I define can be completed in any arbitrary fashion. This concludes the construction of (σ^*, ϕ^*). The next section specifies the beliefs.

3.5.2.5 Specification of the beliefs

A *belief assessment* is a sequence $\mu = (\mu_t^i)_{t\geq 1, i\in N}$ with $\mu_t^i : H_t^i \to \Delta(H_t)$: given a private history h^i of player i, $\mu_t^i(h^i)$ is the probability distribution representing the belief that player i holds on the full history. An *assessment* is an element $((\sigma,\phi),\mu)$ where (σ,ϕ) is a strategy profile and μ a belief assessment.

I consider a restricted set of beliefs, which is strictly included in the set of total histories H_t. I call this set of beliefs $\mathcal{B} = (\mathcal{B}^i)_{i\in N}$. Namely, for each player i in N, every belief in $\mathcal{B}^i$ only assigns positive probability to histories that differ from equilibrium play, (σ^*,ϕ^*), in as much as, and to the minimal extent which, their private history dictates that it does. Formally, for every player i in N and every history $h_t^i \in H_t^i$, I denote by $H_t[h_t^i] \subset H_t$ the set of total histories for which the projection on H_t^i is h_t^i. A total history h_t in $H_t[h_t^i]$ is said to be compatible with private history h_t^i of player i. Now, for every player i in N and every history $h_t^i \in H_t^i$, let $H_t[h_t^i](U|(\sigma^*,\phi^*)) \subseteq H_t[h_t^i]$ be the set containing all the total histories that are compatible with h_t^i and included in $H_t(U|(\sigma^*,\phi^*))$.[20] Formally, for each player i in N and every history $h_t^i \in H_t^i$:

$$H_t[h_t^i](U|(\sigma^*,\phi^*)) = H_t[h_t^i] \cap H_t(U|(\sigma^*,\phi^*)).$$

The set of beliefs $\mathcal{B}^i$ is then the following:

$$\begin{aligned}\mathcal{B}^i \;=\; \Big\{ &(\mu_t^i)_{t\geq 1} \;:\; \forall t \geq 1,\, \forall h_t^i \in H_t^i,\\ &h_t^i \in H_t^i(U|(\sigma^*,\phi^*)) \Rightarrow \operatorname{supp} \mu_t^i(h_t^i) \subseteq H_t[h_t^i](U|(\sigma^*,\phi^*)) \Big\},\end{aligned}$$

where supp stands for the support of $\mu_t^i(h_t^i)$. In other words, the beliefs of the players are such that, if they observe a history compatible with either no deviation or unilateral deviations, then they assign probability one to the fact that the total history is in $H_t(U|(\sigma^*,\phi^*))$ and is compatible with h_t^i.

In the next section, I show that for every $\mu \in \mathcal{B}$, $((\sigma^*,\phi^*),\mu)$ is a PBE with payoff v^*.

3.5.3 The equilibrium property

I now prove the following proposition, which implies the sufficiency part of Theorem 3.4.1.

Proposition 3.5.11. *Assume that G is such Condition DN holds, that g satisfies Assumption P, and that* $\operatorname{int} V^* \neq \emptyset$. *Then, for every $v \in V^*$, there exists $\bar{\delta} \in (0,1)$*

20. Recall that $H_t(U|(\sigma^*,\phi^*))$ is the set of total histories along which only unilateral deviations, if any, have taken place.

such that for all $\delta \in (\bar{\delta}, 1)$, the assessment $((\sigma^*, \phi^*), \mu)$, for each $\mu \in \mathcal{B}$, is a PBE with payoff v in the δ-discounted game.

Remark 3.5.12. Actually, in Proposition 3.5.11, there exists a message set M^i for each player $i \in N$, such that for every $v \in V^*$, there exists $\bar{\delta} \in (0,1)$ such that for all $\delta \in (\bar{\delta}, 1)$, the assessment $((\sigma^*, \phi^*), \mu)$, for each $\mu \in \mathcal{B}$, is a PBE with payoff v in the δ-discounted game. The proof relies on the fact that the cardinality of the set M^i is sufficiently large. More precisely, I use $M^i = N \times (A^i)^{T(\delta)}$ (each player announce his set of suspects and, during phase III, the sequence of pure actions he played. Hence, the proof relies on the assumption that $\sharp M^i \geq n + (\sharp A^i)^{T(\delta)}$.[21]

Proof. Assume that G is such that Condition DN holds, that g satisfies Assumption P, and that $\operatorname{int} V^* \neq \emptyset$. Take $v \in V^*$, $\bar{\delta}$ (defined in Section 3.5.2.4), and consider $\delta > \bar{\delta}$. Recall that $\bar{\delta} > \delta_\epsilon$ so that the sequence of pure actions $\bar{a}_t$ defined in Section 3.5.2.1 exists and the continuation payoffs are within ϵ of v.

Since, play after histories that involve multilateral deviations is irrelevant, consider the strategy (σ^*, ϕ^*) constructed in the previous sections for all private histories in $H_t^i(U|(\sigma^*, \phi^*))$, for each player i in N. Take a belief assessment μ in $\mathcal{B}$. I now prove that $((\sigma^*, \phi^*), \mu)$ is a PBE.

Take first a player j such that $\sharp\mathcal{N}(j) \geq 2$ and $\sharp\mathcal{N}(i) \geq 2$ for each $i \in \mathcal{N}(j)$. By Lemma 3.5.4, player j's communication deviations do not change continuation strategies, provided player j does not deviate in action (recall that the deviation which consists in falsely reporting his pure actions during phase III is regarded as an action deviation). Henceforth, I focus on player j's action deviations. Assume that player j stops playing action $\bar{a}_t^j$ at some stage t during phase I and then conforms; without loss of generality, let $t = 1$. Lemmas 3.5.2 and 3.5.3 imply that player j is identified as the deviator at stage $t + 1 = 2$ by all players and the state becomes $\{j\}$. Player j is thus minmaxed at stage $t + 2 = 3$ during $T(\delta)$ periods. Player j's discounted expected payoff is then no more than:

$$(1-\delta)\sum_{t=1}^{2}\delta^{t-1}\bar{g}^j + (1-\delta)\sum_{t>T(\delta)+3}\delta^{t-1}v_r^j \quad = \quad (1-\delta^2)\bar{g}^j + \delta^{T(\delta)+3}v_r^j.$$

Since $\delta > \bar{\delta} > \delta^*$, Equation (1) ensures that this is less than $v_r^j - \epsilon$ which is a lower bound for player j's continuation payoff for conforming from date t on.

If player j deviates in action during phase III when he is being punished, he obtains at most zero the stage in which he deviates, and then only lengthens his

21. The fact that the length of the punishment phase depends on δ can be traced back to Fudenberg and Maskin ([16]) for repeated games with perfect monitoring without public randomization device. Since no public randomization device is assumed here, the fact that $T(\delta)$ depends on δ is maintained. Nevertheless, this dependency on δ may not be needed.

punishment, postponing the positive payoff v_r^j. The case where player j deviates in action in phase III when a player $d \notin \mathcal{N}(j)$ is being punished is also trivial, since by construction player j's action is arbitrary and may be a best-response.

Assume now that player j deviates in action at stage t during phase III when player $d \in \mathcal{N}(j)$ is being punished and then conforms. Two cases are possible. Assume first that player j deviates at stage t by playing an action which is not in the support of his minmax strategy. By construction, player j is identified as deviator at stage $t+1$. Player j's discounted expected payoff from the beginning of stage t is thus no more than:

$$(1-\delta)\sum_{t=1}^{2}\delta^{t-1}\bar{g}^j + (1-\delta)\sum_{t>T(\delta)+3}\delta^{t-1}v_r^j \quad = \quad (1-\delta^2)\bar{g}^j + \delta^{T(\delta)+3}v_r^j. \tag{3.7}$$

On the contrary, if he conforms, he gets at least:

$$(1-\delta^{T(\delta)-t+2})w^j(d) + \delta^{T(\delta)-t+2}(v_r^j + \rho). \tag{3.8}$$

If $t = 1$, then Equation (2) implies that Equation (8) exceeds Equation (7). If $t = T(\delta)$, it follows from Equation (3) that Equation (8) exceeds Equation (7). Finally, the cases for which $1 < t < T(\delta)$ follow from Equations (2) and (3) combined.

Second, Phase IV is constructed so that player j is indifferent among all actions in the support of his minmax strategy during phase III when player i is punished (if player j conforms during phase IV). Regardless of player j's actions in this phase, his continuation payoff from the beginning of phase III is within ϵ of:

$$(1-\delta)\sum_{t=1}^{T(\delta)}\delta^{t-1}g^j(a_t^j, a_t^{\mathcal{N}(j)}) + \delta^{T(\delta)}(v_r^j + \rho - z^j(i)) = \delta^{T(\delta)}(v_r^j + \rho)$$

Hence, player j has no incentive to deviate in phase III by randomizing according to another distribution, whose support is included in the support of his minmax strategy.

Finally, if player j deviates in action at stage t during phase IV, his discounted expected payoff is no more than:

$$(1-\delta)\sum_{t=1}^{2}\delta^{t-1}\bar{g}^j + (1-\delta)\sum_{t>T(\delta)+3}\delta^{t-1}v_r^j \quad = \quad (1-\delta^2)\bar{g}^j + \delta^{T(\delta)+3}v_r^j$$

If player j conforms, his continuation payoff is at least $v_r^j - \epsilon$, and so Equation (1) ensures that deviation is not profitable.

Take now a pair of players k and ℓ such that $\mathcal{N}(k) = \ell$. If player k (respectively player ℓ) deviates only in communication, then either continuation strategies do not change (in the case in which all players are cleared, see Lemma 3.5.4), or both players k and ℓ are minmaxed (see Lemma 3.5.3) and similar arguments as for action deviations apply. Moreover, similar arguments as before (cases in which player j deviates) show that neither player k nor player ℓ has an incentive to deviate in action during any phase of the game. (Notice that even if neither player k nor ℓ obtain a reward after plan $\{k, \ell\}$ although they are minmaxing players, they still have an incentive to play their minmax strategies, otherwise they would lengthen their punishment, postponing positive payoffs.)

To conclude, notice that the proof of the optimality of (σ^*, ϕ^*) above does not take into account the beliefs. Indeed, since μ^i in $\mathcal{B}^i$, for any history in $H_t^i(U|(\sigma^*, \phi^*))$, each player i in N believes that there is either no deviation or only unilateral deviations. The partial strategy (σ^*, ϕ^*) prevents player i from deviating, no matter what his beliefs (in $\mathcal{B}^i$) are. Indeed, under μ^i, player i believes that if he deviates, it will lead either to his punishment, or to no changes in his continuation payoff (in case of a communication deviate only). □

Remark 3.5.13. One can deduce from the previous proof that the strategy $((\sigma^*, \phi^*), \mu)$ is *belief-free* for each history in $H_t(U|(\sigma^*, \phi^*))$ and for each belief assessment in $\mu \in \mathcal{B}$. Formally, I denote by $((\sigma^i, \phi^i), \mu^i)|h_t^i$ player i's continuation assessment after private history h_t^i. Let also $((\sigma^{-i}, \phi^{-i}), \mu^{-i})|h_t^{-i}$ be the profile of continuation assessments of players $j \neq i$ after private histories h_t^j. The assessment $((\sigma^*, \phi^*), \mu)$ satisfies the following: for every $h_t \in H_t(U|(\sigma^*, \phi^*))$, every $\mu \in \mathcal{B}$ and every $i \in N$:

$$((\sigma^i, \phi^i), \mu^i)|h_t^i \in \mathrm{BR}\Big(\big((\sigma^{-i}, \phi^{-i}), \mu^{-i}\big)|h_t^{-i}\Big),$$

where BR stands for the best-reply function of player i. In other words, $((\sigma^*, \phi^*), \mu)$ is such that, for every player i, every private history $h_t^i \in H_t^i(U|(\sigma^*, \phi^*))$, every $\mu^i \in \mathcal{B}^i$:

$$((\sigma^i, \phi^i), \mu^i)|h_t^i \in \mathrm{BR}\Big(\big((\sigma^{-i}, \phi^{-i}), \mu^{-i}\big)|h_t^{-i}\Big),$$

for every $\mu^{-i} \in \mathcal{B}^{-i}$ and every $h_t^{-i} \in H_t^{-i}(U)$ that are possible given the monitoring structure.

3.6 Necessity of Condition DN

In this section, I prove that Condition DN of Theorem 3.4.1 is necessary for the folk theorem to hold.

Proposition 3.6.1. *Assume that G does not satisfy Condition DN. Then, there exists a payoff function g such that: Assumption P holds,* $\operatorname{int} V^* \neq \emptyset$*, and there* *exists a feasible and strictly individually rational payoff $v \in V^*$ such that v is not a Nash equilibrium payoff of the repeated game.*

Intuitively, when Condition DN is violated, there exists a player i, two of his neighbors j and k, and two deviations of j and k such that: for any action profile (possibly mixed), both j's and k's deviations induce the same distribution over player i's payoffs, and the same distribution over the messages received by player i.[22] The payoffs are constructed so that there exists a feasible and strictly individually rational payoff which is not *jointly rational* (in the sense of Renault and Tomala in [38]; see Example 3.1 therein for a similar phenomenon). Now, I prove Proposition 3.6.1.

Proof. Take a network G such that Condition DN does not hold. It implies that there exists two players j and k in N, who have the same neighbors: $\mathcal{N}(j) \setminus \{k\} \neq \mathcal{N}(k) \setminus \{j\}$. For brevity, I focus on the case where players j and k are not neighbors. The proof can be easily extended to the case where j and k are neighbors.

Take a player $i \in \mathcal{N}(j)$. Notice that all other players, $m \neq i, j, k$ are either neighbors of both j and k, or of none of them. Consider the payoff function for players i, j and k represented by the following table (where player i chooses the row, player j the column, and player k the matrix):[23]

k plays C

$i \setminus j$	C	D
C	1,0,2	0,0,2
D	1,2,0	0,6,0

k plays D

$i \setminus j$	C	D
C	0,0,6	1,0,6
D	0,2,0	1,6,0

I write $u(a^i, a^j, a^k)$ for this payoff vector. Player j's payoff does not depend on k's action, nor does k's payoff depends on j's action. Accordingly, I write $u^j(a^j, a^i)$ and $u^k(a^k, a^i)$ in what follows. To complete the description of g, each player $m \neq i, j, k$ has two actions C and D such that:

- for each player m such that $m \notin \mathcal{N}(j) \cap \mathcal{N}(k)$, player m's payoff at stage t is:

$$g^m(a_t^m, a_t^{\mathcal{N}(m)}) = \ell_t \frac{\epsilon}{n}$$

for some $\epsilon > 0$, and $\ell_t = \sharp\{\ell : \ell \in \mathcal{N}(m) \cup \{m\} \text{ and } a_t^\ell = C\}$ (ℓ_t is the number of m's neighbors including himself who play C at stage t);

22. Following the terminology of Fudenberg, Levine and Maskin ([14]), *pairwise identifiability* fails.

23. In the case where j and k are neighbors, I assume that if player j plays D it adds $\epsilon > 0$ to player k's payoff, and symmetrically if player k plays D it adds $\epsilon > 0$ to player j's payoff. This ensures that Assumption P holds.

- for each player m such that $m \in \mathcal{N}(j) \cap \mathcal{N}(k)$, player m's payoff at stage t is:

$$g^m(a_t^m, a_t^{\mathcal{N}(m)}) = \begin{cases} 1 + \ell_t^{-jk}\frac{\epsilon}{n} & \text{if } a^j = a^k; \\ \ell_t^{-jk}\frac{\epsilon}{n} & \text{otherwise.} \end{cases}$$

for some $\epsilon > 0$, and $\ell_t^{-jk} = \sharp\{\ell \ : \ \ell \in \mathcal{N}(m) \setminus \{j,k\} \cup \{m\} \text{ and } a_t^\ell = C\}$ (ℓ_t is the number of m's neighbors distinct from j and k, including himself, who play C at stage t);

- for players i, j and k:

$$\begin{aligned} g^i(a_t^i, a_t^{\mathcal{N}(i)}) &= u^i(a_t^i, a_t^j, a_t^k) - \ell_t^i\frac{\epsilon}{n}, \\ g^j(a_t^j, a_t^{\mathcal{N}(j)}) &= u^j(a_t^j, a_t^i) + \ell_t^j\frac{\epsilon}{n}, \\ g^k(a_t^k, a_t^{\mathcal{N}(k)}) &= u^k(a_t^k, a_t^i) + \ell_t^k\frac{\epsilon}{n}, \end{aligned}$$

where $\ell_t^i = \sharp\{\ell \ : \ \ell \in \mathcal{N}(i) \setminus \{j,k\} \text{ and } a_t^\ell = D\}$, $\ell_t^j = \sharp\{\ell \ : \ \ell \in \mathcal{N}(j) \setminus \{i\} \text{ and } a_t^\ell = D\}$, $\ell_t^k = \sharp\{\ell \ : \ \ell \in \mathcal{N}(k) \setminus \{i\} \text{ and } a_t^\ell = D\}$ and $u^i(a_t^i, a_t^j, a_t^k)$, $u^j(a_t^j, a_t^i)$, and $u^k(a_t^k, a_t^i)$ are defined by the matrix above.

This payoff function g has the following properties:[24]

(i) g satisfies Assumption P;

(ii) $\operatorname{int} V^* \neq \emptyset$;

(iii) $\underline{v}^i < 0$, $\underline{v}^j = 0$, and $\underline{v}^k = 0$;

(iv) C is a dominant strategy for each player $\ell \neq i, j, k$;

(v) the outcome $(1,1,1)$ (representing the payoffs of players i, j and k) is in V^*;

(vi) if $a^\ell \neq C$ for every $\ell \in \mathcal{N}(i)$, then $g^i(a_t^i, a_t^{\mathcal{N}(i)}) < 1$. Hence, the unique way to obtain the outcome $(1,1,1)$ is that player i chooses between C and D with probability $\frac{1}{2}$- $\frac{1}{2}$, and that all his neighbors (including players j and k) take action C;

(vii) player i cannot punish both players j and k: player i has to play C in order to minmax player j, which yields a payoff of 6 for player k; and player i has to choose action D in order to minmax player k, which yields a payoff of 6 for player j;

(viii) for each player $m \in \mathcal{N}(j) \cap \mathcal{N}(k)$ (including player i), for every $a^m \in \{C, D\}$, and for every $a^{\mathcal{N}(m)\setminus\{j,k\}}$, the following properties hold:

$$g^m(a^m, a^{\mathcal{N}(m)\setminus\{j,k\}}, a^j = C, a^k = C) = g^m(a^m, a^{\mathcal{N}(m)\setminus\{j,k\}}, a^j = D, a^k = D),$$

$$g^m(a^m, a^{\mathcal{N}(m)\setminus\{j,k\}}, a^j = C, a^k = D) = g^m(a^m, a^{\mathcal{N}(m)\setminus\{j,k\}}, a^j = D, a^k = C).$$

24. It is possible to construct a payoff function g which satisfies these properties when players have more than two actions. This construction is described in Appendix 3.8.1.

Assume now that $(1,1,1)$ is a Nash equilibrium of the repeated game, and let the profiles $\bar{\sigma} = (\bar{\sigma}^i, \bar{\sigma}^j, \bar{\sigma}^k, (\bar{\sigma}^m)_{m \neq i,j,k})$ and $\bar{\phi} = (\bar{\phi}^i, \bar{\phi}^j, \bar{\phi}^k, (\bar{\phi}^m)_{m \neq i,j,k})$ be an equilibrium yielding a payoff of $\gamma_\delta = (1,1,1)$ for players i, j and k. I construct deviations (τ^j, ψ^j) and (τ^k, ψ^k) for players j and k such that:

(1) both deviations induce the same probability distributions over the sequences of messages and payoffs received by player i (deviations are indistinguishable).

(2) I will deduce from (1) that: $\gamma^j(\tau^j, \bar{\sigma}^{-j}, \psi^j, \bar{\phi}^{-j}) + \gamma^k(\tau^k, \bar{\sigma}^{-k}, \psi^k, \bar{\phi}^{-k}) \geq 3$.

The latter equation contradicts that $(\bar{\sigma}, \bar{\phi})$ is an equilibrium of the repeated game. Indeed, $\gamma^j(\bar{\sigma}, \bar{\phi}) + \gamma^k(\bar{\sigma}, \bar{\phi}) = 2$ and $\gamma^j(\tau^j, \bar{\sigma}^{-j}, \psi^j, \bar{\phi}^{-j}) + \gamma^k(\tau^k, \bar{\sigma}^{-k}, \psi^k, \bar{\phi}^{-k}) \geq 3$. Therefore, either (τ^j, ψ^j) is a profitable deviation for player j, or (τ^k, σ^k) is a profitable deviation for player k. I now construct these deviations. Define (τ^j, ψ^j) as follows (the construction of (τ^k, ψ^k) is symmetric):

- at each stage, player j chooses between C and D with probability $\frac{1}{2}$-$\frac{1}{2}$ (instead of C with probability 1);
- player j uses the communication strategy $\psi^j = \bar{\phi}^j(h^j(\tau^k, \sigma^{-k}, \psi^k, \phi^{-k}))$, where the element $h^j(\tau^k, \sigma^{-k}, \psi^k, \phi^{-k})$ stands for an arbitrary private history of player j in which player k plays according to (τ^k, ψ^k), and all other players follow (σ^{-k}, ϕ^{-k}): player j follows the equilibrium communication strategy but replaces the true history by the fictitious one in which player k is the deviator.

Under such a deviation, player i has no way to deduce whether j or k deviated, even when $(\bar{\sigma}, \bar{\phi})$ is a mixed strategy. Indeed, for every pure action profile, both player j's and k's deviations induce the same payoffs for player i, since g satisfies property (viii). In addition, notice that player i's payoff is 1 if j and k choose the same action, and 0 otherwise. Therefore, since τ^j and τ^k prescribe to choose C and D with probability $\frac{1}{2}$-$\frac{1}{2}$ at each stage, $(\tau^j, \psi^j, \bar{\sigma}^{-j}, \bar{\phi}^{-j})$ and $(\tau^k, \psi^k, \bar{\sigma}^{-k}, \bar{\phi}^{-k})$ induce the same distribution over the payoffs of player i, even if $\bar{\sigma}^j$ and $\bar{\sigma}^k$ are mixed: player i gets a payoff of 1 or 0 with probability $\frac{1}{2}$-$\frac{1}{2}$.[25] Hence, no player has relevant information on the deviator's identity: each player is either a neighbor of both j and k, or a neighbor of none of them. Moreover, both deviations induce the same distribution over the messages received by player i, even when $\bar{\phi}$ is a mixed strategy.[26]

As a consequence, at every stage t, for each private history h_t^i of player i:

$$\mathbb{P}_{\tau^j, \bar{\sigma}^{-j}, \psi^j, \bar{\phi}^{-j}}(h_t^i) = \mathbb{P}_{\tau^k, \bar{\sigma}^{-k}, \psi^k, \bar{\phi}^{-k}}(h_t^i).$$

25. These deviations of players j and k also induce the same distribution of payoffs for all players who are neighbors of both j and k.

26. Indeed, player j draws a fictitious history, using $\bar{\phi}^j$. If $\bar{\phi}^j$ is mixed, so is ψ^j (and similarly for player k's deviation).

Now, I define the following numbers b_t and c_t:

$$b_t = \mathbb{P}_{\tau^j,\bar{\sigma}^{-j},\psi^j,\bar{\phi}^{-j}}(a_t^i = C) = \mathbb{P}_{\tau^k,\bar{\sigma}^{-k},\psi^k,\bar{\phi}^{-k}}(a_t^i = C),$$

$$c_t = \mathbb{P}_{\tau^j,\bar{\sigma}^{-j},\psi^j,\bar{\phi}^{-j}}(a_t^i = D) = \mathbb{P}_{\tau^k,\bar{\sigma}^{-k},\psi^k,\bar{\phi}^{-k}}(a_t^i = D).$$

Under $(\tau^j, \psi^j, \bar{\sigma}^{-j}, \bar{\phi}^{-j})$, player j's expected payoff at stage t is then: $g_t^j(\tau^j, \bar{\sigma}^{\mathcal{N}(j)}) \geq 4c_t \geq 4(1 - b_t)$. As a consequence, $\gamma_\delta^j(\tau^j, \psi^j, \bar{\sigma}^{-j}, \bar{\phi}^{-j}) \geq 4(1-\delta)\sum_{t=1}^{+\infty} \delta^{t-1}(1 - b_t)$. Since $(\bar{\sigma}, \bar{\phi})$ is a Nash equilibrium of the repeated game, there must exist $\bar{\delta} \in (0,1)$ such that for any $\delta \in (\bar{\delta}, 1)$, $4(1-\delta)\sum_{t=1}^{+\infty} \delta^{t-1}(1 - b_t) \leq 1$, so that $(1 - \delta)\sum_{t=1}^{+\infty} \delta^{t-1} b_t \geq \frac{3}{4}$. In the same way, player k's expected payoff under the profile $(\tau^k, \psi^k, \bar{\sigma}^{-k}, \bar{\phi}^{-k})$ at stage t is: $g_t^k(\tau^k, \bar{\sigma}^{\mathcal{N}(k)}) \geq 4b_t$. Hence, $\gamma_\delta^k(\tau^k, \psi^k, \bar{\sigma}^{-k}, \bar{\phi}^{-k}) \geq 4(1-\delta)\sum_{t=1}^{+\infty} \delta^{t-1} b_t$. However, there exists $\bar{\delta} \in (0,1)$ such that for any $\delta \in (\bar{\delta}, 1)$, $(1-\delta)\sum_{t=1}^{+\infty} \delta^{t-1} b_t \geq \frac{3}{4}$, so $\gamma_\delta^k(\tau^k, \psi^k, \bar{\sigma}^{-k}, \bar{\phi}^{-k}) \geq 3$. This contradicts the fact that $(\bar{\sigma}, \bar{\phi})$ is a Nash equilibrium of the repeated game. □

As a conclusion, Condition DN is necessary for the folk theorem to hold.

3.7 Concluding remarks

In this section, I introduce some extensions of the model, and state some open problems.

Strict incentives to confess. Part of the equilibrium strategy constructed in Section 3.5 is weakly dominated. Indeed, a player who deviates at some stage t is required to confess at stage $t+1$. Nevertheless, it is possible to make the players have strict incentives to confess, by adjusting accordingly the reward in phase IV: indeed, the reward phase can be such that a player who deviates, and confesses publicly his deviation after that, obtains a bonus of $\frac{\rho}{2} > 0$ during phase IV. Hence, a player who deviates has a strict incentive to confess. In addition, this bonus is strictly lower than ρ, otherwise a minmaxing player might have an incentive to deviate. Indeed, a minmaxing player's realized payoff during phase III may be less than if he would get his minmax level instead. However, a minmaxing player has still no incentive to deviate, otherwise he would loose $\frac{\rho}{2}$ at each stage during phase IV.

Correlated minmax. In some repeated games with imperfect monitoring, it is possible to drive equilibrium payoffs below the independent minmax, see Renault and Tomala ([37]) for illuminating examples. It is the case here: Theorem 3.4.1 remains unchanged if I rather consider correlated minmax, defined as follows:

$$\bar{w}^i = \min_{x^{\mathcal{N}(i)} \in \Delta(A^{\mathcal{N}(i)})} \max_{x^i \in \Delta(A^i)} g^i(x^i, x^{\mathcal{N}(i)}).$$

To prove Theorem 3.4.1 in that case, I adapt the strategy constructed in Section 3.5 in the following way. The idea is that players can correlate their actions when punishing player k (respectively players k and ℓ) in phase III, without revealing information to the minmaxed player(s). For that, define $Q^{N(k)}(k) \in \Delta(A^{\mathcal{N}(k)})$ (respectively $Q^{N(\ell)}(\ell) \in \Delta(A^{\mathcal{N}(\ell)})$) a correlated strategy that realizes the minimum in $\bar{w}^k$ (respectively $\bar{w}^\ell$). Choose a player $j \neq k$ (respectively $j \neq k, \ell$). At the beginning of phase III, I add a stage in which player j draws i.i.d. sequences of pure actions according to $Q^{N(k)}(k)$ (respectively $Q^{N(\ell)}(\ell)$ when both players k and ℓ are minmaxed) [27] for the minmaxing players for $T(\delta)$ periods. Player j announces the sequences publicly to all players except k (respectively player ℓ). Deviations of player j are punished as before, and the reward phase makes player j indifferent between the pure actions actually played by him and his neighbors (recall that the reward phase is based on player j's realized payoff).

Private communication An alternative model would be to consider private announcements, *i.e.* the list of receivers of a message is not certifiable. The construction requires the possibility for the players to make public announcements in two cases only.

(i) First, if there exists a player k such that $\mathcal{N}(k) = \{\ell\}$, then players k and ℓ have to make public announcements in phase II of the construction. Otherwise, a communication deviation of player k (respectively player ℓ) could be to send spurious messages to a subset of players only. With the possibility of public communication, the strategy constructed in Section 3.5 ignores such deviations. If public communication is not allowed, this implies a lack of common knowledge of the deviation's date, and there may be a coordination failure with some players starting phase III whereas other do not.

(ii) Second, public announcements are crucial in the punishment phase (phase III). Otherwise, some communication deviations may entail a coordination failure, since players could have different informations on the pure actions chosen by their opponents (recall that pure actions are announced in phase III).

Otherwise, only private communication is required. Hence, the following corollary holds:

Corollary 3.7.1. *Assume that players are only allowed to communicate privately with each other (no certifiability). If Condition DN holds, and if each player has*

27. Notice that if $\mathcal{N}(k) = \{\ell\}$, and both players k and ℓ are minmaxed, there is no need to correlate in order to punish player k since only player ℓ is a neighbor of k.

more than two neighbors, then the folk theorem holds with minmax levels in pure strategies (and PBE as solution concept).

The proof is a straightforward application of the proof of Theorem 3.4.1.

Folk theorem without discounting. Condition DN of Theorem 3.4.1 is also the necessary and sufficient condition for a folk theorem to hold if I consider uniform (sequential) equilibria of the undiscounted repeated game (see [13] and [43]). Namely, every feasible and individually rational payoff is a uniform (sequential) equilibrium payoff for any payoff function g that satisfies Assumption P if and only if Condition DN holds. Moreover, in that case, public announcements are not required, and players can be only allowed to send private messages (coordination in cases (i) and (ii) above is not required). In addition, it is possible to restrict communication along the network: the necessary and sufficient conditions for a folk theorem to hold are known (but are different from Condition DN) if (i) only private communication is allowed (no public announcements) and (ii) I consider Nash equilibrium of the repeated game or uniform sequential equilibrium for the undiscounted case (see Chapter 2). If communication is restricted along the network, finding conditions for a folk theorem to hold is an open problem for (i) public announcements and (ii) sequential equilibria of repeated games with discounting.

3.8 Proofs

3.8.1 Payoff function of Section 3.6 with more than two actions

In this section, I modify the payoff function constructed in Section 3.6 for games in which players have more than two actions. For this purpose, I duplicate rows, columns, matrices…in the following manner. For each player $p \in N$, identify A^p with $\{1, \dots, k_p\}$, where $k_p = \sharp A^p$. The payoff functions are the following.

- For each player $m \neq i, j, k$ such that $m \notin \mathcal{N}(j) \cap \mathcal{N}(k)$, player m's payoff at stage t is $g^m(a_t^m, a_t^{\mathcal{N}(m)}) = \ell_t \frac{\epsilon}{n} + \frac{1}{\sum_{p \in \mathcal{N}(m) \cup \{m\}} k_p}$.
- For each player $m \neq i, j, k$ such that $m \in \mathcal{N}(j) \cap \mathcal{N}(k)$, player m's payoff at stage t is:

$$g^m(a_t^m, a_t^{\mathcal{N}(m)}) = \begin{cases} 1 + \ell_t^{-jk} \frac{\epsilon}{n} + \frac{1}{\sum_{p \in \mathcal{N}(m) \setminus \{j,k\} \cup \{m\}} k_p} & \text{if } a^j = a^k; \\ \ell_t^{-jk} \frac{\epsilon}{n} + \frac{1}{\sum_{p \in \mathcal{N}(m) \setminus \{j,k\} \cup \{m\}} k_p} & \text{otherwise.} \end{cases}$$

– Player i's payoff at stage t is:

$$g^i(a_t^i, a_t^{\mathcal{N}(i)}) = \begin{cases} u^i(a_t^i, a_t^j, a_t^k) & \text{if } a_t^p \in \{1,2\} \text{ for each } \ p \in \mathcal{N}(i), \\ u^i(a_t^i, a_t^j, a_t^k) - \frac{1}{\sum_{p \in \mathcal{N}(i) \cup \{i\}} k_p} & \text{if } a_t^i,\ a_t^j,\ a_t^k \in \{1,2\}, \\ & a_t^m \geq 3\, \forall m \in \mathcal{N}(i) \setminus \{i,j,k\}, \\ 6 - \frac{1}{\sum_{p \in \mathcal{N}(i) \cup \{i\}} k_p} & \text{otherwise.} \end{cases}$$

– For each player $n \in \{j,k\}$, player n's payoff at stage t is:

$$g^n(a_t^n, a_t^{\mathcal{N}(n)}) = \begin{cases} u^n(a_t^i, a_t^j, a_t^k) \text{ if } a_t^p \in \{1,2\} & \text{for each } \ p \in \mathcal{N}(n), \\ u^n(a_t^i, a_t^j, a_t^k) + \frac{1}{\sum_{p \in \mathcal{N}(n) \cup \{n\}} k_p} & \text{if } a_t^i,\ a_t^j,\ a_t^k \in \{1,2\}, \\ & a_t^m \geq 3\, \forall m \neq i,j,k, \\ 6 + \frac{1}{\sum_{p \in \mathcal{N}(n) \cup \{n\}} k_p} & \text{otherwise.} \end{cases}$$

It is easy to see that this payoff function satisfies Assumption P, and that all the desired properties for the proof of necessity in Section 3.6 hold.

3.8.2 Proof of Proposition 3.4.2

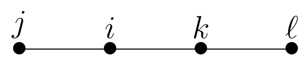
Proof. Take a connected network G that satisfies Condition DN. We first prove point (i). Assume $n = 3$, with $N = \{i,j,k\}$. Since G is connected, there exists a player, say i, who has exactly two neighbors. Therefore, $\mathcal{N}(k) = \{i,j\}$. But then, there is no player in $\mathcal{N}(i) \setminus \{j\} \bigtriangleup \mathcal{N}(j) \setminus \{i\}$ and Condition DN is violated.

We now prove point (ii). Assume $n = 4$, let $N = \{i,j,k,\ell\}$. Since G is connected, at least one player, say i, has at least two neighbors, say j and k. Because of Condition DN, it must be the case that $\ell \in \mathcal{N}(j) \setminus \{k\} \bigtriangleup \mathcal{N}(k) \setminus \{j\}$. Assume without loss of generality that $\ell \in \mathcal{N}(k)$, so $\ell \notin \mathcal{N}(j)$. I prove the following:

(1) j and k are not neighbors;

(2) i and ℓ are not neighbors;

so that: $\mathcal{N}(i) = \{j,k\}$, $\mathcal{N}(j) = \{i\}$, $\mathcal{N}(k) = \{i,\ell\}$ and $\mathcal{N}(l) = \{k\}$, which is represented by the following network:



and proves point (ii) of Proposition 3.4.2.

For (1), assume that j and k are neighbors. Then $\mathcal{N}(j) = \{i,k\}$ (recall that $\ell \notin \mathcal{N}(j)$) and $\{j,k\} \subseteq \mathcal{N}(i)$. For Condition DN to be satisfied, it must be that $\ell \in \mathcal{N}(i)$, so $\mathcal{N}(i) = \{j,k,\ell\}$. Yet, $\mathcal{N}(k) = \{i,j,\ell\}$, hence $\mathcal{N}(i)\setminus\{k\}\bigtriangleup\mathcal{N}(k)\setminus\{i\} = \emptyset$. This contradicts Condition DN, so that j and k cannot be neighbors.

For (2), assume that i and ℓ are neighbors. Then, $\{i,\ell\} \subseteq \mathcal{N}(k)$ and $\mathcal{N}(\ell) = \{i,k\}$ (again, recall that $j \notin \mathcal{N}(\ell)$). Therefore, it must be the case that $j \in \mathcal{N}(k)$ for Condition DN to be satisfied, which is impossible by (i). As a consequence, i and ℓ cannot be neighbors. □

3.8.3 Proof of Lemma 3.5.2

Proof. Take a network G that satisfies Condition DN and a player $k \in N$ such that $\sharp\mathcal{N}(k) \geq 2$. Assume that player k deviates in action at stage t. Take any player $j \neq k$. I first prove that j is cleared at stage $t+1$ by every player $i \in N$. Two cases are possible.

(1) Assume first that for each player $j^1 \in \mathcal{N}(j)$, $\sharp\mathcal{N}(j^1) \geq 2$. Then, each player i in N clears player j at stage $t+1$. Indeed, the following holds:

- first, if j plays ϕ^{*j} at stage $t+1$, then $j \in m^j_{t+1}(N)$. Indeed, either $j \in \mathcal{N}(k)$ and $\tilde{\phi}^j$ prescribes player j to announce publicly $N \setminus \mathcal{N}(j)$ to all players at stage $t+1$, so $j \in \tilde{m}^j_{t+1}(N)$. Or $j \notin \mathcal{N}(k)$ and player j starts the protocol at the end of stage $t+1$,[28] so j is prescribed to announce $\mathcal{N}(j) \cup \{j\}$ publicly to all players at stage $t+1$, so $j \in \bar{m}^j_{t+1}(N)$.
- Second, since Condition DN is satisfied, there exists a player $m \neq j,k$ such that $m \in \mathcal{N}(k) \setminus \{j\} \bigtriangleup \mathcal{N}(j) \setminus \{k\}$. Moreover, if m plays ϕ^{*m} at stage $t+1$, then $j \in m^m_{t+1}(N)$. Indeed, either, $m \in \mathcal{N}(k)$ and $\tilde{\phi}^m$ prescribes player m to announce $N \setminus \mathcal{N}(m)$ publicly to all players at stage $t+1$, so $j \in \tilde{m}^m_{t+1}(N)$. Or $m \notin \mathcal{N}(k)$, and $\bar{\phi}^m$ prescribes player m to announce $\mathcal{N}(m) \cup \{m\}$ publicly to all players at stage $t+1$, so $j \in \bar{\phi}^m(N)$.
- Third, if k follows ϕ^{*k} at stage $t+1$, then $\tilde{\phi}^k$ prescribes player k to announce $N \setminus \{k\}$ publicly to all players at stage $t+1$, so $j \in \tilde{m}^k_{t+1}(N)$.

Since at most one player in $\{j,k,m\}$ deviates in communication at stage $t+1$, then $j \notin X^i_{t+1}$ for each player $i \in N$.

(2) Assume now that there exists $j^1 \in \mathcal{N}(j)$ such that $\mathcal{N}(j^1) = \{j\}$. First, proposition 3.4.3 implies that j^1 is unique. Second, G is connected and $n \geq 3$, so $\sharp\mathcal{N}(j) \geq 2$. Finally, $j^1 \neq k$ since $\sharp\mathcal{N}(k) \geq 2$ by assumption. Hence, j^1 is cleared at stage $t+1$ by every player i in N (see (1) above). In addition, with the same reasoning as before, at least two players in $\{j,k,m\}$ make public announcements

28. Notice that player j could have started the protocol at the end of stage t if (a) one of his neighbor deviated at stage t, or (b) if a player who is not his neighbor deviated at stage $t-1$, or (c) if player k deviated also in communication at stage t by sending to j a message different from $\bar{m}^k_{t+1}(N)$. However, in any case, player j ends this "previous" protocol at the end of stage t, and starts a "new" protocol at the end of stage $t+1$.

including j to all players at stage $t+1$. As a consequence, for each player $i \in N$, $j \notin X^i_{t+1}$.

Finally, I prove that no player $i \in N$ clears player k, *i.e.* $k \in X^i_{t+1}$ for each player $i \in N$. By construction, $k \notin m^j_{t+1}(N)$ for any $j \in N$ who follows ϕ^{*j}. Since at most one player deviates in communication at stage $t+1$, $k \in X^i_{t+1}$ for each player $i \in N$.

Hence, $X^i_{t+1} = \{k\}$ for each player i in N. □

3.8.4 Proof of Lemma 3.5.3

Proof. Take a connected network G that satisfies Condition DN and suppose Assumption P holds. Take a pair of players k and ℓ such that $\mathcal{N}(k) = \{\ell\}$.

Assume first that player k deviates in action at stage t. At stage $t+1$, players k and ℓ should announce N^{-k} and $N \setminus \mathcal{N}(k)$ respectively publicly to all players. All other players $j \neq k, \ell$ should announce $\mathcal{N}(j) \cup \{j\}$ and $k \notin m^j_{t+1}(N)$ publicly to all players since $\mathcal{N}(k) = \{\ell\}$. As a consequence, the name of k appears in at most one public announcement at stage t and $X^i_{t+1} \supseteq \{k, \ell\}$ for each player i in N. In addition, no player $j \neq k$, including player ℓ, deviates in action at stage t under unilateral deviations. So, each player $j \neq k$ such that $\sharp\mathcal{N}(j) \geq 2$ appears in the public announcements of at least two distinct players (among his two neighbors and himself), and $j \notin X^i_{t+1}$ for each player $i \in N$. On the other hand, each player $j \neq k$ such that $\sharp\mathcal{N}(j) = 1$ also appears in the public announcements of at least two distinct players: either j and his single neighbor do not deviate and j appears in both of their public announcements, or one of them deviates at stage $t+1$, which implies that k follows ϕ^{*k}_{t+1} and $j \in m^k_{t+1}(N)$. As a consequence, each player $j \neq k$ is cleared by all players at stage $t+1$. Hence, $X^i_{t+1} = \{k, \ell\}$ for each player i in N.

Assume now that there is no action deviation at stage t and that $m^k_{t+1} = N^{-k}$. Player k thus deviates in communication at stage $t+1$ and no other player does under unilateral deviations. It follows that all other players $j \neq k$ announce $\mathcal{N}(j) \cup \{j\}$ publicly since there was no action deviation at stage t. As a consequence, each player $j \neq k$ is cleared by at least two players at stage $t+1$, and player k appears in the public announcement of player ℓ only. Henceforth, $X^i_{t+1} = \{k, \ell\}$ for each player $i \in N$.

Finally, assume that there is no action deviation at stage t and that $m^\ell_{t+1}(N) = N \setminus \mathcal{N}(\ell)$. Player ℓ thus deviates in communication at stage $t+1$ and no other player does under unilateral deviations. It follows that all other players $j \neq k$ announce $\mathcal{N}(j) \cup \{j\}$ publicly to all players since there was no action deviation at stage t. As a consequence, each player $j \notin \mathcal{N}(\ell)$ is cleared by at least two players at stage $t+1$. Moreover, for each player $j \in \mathcal{N}(\ell) \setminus \{k\}$, $\sharp\mathcal{N}(j) \geq 2$. Indeed, if it was not the case, then $\mathcal{N}(j) = \{\ell\} = \mathcal{N}(k)$ which contradicts Condition DN. Therefore, each player

$j \in \mathcal{N}(\ell) \setminus \{k\}$ also appears in the public announcements of at least two distinct players at stage $t+1$. Finally, player k appears in his own public announcement only. As a conclusion, $X^i_{t+1} = \{k, \ell\}$ for every player i in N. □

3.8.5 Proof of Lemma 3.5.4

Proof. Take a connected network G that satisfies Condition DN, and suppose that Assumption P holds. Take a player k in N and assume that there is no action deviation at stage t.

Assume first that $\mathcal{N}(k) = \{\ell\}$ and that k deviates in communication at stage $t+1$. If $m^k_{t+1} \neq m^k_{t+1}(N)$ (k does not make a public announcement to all players), then the message of player k is ignored. Assume now that $m^k_{t+1} = m^k_{t+1}(N)$ and $\{N^{-k}, \mathcal{N}(k) \cup \{k\}\} \notin m^k_{t+1}(N)$. Under unilateral deviations, all other players $j \neq k$ announce $\mathcal{N}(j) \cup \{j\}$ publicly. Two cases are then possible. Either $N \setminus \mathcal{N}(k) \notin m^k_{t+1}(N)$ and $m^k_{t+1}(N)$ is not taken into account under ϕ^* since it implies that player k's message is different from the kinds of messages regarded during the communication protocol. Then, $X^i_{t+1} = \emptyset$ for every player i in N. Or, $N \setminus \mathcal{N}(k) \in m^k_{t+1}(N)$. Player k is then cleared by all players since his name is in the public announcements of k and ℓ. Player ℓ is also cleared by all players because his name is in at least two public announcements among his own and his other neighbor than k (since $n \geq 3$ and G is connected, $\sharp\mathcal{N}(k) \geq 2$). Each other player i is cleared by all players since his name appears in the public announcements of at least two players among him and his neighbors (each has at least one neighbor since G is connected). Hence, $X^i_{t+1} = \emptyset$ for every player i in N.

Assume now that there exists k' such that $\mathcal{N}(k') = k$, and that k deviates in communication at stage $t+1$. If $m^k_{t+1} \neq m^k_{t+1}(N)$, then the message of player k is ignored. Assume now that $m^k_{t+1} = m^k_{t+1}(N)$ and that $\{N \setminus \mathcal{N}(k), \mathcal{N}(k) \cup \{k\}\} \notin m^k_{t+1}(N)$. Under unilateral deviations, all other players $j \neq k$ announce $\mathcal{N}(j) \cup \{j\}$ publicly to all players. Two cases are possible. Either $N^{-k} \notin m^k_{t+1}(N)$ and player k's public announcement is not taken into account as before. Or $N^{-k} \in m^k_{t+1}(N)$, in which case it is obvious that all players are cleared (recall that player k has at least two neighbors). In any case, $X^i_{t+1} = \emptyset$ for each player i in N.

Finally, assume $\sharp\mathcal{N}(k) \geq 2$, $\sharp\mathcal{N}(j) \geq 2$ for every $j \in \mathcal{N}(k)$, and that k deviates in communication at stage $t+1$. If $m^k_{t+1} \neq m^k_{t+1}(N)$, then player k's message is ignored. Assume now $m^k_{t+1} = m^k_{t+1}(N)$ and $\{\mathcal{N}(k) \cup \{k\}\} \notin m^k_{t+1}(N)$. Under unilateral deviations, all other players $j \neq k$ announce $\mathcal{N}(j) \cup \{j\}$ publicly to all players. Three cases are possible. Assume first that $N^{-k} \in m^k_{t+1}(N)$. Then all players are cleared by everybody at stage $t+1$, since player k has at least two neighbors. Second, assume $N \setminus \mathcal{N}(k) \in m^k_{t+1}(N)$. Since player k's neighbors have more than two neighbors, they are cleared by all players at stage $t+1$. Obviously, so are players other than k's neighbors. Third, assume $\{N^{-k}, N \setminus \mathcal{N}(k)\} \notin m^k_{t+1}(N)$. As before, m^k_{t+1} is not taken into account. Therefore, in any case, $X^i_{t+1} = \emptyset$ for each player i in N. □

Sequential rationality, local monitoring, private or local communication

4

I consider repeated games with local monitoring: each player observes his neighbors' moves only. Hence, monitoring is private and imperfect. Two models of communication are studied. In the first one, communication is global and private: each player can send costless messages to all other players, and he can send different messages to different players.The solution concept is belief-free equilibrium. In this case, a folk theorem holds if and only if each player has two neighbors. In the second model, communication is local and public: communication is restricted to neighbors, and each player sends the same message to each of his neighbors at each stage. Here, both communication and monitoring structures are given by the network. The solution concept is perfect Bayesian equilibrium. In the four-player case, a folk theorem holds if and only if the network is 2-connected. Some examples are given for games with more than four players.

4.1 Introduction

Many papers on folk theorems with imperfect monitoring consider global and public communication (Ben-Porath and Kahneman, [5]; Compte, [8]; Kandori and Matsushima, [26]; Obara, [35]; etc): players can send messages to all their opponents, and they are constrained to send the same message to all of them. However, it seems natural to consider other kinds of communication. In this chapter, I study two models of communication that are distinct from global and public communication: in the first one, communication is *private*: each player can send different messages to different players; in the second, communication is *local*: each player can communicate only with a subset of players, called the neighbors.

I study these models of communication in the context of repeated games with local monitoring: each player observes his neighbors' moves only. This is modeled by a network: nodes represent players, edges link neighbors. The monitoring structure is represented by the network: monitoring is private and local. In addition, players can send costless messages at each stage. This paper addresses the following question: for which networks does a *full* folk theorem hold, *i.e.* under which conditions are *all* feasible, strictly individually rational payoffs are equilibrium payoffs in the repeated game with low discounting?

In my first model, I study global and private communication: at each stage, each player can send costless messages to all other players (*i.e.* communication is global), and each player can send different messages to different players (*i.e.* communication is private). I use a refinement of sequential equilibrium, namely belief-free equilibrium, which requires sequential rationality for all private beliefs. The main result is that a folk theorem holds if and only if each player has at least two neighbors, that is each player's action is monitored by at least two other players.

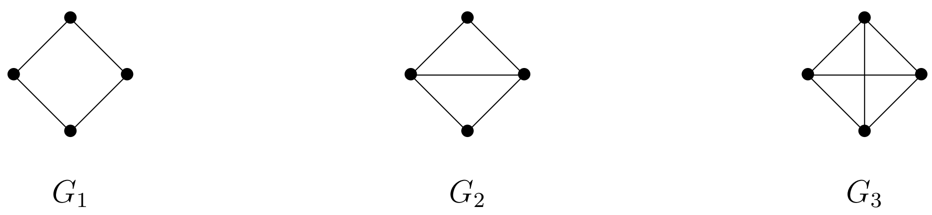
In my second model, I study local and public communication: at each stage, each player can send costless messages only to his neighbors (*i.e.* communication is local), and each player is constrained to send the same message to all his neighbors (*i.e.* communication is public). Hence, communication is restricted by the network structure. The solution concept is perfect Bayesian equilibrium. In the four-player case, the main result is that a folk theorem holds if and only if the network is 2-connected. With more than four players, I exhibit some preliminary results and examples.

Application An application of interest is a partnership game (see [36]). Consider a partnership in which production is conducted by a team whose goal is to maintain a certain level of individual effort by its members. Each member's effort is observable only by his direct colleagues (called the neighbors), and there is moral hazard (effort is costly). If a member is reported by his direct colleagues, the group can punish him by reducing his share in the total profit, which raises other members' shares.

On the one hand, if agents can communicate privately with each others, then the first result (Theorem 4.3.1) shows that cooperation can be achieved for every payoff function if and only if each member's effort is observable by at least two direct colleagues.

On the other hand, agents may not be able to communicate with all the members in the group. If agents can communicate publicly but only with their direct colleagues, I prove in the four-player case (Theorem 4.4.1) that coordination can be sustained if and only if the network is 2-connected, that is if the team's structure takes one of the following forms (see Section 4.4.1):

FIGURE 4.1:



Related literature. The folk theorem was originally established for Nash equilibria ([3, 15, 41, 42]), and extended by Fudenberg and Maskin ([15, 16]) to subgame-perfect equilibria. A key assumption is perfect monitoring. Lots of papers on folk theorems with imperfect monitoring have focused on imperfect public monitoring (see [14]). The model of collusion by Green and Porter ([21]) also considers imperfect public monitoring in that the market price serves as a commonly observable signal. In the undiscounted case, Lehrer (see [31, 32]) provides a fairly comprehensive study of the equilibrium payoff set for two-player games with imperfect private monitoring. With more than two players, the difficulty relies on the necessity for the players to coordinate their behavior to punish potential deviators. Under discounting, as assumed here, much less is known. There is a large recent literature on imperfect private monitoring and belief-free equilibria (see [33] for a general survey). Fudenberg and Levine ([13]) establish a folk theorem with imperfect private monitoring without explicit communication. They consider private random signals induced by the action profile of all other players.

With public and global communication, Compte ([8]), Kandori and Matsushima ([26]), and Obara ([35]), provide sufficient conditions for a folk theorem to hold. Closer to my setting, Ben-Porath and Kahneman ([5]) establish a folk theorem for the case in which each player observes his neighbors' moves. However, they maintain the assumption of public and global communication.

With public and local communication, Renault and Tomala ([37]), and Tomala ([44]), study repeated games with the same signals as here (*i.e.* each player observes his neighbors' moves). Yet, they do not impose sequential rationality. In [39] and [40], Renault and Tomala study local communication. The aim of both papers is to transmit information along a network (with a single possible deviator in [39], extended to k deviators in [40]). Their techniques prove useful for the case of public and local communication studied here. However, they do not apply directly here, since they assume that the set of deviators is fixed through time. On the contrary, since I impose sequential rationality here, all players may deviate unilaterally.

Finally, there is a vast literature on information transmission in networks which consider *broadcast* (*i.e.* public) or *unicast* (*i.e.* private) communication along networks (see for instance [10]). However, with very few exceptions (as [40] mentioned above), both rationality and equilibrium are ignored. Deviant agents are not supposed to respond to incentives. The focus is on properties of communication protocols, such as reliability, security, etc.

The paper is organized as follows. The setup, and the different models of communication, are introduced in Section 4.2. In Section 4.3, I consider private and global communication. Section 4.4 is devoted to local and public commutation. Finally, I raise open questions in Section 4.5.

4.2 The setup

4.2.1 Preliminaries

Consider a repeated game described by the following data:

- a finite set $N = \{1, \ldots, n\}$ of players ($n \geq 2$);
- For each player $i \in N$, a non-empty finite set A^i of actions (with $\sharp A^i \geq 2$). Denote $A = \prod_{i \in N} A^i$.
- An undirected graph $G = (N, E)$ in which the nodes are the players and $E \subseteq N \times N$ is a set of links. Let $\mathcal{N}(i) = \{j \neq i : ij \in E\}$ be the set of neighbors of player i. Since G is undirected, the following holds: $i \in \mathcal{N}(j) \Leftrightarrow j \in \mathcal{N}(i)$.
- A payoff function for each player i in N: $g^i : \prod_{j \in N} A^j \to \mathbb{R}$.
- Finally, a non-empty finite set M^i of player i's messages. The specification of the set M^i depends on the communication structure, hence is described in the following sections. In particular, M^i may be finite or infinite.

I use the following notations: $A^{\mathcal{N}(i)} = \prod_{j \in \mathcal{N}(i)} A^j$, $N^{-i} = N \backslash \{i\}$, and $g = (g^1, \ldots, g^n)$ denote the payoff vector. The repeated game unfolds as follows. At every stage $t \in \mathbb{N}^*$:

(i) simultaneously, players choose actions in their action sets, and send costless messages to other players. Several models of communication are studied in the following sections depending on the nature and on the structure of communication (private or public, local or global). For each player i in N, let $m_t^i(j)$ be the private message player i sends to player j at stage t if player i is able to communicate with player j.

(ii) Let $a_t = (a_t^i)$ be the action profile at stage t. At the end of stage t, each player $i \in N$ observes his neighbors' moves $(a^j)_{j\in\mathcal{N}(i)}$.

Hence, the monitoring possibilities are given by the network G. In addition, I assume perfect recall, and that the whole description of the game is common knowledge. For each stage t, denote by H_t^i the set of private histories of player i up to stage t, that is $H_t^i = (A^i \times (M^i)^{N^{-i}} \times (M^j)_{j\in N^{-i}} \times \{g^i\} \times A^{\mathcal{N}(i)})^t$, where $\{g^i\}$ is the range of g^i (H_0^i is a singleton). An element of h_t^i is called an i-history of length t. An *action strategy* for player i is denoted by $\sigma^i = (\sigma_t^i)_{t\geq 1}$ where for each stage t, σ_t^i is a mapping from H_{t-1}^i to $\Delta(A^i)$ (where $\Delta(A^i)$ denotes the set of probability distributions over A^i). A *communication strategy* for player i is denoted by $\phi^i = (\phi_t^i)_{t\geq 1}$ where for each stage t, ϕ_t^i is a mapping from H_{t-1}^i to $\Delta((M^i)^{N^{-i}})$. Each player can deviate from σ^i or from ϕ^i, henceforth I shall distinguish between action and communication deviations accordingly. I call a *behavior strategy* of a player i the pair (σ^i, ϕ^i). Let Σ^i be the set of action strategies of player i and Φ^i his set of communication strategies. I denote by $\sigma = (\sigma^i)_{i\in N} \in \prod_{i\in N} \Sigma^i$ the joint action strategy of the players and by $\phi = (\phi^i)_{i\in N} \in \prod_{i\in N} \Phi^i$ their joint communication strategy. Let H_t be the set of histories of length t that consists of the sequences of actions, payoffs and messages for t stages. Let H_∞ be the set of all possible infinite histories. A profile (σ, ϕ) defines a probability distribution, $\mathbb{P}_{\sigma,\phi}$, over the set of plays H_∞, and I denote $\mathbb{E}_{\sigma,\phi}$ the corresponding expectation. I study the discounted infinitely repeated game, in which the overall payoff function of each player i in N is the expected sum of discounted payoffs. That is, for each player i in N:

$$\gamma_\delta^i(\sigma, \phi) = \mathbb{E}_{\sigma,\phi}\left[(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} g_t^i(a_t^i, a_t^{\mathcal{N}(i)})\right],$$

where $\delta \in [0, 1)$ is a common discount factor.

Depending on the model considered, I will study belief-free equilibria or perfect Bayesian equilibria (henceforth PBE) of the discounted repeated game (see definitions in Section 1.1.6). The definitions are given in Chapter 1 (see Section 1.1.6). Let $\Gamma_\delta(G, g)$ be the δ-discounted game, and $E_\delta^B(G, g)$ (respectively $E_\delta^{PBE}(G, g)$) its associated set of belief-free equilibrium payoffs (respectively PBE payoffs). For each $a \in A$, let $g(a) = (g^1(a), \ldots, g^n(a))$, and $g(A) = \{g(a) : a \in A\}$. The set of feasible payoffs is the convex hull of $g(A)$, denoted by $\operatorname{co} g(A)$ be the convex hull of $g(A)$.

The (independent) minmax level of player i is defined by:

$$\underline{v}^i = \min_{(x^j)_{j\in N^{-i}}\in\prod_{j\in N^{-i}}\Delta(A^j)} \max_{x^i\in\Delta(A^i)} g^i(x^i,(x^j)_{j\in N^{-i}}).$$

Henceforth, I shall normalize the payoffs of the game such that $(\underline{v}^1,\ldots,\underline{v}^n) = (0,\ldots,0)$. I denote by $IR^*(G,g) = \{g = (g^1,\ldots,g^n) \in \mathbb{R}^n \ : \ \forall i \in N,\ g^i > 0\}$ the set of strictly individually rational payoffs. Finally, let $V^* = \mathrm{co}\, g(A) \cap IR^*(G,g)$ be the set of feasible and strictly individually rational payoffs.

The aim of this paper is, for each model of communication considered, to characterize the networks G for which a folk theorem holds, that is: each feasible and strictly individually rational payoff is an equilibrium (belief-free or PBE) payoff of the repeated game, for a discount factor close enough to one. In the next section, I define different models of communication.

4.2.2 Communication structures

The following definitions specify different communication structures regarding whom each player is allowed to talk to.

Definition 4.2.1. Global communication.
Communication is *global* in the game $\Gamma_\delta(G,g)$ if each player can communicate with all players in N.

Definition 4.2.2. Local communication.
Communication is *local* in the game $\Gamma_\delta(G,g)$ if each player $i \in N$ can communicate only with his neighbors $\mathcal{N}(i) \subset N$ in G.

I now define different communication structures depending on the nature of the communication.

Definition 4.2.3. Public versus private communication.
Assume that each player $i \in N$ can communicate with a set $\mathcal{I}^i$ of players, $\mathcal{I}^i \subseteq N$, in the game $\Gamma_\delta(G,g)$.[1] Communication is *public* if at each stage, each player i is constrained to send the same message to each player in $\mathcal{I}^i$:

$$\forall i \in N,\ \forall j \in \mathcal{I}^i,\ m^i_t(j) = m^i_t.$$

On the contrary, communication is *private* if each player i in N is allowed to send different messages to different players in $\mathcal{I}^i$.

In Section 4.3, I assume that communication is private and global. On the contrary, communication is public and local in Section 4.4.

1. Communication can be either local, $\mathcal{I}^i = \mathcal{N}(i)$, or global, $\mathcal{I}^i = N$.

4.3 Global and private communication

In this section, I assume that communication is private and global. The main result is the following.

Theorem 4.3.1. *Under private and global communication, the following statements are equivalent:*

(1) For any payoff function g such that the interior of V^* is nonempty, and for any vector v in V^*, there exists $\bar{\delta} \in (0,1)$ such that for all $\delta \in (\bar{\delta}, 1)$, v is a belief-free equilibrium vector payoff of the δ-discounted game.[2]

(2) The network G satisfies Condition 2-NEIGHBORS (henceforth, Condition 2N): each player i in N has at least two neighbors.

Ben-Porath and Kahneman (1996) have already proved that Condition 2N is necessary for a Nash folk theorem to hold (page 291 in [5]). Actually, they consider global and public communication instead, but one can easily see that their proof extends to private communication. The main ideas of the proof of Ben-Porath and Kahneman are the following.[3] If a player i has a single neighbor j, then only player j is able to monitor player i. Assume that a Nash equilibrium exists, and whose payoff is given by v. Ben-Porath and Kahneman then construct a particular payoff function for which either (a) player i has an incentive to always deviate in action; or (b) player j has an incentive to deviate in communication by falsely reporting that i deviates, in order to induce all the players (including himself) to minmax player i. Their construction is such that it is not possible to minmax both players i and j. This contradicts the fact that a Nash equilibrium exists. Moreover, the proof does not rely on public communication. Indeed, player j's communication deviation consists in using the equilibrium strategy but replacing his true private history by a fictitious one in which player i always deviates in action. Hence, if the equilibrium strategy uses public communication, then player j's deviation also uses public communication. On the other hand, if it uses private communication, then player j's deviation also uses private communication.

Ben-Porath and Kahneman (1996) prove that Condition 2N is also sufficient for a folk theorem to hold with global and public communication, and sequential equilibrium as solution concept. However, their proof crucially relies on the possibility for the players to make public announcements. I now prove that Condition 2N is also sufficient for a folk theorem to hold with global and private communication,

2. The assumption that $\operatorname{int} V^* \neq \emptyset$ can be relaxed: it could be replaced by the non-equivalent utilities condition introduced by Abreu, Dutta and Smith ([1]). The proof is a straightforward adaptation of [1].

3. Similar arguments can be found in Sections 2.3 and 3.6.

and the solution concept is a belief-free equilibrium. Hence, public communication is not necessary in this setup, provided that communication is global.

Proof of Theorem 4.3.1 As explained above, Ben-Porath and Kahneman have already proved that (2) implies (1) in Theorem 4.3.1. Assume now that each player has at least two neighbors. In other words, each player is monitored by at least two players. Take a payoff function g such that $\operatorname{int} V^* \neq \emptyset$, and a vector $v = (v^1, \dots, v^n)$ in V^*. I construct a belief-free equilibrium of the repeated game (σ^*, ϕ^*) with payoff v for a discount factor close enough to one.

Equilibrium path. For each player i in N and each stage $t > 0$, choose $\bar{a}^i_t \in A^i$ such that

$$(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} g^i_t(\bar{a}_t) = v^i.$$

This is possible when $\delta \geq 1 - \frac{1}{n}$ (existence is proved by Sorin, Proposition 4 p.151 in [42]). Moreover, Fudenberg and Maskin (1991) prove that for every $\epsilon > 0$, there exists $\delta_\epsilon < 1$ such that for all $\delta \geq \delta_\epsilon$ and every $v \in V^*$ such that $v^i \geq v$ for all i, the deterministic sequence of pure actions $\bar{a}_t$ can be constructed so that the continuation payoffs at each stage are within ϵ of v (Lemma 2 p. 432 in [16]).[4]

Communication strategy. Fix a player $i \in N$. The message that player i sends at each stage $t+1$ is a function of his observation at stage t, computed as follows:[5]

- if player i observes the action deviation of some neighbor $k \in \mathcal{N}(i)$ at stage t, then player i sends $k \in m^i_{t+1}(j)$ to each player $j \in N$ at stage $t+1$;[6]
- for each neighbor $\ell \in \mathcal{N}(i)$, if player i observes that ℓ does not deviate in action at stage t, then $\ell \notin m^i_{t+1}(j)$ for every $j \in N$;
- if player i deviates in action at stage t, he should confess his deviation to all players at stage $t+1$, hence $i \in m^i_{t+1}(j)$ for each player $j \in N$;
- if player i did not report an action deviation of some player k (k being his neighbor or himself) at some stage $t' < t$, then he should send $(t+1-t', k) \in m^i_{t+1}(j)$ to each player $j \in N$.

For each stage $t \geq 1$, denote by $X^i_{t+1}(t) \subseteq N$ player i's set of suspected players at stage $t+1$ regarding a possible deviation at stage t. At every stage $t+1$ of the game, each player i in N computes his set of suspects $X^i_{t+1}(t)$ as follows. For every player

4. If it was not the case, some player would prefer to deviate from $\bar{a}$, even if doing so caused his opponents to minmax him thereafter.

5. Recall that, at each stage, messages are sent before observing stage payoffs. This assumption is not required: with a slight modification, the construction of the strategy is still valid for the case in which messages are sent after the observation of stage payoffs.

6. Under multilateral deviations, there can be several neighbors of player i who deviate, hence several neighbors may appear in player i's messages.

$k \in N$, if there exist at least two players k^1 and k^2 such that $k^1 \neq k^2$, $k \in m_{t+1}^{k^1}(i)$ and $k \in m_{t+1}^{k^2}(i)$, then $k \in X_{t+1}^i(t)$ (*i.e.* k is identified as a deviator by player i). Otherwise, $k \notin X_{t+1}^i(t)$ (*i.e.* k is cleared by player i). (Notice that $k^1 = k$ or $k^2 = k$ are possible, in the case in which player k confesses a past deviation.)

With this rule, the following lemma holds.

Lemma 4.3.2. *Assume that each player has at least two neighbors. The following statements hold:*

(i) If some player k deviates in action at some stage t, and if all players follow ϕ^* (except possibly a unique deviator) at stage $t+1$, then $k \in X_{t+1}^i(t)$ for each player i in N.

(ii) If a player j does not deviate in action at some stage t, and if all players follow ϕ^* (except possibly a unique deviator) at stage $t+1$, then $j \notin X_{t+1}^i(t) = \{k\}$ for each player i in N.

Proof of Lemma 4.3.2 Assume that each player has at least two neighbors. Take a player $i \in N$.

(i) Consider first the situation in which some player k deviates in action at stage t, that is $a_t^k \neq a_t^{*k}$ (if the play follows the equilibrium path, then $a_t^k \neq \bar{a}_t^k$). By assumption, player k has at least two neighbors k^1 and k^2. At stage $t+1$, it should be the case that $k \in m_{t+1}^k(i)$, $k \in m_{t+1}^{k^1}(i)$, and $k \in m_{t+1}^{k^2}(i)$. Under unilateral deviations, at most one player deviates at stage $t+1$, hence $k \in X_{t+1}^i(t)$ for each player $i \in N$.

(ii) Consider now the situation in which some player j does not deviate at stage t. Player j has at least two neighbors, called j^1 and j^2. At stage $t+1$, it should be the case that $j \notin m_{t+1}^j(i)$, $j \notin m_{t+1}^{j^1}(i)$ and $j \notin m_{t+1}^{j^2}(i)$ for each player $i \in N$. Under unilateral deviations, at most one player deviates at stage $t+1$, hence $j \notin X_{t+1}^i(t)$ for each player $i \in N$.

□

The following corollary is a straightforward application of Lemma 4.3.2.

Corollary 4.3.3. *Assume that each player has at least two neighbors. The following statements hold:*

(i) Under unilateral deviations from σ^*, if some player k deviates in action at some stage t, and if all players follow ϕ^* (except possibly a unique deviator), then $X_{t+1}^i(t) = \{k\}$ for each player i in N.

(ii) If there is no action deviation at stage t, and if all players follow ϕ^* (except possibly a unique deviator), then $X_{t+1}^i(t) = \varnothing$ for each player i in N.

In the same way of $X_{t+1}^i(t)$ defined above for every player $i \in N$ and every stage $t \geq 1$, define, for every player i in N and all stages t and s such that $s < t$, the set

$X^i_{t+1}(s)$ of suspected players regarding an alleged action deviation at stage s. For each player $i \in N$ and all stages t and s such that $s < t$, the set $X^i_{t+1}(s)$ is computed as follows. For every player $k \in N$, if there exist at least two players k^1 and k^2 such that the following properties hold:

– $k^1 \neq k^2$;
– and, either $k \in m^{k^1}_{s+1}(i)$ or $(t+1-s, k) \in m^{k^1}_{t+1}(i)$;
– and, either $k \in m^{k^2}_{s+1}(i)$ or $(t+1-s, k) \in m^{k^2}_{t+1}(i)$;

then $k \in X^i_{t+1}(s)$ (*i.e.* k is identified as a deviator at stage s by player i). Otherwise, $k \notin X^i_{t+1}(s)$ (*i.e.* k is cleared by player i regarding an alleged deviation at stage s). We then have the following lemma (the proof is a straightforward application of the proof of Lemma 4.3.2).

Lemma 4.3.4. *For every player i and all stages t and s such that $s < t$, if there exists a set $\tilde{N} \subseteq N$ such that all players in $\tilde{N}$ have deviated in action at stage s, and if all player (except possibly a unique deviator) follow ϕ^* at stage t, then $\tilde{N} \subseteq X^i_{t+1}(s)$.*

The previous lemma shows that, after any history h^i_{t-1} containing multilateral deviations from σ^*, and if all players (except possibly a unique deviator) follow ϕ^* at stage t, then all players know at stage $t+1$ that there was multilateral deviations at some stage before t.

The strategy σ^* is the following in case of deviation(s). For each player $i \in N$ and every stage $t \geq 1$:

(i) if $t \geq 2$, and if there exists some stage $s < t-1$ such that $X^i_t(s) = \tilde{N}$ with $\tilde{N} \subseteq N$ and $\sharp\tilde{N} \geq 2$, then player i plays his coordinate for an arbitrary Nash equilibrium of the one-shot game forever;

(ii) otherwise:

  (a) if $X^i_{t+1}(t) = \emptyset$, then player i keeps playing the action prescribed by σ^{*i} according to the phase in which the game is;

  (b) if $X^i_{t+1}(t) = \tilde{N}$ with $\tilde{N} \subseteq N$ and $\sharp\tilde{N} \geq 2$, then player i plays his coordinate for an arbitrary Nash equilibrium of the one-shot game forever;

  (c) if $X^i_{t+1}(t) = \{k\}$ for some player $k \in N$, then player i starts the punishment phase in order to minmax player k (this phase is described in the next paragraph).

The description of the profile of communication strategies ϕ^* is almost complete. In addition, players also make announcements during the punishment phase. This last part of the communication strategy is described jointly with the specification of the action strategy during the punishment phase.

Punishment phase. Now, for each player $i \in N$, if point (c) above happens, then player i enters a punishment phase at stage $t+2$, whose goal is to minmax player k. Each player first plays according to his minmax strategy against k, denoted $(P^i(k))$. Denote by $P(k) = (P^i(k))_{i \in N^{-k}}$ the profile of minmax strategies against player k. For any strategy (σ^k, ψ^k) of player k:

$$\begin{aligned} \gamma_\delta^k(\sigma^k, P(k), \psi^k, (\phi^i)_{i \in N^{-k}}) &\leq \sum_{t=1}^{\infty} (1-\delta)\delta^{t-1} \underline{v}^k \\ &\leq 0, \end{aligned}$$

where ϕ^i stands for any communication strategy of each minmaxing player $i \in N^{-k}$. Player k is compelled to play some pure action $P^k(k)$ at each stage during his punishment.[7]

In addition, at each stage s during the punishment phase, each player i in N, including player k, keeps sending messages regarding alleged action deviations at stage $s-1$. In particular, a player $j \in N^{-k}$ who chooses an action which is not in the support of his minmax strategy is identified as a deviator. Also, it is an action deviation for player k to play a pure action different from $P^k(k)$. Players also update their sets of suspects at each stage as previously. The punishment phase lasts $T(\delta)$ stages (the length $T(\delta)$ has to be adapted), and at stage $t+2+T(\delta)$, each player enters a reward phase.

The reward phase. The goal of this reward phase is twofold:

(i) In order to provide each minmaxing player, who is not minmaxed himself, an incentive to play his minmax strategy in phase III, a reward is added to the form of an additional bonus $\rho > 0$ to his average payoff. If the discount factor is large enough, the potential loss during the punishment is compensated by the future bonus.

(ii) Moreover, to induce each minmaxing player to draw his pure actions according to the right distribution of his minmax strategy, I add a phase so that his payoff in the continuation game varies with his realized payoff in a way that makes him indifferent between the pure actions in the support of his minmax strategy. As in Fudenberg and Maskin ([15, 16]), it is convenient to require that any feasible and strictly individually rational continuation payoff must be exactly attained. Otherwise, a minmaxing player might not be exactly indifferent between his pure actions in the support of his minmax strategy.

7. Otherwise, consider the situation in which some player $i \neq k$ deviates by reporting to player k the sequence of pure actions of his minmax strategy. If this minmax strategy were mixed, player k might benefit from this information by randomizing among his pure actions. However, since player k is restricted to play the same pure action during his punishment, player i's deviation has no impact on player k's punishment.

The possibility of providing such rewards relies on the full dimensionality of the payoff set (recall that $\operatorname{int} V^* \neq \emptyset$). In addition, it is convenient to require that all players know the sequences of pure actions played by each minmaxing player during the punishment phase to handle point (ii). For that reason, at the first stage of the reward phase, each player $i \in N$ sends to all players all the sequences of pure actions that he and his neighbors $j \in \mathcal{N}(i)$ played (recall that each player observes his neighbors' moves). The action of player i at this stage is arbitrary. For each player $j \in N$, at least three players know his sequence of pure actions: his (two or more) neighbors and himself. Hence under unilateral deviations, a simple majority rule implies that, at the end of the first stage of the reward phase, all players know the sequences of actions played by all players. Finally, the construction of the reward phase, as well as the specification of $\mu(\delta)$, is the same as in Fudenberg and Maskin ([16]). Similar arguments can be found in Chapter 3 (Section 3.5.2.4). Moreover, the proof that the strategy constructed is a belief-free equilibrium is a straightforward application of Fudenberg and Maskin ([16]; see also similar ideas in the proof of Proposition 3.5.11 in Section 3.5.3). □

Remark 4.3.5. For the strategy constructed in the proof of Theorem 4.3.1, it suffices that each player has a finite set of messages M^i. The strategy only requires that the players send sets of suspected players, and the sequences of actions played during the punishment phase, whose length is finite.

Remark 4.3.6. As in Chapter 3, the equilibrium strategy constructed is weakly dominated, since a player who deviates in action at some stage t is required to confess at stage $t+1$. Nevertheless, it is possible to give strict incentives for a deviator to confess by giving him a reward thereafter: indeed, the reward phase can be such that a player who deviates, and confesses privately to all players his deviation after that, obtains a bonus of $\frac{\rho}{2} > 0$ after the punishment phase. Hence, a player who deviates has a strict incentive to confess now. In addition, this bonus is strictly lower than ρ, otherwise a minmaxing player might have an incentive to deviate. Indeed, a minmaxing player's realized payoff during the punishment phase may be less than if he would get his minmax level instead. However, a minmaxing player has still no incentive to deviate, otherwise he would loose $\frac{\rho}{2}$ at each stage during the reward phase.

Remark 4.3.7. Condition 2N of Theorem 4.3.1 is also necessary and sufficient for a folk theorem to hold if I consider uniform (sequential) equilibria of the undiscounted repeated game (see [13] and [43]). Namely, every feasible and individually rational payoff is a uniform (sequential) equilibrium payoff for every payoff function g if and only if each player has at least two neighbors. The proof is a straightforward application of the proof of Theorem 4.3.1.

4.4 Public and local communication

Contrary to the previous section, I assume now that communication is public and local (communication is restricted by the network structure). Hence at each stage t, every player i in N sends the same message m_t^i to all his neighbors. In the next section, I study the four-player case. In Section 4.4.2, I give some examples with more than four players. Finally, I develop conjectures and raise open issues in Section 4.4.3.

4.4.1 Four-player case

I assume $n = 4$ throughout this section, and I prove the following theorem.

Theorem 4.4.1. *Assume $n = 4$, and that communication is public and local. Then, the following statements are equivalent.*

(1) The network G is 2-connected.

(2) For any payoff function g such that the interior of V^* is nonempty, for every vector v in V^*, there exists $\bar{\delta} \in (0,1)$ such that for all $\delta \in (\bar{\delta}, 1)$, v is a PBE vector payoff of the δ-discounted game.

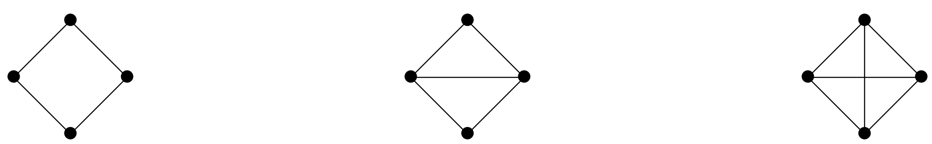
Renault and Tomala ([37]) show that 2-connectedness is necessary for a Nash folk theorem to hold (for any $n \geq 3$).

Remark 4.4.2. Theorem 4.4.1 is also true for $n = 3$ since the unique 2-connected network with three nodes is the complete graph. (Notice that the complete-network case reduces to perfect monitoring.) The 2-player case is also trivial: a folk theorem holds if and only if the graph is connected.

Before proving Theorem 4.4.1, I set out the following lemma.

Lemma 4.4.3. *Assume $n = 4$. Then, the network G is 2-connected if and only if $G \in \{G_1, G_2, G_3\}$ where:*

FIGURE 4.2:



G_1 G_2 G_3

Proof. First, it is obvious that G_1, G_2 and G_3 are 2-connected. Assume now $n = 4$, let $N = \{1, 2, 3, 4\}$. Furthermore, suppose that G is 2-connected, hence each player has at least two neighbors. Two cases are possible: either (i) each player has exactly two neighbors, or (ii) there exists a player who has three neighbors.

(i) Consider the first case in which each player has two neighbors. Without loss of generality, let $\mathcal{N}(1) = \{2, 4\}$. It is neither possible to have $2 \in \mathcal{N}(4)$ nor $4 \in \mathcal{N}(2)$, since it would imply that player 3 has no neighbor. Hence, $\mathcal{N}(3) = \{2, 4\}$ and the unique possible network is G_1.

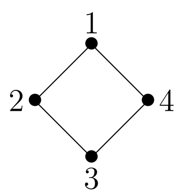
(ii) Assume now that there exists a player who has three neighbors. Notice that $\sum_{i \in N} \sharp\mathcal{N}(i) = 2 \times \sharp E$, where E is the set of edges in G. Hence, the total number of neighbors in G is even. Therefore, either all players have three neighbors and G is complete (network G_3); or, exactly two players have three neighbors, say players 1 and 3. In the latter case, it must be the case that: $\mathcal{N}(1) = \{2, 3, 4\}$ and $\mathcal{N}(3) = \{2, 3, 4\}$. The unique possible network is then G_2. □

I now prove Theorem 4.4.1. As mentioned before, I only have to prove that (1) implies (2) in Theorem 4.4.1. Assume that the network G is 2-connected. By Lemma 4.4.3, I need to prove that a folk theorem holds for the networks G_1, G_2 and G_3. The case of G_3 reduces to perfect monitoring, hence the proofs of Fudenberg and Maskin ([15, 16]) apply (and no communication is required). In the next sections, I prove that a folk theorem holds for the networks G_1 (Section 4.4.1.1) and G_2 (Section 4.4.1.2).

4.4.1.1 Each player has exactly two neighbors

Throughout this section, assume $n = 4$, and that communication is public and local. Furthermore, suppose that the game is played on the following network (G_1):

FIGURE 4.3:



G_1

Now take a payoff function g such that $\operatorname{int} V^*$ is non-empty, and a vector $v = (v^1, \ldots, v^n)$ in V^*. To prove a folk theorem, I construct a PBE of the repeated game (σ^*, ϕ^*) with payoff v for a discount factor close enough to one. More precisely, I construct as in Chapter 3 a restriction of the PBE strategy to a particular class of private histories; namely, the histories along which only unilateral deviations from (σ^*, ϕ^*), if any, have taken place. In addition, the construction has the property that, after such histories, the specified play is optimal no matter what beliefs a player holds about their opponents' play, provided that the beliefs are such that: for every player $i \in N$, if player i observes a private history compatible with a history along which no deviation has taken place (respectively along which only unilateral deviations have taken place), then player i believes that no deviation has taken place (respectively only unilateral deviations have taken place). Plainly, this suffices to ensure optimality. Given that play after other histories (*i.e.*, histories that involve simultaneous deviations) is irrelevant, the partial strategy and beliefs that I define can be completed in any arbitrary fashion. Details are given in Section 3.5.2.

Formally, I denote by $H_t^i(U|(\sigma^*, \phi^*))$ the set of private histories for player i such that: either no deviation (from (σ^*, ϕ^*)) has occurred, or only unilateral deviations have taken place. That is to say, for every history in $H_t^i(U|(\sigma^*, \phi^*))$, no multilateral deviation has occurred. Similarly, denote by $H_t(U|(\sigma^*, \phi^*))$ the set of total histories along which only unilateral deviations, if any, have taken place. I define now, for every history in $H_t(U|(\sigma^*, \phi^*))$, a strategy profile which can be decomposed into four phases. First, there is a stream of pure action profiles that yields the desired payoff. This is how the game starts off and how it unfolds as long as no player deviates. Second, there is a communication phase (the communication protocol previously described) in case of a deviation, whose purpose is to inform the deviator's neighbors of the deviator's identity. Third, there is a punishment phase, and finally, a reward phase.

Phase I: equilibrium path

At each stage t during this phase, each player i should play action $\bar{a}_t^i$ defined as in the proof of Theorem 4.3.1. Moreover, at every period t, player i should send the message $m_t^i = (1, \varnothing, \alpha_t^i)$ to all his neighbors, where α_t^i is uniformly drawn on $[0, 1]$.[8] The number 1 stands for a delay and means "one stage before" (hence stage $t-1$). The part $\varnothing$ of player i's message means that he neither deviated nor detected any action deviation at stage $t-1$.[9] In addition, the number α_t^i can be seen as an encoding key, which is used during the communication phase (phase II).

Phase II: communication phase

This phase aims at identifying the deviator when a deviation occurs. Formally, the

8. Hence, I assume that the message space M^i of player i is uncountable here.

9. Recall that, at each stage, messages are sent before observing stage payoffs. This assumption is not crucial: with a slight modification, the strategy construction is still valid for the case in which messages are sent after the observation of stage payoffs.

strategy of phase II can be seen as a communication protocol: a specification of how players choose their messages, the number of communication rounds and an output rule for each player. Each player $i \in N$ starts the communication protocol every time he detects any kind of deviation from (σ^*, ϕ^*). For instance, when in phase I, player i enters phase II at the end of stage t either if he observes a neighbor's deviation (or if he deviates himself in action), or if he receives a message in M –where M is the set of messages allowed by the protocol (see the construction below)– different from $\bar{m}_t^j = (1, \varnothing, \alpha_t^j)$ from some neighbor $j \in \mathcal{N}(i)$ at stage t (messages that are not in M are disregarded). Players may enter phase II at different stages. Indeed, consider the situation in which there is an action deviation of some player k at stage t. For instance, if $k = 1$, player 1's neighbors start phase II at the end of stage t, whereas player 4 does not start phase 2 before the end of stage $t + 1$. Moreover, players may start a new communication protocol although a previous one has not ended yet. Therefore, there can be several communication protocols running at the same time.

During phase II, players should stick to the action strategy according to the phase in which the play is. For instance, if players are following the equilibrium path at stage t when they enter phase II, they should keep playing $\bar{a}$ when performing the protocol, until a possibly previous protocol ends up and yields to a new phase of the game. This part of the strategy is thereby purely communicative. In what follows, I construct the communication protocol used by player i in phase II, denoted $\tilde{\phi}^i$.

COMMUNICATION PROTOCOL

The message space. All players communicate using the same infinite set M of messages. The messages in M take the following form:

- a delay $s \in \{1, 5\}$, where $T = 5$ is the length of the protocol. The delay s is interpreted as "s stages before" and corresponds to the stage of the alleged deviation.
- An element $x(s)$ in $\{\varnothing\} \cup N$ which represents the name of the alleged (action) deviator "s stages before". Hence, the element $(s, x_t(s))$ at stage s is interpreted as "player $x_t(s)$ deviated in action at stage $t - s$."
- An encoding key α drawn uniformly on $[0, 1]$ (which implies that M is infinite).
- An element of $F(s) = \emptyset \cup \big\{(s', x, \alpha^x) \,:\, s' \in \{0, \dots, s-1\},\, x \in N,\, \alpha^x \in [0, 1]\big\}$. The interpretation of $F(s) = (s', x, \alpha^x)$ sent at stage t is: "player x deviated in communication at stage $t - s'$ when he transmitted information about what happened at stage $t - s$, and the encoding key sent by player x at stage

$t - s'$ was α^x." There are possibly several such messages since several players might deviate in communication at consecutive stages, but the number of such messages is bounded by two (see the construction of the strategy below). The message $\emptyset$ stands for the case in which no communication deviation is detected.

With the previous notations, the set M is the following:

$$M = \Big\{(s, x(s), \alpha, F(s)) \,:\, s \in \{1, \dots, 5\},\ \alpha \in [0,1]\Big\}.$$

If there are several protocols running at the same time, then each player's message is a concatenation of each message in M corresponding to each deviation: different delays correspond to different stages, thus to different protocols. In what follows, I denote by $x_t^i(s)$ (respectively α_t^i, $F_t^i(s)$) the element $x(s)$ (respectively the number $\alpha \in [0,1]$, the element $F(s)$) sent by player i at stage t.

The strategy of player i. When he performs the protocol, player i sends some messages as a function of his observations, as follows:

(i) If player i detects an action deviation of some neighbor $k \in \mathcal{N}(i)$ at stage t, then for each $s \in \{1, \dots, 5\}$, player i's message at stage $t+s$ contains $(s, x_{t+s}^i(s) = k, \alpha_{t+s}^i)$, where α_{t+s}^i is drawn uniformly on $[0,1]$.[10] Moreover, $F_{t+1}^i(1) = F_{t+2}^i(2) = \emptyset$. Three cases are then possible:

- If player i receives $F_{t+2}^k(2) = (1, k^1, \alpha_{t+1}^{k^1})$ at stage $t+2$ from player k (recall that player k is the stage t's deviator), where $k^1 \in \mathcal{N}(k)$ and $k^1 \neq i$, then player i's messages at stages $t+3$, $t+4$ and $t+5$ contain $F_{t+3}^i(3) = (2, k^1, \alpha_{t+1}^{k^1})$, $F_{t+4}^i(4) = (3, k^1, \alpha_{t+1}^{k^1})$ and $F_{t+5}^i(5) = (4, k^1, \alpha_{t+1}^{k^1})$ respectively. In other words, player i transmits the stage $t+1$'s encoding key of player k^1 that he received from player k, after having updating the delay corresponding to stage $t+1$.
- If player i receives $F_{t+2}^k(2) = (1, i, \alpha_{t+1}^i)$ from player k, and if $F_{t+4}^k(4) = (1, k^1, \alpha_{t+3}^{k^1})$ at stage $t+4$, with $k^1 \in \mathcal{N}(k)$ and $k^1 \neq i$, then $F_{t+3}^i(3) = F_{t+4}^i(4) = \emptyset$ and $F_{t+5}^i(5) = (2, k^1, \alpha_{t+3}^{k^1})$.
- Otherwise, $F_{t+s}^i(s) = \emptyset$ for each $s \in 1, \dots, 5$.

(ii) If player i deviates in action at stage t, then for each $s \in \{1, \dots, 5\}$, player i's message at stage $t+s$ contains $(s, x_{t+s}(s) = i, \alpha_{t+s}^i)$, where α_{t+s}^i is drawn uniformly on $[0,1]$. Moreover, $F_{t+1}^i(1) = \emptyset$, and:

- If one of his neighbors, say i^1, does not report stage t's deviation at stage $t+1$, *i.e.* either $m_{t+1}^{i^1} \notin M$ or $x_{t+1}^{i^1}(1) \neq i$, then $F_{t+2}^i(2) = (1, i^1, \alpha_{t+1}^{i^1})$ (recall

10. Notice that player i's message is not complete here, since $F_{t+s}^i(s)$ is not specified yet. Moreover, several communication protocols might be running at the same time.

that player i receives the value $\alpha^{i^1}_{t+1}$ at stage $t+1$) and $F^i_{t+3}(3) = \emptyset$. In addition:

- If player i's neighbor other than i^1, denoted i^2, does not report $F^i_{t+2}(2)$ correctly at stage $t+3$, *i.e.* either $m^{i^2}_{t+3} \notin M$ or $F^{i^2}_{t+3}(3) \neq (2, i^1, \alpha^{i^1}_{t+1})$, then $F^i_{t+4}(4) = (1, i^2, \alpha^{i^2}_{t+3})$ and $F^i_{t+5}(5) = \emptyset$.
- Otherwise, $F^i_{t+4}(4) = \emptyset$ and $F^i_{t+5}(5) = \emptyset$.

• Otherwise, $F^i_{t+2}(2) = F^i_{t+3}(3) = F^i_{t+4}(4) = F^i_{t+5}(5) = \emptyset$.

(iii) Otherwise, *i.e.* if player i does not deviate nor does detect any action deviation at stage t, then for each $s \in \{1, \dots, 5\}$, player i's message at stage $t+s$ contains $= (s, x^i_{t+s} = \varnothing, \alpha^i_{t+s})$, where α^i_{t+s} is drawn uniformly on $[0, 1]$. Moreover, $F^i_{t+1}(1) = \emptyset$, and:

• If there is one of his neighbors, say i^1, such that $m^{i^1}_{t+1} \in M$ and $x^{i^1}_{t+1}(1) = i$ (*i.e.* player i^1 lies about player i deviating in action at stage t), then $F^i_{t+2}(2) = (1, i^1, \alpha^{i^1}_{t+1})$ (recall that player i receives the value $\alpha^{i^1}_{t+1}$ at stage $t+1$) and $F^i_{t+3}(3) = \emptyset$. In addition:

- If player i's neighbor other than i^1, denoted i^2, does not report $F^i_{t+2}(2)$ correctly at stage $t+3$, *i.e.* either $m^{i^2}_{t+3} \notin M$ or $F^{i^2}_{t+3}(3) \neq (2, i^1, \alpha^{i^1}_{t+1})$, then $F^i_{t+4}(4) = (1, i^2, \alpha^{i^2}_{t+3})$ and $F^i_{t+5}(5) = \emptyset$.
- Otherwise, $F^i_{t+4}(4) = \emptyset$ and $F^i_{t+5}(5) = \emptyset$.

• Otherwise, $F^i_{t+2}(2) = F^i_{t+3}(3) = F^i_{t+4}(4) = F^i_{t+5}(5) = \emptyset$.

The output rule. Let $\mathcal{N}(i) = \{i^1, i^2\}$ and $k \in \mathcal{N}(i^1) \cap \mathcal{N}(i^2) \setminus \{i\}$. At each stage t, player i computes his set of suspected players (regarding stage $t-5$), denoted X^i_t, as follows:

• if player i deviates in action at stage $t-5$, then $X^i_t = \{i\}$;

• if player i^1 (respectively player i^2) deviates in action at stage $t-5$, then $X^i_t = \{i^1\}$ (respectively $X^i_t = \{i^2\}$);

• otherwise:

– if $m^{i^1}_{t-5+s} \in M$, $m^{i^2}_{t-5+s} \in M$ and $x^{i^1}_{t-5+s}(s) = x^{i^2}_{t-5+s}(s) = k$ for some stage $t-5+s$ with $s \in \{1, \dots, 5\}$, then $X^i_t = \{k\}$;

- if $m^{i^1}_{t-5+s} \in M$, $m^{i^2}_{t-5+s} \in M$ and $x^{i^1}_{t-5+s}(s) = x^{i^2}_{t-5+s}(s) = \varnothing$ for some stage $t-5-s$ with $s \in \{1, \dots, 5\}$, then $X^i_t = \emptyset$;

- if $m^{i^2}_{t-5+s} \notin M$ and $m^{i^1}_{t-5+s} \in M$, for some stage $t-5-s$ with $s \in \{1, \dots, 5\}$, then $X^i_t = \{x^{i^1}_{t-5+s}(s)\}$ (and symmetrically when exchanging the roles of player i^1 and i^2);

- otherwise:

– if $m^{i^1}_{t-2} \in M$ with $F^{i^1}_{t-2}(3) = (2, i^2, \alpha^{i^2}_{t-4})$ (where $\alpha^{i^2}_{t-4}$ is the correct value of player i^2's encoding key received by player i at stage $t-4$), and if

$m^{i^1}_{t-4} \in M$, then $X^i_t = \{x^{i^1}_{t-4}(s)\}$ (and symetrically when exchanging the roles of player i^1 and i^2);
- if $m^{i^1}_t \in M$ with $F^{i^1}_t(5) = (2, i^2, \alpha^{i^2}_{t-2})$ (where $\alpha^{i^2}_{t-2}$ is the correct value of player i^2's encoding key received by player i at stage $t-2$), and if $m^{i^1}_{t-2} \in M$, then $X^i_t = \{x^{i^1}_{t-2}(s)\}$ (and symetrically when exchanging the roles of player i^1 and i^2);
- otherwise, $X^i_t = \emptyset$ (history incompatible with unilateral deviations).

The number of rounds. The number of rounds of communication is equal to $T = 5$. In the paragraph entitled "The equilibrium property" (page 132), I prove that the following claims hold:

(1) If there is an action deviation of some player k at stage t, then $X^i_{t+T} = \{k\}$ almost surely for each player $i \in N$;

(2) If there is no action deviation at stage t (but possibly a communication deviation), then $X^i_{t+T} = \emptyset$ almost surely for each player $i \in N$.

This ends the description of the communication protocol.

I now describe how transition from phase II to another phase is made. For each player $i \in N$:
- if $X^i_{t+T} = \emptyset$, then keep playing according to the current action phase and use the corresponding communication strategy;
- if $X^i_{t+T} = \{k\}$, then go to phase III in order to minmax player k;
- otherwise, play the coordinate of an arbitrary Nash equilibrium of the one-shot game (history incompatible with unilateral deviations).

Phase III: punishment phase

For each player $i \in N$, if $X^i_{t+5} = \{k\}$ for some player k in N, player i enters a punishment at stage $t+6$, whose goal is to minmax player k. Each player $i \neq k$ first plays according to his minmax strategy against k, denoted $(P^i(k))$, defined as in the proof of Theorem 4.3.1. Player k is compelled to play the same pure action $P^k(k)$ during his punishment (see footnote 7 page 123).

In addition, at each stage s during the punishment phase, each player i in N, including player k, keeps sending messages in M regarding stage $s-1$. In particular, a player $j \in N^{-k}$ who chooses an action which is not in the support of his minmax strategy is identified as a deviator. The punishment phase lasts $T(\delta)$ stages (the length $T(\delta)$ has to be adapted, see below).

Phase IV: reward phase

After the punishment phase, hence at stage $t+5+T(\delta)$, each player enters a reward phase, whose goal is the same as in Section 4.3 (see page 123). The possibility of providing such rewards relies on the full dimensionality of the payoff set (recall that $\operatorname{int} V^* \neq \emptyset$). In addition, it is convenient to require that all players know the sequences of pure actions played by each minmaxing player during the punishment phase. For that reason, at the first stage of the reward phase (*i.e.* at stage $t+5+T(\delta)$), each player $i \in N$ sends all the sequences of pure actions that his neighbors $j \in \mathcal{N}(i)$ played (recall that each player observes his neighbors' moves). The action of player i at this stage is arbitrary. By replacing $x(s)$ by the sequences of pure actions played by one's neighbors in the communication protocol of phase II, each player knows at the end of stage $t+10+T(\delta)$ all the sequences actually played by all players. Finally, the construction of the reward phase as well as the specification of $\mu(\delta)$ is the same as in Fudenberg and Maskin ([16]). Similar arguments can be found in Chapter 3 (Section 3.5.2.4).

Specification of the beliefs

The specification of the beliefs is the same as in Chapter 3 (Section 3.5.2.5).

The equilibrium property

In this section I prove the following proposition.

Proposition 4.4.4. *Assume $n = 4$, public and local communication, and $G = G_1$. The following statements hold.*

(a) If player k deviates in action at stage t and if at all stages $t + s$ with $s \in \{1, \ldots, 5\}$, at least three players follow (σ^*, ϕ^*), then $X^i_{t+5} = \{k\}$ almost surely.

(b) If there is no action deviation at stage t (but possibly an action deviation) and if at all stages $t + s$ with $s \in \{1, \ldots, 5\}$, at least three players follow (σ^*, ϕ^*), then $X^i_{t+5} = \emptyset$ almost surely.

Proof of Proposition 4.4.4 Assume $n = 4$, public and local communication, and $G = G_1$. Recall that $N = \{1, 2, 3, 4\}$ and $\mathcal{N}(1) = \{2, 4\}$. I prove points (a) and (b).

(a) Assume without loss of generality that player 1 deviates in action at stage t and that at least three players follow (σ^*, ϕ^*) at each stage, *i.e.* only unilateral deviations are allowed at each stage. Players 2 and 4 observe player 1's moves, hence $X^1_{t+5} = X^2_{t+5} = X^4_{t+5} = \{1\}$. At stage $t + 1$, three cases are possible.

First, if both players 2 and 4 send $m^2_{t+1} \in M$ and $m^4_{t+1} \in M$, which both contain $(1, x(1) = 1)$, then by construction $X^3_{t+5} = \{1\}$.

Second, if $m^2_{t+1} \notin M$, then player 4 plays according to $\tilde{\phi}^4_{t+1}$ under unilateral deviations, hence $m^4_{t+1} \in M$ and $x^4_{t+1}(1) = 1$. Then by construction, $X^3_{t+5} = \{1\}$. (And symmetrically, if $m^4_{t+1} \notin M$, then $X^3_{t+5} = \{1\}$.)

Finally, assume $m^2_{t+1} \in M$, $m^4_{t+1} \in M$ and that $x^2_{t+1}(1) \neq 1$.[11] At stage $t+2$:

(i) either player 1 deviates, hence players 2 and 4 do not deviate and by construction they both send a message in M containing $(2, x(2) = 1)$: it follows that $X^3_{t+5} = \{1\}$.

(ii) Or player 1 does not deviate and sends $m^1_{t+2} = (2, x(2) = 1, \alpha^2_{t+2}, F^1_{t+2}(2) = (1, 2, \alpha^2_{t+1}))$. At stage $t+3$, several cases are possible:

- either player 4 does not deviate, thus sends a message in M containing $F^4_{t+3}(3) = (2, 2, \alpha^2_{t+1})$. Moreover, under unilateral deviations (recall that player 2 deviates at stage $t+1$), the following holds: $m^4_{t+1} \in M$ and $x^4_{t+1}(1) = 1$. Finally, the probability that player 2 guesses α^4_{t+1} is zero since α^4_{t+1} is uniformly drawn on $[0, 1]$. As a consequence, $X^3_{t+5} = \{1\}$.
- Or, player 4 deviates at stage $t+3$. Notice first that if $m^4_{t+3} \in M$ and $x^4_{t+3}(1) = 1$, the previous reasoning apply and $X^3_{t+5} = \{1\}$. Otherwise, player 1 sends at stage $t+4$ a message in M containing $F^1_{t+4}(4) = (1, 4, \alpha^4_{t+3})$. If player 1 deviates, then both players 2 and 4 follow $\tilde{\phi}$ and as before, $X^i_{t+5} = \{1\}$. Assume now that player 1 does not deviate, then at stage $t+5$, players 2 and 4 should send $F^2_{t+5}(5) = (2, 4, \alpha^4_{t+3})$ and $F^4_{t+5}(5) = (4, 2, \alpha^2_{t+1})$ respectively. Moreover, under unilateral deviations, the following holds: $m^4_{t+1} \in M$, $(1, x(1) = 1) \in m^4_{t+1}$, $m^2_{t+3} \in M$ and $(3, x(3) = 1) \in M$. Finally, either player 2 or player 4 deviates, but not both, hence by construction $X^3_{t+5} = \{1\}$.

Consequently, $X^3_{t+5} = \{1\}$ almost surely. This concludes the proof of point (a).

(b) Assume now that there is no action deviation at stage t. Consider, without loss of generality, player 3's output at stage $t+5$. By construction, 2, 3 and 4 are not in X^3_{t+5} since player 3 neither deviate nor observe any neighbor's deviation at stage t. Moreover, under unilateral deviations, there is no stage $t+s$, with $s \in \{1, \ldots, 5\}$, such that $m^2_{t+s} \in M$, $m^4_{t+s} \in M$ and $x^2_{t+s}(s) = x^4_{t+s}(s) = 1$). Hence, the only possibility for player 3 to output $X^3_{t+5} \neq \emptyset$ is that both following conditions are satisfied:

(1) player 2 (without loss of generality) sends at stage $t+1$ (respectively $t+3$) a message in M containing $(1, x(1) = 1)$ (respectively $(3, x(3) = 1)$);

11. The same reasoning apply if $x^4_{t+1}(1) \neq 1$.

(2) player 2 announces correctly α^4_{t+1} at stage $t+3$ (respectively α^4_{t+3} at stage $t+5$).

The probability of guessing α^4_{t+1} (respectively α^4_{t+3}) is zero. Hence, $X^3_{t+5} = \{1\}$ is possible only if player 1 deviate by announcing α^4_{t+1} (respectively α^4_{t+3}) at stage $t+2$ (respectively $t+4$). However, if player 1 deviates at stage $t+2$ (respectively $t+4$), then both players 2 and 4 send at stage $t+2$ (respectively $t+4$) a message in M containing $(2, x(2) = \varnothing)$ (respectively $(4, x(4) = \varnothing)$). As a consequence, $X^3_{t+5} = \emptyset$ almost surely. □

By replacing $x(s)$ by the sequences of pure actions played by one's neighbors during the punishment phase, it follows that the proof of Proposition 4.4.4 also shows that under (σ^*, ϕ^*), all players know the sequences of pure actions played by others during the punishment phase at stage $t + T(\delta) + 10$.

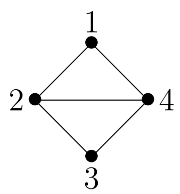
Finally, the proof that the strategy constructed is a PBE is a straightforward application of the proof of Proposition 3.5.11 in Section 3.5.3, and of the proof of Fudenberg and Maskin ([15, 16]).

Remark 4.4.5. Renault and Tomala ([39, 40]) already introduced encoding keys to transmit information through a network. However in their setting, the set of deviators is fixed throughout the repeated interaction, which is not the case here. In my model, the key difficulty is that the set of deviators is not fixed, hence all players may deviate at some stage.

4.4.1.2 Exactly two players have three neighbors

Throughout this section, assume $n = 4$, and that communication is public and local. Furthermore, suppose that the game is played on the following network (G_2):

FIGURE 4.4:



G_2

The insights of the previous section 4.4.1.1 about uniformly distributed keys are not valid anymore, since player 4 knows α^2_t whether he deviates at stage t or not. Indeed, communication is public among neighbors, and player 4 receives player 2's announcements.

However, a folk theorem still holds for this network. Indeed it is possible to extend the insights of Ben-Porath and Kahneman ([5]) to prevent both players 2 and 4 from lying.[12] The main idea is to lower by $\epsilon > 0$ both players 2's and 4's continuation payoffs if their announcements are incompatible at some stage (this is possible because $\text{int}\, V^* \neq \emptyset$).[13] The argument relies on the fact that communication is costless, hence my construction does not require to minmax a liar who does not deviate in action, rather in communication only. Moreover, incompatible announcements of players 2 and 4 are observed by all players.[14] Hence, all players can start at the same stage the phase in which both players 2's and 4's continuation payoffs are decreased.

Nevertheless, decreasing both players 2's and 4's continuation payoffs creates the following issue: whenever player 2 (without loss of generality) is being minmaxed, he might have an incentive to lie about player 1's deviation in order to reach the state in which both players 2's and 4's continuation payoffs are decreased. For that reason, if players 2's and 4's announcements are incompatible in that player 2 reports player 1's deviation while player 4 doesn't, and if in addition player 2 is being minmaxed, then players start minmaxing player 2 again.

However, it might then be profitable for player 4 not to report player 1's deviations –for instance, player 4 may enjoy minmaxing player 2. To circumvent this issue, I require that whenever player 4 is minmaxed, player 2 gets an additional bonus of $\rho > 0$ if he reports player 1's deviation.

Notice also that players 2's and 4's moves are observed by all players, hence if one of them deviates in action at some stage t, each player $i \in N$ identifies the deviator at stage t.

Finally, to design the reward phase –following each punishment phase–, it is convenient that all players know the sequences of pure actions played by all players during the punishment phase.[15] It is not an issue for players 2 and 4, whose actions are observed by all players. Regarding the sequences of pure actions played by players 1 and 3, I add a subphase at the beginning of the reward phase in which players 2 and 4 announce the sequences played by players 1 and 3. Again, incompatible announcements are punished as before. The action is arbitrary during this subphase.

This strategy is a PBE. The proof of the equilibrium property is a straightforward application of Ben-Porath and Kahneman ([5]), and Fudenberg and Maskin

12. Recall that Ben-Porath and Kahneman consider public and global communication.

13. Notice that the non-equivalent utilities condition introduced by Abreu, Dutta and Smith ([1]) is not sufficient to ensure that this is possible.

14. Notice that it would not be the case if $G = G_1$.

15. As in Section 4.4.1.1, the minmaxed player, say k, is compelled to play the same pure action P^k during his punishment (see footnote 7 page 123).

([15, 16]). Hence, a folk theorem holds for the network G_2 under public and local communication.

Remark 4.4.6. The proof does not require an infinite set of messages. As in [5], the proof of the folk theorem is valid if I impose that M be finite.

Remark 4.4.7. Regarding the network G_1 of the previous section, incompatible announcements of players 2 and 4 are not observed by all players. Hence, the insights of Ben-Porath and Kahneman do not apply to the network G_1. Indeed, if $G = G_1$, player 2 does not know if player 4's announcement is compatible with his own. One could imagine that players 1 and 4 can transmit this information, but again, the announcements of players 1 and 4 might be incompatible, etc.

4.4.2 More than four players

With more than four players, it is an open question to characterize the networks for which a folk theorem holds under public and local communication. In this section, I show a folk theorem for a specific class of networks (Section 4.4.2.1), then I introduce an example (Section 4.4.2.2).

4.4.2.1 A class of networks

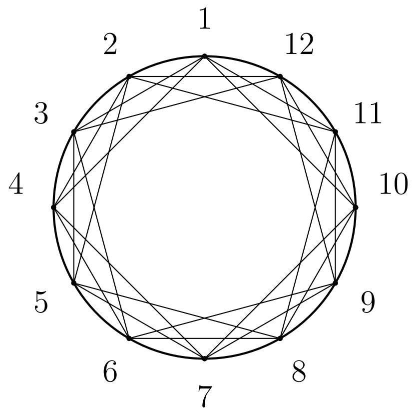
Proposition 4.4.8. *Assume $N = \{x^1, x^2, \ldots, x^n\}$, with $n \geq 7$, and that the network G satisfies the following Condition C:*

$$\forall i,\ j \in \{1, \ldots, n\}, \quad i - j = -3, -2, -1, 1, 2, 3\ [n] \Rightarrow i \in \mathcal{N}(j).$$

Then, for any payoff function g such that the interior of V^* is nonempty, for every vector v in V^*, there exists $\bar{\delta} \in (0, 1)$ such that for all $\delta \in (\bar{\delta}, 1)$, v is a PBE vector payoff of the δ-discounted game.

The following network satisfies Condition C of Proposition 4.4.8:

Figure 4.5:



G_4

Proof of Proposition 4.4.8

Assume public et local communication, $n \geq 7$, and that the network G satisfies Condition C of Theorem 4.4.8. Now take a payoff function g such that $\operatorname{int} V^*$ is non-empty, and a vector $v = (v^1, \ldots, v^n)$ in V^*. To prove a folk theorem, I construct a PBE of the repeated game (σ^*, ϕ^*) with payoff v for a discount factor close enough to one. More precisely, I construct as in Chapter 3 a restriction of the PBE strategy to a particular class of private histories; namely, the histories along which only unilateral deviations from (σ^*, ϕ^*), if any, have taken place. Given that play after other histories (*i.e.*, histories that involve simultaneous deviations) is irrelevant, the partial strategy and beliefs that I define can be completed in any arbitrary fashion.

Formally, I denote by $H_t^i(U|(\sigma^*, \phi^*))$ the set of private histories for player i such that: either no deviation (from (σ^*, ϕ^*)) has occurred, or only unilateral deviations have taken place. That is to say, for any history in $H_t^i(U|(\sigma^*, \phi^*))$, no multilateral deviation has occurred. Similarly, denote by $H_t(U|(\sigma^*, \phi^*))$ the set of total histories along which only unilateral deviations, if any, have taken place. I define now, for every history in $H_t(U|(\sigma^*, \phi^*))$, a strategy profile which can be decomposed into four phases. First, there is a stream of pure action profiles that yields the desired payoff. This is how the game starts off and how it unfolds as long as no player deviates. Second, there is a communication phase (the communication protocol previously described) in case of a deviation, whose purpose is to inform the deviator's neighbors of the deviator's identity. Third, there is a punishment phase, and finally, a reward phase.

Phase I: equilibrium path

During this phase, each player i in N should play action $\bar{a}_t^i$ at stage t defined as in the proof of Theorem 4.3.1. Moreover, at every period t, player i should sends the message $m_t^i = (1, \varnothing)$ to all his neighbors. As before, $(1, \varnothing)$ of player i's message means that he did not deviate nor detected any action deviation at stage $t - 1$.[16]

Phase II: communication phase

Fix a player $i \in N$. If player i in N receives a message different from $(1, \varnothing)$ at some stage t, or if player i observes a neighbor's deviation, then player i starts a communication protocol, described as follows:

– if a player i observes an action deviation of a neighbor $k \in \mathcal{N}(i)$ at stage t, then he sends the message $(1, k)$ at stage $t + 1$;

– if a player i receives from two distinct neighbors, denoted i^1 and i^2, the messages $m_t^{i^1} = m_t^{i^2} = (s, k)$ for some $k \in N$ and $s \in \{1, T\}$, where T is the length of the protocol (see below for the specification of T), then $m_{t+1}^i = (s + 1, k)$.

As in Section 4.4.1.1, several communication protocols may be running at the same time: different delays refer to different stages, hence to different protocols.

At each stage t, player i computes his set of suspects regarding the alleged deviation stage $t - s$ for each $s \in \{0, \ldots, T\}$, denoted $X_t^i(s)$, as follows:

– if player i deviates at stage $t - s$, then $X_t^i(s) = \{i\}$;

– if player i observes the action deviation of some neighbor $k \in \mathcal{N}(i)$ at stage s, then $X_t^i(s) = \{k\}$;

– if there exists some $s' \in \{1, \ldots, T\}$ such that there exists two distinct neighbors i^1 and i^2 in $\mathcal{N}(i)$ such that $m_{t-s+s'}^{i^1} = m_{t-s+s'}^{i^2} = (s', k)$ for some $k \in N$, then $X_t^i(s) = \{k\}$.

The number of rounds is $T = \lfloor n \rfloor - 3$, where $\lfloor x \rfloor$ stands for the integer part of x. For each player $i \in N$, the transition rule from phase II to another phase is then the following:

- if $X_t^i(T) = \emptyset$, then keep playing according to the current action phase and use the corresponding communication strategy;
- if $X_t^i(T) = \{k\}$, then go to phase III in order to minmax player k;
- otherwise, play the coordinate of an arbitrary Nash equilibrium of the one-shot game (history incompatible with unilateral deviations).

Lemma 4.4.9. *Assume public and local communication, $n \geq 7$, and that G satisfies Condition C. The following statements hold.*

(a) If player k deviates in action at stage t, and if at all stages $t + s$ with $s \in \{1, \ldots, T\}$, at least $n - 1$ players follow (σ^*, ϕ^*), then $X_{t+T}^i = \{k\}$.

16. Recall that, at each stage, messages are sent before observing stage payoffs. This assumption is not crucial: with a slight modification, the strategy construction is still valid for the case in which messages are sent after the observation of stage payoffs.

(b) *If there is no action deviation at stage t (but possibly an action deviation), and if at all stages $t+s$ with $s \in \{1,\dots,T\}$, at least $n-1$ players follow (σ^*,ϕ^*), then $X^i_{t+T} = \emptyset$.*

Proof of Lemma 4.4.9 Assume public and local communication, $n \geq 7$, and that G satisfies Condition C.

(a) Suppose first that some player k deviates in action at stage t, and denote by x^2, x^3, x^4, x^{n-2}, x^{n-1}, x^n his neighbors. Then $X^i_t(0) = \{k\}$ for each player $i \in \{k, x^2, x^3, x^4, x^{n-2}, x^{n-1}, x^n\}$. Denote by x^5 (respectively x^{n-3}) the player such that $x^5 \in \mathcal{N}(x^1) \cap \mathcal{N}(x^2) \cap \mathcal{N}(x^3)$ and $x^5 \neq k$ (respectively $x^{n-3} \in \mathcal{N}(x^{n-2}) \cap \mathcal{N}(x^{n-1}) \cap \mathcal{N}(x^n)$ and $x^{n-3} \neq k$). Players x^5 and x^{n-3} exist because of Condition C of Proposition 4.4.8 (if $n = 7$, then $x^5 = x^{n-3}$). At stage $t+1$, at least two players among x^2, x^3 and x^4 (respectively among x^{n-2}, x^{n-1} and x^n) send $(1,k)$. Hence $X^{x^5}_{t+1}(1) = \{k\}$, and symmetrically $X^{x^{n-3}}_{t+1}(1) = \{k\}$. By induction, it is easy to see that $X^i_{t+T}(T) = \{k\}$ for each player in N.

(b) Assume now that there is no action deviation at stage t. It is easy to see that under unilateral deviations, $X^i_{t+T}(T) = \emptyset$ for each player $i \in N$. □

Phase III: punishment phase

For each player $i \in N$, if $X^i_{t+T} = \{k\}$ for some player k in N, player i enters a punishment at stage $t+T+1$, whose goal is to minmax player k. Each player $i \neq k$ first plays according to his minmax strategy against k, denoted $(P^i(k))$, defined as in the proof of Theorem 4.3.1. Player k is compelled to play the same pure action $P^k(k)$ during his punishment (see footnote [7] page 123).

In addition, at each stage s during the punishment phase, each player i in N, including player k, keeps sending messages in M regarding stage $s-1$. In particular, a player $j \in N^{-k}$ who chooses an action which is not in the support of his minmax strategy is identified as a deviator. The punishment phase lasts $T(\delta)$ stages (the length $T(\delta)$ has to be adapted, see below).

Phase IV: reward phase

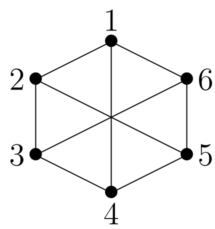
After the punishment phase, hence at stage $t+T+T(\delta)$, each player enters a reward phase, whose goal is the same as in Section 4.3 (see page 123). The possibility of providing such rewards relies on the full dimensionality of the payoff set (recall that $\operatorname{int} V^* \neq \emptyset$). In addition, it is convenient to require that all players know the sequences of pure actions played by each minmaxing player during the punishment phase. For that reason, at the first stage of the reward phase (*i.e.* at stage $t+T+T(\delta)$), each player $i \in N$ sends all the sequences of pure actions that his neighbors

$j \in \mathcal{N}(i)$ played (recall that each player observes his neighbors' moves). Then, players use the previous protocol but replace the name of the deviator k by all the sequences of actions received. By induction, a majority rule at each stage proves that each player knows at the end of stage $t + 2T + T(\delta)$ all the sequences of pure actions actually played by all players. Players choose $\bar{a}_t$ at each stage $t \in \{t + T + T(\delta), \ldots, t + 2T + T(\delta) - 1$. Then, players start the reward phase at stage $t + 2T + T(\delta)$. The construction of the reward phase as well as the specification of $\mu(\delta)$ is the same as in Fudenberg and Maskin ([15, 16]). Similar arguments can be found in Chapter 3 (Section 3.5.2.4). The specification of the beliefs is the same as in Chapter 3 (Section 3.5.2.5). The folk theorem directly follows. □

4.4.2.2 An example

In the previous section, I show a folk theorem for a specific class of networks (see Figure 4.4.2.1). However, Condition C of Theorem 4.4.8 is not necessary for a folk theorem to hold with more than four players. Consider the following network:

FIGURE 4.6:



for which Condition C is not satisfied. Nevertheless, it is easy to see that a folk theorem holds for this network under local and public communication. Indeed, the strategy constructed in the proof of Proposition 4.4.8 is a PBE for this network too. Notice that if player 1 deviates in action at some stage t, then players 2, 4 and 6 know that the deviator is player 1, and transmit this information at stage $t + 1$ to both players 3 and 5. A simple majority rule implies that players 3 and 5 know who the deviator is at the end of stage $t + 1$.

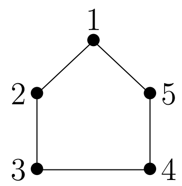
4.4.3 Conjectures and open questions

Under public and local communication, the characterization of the networks G for which a folk theorem holds remains open if sequential rationality is imposed.

A conjecture is that a folk theorem holds for all circle networks. Indeed, I believe that the strategy in Section 4.4.1.1 may be extended to more than $n = 4$ players.

However, this is still an open question, and the strategy in Section 4.4.1.1 needs main modifications to be adapted to more than four players. Indeed, consider the following network:

FIGURE 4.7:



The insights of Section 4.4.1.1 rely on the following observation: if the announcements of players 2 and 5 regarding player 1's action are incompatible, then one of them is lying, hence player 1 is not, and player 1 thus transmits the liar's encoding key. The key issue here is that player 3 may not distinguish between the following histories: "the announcements of players 2 and 5 are incompatible", and "the announcements of players 2 and 5 are compatible, but player 4 is lying about player 5's announcement". Hence, if player 3 receives player 2's encoding key α_t^2 from player 4, player 3 may not know if player 1 deviates at stage $t+1$ when reporting player 2's key, or if player 2 indeed lied at stage t. Indeed, since player 3 does not know if the announcements of players 2 and 5 are compatible or not, he cannot concludes that player 1 deviates or not when transmitting player 2's encoding key. As a consequence, the strategy constructed in Section 4.4.1.1 does not enable to identify the deviator in case of a deviation.

4.5 Concluding remarks

A lot of work still has to be done, especially regarding public and local communication: in this setting, the characterization of the networks for which a folk theorem holds remains an open question if sequential rationality is imposed. It is also an open question to characterize the networks for which a folk theorem holds under private and local communication, if again sequential rationality is imposed.

Under local interaction and payoff monitoring (see Chapters 2 and 3), the characterization of the networks for which a folk theorem holds remains unknown in the following cases: (i) private and global communication, and (ii) public and local communication (and obviously, it is also an open question with private and local communication). In Chapter 3, the proof of the folk theorem relies on the fact that public and global announcements are possible (the list of receivers is certifiable).

Finally, a question of interest is to study previous works on folk theorems with imperfect monitoring, and public and global communication ([8, 26, 35] among others), and to specify to what extend the assumption of public and global communication can be relaxed. This is left to future research.

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Résumé

Jeux répétés en réseaux et communication

On s'intéresse aux jeux infiniment répétés avec observation imparfaite et privée, joués dans des réseaux. Différents réseaux peuvent modéliser les structures d'interaction, d'observation ou de communication d'un jeu répété. La communication est gratuite, et peut être privée ou publique. Différents modèles de jeux répétés sont étudiés, en fonction des réseaux considérés, de la nature de la communication, et du concept d'équilibre étudié. L'objectif principal de cette thèse est d'établir des conditions nécessaires et suffisantes sur les réseaux considérés afin d'obtenir un folk théorème.

Mots-clés : jeux répétés, observation imparfaite (privée), réseaux, théorème folk, communication, protocoles.

Abstract

Repeated games on networks and communication

I study infinitely repeated games with imperfect private monitoring played on networks. Different networks may represent the structures of interaction, of monitoring, and of communication of the repeated game. Communication is costless, and may be either private or public. I study different models of repeated games, depending on the networks considered, on the nature of communication, and on the solution concept. The aim of this thesis is to establish necessary and sufficient conditions on the networks for a folk theorem to hold.

Keywords: repeated games, imperfect (private) monitoring, networks, folk theorem, communication, protocols.