# Uniqueness, reconstruction, stability for some two-dimensional inverse problems

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DOCTEUR DE L'ÉCOLE POLYTECHNIQUE

Specialité : *Mathématiques appliquées*

par

Matteo Santacesaria

# Unicité, reconstruction, stabilité pour des problèmes inverses bidimensionnels

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# Contents

## Abstract

In this thesis some inverse boundary value problems in two dimensions are studied. The problems considered are the Calderón problem and the Gel'fand-Calderón problem in the single and multi-channel (i.e. matrix-valued) case. The latter can be seen as a non-overdetermined approximation of the three-dimensional case. We begin with some results for the anisotropic Calderón problem: a new formulation of the uniqueness result on the plane is presented as well as the first global uniqueness on two-dimensional surfaces with boundary. Next, we prove new global stability estimates for the Gel'fand-Calderón problem in the single and multi-channel cases. Similar techniques also give a global reconstruction procedure for the same problem in the multi-channel case. A rapidly converging approximation algorithm for the multi-channel Gel'fand-Calderón problem is presented afterwards. This algorithm is inspired mostly by results from multi-dimensional inverse scattering theory. Finally we present new global stability estimates for the two aforementioned problems which explicitly depend on regularity and energy.



## Résumé

Dans cette thèse nous étudions quelques problèmes inverses de valeurs au bord en dimension deux. Les problèmes considérés sont le problème de Calderón et le problème de Gel'fand-Calderón dans le cas scalaire et multi-canal, c'est-à-dire matriciel : ce dernier peut être vu notamment comme une approximation non-surdéterminée du cas tridimensionnel. Nous montrons d'abord quelques résultats pour le problème de Calderón anisotrope : une nouvelle formulation du résultat d'unicité sur le plan ainsi que le premier résultat d'unicité globale pour le cas des surfaces à bord. Après, nous démontrons une nouvelle estimation de stabilité globale pour le problème de Gel'fand-Calderón dans le cas scalaire et multi-canal. Des techniques similaires donnent aussi une procédure de reconstruction globale pour le même problème. Nous proposons ensuite un algorithme d'approximation rapidement convergent pour le problème de Gel'fand-Calderón multi-canal : cet algorithme est principalement motivé par des résultats de la théorie de diffusion inverse multi-dimensionnelle. Comme derniers résultats nous présentons des nouvelles estimations de stabilité globale pour les deux problèmes mentionnés plus haut qui dépendent explicitement de la régularité et de l'énergie.

# Introduction

This thesis consists of several papers which focus on different aspects of some inverse boundary value problems in two dimensions. The problems under consideration are the Calderón problem and the Gel'fand-Calderón problem in the single and multi-channel case.

To summarize, papers are divided into four groups. The first covers the anisotropic Calderón problem on the plane and on two-dimensional surfaces.

**A.** G. Henkin, M. Santacesaria, *On an inverse problem for anisotropic conductivity in the plane*, Inverse Problems **26**, 2010, no. 9, 095011.

**B.** G. Henkin, M. Santacesaria, *Gel'fand–Calderón's Inverse Problem for Anisotropic Conductivities on Bordered Surfaces in* $\mathbb{R}^3$, Int. Math. Res. Not. IMRN **2012**, 2012, no. 4, 781–809.

The second group deals with stability estimates for the Gel'fand-Calderón problem in the single and multi-channel case.

**C.** R. G. Novikov, M. Santacesaria, *A global stability estimate for the Gel'fand–Calderón inverse problem in two dimensions*, J. Inverse Ill-Posed Probl. **18**, 2010, no. 7, 765–785.

**D.** M. Santacesaria, *Global stability for the multi-channel Gel'fand–Calderón inverse problem in two dimensions*, Bull. Sci. Math., 2012, doi:10.1016/j.bulsci.2012.02.004.

The third group of papers studies global uniqueness and reconstruction algorithms for the multichannel Gel'fand-Calderón problem.

**E.** R. G. Novikov, M. Santacesaria, *Global uniqueness and reconstruction for the multi-channel Gel'fand–Calderón inverse problem in two dimensions*, Bull. Sci. Math. **135**, 2011, no. 5, 421–434.

**F.** R. G. Novikov, M. Santacesaria, *Monochromatic Reconstruction Algorithms for Two-dimensional Multi-channel Inverse Problems*, Int. Math. Res. Not. IMRN, 2012, doi:10.1093/imrn/rns025.

In the last group some stability estimates for both problems depending on energy and regularity are presented.

**G.** M. Santacesaria, *New global stability estimates for the Calderón problem in two dimensions*, J. Inst. Math. Jussieu, 2012, doi:10.1017/S147474801200076X.

**H.** M. Santacesaria, *Stability estimates for an inverse problem for the Schrödinger equation at negative energy in two dimensions*, Applicable Analysis, 2012, doi:10.1080/00036811.2012.698006.

The problem of the recovery of an electrical conductivity from boundary measurements of voltage and current was proposed, among others, by Calderón [**23**] in 1980 and has a direct application in electrical impedance tomography. Let us formulate the problem mathematically.

Let $D \subset \mathbb{R}^d$, $d \geq 2$, be an open bounded domain with smooth boundary and $\sigma \in L^\infty(D, M_n(\mathbb{R}))$ be a matrix-valued function representing an electrical conductivity. It is customary to assume that $\sigma(x)$ is positive definite (with an uniform lower bound) and symmetric for every $x \in D$. The Dirichlet-to-Neumann map corresponding to $\sigma$ is the operator $\Lambda_\sigma : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ defined as

$$\Lambda_\sigma f = \sigma \nabla u \cdot \nu|_{\partial D}, \tag{0.1}$$

where $f \in H^{1/2}(\partial D)$, $\nu$ is the outer normal to $\partial D$ and $u$ is the unique $H^1(D)$ solution of the Dirichlet problem

$$\nabla \cdot (\sigma \nabla u) = 0 \text{ on } D, \qquad u|_{\partial D} = f. \tag{0.2}$$

To be more precise,

$$\langle \Lambda_\sigma f, g \rangle = \int_D \sigma \nabla u \cdot \nabla g, \tag{0.3}$$

for any $g \in H^1(D)$ and $\langle \cdot, \cdot \rangle$ is the pairing between $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$.

Equation (0.2) represents the conservation of the electrical charge on $D$ if the voltage $f$ is applied to $\partial D$, and $\Lambda_\sigma f$ is the current flux at the boundary. The following is called Calderón problem.

PROBLEM 1. *Given $\Lambda_\sigma$, determine $\sigma$ on $D$.*

Calderón originally proposed this problem for the special class of isotropic conductivities. A conductivity $\sigma$ is *isotropic* if there exists a positive function $\sigma_0$ such that $\sigma(x) = \sigma_0(x) I$ for every $x \in D$, where $I$ is the identity matrix; we will call a conductivity *anisotropic* if it does not satisfy this condition.

One of the first and most studied strategies to solve Problem 1, for smooth isotropic conductivities, is to substitute $\tilde{u} = u\sqrt{\sigma}$ into equation (0.2), in order to obtain

$$(-\Delta + v)\tilde{u} = 0 \text{ on } D, \quad \text{where } v = \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}}. \tag{0.4}$$

This is the Schrödinger equation at zero energy for the potential $v$, for which it is possible to define the corresponding Dirichlet-to-Neumann operator $\Phi_v$ as

$$\Phi_v \tilde{f} = \left. \frac{\partial \tilde{u}}{\partial \nu} \right|_{\partial D},$$

where $\tilde{f} \in H^{1/2}(\partial D)$ and $\tilde{u}$ is the unique solution of the corresponding Dirichlet problem for the above Schrödinger equation. We also have the following formula which relates the two boundary operators:

$$\Phi_v = \sigma^{-1/2}\left(\Lambda_\sigma \sigma^{-1/2} + \frac{\partial \sigma^{1/2}}{\partial \nu}\right). \tag{0.5}$$

Thus if $\sigma$ is isotropic and sufficiently regular (for instance $C^2$) and if one knows the boundary values of $\sigma$ and $\partial\sigma/\partial\nu$, Problem 1 is reduced to the problem of the recovery of a potential in the Schrödinger equation from the Dirichlet-to-Neumann map $\Phi_v$. Potentials of the form (0.4) are called *conductivity type* potentials.

This is a particular case of a more general problem. Consider the Schrödinger equation at fixed energy $E \in \mathbb{R}$,

$$(-\Delta + v)\psi = E\psi \quad \text{on } D, \tag{0.6}$$

where $D$ is a open bounded domain in $\mathbb{R}^d$, $d \geq 2$ and $v \in L^\infty(D)$. Under the assumption that

$$0 \text{ is not a Dirichlet eigenvalue for the operator } -\Delta + v - E \text{ in } D, \tag{0.7}$$

we can define the Dirichlet-to-Neumann operator $\Phi_v(E) : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$, corresponding to the potential $v$, as follows:

$$\Phi_v(E) f = \left.\frac{\partial u}{\partial \nu}\right|_{\partial D}, \tag{0.8}$$

where $f \in H^{1/2}(\partial D)$, and $u$ is the $H^1(D)$ solution of the Dirichlet problem

$$(-\Delta + v)u = Eu \text{ on } D, \quad u|_{\partial D} = f. \tag{0.9}$$

This construction gives rise to the so-called Gel'fand-Calderón problem.

Problem 2. *Given $\Phi_v(E)$ for a fixed $E \in \mathbb{R}$, determine $v$ on $D$.*

The name, besides the aforementioned connection to Calderón problem, comes from an inverse problem proposed by Gel'fand [**34**] in 1954; indeed, Problem 2 can be considered as the fixed-energy version of the original Gel'fand problem.

Note that Problem 1 in the isotropic case and Problem 2 are formally non-overdetermined in the two-dimensional case. Indeed, for dimension $d$, the functions to be reconstructed depends on $d$ variables while the inverse problem data depends on $2(d-1)$ variables.

In addition, the history of inverse problems for the two-dimensional Schrödinger equation at fixed energy goes back to [**30**] (see also [**69, 35**] and reference therein). Problems 1 and 2 can also be considered as examples of ill-posed problems: see [**59**], [**17**] for an introduction to this theory.

Both problems can be formulated with the additional condition that the data are given only on a part of the boundary. This case was not taken into consideration in the present work, thus we will not give precise references to results with such conditions.

In every result concerning Problem 2 we will always assume condition (0.7). It is possible to ignore it by considering the Cauchy data space $\{(\psi|_{\partial D}, \partial\psi/\partial\nu|_{\partial D}) : \psi \text{ solution of (0.6)}\}$ as boundary data instead of the Dirichlet-to-Neumann map. Under these conditions, several uniqueness, reconstruction and stability results are obtained, for instance in [**21**], [**52**], [**53**].

Problem 1 can also be formulated in a discrete way: the reconstruction of a resistor network from boundary measurements. A resistor network is a graph endowed with a function representing an electrical conductance on the edges; in this case the Dirichlet problem (0.2) is replaced with a discrete analogue, using a discrete Laplacian. Fundamental results are obtained in [**26**], [**27**], where, in particular, the Dirichlet-to-Neumann operators are classified.

We also obtain a special case of Problem 1 when the conductivity is piecewise constant. The problem is then to determine the shape or the volume of the subdomains in which it is constant or, in other words, to reconstruct some inclusions from boundary measurements (even a single one). Many theoretical and numerical works have been done for this problem: [**7**] gives a stability result and [**43**] provides recent numerical algorithm.

There is a wide literature on Problems 1 and 2. In order to introduce the results of our Papers **A** and **B**, we just mention here the most significant results for Problem 1 in the full data case. Results for Problem 2 will be mentioned afterwards.

In the isotropic case, it was shown that the boundary measurements uniquely determines a conductivity. After Calderón's paper [**23**], which considered the linearised problem, it was proved in [**56**] that the boundary values of a conductivity, as well as its derivatives, are uniquely determined by the Dirichlet-to-Neumann map in dimension $d \geq 2$ (see also [**85**], [**5**]). For piecewise constant conductivities uniqueness was obtained for the first time in [**29**] and for piecewise real-analytic in [**57**]. The first global uniqueness result in dimension $n \geq 3$ was given in [**84**] for $C^\infty$ conductivities; this was soon improved to the $C^2$ case, first in [**68**] (which gives also the first global reconstruction), then in [**65**], [**63**]. In two dimensions, global uniqueness was proved in [**64**] for $W^{2,p}$ conductivities, $p > 1$. Some of the most recent references are the following: in [**42**], uniqueness is proved for $C^1$ conductivities for dimension $d \geq 3$ and in [**9**] for $L^\infty$ conductivities in two dimensions (see also [**11**] for recent developments and [**12**], [**13**] for related problems on the plane).

A global logarithmic stability estimate for the Calderón problem was given for the first time in [**4**] for dimension $d \geq 3$ and in [**60**] for $d = 2$. Instability estimates of [**62**] showed that the logarithmic estimates were optimal, up to the value of some exponent. Nevertheless, they were improved in several essential ways: logarithmic stability was obtained for less regular conductivities (see [**24**], [**15**], [**25**]), Lipschitz stability for piecewise constant conductivities (see [**8**], [**79**], [**19**]) and regularity-dependent estimates were also obtained (see [**75**], [**81**] = Paper **E**). See also [**6**] for an additional survey on the subject.

For anisotropic conductivities the situation is quite different: it is not possible to determine $\sigma$ uniquely from $\Lambda_\sigma$. This was discovered by L. Tartar (see [**57**]). Indeed, let $F : \bar{D} \to \bar{D}$ be a diffeomorphism with $F|_{\partial D} = \mathrm{Id}$, where Id is the identity map. Then we can define the push-forward of $\sigma$ as

$$F_*\sigma = \left( \frac{{}^t(DF)\sigma(DF)}{|\det(DF)|} \right) \circ F^{-1}, \tag{0.10}$$

where $DF$ is the matrix differential of $F$, and one verifies that $\Lambda_{F_*\sigma} = \Lambda_\sigma$.

It happens that in dimension two this is the only obstruction to unique identifiability of the conductivity. The anisotropic problem can be reduced to the isotropic one by using isothermal coordinates [**83**], and combining this technique with the result of [**64**] for isotropic conductivities one obtains a uniqueness result for anisotropic conductivities with two derivatives. Uniqueness for $L^\infty$-conductivities was later obtained in [**10**] (see also [**11**]): for an anisotropic conductivity $\sigma \in L^\infty(D)$ ($D \subset \mathbb{R}^2$ bounded simply connected domain) the Dirichlet-to-Neumann map determines the equivalence class of conductivities $\sigma'$ such that there exists a diffeomorphism $F : D \to D$ in the $W^{1,2}$ class with $F|_{\partial D} = \mathrm{Id}$ and $\sigma' = F_*\sigma$.

The main purpose of our Paper **A** is to clarify and show what one can explicitly reconstruct from the Dirichlet-to-Neumann operator in the anisotropic case, on the plane. The main result is the following.

THEOREM 0.1 ([Paper **A**]). *Let $\hat{D} \subset \mathbb{R}^2$ be a bounded domain with $C^1$ boundary and let $\hat{\sigma}$ be a $C^2$-anisotropic conductivity on $\hat{D}$, isotropic in a neighbourhood of $\partial\hat{D}$. Suppose we know $\Lambda_{\hat{\sigma}} : H^{1/2}(\partial\hat{D}) \to H^{-1/2}(\partial\hat{D})$.*

*Then we can reconstruct a unique domain $D \subset \mathbb{R}^2 \sim \mathbb{C}$ (up to a biholomorphism), an isotropic conductivity $\sigma$ on $D$ and the boundary values $F|_{\partial\hat{D}}$ of a quasiconformal $C^1$-diffeomorphism $F : \hat{D} \to D$ such that $\sigma = F_*\hat{\sigma}$.*

This was motivated by results of [**83**], [**68**], [**64**], [**41**] and yields, as a corollary, earlier global uniqueness result of [**64**]. The new point in this statement is the existence of $F : \hat{D} \to D$ (and its explicit reconstruction at the boundary) without any assumption on the topology of $\hat{D}$. Earlier in [**10**] this result was proved for simply connected domains.

The main idea behind the paper is that, since the class of isotropic conductivities is preserved by conformal maps, to any anisotropic conductivity $\hat{\sigma}$ we can associate a complex structure given by a Beltrami coefficient $\mu_{\hat{\sigma}}$. The conductivity, in this new complex structure (and generally on a different domain), becomes isotropic and thus is it possible to apply the techniques of the isotropic case. Theorem 0.1 yields in particular that the Dirichlet-to-Neumann operator uniquely determines this complex structure.

In Paper **B** we generalised the result to the case of surfaces with boundary. Before stating the result, some remarks on the Calderón problem on manifolds are useful.

On a $d$-dimensional manifold $M$ with boundary, with $d \geq 2$, an anisotropic conductivity $\sigma$ can be represented by a mapping from 1-forms to $(d-1)$-forms, symmetric and positive definite, in a sense to be made precise (see **[83]** for more details). The conductivity equation becomes $d\sigma\, du = 0$ (where $d$ is the exterior derivative acting on differential forms) and the Dirichlet-to-Neumann operator $\Lambda_\sigma$ is defined as

$$\Lambda_\sigma f = \sigma\, du|_{\partial M}, \tag{0.11}$$

where $u$ is the unique solution of the Dirichlet problem

$$d\sigma\, du = 0 \text{ in } M, \qquad u|_{\partial M} = f. \tag{0.12}$$

Thus the Calderón problem becomes: given a manifold $M$ (with a fixed metric) and $\Lambda_\sigma$ on $\partial M$, determine $\sigma$ or the class of conductivities which produces the same boundary data as $\sigma$.

A closely related problem is the reconstruction of a Riemannian manifold, without knowing its metric, from the knowledge of its boundary and of the Dirichlet-to-Neumann map of the Laplacian. More precisely, let $(M, g)$ be a smooth Riemannian manifold with boundary. We can consider the Dirichlet problem

$$\Delta_g u = 0 \text{ in } M, \qquad u|_{\partial M} = f, \tag{0.13}$$

where $\Delta_g$ is the Laplace-Beltrami operator associated to the metric $g$. The Dirichlet-to-Neumann map is defined in this case as the normal derivative

$$\Lambda_g f = \sum_{j,k} g^{jk} \frac{\partial u}{\partial x_j} \nu_k \bigg|_{\partial M}, \tag{0.14}$$

where $\nu = \sum_l \nu^l \partial/\partial x_l$ denotes the unit outer normal to $\partial M$, $\nu_k = \sum_l g_{kl}\nu^l$ is the conormal. Here $(g^{jk})$ is the inverse of $(g_{jk})$, the local coordinates expression of $g$. The problem is then to reconstruct the metric $g$ (up to some transformations) from $\Lambda_g$.

Another close problem, similar to Problem 2, is the reconstruction of a potential from the Schrödinger equation on a manifold. Let $(M, g)$ as above, and $v$ a smooth complex-valued function on $(M, g)$. Suppose that 0 is not a Dirichlet eigenvalue for the operator $-\Delta_g + v$ in $M$. Then the Dirichlet problem

$$(-\Delta_g + v)u = 0 \text{ in } M, \qquad u|_{\partial M} = f, \tag{0.15}$$

has a unique solution for any $f \in H^{1/2}(\partial M)$ (for instance) and we define the Dirichlet-to-Neumann map

$$\Lambda_{g,q} : f \mapsto \sum_{j,k} g^{jk} \frac{\partial u}{\partial x_j} \nu_k \bigg|_{\partial M}. \tag{0.16}$$

These problems on manifolds are mostly motivated by the anisotropic Calderón problem for euclidean domains in $\mathbb{R}^d$, $d \geq 3$. This problem is still open and the results obtained on manifolds can be seen as partial results towards its solution. For results obtained in dimension $d \geq 3$ we refer to [**28**].

Uniqueness for the reconstruction of a Riemann surface from the Dirichlet-to-Neumann map was proved in [**58**], [**18**], [**44**]. For the Schrödinger equation and the isotropic Calderón problem on bordered Riemann surfaces, uniqueness results were given in [**38**] and an explicit reconstruction procedure in [**46**]. See also [**40**], [**3**] for related problems on surfaces and [**39**] for other references.

In our Paper **B** the anisotropic Calderón problem is solved for the first time on bordered surfaces in $\mathbb{R}^3$, or equivalently on Riemann surfaces with boundary. The main statement is very close to Theorem 0.1.

THEOREM 0.2 ([Paper **B**]). *Let $X$ be a bordered, $C^3$, oriented, two-dimensional manifold in $\mathbb{R}^3$ with $C^3$ boundary and let $\hat{\sigma}$ be a $C^3$ anisotropic conductivity on $X$. From the Dirichlet-to-Neumann operator $\Lambda_{\hat{\sigma}}$ and from the knowledge of the genus of $X$, we can find by an explicit procedure:*

*i) a bordered Riemann surface $Y$,*

*ii) an isotropic conductivity $\sigma$ on $Y$,*

*iii) a $C^3$ diffeomorphism $F : X \to Y$ such that $F_*\hat{\sigma} = \sigma$.*

*Moreover, if $\tilde{Y}$ is another Riemann surface, $\tilde{\sigma}$ an isotropic conductivity on $\tilde{Y}$ and $\tilde{F} : X \to \tilde{Y}$ a $C^3$ diffeomorphism such that $\tilde{F}_*\hat{\sigma} = \tilde{\sigma}$, then $\Psi = \tilde{F} \circ F^{-1} : Y \to \tilde{Y}$ is a biholomorphism such that $\Psi_*\sigma = \tilde{\sigma}$.*

Here the push-forward of a conductivity $\sigma$ by a diffeomorphism $F : \overline{X} \to \overline{Y}$ is defined, following [**83**, §1], as

$$(F_*\sigma)\alpha = F_*(\sigma(F^*\alpha)), \tag{0.17}$$

where $F^*\alpha$ denotes the pull-back of the 1-form $\alpha$ and $F_* = (F^{-1})^*$ denotes the pull-back by $F^{-1}$ acting on the 1-form $\sigma(F^*\alpha)$. In local coordinates it coincides with the previous definition (0.10).

Theorem 0.2 yields the following uniqueness result as a corollary.

COROLLARY 0.3 ([Paper **B**]). *Let $X$ be a bordered, $C^3$, oriented, two-dimensional manifold in $\mathbb{R}^3$ with $C^3$ boundary and let $\sigma_1, \sigma_2$ be two $C^3$-anisotropic conductivities on $X$. If $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$ then there exists a $C^3$ diffeomorphism $G : \overline{X} \to \overline{X}$ such that $G|_{\partial X} = \mathrm{Id}$ and $\sigma_2 = G_*\sigma_1$.*

Despite the statement of Theorem 0.2 being very similar to that of Theorem 0.1, the proof involves much more sophisticated techniques.

The basic idea is to reduce the problem to the isotropic case. On the plane this is done by solving a Beltrami equation with special asymptotics; a systematic study of this equation was already done by Ahlfors [**2**] and Vekua [**86**]. For surfaces, although the study of the Beltrami equation on Riemann surfaces is quite developed and has several applications – for instance Teichmüller theory – we could not find in the literature results about global solutions with prescribed conditions. Thus an important part of the paper is devoted to the construction of such global solutions.

This is done in several steps: we first embed the surface in a canonical way into the complex affine space $\mathbb{C}^3$ as a domain on a nonsingular affine algebraic curve, and then we study the Beltrami equation on the whole algebraic curve, using in particular results of Ahlfors [**2**] and Vekua [**86**] as well as the Hodge-Riemann decomposition [**49**]. In the rest of the paper we use these solutions of the Beltrami equation in order to deform the surface and reduce the problem to the isotropic case. We use the reconstruction procedure of [**46**] and also Cauchy-type formulas to reconstruct the deformed Riemann surface given only the points of its boundary (solving a kind of Plateau's problem).

We now focus on Problem 2. For dimension $d \geq 3$ the first global uniqueness and reconstruction results were obtained in [**68**] and a stability estimate in [**4**]. Principal improvements of these global reconstruction and stability results were recently given in [**74**], [**75**].

The method used to solve Problem 2 in dimension $d \geq 3$ (but also Problem 1 for smooth conductivities, in dimension $d \geq 2$) is to introduce an intermediary object between the boundary data and the potential: the (generalised) scattering amplitude.

The inverse scattering problem, i.e. the reconstruction of a potential in the Schrödinger equation from its (generalised) scattering amplitude, was proposed and studied much earlier then Problem 1 or 2. This problem comes initially from quantum mechanics (see [**33**]), but afterwards it appeared in several other context, for instance nonlinear evolution equations (see [**16**], [**45**, Chapter 1], [**35**] for a survey of results).

The fundamental object used to solve Problem 2 in dimension $d \geq 3$ is a special family of solutions $\psi(x,k)$ of equation (0.6), depending on a complex parameter $k \in \mathbb{C}^n$ such that $k^2 = k_1^2 + \ldots + k_n^2 = E \in \mathbb{R}$. These functions were originally introduced by Faddeev [**32**] in quantum scattering and are also called *complex geometrical optics* solutions: their main property is an exponential asymptotic condition with a linear phase depending on the complex parameter $k$. Without enter into details we just mention the fact that, in order to use such functions to solve Problem 2 in two dimensions, it is required their existence (and uniqueness) for every complex parameter $k \in \mathbb{C}^2$, $k^2 = E$. This was proved to be the case in [**69**], but only for high energies (in modulus) and yielded the solution to the inverse scattering problem on the plane; it was also proved in [**64**] in the zero energy case, but only for conductivity type potentials.

Global uniqueness in two dimensions at zero energy was obtained quite recently in [**21**] and the proof is based on totally new ideas. In our Papers **C, D**, we developed these ideas, together with results going back to [**4**], in order to obtain stability estimates for Problem 2 at zero energy in two dimensions; Paper **C** deals with the scalar case while Paper **D** with the multi-channel case, i.e. the case where the potential in the Schrödinger equation is matrix-valued. The main result of Paper **C** is the following.

THEOREM 0.4 ([Paper **C**]). *Let $D \subset \mathbb{R}^2$ be an open bounded domain with $C^2$ boundary, let $v_1, v_2 \in C^2(\bar{D})$ which satisfy (0.7), with $\|v_j\|_{C^2(\bar{D})} \leq N$ for $j = 1, 2$, and $\Phi_1, \Phi_2$ the corresponding Dirichlet-to-Neumann operators. For simplicity we assume also that $v_j|_{\partial D} = 0$ and $\frac{\partial}{\partial \nu} v_j|_{\partial D} = 0$ for $j = 1, 2$. Then there exists a constant $C = C(D, N)$ such that*

$$\|v_2 - v_1\|_{L^\infty(D)} \leq C(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{-\frac{1}{2}} \log(3 \log(3 + \|\Phi_2 - \Phi_1\|^{-1})),$$

*where $\|A\|$ denotes the norm of an operator $A : L^\infty(\partial D) \to L^\infty(\partial D)$.*

This is the first global stability result for the Gel'fand-Calderón inverse problem in two dimensions, for general real-valued or complex-valued potentials. Results of such a type were only known for special kinds of potentials, e.g. conductivity type potentials. It is possible to ignore the condition on the boundary values of the potential and the normal derivatives; in the same paper we give indeed a more general, but weaker, log-type stability estimate.

In our Paper **D** this result was improved and generalised to the multi-channel case: this means that we consider the Schrödinger equation (0.6) at zero energy where the potential $v$, as well as $\psi$, is a $M_n(\mathbb{C})$-valued function; here $M_n(\mathbb{C})$ is the set of $n \times n$ complex matrices.

One of the main motivations to study of the 2D multi-channel case is that it can be seen as an approximation to the 3D equation. As already mentioned before, the two dimensional case is non overdetermined, thus for different applications it is useful to study a 2D multi-channel approximation of the 3D equation. Indeed, consider the 3D Schrödinger equation on a cylindrical domain $D \times L$, $D \subset \mathbb{R}^2$, $L = [a, b] \subset \mathbb{R}$. Using the orthonormal basis in $L^2(L)$ given by the eigenfunctions of the Laplacian it is straightforward to see that this 3D equation it is equivalent to an infinite dimensional system of the form

$$-\Delta_x \psi_i(x) + \sum_{j=1}^{\infty} V_{ij}(x)\psi_j(x) = -\lambda_i \psi_j(x),\ x \in D,\ i = 1, \ldots.$$

If we impose $1 \leq i, j \leq n$, for some $n \in \mathbb{N}$ we obtain a finite system (for full details see §2 of Paper **E**). In Paper **D** the following estimates is proved.

THEOREM 0.5 ([Paper **D**]). *Let $D \subset \mathbb{R}^2$ be an open bounded domain with a $C^2$ boundary, $v_1, v_2 \in C^2(\bar{D}, M_n(\mathbb{C}))$ two matrix-valued potentials which satisfy (0.7), with $\|v_j\|_{C^2(\bar{D})} \leq N$ for $j = 1, 2$, and $\Phi_1, \Phi_2$ the corresponding Dirichlet-to-Neumann operators. For simplicity we also assume that $v_1|_{\partial D} = v_2|_{\partial D}$ and $\frac{\partial}{\partial \nu} v_1|_{\partial D} = \frac{\partial}{\partial \nu} v_2|_{\partial D}$. Then there exists a constant $C = C(D, N, n)$ such that*

$$\|v_2 - v_1\|_{L^\infty(D)} \leq C \left(\log(3 + \|\Phi_2 - \Phi_1\|^{-1})\right)^{-\frac{3}{4}} \left(\log(3 \log(3 + \|\Phi_2 - \Phi_1\|^{-1}))\right)^2,$$

*where $\|\cdot\|$ is the induced operator norm on $L^\infty(\partial D, M_n(\mathbb{C}))$ and $\|v\|_{L^\infty(D)} = \max_{1 \leq i,j \leq n} \|v_{i,j}\|_{L^\infty(D)}$ (likewise for $\|v\|_{C^2(\bar{D})}$) for a matrix-valued potential $v$.*

The proof of this result is based on the following construction. Using the standard identification of $\mathbb{R}^2$ with $\mathbb{C}$, with the coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, where $(x_1, x_2) \in \mathbb{R}^2$, we define a special family of solutions of equation (0.6), which we call the Buckhgeim analogues of the Faddeev solutions: $\psi_{z_0}(z, \lambda)$, for $z, z_0 \in \bar{D}, \lambda \in \mathbb{C}$, such that $-\Delta\psi + v(x)\psi = 0$ over $D$, where in particular $\psi_{z_0}(z, \lambda) \to e^{\lambda(z-z_0)^2} I$ for $\lambda \to \infty$ (i.e. for $|\lambda| \to +\infty$) and $I$ is the identity matrix.

More precisely, for a matrix valued potential $v$ of size $n$, we define $\psi_{z_0}(z, \lambda)$ as $\psi_{z_0}(z, \lambda) = e^{\lambda(z-z_0)^2} \mu_{z_0}(z, \lambda)$, where $\mu_{z_0}(\cdot, \lambda)$ solves the integral equation

$$\mu_{z_0}(z, \lambda) = I + \int_D g_{z_0}(z, \zeta, \lambda) v(\zeta) \mu_{z_0}(\zeta, \lambda) d\mathrm{Re}\zeta \, d\mathrm{Im}\zeta, \tag{0.18}$$

$I$ is the identity matrix of size $n \in \mathbb{N}$, $z, z_0 \in \bar{D}$, $\lambda \in \mathbb{C}$ and

$$g_{z_0}(z, \zeta, \lambda) = \frac{e^{\lambda(\zeta-z_0)^2 - \bar{\lambda}(\bar{\zeta}-\bar{z}_0)^2}}{4\pi^2} \int_D \frac{e^{-\lambda(\eta-z_0)^2 + \bar{\lambda}(\bar{\eta}-\bar{z}_0)^2}}{(z-\eta)(\bar{\eta}-\bar{\zeta})} d\mathrm{Re}\eta \, d\mathrm{Im}\eta \tag{0.19}$$

is a Green function of the operator $4\left(\frac{\partial}{\partial z} + 2\lambda(z - z_0)\right)\frac{\partial}{\partial \bar{z}}$ in $D$, for $z_0 \in D$ (we used here the complex derivatives $\partial/\partial z, \partial/\partial \bar{z}$). Equation (0.18), at fixed $z_0$ and $\lambda$, is considered as a linear integral equation for $\mu_{z_0}(\cdot, \lambda)$ in some function space. In Papers **C** and **D** we prove several estimates for the green function $g_{z_0}$ which yield in particular existence and uniqueness for solution of integral equation (0.18) for sufficiently large $|\lambda|$. The results of Paper **D** are obtained using this construction and an appropriate matrix-valued version of Alessandrini's identity, along with stationary phase techniques.

Combining the above construction with the approach of [**68**], in Paper **E** we gave a global uniqueness result and an exact reconstruction method for Problem 2 in the multi-channel case. This will clarify the usefulness of these special solutions with exponential asymptotic with a quadratic phase.

In order to state the results we introduce the Bukhgeim analogue of the Faddeev generalized scattering amplitude

$$h_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} v(z) \mu_{z_0}(z, \lambda) d\mathrm{Re}z \, d\mathrm{Im}z, \tag{0.20}$$

for $z_0 \in \bar{D}$, $\lambda \in \mathbb{C}$.

Theorem 0.6 ([Paper **E**]). *Let $D \subset \mathbb{R}^2$ be an open bounded domain with $C^2$ boundary and let $v \in C^1(\bar{D}, M_n(\mathbb{C}))$ be a matrix-valued potential which satisfies (0.7) and $v|_{\partial D} = 0$. Consider, for $z_0 \in D$, the functions $h_{z_0}$, $\psi_{z_0}$, $g_{z_0}$ defined above* *and* $\Phi, \Phi_0$ *the Dirichlet-to-Neumann maps associated to the potentials* $v$ *and* $0$, *respectively. Then the following reconstruction formulas and equation hold:*

$$v(z_0) = \lim_{\lambda\to\infty} \frac{2}{\pi}|\lambda| h_{z_0}(\lambda), \tag{0.21}$$

$$h_{z_0}(\lambda) = \int_{\partial D} e^{-\bar\lambda(\bar z - \bar z_0)^2}(\Phi - \Phi_0)\psi_{z_0}(z,\lambda)|dz|, \tag{0.22}$$

$$\psi_{z_0}(z,\lambda)|_{\partial D} = e^{\lambda(z-z_0)^2} I + \int_{\partial D} G_{z_0}(z,\zeta,\lambda)(\Phi - \Phi_0)\psi_{z_0}(\zeta,\lambda)|d\zeta|, \tag{0.23}$$

*where*

$$G_{z_0}(z,\zeta,\lambda) = e^{\lambda(z-z_0)^2} g_{z_0}(z,\zeta,\lambda) e^{-\lambda(\zeta-z_0)^2}, \tag{0.24}$$

$z_0 \in D$, $z,\zeta \in \partial D$, $\lambda \in \mathbb{C}$, $|\lambda| > \rho_1(D, N_1, n)$, *where* $\|v\|_{C^1_{\bar z}(\bar D, M_n(\mathbb{C}))} < N_1$.

*In addition, if* $v \in C^2(\bar D, M_n(\mathbb{C}))$ *with* $\|v\|_{C^2(\bar D, M_n(\mathbb{C}))} < N_2$ *and* $\frac{\partial v}{\partial \nu}|_{\partial D} = v|_{\partial D} = 0$ *then the following estimates hold:*

$$\left| v(z_0) - \frac{2}{\pi}|\lambda| h_{z_0}(\lambda)\right| \le a(D,n)\frac{\log(3|\lambda|)}{|\lambda|^{1/2}} N_2(N_2+1), \tag{0.25a}$$

$$\left| v(z_0) - \frac{2}{\pi}|\lambda| h_{z_0}(\lambda)\right| \le b(D,n)\frac{(\log(3|\lambda|))^2}{|\lambda|^{3/4}} N_2(N_2^2+1), \tag{0.25b}$$

*for* $|\lambda| > \rho_2(D, N_1, n)$, $z_0 \in D$.

In order to make use of this reconstruction procedure, the following two propositions are also necessary:

PROPOSITION 0.7 ([Paper **E**]). *Under the assumptions of Theorem 0.6 (without the additional assumptions used for* (0.25)*), equation* (0.23) *is a Fredholm linear integral equation of the second kind for* $\psi_{z_0} \in C(\partial D)$.

PROPOSITION 0.8 ([Paper **E**]). *Under the assumptions of Theorem 0.6 (without the additional assumptions used for* (0.25)*), for* $|\lambda| > \rho_1(D, N_1, n)$, *where* $\|v\|_{C^1_{\bar z}(\bar D, M_n(\mathbb{C}))} < N_1$, *equations* (0.18) *and* (0.23) *are uniquely solvable in the spaces of continuous functions on* $\bar D$ *and* $\partial D$, *respectively.*

Note that if $v|_{\partial D} \neq 0$ but $v \equiv \Lambda \in M_n(\mathbb{C})$ on some open neighborhood of $\partial D$ in $\bar D$, then estimates (0.25) hold with some minor modifications in the definition of the generalised scattering amplitude.

Theorem 0.6 and Propositions 0.7, 0.8 yield the following uniqueness result.

COROLLARY 0.9 ([Paper **E**]). *Let* $D \subset \mathbb{R}^2$ *be an open bounded domain with* $C^2$ *boundary, let* $v_1, v_2 \in C^1(\bar D, M_n(\mathbb{C}))$ *be two matrix-valued potentials which satisfy* (0.7) *and* $\Phi_1, \Phi_2$ *the corresponding Dirichlet-to-Neumann operators. If* $\Phi_1 = \Phi_2$ *then* $v_1 = v_2$.

Note that global uniqueness for the more general problem of the Schrödinger equation with external Yang-Mills potentials was obtained in [**31**] in dimension $d \geq 3$ and recently in [**3**] on Riemann surfaces.

The global reconstruction of Paper **E** is fine, in the sense that it consists in solving Fredholm linear integral equations of the second type and using explicit formulas; nevertheless this reconstruction is not optimal with respect to its stability properties: see [**22**], [**75**], [**20**], Paper **F** for discussions and numerical implementations of the aforementioned similar (but overdetermined) reconstruction of [**68**] for Problem 2 for dimension $d = 3$ in the single-channel case.

In paper **F** we presented a stable and rapidly converging algorithm which gives an approximate solution to Problem 2 in the multichannel case. At first sight this could seem contradicting all the aforementioned stability estimates: indeed, at any fixed energy, Problem 2 is severely ill posed, as shown by the logarithmic estimates. What happens is that at high energies the stability becomes approximately Lipschitz: we can indeed construct – at any fixed sufficiently large energy $E$ and with Lipschitz stability – an approximated potential which converges to the exact one with an error of size $O(E^{-(m-2)})$, where $m$ is related to the regularity of the potential.

This reconstruction is mainly inspired by a similar algorithm developed in [**72**], [**71**] to solve the inverse scattering problem in two dimensions (see [**73**] for similar results in dimension 3); we now state this problem in the multi-channel case. The scattering amplitude $f$ is defined as follows: we consider the continuous solutions $\psi^+(x,k)$ of the multi-channel equation (0.6) (with $E > 0$), where $k$ is a parameter, $k \in \mathbb{R}^d, k^2 = E$, such that

$$\psi^+(x,k) = e^{ikx}I - i\pi\sqrt{2\pi}e^{-i\frac{\pi}{4}} f\left(k, |k|\frac{x}{|x|}\right)\frac{e^{i|k||x|}}{\sqrt{|k||x|}} + o\left(\frac{1}{\sqrt{|x|}}\right), \qquad \text{as } |x| \to \infty, \tag{0.26}$$

for some *a priori* unknown $M_n(\mathbb{C})$-valued function $f$, where $I$ is the identity matrix. The function $f$ on $\mathscr{M}_E = \{(k,l) \in \mathbb{R}^2 \times \mathbb{R}^2 : k^2 = l^2 = E\}$ arising in (0.26) is the scattering amplitude for the potential $v$ in the framework of equation (0.6). This construction gives rise to the following inverse scattering problem on $\mathbb{R}^2$:

Problem 3. *Given $f$ on $\mathscr{M}_E$, find $v$ on $\mathbb{R}^2$.*

Approximate reconstruction algorithms for Problem 2 (Algorithm 1) and Problem 3 (Algorithm 2), in the multichannel case, are given in Paper **F**. As well as in [**72**], [**71**] our functional analytic approach gives an efficient nonlinear approximation $v_{appr}(x,E)$ to the unknown $v(x)$ of the two problems. The reconstruction of $v_{appr}(x,E)$ from $\Phi(E)$ for Problem 2 and from $f$ on $\mathscr{M}_E$ for Problem 3 is realized with some Lipschitz stability and is based on solving linear integral equations; we refer to the paper for full details but we sketch here the main ideas.

The first part of the algorithms is the following: starting from $\Phi(E)$, for Problem 2, and $f$ on $\mathscr{M}_E$, for Problem 3, we find $M_n(\mathbb{C})$-valued functions $h_\pm$ on $\mathscr{M}_E$ through some linear integral equations. These functions, which can be stably reconstructed, are closely related to the generalised scattering amplitude $r$, first introduced by Faddeev, which is an analytic extension of the classical scattering amplitude $f$.

The second part of both algorithms consists in the reconstruction of $v_{appr}$ starting from $h_\pm$. This is done using the method of the non-local Riemann-Hilbert problem going back to [**61**] and $\bar\partial$ techniques of [**1**] developed in [**67**], [**36**], [**69**] for solving the inverse scattering problem at fixed energy in the plane. The reconstruction of the potential is based on properties of Faddeev-type solutions of the Schrödinger equation at fixed energy, which are obtained via some function $\mu(z,\lambda,E)$, where $z,\lambda\in\mathbb{C}$, $E>0$. This function satisfies a non-local Riemann-Hilbert problem with respect to the complex parameter $\lambda$, which can be solved for sufficiently large $E$. In particular, we have the following exact formula for the potential:

$$v(z) = 2iE^{1/2}\frac{\partial}{\partial z}\left(\frac{1}{\pi}\int\limits_{|\zeta|>1}\mu(z,-\frac{1}{\bar\zeta},E)r(\zeta,z,E)d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\mu_-(z,\zeta,E)i\zeta|d\zeta|\right), \tag{0.27}$$

for $z\in\mathbb{C}$ and $E$ sufficiently large ($\partial/\partial z$ is the complex derivatives with respect to $z$). Here $\mu_-$ is some restriction of $\mu$ to the unit circle, which can be stably reconstructed from $h_\pm$; in contrast, the reconstruction of $r$ from $\Phi(E)$ is not stable (and generally it is not even possible from $f$ on $\mathscr{M}_E$). The idea is to get rid of the first part and define

$$v_{appr}(z,E) = 2iE^{1/2}\frac{\partial}{\partial z}\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\mu_-(z,\zeta,E)i\zeta|d\zeta|\right), \tag{0.28}$$

because, thanks to estimates of the same type as in [**72**], the error $\|v-v_{appr}(\cdot,E)\|_{L^\infty}$ goes to zero for $E\to+\infty$.

More precisely, we introduce the following function spaces, for $m\in\mathbb{N}$, $\varepsilon>0$,

$$W^{m,1}(\mathbb{R}^2,M_n(\mathbb{C})) = \{u : \partial^k u\in L^1(\mathbb{R}^2,M_n(\mathbb{C})) \text{ for } |k|\le m\},$$
$$W^{m,1}_\varepsilon(\mathbb{R}^2,M_n(\mathbb{C})) = \{u : \varkappa^\varepsilon\partial^k u\in L^1(\mathbb{R}^2,M_n(\mathbb{C})) \text{ for } |k|\le m\},$$
$$(\varkappa^\varepsilon u)(x) = (1+|x|^2)^{\varepsilon/2}u(x),\qquad k\in(\mathbb{N}\cup 0)^2,\qquad |k|=k_1+k_2,$$
$$\partial^k = \partial_1^{k_1}\partial_2^{k_2},\qquad \partial_j = \frac{\partial}{\partial x_j}.$$

Then we have the following theorems.

THEOREM 0.10 ([Paper **F**]). *Let* $v \in W^{m,1}(\mathbb{R}^2, M_n(\mathbb{C}))$, $m \geq 3$, $\operatorname{supp} v \subset D$ *and let* $\Phi(E)$ *be the corresponding Dirichlet-to-Neumann operator at fixed energy* $E$, *where* $E \geq E_2$ *and* $E$ *is not a Dirichlet eigenvalue of* $-\Delta+v$ *and* $-\Delta$ *in* $D$. *Then* $v$ *is reconstructed from* $\Phi(E)$ *with Lipschitz stability via Algorithm 1 up to* $O(E^{-(m-2)/2})$ *in the uniform norm as* $E \to +\infty$.

THEOREM 0.11 ([Paper **F**]). *Let* $v \in W^{m,1}_{\varepsilon}(\mathbb{R}^2, M_n(\mathbb{C}))$, *for* $m \geq 3$, *and let* $f$ *be the corresponding scattering amplitude at fixed energy* $E \geq E_2$. *Then* $v$ *is reconstructed from* $f$ *on* $\mathscr{M}_E$ *with Lipschitz stability via Algorithms 2 up to* $O(E^{-(m-2)/2})$ *in the uniform norm as* $E \to +\infty$.

The constant $E_2$ in Theorems 0.10 and 0.11 is precisely stated in the paper, as well as the error term $O(E^{-(m-2)/2})$.

Next, motivated by these algorithms (as well as [**72**], [**71**]) and by instability estimates of Mandache [**62**] (improved in [**50**]), we focused again on global stability estimates for Problems 1 and 2 (in the single channel case). On one hand Theorem 0.10 indicates that at high energies the stability estimates can be decomposed into a stable part (Lipschitz) and an unstable part (logarithmic) where the latter goes to zero as $E \to +\infty$. On the other hand instability estimates show that, at any fixed energy, an inequality like

$$\|v_2 - v_1\|_{L^\infty(D)} \leq c(\log(3 + \|\Phi_2(E) - \Phi_1(E)\|_*^{-1}))^{-\alpha}, \tag{0.29}$$

cannot hold for $\alpha > m$, for complex-valued $v_1, v_2 \in C^m(D)$ and $\Phi_1(E), \Phi_2(E)$ the corresponding Dirichlet-to-Neumann operators (here we used the notation $\|\cdot\|_* = \|\cdot\|_{H^{1/2}(\partial D)\to H^{-1/2}(\partial D)}$). Bearing in mind these considerations, we obtained new estimates which depend on regularity and energy.

In Paper **G** we tackled Problem 1 and Problem 2 at zero energy. In that paper it is assumed for simplicity that

$$\begin{aligned} &D \text{ is an open bounded domain in } \mathbb{R}^2, \qquad \partial D \in C^2, \\ &v \in W^{m,1}(\mathbb{R}^2) \text{ for some } m > 2, \qquad \operatorname{supp} v \subset D, \end{aligned} \tag{0.30}$$

where $W^{m,1}(\mathbb{R}^2)$ was defined above; let also $\|v\|_{m,1} = \max_{|J|\leq m} \|\partial^J v\|_{L^1(\mathbb{R}^2)}$. The main hypothesis is that only potentials of conductivity type are considered, i.e.

$$v = \frac{\Delta\sigma^{1/2}}{\sigma^{1/2}}, \text{ for some } \sigma \in L^\infty(D), \text{ with } \sigma \geq \sigma_{\min} > 0. \tag{0.31}$$

The main results are the following.

THEOREM 0.12 ([Paper **G**]). *Let the conditions* (0.7), (0.30), (0.31) *hold for the potentials* $v_1, v_2$, *where* $D$ *is fixed, and let* $\Phi_1$ , $\Phi_2$ *be the corresponding Dirichlet-to-Neumann operators. Let* $\|v_j\|_{m,1} \leq N$, $j = 1, 2$, *for some* $N > 0$. *Then there exists a constant* $C = C(D, N, m)$ *such that*

$$\|v_2 - v_1\|_{L^\infty(D)} \leq C(\log(3 + \|\Phi_2 - \Phi_1\|_*^{-1}))^{-\alpha}, \tag{0.32}$$

*where* $\alpha = m - 2$.

Theorem 0.13 ([Paper **G**]). *Let $\sigma_1, \sigma_2$ be two isotropic conductivities such that $\Delta(\sigma_j^{1/2})/\sigma_j^{1/2}$ satisfies conditions* (0.30)*, where $D$ is fixed and $0 < \sigma_{\min} \leq \sigma_j \leq \sigma_{\max} < +\infty$ for $j = 1, 2$ and some constants $\sigma_{\min}$ and $\sigma_{\max}$. Let $\Lambda_1$ , $\Lambda_2$ be the corresponding Dirichlet-to-Neumann operators and $\|\Delta(\sigma_j^{1/2})/\sigma_j^{1/2}\|_{m,1} \leq N$, $j = 1, 2$, for some $N > 0$. We suppose, for simplicity, that* $\operatorname{supp}(\sigma_j - 1) \subset D$ *for $j = 1, 2$. Then, for any $\alpha < m$ there exists a constant $C = C(D, N, \sigma_{\min}, \sigma_{\max}, m, \alpha)$ such that*

$$\|\sigma_2 - \sigma_1\|_{L^\infty(D)} \leq C(\log(3 + \|\Lambda_2 - \Lambda_1\|_*^{-1}))^{-\alpha}. \tag{0.33}$$

The main feature of these estimates is that, as $m \to +\infty$, we have $\alpha \to +\infty$. Indeed in every previous global logarithmic stability estimates (in the uniform norm) for Problems 1 and 2 in two dimensions, the exponent $\alpha$ satisfied $\alpha < 1$, even for infinitely smooth potentials/conductivities. In dimension $d \geq 3$ a global stability estimate similar to our result (with respect to dependence on smoothness) was proved in [**75**]: more precisely, an estimate of the same type of (0.32) holds with the exponent $\alpha = m - d$. The assumptions we made on the support of our potentials and conductivities can be taken away by the use of boundary determination results (see, for instance, [**14**, Proposition 2.11] for the Calderón problem); however, in that case, the exponent in the estimates will be generally smaller than the $\alpha$ of our theorems (but always increasing with regularity).

The proof of these results is based on $\bar\partial$ techniques and Faddeev-type functions already mentioned. We use fundamental results from [**64**], [**70**], [**72**] and some useful lemma from [**14**]. For Theorem 0.12, for instance, the idea is to study the stability of the map $\Phi \to v$ as the composition of $\Phi \to h$ and $h \to v$ where $h$ is the generalised Faddeev scattering amplitude at zero energy. The first map is found to satisfy a logarithmic stability (with explicit dependence on the regularity of the potentials) and the second one an Holdĕr stability; in order to prove the latter we obtain in particular some new estimates on the Faddeev function $\mu$ at zero energy.

In Paper **H** we started studies of stability estimates for Problem 2 at non-zero energies. In this paper we restricted ourself to the negative energy case just for the simplicity of the proofs. There we give three estimates: the first one depends on the regularity of potentials like in (0.32), while the others depends also on the energy. Exact energy-dependent Hölder-logarithmic stability estimates for the Schrödinger equations have been studied only recently (see [**55**], [**54**], [**66**], [**82**], [**51**]) and this is the first result in two dimensions. Like in Paper **G** it is assumed that

$$\begin{aligned} &D \text{ is an open bounded domain in } \mathbb{R}^2, \qquad \partial D \in C^2, \\ &v \in W^{m,1}(\mathbb{R}^2) \text{ for some } m > 2, \quad \bar v = v, \quad \operatorname{supp} v \subset D, \end{aligned} \tag{0.34}$$

We will need the following regularity condition:

$$|E| > E_1, \tag{0.35}$$

where $E_1 = E_1(\|v\|_{m,1}, D)$. This condition implies, in particular, that the Faddeev functions are well-defined on the entire fixed-energy surface in the spectral parameter. The main result is the following.

Theorem 0.14 ([Paper **H**]). *Let the conditions* (0.7)*,* (0.34)*,* (0.35) *hold for the potentials* $v_1, v_2$*, where* $D$ *is fixed, and let* $\Phi_1(E)$ , $\Phi_2(E)$ *be the corresponding Dirichlet-to-Neumann operators at fixed negative energy* $E < 0$*. Let* $\|v_j\|_{m,1} \le N$*,* $j = 1, 2$*, for some* $N > 0$*. Then there exists a constant* $c_1 = c_1(E, D, N, m)$ *such that*

$$\|v_2 - v_1\|_{L^\infty(D)} \le c_1(\log(3 + \|\Phi_2(E) - \Phi_1(E)\|_*^{-1}))^{-\alpha}, \tag{0.36}$$

*where* $\alpha = m - 2$*.*

*Moreover, there exists a constant* $c_2 = c_2(D, N, m)$ *such that for any* $0 < \kappa < 1/(l + 2)$*, where* $l = \mathrm{diam}(D)$*, we have*

$$\|v_2 - v_1\|_{L^\infty(D)} \le c_2\Big[\big(|E|^{1/2} + \kappa \log(3 + \delta^{-1})\big)^{-(m-2)} + \delta(3 + \delta^{-1})^{\kappa(l+2)} e^{|E|^{1/2}(l+3)}\Big], \tag{0.37}$$

*where* $\delta = \|\Phi_2(E) - \Phi_1(E)\|_*$*.*

*In addition, there exists a constant* $c_3 = c_3(D, N, m)$ *such that for* $E, \delta$ *which satisfy*

$$|E|^{1/2} > \log(3 + \delta^{-1}), \quad |E| > 1, \tag{0.38}$$

*we have*

$$\|v_2 - v_1\|_{L^\infty(D)} \le c_3\Big[|E|^{-(m-2)/2} \log(3 + \delta^{-1})^{-(m-2)} + \delta e^{|E|(l+3)}\Big]. \tag{0.39}$$

Estimate (0.36) is the negative-energy version of (0.32) and shares the same properties. As regards (0.37) and (0.39), their main feature is the explicit dependence on the energy $E$. These estimates consist each one of two parts, the first logarithmic and the second Hölder or Lipschitz; when $|E|$ increases, the logarithmic part decreases and the Hölder/Lipschitz part becomes dominant. These estimates are coherent with the approximate reconstruction algorithms of [**71**] and Paper **F** at positive energy. In fact inequalities like (0.36), (0.37) and (0.39) should be valid also for the Schrödinger equation at positive energy.

The proof of Theorem 0.14 follows the scheme of Paper **G** and it is based on similar $\bar\partial$ techniques. However here the most important references are [**37**], [**69**], where the inverse scattering problem at fixed non-zero energy is studied.

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# PAPER **A**

## PAPER A

# On an inverse problem for anisotropic conductivity in the plane

Gennadi Henkin and Matteo Santacesaria

Abstract. Let $\hat{\Omega} \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $\hat{\sigma}$ a smooth anisotropic conductivity on $\hat{\Omega}$. Starting from the Dirichlet-to-Neumann operator $\Lambda_{\hat{\sigma}}$ on $\partial\hat{\Omega}$, we give an explicit procedure to find a unique (up to a biholomorphism) domain $\Omega$, an isotropic conductivity $\sigma$ on $\Omega$ and the boundary values of a quasiconformal diffeomorphism $F : \hat{\Omega} \to \Omega$ which transforms $\hat{\sigma}$ into $\sigma$.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and let $\sigma$ be a $C^2$-anisotropic conductivity defined over $\Omega$, i.e. $\sigma = (\sigma^{ij})$ is a positive definite symmetric matrix on $\overline{\Omega}$ in the $C^2$ class. The corresponding Dirichlet-to-Neumann map is the operator $\Lambda_\sigma : C^1(\partial\Omega) \to L^p(\partial\Omega)$, $p < \infty$ defined by

$$\Lambda_\sigma f = \sigma \frac{\partial u}{\partial \nu}|_{\partial\Omega}, \tag{1.1}$$

where $f \in C^1(\partial\Omega)$, $\nu$ is the outer normal of $\partial\Omega$, and $u$ is the $C^1(\overline{\Omega})$-solution of the Dirichlet problem

$$\nabla \cdot (\sigma \nabla u) = 0 \text{ on } \Omega, \quad u|_{\partial\Omega} = f. \tag{1.2}$$

The equation (1.2) represents the conservation of the electrical charge on $\Omega$ if the voltage potential $f$ is applied to $\partial\Omega$, and $\Lambda_\sigma f$ is the current flux at the boundary. The following inverse problem arises from this construction: how much information about $\sigma$ can be detected from the knowledge of the mapping $\Lambda_\sigma$?

Inverse boundary values problems of such a type were formulated in precise mathematical terms by I. Gel'fand [**10**] and by A. Calderon [**5**]. These problems arise naturally in several areas: geophysical electrical prospecting (L. Slichter [**17**], V. Druskin [**6**]), medical imaging (D. Barber, B. Brown [**3**]), nondestructive testing of materials (A. Friedman, M. Vogelius [**9**]), etc.

It is not possible to determine $\sigma$ uniquely from $\Lambda_\sigma$. This was discovered by L. Tartar (see [**12**]). Indeed, let $\Phi : \overline{\Omega} \to \overline{\Omega}$ be a diffeomorphism with $\Phi|_{\partial\Omega} = \mathrm{Id}$, where Id is the identity map. Then we can define the push-forward of $\sigma$ as

$$\Phi_*\sigma = \left( \frac{{}^t(D\Phi)\sigma(D\Phi)}{|\det(D\Phi)|} \right) \circ \Phi^{-1},$$

where $D\Phi$ is the matrix differential of $\Phi$, and one verifies that $\Lambda_{\Phi_*\sigma} = \Lambda_\sigma$. In dimension two this is the only obstruction to unique identifiability of the conductivity. The anisotropic problem can be reduced to the isotropic one by using isothermal coordinates (Sylvester [**18**]), and combining this technique with the result of Nachman for isotropic conductivities ([**15**]) we obtain the uniqueness result for anisotropic conductivities with two derivatives. The optimal regularity condition was later obtained by Astala-Lassas-Paivarinta, who proved the uniqueness for $L^\infty$-conductivities in [**2**]: for an anisotropic conductivity $\sigma \in L^\infty(\Omega)$ ($\Omega \subset \mathbb{R}^2$ bounded simply connected domain) the Dirichlet-to-Neumann map determines the equivalence class of conductivities $\sigma'$ such that there exists a diffeomorphism $\Phi : \Omega \to \Omega$ in the $W^{1,2}$ class with $\Phi|_{\partial\Omega} = \mathrm{Id}$ and $\sigma' = \Phi_*\sigma$.

The main purpose of this article is to clarify and show what one can explicitly reconstruct from a given Dirichlet-to-Neumann operator in the anisotropic case. From the results obtained in [**18**], [**16**], [**15**], [**11**] we have deduced

THEOREM 1.1. *Let $\hat{\Omega} \subset \mathbb{R}^2$ be a bounded domain with $C^1$ boundary and let $\hat{\sigma}$ be a $C^2$-anisotropic conductivity on $\hat{\Omega}$, isotropic in a neighbourhood of $\partial\hat{\Omega}$. Suppose we know $\Lambda_{\hat{\sigma}} : C^1(\partial\hat{\Omega}) \to L^p(\partial\hat{\Omega})$, $p < \infty$.*

*Then we can reconstruct a unique domain $\Omega \subset \mathbb{R}^2 \sim \mathbb{C}$ (up to a biholomorphism), an isotropic conductivity $\sigma$ on $\Omega$ and the boundary values $F|_{\partial\hat{\Omega}}$ of a quasiconformal $C^1$-diffeomorphism $F : \hat{\Omega} \to \Omega$ such that $\sigma = F_*\hat{\sigma}$.*

The new point in this statement is the existence of $F : \hat{\Omega} \to \Omega$ (and its explicit reconstruction at the boundary) without any assumption on the topology of $\hat{\Omega}$. Early in [**2**] this result was proved for simply connected domains, a situation in which the question about deformations of complex structures of $\hat{\Omega}$ does not make sense.

Our main tool, as in [**18**] and [**2**], is the global solution $F$ of a certain Beltrami equation equipped with an asymptotic condition, which takes our anisotropic conductivity $\hat{\sigma}$ into an isotropic one, $\sigma$, defined in general over a different domain $\Omega = F(\hat{\Omega})$. With the help of $F$ we then show the existence and uniqueness of a family of solutions $\hat{\psi}(z, \lambda)$ of the anisotropic conductivity equation, with special asymptotics at infinity, using also the existence of such type of functions in the isotropic case, that we call $\psi(w, \lambda)$ (firstly introduced by Faddeev in [**7**]; see [**16**], [**15**] for the main properties). Then we show how one can reconstruct the boundary values of $\hat{\psi}$ from the Dirichlet-to-Neumann operator $\Lambda_{\hat{\sigma}}$, for any $\lambda$, with a Fredholm-type integral equation, following the work of Gutarts ([**11**]). This is a generalization of R. Novikov's method for isotropic conductivities ([**16**]). We also show how to find the boundary values of $F$ from the knowledge of $\hat{\psi}|_{\partial\hat{\Omega}}$ (generalizing the result in [**18**] and [**2**]), and so we find $F(\partial\hat{\Omega}) = \partial\Omega$ (therefore also $\Omega$).

After this, we explain how the knowledge of $\Lambda_{\hat{\sigma}}$, $\hat{\psi}|_{\partial\hat{\Omega}}$ and $F|_{\partial\hat{\Omega}}$ suffices to reconstruct the isotropic scattering amplitude $b(\lambda)$. We give also another method: we define the anisotropic scattering amplitude $\hat{b}(\lambda)$, and we show that it is equal to the isotropic one, proving that it is essentially a quasiconformal invariant. This result was already included in [**11**]; here we give a new simpler proof.

Thus with both methods, starting from $\hat{\psi}|_{\partial\hat{\Omega}}$ we can reconstruct the isotropic scattering amplitude: this allows us to write the $\overline{\partial}$-equation which will permit us to find the isotropic conductivity $\sigma$ on $\Omega$, by the Novikov-Nachman reconstruction scheme ([**16**], [**15**]).

Our scheme can be summarized in the following diagram

$$\Lambda_{\hat{\sigma}} \to \hat{\psi}|_{\partial\hat{\Omega}} \to \begin{cases} b(\lambda) \\ F|_{\partial\hat{\Omega}} \end{cases} \to \begin{cases} \sigma \\ \Omega \end{cases}$$

All steps of this reconstruction scheme are explicit and can be numerically implemented using the Novikov-Nachman reconstruction-type algorithm [**16**], [**15**]. Therefore, our paper admits potential practical applications.

Remark 1.1. Although we cannot reconstruct $\hat{\sigma}$ uniquely, for the applications it may be useful to find one representative of the equivalence class of $\hat{\sigma}$. To do this, using our theorem it suffices to find a diffeomorphism $G : \hat{\Omega} \to \Omega$ with fixed boundary values (which are the boundary values of a quasiconformal mapping, in our notation $F|_{\partial\hat{\Omega}}$), and no other particular restriction: in this way $(G^{-1})_*\sigma$ will be a representative of $\hat{\sigma}$. If $\Omega$ is simply connected one can use the Ahlfors-Beurling extension theorem for quasi-symmetric homeomorphism of the circle ([**1**, Thm. 2, p.69]).

Remark 1.2. An analogous result to our Theorem 1 is valid also on bordered surfaces in $\mathbb{R}^3$.

Remark 1.3. One of the referees has drown our attention to the possible relation of our paper to the publications [**13**] and [**14**]. In these papers is shown that for the inverse isotropic-conductivity problem in an inaccurately modelled (simply connected) domain there is a unique anisotropic conductivity, corresponding to the boundary measurements, which has the minimal possible anisotropy; this minimally-anisotropic conductivity can be «isotropized», using Beltrami equation, in order to obtain the original isotropic conductivity (up to biholomorphisms of simply connected domains). These papers have certainly some common parts with [**2**], where the inverse anisotropic-conductivity problem on simply connected domains is studied. But these publications have no common points with our paper; indeed our main novelty consists in the complete study of the inverse anisotropic-conductivity problem in arbitrary domains (not necessarily simply connected) with smooth boundaries.

Nevertheless, our results can be applied to extend the above-mentioned publications to the case of non simply connected domains.

## 2. The Beltrami equation and Faddeev-type anisotropic solutions

We identify $\mathbb{R}^2$ with $\mathbb{C}$ by the map $(x,y) \mapsto x+iy = z$ and we use the notation

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$$

where $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$. We will also use the differential operators $\partial$, $\overline{\partial}$ such that $\partial f = \partial_z f dz$, $\overline{\partial} f = \partial_{\overline{z}} f d\overline{z}$, with $dz = dx + idy$, $d\overline{z} = dx - idy$. We also recall the identity $d = \partial + \overline{\partial}$.

We can suppose that $\hat{\sigma}$, already isotropic near $\partial\hat{\Omega}$, is the identity near $\partial\hat{\Omega}$ (see [**15**] for the reduction to this case). Besides, we extend $\hat{\sigma}$ to the whole complex plane by putting $\hat{\sigma} = I$ for $z \in \mathbb{C} \setminus \hat{\Omega}$. Then, for the conductivity $\hat{\sigma} = \hat{\sigma}^{ij}$ we define the following Beltrami coefficient

$$\mu_1(z) = \frac{-\hat{\sigma}^{11}(z) + \hat{\sigma}^{22}(z) - 2i\hat{\sigma}^{12}(z)}{\hat{\sigma}^{11}(z) + \hat{\sigma}^{22}(z) + 2\sqrt{\det(\hat{\sigma})}}$$

which satisfies $|\mu_1(z)| \leq k < 1$ and is compactly supported in $\hat{\Omega}$. We now recall the existence of a diffeomorphism that transforms $\hat{\sigma}$ into an isotropic conductivity.

Proposition 2.1. *(Sylvester* [**18**, Prop. 2.1]*) There is a quasiconformal $C^1$-diffeomorphism*
*$F : \mathbb{C} \to \mathbb{C}$ such that*

$$F(z) = z + O\left(\frac{1}{z}\right) \quad \text{as } |z| \to \infty,$$

*and for which*

$$(F_*\hat{\sigma})(z) = \sigma(z)I := (\det(\hat{\sigma}))^{1/2} \circ F^{-1}(z)I.$$

Thanks to results by Ahlfors and Vekua ([**1**], [**19**]), $F$ is obtained as the solution of the Beltrami equation $\partial_{\overline{z}} F = \mu_1 \partial_z F$, so $F$ is holomorphic in $\mathbb{C} \setminus \hat{\Omega}$.

Proposition 2.2. *There exist unique Faddeev-type solutions of the anisotropic conductivity equation, i.e. functions $\hat{\psi}(z,\lambda)$ such that*

$$\nabla \cdot (\hat{\sigma}(\nabla\hat{\psi})) = 0 \tag{2.1}$$

*for all $z \in \mathbb{C}$, $\lambda \in \mathbb{C}$, and $\hat{\psi}(z,\lambda) = e^{\lambda z}(1 + O(\frac{1}{z}))$ when $z \to \infty$.*

Proposition 2.2 for the case $\det\hat{\sigma}$ close to a constant was obtained firstly in [**18**].

Proof. We define $\Omega = F(\hat{\Omega})$ and $q = \frac{\Delta\sigma^{1/2}}{\sigma^{1/2}}$. It is known that if $u$ is a solution of $\nabla \cdot (\sigma\nabla u) = 0$ in $\Omega$, then $\tilde{u} = \sigma^{1/2} u$ is a solution of

$$-\Delta\tilde{u} + q\tilde{u} = 0 \tag{2.2}$$

in $\Omega$. From [**4**], [**15**] and [**16**], we have that for every $\lambda \in \mathbb{C}$ there is a unique solution $\tilde{\psi}(w,\lambda)$ of (2.2) with the asymptotic behaviour $\tilde{\psi}(w,\lambda) = e^{\lambda w}(1 + O(\frac{1}{w}))$ when $w \to \infty$. So we directly have that $\psi(w,\lambda) := \sigma^{-1/2}\tilde{\psi}(w,\lambda)$ is a solution of $\nabla \cdot (\sigma \nabla \psi) = 0$ with the same asymptotic (because $\sigma = 1$ outside $\Omega$).

Now let $\hat{\psi}(z,\lambda)$ be a Faddeev-type anisotropic solution. If we consider $\psi'(w,\lambda) = \hat{\psi}(F^{-1}(w),\lambda)$, we have that $\nabla \cdot (\sigma \nabla \psi') = 0$ from the construction of $\sigma$. Using the properties of $F$ and $\hat{\psi}$, we get, for $w \to \infty$,

$$\begin{aligned}\psi'(w,\lambda) = \hat{\psi}(F^{-1}(w),\lambda) &= e^{\lambda F^{-1}(w)}\left(1 + O\left(\frac{1}{|F^{-1}(w)|}\right)\right)\\ &= e^{\lambda w}\left(1 + O\left(\frac{1}{1+|w|}\right)\right)\end{aligned}$$

showing that $\psi'(w,\lambda)$ satisfies the same asymptotic of $\psi(w,\lambda)$. From the uniqueness of $\psi(w,\lambda)$ we obtain

$$\hat{\psi}(z,\lambda) = \psi(F(z),\lambda), \tag{2.3}$$

which proves both existence and uniqueness. □

From the equality (2.3) we can also derive a useful formula to calculate $F|_{\partial\hat{\Omega}}$. In fact, results in [**8**] also indicate how the family of Faddeev-type solutions behaves with respect to $\lambda$. We have indeed $|e^{-w\lambda}\tilde{\psi}(w,\lambda) - 1| \to 0$ as $|\lambda| \to \infty$ for every fixed $w \in \mathbb{C}$. If we take $w \in \mathbb{C} \setminus \Omega$ the same limit is also valid for $\psi(w,\lambda)$; combining this with (2.3) we deduce the following formula.

PROPOSITION 2.3. *(*[**18**, Prop. 2.7]*) For all $z \in \mathbb{C} \setminus \hat{\Omega}$ (in particular for $z \in \partial\hat{\Omega}$) we have*

$$\lim_{|\lambda|\to\infty} \frac{\log(\hat{\psi}(z,\lambda))}{\lambda} = \lim_{|\lambda|\to\infty} \frac{\log(\psi(F(z),\lambda))}{\lambda} = F(z).$$

## 3. An integral equation for $\hat{\psi}|_{\partial\hat{\Omega}}$

Following the approach of [**11**], we show that, as in the isotropic case, we can find $\hat{\psi}|_{\partial\hat{\Omega}}$ through a Fredholm-type integral equation.

The main idea is to decompose the differential operator $-\nabla \cdot \hat{\sigma}\nabla$ as $-\Delta + M$, where $M$ is a compactly supported operator. So we can characterize $\hat{\psi}(z,\lambda)$ as the solution of the following integral equation:

$$\hat{\psi}(z,\lambda) = e^{z\lambda} - \frac{i}{2}\int_{\hat{\Omega}} G(z-w,\lambda) M \hat{\psi}(w,\lambda)\, dw \wedge d\overline{w},$$

where

$$G(z,\lambda) = \frac{ie^{\lambda z}}{2(2\pi)^2}\int_{\mathbb{C}} \frac{e^{i(w\overline{z}+\overline{w}z)}dw \wedge d\overline{w}}{w(\overline{w}-i\lambda)}, \quad z \in \mathbb{C},\ \lambda \in \mathbb{C}$$

is the Faddeev-Green function for the Laplacian.

PROPOSITION 3.1. *([11, Lemma 2.4]) For every $\lambda \in \mathbb{C}$ the boundary value of $\hat{\psi}$ satisfies*

$$\hat{\psi}(z,\lambda)|_{\partial\hat{\Omega}} = e^{z\lambda} - \int_{\partial\hat{\Omega}} G(z-w,\lambda)(\Lambda_{\hat{\sigma}} - \Lambda_0)\hat{\psi}(w,\lambda)dw, \tag{3.1}$$

*where $\Lambda_0$ is the Dirichlet-to-Neumann operator of the standard Laplacian (or for the case of constant conductivity).*

This follows from the identity

$$\int_{\partial\hat{\Omega}} u_0(\Lambda_{\hat{\sigma}} - \Lambda_0)u = \int_{\hat{\Omega}} u_0 M u, \tag{3.2}$$

where $u_0, u \in W^{1,2}(\hat{\Omega}),\ \nabla\cdot(\hat{\sigma}\nabla u) = 0,\ \Delta u_0 = 0$ in $\hat{\Omega}$.

The fact that the integral equation (3.1) is of Fredholm type in the Sobolev space $W^{s,2}(\partial\hat{\Omega})$ is the content of [**11**, Lemma 2.5], and it is uniquely solvable by [**11**, Lemma 2.6] (these properties are implied by the same results in the isotropic case [**15**]).

## 4. Reconstruction of the scattering amplitude

Following [**8**], we define the *non-physical* scattering amplitude for the isotropic inverse problem as

$$b(\lambda) = \int_{\Omega} e^{-\overline{\lambda}\overline{w}} q(w)\tilde{\psi}(w,\lambda)dw. \tag{4.1}$$

From [**16**] we have

$$b(\lambda) = \int_{\partial\Omega} e^{-\overline{\lambda}\overline{w}} (\Lambda_q - \Lambda_0)\tilde{\psi}(w,\lambda)dw,$$

where $\Lambda_q$ is the Dirichlet-to-Neumann operator of the Schrödinger equation (2.2).

Since $\sigma$ is the identity near $\partial\Omega$, equation $\Lambda_q = \sigma^{-1/2}(\Lambda_\sigma + \frac{1}{2}\frac{\partial\sigma}{\partial\nu})\sigma^{-1/2}$ reads $\Lambda_q = \Lambda_\sigma$, and $\tilde{\psi}|_{\partial\Omega} = \psi|_{\partial\Omega}$, so

$$b(\lambda) = \int_{\partial\Omega} e^{-\overline{\lambda}\overline{w}} (\Lambda_\sigma - \Lambda_0)\psi(w,\lambda)dw. \tag{4.2}$$

Thus, for the reconstruction of $b$, it is sufficient to determine $\Lambda_\sigma$ and $\psi|_{\partial\Omega}$. By (2.3) we already know $\psi|_{\partial\Omega}$; for the determination of $\Lambda_\sigma$, by arguments of [**2**], we obtain the identity

$$\int_{\partial\hat{\Omega}} \hat{u}\Lambda_{\hat{\sigma}}\hat{v} = \int_{\partial\Omega} u\Lambda_\sigma v \tag{4.3}$$

which holds for any $\hat{u}, \hat{v} \in C^1(\partial\hat{\Omega})$ and $u, v \in C^1(\partial\Omega)$ such that $\hat{u} = u \circ F$ and $\hat{v} = v \circ F$ (this follows directly from the properties of $F$ and the symmetry of the two Dirichlet-to-Neumann operators). So we find $\Lambda_\sigma$ from $\Lambda_{\hat{\sigma}}$ and $F|_{\partial\hat{\Omega}}$.

**4.1. Complementary result.** We give here another method to find $b(\lambda)$. Inspired by [**11**], we define the anisotropic scattering amplitude as

$$\hat{b}(\lambda) = \int_{\hat{\Omega}} e^{-\overline{\lambda}\overline{z}} M\hat{\psi}(z,\lambda)dz \tag{4.4}$$

and we have the following result.

Proposition 4.1. $b(\lambda) = \hat{b}(\lambda)$

We will need the following lemma

Lemma 4.2. *For every $\phi \in C^1(\partial\hat{\Omega})$, $\psi \in C^1(\hat{\Omega})$ solution of $\nabla\cdot(\hat{\sigma}\nabla\psi) = (\Delta - M)\psi = 0$ in $\hat{\Omega}$, we have*

$$\int_{\partial\hat{\Omega}} \phi(\Lambda_{\hat{\sigma}} - \Lambda_0)\psi = 2i\int_{\partial\hat{\Omega}} \phi(\overline{\partial}\psi - \overline{\partial}\psi_0), \tag{4.5}$$

*where $\Delta\psi_0 = 0$ in $\hat{\Omega}$ and $\psi_0|_{\partial\hat{\Omega}} = \psi|_{\partial\hat{\Omega}}$.*

Proof. Let $a \in C^1(\hat{\Omega})$ such that $a|_{\partial\hat{\Omega}} = \phi$, and $w = x+iy$. From the definition of the Dirichlet-to-Neumann operator and from Stokes' theorem, one has

$$\int_{\partial\hat{\Omega}} \phi(\Lambda_{\hat{\sigma}} - \Lambda_0)\psi = \int_{\hat{\Omega}} (\nabla a\cdot\nabla(\psi-\psi_0) + aM\psi)dxdy,$$

and by Stokes' theorem and by the identity $\Delta = 4\frac{\partial^2}{\partial z\partial\overline{z}}$

$$2i\int_{\partial\hat{\Omega}} \phi(\overline{\partial}\psi - \overline{\partial}\psi_0) = 2i\int_{\hat{\Omega}} \partial a\wedge(\overline{\partial}\psi - \overline{\partial}\psi_0) + \int_{\hat{\Omega}} aM\psi\; dxdy.$$

Writing in coordinates we get

$$\partial a\wedge\overline{\partial}(\psi-\psi_0) = \frac{1}{2i}\nabla a\cdot\nabla(\psi-\psi_0)dxdy + \frac{1}{2}da\wedge d(\psi-\psi_0).$$

Again by Stokes' thorem we have

$$\int_{\hat{\Omega}} da\wedge d(\psi-\psi_0) = -\int_{\partial\hat{\Omega}} (\psi-\psi_0)da = 0$$

because $\psi|_{\partial\hat{\Omega}} = \psi_0|_{\partial\hat{\Omega}}$. The proof follows. □

Proof of Proposition 4.1. From identity (3.2) we find

$$\hat{b}(\lambda) = \int_{\partial\hat{\Omega}} e^{-\overline{\lambda}\overline{z}}(\Lambda_{\hat{\sigma}} - \Lambda_0)\hat{\psi}(z,\lambda)dz.$$

Using the lemma we find

$$\hat{b}(\lambda) = 2i\int_{\partial\hat{\Omega}} e^{-\overline{\lambda}\overline{z}}(\overline{\partial}\hat{\psi} - \overline{\partial}\hat{\psi}_0) = 2i\int_{\partial\hat{\Omega}} e^{-\overline{\lambda}\overline{z}}\overline{\partial}\hat{\psi}, \tag{4.6}$$

$$b(\lambda) = 2i\int_{\partial\Omega} e^{-\overline{\lambda}\overline{w}}(\overline{\partial}\psi - \overline{\partial}\psi_0) = 2i\int_{\partial\Omega} e^{-\overline{\lambda}\overline{w}}\overline{\partial}\psi, \tag{4.7}$$

where the second equalities follows from Stokes' theorem, the fact that $e^{-\overline{\lambda}\overline{z}}$ (resp. $e^{-\overline{\lambda}\overline{w}}$) is antiholomorphic and $\hat{\psi}_0$ (resp. $\psi_0$) is harmonic in $\hat{\Omega}$ (resp. in $\Omega$).

If we call $z = G(w) = F^{-1}(w)$ we find, from (4.6),

$$\begin{aligned} \hat{b}(\lambda) &= 2i \int_{\partial\hat{\Omega}} e^{-\overline{\lambda}\overline{z}} \frac{\partial\hat{\psi}}{\partial\overline{z}} d\overline{z} \\ &= 2i \int_{\partial\Omega} e^{-\overline{\lambda G(w)}} \overline{\left(\frac{\partial F}{\partial z}\right)} \frac{\partial\psi}{\partial\overline{w}}(w,\lambda) \overline{\left(\frac{\partial G}{\partial w}\right)} d\overline{w} \\ &= 2i \int_{\partial\Omega} e^{-\overline{\lambda G(w)}} \overline{\partial}\psi, \end{aligned} \tag{4.8}$$

because $F$ (resp. $G$) is holomorphic in a neighbourhood of $\partial\hat{\Omega}$ (resp. $\partial\Omega$), and from the equality $\psi \circ F = \hat{\psi}$.

To see that (4.8) is equal to (4.7) we proceed as follows. Let $\Omega_R = \{\, z \in \mathbb{C} : \ |z| < R \,\}$ the disk of radius $R$, and let $R$ be sufficiently large to have $\overline{\Omega} \subset \Omega_R$. We apply Stokes' theorem to $\Omega_R \setminus \Omega$ and we obtain, for every quasiconformal homeomorphism $E : \mathbb{C} \to \mathbb{C}$, holomorphic in $\mathbb{C} \setminus \Omega$,

$$\int_{\partial\Omega} e^{-\overline{\lambda E(w)}} \overline{\partial}\psi = \int_{\partial\Omega_R} e^{-\overline{\lambda E(w)}} \overline{\partial}\psi + \int_{\Omega_R \setminus \Omega} \partial(e^{-\overline{\lambda E(w)}} \overline{\partial}\psi)$$

but the last term vanishes, because $e^{-\overline{\lambda E(w)}}$ is anti-holomorphic and $\partial\overline{\partial}\psi = 0$ in $\mathbb{C} \setminus \Omega$.

So the identity

$$\int_{\partial\Omega} e^{-\overline{\lambda E(w)}} \overline{\partial}\psi = \int_{\partial\Omega_R} e^{-\overline{\lambda E(w)}} \overline{\partial}\psi$$

is true for $R \gg 0$, $E(w) = G(w)$ and $E(w) = w$. As we have $G(w) = w + O(\frac{1}{|w|})$ for $w \to \infty$, using the lemma we deduce

$$\begin{aligned} \hat{b}(\lambda) &= 2i \int_{\partial\Omega} e^{-\overline{\lambda G(w)}} \overline{\partial}\psi = \lim_{R\to\infty} 2i \int_{\partial\Omega_R} e^{-\overline{\lambda G(w)}} \overline{\partial}\psi \\ &= \lim_{R\to\infty} 2i \int_{\partial\Omega_R} e^{-\overline{\lambda w}} \overline{\partial}\psi = 2i \int_{\partial\Omega} e^{-\overline{\lambda w}} \overline{\partial}\psi = b(\lambda) \end{aligned}$$

□

## 5. The $\overline{\partial}$-equation and the reconstruction of $\sigma$

Here we follow the steps of **[16]** to reconstruct isotropic conductivities. The function $\mu(w,\lambda) = \tilde{\psi}(w,\lambda)e^{-\lambda w}$ satisfies the following $\overline{\partial}$-equation with respect to $\lambda$

$$\frac{\partial\mu(w,\lambda)}{\partial\overline{\lambda}} = \frac{b(\lambda)}{4\pi\overline{\lambda}} e^{\overline{\lambda}\overline{w} - \lambda w} \overline{\mu(w,\lambda)}. \tag{5.1}$$

This is equivalent to the integral equation:

$$\mu(w,\lambda) = 1 + \frac{1}{8\pi^2 i}\int_{\mathbb{C}} \frac{b(\lambda')}{(\lambda'-\lambda)\overline{\lambda}'} e^{\overline{\lambda}'\overline{w}-\lambda' w}\overline{\mu(w,\lambda')}d\lambda' \wedge d\overline{\lambda}' \tag{5.2}$$

because $\mu \to 1$ when $w \to \infty$. By results of [**15**], equation (5.2) is solvable, and one can find $\sigma(w)$ from the integral formula

$$\sigma^{1/2}(w) = \mu(w,0) = 1 + \frac{1}{8\pi^2 i}\int_{\mathbb{C}} \frac{b(\lambda)}{|\lambda|^2} e^{\overline{\lambda}\overline{w}-\lambda w}\overline{\mu(w,\lambda)}d\lambda \wedge d\overline{\lambda}, \quad \forall w \in \mathbb{C} \tag{5.3}$$

or from the more stable general formula

$$\frac{\Delta\sigma^{1/2}(w)}{\sigma^{1/2}(w)} = \frac{\Delta\tilde{\psi}(w,\lambda)}{\tilde{\psi}(w,\lambda)}, \quad \forall w \in \mathbb{C},\ \forall \lambda \in \mathbb{C}. \tag{5.4}$$

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# Paper B

**PAPER B**

# Gel'fand-Calderón's inverse problem for anisotropic conductivities on bordered surfaces in $\mathbb{R}^3$

Gennadi Henkin and Matteo Santacesaria

Abstract. Let $X$ be a smooth bordered surface in $\mathbb{R}^3$ with smooth boundary and $\hat{\sigma}$ a smooth anisotropic conductivity on $X$. If the genus of $X$ is given, then starting from the Dirichlet-to-Neumann operator $\Lambda_{\hat{\sigma}}$ on $\partial X$, we give an explicit procedure to find a unique Riemann surface $Y$ (up to a biholomorphism), an isotropic conductivity $\sigma$ on $Y$ and a quasiconformal diffeomorphism $F : X \to Y$ which transforms $\hat{\sigma}$ into $\sigma$.

As a corollary we obtain the following uniqueness result: if $\sigma_1, \sigma_2$ are two smooth anisotropic conductivities on $X$ with $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, then there exists a smooth diffeomorphism $\Phi : \overline{X} \to \overline{X}$ such that $\Phi|_{\partial X} = \mathrm{Id}$ and $\Phi_* \sigma_1 = \sigma_2$.

## 1. Introduction

Let $X$ be a bordered, oriented, two-dimensional manifold in $\mathbb{R}^3$. We suppose that $X$ possesses a conductivity $\sigma$: this means that we have the following relation

$$j(x) = \sigma(x)\, du(x),\ x \in X, \tag{1.1}$$

where $u(x)$ is the voltage potential at $x$, $du(x)$ its differential, and $j(x)$ is the current flowing through $x$. Equation (1.1) is just a differential version of Ohm's law. As $j$ is a 1-form, $\sigma$ represents a mapping from 1-forms to 1-forms, i.e. $\sigma$ is a global section of the vector bundle $T(X)^* \otimes T(X)$ (where $T(X)$, $T(X)^*$ are respectively the tangent and the cotangent bundle of $X$). It is customary to assume that $\sigma(x)$ is both positive definite and symmetric, in a sense that will be explained later.

We shall also assume that there is no displacement current; thus for any smooth subdomain $X' \subset X$ we have Green's theorem

$$0 = \int_{\partial X'} j = \int_{X'} dj.$$

Since $X'$ is arbitrary, we conclude that $dj = d\sigma\, du = 0$ in $X$.

In general, conductivities are anisotropic; we say that a conductivity is isotropic if the relationship between voltage and current is independent of the direction.

In order to introduce the problem, we define the Dirichlet-to-Neumann operator $\Lambda_\sigma : C^1(\partial X) \to L^p(T(\partial X)^*)$, $p < \infty$ as

$$\Lambda_\sigma f = \sigma\, du|_{\partial X}, \tag{1.2}$$

where $\sigma \in C^3(T(X)^* \otimes T(X))$, $f \in C^1(\partial X)$ and $u$ is the unique $W^{1,p}(\overline{X})$-solution of the Dirichlet problem

$$d\sigma\, du = 0 \text{ on } X,\ u|_{\partial X} = f. \tag{1.3}$$

More details about definitions (1.2), (1.3) can be found in [**28**] and in section 2.1 below.

Our aim is to answer the following question, that is a variation of an inverse boundary value problem posed by Gel'fand [**12**] and Calderón [**6**]: which information about $X$ and $\sigma$ can be extracted from the mapping $\Lambda_\sigma$?

The main result of this paper is:

THEOREM 1.1. *Let $X$ be a bordered, $C^3$, oriented, two-dimensional manifold in $\mathbb{R}^3$ with $C^3$ boundary and let $\hat\sigma$ be a $C^3$-anisotropic conductivity on $X$. From the Dirichlet-to-Neumann operator $\Lambda_{\hat\sigma} : C^1(\partial X) \to L^p(T(\partial X)^*)$, $p < \infty$, and from the knowledge of the genus of $X$, we can find by an explicit procedure:*
*i) a bordered Riemann surface $Y$,*
*ii) an isotropic conductivity $\sigma$ on $Y$,*
*iii) a $C^3$ diffeomorphism $F : X \to Y$ such that $F_*\hat\sigma = \sigma$.*
*Moreover, if $\tilde Y$ is another Riemann surface, $\tilde\sigma$ an isotropic conductivity on $\tilde Y$ and $\tilde F : X \to \tilde Y$ a $C^3$ diffeomorphism such that $\tilde F_*\hat\sigma = \tilde\sigma$, then $\Psi = \tilde F \circ F^{-1} : Y \to \tilde Y$ is a biholomorphism such that $\Psi_*\sigma = \tilde\sigma$.*

Note that the hypothesis that $X \subset \mathbb{R}^3$ is not restrictive. Indeed, by classical theorems of Garsia and Rüedy (see [**10**], [**27**]), any Riemann surface is conformally equivalent to a complete classical surface in $\mathbb{R}^3$.

The push-forward of a conductivity $\sigma$ by a diffeomorphism $\Phi : \overline{X} \to \overline{Y}$ is defined, following [**28**, §1], as

$$(\Phi_*\sigma)\alpha = \Phi_*(\sigma(\Phi^*\alpha)), \tag{1.4}$$

where $\Phi^*\alpha$ denotes the pull-back of the 1-form $\alpha$ and $\Phi_* = (\Phi^{-1})^*$ denotes the pull-back by $\Phi^{-1}$ acting on the 1-form $\sigma(\Phi^*\alpha)$.

L. Tartar was the first to remark (see [**23**]) that, when $\Phi : \overline{X} \to \overline{X}$, this new conductivity $\Phi_*\sigma$ has the same boundary measurements as $\sigma$ if $\Phi|_{\partial X} = \mathrm{Id}$, where Id is the identity map. Thus, it is clearly not possible to determine $\sigma$ uniquely from $\Lambda_\sigma$; more specifically we cannot find more than *i)–iii)* from $\Lambda_\sigma$. This is pointed out in the following corollary of our main result.

COROLLARY 1.2. *Let $X$ be a bordered, $C^3$, oriented, two-dimensional manifold in $\mathbb{R}^3$ with $C^3$ boundary and let $\sigma_1, \sigma_2$ be two $C^3$-anisotropic conductivities on $X$. If $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$ then there exists a $C^3$ diffeomorphism $\Phi : \overline{X} \to \overline{X}$ such that $\Phi|_{\partial X} = \mathrm{Id}$ and $\sigma_2 = \Phi_*\sigma_1$.*

**Historical remarks.** These results were obtained earlier only for $X \subset \mathbb{R}^2$. Even for this case, Theorem 1.1 was obtained only recently by the authors [**20**], using arguments and results taken from [**26**], [**25**], [**28**], [**2**], [**15**].

Corollary 1.2, for $X \subset \mathbb{R}^2$, was proved in an original paper by Sylvester [**28**] for $C^3$ conductivities close to constants (the last restriction was eliminated in [**25**]). From [**28**] one can deduce (see [**18**]) that for any bordered surface $X$, equipped with an anisotropic conductivity, there exists a unique complex structure, i.e. $d = \overline{\partial} + \partial$, for which the equation $d\hat{\sigma}\, du = 0$ transforms into $d\sigma\, d^c u = 0$, where $\sigma$ is a positive function (which represents an isotropic conductivity) and $d^c = i(\overline{\partial} - \partial)$.

In the context of surfaces Corollary 1.2 is the first uniqueness result for the inverse anisotropic-conductivity problem. A uniqueness result (even with partial data) for the inverse isotropic-conductivity problem was recently obtained in [**13**], using stationary phase techniques from [**5**].

From the reconstruction viewpoint, Theorem 1.1 is the first result on the recovering of the above-mentioned complex structure of a bordered surface with known genus from its Dirichlet-to-Neumann operator. A method for recovering isotropic conductivities on surfaces with known genus was recently developed in [**19**]. In addition, a reconstruction procedure for complex surfaces with constant conductivity was obtained in [**18**].

**Scheme of the proofs.** The main idea behind this paper is the same as in [**28**], i.e. to reduce the problem to the isotropic case.

We first equip our real surface with some complex structure (e.g. the complex structure induced by the euclidean metric of $\mathbb{R}^3$) and then we embed the surface in the complex affine space $\mathbb{C}^3$ as a domain $X$ on a nonsingular affine algebraic curve $V$. Without loss of generality (see section 2.4) we can suppose that in a neighbourhood of $\partial X$ we have $\hat{\sigma} du = d^c u$ and so we uniquely extend $\hat{\sigma}$ on $V \setminus X$ keeping this property. Successively, we find a global analogue of isothermal coordinates, uniquely determined on $V$ by a given anisotropic conductivity and natural asymptotic conditions. This is accomplished by proving existence and uniqueness of special solutions of a certain Beltrami equation; here we follow the works started by Gauss [**11**] and fully developed by Ahlfors [**1**] and Vekua [**29**], along with the Hodge-Riemann decomposition [**21**] and the generalization of related operator estimates.

We cannot expect, like in the plane, that the deformed surface will live in the same compactified surface after the change of coordinates. Thus, we will find a new surface $W$ where our conductivity is isotropic; in general this surface will be algebraic, but possibly with intersection points.

Thanks to this global Beltrami solution $F$ and results in [**19**] for the isotropic case, we can prove existence and uniqueness of Faddeev-type anisotropic functions $\hat{\psi}_\theta(z,\lambda)$ on $V$: a two-parameter family of solutions of the anisotropic conductivity equation (1.3) on $V$ with exponential asymptotics (see (4.1)), originally introduced in [**8**]. We will also prove a formula, inspired by [**28**], that allows us to reconstruct the boundary values $F|_{\partial X}$ of our Beltrami solution starting from $\hat{\psi}|_{\partial X}$. We then show how to reconstruct $\hat{\psi}|_{\partial X}$ from the knowledge of $\Lambda_{\hat{\sigma}}$, through a Fredholm-type integral equation.

The reconstruction procedure works as follows: starting from $\Lambda_{\hat{\sigma}}$ one reconstructs $\hat{\psi}|_{\partial X}$ and then $F|_{\partial X}$; thus one recovers $\Gamma = F(\partial X)$ and also $\Lambda_{F_*\hat{\sigma}} = \Lambda_\sigma$. Since $\Gamma$ has to be the boundary $\partial Y$ of a Riemann surface $Y$, one recovers that surface through Cauchy-type formulas. Finally, from the knowledge of $\Lambda_\sigma$, the application of results in [**19**] yields $F_*\hat{\sigma} = \sigma$ on $Y$.

Our scheme can be summarized in the following diagram:

$$\Lambda_{\hat{\sigma}} \to \hat{\psi}|_{\partial X} \to F|_{\partial X} \to \partial Y \to Y \to \sigma.$$

**An open problem.** It is known that, for constant conductivities, the Dirichlet-to-Neumann operator for $dd^c u = 0$ determines the genus of a surface; this is a consequence of results in [**24**], [**3**], [**18**] and [**14**]. These results can be generalized to the case of conductivities close to constants.

In the general case of non-constant conductivities, the unique determination of the genus of a bordered surface from its Dirichlet-to-Neumann operator is still an open question.

## 2. Preliminaries

**2.1. Basic definitions.** Let us provide more details about the objects discussed in the introduction.

We say that a conductivity $\sigma$ is positive definite and symmetric, if, for $a, b \in T_x(X)^*$, $x \in X$,

$$a \wedge \sigma b = b \wedge \sigma a, \tag{2.1}$$

$$a \wedge \sigma a = \varphi(x) dx^1 \wedge dx^2, \ \ \varphi(x) \geq C_\sigma |a|^2 > 0, \tag{2.2}$$

where $x^1, x^2$ are positively oriented coordinates and $|\,|$ is the euclidean norm. From (2.1) and (2.2) one sees that locally, in the chart $(U_\alpha, x_\alpha)$, our conductivity can be written as

$$\sigma|_{U_\alpha} = \sum_{i,j=1}^{2} \sigma_\alpha^{ij} (-1)^{j-1} dx_\alpha^{3-j} \wedge \frac{\partial}{\partial x_\alpha^i}, \tag{2.3}$$

where the matrix $(\sigma_\alpha^{ij})$ is positive definite and symmetric ($> C_\sigma I$).

With this notation, an isotropic conductivity $\sigma$ is just a conductivity whose associated matrix has the form $(\sigma^{ij}) = \sigma_0 I$, where $\sigma_0 : X \to \mathbb{R}$ is a bounded positive function and $I$ is the identity matrix.

The Dirichlet-to-Neumann operator $\Lambda_\sigma$, defined in (1.2), reads locally, for $f \in C^1(\partial X)$ and $u$ solution of (1.3),

$$\Lambda_\sigma f|_{U_\alpha} = \sum_{i,j=1}^{2} (-1)^{i-1} \sigma_\alpha^{ij} \frac{\partial u}{\partial x_\alpha^j} dx_\alpha^{3-i} = (\sigma \nabla u) \cdot (dx_\alpha^2, -dx_\alpha^1), \tag{2.4}$$

where the last expression shows that (1.2) is coherent with the definition of the Dirichlet-to-Neumann operator on the plane.

Equation (1.3) now reads locally

$$d\sigma\, du = \left( \sum_{i,j=1}^{2} \frac{\partial}{\partial x^i} \left( \sigma^{ij} \frac{\partial u}{\partial x^j} \right) \right) dx^1 \wedge dx^2 = 0. \tag{2.5}$$

Let us now explain some general properties of the push-forward of a conductivity. Let $\Phi : \overline{X} \to \overline{Y}$ be a diffeomorphism between bordered surfaces and $\sigma$ a conductivity on $X$. We define the push-forward $\Phi_* \sigma$ of $\sigma$ as in (1.4); locally, it reads

$$\Phi_* \sigma = \left( \frac{{}^t(D\Phi) \sigma (D\Phi)}{|\det(D\Phi)|} \right) \circ \Phi^{-1},$$

where $D\Phi$ is the matrix differential of $\Phi$ and $\sigma$ is seen as its associated matrix.

We recall that if $\Phi$ is conformal, then $\frac{{}^t(D\Phi)}{|\det(D\Phi)|} = (D\Phi)^{-1}$, thus the push-forward of an isotropic conductivity by a conformal diffeomorphism is still isotropic.

We would also like to compare the two Dirichlet-to-Neumann operators $\Lambda_\sigma$ and $\Lambda_{\Phi_*\sigma}$. By pull-back properties, if $u$ satisfies $d\sigma\, du = 0$, then $\Phi_* u = u \circ \Phi^{-1}$ satisfies $d\Phi_*\sigma\, d(\Phi_* u) = 0$. This fact implies that the unique solution of the Dirichlet problem

$$d(\Phi_*\sigma)\, dv = 0 \text{ on } Y,\ v|_{\partial Y} = f \circ (\Phi|_{\partial X})^{-1} \tag{2.6}$$

is just $v = \Phi_* u$, where $u$ is the unique solution of

$$d\sigma\, du = 0 \text{ on } X,\ u|_{\partial X} = f. \tag{2.7}$$

So if $Y = X$ and $\Phi|_{\partial X} = \mathrm{Id}$ we see that $\Lambda_{\Phi_*\sigma} = \Lambda_\sigma$; in general, it is important to underline the fact that $\Lambda_{\Phi_*\sigma}$ is completely determined by $\Lambda_\sigma$ and $\Phi|_{\partial X}$.

**2.2. Complex viewpoint.** Here we will introduce some complex notation. We define standard complex coordinates $z = x^1 + ix^2$, $\overline{z} = x^1 - ix^2$, $dz = dx^1 + idx^2$, $d\overline{z} = dx^1 - idx^2$, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right)$, $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right)$.

We can now rewrite the conductivity $\sigma$ with the complex coordinates; we obtain

$$\sigma|_{U_\alpha} = (\sigma_\alpha^0(-idz_\alpha) + \overline{\sigma}_\alpha^1(id\overline{z}_\alpha)) \wedge \frac{\partial}{\partial z_\alpha} + (\sigma_\alpha^1(-idz_\alpha) + \sigma_\alpha^0(id\overline{z}_\alpha)) \wedge \frac{\partial}{\partial \overline{z}_\alpha} \tag{2.8}$$

where

$$\sigma^0 = \frac{\sigma^{22} + \sigma^{11}}{2}, \quad \sigma^1 = \frac{\sigma^{11} - \sigma^{22}}{2} - i\sigma^{12}. \tag{2.9}$$

We have chosen to represent the image of $\sigma$ in the basis $\{-i\,dz,\ id\overline{z}\}$ in order to have the hermitian matrix $\begin{pmatrix}\sigma^0 & \sigma^1\\ \overline{\sigma}^1 & \sigma^0\end{pmatrix}$.

One verifies that these new coefficients satisfy the following transformation rules

$$\sigma^0_\alpha = \sigma^0_\beta, \text{ and } \sigma^1_\alpha = \sigma^1_\beta \frac{dz_\beta}{dz_\alpha}\overline{\left(\frac{dz_\beta}{dz_\alpha}\right)}^{-1} \tag{2.10}$$

Let us remark that, if $\sigma$ is isotropic, represented by the matrix $\sigma_0 I$, then equation (2.5) reads

$$d\sigma\, du = d\sigma_0\, d^c u = 0.$$

Throughout all the paper we will always identify an isotropic conductivity $\sigma$ with its associated function $\sigma_0$ to simplify notation; thus the conductivity equation, in this case, will always be written $d\sigma\, d^c u = 0$ and $\Lambda_\sigma f = \sigma d^c u|_{\partial X}$, with $u$ the solution of (1.3).

**2.3. Embedding in projective space.** Let $\mathbb{C}P^3$ be the complex projective space with homogeneous coordinates $w = (w_0 : w_1 : w_2 : w_3)$ and let $\mathbb{C}P^2_\infty = \{w \in \mathbb{C}P^3 \ :\ w_0 = 0\}$. Then $\mathbb{C}P^3 \setminus \mathbb{C}P^2_\infty$ can be considered as a complex affine space with coordinates $z_k = w_k/w_0$, $k = 1, 2, 3$. Thanks to a classical result of G. Halphen (cfr. [**16**, Cap. IV, Prop. 6.1]) any compact Riemann surface of genus $g$ can be embedded in $\mathbb{C}P^3$ as a projective algebraic curve $\tilde{V}$, which intersects $\mathbb{C}P^2_\infty$ transversally in $d > g$ points, where $d \geq 1$ if $g = 0$, $d \geq 3$ if $g = 1$ and $d \geq g + 3$ if $g \geq 2$.

Without loss of generality we can assume the following facts:

i) $V = \tilde{V} \setminus \mathbb{C}P^2_\infty$ is a connected affine algebraic curve in $\mathbb{C}^3$ defined by polynomial equations $V = \{z \in \mathbb{C}^3 : p_1(z) = p_2(z) = p_3(z) = 0\}$ such that $\text{rank}\left[\frac{\partial p_1}{\partial z}(z), \frac{\partial p_2}{\partial z}(z), \frac{\partial p_3}{\partial z}(z)\right] \equiv 2,\ \forall z \in V$;

ii) $\tilde{V} \cap \mathbb{C}P^2_\infty = \{\beta_1, \ldots, \beta_d\}$, where

$$\beta_l = (0 : \beta^1_l : \beta^2_l : \beta^3_l), \left(\frac{\beta^2_l}{\beta^1_l}, \frac{\beta^3_l}{\beta^1_l}\right) \in \mathbb{C}^2,\ l = 1, 2, \ldots, d;$$

From the choice of the coordinates and properties i), ii), it follows that, for $|z_1|$ large enough, on each connected component $V_l$ of $V \setminus V_0$ we have:

$$z_2 = \gamma_l z_1 + c_l + O\left(\frac{1}{|z_1|}\right),$$
$$z_3 = \tilde{\gamma}_l z_1 + \tilde{c}_l + O\left(\frac{1}{|z_1|}\right), \quad l = 1, \ldots, d.$$

This imply the following additional facts:

iii) for $r_0 > 0$ large enough

$$\det\begin{bmatrix} \frac{\partial p_\alpha}{\partial z_2} & \frac{\partial p_\alpha}{\partial z_3} \\ \frac{\partial p_\beta}{\partial z_2} & \frac{\partial p_\beta}{\partial z_3}\end{bmatrix} \neq 0, \text{ for } z \in V : |z_1| \geq r_0 \text{ and } \alpha \neq \beta;$$

iv) for $|z|$ sufficiently large we have

$$\frac{dz_2}{dz_1}|_{V_l} = \gamma_l + O\left(\frac{1}{|z_1|^2}\right), \ \frac{dz_3}{dz_1}|_{V_l} = \tilde{\gamma}_l + O\left(\frac{1}{|z_1|^2}\right),$$

where $\gamma_l, \tilde{\gamma}_l \neq 0$, for $l = 1, \dots, d$, $d \geq 2$, $V_0 = \{z \in V : |z_1| \leq r_0\}$ and $V \setminus V_0 = \cup_{l=1}^d V_l$ (the $V_l$ are the connected components of $V \setminus V_0$, for $l = 1, \dots, d$).

We equip $\tilde{V}$ with the projective volume form $dd^c \log(1+|z|^2)$ and $V$ with the euclidean volume form $dd^c|z|^2$; we can thus consider the spaces $L^p_{0,1}(\tilde{V})$ and $L^p_{0,1}(V)$ of $L^p$ (0,1)-forms, equipped with the norms $\| \ \|_{L^p_{0,1}(\tilde{V})}$ and $\| \ \|_{L^p_{0,1}(V)}$, respectively. There is a canonical surjective map $C^\infty_{0,1}(\tilde{V}) \to C^\infty_{0,1}(V)$, so that we can compare the two above-defined norms; indeed, for $p \geq 2$ and $f \in L^p_{0,1}(\tilde{V})$, we have that $\|f\|_{L^p_{0,1}(V)} \leq \|f\|_{L^p_{0,1}(\tilde{V})}$ (in particular $\|f\|_{L^2_{0,1}(V)} = \|f\|_{L^2_{0,1}(\tilde{V})}$). This yields the inclusion $L^p_{0,1}(\tilde{V}) \subset L^p_{0,1}(V)$, for $p \geq 2$ (through the canonical map), and the same result is true for (1,0)-forms, i.e. $L^p_{1,0}(\tilde{V}) \subset L^p_{1,0}(V)$, for $p \geq 2$.

In section 3, the norm $\| \ \|_p$ will always stand for the affine norm $\| \ \|_{L^p_{0,1}(V)}$ (or $\| \ \|_{L^p_{1,0}(V)}$), although it will be use to make some estimates on forms defined on the whole compact surface $\tilde{V}$.

We now define the spaces $\tilde{W}^{1,p}(\tilde{V}) = \{F \in L^\infty(\tilde{V}) : \partial F \in L^p_{1,0}(\tilde{V})\}$, $\tilde{W}^{1,p}_{0,1}(\tilde{V}) = \{F \in L^\infty_{0,1}(\tilde{V}) : \partial F \in L^p_{1,1}(\tilde{V})\}$ for $1 < p < \infty$ and $H_{0,1}(\tilde{V})$ the space of antiholomorphic (0,1)-forms on $\tilde{V}$.

From the Hodge-Riemann decomposition theorem we have, for every $\Phi_0 \in W^{1,p}_{0,1}(\tilde{V})$, $\Phi_0 = \overline{\partial}(\overline{\partial}^* G\Phi_0) + \mathcal{H}\Phi_0$, where $\mathcal{H}\Phi_0 \in H_{0,1}(\tilde{V})$ is defined as

$$\mathcal{H}\Phi_0 = \sum_{j=1}^{g} \left( \int_V \Phi_0 \wedge \omega_j \right) \overline{\omega}_j,$$

with $\{\omega_j\}$ an orthonormal basis of holomorphic (1,0)-forms on $\tilde{V}$ and $G$ is the Hodge-Green operator for the Laplacian $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ on $\tilde{V}$ with the following properties: $G(H_{0,1}(\tilde{V})) = 0$, $\overline{\partial}G = G\overline{\partial}$, $\overline{\partial}^* G = G\overline{\partial}^*$.

We also define the operator $R$, for $f \in C^\infty_{0,1}(\tilde{V})$, as $Rf(x) = \overline{\partial}^* Gf(x) - \overline{\partial}^* Gf(\beta_1)$; we will see, as a consequence of Lemma 3.2, that $R : L^p_{0,1}(\tilde{V}) \to \tilde{W}^{1,p}(\tilde{V})$, for $2 < p < \infty$.

In the rest of the paper we will suppose for simplicity that $V = \{z \in \mathbb{C}^2 : P(z) = 0\}$ is an affine algebraic curve in $\mathbb{C}^2$.

**2.4. Remarks on the extension of $\hat{\sigma}$ on $V \setminus X$.** In the following of the paper, we will always suppose that $\hat{\sigma}$ is the identity in a neighbourhood of $\partial X$ (i.e. its associated matrix is the identity). In this way we could easily extend $\hat{\sigma}$ to $V$ by putting $(\hat{\sigma}^{ij}) = I$ on $V \setminus X$, and this new conductivity will still be $C^3$.

This simplification is possible thanks to the following construction. After embedding $X = X_1$ as an open set of the affine algebraic curve $V \subset \mathbb{C}^2$ above, we can find an open set $X_2 \subset V$ with the following properties:

i) $X_1 \subset X_2 \subset V$,

ii) $X_2$ has a $C^3$ boundary (the same smoothness as $\partial X$),

iii) $\hat{\sigma}$ can be extended to $X_2$ as a $C^3$ conductivity $\hat{\sigma}'$ such that $\hat{\sigma}' \equiv I$ in a neighbourhood of $\partial X_2$.

This is possible because one can reconstruct $\hat{\sigma}|_{\partial X_1}$ and its derivatives at the boundary from $\Lambda_{\hat{\sigma}}$ as in [**22**].

Thus we only have to show that $\Lambda_{\hat{\sigma}'}$ can be determined by $\Lambda_{\hat{\sigma}}$ and $\hat{\sigma}'|_{X_2 \setminus X_1}$. This can be done as in [**25**, Sec. 6].

The Dirichlet-to-Neumann maps $\Lambda^{ij}$ are defined as follows: we consider, for $i, j = 1, 2$, $f_j \in C^1(\partial X_j)$ and $u_j \in H^1(X_2 \setminus X_1)$ the solution of the Dirichlet problem $d\hat{\sigma}'\, du_i = 0$ in $X_2 \setminus \overline{X}_1$ such that $u_1|_{\partial X_1} = f_1$, $u_1|_{\partial X_2} = 0$, respectively $u_2|_{\partial X_1} = 0$, $u_2|_{\partial X_2} = f_2$. Then we define

$$\Lambda^{ij} f_j = \hat{\sigma}' du_j|_{\partial X_i}$$

and we have the following relation.

PROPOSITION 2.1. *Under our assumption,* $\Lambda_{\hat{\sigma}} - \Lambda^{11}$ *is an invertible operator* $\Lambda_{\hat{\sigma}} - \Lambda^{11} : C^1(\partial X_1) \to C^0(\partial X_1)$ *and*

$$\Lambda_{\hat{\sigma}'} = \Lambda^{22} + \Lambda^{21}(\Lambda_{\hat{\sigma}} - \Lambda^{11})^{-1}\Lambda^{12}.$$

The proof of this formula follows from the definition of the operators. The fact that $\Lambda_{\hat{\sigma}} - \Lambda^{11}$ is invertible comes from an explicit construction of its inverse, which turns out to be the single-layer operator on $\partial X_1$ corresponding to the Green function $G$ for the Dirichlet problem on $X_2$. More explicitly, it is the operator

$$Sf(x) = \int_{\partial X_1} G(x, y) f(y) dy,$$

where $G$ satisfies $d\hat{\sigma}'\, dG = -\delta(x - y)$ in $X_2$ and $G(\cdot, 0)|_{\partial X_2} = 0$.

## 3. The Beltrami Equation

In this section we will study the equation

$$\overline{\partial} w = \mu \partial w, \tag{3.1}$$

called the Beltrami equation, on a Riemann surface. Here $\mu$ is a bounded (-1,1)-form, namely a Beltrami differential, whose definition we will recall.

DÉFINITION 3.1. A Beltrami differential $\mu(z)\frac{\overline{dz}}{dz}$ on a Riemann surface $V$, equipped with an atlas $\{U_\alpha, z_\alpha\}$, is a collection of $L^\infty$ complex-valued functions $\mu^\alpha$ defined on $z_\alpha(U_\alpha)$ such that

$$\mu^\alpha(z_\alpha) = \mu^\beta(z_\beta)\frac{\overline{\left(\frac{dz_\beta}{dz_\alpha}\right)}}{\frac{dz_\beta}{dz_\alpha}} \tag{3.2}$$

and $\|\mu\|_\infty = \sup_\alpha \|\mu^\alpha\|_\infty < 1$.

With this definition, equation (3.1) is valid globally.

The main result of this section is:

THEOREM 3.1. *Let $X \subset V$ be an open subset of an affine Riemann surface $V$, let $\tilde{V} \supset V$ be its compactification, as in section 2, and let $\mu$ be a Beltrami differential on $\tilde{V}$ with $\mathrm{supp}(\mu) \subset X$ and $\|\mu\|_\infty \le k < 1$. Then, for $j = 1, 2$, there is a unique solution $w_j(z)$ of equation (3.1) on $V$ such that $w_j(z) = z_j + w_{0j}(z)$, $w_{0j} \in \tilde{W}^{1,p}(\tilde{V})$ for $p > 2$ and $w_{0j}(\beta_1) = 0$.*

In order to prove this theorem we introduce the operator $\Pi = \partial R$, initially defined on smooth forms, and we show some estimates which slightly generalize a result by Calderón and Zygmund; these will yield in particular that $\Pi : L^p_{0,1}(\tilde{V}) \to L^p_{1,0}(\tilde{V})$, for $2 \le p < \infty$.

We recall (see section 2.3 for further explanations) that the norm $\| \ \|_p$ stand for the affine norm $\| \ \|_{L^p_{0,1}(V)}$ (or $\| \ \|_{L^p_{1,0}(V)}$).

LEMMA 3.2. *For $f \in L^2_{0,1}(\tilde{V}) \cap \ker(\mathcal{H})$ we have*

$$\|\Pi f\|_2 = \|f\|_2 \tag{3.3}$$

*and, for $f \in L^p_{0,1}(\tilde{V}) \cap \ker(\mathcal{H})$, $p > 2$*

$$\|\Pi f\|_p \le C_p \|f\|_p, \text{ and } \lim_{p\to 2^+} C_p = 1. \tag{3.4}$$

PROOF. The proof is given for $f \in C^2_{0,1}(\tilde{V}) \cap \ker(\mathcal{H})$; the original statement will follow by a density argument. We have the following chain of equalities, where by Stokes' theorem and the Hodge decomposition on $\tilde{V}$

$$\begin{aligned}\|\Pi f\|_2^2 &= \int_V \partial Rf \wedge \overline{\partial Rf} = -\int_V Rf \wedge \partial\overline{\partial Rf} = -\int_V Rf \wedge \partial \overline{R\partial f} \\ &= -\int_V Rf \wedge \overline{\partial f} = \int_V \overline{\partial} Rf \wedge \overline{f} = \int_V f \wedge \overline{f} = \|f\|_2^2.\end{aligned}$$

To prove (3.4) we first decompose the operator $\Pi$ in the following way

$$\Pi f = \Pi_1 f + \Pi_2 f = \int_{|\zeta - z|\le\delta} f(\zeta)\Pi_1(\zeta, z) + \int_{|\zeta-z|>\delta} f(\zeta)\Pi_2(\zeta, z), \tag{3.5}$$

for $\delta$ sufficiently small, where in affine coordinate form

$$\Pi_1(\zeta, z) = \frac{d\zeta \wedge dz}{2\pi i(\zeta - z)^2}(1 + \varepsilon(\delta)),\ \varepsilon(\delta) \to 0 \text{ when } \delta \to 0 \tag{3.6}$$

and $\Pi_2$ is bounded. Decomposition (3.5) gives a so-called *parametrix* for the operator $\Pi$. From the Calderón-Zygmund result for the operator

$$F \mapsto \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{|\zeta - z|>\varepsilon} \frac{F(z)}{(z-\zeta)^2} dz d\overline{z}$$

(see [**1**, p.106]) where $f = F d\overline{z}$ , we have the estimate $\|\Pi_1 f\|_p \leq (1+\varepsilon(\delta))\tilde{C}_p \|f\|_p$. In addition, we also have $\|\Pi_2 f\|_p \leq \|\Pi_2\|_{L^\infty(|\zeta - z|>\delta)} \|f\|_p = K(\delta)\|f\|_p$. Putting it all together we find that

$$\|\Pi f\|_p \leq ((1+\varepsilon(\delta))\tilde{C}_p + K(\delta))\|f\|_p = C_p\|f\|_p. \tag{3.7}$$

The fact that $C_p \to 1$ when $p \to 2$ is a consequence of the Riesz-Thorin interpolation theorem (see [**4**, Thm. 1.1.1, p.2]) and of (3.3). □

Now, using the last lemma, we fix $p > 2$ such that $kC_p < 1$. The proof of the theorem will be given for the case $j = 1$; the other case is completely analogous.

Proof of Theorem 3.1. Let us begin with the existence statement. We look for solutions of the form $w(z) = z_1 + Rf$. Thus

$$\begin{aligned} \overline{\partial} w &= \overline{\partial} R f = f - \mathcal{H} f, \\ \partial w &= dz_1 + \partial R f = dz_1 + \Pi f = dz_1 + \Pi(f - \mathcal{H} f), \end{aligned}$$

for $R\mathcal{H} f = 0$ (and so $\Pi \mathcal{H} f = 0$). If we impose equation (3.1), we obtain an integral equation for $f_0 = f - \mathcal{H} f$:

$$f_0 - \mu \Pi f_0 = \mu dz_1. \tag{3.8}$$

Under our assumptions, the linear operator $f \mapsto \mu\Pi f$ is a contraction in $L^p_{0,1}(\tilde{V}) \cap \ker(\mathcal{H})$ (its norm is $\leq kC_p < 1$), so the series

$$f_0 = \mu dz_1 + \mu\Pi\mu dz_1 + \mu\Pi\mu\Pi\mu dz_1 + \dots$$

converges in $L^p_{0,1}(\tilde{V}) \cap \ker(\mathcal{H})$ to a solution of (3.8). Then we define $w_{01} = Rf_0$ which satisfies $\partial w_{01} = \Pi f_0 \in L^p_{1,0}(\tilde{V})$ and $w_{01} \in L^\infty(\tilde{V})$ (the latter follows from properties of $R$). Thus the function $w(z) = z_1 + w_{01}(z)$ is a solution of (3.1).

To show uniqueness, we first remark that $w_{01} = R\overline{\partial} w_{01}$. This follows from the fact that $\overline{\partial} w_{01} = \overline{\partial} w = \mu \partial w = \mu(dz_1 + \partial w_{01}) \in L^p_{0,1}(\tilde{V})$ because the support of $\mu$ is contained in $X$; we can thus calculate $R\overline{\partial} w_{01}$ and see that $\overline{\partial}(w_{01} - R\overline{\partial} w_{01}) = 0$. Now $w_{01} - R\overline{\partial} w_{01}$ is a bounded holomorphic function which goes to zero for $z \to \beta_1$, so it vanishes. In particular, this yields $w = z_1 + R\overline{\partial} w$.

Now, if $w' = z_1 + w'_{01} = z_1 + R\overline{\partial} w'$ is another solution, we obtain

$$\partial(w - w') = \Pi\mu(\partial(w - w')),$$

which gives $\partial(w - w') = 0$ thanks to our estimates, and also $\overline{\partial}(w - w') = 0$ because of the Beltrami equation. So $w - w'$ must be constant, and in fact it vanishes because of our normalisation. □

**3.1. Properties of the solution.** We now consider the application $F : V \to \mathbb{C}^2$ defined as $F(z) = (w_1(z), w_2(z))$ where $w_1$, $w_2$ are the solutions of the Beltrami equation given by Theorem 3.1. In particular, we want to understand the image surface $W = F(V)$.

By [**1**, Thm. 2, p.97] one has that $F$ is a local homeomorphism; besides, since $w_1$ and $w_2$ are solutions of the Beltrami equation, $W$ has a holomorphic atlas. Thus, by classical results, it is an algebraic curve as well, but possibly with intersection points. Let us note that, by the properties of $F$, we have $W \cap \mathbb{C}P^1_\infty = V \cap \mathbb{C}P^1_\infty$.

**3.2. Applications to anisotropic conductivities.** The most important consequence of Theorem 3.1, for this paper, is the following proposition about the existence of global isothermal coordinates which transforms an anisotropic conductivity into an isotropic one.

PROPOSITION 3.3. *Let $X \subset V$ be an open subset of an affine Riemann surface $V$, let $\tilde{V} \supset V$ be its compactification, as in section 2, and $\hat{\sigma}$ a $C^k$-anisotropic conductivity on $V$ ($k \geq 1$), represented by the identity matrix on $V \backslash X$. Then there exists a unique affine algebraic curve $W$, and a unique $C^k$ immersion $F : V \to W$, $F = (w_1, w_2)$ such that $F_* \hat{\sigma} = \sigma$ is isotropic on $W$ (where $F^{-1}$ exists) and $w_j(z) = z_j + w_{0j}(z)$ with $w_{0j} \in \tilde{W}^{1,p}(\tilde{V})$ and $w_{0j}(\beta_1) = 0$, for $j = 1, 2$.*

We will need the following lemma:

LEMMA 3.4. *Let $X \subset V \subset \tilde{V}$ as in proposition 3.3. Then every conductivity $\sigma$ on $X$, extended on $V \setminus X$ by the identity matrix, determines a Beltrami differential $\mu_\sigma \frac{\overline{dz}}{dz}$ with support contained in $X$ given locally by*

$$\mu^\alpha_\sigma = \mu_\sigma|_{U_\alpha} = \frac{\sigma^{22}_\alpha - \sigma^{11}_\alpha - 2i\sigma^{12}_\alpha}{\sigma^{11}_\alpha + \sigma^{22}_\alpha + 2\sqrt{\det(\sigma_\alpha)}} = \frac{-\overline{\sigma}^1_\alpha}{\sigma^0_\alpha + \sqrt{(\sigma^0_\alpha)^2 - |\sigma^1_\alpha|^2}}. \tag{3.9}$$

PROOF. From the transformation rules (2.10) one immediately has the relation

$$\mu^\beta_\sigma = \frac{-\overline{\sigma}^1_\alpha}{\sigma^0_\alpha + \sqrt{(\sigma^0_\alpha)^2 - |\sigma^1_\alpha|^2}} \frac{\overline{\left(\frac{dz_\alpha}{dz_\beta}\right)}}{\frac{dz_\alpha}{dz_\beta}} = \mu^\alpha_\sigma \frac{\overline{\left(\frac{dz_\alpha}{dz_\beta}\right)}}{\frac{dz_\alpha}{dz_\beta}}.$$

In addition, we have that

$$|\mu_\sigma|^2 = \frac{\sigma^{11} + \sigma^{22} - 2\sqrt{\det \sigma}}{\sigma^{11} + \sigma^{22} + 2\sqrt{\det \sigma}} \leq k < 1 \tag{3.10}$$

and $\mu_\sigma \equiv 0$ outside $X$. ☐

**Proof of Proposition 3.3.** We define $\mu_{\hat{\sigma}}$ as in lemma 3.4; by Theorem 3.1 we can construct $F(z) = (w_1(z), w_2(z))$, $F : \mathbb{C}^2 \to \mathbb{C}^2$, where $w_1, w_2$ are the special solutions of the Beltrami equation $\overline{\partial} w_j = \mu_{\hat{\sigma}} \partial w_j$. Using [**28**, Prop. 1.3], we have that $F_*\hat\sigma = \sigma I$ is isotropic on $F(V) = W \subset \tilde W$ but defined only where $F^{-1}$ is. In particular we have

$$(F_*\hat\sigma)(w) = (\det\hat\sigma)^{1/2} \circ F^{-1}(w) I = \sigma(w) I, \tag{3.11}$$

where $\sigma(w) = \sqrt{\sigma_0^2(z(w)) - |\sigma_1(z(w))|^2}$.

By remarks of section 3.1 we have that $F$ is an immersion: it is a $C^k$ immersion because of smoothness assumptions on $\hat\sigma$. □

## 4. Faddeev-type Anisotropic Solutions

In this section we generalise the results of [**19**], by proving existence and uniqueness of a family of special solutions of the anisotropic conductivity equation, so-called Faddeev-type solutions.

Let us recall from [**19**] the definitions of a few operators. We equip $V$ with the Euclidean volume form $dd^c|z|^2$, and let $\varphi \in L^1_{1,1}(V) \cap L^\infty_{1,1}(V)$, $f \in \tilde W^{1,p}_{1,0}(V) = \{F \in L^\infty_{1,0}(V) : \overline\partial F \in L^p_{1,1}(V)\}$, for $p > 2$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $\theta \in \mathbb{C}$. We define

$$\hat R_\theta \varphi = R((dz_1 + \theta dz_2)\rfloor\varphi) \wedge (dz_1 + \theta dz_2),$$

$$R_{\lambda,\theta} f = e_{-\lambda,\theta}\overline{R(\overline{e_{\lambda,\theta} f})}, \text{ where } e_{\lambda,\theta}(z) = e^{\lambda(z_1+\theta z_2) - \overline\lambda(\overline z_1 + \overline\theta \overline z_2)}.$$

Let $\hat\sigma$ be a $C^3$ anisotropic conductivity on $V$ with $\hat\sigma \equiv I$ on $V \setminus X$ and $\hat a_1, \ldots, \hat a_g \in V \setminus X$ an effective divisor.

Définition 4.1. A function $\hat\psi_\theta(z,\lambda)$, with $\theta, \lambda \in \mathbb{C}$, $z \in V$, is called a Faddeev-type function on $V$ associated with $\hat\sigma, \theta, \lambda$ and $\{\hat a_1, \ldots, \hat a_g\} \subset V \setminus X$, if

$$d\hat\sigma\, d\hat\psi_\theta(z,\lambda) = 2\left(\sum_{j=1}^g \hat C_{j,\theta}(\lambda)\delta(z,\hat a_j)\right) e^{\lambda(z_1+\theta z_2)}, \; z \in V, \tag{4.1}$$

and $\hat\psi_\theta(z,\lambda)e^{-\lambda(z_1+\theta z_2)} \to \hat K_l$ (constant), when $z \in V_l$, $z \to \infty$, for $l = 1, \ldots, d$ with the normalisation $\hat K_1 = 1$.

Let $F : V \to W$ be the mapping constructed in Proposition 3.3, $Y = F(X)$, $a_j = F(\hat a_j)$ for $j = 1, \ldots, g$ and $\sigma = F_*\hat\sigma$ the isotropic conductivity on $W$. Let $\psi_\theta(w,\lambda)$ be the Faddeev-type isotropic functions on $W$ constructed in [**19**] as the solutions of

$$d\sigma d^c \psi_\theta(w,\lambda) = 2\left(\sum_{j=1}^g C_{j,\theta}(\lambda)\delta(w,a_j)\right) e^{\lambda(w_1+\theta w_2)} \tag{4.2}$$

with $\psi_\theta e^{-\lambda(w_1+\theta w_2)} \to K_l$ (constants, with $K_1 = 1$), when $w \in W_l$, $w \to \infty$, for $l = 1, \ldots, d$, where $W_l = F(V_l)$.

We also define

$$\Delta_\theta(\lambda) = \det\left[\int_{\eta\in W} \hat R_\theta(\delta(\eta,a_j)) \wedge \overline\omega_k(\eta) e_{\lambda,\theta}(\eta)\right]_{j,k=1,\ldots,g} \tag{4.3}$$

where $\{\omega_k\}$ is an orthonormal basis of holomorphic (1,0)-forms on $\tilde{W}$, and we call $E_\theta = \{\lambda \in \mathbb{C} : \Delta_\theta(\lambda) = 0\}$.

THEOREM 4.1. *For any generic $\theta$, $\{\hat{a}_1, \ldots, \hat{a}_g\}$ and $\lambda \in \mathbb{C} \setminus E_\theta$, $|\lambda| \geq const(V, \{\hat{a}_j\}, \theta, \hat{\sigma})$ there exists a unique Faddeev-type solution $\hat{\psi}_\theta(z, \lambda)$ associated with $\hat{\sigma}$, $\theta$, $\lambda$ and $\{\hat{a}_1, \ldots, \hat{a}_g\}$. Moreover $E_\theta$ is a closed, nowhere dense subset of $\mathbb{C}$ and we have the equality*

$$\hat{\psi}_\theta(z, \lambda) = \psi_\theta(F(z), \lambda), \ z \in V \tag{4.4}$$

PROOF. We will here provide a complete proof of Theorem 4.1 when the Beltrami solution $F$, given by proposition 3.3, is an embedding; at the end we will indicate necessary corrections for the proof of the general case.

With this assumption, by proposition 3.3 there exists a unique diffeomorphism $F(z) = (w_1(z), w_2(z))$ such that $w_j(z) = z_j + w_{0j}(z)$, $w_{0j} \in \tilde{W}^{1,p}(\tilde{V})$, $p > 2$ and $F_*\hat{\sigma} = \sigma$ is isotropic on the image.

By [**19**, Prop. 1.1], the set $E_\theta$ is closed and nowhere dense and by [**19**, Thm. 1.1], for every $\lambda \in \mathbb{C} \setminus E_\theta, |\lambda| \geq const(W, \{a_j\}, \theta, \sigma)$ there exists a unique Faddeev-type isotropic function $\psi_\theta(w, \lambda)$ as defined in (4.2).

Now let $\hat{\psi}_\theta(z, \lambda)$ be an anisotropic Faddeev-type solution. We consider $\psi'_\theta(w, \lambda) = \hat{\psi}_\theta(F^{-1}(w), \lambda)$ and see that

$$d\sigma d^c \psi'_\theta(w, \lambda) = 2 \left( \sum_{j=1}^{g} C'_{j,\theta}(\lambda) \delta(z, a_j) \right) e^{\lambda(F_1^{-1}(w) + \theta F_2^{-1}(w))}$$

from the construction of $\sigma$ and the definition of $a_j$. Using the properties of $F$ (in particular that $F \to \mathrm{Id}$ for $z \to \infty$) and of $\hat{\psi}_\theta$, we have that $\psi'_\theta e^{-\lambda(w_1 + \theta w_2)} \to K_l$ with $K_1 = 1$; this shows that $\psi'_\theta$ and $\psi_\theta$ satisfy the same asymptotic conditions. Thus, by the uniqueness of $\psi_\theta(w, \lambda)$ we obtain the identity (4.4), which proves both existence and uniqueness for the case where $F$ is a diffeomorphism.

If $F$ is just an immersion the result is still valid; we can follow the same outline of the proof, taking into account the following:

i) in the definition (4.3) we have to use weakly holomorphic forms $\omega_k$, i.e. forms such that $\omega_k \in H^{1,0}(W \setminus Sing\, W)$ and $\omega_k$ are bounded on $W$ in a neighbourhood of $Sing\, W$;

ii) we say that $u$ is a solution of $d\sigma d^c u = 0$ on $W \setminus \{a_1, \ldots, a_g\}$, for $a_1, \ldots, a_g \in Reg\, W$ if $u$ is locally bounded on $W \setminus \{a_1, \ldots, a_g\}$ and $d\sigma d^c u = 0$ on $Reg\, W \setminus \{a_1, \ldots, a_g\}$,

iii) Proposition 1.1 and Theorem 1.1 of [**19**] are still valid for $W$ with points of simple self-intersection, but in the proofs one has to make some minor modifications in order to make estimates for operators $\hat{R}$ and $R_\lambda$.

The properties i) and ii) show that the holomorphic forms $\omega_k$ and the functions $u$ can be smoothly extended to a normalization of $W$. □

We now prove a formula, motivated by [**28**, Prop. 2.7], which will play a key role in the reconstruction procedure.

THEOREM 4.2. *Let $\hat{\psi}_\theta$ be the Faddeev-type anisotropic functions constructed above. Then for every $z \in V \setminus X$ (in particular for $z \in \partial X$), for every $\varepsilon > 0$ and generic $\theta \in \mathbb{C}$ we have*

$$\lim_{\lambda\to\infty} \inf_{\{\lambda' : |\lambda'-\lambda|\le\varepsilon\}} \frac{\log \hat{\psi}_\theta(z,\lambda')}{\lambda'} = w_1(z) + \theta w_2(z) \tag{4.5}$$

PROOF. We will use the following essential property of $\Delta_\theta(\lambda)$ from [**19**, Prop. 1.1], i.e., for every $\varepsilon > 0$

$$\underline{\lim}_{\lambda\to\infty} \sup_{\{\lambda' : |\lambda'-\lambda|\le\varepsilon\}} |\Delta_\theta(\lambda')||\lambda|^g > 0. \tag{4.6}$$

Using [**19**, Prop. 3.1] and (4.6), for $z \in V \setminus X$ we have $\sigma(F(z)) = 1$,

$$\hat{\psi}_\theta(z,\lambda) = e^{\lambda(F_1(z)+\theta F_2(z))}\mu_\theta(F(z),\lambda),$$

$$\inf_{\{\lambda' : |\lambda'-\lambda|\le\varepsilon\}} |\mu_\theta(\lambda') - 1| = O\left(\frac{1}{\lambda^{1-0}}\right),\ \lambda\to\infty.$$

Thus one obtains

$$\inf_{\{\lambda' : |\lambda'-\lambda|\le\varepsilon\}} \frac{\log \hat{\psi}_\theta(z,\lambda')}{\lambda'} = w_1(z) + \theta w_2(z) + \inf_{\{\lambda' : |\lambda'-\lambda|\le\varepsilon\}} \frac{\log \mu_\theta(w(z),\lambda')}{\lambda'}$$
$$= w_1(z) + \theta w_2(z) + O\left(\frac{\log\lambda}{\lambda}\right) \to w_1(z) + \theta w_2(z),\ \text{as } \lambda\to\infty. \qquad \square$$

## 5. An Integral Equation for $\hat{\psi}_\theta|_{\partial X}$

In this section we show how one can reconstruct the boundary values $\hat{\psi}_\theta|_{\partial X}$ from the Dirichlet-to-Neumann operator through a Fredholm-type integral equation.

Following the approach of Gutarts [**15**], based on Eskin [**7**, Thm. 18.5], we decompose the differential operator $d\hat{\sigma}d$ as $dd^c - Q$, where $Q$ is a compactly supported operator. Faddeev-type anisotropic functions, $\hat{\psi}_\theta(z,\lambda) = e^{\lambda(z_1+\theta z_2)}\hat{\mu}_\theta(z,\lambda)$, then satisfy

$$dd^c\hat{\psi}_\theta(z,\lambda) = Q\hat{\psi}_\theta(z,\lambda) + 2\left(\sum_{j=1}^{g} \hat{C}_{j,\theta}(\lambda)\delta(z,\hat{a}_j)\right) e^{\lambda(z_1+\theta z_2)}, \tag{5.1}$$

$$\overline{\partial}\left(\partial + \lambda(dz_1 + \theta dz_2)\right)\hat{\mu}_\theta = \frac{i}{2}Q\hat{\mu}_\theta + i\sum_{j=1}^{g} \hat{C}_{j,\theta}(\lambda)\delta(z,\hat{a}_j). \tag{5.2}$$

THEOREM 5.1. *We have*

*i) For every $\lambda \in \mathbb{C} \setminus E_\theta$, $|\lambda| \geq const(V, \{a_j\}, \theta, \hat{\sigma})$ the boundary values of $\hat{\psi}_\theta$ satisfy the following integral equation:*

$$\begin{aligned}\hat{\psi}_\theta(z,\lambda)|_{\partial X} &= \frac{i}{2}\int_{\zeta\in\partial X} G_{\lambda,\theta}(z,\zeta)(\Lambda_{\hat{\sigma}} - \Lambda_0)\hat{\psi}_\theta(\zeta,\lambda) \\ &+ ie^{\lambda(z_1+\theta z_2)}\sum_{j=1}^{g}\hat{C}_{j,\theta}(\lambda)g_{\lambda,\theta}(z,\hat{a}_j) \\ &- \lim_{\varepsilon\to 0}\frac{i}{2}\int\limits_{\{\zeta\in V:|\zeta-z|\geq\varepsilon\}}\hat{\psi}^0_\theta(\zeta,\lambda)dd^cG_{\lambda,\theta}(z,\zeta) \\ &- \lim_{R\to\infty}\int\limits_{|\zeta_1|=R}[\bar{\partial}G_{\lambda,\theta}(z,\zeta)\hat{\psi}^0_\theta(\zeta,\lambda) + G_{\lambda,\theta}(z,\zeta)\partial\hat{\psi}^0_\theta(\zeta,\lambda)],\end{aligned} \tag{5.3}$$

*with*

$$i\sum_{j=1}^{g}(\hat{a}_{j,1}+\theta\hat{a}_{j,2})^{-k}\hat{C}_{j,\theta}(\lambda) = -\int_{z\in\partial X}(z_1+\theta z_2)^{-k}e^{-\lambda(z_1+\theta z_2)}\overline{\Lambda_{\hat{\sigma}}\overline{\hat{\psi}_\theta}(z,\lambda)}, \tag{5.4}$$

*for $k = 2, \ldots, g+1$,*

$$G_{\lambda,\theta}(z,\zeta) = e^{\lambda[(z_1-\zeta_1)+\theta(z_2-\zeta_2)]}g_{\lambda,\theta}(z,\zeta),$$

*$g_{\lambda,\theta}(z,\zeta)$ is the kernel of the operator $R_{\lambda,\theta}\circ\hat{R}_\theta$,*
*$\Lambda_0 f = d^c u|_{\partial X}$ where $dd^c u = 0$ on $X$ and $u|_{\partial X} = f$,*
*$\hat{\psi}^0_\theta(\zeta,\lambda)$ is a continuous function for $\zeta \in V \setminus (\bigcup_j\{a_j\})$ such that*

$$\hat{\psi}^0_\theta(\cdot,\lambda)|_{V\setminus X} = \hat{\psi}_\theta(\cdot,\lambda)|_{V\setminus X},$$
$$dd^c\hat{\psi}^0_\theta = 0 \text{ on } X.$$

*ii) Equation (5.3) is a Fredholm-type integral equation and has a unique solution in*
*$W^{1,2}(\partial X)$, $\forall\lambda \in \mathbb{C}\setminus E_\theta$, $|\lambda| \geq const(V, \{a_j\}, \theta, \hat{\sigma})$.*

REMARK 5.1. Theorem 5.1 is a generalization of [**19**, Thm. 1.2A] to the anisotropic case. Note that the term $e^{\lambda(z_1+\theta z_2)}$ in the right hand side of the integral equation in [**19**, Thm. 1.2A] must be replaced by the term

$$\begin{aligned}&- \lim_{\varepsilon\to 0}\frac{i}{2}\int\limits_{\{\zeta\in V:|\zeta-z|\geq\varepsilon\}}\psi^0_\theta(\zeta,\lambda)dd^cG_{\lambda,\theta}(z,\zeta) \\ &- \lim_{R\to\infty}\int\limits_{|\zeta_1|=R}[\bar{\partial}G_{\lambda,\theta}(z,\zeta)\psi^0_\theta(\zeta,\lambda) + G_{\lambda,\theta}(z,\zeta)\partial\psi^0_\theta(\zeta,\lambda)],\end{aligned}$$

like in formula (5.3) above. It is important to note that the function $\hat{\psi}^0_\theta$ in (5.3) can be represented using $\hat{\psi}_\theta(\cdot,\lambda)|_{\partial X}$ by Poisson-type formulas on $X$ and $V\setminus X$:

$$\hat{\psi}^0_\theta(\zeta,\lambda)=\int\limits_{w\in\partial X}\hat{\psi}_\theta(w,\lambda)\partial g^0_+(\zeta,w),\quad \text{if } \zeta\in X, \tag{5.5}$$

$$\hat{\psi}^0_\theta(\zeta,\lambda)=-\int\limits_{w\in\partial X}\hat{\psi}_\theta(w,\lambda)\partial g^0_-(\zeta,w)-i\sum_{j=1}^{g}e^{\lambda(a_{j,1}+\theta a_{j,2})}C_{j,\theta}(\lambda)g^0_-(\zeta,a_j), \tag{5.6}$$

if $\zeta\in V\setminus X$, where $g^0_+$ is the Green function for the Laplacian $\bar\partial\partial$ on $X$ such that $g^0_+(\cdot,0)|_{\partial X}=0$, and $g^o_-$ is a Green function for $\bar\partial\partial\psi=0$ on $V\setminus X$ with the condition $g^0_-(\cdot,0)|_{\partial X}=0$ and $\psi(\zeta)=e^{\lambda(\zeta_1+\theta\zeta_2)}O(1)$, $\zeta\to\infty$. The existence of such a Green function on $V\setminus X$ follows from [**19**, Lemma 4.1].

In order to prove Theorem 5.1 we will need the following equality:

LEMMA 5.2. *For $\lambda\in\mathbb{C}\setminus E_\theta$, $|\lambda|\ge const(V,\{a_j\},\theta,\hat\sigma)$ and $z\in V$ we have*

$$e^{\lambda(z_1+\theta z_2)}+\lim_{\varepsilon\to 0}\frac{i}{2}\int\limits_{\{\zeta\in V:|\zeta-z|\ge\varepsilon\}}\hat{\psi}_\theta(\zeta,\lambda)dd^cG_{\lambda,\theta}(z,\zeta)$$
$$+\lim_{R\to\infty}\int\limits_{|\zeta_1|=R}[\bar\partial G_{\lambda,\theta}(z,\zeta)\hat{\psi}_\theta(\zeta,\lambda)+G_{\lambda,\theta}(z,\zeta)\partial\hat{\psi}_\theta(\zeta,\lambda)]=0 \tag{5.7}$$

PROOF. We write $\hat\mu_\theta$ as the solution of the integral equation

$$\hat\mu_\theta(z,\lambda)=1+\frac{i}{2}\int_{\zeta\in X}g_{\lambda,\theta}(z,\zeta)Q\hat\mu_\theta(\zeta,\lambda)+i\sum_{j=1}^{g}\hat C_{j,\theta}(\lambda)g_{\lambda,\theta}(z,a_j), \tag{5.8}$$

for $z\in V$. The equivalence between (5.2) and (5.8) implies the equality

$$\hat\mu_\theta(z,\lambda)=1+\int_{\zeta\in V}g_{\lambda,\theta}(z,\zeta)\bar\partial(\partial+\lambda(d\zeta_1+\theta d\zeta_2))\hat\mu_\theta(\zeta,\lambda),$$

which becomes, using integration by parts,

$$\hat\mu_\theta(z,\lambda)=1+\int_{\zeta\in V}\bar\partial(\partial-\lambda(d\zeta_1+\theta d\zeta_2))g_{\lambda,\theta}(z,\zeta)\hat\mu_\theta(\zeta,\lambda)$$
$$+\lim_{R\to\infty}\int\limits_{|\zeta_1|=R}[\bar\partial g_{\lambda,\theta}(z,\zeta)\hat\mu_\theta(\zeta,\lambda)+g_{\lambda,\theta}(z,\zeta)(\partial+\lambda(d\zeta_1+\theta d\zeta_2))\hat\mu_\theta(\zeta,\lambda)].$$

Now, in order to obtain (5.7), it is sufficient to prove the following limit:

$$\lim_{\varepsilon\to 0}\int\limits_{\{\zeta\in V:|\zeta-z|\le\varepsilon\}}\bar\partial(\partial-\lambda(d\zeta_1+\theta d\zeta_2))g_{\lambda,\theta}(z,\zeta)\hat\mu_\theta(\zeta,\lambda)=\hat\mu_\theta(z,\lambda). \tag{5.9}$$

This limit is based on the following formula

$$
\begin{aligned}
(5.10)\quad & G_{\lambda,\theta}(z,\zeta) - \overline{G_{-\lambda,\theta}(\zeta,z)} \\
&= -\int_{w\in V} G_{\lambda,\theta}(w,\zeta) e^{\bar\lambda[(\bar w_1-\bar z_1)+\bar\theta(\bar w_2-\bar z_2)]}\overline{\mathcal{H}_{-\lambda,\theta}(\hat R(\delta(\cdot,z)))}\wedge\lambda(dw_1+\theta dw_2) \\
&\quad + \int_{w\in V}\overline{G_{-\lambda,\theta}(w,z)} e^{\lambda[(w_1-\zeta_1)+\theta(w_2-\zeta_2)]}\mathcal{H}_{\lambda,\theta}(\hat R(\delta(\cdot,\zeta)))\wedge\bar\lambda(d\bar w_1+\bar\theta d\bar w_2),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}_{\lambda,\theta}(\hat R(\delta(\cdot,\zeta))) &= e_{-\lambda,\theta}\mathcal{H}(e_{\lambda,\theta}(\hat R(\delta(\cdot,\zeta)))),\\
e_{\lambda,\theta}(w) &= e^{\lambda(w_1+\theta w_2)-\bar\lambda(\bar w_1+\bar\theta\bar w_2)}.
\end{aligned}
$$

The proof of (5.10) follows the proof of a classical theorem about the symmetry of the classical Green function (see [**9**, p.434]), combined with the following statement from [**19**, Remark 1.2]

$$
(5.11)\quad \bar\partial(\partial+\lambda(dz_1+\theta dz_2))g_{\lambda,\theta}(z,\zeta) = \delta(z,\zeta)+\bar\lambda(d\bar z_1+\bar\theta d\bar z_2)\wedge\mathcal{H}_{\lambda,\theta}(\hat R(\delta(z,\zeta))).
$$

Limit (5.9) is now given by formula (5.10) and the following estimates:

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int\limits_{\{\zeta\in V:|\zeta-z|\le\varepsilon\}}\bar\lambda(d\bar\zeta_1+\bar\theta d\bar\zeta_2)\wedge\mathcal{H}_{\lambda,\theta}(\hat R(\delta(\zeta,z)))\hat\mu_\theta(\zeta,\lambda)=0,\\
&\int_V \bar\lambda(d\bar\zeta_1+\bar\theta d\bar\zeta_2)\wedge\mathcal{H}_{\lambda,\theta}(\hat R(\delta(\zeta,z)))\hat\mu_\theta(\zeta,\lambda)<\infty. \qquad\square
\end{aligned}
$$

Proof of Theorem 5.1. *i)* Like in the isotropic case (see [**19**, Lemmas 3.1, 3.3]) a solution $\hat\psi_\theta$ of the differential equation (5.1) can be characterized as a solution of the integral equation

$$
\begin{aligned}
(5.12)\qquad \hat\psi_\theta(z,\lambda) &= \frac{i}{2}\int_{\zeta\in X} G_{\lambda,\theta}(z,\zeta)Q\hat\psi_\theta(\zeta,\lambda)\\
&\quad + e^{\lambda(z_1+\theta z_2)} + ie^{\lambda(z_1+\theta z_2)}\sum_{j=1}^{g}\hat C_{j,\theta}(\lambda)g_{\lambda,\theta}(z,\hat a_j),
\end{aligned}
$$

where $\{\hat C_{j,\theta}(\lambda)\}$ satisfy (5.4). Indeed (5.1) implies that $\partial\hat\psi_\theta$ is holomorphic on $V\setminus(X\cup\bigcup_j\{\hat a_j\})$, the estimate

$$
\partial\hat\psi_\theta = e^{z_1+\theta z_2}O(1),\quad \text{for } z\to\infty
$$

and the equality

$$
\mathrm{Res}_{\hat a_j}\partial\hat\psi_\theta = \frac{\hat C_{j,\theta}}{2\pi}e^{\hat a_{j,1}+\theta\hat a_{j,2}}.
$$

The residue theorem applied to the form

$$
\frac{e^{-(z_1+\theta z_2)}\partial\hat\psi_\theta}{(z_1+\theta z_2)^k}
$$

gives (5.4).

Now, using equality (5.7), for $z \in V \setminus X$ we obtain

$$\begin{aligned}
&\frac{i}{2}\int_{\partial X} e^{\lambda[(z_1-\zeta_1)+\theta(z_2-\zeta_2)]} g_{\lambda,\theta}(z,\zeta)(\Lambda_{\hat\sigma}-\Lambda_0)\hat\psi_\theta(\zeta,\lambda)\\
&\quad= \frac{i}{2}\int_{\partial X} G_{\lambda,\theta}(z,\zeta)(\Lambda_{\hat\sigma}-\Lambda_0)\hat\psi_\theta(\zeta,\lambda)\\
&\quad= \frac{i}{2}\int_X G_{\lambda,\theta}(z,\zeta)dd^c\hat\psi_\theta(\zeta,\lambda) - \frac{i}{2}\int_X dd^cG_{\lambda,\theta}(z,\zeta)\,[\hat\psi_\theta(\zeta,\lambda)-\hat\psi^0_\theta(\zeta,\lambda)]\\
&\quad= \frac{i}{2}\int_X G_{\lambda,\theta}(z,\zeta)Q\hat\psi_\theta + \lim_{\varepsilon\to 0}\frac{i}{2}\int\limits_{\{\zeta\in V:|\zeta-z|\ge\varepsilon\}} \hat\psi^0_\theta(\zeta,\lambda)dd^cG_{\lambda,\theta}(z,\zeta) + e^{\lambda(z_1+\theta z_2)}\\
&\qquad+ \lim_{R\to\infty}\int\limits_{|\zeta_1|=R} [\bar\partial G_{\lambda,\theta}(z,\zeta)\hat\psi^0_\theta(\zeta,\lambda) + G_{\lambda,\theta}(z,\zeta)\partial\hat\psi^0_\theta(\zeta,\lambda)]\\
&\quad= \hat\psi_\theta(z,\lambda) - ie^{\lambda(z_1+\theta z_2)}\sum_{j=1}^{g}\hat C_{j,\theta}(\lambda)g_{\lambda,\theta}(z,\hat a_j)\\
&\qquad+ \lim_{\varepsilon\to 0}\frac{i}{2}\int\limits_{\{\zeta\in V:|\zeta-z|\ge\varepsilon\}} \hat\psi^0_\theta(\zeta,\lambda)dd^cG_{\lambda,\theta}(z,\zeta)\\
&\qquad+ \lim_{R\to\infty}\int\limits_{|\zeta_1|=R} [\bar\partial G_{\lambda,\theta}(z,\zeta)\hat\psi^0_\theta(\zeta,\lambda) + G_{\lambda,\theta}(z,\zeta)\partial\hat\psi^0_\theta(\zeta,\lambda)].
\end{aligned}$$

The restriction of the last equation to the boundary $\partial X$ from outside yields (5.3).

*ii)* To prove that (5.3) is a Fredholm-type equation, for fixed $\lambda \in \mathbb{C}\setminus E_\theta$, $|\lambda| \ge const(V,\{a_j\},\theta,\hat\sigma)$, we proceed as follows. Let $f(z) = \hat\psi_\theta(z,\lambda) - e^{\lambda(z_1+\theta z_2)}$ and $f^0(z) = \hat\psi^0_\theta(z,\lambda) - e^{\lambda(z_1+\theta z_2)}$; we can write equation (5.3) as

$$f + Tf = g, \tag{5.13}$$

where

$$\begin{aligned}
g(z) = {}&\frac{i}{2}\int_{\zeta\in\partial X} G_{\lambda,\theta}(z,\zeta)(\Lambda_{\hat\sigma}-\Lambda_0)e^{\lambda(\zeta_1+\theta\zeta_2)}\\
&+ ie^{\lambda(z_1+\theta z_2)}\sum_{j=1}^{g}\hat C^0_{j,\theta}(\lambda)g_{\lambda,\theta}(z,\hat a_j),
\end{aligned}\tag{5.14}$$

$$\begin{aligned}
Tf(z) = {}&-\frac{i}{2}\int_{\zeta\in\partial X} G_{\lambda,\theta}(z,\zeta)(\Lambda_{\hat\sigma}-\Lambda_0)f(\zeta)\\
&- ie^{\lambda(z_1+\theta z_2)}\sum_{j=1}^{g}\hat C^1_{j,\theta}(\lambda)g_{\lambda,\theta}(z,\hat a_j)
\end{aligned}\tag{5.15}$$

$$+\lim_{\varepsilon\to 0}\frac{i}{2}\int\limits_{\{\zeta\in V:|\zeta-z|\geq\varepsilon\}} f^0(\zeta)dd^cG_{\lambda,\theta}(z,\zeta)$$
$$+\lim_{R\to\infty}\int\limits_{|\zeta_1|=R}[\bar\partial G_{\lambda,\theta}(z,\zeta)f^0(\zeta)+G_{\lambda,\theta}(z,\zeta)\partial f^0(\zeta)],$$

where $\hat{C}^0_{j,\theta}+\hat{C}^1_{j,\theta}=\hat{C}_{j,\theta}$ ($C^0_{j,\theta}$ is obtained from (5.4) with $e^{\lambda(z_1+\theta z_2)}$ instead of $\hat\psi_\theta(z,\lambda)$, so it is independent from $f$).

We have now that equation (5.13) is a Fredholm-type integral equation for $f\in W^{1,2}(\partial X)$. Indeed $g\in W^{1,2}(\partial X)$ and $T$ is a compact operator: this follows from the compactness of $\Lambda_{\hat\sigma}-\Lambda_0$ for the first term in (5.15), from formulas (5.5), (5.6) and (5.11) for the third term, while the second and the fourth term are operators with finite-dimensional range.

The existence, for $\lambda\in\mathbb{C}\setminus E_\theta$, $|\lambda|\geq const(V,\{a_j\},\theta,\hat\sigma)$, of a unique Faddeev-type function $\hat\psi_\theta(z,\lambda)$ imply the existence, for such $\lambda$, of a solution of (5.3) with residue data $i\hat{C}_{j,\theta}(\lambda)$, $j=1,\ldots,g$.

Let us prove the uniqueness, with $\lambda$ as above, of the solution of (5.3) in $W^{1,2}(\partial X)$. Suppose that $\hat\psi_\theta\in W^{1,2}(\partial X)$ solves (5.3), and consider $\hat\mu_\theta=e^{-\lambda(z_1+\theta z_2)}\hat\psi_\theta$ as the Dirichlet data for

$$\overline\partial(\partial+\lambda(dz_1+\theta dz_2))\hat\mu_\theta=\frac{i}{2}Q\hat\mu_\theta$$

on X; thanks to this equation we can well define $\hat\mu_\theta$ on $\overline{X}$. We also define $\hat\mu_\theta$ on $V\setminus\overline{X}$ by (5.3). The function $\hat\mu_\theta$ then defined on $V$ belongs to $C(V\setminus\cup_{j=1}^g\{a_j\})$.

To show that $\hat\psi_\theta=e^{\lambda(z_1+\theta z_2)}\hat\mu_\theta$ satisfies (4.1), (5.1) globally, we can follow without modification the arguments of [**19**, Prop. 5.1], based on the Sohotsky-Plemelj jump formula.

The uniqueness of the solution of (5.3) in $W^{1,2}(\partial X)$ with residue data $\{\hat{C}_{j,\theta}\}$ now follows from the uniqueness for Faddeev-type functions for $\lambda\in\mathbb{C}\setminus E_\theta$, $|\lambda|\geq const(V,\{a_j\},\theta,\hat\sigma)$. □

## 6. Cauchy-type Formulas

Following our reconstruction scheme, after recovering the boundary value of the Beltrami solution $F$, we obtain $F(\partial X)=\Gamma$.

Thus the remaining problem is reconstructing the interior points of a bordered Riemann surface $Y$ given the boundary $\Gamma$.

We will use the coordinates $z=(z_1,z_2)\in\mathbb{C}^2$ and the projection $p:\mathbb{C}^2\to\mathbb{C}$ on the first factor, $p(z)=z_1$. For $a\in\mathbb{C}$ we define

$$N_a=\frac{1}{2\pi i}\int_\Gamma\frac{dz_1}{z_1-a}\in\mathbb{N},$$

which counts the number of intersection points of the line $\{z_1=a\}$ with the surface $Y$ that we are going to reconstruct. Let us remark that, if we call $Y_1,\ldots,Y_s$ the bounded connected components of $\mathbb{C} \setminus p(\Gamma)$, we have that $N_a$ is constant on every $Y_h$, $h = 1, \ldots, s$.

We have the following proposition, the first part of which is a special case of a result by Harvey-Shiffman [**17**], while the second part goes back to Cauchy.

PROPOSITION 6.1. *Let $\Gamma$ be a $C^1$-closed curve in $\mathbb{C}^2$:*

*i) if $Y_1, Y_2$ are two bordered Riemann surfaces in $\mathbb{C}^2$ with the same boundary $\Gamma$, then $Y_1 = Y_2$;*

*ii) the interior points of the unique Riemann surface $Y$ whose boundary is $\Gamma$ can be explicitly found from the system of equation*

$$\frac{1}{2\pi i} \int_\Gamma z_2^k(z_1) \frac{dz_1}{z_1 - a} = \sum_{j=1}^{N_a} (z_2^{(j)})^k(a), \quad k = 1, \ldots, N_a. \tag{6.1}$$

*The points of the surface are the pairs $(a, z_2^{(j)}(a))$, for $j = 1, \ldots, N_a$, $a \in Y_h$, $h = 1, \ldots, s$.*

By $i)$ we have that $F(X) = Y$; then, from the regularity assumptions on $X$ and $F$ we deduce that $Y$ is a Riemann surface with $C^1$ boundary.

PROOF. *i)* Consider the current $Y = [Y_1] - [Y_2]$ where $[Y_1], [Y_2]$ are the currents of integration associated to $Y_1, Y_2$, respectively. As $\partial Y_1 = \partial Y_2$ we have that $Y$ is closed (i.e. $dY = 0$), of bidegree (1,1) and with $\dim_{\mathbb{R}} \operatorname{supp}(Y) \leq 2$. By the structure theorem of Harvey-Shiffman (see [**17**]), $Y$ is the current associated to a compact complex manifold $Y' \subset \mathbb{C}^2$, which can be at most a single point; thus $Y = 0$ (as (1,1)-current) and $Y_1 = Y_2$.

*ii)* Formulas (6.1) are true by residue theorem. Now, if $a \in Y_h$ for some $h$, since we know the Newton sums $\sum_{j=1}^{N_a} (z_2^{(j)})^k(a)$ for every $k$, we can find $(z_2^{(j)})(a)$ by a well-known algebra result. □

## 7. Reconstruction of $\sigma$

Thanks to the integral equation (5.3) and formulas (4.5), (4.4), we can find $\psi_\theta(w, \lambda)|_{\partial Y}$ from $\Lambda_{\hat{\sigma}}$, where $\psi_\theta$ is a Faddeev-type isotropic solution as in the proof of Theorem 4.1 and $Y$ is the reconstructed surface in section 6. By the remarks in section 2.1, from $\Lambda_{\hat{\sigma}}$ and $F|_{\partial X}$ we can also find $\Lambda_\sigma$ on $\partial Y$.

Thus we have that $\Lambda_{\hat{\sigma}}$ determines $\Lambda_\sigma$ uniquely and $\psi_\theta(w, \lambda)|_{\partial Y}$, for $\lambda \in \mathbb{C} \setminus E_\theta$, $|\lambda| \geq const(V, \{a_j\}, \theta, \hat{\sigma})$ and for $\theta \in \mathbb{C}$. This will be sufficient to recover $\sigma$ on $Y$.

We define $\tilde{\psi}_\theta = \sqrt{\sigma}\psi_\theta$, so that by (4.2) $dd^c\tilde{\psi}_\theta = q\tilde{\psi}_\theta + \sum_{j=1}^{g} C_{j,\theta}(\lambda)\delta(z, a_j)$, where $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$, and we have the following theorem:

THEOREM 7.1 (Thm. 1.2B [**19**]). *The function $\sigma(w)$, $w \in Y$, can be reconstructed from the Dirichlet-to-Neumann data*

$$\tilde{\psi}_\theta|_{\partial Y} = \mu_\theta|_{\partial Y} e^{\lambda(z_1 + \theta z_2)} \to \overline{\partial}\tilde{\psi}_\theta|_{\partial Y}$$

*using an explicit formula. In particular, for the case* $W = \{z \in \mathbb{C}^2 : P(z) = 0\}$, *where* $P$ *is a polynomial of degree* $N$, *this formula is as follows. Let* $\{w_m\}$ *be the points of* $W$ *where* $(dz_1 + \theta dz_2)|_W(w_m) = 0$, $m = 1, \ldots, M$. *Then, for almost every* $\theta$, *the value* $\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}dd^c|z|^2}|_W(w_m)$ *can be found from the following linear system:*

$$
\begin{aligned}
(7.1) \qquad & \tau(1+o(1))\frac{d^k}{d\tau^k}\left(\int_{z\in\partial Y} e_{i\tau,\theta}\overline{\partial}\mu_\theta(z,i\tau)\right) \\
&= \tau(1+o(1))\frac{d^k}{d\tau^k}\left(\int_{z\in Y} e_{i\tau,\theta} q\mu_\theta(z,i\tau)\right) \\
&= \sum_{m=1}^{M} \frac{i\pi(1+|\theta|^2)}{2}\left.\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}dd^c|z|^2}\right|_W (w_m) \\
&\times \frac{|\frac{\partial P}{\partial z_1}(w_m)|^3 \frac{d^k}{d\tau^k}\exp i\tau[(w_{m,1}+\theta w_{m,2}) + (\overline{w}_{m,1}+\overline{\theta}\overline{w}_{m,2})]}{|\frac{\partial^2 P}{\partial z_1^2}(\frac{\partial P}{\partial z_2})^2 - 2\frac{\partial^2 P}{\partial z_1\partial z_2}(\frac{\partial P}{\partial z_2})(\frac{\partial P}{\partial z_1}) + \frac{\partial^2 P}{\partial z_2^2}(\frac{\partial P}{\partial z_1})^2|(w_m)}
\end{aligned}
$$

*where* $m, k = 1, \ldots, M$; $M = N(N-1)$, $\tau \in \mathbb{R}$, $\tau \to \infty$ *such that* $|\tau|^g|\Delta_\theta(i\tau)| \geq \varepsilon > 0$, *with* $\varepsilon$ *small enough. The determinant of system* (7.1) *is proportional to the Vandermonde determinant of the points* $\{(w_{m,1} + \theta w_{m,2}) + (\overline{w}_{m,1} + \overline{\theta}\overline{w}_{m,2})\}$.

The proof of this theorem is given in **[19]**, under the condition that $Sing\,Y = \emptyset$; nevertheless, the proof is still valid if $Y$ contains self-intersection-type singularities.

To apply Theorem 7.1, since $\tilde{\psi}_\theta|_{\partial Y} = \psi_\theta|_{\partial Y}$ we only need to show that the integral

$$\int_{\partial Y} e_{\lambda,\theta}\overline{\partial}\mu_\theta(z,\lambda) = \int_{\partial Y} e^{-\overline{\lambda}(\overline{z}_1+\overline{\theta}\overline{z}_2)}\overline{\partial}\psi_\theta(z,\lambda), \quad \lambda \in \mathbb{C}$$

can be expressed in terms of $\Lambda_\sigma$. This is a consequence of the following lemma.

Lemma 7.2. *For every* $\phi \in C^1(\partial Y)$ *and every* $\psi \in C^1(Y)$ *solution of* $d\sigma d^c\psi = (dd^c - M)\psi = 0$ *in* $Y$, *we have*

$$\int_{\partial Y} \phi(\Lambda_\sigma - \Lambda_0)\psi = 2i\int_{\partial Y} \phi(\overline{\partial}\psi - \overline{\partial}\psi_0), \tag{7.2}$$

*where* $dd^c\psi_0 = 0$ *in* $Y$ *and* $\psi_0|_{\partial Y} = \psi|_{\partial Y}$.

Proof. Let $a \in C^1(Y)$ such that $a|_{\partial Y} = \phi$. From the definition of the Dirichlet-to-Neumann operator and from Stokes' theorem, one has

$$\int_{\partial Y} \phi(\Lambda_\sigma - \Lambda_0)\psi = \int_Y (da \wedge d^c(\psi - \psi_0) + aM\psi),$$

and, with the identity $dd^c = 2i\partial\overline{\partial}$, Stokes' theorem gives

$$2i\int_{\partial Y} \phi(\overline{\partial}\psi - \overline{\partial}\psi_0) = 2i\int_Y \partial a \wedge (\overline{\partial}\psi - \overline{\partial}\psi_0) + \int_Y aM\psi\, dxdy.$$

Expressing the first integrand on the right in coordinate form we get

$$\partial a \wedge \overline{\partial}(\psi - \psi_0) = da \wedge \overline{\partial}(\psi - \psi_0) = \frac{1}{2i} da \wedge d^c(\psi - \psi_0) + \frac{1}{2} da \wedge d(\psi - \psi_0).$$

Again by Stokes' thorem we have

$$\int_Y da \wedge d(\psi - \psi_0) = -\int_{\partial Y} (\psi - \psi_0) da = 0$$

because $\psi|_{\partial Y} = \psi_0|_{\partial Y}$. The proof follows. □

If we put $\phi = e^{-\overline{\lambda}(\overline{z}_1 + \overline{\theta}\overline{z}_2)}|_{\partial Y}$, we find that

$$\begin{aligned}\frac{1}{2i}\int_{\partial Y} e^{-\overline{\lambda}(\overline{z}_1+\overline{\theta}\overline{z}_2)}(\Lambda_\sigma - \Lambda_0)\psi_\theta &= \int_{\partial Y} e^{-\overline{\lambda}(\overline{z}_1+\overline{\theta}\overline{z}_2)}(\overline{\partial}\psi_\theta - \overline{\partial}\psi_0)\\ &= \int_{\partial Y} e^{-\overline{\lambda}(\overline{z}_1+\overline{\theta}\overline{z}_2)}\overline{\partial}\psi_\theta,\end{aligned}$$

because $\partial e^{-\overline{\lambda}(\overline{z}_1+\overline{\theta}\overline{z}_2)} = 0$ and $\partial\overline{\partial}\psi_0 = 0$ on $Y$.

## 8. Proof of Theorem 1.1

We now put together all the results of this paper to prove the main theorem and his corollary.

**Proof of Theorem 1.1.** We start finding a complex structure on $X$. This is done by a standard construction, as suggested in the introduction. We consider the local form of the Euclidean metric of $\mathbb{R}^3$ restricted to $X$:

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2$$

where $x, y$ are oriented coordinates. Let $z = x + iy$, and define

$$\mu(z) = \frac{\frac{1}{2}(E - G) + iF}{\frac{1}{2}(E + G) + \sqrt{EG - F^2}}.$$

Then the local homeomorphic solutions of the Beltrami equation $\frac{\partial w}{\partial \bar{z}} = \mu(z)\frac{\partial w}{\partial z}$ form a holomorphic atlas on $X$, which then becomes a Riemann surface.

We now embed $X$ in $\mathbb{C}P^3$ – as explained in section 2 – as an open set of a nonsingular affine algebraic curve $V$. By proposition 3.3, there exists a unique $C^1$-quasiconformal diffeomorphism $F : V \to W$ with special asymptotic conditions such that $F_*\hat{\sigma} = \sigma$ is isotropic on $W$.

Starting from $\Lambda_{\hat{\sigma}}$ we first recover $\hat{\psi}_\theta(z, \lambda)|_{\partial X}$ by integral equation (5.3), and then $F|_{\partial X}$ by formula (4.5).

Successively, from the knowledge of $F(\partial X) = \partial Y$, we reconstruct $Y$ using the formulas (6.1). Finally we can reconstruct $\sigma$ on $Y \backslash Sing(Y)$ with the help of Theorem 7.1 and the remarks in section 7.

If $\tilde{Y}$, $\tilde{\sigma}$ and $\tilde{F} : X \to \tilde{Y}$ are as in the statement of the theorem, then $\Psi = \tilde{F} \circ F^{-1} : Y \to \tilde{Y}$ is weakly holomorphic because $\tilde{F}$ satisfies the same Beltrami equation as

$F$. By properties of $F$ we have that $\Psi : Y \setminus Sing(Y) \to \tilde{Y} \setminus \Psi(Sing(Y))$ is a biholomorphism which can be uniquely extended to a biholomorphism $\Psi' : Y' \to \tilde{Y}'$, where $Y'$ and $\tilde{Y}'$ are normalizations of $Y$ and $\tilde{Y}$ respectively. Properties of $F$ allow us also to extend smoothly $\sigma$ and $\tilde{\sigma}$ on $Y'$ and $\tilde{Y}'$ as $\sigma'$ and $\tilde{\sigma}'$ respectively. Finally we obtain $\Psi'_*\sigma' = \tilde{\sigma}'$, which ends the proof. □

**Proof of Corollary 1.2.** If we require that $F$ has the special asymptotics as in proposition 3.3, then the whole construction in Theorem 1.1 is unique.

Taking account of this, if $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$ we have $F_1|_{\partial X} = F_2|_{\partial X}$, where $F_1, F_2$ are the special quasiconformal solutions given by proposition 3.3 associated to $\sigma_1$ and $\sigma_2$ respectively. Thus we also obtain, from $F_1(\partial X) = F_2(\partial X) = \partial Y$ and the formulas (6.1), that $F_1(X) = F_2(X) = Y$. Let $G : Y' \to Y$ be a normalization of $Y$, and $F'_j = G^{-1} \circ F_j : X \setminus F_j^{-1}(Sing(Y)) \to Y' \setminus G^{-1}(Sing(Y))$, for $j = 1, 2$. Then, by construction, $F'_j$ can be extended as a global diffeomorphism between $X$ and $Y'$, for $j = 1, 2$. Now, if we define the smooth isotropic conductivities on $Y'$ as $\sigma'_j = (F'_j)_*\sigma_j$, $j = 1, 2$, we find $\Lambda_{\sigma'_1} = \Lambda_{\sigma'_2}$, and the boundary values of the respective Faddeev-type anisotropic (resp. isotropic) solutions coincide on $\partial X$ (resp. $\partial Y'$). Consequently $\sigma'_1 = \sigma'_2$ on $Y'$.

We finally define $\Phi = {F'}_2^{-1} \circ F'_1 : \overline{X} \to \overline{X}$ which satisfies $\Phi|_{\partial X} = \mathrm{Id}$ and $\Phi_*\sigma_1 = \sigma_2$. □

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# PAPER **C**

## PAPER C

# A global stability estimate for the Gel'fand-Calderón inverse problem in two dimensions

Roman G. Novikov and Matteo Santacesaria

Abstract. We prove a global logarithmic stability estimate for the Gel'fand-Calderón inverse problem on a two-dimensional domain.

## 1. Introduction

Let $D$ be an open bounded domain in $\mathbb{R}^2$ with with $C^2$ boundary and let $v \in C^1(\bar{D})$. The Dirichlet-to-Neumann map associated to $v$ is the operator $\Phi : C^1(\partial D) \to L^p(\partial D),\ p < \infty$ defined by:

$$\Phi(f) = \frac{\partial u}{\partial \nu}\Big|_{\partial D} \tag{1.1}$$

where $f \in C^1(\partial D)$, $\nu$ is the outer normal of $\partial D$ and $u$ is the $H^1(\bar{D})$-solution of the Dirichlet problem

$$-\Delta u + v(x)u = 0 \text{ on } D,\ \ u|_{\partial D} = f; \tag{1.2}$$

here we assume that 0 is not a Dirichlet eigenvalue for the operator $-\Delta + v$ in $D$.

Equation (1.2) arises, in particular, in quantum mechanics, acoustics, electrodynamics; formally, it looks like the Schrödinger equation with potential $v$ at zero energy.

The following inverse boundary value problem arises from this construction: given $\Phi$ on $\partial D$, find $v$ on $D$.

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at zero energy (see **[4]**, **[9]**) and can also be seen as a generalization of the Calderón problem for the electrical impedance tomography (see **[3]**, **[9]**).

The global injectivity of the map $v \to \Phi$ was firstly proved in **[9]** for $D \subset \mathbb{R}^d$ with $d \geq 3$ and in **[2]** for $d = 2$ with $v \in L^p$. A global stability estimate for the Gel'fand-Calderón problem for $d \geq 3$ was firstly proved by Alessandrini in **[1]**; this result was recently improved in **[10]**.

In this paper we show that, also in the two dimensional case, an estimate of the same type as in [**1**] is valid. Indeed out main theorem is the following:

THEOREM 1.1. *Let $D \subset \mathbb{R}^2$ be an open bounded domain with $C^2$ boundary, let $v_1, v_2 \in C^2(\bar{D})$ with $\|v_j\|_{C^2(\bar{D})} \le N$ for $j = 1, 2$, and $\Phi_1, \Phi_2$ the corresponding Dirichlet-to-Neumann operators. For simplicity we assume also that $v_j|_{\partial D} = 0$ and $\frac{\partial}{\partial \nu} v_j|_{\partial D} = 0$ for $j = 1, 2$. Then there exists a constant $C = C(D, N)$ such that*

$$\|v_2 - v_1\|_{L^\infty(D)} \le C(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{-\frac{1}{2}} \log(3 \log(3 + \|\Phi_2 - \Phi_1\|^{-1})), \tag{1.3}$$

*where $\|A\|$ denotes the norm of an operator $A : L^\infty(\partial D) \to L^\infty(\partial D)$.*

This is the first result about the global stability of the Gel'fand-Calderón inverse problem in two dimension, for general potentials. Results of such a type were only known for special kinds of potentials, e.g. potentials coming from conductivities (see [**6**] for example). Note also that for the Calderón problem (of the electrical impedance tomography) in its initial formulation the global injectivity was firstly proved in [**11**] for $d \ge 3$ and in [**8**] for $d = 2$.

Instability estimates complementing the stability estimates of [**1**], [**6**], [**10**] and of the present work are given in [**7**].

The proof of Theorem 1.1 takes inspiration mostly from [**2**] and [**1**]. For $z_0 \in D$ we show existence and uniqueness of a family of solution $\psi_{z_0}(z, \lambda)$ of equation (1.2) where in particular $\psi_{z_0} \to e^{\lambda(z - z_0)^2}$, for $\lambda \to \infty$. This is accomplished by introducing a special Green's function for the Laplacian which satisfies precise estimates. Then, using Alessandrini's identity along with stationary phase techniques, we obtain the result.

An extension of Theorem 1.1 for the case when we do not assume that $v_j|_{\partial D} = 0$ and $\frac{\partial}{\partial \nu} v_j|_{\partial D} = 0$ for $j = 1, 2$ is given in section 6.

## 2. Bukhgeim-type analogues of the Faddeev functions

In this section we introduce the above-mentioned family of solutions of equation (1.2), which will be used throughout all the paper.

We identify $\mathbb{R}^2$ with $\mathbb{C}$ and use the coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ where $(x_1, x_2) \in \mathbb{R}^2$. Let us define the function spaces $C^1_{\bar{z}}(\bar{D}) = \{u : u, \frac{\partial u}{\partial \bar{z}} \in C(\bar{D})\}$ with the norm $\|u\|_{C^1_{\bar{z}}(\bar{D})} = \max(\|u\|_{C(\bar{D})}, \|\frac{\partial u}{\partial \bar{z}}\|_{C(\bar{D})})$, $C^1_z(\bar{D}) = \{u : u, \frac{\partial u}{\partial z} \in C(\bar{D})\}$ with an analogous norm and the following functions:

$$G_{z_0}(z,\zeta,\lambda) = e^{\lambda(z-z_0)^2} g_{z_0}(z,\zeta,\lambda) e^{-\lambda(\zeta-z_0)^2}, \tag{2.1}$$

$$g_{z_0}(z,\zeta,\lambda) = \frac{e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2}}{4\pi^2}\int_D \frac{e^{-\lambda(\eta-z_0)^2+\bar\lambda(\bar\eta-\bar z_0)^2}}{(z-\eta)(\bar\eta-\bar\zeta)} d\mathrm{Re}\eta\, d\mathrm{Im}\eta, \tag{2.2}$$

$$\psi_{z_0}(z,\lambda) = e^{\lambda(z-z_0)^2}\mu_{z_0}(z,\lambda), \tag{2.3}$$

$$\mu_{z_0}(z,\lambda) = 1 + \int_D g_{z_0}(z,\zeta,\lambda) v(\zeta)\mu_{z_0}(\zeta,\lambda) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \tag{2.4}$$

$$h_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2} v(z)\mu_{z_0}(z,\lambda) d\mathrm{Re}z\, d\mathrm{Im}z, \tag{2.5}$$

where $z, z_0, \zeta \in D$ and $\lambda \in \mathbb{C}$. In addition, equation (2.4) at fixed $z_0$ and $\lambda$, is considered as a linear integral equation for $\mu_{z_0}(\cdot,\lambda) \in C^1_{\bar z}(\bar D)$.

We have that

$$4\frac{\partial^2}{\partial z \partial \bar z} G_{z_0}(z,\zeta,\lambda) = \delta(z-\zeta), \tag{2.6}$$

$$4\left(\frac{\partial}{\partial z} + 2\lambda(z-z_0)\right)\frac{\partial}{\partial \bar z} g_{z_0}(z,\zeta,\lambda) = \delta(z-\zeta), \tag{2.7}$$

$$-4\frac{\partial^2}{\partial z \partial \bar z}\psi_{z_0}(z,\lambda) + v(z)\psi_{z_0}(z,\lambda) = 0, \tag{2.8}$$

$$-4\left(\frac{\partial}{\partial z} + 2\lambda(z-z_0)\right)\frac{\partial}{\partial \bar z}\mu_{z_0}(z,\lambda) + v(z)\mu_{z_0}(z,\lambda) = 0, \tag{2.9}$$

where $z, z_0, \zeta \in D$, $\lambda \in \mathbb{C}$, $\delta$ is the Dirac's delta. Formulas (2.6)-(2.9) follow from (2.1)-(2.4) and from

$$\frac{\partial}{\partial \bar z}\frac{1}{\pi z} = \delta(z), \quad \left(\frac{\partial}{\partial z} + 2\lambda(z-z_0)\right)\frac{e^{-\lambda(z-z_0)^2+\bar\lambda(\bar z-\bar z_0)^2}}{\pi \bar z} e^{\lambda z_0^2-\bar\lambda\bar z_0^2} = \delta(z),$$

where $z, z_0, \lambda \in \mathbb{C}$.

We say that the functions $G_{z_0}$, $g_{z_0}$, $\psi_{z_0}$, $\mu_{z_0}$, $h_{z_0}$ are the Bukhgeim-type analogues of the Faddeev functions (see **[9]**, **[8]**, **[2]**).

## 3. Estimates for $g_{z_0}, \mu_{z_0}, h_{z_0}$

This section is devoted to crucial estimates concerning the functions defined in section 2.

Let

$$g_{z_0,\lambda}u(z) = \int_D g_{z_0}(z,\zeta,\lambda)u(\zeta) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta,\ z \in \bar D,\ z_0, \lambda \in \mathbb{C}, \tag{3.1}$$

where $g_{z_0}(z,\zeta,\lambda)$ is defined by (2.2) and $u$ is a test function.

LEMMA 3.1. *Let $g_{z_0,\lambda}u$ be defined by* (3.1), *where $u \in C^1_{\bar z}(\bar D)$, $z_0, \lambda \in \mathbb{C}$. Then the following estimates hold:*

$$g_{z_0,\lambda}u \in C^1_{\bar z}(\bar D),$$

$$\|g_{z_0,\lambda}u\|_{C^1_{\bar z}(\bar D)} \le \frac{c_1(D)}{|\lambda|^{\frac{1}{2}}}\|u\|_{C^1_{\bar z}(\bar D)},\ |\lambda| \ge 1, \tag{3.2}$$

$$\|\frac{\partial}{\partial z}g_{z_0,\lambda}u\|_{L^p(\bar D)} \le \frac{c_2(D,p)}{|\lambda|^{\frac{1}{2}}}\|u\|_{C^1_{\bar z}(\bar D)},\ |\lambda| \ge 1,\ 1 < p < \infty. \tag{3.3}$$

Lemma 3.1 is proved in section 5.

Given a potential $v \in C^1_{\bar z}(\bar D)$ we define the operator $g_{z_0,\lambda}v$ simply as $(g_{z_0,\lambda}v)u(z) = g_{z_0,\lambda}w(z),\ w = vu$, for a test function $u$. If $u \in C^1_{\bar z}(\bar D)$, by Lemma 3.1 we have that $g_{z_0,\lambda}v : C^1_{\bar z}(\bar D) \to C^1_{\bar z}(\bar D)$,

$$\|g_{z_0,\lambda}v\|^{op}_{C^1_{\bar z}(\bar D)} \le 2\|g_{z_0,\lambda}\|^{op}_{C^1_{\bar z}(\bar D)}\|v\|_{C^1_{\bar z}(\bar D)}, \tag{3.4}$$

where $\|\cdot\|^{op}_{C^1_{\bar z}(\bar D)}$ denotes the operator norm in $C^1_{\bar z}(\bar D)$, $z_0, \lambda \in \mathbb{C}$. In addition, $\|g_{z_0,\lambda}\|^{op}_{C^1_{\bar z}(\bar D)}$ is estimated in Lemma 3.1. Inequality (3.4) and Lemma 3.1 implies existence and uniqueness of $\mu_{z_0}(z,\lambda)$ (and thus also $\psi_{z_0}(z,\lambda)$) for $|\lambda|$ sufficiently large.

Let

$$\mu^{(k)}_{z_0}(z,\lambda) = \sum_{j=0}^{k}(g_{z_0,\lambda}v)^j 1,$$

$$h^{(k)}_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}v(z)\mu^{(k)}_{z_0}(z,\lambda)d\mathrm{Re}z\,d\mathrm{Im}z,$$

where $z, z_0 \in D$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N}\cup\{0\}$.

LEMMA 3.2. *For $v \in C^1_{\bar z}(\bar D)$ such that $v|_{\partial D} = 0$ the following formula holds:*

$$v(z_0) = \frac{2}{\pi}\lim_{\lambda\to\infty}|\lambda|h^{(0)}_{z_0}(\lambda),\ z_0 \in D. \tag{3.5}$$

*In addition, if $v \in C^2(\bar D)$, $v|_{\partial D} = 0$ and $\frac{\partial v}{\partial \nu}|_{\partial D} = 0$ then*

$$|v(z_0) - \frac{2}{\pi}|\lambda|h^{(0)}_{z_0}(\lambda)| \le c_3(D)\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar D)}, \tag{3.6}$$

*for $z_0 \in D$, $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$.*

Lemma 3.2 is proved in section 5.

Let

$$W_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}w(z)d\mathrm{Re}\,zd\mathrm{Im}\,z,$$

where $z_0 \in \bar D$, $\lambda \in \mathbb{C}$ and $w$ is some function on $\bar D$. (One can see that $W_{z_0} = h^{(0)}_{z_0}$ for $w = v$.)

LEMMA 3.3. *For $w \in C^1_{\bar z}(\bar D)$ the following estimate holds:*

$$|W_{z_0}(\lambda)| \le c_4(D)\frac{\log(3|\lambda|)}{|\lambda|}\|w\|_{C^1_{\bar z}(\bar D)},\ z_0 \in \bar D,\ |\lambda| \ge 1, \tag{3.7a}$$

$$|W_{z_0}(\lambda)| \le c_{4,1}(D)\frac{\log(3|\lambda|)}{|\lambda|}\|w\|_{C(\bar D)} + \frac{c_{4,2}(D,p)}{|\lambda|}\|\frac{\partial}{\partial z}w\|_{L^p(\bar D)}, \tag{3.7b}$$

*for $2 < p < \infty$.*

Lemma 3.3 is proved in Section 5.

LEMMA 3.4. *For $v \in C^1_{\bar z}(\bar D)$ and for $\|g_{z_0,\lambda}v\|^{op}_{C^1_{\bar z}(\bar D)} \le \delta < 1$ we have that*

$$\|\mu_{z_0}(\cdot,\lambda) - \mu^{(k)}_{z_0}(\cdot,\lambda)\|_{C^1_{\bar z}(\bar D)} \le \frac{\delta^{k+1}}{1-\delta}, \tag{3.8}$$

$$|h_{z_0}(\lambda) - h^{(k)}_{z_0}(\lambda)| \le c_4(D)\frac{\log(3|\lambda|)}{|\lambda|}\frac{\delta^{k+1}}{1-\delta}\|v\|_{C^1_{\bar z}(\bar D)}, \tag{3.9}$$

*where $z_0 \in D \setminus \{0\},\ \lambda \in \mathbb{C},\ |\lambda| \ge 1,\ k \in \mathbb{N} \cup \{0\}$.*

Lemma 3.4 is proved in section 5.

## 4. Proof of Theorem 1.1

We start from Alessandrini's identity

$$\int_D (v_2(z) - v_1(z))\psi_2(z)\psi_1(z) d\mathrm{Re}z\, d\mathrm{Im}z = \int_{\partial D}\int_{\partial D}\psi_1(z)(\Phi_2 - \Phi_1)(z,\zeta)\psi_2(\zeta)|d\zeta||dz|,$$

which holds for every $\psi_j$ solution of $(-\Delta + v_j)\psi_j = 0$ on $D$, $j = 1,2$. Here $(\Phi_2 - \Phi_1)(z,\zeta)$ is the kernel of the operator $\Phi_2 - \Phi_1$.

Let $\bar\mu_{z_0}$ denote the complex conjugated of $\mu_{z_0}$ for real-valued $v$ and, more generally, the solution of (2.4) with $g_{z_0}(z,\zeta,\lambda)$ replaced by $\overline{g_{z_0}(z,\zeta,\lambda)}$for complex-valued $v$. Put $\psi_1(z) = \bar\psi_{1,z_0}(z,-\lambda) = e^{-\bar\lambda(\bar z - \bar z_0)^2}\bar\mu_1(z,-\lambda)$, $\psi_2(z) = \psi_{2,z_0}(z,\lambda) = e^{\lambda(z-z_0)^2}\mu_2(z,\lambda)$, where we called for simplicity $\bar\mu_1 = \bar\mu_{1,z_0}$, $\mu_2 = \mu_{2,z_0}$. This gives

$$\int_D e_{\lambda,z_0}(z)(v_2(z) - v_1(z))\mu_2(z,\lambda)\bar\mu_1(z,\lambda) d\mathrm{Re}z\, d\mathrm{Im}z = \int_{\partial D}\int_{\partial D} e^{-\bar\lambda(\bar z - \bar z_0)^2}\bar\mu_1(z,-\lambda)(\Phi_2 - \Phi_1)(z,\zeta)e^{\lambda(\zeta - z_0)^2}\mu_2(\zeta,\lambda)|d\zeta||dz|, \tag{4.1}$$

where $e_{\lambda,z_0}(z) = e^{\lambda(z-z_0)^2-\bar{\lambda}(\bar{z}-\bar{z}_0)^2}$. The left side $I(\lambda)$ of (4.1) can be written as the sum of four integrals, namely

$$I_1(\lambda) = \int_D e_{\lambda,z_0}(z)(v_2(z) - v_1(z))d\mathrm{Re}z\, d\mathrm{Im}z,$$
$$I_2(\lambda) = -\int_D e_{\lambda,z_0}(z)(v_2(z) - v_1(z))(\mu_2 - 1)(\bar{\mu}_1 - 1)d\mathrm{Re}z\, d\mathrm{Im}z,$$
$$I_3(\lambda) = -I_2(\lambda) + \int_D e_{\lambda,z_0}(z)(v_2(z) - v_1(z))(\mu_2 - 1)d\mathrm{Re}z\, d\mathrm{Im}z,$$
$$I_4(\lambda) = -I_2(\lambda) + \int_D e_{\lambda,z_0}(z)(v_2(z) - v_1(z))(\bar{\mu}_1 - 1)d\mathrm{Re}z\, d\mathrm{Im}z,$$

for $z_0 \in D$. By Lemma 3.1, 3.2, 3.3, 3.4 we have the following estimates:

$$\left|\frac{2}{\pi}|\lambda|I_1 - (v_2(z_0) - v_1(z_0))\right| \le c_3(D)\frac{\log(3|\lambda|)}{|\lambda|}\|v_2 - v_1\|_{C^2(\bar{D})}, \tag{4.2}$$

$$|I_2| \le c_5(D)\frac{\log(3|\lambda|)}{|\lambda|^{\frac{3}{2}}}\|v_2 - v_1\|_{C^1(\bar{D})}\|v_1\|_{C^1_{\bar{z}}(\bar{D})}\|v_2\|_{C^1_{\bar{z}}(\bar{D})}, \tag{4.3}$$

$$|I_3| \le |I_2| + c_6(D)\frac{\log(3|\lambda|)}{|\lambda|^{\frac{3}{2}}}\|v_2 - v_1\|_{C^1_{\bar{z}}(\bar{D})}\|v_2\|_{C^1_{\bar{z}}(\bar{D})}, \tag{4.4}$$

$$|I_4| \le |I_2| + c_6(D)\frac{\log(3|\lambda|)}{|\lambda|^{\frac{3}{2}}}\|v_2 - v_1\|^2_{C^1_z(\bar{D})}\|v_1\|_{C^1_z(\bar{D})}, \tag{4.5}$$

for $|\lambda|$ sufficiently large for example, for $\lambda$ such that

$$\frac{2\, c_1(D)}{|\lambda|^{\frac{1}{2}}}\max\left(\|v_1\|_{C^1_{\bar{z}}(\bar{D})}, \|v_1\|_{C^1_z(\bar{D})}, \|v_2\|_{C^1_{\bar{z}}(\bar{D})}, \|v_2\|_{C^1_z(\bar{D})}\right) \le \frac{1}{2},\ |\lambda| \ge 1. \tag{4.6}$$

The right side $J(\lambda)$ of (4.1) can be estimated as follows:

$$|\lambda||J(\lambda)| \le c_7(D)e^{(2L^2+1)|\lambda|}\|\Phi_2 - \Phi_1\|, \tag{4.7}$$

where we called $L = \max_{z\in\partial D,\, z_0\in D}|z - z_0|$.

Putting together estimates (4.2)-(4.7) we obtain

$$|v_2(z_0) - v_1(z_0)| \le c_8(D)\frac{\log(3|\lambda|)}{|\lambda|^{\frac{1}{2}}}N^3 + \frac{2}{\pi}c_7(D)e^{(2L^2+1)|\lambda|}\|\Phi_2 - \Phi_1\| \tag{4.8}$$

for $z_0 \in D$ and $N$ is the costant in the statement of Theorem 1.1. We call $\varepsilon = \|\Phi_2 - \Phi_1\|$ and impose $|\lambda| = \gamma\log(3 + \varepsilon^{-1})$, where $0 < \gamma < (2L^2 + 1)^{-1}$ so that (4.8) reads

$$\begin{aligned}|v_2(z_0) - v_1(z_0)| \le\ & c_8(D)N^3(\gamma\log(3 + \varepsilon^{-1}))^{-\frac{1}{2}}\log(3\gamma\log(3 + \varepsilon^{-1}))\\ &+ \frac{2}{\pi}c_7(D)(3 + \varepsilon^{-1})^{(2L^2+1)\gamma}\varepsilon,\end{aligned} \tag{4.9}$$

for every $z_0 \in D$, with

$$0 < \varepsilon \le \varepsilon_1(D, N, \gamma), \tag{4.10}$$

where $\varepsilon_1$ is sufficiently small or, more precisely, where (4.10) implies that $|\lambda| = \gamma \log(3 + \varepsilon^{-1})$ satisfies (4.6).

As $(3+\varepsilon^{-1})^{(2L^2+1)\gamma}\varepsilon \to 0$ for $\varepsilon \to 0$ more rapidly then the other term, we obtain that

$$\|v_2 - v_1\|_{L^\infty(D)} \le c_9(D, N, \gamma) \frac{\log(3 \log(3 + \|\Phi_2 - \Phi_1\|^{-1}))}{(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{\frac{1}{2}}} \tag{4.11}$$

for $\varepsilon = \|\Phi_2 - \Phi_1\| \le \varepsilon_1(D, N, \gamma)$.

Estimate (4.11) for general $\varepsilon$ (with modified $c_{10}$) follows from (4.11) for $\varepsilon \le \varepsilon_1(D, N, \gamma)$ and the assumption that $\|v_j\|_{L^\infty(D)} \le N,\ j = 1, 2$. This completes the proof of Theorem 1.1.

## 5. Proofs of the Lemmata

Proof of Lemma 3.1. One can see that $g_{z_0,\lambda} = \frac{1}{4} T \bar{T}_{z_0,\lambda}$, for $z_0, \lambda \in \mathbb{C}$, where

$$Tu(z) = -\frac{1}{\pi} \int_D \frac{u(\zeta)}{\zeta - z} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \tag{5.1}$$

$$\bar{T}_{z_0,\lambda} u(z) = -\frac{e^{-\lambda(z-z_0)^2 + \bar{\lambda}(\bar{z}-\bar{z}_0)^2}}{\pi} \int_D \frac{e^{\lambda(\zeta-z_0)^2 - \bar{\lambda}(\bar{\zeta}-\bar{z}_0)^2}}{\bar{\zeta} - \bar{z}} u(\zeta) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \tag{5.2}$$

for $z \in \bar{D}$ and $u$ a test function. Estimates (3.2), (3.3) now follow from

$$Tw \in C^1_{\bar{z}}(\bar{D}), \tag{5.3}$$

$$\|Tw\|_{C^1_{\bar{z}}(\bar{D})} \le n_1(D) \|w\|_{C(\bar{D})}, \text{ where } w \in C(D), \tag{5.4a}$$

$$\|\frac{\partial T}{\partial z} w\|_{L^p(\bar{D})} \le n(D, p) \|w\|_{L^p(\bar{D})},\ 1 < p < \infty, \tag{5.4b}$$

$$\bar{T}_{z_0,\lambda} u \in C(\bar{D}), \tag{5.5}$$

$$\|\bar{T}_{z_0,\lambda} u\|_{C(\bar{D})} \le \frac{n_2(D)}{|\lambda|^{\frac{1}{2}}} \|u\|_{C^1_{\bar{z}}(\bar{D})},\ |\lambda| \ge 1, \tag{5.6}$$

$$\|\bar{T}_{z_0,\lambda} u\|_{C(\bar{D})} \le \frac{\log(3|\lambda|)(1 + |z - z_0|) n_3(D)}{|\lambda||z - z_0|^2} \|u\|_{C^1_{\bar{z}}(\bar{D})},\ |\lambda| \ge 1, \tag{5.7}$$

where $u \in C^1_{\bar{z}}(\bar{D})$, $z_0, \lambda \in \mathbb{C}$. Estimates (5.3), (5.4) are well-known (see **[12]**).

The assumption $u \in C^1_{\bar{z}}(\bar{D})$ is not necessary at all for (5.5): indeed, using well-known arguments it is sufficient to take $u \in C(\bar{D})$.

Let us prove (5.6) and (5.7). We have that

$$-\pi e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} \bar{T}_{z_0,\lambda} u(z) = I_{z_0,\lambda,\varepsilon}(z) + J_{z_0,\lambda,\varepsilon}(z),$$

where

$$I_{z_0,\lambda,\varepsilon}(z) = \int_{D\cap(B_{z,\varepsilon}\cup B_{z_0,\varepsilon})} \frac{e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2}}{\bar\zeta-\bar z} u(\zeta) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \tag{5.8}$$

$$J_{z_0,\lambda,\varepsilon}(z) = \int_{D_{z,z_0,\varepsilon}} \frac{e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2}}{\bar\zeta-\bar z} u(\zeta) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \tag{5.9}$$

and $B_{z,\varepsilon} = \{\zeta\in\mathbb{C} : |\zeta - z| < \varepsilon\}$, $D_{z,z_0,\varepsilon} = D \setminus (B_{z,\varepsilon}\cup B_{z_0,\varepsilon})$. One sees that

$$|I_{z_0,\lambda,\varepsilon}(z)| \le 2\int_{B_{z,\varepsilon}} \frac{\|u\|_{C(\bar D)}}{|\zeta - z|} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta = 4\pi\varepsilon\|u\|_{C(\bar D)}, \tag{5.10}$$

with $z, z_0, \lambda\in\mathbb{C}$, $\varepsilon > 0$. Further, we have that

$$\begin{aligned} J_{z_0,\lambda,\varepsilon}(z) &= -\frac{1}{2\bar\lambda}\int_{D_{z,z_0,\varepsilon}} \frac{\partial}{\partial\bar\zeta}\left(e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2}\right) \frac{u(\zeta)}{(\bar\zeta-\bar z)(\bar\zeta-\bar z_0)} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \\ &= J^1_{z_0,\lambda,\varepsilon}(z) + J^2_{z_0,\lambda,\varepsilon}(z), \end{aligned}$$

where

$$\begin{aligned} J^1_{z_0,\lambda,\varepsilon}(z) &= -\frac{1}{4i\bar\lambda}\int_{\partial D_{z,z_0,\varepsilon}} \frac{e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2}}{(\bar\zeta-\bar z)(\bar\zeta-\bar z_0)} u(\zeta) d\zeta, \\ J^2_{z_0,\lambda,\varepsilon}(z) &= \frac{1}{2\bar\lambda}\int_{D_{z,z_0,\varepsilon}} e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2} \frac{\partial}{\partial\bar\zeta}\left(\frac{u(\zeta)}{(\bar\zeta-\bar z)(\bar\zeta-\bar z_0)}\right) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \end{aligned}$$

Now we get

$$|J^1_{z_0,\lambda,\varepsilon}(z)| \le M^1_{z,z_0,\lambda,\varepsilon} := \frac{1}{4|\lambda|}\int_{\partial D_{z,z_0,\varepsilon}} \frac{|u(\zeta)||d\zeta|}{|\bar\zeta-\bar z||\bar\zeta-\bar z_0|},$$

$$M^1_{z,z_0,\lambda,\varepsilon} \le \frac{1}{8|\lambda|}\int_{\partial D_{z,z_0,\varepsilon}} \left(\frac{1}{|\bar\zeta-\bar z|^2} + \frac{1}{|\bar\zeta-\bar z_0|^2}\right) |d\zeta|\|u\|_{C(D)}, \tag{5.11}$$

$$M^1_{z,z_0,\lambda,\varepsilon} \le \frac{1}{2|z-z_0||\lambda|}\int_{\partial D_{z,z_0,\varepsilon}} \left(\frac{1}{|\bar\zeta-\bar z|} + \frac{1}{|\bar\zeta-\bar z_0|}\right) |d\zeta|\|u\|_{C(D)}. \tag{5.12}$$

We also have

$$\begin{aligned} |J^2_{z_0,\lambda,\varepsilon}(z)| \le M^2_{z,z_0,\lambda,\varepsilon} :&= \frac{1}{2|\lambda|}\int_{D_{z,z_0,\varepsilon}} \frac{|\frac{\partial u}{\partial\bar\zeta}(\zeta)|}{|\bar\zeta-\bar z||\bar\zeta-\bar z_0|} + \frac{|u(\zeta)|}{|\bar\zeta-\bar z|^2|\bar\zeta-\bar z_0|} \\ &\quad + \frac{|u(\zeta)|}{|\bar\zeta-\bar z||\bar\zeta-\bar z_0|^2} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \end{aligned}$$

$$\begin{aligned} M^2_{z,z_0,\lambda,\varepsilon} \le \frac{1}{2|\lambda|}\int_{D_{z,z_0,\varepsilon}} &\frac{|\frac{\partial u}{\partial\bar\zeta}(\zeta)|}{|\bar\zeta-\bar z|^2} + \frac{|\frac{\partial u}{\partial\bar\zeta}(\zeta)|}{|\bar\zeta-\bar z_0|^2} + 2\frac{|u(\zeta)|}{|\bar\zeta-\bar z|^3} \\ &+ 2\frac{|u(\zeta)|}{|\bar\zeta-\bar z_0|^3} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \end{aligned} \tag{5.13}$$

$$M^2_{z,z_0,\lambda,\varepsilon} \le \frac{1}{2|\lambda|}\int_{D_{z,z_0,\varepsilon}} \frac{2|\frac{\partial u}{\partial \zeta}(\zeta)|}{|\bar\zeta-\bar z||z-z_0|} + \frac{2|\frac{\partial u}{\partial \zeta}(\zeta)|}{|\bar\zeta-\bar z_0||z-z_0|} + \frac{2|u(\zeta)|}{|\bar\zeta-\bar z|^2|z-z_0|} + \frac{4|u(\zeta)|}{|\bar\zeta-\bar z||z-z_0|^2} + \frac{2|u(\zeta)|}{|\bar\zeta-\bar z_0|^2|z-z_0|} + \frac{4|u(\zeta)|}{|\bar\zeta-\bar z_0||z-z_0|^2} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta. \tag{5.14}$$

Using (5.11) and (5.13) we obtain that

$$|J^1_{z_0,\lambda,\varepsilon}(z)| \le |\lambda|^{-1} n_4(D)\varepsilon^{-1}\|u\|_{C(D)}, \tag{5.15}$$

$$|J^2_{z_0,\lambda,\varepsilon}(z)| \le |\lambda|^{-1} n_5(D)\varepsilon^{-1}\|u\|_{C(D)} + |\lambda|^{-1} n_6(D)\log(3\varepsilon^{-1})\|\frac{\partial u}{\partial \bar z}\|_{C(D)}, \tag{5.16}$$

where $z, z_0, \lambda \in \mathbb{C}$, $|\lambda| \ge 1$, $0 < \varepsilon < 1$.

If $z_0 \ne z$ we can use (5.12) and (5.14) in order to obtain

$$|J^1_{z_0,\lambda,\varepsilon}(z)| \le |\lambda|^{-1}|z-z_0|^{-1} n_7(D)\log(3\varepsilon^{-1})\|u\|_{C(D)}, \tag{5.17}$$

$$|J^2_{z_0,\lambda,\varepsilon}(z)| \le |\lambda|^{-1}|z-z_0|^{-2} n_8(D)\log(3\varepsilon^{-1})\|u\|_{C(D)} + |\lambda|^{-1}|z-z_0|^{-1} n_9(D)\|\frac{\partial u}{\partial \bar z}\|_{C(D)}, \tag{5.18}$$

Finally, putting $\varepsilon = |\lambda|^{-\frac{1}{2}}$ into (5.10), (5.15), (5.16) we obtain (5.6), while putting $\varepsilon = |\lambda|^{-1}$ into (5.10), (5.17), (5.18) we obtain (5.7). The proof follows. □

PROOF OF LEMMA 3.2. First we extend our potential $v$ to a larger domain $D_1 \supset D$ (always with $C^2$ boundary) such that $dist(\partial D_1, \partial D) \ge \delta > 0$ (for some $\delta$) by putting $v|_{D_1\setminus D} \equiv 0$. In such a way $v \in C^1(D_1) \cap C^2(D_1 \setminus \partial D)$ with $\|v\|_{C^k(D_1)} = \|v\|_{C^k(D)}$ for $k = 1, 2$.

Now let $\chi_\delta$ be a real-valued function on $\mathbb{C}$, with $\delta > 0$, constructed as follows:

$$\begin{array}{l} \chi_\delta(z) = \chi(z/\delta), \text{ where} \\ \chi \in C^\infty(\mathbb{C}), \ \chi \text{ is real valued,} \\ \chi(z) = \chi(|z|), \\ \chi(z) \equiv 1 \text{ for } |z| \le 1/2, \\ \chi(z) \equiv 0 \text{ for } |z| \ge 1. \end{array}$$

Let

$$v_{lin}(z, z_0) = v(z_0) + v_z(z_0)(z - z_0) + v_{\bar z}(z_0)(\bar z - \bar z_0),$$

for $z, z_0 \in D_1$, $v_z = \frac{\partial v}{\partial z}$ and $v_{\bar z} = \frac{\partial v}{\partial \bar z}$.

We can write $h^{(0)}_{z_0}(\lambda) = S_{z_0,\delta}(\lambda) + R_{z_0,\delta}(\lambda)$, where

$$\begin{aligned} S_{z_0,\delta}(\lambda) &= \int_{\mathbb{C}} e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z-\bar z_0)^2} v_{lin}(z, z_0)\chi_\delta(z - z_0) d\mathrm{Re}z\, d\mathrm{Im}z \\ &= \int_{\mathbb{C}} e^{i|\lambda|(z^2+\bar z^2)} v_{lin}(e^{-i\varphi(\lambda)}z + z_0, z_0)\chi_\delta(z) d\mathrm{Re}z\, d\mathrm{Im}z, \\ R_{z_0,\delta}(\lambda) &= \int_{D_1} e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z-\bar z_0)^2}\left(v(z) - v_{lin}(z, z_0)\chi_\delta(z - z_0)\right) d\mathrm{Re}z\, d\mathrm{Im}z \end{aligned}$$

where $\varphi(\lambda) = \frac{1}{2}(\arg(\lambda) - \frac{\pi}{2})$, $z_0 \in D$, $\lambda \in \mathbb{C}$.

Using the stationary phase method we obtain that

$$v(z_0) = \frac{2}{\pi} \lim_{\lambda\to\infty} |\lambda| S_{z_0,\delta}(\lambda), \tag{5.19}$$

$$\left|v(z_0) - \frac{2}{\pi}|\lambda| S_{z_0,\delta}(\lambda)\right| \le q_1(D,\delta)\|v\|_{C^1(\bar D)}|\lambda|^{-1}, \tag{5.20}$$

$z_0 \in D$, $\delta > 0$, $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$. Integrating by parts we can write

$$\begin{aligned} R_{z_0,\delta}(\lambda) &= -\frac{1}{2\bar\lambda}\int_{D_1} \frac{\partial}{\partial \bar z}\left(e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2}\right) \\ &\quad \times \frac{(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z - \bar z_0} d\mathrm{Re}z\, d\mathrm{Im}z = R^1_{z_0,\delta}(\lambda) + R^2_{z_0,\delta}(\lambda), \\ R^1_{z_0,\delta}(\lambda) &= \frac{-1}{4i\bar\lambda}\int_{\partial D_1} e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2}\frac{(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z - \bar z_0} dz, \\ R^2_{z_0,\delta}(\lambda) &= \frac{1}{2\bar\lambda}\int_{D_1} e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} \\ &\quad \times \frac{\partial}{\partial \bar z}\left(\frac{(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z - \bar z_0}\right) d\mathrm{Re}z\, d\mathrm{Im}z, \end{aligned}$$

for $z_0 \in D$, $\lambda \in \mathbb{C}\setminus\{0\}$. In addition, we have that

$$\lim_{\lambda\to\infty} |\lambda| R^1_{z_0,\delta}(\lambda) = 0, \tag{5.21}$$

$$\lim_{\lambda\to\infty} |\lambda| R^2_{z_0,\delta}(\lambda) = 0. \tag{5.22}$$

Formula (5.21) follows from properties of $\chi_\delta$, the assumption that $z_0 \in D$ and that $v|_{\partial D_1} \equiv 0$. Actually, as a corollary of this properties we have that $v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0) \equiv 0$ for $z \in \partial D_1$ and, therefore, $R^1_{z_0,\delta}(\lambda) \equiv 0$ for $\lambda \in \mathbb{C}\setminus\{0\}$.

Formula (5.22) for $v \in C^1(\bar D_1)$ is a consequence of the estimates

$$\begin{aligned} R^{2,1}_{z_0,\delta,\varepsilon}(\lambda) &:= \int_{B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} \\ &\quad \times \frac{\partial}{\partial \bar z}\left(\frac{(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z - \bar z_0}\right) d\mathrm{Re}z\, d\mathrm{Im}z = O(\varepsilon) \text{ as } \varepsilon \to 0 \end{aligned} \tag{5.23}$$

$$\begin{aligned} R^{2,2}_{z_0,\delta,\varepsilon}(\lambda) &:= \int_{D_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} \\ &\quad \times \frac{\partial}{\partial \bar z}\left(\frac{(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z - \bar z_0}\right) d\mathrm{Re}z\, d\mathrm{Im}z \to 0 \text{ as } \lambda \to \infty \end{aligned} \tag{5.24}$$

where $B_{z_0,\varepsilon} = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$, $D_{z_0,\varepsilon} = D_1 \setminus B_{z_0,\varepsilon}$. In (5.23)-(5.24) we assume that $z_0 \in D$, $0 < \varepsilon < \delta$, $\lambda \in \mathbb{C}$.

Estimate (5.23) is obtained by standard arguments using that

$$|v(z) - v(z_0)| \le \|v\|_{C^1(\bar D)}|z - z_0|, \ z_0 \in D, \ z \in B_{z_0,\delta},$$

while (5.24) is a variation of the Riemann-Lebesgue Lemma.

Formula (3.5) now follows from (5.19), (5.21), (5.22).

Under the assumptions mentioned in Lemma 3.2, the final part of the proof of estimate (3.6) consists in the following. We have, for $\varepsilon < \delta/2$,

$$
\begin{aligned}
(5.25)\qquad |R^{2,1}_{z_0,\delta,\varepsilon}(\lambda)| &\le \int_{B_{z_0,\varepsilon}} \frac{|v(z)-v_{lin}(z,z_0)|}{|z-z_0|^2} d\mathrm{Re}z\, d\mathrm{Im}z \\
&+ \int_{B_{z_0,\varepsilon}} \frac{|v_{\bar z}(z)-v_{\bar z}(z_0)|}{|z-z_0|} d\mathrm{Re}z\, d\mathrm{Im}z \le \frac{7}{2}\pi \|v\|_{C^2(\bar D)}\varepsilon^2, \\
R^{2,2}_{z_0,\delta,\varepsilon}(\lambda) &= \frac{-1}{2\bar\lambda} \int_{D_{z_0,\varepsilon}} \frac{\partial}{\partial \bar z}\left(e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\right)\frac{1}{\bar z-\bar z_0} \\
&\times \frac{\partial}{\partial \bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right) d\mathrm{Re}z\, d\mathrm{Im}z \\
&= \frac{-1}{2\bar\lambda}(R^{2,2,1}_{z_0,\delta,\varepsilon}(\lambda)+R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda)), \\
R^{2,2,1}_{z_0,\delta,\varepsilon}(\lambda) &= \frac{1}{2i}\int_{\partial D_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\frac{1}{\bar z-\bar z_0} \\
&\times \frac{\partial}{\partial \bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right) dz \\
= \frac{-1}{2i}\int_{\partial B_{z_0,\varepsilon}} & e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\frac{1}{\bar z-\bar z_0}\frac{\partial}{\partial \bar z}\left(\frac{v(z)-v_{lin}(z,z_0)}{\bar z-\bar z_0}\right) dz,
\end{aligned}
$$

where we used in particular that $v|_{\partial D_1} \equiv 0$, $\frac{\partial}{\partial\nu}v|_{\partial D_1} \equiv 0$,

$$
\begin{aligned}
R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda) = -\int_{D_{z_0,\varepsilon}} & e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2} \\
&\times \frac{\partial}{\partial \bar z}\left(\frac{1}{\bar z-\bar z_0}\frac{\partial}{\partial \bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right)\right) d\mathrm{Re}z\, d\mathrm{Im}z.
\end{aligned}
$$

We have, for $\varepsilon < \delta/2$

$$
\begin{aligned}
(5.26)\qquad |R^{2,2,1}_{z_0,\delta,\varepsilon}(\lambda)| &\le \frac{1}{2}\int_{\partial B_{z_0,\varepsilon}} \frac{|v(z)-v_{lin}(z,z_0)|}{|z-z_0|^3}|dz| \\
&+ \frac{1}{2}\int_{\partial B_{z_0,\varepsilon}} \frac{|v_{\bar z}(z)-v_{\bar z}(z_0)|}{|z-z_0|^2}|dz| \le \frac{7}{2}\pi\|v\|_{C^2(\bar D)},
\end{aligned}
$$

$$(5.27)\qquad |R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda)| \le |R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| + |R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda) - R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)|,$$

$$(5.28)\qquad |R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| \le q_2(D,\delta)\|v\|_{C^2(\bar D)},$$

$$|R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda) - R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| \le \sum_{j=1}^{5}\int_{B_{z_0,\delta/2}\setminus B_{z_0,\varepsilon}} u_j(z,z_0)\, d\mathrm{Re}z\, d\mathrm{Im}z,$$

with

$$u_1(z,z_0)=\frac{1}{|z-z_0|^2}\left|\frac{v_{\bar z}(z)-v_{\bar z}(z_0)}{\bar z-\bar z_0}\right|, \tag{5.29}$$

$$u_2(z,z_0)=\frac{1}{|z-z_0|^2}\left|\frac{v(z)-v_{lin}(z,z_0)}{(\bar z-\bar z_0)^2}\right|, \tag{5.30}$$

$$u_3(z,z_0)=\frac{1}{|z-z_0|}\left|\frac{v_{\bar z\bar z}(z)}{\bar z-\bar z_0}\right|, \tag{5.31}$$

$$u_4(z,z_0)=\frac{2}{|z-z_0|}\left|\frac{v_{\bar z}(z)-v_{\bar z}(z_0)}{(\bar z-\bar z_0)^2}\right|, \tag{5.32}$$

$$u_5(z,z_0)=\frac{2}{|z-z_0|}\left|\frac{v(z)-v_{lin}(z,z_0)}{(\bar z-\bar z_0)^3}\right|. \tag{5.33}$$

This yields

$$|R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda)-R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)|\le q_3\log(\frac{\delta}{2\varepsilon})\|v\|_{C^2(\bar D)}, \tag{5.34}$$

where $z_0\in D$, $0<\varepsilon<\delta/2$. $\lambda\in\mathbb{C}\setminus\{0\}$. Using (5.20), (5.25)-(5.34) with $\varepsilon=|\lambda|^{-1}$ we obtain (3.6). Lemma 3.2 is proved. □

Proof of Lemma 3.3. We write

$$W_{z_0}(\lambda)=W^1_{z_0,\varepsilon}(\lambda)+W^2_{z_0,\varepsilon}(\lambda),$$
$$W^1_{z_0,\varepsilon}(\lambda)=\int_{D\cap B_{z_0,\varepsilon}}e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}w(z)d\mathrm{Re}z\,d\mathrm{Im}z,$$
$$W^2_{z_0,\varepsilon}(\lambda)=\int_{D\setminus B_{z_0,\varepsilon}}e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}w(z)d\mathrm{Re}z\,d\mathrm{Im}z,$$

where $B_{z_0,\varepsilon}=\{z\in\mathbb{C}:|z-z_0|<\varepsilon\}$. One sees that

$$|W^1_{z_0,\varepsilon}(\lambda)|\le\int_{D\cap B_{z_0,\varepsilon}}\|w\|_{C(D)}d\mathrm{Re}z\,d\mathrm{Im}z=\pi\|w\|_{C(D)}\varepsilon^2, \tag{5.35}$$
$$W^2_{z_0,\varepsilon}(\lambda)=\frac{-1}{2\bar\lambda}\int_{D\setminus B_{z_0,\varepsilon}}\frac{\partial}{\partial\bar z}\left(e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\right)\frac{w(z)}{\bar z-\bar z_0}d\mathrm{Re}z\,d\mathrm{Im}z$$
$$=W^{2,1}_{z_0,\varepsilon}(\lambda)+W^{2,2}_{z_0,\varepsilon}(\lambda),$$
$$W^{2,1}_{z_0,\varepsilon}(\lambda)=\frac{-1}{4i\bar\lambda}\int_{\partial(D\setminus B_{z_0,\varepsilon})}e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\frac{w(z)}{\bar z-\bar z_0}dz,$$
$$W^{2,2}_{z_0,\varepsilon}(\lambda)=\frac{1}{2\bar\lambda}\int_{D\setminus B_{z_0,\varepsilon}}e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\frac{\partial}{\partial\bar z}\left(\frac{w(z)}{\bar z-\bar z_0}\right)d\mathrm{Re}z\,d\mathrm{Im}z.$$

We have

$$|W^{2,1}_{z_0,\varepsilon}(\lambda)|\le|\lambda|^{-1}a_1(D)\|w\|_{C(\bar D)}\log(3\varepsilon^{-1}), \tag{5.36}$$

(5.37a) $$|W^{2,2}_{z_0,\varepsilon}(\lambda)| \le |\lambda|^{-1} a_2(D)\|w\|_{C^1_{\bar z}(\bar D)} \log(3\varepsilon^{-1})$$

(5.37b) $$|W^{2,2}_{z_0,\varepsilon}(\lambda)| \le |\lambda|^{-1} a_2(D)\|w\|_{C(\bar D)} \log(3\varepsilon^{-1}) + |\lambda|^{-1} a_3(D,p)\|\frac{\partial w}{\partial \bar z}\|_{L^p(\bar D)},$$

for $z_0 \in D,\ \lambda \in \mathbb{C}\setminus\{0\},\ 0<\varepsilon\le 1,\ 2<p<\infty$.

Using (5.35), (5.36), (5.37) with $\varepsilon = |\lambda|^{-1}$ we obtain (3.7). This finishes the proof. □

Proof of Lemma 3.4. Formula (3.8) follows from the assumption on $\|g_{z_0,\lambda} v\|$ and from solving (2.4) by the method of successive approximations. The proof of estimate (3.9) follows from (3.8) and Lemma 3.3. The proof follows. □

## 6. An extension of Theorem 1.1

As an extension of Theorem 1.1 for the case when we do not assume that $v_j|_{\partial D} \equiv 0,\ \frac{\partial}{\partial\nu} v_j|_{\partial D} \equiv 0,\ j=1,2$, we give the following result.

Proposition 6.1. *Let $D \subset \mathbb{R}^2$ be an open bounded domain with $C^2$ boundary, let $v_1, v_2 \in C^2(\bar D)$ with $\|v_j\|_{C^2(\bar D)} \le N$ for $j=1,2$, and $\Phi_1,\Phi_2$ the corresponding Dirichlet-to-Neumann operators. Then, for any $0<\alpha<\frac{1}{5}$, there exists a constant $C = C(D,N,\alpha)$ such that the following inequality holds*

(6.1) $$\|v_2 - v_1\|_{L^\infty(D)} \le C \log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})^{-\alpha},$$

*where $\|A\|_1$ is the norm for an operator $A : L^\infty(\partial D) \to L^\infty(\partial D)$, with kernel $A(x,y)$, defined as $\|A\|_1 = \sup_{x,y\in\partial D} |A(x,y)|(\log(3+|x-y|^{-1}))^{-1}$.*

All we need to know about $\|\cdot\|_1$ consists of the following:
*i)* $\|A\|_{L^\infty(\partial D)\to L^\infty(\partial D)} \le const(D)\|A\|_1$;
*ii)* by formula (4.9) of [**9**] one has

$$\|v\|_{L^\infty(\partial D)} \le const\|\Phi_v - \Phi_0\|_1.$$

In order to prove Proposition 6.1 we need the following modified version of Lemma 3.2. We will call $(\partial D)_\delta = \{z \in \mathbb{C} : dist(z,\partial D) < \delta\}$.

Lemma 6.2. *For $v \in C^2(\bar D)$ we have that*

(6.2) $$|v(z_0) - \frac{2}{\pi}|\lambda| h^{(0)}_{z_0}(\lambda)| \le \kappa_1(D)\delta^{-4}\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar D)} + \kappa_2(D)\log(3+\delta^{-1})\|v\|_{C(\partial D)},$$

*for $z_0 \in D\setminus(\partial D)_\delta,\ 0<\delta<1,\ \lambda\in\mathbb{C},\ |\lambda|\ge 1$.*

Proof of Lemma 6.2. Let $\chi_\delta$ be as in the proof of Lemma 3.2. We have in particular that

(6.3) $$\|\chi_\delta\|_{C^k(\mathbb{C})} \le \delta^{-k}\|\chi\|_{C^k(\mathbb{C})},\ k\in\mathbb{N}.$$

Let

$$v_{lin}(z,z_0) = v(z_0) + v_z(z_0)(z-z_0) + v_{\bar z}(z_0)(\bar z - \bar z_0),$$

for $z, z_0 \in D$, $v_z = \frac{\partial v}{\partial z}$ and $v_{\bar z} = \frac{\partial v}{\partial \bar z}$.

We can write $h^{(0)}_{z_0}(\lambda) = S_{z_0,\delta}(\lambda) + R_{z_0,\delta}(\lambda)$, where

$$\begin{aligned} S_{z_0,\delta}(\lambda) &= \int_{\mathbb{C}} e_{\lambda,z_0}(z) v_{lin}(z,z_0)\chi_\delta(z-z_0) d\mathrm{Re} z\, d\mathrm{Im} z \\ &= \int_{\mathbb{C}} e^{i|\lambda|(z^2+\bar z^2)} v_{lin}(e^{-i\varphi(\lambda)}z + z_0, z_0)\chi_\delta(z) d\mathrm{Re} z\, d\mathrm{Im} z, \\ R_{z_0,\delta}(\lambda) &= \int_D e_{\lambda,z_0}(z)\left(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0)\right) d\mathrm{Re} z\, d\mathrm{Im} z \end{aligned}$$

where $\varphi(\lambda) = \frac{1}{2}(\arg(\lambda) - \frac{\pi}{2})$, $e_{\lambda,z_0}(z) = e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2}$, $z_0 \in D \setminus (\partial D)_\delta$, $\lambda \in \mathbb{C}$.

Using the stationary phase method and the explicit construction of $\chi_\delta$ we obtain that

$$v(z_0) = \frac{2}{\pi}\lim_{\lambda\to\infty} |\lambda| S_{z_0,\delta}(\lambda), \tag{6.4}$$

$$|v(z_0) - \frac{2}{\pi}|\lambda| S_{z_0,\delta}(\lambda)| \le \frac{\rho_1(D)}{\delta^4}\|v\|_{C^1(\bar D)}\|\chi\|_{C^4(\mathbb{C})}|\lambda|^{-1}, \tag{6.5}$$

$z_0 \in D \setminus (\partial D)_\delta$, $0 < \delta < 1$, $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$. Inequality (6.5) follows from

$$\begin{aligned} |v(z_0) - \frac{2}{\pi}|\lambda| S_{z_0,\delta}(\lambda)| &\le \frac{\rho_1(D)}{|\lambda|}\|v_{lin}\|_{C^4(\bar D)}\|\chi_\delta\|_{C^4(\mathbb{C})} \\ &\le \frac{\rho_1(D)}{|\lambda|\delta^4}\|v\|_{C^1(\bar D)}\|\chi\|_{C^4(\mathbb{C})}, \end{aligned}$$

where we used [**5**, Lemma 7.7.3] and (6.3).

Integrating by parts we can write

$$\begin{aligned} R_{z_0,\delta}(\lambda) &= -\frac{1}{2\bar\lambda}\int_D \frac{\partial}{\partial \bar z}\left(e_{\lambda,z_0}(z)\right)\frac{\left(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0)\right)}{\bar z - \bar z_0} d\mathrm{Re} z\, d\mathrm{Im} z \\ &= R^1_{z_0,\delta}(\lambda) + R^2_{z_0,\delta}(\lambda), \\ R^1_{z_0,\delta}(\lambda) &= \frac{-1}{4i\bar\lambda}\int_{\partial D} e_{\lambda,z_0}(z)\frac{\left(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0)\right)}{\bar z - \bar z_0} dz, \\ R^2_{z_0,\delta}(\lambda) &= \frac{1}{2\bar\lambda}\int_D e_{\lambda,z_0}(z)\frac{\partial}{\partial \bar z}\left(\frac{\left(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0)\right)}{\bar z - \bar z_0}\right) d\mathrm{Re} z\, d\mathrm{Im} z, \end{aligned}$$

for $z_0 \in D \setminus (\partial D)_\delta$, $\lambda \in \mathbb{C} \setminus \{0\}$. In addition, we have that

$$\frac{2}{\pi}|\lambda||R^1_{z_0,\delta}(\lambda)| \le \kappa_2(D)\log(3 + \delta^{-1})\|v\|_{C(\partial D)}. \tag{6.6}$$

Formula (6.6) follows from the fact that $\chi_\delta(z-z_0)=0$ for $z\in\partial D,\ z_0\in D\backslash(\partial D)_\delta$ and from the estimate

$$\frac{2}{\pi}|R^1_{z_0,\delta}(\lambda)|\le\frac{2}{\pi}\frac{1}{|\lambda|}\int_{\partial D}\frac{|v(z)|}{|\bar z-\bar z_0|}|dz|\le\frac{\kappa_2(D)\log(3+\delta^{-1})}{|\lambda|}\|v\|_{C(\partial D)}.$$

We now write $R^2_{z_0,\delta}(\lambda)=\frac{1}{2\lambda}(R^{2,1}_{z_0,\delta,\varepsilon}(\lambda)+R^{2,2}_{z_0,\delta,\varepsilon}(\lambda))$, with

$$R^{2,1}_{z_0,\delta,\varepsilon}(\lambda)=\int_{B_{z_0,\varepsilon}}e_{\lambda,z_0}(z)\frac{\partial}{\partial\bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right)d\mathrm{Re}z\,d\mathrm{Im}z \tag{6.7}$$

$$R^{2,2}_{z_0,\delta,\varepsilon}(\lambda)=\int_{D_{z_0,\varepsilon}}e_{\lambda,z_0}(z)\frac{\partial}{\partial\bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right)d\mathrm{Re}z\,d\mathrm{Im}z, \tag{6.8}$$

where $B_{z_0,\varepsilon}=\{z\in\mathbb{C}:|z-z_0|<\varepsilon\}$, $D_{z_0,\varepsilon}=D\setminus B_{z_0,\varepsilon}$. In (6.7)-(6.8) we assume that $z_0\in D\setminus(\partial D)_\delta,\ 0<\varepsilon<\delta,\ \lambda\in\mathbb{C}$.

The final part of the proof of estimate (6.2) consists in the following. We have, for $\varepsilon<\delta/2$,

$$|R^{2,1}_{z_0,\delta,\varepsilon}(\lambda)|\le\frac{7}{2}\pi\|v\|_{C^2(\bar D)}\varepsilon^2, \tag{6.9}$$

exactly as in (5.25),

$$\begin{aligned}R^{2,2}_{z_0,\delta,\varepsilon}(\lambda)&=-\frac{1}{2\bar\lambda}\int_{D_{z_0,\varepsilon}}\frac{\partial}{\partial\bar z}\left(e_{\lambda,z_0}(z)\right)\frac{1}{\bar z-\bar z_0}\\&\quad\times\frac{\partial}{\partial\bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right)d\mathrm{Re}z\,d\mathrm{Im}z\\&=-\frac{1}{2\bar\lambda}(R^{2,2,1}_{z_0,\delta,\varepsilon}(\lambda)+R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda)),\end{aligned}$$

$$\begin{aligned}R^{2,2,1}_{z_0,\delta,\varepsilon}(\lambda)&=\frac{1}{2i}\int_{\partial D_{z_0,\varepsilon}}e_{\lambda,z_0}(z)\frac{1}{\bar z-\bar z_0}\frac{\partial}{\partial\bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right)dz\\&=-\frac{1}{2i}\int_{\partial B_{z_0,\varepsilon}}e_{\lambda,z_0}(z)\frac{1}{\bar z-\bar z_0}\frac{\partial}{\partial\bar z}\left(\frac{v(z)-v_{lin}(z,z_0)}{\bar z-\bar z_0}\right)dz\\&\quad-\frac{1}{2i}\int_{\partial D}e_{\lambda,z_0}(z)\frac{1}{\bar z-\bar z_0}\frac{\partial}{\partial\bar z}\left(\frac{v(z)}{\bar z-\bar z_0}\right)dz,\end{aligned}$$

$$\begin{aligned}R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda)&=-\int_{D_{z_0,\varepsilon}}e_{\lambda,z_0}(z)\\&\quad\times\frac{\partial}{\partial\bar z}\left(\frac{1}{\bar z-\bar z_0}\frac{\partial}{\partial\bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right)\right)d\mathrm{Re}z\,d\mathrm{Im}z.\end{aligned}$$

We have, for $\varepsilon < \delta/2$

$$|R^{2,2,1}_{z_0,\delta,\varepsilon}(\lambda)| \leq \frac{1}{2}\int_{\partial B_{z_0,\varepsilon}} \frac{|v(z)-v_{lin}(z,z_0)|}{|z-z_0|^3}|dz| + \frac{1}{2}\int_{\partial B_{z_0,\varepsilon}} \frac{|v_{\bar z}(z)-v_{\bar z}(z_0)|}{|z-z_0|^2}|dz| + \frac{1}{2}\int_{\partial D} \frac{|v(z)|}{|z-z_0|^3}|dz| + \frac{1}{2}\int_{\partial D} \frac{|v_{\bar z}(z)|}{|z-z_0|^2}|dz| \leq \frac{7}{2}\pi\|v\|_{C^2(\bar D)} + \frac{\rho_2(D)}{\delta^2}\|v\|_{C^1(\bar D)}, \tag{6.10}$$

$$|R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda)| \leq |R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| + |R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda) - R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)|, \tag{6.11}$$

$$|R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| \leq \frac{\rho_3(D)}{\delta^3}\|v\|_{C^2(\bar D)}, \tag{6.12}$$

$$|R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda) - R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| \leq \sum_{j=1}^{5}\int_{B_{z_0,\delta/2}\setminus B_{z_0,\varepsilon}} u_j(z,z_0)d\mathrm{Re}z\, d\mathrm{Im}z,$$

with $u_j$ defined as in (5.29)-(5.33). This yields

$$|R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda) - R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| \leq \rho_4(D)\log(\frac{\delta}{2\varepsilon})\|v\|_{C^2(\bar D)}, \tag{6.13}$$

where $z_0 \in D\setminus(\partial D)_\delta$, $0<\varepsilon<\delta/2$, $\lambda\in\mathbb{C}\setminus\{0\}$. Using (6.5), (6.6), (6.9)-(6.13) with $\varepsilon = |\lambda|^{-1}$ we obtain (6.2) for $|\lambda| > \frac{2}{\delta}$.

Notice that only the estimation of $|\lambda||R^2_{z_0,\delta}(\lambda)|$ requires $|\lambda| > \frac{2}{\delta}$. In that case one has

$$\frac{2}{\pi}|\lambda||R^2_{z_0,\delta}(\lambda)| \leq \rho_5(D)\delta^{-4}\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar D)}.$$

If $1 \leq |\lambda| \leq \frac{2}{\delta}$ we have that

$$\frac{2}{\pi}|\lambda||R^2_{z_0,\delta}(\lambda)| \leq \frac{\rho_6(D)N}{\delta} \tag{6.14}$$

and

$$\rho_5(D)\delta^{-4}\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar D)} \geq \frac{\rho_5(D)}{2\delta^3}\log(6\delta^{-1})\|v\|_{C^2(\bar D)}, \tag{6.15}$$

where we used the fact that the function $\frac{\log(3s)}{s}$ is decreasing for $s > \frac{e}{3}$.

We now define

$$c' = \frac{2\rho_6(D)N}{\rho_5(D)\log(6)\|v\|_{C^2(\bar D)}},$$

in order to have

$$\frac{2}{\pi}|\lambda||R^2_{z_0,\delta}(\lambda)| \leq c'\rho_5(D)\delta^{-4}\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar D)},$$

for $1 \leq |\lambda| \leq \frac{2}{\delta}$, $0<\delta<1$.

Thus, taking $\kappa_1 = \max(\rho_5, c'\rho_5, \rho_1\|\chi\|_{C^4(\mathbb{C})})$, we obtain estimation (6.2) for $|\lambda| \geq 1$ and $0<\delta<1$. This finish the proof of Lemma 6.2. □

Proof of Proposition 6.1. Fix $0 < \alpha < \frac{1}{5}$, and $0 < \delta < 1$. We have the following chain of inequalities

$$\begin{aligned}\|v_2 - v_1\|_{L^\infty(D)} &= \max(\|v_2 - v_1\|_{L^\infty(D\cap(\partial D)_\delta)}, \|v_2 - v_1\|_{L^\infty(D\setminus(\partial D)_\delta)}) \\ &\le C_1 \max\Bigg(2N\delta + \|\Phi_2 - \Phi_1\|_1, \frac{\log(3\log(3+\|\Phi_2-\Phi_1\|^{-1}))}{\delta^4\log(3+\|\Phi_2-\Phi_1\|^{-1})} \\ &\quad + \log(3+\frac{1}{\delta})\|\Phi_2-\Phi_1\|_1 + \frac{\log(3\log(3+\|\Phi_2-\Phi_1\|^{-1}))}{(\log(3+\|\Phi_2-\Phi_1\|^{-1}))^{\frac{1}{2}}}\Bigg) \\ &\le C_2 \max\Bigg(2N\delta + \|\Phi_2 - \Phi_1\|_1, \frac{1}{\delta^4}\log(3+\|\Phi_2-\Phi_1\|_1^{-1})^{-5\alpha} \\ &\quad + \log(3+\frac{1}{\delta})\|\Phi_2-\Phi_1\|_1 + \frac{\log(3\log(3+\|\Phi_2-\Phi_1\|_1^{-1}))}{(\log(3+\|\Phi_2-\Phi_1\|_1^{-1}))^{\frac{1}{2}}}\Bigg),\end{aligned}$$

where we followed the scheme of the proof of Theorem 1.1 with the following modifications: we make use of Lemma 6.2 instead of Lemma 3.2 and we also use i)-ii); note that $C_1 = C_1(D, N)$ and $C_2 = C_2(D, N, \alpha)$.

Putting $\delta = \log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})^{-\alpha}$ we obtain the desired inequality

$$\|v_2 - v_1\|_{L^\infty(D)} \le C_3 \log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})^{-\alpha}, \tag{6.16}$$

with $C_3 = C_3(D, N, \alpha)$, $\|\Phi_2 - \Phi_1\|_1 = \varepsilon \le \varepsilon_1(D, N, \alpha)$ with $\varepsilon_1$ sufficiently small or, more precisely when $\delta_1 = \log(3 + \varepsilon_1^{-1})^{-\alpha}$ satisfies:

$$\delta_1 < 1, \qquad \varepsilon_1 \le 2N\delta_1, \qquad \log(3 + \frac{1}{\delta_1})\varepsilon_1 \le \delta_1.$$

Estimate (6.16) for general $\varepsilon$ (with modified $C_3$) follows from (6.16) for $\varepsilon \le \varepsilon_1(D, N, \alpha)$ and the assumption that $\|v_j\|_{L^\infty(\bar{D})} \le N$ for $j = 1, 2$. This completes the proof of Proposition 6.1. □

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# Paper D

**PAPER D**

# Global stability for the multi-channel Gel'fand-Calderón inverse problem in two dimensions

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Abstract. We prove a global logarithmic stability estimate for the multi-channel Gel'fand-Calderón inverse problem on a two-dimensional bounded domain, i.e., the inverse boundary value problem for the equation $-\Delta\psi + v\,\psi = 0$ on $D$, where $v$ is a smooth matrix-valued potential defined on a bounded planar domain $D$.

## 1. Introduction

The Schrödinger equation at zero energy,

$$-\Delta\psi + v(x)\psi = 0 \ \text{ on } D \subset \mathbb{R}^2, \tag{1.1}$$

arises in quantum mechanics, acoustics and electrodynamics. The reconstruction of the complex-valued potential $v$ in equation (1.1) through the Dirichlet-to-Neumann operator is one of the most studied inverse problems (see **[11]**, **[10]**, **[4]**, **[12]**, **[13]**, **[14]** and references therein).

In this article we consider the multi-channel two-dimensional Schrödinger equation, i.e., equation (1.1) with matrix-valued potentials and solutions; this case was already studied in **[15, 14]**. One of the motivations for studying the multi-channel equation is that it comes up as a 2D-approximation for the 3D equation (see **[14**, Sec. 2**]**).

The main purpose of this paper is to give a global stability estimate for this inverse problem in the multi-channel case.

Let $D$ be an open bounded domain in $\mathbb{R}^2$ with $C^2$ boundary and $v \in C^1(\bar{D}, M_n(\mathbb{C}))$, where $M_n(\mathbb{C})$ is the set of the $n \times n$ complex-valued matrices. The Dirichlet-to-Neumann map associated with $v$ is the operator $\Phi : C^1(\partial D, M_n(\mathbb{C})) \to L^p(\partial D, M_n(\mathbb{C}))$, $p < \infty$, defined by

$$\Phi(f) = \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial D}, \tag{1.2}$$

where $f \in C^1(\partial D, M_n(\mathbb{C}))$, $\nu$ is the outer normal of $\partial D$ and $\psi$ is the $H^1(\bar{D}, M_n(\mathbb{C}))$-solution of the Dirichlet problem

$$-\Delta\psi + v(x)\psi = 0 \text{ on } D, \quad \psi|_{\partial D} = f; \tag{1.3}$$

here we assume that

$$0 \text{ is not a Dirichlet eigenvalue of the operator } -\Delta + v \text{ in } D. \tag{1.4}$$

This construction gives rise to the following inverse boundary value problem: given $\Phi$, find $v$.

This problem can be considered as the Gel'fand inverse boundary value problem for the multi-channel Schrödinger equation at zero energy (see [**8**], [**11**]) and can also be seen as a generalization of the Calderón problem for the electrical impedance tomography (see [**5**], [**11**]). Note also that we can think of this problem as a model for monochromatic ocean tomography (e.g., see [**2**] for similar problems arising in this type of tomography).

In the case of complex-valued potentials the global injectivity of the map $v \to \Phi$ was first proved for $D \subset \mathbb{R}^d$ with $d \geq 3$ in [**11**] and for $d = 2$ with $v \in L^p$ in [**4**]: in particular, these results were obtained by the use of global reconstructions developed in the same papers. The first global uniqueness result (along with an exact reconstruction method) for matrix-valued potentials was given in [**14**], which deals with $C^1$ matrix-valued potentials defined on a domain in $\mathbb{R}^2$. A global stability estimate for the Gel'fand-Calderón problem with $d \geq 3$ was first found by Alessandrini in [**1**]; this result was recently improved in [**12**]. In the two-dimensional case the first global stability estimate was given in [**13**].

In this paper we extend the results of [**13**] to the matrix-valued case. We do not discuss global results for special real-valued potentials arising from conductivities: for this case the reader is referred to the references given in [**1**], [**4**], [**10**], [**11**], [**12**], [**13**].

Our main result is the following:

THEOREM 1.1. *Let $D \subset \mathbb{R}^2$ be an open bounded domain with a $C^2$ boundary, $v_1, v_2 \in C^2(\bar{D}, M_n(\mathbb{C}))$ two matrix-valued potentials which satisfy* (1.4)*, with $\|v_j\|_{C^2(\bar{D})} \leq N$ for $j = 1, 2$, and $\Phi_1, \Phi_2$ the corresponding Dirichlet-to-Neumann operators. For simplicity we also assume that $v_1|_{\partial D} = v_2|_{\partial D}$ and $\frac{\partial}{\partial\nu}v_1|_{\partial D} = \frac{\partial}{\partial\nu}v_2|_{\partial D}$. Then there exists a constant $C = C(D, N, n)$ such that*

$$\|v_2 - v_1\|_{L^\infty(D)} \leq C \left(\log(3 + \|\Phi_2 - \Phi_1\|^{-1})\right)^{-\frac{3}{4}} \left(\log(3 \log(3 + \|\Phi_2 - \Phi_1\|^{-1}))\right)^2, \tag{1.5}$$

*where $\|\cdot\|$ is the induced operator norm on $L^\infty(\partial D, M_n(\mathbb{C}))$ and $\|v\|_{L^\infty(D)} = \max_{1\leq i,j\leq n} \|v_{i,j}\|_{L^\infty(D)}$ (likewise for $\|v\|_{C^2(\bar{D})}$) for a matrix-valued potential $v$.*

This is the first global stability result for the multi-channel ($n \geq 2$) Gel'fand-Calderón inverse problem in two dimensions. In addition, Theorem 1.1 is new also for the scalar case, as the estimate obtained in **[13]** is weaker. We remark, in particular, that this result is true in the special case when $v_1 \equiv v_2 \equiv \Lambda \in M_n(\mathbb{C})$ in a neighborhood of $\partial D$ (situation which appears in the approximation of the 3D equation, see [**14**, Remark 3 and Section 2]).

Instability estimates complementing the stability estimates of **[1]**, **[12]**, **[13]** and of the present work are given in **[10]**, **[9]**.

The proof of Theorem 1.1 is based on results obtained in **[13]**, **[14]**, which take inspiration mostly from **[4]** and **[1]**. In particular, for $z_0 \in D$ we use the existence and uniqueness of a family of solutions $\psi_{z_0}(z,\lambda)$ of equation (1.1) where in particular $\psi_{z_0} \to e^{\lambda(z-z_0)^2} I$, for $\lambda \to \infty$ (where $I$ is the identity matrix). Then, using an appropriate matrix-valued version of Alessandrini's identity along with stationary phase techniques, we obtain the result. Note that this matrix-valued identity is one of the new results of this paper.

A generalizations of Theorem 1.1 in the case where we do not assume that $v_1|_{\partial D} = v_2|_{\partial D}$ and $\frac{\partial}{\partial\nu} v_1|_{\partial D} = \frac{\partial}{\partial\nu} v_2|_{\partial D}$, is given in section 5.

This work was fulfilled in the framework of research under the direction of R. G. Novikov.

## 2. Preliminaries

In this section we introduce and give details on the above-mentioned family of solutions of equation (1.1), which will be used throughout the paper.

We identify $\mathbb{R}^2$ with $\mathbb{C}$ and use the coordinates $z = x_1 + ix_2$, $\bar z = x_1 - ix_2$ where $(x_1, x_2) \in \mathbb{R}^2$. Let us define the function spaces $C^1_{\bar z}(\bar D) = \{u : u, \frac{\partial u}{\partial \bar z} \in C(\bar D, M_n(\mathbb{C}))\}$ with the norm $\|u\|_{C^1_{\bar z}(\bar D)} = \max(\|u\|_{C(\bar D)}, \|\frac{\partial u}{\partial \bar z}\|_{C(\bar D)})$, where $\|u\|_{C(\bar D)} = \sup_{z\in\bar D} |u|$ and $|u| = \max_{1\le i,j\le n} |u_{i,j}|$; we also define $C^1_{z}(\bar D) = \{u : u, \frac{\partial u}{\partial z} \in C(\bar D, M_n(\mathbb{C}))\}$ with an analogous norm. Following **[13]**, **[14]**, we consider the functions:

$$G_{z_0}(z,\zeta,\lambda) = e^{\lambda(z-z_0)^2} g_{z_0}(z,\zeta,\lambda) e^{-\lambda(\zeta-z_0)^2}, \tag{2.1}$$

$$g_{z_0}(z,\zeta,\lambda) = \frac{e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2}}{4\pi^2} \int_D \frac{e^{-\lambda(\eta-z_0)^2+\bar\lambda(\bar\eta-\bar z_0)^2}}{(z-\eta)(\bar\eta-\bar\zeta)} d\mathrm{Re}\eta\, d\mathrm{Im}\eta, \tag{2.2}$$

$$\psi_{z_0}(z,\lambda) = e^{\lambda(z-z_0)^2}\mu_{z_0}(z,\lambda), \tag{2.3}$$

$$\mu_{z_0}(z,\lambda) = I + \int_D g_{z_0}(z,\zeta,\lambda) v(\zeta)\mu_{z_0}(\zeta,\lambda) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \tag{2.4}$$

$$h_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2} v(z)\mu_{z_0}(z,\lambda) d\mathrm{Re}z\, d\mathrm{Im}z, \tag{2.5}$$

where $z, z_0, \zeta \in D$, $\lambda \in \mathbb{C}$ and $I$ is the identity matrix. In addition, equation (2.4) at fixed $z_0$ and $\lambda$, is considered as a linear integral equation for $\mu_{z_0}(\cdot,\lambda) \in C^1_{\bar z}(\bar D)$. The functions $G_{z_0}(z,\zeta,\lambda)$, $g_{z_0}(z,\zeta,\lambda)$, $\psi_{z_0}(z,\lambda)$, $\mu_{z_0}(z,\lambda)$ defined above, satisfy the following equations (see **[13]**, **[14]**):

$$4\frac{\partial^2}{\partial z\partial\bar z}G_{z_0}(z,\zeta,\lambda)=\delta(z-\zeta), \tag{2.6}$$

$$4\frac{\partial^2}{\partial \zeta\partial\bar \zeta}G_{z_0}(z,\zeta,\lambda)=\delta(\zeta-z), \tag{2.7}$$

$$4\left(\frac{\partial}{\partial z}+2\lambda(z-z_0)\right)\frac{\partial}{\partial\bar z}g_{z_0}(z,\zeta,\lambda)=\delta(z-\zeta), \tag{2.8}$$

$$4\frac{\partial}{\partial\bar\zeta}\left(\frac{\partial}{\partial\zeta}-2\lambda(\zeta-z_0)\right)g_{z_0}(z,\zeta,\lambda)=\delta(\zeta-z), \tag{2.9}$$

$$-4\frac{\partial^2}{\partial z\partial\bar z}\psi_{z_0}(z,\lambda)+v(z)\psi_{z_0}(z,\lambda)=0, \tag{2.10}$$

$$-4\left(\frac{\partial}{\partial z}+2\lambda(z-z_0)\right)\frac{\partial}{\partial\bar z}\mu_{z_0}(z,\lambda)+v(z)\mu_{z_0}(z,\lambda)=0, \tag{2.11}$$

where $z,z_0,\zeta\in D$, $\lambda\in\mathbb{C}$, $\delta$ is the Dirac delta. (In addition, we assume that (2.4) is uniquely solvable for $\mu_{z_0}(\cdot,\lambda)\in C^1_{\bar z}(\bar D)$ at fixed $z_0$ and $\lambda$.)

We say that the functions $G_{z_0}$, $g_{z_0}$, $\psi_{z_0}$, $\mu_{z_0}$, $h_{z_0}$ are the Bukhgeim-type analogues of the Faddeev functions (see [**14**]). We recall that the history of these functions goes back to [**7**] and [**3**].

Now we state some fundamental lemmata. Let

$$g_{z_0,\lambda}u(z)=\int_D g_{z_0}(z,\zeta,\lambda)u(\zeta)d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta,\ z\in\bar D,\ z_0,\lambda\in\mathbb{C}, \tag{2.12}$$

where $g_{z_0}(z,\zeta,\lambda)$ is defined by (2.2) and $u$ is a test function.

Lemma 2.1 ([**13**]). *Let $g_{z_0,\lambda}u$ be defined by* (2.12)*. Then, for $z_0,\lambda\in\mathbb{C}$, the following estimates hold:*

$$g_{z_0,\lambda}u\in C^1_{\bar z}(\bar D),\quad \textit{for } u\in C(\bar D), \tag{2.13}$$

$$\|g_{z_0,\lambda}u\|_{C^1(\bar D)}\le c_1(D,\lambda)\|u\|_{C(\bar D)},\quad \textit{for } u\in C(\bar D), \tag{2.14}$$

$$\|g_{z_0,\lambda}u\|_{C^1_{\bar z}(\bar D)}\le\frac{c_2(D)}{|\lambda|^{\frac12}}\|u\|_{C^1_{\bar z}(\bar D)},\quad \textit{for } u\in C^1_{\bar z}(\bar D),\ |\lambda|\ge 1. \tag{2.15}$$

Given a potential $v\in C^1_{\bar z}(\bar D)$ we define the operator $g_{z_0,\lambda}v$ simply as $(g_{z_0,\lambda}v)u(z)=g_{z_0,\lambda}w(z),\ w=vu$, for a test function $u$. If $u\in C^1_{\bar z}(\bar D)$, by Lemma 2.1 we have that $g_{z_0,\lambda}v: C^1_{\bar z}(\bar D)\to C^1_{\bar z}(\bar D)$,

$$\|g_{z_0,\lambda}v\|^{op}_{C^1_{\bar z}(\bar D)}\le 2n\|g_{z_0,\lambda}\|^{op}_{C^1_{\bar z}(\bar D)}\|v\|_{C^1_{\bar z}(\bar D)}, \tag{2.16}$$

where $\|\cdot\|^{op}_{C^1_{\bar z}(\bar D)}$ denotes the operator norm in $C^1_{\bar z}(\bar D)$, $z_0,\lambda\in\mathbb{C}$. In addition, $\|g_{z_0,\lambda}\|^{op}_{C^1_{\bar z}(\bar D)}$ is estimated in Lemma 2.1. Inequality (2.16) and Lemma 2.1 imply the existence and uniqueness of $\mu_{z_0}(z,\lambda)$ (and thus also of $\psi_{z_0}(z,\lambda)$) for $|\lambda|>\rho(D,K,n)$, where $\|v\|_{C^1_{\bar z}(\bar D)}<K$.

Let

$$\mu_{z_0}^{(k)}(z,\lambda) = \sum_{j=0}^{k} (g_{z_0,\lambda} v)^j I,$$

$$h_{z_0}^{(k)}(\lambda) = \int_D e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} v(z) \mu_{z_0}^{(k)}(z,\lambda) d\mathrm{Re} z\, d\mathrm{Im} z,$$

where $z, z_0 \in D$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N} \cup \{0\}$.

LEMMA 2.2 ([**13**]). *For $v \in C^1_{\bar z}(\bar D)$ such that $v|_{\partial D} = 0$ the following formula holds:*

$$v(z_0) = \frac{2}{\pi} \lim_{\lambda\to\infty} |\lambda| h_{z_0}^{(0)}(\lambda),\ z_0 \in D. \tag{2.17}$$

*In addition, if $v \in C^2(\bar D)$, $v|_{\partial D} = 0$ and $\frac{\partial v}{\partial \nu}|_{\partial D} = 0$ then*

$$\left| v(z_0) - \frac{2}{\pi}|\lambda| h_{z_0}^{(0)}(\lambda) \right| \le c_3(D,n) \frac{\log(3|\lambda|)}{|\lambda|} \|v\|_{C^2(\bar D)}, \tag{2.18}$$

*for $z_0 \in D$, $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$.*

Let

$$W_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} w(z) d\mathrm{Re}\, z d\mathrm{Im}\, z,$$

where $z_0 \in \bar D$, $\lambda \in \mathbb{C}$ and $w$ is some $M_n(\mathbb{C})$-valued function on $\bar D$. (One can see that $W_{z_0} = h_{z_0}^{(0)}$ for $w = v$.)

LEMMA 2.3 ([**13**]). *For $w \in C^1_{\bar z}(\bar D)$ the following estimate holds:*

$$|W_{z_0}(\lambda)| \le c_4(D) \frac{\log(3|\lambda|)}{|\lambda|} \|w\|_{C^1_{\bar z}(\bar D)},\ z_0 \in \bar D,\ |\lambda| \ge 1. \tag{2.19}$$

LEMMA 2.4 ([**14**]). *For $v \in C^1_{\bar z}(\bar D)$ and for $\|g_{z_0,\lambda} v\|^{op}_{C^1_{\bar z}(\bar D)} \le \delta < 1$ we have that*

$$\|\mu_{z_0}(\cdot,\lambda) - \mu_{z_0}^{(k)}(\cdot,\lambda)\|_{C^1_{\bar z}(\bar D)} \le \frac{\delta^{k+1}}{1-\delta}, \tag{2.20}$$

$$|h_{z_0}(\lambda) - h_{z_0}^{(k)}(\lambda)| \le c_5(D,n) \frac{\log(3|\lambda|)}{|\lambda|} \frac{\delta^{k+1}}{1-\delta} \|v\|_{C^1_{\bar z}(\bar D)}, \tag{2.21}$$

*where $z_0 \in D$, $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$, $k \in \mathbb{N} \cup \{0\}$.*

The proofs of Lemmata 2.1-2.4 can be found in the references given.
We will also need the following two new lemmata.

LEMMA 2.5. *Let $g_{z_0,\lambda} u$ be defined by (2.12), where $u \in C^1_{\bar z}(\bar D)$, $z_0, \lambda \in \mathbb{C}$. Then the following estimate holds:*

$$\|g_{z_0,\lambda} u\|_{C(\bar D)} \le c_6(D) \frac{\log(3|\lambda|)}{|\lambda|} \|u\|_{C^1_{\bar z}(\bar D)},\ |\lambda| \ge 1. \tag{2.22}$$

Lemma 2.6. *The expression*

$$W(u,v)(\lambda) = \int_D e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2} u(z)(g_{z_0,\lambda}v)(z)d\mathrm{Re}z\, d\mathrm{Im}z, \tag{2.23}$$

*defined for* $u, v \in C^1_{\bar z}(\bar D)$ *with* $\|u\|_{C^1_{\bar z}(\bar D)}, \|v\|_{C^1_{\bar z}(\bar D)} \le N_1$, $\lambda \in \mathbb{C}$, $z_0 \in D$, *satisfies the estimate*

$$|W(u,v)(\lambda)| \le c_7(D,N_1,n)\frac{(\log(3|\lambda|))^2}{|\lambda|^{1+3/4}}, \qquad |\lambda| \ge 1. \tag{2.24}$$

The proofs of Lemmata 2.5, 2.6 are given in section 4.

## 3. Proof of Theorem 1.1

We begin with a technical lemma, which will prove useful when generalising Alessandrini's identity.

Lemma 3.1. *Let* $v \in C^1(\bar D, M_n(\mathbb{C}))$ *be a matrix-valued potential which satisfies condition* (1.4) *(i.e.,* 0 *is not a Dirichlet eigeinvalue for the operator* $-\Delta + v$ *in* $D$*). Then* ${}^t v$*, the transpose of* $v$*, also satisfies condition* (1.4)*.*

The proof of Lemma 3.1 is given in section 4.

We can now state and prove a matrix-valued version of Alessandrini's identity (see [**1**] for the scalar case).

Lemma 3.2. *Let* $v_1, v_2 \in C^1(\bar D, M_n(\mathbb{C}))$ *be two matrix-valued potentials which satisfy* (1.4)*,* $\Phi_1, \Phi_2$ *their associated Dirichlet-to-Neumann operators, respectively, and* $u_1, u_2 \in C^2(\bar D, M_n(\mathbb{C}))$ *two matrix-valued functions such that*

$$(-\Delta + v_1)u_1 = 0, \quad (-\Delta + {}^t v_2)u_2 = 0 \quad \text{on } D,$$

*where* ${}^t A$ *stand for the transpose of* $A$*. Then we have the identity*

$$\int_{\partial D} {}^t u_2(z)(\Phi_2 - \Phi_1)u_1(z)|dz| = \int_D {}^t u_2(z)(v_2(z) - v_1(z))u_1(z)d\mathrm{Re}z\, d\mathrm{Im}z. \tag{3.1}$$

Proof. If $v \in C^1(\bar D, M_n(\mathbb{C}))$ is any matrix-valued potential (which satisfies (1.4)) and $f_1, f_2 \in C^1(\partial D, M_n(\mathbb{C}))$ then we have

$$\int_{\partial D} {}^t f_2 \Phi f_1 |dz| = \int_{\partial D} {}^t\left({}^t f_1 \Phi^* f_2\right)|dz|, \tag{3.2}$$

where $\Phi$ and $\Phi^*$ are the Dirichlet-to-Neumann operators associated with $v$ and ${}^t v$, respectively (these operators are well-defined thanks to Lemma 3.1). Indeed, it is sufficient to extend $f_1$ and $f_2$ in $D$ as the solutions of the Dirichlet problems $(-\Delta + v)\tilde f_1 = 0$, $(-\Delta + {}^t v)\tilde f_2 = 0$ on $D$ and $\tilde f_j|_{\partial D} = f_j$, for $j = 1, 2$, so that one obtains

$$\begin{aligned}
&\int_{\partial D} \left({}^t f_2 \Phi f_1 - {}^t\left({}^t f_1 \Phi^* f_2\right)\right) |dz| \\
&\quad = \int_{\partial D} \left( {}^t f_2 \frac{\partial \tilde f_1}{\partial \nu} - {}^t\left(\frac{\partial \tilde f_2}{\partial \nu}\right) f_1 \right) |dz| \\
&\quad = \int_D \left( {}^t \tilde f_2 \, \Delta \tilde f_1 - {}^t\left(\Delta \tilde f_2\right) \tilde f_1 \right) d\mathrm{Re} z \, d\mathrm{Im} z \\
&\quad = \int_D \left( {}^t \tilde f_2 \, v \, \tilde f_1 - {}^t\left({}^t v \, \tilde f_2\right) \tilde f_1 \right) d\mathrm{Re} z \, d\mathrm{Im} z = 0,
\end{aligned}$$

where for the second equality we used the following matrix-valued version of the classical scalar Green's formula:

$$\int_{\partial D} \left( {}^t\left(\frac{\partial f}{\partial \nu}\right) g - {}^t f \frac{\partial g}{\partial \nu} \right) |dz| = \int_D \left( {}^t(\Delta f)\, g - {}^t f \Delta g \right) d\mathrm{Re} z \, d\mathrm{Im} z, \tag{3.3}$$

for any $f, g \in C^2(D, M_n(\mathbb{C})) \cap C^1(\bar D, M_n(\mathbb{C}))$.

Identities (3.2) and (3.3) imply

$$\begin{aligned}
&\int_{\partial D} {}^t u_2(z)(\Phi_2 - \Phi_1) u_1(z) |dz| \\
&\quad = \int_{\partial D} \left( {}^t\left({}^t u_1(z) \Phi_2^* u_2(z)\right) - {}^t u_2(z) \Phi_1 u_1(z) \right) |dz| \\
&\quad = \int_{\partial D} \left( {}^t\left(\frac{\partial u_2(z)}{\partial \nu}\right) u_1(z) - {}^t u_2(z) \frac{\partial u_1(z)}{\partial \nu} \right) |dz| \\
&\quad = \int_D \left( {}^t(\Delta u_2(z))\, u_1(z) - {}^t u_2(z) \Delta u_1(z) \right) d\mathrm{Re} z \, d\mathrm{Im} z \\
&\quad = \int_D \left( {}^t\left({}^t v_2(z)\, u_2(z)\right) u_1(z) - {}^t u_2(z)\, v_1(z)\, u_1(z) \right) d\mathrm{Re} z \, d\mathrm{Im} z \\
&\quad = \int_D {}^t u_2(z)(v_2(z) - v_1(z)) u_1(z) d\mathrm{Re} z \, d\mathrm{Im} z. \qquad \square
\end{aligned}$$

Now let $\bar\mu_{z_0}$ denote the complex conjugate of $\mu_{z_0}$ (the solution of (2.4)) for a $M_n(\mathbb{R})$-valued potential $v$ and, more generally, the solution of (2.4) with $g_{z_0}(z, \zeta, \lambda)$ replaced by $\overline{g_{z_0}(z, \zeta, \lambda)}$ for a $M_n(\mathbb{C})$-valued potential $v$. In order to make use of (3.1) we define

$$\begin{aligned}
u_1(z) &= \psi_{1,z_0}(z, \lambda) = e^{\lambda(z - z_0)^2} \mu_1(z, \lambda), \\
u_2(z) &= \overline{\psi}_{2,z_0}(z, -\lambda) = e^{-\bar\lambda(\bar z - \bar z_0)^2} \bar\mu_2(z, -\lambda),
\end{aligned}$$

for $z_0 \in D$, $\lambda \in C$, $|\lambda| > \rho$ ($\rho$ is mentioned in section 2), where we set $\mu_1 = \mu_{1,z_0}$, $\mu_2 = \mu_{2,z_0}$ for simplicity's sake and $\mu_{1,z_0}$, $\mu_{2,z_0}$ are the solutions of (2.4) with $v$ replaced by $v_1$, ${}^t v_2$, respectively.

Equation (3.1), with the above-defined $u_1, u_2$, now reads

$$\int_{\partial D}\int_{\partial D} e^{-\bar\lambda(\bar z-\bar z_0)^2}\,{}^t\bar\mu_2(z,-\lambda)(\Phi_2-\Phi_1)(z,\zeta)e^{\lambda(\zeta-z_0)^2}\mu_1(\zeta,\lambda)|d\zeta||dz| \\ = \int_D e_{\lambda,z_0}(z)\,{}^t\bar\mu_2(z,-\lambda)(v_2-v_1)(z)\mu_1(z,\lambda)d\mathrm{Re}z\,d\mathrm{Im}z. \tag{3.4}$$

with $e_{\lambda,z_0}(z) = e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}$ and $(\Phi_2-\Phi_1)(z,\zeta)$ is the Schwartz kernel of the operator $\Phi_2-\Phi_1$.

The right side $I(\lambda)$ of (3.4) can be written as the sum of four integrals, namely

$$\begin{aligned} I_1(\lambda) &= \int_D e_{\lambda,z_0}(z)(v_2-v_1)(z)d\mathrm{Re}z\,d\mathrm{Im}z,\\ I_2(\lambda) &= \int_D e_{\lambda,z_0}(z)\,{}^t(\bar\mu_2-I)(v_2-v_1)(z)(\mu_1-I)d\mathrm{Re}z\,d\mathrm{Im}z,\\ I_3(\lambda) &= \int_D e_{\lambda,z_0}(z)\,{}^t(\bar\mu_2-I)(v_2-v_1)(z)\,d\mathrm{Re}z\,d\mathrm{Im}z,\\ I_4(\lambda) &= \int_D e_{\lambda,z_0}(z)\,(v_2-v_1)(z)(\mu_1-I)d\mathrm{Re}z\,d\mathrm{Im}z, \end{aligned}$$

for $z_0\in D$.

Since $(v_2-v_1)|_{\partial D} = \frac{\partial}{\partial\nu}(v_2-v_1)|_{\partial D}=0$, the first term, $I_1$, can be estimated using Lemma 2.2 as

$$\left|\frac{2}{\pi}|\lambda|I_1-(v_2(z_0)-v_1(z_0))\right| \le c_3(D,n)\frac{\log(3|\lambda|)}{|\lambda|}\|v_2-v_1\|_{C^2(\bar D)}, \tag{3.5}$$

for $|\lambda|\ge 1$. The other terms, $I_2, I_3, I_4$, satisfy, by Lemmata 2.1 and 2.4,

$$|I_2| \le \left|\int_D e_{\lambda,z_0}(z)\,{}^t(\overline{g_{z_0,\lambda}{}^t v_2})(v_2-v_1)(z)(g_{z_0,\lambda}v_1)d\mathrm{Re}z\,d\mathrm{Im}z\right| \\ + O\left(\frac{\log(3|\lambda|)}{|\lambda|^2}\right)c_8(D,N,n), \tag{3.6}$$

$$|I_3| \le \left|\int_D e_{\lambda,z_0}(z)\,{}^t(\overline{g_{z_0,\lambda}{}^t v_2})(v_2-v_1)(z)d\mathrm{Re}z\,d\mathrm{Im}z\right| \\ + O\left(\frac{\log(3|\lambda|)}{|\lambda|^2}\right)c_9(D,N,n), \tag{3.7}$$

$$|I_4| \le \left|\int_D e_{\lambda,z_0}(z)\,(v_2-v_1)(z)(g_{z_0,\lambda}v_1)d\mathrm{Re}z\,d\mathrm{Im}z\right| \\ + O\left(\frac{\log(3|\lambda|)}{|\lambda|^2}\right)c_{10}(D,N,n), \tag{3.8}$$

where $N$ is the constant in the statement of Theorem 1.1 and $|\lambda|$ is sufficiently large, for example for $\lambda$ such that

$$2n\frac{c_2(D)}{|\lambda|^{\frac{1}{2}}} \leq \frac{1}{2}, \qquad |\lambda| \geq 1. \tag{3.9}$$

Lemmata 2.5, 2.6, applied to (3.6)-(3.8), give us

$$|I_2| \leq c_{11}(D,N,n)\frac{(\log(3|\lambda|))^2}{|\lambda|^2}, \tag{3.10}$$

$$|I_3| \leq c_{12}(D,N,n)\frac{(\log(3|\lambda|))^2}{|\lambda|^{1+3/4}}, \tag{3.11}$$

$$|I_4| \leq c_{13}(D,N,n)\frac{(\log(3|\lambda|))^2}{|\lambda|^{1+3/4}}. \tag{3.12}$$

The left side $J(\lambda)$ of (3.4) can be estimated as follows:

$$|\lambda||J(\lambda)| \leq c_{14}(D,n)e^{(2L^2+1)|\lambda|}\|\Phi_2 - \Phi_1\|, \tag{3.13}$$

for $\lambda$ which satisfies (3.9), and $L = \max_{z\in\partial D,\, z_0\in D}|z - z_0|$.

Putting together estimates (3.5)-(3.13) we obtain

(3.14)

$$|v_2(z_0) - v_1(z_0)| \leq c_{15}(D,N,n)\frac{(\log(3|\lambda|))^2}{|\lambda|^{3/4}} + \frac{2}{\pi}c_{14}(D,n)e^{(2L^2+1)|\lambda|}\|\Phi_2 - \Phi_1\|$$

for any $z_0 \in D$. We call $\varepsilon = \|\Phi_2 - \Phi_1\|$ and impose $|\lambda| = \gamma\log(3+\varepsilon^{-1})$, where $0 < \gamma < (2L^2+1)^{-1}$ so that (3.14) reads

$$\begin{aligned}|v_2(z_0) - v_1(z_0)| \leq\ & c_{15}(D,N,n)(\gamma\log(3+\varepsilon^{-1}))^{-\frac{3}{4}}\left(\log(3\gamma\log(3+\varepsilon^{-1}))\right)^2 \\ & + \frac{2}{\pi}c_{14}(D,n)(3+\varepsilon^{-1})^{(2L^2+1)\gamma}\varepsilon,\end{aligned} \tag{3.15}$$

for every $z_0 \in D$, with

$$0 < \varepsilon \leq \varepsilon_1(D,N,\gamma,n), \tag{3.16}$$

where $\varepsilon_1$ is sufficiently small or, more precisely, where (3.16) implies that $|\lambda| = \gamma\log(3+\varepsilon^{-1})$ satisfies (3.9).

As $(3+\varepsilon^{-1})^{(2L^2+1)\gamma}\varepsilon \to 0$ for $\varepsilon \to 0$ more rapidly then the other term, we obtain that

$$\|v_2 - v_1\|_{L^\infty(D)} \leq c_{16}(D,N,\gamma,n)\frac{(\log(3\log(3+\|\Phi_2-\Phi_1\|^{-1})))^2}{(\log(3+\|\Phi_2-\Phi_1\|^{-1}))^{\frac{3}{4}}} \tag{3.17}$$

for any $\varepsilon = \|\Phi_2 - \Phi_1\| \leq \varepsilon_1(D,N,\gamma,n)$.

Estimate (3.17) for general $\varepsilon$ (with modified $c_{16}$) follows from (3.17) for $\varepsilon \leq \varepsilon_1(D,N,\gamma,n)$ and the assumption that $\|v_j\|_{L^\infty(D)} \leq N,\ j = 1,2$. This completes the proof of Theorem 1.1. □

## 4. Proofs of Lemmata 2.5, 2.6, 3.1.

Proof of Lemma 2.5. We decompose the operator $g_{z_0,\lambda}$, defined in (2.12), as the product $\frac{1}{4}T_{z_0,\lambda}\bar{T}_{z_0,\lambda}$, where

$$T_{z_0,\lambda}u(z) = \frac{1}{\pi}\int_D \frac{e^{-\lambda(\zeta-z_0)^2+\bar{\lambda}(\bar{\zeta}-\bar{z}_0)^2}}{z-\zeta}u(\zeta)d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \tag{4.1}$$

$$\bar{T}_{z_0,\lambda}u(z) = \frac{1}{\pi}\int_D \frac{e^{\lambda(\zeta-z_0)^2-\bar{\lambda}(\bar{\zeta}-\bar{z}_0)^2}}{\bar{z}-\bar{\zeta}}u(\zeta)d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \tag{4.2}$$

for $z_0, \lambda \in \mathbb{C}$. From the proof of [**13**, Lemma 3.1] we have the estimate

$$\|\bar{T}_{z_0,\lambda}u\|_{C(\bar{D})} \leq \frac{\eta_1(D)}{|\lambda|^{1/2}}\|u\|_{C(\bar{D})} + \eta_2(D)\frac{\log(3|\lambda|)}{|\lambda|}\left\|\frac{\partial u}{\partial \bar{z}}\right\|_{C(\bar{D})}, \tag{4.3}$$

for $u \in C^1_{\bar{z}}(\bar{D})$, $z_0 \in D$, $|\lambda| \geq 1$. As the kernels of $T_{z_0,\lambda}$ and $\bar{T}_{z_0,\lambda}$ are conjugates of each other we deduce immediately that

$$\|T_{z_0,\lambda}u\|_{C(\bar{D})} \leq \frac{\eta_1(D)}{|\lambda|^{1/2}}\|u\|_{C(\bar{D})} + \eta_2(D)\frac{\log(3|\lambda|)}{|\lambda|}\left\|\frac{\partial u}{\partial z}\right\|_{C(\bar{D})},\ |\lambda| \geq 1, \tag{4.4}$$

for $u \in C^1_z(\bar{D})$. Combining the two estimates we obtain

$$\begin{aligned}
\|g_{\lambda,z_0}u\|_{C(\bar{D})} &= \frac{1}{4}\|T_{z_0,\lambda}\bar{T}_{z_0,\lambda}u\|_{C(\bar{D})}\\
&\leq \frac{1}{4}\left(\eta_1(D)\frac{\|\bar{T}_{z_0,\lambda}u\|_{C(\bar{D})}}{|\lambda|^{1/2}} + \eta_2(D)\frac{\log(3|\lambda|)}{|\lambda|}\left\|\frac{\partial}{\partial z}\bar{T}_{z_0,\lambda}u\right\|_{C(\bar{D})}\right)\\
&\leq \eta_3(D)\left(\frac{\|u\|_{C(\bar{D})}}{|\lambda|} + \frac{\log(3|\lambda|)}{|\lambda|^{3/2}}\left\|\frac{\partial u}{\partial \bar{z}}\right\|_{C(\bar{D})} + \frac{\log(3|\lambda|)}{|\lambda|}\|u\|_{C(\bar{D})}\right)\\
&\leq \eta_4(D)\frac{\log(3|\lambda|)}{|\lambda|}\|u\|_{C^1_{\bar{z}}(\bar{D})}, \qquad |\lambda| \geq 1,
\end{aligned}$$

where we use the fact that $\|\frac{\partial}{\partial z}\bar{T}_{z_0,\lambda}u\|_{C(D)} = \|u\|_{C(D)}$. ☐

Proof of Lemma 2.6. For $0 < \varepsilon \leq 1$, $z_0 \in D$, let $B_{z_0,\varepsilon} = \{z \in \mathbb{C} : |z-z_0| \leq \varepsilon\}$. We write $W(u,v)(\lambda) = W^1(\lambda) + W^2(\lambda)$, where

$$W^1(\lambda) = \int_{D\cap B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar{\lambda}(\bar{z}-\bar{z}_0)^2}u(z)g_{z_0,\lambda}v(z)d\mathrm{Re}z\, d\mathrm{Im}z,$$

$$W^2(\lambda) = \int_{D\setminus B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar{\lambda}(\bar{z}-\bar{z}_0)^2}u(z)g_{z_0,\lambda}v(z)d\mathrm{Re}z\, d\mathrm{Im}z.$$

The first term, $W^1$, can be estimated as follows:

$$|W^1(\lambda)| \leq \sigma_1(D,n)\|u\|_{C(\bar{D})}\|v\|_{C^1_{\bar{z}}(\bar{D})}\frac{\varepsilon^2\log(3|\lambda|)}{|\lambda|}, \qquad |\lambda| \geq 1, \tag{4.5}$$

where we use estimates (2.16) and (2.22).

For the second term, $W^2$, we proceed using integration by parts, in order to obtain

$$W^2(\lambda) = \frac{1}{4i\bar{\lambda}} \int_{\partial(D\setminus B_{z_0,\varepsilon})} e^{\lambda(z-z_0)^2-\bar{\lambda}(\bar{z}-\bar{z}_0)^2} \frac{u(z)g_{z_0,\lambda}v(z)}{\bar{z}-\bar{z}_0} dz - \frac{1}{2\bar{\lambda}} \int_{D\setminus B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar{\lambda}(\bar{z}-\bar{z}_0)^2} \frac{\partial}{\partial \bar{z}} \left( \frac{u(z)g_{z_0,\lambda}v(z)}{\bar{z}-\bar{z}_0} \right) d\mathrm{Re}z\, d\mathrm{Im}z.$$

This implies that

$$(4.6) \qquad |W^2(\lambda)| \le \frac{1}{4|\lambda|} \int_{\partial(D\setminus B_{z_0,\varepsilon})} \frac{\|u(z)g_{z_0,\lambda}v(z)\|_{C(\bar{D})}}{|\bar{z}-\bar{z}_0|} |dz| + \frac{1}{2|\lambda|} \left| \int_{D\setminus B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar{\lambda}(\bar{z}-\bar{z}_0)^2} \frac{\partial}{\partial \bar{z}} \left( \frac{u(z)g_{z_0,\lambda}v(z)}{\bar{z}-\bar{z}_0} \right) d\mathrm{Re}z\, d\mathrm{Im}z \right|,$$

for $\lambda \neq 0$. Again by estimates (2.16) and (2.22) we obtain

$$(4.7) \qquad |W^2(\lambda)| \le \sigma_2(D,n) \|u\|_{C^1_{\bar{z}}(\bar{D})} \|v\|_{C^1_{\bar{z}}(\bar{D})} \frac{\log(3\varepsilon^{-1})\log(3|\lambda|)}{|\lambda|^2} + \frac{1}{8|\lambda|} \left| \int_{D\setminus B_{z_0,\varepsilon}} u(z) \frac{\bar{T}_{z_0,\lambda}v(z)}{\bar{z}-\bar{z}_0} d\mathrm{Re}z\, d\mathrm{Im}z \right|, \qquad |\lambda| \ge 1,$$

where we used the fact that $\frac{\partial}{\partial \bar{z}} g_{z_0,\lambda}v(z) = \frac{1}{4} e^{-\lambda(z-z_0)^2+\bar{\lambda}(\bar{z}-\bar{z}_0)^2} \bar{T}_{z_0,\lambda}v(z)$, with $\bar{T}_{z_0,\lambda}$ defined in (4.2).

The last term in (4.7) can be estimated independently of $\varepsilon$ by

$$(4.8) \qquad \sigma_3(D,n) \|u\|_{C(\bar{D})} \|v\|_{C^1_{\bar{z}}(\bar{D})} \frac{\log(3|\lambda|)}{|\lambda|^{1+3/4}}.$$

This is a consequence of (4.3) and of the estimate

$$(4.9) \qquad |\bar{T}_{z_0,\lambda}u(z)| \le \frac{\log(3|\lambda|)(1+|z-z_0|)\tau_1(D)}{|\lambda||z-z_0|^2} \|u\|_{C^1_{\bar{z}}(\bar{D})},\ |\lambda| \ge 1,$$

for $u \in C^1_{\bar{z}}(\bar{D})$, $z, z_0 \in D$ (a proof of (4.9) can be found in the proof of [**13**, Lemma 3.1]).

Indeed, for $0 < \delta \le \frac{1}{2}$ we have

$$\begin{aligned}
&\left|\int_D u(z)\frac{\bar{T}_{z_0,\lambda}v(z)}{\bar{z}-\bar{z}_0}d\mathrm{Re}z\,d\mathrm{Im}z\right| \\
&\le \int_{B_{z_0,\delta}\cap D}|u(z)|\frac{|\bar{T}_{z_0,\lambda}v(z)|}{|z-z_0|}d\mathrm{Re}z\,d\mathrm{Im}z + \int_{D\setminus B_{z_0,\delta}}|u(z)|\frac{|\bar{T}_{z_0,\lambda}v(z)|}{|z-z_0|}d\mathrm{Re}z\,d\mathrm{Im}z \\
&\le \|u\|_{C(\bar{D})}\|v\|_{C^1_{\bar{z}}(\bar{D})}\frac{\tau_2(D,n)}{|\lambda|^{1/2}}\int_{B_{z_0,\delta}\cap D}\frac{d\mathrm{Re}z\,d\mathrm{Im}z}{|z-z_0|} \\
&\quad + \|u\|_{C(\bar{D})}\|v\|_{C^1_{\bar{z}}(\bar{D})}\frac{\log(3|\lambda|)}{|\lambda|}\tau_3(D,n)\int_{D\setminus B_{z_0,\delta}}\frac{d\mathrm{Re}z\,d\mathrm{Im}z}{|z-z_0|^3} \\
&\le 2\pi\|u\|_{C(\bar{D})}\|v\|_{C^1_{\bar{z}}(\bar{D})}\tau_2(D,n)\frac{\delta}{|\lambda|^{\frac{1}{2}}} + \|u\|_{C(\bar{D})}\|v\|_{C^1_{\bar{z}}(\bar{D})}\tau_4(D,n)\frac{\log(3|\lambda|)}{|\lambda|\delta},
\end{aligned}$$

for $|\lambda| \ge 1$. Putting $\delta = \frac{1}{2}|\lambda|^{-1/4}$ in the last inequality gives (4.8).

Finally, defining $\varepsilon = |\lambda|^{-1/2}$ in (4.7), (4.5) and using (4.8), we obtain the main estimate (2.24), which thus finishes the proof of Lemma 2.6. □

Proof of Lemma 3.1. Take $u \in H^1(D, M_n(\mathbb{C}))$ such that $(-\Delta + {}^t v)u = 0$ on $D$ and $u|_{\partial D} = 0$. We want to prove that $u \equiv 0$ on $D$.

By our hypothesis, for any $f \in C^1(\partial D, M_n(\mathbb{C}))$ there exists a unique $\tilde{f} \in H^1(D, M_n(\mathbb{C}))$ such that $(-\Delta + v)\tilde{f} = 0$ on $D$ and $\tilde{f}|_{\partial D} = f$. Thus we have, using Green's formula (3.3),

$$\begin{aligned}
\int_{\partial D} {}^t\!\left(\frac{\partial u}{\partial \nu}\right) f|dz| &= \int_D \left({}^t(\Delta u)\,\tilde{f} - {}^t u\Delta\tilde{f}\right) d\mathrm{Re}z\,d\mathrm{Im}z \\
&= \int_D \left({}^t\!\left({}^t v\, u\right)\tilde{f} - {}^t u\, v\,\tilde{f}\right) d\mathrm{Re}z\,d\mathrm{Im}z = 0,
\end{aligned}$$

which yields $\frac{\partial u}{\partial \nu}|_{\partial D} = 0$. Now consider the following straightforward generalization of Green's formula (3.3),

$$\int_{\partial D}\left({}^t\!\left(\frac{\partial f}{\partial \nu}\right) g - {}^t f\frac{\partial g}{\partial \nu}\right)|dz| = \int_D {}^t\!\left((\Delta - {}^t v)f\right) g - {}^t f\left((\Delta - v)g\right) d\mathrm{Re}z\,d\mathrm{Im}z, \tag{4.10}$$

which holds (weakly) for any $f, g \in H^1(D, M_n(\mathbb{C}))$. If we put $f = u$ we obtain

$$\int_D {}^t u\,(-\Delta + v)g\,d\mathrm{Re}z\,d\mathrm{Im}z = 0, \tag{4.11}$$

for any $g \in H^1(D, M_n(\mathbb{C}))$. By Fredholm alternative (see [**6**, Sec. 6.2]), for each $h \in L^2(D, M_n(\mathbb{C}))$ there exists a unique $g \in H^1_0(D, M_n(\mathbb{C})) = \{g \in H^1(D, M_n(\mathbb{C})) : g|_{\partial D} = 0\}$ such that $(-\Delta + v)g = h$. This yields $u \equiv 0$ on $D$ and thus Lemma 3.1 is proved. □

## 5. An extensions of Theorem 1.1

As an extension of Theorem 1.1 to the case where we do not assume that $v_1|_{\partial D} = v_2|_{\partial D}$ and $\frac{\partial}{\partial\nu}v_1|_{\partial D} = \frac{\partial}{\partial\nu}v_2|_{\partial D}$, we give the following proposition:

PROPOSITION 5.1. *Let $D \subset \mathbb{R}^2$ be an open bounded domain with a $C^2$ boundary, $v_1, v_2 \in C^2(\bar{D}, M_n(\mathbb{C}))$ two matrix-valued potentials which satisfy* (1.4)*, with $\|v_j\|_{C^2(\bar{D})} \le N$ for $j = 1, 2$, and $\Phi_1, \Phi_2$ the corresponding Dirichlet-to-Neumann operators. Then, for any $0 < \alpha < \frac{1}{5}$, there exists a constant $C = C(D, N, n, \alpha)$ such that*

$$\|v_2 - v_1\|_{L^\infty(D)} \le C\left(\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})\right)^{-\alpha}, \tag{5.1}$$

*where, for an operator $A$ which acts on $L^\infty(\partial D, M_n(\mathbb{C}))$ with kernel $A(x,y)$, $\|A\|_1$ is the norm defined as $\|A\|_1 = \sup_{x,y\in\partial D}|A(x,y)|(\log(3 + |x-y|^{-1}))^{-1}$ and $|A(x,y)| = \max_{1\le i,j\le n}|A_{i,j}(x,y)|$.*

The only properties of $\|\cdot\|_1$ we will use are the following:
*i)* $\|A\|_{L^\infty(\partial D)\to L^\infty(\partial D)} \le const(D,n)\|A\|_1$;
*ii)* In a similar way as in formula (4.9) of **[11]** one can deduce

$$\|v\|_{L^\infty(\partial D)} \le const(n)\|\Phi_v - \Phi_0\|_1,$$

for a matrix-valued potential $v$, $\Phi_v$ its associated Dirichlet-to-Neumann operator and $\Phi_0$ the Dirichlet-to-Neumann operator of the 0 potential.

We recall a lemma from **[13]**, which generalizes Lemma 2.2 to the case of potentials without boundary conditions. We then define $(\partial D)_\delta = \{z \in \mathbb{C} : dist(z, \partial D) < \delta\}$.

LEMMA 5.2. *For $v \in C^2(\bar{D})$ we have that*

$$\left|v(z_0) - \frac{2}{\pi}|\lambda| h^{(0)}_{z_0}(\lambda)\right| \le \kappa_1(D,n)\delta^{-4}\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar{D})} + \kappa_2(D,n)\log(3+\delta^{-1})\|v\|_{C(\partial D)}, \tag{5.2}$$

*for $z_0 \in D \setminus (\partial D)_\delta$, $0 < \delta < 1$, $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$.*

The proof of Lemma 5.2 for the scalar case can be found in **[13]** and its generalization to the matrix-valued case is straightforward.

Proof of Proposition 5.1. Fix $0 < \alpha < \frac{1}{5}$ and $0 < \delta < 1$. We then have the following chain of inequalities

$$\begin{aligned}
&\|v_2 - v_1\|_{L^\infty(D)} \\
&\quad = \max(\|v_2 - v_1\|_{L^\infty(D\cap(\partial D)_\delta)}, \|v_2 - v_1\|_{L^\infty(D\setminus(\partial D)_\delta)}) \\
&\quad \le C_1 \max\Bigg(2N\delta + \|\Phi_2 - \Phi_1\|_1, \frac{\log(3\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))}{\delta^4 \log(3 + \|\Phi_2 - \Phi_1\|^{-1})} \\
&\qquad + \log(3 + \frac{1}{\delta})\|\Phi_2 - \Phi_1\|_1 + \frac{(\log(3\log(3 + \|\Phi_2 - \Phi_1\|^{-1})))^2}{(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{\frac{3}{4}}}\Bigg) \\
&\quad \le C_2 \max\Bigg(2N\delta + \|\Phi_2 - \Phi_1\|_1, \frac{1}{\delta^4}\left(\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})\right)^{-5\alpha} \\
&\qquad + \log(3 + \frac{1}{\delta})\|\Phi_2 - \Phi_1\|_1 + \frac{\left(\log(3\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1}))\right)^2}{(\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1}))^{\frac{3}{4}}}\Bigg),
\end{aligned}$$

where we followed the outline of the proof of Theorem 1.1 with the following modifications: we made use of Lemma 5.2 instead of Lemma 2.2 and we also used i)-ii); note that $C_1 = C_1(D, N, n)$ and $C_2 = C_2(D, N, n, \alpha)$.

Putting $\delta = \left(\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})\right)^{-\alpha}$ we obtain the desired inequality

$$\|v_2 - v_1\|_{L^\infty(D)} \le C_3\left(\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})\right)^{-\alpha}, \tag{5.3}$$

with $C_3 = C_3(D, N, n, \alpha)$, $\|\Phi_2 - \Phi_1\|_1 = \varepsilon \le \varepsilon_1(D, N, n, \alpha)$ with $\varepsilon_1$ sufficiently small or, more precisely when $\delta_1 = \left(\log(3 + \varepsilon_1^{-1})\right)^{-\alpha}$ satisfies:

$$\delta_1 < 1, \qquad \varepsilon_1 \le 2N\delta_1, \qquad \log(3 + \frac{1}{\delta_1})\varepsilon_1 \le \delta_1.$$

Estimate (5.3) for general $\varepsilon$ (with modified $C_3$) follows from (5.3) for $\varepsilon \le \varepsilon_1(D, N, n, \alpha)$ and the assumption that $\|v_j\|_{L^\infty(\bar{D})} \le N$ for $j = 1, 2$. This completes the proof of Proposition 5.1. □

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# Paper **E**

**PAPER E**

# Global uniqueness and reconstruction for the multi-channel Gel'fand-Calderón inverse problem in two dimensions

Roman G. Novikov and Matteo Santacesaria

Abstract. We study the multi-channel Gel'fand-Calderón inverse problem in two dimensions, i.e. the inverse boundary value problem for the equation $-\Delta\psi + v(x)\psi = 0$, $x \in D$, where $v$ is a smooth matrix-valued potential defined on a bounded planar domain $D$. We give an exact global reconstruction method for finding $v$ from the associated Dirichlet-to-Neumann operator. This also yields a global uniqueness results: if two smooth matrix-valued potentials defined on a bounded planar domain have the same Dirichlet-to-Neumann operator then they coincide.

## 1. Introduction

Let $D$ be an open bounded domain in $\mathbb{R}^2$ with with $C^2$ boundary and let $v \in C^1(\bar{D}, M_n(\mathbb{C}))$, where $M_n(\mathbb{C})$ is the set of the $n \times n$ complex-valued matrices. The Dirichlet-to-Neumann map associated to $v$ is the operator $\Phi : C^1(\partial D, M_n(\mathbb{C})) \to L^p(\partial D, M_n(\mathbb{C})),\ p < \infty$ defined by:

$$\Phi(f) = \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial D} \tag{1.1}$$

where $f \in C^1(\partial D, M_n(\mathbb{C}))$, $\nu$ is the outer normal of $\partial D$ and $\psi$ is the $H^1(\bar{D}, M_n(\mathbb{C}))$-solution of the Dirichlet problem

$$-\Delta\psi + v(x)\psi = 0 \ \text{ on } D, \ \ \psi|_{\partial D} = f; \tag{1.2}$$

here we assume that

(1.3) 0 is not a Dirichlet eigenvalue for the operator $-\Delta + v$ in $D$.

Equation (1.2) arises, in particular, in quantum mechanics, acoustics, electrodynamics; formally, it looks like the Schrödinger equation with potential $v$ at zero energy.

In addition, (1.2) comes up as a 2D-approximation for the 3D equation (see section 2).

The following inverse boundary value problem arises from this construction.

**Problem 1.** Given $\Phi$, find $v$.

This problem can be considered as the Gel'fand inverse boundary value problem for the multi-channel 2D Schrödinger equation at zero energy (see [**11**], [**13**]) and can also be seen as a generalization of the 2D Calderón problem for the electrical impedance tomography (see [**8**], [**13**]). In addition, the history of inverse problems for the two-dimensional Schrödinger equation at fixed energy goes back to [**9**] (see also [**14**], [**12**] and references therein). Note also that Problem 1 can be considered as a model problem for the monochromatic ocean tomography (e.g. see [**3**] for similar problems arising in this tomography).

In the case of complex-valued potentials the global injectivity of the map $v \to \Phi$ was firstly proved in [**13**] for $D \subset \mathbb{R}^d$ with $d \geq 3$ and in [**6**] for $d = 2$ with $v \in L^p$: in particular, these results were obtained by the use of global reconstructions developed in the same papers.

This is the first paper which gives global (uniqueness and reconstruction) results for Problem 1 with $M_n(\mathbb{C})$-valued potentials with $n \geq 2$. Results in this direction were only known for potentials with many restrictions (e.g. see [**19**]).

We emphasize that Problem 1 is not overdetermined, in the sense that we consider the reconstruction of a $M_n(\mathbb{C})$-valued function $v(x)$ of two variables, $x \in D \subset \mathbb{R}^2$, from a $M_n(\mathbb{C})$-valued function $\Phi(\theta, \theta')$ of two variables, $(\theta, \theta') \in \partial D \times \partial D$, where $\Phi(\theta, \theta')$ is the Schwartz kernel of the Dirichlet-to-Neumann operator $\Phi$: this is one of the principal differences between Problem 1 and its analogue for $D \subset \mathbb{R}^d$ with $d \geq 3$. At present, very few global results are proved for non-overdetermined inverse problems for the Schrödinger equation on $D \subset \mathbb{R}^d$ with $d \geq 2$. Concerning these results, our paper develops the two-dimensional works [**6**], [**17**] and indicates 3D applications of the method. The non-overdetermined inverse problems, including multi-channel ones, are much more developed for the Schrödinger equation in dimension $d = 1$ (e.g. see [**1**], [**20**]).

We recall that in global results one does not assume that the potential $v$ is small in some sense or is (piecewise) real analytic or is subject to some other serious restrictions.

Our global reconstruction procedure for Problem 1 follows the same scheme as in the scalar case given in [**13**], with some fundamental modifications inspired by [**6**].

Let us identify $\mathbb{R}^2$ with $\mathbb{C}$ and use the coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, where $(x_1, x_2) \in \mathbb{R}^2$. We define a special family of solutions of equation (1.2), which we call the Buckhgeim analogues of the Faddeev solutions: $\psi_{z_0}(z, \lambda)$, for $z, z_0 \in \bar{D}, \lambda \in \mathbb{C}$, such that $-\Delta\psi + v(x)\psi = 0$ over $D$, where in particular $\psi_{z_0}(z, \lambda) \to e^{\lambda(z-z_0)^2} I$ for $\lambda \to \infty$ (i.e. for $|\lambda| \to +\infty$) and $I$ is the identity matrix.

More precisely, for a matrix valued potential $v$ of size $n$, we define $\psi_{z_0}(z, \lambda)$ as

$$\psi_{z_0}(z, \lambda) = e^{\lambda(z-z_0)^2} \mu_{z_0}(z, \lambda), \tag{1.4}$$

where $\mu_{z_0}(\cdot, \lambda)$ solves the integral equation

$$\mu_{z_0}(z, \lambda) = I + \int_D g_{z_0}(z, \zeta, \lambda) v(\zeta) \mu_{z_0}(\zeta, \lambda) d\mathrm{Re}\zeta \, d\mathrm{Im}\zeta, \tag{1.5}$$

$I$ is the identity matrix of size $n \in \mathbb{N}$, $z, z_0 \in \bar{D}$, $\lambda \in \mathbb{C}$ and

$$g_{z_0}(z,\zeta,\lambda) = \frac{e^{\lambda(\zeta-z_0)^2-\bar{\lambda}(\bar{\zeta}-\bar{z}_0)^2}}{4\pi^2}\int_D \frac{e^{-\lambda(\eta-z_0)^2+\bar{\lambda}(\bar{\eta}-\bar{z}_0)^2}}{(z-\eta)(\bar{\eta}-\bar{\zeta})} d\mathrm{Re}\eta\, d\mathrm{Im}\eta \tag{1.6}$$

is a Green function of the operator $4\left(\frac{\partial}{\partial z}+2\lambda(z-z_0)\right)\frac{\partial}{\partial \bar{z}}$ in $D$, for $z_0 \in D$. We consider equation (1.5), at fixed $z_0$ and $\lambda$, as a linear integral equation for $\mu_{z_0}(\cdot,\lambda) \in C^1_{\bar{z}}(\bar{D})$: we will see that it is uniquely solvable for $|\lambda| > \rho_1(D, N_1, n)$, where $\|v\|_{C^1_{\bar{z}}(\bar{D},M_n(\mathbb{C}))} < N_1$ (see Proposition 1.3).

In order to state the reconstruction method we also define the Bukhgeim analogue of the Faddeev generalized scattering amplitude

$$h_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2-\bar{\lambda}(\bar{z}-\bar{z}_0)^2} v(z)\mu_{z_0}(z,\lambda) d\mathrm{Re}z\, d\mathrm{Im}z, \tag{1.7}$$

for $z_0 \in \bar{D}$, $\lambda \in \mathbb{C}$.

Theorem 1.1. *Let $D \subset \mathbb{R}^2$ be an open bounded domain with $C^2$ boundary and let $v \in C^1(\bar{D}, M_n(\mathbb{C}))$ be a matrix-valued potential which satisfies* (1.3) *and $v|_{\partial D} = 0$. Consider, for $z_0 \in D$, the functions $h_{z_0}$, $\psi_{z_0}$, $g_{z_0}$ defined above and $\Phi, \Phi_0$ the Dirichlet-to-Neumann maps associated to the potentials $v$ and* 0*, respectively. Then the following reconstruction formulas and equation hold:*

$$v(z_0) = \lim_{\lambda\to\infty} \frac{2}{\pi}|\lambda| h_{z_0}(\lambda), \tag{1.8}$$

$$h_{z_0}(\lambda) = \int_{\partial D} e^{-\bar{\lambda}(\bar{z}-\bar{z}_0)^2}(\Phi-\Phi_0)\psi_{z_0}(z,\lambda)|dz|, \tag{1.9}$$

$$\psi_{z_0}(z,\lambda)|_{\partial D} = e^{\lambda(z-z_0)^2} I + \int_{\partial D} G_{z_0}(z,\zeta,\lambda)(\Phi-\Phi_0)\psi_{z_0}(\zeta,\lambda)|d\zeta|, \tag{1.10}$$

*where*

$$G_{z_0}(z,\zeta,\lambda) = e^{\lambda(z-z_0)^2} g_{z_0}(z,\zeta,\lambda) e^{-\lambda(\zeta-z_0)^2}, \tag{1.11}$$

*$z_0 \in D$, $z, \zeta \in \partial D$, $\lambda \in \mathbb{C}$, $|\lambda| > \rho_1(D, N_1, n)$, where $\|v\|_{C^1_{\bar{z}}(\bar{D},M_n(\mathbb{C}))} < N_1$.*

*In addition, if $v \in C^2(\bar{D}, M_n(\mathbb{C}))$ with $\|v\|_{C^2(\bar{D},M_n(\mathbb{C}))} < N_2$ and $\frac{\partial v}{\partial \nu}|_{\partial D} = v|_{\partial D} = 0$ then the following estimates hold:*

$$\left| v(z_0) - \frac{2}{\pi}|\lambda| h_{z_0}(\lambda)\right| \le a(D,n)\frac{\log(3|\lambda|)}{|\lambda|^{1/2}} N_2(N_2+1), \tag{1.12a}$$

$$\left| v(z_0) - \frac{2}{\pi}|\lambda| h_{z_0}(\lambda)\right| \le b(D,n)\frac{(\log(3|\lambda|))^2}{|\lambda|^{3/4}} N_2(N_2^2+1), \tag{1.12b}$$

*for $|\lambda| > \rho_2(D, N_1, n)$, $z_0 \in D$.*

REMARK 1.1. Note that in Theorem 1.1, $\rho_j = \rho_j(D, N_1, n)$, $j = 1, 2$ (where $\|v\|_{C^1_{\bar z}(\bar D, M_n(\mathbb{C}))} < N_1$), are arbitrary fixed positive constants such that

$$
\begin{aligned}
&2n\frac{c_2(D)}{|\lambda|^{\frac{1}{2}}}\|v\|_{C^1_{\bar z}(\bar D)} < 1,\ |\lambda| \geq 1,\ \text{if } |\lambda| > \rho_1,\\
&2n\frac{c_2(D)}{|\lambda|^{\frac{1}{2}}}\|v\|_{C^1_{\bar z}(\bar D)} \leq \frac{1}{2},\ |\lambda| \geq 1,\ \text{if } |\lambda| > \rho_2,
\end{aligned}
\tag{1.13}
$$

where $c_2$ is the constant in Lemma 3.1.

REMARK 1.2. Note that estimate (1.12b) is not strictly stronger than (1.12a) because of the presence of the $N_2^3$ factor.

In order to make use of the reconstruction given by Theorem 1.1, the following two propositions are necessary:

PROPOSITION 1.2. *Under the assumptions of Theorem 1.1 (without the additional assumptions used for* (1.12)*), equation* (1.10) *is a Fredholm linear integral equation of the second kind for $\psi_{z_0} \in C(\partial D)$.*

PROPOSITION 1.3. *Under the assumptions of Theorem 1.1 (without the additional assumptions used for* (1.12)*), for $|\lambda| > \rho_1(D, N_1, n)$, where $\|v\|_{C^1_{\bar z}(\bar D, M_n(\mathbb{C}))} < N_1$, equations* (1.5) *and* (1.10) *are uniquely solvable in the spaces of continuous functions on $\bar D$ and $\partial D$, respectively.*

REMARK 1.3. Note that the assumption that $v|_{\partial D} = 0$ is unnecessary for formula (1.9), equation (1.10) and Propositions 1.2, 1.3. In addition, formula (1.8) also holds without this assumption if

$$
\int_{\partial D} e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} w(z)|dz| \to 0 \qquad \text{as } |\lambda| \to \infty,
\tag{1.14}
$$

for fixed $z_0 \in D$ and each $w \in C^1(\partial D)$. The class of domains $D$ for which (1.14) holds for each $z_0 \in D$ is large and includes, for example, all ellipses.

Note also that if $v|_{\partial D} \neq 0$ but $v \equiv \Lambda \in M_n(\mathbb{C})$ on some open neighborhood of $\partial D$ in $\bar D$, then estimates (1.12) hold with $h_{z_0}(\lambda)$ replaced by

$$
h^+_{z_0}(\lambda) = h_{z_0}(\lambda) + \int_{\mathbb{R}^2\setminus D} e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} \Lambda \chi(z) d\mathrm{Re} z\, d\mathrm{Im} z,
\tag{1.15}
$$

where $\chi \in C^2(\mathbb{R}^2, \mathbb{R})$, $\chi \equiv 1$ on $D$, $\mathrm{supp}\chi$ is compact, and with the constants $a, b$ depending also on $\chi$. The aforementioned matrix $\Lambda$, for example, can be related with a diagonal matrix composed by the eigenvalues $\{\lambda_i\}_{1\leq i\leq n}$ arising in section 2.

Theorem 1.1 and Propositions 1.2, 1.3 yield the following corollary:

COROLLARY 1.4. *Let $D \subset \mathbb{R}^2$ be an open bounded domain with $C^2$ boundary, let $v_1, v_2 \in C^1(\bar D, M_n(\mathbb{C}))$ be two matrix-valued potentials which satisfy* (1.3) *and $\Phi_1, \Phi_2$ the corresponding Dirichlet-to-Neumann operators. If $\Phi_1 = \Phi_2$ then $v_1 = v_2$.*

Theorem 1.1, Propositions 1.2, 1.3 and Corollary 1.4 are proved in section 4.

The global reconstruction of Theorem 1.1 is fine in the sense that is consists in solving Fredholm linear integral equations of the second type and using explicit formulas; nevertheless this reconstruction is not optimal with respect to its stability properties: see [**7**], [**16**], [**5**] for discussions and numerical implementations of the aforementioned similar (but overdetermined) reconstruction of [**13**] for $d = 3$ and $n = 1$. An approximate but more stable reconstruction method for Problem 1 will be published in another paper.

The present paper is focused on global uniqueness and reconstruction for Problem 1 for $n \geq 2$. In addition, using the techniques developed in the present work and following the scheme of [**17**] it is also possible to obtain a global logarithmic stability estimate for Problem 1 in the multi-channel case. Following inverse problem traditions (e.g. see [**2**], [**16**], [**17**]) this result will be published in another paper.

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## 2. Approximation of the 3D equation

In this section we recall how the multi-channel two-dimensional Schrödinger equation can be seen as an approximation of the scalar 3D equation in a cylindrical domain; in this framework, three-dimensional inverse problems can be approximated by two-dimensional ones.

Let $L = [a, b]$ for some $a, b \in \mathbb{R}$ and consider the complex-valued potential $v(x, z)$ defined on the set $D \times L$, where $x = (x_1, x_2) \in D \subset \mathbb{R}^2$, $z \in L$. We consider the equation

$$-\Delta\psi(x, z) + v(x, z)\psi(x, z) = 0 \quad \text{in } D \times L. \tag{2.1}$$

Now, for every $x \in D$ we can write $\psi(x, z) = \sum_{j=1}^{\infty} \psi_j(x)\phi_j(z)$, where $\{\phi_j\}$ is the orthonormal basis of $L^2(L)$ given by the eigenfunctions of $-\frac{d^2}{dz^2}$: more precisely

$$-\frac{d^2}{dz^2}\phi_j(z) = \lambda_j\phi_j(z) \text{ for } z \in L, \tag{2.2}$$

$$\phi_j|_{\partial L} = 0 \quad \text{(for example)} \tag{2.3}$$

$$\int_L \bar{\phi}_i(z)\phi_j(z)dz = \delta_{ij}$$

and $\psi_j(x) = \int_L \psi(x, z)\bar{\phi}_j(z)dz$. Now equation (2.1) reads

$$\sum_{j=1}^{\infty} (-\Delta_x\psi_j(x)\phi_j(z) - \psi_j(x)\Delta_z\phi_j(z)) + v(x, z)\sum_{j=1}^{\infty} \psi_j(x)\phi_j(z) = 0. \tag{2.4}$$

Using (2.2)-(2.4) and the properties of $\{\phi_j(z)\}$, we obtain that equation (2.1) is equivalent to the following infinite-dimensional system

$$-\Delta_x\psi_i(x)+\lambda_i\psi_i(x)+\sum_{j=1}^{\infty}V_{ij}(x)\psi_j(x)=0,\ \text{for } i=1,\dots, \tag{2.5}$$

where

$$V_{ij}(x)=\int_L\bar{\phi}_i(z)v(x,z)\phi_j(z)dz.$$

Notice that if $\bar{v}=v$ then $V^*=V$. Now, if we impose $1\le i,j\le n$ for some $n\in\mathbb{N}$, we find equation (1.2).

We also give here the relation between the Dirichlet-to-Neumann (D-t-N) operators of the 3D equation and that of the 2D multi-channel equation. If $\Phi(\theta,z,\theta',z')$ is the Schwartz kernel of the D-t-N operator of the 3D problem, and $(\Phi_{ij}(\theta,\theta'))_{i,j\ge1}$ that of the 2D infinity-channel problem, we have

$$\Phi_{ij}(\theta,\theta')=\int_{L\times L}\Phi(\theta,z,\theta',z')\bar{\phi}_i(z)\phi_j(z')dz\,dz', \tag{2.6}$$

where $\theta,\theta'\in\partial D,\ z,z'\in L$. This follows from

$$\int_{\partial D\times L}\Phi(\theta,z,\theta',z')f(\theta',z')d\theta'\,dz'=\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\int_{\partial D}\Phi_{ij}(\theta,\theta')f_j(\theta')d\theta'\right)\phi_i(z), \tag{2.7}$$

for every $f\in C^1(\partial(D\times L))$ such that $f|_{D\times\partial L}=0$ and $f(\theta,z)=\sum_{j=1}^{\infty}f_j(\theta)\phi_j(z)$.

Let us remark that reductions of 3D direct and inverse problems to multi-channel 2D problems are well known in the physical literature for a long time (e.g. see **[3]**). Nevertheless, we do not know a reference containing formula (2.6) in its precise form.

## 3. Preliminaries

In this section we introduce and give details about the above-mentioned family of solutions of equation (1.2), which will be used throughout all the paper.

Let us define the function spaces $C^1_{\bar{z}}(\bar{D})=\{u\,:\,u,\frac{\partial u}{\partial\bar{z}}\in C(\bar{D},M_n(\mathbb{C}))\}$ with the norm $\|u\|_{C^1_{\bar{z}}(\bar{D})}=\max(\|u\|_{C(\bar{D})},\|\frac{\partial u}{\partial\bar{z}}\|_{C(\bar{D})})$, $\|u\|_{C(\bar{D})}=\sup_{z\in\bar{D}}|u|$ and $|u|=\max_{1\le i,j\le n}|u_{i,j}|$; we define also $C^1_z(\bar{D})=\{u\,:\,u,\frac{\partial u}{\partial z}\in C(\bar{D},M_n(\mathbb{C}))\}$ with an analogous norm.

The functions $G_{z_0}(z,\zeta,\lambda),\ g_{z_0}(z,\zeta,\lambda),\ \psi_{z_0}(z,\lambda),\ \mu_{z_0}(z,\lambda)$ defined in Section 1, satisfy

$$4\frac{\partial^2}{\partial z\partial\bar{z}}G_{z_0}(z,\zeta,\lambda)=\delta(z-\zeta), \tag{3.1}$$

$$4\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}G_{z_0}(z,\zeta,\lambda)=\delta(\zeta-z), \tag{3.2}$$

$$4\left(\frac{\partial}{\partial z}+2\lambda(z-z_0)\right)\frac{\partial}{\partial\bar{z}}g_{z_0}(z,\zeta,\lambda)=\delta(z-\zeta), \tag{3.3}$$

$$4\frac{\partial}{\partial\bar{\zeta}}\left(\frac{\partial}{\partial\zeta}-2\lambda(\zeta-z_0)\right)g_{z_0}(z,\zeta,\lambda)=\delta(\zeta-z), \tag{3.4}$$

$$-4\frac{\partial^2}{\partial z\partial\bar{z}}\psi_{z_0}(z,\lambda)+v(z)\psi_{z_0}(z,\lambda)=0, \tag{3.5}$$

$$-4\left(\frac{\partial}{\partial z}+2\lambda(z-z_0)\right)\frac{\partial}{\partial\bar{z}}\mu_{z_0}(z,\lambda)+v(z)\mu_{z_0}(z,\lambda)=0, \tag{3.6}$$

where $z, z_0, \zeta \in D$, $\lambda \in \mathbb{C}$, $\delta$ is the Dirac's delta. (In addition, it is assumed that (1.5) is uniquely solvable for $\mu_{z_0}(\cdot,\lambda) \in C^1_{\bar{z}}(\bar{D})$ at fixed $z_0$ and $\lambda$.) Formulas (3.1)-(3.6) follow from (1.5), (1.6), (1.11) and from

$$\frac{\partial}{\partial\bar{z}}\frac{1}{\pi(z-\zeta)}=\delta(z-\zeta),$$
$$\left(\frac{\partial}{\partial z}+2\lambda(z-z_0)\right)\frac{e^{-\lambda(z-z_0)^2+\bar{\lambda}(\bar{z}-\bar{z}_0)^2}}{\pi(\bar{z}-\bar{\zeta})}e^{\lambda(\zeta-z_0)^2-\bar{\lambda}(\bar{\zeta}-\bar{z}_0)^2}=\delta(z-\zeta),$$

where $z, \zeta, z_0, \lambda \in \mathbb{C}$.

We say that the functions $G_{z_0}$, $g_{z_0}$, $\psi_{z_0}$, $\mu_{z_0}$, $h_{z_0}$ are the Bukhgeim-type analogues of the Faddeev functions (see [**17**]). We recall that the history of these functions goes back to [**10**] and [**4**].

Now we state some fundamental lemmata. Let

$$g_{z_0,\lambda}u(z)=\int_D g_{z_0}(z,\zeta,\lambda)u(\zeta)d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta,\ z\in\bar{D},\ z_0,\lambda\in\mathbb{C}, \tag{3.7}$$

where $g_{z_0}(z,\zeta,\lambda)$ is defined by (1.6) and $u$ is a test function.

LEMMA 3.1 ([**17**]). *Let $g_{z_0,\lambda}u$ be defined by (3.7). Then, for $z_0, \lambda \in \mathbb{C}$, the following estimates hold:*

$$g_{z_0,\lambda}u \in C^1_{\bar{z}}(\bar{D}), \quad \text{for } u \in C(\bar{D}), \tag{3.8}$$

$$\|g_{z_0,\lambda}u\|_{C^1(\bar{D})} \le c_1(D,\lambda)\|u\|_{C(\bar{D})}, \quad \text{for } u \in C(\bar{D}), \tag{3.9}$$

$$\|g_{z_0,\lambda}u\|_{C^1_{\bar{z}}(\bar{D})} \le \frac{c_2(D)}{|\lambda|^{\frac{1}{2}}}\|u\|_{C^1_{\bar{z}}(\bar{D})}, \quad \text{for } u \in C^1_{\bar{z}}(\bar{D}),\ |\lambda|\ge 1. \tag{3.10}$$

Given a potential $v \in C^1_{\bar{z}}(\bar{D})$ we define the operator $g_{z_0,\lambda}v$ simply as $(g_{z_0,\lambda}v)u(z) = g_{z_0,\lambda}w(z),\ w = vu$, for a test function $u$. If $u \in C^1_{\bar{z}}(\bar{D})$, by Lemma 3.1 we have that

$g_{z_0,\lambda}v : C^1_{\bar z}(\bar D) \to C^1_{\bar z}(\bar D)$,

$$\|g_{z_0,\lambda}v\|^{op}_{C^1_{\bar z}(\bar D)} \le 2n\|g_{z_0,\lambda}\|^{op}_{C^1_{\bar z}(\bar D)}\|v\|_{C^1_{\bar z}(\bar D)}, \tag{3.11}$$

where $\|\cdot\|^{op}_{C^1_{\bar z}(\bar D)}$ denotes the operator norm in $C^1_{\bar z}(\bar D)$, $z_0, \lambda \in \mathbb{C}$. In addition, $\|g_{z_0,\lambda}\|^{op}_{C^1_{\bar z}(\bar D)}$ is estimated in Lemma 3.1. Inequality (3.11) and Lemma 3.1 implies existence and uniqueness of $\mu_{z_0}(z,\lambda)$ (and thus also $\psi_{z_0}(z,\lambda)$) for $|\lambda| > \rho_1(D, N_1, n)$.

Let

$$\mu^{(k)}_{z_0}(z,\lambda) = \sum_{j=0}^{k}(g_{z_0,\lambda}v)^j I,$$
$$h^{(k)}_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}v(z)\mu^{(k)}_{z_0}(z,\lambda)d\mathrm{Re}z\, d\mathrm{Im}z,$$

where $z, z_0 \in D$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N}\cup\{0\}$.

Lemma 3.2 ([**17**]). *For $v \in C^1_{\bar z}(\bar D)$ such that $v|_{\partial D} = 0$ the following formula holds:*

$$v(z_0) = \frac{2}{\pi}\lim_{\lambda\to\infty}|\lambda|h^{(0)}_{z_0}(\lambda), \qquad z_0 \in D. \tag{3.12}$$

*In addition, if $v \in C^2(\bar D)$, $v|_{\partial D} = 0$ and $\frac{\partial v}{\partial \nu}|_{\partial D} = 0$ then*

$$\left|v(z_0) - \frac{2}{\pi}|\lambda|h^{(0)}_{z_0}(\lambda)\right| \le c_3(D,n)\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar D)}, \tag{3.13}$$

*for $z_0 \in D$, $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$.*

Following the proof of [**17**, Lemma 6.2] and assuming (1.14), we have that limit (3.12) is valid without the assumption that $v|_{\partial D} = 0$. In addition, if $v|_{\partial D} \ne 0$ but $v \equiv \Lambda \in M_n(\mathbb{C})$ on some open neighborhood of $\partial D$ in $\bar D$, then estimate (3.13) holds with $h^{(0)}_{z_0}(\lambda)$ replaced by

$$h^{(0),+}_{z_0}(\lambda) = h^{(0)}_{z_0}(\lambda) + \int_{\mathbb{R}^2\setminus D} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\Lambda\chi(z)d\mathrm{Re}z\, d\mathrm{Im}z, \tag{3.14}$$

where $\chi \in C^2(\mathbb{R}^2, \mathbb{R})$, $\chi \equiv 1$ on $D$, supp$\chi$ is compact, and the constant $c_3$ depending also on $\chi$.

Let

$$W_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}w(z)d\mathrm{Re}\, zd\mathrm{Im}\, z, \tag{3.15}$$

where $z_0 \in \bar D$, $\lambda \in \mathbb{C}$ and $w$ is some $M_n(\mathbb{C})$-valued function on $\bar D$. (One can see that $W_{z_0} = h^{(0)}_{z_0}$ for $w = v$.)

LEMMA 3.3 ([**17**]). *For $w \in C^1_{\bar z}(\bar D)$ the following estimate holds:*

$$|W_{z_0}(\lambda)| \le c_4(D)\frac{\log(3|\lambda|)}{|\lambda|}\|w\|_{C^1_{\bar z}(\bar D)},\ z_0 \in \bar D,\ |\lambda| \ge 1. \tag{3.16}$$

LEMMA 3.4. *For $v \in C^1_{\bar z}(\bar D)$ and for $\|g_{z_0,\lambda}v\|^{op}_{C^1_{\bar z}(\bar D)} \le \delta < 1$ we have that*

$$\|\mu_{z_0}(\cdot,\lambda) - \mu^{(k)}_{z_0}(\cdot,\lambda)\|_{C^1_{\bar z}(\bar D)} \le \frac{\delta^{k+1}}{1-\delta}, \tag{3.17}$$

$$|h_{z_0}(\lambda) - h^{(k)}_{z_0}(\lambda)| \le c_5(D,n)\frac{\log(3|\lambda|)}{|\lambda|}\frac{\delta^{k+1}}{1-\delta}\|v\|_{C^1_{\bar z}(\bar D)}, \tag{3.18}$$

*where $z_0 \in D,\ \lambda \in \mathbb{C},\ |\lambda| \ge 1,\ k \in \mathbb{N} \cup \{0\}$.*

The proof of Lemma 3.4 in the scalar case can be found in [**17**]: the generalization to the matrix-valued case is straightforward.

LEMMA 3.5. *The function $g_{z_0}(z,\zeta,\lambda)$ satisfies the following properties:*

$$g_{z_0}(z,\zeta,\lambda) \quad \text{is continuous for } z,\zeta \in \bar D,\ z \ne \zeta,\ z_0 \in D, \tag{3.19}$$

$$|g_{z_0}(z,\zeta,\lambda)| \le c_6(D)|\log|z-\zeta||, \qquad z,\zeta \in \bar D,\ z_0 \in D, \tag{3.20}$$

*where $\lambda \in \mathbb{C}$.*

These properties follow from the definition (1.6) and from classical estimates (see [**18**]).

LEMMA 3.6. *Under the assumptions of Proposition 1.2, the Schwartz kernel $(\Phi - \Phi_0)(z,\zeta)$ of the operator $\Phi - \Phi_0$ satisfies the following properties:*

$$(\Phi - \Phi_0)(z,\zeta) \quad \text{is continuous for } z,\zeta \in \partial D,\ z \ne \zeta, \tag{3.21}$$

$$|(\Phi - \Phi_0)(z,\zeta)| \le c_7(D,v,n)|\log|z-\zeta||, \qquad z,\zeta \in \partial D. \tag{3.22}$$

For a proof of this Lemma in the scalar case we refer to [**68, 75**]: the generalization to the matrix-valued case is straightforward.

## 4. Proofs of Theorem 1.1, Propositions 1.2, 1.3 and Corollary 1.4

We begin with a matrix version of Alessandrini's identity (see [**2**] for the scalar case):

$$\int_{\partial D} u_0(z)(\Phi - \Phi_0)u(z)|dz| = \int_D u_0(z)v(z)u(z)d\mathrm{Re}z\, d\mathrm{Im}z \tag{4.1}$$

for any sufficiently regular $M_n(\mathbb{C})$-valued function $u$ (resp. $u_0$) such that $\Delta u_0 = 0$ (resp. $(-\Delta + v)u = 0$) in $D$. This follows from Stokes's theorem, exactly as in the scalar case.

The general matrix version of Alessandrini's identity (that will not be used)

$$\int_{\partial D} u_1(z)(\Phi_2 - \Phi_1)u_2(z)|dz| = \int_D u_1(z)(v_2(z) - v_1(z))u_2(z)d\mathrm{Re}z\, d\mathrm{Im}z \tag{4.2}$$

for $u_1, u_2 \in C^2(\bar{D}, M_n(\mathbb{C}))$ such that $(-\Delta + v_j)u_j = 0$ in $D$, works if $u_1$ and $v_1$ commute each other (but does not work in general).

Proof of Theorem 1.1. Let us begin with the proof of formulas (1.8) and (1.12): we have indeed

$$\left| v(z_0) - \frac{2}{\pi}|\lambda| h_{z_0}(\lambda) \right| \le \left| v(z_0) - \frac{2}{\pi}|\lambda| h^{(0)}_{z_0}(\lambda) \right| + \frac{2}{\pi}|\lambda||h_{z_0}(\lambda) - h^{(0)}_{z_0}(\lambda)|. \tag{4.3}$$

The first term in the right side goes to zero as $|\lambda| \to \infty$ by Lemma 3.2, while the other by Lemmata 3.1 and 3.4. In addition, for $v \in C^2(\bar{D}, M_n(\mathbb{C}))$ with $\|v\|_{C^2(\bar{D})} < N_2$ and $\frac{\partial v}{\partial \nu}|_{\partial D} = 0$, using (3.10), (3.11), (3.13) and (3.18) we obtain, from (4.3):

$$\begin{aligned}\left| v(z_0) - \frac{2}{\pi}|\lambda| h_{z_0}(\lambda) \right| &\le c_3(D,n)\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar{D})} \\ &\quad + c_5(D,n)\frac{\log(3|\lambda|)}{|\lambda|^{1/2}}\|v\|^2_{C^1_{\bar{z}}(\bar{D})} \\ &\le c_8(D,n)\frac{\log(3|\lambda|)}{|\lambda|^{1/2}}(\|v\|_{C^2(\bar{D})} + \|v\|^2_{C^1_{\bar{z}}(\bar{D})}),\end{aligned}$$

for $\lambda$ such that

$$2n\frac{c_2(D)}{|\lambda|^{\frac{1}{2}}}\|v\|_{C^1_{\bar{z}}(\bar{D})} \le \frac{1}{2},\ |\lambda| \ge 1,$$

which implies (1.12a). In order to prove (1.12b) we will need the following lemma:

Lemma 4.1. *Let $g_{z_0,\lambda}u$ be defined by (3.7), where $u \in C^1_{\bar{z}}(\bar{D})$, $z_0, \lambda \in \mathbb{C}$. Then the following estimate holds:*

$$\|g_{z_0,\lambda}u\|_{C(\bar{D})} \le \eta(D)\frac{\log(3|\lambda|)}{|\lambda|^{\frac{3}{4}}}\|u\|_{C^1_{\bar{z}}(\bar{D})},\ |\lambda| \ge 1. \tag{4.4}$$

*Proof of Lemma 4.1.* As in the proof of [**17**, Lemma 3.1], we can write $g_{z_0,\lambda} = \frac{1}{4}T\bar{T}_{z_0,\lambda}$, for $z_0, \lambda \in \mathbb{C}$, where

$$\begin{aligned} Tu(z) &= -\frac{1}{\pi}\int_D \frac{u(\zeta)}{\zeta - z} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \\ \bar{T}_{z_0,\lambda}u(z) &= -\frac{e^{-\lambda(z-z_0)^2 + \bar{\lambda}(\bar{z}-\bar{z}_0)^2}}{\pi}\int_D \frac{e^{\lambda(\zeta-z_0)^2 - \bar{\lambda}(\bar{\zeta}-\bar{z}_0)^2}}{\bar{\zeta} - \bar{z}}u(\zeta)d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta,\end{aligned}$$

for $z \in \bar{D}$ and $u$ a test function. We have that (see [**17**]):

$$Tw \in C^1_{\bar{z}}(\bar{D}), \tag{4.5}$$

$$\|Tw\|_{C^1_{\bar{z}}(\bar{D})} \le \eta_1(D)\|w\|_{C(\bar{D})}, \text{ where } w \in C(D), \tag{4.6}$$

$$\bar{T}_{z_0,\lambda}u \in C(\bar{D}), \tag{4.7}$$

$$\|\bar{T}_{z_0,\lambda}u\|_{C(\bar{D})} \le \frac{\eta_2(D)}{|\lambda|^{\frac{1}{2}}}\|u\|_{C^1_{\bar{z}}(\bar{D})},\ |\lambda| \ge 1, \tag{4.8}$$

$$\|\bar{T}_{z_0,\lambda}u\|_{C(\bar{D})} \le \frac{\log(3|\lambda|)(1+|z-z_0|)\eta_3(D)}{|\lambda||z-z_0|^2}\|u\|_{C^1_{\bar{z}}(\bar{D})},\ |\lambda| \ge 1, \tag{4.9}$$

where $u \in C^1_{\bar{z}}(\bar{D})$, $z_0, \lambda \in \mathbb{C}$.

Let $z_0 \in D$, $0 < \delta < \frac{1}{2}$ and $B_{z_0,\delta} = \{z \in \mathbb{C} : |z - z_0| < \delta\}$. We have

(4.10)

$$\begin{aligned}
|4\pi g_{z_0,\lambda}u(z)| &= \left|\int_D \frac{\bar{T}_{z_0,\lambda}u(\zeta)}{\zeta - z} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta\right| \\
&\le \int_{B_{z_0,\delta}\cap D} \frac{|\bar{T}_{z_0,\lambda}u(\zeta)|}{|\zeta - z|} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + \int_{D\setminus B_{z_0,\delta}} \frac{|\bar{T}_{z_0,\lambda}u(\zeta)|}{|\zeta - z|} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \\
&\le 2\pi\delta\frac{\eta_2(D)}{|\lambda|^{\frac{1}{2}}}\|u\|_{C^1_{\bar{z}}(\bar{D})} + \frac{\log(3|\lambda|)\eta_4(D)}{|\lambda|\delta}\|u\|_{C^1_{\bar{z}}(\bar{D})},
\end{aligned}$$

where we used the following estimate:

$$\begin{aligned}
&\int_{D\setminus B_{z_0,\delta}} \frac{1}{|\zeta - z||\zeta - z_0|^2} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \\
&\quad = \int_{B_{z,\delta}\cap(D\setminus B_{z_0,\delta})} \frac{1}{|\zeta - z||\zeta - z_0|^2} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \\
&\qquad + \int_{D\setminus(B_{z,\delta}\cup B_{z_0,\delta})} \frac{1}{|\zeta - z||\zeta - z_0|^2} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \\
&\quad \le \frac{2\pi}{\delta} + \int_{D\setminus(B_{z,\delta}\cup B_{z_0,\delta})} \frac{1}{|\zeta - z|^3} + \frac{1}{|\zeta - z_0|^3} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \\
&\quad \le \frac{\eta_5(D)}{\delta}.
\end{aligned}$$

Putting $\delta = \frac{1}{2}|\lambda|^{-\frac{1}{4}}$ in (4.10) we obtain the result. Thus Lemma 4.1 is proved.

We now come back to the proof of (1.12b). Proceeding from (4.3) and Lemma 3.2 we obtain:

$$\left|v(z_0) - \frac{2}{\pi}|\lambda| h_{z_0}(\lambda)\right| \le c_3(D,n)\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar{D})} + \frac{2}{\pi}|\lambda||h_{z_0}(\lambda) - h^{(0)}_{z_0}(\lambda)|, \tag{4.11}$$

for $|\lambda| \geq 1$. In addition, from the definitions of $h^{(k)}$, $\mu^{(k)}$, Lemmata 3.1 and 3.4, we have

$$|h_{z_0}(\lambda) - h^{(0)}_{z_0}(\lambda)| \\ \leq \left| \int_D e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} v(z) g_{z_0,\lambda} v(z) d\mathrm{Re}z \, d\mathrm{Im}z \right| + O\left(\frac{\log(3|\lambda|)}{|\lambda|^2}\right) n^2 \|v\|^3_{C^1_{\bar z}(\bar D)},$$

for $\lambda$ such that $2n\frac{c_2(D)}{|\lambda|^{1/2}}\|v\|_{C^1_{\bar z}(\bar D)} \leq \frac{1}{2}$, $|\lambda| \geq 1$.

Repeating the proof of [**17**, Lemma 3.3] and using also Lemma 4.1, we have, for $0 < \varepsilon \leq 1$,

$$\begin{aligned} &\left| \int_D e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} v(z) g_{z_0,\lambda} v(z) d\mathrm{Re}z \, d\mathrm{Im}z \right| \\ &\leq \int_{D\cap B_{z_0,\varepsilon}} \|v(z) g_{z_0,\lambda} v(z)\|_{C(\bar D)} d\mathrm{Re}z \, d\mathrm{Im}z + \frac{1}{4|\lambda|} \int_{\partial(D\setminus B_{z_0,\varepsilon})} \frac{\|v(z) g_{z_0,\lambda} v(z)\|_{C(\bar D)}}{|\bar z - \bar z_0|} |dz| \\ &\quad + \frac{1}{2|\lambda|} \int_{D\setminus B_{z_0,\varepsilon}} \left| \frac{\partial}{\partial \bar z} \left( \frac{v(z) g_{z_0,\lambda} v(z)}{\bar z - \bar z_0} \right) \right| d\mathrm{Re}z \, d\mathrm{Im}z \\ &\leq \sigma_1(D,n) \|v\|_{C(\bar D)} \|v\|_{C^1_{\bar z}(\bar D)} \frac{\varepsilon^2 \log(3|\lambda|)}{|\lambda|^{3/4}} \\ &\quad + \sigma_2(D,n) \|v\|_{C(\bar D)} \|v\|_{C^1_{\bar z}(\bar D)} \frac{\log(3\varepsilon^{-1}) \log(3|\lambda|)}{|\lambda|^{1+3/4}} \\ &\quad + \frac{1}{8|\lambda|} \left| \int_{D\setminus B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} v(z) \frac{\bar T_{z_0,\lambda} v(z)}{\bar z - \bar z_0} d\mathrm{Re}z \, d\mathrm{Im}z \right|, \quad |\lambda| \geq 1, \end{aligned} \tag{4.12}$$

where we also used integration by parts and the fact that $\frac{\partial}{\partial \bar z} g_{\lambda,z_0} u(z) = \frac{1}{4}\bar T_{z_0,\lambda} u(z)$. The last term in (4.12) can be estimated independently on $\varepsilon$ by

$$\sigma_3(D,n) \frac{\log(3|\lambda|)}{|\lambda|^{1+3/4}} \|v\|_{C(\bar D)} \|v\|_{C^1_{\bar z}(\bar D)} \tag{4.13}$$

using the same argument as in the proof of Lemma 4.1 (see estimate (4.10)). Now putting $\varepsilon = |\lambda|^{-1/2}$ in (4.12) we obtain

$$|\lambda| |h_{z_0}(\lambda) - h^{(0)}_{z_0}(\lambda)| \leq \sigma_4(D,n) \frac{(\log(3|\lambda|))^2}{|\lambda|^{3/4}} \|v\|^2_{C^1_{\bar z}(\bar D)} (\|v\|_{C^1_{\bar z}(\bar D)} + 1),$$

for $|\lambda| > \rho_2(D, N_1, n)$, which, together with (4.11), gives us (1.12b).

The proofs of the other formulas of Theorem 1.1 are based on identity (4.1). As $\mu_{z_0}(z,\lambda) = e^{-\lambda(z-z_0)^2}\psi_{z_0}(z,\lambda)$, we can write the generalized scattering amplitude as

$$h_{z_0}(\lambda) = \int_D e^{-\bar\lambda(\bar z - \bar z_0)^2} v(z) \psi_{z_0}(z,\lambda) d\mathrm{Re}z \, d\mathrm{Im}z.$$

Now identity (4.1) with $u_0(z) = e^{-\bar\lambda(\bar z-\bar z_0)^2}I$ and $u(z) = \psi_{z_0}(z,\lambda)$ reads

$$\int_{\partial D} e^{-\bar\lambda(\bar z-\bar z_0)^2}(\Phi-\Phi_0)\psi_{z_0}(z,\lambda)|dz| = \int_D e^{-\bar\lambda(\bar z-\bar z_0)^2}v(z)\psi_{z_0}(z,\lambda)d\mathrm{Re}z\,d\mathrm{Im}z$$

which gives formula (1.9).

Since $\mu_{z_0}$ is a solution of equation (1.5), $\psi_{z_0}(z,\lambda)$ satisfies the equation

$$\psi_{z_0}(z,\lambda) = e^{\lambda(z-z_0)^2}I + \int_D G_{z_0}(z,\zeta,\lambda)v(\zeta)\psi_{z_0}(\zeta,\lambda)d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta, \tag{4.14}$$

for $z_0, z \in \bar D$, $\lambda\in\mathbb{C}$, $|\lambda| > \rho_1(D,N_1,n)$. Thus again by identity (4.1), with $u_0 = G_{z_0}(z,\zeta,\lambda)I$ and $u(z) = \psi_{z_0}(\zeta,\lambda)$, by (3.2) and (4.14) we obtain, for $z\in\partial D$,

$$\begin{aligned}\int_{\partial D} G_{z_0}(z,\zeta,\lambda)(\Phi-\Phi_0)\psi_{z_0}(\zeta,\lambda)|d\zeta| &= \int_D G_{z_0}(z,\zeta,\lambda)v(\zeta)\psi_{z_0}(\zeta,\lambda)d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta\\ &= \psi_{z_0}(z,\lambda) - e^{\lambda(z-z_0)^2}I.\end{aligned}$$

This finish the proof of Theorem 1.1. □

Proof of Proposition 1.2. By (1.11) we have that $G_{z_0}(z,\zeta,\lambda)$ satisfies the same properties as $g_{z_0}(z,\zeta,\lambda)$ in Lemma 3.5, with the difference that the constant in (3.20) depends also on $\lambda$. This observation, along with Lemma 3.6, implies that the operator $A(\lambda)$ defined as

$$A(\lambda)u(z) = \int_{\partial D} G_{z_0}(z,\zeta,\lambda)(\Phi-\Phi_0)u(\zeta)|d\zeta|, \qquad z\in\partial D,$$

for a test function $u$, is compact on the space of continuous functions on $\partial D$. Thus equation (1.10) is a Fredholm linear integral equation of the second kind in the space of continuous functions on $\partial D$. □

Proof of Proposition 1.3. First we have that equations (1.5) and (1.10) are well defined (i.e. Fredholm linear integral equations of the second type) on the spaces of continuous functions on $\bar D$ and $\partial D$ respectively. This follows from (3.9) for the first equation and from Proposition 1.2 for the second one.

Now if (1.5) admits a solution $\mu_{z_0}(z,\lambda)\in C(\bar D)$, then by (3.8) and (1.5) one readily obtains $\mu_{z_0}(z,\lambda)\in C^1_{\bar z}(\bar D)$. This solution is unique by Lemma 3.1 for $|\lambda| > \rho_1(D,N_1,n)$ and by the same arguments as in the proof of Theorem 1.1 one has that $\psi_{z_0}(z,\lambda)|_{z\in\partial D}$ satisfies equation (1.10).

Conversely, suppose that $\psi_{z_0}(z,\lambda)\in C(\partial D)$ satisfies equation (1.10): we have to show that $\psi_{z_0}(z,\lambda)$, defined on $\bar D$ as the solution of the Dirichlet problem $(-\Delta + v)\psi_{z_0}(z,\lambda) = 0$ with boundary values given by a solution of equation (1.10), satisfies (4.14).

By identity (4.1), $\psi_{z_0}(z,\lambda)$ satisfies already equation (4.14) with $z\in\partial D$. Now, the function

$$\varphi(z) = \psi_{z_0}(z,\lambda) - e^{\lambda(z-z_0)^2}I - \int_D G_{z_0}(z,\zeta,\lambda)v(\zeta)\psi_{z_0}(\zeta,\lambda)d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta \tag{4.15}$$

satisfies $\Delta\varphi = 0$ in $D$ and $\varphi|_{\partial D} = 0$, so $\varphi \equiv 0$ in $D$. Proposition 1.3 is proved. □

PROOF OF COROLLARY 1.4. If $v_j|_{\partial D} = 0$, for $j = 1, 2$, then we can apply Theorem 1.1 and Propositions 1.2, 1.3. As $\Phi_1 = \Phi_2$, then $\psi^1_{z_0}(\cdot, \lambda)|_{\partial D} = \psi^2_{z_0}(\cdot, \lambda)|_{\partial D}$ for $|\lambda| > \rho_1(D, N_1, n)$ (where we called $\psi^j_{z_0}(z, \lambda)$ the Bukhgeim analogues of the Faddeev solutions corresponding to $v_j$, for $j = 1, 2$). Thus we also have equality between the corresponding generalized scattering amplitudes, $h^1_{z_0}(\lambda) = h^2_{z_0}(\lambda)$ for $|\lambda| > \rho_1(D, N_1, n)$, which yields $v_1(z_0) = v_2(z_0)$ for $z_0 \in D$.

If $v_j|_{\partial D} \neq 0$, for $j = 1, 2$, and $D$ is such that (1.14) holds, then by Remark 1.3 we can apply Theorem 1.1 and argue as above.

The general case follows from stability estimates which will be published in another paper, following the scheme of [**17**]. □

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# Paper **F**

**PAPER F**

# Monochromatic reconstruction algorithms for two-dimensional multi-channel inverse problems

Roman G. Novikov and Matteo Santacesaria

Abstract. We consider two inverse problems for the multi-channel two-dimensional Schrödinger equation at fixed positive energy, i.e. the equation $-\Delta\psi + V(x)\psi = E\psi$ at fixed positive $E$, where $V$ is a matrix-valued potential. The first is the Gel'fand inverse problem on a bounded domain $D$ at fixed energy and the second is the inverse fixed-energy scattering problem on the whole plane $\mathbb{R}^2$. We present in this paper two algorithms which give efficient approximate solutions to these problems: in particular, in both cases we show that the potential $V$ is reconstructed with Lipschitz stability by these algorithms up to $O(E^{-(m-2)/2})$ in the uniform norm as $E \to +\infty$, under the assumptions that $V$ is $m$-times differentiable in $L^1$, for $m \geq 3$, and has sufficient boundary decay.

## 1. Introduction

We consider the equation

$$-\Delta\psi + V(x)\psi = E\psi, \qquad x \in \mathbb{R}^2, \qquad E > 0, \tag{1.1}$$

where

$$V \text{ is a sufficiently regular } M_n(\mathbb{C})\text{-valued function on } \mathbb{R}^2 \text{ with sufficient decay at infinity,} \tag{1.2}$$

$M_n(\mathbb{C})$ is the set of the $n \times n$ complex matrices. This equation will also be considered on a domain $D$, where

$$D \text{ is an open bounded domain in } \mathbb{R}^2 \text{ with a } C^2 \text{ boundary.} \tag{1.3}$$

Equation (1.1) at fixed $E$ can be considered as rather general multi-channel Schrödinger (resp. acoustic) equation on $D$ at a fixed energy (resp. frequency) related to $E$. It arises, in particular, as a 2D approximation to the following 3D equation

$$-\Delta_{x,z}\psi + v(x,z)\psi = E\psi, \quad (x,z) \in \Omega = D \times L, \tag{1.4}$$

where $L = [a,b]$, $a, b \in \mathbb{R}$, $v$ is a sufficiently regular complex-valued function on $\Omega$ and $\psi|_{D\times\partial L} = 0$ (for example): see [**23**, Sec. 2]. In this framework, the approximate 2D matrix-valued potential $V$ is given by

$$V_{ij}(x) = \lambda_i \delta_{ij} + \int_L \bar{\phi}_i(z) v(x,z) \phi_j(z) dz, \qquad x \in D, \tag{1.5}$$

for $1 \le i,j \le n$, where $n \in \mathbb{N}$, $\{\phi_j\}_{j\in\mathbb{N}}$ is the orthonormal basis of $L^2(L)$ given by the eigenfunctions of $-\frac{d^2}{dz^2}$ such that $\phi_j|_{\partial L} = 0$, $-\frac{d^2\phi_j}{dz^2} = \lambda_j \phi_j$, for $j \in N$, and $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

In addition, equation (1.1) can be seen as a particular case of the 2D Schrödinger equation in an external Yang-Mills field.

For equation (1.1) on $D$ we consider the Dirichlet-to-Neumann map $\Phi(E)$ such that

$$\Phi(E)(\psi|_{\partial D}) = \left.\frac{\partial \psi}{\partial \nu}\right|_{\partial D} \tag{1.6}$$

for all sufficiently regular solution $\psi$ of (1.1) on $\bar{D} = D \cup \partial D$, where $\nu$ is the outer normal of $\partial D$. Here we assume also that

(1.7) $E$ is not a Dirichlet eigenvalue for the operator $-\Delta + V$ in $D$.

This construction gives rise to the following inverse boundary value problem on $D$:

Problem 4. *Given $\Phi(E)$, find $V$ on $D$.*

On the other hand, for equation (1.1) on $\mathbb{R}^2$, under assumptions (1.2), we consider the scattering amplitude $f$ defined as follows: we consider the continuous solutions $\psi^+(x,k)$ of (1.1), where $k$ is a parameter, $k \in \mathbb{R}^2, k^2 = E$, such that

$$\psi^+(x,k) = e^{ikx} I - i\pi\sqrt{2\pi} e^{-i\frac{\pi}{4}} f\left(k, |k|\frac{x}{|x|}\right) \frac{e^{i|k||x|}}{\sqrt{|k||x|}} + o\left(\frac{1}{\sqrt{|x|}}\right), \qquad \text{as } |x| \to \infty, \tag{1.8}$$

for some *a priori* unknown $M_n(\mathbb{C})$-valued function $f$, where $I$ is the identity matrix. The function $f$ on $\mathscr{M}_E = \{(k,l) \in \mathbb{R}^2 \times \mathbb{R}^2 : k^2 = l^2 = E\}$ arising in (1.8) is the scattering amplitude for the potential $V$ in the framework of equation (1.1).

This construction gives rise to the following inverse scattering problem on $\mathbb{R}^2$:

Problem 5. *Given $f$ on $\mathscr{M}_E$, find $V$ on $\mathbb{R}^2$.*

Problems 4 and 5 can be considered as multi-channel fixed-energy analogues in dimension $d = 2$ of inverse problems formulated in [**10**] in dimension $d \ge 2$. Note that Problems 1 and 2 are not overdetermined, in the sense that we consider the reconstruction of a $M_n(\mathbb{C})$-valued function $V$ of two variables from $M_n(\mathbb{C})$-valued inverse problem data dependent on two variables. In addition, the history of inverse problems for the two-dimensional Schrödinger equation at fixed energy goes back to [**7**] (see also [**17, 11**] and reference therein). Note also that Problem 4 can be considered as a model problem for the monochromatic ocean tomography (e.g. see [**2**] for similar problems arising in this tomography).

As regards efficient algorithms for solving Problems 4 and 5 for the scalar case, i.e. for $n = 1$, see [**16, 17, 18, 19**]. In addition, as concerns numerical implementations of these algorithms for Problem 5 for $n = 1$, see [**4, 6**], and references therein.

Nevertheless, the fixed-energy global uniqueness for Problem 4 (and for Problem 5 with compactly supported $V$) for $n = 1$ was completely proved only recently in [**5**]. The reconstruction scheme of [**5**] is not optimal with respect to its stability properties, and, therefore, is not efficient numerically in comparison with the aforementioned 2D reconstructions of [**16, 17, 18, 19**], but it is very efficient for proving some global mathematical results. In particular: a related global logarithmic stability estimate for Problem 4 for $n = 1$ was proved in [**22**]; global uniqueness and reconstruction results for Problem 4 for $n \geq 2$ were obtained in [**23**]; a global logarithmic stability estimate for Problem 4 for $n \geq 2$ was proved in [**24**]. In addition, Problem 5 with compactly supported $V$ can be reduced, for $n \geq 2$, to Problem 4, as in [**16**] for $n = 1$. This implies, at least, global uniqueness for Problem 5 (in the compactly supported case). On the other hand, the uniqueness for Problem 5 fails already for scalar ($n = 1$) real-valued spherically-symmetric potentials $V$ of the Schwartz class on $\mathbb{R}^2$ (see [**12**]).

The main purpose of the present work consists in generalizing the aforementioned reconstruction approach of [**18, 19**] to the case of Problems 4 and 5 for $n \geq 2$. As well as for $n = 1$ this functional analytic approach gives an efficient non-linear approximation $V_{appr}(x, E)$ to the unknown $V(x)$ of Problems 4 and 5. The reconstruction of $V_{appr}(x, E)$ from $\Phi(E)$ for Problem 4 and from $f$ on $\mathscr{M}_E$ for Problem 5 is realized with some Lipschitz stability and is based on solving linear integral equations; see Algorithms 1 and 2 of Section 3, Theorems 6.1, 6.2 and Remarks 6.2, 6.3 of Section 6. Among these linear integral equations, the most important ones arise from a non-local Riemann-Hilbert problem. For the scalar case, Riemann-Hilbert problems of such a type go back to [**15**]. Another important part of these equations is used for transforming $\Phi(E)$ for Problem 4 and $f$ on $\mathscr{M}_E$ for Problem 5 into $M_n(\mathbb{C})$-valued Faddeev function analogues $h_\pm$ on $\mathscr{M}_E$, involved in the formulation of the above-mentioned Riemann-Hilbert problem. In addition,

$$\|V_{appr}(\cdot, E) - V\| = \varepsilon(E)$$

rapidly decays as $E \to +\infty$, where $\|\cdot\|$ denotes an appropriate norm. In particular, $\varepsilon(E) = O(E^{-\infty})$ as $E \to +\infty$ if $\|\cdot\|$ is specified as $\|\cdot\|_{L^\infty(D)}$ and $V \in C^\infty(\mathbb{R}^2, M_n(\mathbb{C}))$, $\operatorname{supp} V \subset D$, for Problem 4 and if $\|\cdot\|$ is specified as $\|\cdot\|_{L^\infty(\mathbb{R}^2)}$ and $V \in \mathcal{S}(\mathbb{R}^2, M_n(\mathbb{C}))$, for Problem 5, where $\mathcal{S}$ denotes the Schwartz class. In addition, no reconstruction algorithms for Problems 1 and 2 — comparable, with respect to their stability, with Algorithms 1 and 2 and with an approximation error decaying more rapidly than $O(E^{-\frac{1}{2}})$ as $E \to \infty$ — are available in the preceding literature, even for $V \in C^\infty(\mathbb{R}^2, M_n(\mathbb{C}))$, $\mathrm{supp} V \subset D \subset \mathbb{R}^2$, when $n \geq 2$ (in general).

In spite of the fact that some excellent properties of Algorithms 1 and 2 are proved assuming that $V$ is sufficiently smooth and that $E$ is sufficiently great in comparison with (some norm of) $V$, we expect that these algorithms will work rather well even for $V$ with discontinuities and for the case when $E$ is not very big in comparison with $V$. This expectation is based on numerical results for Algorithm 2 for the case $n = 1$; see [**6**] and references therein. Numerical implementations of Algorithm 1 for $n \geq 1$ and Algorithm 2 for $n \geq 2$ are in preparation.

Let us emphasize that in the present work we also develop studies of [**23**] on the 2D multi-channel approach to 3D monochromatic inverse problems for equation (1.4). In this connection, the principal advantage of the 2D multi-channel Algorithm 1 (see section 3) in comparison with the 3D algorithm of [**21**] is that Algorithm 1 deals with non-overdetermined data and is only based on linear integral equations. High energy error estimates for both cases are similar. However, properties of Algorithm 1 of the present work are not estimated yet with respect to the approximation level $n$ in the framework of 3D applications.

Finally, note that multi-channel inverse problems and their applications to inverse problems in greater dimensions were initially considered for the one-dimensional multi-channel case, see [**1**], [**26**]. As one of the most recent result in this direction see [**14**].

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## 2. Faddeev functions

In this section we recall some preliminary definitions.

Under assumptions (1.2), we consider the Faddeev functions $G(x,k) = e^{ikx}g(x,k)$, $\psi(x,k)$, $h(k,l)$ and related function $R(x,y,k)$ (see [**8, 9, 16, 20**] for $n = 1$):

$$g(x,k) = -\left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{R}^2} \frac{e^{i\xi x}}{\xi^2 + 2k\xi} d\xi, \tag{2.1}$$

$$\psi(x,k) = e^{ikx} I + \int_{\mathbb{R}^2} G(x-y,k)V(y)\psi(y,k)dy, \tag{2.2}$$

$$h(k,l) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{R}^2} e^{-ilx} V(x)\psi(x,k)dx, \tag{2.3}$$

$$R(x,y,k) = G(x-y,k) + \int_{\mathbb{R}^2} G(x-\xi,k)V(\xi)R(\xi,y,k)d\xi \tag{2.4}$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, $k = (k_1, k_2) \in \mathbb{C}^2 \setminus \mathbb{R}^2$, $l = (l_1, l_2) \in \mathbb{C}^2$, $\mathrm{Im}k = \mathrm{Im}l \neq 0$ and $I$ is the identity matrix. We recall that

$$(\Delta + k^2)G(x,k) = \delta(x), \tag{2.5}$$

for $x \in \mathbb{R}^2$, $k \in \mathbb{C}^2 \setminus \mathbb{R}^2$, where $\delta$ is the Dirac delta. In addition: formula (2.2) at fixed $k$ is considered as an equation for

$$\psi(x,k) = e^{ikx}\mu(x,k), \tag{2.6}$$

where $\mu$ is sought in $L^\infty(\mathbb{R}^2, M_n(\mathbb{C}))$; formula (2.4) at fixed $k$ and $y$ is considered as an equation for

$$R(x,y,k) = e^{ik(x-y)}r(x,y,k), \tag{2.7}$$

where $r$ is sought in $L^2_{\mathrm{loc}}(\mathbb{R}^2, M_n(\mathbb{C}))$, with the property that $|r(x,y,k)| \to 0$ as $|x| \to \infty$. As a corollary of (2.1), (2.2) and (2.5), $\psi$ satisfies (1.1) for $E = k^2 = k_1^2 + k_2^2$ and

$$(\Delta + k^2 - V(x))R(x,y,k) = \delta(x-y), \tag{2.8}$$

for $x, y, \in \mathbb{R}^2$, $k \in \mathbb{C} \setminus \mathbb{R}^2$. In addition, $h$ in (2.3) is a generalised *scattering amplitude* in the complex domain for the potential $V$.

For $\gamma \in S^1 = \{\gamma \in \mathbb{R}^2 : |\gamma| = 1\}$, we consider

$$G_\gamma(x,k) = G(x, k + i0\gamma), \tag{2.9}$$

$$R_\gamma(x,y,k) = R(x,y,k + i0\gamma), \tag{2.10}$$

$$\psi_\gamma(x,k) = e^{ikx}\mu_\gamma(x,k), \qquad \mu_\gamma(x,k) = \mu(x, k + i0\gamma), \tag{2.11}$$

$$h_\gamma(k,l) = h(k + i0\gamma, l + i0\gamma), \tag{2.12}$$

where $x, y \in \mathbb{R}^2$, $k \in \mathbb{R}^2$, $l \in \mathbb{R}^2$.

In addition, the functions

$$G^+(x,k) = G_{k/|k|}(x,k) = -\frac{i}{4}H_0^1(|x||k|), \tag{2.13}$$

$$R^+(x,y,k) = R_{k/|k|}(x,y,k) \tag{2.14}$$

$$\psi^+(x,k) = e^{ikx}\mu^+(x,k), \qquad \mu^+(x,k) = \mu_{k/|k|}(x,k), \tag{2.15}$$

$$f(k,l) = h_{k/|k|}(k,l), \tag{2.16}$$

for $x, y, k, l \in \mathbb{R}^2$, $|k| = |l|$, are functions from the classical scattering theory; in particular, $f$ is the scattering amplitude of (1.8) and $H_0^1$ is the Hankel function of the first type. We also define

$$h_\pm(k,l) = h_{\pm\hat{k}_\perp}(k,l), \tag{2.17}$$

$$\mu_\pm(x,k) = \mu_{\pm\hat{k}_\perp}(x,k), \qquad \psi_\pm(x,k) = \psi_{\pm\hat{k}_\perp}(x,k), \tag{2.18}$$

$$R_\pm(x,y,k) = R_{\pm\hat{k}_\perp}(x,y,k), \tag{2.19}$$

where $k, l, x, y \in \mathbb{R}^2$, $|k| = |l|$, $\hat{k}_\perp = |k|^{-1}(-k_2, k_1)$ for $k = (k_1, k_2)$. Note that $\mu_+ \neq \mu^+$, $\psi_+ \neq \psi^+$ and $R_+ \neq R^+$ in general. We shall consider, in particular, the following restriction of the function $h$:

$$b(k) = h(k, -\bar{k}), \qquad \text{for } k \in \mathbb{C}^2,\ k^2 = E > 0. \tag{2.20}$$

We now introduce the notations

$$\begin{aligned} &z = x_1 + ix_2, && \bar{z} = x_1 - ix_2, \\ &\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), && \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right), \\ &\lambda = E^{-1/2}(k_1 + ik_2), && \lambda' = E^{-1/2}(l_1 + il_2), \end{aligned} \tag{2.21}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $k = (k_1, k_2), l = (l_1, l_2) \in \mathbb{C}^2$, $k^2 = l^2 = E \in \mathbb{R}_+$. In the new notations

$$k_1 = \frac{1}{2}E^{1/2}(\lambda + \lambda^{-1}), \qquad k_2 = \frac{i}{2}E^{1/2}(\lambda^{-1} - \lambda), \tag{2.22a}$$

$$l_1 = \frac{1}{2}E^{1/2}(\lambda' + \lambda'^{-1}), \qquad l_2 = \frac{i}{2}E^{1/2}(\lambda'^{-1} - \lambda'), \tag{2.22b}$$

$$\exp(ikx) = \exp[\frac{i}{2}E^{1/2}(\lambda\bar{z} + \lambda^{-1}z)], \tag{2.22c}$$

where $\lambda, \lambda' \in \mathbb{C} \setminus \{0\}$, $z \in \mathbb{C}$ and the Schrödinger equation (1.1) takes the form

$$-4\frac{\partial^2}{\partial z \partial \bar{z}}\psi + V(z)\psi = E\psi, \qquad z \in \mathbb{C}. \tag{2.23}$$

In addition, the functions $f$ from (1.8) and (2.16), $h_\pm$ from (2.17), $\mu^+, \psi^+$ from (2.15), $\mu_\pm, \psi_\pm$ from (2.18), $\psi$ from (2.2), $\mu$ from (2.6) and $b$ from (2.20) take the form

$$\begin{aligned} f &= f(\lambda, \lambda', E), & h_\pm &= h_\pm(\lambda, \lambda', E), \\ \mu^+ &= \mu^+(z, \lambda, E), & \psi^+ &= \psi^+(z, \lambda, E), \\ \mu_\pm &= \mu_\pm(z, \lambda, E), & \psi_\pm &= \psi_\pm(z, \lambda, E), \end{aligned} \tag{2.24}$$

where $\lambda, \lambda' \in T,\ z \in \mathbb{C},\ E \in \mathbb{R}_+$,

$$\mu = \mu(z, \lambda, E), \qquad \psi = \psi(z, \lambda, E), \qquad b = b(\lambda, E), \tag{2.25}$$

where $\lambda \in \mathbb{C} \setminus T,\ z \in \mathbb{C}, E \in \mathbb{R}_+$. Here

$$T = \{\zeta \ :\ \zeta \in \mathbb{C}, |\zeta| = 1\}. \tag{2.26}$$

Under assumption (1.2), for $E$ sufficiently large the function $\mu(z,\lambda,E)$ has the following properties (see [**18, 19**] for $n=1$ and Section 4 for $n\geq 2$):

$$\mu(z,\lambda,E) \text{ is continuous in } \lambda\in\mathbb{C}\setminus T; \tag{2.27}$$

$$\mu(z,\lambda(1\mp 0),E)=\mu_\pm(z,\lambda,E) \text{ for } \lambda\in T; \tag{2.28}$$

$$\begin{aligned}\mu_\pm(z,\lambda,E)&=\mu^+(z,\lambda,E)\\ &+\pi i\int_T \mu^+(z,\lambda'',E)\chi_+\left(\pm i\left(\frac{\lambda}{\lambda''}-\frac{\lambda''}{\lambda}\right)\right)h_\pm(\lambda,\lambda'',z,E)|d\lambda''|,\end{aligned} \tag{2.29}$$

for $\lambda\in T$, where

$$\chi_+(s)=0 \text{ for } s<0,\quad \chi_+(s)=1 \text{ for } s\geq 0, \tag{2.30}$$

$$h_\pm(\lambda,\lambda',z,E)=\exp\left[-\frac{i}{2}E^{1/2}\left(\lambda\bar z+\frac{z}{\lambda}-\lambda'\bar z-\frac{z}{\lambda'}\right)\right]h_\pm(\lambda,\lambda',E); \tag{2.31}$$

$$\frac{\partial}{\partial\bar\lambda}\mu(z,\lambda,E)=\mu\left(z,-\frac{1}{\bar\lambda},E\right)r(\lambda,z,E), \tag{2.32}$$

for $\lambda\in\mathbb{C}\setminus T$, where

$$r(\lambda,z,E)=\exp\left[-\frac{i}{2}E^{1/2}\left(\lambda\bar z+\frac{z}{\lambda}+\bar\lambda z+\frac{\bar z}{\bar\lambda}\right)\right]\frac{\pi}{\bar\lambda}\mathrm{sign}(\lambda\bar\lambda-1)b(\lambda,E), \tag{2.33}$$

where $b$ is defined by means of (2.20) and (2.22a);

$$\mu(z,\lambda,E)=I+\frac{\mu_{-1}(z,E)}{\lambda}+o\left(\frac{1}{\lambda}\right),\qquad \lambda\to\infty, \tag{2.34}$$

$$V(z)=2iE^{1/2}\frac{\partial}{\partial z}\mu_{-1}(z,E). \tag{2.35}$$

The following formula is valid (see [**19**] for $n=1$ and Section 4 for $n\geq 2$):

$$\begin{aligned}V(z)=2iE^{1/2}\frac{\partial}{\partial z}\Bigg(&\frac{1}{\pi}\int\limits_{D_-}\mu(z,-\frac{1}{\bar\zeta},E)r(\zeta,z,E)d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta\\ &+\frac{1}{2\pi i}\int\limits_T\mu_-(z,\zeta,E)i\zeta|d\zeta|\Bigg),\end{aligned} \tag{2.36}$$

for $z\in\mathbb{C}$, $E$ sufficiently large and $D_-=\{\zeta\ :\ \zeta\in\mathbb{C},|\zeta|>1\}$.

## 3. Reconstruction algorithms

We present here Algorithms 1 and 2, which yield approximate but sufficiently stable solutions to Problems 1 and 2, respectively. These algorithms have a final common part: the reconstruction of the approximate potential $V_{appr}$ starting from $h_\pm$ of (2.17). Thus, for the sake of clarity, we first give the different initial parts of the algorithms—that is, the reconstruction of $h_\pm$ starting from $\Phi(E)$ for Algorithm 1 and from $f$ for Algorithm 2—and then the final common part.

Note that in both algorithms we consider in particular the functions $\psi_\pm, h_\pm, \mu_-$ of (2.17), (2.18) and $\mu^+$ of (2.15). In addition, in Algorithm 1, in the definitions of these functions we assume that $V \equiv 0$ on $\mathbb{R}^2 \setminus D$.

ALGORITHM 1 ($\Phi(E) \longrightarrow h_\pm$). *Given $\Phi(E)$, for $E$ sufficiently large, we first reconstruct $\psi_\pm(x,k)|_{\partial D}, k \in \mathbb{R}^2, k^2 = E$, with the help of the following Fredholm linear integral equation (see* **[16]** *for $n = 1$ and Section 4 for $n \geq 2$):*

$$\psi_\pm(x,k)|_{\partial D} = e^{ikx} I + \int_{\partial D} A_\pm(x,y,k)\psi_\pm(y,k)dy, \quad k \in \mathbb{R}^2,\ k^2 = E, \tag{3.1}$$

*where*

$$A_\pm(x,y,k) = \int_{\partial D} G_\pm(x-\xi,k)\,(\Phi - \Phi_0)\,(\xi,y,E)d\xi, \quad x,y \in \partial D, \tag{3.2}$$

$$G_\pm(x,k) = G^+(x,k) - \frac{1}{4\pi i}\int_{S^1} e^{i|k|\theta x}\chi_+(\pm\theta k_\perp)d\theta, \tag{3.3}$$

*$I$ is the identity matrix, $(\Phi - \Phi_0)(x,y,E)$ is the Schwartz kernel of the operator $\Phi(E) - \Phi_0(E)$, $\Phi_0(E)$ is the Dirichlet-to-Neumann operator associated to the zero potential in $D$ at fixed energy $E$, $G^+(x,k)$ is defined in (2.13), $k_\perp = (-k_2,k_1)$ for $k = (k_1,k_2)$, $dy$, $d\xi$ denote the standard Euclidean measure on the boundary $\partial D$ and $d\theta$ denotes the standard Euclidean measure on $S^1$.*

*Then, in order to obtain $h_\pm$, it is sufficient to use the following formula (see* **[16]** *for $n = 1$ and Section 4 for $n \geq 2$):*

$$h_\pm(k,l) = \frac{1}{(2\pi)^2}\int_{\partial D}\int_{\partial D} e^{-ilx}(\Phi - \Phi_0)(x,y,E)\psi_\pm(y,k)dydx, \quad (k,l) \in \mathscr{M}_E. \tag{3.4}$$

ALGORITHM 2 ($f \longrightarrow h_\pm$). *Starting from $f$ on $\mathscr{M}_E$ (for $E$ sufficiently large), one directly recovers $h_\pm$ solving the following integral equation (see* **[17, 19]** *for $n = 1$ and Section 4 for $n \geq 2$):*

$$h_\pm(\lambda,\lambda',E) - \pi i \int_T f(\lambda'',\lambda',E)\chi_+\left(\pm i\left(\frac{\lambda}{\lambda''} - \frac{\lambda''}{\lambda}\right)\right) h_\pm(\lambda,\lambda'',E)|d\lambda''| = f(\lambda,\lambda',E), \qquad (\lambda,\lambda') \in T \times T. \tag{3.5}$$

ALGORITHMS 1 AND 2 ($h_\pm \longrightarrow V_{appr}$). *We begin with the construction of $\tilde{\mu}^+$, an approximation to $\mu^+$ of (2.15); this is done by solving the following integral equation arising from the non-local Riemann-Hilbert problem (2.27)-(2.34) for $\mu$ in the approximation that $b \equiv 0$ at fixed $E$ (see* **[19]** *for $n = 1$ and Section 4 for $n \geq 2$):*

$$\tilde{\mu}^+(z,\lambda,E) + \int_T \tilde{\mu}^+(z,\lambda',E)B(\lambda,\lambda',z,E)|d\lambda'| = I,\ \lambda \in T,\ z \in \mathbb{C}, \tag{3.6}$$

*where $E$ is sufficiently large and*

$$B(\lambda,\lambda',z,E)=\frac{1}{2}\int_T h_-(\zeta,\lambda',z,E)\chi_+\left(-i\left(\frac{\zeta}{\lambda'}-\frac{\lambda'}{\zeta}\right)\right)\frac{d\zeta}{\zeta-\lambda(1-0)}$$
$$-\frac{1}{2}\int_T h_+(\zeta,\lambda',z,E)\chi_+\left(i\left(\frac{\zeta}{\lambda'}-\frac{\lambda'}{\zeta}\right)\right)\frac{d\zeta}{\zeta-\lambda(1+0)}, \tag{3.7}$$

*where $\chi_+$, $h_\pm$ are defined in* (2.30), (2.31). *Then one can obtain an approximation $\tilde\mu_-$ to $\mu_-$ via* (2.29)*, used as follows:*

$$\tilde\mu_-(z,\lambda,E)=\tilde\mu^+(z,\lambda,E)$$
$$+\pi i\int_T \tilde\mu^+(z,\lambda'',E)\chi_+\left(-i\left(\frac{\lambda}{\lambda''}-\frac{\lambda''}{\lambda}\right)\right)h_-(\lambda,\lambda'',z,E)|d\lambda''|, \tag{3.8}$$

*for $\lambda\in T$, $z\in\mathbb{C}$. Finally, the approximate potential $V_{appr}(\cdot,E)$ can be obtained using the following formula (see* **[18, 19]** *for $n=1$ and Section 4 for $n\geq 2$):*

$$V_{appr}(z,E)=2iE^{1/2}\frac{\partial}{\partial z}\left(\frac{1}{2\pi i}\int_T \tilde\mu_-(z,\zeta,E)i\zeta|d\zeta|\right). \tag{3.9}$$

The approximate potential $V_{appr}$ depends in a non-linear way on $\Phi(E)$ in Algorithm 1 and on $f$ on $\mathscr{M}_E$ in Algorithm 2, in spite of the fact that both algorithms are based on solving linear integral equation. In the linear approximation near zero potential, the following formulas hold:

$$h_\pm(k,l)\approx\frac{1}{(2\pi)^2}\int_{\partial D}\int_{\partial D}e^{i(-lx+ky)}(\Phi-\Phi_0)(x,y,E)dx\,dy,\quad (k,l)\in\mathscr{M}_E, \tag{3.10}$$

for linearised Algorithm 1;

$$h_\pm(\lambda,\lambda',E)\approx f(\lambda,\lambda',E),\qquad \lambda,\lambda'\in T, \tag{3.11}$$

for linearised Algorithm 2;

$$V_{appr}(z,E)\approx\frac{1}{\pi}E^{1/2}\int_T w(z,\lambda,E)i\lambda|d\lambda|, \tag{3.12}$$

where $z\in D$ for linearised Algorithm 1 and $z\in\mathbb{C}$ for linearised Algorithm 2, and

$$w(z,\lambda,E)=\frac{\partial}{\partial z}\left(\pi i\int_T \exp\left[-\frac{i}{2}E^{1/2}\left(\lambda\bar z+\frac{z}{\lambda}-\lambda'\bar z-\frac{z}{\lambda'}\right)\right]\right.$$
$$\left.\times\,\mathrm{sign}\left(-i\left(\frac{\lambda}{\lambda'}-\frac{\lambda'}{\lambda}\right)\right)h_\pm(\lambda,\lambda',E)|d\lambda'|\right), \tag{3.13}$$

for $z\in\mathbb{C}$, $\lambda\in T$, $E>0$.

**3.1. Algorithm 1 with a non-zero background potential $\Lambda$.** Consider a potential $V$ defined as in (1.5), where the diagonal matrix $\Lambda$, defined as $\Lambda_{ij} = \lambda_i \delta_{ij}$, is supposed to be a known background potential. In this case Algorithm 1 admits the following effectivisations.

Let $V_1 \equiv \Lambda$ on $\bar{D}$, $V_1 \equiv 0$ on $\mathbb{R}^2 \setminus \bar{D}$. The following parts A and B provide two different approaches to the reconstruction of $\psi_\pm(x,k)|_{\partial D}$ from $\Phi(E)$ and of $h_\pm(k,l)$ from $\psi_\pm(x,k)|_{\partial D}$; the reconstruction of $V_{appr}$ from $h_\pm$ is given after in steps C and D.

**A.** $\Phi(E) \longrightarrow h_\pm$. Starting from $\Phi(E)$, for $E$ sufficiently large, we first reconstruct $\psi_\pm(x,k)|_{\partial D}, k \in \mathbb{R}^2, k^2 = E$, with the help of the following Fredholm linear integral equation (see Section 4):

$$(\mathrm{Id} + (\mathrm{Id} - A^1_\pm)^{-1} \delta A_\pm) \psi_\pm(x,k)|_{\partial D} = \psi^1_\pm(x,k)|_{\partial D}, \tag{3.14}$$

where

$$A^1_\pm u(x) = \int_{\partial D} A^1_\pm(x,y,k) u(y) dy, \quad x \in \partial D, \tag{3.15}$$

$$A^1_\pm(x,y,k) = \int_{\partial D} G_\pm(x-\xi,k)\,(\Phi_1 - \Phi_0)\,(\xi,y,E) d\xi, \quad x,y \in \partial D, \tag{3.16}$$

$$\delta A_\pm u(x) = \int_{\partial D \times \partial D} G_\pm(x-\xi,k)(\Phi_1 - \Phi)(\xi,y,E) u(y) dy\, d\xi,\ x \in \partial D, \tag{3.17}$$

$\psi^1_\pm(x,k)|_{\partial D} = (\mathrm{Id} - A^1_\pm)^{-1}(e^{ikx} I)$ are the functions $\psi_\pm(x,k)|_{\partial D}$ for $V = V_1$, $(\Phi_1 - \Phi_0)(x,y,E)$ is the Schwartz kernel of the operator $\Phi_1(E) - \Phi_0(E)$, $(\Phi_1 - \Phi)(x,y,E)$ is the Schwartz kernel of the operator $\Phi_1(E) - \Phi(E)$, $\Phi_0(E)$ is the Dirichlet-to-Neumann operator associated to the zero potential in $D$ at fixed energy $E$, $\Phi_1(E)$ is the Dirichlet-to-Neumann operator associated to the potential $V_1$ in $D$ at fixed energy $E$ and $u$ is a $M_n(\mathbb{C})$-valued test function on $\partial D$.

In order to obtain $h_\pm$ we use the following formula (see Section 4):

$$\begin{aligned} h_\pm(k,l) = h^1_\pm(k,l) &+ \frac{1}{(2\pi)^2} \int_{\partial D} \int_{\partial D} e^{-ilx} (\Phi - \Phi_1)(x,y,E) \psi_\pm(y,k) dy dx \\ &+ \frac{1}{(2\pi)^2} \int_{\partial D} \int_{\partial D} e^{-ilx} (\Phi_1 - \Phi_0)(x,y,E) \delta\psi_\pm(y,k) dy dx, \end{aligned} \tag{3.18}$$

for $(k,l) \in \mathscr{M}_E$, where $h^1_\pm(k,l)$ is defined as in (2.3), (2.17) with $V = V_1$, $\delta\psi_\pm(x,k) = \psi_\pm(x,k) - \psi^1_\pm(x,k)$ and $\psi^1_\pm(x,k)$ is defined as $\psi_\pm(x,k)$ in (2.2), (2.11), (2.18) with $V = V_1$.

**B.** $\Phi(E) \longrightarrow h_\pm$. As above, starting from $\Phi(E)$, for $E$ sufficiently large, we first reconstruct $\psi_\pm(x,k)|_{\partial D}, k \in \mathbb{R}^2, k^2 = E$, with the help of the following Fredholm linear integral equation (see **[20]** for $n = 1$ and Section 4 for $n \geq 2$):

$$\psi_\pm(x,k)|_{\partial D} = \psi^1_\pm(x,k)|_{\partial D} + \int_{\partial D} A_\pm(x,y,k) \psi_\pm(y,k) dy, \tag{3.19}$$

for $k \in \mathbb{R}^2,\ k^2 = E$, where

$$A_\pm(x,y,k) = \int_{\partial D} R^1_\pm(x,\xi,k)\,(\Phi - \Phi_1)\,(\xi,y,E)d\xi, \quad x,y \in \partial D, \tag{3.20}$$

$\psi^1_\pm$, $R^1_\pm$ are defined as $\psi_\pm$, $R_\pm$ of (2.2), (2.4), (2.10), (2.11), (2.18), (2.19) with $V = V_1$, $(\Phi - \Phi_1)(x,y,E)$ is the Schwartz kernel of the operator $\Phi(E) - \Phi_1(E)$, $\Phi_1(E)$ is the Dirichlet-to-Neumann operator associated to the potential $V_1$ in $D$ at fixed energy $E$.

In order to obtain $h_\pm$ we use the following formula (see **[20]** for $n = 1$ and Section 4 for $n \geq 2$):

$$h_\pm(k,l) = h^1_\pm(k,l) + \frac{1}{(2\pi)^2}\int_{\partial D}\int_{\partial D} \psi^1_\mp(x,-k,-l)(\Phi - \Phi_1)(x,y,E)\psi_\pm(y,k)dydx, \tag{3.21}$$

for $(k,l) \in \mathscr{M}_E$, where $h^1_\pm(k,l)$ is defined as in (2.3), (2.17) with $V = V_1$, $\psi^1_\mp(x,k,l)$ is defined as the solution of the following linear integral equation (see **[20]** for $n = 1$ and Section 4 for $n \geq 2$)

$$\psi^1_\mp(x,k,l) = e^{ilx}I + \int_{\mathbb{R}^2} G_\mp(x-y,k)V_1(y)\,\psi^1_\mp(y,k,l)dy, \tag{3.22}$$

where $x,k,l \in R^2$, $k^2 = l^2 > 0$ and $G_\mp$ is defined in (3.3).

**C.** $h_\pm \longrightarrow \tilde\mu^+$. We construct an approximation $\tilde\mu^+$ to $\mu^+$ of (2.15) via the following integral equation which generalises (3.6) (see Section 4):

$$(\mathrm{Id} + (\mathrm{Id} + B^1)^{-1}\delta B)\tilde\mu^+(z,\lambda,E) = \mu^{1,+}(z,\lambda,E), \quad \lambda \in T,\ z \in \mathbb{C}, \tag{3.23}$$

for $E$ sufficiently large, where

$$B^1 u(\lambda) = \int_T u(\lambda')B^1(\lambda,\lambda',z,E)|d\lambda'|, \tag{3.24}$$

$$\delta B u(\lambda) = \int_T u(\lambda')[B(\lambda,\lambda',z,E) - B^1(\lambda,\lambda',z,E)]|d\lambda'|, \tag{3.25}$$

for $\lambda \in T,\ z \in \mathbb{C},\ B^1(\lambda,\lambda',z,E)$ is defined as $B(\lambda,\lambda',z,E)$ in (3.7) with $h_\pm = h^1_\pm$, $\mu^{1,+}$ is defined as $\mu^+$ in (2.6), (2.15) with $V = V_1$ and $u$ is a $M_n(\mathbb{C})$-valued test function on $T$.

**D.** $\tilde\mu^+ \longrightarrow V_{appr}$. The final part of the algorithm is the same as for Algorithm 1 with zero background potential. We construct an approximation $\tilde\mu_-$ to $\mu_-$ using formula (3.8) and then the approximate potential $V_{appr}(z,E)$ via formula (3.9).

In the linear approximation near the potential $V_1$, the following formulas hold:

$$h_\pm(k,l) \approx h^1_\pm(k,l) + \frac{1}{(2\pi)^2}\int_{\partial D}\int_{\partial D} e^{-ilx}(\Phi-\Phi_1)(x,y,E)\psi^1_\pm(y,k)dydx - \frac{1}{(2\pi)^2}\int_{\partial D}\int_{\partial D} e^{-ilx}(\Phi_1-\Phi_0)(x,y,E)(\mathrm{Id}-A^1_\pm)^{-1}\delta A_\pm\psi^1_\pm(y,k)dy\,dx, \tag{3.26a}$$

$$h_\pm(k,l) \approx h^1_\pm(k,l) + \frac{1}{(2\pi)^2}\int_{\partial D}\int_{\partial D} \psi^1_\mp(x,-k,-l)(\Phi-\Phi_1)(x,y,E)\psi^1_\pm(y,k)dydx, \tag{3.26b}$$

for $(k,l)\in\mathscr{M}_E$,

$$\tilde\mu^+(z,\lambda,E) \approx \mu^{1,+}(z,\lambda,E) - (\mathrm{Id}+B^1)^{-1}\delta B\mu^{1,+}(z,\lambda,E), \tag{3.27}$$

for $\lambda\in T,\ z\in\mathbb{C}$,

$$\tilde\mu_-(z,\lambda,E) \approx \mu^1_- - (\mathrm{Id}+B^1)^{-1}\delta B\mu^{1,+}(z,\lambda,E) + \pi i\int_T \mu^{1,+}(z,\lambda'',E)\chi_+\left(-i\left(\frac{\lambda}{\lambda''}-\frac{\lambda''}{\lambda}\right)\right)(h_- - h^1_-)(\lambda,\lambda'',z,E)|d\lambda''|, \tag{3.28}$$

$$V_{appr}(z,E) \approx V_1 - \frac{1}{\pi}E^{1/2}\int_T \frac{\partial}{\partial z}\left((\mathrm{Id}+B^1)^{-1}\delta B\mu^{1,+}(z,\lambda,E)\right) i\lambda|d\lambda| + iE^{1/2}\int_T\int_T \frac{\partial}{\partial z}\Big[\mu^{1,+}(z,\lambda'',E)\chi_+\left(-i\left(\frac{\lambda}{\lambda''}-\frac{\lambda''}{\lambda}\right)\right) \times (h_- - h^1_-)(\lambda,\lambda'',z,E)\Big]|d\lambda''|i\lambda|d\lambda|, \tag{3.29}$$

for $z\in D$ and $E$ sufficiently large.

## 4. Derivation of some formulas and equations of Section 2 and 3 for the matrix case

The following formula and equations will be useful:

$$\psi_\gamma(x,k) = \psi^+(x,k) + 2\pi i\int_{\mathbb{R}^2}\psi^+(x,l)\delta(l^2-k^2)\chi_+((l-k)\gamma)h_\gamma(k,l)dl, \tag{4.1}$$

$$h_\gamma(k,l) = f(k,l) + 2\pi i\int_{\mathbb{R}^2} f(m,l)\delta(m^2-k^2)\chi_+((m-k)\gamma)h_\gamma(k,m)dm, \tag{4.2}$$

for $\gamma \in S^1$, $x, k, l \in \mathbb{R}^2$, $k^2 = E \in \mathbb{R}_+$ sufficiently large,

$$\frac{\partial \mu}{\partial \bar{k}_j}(x,k) = -2\pi \int_{\mathbb{R}^2} \xi_j e^{i\xi x} \mu(x, k+\xi) H(k,-\xi)\delta(\xi^2 + 2k\xi) d\xi, \tag{4.3}$$

$$\frac{\partial H}{\partial \bar{k}_j}(k,p) = -2\pi \int_{\mathbb{R}^2} \xi_j H(k+\xi, p+\xi) H(k,-\xi)\delta(\xi^2 + 2k\xi) d\xi, \tag{4.4}$$

for $j = 1, 2$, $k \in \mathbb{C}^2 \setminus \mathbb{R}^2$, $k^2 = E \in \mathbb{R}_+$ sufficiently large, $x, p \in \mathbb{R}^2$, where $H(k,p) = h(k, k-p)$, $\delta$ is the Dirac delta and the other functions were already defined in Section 2. Formula (4.1) and equations (4.2)-(4.4) are proved in [**9, 3, 13**] for the scalar case: the proof can be straightforwardly generalized to the matrix case, where one only has to pay attention to the order of factors (which is indeed different from the formulation given in the quoted papers, but coherent with similar results obtained in [**25**]).

Now formula (2.29) follows directly from (4.1), (2.6), (2.31) using notations (2.21); equation (2.32) follows from (4.3) taking into account (2.20), (2.33) and notations (2.21). In addition, equation (3.5) is a direct consequence of (4.2) (with $\gamma = \pm\hat{k}_\perp$) using notations (2.21).

Formula (2.36) follows from (2.35), (2.34), (2.32), (2.28), (2.27) (these can be proved exactly as in the scalar case) and the Cauchy–Pompeiu formula

$$u(\lambda) = \frac{1}{2\pi i}\int_{\partial\mathcal{D}} u(\zeta)\frac{d\zeta}{\zeta - \lambda} - \frac{1}{\pi}\int_{\mathcal{D}} \frac{\partial u(\zeta)}{\partial\bar{\zeta}} \frac{d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta}{\zeta - \lambda}, \quad \lambda \in \mathcal{D}, \tag{4.5}$$

for any sufficiently regular $M_n(\mathbb{C})$-valued $u$ in $\mathcal{D}$, where $\partial\mathcal{D}$ is sufficiently regular. In addition, formula (3.9) is just formula (2.36) without the first term in the sum.

Equation (3.6) is an approximation of the following (exact) equation for $\mu^+$:

$$\mu^+(z,\lambda,E) + \int_T \mu^+(z,\lambda',E) B(\lambda,\lambda',z,E)|d\lambda'| = I + \varphi(z,\lambda,E), \tag{4.6}$$

for $\lambda \in T,\ z \in \mathbb{C}$, where

$$\varphi(z,\lambda,E) = -\frac{1}{\pi}\int_{\mathbb{C}} \mu\left(z, -\frac{1}{\bar{\zeta}}, E\right) r(\zeta, z, E) \frac{d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta}{\zeta - \lambda}. \tag{4.7}$$

The derivation of (4.6) can be found in [**19**] for the scalar case and its generalisation to the matrix case is straightforward (paying attention to the order of factors).

Formula (3.3) is a result of [**9**], while formulas (3.1), (3.2), (3.4) are results of [**16**] for the scalar case and can be proved for the matrix case following the scheme of [**23**], where similar formulas appear.

Formulas (3.14) and (3.18) follows from (3.1) and (3.4).

Formulas (3.19)-(3.22) are results of [**20**] for the scalar case and can directly extended to the matrix case following the scheme of [**23**] because, in particular, the general matrix version of Alessandrini's identity in [**23**] works for our diagonal background potential $\Lambda$.

Finally, equation (3.23) follows from (4.6).

## 5. Function spaces and some estimates

We introduce some function spaces, which will be useful to prove the high-energy convergence of our algorithms. For $m \in \mathbb{N}$, $\varepsilon > 0$ we consider

$$W^{m,1}(\mathbb{R}^2, M_n(\mathbb{C})) = \{u : \partial^k u \in L^1(\mathbb{R}^2, M_n(\mathbb{C})) \text{ for } |k| \le m\},$$
$$W^{m,1}_\varepsilon(\mathbb{R}^2, M_n(\mathbb{C})) = \{u : \varkappa^\varepsilon \partial^k u \in L^1(\mathbb{R}^2, M_n(\mathbb{C})) \text{ for } |k| \le m\},$$
$$(\varkappa^\varepsilon u)(x) = (1+|x|^2)^{\varepsilon/2} u(x), \qquad k \in (\mathbb{N} \cup 0)^2, \qquad |k| = k_1 + k_2,$$
$$\partial^k = \partial_1^{k_1}\partial_2^{k_2}, \qquad \partial_j = \frac{\partial}{\partial x_j};$$

for $\alpha \in ]0,1]$, $s \in \mathbb{R}$ we consider

$$C^{\alpha,s}(\mathbb{R}^2, M_n(\mathbb{C})) = \{u \;:\; \|u\|_{\alpha,s} < \infty\},$$

where

$$\|u\|_{\alpha,s} = \|\varkappa^s u\|_\alpha$$
$$\|w\|_\alpha = \sup_{p,\xi \in \mathbb{R}^2,\ |\xi| \le 1} \left( |w(p)| + \frac{|w(p+\xi) - w(p)|}{|\xi|^\alpha} \right),$$
$$(\varkappa^s u)(p) = (1+|p|^2)^{s/2} u(p), \qquad |u(p)| = \max_{1 \le i,j \le n} |u_{ij}(p)|;$$

in addition we consider $\mathcal{H}_{\alpha,s}(\mathbb{R}^2, M_n(\mathbb{C}))$, defined as the closure of $C_0^\infty(\mathbb{R}^2, M_n(\mathbb{C}))$ (the space of infinitely smooth functions with compact support) in $\|\cdot\|_{\alpha,s}$.

Let

$$\widehat{V}(p) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ipx} V(x) dx, \qquad p \in \mathbb{R}^2. \tag{5.1}$$

If a matrix-valued potential $V$ satisfies

$$V \in W^{m,1}_\varepsilon(\mathbb{R}^2, M_n(\mathbb{C})) \text{ for some } \varepsilon > 0,\ m \in \mathbb{N}, \tag{5.2}$$

then

$$\widehat{V} \in \mathcal{H}_{\alpha,s}(\mathbb{R}^2, M_n(\mathbb{C})), \qquad \alpha \in ]0,1], \qquad s \in \mathbb{R}_+, \tag{5.3}$$

where $\alpha = \min(1,\varepsilon)$, $s = m$. Let

$$\Sigma(r) = (1-r)^{-1} r. \tag{5.4}$$

We have the following results:

Proposition 5.1. *Let the condition* (5.3) *be valid. Then*

$$|f(k,l) - \widehat{V}(k-l)| \le \Sigma(r) \|\widehat{V}\|_{\alpha,s} (1+|k-l|^2)^{-s/2}, \tag{5.5a}$$
$$|H_\gamma(k,p) - \widehat{V}(p)| \le \Sigma(r) \|\widehat{V}\|_{\alpha,s} (1+p^2)^{-s/2}, \tag{5.5b}$$

*for* $r = |k|^{-\sigma} c_1(\alpha,s,\sigma,n) \|\widehat{V}\|_{\alpha,s} < 1,\ k,l,p \in \mathbb{R}^2,\ \gamma \in S^1,\ k^2 \ge 1,$

$$|H(k,p) - \widehat{V}(p)| \le \Sigma(r) \|\widehat{V}\|_{\alpha,s} (1+p^2)^{-s/2}, \tag{5.5c}$$

*for* $r = |\mathrm{Re}k|^{-\sigma} c_1(\alpha, s, \sigma, n)\|\widehat{V}\|_{\alpha,s} < 1$, $k \in \mathbb{C}^2 \setminus \mathbb{R}^2$, $p \in \mathbb{R}^2$, $\mathbb{R} \ni k^2 \geq 1$. *In particular*

$$|f(k,l)| \leq 2\|\widehat{V}\|_{\alpha,s}(1+|k-l|^2)^{-s/2}, \qquad k,l \in \mathbb{R}^2, \tag{5.6a}$$

$$|H_\gamma(k,p)| \leq 2\|\widehat{V}\|_{\alpha,s}(1+p^2)^{-s/2}, \qquad k,p \in \mathbb{R}^2, \qquad \gamma \in S^1, \tag{5.6b}$$

$$|H(k,p)| \leq 2\|\widehat{V}\|_{\alpha,s}(1+p^2)^{-s/2}, \qquad k \in \mathbb{C}^2 \setminus \mathbb{R}^2, \qquad p \in \mathbb{R}^2, \tag{5.6c}$$

*for* $k^2 \geq E_1 = \max(1, (2c_1(\alpha, s, \sigma, n)\|\widehat{V}\|_{\alpha,s})^{2/\sigma})$, *where* $H_\gamma(k,l) = h_\gamma(k, k-l)$, $0 < \alpha < 1$, $s > 0$, $0 < \sigma < \min(1, s)$.

LEMMA 5.2. *Under condition* (5.3), *we have the following estimates:*

$$|\mu_\gamma(x,k) - I| + \left|\frac{\partial \mu_\gamma(x,k)}{\partial x_1}\right| + \left|\frac{\partial \mu_\gamma(x,k)}{\partial x_2}\right| \leq |k|^{-\sigma} c_2(\alpha, s, \sigma, n)\|\widehat{V}\|_{\alpha,s}, \tag{5.7a}$$

*for* $x = (x_1, x_2) \in \mathbb{R}^2$, $k \in \mathbb{R}^2$, $\gamma \in S^1$,

$$|\mu(x,k) - I| + \left|\frac{\partial \mu(x,k)}{\partial x_1}\right| + \left|\frac{\partial \mu(x,k)}{\partial x_2}\right| \leq |\mathrm{Re}k|^{-\sigma} c_2(\alpha, s, \sigma, n)\|\widehat{V}\|_{\alpha,s}, \tag{5.7b}$$

*for* $x = (x_1, x_2) \in \mathbb{R}^2$, $k \in \mathbb{C}^2 \setminus \mathbb{R}^2$, $k^2 \geq E_1(\alpha, s, \sigma, n, \|\widehat{V}\|_{\alpha,s})$, *where* $0 < \alpha < 1$, $s > 1$, $0 < \sigma < \min(1, s-1)$.

Proposition 5.1 and Lemma 5.2 for the scalar case ($n = 1$) were given in **[19]** and their generalisation to the matrix case ($n \geq 2$) is straightforward.

## 6. Lipschitz stability and rapid convergence of Algorithms 1 and 2 for $E \to +\infty$

We present here main rigorous results concerning stability and convergence of our algorithms in the case of zero background potential for simplicity. In addition, we expect that, for potentials of the form (1.5), Algorithm 1 with non-zero background potential $\Lambda$ (see subsection 3.1) will work even better than its version with zero background potential.

THEOREM 6.1 (Stability and convergence of Algorithm 1). *Let* $m \geq 3$, $V \in W^{m,1}(\mathbb{R}^2, M_n(\mathbb{C}))$, $\operatorname{supp} V \subset D$ *and let* $\Phi(E)$ *be the Dirichlet-to-Neumann operator of* (1.6) *at fixed energy* $E$, *where* $E \geq E_2(\alpha, s, \sigma, n, \|\widehat{V}\|_{\alpha,s})$, $0 < \alpha \leq 1$, $s = m$, $0 < \sigma < 1$ *and* $E$ *is not a Dirichlet eigenvalue of* $-\Delta + V$ *and* $-\Delta$ *in* $D$. *Then* $V$ *is reconstructed from* $\Phi(E)$ *with Lipschitz stability via Algorithm 1 up to* $O(E^{-(m-2)/2})$ *in the uniform norm as* $E \to +\infty$.

THEOREM 6.2 (Stability and convergence of Algorithm 2). *Let* $V$ *satisfy* (5.2), *for* $m \geq 3$, *and let* $f$ *be the scattering amplitude of* (1.8) *at fixed energy* $E \geq E_2(\alpha, s, \sigma, n, \|\widehat{V}\|_{\alpha,s})$, *where* $\alpha = \min(1, \varepsilon)$, $s = m$ *and* $0 < \sigma < 1$. *Then* $V$ *is reconstructed from* $f$ *on* $\mathscr{M}_E$ *with Lipschitz stability via Algorithms 2 up to* $O(E^{-(m-2)/2})$ *in the uniform norm as* $E \to +\infty$.

The constant $E_2$ of Theorems 6.1 and 6.2 is precisely stated in Remark 6.1. The Lipschitz stability of Theorems 6.1 and 6.2 is specified in the proofs of these theorems and is summarized in Remarks 6.2 and 6.3. The error term $O(E^{-(m-2)/2})$ of Theorems 6.1 and 6.2 is made explicit in formula (6.9).

Similarly with the presentation of Algorithms 1 and 2 in section 3, we separate the proofs of Theorems 6.1 and 6.2 in several steps.

Proof of Theorem 6.1 ($\Phi(E) \longrightarrow h_\pm$). We have that equation (3.1) is a Fredholm linear integral equation of second kind for $\psi_\pm|_{\partial D} \in L^2(\partial D)$, which is uniquely solvable with precise data $\Phi - \Phi_0$ (the proof of the latter fact is the same as in the scalar case; see **[16]**). Therefore the reconstruction of $\psi_\pm$ via (3.1) is Lipschitz stable, with respect to small errors in $\Phi - \Phi_0$ (in the $L^2$ norm of the Schwartz kernel),

As a corollary, the reconstruction of $h_\pm$ in $L^2(\mathscr{M}_E)$ from $\Phi(E) - \Phi_0(E)$ via equation (3.1) and formula (3.4) is also Lipschitz stable. ☐

Proof of Theorem 6.2 ($f \longrightarrow h_\pm$). Estimates (5.6) and notations (2.21) give

$$|f(\lambda,\lambda',E)| \le 2\|\widehat{V}\|_{\alpha,s}(1+E|\lambda-\lambda'|^2)^{-s/2}, \qquad \lambda,\lambda' \in T, \tag{6.1a}$$

$$\|f\|_{L^2(T\times T)} \le c_3 n \|\widehat{V}\|_{\alpha,s} E^{-1/4}, \tag{6.1b}$$

for $E \ge E_1$, $\alpha = \min(1,\varepsilon)$, $s = m$. Now, under the assumptions of Theorem 6.2, integral equation (3.5) is uniquely solvable for $h_\pm(\lambda,\cdot,E) \in L^2(T)$ for $\lambda \in T$, $E \ge E_1$ (this is a consequence of the unique solvability of integral equation (2.2) for $E \ge E_1$). In addition, by estimate (6.1b), for $E \ge \max(E_1, (\pi c_3 n \|\widehat{V}\|_{\alpha,s})^4)$, equation (3.5) is uniquely solvable for $h_\pm(\lambda,\cdot,E) \in L^2(T)$, $\lambda \in T$, and for $h_\pm(\cdot,\cdot,E) \in L^2(T\times T)$ by the method of successive approximations. This implies the Lipschitz stability of the reconstruction of $h_\pm$ on $T\times T$ from $f$ on $T \times T$, with respect to small errors in the $L^2$ norm. ☐

Proof of Theorems 6.1 and 6.2 ($h_\pm \longrightarrow V_{appr}$). The proof follows as in the scalar case (that was treated in **[19]**), except for the order of the terms in formulas and integral equations.

Estimates (5.6), formula (2.16) and notations (2.21) give

$$|h_\pm(\lambda,\lambda',E)| \le 2\|\widehat{V}\|_{\alpha,s}(1+E|\lambda-\lambda'|^2)^{-s/2}, \qquad \lambda,\lambda' \in T, \tag{6.2a}$$

$$\|h_\pm\|_{L^2(T\times T)} \le c_3 n \|\widehat{V}\|_{\alpha,s} E^{-1/4}, \tag{6.2b}$$

for $E \ge E_1$, $s = m$ and

$$0 < \alpha \le 1 \text{ for Theorem 6.1 and } \alpha = \min(1,\varepsilon) \text{ for Theorem 6.2.} \tag{6.3}$$

We define the integral operator $B(z,E)$ as

$$(B(z,E)u)(\lambda) = \int_T u(\lambda')B(\lambda,\lambda',z,E)|d\lambda'|, \tag{6.4}$$

for $\lambda \in T$, where $B(\lambda, \lambda', z, E)$ is defined in (3.7) and $u$ is a test matrix function. The following decomposition holds

$$B(z, E) = C_+ Q_-(z, E) - C_- Q_+(z, E), \tag{6.5}$$

where

$$(C_\pm u)(\lambda) = \frac{1}{2\pi i} \int_T \frac{u(\zeta)}{\zeta - \lambda(1 \mp 0)} d\zeta, \tag{6.6}$$

$$(Q_\pm u)(\lambda) = \pi i \int_T u(\lambda') \chi_+ \left( \pm i \left( \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right) \right) h_\pm(\lambda, \lambda', z, E) |d\lambda'|, \tag{6.7}$$

$z \in \mathbb{C}$, $\lambda \in T$, $\chi_+, h_\pm$ are defined in (2.30) and (2.31) and $u$ is a test matrix function. Thanks to (6.2), (6.5) and properties of the Cauchy projectors $C_\pm$ (see **[19]** for more details), $B(z, E)$ satisfies the estimates

$$\|B(z, E) u\|_{L^2(T)} \le c_4 n \|\widehat{V}\|_{\alpha,s} E^{-1/4} \|u\|_{L^2(T)}, \tag{6.8a}$$

$$\left\| \frac{\partial}{\partial z} B(z, E) u \right\|_{L^2(T)} \le c_4 n \|\widehat{V}\|_{\alpha,s} E^{-1/4} \|u\|_{L^2(T)}, \tag{6.8b}$$

for $z \in \mathbb{C}$, $E \ge E_1$, $s = m$, $\alpha$ as in (6.3).

Now by estimate (6.8a), for $E \ge \max(E_1, (c_4 n \|\widehat{V}\|_{\alpha,s})^4)$, integral equation (3.6) is uniquely solvable for $\tilde{\mu}^+(z, \cdot, E) \in L^2(T)$, at fixed $z \in \mathbb{C}$, by the method of successive approximations. This implies the Lipschitz stability of the reconstruction of $\tilde{\mu}^+(z, \cdot, E)$ on $T$, at fixed $z \in \mathbb{C}$, from $h_\pm$ on $T \times T$ with respect to small errors in the $L^2$ norm. □

Proof of Theorems 6.1 and 6.2 ($V_{appr} \longrightarrow V$). Our high-energy convergence estimate is as follows:

$$|V(z) - V_{appr}(z, E)| \le c_5 n \|\widehat{V}\|_{\alpha,s} E^{-(s-2)/2}, \tag{6.9}$$

where $z \in \mathbb{C}$, $E \ge E_2(\alpha, s, \sigma, n, \|\widehat{V}\|_{\alpha,s})$, $\alpha$ as in (6.3), $s = m$, $0 < \sigma < 1$ (see **[19]** for complete details). This estimate follows from (2.29), (2.36), (3.6)–(3.9), (6.2b) and the following estimates (whose proofs for $n = 1$ can be found in **[19]**):

$$\left| 2iE^{1/2} \frac{\partial}{\partial z} \left( \int_{D_-} \mu(z, -\frac{1}{\bar{\zeta}}, E) r(\zeta, z, E) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \right) \right| \le c_6 n \|\widehat{V}\|_{\alpha,s} E^{-(s-2)/2}, \tag{6.10}$$

$$\|\mu^+(z, \cdot, E) - \tilde{\mu}^+(z, \cdot, E)\|_{L^2(T, M_n(\mathbb{C}))} \le c_7 n \|\widehat{V}\|_{\alpha,s} E^{-s/2}, \tag{6.11}$$

$$\left\| \frac{\partial \mu^+}{\partial z}(z, \cdot, E) - \frac{\partial \tilde{\mu}^+}{\partial z}(z, \cdot, E) \right\|_{L^2(T, M_n(\mathbb{C}))} \le c_7 n \|\widehat{V}\|_{\alpha,s} E^{-(s-1)/2}, \tag{6.12}$$

for $z \in \mathbb{C}$, $s = m \ge 3$, $E \ge E_2(\alpha, s, \sigma, n, \|\widehat{V}\|_{\alpha,s})$, $\alpha$ as in (6.3). □

Remark 6.1. The constant $E_2$ of Theorems 6.1 and 6.2 can be fixed as some constant such that $E \geq E_2$ implies that

$$E \geq E_1, \qquad |\mu(z,\lambda,E)| \leq 2, \qquad \left|\frac{\partial}{\partial z}\mu(z,\lambda,E)\right| \leq 1,$$
$$\|B(z,E)\|^{op}_{L^2(T)} \leq \frac{1}{2}, \qquad \left\|\frac{\partial}{\partial z}B(z,E)\right\|^{op}_{L^2(T)} \leq \frac{1}{2},$$

for $z \in \mathbb{C}$, $\lambda \in \mathbb{C}$, where $\mu$ and $B$ are estimated in (5.7) and (6.8).

Now, let $\Phi_{V,0}(x,y,E)$, $x,y \in \partial D$, denote the Schwartz kernel of the operator $\Phi(E) - \Phi_0(E)$ considered as precise data for Problem 1. Let $\Phi'_{V,0}$ denote $\Phi_{V,0}$ with some small errors (for the case of Problem 1) and $f'$ denote $f$ with some small errors (for the case of Problem 2). Let $V'_{appr}$ denote $V_{appr}$ reconstructed from $\Phi'_{V,0}$ via Algorithm 1 (for Problem 1) and from $f'$ via Algorithm 2 (for Problem 2).

The Lipschitz stability of Theorems 6.1 and 6.2 is summarized in the following remarks:

Remark 6.2. Let the assumptions of Theorem 6.1 hold and let

$$\delta = \|\Phi'_{V,0}(\cdot,\cdot,E) - \Phi_{V,0}(\cdot,\cdot,E)\|_{L^2(\partial D\times\partial D)} \leq \delta_1(V,E,D,n). \tag{6.13}$$

Then

$$\varepsilon = \|V'_{appr} - V_{appr}\|_{L^\infty(D)} \leq \eta_1(V,E,D,n)\delta. \tag{6.14}$$

Here $\delta_1$ and $\eta_1$ are some positive constants summarizing the Lipschitz stability of Algorithm 1. In particular,

$$\delta_1(V,E,D,n) \geq \delta_1^0, \tag{6.15}$$

$$\eta_1(V,E,D,n) \leq \eta_1^0 E, \tag{6.16}$$

as $\|\Phi_{V,0}(\cdot,\cdot,E)\|_{L^2(\partial D\times\partial D)} \to 0$, for some positive (sufficiently small) $\delta_1^0$ and (sufficiently big) $\eta_1^0$, where $\delta_1^0$ and $\eta_1^0$ are independent of $V$ and $E$ for fixed $D$ and $n$.

Remark 6.3. Let the assumptions of Theorem 6.2 hold and let

$$\delta = \|f - f'\|_{L^2(\mathscr{M}_E)} \leq \delta_2(V,E,n). \tag{6.17}$$

Then

$$\varepsilon = \|V_{appr} - V'_{appr}\|_{L^\infty(\mathbb{R}^2)} \leq \eta_2(V,E,n)\delta. \tag{6.18}$$

Here $\delta_2$ and $\eta_2$ are suitable constants summarizing the Lipschitz stability of Algorithms 2. In particular,

$$\delta_2(V,E,n) \geq \delta_2^0, \tag{6.19}$$

$$\eta_2(V,E,n) \leq \eta_2^0 E, \tag{6.20}$$

as $\|f\|_{L^2(\mathscr{M}_E)} \to 0$, for some positive (sufficiently small) $\delta_2^0$ and (sufficiently big) $\eta_2^0$, where $\delta_2^0$ and $\eta_2^0$ are independent of $V$ and $E$ for fixed $n$.

Note that in Remark 6.3, the norm $\|\cdot\|_{L^2(\mathscr{M}_E)}$ is identified with $\|\cdot\|_{L^2(T\times T)}$.

The property that $\|f\|_{L^2(\mathscr{M}_E)} \to 0$, mentioned in Remark 6.3, is fulfilled, in particular, for $E \to +\infty$, as a consequence of estimate (6.1b). On the contrary, the property that $\|\Phi_{V,0}(\cdot,\cdot,E)\|_{L^2(\partial D\times\partial D)} \to 0$, mentioned in Remark 6.2, is not fulfilled for $E = E_j$, $j \to \infty$, for any sequence $\{E_j\}_{j\in\mathbb{N}}$ of positive real numbers such that $E_j \to +\infty$ as $j \to \infty$, if $V \not\equiv 0$. In this connection, our high-energy conjecture is that

$$\sup_{j\in\mathbb{N}} \|\Phi_{V,0}(\cdot,\cdot,E_j)\|_{L^2(\partial D\times\partial D)} < +\infty, \tag{6.21}$$

for some $\{E_j\}_{j\in\mathbb{N}}$ dependent on $V$, where $E_j \to +\infty$ as $j \to \infty$.

Note that the $E$ factor in the right side of (6.16) and of (6.20) is related with the choice of the $L^2$ norm for estimates of the inverse problem data. For example, for Algorithm 2, at least in the linear approximation (3.11)-(3.13), this factor disappear if $\|\cdot\|_{L^2(\mathscr{M}_E)}$ is replaced by $\|\cdot\|_{L^\infty_s(\mathscr{M}_E)}$, $s = m$, where

$$\|u\|_{L^\infty_s(\mathscr{M}_E)} = \sup_{(\lambda,\lambda')\in T\times T} (1 + E|\lambda - \lambda'|^2)^{s/2}|u(\lambda,\lambda')|. \tag{6.22}$$

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# Paper G

**PAPER G**

# New global stability estimates for the Calderón problem in two dimensions

Matteo Santacesaria

Abstract. We prove a new global stability estimate for the Gel'fand-Calderón inverse problem on a two-dimensional bounded domain. Specifically, the inverse boundary value problem for the equation $-\Delta\psi + v\,\psi = 0$ on $D$ is analysed, where $v$ is a smooth real-valued potential of conductivity type defined on a bounded planar domain $D$. The main feature of this estimate is that it shows that the more a potential is smooth, the more its reconstruction is stable. Furthermore, the stability is proven to depend exponentially on the smoothness, in a sense to be made precise. The same techniques yield a similar estimate for the Calderón problem for the electrical impedance tomography.

## 1. Introduction

Let $D \subset \mathbb{R}^2$ be a bounded domain equipped with a potential given by a function $v \in L^\infty(D)$. The corresponding Dirichlet-to-Neumann map is the operator $\Phi : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$, defined by

$$\Phi(f) = \left.\frac{\partial u}{\partial \nu}\right|_{\partial D}, \tag{1.1}$$

where $f \in H^{1/2}(\partial D)$, $\nu$ is the outer normal of $\partial D$, and $u$ is the $H^1(D)$-solution of the Dirichlet problem

$$(-\Delta + v)u = 0 \text{ on } D, \quad u|_{\partial D} = f. \tag{1.2}$$

Here we have assumed that

$$0 \text{ is not a Dirichlet eigenvalue for the operator } -\Delta + v \text{ in } D. \tag{1.3}$$

The following inverse boundary value problem arises from this construction:

Problem 6. *Given $\Phi$, find $v$ on $D$.*

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at zero energy (see [**12**], [**21**]) as well as a generalization of the Calderón problem for the electrical impedance tomography (see [**9**], [**21**]), in two dimensions.

It is convenient to recall how the above problem generalises the inverse conductivity problem proposed by Calderón. In the latter, $D$ is a body equipped with an isotropic conductivity $\sigma(x) \in C^2(\bar{D})$ (with $\sigma \geq \sigma_{\min} > 0$),

$$v(x) = \frac{\Delta \sigma^{1/2}(x)}{\sigma^{1/2}(x)}, \qquad x \in D, \tag{1.4}$$

$$\Phi = \sigma^{-1/2}\left(\Lambda \sigma^{-1/2} + \frac{\partial \sigma^{1/2}}{\partial \nu}\right), \tag{1.5}$$

where $\sigma^{-1/2}$, $\partial\sigma^{1/2}/\partial\nu$ in (1.5) denote the multiplication operators by the functions $\sigma^{-1/2}|_{\partial D}$, $\partial\sigma^{1/2}/\partial\nu|_{\partial D}$, respectively and $\Lambda$ is the voltage-to-current map on $\partial D$, defined as

$$\Lambda f = \sigma \left.\frac{\partial u}{\partial \nu}\right|_{\partial D}, \tag{1.6}$$

where $f \in H^{1/2}(\partial D)$, $\nu$ is the outer normal of $\partial D$, and $u$ is the $H^1(D)$-solution of the Dirichlet problem

$$\operatorname{div}(\sigma \nabla u) = 0 \text{ on } D, \quad u|_{\partial D} = f. \tag{1.7}$$

Indeed, the substitution $u = \tilde{u}\sigma^{-1/2}$ in (1.7) yields $(-\Delta + v)\tilde{u} = 0$ in $D$ with $v$ given by (1.4). The following problem is called the Calderón problem:

Problem 7. *Given $\Lambda$, find $\sigma$ on $D$.*

We underline the fact that in order to reduce Problem 2 to Problem 1 the conductivity $\sigma$ must have some regularity and the boundary values of $\sigma$ and $\partial\sigma/\partial\nu$ have to be determined in advance (this was shown for the first time in [**16**]). We also remark that Problems 1 and 2 are not overdetermined, in the sense that we consider the reconstruction of a real-valued function of two variables from real-valued inverse problem data dependent on two variables. In addition, the history of inverse problems for the two-dimensional Schrödinger equation at fixed energy goes back to [**10**].

There are several questions to be answered in these inverse problems: to prove the uniqueness of their solutions (e.g. the injectivity of the map $v \to \Phi$ for Problem 1), the reconstruction and the stability of the inverse map.

In this paper we study interior stability estimates for the two problems. Let us consider, for instance, Problem 1 with a potential of conductivity type. We want to prove that given two Dirichlet-to-Neumann operators, respectively $\Phi_1$ and $\Phi_2$, corresponding to potentials, respectively $v_1$ and $v_2$ on $D$, we have that

$$\|v_1 - v_2\|_{L^\infty(D)} \leq \omega\left(\|\Phi_1 - \Phi_2\|_{H^{1/2} \to H^{-1/2}}\right),$$

where the function $\omega(t) \to 0$ as fast as possible as $t \to 0$. For Problem 2 similar estimates are considered.

There is a wide literature on the Gel'fand-Calderón inverse problem. In the case of complex-valued potentials the global injectivity of the map $v \to \Phi$ was firstly proved in [**21**] for $D \subset \mathbb{R}^d$ with $d \geq 3$ and in [**8**] for $d = 2$: in particular, these results were obtained by the use of global reconstructions developed in the same papers. A global stability estimate for Problem 1 and 2 for $d \geq 3$ was first found by Alessandrini in [**1**]; this result was recently improved in [**25**]. In the two-dimensional case the first global stability estimate for Problem 1 was given in [**27**].

Global results for Problem 2 in the two dimensional case have been found much earlier than for Problem 1. In particular, global uniqueness was first proved in [**20**] for conductivities in the $W^{2,p}(D)$ class ($p > 1$) and after in [**3**] for $L^\infty$ conductivities. Note that in dimension $d \geq 3$ the first global uniqueness result for the Calderón problem was given in [**28**]. In addition, for piecewise real analytic conductivities the first uniqueness result in dimension $d \geq 2$ was given in [**17**]. Moreover, in the case of piecewise constant conductivities, a Lipschitz stability estimate was proved in [**2**] (see [**7**] for a generalisation to complex-valued conductivities).

The first global stability result in two dimensions was given in [**18**], where a logarithmic estimate is obtained for conductivities with two continuous derivatives. This result was improved in [**5**], where the same kind of estimate is obtained for Hölder continuous conductivities.

The research line delineated above is devoted to prove stability estimates for the least regular potentials/conductivities possible. Here, instead, we focus on the opposite situation, i.e. smooth potentials/conductivities, and try to answer another question: how the stability estimates vary with respect to the smoothness of the potentials/conductivities.

The results, detailed below, also constitute a progress for the case of non-smooth potentials: they indicate stability dependence of the smooth part of a singular potential with respect to boundary value data.

We will assume for simplicity that

$$\begin{aligned} &D \text{ is an open bounded domain in } \mathbb{R}^2, \qquad \partial D \in C^2, \\ &v \in W^{m,1}(\mathbb{R}^2) \text{ for some } m > 2, \qquad \operatorname{supp} v \subset D, \end{aligned} \tag{1.8}$$

where

$$\begin{aligned} &W^{m,1}(\mathbb{R}^2) = \{v \;:\; \partial^J v \in L^1(\mathbb{R}^2),\ |J| \leq m\}, \qquad m \in \mathbb{N} \cup \{0\}, \\ &J \in (\mathbb{N} \cup \{0\})^2, \qquad |J| = J_1 + J_2, \qquad \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \partial x_2^{J_2}}. \end{aligned} \tag{1.9}$$

Let

$$\|v\|_{m,1} = \max_{|J| \leq m} \|\partial^J v\|_{L^1(\mathbb{R}^2)}.$$

The last (strong) hypothesis is that we will consider only potentials of conductivity type, i.e.

$$v = \frac{\Delta \sigma^{1/2}}{\sigma^{1/2}}, \text{ for some } \sigma \in L^\infty(D), \text{ with } \sigma \geq \sigma_{\min} > 0. \tag{1.10}$$

The main results are the following.

THEOREM 1.1. *Let the conditions* (1.3), (1.8), (1.10) *hold for the potentials* $v_1, v_2$, *where* $D$ *is fixed, and let* $\Phi_1$ , $\Phi_2$ *be the corresponding Dirichlet-to-Neumann operators. Let* $\|v_j\|_{m,1} \le N$, $j = 1, 2$, *for some* $N > 0$. *Then there exists a constant* $C = C(D, N, m)$ *such that*

$$\|v_2 - v_1\|_{L^\infty(D)} \le C(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{-\alpha}, \tag{1.11}$$

*where* $\alpha = m - 2$ *and* $\|\Phi_2 - \Phi_1\| = \|\Phi_2 - \Phi_1\|_{H^{1/2} \to H^{-1/2}}$.

THEOREM 1.2. *Let* $\sigma_1, \sigma_2$ *be two isotropic conductivities such that* $\Delta(\sigma_j^{1/2})/\sigma_j^{1/2}$ *satisfies conditions* (1.8), *where* $D$ *is fixed and* $0 < \sigma_{\min} \le \sigma_j \le \sigma_{\max} < +\infty$ *for* $j = 1, 2$ *and some constants* $\sigma_{\min}$ *and* $\sigma_{\max}$. *Let* $\Lambda_1$ , $\Lambda_2$ *be the corresponding Dirichlet-to-Neumann operators and* $\|\Delta(\sigma_j^{1/2})/\sigma_j^{1/2}\|_{m,1} \le N$, $j = 1, 2$, *for some* $N > 0$. *We suppose, for simplicity, that* $\mathrm{supp}\,(\sigma_j - 1) \subset D$ *for* $j = 1, 2$. *Then, for any* $\alpha < m$ *there exists a constant* $C = C(D, N, \sigma_{\min}, \sigma_{\max}, m, \alpha)$ *such that*

$$\|\sigma_2 - \sigma_1\|_{L^\infty(D)} \le C(\log(3 + \|\Lambda_2 - \Lambda_1\|^{-1}))^{-\alpha}, \tag{1.12}$$

*where* $\|\Lambda_2 - \Lambda_1\| = \|\Lambda_2 - \Lambda_1\|_{H^{1/2} \to H^{-1/2}}$.

The main feature of these estimates is that, as $m \to +\infty$, we have $\alpha \to +\infty$. In addition we would like to mention that, under the assumptions of Theorems 1.1 and 1.2, according to instability estimates of Mandache [**19**] and Isaev [**15**], our results are almost optimal. Note that, in the linear approximation near the zero potential, Theorem 1.1 (without condition (1.10)) was proved in [**26**]. In dimension $d \ge 3$ a global stability estimate similar to our result (with respect to dependence on smoothness) was proved in [**25**]. More precisely, it was proved that in dimension $d \ge 3$ a stability estimate of the same type of (1.11) holds with the exponent $\alpha = m - d$.

In both theorems we made some assumptions on the support of our potentials and conductivities. These hypothesis can be taken away by the use of boundary determination results (see, for instance, [**4**, Proposition 2.11] for the Calderón problem); however, in that case, the exponent in the estimates will be generally smaller than the $\alpha$ of our theorems.

The proof of Theorem 1.1 relies on the $\bar\partial$-techniques introduced by Beals–Coifman [**6**], Henkin–R. Novikov [**14**], Grinevich–S. Novikov [**13**] and developed by R. Novikov [**21**] and Nachman [**20**] for solving the Calderón problem in two dimensions.

The Novikov–Nachman method starts with the construction of a special family of solutions $\psi(x, \lambda)$ of equation (1.2), which was originally introduced by Faddeev in [**11**]. These solutions have an exponential behaviour depending on the complex parameter $\lambda$ and they are constructed via some function $\mu(x, \lambda)$ (see (2.5)). One of the most important property of $\mu(x, \lambda)$ is that it satisfies a $\bar\partial$-equation with respect to the variable $\lambda$ (see equation (2.8)), in which appears the so-called Faddeev generalized scattering amplitude $h(\lambda)$ (defined in (2.6)). On the contrary, if one knows $h(\lambda)$ for every $\lambda \in \mathbb{C}$, it is possible to recover $\mu(x, \lambda)$ via this $\bar\partial$-equation. Starting from these arguments we will prove that the map $h(\lambda) \to \mu(z, \lambda)$ satisfies an Hölder condition, uniformly in the space variable $z$. This is done in Section 4.

Another part of the method relates the scattering amplitude $h(\lambda)$ to the Dirichlet-to-Neumann operator $\Phi$. In the present paper this is done using the Alessandrini identity (see [**1**]) and an estimate of $h(\lambda)$ for high values of $|\lambda|$ given in [**23**]. We find that the map $\Phi \to h$ has logarithmic stability in some natural norm (Proposition 3.3). This is explained in Section 3.

The final part of the method for the two problems is quite different. For Problem 2, in order to recover $\sigma(x)$ from $\mu(x, \lambda)$, we use a limit found for the first time in [**20**]. Instead, for Problem 1, we use an explicit formula for $v(x)$ which involves the scattering amplitude $h(\lambda)$, $\mu(x, \lambda)$ and its first (complex) derivative with respect to $z = x_1 + ix_2$ (see formula (5.3)). The two results are presented in section 5 and yield the proofs of Theorems 1.1 and 1.2.

This work was fulfilled in the framework of researches under the direction of R. G. Novikov.

## 2. Preliminaries

In this section we recall some definitions and properties of the Faddeev functions, the above-mentioned family of solutions of equation (1.2), which will be used throughout all the paper.

Following [**20**], we fix some $1 < p < 2$ and define $\psi(x, k)$ to be the solution of

$$(-\Delta + v)\psi(x, k) = 0 \text{ in } \mathbb{R}^2, \tag{2.1}$$

satisfying the condition $e^{-ixk}\psi(x, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2) = \{u \,:\, \partial^J u \in L^{\tilde{p}}(\mathbb{R}^2),\ |J| \leq 1\}$, where $x = (x_1, x_2) \in \mathbb{R}^2$, $k = (k_1, k_2) \in \mathcal{V} \subset \mathbb{C}^2$,

$$\mathcal{V} = \{k \in \mathbb{C}^2 \,:\, k^2 = k_1^2 + k_2^2 = 0\} \tag{2.2}$$

and

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{2}. \tag{2.3}$$

Condition (1.10) indeed guarantees that for every parameter $k \in \mathcal{V}$ there exists a unique solution $\psi(x, k)$ with the wanted properties (see Proposition 2.1 below).

The variety $\mathcal{V}$ can be written as $\{(\lambda, i\lambda) : \lambda \in \mathbb{C}\} \cup \{(\lambda, -i\lambda) : \lambda \in \mathbb{C}\}$. We henceforth denote $\psi(x, (\lambda, i\lambda))$ by $\psi(x, \lambda)$ and observe that, since $v$ is real-valued, uniqueness for (2.1) yields $\psi(x, (-\bar{\lambda}, i\bar{\lambda})) = \overline{\psi(x, (\lambda, i\lambda))} = \overline{\psi(x, \lambda)}$ so that, for reconstruction and stability purposes, it is sufficient to work on the sheet $k = (\lambda, i\lambda)$.

We now identify $\mathbb{R}^2$ with $\mathbb{C}$ and use the coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right),$$

where $(x_1, x_2) \in \mathbb{R}^2$.

Then we define

$$\psi(z,\lambda) = \psi(x,\lambda), \tag{2.4}$$

$$\mu(z,\lambda) = e^{-iz\lambda}\psi(z,\lambda), \tag{2.5}$$

$$h(\lambda) = \int_D e^{i\bar{z}\bar{\lambda}} v(z)\psi(z,\lambda) d\mathrm{Re}z\, d\mathrm{Im}z, \tag{2.6}$$

for $z, \lambda \in \mathbb{C}$.

Throughout all the paper $c(\alpha, \beta, \ldots)$ is a positive constant depending on parameters $\alpha, \beta, \ldots$

We now restate some fundamental results about Faddeev functions. In the following statement $\psi_0$ denotes $\sigma^{1/2}$.

PROPOSITION 2.1 (see [**20**]). *Let $D \subset \mathbb{R}^2$ be an open bounded domain with $C^2$ boundary, $v \in L^p(\mathbb{R}^2)$, $1 < p < 2$, $\operatorname{supp} v \subset D$, $\|v\|_{L^p(\mathbb{R}^2)} \le N$, be such that there exists a real-valued $\psi_0 \in L^\infty(\mathbb{R}^2)$ with $v = (\Delta\psi_0)/\psi_0$, $\psi_0(x) \ge c_0 > 0$ and $\psi_0 \equiv 1$ outside $D$. Then, for any $\lambda \in \mathbb{C}$ there is a unique solution $\psi(z,\lambda)$ of (2.1) with $e^{-iz\lambda}\psi(\cdot,\lambda) - 1$ in $L^{\tilde{p}} \cap L^\infty$ ($\tilde{p}$ is defined in (2.3)). Furthermore, $e^{-iz\lambda}\psi(\cdot,\lambda) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)$ and*

$$\|e^{-iz\lambda}\psi(\cdot,\lambda) - 1\|_{W^{s,\tilde{p}}} \le c(p,s)N|\lambda|^{s-1}, \tag{2.7}$$

*for $0 \le s \le 1$ and $\lambda$ sufficiently large.*

*The function $\mu(z,\lambda)$ defined in (2.5) satisfies the equation*

$$\frac{\partial \mu(z,\lambda)}{\partial \bar{\lambda}} = \frac{1}{4\pi\bar{\lambda}} h(\lambda) e_{-\lambda}(z)\overline{\mu(z,\lambda)}, \qquad z,\lambda \in \mathbb{C}, \tag{2.8}$$

*in the $W^{1,\tilde{p}}$ topology, where $h(\lambda)$ is defined in (2.6) and the function $e_{-\lambda}(z)$ is defined as follows:*

$$e_\lambda(z) = e^{i(z\lambda + \bar{z}\bar{\lambda})}. \tag{2.9}$$

*In addition, the functions $h(\lambda)$ and $\mu(z,\lambda)$ satisfy*

$$\left\| \frac{h(\lambda)}{\bar{\lambda}} \right\|_{L^r(\mathbb{R}^2)} \le c(r,N), \text{ for all } r \in (\tilde{p}', \tilde{p}), \quad \frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = 1, \tag{2.10}$$

$$\sup_{z\in\mathbb{C}} \|\mu(z,\cdot) - 1\|_{L^r(\mathbb{C})} \le c(r,D,N), \qquad \text{for all } r \in (p', \infty] \tag{2.11}$$

*and*

$$|h(\lambda)| \le c(p,D,N)|\lambda|^\varepsilon, \tag{2.12}$$

$$\|\mu(\cdot,\lambda) - \psi_0\|_{W^{1,\tilde{p}}} \le c(p,D,N)|\lambda|^\varepsilon, \tag{2.13}$$

*for $\lambda \le \lambda_0(p,D,N)$ and $0 < \varepsilon < \frac{2}{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

REMARK 2.1. Equation (2.8) means that $\mu$ is a generalised analytic function in $\lambda \in \mathbb{C}$ (see [**29**]). In two-dimensional inverse scattering for the Schrödinger equation, the theory of generalised analytic functions was used for the first time in [**13**].

We recall that if $v \in W^{m,1}(\mathbb{R}^2)$ with $\operatorname{supp} v \subset D$, then $\|\hat{v}\|_m < +\infty$, where

$$\hat{v}(p) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ipx} v(x) dx, \qquad p \in \mathbb{C}^2, \tag{2.14}$$

$$\|u\|_m = \sup_{p \in \mathbb{R}^2} |(1 + |p|^2)^{m/2} u(p)|, \tag{2.15}$$

for a test function $u$.

In addition, if $v \in W^{m,1}(\mathbb{R}^2)$ with $\operatorname{supp} v \subset D$ and $m > 2$, we have, by Sobolev embedding, that

$$\|v\|_{L^\infty(D)} \le c(D) \|v\|_{m,1}, \tag{2.16}$$

so, in particular, the hypothesis $v \in L^p(\mathbb{R}^2)$, $\operatorname{supp} v \subset D$, in the statement of Proposition 2.1 is satisfied for every $1 < p < 2$ (since $D$ is bounded).

The following lemma is a variation of a result in [**23**]:

LEMMA 2.2. *Under the assumption* (1.8)*, there exists* $R = R(m, \|\hat{v}\|_m) > 0$ *such that*

$$|h(\lambda)| \le 8\pi^2 \|\hat{v}\|_m (1 + 4|\lambda|^2)^{-m/2}, \qquad \textit{for } |\lambda| > R. \tag{2.17}$$

PROOF. We consider the function $H(k,p)$ defined as

$$H(k,p) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(p-k)x} v(x) \psi(x,k) dx, \tag{2.18}$$

for $k \in \mathcal{V}$ (where $\mathcal{V}$ is defined in (2.2)), $p \in \mathbb{R}^2$ and $\psi(x,k)$ as defined at the beginning of this section.

We deduce that $h(\lambda) = (2\pi)^2 H(k(\lambda), k(\lambda) + \overline{k(\lambda)})$, for $k(\lambda) = (\lambda, i\lambda)$. By [**23**, Corollary 1.1] we have

$$|H(k,p)| \le 2\|\hat{v}\|_m (1 + p^2)^{-m/2} \qquad \text{for } |\lambda| > R, \tag{2.19}$$

for $R = R(m, \|\hat{v}\|_m) > 0$ and then the proof follows. ☐

We restate [**4**, Lemma 2.6], which will be useful in section 4.

LEMMA 2.3 ([**4**]). *Let* $a \in L^{s_1}(\mathbb{R}^2) \cap L^{s_2}(\mathbb{R}^2)$, $1 < s_1 < 2 < s_2 < \infty$ *and* $b \in L^s(\mathbb{R}^2)$, $1 < s < 2$. *Assume* $u$ *is a function in* $L^{\tilde{s}}(\mathbb{R}^2)$, *with* $\tilde{s}$ *defined as in* (2.3), *which satisfies*

$$\frac{\partial u(\lambda)}{\partial \bar{\lambda}} = a(\lambda) \bar{u}(\lambda) + b(\lambda), \qquad \lambda \in \mathbb{C}. \tag{2.20}$$

*Then there exists* $c > 0$ *such that*

$$\|u\|_{L^{\tilde{s}}} \le c \|b\|_{L^s} \exp(c(\|a\|_{L^{s_1}} + \|a\|_{L^{s_2}})). \tag{2.21}$$

We will make also use of the well-known Hölder's inequality, which we recall in a special case: for $f \in L^p(\mathbb{C})$, $g \in L^q(\mathbb{C})$ such that $1 \le p, q \le \infty$, $1 \le r < \infty$, $1/p + 1/q = 1/r$, we have

$$\|fg\|_{L^r(\mathbb{C})} \le \|f\|_{L^p(\mathbb{C})} \|g\|_{L^q(\mathbb{C})}.$$

## 3. From $\Phi$ to $h(\lambda)$

LEMMA 3.1. *Let the condition* (1.8) *holds. Then we have, for* $p \geq 1$,

$$\left\| \frac{h(\lambda)}{\bar{\lambda}} \right\|_{L^p(|\lambda|>R)} \leq c(p,m)\|\hat{v}\|_m \frac{1}{R^{m+1-2/p}}, \tag{3.1}$$

$$\|h\|_{L^p(|\lambda|>R)} \leq c(p,m)\|\hat{v}\|_m \frac{1}{R^{m-2/p}}, \tag{3.2}$$

*where* $R$ *is as in Lemma 2.2.*

PROOF. It's a corollary of Lemma 2.2. Indeed we have

$$\left\| \frac{h(\lambda)}{\bar{\lambda}} \right\|^p_{L^p(|\lambda|>R)} \leq c\|\hat{v}\|^p_m \int_{r>R} r^{1-mp-p} dr = \frac{c(p,m)\|\hat{v}\|^p_m}{R^{(m+1)p-2}}, \tag{3.3}$$

which gives (3.1). The proof of (3.2) is analogous. □

LEMMA 3.2. *Let* $D \subset \{x \in \mathbb{R}^2 : |x| \leq l\}$, $v_1, v_2$ *be two potentials satisfying* (1.3), (1.8), (1.10), *let* $\Phi_1, \Phi_2$ *the corresponding Dirichlet-to-Neumann operator and* $h_1, h_2$ *the corresponding generalised scattering amplitude. Let* $\|v_j\|_{m,1} \leq N$, $j = 1, 2$. *Then we have*

$$|h_2(\lambda) - h_1(\lambda)| \leq c(D,N)e^{2l|\lambda|}\|\Phi_2 - \Phi_1\|_{H^{1/2}\to H^{-1/2}}, \quad \lambda \in \mathbb{C}. \tag{3.4}$$

PROOF. We have the following identity:

$$h_2(\lambda) - h_1(\lambda) = \int_{\partial D} \overline{\psi_1(z,\lambda)}(\Phi_2 - \Phi_1)\psi_2(z,\lambda)|dz|, \tag{3.5}$$

where $\psi_j(z,\lambda)$ are the Faddeev functions associated to the potential $v_j$, $j = 1, 2$. This identity is a particular case of the one in [**24**, Theorem 1]: we refer to that paper for a proof.

From this identity we have:

$$|h_2(\lambda) - h_1(\lambda)| \leq \|\psi_1(\cdot,\lambda)\|_{H^{1/2}(\partial D)}\|\Phi_2 - \Phi_1\|_{H^{1/2}\to H^{-1/2}}\|\psi_2(\cdot,\lambda)\|_{H^{1/2}(\partial D)}. \tag{3.6}$$

Now take $\tilde{p} > 2$ and use the trace theorem to get

$$\begin{aligned}\|\psi_j(\cdot,\lambda)\|_{H^{1/2}(\partial D)} &\leq C\|\psi_j(\cdot,\lambda)\|_{W^{1,\tilde{p}}(D)} \leq Ce^{l|\lambda|}\|e^{-iz\lambda}\psi_j(\cdot,\lambda)\|_{W^{1,\tilde{p}}(D)} \\ &\leq Ce^{l|\lambda|}\left(\|e^{-iz\lambda}\psi_j(\cdot,\lambda) - 1\|_{W^{1,\tilde{p}}(D)} + \|1\|_{W^{1,\tilde{p}}(D)}\right), \qquad j = 1, 2,\end{aligned}$$

which from (2.7) and (2.11) is bounded by $C(D,N)e^{l|\lambda|}$. These estimates together with (3.6) give (3.4). □

The main results of this section are the following propositions:

PROPOSITION 3.3. *Let* $v_1, v_2$ *be two potentials satisfying* (1.3), (1.8), (1.10), *let* $\Phi_1, \Phi_2$ *the corresponding Dirichlet-to-Neumann operator and* $h_1, h_2$ *the corresponding* *generalised scattering amplitude. Let* $0<\varepsilon<1$, $1<p<\frac{2}{1-\varepsilon}$ *and* $\|v_j\|_{m,1}\le N$, $j=1,2$. *Then there exists a constant* $c=c(D,N,m,p)$ *such that*

$$\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p(\mathbb{C})}\le c\log(3+\|\Phi_2-\Phi_1\|^{-1}_{H^{1/2}\to H^{-1/2}})^{-(m+1-2/p)}. \tag{3.7}$$

Proposition 3.4. *Let* $v_1,v_2,\Phi_1,\Phi_2,h_1,h_2$ *be as in Proposition 3.3. Let* $p\ge 1$ *and* $\|v_j\|_{m,1}\le N$, $j=1,2$. *Then there exists a constant* $c=c(D,N,m,p)$ *such that*

$$\|h_2-h_1\|_{L^p(\mathbb{C})}\le c\log(3+\|\Phi_2-\Phi_1\|^{-1}_{H^{1/2}\to H^{-1/2}})^{-(m-2/p)}. \tag{3.8}$$

Proof of Proposition 3.3. Choose $a,b>0$, $a$ close to 0 and $b$ big to be determined and let

$$\delta=\|\Phi_2-\Phi_1\|_{H^{1/2}\to H^{-1/2}}. \tag{3.9}$$

We split down the left term of (3.7) as follows:

$$\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p(\mathbb{C})}\le\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p(|\lambda|<a)}+\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p(a<|\lambda|<b)}+\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p(|\lambda|>b)}.$$

From (2.12) we obtain, for $a\le\lambda_0(p,D,N)$,

$$\begin{aligned}\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p(|\lambda|<a)}&\le c(D,N,p)\left(\int_{|\lambda|<a}|\lambda|^{(\varepsilon-1)p}d\mathrm{Re}\lambda\, d\mathrm{Im}\lambda\right)^{\frac1p}\\&=c(D,N,p)a^{\varepsilon-1+2/p}.\end{aligned} \tag{3.10}$$

From Lemma 3.2 and (3.9) we get, for $0<a<1<b$,

$$\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p(a<|\lambda|<b)}\le c(D,N)\left(\frac{\delta}{a^{1-2/p}}+\delta e^{2lb}\right), \tag{3.11}$$

where the right side is obtained as the sum of the $L^p$ norm for $a<|\lambda|<1$ and $1<|\lambda|<b$, taking into account (3.4). From Lemma 3.1

$$\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p(|\lambda|>b)}\le\frac{c(N)}{b^{m+1-2/p}}. \tag{3.12}$$

We now define

$$a=\log(3+\delta^{-1})^{-\frac{m+1-2/p}{\varepsilon-1+2/p}},\qquad b=\beta\log(3+\delta^{-1}), \tag{3.13}$$

for $0<\beta<1/(2l)$, in order to have (3.10) and (3.12) of the order $\log(3+\delta^{-1})^{-(m+1-2/p)}$. We also choose $\bar\delta<1$ such that for every $\delta\le\bar\delta$, $a$ is sufficiently small in order to have (2.12) (which yields (3.10)), $b\ge R$ (with $R$ as in Lemma 2.2) and also

$$\frac{\delta}{a^{1-2/p}}=\delta\log(3+\delta^{-1})^{\left(\frac{m+1-2/p}{\varepsilon-1+2/p}\right)(1-2/p)}<\log(3+\delta^{-1})^{-(m+1-2/p)}. \tag{3.14}$$

Thus we obtain

$$\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p(\mathbb{C})} \le \frac{c(D,N,p)}{\log(3+\delta^{-1})^{m+1-2/p}} + c(D,N)\delta(3+\delta^{-1})^{2l\beta}, \tag{3.15}$$

for $\delta \le \bar\delta$, $0<\beta<1/(2l)$. As $\delta(3+\delta^{-1})^{2l\beta}\to 0$ for $\delta\to 0$ more rapidly than the other term, we obtain that

$$\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p(\mathbb{C})} \le \frac{c(D,N,m,p,\beta)}{\log(3+\delta^{-1})^{m+1-2/p}}, \tag{3.16}$$

for $\delta \le \bar\delta$, $0<\beta<1/(2l)$.

Estimate (3.16) for general $\delta$ (with modified constant) follows from (3.16) for $\delta\le\bar\delta$ and the property (2.10) of the scattering amplitude. This completes the proof of Proposition 3.3. □

Proof of Proposition 3.4. We follow almost the same scheme as in the proof of Proposition 3.3. Let choose $b>0$ big to be determined and let

$$\delta = \|\Phi_2-\Phi_1\|_{H^{1/2}\to H^{-1/2}}. \tag{3.17}$$

We split down the left term of (3.8) as follows:

$$\|h_2-h_1\|_{L^p(\mathbb{C})} \le \|h_2-h_1\|_{L^p(|\lambda|<b)} + \|h_2-h_1\|_{L^p(|\lambda|\ge b)}.$$

From Lemma 3.2 we obtain

$$\|h_2-h_1\|_{L^p(|\lambda|<b)} \le c(D,N,p)\delta b^{1/p}e^{2lb}, \tag{3.18}$$

and from (3.2)

$$\|h_2-h_1\|_{L^p(|\lambda|\ge b)} \le c(N,p,m)\frac{1}{b^{m-2/p}}. \tag{3.19}$$

Define $b=\beta\log(3+\delta^{-1})$ for $0<\beta<1/(2l)$. Let $\bar\delta<1$ such that for $\delta\le\bar\delta$ we have that $b>R$, where $R$ is defined in Lemma 2.2.

Then we have, for $\delta\le\bar\delta$,

$$\|h_2-h_1\|_{L^p(\mathbb{C})} \le c(D,N,m,p)\delta(1+\delta^{-1})^{2l\beta}(\beta\log(3+\delta^{-1}))^{1/p} + c(N,m,p)(\log(3+\delta^{-1}))^{-(m-2/p)}.$$

Since $2l\beta<1$, we have that

$$\delta(1+\delta^{-1})^{2l\beta}(\beta\log(3+\delta^{-1}))^{1/p}\to 0 \quad \text{for } \delta\to 0$$

more rapidly than the other term. Thus

$$\|h_2-h_1\|_{L^p(\mathbb{C})} \le c(D,N,m,p,\beta)(\log(3+\delta^{-1}))^{-(m-2/p)}, \tag{3.20}$$

for $\delta\le\bar\delta$, $0<\beta<1/(2l)$.

Estimate (3.20) for general $\delta$ (with modified constant) follows from (3.20) for $\delta\le\bar\delta$ and the $L^p$-boundedness of the scattering amplitude (this because it is continuous and decays at infinity like in Lemma 3.1). This completes the proof of Proposition 3.4. □

## 4. Estimates of the Faddeev functions

LEMMA 4.1. *Let $v_1, v_2$ be two potentials satisfying* (1.8), (1.10), *with $\|v_j\|_{m,1} \leq N$, $h_1, h_2$ the corresponding scattering amplitude and $\mu_1(z,\lambda), \mu_2(z,\lambda)$ the corresponding Faddeev functions. Let $1 < s < 2$, and $\tilde{s}$ be as in* (2.3). *Then*

$$\sup_{z\in\mathbb{C}} \|\mu_2(z,\cdot) - \mu_1(z,\cdot)\|_{L^{\tilde{s}}(\mathbb{C})} \leq c(D,N,s) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})}, \tag{4.1}$$

$$\sup_{z\in\mathbb{C}} \left\| \frac{\partial \mu_2(z,\cdot)}{\partial z} - \frac{\partial \mu_1(z,\cdot)}{\partial z} \right\|_{L^{\tilde{s}}(\mathbb{C})} \leq c(D,N,s) \left[ \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})} + \|h_2 - h_1\|_{L^s(\mathbb{C})} \right] \tag{4.2}$$

PROOF. We begin with the proof of (4.1). Let

$$\nu(z,\lambda) = \mu_2(z,\lambda) - \mu_1(z,\lambda). \tag{4.3}$$

From the $\bar{\partial}$-equation (2.8) we deduce that $\nu$ satisfies the following non-homogeneous $\bar{\partial}$-equation:

$$\frac{\partial}{\partial \bar{\lambda}} \nu(z,\lambda) = \frac{e_{-\lambda}(z)}{4\pi} \left( \frac{h_1(\lambda)}{\bar{\lambda}} \overline{\nu(z,\lambda)} + \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \overline{\mu_2(z,\lambda)} \right), \tag{4.4}$$

for $\lambda \in \mathbb{C}$, where $e_{-\lambda}(z)$ is defined in (2.9). Note that since, by Sobolev embedding, $v \in L^\infty(D) \subset L^s(D)$, we have that $\nu(z,\cdot) \in L^{\tilde{s}}(\mathbb{C})$ for every $\tilde{s} > 2$ (see (2.11)). In addition, from Proposition 2.1 (see (2.10)) we have that $h(\lambda)/\bar{\lambda} \in L^p(\mathbb{C})$, for $1 < p < \infty$. Then it is possible to use Lemma 2.3 in order to obtain

$$\begin{aligned} \|\nu(z,\cdot)\|_{L^{\tilde{s}}} &\leq c(D,N,s) \left\| \overline{\mu_2(z,\lambda)} \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})} \\ &\leq c(D,N,s) \sup_{z\in\mathbb{C}} \|\mu_2(z,\cdot)\|_{L^\infty} \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})} \\ &\leq c(D,N,s) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})}, \end{aligned}$$

where we used again the property (2.11) of $\mu_2(z,\lambda)$.

Now we pass to (4.2). To simplify notations we write, for $z, \lambda \in \mathbb{C}$,

$$\mu_z^j(z,\lambda) = \frac{\partial \mu_j(z,\lambda)}{\partial z}, \quad \mu_{\bar{z}}^j(z,\lambda) = \frac{\partial \mu_j(z,\lambda)}{\partial \bar{z}}, \quad j = 1,2.$$

From the $\bar{\partial}$-equation (2.8) we have that $\mu_z^j$ and $\mu_{\bar{z}}^j$ satisfy the following system of non-homogeneous $\bar{\partial}$-equations, for $j = 1, 2$:

$$\frac{\partial}{\partial\bar{\lambda}}\mu_z^j(z,\lambda) = \frac{e_{-\lambda}(z)}{4\pi}\frac{h_j(\lambda)}{\bar{\lambda}}\left(\overline{\mu_{\bar{z}}^j(z,\lambda)} - i\lambda\overline{\mu_j(z,\lambda)}\right),$$
$$\frac{\partial}{\partial\bar{\lambda}}\mu_{\bar{z}}^j(z,\lambda) = \frac{e_{-\lambda}(z)}{4\pi}\frac{h_j(\lambda)}{\bar{\lambda}}\left(\overline{\mu_z^j(z,\lambda)} - i\bar{\lambda}\overline{\mu_j(z,\lambda)}\right).$$

Define now $\mu_\pm^j(z,\lambda) = \mu_z^j(z,\lambda) \pm \mu_{\bar{z}}^j(z,\lambda)$, for $j = 1, 2$. Then they satisfy the following two non-homogeneous $\bar{\partial}$-equations:

$$\frac{\partial}{\partial\bar{\lambda}}\mu_\pm^j(z,\lambda) = \pm\frac{e_{-\lambda}(z)}{4\pi}\frac{h_j(\lambda)}{\bar{\lambda}}\left(\overline{\mu_\pm^j(z,\lambda)} \mp i(\lambda \pm \bar{\lambda})\overline{\mu_j(z,\lambda)}\right).$$

Finally define $\tau_\pm(z,\lambda) = \mu_\pm^2(z,\lambda) - \mu_\pm^1(z,\lambda)$. They satisfy the two non-homogeneous $\bar{\partial}$-equations below:

$$\frac{\partial}{\partial\bar{\lambda}}\tau_\pm(z,\lambda) = \pm\frac{e_{-\lambda}(z)}{4\pi}\left[\frac{h_1(\lambda)}{\bar{\lambda}}\overline{\tau_\pm(z,\lambda)} + \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}}\overline{\mu_\pm^2(z,\lambda)}\right.$$
$$\left.\mp i\frac{\lambda \pm \bar{\lambda}}{\bar{\lambda}}\left((h_2(\lambda) - h_1(\lambda))\,\overline{\mu_2(z,\lambda)} + h_1(\lambda)\overline{\nu(z,\lambda)}\right)\right],$$

where $\nu(z,\lambda)$ was defined in (4.3).

Now remark that by [**23**, Lemma 2.1] and regularity assumptions on the potentials we have that $\mu_z^j(z,\cdot), \mu_{\bar{z}}^j(z,\cdot) \in L^{\tilde{s}}(\mathbb{C}) \cap L^\infty(\mathbb{C})$ for any $\tilde{s} > 2$, $j = 1, 2$. This, in particular, yields $\tau_\pm(z,\cdot) \in L^{\tilde{s}}(\mathbb{C})$. These arguments, along with the above remarks on the $L^p$ boundedness of $h_j(\lambda)/\bar{\lambda}$, make possible to use Lemma 2.3, which gives

$$\begin{aligned}
\|\tau_\pm(z,\cdot)\|_{L^{\tilde{s}}(\mathbb{C})} &\le c(D,N,s)\left[\left\|\frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}}\overline{\mu_\pm^2(z,\cdot)}\right\|_{L^s(\mathbb{C})}\right. \\
&\qquad \left. + \|(h_2(\cdot) - h_1(\cdot))\overline{\mu_2(z,\cdot)}\|_{L^s(\mathbb{C})} + \|h_1(\cdot)\overline{\nu(z,\cdot)}\|_{L^s(\mathbb{C})}\right] \\
&\le c(D,N,s)\left[\left\|\frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}}\right\|_{L^s(\mathbb{C})} + \|h_2 - h_1\|_{L^s(\mathbb{C})}\right. \\
&\qquad \left. + \|h_1\|_{L^2(\mathbb{C})}\|\nu(z,\cdot)\|_{L^{\tilde{s}}(\mathbb{C})}\right] \\
&\le c(D,N,s)\left[\left\|\frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}}\right\|_{L^s(\mathbb{C})} + \|h_2 - h_1\|_{L^s(\mathbb{C})}\right],
\end{aligned}$$

where we used Hölder's inequality (since $1/s = 1/2 + 1/\tilde{s}$) and estimate (4.1). The proof of (4.2) now follows from this last inequality and the fact that $\mu_z^2 - \mu_z^1 = \frac{1}{2}(\tau_+ + \tau_-)$. □

REMARK 4.1. We also have proved that

$$\sup_{z\in\mathbb{C}}\left\|\frac{\partial\mu_2(z,\cdot)}{\partial\bar z}-\frac{\partial\mu_1(z,\cdot)}{\partial\bar z}\right\|_{L^{\tilde s}(\mathbb{C})}\le c(D,N,s)\left[\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^s(\mathbb{C})}+\|h_2-h_1\|_{L^s(\mathbb{C})}\right].$$

We will need the following consequence of Lemma 4.1.

LEMMA 4.2. *Let $v_1, v_2$ be two potentials satisfying* (1.3), (1.8), (1.10), *with $\|v_j\|_{m,1}\le N$. Let $h_1, h_2$ be the corresponding scattering amplitude and $\mu_1(z,\lambda), \mu_2(z,\lambda)$ the corresponding Faddeev functions. Let $p, p'$ such that $1<p<2<p'<\infty$, $1/p+1/p'=1$. Then*

$$\|\mu_2(\cdot,0)-\mu_1(\cdot,0)\|_{L^\infty(D)}\le c(D,N,p)\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p(\mathbb{C})\cap L^{p'}(\mathbb{C})}. \tag{4.5}$$

PROOF. We recall again that if $v\in W^{m,1}(\mathbb{R}^2)$, $m>2$, with $\operatorname{supp} v\subset D$ then $v\in L^p(D)$ for $p\in[1,\infty]$; in particular, from Proposition 2.1, this yields $h(\lambda)/\bar\lambda\in L^p(\mathbb{C})$, for $1<p<\infty$.

We write, as in the preceding proof,

$$\nu(z,\lambda)=\mu_2(z,\lambda)-\mu_1(z,\lambda), \tag{4.6}$$

which satisfies the non-homogeneous $\bar\partial$-equations (4.4). From this equation we obtain

$$\begin{aligned}|\nu(z,0)|&=\frac{1}{\pi}\left|\int_{\mathbb{C}}\frac{e_{-\lambda}(z)}{4\pi\lambda}\frac{h_1(\lambda)}{\bar\lambda}\overline{\nu(z,\lambda)}d\mathrm{Re}\lambda\,d\mathrm{Im}\lambda\right.\\&\qquad\left.+\int_{\mathbb{C}}\frac{e_{-\lambda}(z)}{4\pi\lambda}\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\overline{\mu_2(z,\lambda)}d\mathrm{Re}\lambda\,d\mathrm{Im}\lambda\right|\\&\le\frac{1}{4\pi^2}\sup_{z\in\mathbb{C}}\|\nu(z,\cdot)\|_{L^r}\left\|\frac{h_1(\lambda)}{\lambda\bar\lambda}\right\|_{L^{r'}}\\&\qquad+\frac{1}{4\pi^2}\sup_{z\in\mathbb{C}}\|\mu_2(z,\cdot)\|_{L^\infty}\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\lambda\bar\lambda}\right\|_{L^1}\end{aligned} \tag{4.7}$$

where $1/r+1/r'=1$, $1<r'<2<r<\infty$. The number $s=2r/(r+2)$ can be chosen $s<2$ and as close to 2 as wanted, by taking $r$ big enough.

Then

$$\left\|\frac{h_1(\lambda)}{\lambda\bar\lambda}\right\|_{L^{r'}(|\lambda|<R)}\le\left\|\frac{h_1(\lambda)}{\bar\lambda}\right\|_{L^p}\left\|\frac{1}{\lambda}\right\|_{L^q(|\lambda|<R)}\le c(N,r), \tag{4.8}$$

where we have chosen $p>2$ such that $\left\|h_1(\lambda)/\bar\lambda\right\|_{L^p}\le c(N,p)$ from (2.10) and also, since $1/q=1/r'-1/p=1-1/r-1/p$, $q$ can be chosen less than 2 by taking $r$ big enough depending on $p$. With the same choice of $p,q$ we also obtain

$$\left\|\frac{h_1(\lambda)}{\lambda\bar{\lambda}}\right\|_{L^{r'}(|\lambda|>R)} \le \left\|\frac{h_1(\lambda)}{\bar{\lambda}}\right\|_{L^q}\left\|\frac{1}{\lambda}\right\|_{L^p(|\lambda|>R)} \le c(N,r). \tag{4.9}$$

From Lemma 4.1 with $r=\tilde{s}=2s/(2-s)$ we get

$$\sup_{z\in\mathbb{C}}\|\nu(z,\cdot)\|_{L^r} \le c(D,N,r)\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar{\lambda}}\right\|_{L^s(\mathbb{C})}, \tag{4.10}$$

and from (2.11)

$$\sup_{z,\lambda\in\mathbb{C}}|\mu_2(z,\lambda)| \le c(D,N). \tag{4.11}$$

Finally

$$\begin{aligned}\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\lambda\bar{\lambda}}\right\|_{L^1} \le & \left\|\frac{1}{\lambda}\right\|_{L^p(|\lambda|>R)}\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar{\lambda}}\right\|_{L^{p'}} \\ & + \left\|\frac{1}{\lambda}\right\|_{L^{p'}(|\lambda|<R)}\left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar{\lambda}}\right\|_{L^p},\end{aligned} \tag{4.12}$$

by taking $p'=s$ and $p$ such that $1/p+1/p'=1$. Now (4.5) follow from (4.6)–(4.12); this finishes the proof of Lemma 4.2. □

## 5. Proof of Theorems 1.1 and 1.2

PROOF OF THEOREM 1.1. We begin with a remark, which takes inspiration from Problem 1 at non-zero energy (see, for instance, [**22**]).

Let $v(z)$ be a potential which satisfies the hypothesis of Theorem 1.1 and $\mu(z,\lambda)$ the corresponding Faddeev functions. Since $\mu(z,\lambda)$ satisfies (2.11), the $\bar{\partial}$-equation (2.8) and $h(\lambda)$ decreases at infinity like in Lemma 2.2, it is possible to write the following development:

$$\mu(z,\lambda) = 1+\frac{\mu_{-1}(z)}{\lambda}+O\left(\frac{1}{|\lambda|^2}\right), \qquad \lambda\to\infty, \tag{5.1}$$

for some function $\mu_{-1}(z)$. If we insert (5.1) into equation (2.1), for $\psi(z,\lambda)=e^{iz\lambda}\mu(z,\lambda)$, we obtain, letting $\lambda\to\infty$,

$$v(z) = 4i\frac{\partial\mu_{-1}(z)}{\partial\bar{z}}, \qquad z\in\mathbb{C}. \tag{5.2}$$

We can write this in a more explicit form, using the following integral equation (a consequence of (2.8)):

$$\mu(z,\lambda)-1 = \frac{1}{8\pi^2 i}\int_{\mathbb{C}}\frac{h(\lambda')}{(\lambda'-\lambda)\bar{\lambda}'}e_{-\lambda'}(z)\overline{\mu(z,\lambda')}d\lambda'\,d\bar{\lambda}'.$$

By Lebesgue's dominated convergence (using (2.12)) we obtain

$$\mu_{-1}(z) = -\frac{1}{8\pi^2 i}\int_{\mathbb{C}} \frac{h(\lambda)}{\bar\lambda} e_{-\lambda}(z)\overline{\mu(z,\lambda)}d\lambda\, d\bar\lambda,$$

and the explicit formula

$$v(z) = \frac{1}{2\pi^2}\int_{\mathbb{C}} e_{-\lambda}(z)\left(ih(\lambda)\overline{\mu(z,\lambda)} - \frac{h(\lambda)}{\bar\lambda}\overline{\left(\frac{\partial\mu(z,\lambda)}{\partial z}\right)}\right)d\lambda\, d\bar\lambda. \tag{5.3}$$

Formula (5.3) for $v_1$ and $v_2$ yields

$$\begin{aligned}
v_2(z) - v_1(z) = \frac{1}{2\pi^2}\int_{\mathbb{C}} e_{-\lambda}(z)\Bigg[& i(h_2(\lambda) - h_1(\lambda))\overline{\mu_2(z,\lambda)}\\
&+ ih_1(\lambda)(\overline{\mu_2(z,\lambda)} - \overline{\mu_1(z,\lambda)})\\
&- \frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\overline{\left(\frac{\partial\mu_2(z,\lambda)}{\partial z}\right)}\\
&- \frac{h_1(\lambda)}{\bar\lambda}\overline{\left(\frac{\partial\mu_2(z,\lambda)}{\partial z} - \frac{\partial\mu_1(z,\lambda)}{\partial z}\right)}\Bigg]d\lambda\, d\bar\lambda.
\end{aligned}$$

Then, using several times Hölder's inequality, we find

$$\begin{aligned}
|v_2(z) - v_1(z)| \le \frac{1}{2\pi^2}\Bigg(& \|\mu_2(z,\cdot)\|_{L^\infty}\|h_2 - h_1\|_{L^1}\\
&+ \|h_1\|_{L^{\tilde p'}}\|\mu_2(z,\cdot) - \mu_1(z,\cdot)\|_{L^{\tilde p}}\\
&+ \left\|\frac{h_2(\lambda)-h_1(\lambda)}{\bar\lambda}\right\|_{L^p}\left\|\frac{\partial\mu_2(z,\cdot)}{\partial z}\right\|_{L^{p'}}\\
&+ \left\|\frac{h_1(\lambda)}{\bar\lambda}\right\|_{L^{\tilde p'}}\left\|\frac{\partial\mu_2(z,\cdot)}{\partial z} - \frac{\partial\mu_1(z,\cdot)}{\partial z}\right\|_{L^{\tilde p}}\Bigg),
\end{aligned}$$

for $1 < p < 2$, $\tilde p$ defined as in (2.3) and $1/p + 1/p' = 1/\tilde p + 1/\tilde p' = 1$. From (2.11), (2.10), the continuity of $h_j$ and Lemma 2.2, [**23**, Lemma 2.1] (see the end of the proof of Lemma 4.1 for more details), Lemma 4.1, Propositions 3.4 and 3.3 we finally obtain

$$\begin{aligned}
\|v_2 - v_1\|_{L^\infty(D)} \le c(D,N,m,p)\Bigg(& \log(3 + \|\Phi_2 - \Phi_1\|^{-1}_{H^{1/2}\to H^{-1/2}})^{-(m-2)}\\
&+ \log(3 + \|\Phi_2 - \Phi_1\|^{-1}_{H^{1/2}\to H^{-1/2}})^{-(m+1-2/p)}\\
&+ \log(3 + \|\Phi_2 - \Phi_1\|^{-1}_{H^{1/2}\to H^{-1/2}})^{-(m-2/p)}\Bigg)\\
\le c(D,N,m,p)\log(3 + \|&\Phi_2 - \Phi_1\|^{-1}_{H^{1/2}\to H^{-1/2}})^{-(m-2)}.
\end{aligned}$$

This finishes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We first extend $\sigma$ on the whole plane by putting $\sigma(x) = 1$ for $x \in \mathbb{R}^2 \setminus D$ (this extension is smooth by our hypothesis on $\sigma$). Now since $\sigma_j|_{\partial D} = 1$ and $\frac{\partial \sigma_j}{\partial \nu}|_{\partial D} = 0$ for $j = 1, 2$, from (1.5) we deduce that

$$\Phi_j = \Lambda_j, \qquad j = 1, 2. \tag{5.4}$$

In addition, from (2.13) we get

$$\lim_{\lambda \to 0} \mu_j(z, \lambda) = \sigma_j^{1/2}(z), \qquad j = 1, 2; \tag{5.5}$$

thus we obtain, using the fact that $\sigma_j$ is bounded from above and below, for $j = 1, 2$,

$$\begin{aligned} \|\sigma_2 - \sigma_1\|_{L^\infty(D)} &\le c(N) \|\sigma_2^{1/2} - \sigma_1^{1/2}\|_{L^\infty(D)} \\ &= c(N) \|\mu_2(\cdot, 0) - \mu_1(\cdot, 0)\|_{L^\infty(D)}. \end{aligned} \tag{5.6}$$

Now fix $\alpha < m$ and take $p$ such that

$$\max\left(1, \frac{2}{m - \alpha + 1}\right) < p < 2.$$

From Lemma 4.2 we have

$$\|\mu_2(\cdot, 0) - \mu_1(\cdot, 0)\|_{L^\infty(D)} \le c(D, N, p) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C}) \cap L^{p'}(\mathbb{C})}, \tag{5.7}$$

where $1/p + 1/p' = 1$. From Proposition 3.3

$$\begin{aligned} \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C}) \cap L^{p'}(\mathbb{C})} &\le c(D, N, p) \log(3 + \|\Phi_2 - \Phi_1\|^{-1}_{H^{1/2} \to H^{-1/2}})^{-(m+1-2/p)} \\ &\le c(D, N, p) \log(3 + \|\Phi_2 - \Phi_1\|^{-1}_{H^{1/2} \to H^{-1/2}})^{-\alpha} \\ &= c(D, N, p) \log(3 + \|\Lambda_2 - \Lambda_1\|^{-1}_{H^{1/2} \to H^{-1/2}})^{-\alpha}, \end{aligned}$$

from (5.4) and since $\alpha < m + 1 - \frac{2}{p}$. Theorem 1.2 is thus proved. □

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# Paper **H**

PAPER H

# Stability estimates for an inverse problem for the Schrödinger equation at negative energy in two dimensions

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ABSTRACT. We study the inverse problem of determining a real-valued potential in the two-dimensional Schrödinger equation at negative energy from the Dirichlet-to-Neumann map. It is known that the problem is ill-posed and a stability estimate of logarithmic type holds. In this paper we prove three new stability estimates. The main feature of the first one is that the stability increases exponentially with respect to the smoothness of the potential, in a sense to be made precise. The others show how the first estimate depends on the energy. In particular it is found that for high energies the stability estimate changes, in some sense, from logarithmic type to Lipschitz type: in this sense the ill-posedness of the problem decreases when increasing the energy (in modulus).

## 1. Introduction

The problem of the recovery of a potential in the Schrödinger equation from boundary measurements, the Dirichlet-to-Neumann map, has been studied since the 1980s, namely in connection with Calderón's inverse conductivity problem. The aim of this paper is to give new insights about its stability issues.

It is well known that the problem is ill-posed: Alessandrini [**1**] proved that a logarithmic stability holds and Mandache [**20**] showed that it was optimal, in some sense. Nevertheless, Mandache's result provided also the information that stability could be increased in a way depending on the smoothness of potentials. Optimal stability estimates, with respect to smoothness of potentials, were indeed recently obtained in [**27**] and [**31**] in dimensions $d \geq 3$ and $d = 2$, respectively (at zero energy). However, even for smooth potentials the problem remains ill-posed.

It was observed that one way to increase stability is to modify another factor in the equation: the energy. Indeed, at high energies the ill-posedness diminishes considerably: this motivated some rapidly converging approximation algorithms in two and three dimensions [**24**], [**26**], [**29**] and stability estimates of Lipschitz-logarithmic type explicitly depending on the energy in three dimensions [**16**].

In this paper we continue the work started in [**31**], at zero energy (the Calderón problem), and give new stability estimates depending on the smoothness of potentials and the energy. We restricted ourself to the negative energy case, for the simplicity of the proofs. Results for the positive energy case are indeed similar in many respects and will be published in a subsequent paper.

We consider the Schrödinger equation at fixed energy $E$,

$$(-\Delta + v)\psi = E\psi \quad \text{on } D, \quad E \in \mathbb{R}, \tag{1.1}$$

where $D$ is a open bounded domain in $\mathbb{R}^2$ and $v \in L^\infty(D)$ (we will refer to $v$ as a *potential*). Under the assumption that

$$0 \text{ is not a Dirichlet eigenvalue for the operator } -\Delta + v - E \text{ in } D, \tag{1.2}$$

we can define the Dirichlet-to-Neumann operator $\Phi(E) : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$, corresponing to the potential $v$, as follows:

$$\Phi(E)f = \left.\frac{\partial u}{\partial \nu}\right|_{\partial D}, \tag{1.3}$$

where $f \in H^{1/2}(\partial D)$, $\nu$ is the outer normal of $\partial D$, and $u$ is the $H^1(D)$-solution of the Dirichlet problem

$$(-\Delta + v)u = Eu \text{ on } D, \quad u|_{\partial D} = f. \tag{1.4}$$

The following inverse problem arises from this construction.

**Problem 1.** Given $\Phi(E)$ for a fixed $E \in \mathbb{R}$, find $v$ on $D$.

This problem can be considered as the Gel'fand inverse boundary value problem for the two-dimensional Schrödinger equation at fixed energy (see [**11**], [**22**]). At zero energy this problem can be seen also as a generalization of the Calderón problem of the electrical impedance tomography (see [**7**], [**22**]). In addition, the history of inverse problems for the two-dimensional Schrödinger equation at fixed energy goes back to [**9**] (see also [**23, 12**] and reference therein). Problem 1 can also be considered as an example of ill-posed problem: see [**18**], [**5**] for an introduction to this theory.

Note that this problem is not overdetermined, in the sense that we consider the reconstruction of a function $v$ of two variables from inverse problem data dependent on two variables.

In this paper we study interior stability estimates, i.e. we want to prove that given two Dirichlet-to-Neumann operators $\Phi_1(E)$ and $\Phi_2(E)$, corresponding to potentials $v_1$ and $v_2$ on $D$, we have that

$$\|v_1 - v_2\|_{L^\infty(D)} \le \omega\left(\|\Phi_1(E) - \Phi_2(E)\|_{H^{1/2}(\partial D) \to H^{-1/2}(\partial D)}\right),$$

where the function $\omega(t) \to 0$ as fast as possible as $t \to 0$ at any fixed $E$. The explicit dependence of $\omega$ on $E$ is analysed as well.

There is a wide literature on the Gel'fand inverse problem at fixed energy (i.e. Problem 1 in multidimensions). In the case of complex-valued potentials the global injectivity of the map $v \to \Phi$ was firstly proved in [**22**] for $D \subset \mathbb{R}^d$ with $d \ge 3$ and in [**6**] for $d = 2$: in particular, these results were obtained by the use of global reconstructions developed in the same papers. A global stability estimate for

Problem 1 for $d \geq 3$ was first found by Alessandrini in [**1**]; a principal improvement of this result was given recently in [**27**]. In the two-dimensional case the first global stability estimate was given in [**28**]. Note that for the Calderón problem (of the electrical impedance tomography) in its initial formulation the global uniqueness was firstly proved in [**32**] for $d \geq 3$ and in [**21**] for $d = 2$. In addition, for the case of piecewise constant or piecewise real analytic conductivity the first uniqueness results for the Calderón problem in dimension $d \geq 2$ were given in [**8**] and [**17**]. In the case of piecewise constant conductivities a Lipschitz stability estimate was proved in [**2**] (see [**30**] for additional studies in this direction).

Most stability results for the Calderón problem in two dimensions have been formulated with the goal of proving stability estimates using the least regular conductivities possible (see [**19**], [**4**]). Instead, we have tried to address different questions: how the estimates vary with respect to the smoothness of the potentials and the energy.

The results, detailed below, constitute also a progress in the non-smooth case: they indicate stability dependence of the smooth part of a singular potential with respect to boundary value data.

We will assume for simplicity that

$$\begin{aligned} &D \text{ is an open bounded domain in } \mathbb{R}^2, \qquad \partial D \in C^2, \\ &v \in W^{m,1}(\mathbb{R}^2) \text{ for some } m > 2, \quad \bar{v} = v, \quad \operatorname{supp} v \subset D, \end{aligned} \tag{1.5}$$

where

$$W^{m,1}(\mathbb{R}^2) = \{v \,:\, \partial^J v \in L^1(\mathbb{R}^2),\ |J| \leq m\}, \qquad m \in \mathbb{N} \cup \{0\}, \tag{1.6}$$

$$J \in (\mathbb{N} \cup \{0\})^2, \qquad |J| = J_1 + J_2, \qquad \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \partial x_2^{J_2}}.$$

Let

$$\|v\|_{m,1} = \max_{|J| \leq m} \|\partial^J v\|_{L^1(\mathbb{R}^2)}.$$

We will need the following regularity condition:

$$|E| > E_1, \tag{1.7}$$

where $E_1 = E_1(\|v\|_{m,1}, D)$. This condition implies, in particular, that the Faddeev eigenfunctions are well-defined on the entire fixed-energy surface in the spectral parameter.

THEOREM 1.1. *Let the conditions* (1.2), (1.5), (1.7) *hold for the potentials* $v_1, v_2$, *where* $D$ *is fixed, and let* $\Phi_1(E)$ , $\Phi_2(E)$ *be the corresponding Dirichlet-to-Neumann operators at fixed negative energy* $E < 0$. *Let* $\|v_j\|_{m,1} \leq N$, $j = 1, 2$, *for some* $N > 0$. *Then there exists a constant* $c_1 = c_1(E, D, N, m)$ *such that*

$$\|v_2 - v_1\|_{L^\infty(D)} \leq c_1 (\log(3 + \|\Phi_2(E) - \Phi_1(E)\|_*^{-1}))^{-\alpha}, \tag{1.8}$$

*where* $\alpha = m - 2$ *and* $\|\Phi_2 - \Phi_1\|_* = \|\Phi_2 - \Phi_1\|_{H^{1/2}(\partial D) \to H^{-1/2}(\partial D)}$.

*Moreover, there exists a constant* $c_2 = c_2(D, N, m)$ *such that for any* $0 < \kappa < 1/(l+2)$, *where* $l = \mathrm{diam}(D)$, *we have*

$$\|v_2 - v_1\|_{L^\infty(D)} \leq c_2 \Big[ \big(|E|^{1/2} + \kappa \log(3 + \delta^{-1})\big)^{-(m-2)} + \delta(3+\delta^{-1})^{\kappa(l+2)} e^{|E|^{1/2}(l+3)} \Big], \tag{1.9}$$

*where* $\delta = \|\Phi_2(E) - \Phi_1(E)\|_*$.

*In addition, there exists a constant* $c_3 = c_3(D, N, m)$ *such that for* $E, \delta$ *which satisfy*

$$|E|^{1/2} > \log(3 + \delta^{-1}), \quad |E| > 1, \tag{1.10}$$

*we have*

$$\|v_2 - v_1\|_{L^\infty(D)} \leq c_3 \Big[ |E|^{-(m-2)/2} \log(3+\delta^{-1})^{-(m-2)} + \delta e^{|E|(l+3)} \Big]. \tag{1.11}$$

The novelty of estimate (1.8), with respect to [**28**], is that, as $m \to +\infty$, we have $\alpha \to +\infty$. Moreover, under the assumption of Theorem 1.1, according to instability estimates of Mandache [**20**] and Isaev [**14**], our result is almost optimal. To be more precise, it was proved that stability estimate (1.8) cannot hold for $\alpha > 2m$ for real-valued potentials and $\alpha > m$ for complex-valued potentials. Indeed, our estimates are still valid for complex-valued potentials, if $|E|$ is sufficiently large with respect to $\|v\|_{C(\bar{D})}$.

In addition, estimate (1.8) extends the result obtained in [**31**] for the same problem at zero energy. In dimension $d \geq 3$ a global stability estimate similar to (1.8) was proved in [**27**], at zero energy.

As regards (1.9) and (1.11), their main feature is the explicit dependence on the energy $E$. These estimates consist each one of two parts, the first logarithmic and the second Hölder or Lipschitz; when $|E|$ increases, the logarithmic part decreases and the Hölder/Lipschitz part becomes dominant.

These estimates, namely (1.11), are coherent with the approximate reconstruction algorithm developed in [**24**] and [**29**] at positive energy. In fact, inequalities like (1.8), (1.9) and (1.11) should be valid also for the Schrödinger equation at positive energy.

Note that, for Problem 1 in three dimensions, global energy-dependent stability estimates changing from logarithmic type to Lipschitz type for high energies were given recently in [**16**]. However these estimates are given in the $L^2(D)$ norm and without any dependence on the smoothness of the potentials.

The proof of Theorem 1.1 follows the scheme of [**31**] and it is based on the same $\bar{\partial}$ techniques. The map $\Phi(E) \to v(x)$ is considered as the composition of $\Phi(E) \to r(\lambda)$ and $r(\lambda) \to v(x)$, where $r(\lambda)$ is a complex valued function, closely related to the so-called generalised scattering amplitude (see Section 2 for details).

The stability of $\Phi(E) \to r(\lambda)$ – previously known only for $E = 0$ – relies on an identity of [**25**] (based in particular on [**1**]), and estimates on $r(\lambda)$ for $\lambda$ near 0 and

$\infty$. The estimate is of logarithmic type, with respect to $\Phi$ (at fixed $E$): it is proved in section 3.

The stability of $r(\lambda) \to v(x)$ is of Hölder type and follows the same arguments as in [**31**, Section 4]. The composition of the two above-mentioned maps gives the result of Theorem 1.1, as showed in Section 4.

Remark 1.1. We point out another possible approach to obtain inequality (1.8). The approach is based on the following observation (which follows from [**13**, Basic Lemma]): for potentials $v$ satisfying the assumptions of Theorem 1.1 we have that $v - E$ is of conductivity type, i.e. there exists a positive real-valued function $\psi_0 \in L^\infty(D)$ bounded from below such that

$$v - E = \frac{\Delta \psi_0}{\psi_0}. \tag{1.12}$$

Thus Problem 1 at fixed negative energy is reduced to the the same problem at zero energy for the conductivity-type potential $\frac{\Delta\psi_0}{\psi_0}$. It is then possible to apply the result of [**31**] and find the same stability estimate.

Remark 1.2. In a similar way as in [**15**], the stability estimates of Theorem 1.1 can be extended to the case when we do not assume that condition (1.2) is fulfilled and consider the Cauchy data set instead of the Dirichlet-to-Neumann map $\Phi(E)$.

This work was fulfilled in the framework of research carried out under the supervision of R.G. Novikov.

## 2. Preliminaries

We recall the definition of the Faddeev eigenfunctions $\psi(x,k)$ of equation (1.1), for $x = (x_1, x_2) \in \mathbb{R}^2$, $k = (k_1, k_2) \in \Sigma_E \subset \mathbb{C}^2$, $\Sigma_E = \{k \in \mathbb{C}^2 : k^2 = k_1^2 + k_2^2 = E\}$ for $E \neq 0$ (see [**10**], [**23**], [**12**]). We first extend $v \equiv 0$ on $\mathbb{R}^2 \setminus D$ and define $\psi(x,k)$ as the solution of the following integral equation:

$$\psi(x,k) = e^{ikx} + \int_{y\in\mathbb{R}^2} G(x-y,k)v(y)\psi(y,k)dy, \tag{2.1}$$

$$G(x,k) = g(x,k)e^{ikx}, \tag{2.2}$$

$$g(x,k) = -\left(\frac{1}{2\pi}\right)^2 \int_{\xi\in\mathbb{R}^2} \frac{e^{i\xi x}}{\xi^2 + 2k\xi} d\xi, \tag{2.3}$$

where $x \in \mathbb{R}^2$, $k \in \Sigma_E \setminus \mathbb{R}^2$. It is convenient to write (2.1) in the following form

$$\mu(x,k) = 1 + \int_{y\in\mathbb{R}^2} g(x-y,k)v(y)\mu(y,k)dy, \tag{2.4}$$

where $\mu(x,k)e^{ikx} = \psi(x,k)$.

We define $\mathcal{E}_E \subset \Sigma_E \setminus \mathbb{R}^2$ the set of exceptional points of integral equation (2.4): $k \in \Sigma_E \setminus (\mathcal{E}_E \cup \mathbb{R}^2)$ if and only if equation (2.4) is uniquely solvable in $L^\infty(\mathbb{R}^2)$.

REMARK 2.1. From [**24**, Proposition 1.1] we have that there exists $E_0 = E_0(\|v\|_{m,1}, D)$ such that for $|E| \geq E_0(\|v\|_{m,1}, D)$ there are no exceptional points for equation (2.4), i.e. $\mathcal{E}_E = \emptyset$: thus the Faddeev eigenfunctions exist (unique) for all $k \in \Sigma_E \setminus \mathbb{R}^2$.

Following [**13**], [**23**], we make the following change of variables

$$z = x_1 + ix_2, \qquad \lambda = \frac{k_1 + ik_2}{\sqrt{E}},$$

and write $\psi, \mu$ as functions of these new variables. For $k \in \Sigma_E \setminus (\mathcal{E}_E \cup \mathbb{R}^2)$ we can define, for the corresponding $\lambda$, the following generalised scattering amplitude:

$$b(\lambda, E) = \frac{1}{(2\pi)^2}\int_{\mathbb{C}} \exp\left[\frac{i}{2}\sqrt{E}\left(1 + (\operatorname{sgn} E)\frac{1}{\lambda\bar{\lambda}}\right)\right.$$
$$\left.\times\left((\operatorname{sgn} E)z\bar{\lambda} + \lambda\bar{z}\right)\right] v(z)\mu(z,\lambda)d\mathrm{Re}z\, d\mathrm{Im}z. \tag{2.5}$$

This function plays an important role in the inverse problem because of the following $\bar{\partial}$-equation, which holds when $v$ is real-valued (see [**23**] for more details):

$$\frac{\partial}{\partial\bar{\lambda}}\mu(z,\lambda) = r(z,\lambda)\overline{\mu(z,\lambda)}, \tag{2.6}$$

for $\lambda$ not an exceptional point (i.e. $k(\lambda) \in \Sigma_E \setminus (\mathcal{E}_E \cup \mathbb{R}^2)$), where

$$r(z,\lambda) = r(\lambda)\exp\left[\frac{i}{2}\sqrt{E}\left(1 + (\operatorname{sgn} E)\frac{1}{\lambda\bar{\lambda}}\right)\left((\operatorname{sgn} E)z\bar{\lambda} + \lambda\bar{z}\right)\right], \tag{2.7}$$

$$r(\lambda) = \frac{\pi}{\bar{\lambda}}\operatorname{sgn}(\lambda\bar{\lambda} - 1)b(\lambda, E). \tag{2.8}$$

We recall that if $v \in W^{m,1}(\mathbb{R}^2)$ with $\operatorname{supp} v \subset D$, then $\|\hat{v}\|_m < +\infty$, where

$$\hat{v}(p) = (2\pi)^{-2}\int_{\mathbb{R}^2} e^{ipx}v(x)dx, \qquad p \in \mathbb{C}^2, \tag{2.9}$$

$$\|u\|_m = \sup_{p\in\mathbb{R}^2} |(1+|p|^2)^{m/2}u(p)|, \tag{2.10}$$

for a test function $u$.

The following lemma is a variation of a result in [**24**]:

LEMMA 2.1. *Let the conditions* (1.5), (1.7) *hold for a potentials $v$ and let $E \in \mathbb{R}\setminus\{0\}$. Then there exists an $R = R(m, \|\hat{v}\|_m) > 1$, such that*

$$|b(\lambda, E)| \leq 2\|\hat{v}\|_m \left(1 + |E|\left(|\lambda| + \operatorname{sgn}(E)/|\lambda|\right)^2\right)^{-m/2}, \tag{2.11}$$

*for $|\lambda| > \frac{2R}{|E|^{1/2}}$ and $|\lambda| < \frac{|E|^{1/2}}{2R}$.*

PROOF. We consider the function $H(k,p)$ defined as

$$H(k,p) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i(p-k)x}v(x)\psi(x,k)dx, \tag{2.12}$$

for

$$k = k(\lambda) = \left(\frac{\sqrt{E}}{2}(\lambda+\lambda^{-1}), \frac{i\sqrt{E}}{2}(\lambda^{-1}-\lambda)\right), \tag{2.13}$$

$\lambda \in \mathbb{C}\setminus\{0\}$, $\operatorname{Im} k(\lambda) \neq 0$, $p \in \mathbb{R}^2$ and $\psi(x,k)$ as defined at the beginning of this section. Since $\mathcal{E}_E = \emptyset$ (see Remark 2.1), the function $H(k(\lambda), k(\lambda)+\overline{k(\lambda)}) = b(\lambda, E)$ is defined for every $\lambda \in \mathbb{C}\setminus\{0\}$. Then, by [**24**, Proposition 1.1, Corollary 1.1] (see also Remark 2.2) we have

$$|H(k,p)| \leq 2\|\hat{v}\|_m(1+p^2)^{-m/2}, \qquad \text{for } |k| > R(m, \|\hat{v}\|_m), \tag{2.14}$$

where $|k| = (|\mathrm{Re}k|^2 + |\mathrm{Im}k|^2)^{1/2}$. This finishes the proof of Lemma 2.1. □

At several points in the paper we will use [**24**, Lemma 2.1], which we restate in an adapted form.

LEMMA 2.2. *Let the conditions* (1.5), (1.7) *hold for a potentials* $v$*. Let* $\mu(x,k)$ *be the associated Faddeev functions. Then, for any* $0<\sigma<1$*, we have*

$$|\mu(x,k)-1| + \left|\frac{\partial\mu(x,k)}{\partial x_1}\right| + \left|\frac{\partial\mu(x,k)}{\partial x_2}\right| \leq |k|^{-\sigma}c(m,\sigma)\|\hat{v}\|_m, \tag{2.15}$$

*for* $k \in \mathbb{C}^2$ *such that* $k^2 < 0$ *and* $|k| \geq R$*, where* $R$ *is defined in Lemma 2.1.*

Throughout all the paper $c(\alpha,\beta,\ldots)$ is a positive constant depending on parameters $\alpha,\beta,\ldots$

REMARK 2.2. Even if [**24**, Proposition 1.1, Corollary 1.1, Lemma 2.1] were proved for $E>0$, they are still valid in the negative energy case (and zero energy case).

We also restate [**3**, Lemma 2.6], which will be useful in section 4.

LEMMA 2.3 ([**3**]). *Let* $q_1 \in L^{s_1}(\mathbb{C})\cap L^{s_2}(\mathbb{C})$, $1<s_1<2<s_2<\infty$ *and* $q_2 \in L^s(\mathbb{C})$, $1<s<2$*. Assume* $u$ *is a function in* $L^{\tilde{s}}(\mathbb{C})$*, with* $1/\tilde{s} = 1/s - 1/2$*, which satisfies*

$$\frac{\partial u(\lambda)}{\partial\bar{\lambda}} = q_1(\lambda)\bar{u}(\lambda) + q_2(\lambda), \qquad \lambda \in \mathbb{C}. \tag{2.16}$$

*Then there exists* $c = c(s,s_1,s_2) > 0$ *such that*

$$\|u\|_{L^{\tilde{s}}} \leq c\|q_2\|_{L^s}\exp(c(\|q_1\|_{L^{s_1}} + \|q_1\|_{L^{s_2}})). \tag{2.17}$$

We will make also use of the well-known Hölder's inequality, which we recall in a special case: for $f \in L^p(\mathbb{C})$, $g \in L^q(\mathbb{C})$ such that $1 \leq p,q \leq \infty$, $1 \leq r < \infty$, $1/p+1/q = 1/r$, we have

$$\|fg\|_{L^r(\mathbb{C})} \leq \|f\|_{L^p(\mathbb{C})}\|g\|_{L^q(\mathbb{C})}. \tag{2.18}$$

## 3. From $\Phi(E)$ to $r(\lambda)$

LEMMA 3.1. *Let the conditions* (1.5), (1.7) *hold and take* $0 < a_1 < \min\left(1, \frac{|E|^{1/2}}{2R}\right)$, $a_2 > \max\left(1, \frac{2R}{|E|^{1/2}}\right)$, *for* $E \in \mathbb{R} \setminus \{0\}$ *and* $R$ *as defined in Lemma 2.1. Then for* $p \geq 1$ *we have*

$$\big\||\lambda|^j r(\lambda)\big\|_{L^p(|\lambda|<a_1)} \leq c(p,m)\|\hat{v}\|_m |E|^{-m/2} a_1^{m-1+j+2/p}, \tag{3.1}$$

$$\big\||\lambda|^j r(\lambda)\big\|_{L^p(|\lambda|>a_2)} \leq c(p,m)\|\hat{v}\|_m |E|^{-m/2} a_2^{-m-1+j+2/p}, \tag{3.2}$$

*where* $j = 1, 0, -1$ *and* $r$ *was defined in* (2.8).

PROOF. It is a corollary of Lemma 2.1. Indeed $|r(\lambda)| = \pi|b(\lambda,E)|/|\lambda|$ and

$$\begin{aligned}\||\lambda|^j r\|^p_{L^p(|\lambda|<a_1)} &\leq c\left(\frac{\|\hat{v}\|_m}{|E|^{m/2}}\right)^p \int_{t<a_1} t^{1+(m-1+j)p} dt \\ &= c(p,m)\left(\frac{\|\hat{v}\|_m}{|E|^{m/2}}\right)^p a_1^{(m-1+j)p+2},\end{aligned}$$

which gives (3.1). The proof of (3.2) is analogous. □

LEMMA 3.2. *Let* $D \subset \{x \in \mathbb{R}^2 : |x| \leq l\}$, $E < 0$, $v_1, v_2$ *be two potentials satisfying* (1.2), (1.5), (1.7), $\Phi_1(E), \Phi_2(E)$ *the corresponding Dirichlet-to-Neumann operator and* $b_1, b_2$ *the corresponding generalised scattering amplitude. Let* $\|v_j\|_{m,1} \leq N$, $j = 1, 2$. *Then we have*

$$|b_2(\lambda) - b_1(\lambda)| \leq c(D,N) e^{(l+1)\sqrt{|E|}(|\lambda|+1/|\lambda|)} \|\Phi_2(E) - \Phi_1(E)\|_*, \ \lambda \in \mathbb{C} \setminus \{0\}. \tag{3.3}$$

PROOF. We have the following identity:

$$b_2(\lambda) - b_1(\lambda) = \left(\frac{1}{2\pi}\right)^2 \int_{\partial D} \psi_1(x, \overline{k(\lambda)})(\Phi_2(E) - \Phi_1(E))\psi_2(x, k(\lambda))dx, \tag{3.4}$$

where $\psi_i(x,k)$ are the Faddeev functions associated to the potential $v_i$, $i = 1, 2$. This identity is a particular case of the one in [**25**, Theorem 1]: we refer to that paper for a proof.

From this identity we obtain:

$$|b_2(\lambda) - b_1(\lambda)| \leq \frac{1}{(2\pi)^2} \|\psi_1(\cdot,k)\|_{H^{1/2}(\partial D)} \|\Phi_2(E) - \Phi_1(E)\|_* \|\psi_2(\cdot,k)\|_{H^{1/2}(\partial D)}. \tag{3.5}$$

Now, for $\tilde{p} > 2$, using the trace theorem and Lemma 2.2 we get

$$\begin{aligned}\|\psi_j(\cdot,k(\lambda))\|_{H^{1/2}(\partial D)} &\leq c\|\psi_j(\cdot,k(\lambda))\|_{W^{1,\tilde{p}}(D)} \\ &\leq c\frac{\sqrt{|E|}}{2} l(|\lambda| + 1/|\lambda|) e^{\frac{\sqrt{|E|}}{2} l(|\lambda|+1/|\lambda|)} \|\mu_j(\cdot,k(\lambda))\|_{W^{1,\tilde{p}}(D)} \\ &\leq c\, e^{\frac{\sqrt{|E|}}{2}(l+1)(|\lambda|+1/|\lambda|)} \|\mu_j(\cdot,k(\lambda))\|_{W^{1,\tilde{p}}(D)} \leq c(D,N,m) e^{\frac{\sqrt{|E|}}{2}(l+1)(|\lambda|+1/|\lambda|)},\end{aligned}$$

for $j = 1, 2$. This, combined with (3.5), gives (3.3). □

Now we turn to the main result of the section.

PROPOSITION 3.3. *Let $E<0$ be such that $|E|\geq E_1=\max((2R)^2,E_0)$, where $R$ is defined in Lemma 2.1 and $E_0$ in Remark 2.1, let $v_1,v_2$ be two potentials satisfying (1.2), (1.5), (1.7), $\Phi_1(E),\Phi_2(E)$ the corresponding Dirichlet-to-Neumann operator and $r_1,r_2$ as defined in (2.8). Let $\|v_k\|_{m,1}\leq N$, $k=1,2$. Then for every $p\geq 1$ there exists a constant $\theta_1=\theta_1(E,D,N,m,p)$ such that*

$$\||\lambda|^j|r_2-r_1|\|_{L^p(\mathbb{C})}\leq \theta_1\log(3+\delta^{-1})^{-(m-2)}, \tag{3.6}$$

*for $j=-1,0,1$, $\delta=\|\Phi_2(E)-\Phi_1(E)\|_*$. Moreover, there exists a constant $\theta_2=\theta_2(D,N,m,p)$ such that for any $0<\kappa<\frac{1}{l+2}$, where $l=\mathrm{diam}(D)$, and for $|E|\geq E_1$ we have*

$$\||\lambda|^j|r_2-r_1|\|_{L^p(\mathbb{C})}\leq \theta_2\Bigg[|E|^{-1}\left(|E|^{1/2}+\kappa\log(3+\delta^{-1})\right)^{-(m-2)} + \frac{\delta(3+\delta^{-1})^{\kappa(l+2)}}{|E|^{1/2p}}e^{|E|^{1/2}(l+2)}\Bigg],\quad j=-1,0,1. \tag{3.7}$$

*In addition, there exists a constant $\theta_3=\theta_3(D,N,m,p)$ such that for $E,\delta$ which satisfy*

$$|E|^{1/2}>\log(3+\delta^{-1}), \tag{3.8}$$

*we have*

$$\||\lambda|^j|r_2-r_1|\|_{L^p(\mathbb{C})}\leq \theta_3\Bigg[|E|^{-m/2}\log(3+\delta^{-1})^{-(m-2)}+\frac{\delta}{|E|^{1/2p}}e^{|E|(l+2)}\Bigg], \tag{3.9}$$

*for $j=-1,0,1$.*

PROOF. Let choose $0<a_1\leq 1\leq a_2$ to be determined and let

$$\delta=\|\Phi_2(E)-\Phi_1(E)\|_*. \tag{3.10}$$

We split down the left term of (3.6) as follows:

$$\||\lambda|^j|r_2-r_1|\|_{L^p(\mathbb{C})}\leq \||\lambda|^j|r_2-r_1|\|_{L^p(|\lambda|<a_1)}+\||\lambda|^j|r_2-r_1|\|_{L^p(a_1<|\lambda|<a_2)} + \||\lambda|^j|r_2-r_1|\|_{L^p(|\lambda|>a_2)}.$$

From (3.1) and (3.2) we have

$$\||\lambda|^j|r_2-r_1|\|_{L^p(|\lambda|<a_1)}\leq c(N,p,m)|E|^{-m/2}a_1^{m-1+j+2/p}, \tag{3.11}$$

$$\||\lambda|^j|r_2-r_1|\|_{L^p(|\lambda|>a_2)}\leq c(N,p,m)|E|^{-m/2}a_2^{-m-1+j+2/p}. \tag{3.12}$$

From Lemma 3.2 and (3.10) we obtain, for $j=-1,0,1$,

$$\||\lambda|^j|r_2-r_1|\|_{L^p(a_1<|\lambda|<a_2)}\leq c(D,N,p)\frac{\delta}{|E|^{1/2p}}\left(e^{(\sqrt{|E|}l+2)/a_1}+e^{(\sqrt{|E|}l+2)a_2}\right). \tag{3.13}$$

We now prove (3.6). Fix an energy $E < 0$ satisfying the hypothesis and define

$$a_2 = \frac{1}{a_1} = \beta \log(3 + \delta^{-1}), \tag{3.14}$$

for $0 < \beta < 1/(l\sqrt{|E|} + 2)$. We choose $\delta_\beta(E) < 1$ such that for every $\delta \leq \delta_\beta(E)$, $a_2 > 1$ (and so $a_1 < 1$). Note that since $E_1 > (2R)^2$, the estimates in Lemma 3.1 hold for $a_1 < 1$ and $a_2 > 1$.

The aim is to have (3.11), (3.12) of the order $\log(3+\delta^{-1})^{-(m-2)}$. Indeed we have, for every $p \geq 1$ and $\delta \leq \delta_\beta(E)$,

$$a_1^{m-1+j+2/p} \leq c(\beta) \log(3 + \delta^{-1})^{-(m-2)}, \quad a_2^{-m-1+j+2/p} \leq c(\beta) \log(3 + \delta^{-1})^{-(m-2)},$$

for $j = -1, 0, 1$. Thus, for $\delta \leq \delta_\beta(E)$,

$$\||\lambda|^j |r_2 - r_1|\|_{L^p(\mathbb{C})} \leq c(D, N, m, p, \beta) \Big[ |E|^{-m/2} \log(3 + \delta^{-1})^{-(m-2)} + \frac{\delta}{|E|^{1/2p}} (3 + \delta^{-1})^{\beta(\sqrt{|E|}l+2)} \Big].$$

Since by construction $\beta(\sqrt{|E|}l + 2) < 1$, we have that

$$\frac{\delta}{|E|^{1/2p}} (3 + \delta^{-1})^{\beta(\sqrt{|E|}l+2)} \to 0 \quad \text{for } \delta \to 0 \tag{3.15}$$

more rapidly than the other term, at fixed $E$. This gives

$$\||\lambda|^j |r_2 - r_1\|_{L^p(\mathbb{C})} \leq c(E, D, N, m, p, \beta) \left( \log(3 + \delta^{-1}) \right)^{-(m-2)}, \tag{3.16}$$

for $\delta \leq \tilde{\delta}_\beta(E)$ (where $\tilde{\delta}_\beta(E)$ is sufficiently small in order to estimate the term in (3.15)). Estimate (3.16) for general $\delta$ (with modified constant) follows from (3.16) for $\delta \leq \tilde{\delta}_\beta(E)$ and the fact that $\||\lambda|^k |r_j|\|_{L^p(D)} < c(D, N, p)$, for $j = 1, 2$, $k = -1, 0, 1$ and $p \geq 1$: this follows from Lemma 3.1 (using the fact that $|E| > R$): indeed the estimate of Lemma 2.1 hold for every $\lambda \in \mathbb{C}$, since $|E| > R$.

In order to prove (3.7) we define, in (3.11)-(3.13),

$$a_2 = \frac{1}{a_1} = 1 + \frac{\kappa \log(3 + \delta^{-1})}{|E|^{1/2}}, \tag{3.17}$$

for any $0 < \kappa < \frac{1}{l+2}$. Note that we have $a_2 > 1$ and $a_1 < 1$. Thus we find, for every $p \geq 1$, $j = -1, 0, 1$,

$$a_1^{m-1+j+2/p} \leq \frac{|E|^{(m-2)/2}}{\left(|E|^{1/2} + \kappa \log(3 + \delta^{-1})\right)^{m-2}},$$

$$a_2^{-m-1+j+2/p} \leq \frac{|E|^{(m-2)/2}}{\left(|E|^{1/2} + \kappa \log(3 + \delta^{-1})\right)^{m-2}}.$$

We have also that

$$e^{(\sqrt{|E|}l+2)/a_1}+e^{(\sqrt{|E|}l+2)a_2}\le 2e^{(l+2)\left(|E|^{1/2}+\kappa\log(3+\delta^{-1})\right)}$$
$$=2(3+\delta^{-1})^{\kappa(l+2)}e^{(l+2)|E|^{1/2}}.$$

Repeating the same arguments as above we obtain, for $\delta>0$,

$$\||\lambda|^j|r_2-r_1|\|_{L^p(\mathbb{C})}\le c(D,N,m,p)\Bigg[|E|^{-1}\left(|E|^{1/2}+\kappa\log(3+\delta^{-1})\right)^{-(m-2)}$$
$$+\frac{\delta(3+\delta^{-1})^{\kappa(l+2)}}{|E|^{1/2p}}e^{(l+2)|E|^{1/2}}\Bigg],$$

which proves estimate (3.7).

We pass to estimate (3.9). Take, in (3.11)-(3.13),

$$a_2=\frac{1}{a_1}=\log(3+\delta^{-1}). \tag{3.18}$$

Define $\tilde{\delta}<1$ such that for $\delta\le\tilde{\delta}$ we have $a_2>1$ (so $a_1<1$). From our assumption (3.8) we have that $e^{(\sqrt{|E|}l+2)/a_1}+e^{(\sqrt{|E|}l+2)a_2}<2e^{|E|(l+2)}$. Then we obtain, using the same arguments as above,

$$\||\lambda|^j|r_2-r_1|\|_{L^p(\mathbb{C})}\le c(D,N,m,p)\left[|E|^{-m/2}\log(3+\delta^{-1})^{-(m-2)}+\frac{\delta}{|E|^{1/2p}}e^{|E|(l+2)}\right],$$

for $\delta\le\tilde{\delta}$. To remove this last assumption we argue as for (3.6). This completes the proof of Proposition 3.3. □

## 4. Proof of Theorem 1.1

We begin with a lemma which generalises [**31**, Proposition 4.2] to negative energy.

LEMMA 4.1. *Let $E<0$ be such that $|E|\ge E_1$, where $E_1$ is defined in Proposition 3.3; let $v_1,v_2$ be two potentials satisfying* (1.5), (1.7), *with $\|v_j\|_{m,1}\le N$, $\mu_1(z,\lambda),\mu_2(z,\lambda)$ the corresponding Faddeev functions and $r_1,r_2$ as defined in* (2.8), (2.7). *Let $1<s<2$, and $\tilde{s}$ such that $1/\tilde{s}=1/s-1/2$. Then*

$$\sup_{z\in\mathbb{C}}\|\mu_2(z,\cdot)-\mu_1(z,\cdot)\|_{L^{\tilde{s}}(\mathbb{C})}\le c(D,N,s,m)\|r_2-r_1\|_{L^s(\mathbb{C})}, \tag{4.1}$$

$$\sup_{z\in\mathbb{C}}\left\|\frac{\partial\mu_2(z,\cdot)}{\partial\bar{z}}-\frac{\partial\mu_1(z,\cdot)}{\partial\bar{z}}\right\|_{L^{\tilde{s}}(\mathbb{C})}\le c(D,N,s,m)\Bigg[\|r_2-r_1\|_{L^s(\mathbb{C})} \tag{4.2}$$
$$+|E|^{1/2}\left(\left\|\left(|\lambda|+\frac{1}{|\lambda|}\right)|r_2-r_1|\right\|_{L^s(\mathbb{C})}+\|r_2-r_1\|_{L^s(\mathbb{C})}\right)\Bigg].$$

PROOF. We begin with the proof of (4.1). Let

$$\nu(z,\lambda)=\mu_2(z,\lambda)-\mu_1(z,\lambda). \tag{4.3}$$

From the $\bar{\partial}$-equation (2.6) we deduce that $\nu$ satisfies the following non-homogeneous $\bar{\partial}$-equation:

$$\frac{\partial}{\partial\bar{\lambda}}\nu(z,\lambda) = r_1(z,\lambda)\overline{\nu(z,\lambda)} + (r_2(z,\lambda) - r_1(z,\lambda))\overline{\mu_2(z,\lambda)}, \tag{4.4}$$

for $\lambda \in \mathbb{C}$. Note that from Lemma 2.2 we have that $\nu(z,\cdot) \in L^{\tilde{s}}(\mathbb{C})$ for every $2 < \tilde{s} \leq \infty$. In addition, from Lemma 2.1 (using the fact that $|E| > R$), we have that $\|r_j\|_{L^p(D)} < c(D,N,p,m)$, for $1 < p < \infty$, $j = 1,2$. Then it is possible to use Lemma 2.3 in order to obtain

$$\begin{aligned}
\|\nu(z,\cdot)\|_{L^{\tilde{s}}} &\leq c(D,N,s,m)\left\|\overline{\mu_2(z,\lambda)}(r_2(\lambda) - r_1(\lambda))\right\|_{L^s(\mathbb{C})} \\
&\leq c(D,N,s,m)\sup_{z\in\mathbb{C}}\|\mu_2(z,\cdot)\|_{L^\infty}\|r_2 - r_1\|_{L^s(\mathbb{C})} \\
&\leq c(D,N,s,m)\|r_2 - r_1\|_{L^s(\mathbb{C})},
\end{aligned}$$

and the constant is independent from $E$ for $|E| > R$, because of Lemma 2.2 and Lemma 2.1.

Now we pass to (4.2). To simplify notations we write, for $z,\lambda \in \mathbb{C}$,

$$\mu_z^j(z,\lambda) = \frac{\partial\mu_j(z,\lambda)}{\partial z}, \quad \mu_{\bar{z}}^j(z,\lambda) = \frac{\partial\mu_j(z,\lambda)}{\partial\bar{z}}, \quad j = 1,2.$$

From the $\bar{\partial}$-equation (2.6) we have that $\mu_z^j$ and $\mu_{\bar{z}}^j$ satisfy the following system of non-homogeneous $\bar{\partial}$-equations, for $j = 1,2$:

$$\begin{aligned}
\frac{\partial}{\partial\bar{\lambda}}\mu_z^j(z,\lambda) &= r_j(z,\lambda)\left(\overline{\mu_{\bar{z}}^j(z,\lambda)} + \frac{\sqrt{|E|}}{2}\left(\bar{\lambda} - \frac{1}{\lambda}\right)\overline{\mu_j(z,\lambda)}\right), \\
\frac{\partial}{\partial\bar{\lambda}}\mu_{\bar{z}}^j(z,\lambda) &= r_j(z,\lambda)\left(\overline{\mu_z^j(z,\lambda)} + \frac{\sqrt{|E|}}{2}\left(\frac{1}{\bar{\lambda}} - \lambda\right)\overline{\mu_j(z,\lambda)}\right).
\end{aligned}$$

Define now $\mu_\pm^j(z,\lambda) = \mu_z^j(z,\lambda) \pm \mu_{\bar{z}}^j(z,\lambda)$, for $j = 1,2$. Then they satisfy the following two non-homogeneous $\bar{\partial}$-equations:

$$\frac{\partial}{\partial\bar{\lambda}}\mu_\pm^j(z,\lambda) = r_j(z,\lambda)\left(\pm\overline{\mu_\pm^j(z,\lambda)} + \frac{\sqrt{|E|}}{2}\left(\left(\bar{\lambda} - \frac{1}{\lambda}\right) \pm \left(\frac{1}{\bar{\lambda}} - \lambda\right)\right)\overline{\mu_j(z,\lambda)}\right).$$

Finally define $\tau_\pm(z,\lambda) = \mu_\pm^2(z,\lambda) - \mu_\pm^1(z,\lambda)$. They satisfy the two non-homogeneous $\bar{\partial}$-equations below:

$$\begin{aligned}
\frac{\partial}{\partial\bar{\lambda}}\tau_\pm(z,\lambda) = \Bigg[&\pm\left(r_1(z,\lambda)\overline{\tau_\pm(z,\lambda)} + (r_2(z,\lambda) - r_1(z,\lambda))\overline{\mu_\pm^2(z,\lambda)}\right) \\
&+ \frac{\sqrt{|E|}}{2}\left(\left(\bar{\lambda} - \frac{1}{\lambda}\right) \pm \left(\frac{1}{\bar{\lambda}} - \lambda\right)\right)\left((r_2(z,\lambda) - r_1(z,\lambda))\overline{\mu_2(z,\lambda)} + r_1(z,\lambda)\overline{\nu(z,\lambda)}\right)\Bigg],
\end{aligned}$$

where $\nu(z,\lambda)$ was defined in (4.3).

Now remark that by Lemma 2.2 and regularity assumptions on the potentials we have that $\mu_z^j(z,\cdot), \mu_{\bar{z}}^j(z,\cdot) \in L^{\tilde{s}}(\mathbb{C}) \cap L^\infty(\mathbb{C})$ for any $\tilde{s} > 2$, $j = 1, 2$ (and their norms are bounded by a constant $C(D, N, p, m)$ thanks to Lemma 2.2). This, in particular, yields $\tau_\pm(z,\cdot) \in L^{\tilde{s}}(\mathbb{C})$. These arguments, along with the above remarks on the $L^p$ boundedness of $r_j$, make possible to use Lemma 2.3, which gives

$$\begin{aligned}
&\|\tau_\pm(z,\cdot)\|_{L^{\tilde{s}}(\mathbb{C})} \\
&\quad \le c(D,N,s,m)\Bigg[\|r_2 - r_1\|_{L^s(\mathbb{C})}\|\mu_\pm^2(z,\cdot)\|_{L^\infty(\mathbb{C})} \\
&\qquad + \sqrt{|E|}\Bigg(\left\|\left(|\lambda| + \frac{1}{|\lambda|}\right)|r_2 - r_1|\right\|_{L^s(\mathbb{C})}\|\mu_2(z,\cdot)\|_{L^\infty(\mathbb{C})} \\
&\qquad + \|(|\lambda| + |\lambda|^{-1})r_1\|_{L^2(\mathbb{C})}\|\nu(z,\cdot)\|_{L^{\tilde{s}}(\mathbb{C})}\Bigg)\Bigg] \\
&\quad \le c(D,N,s,m)\Bigg[\|r_2 - r_1\|_{L^s(\mathbb{C})} + \sqrt{|E|}\Bigg(\left\|\left(|\lambda| + \frac{1}{|\lambda|}\right)|r_2 - r_1|\right\|_{L^s(\mathbb{C})} \\
&\qquad + \||\lambda||r_2 - r_1|\|_{L^s(\mathbb{C})}\Bigg)\Bigg],
\end{aligned}$$

where we used Hölder's inequality (2.18) (since $1/s = 1/2 + 1/\tilde{s}$) and estimate (4.1). Again, the constants are independent from $E$ since $|E| > R$.

The proof of (4.2) now follows from this last inequality and the fact that $\mu_{\bar{z}}^2 - \mu_{\bar{z}}^1 = \frac{1}{2}(\tau_+ - \tau_-)$. □

Remark 4.1. We also have proved that

$$\begin{aligned}
\sup_{z\in\mathbb{C}}\left\|\frac{\partial\mu_2(z,\cdot)}{\partial z} - \frac{\partial\mu_1(z,\cdot)}{\partial z}\right\|_{L^{\tilde{s}}(\mathbb{C})} &\le c(D,N,s,m)\Bigg[\|r_2 - r_1\|_{L^s(\mathbb{C})} \\
&+ |E|^{1/2}\Bigg(\left\|\left(|\lambda| + \frac{1}{|\lambda|}\right)|r_2 - r_1|\right\|_{L^s(\mathbb{C})} + \|r_2 - r_1\|_{L^s(\mathbb{C})}\Bigg)\Bigg].
\end{aligned}$$

Proof of Theorem 1.1. We recall the derivation of an explicit formula for the potential, taken from [**23**].

Let $v(z)$ be a potential which satisfies the hypothesis of Theorem 1.1 and $\mu(z,\lambda)$ the corresponding Faddeev functions. Since $\mu(z,\lambda)$ satisfies the estimates of Lemma 2.2, the $\bar{\partial}$-equation (2.6) and $b(\lambda, E)$ decreases at infinity like in Lemma 2.1, it is possible to write the following development:

$$\mu(z,\lambda) = 1 + \frac{\mu_{-1}(z)}{\lambda} + O\left(\frac{1}{|\lambda|^2}\right), \qquad \lambda \to \infty, \tag{4.5}$$

for some function $\mu_{-1}(z)$. If we insert (4.5) into equation (1.1), for $\psi(z,\lambda) = e^{-\frac{\sqrt{|E|}}{2}(z/\lambda+\bar{z}\lambda)}\mu(z,\lambda)$, we obtain, letting $\lambda \to \infty$,

$$v(z) = -2|E|^{1/2}\frac{\partial \mu_{-1}(z)}{\partial z}, \qquad z \in \mathbb{C}. \tag{4.6}$$

More explicitly, we have, as a consequence of (2.6),

$$\mu(z,\lambda) - 1 = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{r(z,\lambda')}{\lambda'-\lambda}\overline{\mu(z,\lambda')}d\lambda'\, d\bar{\lambda}'.$$

By Lebesgue's dominated convergence (using Lemma 2.1) we obtain

$$\mu_{-1}(z) = -\frac{1}{2\pi i}\int_{\mathbb{C}} r(z,\lambda)\overline{\mu(z,\lambda)}d\lambda\, d\bar{\lambda},$$

and the explicit formula

$$v(z) = \frac{|E|^{1/2}}{\pi i}\int_{\mathbb{C}} r(z,\lambda)\left(\frac{|E|^{1/2}}{2}\left(\bar{\lambda}-\frac{1}{\lambda}\right)\overline{\mu(z,\lambda)} + \overline{\left(\frac{\partial \mu(z,\lambda)}{\partial \bar{z}}\right)}\right)d\lambda\, d\bar{\lambda}. \tag{4.7}$$

Formula (4.7) for $v_1$ and $v_2$ yields

$$\begin{aligned} v_2(z) - v_1(z) = \frac{|E|^{1/2}}{\pi i}\int_{\mathbb{C}} \Bigg[ & \frac{|E|^{1/2}}{2}\left(\bar{\lambda}-\frac{1}{\lambda}\right)\big((r_2-r_1)\overline{\mu_2} + r_1(\overline{\mu_1}-\overline{\mu_2})\big) \\ & + (r_2-r_1)\overline{\left(\frac{\partial \mu_2}{\partial \bar{z}}\right)} + r_1\overline{\left(\frac{\partial \mu_2}{\partial \bar{z}} - \frac{\partial \mu_1}{\partial \bar{z}}\right)}\Bigg]d\lambda\, d\bar{\lambda}. \end{aligned}$$

Then, using several times Hölder's inequality (2.18), we find

$$\begin{aligned} |v_2(z) - v_1(z)| \le \frac{|E|^{1/2}}{\pi}\Bigg[ & \frac{|E|^{1/2}}{2}\Bigg( \left\|\left(\bar{\lambda}-\frac{1}{\lambda}\right)(r_2-r_1)\right\|_{L^1} \|\mu_2(z,\cdot)\|_{L^\infty} \\ & + \left\|\left(\bar{\lambda}-\frac{1}{\lambda}\right) r_1\right\|_{L^{\tilde{p}'}} \|\mu_2(z,\cdot)-\mu_1(z,\cdot)\|_{L^{\tilde{p}}}\Bigg) \\ & + \|r_2-r_1\|_{L^p}\left\|\frac{\partial \mu_2(z,\cdot)}{\partial \bar{z}}\right\|_{L^{p'}} \\ & + \|r_1\|_{L^{\tilde{p}'}}\left\|\frac{\partial \mu_2(z,\cdot)}{\partial \bar{z}} - \frac{\partial \mu_1(z,\cdot)}{\partial \bar{z}}\right\|_{L^{\tilde{p}}}\Bigg], \end{aligned}$$

for $1 < p < 2$, $\tilde{p}$ such that $1/\tilde{p} = 1/p - 1/2$ and $1/p + 1/p' = 1/\tilde{p} + 1/\tilde{p}' = 1$. From Lemmas 4.1, 2.2 and 3.1 we obtain

$$|v_2(z) - v_1(z)| \le c(D, N, m, p)|E|^{1/2} \left[ |E|^{1/2} \left( \left\| \left( \bar{\lambda} - \frac{1}{\lambda} \right) (r_2 - r_1) \right\|_{L^1} + \sum_{k=-1}^{1} \| |\lambda|^k |r_2 - r_1| \|_{L^p(\mathbb{C})} \right) + \|r_2 - r_1\|_{L^p} \right].$$

Now Proposition 3.3 gives

$$\|v_2 - v_1\|_{L^\infty(D)} \le c(E, D, N, m)(\log(3 + \|\Phi_2(E) - \Phi_1(E)\|_*^{-1}))^{-(m-2)}, \tag{4.8}$$

which is (1.8). Estimates (1.9) and (1.11) are also obtained as a consequence of the above inequality and Proposition 3.3. This finishes the proof of Theorem 1.1. □

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