Stability and instability in inverse problems

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November 27, 2013.

Plan of the presentation

- The Gel'fand inverse problem with boundary measurements represented as a Dirichlet-to-Neumann map
- The Gel'fand inverse problem with boundary measurements represented as an impedance boundary map (Robin-to-Robin map)
- Inverse scattering problems

Basic assumptions

Consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi \quad \text{for } x \in D, \tag{1}$$

where

- D is an open bounded domain in $\mathbb{R}^d$,
- $d \geq 2$,
- $\partial D \in C^2$,
- $v \in L^\infty(D)$.

Statement of the problem

Let

$$\mathcal{C}_v(E) = \left\{ \left(\psi|_{\partial D}, \frac{\partial \psi}{\partial \nu}|_{\partial D} \right) : \begin{array}{r} \text{for all sufficiently regular solutions } \psi \text{ of} \\ \text{equation (1) in } \bar{D} = D \cup \partial D \end{array} \right\}.$$

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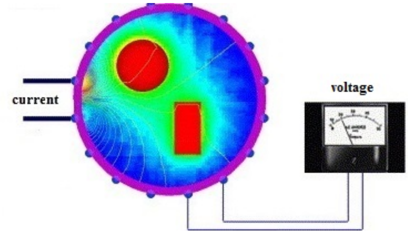
Problem 1.

- Given $\mathcal{C}_v(E)$.
- Find v.

Problem 1 was formulated for the first time by Gel'fand (1954). In this first formulation energy E was not yet fixed.

Physical interpretation of the problem

Electrical impedance tomography (EIT)



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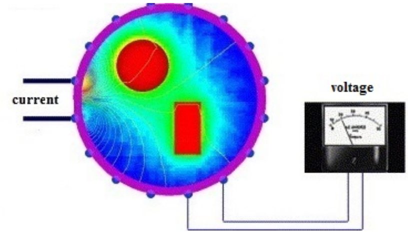

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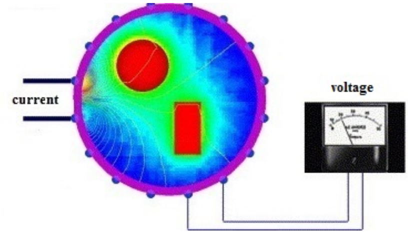

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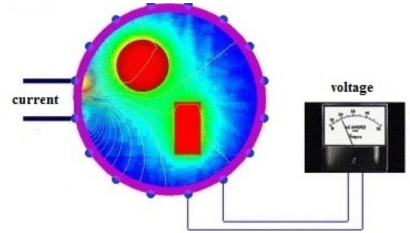

The equation of the conductivity: $\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } D.$

For σ isotrope: $\psi = u\sqrt{\sigma}, \quad v = \frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}} \implies -\Delta\psi + v(x)\psi = 0 \quad \text{in } D.$

Standard representation of the Cauchy data

The Dirichlet-to-Neumann map $\hat{\Phi}_v(E)$ is defined by

$$\hat{\Phi}_v(E)(\psi|_{\partial D}) = \frac{\partial \psi}{\partial \nu}|_{\partial D}.$$

Here we assume also that

E is not a Dirichlet eigenvalue for the operator $-\Delta + v$ in D.

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Problem 1a.

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- Reconstruction.
- Stability: there is some function ϕ such that

$$\|v_2 - v_1\|_{L^\infty(D)} \leq \phi(\|\hat{\Phi}_{v_2}(E) - \|\hat{\Phi}_{v_1}(E)\|),$$
$$\phi(t) \to 0 \text{ as } t \to +0.$$

Historical remarks

First global results (the case of fixed energy):

	$\boldsymbol{d \geq 3}$	$\boldsymbol{d = 2}$
Uniqueness:	Novikov (1988)	Bukhgeim (2008)
Reconstruction:	Novikov (1988)	Bukhgeim (2008)
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The Calderón inverse problem (of the electrical impedance tomography):

Slichter (1933), Tikhonov (1949), Calderón (1980), Druskin (1982), Kohn-Vogelius (1984), Sylvester-Uhlmann (1987), Nachman (1996), Liu (1997).

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- **Instability results:**
  - Mandache (2001), optimality of logarithmic stability results in the case of zero energy (the Gel'fand-Calderón inverse problem) up to the value of some exponent.
  - Cristo-Rondi (2003), some general schema for investigating questions of this type of instability.

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- **Lipschitz stability in the case of piecewise constant potentials:**
    - Alessandrini-Vessella (2005), the Calderón inverse problem.
    - Rondi (2006), exponential growth of the Lipschitz constant.
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  - Bourgeois (2013), some general scheme for investigating similar stability questions.
- **Regularity and/or energy dependent stability estimates:**
  - Novikov (2011), effectivization of the result of Alessandrini (1988).
  - Novikov (1998, 2005, 2008), Isakov (2011), Santacesaria (2013), the phenomena of increasing stability for the high-energy case.

Hölder-logarithmic stability

Hölder-logarithmic stability

Theorem 1 (Isaev, Novikov (2012), see [IN1])

Let the basic assumptions hold and

- $d \geq 3$, $m > d$, $N > 0$ et $\operatorname{supp} v_j \subset D$,
- E is not a Dirichlet eigenvalue for the operator $-\Delta + v_j$ in D,
- $v_j \in W^{m,1}(\mathbb{R}^d)$ and $||v_j||_{m,1} \leq N$, $j = 1, 2$,

Then, for $E \geq 0$, $\tau \in (0,1)$ and $\boxed{\text{for any } \alpha, \beta \geq 0,\ \alpha + \beta \leq (m-d)/d}$:

$$||v_1 - v_2||_{L^\infty(D)} \leq A(1+\sqrt{E})\delta^\tau + B(1+\sqrt{E})^{-\alpha}\left(\ln\left(3+\delta^{-1}\right)\right)^{-\beta}, \qquad (2)$$

where

$$\delta = ||\hat{\Phi}_{v_1}(E) - \hat{\Phi}_{v_2}(E)||_{L^\infty(\partial D) \to L^\infty(\partial D)}$$

and constants $A, B > 0$ depend only on N, D, m, τ.

For the case of the dimension $d = 2$, see Santacesaria (2013).

Logarithmic stability

Consider the estimates of the following type:

$$||v_1 - v_2||_{L^\infty(D)} \leq C \left(\ln\left(3 + \delta^{-1}\right)\right)^{-s}, \tag{3}$$

where $C > 0$ depends only on N, D, m, s, E.

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$$s_0 = \frac{m-d}{m}, \quad s_1 = \frac{m-d}{d}, \quad s_2 = m - d.$$

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- Isaev-Novikov (2012), see [IN1], estimate (3) with $s = s_1$ and $d \ge 3$.

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- Isaev-Novikov (2012), see [IN1], estimate (3) with $s = s_1$ and $d \geq 3$.

The principal advantage:

$$s_1 \to +\infty \text{ and } s_2 \to +\infty \quad \text{as} \quad m \to +\infty.$$

Approximate stability

The potential can be found approximately but with good stability!

Novikov (1998, 2005, 2008) showed that for for inverse problems for the Schrödinger equation at fixed energy E in dimension $d \geq 2$ (like Problem 1), the potential v can be reconstructed approximately, i.e.

$$v = v_{approx} + v_{err}$$

- reconstruction of v_{approx} is Hölder stable,
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- reconstruction of v_{approx} is Hölder stable,
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If we put $\alpha = \dfrac{m-d}{d}$, $\beta = 0$ in estimate (2), we get that

$$||v_1 - v_2||_{L^\infty(D)} \leq A(1+\sqrt{E})\delta^\tau + B(1+\sqrt{E})^{-\frac{m-d}{d}}.$$

Instability results of [Isaev1]

Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa}, \boldsymbol{\tau} \geq \mathbf{0}$. We consider class of estimates of the type

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \leq A(1+\sqrt{E})^{\kappa}\delta^{\tau} + B(1+\sqrt{E})^{-\alpha}\left(\ln\left(3+\delta^{-1}\right)\right)^{-\beta}.$$

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In particular, results of [Isaev1] show the optimality of the estimate

$$||v_1 - v_2||_{L^\infty(D)} \leq A(1+\sqrt{E})\delta^\tau + B(1+\sqrt{E})^{-\frac{m-d}{d}}.$$

Exponential instability

Mandache (2001) for the case of $E = 0$ and $d \geq 2$ showed that the estimate

$$||v_1 - v_2||_{L^\infty(D)} \leq C \left(\ln\left(3 + \delta^{-1}\right)\right)^{-s},$$

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- when $s > 2m - \frac{m}{d}$ for real-valued potentials,
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Instability results of [Isaev2]

It was shown in [Isaev2] that for any $\boldsymbol{C = C(N, D, m, s, S)}$ there are two potentials such that

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} > C \sup_{E \in S} \left(\ln(3 + \delta(E)^{-1}) \right)^{-s},$$

- when $s > 2m$ for real-valued potentials,
- when $s > m$ for complex-valued potentials,

where $S = \bigcup_{j=1}^{K} I_j$ denotes the union of energy intervals such that the DtN maps $\hat{\Phi}_{v_1}(E)$, $\hat{\Phi}_{v_2}(E)$ are correctly defined for any $E \in S$.

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If S consists of one point only $\Longrightarrow$ optimality of estimate (3).

The weakness

Bad news: stability estimates given earlier make no sense if

$$E \text{ is a Dirichlet eigenvalue for } -\Delta + v \text{ in } D,$$

or too weak if energy E is close to the Dirichlet spectrum.

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Idea: let us consider another operator representation of the Cauchy data set

$$\mathcal{C}_v(\boldsymbol{E}) = \left\{ \left(\psi|_{\partial D}, \frac{\partial \psi}{\partial \nu}|_{\partial D} \right) : \begin{array}{r} \text{for all sufficiently regular solutions } \psi \text{ of} \\ \text{equation (1) in } \bar{D} = D \cup \partial D \end{array} \right\} :$$

$$\hat{M}_{c_1,c_2,c_3,c_4}\left(c_1 \psi|_{\partial D} + c_2 \frac{\partial \psi}{\partial \nu}|_{\partial D} \right) = \left(c_3 \psi|_{\partial D} + c_4 \frac{\partial \psi}{\partial \nu}|_{\partial D} \right).$$

Impedance boundary map (Robin-to-Robin map)

Let consider the map $\hat{M}_{\alpha,v}(E)$ defined by

$$\hat{M}_{\alpha,v}[\psi]_\alpha = [\psi]_{\alpha-\pi/2}$$

for all suffuciently regular solutions ψ of equation (1) in $\bar{D} = D \cup \partial D$, where

$$[\psi]_\alpha = \cos\alpha\,\psi|_{\partial D} - \sin\alpha\,\frac{\partial\psi}{\partial\nu}|_{\partial D},$$

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Problem 1b.

- Given $\hat{M}_{\alpha,v}(E)$ for some fixed α.
- Find v.

Impedance boundary map (Robin-to-Robin map)

We have that

- there can not be more than a countable number of α such that E is an eigenvalue for the operator $-\Delta + v$ in D with the boundary condition
$$\cos\alpha\, \psi|_{\partial D} - \sin\alpha\, \frac{\partial\psi}{\partial\nu}|_{\partial D} = 0,$$
- the map $\hat{M}_\alpha$ is reduced to the Dirichlet-to-Neumann map if $\alpha = 0$ and to the Neumann-to-Dirichlet map if $\alpha = \pi/2$.

Stability estimates for $d \geq 3$

Theorem 2 (Isaev, Novikov [IN2]).

Let the assumptions of Problem 1b hold and

- $d \geq 3$, $m > d$, $N > 0$ and $\operatorname{supp} v_j \subset D$,
- $v_j \in W^{m,1}(\mathbb{R}^d)$ and $||v_i||_{m,1} \leq N$, $j = 1, 2$,

Then, for any $s \geq 0$, $s \leq (m-d)/m$,

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} \leq C_\alpha \left(\ln\left(3 + \delta_\alpha^{-1}\right)\right)^{-s},$$

where constant $C_\alpha = C_\alpha(N, D, m, s, E)$,

$$\delta_\alpha = ||\hat{M}_{\alpha,v_1}(E) - \hat{M}_{\alpha,v_2}(E)||_{\mathbb{L}^\infty(\partial D)\to\mathbb{L}^\infty(\partial D)}.$$

For $\alpha = 0$ it is a variation of the result of Alessandrini (1988).

Stability estimates for $d = 2$

Theorem 3 (Isaev, Novikov [IN2]).

Let the assumptions of Problem 1b hold and

- $d = 2$, $N > 0$ and $\operatorname{supp} v_j \subset D$,
- $v_j \in C^2(\bar{D})$ and $||v_j||_{C^2(\bar{D})} \leq N$, $j = 1, 2$,

Then, for any $0 < s \leq 3/4$,

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} \leq C_\alpha \left(\ln\left(3 + \delta_\alpha^{-1}\right)\right)^{-s} \left(\ln\left(3 \ln\left(3 + \delta_\alpha^{-1}\right)\right)\right)^2,$$

where constant $C_\alpha = C_\alpha(N, D, s, E)$,

$$\delta_\alpha = ||\hat{M}_{\alpha,v_1}(E) - \hat{M}_{\alpha,v_2}(E)||_{\mathbb{L}^\infty(\partial D)\to\mathbb{L}^\infty(\partial D)}.$$

Theorem 3 for $\alpha = 0$ was given by Novikov-Santacesaria (2010) with $s = 1/2$ and by Santacesaria (2012) with $s = 3/4$.

Stability of determining a potential from its Cauchy data

Theorems 2 and 3 imply, in particular, that

- For $d \geq 3$ and $0 < s \leq (m-d)/m$

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} \leq \min_{\alpha \in \mathbb{R}} C_\alpha \left(\ln\left(3 + \delta_\alpha^{-1}\right)\right)^{-s}.$$

- For $d = 2$ and $0 < s \leq 3/4$,

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} \leq \min_{\alpha \in \mathbb{R}} C_\alpha \left(\ln\left(3 + \delta_\alpha^{-1}\right)\right)^{-s} \left(\ln\left(3 \ln\left(3 + \delta_\alpha^{-1}\right)\right)\right)^{2}.$$

Idea of the proofs

For any suffuciently regular solutions ψ_1 and ψ_2 of equation (1) in $\bar{D} = D \cup \partial D$ with $v = v_1$ and $v = v_2$, respectively, the following identity holds (see [IN2]):

$$\int_D (v_1 - v_2)\, \psi_1 \psi_2 \, dx = \int_{\partial D} [\psi_1]_\alpha \left(\hat{M}_{\alpha, v_1}(E) - \hat{M}_{\alpha, v_2}(E) \right) [\psi_2]_\alpha dx. \tag{4}$$

Identity (4) for $\alpha = 0$ is reduced to Alessandrini's identity.

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Corollary.

Under basic assumptions real-valued potential v is uniquely determined by its Cauchy data $\mathcal{C}_v(E)$ at fixed real energy E in dimension $d \geq 2$.

To our knowledge the result of this corollary for $d \geq 3$ was not yet completely proved in the literature.

Schema of reconstruction of a potential v from $\hat{M}_{v,\alpha}(E)$

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3. $S_E \to v$ as described by Grinevich (1988, 2000), Henkin-Novikov (1987), Novikov (1992 – 2009), Novikov-Santacesaria (2013).

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3. $S_E \to v$ as described by Grinevich (1988, 2000), Henkin-Novikov (1987), Novikov (1992 – 2009), Novikov-Santacesaria (2013).

In addition, numerical efficiency of related inverse scattering techniques was shown by the research group at MSU headed by Burov (2000, 2008, 2009, 2012), see also Bikowski-Knudsen-Mueller (2011).

Basic assumptions

Consider the three-dimensional stationary acoustic equation at frequency ω in an inhomogeneous medium with refractive index n

$$\Delta\psi + \omega^2 n(x)\psi = 0, \quad x \in \mathbb{R}^3,\ \omega > 0, \tag{5}$$

where

- $(1-n) \in W^{m,1}(\mathbb{R}^3)$ for some $m > 3$,
- $\operatorname{Im} n(x) \geq 0, \quad x \in \mathbb{R}^3$,
- $\operatorname{supp}(1-n) \subset B_{r_1}$ for some $r_1 > 0$,

where $W^{m,1}(\mathbb{R}^3)$ denotes the Sobolev space of m-times smooth functions in $\mathbb{L}^1$ and B_r is the open ball of radius r centered at 0.

The Green function

Let $G^+(x, y, \omega)$ denote the Green function for the operator $\Delta + \omega^2 n(x)$ with the Sommerfeld radiation condition:

$$\left(\Delta + \omega^2 n(x)\right) G^+(x, y, \omega) = \delta(x - y),$$

$$\lim_{|x|\to\infty} |x| \left(\frac{\partial G^+}{\partial |x|}(x, y, \omega) - i\omega G^+(x, y, \omega) \right) = 0,$$

uniformly for all directions $\hat{x} = x/|x|$,

$$x, y \in \mathbb{R}^3,\ \omega > 0.$$

It is know that, under basic assumptions, the function G^+ is uniquely specified, see, for example, Colton-Kress (1998), Hähner-Hohage (2001).

Near-field inverse scattering problem

We consider, in particular, the following near-field inverse scattering problem for equation (5):

Problem 2.

- Given G^+ on $\partial B_r \times \partial B_r$ for fixed $\omega > 0$ and $r > r_1$.
- Find n on B_{r_1}.

Scattering amplitude

Consider also the solutions $\psi^+(x,k)$, $x \in \mathbb{R}^3,\ k \in \mathbb{R}^3,\ k^2 = \omega^2$, of equation (5) specified by the following asymptotic condition:

$$\psi^+(x,k) = e^{ikx} - 2\pi^2 \frac{e^{i|k||x|}}{|x|} f\left(k, |k|\frac{x}{|x|}\right) + o\left(\frac{1}{|x|}\right)$$
$$\text{as } \ |x| \to \infty \left(\text{uniformly in } \frac{x}{|x|}\right), \tag{6}$$

with some a priory unknown f.

The function f on $\mathcal{M}_\omega = \{k \in \mathbb{R}^3, l \in \mathbb{R}^3 : k^2 = l^2 = \omega^2\}$ arising in (6) is the classical scattering amplitude for equation (5).

Far-field inverse scattering problem

In addition to Problem 2, we consider also the following far-field inverse scattering problem for equation (5):

Problem 3.

- Given f on $\mathcal{M}_\omega$ for some fixed $\omega > 0$.
- Find n on B_{r_1}.

Far-field inverse scattering problem

In addition to Problem 2, we consider also the following far-field inverse scattering problem for equation (5):

Problem 3.

- Given $\boldsymbol{f}$ on $\boldsymbol{\mathcal{M}_\omega}$ for some fixed $\boldsymbol{\omega > 0}$.
- Find $\boldsymbol{n}$ on $\boldsymbol{B_{r_1}}$.

- It was shown by Berezanskii (1958) that the near-field scattering data of Problem 2 are uniquely determined by the far-field scattering data of Problem 3 and vice versa.
- Global uniqueness for Problems 2 and 3 was proved for the first time in Novikov (1988); in addition, this proof is constructive.
- Stability estimates were given for the first time by Stefanov (1990).

Stability estimates of [IN4]

Theorem 4 (Isaev, Novikov [IN4]).

Let $N > 0$ and $r > r_1$ be fixed constants. Then there exists a positive constant C (depending only on m, ω, r_1, r and N) such that for all refractive indices n_1, n_2 satisfying

- $\|1 - n_1\|_{m,1}, \|1 - n_2\|_{m,1} < N$,
- $\operatorname{supp}(1 - n_1), \operatorname{supp}(1 - n_2) \subset B_{r_1}$,

the following estimate holds:

$$||n_1 - n_2||_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C \left(\ln\left(3 + \delta^{-1}\right)\right)^{-s}, \quad s = \frac{m-3}{3}, \tag{7}$$

where $\delta = ||G_1^+ - G_2^+||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}$ and G_1^+, G_2^+ are the near-field scattering data for the refractive indices n_1, n_2, respectively, at fixed frequency ω.

For some regularity dependent s but always smaller than 1 the stability estimate of Theorems 4 was proved by Hähner-Hohage (2001).

Stability estimates of [IN4]

Theorem 5 (Isaev, Novikov [IN4]).

Let $N > 0$ and $0 < \epsilon < \frac{m-3}{3}$ be fixed constants. Then there exists a positive constant C (depending only on m, ϵ, ω, r_1 and N) such that for all refractive indices n_1, n_2 satisfying

- $\|1 - n_1\|_{m,1}, \|1 - n_2\|_{m,1} < N$,
- $\mathrm{supp}\,(1 - n_1), \mathrm{supp}\,(1 - n_2) \subset B_{r_1}$,

the following estimate holds:

$$||n_1 - n_2||_{\mathbb{L}^\infty(\mathbb{R}^3)} \leq C \left(\ln\left(3 + \delta^{-1}\right)\right)^{-s+\epsilon}, \quad s = \frac{m-3}{3}, \tag{8}$$

where $\delta = ||f_1 - f_2||_{\mathbb{L}^2(\mathcal{M}_\omega)}$ and f_1, f_2 denote the scattering amplitudes for the refractive indices n_1, n_2, respectively, at fixed frequency ω.

For some regularity dependent s but always smaller than 1 the stability estimate of Theorems 4 was proved by Hähner-Hohage (2001).

Solution of the open problem

Possibility of estimates (7), (8) with $\boldsymbol{s > 1}$ was formulated by Hähner-Hohage (2001) as an open problem.

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Possibility of estimates (7), (8) with $\boldsymbol{s > 1}$ was formulated by Hähner-Hohage (2001) as an open problem.

Our estimates (7), (8) with $\boldsymbol{s = \frac{m-3}{3}}$ give a solution of this problem. Indeed,

$$\boldsymbol{s = \frac{m-3}{3} \to +\infty \quad \text{as} \quad m \to +\infty.}$$

Instability result of [Isaev3]

Result of Stefanov (1990): for some s always smaller than 1

$$||n_1 - n_2||_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C\left(\ln\left(3 + \|f_1 - f_2\|_S^{-1}\right)\right)^{-s},$$

where some special norm $\|f_1 - f_2\|_S$ is used and

$$\|f_1 - f_2\|_{\mathbb{L}^2(\mathcal{M}_\omega)} \le c\,\|f_1 - f_2\|_S.$$

Instability result of [Isaev3]

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where some special norm $\|f_1 - f_2\|_S$ is used and

$$\|f_1 - f_2\|_{\mathbb{L}^2(\mathcal{M}_\omega)} \le c \, \|f_1 - f_2\|_S.$$

It was shown in [Isaev3] that for any interval $I = [\omega_1, \omega_2]$, $\omega_1 > 0$, estimate

$$||n_1 - n_2||_{\mathbb{L}^\infty(D)} \le C \sup_{\omega \in I} \left(\ln(3 + \|f_1 - f_2\|_S^{-1}) \right)^{-s}$$

where $C = C(N, D, m, I)$, **can not hold** with $s > 2m$ in the case of the scattering amplitude given on the interval of frequencies and with $s > 5m/3$ in the case of fixed frequency.

Basic assumptions

Now we focus on inverse scattering for the Schrödinger equation

$$\boldsymbol{L\psi = E\psi, \quad L = -\Delta + v(x), \quad x \in \mathbb{R}^d, \ \ d \geq 2,} \tag{9}$$

where

- $\boldsymbol{v}$ is real-valued, $\boldsymbol{v} \in \mathbb{L}^{\infty}(\mathbb{R}^{\boldsymbol{d}})$
- $\boldsymbol{v(x) = O(|x|^{-d-\varepsilon}), \ \ |x| \to \infty,}$ for some $\boldsymbol{\varepsilon > 0.}$

The Green function

Consider the resolvent $\boldsymbol{R(E)}$ of the Schrödinger operator $\boldsymbol{L}$ in $\mathbb{L}^2(\mathbb{R}^d)$:

$$\boldsymbol{R(E) = (L - E)^{-1}, \quad E \in \mathbb{C} \setminus \sigma(L).}$$

Let $\boldsymbol{R(x, y, E)}$ denote the Schwartz kernel of $\boldsymbol{R(E)}$ as an integral operator. Consider also

$$\boldsymbol{R^+(x, y, E) = R(x, y, E + i0), \quad x, y \in \mathbb{R}^d, \ \ E \in \mathbb{R}_+.}$$

We recall that in the framework of equation (9) the function $\boldsymbol{R^+(x, y, E)}$ describes scattering of the spherical waves

$$\boldsymbol{R_0^+(x, y, E) = -\frac{i}{4}\left(\frac{\sqrt{E}}{2\pi|x - y|}\right)^{\frac{d-2}{2}} H^{(1)}_{\frac{d-2}{2}}(\sqrt{E}|x - y|),}$$

generated by a source at $\boldsymbol{y}$ (where $\boldsymbol{H^{(1)}_\mu}$ is the Hankel function of the first kind of order $\boldsymbol{\mu}$). We recall also that $\boldsymbol{R^+(x, y, E)}$ is the Green function for $\boldsymbol{L - E}$, $\boldsymbol{E} \in \mathbb{R}_+$, with the Sommerfeld radiation condition at infinity.

Near-field inverse scattering problem

In addition, the function

$$\boldsymbol{S^+(x,y,E) = R^+(x,y,E) - R_0^+(x,y,E),}$$
$$\boldsymbol{x, y \in \partial B_r,\ E \in \mathbb{R}_+,\ r \in \mathbb{R}_+,}$$

is considered as near-field scattering data for equation (9).

Near-field inverse scattering problem

In addition, the function

$$S^{+}(x, y, E) = R^{+}(x, y, E) - R_0^{+}(x, y, E),$$
$$x, y \in \partial B_r,\ E \in \mathbb{R}_+,\ r \in \mathbb{R}_+,$$

is considered as near-field scattering data for equation (9).

We consider, in particular, the following near-field inverse scattering problem for equation (9):

Problem 4.

- Given S^{+} on $\partial B_r \times \partial B_r$ for some fixed $r, E \in \mathbb{R}_+$.
- Find v on B_r.

This problem can be considered under the assumption that v is a priori known on $\mathbb{R}^d \setminus B_r$. We consider Problem 4 under the assumption that $v \equiv 0$ on $\mathbb{R}^d \setminus B_r$ for some fixed $r \in \mathbb{R}_+$.

Approaches to the problem

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- It is well-known that the near-field scattering data of Problem 4 uniquely and efficiently determine the scattering amplitude f for equation (9) at fixed energy E, see Berezanskii (1958).

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- It is well-known that the near-field scattering data of Problem 4 uniquely and efficiently determine the scattering amplitude f for equation (9) at fixed energy E, see Berezanskii (1958).

- It is also known that the near-field data of Problem 4 uniquely determine the Dirichlet-to-Neumann map in the case when E is not a Dirichlet eigenvalue for operator L in B_r, see Nachman (1988), Novikov (1988).

Hölder-logarithmic stability for $d \geq 3$

Theorem 6 ([Isaev4]).

Let $E > 0$ and $r > r_1 > 0$ be given constants. Let dimension $d \geq 3$ and potentials v_1, v_2 be real-valued such that

- $v_j \in W^{m,1}(\mathbb{R}^d)$, $m > d$, $\operatorname{supp} v_j \subset B_{r_1}$,
- $||v_j||_{m,1} \leq N$ for some $N > 0$, $j = 1, 2$.

Let $S_1^+(E)$ and $S_2^+(E)$ denote the near-field scattering data for v_1 and v_2, respectively. Then for $\tau \in (0,1)$ and any $s \in [0, s_1]$ the following estimate holds:

$$||v_2 - v_1||_{\mathbb{L}^\infty(B_r)} \leq A(1+E)^{\frac{5}{2}}\delta^\tau + B(1+E)^{\frac{s-s_1}{2}}\left(\ln\left(3+\delta^{-1}\right)\right)^{-s},$$

where $s_1 = \frac{m-d}{d}$, $\delta = ||S_1^+(E) - S_2^+(E)||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}$, and constants $A, B > 0$ depend only on N, m, d, r, τ.

Logarithmic stability for $d = 2$

Theorem 7 ([Isaev4]).

Let $E > 0$ and $r > r_1 > 0$ be given constants. Let dimension $d = 2$ and and potentials v_1, v_2 be real-valued such that

- $v_j \in C^2(\mathbb{R}^d)$, $\operatorname{supp} v_j \subset B_{r_1}$,
- $||v_j||_{m,1} \leq N$ for some $N > 0$, $j = 1, 2$.

Let $S_1^+(E)$ and $S_2^+(E)$ denote the near-field scattering data for v_1 and v_2, respectively. Then

$$||v_1 - v_2||_{\mathbb{L}^\infty(B_r)} \leq C \left(\ln\left(3 + \delta^{-1}\right)\right)^{-3/4} \left(\ln\left(3 \ln\left(3 + \delta^{-1}\right)\right)\right)^2,$$

where $\delta = ||S_1^+(E) - S_2^+(E)||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}$ and $C > 0$ depends only on N, m, r.

Publications

[IN1] M.I. Isaev, R.G. Novikov, *Energy and regularity dependent stability estimates for the Gel'fand inverse problem in multidimensions*, J. of Inverse and Ill-posed Probl., Vol. 20(3), 2012, 313–325.

[Isaev1] M.I. Isaev, *Instability in the Gel'fand inverse problem at high energies*, Applicable Analysis, 2012, DOI:10.1080/00036811.2012.731501.

[Isaev2] M.I. Isaev, *Exponential instability in the Gel'fand inverse problem on the energy intervals*, J. Inverse Ill-Posed Probl., Vol. 19(3), 2011, 453–473.

[IN2] M.I. Isaev, R.G. Novikov, *Stability estimates for determination of potential from the impedance boundary map*, Algebra and Analysis, Vol. 25(1), 2013, 37–63.

[IN3] M.I. Isaev, R.G. Novikov, *Reconstruction of a potential from the impedance boundary map*, Eurasian Journal of Mathematical and Computer Applications, Vol. 1(1), 2013, 5–28.

[IN4] M.I. Isaev, R.G. Novikov, *New global stability estimates for monochromatic inverse acoustic scattering*, SIAM Journal on Mathematical Analysis, Vol. 45(3), 2013, 1495–1504.

[Isaev3] M.I. Isaev, *Exponential instability in the inverse scattering problem on the energy interval*, Func. Anal. i ego Pril., Vol. 47(3), 2013, 28–36.

[Isaev4] M.I. Isaev, *Energy and regularity dependent stability estimates for near-field inverse scattering in multidimensions*, Journal of Mathematics, Hindawi Publishing Corp., 2013, DOI:10.1155/2013/318154.

The end

Thank you for your attention!