Contrôle d'équations de Schrödinger et d'équations paraboliques dégénérées singulières

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Préface

Ce mémoire présente les différents travaux réalisés au cours de ma thèse avec pour ligne directrice l'étude de propriétés de contrôlabilité et non contrôlabilité d'équations aux dérivées partielles.

La première partie est consacrée à l'étude de systèmes quantiques bilinéaires unidimensionnels principalement autour de deux axes : la non contrôlabilité en temps petit avec des contrôles petits et la contrôlabilité simultanée. Ce thème possède de nombreuses applications physiques, comme la manipulation de molécules avec des champs électriques, ou la réalisation de portes logiques quantiques. On établit un cadre pour lequel, bien que la vitesse de propagation du système considéré soit infinie, la contrôlabilité exacte locale avec des contrôles petits est vérifiée si et seulement si le temps est suffisamment grand. Ces résultats, basés sur la coercivité d'une forme quadratique associée au développement de l'état à l'ordre deux, sont étendus dans le contexte de la contrôlabilité simultanée. On montre alors, en utilisant la méthode du retour de J.-M. Coron, la contrôlabilité exacte locale simultanée pour deux ou trois équations, à phase globale et/ou à retard global près. La trajectoire de référence utilisée est construite via des résultats de contrôle partiel. En utilisant un argument de perturbation, on étend cette idée pour montrer la contrôlabilité exacte globale d'un nombre quelconque d'équations sans hypothèses sur le potentiel.

Dans la deuxième partie, on prend en compte dans le modèle un terme supplémentaire, quadratique en le contrôle. Ce terme, dit de polarisabilité, généralement négligé, présente un intérêt physique dans la modélisation, mais aussi mathématique dans le cas où le terme bilinéaire est insuffisant pour conclure à la contrôlabilité. En dimension quelconque, on construit des contrôles explicites réalisant la contrôlabilité approchée de l'état fondamental. En adaptant conjointement l'argument de perturbation précédent et certains résultats du contrôle bilinéaire, on prouve aussi la contrôlabilité globale exacte du modèle avec polarisabilité 1D.

La dernière partie de ce mémoire est consacrée à l'étude de la continuation unique pour un opérateur de type Grushin sur un rectangle 2D. Cet opérateur présente une singularité et une dégénérescence sur un segment séparant le domaine en deux composantes. On donne une condition nécessaire et suffisante sur le coefficient du potentiel singulier pour obtenir la continuation unique.

Mots clés : équation de Schrödinger bilinéaire ; contrôle exact ; contrôle simultané ; polarisabilité ; équation de Grushin singulière

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Introduction

Sommaire

1.1 Introduction générale

La problématique de la théorie du contrôle réside dans la question suivante : étant donnée la loi d'évolution d'un système (physique, biologique ou autre), étant donnés un état initial et une cible, est-il possible de guider l'évolution de ce système, via une action extérieure, de l'état initial vers la cible ? Il est alors possible de vouloir atteindre cette cible de manière exacte ou de manière approchée, en temps fini (contrôle) ou asymptotiquement (stabilisation).

Mathématiquement, on considère le système

$$\begin{cases} x' = f(x, u), \\ x(0) = x_0, \end{cases} \tag{1.1}$$

où l'état x est à valeurs dans un espace $\mathcal{X}$ et le contrôle u dans un espace $\mathcal{U}$.
Si l'espace $\mathcal{X}$ est de dimension finie, les propriétés de contrôlabilité de (1.1) sont relativement bien comprises (voir par exemple les livres [54, 129] par J.-M. Coron et E.D. Sontag). Par exemple, le critère de Kalman assure la contrôlabilité en tout temps de $x' = Ax + Bu$ avec A matrice réelle de taille $n \times n$ et B matrice réelle de taille $n \times m$ si et seulement si la matrice $(B, AB, \dots, A^{n-1}B)$ est de rang maximal. Pour un système affine en le contrôle i.e. $f(x,u) = f_0(x) + \sum_{j=1}^m u_j f_j(x)$ les propriétés de contrôlabilité sont reliées aux propriétés de l'algèbre de Lie engendrée par $\{f_0, f_1, \dots, f_m\}$.

Si l'espace $\mathcal{X}$ est de dimension infinie, les propriétés de contrôlabilité de (1.1) sont plus subtiles. Les notions de contrôlabilité envisagées dépendent des propriétés des opérateurs différentiels apparaissant dans (1.1). En effet, il est par exemple vain de chercher à montrer la contrôlabilité exacte globale pour un système régularisant. Les espaces fonctionnels dans lesquels on examine les propriétés de contrôlabilité peuvent aussi être déterminants. Cette dépendance vis à vis du cadre fonctionnel est soulignée dans la sous-section suivante pour un système de contrôle bilinéaire. Dans ce mémoire, on met aussi en évidence l'existence d'un temps minimal pour la contrôlabilité exacte locale bien que le système considéré ait une vitesse de propagation infinie. Il n'y a donc pas de théorie générale unifiée pour étudier la contrôlabilité en dimension infinie : chaque situation doit être examinée au cas par cas. Ce mémoire s'intéresse aux propriétés de contrôlabilité pour des systèmes de contrôle de Schrödinger bilinéaire, des systèmes de Schrödinger avec un terme de polarisabilité (i.e. avec un terme quadratique en le contrôle en plus du terme bilinéaire) et des systèmes paraboliques dégénérés et singuliers de type Grushin. Ces trois modèles constituent les trois parties de ce mémoire.

La suite de cette introduction s'attache à présenter, pour chacune des parties, le modèle étudié et certains résultats de la littérature liés à ce modèle. Le but n'est pas de donner un panorama exhaustif de la littérature existante mais de mentionner des résultats ou des idées, soit pour situer les résultats de cette thèse parmi cette littérature, soit pour introduire certaines notions utilisées dans ce mémoire. On présente ensuite les principaux résultats obtenus durant cette thèse ainsi que les idées et les stratégies mises en place pour résoudre les difficultés qui se sont présentées. On conclut en donnant de nouveaux problèmes ouverts apparus durant cette étude.

1.2 Contrôle bilinéaire d'équations de Schrödinger

1.2.1 Modèle

On considère une particule quantique évoluant dans un espace de dimension d. Cette particule est décrite par sa fonction d'onde $\psi : (t,x) \in \mathbb{R} \times \mathbb{R}^d \mapsto \psi(t,x) \in \mathbb{C}$. Pour un sous-domaine ω de $\mathbb{R}^d$, la probabilité que la particule se trouve au temps t dans ω est donnée par $\int_\omega |\psi(t,x)|^2 \mathrm{d}x$.

On note que pour toute fonction $\theta : \mathbb{R} \to \mathbb{R}$, les fonctions d'onde ψ et $e^{i\theta(t)}\psi$ décrivent le même état : la phase globale n'a pas de signification physique. L'évolution de la fonction d'onde est donnée par l'équation de Schrödinger

$$i\hbar\partial_t\psi(t,x) = H(t)\psi(t,x) \tag{1.2}$$

où H est l'hamiltonien du système et $\hbar$ la constante de Planck réduite. Dans ce mémoire, on considère que la particule se trouve dans un potentiel $V(x)$ infini hors d'un domaine borné régulier $D \subset \mathbb{R}^d$. L'hamiltonien du système est indépendant du temps et on trouve après un changement d'échelle

$$\begin{cases} i\partial_t\psi(t,x) = (-\Delta + V(x))\,\psi(t,x), & (t,x) \in (0,T) \times D, \\ \psi(t,x) = 0, & (t,x) \in (0,T) \times \partial D, \\ \psi(0,x) = \psi_0(x), & x \in D. \end{cases} \tag{1.3}$$

Les conditions au bord de Dirichlet homogène modélisent le fait que la probabilité de trouver la particule au bord est nulle. Au vu de l'interprétation probabiliste précédente de la fonction d'onde, celle-ci évolue sur la sphère unité de $L^2(D,\mathbb{C})$ notée $\mathcal{S}$ dans la suite. On note A_V l'hamiltonien du système libre (1.3) défini par

$$A_V\psi := (-\Delta + V)\,\psi, \qquad D\,(A_V) := H^2 \cap H_0^1(D,\mathbb{C}). \tag{1.4}$$

Pour $V \in L^2(D,\mathbb{C})$, on note $(\lambda_{k,V})_{k\in\mathbb{N}^*}$ la suite croissante des valeurs propres de A_V et $\varphi_{k,V}$ les vecteurs propres associés dans $\mathcal{S}$. On obtient alors des solutions évidentes du système libre (1.3) données par les états propres

$$\Phi_{k,V}(t,x) := e^{-i\lambda_{k,V}t}\varphi_{k,V}(x), \quad (t,x) \in \mathbb{R} \times D. \tag{1.5}$$

L'état propre de plus basse énergie $\Phi_{1,V}$ est appelé état fondamental. Concernant le cadre fonctionnel, on note pour $s > 0$, $H^s_{(V)} := D\left(A_V^{\frac{s}{2}}\right)$.

Ayant en vue des applications comme la manipulation de liens chimiques de certaines molécules ou la construction de portes quantiques logiques comme préalable à l'ordinateur quantique, on souhaite contrôler l'évolution de la fonction d'onde via un champ électrique extérieur. L'hamiltonien du système est alors la somme de l'hamiltonien libre précédent et de l'hamiltonien d'interaction. En première approximation, l'hamiltonien d'interaction est donné par $-u(t)\mu(x)$ où $u(t) \in \mathbb{R}$ est l'amplitude du champ extérieur appliqué et $\mu(x)$ est le moment dipolaire de la particule considérée. Les différentes approximations conduisant à cette expression sont détaillées par exemple dans [63, Chapitre 2]. On considère alors le système

$$\begin{cases} i\partial_t\psi(t,x) = (-\Delta + V(x))\,\psi(t,x) - u(t)\mu(x)\psi(t,x), & (t,x) \in (0,T) \times D, \\ \psi(t,x) = 0, & (t,x) \in (0,T) \times \partial D, \\ \psi(0,x) = \psi_0(x), & x \in D. \end{cases} \tag{1.6}$$

Le contrôle est la fonction réelle $t \mapsto u(t)$ et l'état est la fonction d'onde $\psi(t,\cdot) \in \mathcal{S}$. Lorsque le système (1.6) est bien posé on note $\psi(t,\psi_0,u)$ la solution au temps t. Bien que cette équation aux dérivées partielles soit linéaire, ce problème de contrôle est non linéaire i.e. l'application $(u, \psi_0) \mapsto \psi(T)$ est non linéaire. L'appellation bilinéaire est relative à la bilinéarité du terme "$u(t)\mu(x)\psi$" par rapport au couple état, contrôle (ψ, u).

Un cas particulier, très étudié, est celui du puits de potentiel unidimensionnel i.e.

$$\begin{cases} i\partial_t \psi = -\partial_{xx}^2 \psi - u(t)\mu(x)\psi, & (t,x) \in (0,T) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0. \end{cases} \tag{1.7}$$

Sans autre précision, les éléments propres $\lambda_{k,0}$, $\varphi_{k,0}$ et $\Phi_{k,0}$ seront notés λ_k, φ_k et Φ_k.

1.2.2 Résultats précédents

Le contrôle de systèmes quantiques est un domaine en plein essor. Pour preuve, le Prix Nobel de Physique 2012 a été décerné à S. Haroche pour ses résultats expérimentaux sur l'observation de particules quantiques sans les détruire. Si le fossé entre résultats théoriques et expérimentations physiques reste encore à combler, les idées de mesures de particules quantiques développées par l'équipe de S. Haroche ont été utilisées pour la contrôlabilité d'un système quantique [126].

Mathématiquement, le problème de contrôle linéaire de l'équation de Schrödinger a été largement étudié. La propriété de réversibilité en temps et la méthode HUM (dont le principe est rappelé en Section 1.4.2) permettent de réduire la contrôlabilité exacte à l'observabilité du système adjoint. De nombreux outils ont été utilisés pour étudier cette inégalité d'observabilité comme la méthode des multiplicateurs (voir [71] par C. Fabre, [99] par E. Machtyngier), l'analyse microlocale (voir [96] par G. Lebeau, [34] par N. Burq), les inégalités de Carleman (voir [93, 94] par I. Lasiecka, R. Triggiani et X. Zhang) ou la théorie des nombres (voir [119, 131] par K. Ramdani, T. Takahashi, G. Tenenbaum et M. Tucsnak). Pour de plus amples détails, ainsi que des résultats pour des équations de Schrödinger non linéaires, on renvoie aux articles de synthèse [143, 95] par E. Zuazua et C. Laurent. Au vu de la modélisation précédente, ce cadre linéaire est insuffisant pour viser les applications physiques mentionnées. On se focalise donc, pour la suite, sur le cadre bilinéaire du système (1.6).

Historiquement, le premier résultat concernant le contrôle bilinéaire en dimension infinie revêt une importance particulière. J.M. Ball, J.E. Marsden et M. Slemrod ont démontré dans [5] que pour un système de contrôle bilinéaire abstrait,

$$\begin{cases} w'(t) = \mathcal{A}w(t) + p(t)\mathcal{B}w(t), \\ w(0) = w_0, \end{cases} \tag{1.8}$$

l'ensemble des états atteignables à partir de n'importe quel w_0 est d'intérieur vide dans l'espace ambiant. Ce théorème s'énonce comme suit.

Théorème 1.1. *Soit X un espace de Banach de dimension infinie. Soit $\mathcal{A}$ le générateur infinitésimal d'un semigroupe continu sur X et $\mathcal{B} : X \to X$ un opérateur linéaire borné. Soit $w_0 \in X$. Pour $p \in L^1_{loc}([0,+\infty),\mathbb{R})$, on note $w(t,w_0,p)$ l'unique solution faible de (1.8). Alors, l'ensemble $R(w_0)$ des états atteignables à partir de w_0 défini par*

$$R(w_0) := \{w(t,w_0,p)\,;\, t \geq 0, r > 1, p \in L^r_{loc}([0,\infty),\mathbb{R})\},$$

est contenu dans une union dénombrable de compacts de X. En particulier, le complémentaire de $R(w_0)$ dans X est dense dans X.

Ainsi, arbitrairement proche (pour une norme naturelle) de tout état atteignable, il existe des cibles que l'on ne pourra pas atteindre. Dans le cadre d'équations de Schrödinger bilinéaire, l'évolution de (1.7) ayant lieu sur la sphère $\mathcal{S}$, ce résultat n'est pas surprenant. Cependant, en adaptant le Théorème 1.1 pour prendre en compte cette conservation de la norme, G. Turinici a montré dans [134] le théorème suivant.

Théorème 1.2. *Soit $\psi_0 \in \mathcal{S} \cap H^2_{(0)}$. Pour $u \in L^2(0,T)$, on note $\psi(t,\psi_0,u)$ la solution de (1.7) issue de ψ_0. Soit $R(\psi_0)$ l'ensemble des états atteignables à partir de ψ_0 défini par*

$$R(\psi_0) := \left\{\psi(t,\psi_0,u)\,;\, t \geq 0, u \in L^2((0,T),\mathbb{R})\right\}.$$

Alors le complémentaire de $R(\psi_0)$ dans $\mathcal{S} \cap H^2_{(0)}$ est dense dans $\mathcal{S} \cap H^2_{(0)}$.

Ces résultats ne sont donc pas engageants du point de vue de la contrôlabilité exacte pour les systèmes bilinéaires de dimension infinie.

Partant de ce constat, deux possibilités sont alors envisagées pour l'étude de tels systèmes : considérer des systèmes de dimension finie ou s'intéresser au contrôle approché du système de dimension infinie. Néanmoins, comme explicité dans la Section 1.2.2.2, ces résultats négatifs sont fortement dépendants du cadre fonctionnel et tombent en défaut pour des espaces ambiants plus réguliers.

Un système quantique bilinéaire de dimension finie est de la forme

$$i\frac{\mathrm{d}X}{\mathrm{d}t} = H_0 X + u(t) H_1 X, \tag{1.9}$$

où $X \in \mathbb{C}^n$ et H_0, H_1 sont des matrices hermitiennes. Ce système entre dans le formalisme des systèmes de dimension finie affines en le contrôle dont les propriétés de contrôlabilité sont reliées à l'algèbre de Lie engendrée par H_0 et H_1 (voir par exemple [63, Chapitre 3] par D. D'Alessandro).

En dimension infinie, les crochets de Lie itérés peuvent être mal définis et leur application éventuelle à la contrôlabilité est mal comprise. Pour ces raisons, on ne détaille pas les résultats relatifs à la dimension finie qui sont assez éloignés des problématiques de ce mémoire.

1.2.2.1 Contrôle approché de systèmes quantiques bilinéaires de dimension infinie

Contrôle exact des approximations de Galerkin. Concernant le contrôle approché de systèmes quantiques bilinéaires de dimension infinie, plusieurs stratégies cohabitent. Une des méthodes les plus prolifiques consiste à utiliser les outils du contrôle géométrique. Le premier résultat pour un système abstrait

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(t) = A\psi(t) + u(t)B\psi(t), \quad u(t) \in U \subset \mathbb{R}, \tag{1.10}$$

a été obtenu par T. Chambrion, P. Mason, M. Sigalotti et U. Boscain dans [45]. L'idée principale pour montrer la contrôlabilité approchée dans L^2 consiste à contrôler de manière exacte les approximations de Galerkin du système (1.10) puis d'obtenir de bonnes propriétés d'approximation du système de dimension infinie par les systèmes de Galerkin de dimension finie. Il est à noter que, si cette stratégie semble naturelle, les propriétés de contrôlabilité d'un système de dimension infinie sont parfois très différentes de celles de ses approximations de Galerkin. A cet égard, il est important de garder à l'esprit le cas de l'oscillateur harmonique quantique

$$i\partial_t\psi = \left(\frac{1}{2}\left(-\partial_{xx}^2 + x^2\right) - u(t)x\right)\psi, \quad x \in \mathbb{R}. \tag{1.11}$$

M. Mirrahimi et P. Rouchon ont montré dans [104] que seule la position moyenne, définie par $\int_{\mathbb{R}} x|\psi(t,x)|^2 \mathrm{d}x$ (de dimension 2), est contrôlable bien que toutes les approximations de Galerkin soient contrôlables (voir [127]).
La méthode mise en œuvre dans [45] est appliquée au cas du puits de potentiel de dimension 3 et à des variantes de l'oscillateur harmonique quantique où $\mu(x) \neq x$. Il est à noter que le fait de considérer des opérateurs abstraits dans (1.10) permet de traiter des problèmes plus généraux que (1.6), en toutes dimensions, ou des problèmes posés sur des variétés. Cette méthode permet aussi d'étudier un problème très important du point de vue physique : la contrôlabilité (approchée) des matrices densité. Le résultat de [45] a été raffiné afin d'affaiblir les hypothèses.

Hypothèses 1.1. On suppose que $\mathcal{H}$ est un espace de Hilbert et que (A, B, δ, Φ) sont tels que

i) A et B sont des opérateurs antihermitiens de $\mathcal{H}$,

ii) $\Phi = (\phi_k)_{k\in\mathbb{N}}$ est une base hilbertienne de $\mathcal{H}$ de vecteurs propres de A associés aux valeurs propres $(i\lambda_k)_{k\in\mathbb{N}}$,

iii) $\phi_k \in D(B)$ pour tout $k \in \mathbb{N}$,

iv) $A + uB : \mathrm{Vect}\{\phi_k\,;\, k \in \mathbb{N}\} \to \mathcal{H}$ est essentiellement antihermitien pour tout $u \in [0, \delta]$,

v) si $j \neq k$ et $\lambda_j = \lambda_k$ alors $\langle \phi_j, B\phi_k \rangle = 0$.

Pour ce système, on appelle chaîne de connexité un sous-ensemble S de $\mathbb{N}^2$ tel que, pour tout $n, p \in \mathbb{N}$, il existe $L \in \mathbb{N}^*$, $s_1^1, \dots, s_1^L \in \mathbb{N}$, $s_2^1, \dots, s_2^L \in \mathbb{N}$, tels que $(s_1^j, s_2^j) \in S$, $s_1^1 = n$, $s_2^L = p$, $s_1^{j+1} = s_2^j$ et $\langle \phi_{s_1^j}, B\phi_{s_2^j} \rangle \neq 0$. En utilisant cette stratégie de contrôle des approximations de Galerkin, U. Boscain, T. Chambrion, M. Caponigro et M. Sigalotti ont prouvé dans [25] le théorème suivant.

Théorème 1.3. *Supposons que (A, B, δ, Φ) vérifient les Hypothèses 1.1. Supposons qu'il existe une chaîne de connexité S non résonnante i.e. pour tout $(s_1, s_2) \in S$, $|\lambda_{s_1} - \lambda_{s_2}| \neq |\lambda_{t_1} - \lambda_{t_2}|$ pour tout $(t_1, t_2) \in \mathbb{N}^2 \backslash \{(s_1, s_2), (s_2, s_1)\}$ vérifiant $\langle \phi_{t_2}, B\phi_{t_1} \rangle \neq 0$. Alors, pour tout $r \in \mathbb{N}^*$, $\psi_0^1, \dots, \psi_0^r \in \mathcal{H}$, $\Upsilon : \mathcal{H} \to \mathcal{H}$ unitaire et $\varepsilon > 0$, il existe $u : [0, T] \to [0, \delta]$ constant par morceaux tel que si ψ^j est la solution de (1.10) avec condition initiale ψ_0^j et contrôle u alors*

$$\|\psi^j(T) - \Upsilon\psi_j^0\|_{\mathcal{H}} < \varepsilon, \quad \forall k = 1, \dots, r.$$

Ce résultat de contrôle approché simultané est valable quel que soit le nombre d'équations ; la notion de contrôle simultané est au cœur des Chapitres 3 et 4 présentés dans cette introduction en Section 1.2.3.2. Ce résultat a été étendu par N. Boussaïd, M. Caponigro et T. Chambrion, à des normes plus fortes pour des système dits faiblement couplés dans

[30], pour des contrôles à variation bornée dans [33] ou pour des modèles avec un terme de polarisabilité dans [31] (voir la Section 1.3 pour de plus amples détails sur ce modèle). Pour un article de synthèse sur les différents résultats obtenus par ces techniques et quelques problèmes ouverts, on renvoie à [27].

Concernant la contrôlabilité approchée en temps fini de certains modèles particuliers, on mentionne les résultats de R. Adami et U. Boscain [1] utilisant la théorie adiabatique et l'intersection de valeurs propres dans l'espace des contrôles et ceux de S. Ervedoza et J.-P. Puel [70] basés sur des contrôles explicites pour la contrôlabilité approchée de sommes finies de fonctions propres.

Méthode de Lyapunov. Une autre méthode pour montrer des résultats de contrôlabilité approchée est l'adaptation de la méthode de Lyapunov et du principe d'invariance de LaSalle. Cette méthode sera utilisée à plusieurs reprises dans ce mémoire sous la forme suivante : on cherche à stabiliser

$$i\partial_t\psi = \left(-\partial^2_{xx} + V(x)\right) - u(t)\mu(x)\psi$$

vers un état $\tilde{\psi}$. L'idée est alors de construire une fonction de Lyapunov $\mathcal{L}$ c'est-à-dire une fonction positive, minimale en $\tilde{\psi}$ et décroissante le long des trajectoires du système considéré. La décroissance le long des trajectoires est assurée par le choix du contrôle u sous la forme d'une loi de rétroaction. Généralement, la fonction $\mathcal{L}$ est choisie telle que

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi(t)) = u(t)F(\psi(t)). \tag{1.12}$$

La loi de rétroaction $u(t) := -F(\psi(t))$ assure alors la décroissance. Ainsi, si de plus $\mathcal{L}$ est une borne supérieure d'une certaine norme $\|\cdot\|$, la fonction de Lyapunov étant décroissante le long des trajectoires, celles-ci sont bornées pour $\|\cdot\|$. Moyennant quelques détails techniques si l'ensemble invariant est réduit à $\tilde{\psi}$, on peut espérer obtenir la convergence faible vers $\tilde{\psi}$. Cette notion de stabilisation fournit alors un contrôle approché en temps grand pour des normes plus faibles que $\|\cdot\|$.

Cette méthode a d'abord été utilisée dans le cadre L^2 par M. Mirrahimi dans [103] pour un opérateur libre présentant un spectre mixte puis par K. Beauchard et M. Mirrahimi dans [17] pour (1.7) dans le cas $\mu(x) = x - \frac{1}{2}$. La convergence faible L^2 ne permettant pas de conclure à la stabilisation, ces deux articles démontrent la stabilisation approchée dans L^2 de l'état fondamental en combinant la stratégie de Lyapunov et des arguments de perturbation.

En utilisant une fonction de Lyapunov adaptée au cadre H^2 construite par V. Nersesyan [111] et la stratégie précédemment décrite, K. Beauchard et V. Nersesyan ont prouvé dans [19] la stabilisation semi-globale faible H^2 de l'état fondamental, pour (1.6), sous des hypothèses favorables sur V et μ. La loi de rétroaction proposée dans [19] contient une constante choisie uniformément pour toute condition initiale dans une boule pour la norme H^2 : c'est le caractère semi-global de ce résultat. L'utilisation de cette fonction de Lyapunov a aussi permis dans [111] de conclure à la contrôlabilité approchée globale dans L^2.

Enfin, cette fonction de Lyapunov a été étendue par V. Nersesyan dans [112] pour toute norme de Sobolev H^s avec $s > 0$ conduisant à la contrôlabilité approchée de l'état fondamental dans $H^{s-\epsilon}$. Dans ce cas, la dérivée de la fonction de Lyapunov le long des trajectoires ne prend pas la forme (1.12). La décroissance de la fonction de Lyapunov est assurée par un argument variationnel : V. Nersesyan prouve l'existence d'une direction dans laquelle la dérivée de la fonction de Lyapunov le long de la trajectoire est non nulle. Ceci conduit à l'existence d'une suite de temps et de contrôles le long desquelles la trajectoire associée converge vers l'état fondamental. A la différence de la méthode de contrôle des approximations de Galerkin, l'utilisation de fonctions de Lyapunov permet donc d'obtenir des résultats de contrôle approché dans des espaces très réguliers. En contrepartie, les informations sur le temps nécessaire à la contrôlabilité approchée sont perdues. Ces fonctions de Lyapunov (pour les normes H^2 et H^s) sont utilisées et adaptées dans ce mémoire (voir les Sections 1.2.3.3 et 1.3.3).

1.2.2.2 Contrôle exact de systèmes quantiques bilinéaires de dimension infinie

Nous allons voir que les résultats négatifs précédents (voir Théorème 1.2) pour la contrôlabilité exacte de systèmes bilinéaires tombent en défaut pour d'autres cadres fonctionnels. Ainsi, K. Beauchard a démontré dans [10] le résultat suivant.

Théorème 1.4. *Soit $\mu(x) = x$. Soient $\theta_0, \theta_f \in \mathbb{R}$. Il existe $T > 0$ et $\eta > 0$ tels que pour tout $\psi_0, \psi_f \in \mathcal{S} \cap H^7_{(0)}$ vérifiant*

$$\|\psi_0 - e^{i\theta_0}\varphi_1\|_{H^7} < \eta, \quad \|\psi_f - e^{i\theta_f}\varphi_1\|_{H^7} < \eta,$$

il existe $u \in H^1_0((0,T),\mathbb{R})$ tel que la solution de (1.7) satisfasse $\psi(T) = \psi_f$.

Ainsi, l'ensemble des états atteignables à partir de l'état fondamental φ_1 contient une boule pour la norme H^7, qui est bien d'intérieur vide dans $H^2_{(0)}$. Alors que, suite au résultat de J.M. Ball, J.E. Marsden et M. Slemrod, le système (1.7) était considéré comme non contrôlable, il apparait dès lors clair que la bonne lecture de ce résultat négatif est la non contrôlabilité dans $H^2_{(0)}$. La preuve relativement technique de ce résultat est basée sur la méthode du retour (introduite par J.-M. Coron et dont le principe sera détaillé en Section 1.2.3.2), des déformations quasi-statiques induisant un temps long pour la contrôlabilité, la méthode des moments et le théorème de Nash-Moser. Le recours au théorème de Nash-Moser est dû à une apparente perte de régularité.
Ce résultat a ensuite été raffiné dans [16] par K. Beauchard et C. Laurent pour des modèles plus généraux, dans un cadre fonctionnel optimal. La mise en lumière d'un effet régularisant a aussi permis de déduire la contrôlabilité locale du système bilinéaire à partir de la contrôlabilité du linéarisé en utilisant le théorème d'inversion locale et non plus le théorème de Nash-Moser, ce qui conduit à une simplification de la preuve. Ce théorème s'énonce comme suit.

Théorème 1.5. *Soit $\mu \in H^3((0,1),\mathbb{R})$ tel que*

$$\exists C > 0 \text{ tel que } |\langle \mu\varphi_1, \varphi_k\rangle| \geq \frac{C}{k^3}, \quad \forall k \in \mathbb{N}^*. \tag{1.13}$$

Soit $T > 0$. Il existe $\delta > 0$ et une application C^1,

$$\Gamma : \mathcal{V}_T \to L^2((0,T),\mathbb{R}),$$

où
$$\mathcal{V}_T := \left\{\psi_f \in \mathcal{S} \cap H^3_{(0)}\,;\, \|\psi_f - \Phi_1(T)\|_{H^3_{(0)}} < \delta\right\}$$
telle que $\Gamma(\Phi_1(T)) = 0$ *et, pour tout* $\psi_f \in \mathcal{V}_T$*, la solution de (1.7) avec contrôle* $u = \Gamma(\psi_f)$ *et condition initiale* $\psi_0 = \varphi_1$ *satisfasse* $\psi(T) = \psi_f$*.*

La norme $\|\cdot\|_{H^3_{(0)}}$ est définie par

$$\|\psi\|_{H^3_{(0)}} := \left(\sum_{k=1}^{\infty} |k^3\langle\psi, \varphi_k\rangle|^2\right)^{\frac{1}{2}}.$$

Ce théorème appelle quelques commentaires au vu du résultat négatif du Théorème 1.1. Il est clair que l'opérateur
$$\begin{array}{ccc} H^2_{(0)} & \to & H^2_{(0)} \\ \psi & \mapsto & \mu\psi \end{array}$$
est un opérateur borné et que l'on est donc dans le cadre d'application du Théorème 1.1. Par intégrations par parties, on obtient

$$\langle\mu\varphi_1, \varphi_k\rangle = \frac{4}{\pi^2 k^3}\big((-1)^{k+1}\mu'(1) - \mu'(0)\big) + \underset{k\to\infty}{o}\left(\frac{1}{k^3}\right). \tag{1.14}$$

Ainsi, l'hypothèse (1.13) implique $\mu'(0) \pm \mu'(1) \neq 0$. Comme
$$H^3_{(0)} = \left\{\psi \in H^3((0,1),\mathbb{C})\,;\, \psi(0) = \psi(1) = \psi''(0) = \psi''(1) = 0\right\},$$
on a $\mu.H^3_{(0)} \not\subset H^3_{(0)}$. Ceci explique que le résultat négatif du Théorème 1.1 ne couvre pas le cadre fonctionnel $H^3_{(0)}$ sous l'hypothèse (1.13).
Les idées du Théorème 1.5 sont essentielles pour les résultats de la Partie I. En plus de la mise en lumière de l'effet régularisant [16, Lemme 1] qui permet de démontrer le caractère bien posé du système (1.7) dans $H^3_{(0)}$, la preuve repose sur la contrôlabilité du linéarisé autour de la trajectoire $(\Phi_1, u \equiv 0)$,

$$\begin{cases} i\partial_t\Psi = -\partial^2_{xx}\Psi - v(t)\mu(x)\Phi_1, & (t,x) \in (0,T)\times(0,1), \\ \Psi(t,0) = \Psi(t,1) = 0, \\ \Psi(0,x) = 0. \end{cases} \tag{1.15}$$

Des calculs explicites conduisent à l'expression

$$\Psi(T) = i\sum_{k=1}^{\infty}\langle\mu\varphi_1, \varphi_k\rangle \int_0^T v(t)e^{i(\lambda_k-\lambda_1)t}\mathrm{d}t\,\Phi_k(T). \tag{1.16}$$

Ainsi, trouver v tel que $\Psi(T) = \Psi_f$ est équivalent à résoudre le problème de moments trigonométriques
$$\int_0^T v(t)e^{i(\lambda_k-\lambda_1)t}\mathrm{d}t = \frac{\langle\Psi_f, \Phi_k(T)\rangle}{i\langle\mu\varphi_1, \varphi_k\rangle}, \quad \forall k \in \mathbb{N}^*. \tag{1.17}$$
Sous cette forme, on voit l'importance du comportement des coefficients $\langle\mu\varphi_1, \varphi_k\rangle$ et de l'hypothèse (1.13). Indépendamment du problème de contrôle, la résolution d'un tel problème

de moments est liée à l'existence d'une condition de saut pour les fréquences considérées (voir par exemple les livres [141, 92] de R.M. Young et de V. Komornik et P. Loreti). Le temps T nécessaire à la résolution d'un tel problème de moments est relié au saut minimal entre les fréquences. Le fait que $\lambda_k = k^2\pi^2$ entraine que le saut entre les fréquences tend vers l'infini, ce qui permet de résoudre le problème de moments en temps arbitraire.
En utilisant l'asymptotique de Weyl, on obtient que pour un domaine Ω de $\mathbb{R}^d$ suffisamment régulier les valeurs propres du Laplacien Dirichlet vérifient

$$\lambda_k \underset{k\to\infty}{\sim} c(\Omega, d)k^{\frac{2}{d}}.$$

Bien qu'encore ouverte, la question de l'extension de la stratégie de K. Beauchard et C. Laurent en dimension supérieure, pourrait trouver une réponse positive en dimension deux. Le cas de la dimension supérieure ou égale à trois est lui hors de portée par ces techniques, le linéarisé (1.15) étant fortement non contrôlable. Ainsi, les résultats basés sur la résolution d'un problème de moments trigonométriques de la forme (1.17) sont restreints à des modèles unidimensionnels.

En couplant le Théorème 1.5 aux résultats de contrôle approché vers l'état fondamental dans des espaces réguliers de V. Nersesyan [112], on obtient la contrôlabilité exacte globale de (1.7) dans $\mathcal{S}\cap H^{3+\epsilon}_{(0)}$ avec des hypothèses génériques sur μ. Comme souligné dans [27, Section IV.A], ce résultat de contrôle exact global ne peut être obtenu en couplant le Théorème 1.5 et la contrôlabilité globale approchée donnée par la méthode de contrôle des approximations de Galerkin (Théorème 1.3). En effet, même si cette technique a été étendue à certains espaces de Sobolev, via la notion de système faiblement couplé par N. Boussaïd, M. Caponigro et T. Chambrion [30], cette stratégie impose des restrictions sur l'espace ambiant. Ainsi, si ces résultats s'appliquent pour la norme d'un espace de Sobolev, l'ensemble atteignable à partir de certaines conditions initiales (par exemple l'état fondamental) est d'intérieur vide dans cet espace (voir [27, Propositions 2 et 8]). Les cadres fonctionnels pour la contrôlabilité exacte locale et la notion de système faiblement couplé sont donc incompatibles.

Avec des arguments semblables à ceux du Théorème 1.4 deux autres résultats de contrôle exact ont été montrés pour des systèmes quantiques particuliers de la forme (1.7). Le cas d'un puits de potentiel dynamique a été considéré dans [15] par K. Beauchard et J.-M. Coron. Dans ce modèle, $\mu(x) := x$, le contrôle est l'accélération du puits de potentiel et l'état est le triplé fonction d'onde, vitesse et position du puits de potentiel. Dans [11], K. Beauchard a considéré le cas d'un domaine de longueur variable. Après changement de variables, on trouve un système de la forme (1.7) où $\mu(x) = x^2$ avec un contrôle $(4v^2(t) - \dot{v}(t))$ en lieu et place de $u(t)$.

Si l'asymptotique de Weyl pour les valeurs propres du Laplacien semblent être un frein à l'extension de la contrôlabilité exacte à des dimensions supérieures, les travaux de V. Nersesyan et H. Nersisyan [113, 114] montrent que ces obstructions ne sont pas valables en temps infini. Ils ont en effet montré la contrôlabilité exacte en temps infini dans des espaces réguliers (contenant $H^{3d}_{(V)}$) du système (1.6) pour D un rectangle de $\mathbb{R}^d$. En considérant un temps infini, la condition de saut n'est plus nécessaire : l'indépendance des fréquences considérées suffit pour la contrôlabilité du système linéarisé.

On se tourne maintenant vers les apports de ce mémoire concernant les problèmes de contrôle quantique bilinéaire.

1.2.3 Principaux résultats

1.2.3.1 Temps minimal pour la contrôlabilité

La première question discutée dans ce mémoire concerne une généralisation du Théorème 1.5 où l'on affaiblit les hypothèses sur μ conduisant à l'existence d'un temps minimal pour la contrôlabilité exacte locale de (1.7).

Les résultats présentés ici sont exposés en détails dans le Chapitre 2. L'article [18], écrit en collaboration avec K. Beauchard, dont s'inspire le Chapitre 2 a été accepté pour publication dans le journal *Mathematical Control and Related Fields*.
Nous avons vu que la résolution du problème de moments (1.17) et donc la preuve du Théorème 1.5 nécessitent l'hypothèse (1.13). On s'intéresse à la question de savoir ce qui persiste du résultat de contrôlabilité exacte locale si l'un des coefficients $\langle \mu\varphi_1, \varphi_K\rangle$ est nul i.e., d'après (1.16), si la direction Φ_K est perdue sur le linéarisé. On précise la notion de contrôlabilité étudiée pour ce problème.

Définition 1.1. Soient $T > 0$, X et Y espaces normés tels que $X \subset L^2((0,1),\mathbb{C})$ et $Y \subset L^2((0,T),\mathbb{R})$. Le système (1.7) est dit contrôlable dans X, localement autour du fondamental, avec des contrôles dans Y, en temps T si pour tout $\varepsilon > 0$, il existe $\delta > 0$ tel que pour tout $\psi_f \in \mathcal{S} \cap X$ vérifiant $\|\psi_f - \Phi_1(T)\|_X < \delta$, il existe un contrôle $u \in Y$ avec $\|u\|_Y < \varepsilon$ tel que la solution associée, ψ, de (1.7) avec condition initiale $\psi_0 = \varphi_1$ satisfasse $\psi(T) = \psi_f$.

Cette notion implique donc la contrôlabilité exacte locale au voisinage de l'état fondamental mais requiert en plus que des mouvements arbitrairement petits se fassent avec des contrôles arbitrairement petits. Au regard de la dépendance C^1 (continue suffirait) entre la cible et le contrôle obtenue par K. Beauchard et C. Laurent dans le Théorème 1.5, cette notion de contrôlabilité est tout a fait justifiée.

Non contrôlabilité en temps petit : formes quadratiques coercives. Le premier résultat que nous avons obtenu est le suivant (voir le Théorème 2.3 page 51).

Théorème 1.6. *Soient $K \in \mathbb{N}^*$ et $\mu \in H^3((0,1),\mathbb{R})$ tels que*

$$\langle \mu\varphi_1, \varphi_K\rangle = 0, \quad \text{et} \quad A_K := \langle (\mu')^2\varphi_1, \varphi_K\rangle \neq 0. \tag{1.18}$$

Soit $\alpha_K \in \{-1, 1\}$ défini par $\alpha_K := sign(A_K)$. Il existe $T_K^* > 0$ tel que, pour tout $T < T_K^*$, il existe $\varepsilon > 0$ tel que pour tout contrôle $u \in L^2((0,T),\mathbb{R})$ vérifiant

$$\|u\|_{L^2(0,T)} < \varepsilon,$$

la solution associée, ψ, de (1.7) avec condition initiale $\psi_0 = \varphi_1$ vérifie

$$\psi(T) \neq \sqrt{1-\delta^2}\Phi_1(T) + i\alpha_K\delta e^{-i\lambda_1 T}\varphi_K, \quad \forall \delta > 0. \tag{1.19}$$

Ainsi, (1.7) n'est pas contrôlable dans $H^3_{(0)}$, localement autour du fondamental, avec des contrôles $u \in L^2((0,T),\mathbb{R})$ en temps $T < T_K^*$.

Il est à noter que la cible (1.19) non atteignable est arbitrairement proche de $\Phi_1(T)$. Contrairement au Théorème 1.1, ce résultat de non contrôlabilité n'est pas de nature topologique. En effet, il n'est pas lié au cadre fonctionnel puisque la cible (1.19) est analytique et arbitrairement proche de l'état fondamental pour toute norme.
Si l'existence d'un temps minimal pour la contrôlabilité est naturelle pour des équations à vitesse finie de propagation (comme l'équation de transport ou l'équation des ondes), ce phénomène peut sembler moins naturel, au premier abord, pour des équations à vitesse infinie de propagation (comme l'équation de Schrödinger considérée ici). Ce temps minimal est ici dû à la nonlinéarité du système de contrôle. Ce phénomène avait déjà été mis en évidence par J.-M. Coron, dans le cas particulier du potentiel dynamique [53] (contrôlable en temps grand d'après [15]), en démontrant le théorème suivant.

Théorème 1.7. *Il existe $\varepsilon > 0$ tel que, pour tout $\overline{D} \neq 0$, il n'existe pas de contrôle $u \in L^2((0,\varepsilon),(-\varepsilon,\varepsilon))$ tel que la solution $(\psi,S,D) \in C^0\big([0,\varepsilon],H^1_0((0,1),\mathbb{C})\big) \times C^0([0,\varepsilon],\mathbb{R}) \times C^1([0,\varepsilon],\mathbb{R})$ du problème de Cauchy*

$$\begin{cases} i\partial_t\psi = -\partial^2_{xx}\psi - u(t)\left(x-\dfrac{1}{2}\right)\psi, & (t,x)\in(0,T)\times(0,1),\\ \psi(t,0)=\psi(t,1)=0, \\ \dot{S}(t)=u(t), \quad \dot{D}(t)=S(t), \\ \big(\psi,S,D\big)(0)=\big(\varphi_1,0,0\big), \end{cases}$$

satisfasse $\big(\psi,S,D)(T)=\big(\Phi_1(\varepsilon),0,\overline{D}\big)$.

Ainsi, le Théorème 1.6 généralise ce résultat. En effet, pour $K=1$, on a bien

$$\Big\langle \left(x-\frac{1}{2}\right)\varphi_1,\varphi_K\Big\rangle = 0, \quad A_K = \langle\varphi_1,\varphi_K\rangle \neq 0.$$

La non contrôlabilité en temps petit ne provient donc pas de la contrainte supplémentaire de contrôler, en même temps que la fonction d'onde, la vitesse et la position du puits de potentiel mais du fait que $\mu(x)=x-\frac{1}{2}$. De plus, notre hypothèse sur la taille des contrôles est exprimée pour la norme L^2 et non plus la norme L^∞ ce qui est plus naturel vis à vis du cadre fonctionnel.

On explique ici l'heuristique de la preuve du Théorème 1.6. Pour étudier la contrôlabilité autour de la trajectoire $(\Phi_1, u\equiv 0)$, on considère un contrôle de la forme $u=0+\epsilon v+\epsilon^2 w$ et on développe la fonction d'onde suivant les puissances de ϵ

$$\psi = \Phi_1 + \epsilon\Psi + \epsilon^2\xi + o(\epsilon^2),$$

où Ψ est solution du linéarisé (1.15) et ξ vérifie

$$\begin{cases} i\partial_t\xi = -\partial^2_{xx}\xi - v(t)\mu(x)\Psi - w(t)\mu(x)\Phi_1, & (t,x)\in(0,T)\times(0,1),\\ \xi(t,0)=\xi(t,1)=0, \\ \xi(0,x)=0. \end{cases} \tag{1.20}$$

D'après (1.16), $\Psi(T)$ appartient à $\text{Adh}_{H^3_{(0)}}\big(\text{Vect}\{\varphi_k\,;\, k\neq K\}\big)$. Donc, si l'on déplace $\epsilon\Psi(T)+\epsilon^2\xi(T)$ dans la direction $i\alpha_K e^{-i\lambda_1 T}\varphi_K$ alors $\Psi(T)=0$ i.e.

$$v\in V_T := \Big\{v\in L^2((0,T),\mathbb{R})\,;\, \int_0^T v(t)e^{i(\lambda_k-\lambda_1)t}\mathrm{d}t=0,\ \text{ pour tout } k\in\mathbb{N}^* \\ \text{tel que } \langle\mu\varphi_1,\varphi_k\rangle\neq 0\Big\}$$

et $\xi(T)=i\delta\alpha_K e^{-i\lambda_1 T}\varphi_K$ avec $\delta>0$. Nécessairement, la forme quadratique

$$\tilde{Q}_{K,T}(v) := \text{Im}(\langle\xi(T), e^{-i\lambda_1 T}\varphi_K\rangle) \tag{1.21}$$

est alors du signe de α_K. L'essence du Théorème 1.6 repose sur le fait que cette forme quadratique $\tilde{Q}_{K,T}$ ait un signe strict en temps petit. Ainsi, pour α_K de signe contraire à celui de cette forme quadratique (ici $\alpha_K := \text{sign}(A_K)$ avec A_K défini en (1.18)) des cibles de la forme (1.19) sont inatteignables.

Supposons que $A_K>0$. En utilisant des intégrations par parties, si $s(t):=\int_0^t v(\tau)\mathrm{d}\tau$ on obtient l'existence d'un temps explicite T_K^* tel que pour tout $T<T_K^*$

$$\tilde{Q}_{K,T}(v)\leq -\frac{A_K}{4}\int_0^T s(t)^2\mathrm{d}t, \quad \forall v\in V_T. \tag{1.22}$$

Ce signe strict de la forme quadratique conclut donc cette heuristique et justifie la restriction du Théorème 1.6 au cas de contrôles petits dans L^2. En effet, faire le lien entre le signe de $\tilde{Q}_{K,T}$ et celui de $\text{Im}(\langle\psi(T), e^{-i\lambda_1 T}\varphi_K\rangle)$ n'est possible que si ξ est une bonne approximation de ψ. Ceci n'est réalisé que pour des contrôles petits.

Pour des raisons techniques, l'obtention du signe strict de $\text{Im}(\langle\psi(T), e^{-i\lambda_1 T}\varphi_K\rangle)$ nécessite, dans la preuve du Théorème 1.6, la coercivité de la forme quadratique $\tilde{Q}_{K,T}$. L'inégalité (1.22) implique la coercivité vis à vis de s et non de v. L'inégalité de coercivité vis à vis de v n'est pas vérifiée. Pour cette raison, l'analyse conduite dans le Chapitre 2, repose sur la même étude pour le système auxiliaire avec contrôle s

$$\begin{cases} i\partial_t\tilde{\psi} = -\partial^2_{xx}\tilde{\psi} - is(t)\big(2\mu'(x)\partial_x\tilde{\psi}+\mu''(x)\tilde{\psi}\big)+s(t)^2\mu'(x)^2\tilde{\psi}, \\ \tilde{\psi}(t,0)=\tilde{\psi}(t,1)=0, \\ \tilde{\psi}(0,x)=\varphi_1(x), \end{cases} \tag{1.23}$$

obtenu à partir de (1.7) par le changement de variable

$$\psi(t,x)=\tilde{\psi}(t,x)e^{is(t)\mu(x)}, \quad s(t)=\int_0^t v(\tau)\mathrm{d}\tau.$$

Contrôlabilité en temps grand : développement aux ordres supérieurs. Comme annoncé, le résultat principal du Chapitre 2 est l'existence d'un temps minimal pour la contrôlabilité exacte locale de (1.7). Au vu du Théorème 1.6, il suffit maintenant de montrer la contrôlabilité en temps grand, ce qui est fait dans le théorème suivant (voir le Théorème 2.4 page 51).

Théorème 1.8. *Soit $\mu \in H^3((0,1),\mathbb{R})$ tel que*

$$\mu'(0) \pm \mu'(1) \neq 0. \tag{1.24}$$

Alors, le système (1.7) est contrôlable dans $H^3_{(0)}$, localement autour du fondamental, avec des contrôles $u \in L^2((0,T),\mathbb{R})$, en temps T suffisamment grand.

Nous avons vu en (1.14) que l'hypothèse (1.24) est une condition nécessaire pour que l'hypothèse (1.13) soit vérifiée ; et obtenir ainsi la contrôlabilité exacte locale en temps arbitraire. Cette hypothèse (1.24) est en fait suffisante pour conclure à la contrôlabilité, mais uniquement en temps suffisamment grand.

La preuve du Théorème 1.8 repose aussi sur le développement en puissances de ϵ du contrôle et de la fonction d'onde et sur l'analyse des termes d'ordre un et deux, respectivement Ψ et ξ solutions de (1.15) et (1.20). En utilisant l'asymptotique (1.14) et l'hypothèse (1.24) on obtient l'existence de $N \in \mathbb{N}$, $K_1, \dots, K_N \in \mathbb{N}^*$ et de $C > 0$ tels que

$$\begin{aligned} |\langle \mu\varphi_1, \varphi_k \rangle| &\geq \frac{C}{k^3}, \quad \forall k \notin \{K_1, \dots, K_N\}, \\ \langle \mu\varphi_1, \varphi_{K_j} \rangle &= 0, \quad \forall j \in \{1, \dots, N\}. \end{aligned} \tag{1.25}$$

Ainsi, cette estimée et la stratégie du Théorème 1.5 permettent de conclure à la contrôlabilité de l'ordre un Ψ à codimension N près. Le reste de la preuve consiste à prouver que l'on récupérer, à l'ordre deux ξ, ce nombre fini de directions perdues. Pour cela, on adapte une idée due à E. Cerpa et E. Crépeau pour une équation de Korteweg-de Vries [44]. Le point clé de leur stratégie est que, dans une certaine base, les composantes de la solution du système libre (i.e. associée au contrôle nul) tournent toutes, et avec des vitesses distinctes. Dans notre cas, en décomposant la solution du système (1.7) libre, dans la base des états propres

$$\psi(t) = \sum_{k=1}^{+\infty} \langle \psi_0, \varphi_k \rangle e^{-i\lambda_k t} \varphi_k,$$

on constate que l'on est bien dans ce cas favorable (voir le Lemme 2.5 page 68).

Cependant, sous l'hypothèse $\langle \mu\varphi_1, \varphi_1 \rangle = 0$, lorsque l'on travaille au voisinage du fondamental, la première direction $i\mathbb{R}\Phi_1$ est perdue à l'ordre un et ne tourne pas à l'ordre deux. Ceci est dû au fait que la trajectoire $(\Phi_1, u \equiv 0)$, autour de laquelle on linéarise, n'est pas constante contrairement à celle considérée dans [44]. Il est alors de nécessaire de procéder à quelques adaptations. La stratégie globale est la même que celle de [44] : on combine des phases successives de contrôle et de rotation (contrôle nul), mais un traitement différent est réservé à la première composante si elle est perdue.

Ces idées sont aussi utilisées pour montrer la contrôlabilité de certaines directions sous l'hypothèse $\mu'(0) = \mu'(1) \neq 0$ (resp. $\mu'(0) = -\mu'(1) \neq 0$) dans le Théorème 2.6 page 52. Signalons que, lorsque $\mu'(0) = \mu'(1) = 0$, le résultat négatif de J.M. Ball, J.E. Marsden et M. Slemrod (Théorème 1.1) permet de démontrer qu'il n'y a pas contrôlabilité exacte dans $H^3_{(0)}$ avec des contrôles $L^2((0,T),\mathbb{R})$. Le cadre fonctionnel adapté à la contrôlabilité exacte locale n'est plus le même. Supposons qu'il existe $L \in \mathbb{N}$ tel que $\mu \in H^{2L+3}((0,1),\mathbb{R})$ et

$$\mu^{(2k+1)}(0) = \mu^{(2k+1)}(1) = 0, \ \forall k \in \{0, \dots, L-1\} \text{ et } \mu^{(2L+1)}(0) \pm \mu^{(2L+1)}(1) \neq 0.$$

Alors, en exploitant les techniques développées pour le Théorème 1.8, on peut prouver la contrôlabilité de (1.7), dans $H^{2L+3}_{(0)}$, localement autour du fondamental, avec des contrôles dans $L^2((0,T),\mathbb{R})$, en temps T suffisamment grand.

Temps minimal. En combinant les Théorèmes 1.6 et 1.8, on obtient l'existence d'un temps minimal pour la contrôlabilité locale.

Théorème 1.9. *Soit $\mu \in H^3((0,1),\mathbb{R})$ tel que les hypothèses (1.18) pour $K \in \mathbb{N}^*$ et (1.24) soient vérifiées. Alors, il existe $T_{min} > 0$ tel que la propriété de contrôlabilité du système (1.7), dans $H^3_{(0)}$, localement autour du fondamental, avec des contrôles $u \in L^2((0,T),\mathbb{R})$, soit vérifiée si $T > T_{min}$ et non vérifiée si $T < T_{min}$.*

Toutefois, la caractérisation de ce temps minimal est un problème largement ouvert. Dans un cadre favorable, où seule la première direction est perdue à l'ordre un, un encadrement de ce temps minimal est proposé en Section 2.6.

Dans la suite, sauf mention explicite, on n'impose plus les restrictions sur la norme du contrôle établies dans la Définition 1.1.

1.2.3.2 Contrôle exact local simultané

Une deuxième question d'intérêt concernant l'équation de Schrödinger bilinéaire est celle du contrôle simultané. Plus précisément on considère, pour $N \in \mathbb{N}^*$, le système

$$\begin{cases} i\partial_t \psi^j = -\partial^2_{xx}\psi^j - u(t)\mu(x)\psi^j, & (t,x) \in (0,T)\times(0,1), \\ \psi^j(t,0) = \psi^j(t,1) = 0, & j \in \{1,\dots,N\}, \\ \psi^j(0,x) = \varphi_j(x). \end{cases} \tag{1.26}$$

C'est un modèle simplifié de l'évolution des fonctions d'onde de N particules identiques soumises au même champ extérieur. Les particules sont supposées indépendantes et l'on néglige l'intrication.

Les résultats discutés ici sont exposés en détails dans le Chapitre 3. L'article [109], dont s'inspire le Chapitre 3, est publié dans le journal *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*.
L'objectif est ici de contrôler localement, autour des N premiers états propres, le système (1.26), avec un seul contrôle u. La notion de contrôlabilité simultanée considérée ici n'est pas à confondre avec celle présentée dans le livre [98] de J.-L. Lions et étudiée par de nombreux auteurs (voir par exemple [132, 4, 3, 88] par M. Tucsnak, G. Weiss, S.A. Avdonin, W. Moran et B. Kapitonov). Contrairement à ces références, les N opérateurs d'évolution et de contrôle considérés dans (1.26) sont identiques.
La première remarque est que l'évolution de (1.7) étant unitaire, les trajectoires de (1.26) vérifient les invariants suivants

$$\langle \psi^j(t), \psi^k(t)\rangle \equiv \langle \psi^j(0), \psi^k(0)\rangle, \quad \forall t \in [0,T], \forall j,k \in \{1,\dots,N\}. \tag{1.27}$$

Les cibles doivent donc vérifier des conditions de compatibilité avec les conditions initiales. Une première étape pour obtenir la contrôlabilité simultanée de (1.26) est que chaque équation, considérée séparément, soit contrôlable. D'après le Théorème 1.5, ceci est assuré par l'hypothèse suivante

$$\exists c > 0 \text{ tel que } |\langle \mu\varphi_j, \varphi_k\rangle| \geq \frac{c}{k^3}, \quad \forall k \in \mathbb{N}^*, \forall j \in \{1, \ldots, N\}. \tag{1.28}$$

Non contrôlabilité pour deux équations. La stratégie naturelle pour montrer la contrôlabilité du système (1.26) au voisinage de la trajectoire $(\Phi_1, \ldots, \Phi_N, u \equiv 0)$ est de montrer la contrôlabilité du linéarisé

$$\begin{cases} i\partial_t \Psi^j = -\partial_{xx}^2 \Psi^j - v(t)\mu(x)\Phi_j, & (t,x) \in (0,T) \times (0,1), \\ \Psi^j(t,0) = \Psi^j(t,1) = 0, & j \in \{1, \ldots, N\}, \\ \Psi^j(0,x) = 0. \end{cases} \tag{1.29}$$

Dans la base des états propres, on obtient alors

$$\Psi^j(T) = i \sum_{k=1}^{+\infty} \langle \mu\varphi_j, \varphi_k\rangle \int_0^T v(t) e^{i(\lambda_k - \lambda_j)t} \mathrm{d}t \, \Phi_k(T). \tag{1.30}$$

La contrôlabilité du linéarisé est donc équivalente à la résolution du problème de moments trigonométriques

$$\int_0^T v(t) e^{i(\lambda_k - \lambda_j)t} \mathrm{d}t = \frac{\langle \Psi_f^j, \Phi_k(T)\rangle}{i\langle \mu\varphi_j, \varphi_k\rangle}, \quad \forall j \in \{1, \ldots, N\}, \forall k \in \mathbb{N}^*, \tag{1.31}$$

où $(\Psi_f^1, \ldots, \Psi_f^N)$ est la cible visée pour le système (1.29) au temps T.
Supposons que $N = 2$. La linéarisation des invariants (1.27) conduit à considérer pour le linéarisé (1.29) des cibles vérifiant

$$\mathrm{Re}(\langle \Psi_f^1, \Phi_1(T)\rangle) = \mathrm{Re}(\langle\langle \Psi_f^2, \Phi_2(T)\rangle) = 0, \tag{1.32}$$

$$\langle \Psi_f^1, \Phi_2(T)\rangle + \langle \Phi^1(T), \Psi_f^2\rangle = 0. \tag{1.33}$$

On peut donc oublier le coefficient $\langle \Psi^2(T), \Phi_1(T)\rangle$ dans le problème de moments : il est contrôlé directement par $\langle \Psi^1(T), \Phi_2(T)\rangle$ et la contrainte (1.33). Les fréquences

$$\{\lambda_k - \lambda_j \, ; \, j \in \{1,2\}, k \geq j+1 \text{ et } k = j = 2\}$$

vérifient une condition de saut suffisante à la résolution en tout temps du problème de moments trigonométriques associé. Finalement, le seul obstacle à la contrôlabilité de (1.29) est le fait que la fréquence 0 soit de multiplicité deux ($\lambda_k - \lambda_j$ pour $j = k = 1$ et $j = k = 2$).

En effet, les coefficients diagonaux $\langle \Psi^j(T), \Phi_j(T)\rangle$ sont liés par la relation suivante

$$\frac{\langle \Psi^j(T), \Phi_j(T)\rangle}{i\langle \mu\varphi_j, \varphi_j\rangle} = \int_0^T v(t)\mathrm{d}t.$$

Pour $N \geq 2$, on a donc au moins une direction perdue à l'ordre un. On est alors dans un contexte similaire à celui du Théorème 1.6. Le premier résultat concernant la contrôlabilité simultanée est donc négatif (voir le Théorème 3.1 page 90) et établit la non contrôlabilité autour de (Φ_1, Φ_2) avec des contrôles petits dans L^2 en temps petit.

Théorème 1.10. *Soit $N \geq 2$. On suppose que $\mu \in H^3((0,1),\mathbb{R})$ vérifie*

$$\mathcal{A} := \langle \mu\varphi_1, \varphi_1\rangle\langle (\mu')^2\varphi_2, \varphi_2\rangle - \langle \mu\varphi_2, \varphi_2\rangle\langle (\mu')^2\varphi_1, \varphi_1\rangle \neq 0. \tag{1.34}$$

On définit $\alpha \in \{-1, 1\}$ par $\alpha := sign(\mathcal{A}\langle \mu\varphi_1, \varphi_1\rangle)$. Il existe $T_* > 0$ et $\varepsilon > 0$ tels que pour tout $T < T_*$, pour tout $u \in L^2((0,T),\mathbb{R})$ vérifiant $||u||_{L^2(0,T)} < \varepsilon$, la solution de (1.26) satisfasse

$$\big(\psi^1(T), \psi^2(T)\big) \neq \Big(\Phi_1(T), \Big(\sqrt{1-\delta^2} + i\alpha\delta\Big)\, \Phi_2(T)\Big), \quad \forall \delta > 0. \tag{1.35}$$

On note que la cible (1.35) vérifie bien les invariants (1.27) et, pour δ petit, est arbitrairement proche de $(\Phi_1(T), \Phi_2(T))$. L'hypothèse (1.34) est technique : son rôle est discuté en Section 1.2.4. Similairement au Théorème 1.6, le Théorème 1.10 est prouvé par l'étude du signe d'une forme quadratique appropriée. Pour $N = 1$, le Théorème 1.5 montre la contrôlabilité exacte au voisinage de Φ_1 en temps arbitraire avec une dépendance C^1 entre la cible et le contrôle. Ce résultat ne s'étend donc pas au cas $N \geq 2$ sous l'hypothèse (1.34).

Contrôlabilité simultanée de deux équations : méthode du retour. Il est important de garder à l'esprit que, du point de vue physique, la densité de probabilité de présence étant donnée par le module de la fonction d'onde, la phase globale n'a aucune signification. Ainsi, pour tout $\theta \in \mathbb{R}$, (ψ^1, ψ^2) et $e^{i\theta}(\psi^1, \psi^2)$ décrivent le même état et sont donc physiquement équivalents. En travaillant à phase globale près, on montre le résultat suivant (voir le Théorème 3.2 page 90).

Théorème 1.11. *Soit $T > 0$. On suppose que $N = 2$ et $\mu \in H^3((0,1),\mathbb{R})$ vérifie (1.28) et $\langle \mu\varphi_1, \varphi_1\rangle \neq \langle \mu\varphi_2, \varphi_2\rangle$. Il existe $\theta \in \mathbb{R}$, $\varepsilon_0 > 0$ et une application C^1*

$$\Gamma : \mathcal{O}_{\varepsilon_0} \to L^2((0,T),\mathbb{R})$$

où

$$\mathcal{O}_{\varepsilon_0} := \left\{ \big(\psi_f^1, \psi_f^2\big) \in \Big(H^3_{(0)}\Big)^2 \; ; \; \langle \psi_f^j, \psi_f^k\rangle = \delta_{j=k} \; \text{ et } \; \sum_{j=1}^{2} ||\psi_f^j - e^{i\theta}\Phi_j(T)||_{H^3_{(0)}} < \varepsilon_0 \right\},$$

telle que pour tout $\big(\psi_f^1, \psi_f^2\big) \in \mathcal{O}_{\varepsilon_0}$, la solution de (1.26) avec contrôle $u = \Gamma\big(\psi_f^1, \psi_f^2\big)$ satisfasse

$$(\psi^1(T), \psi^2(T)) = (\psi_f^1, \psi_f^2).$$

On obtient alors la contrôlabilité au voisinage de (Φ_1, Φ_2) à phase globale près, en temps arbitraire, sous des hypothèses génériques sur le moment dipolaire μ. Nous avons vu au Théorème 1.8 que dans ce cadre, une direction perdue à l'ordre un peut être récupérée à l'ordre deux en utilisant des phases successives de contrôle et de rotation (contrôle nul). Néanmoins, comme pour le cas d'une équation, les directions diagonales $\langle \Psi^j(T), \Phi_j(T)\rangle$ perdues ici sont précisément les seules directions qui ne présentent pas ce phénomène de rotation (voir le Lemme 2.5 page 68). Pour cette raison, les résultats positifs de contrôlabilité présentés au Chapitre 3 sont basés sur une stratégie différente : la méthode du retour.
Cette méthode, introduite par J.-M. Coron pour un problème de stabilisation, a été utilisée à de nombreuses reprises pour montrer la contrôlabilité de systèmes non linéaires dont

le linéarisé est non contrôlable. Un panorama non exhaustif est présenté en Section 3.1.4 page 93. Pour une présentation pédagogique, on renvoie au livre de J.-M. Coron [54, Chapitre 6]. Dans notre contexte, l'idée principale de cette méthode est la construction d'un contrôle de référence u_{ref} tel que la solution $(\psi^1_{ref}, \psi^2_{ref})$ de (1.26) associée vérifie

$$(\psi^1_{ref}, \psi^2_{ref})(T) = e^{i\theta}(\Phi_1(T), \Phi_2(T))$$

et que le linéarisé au voisinage de cette trajectoire de référence soit contrôlable. La contrôlabilité de ce linéarisé induit alors (via le théorème d'inversion locale) la contrôlabilité locale de (1.26) au voisinage de $(\psi^1_{ref}, \psi^2_{ref})(T)$ et conclut la preuve du Théorème 1.11. Si le système considéré est réversible en temps alors la construction d'une trajectoire de référence est immédiate. La plupart des trajectoires de références construites pour des systèmes non réversibles en temps sont basées sur des calculs explicites (voir par exemple [59, 60]). Le système (1.7) vérifie la propriété suivante, souvent qualifiée de réversibilité en temps. Pour $u \in L^2((0,T),\mathbb{R})$ et $v := u(T-\cdot)$, on a

$$\psi(T, \overline{\psi(T,\psi_0,u)}, v) = \overline{\psi_0}. \tag{1.36}$$

Bien qu'utilisée à plusieurs reprises dans ce mémoire, la conjugaison complexe rend cette propriété de réversibilité en temps difficile à exploiter pour la construction d'une trajectoire de référence. L'idée principale de la construction de la trajectoire de référence, en Section 3.3, consiste à utiliser des résultats de contrôle partiel conjointement aux invariants (1.27). La propriété de contrôlabilité du linéarisé au voisinage de la trajectoire de référence repose sur deux idées :

- la trajectoire de référence $(\psi^1_{ref}, \psi^2_{ref})$ est suffisamment proche de (Φ_1, Φ_2) pour que toutes les directions $\langle \Psi^j(T), \Phi_k(T)\rangle$ dans (1.30) contrôlables par résolution d'un problème de moments trigonométriques soient encore contrôlables (voir le Lemme 3.1 page 102),
- les coefficients diagonaux $\langle \Psi^j(T), \Phi_j(T)\rangle$ sont indépendants (voir le Lemme 3.3 page 104).

Intuitivement, la phase globale θ introduit un degré de liberté supplémentaire qui permet de gérer le fait que la fréquence 0 soit de multiplicité deux (pour les deux coefficients diagonaux $\langle \Psi^1(T), \Phi_1(T)\rangle$ et $\langle \Psi^2(T), \Phi_2(T)\rangle$) dans la résolution du problème de moments trigonométriques.

En modifiant légèrement la construction de la trajectoire de référence, on montre aussi pour $N=2$ la contrôlabilité exacte à un retard global près (voir le Théorème 3.3 page 90).

Théorème 1.12. *On suppose que $N=2$ et $\mu \in H^3((0,1),\mathbb{R})$ vérifie (1.28) et $4\langle \mu\varphi_1, \varphi_1\rangle - \langle \mu\varphi_2,\varphi_2\rangle \neq 0$. Il existe $T^* > 0$ tel que, pour tout $T \geq 0$, il existe $\varepsilon_0 > 0$ et une application C^1*

$$\Gamma : \mathcal{O}_{\varepsilon_0,T} \to L^2((0,T^*+T),\mathbb{R})$$

où

$$\mathcal{O}_{\varepsilon_0,T} := \left\{ (\psi^1_f, \psi^2_f) \in \left(H^3_{(0)}\right)^2 \; ; \; \langle \psi^j_f, \psi^k_f\rangle = \delta_{j=k} \text{ et } \sum_{j=1}^2 ||\psi^j_f - \Phi_j(T)||_{H^3_{(0)}} < \varepsilon_0 \right\},$$

telle que pour tout $\big(\psi^1_f, \psi^2_f\big) \in \mathcal{O}_{\varepsilon_0,T}$*, la solution de (1.26) avec contrôle* $u = \Gamma\big(\psi^1_f, \psi^2_f\big)$ *satisfasse*

$$\big(\psi^1(T^* + T), \psi^2(T^* + T)\big) = \big(\psi^1_f, \psi^2_f\big).$$

Ainsi, ce théorème ne fait plus intervenir de phase globale et prouve la contrôlabilité exacte pour deux equations. Là encore, les hypothèses sur μ sont génériques. L'appellation "à retard global près" provient du fait que les cibles proches de $(\Phi_1(T), \Phi_2(T))$ sont atteintes non pas au temps T, mais au temps $T^* + T$; le retard T^* étant le même pour tout T et toute cible. Comme précédemment, le retard global joue le rôle d'un degré de liberté supplémentaire pour gérer la multiplicité double de la fréquence 0 dans le problème de moments trigonométriques associé.

Adaptation au cas de trois équations. Si l'on considère maintenant $N = 3$ alors, pour le système linéarisé (1.29), la fréquence 0 est de multiplicité trois dans le problème de moments trigonométriques associé. Le système (1.29) présente donc de nouvelles directions perdues et l'on montre alors que, même à phase globale près, on n'a pas contrôlabilité locale au voisinage de (Φ_1, Φ_2, Φ_3) avec des contrôles petits dans L^2 en temps petit. Plus précisément (voir le Théorème 3.4 page 91), l'adaptation de la preuve du Théorème 1.10 conduit au résultat suivant.

Théorème 1.13. *Soit* $N \geq 3$*. On suppose que* $\mu \in H^3((0,1), \mathbb{R})$ *vérifie*

$$\begin{aligned}\mathcal{B} := \;& \big(\langle \mu\varphi_3, \varphi_3\rangle - \langle \mu\varphi_2, \varphi_2\rangle\big)\langle (\mu')^2\varphi_1, \varphi_1\rangle \\ &+ \big(\langle \mu\varphi_1, \varphi_1\rangle - \langle \mu\varphi_3, \varphi_3\rangle\big)\langle (\mu')^2\varphi_2, \varphi_2\rangle \\ &+ \big(\langle \mu\varphi_2, \varphi_2\rangle - \langle \mu\varphi_1, \varphi_1\rangle\big)\langle (\mu')^2\varphi_3, \varphi_3\rangle \neq 0. \end{aligned} \tag{1.37}$$

On définit $\beta \in \{-1, 1\}$ *par* $\beta = sign\big(\mathcal{B}(\langle \mu\varphi_2, \varphi_2\rangle - \langle \mu\varphi_1, \varphi_1\rangle)\big)$*. Il existe* $T_* > 0$ *et* $\varepsilon > 0$ *tels que, pour tout* $T < T_*$*, pour tout* $u \in L^2((0,T), \mathbb{R})$ *vérifiant* $||u||_{L^2(0,T)} < \varepsilon$*, la solution de (1.26) satisfasse*

$$\big(\psi^1(T), \psi^2(T), \psi^3(T)\big) \neq e^{i\nu}\left(\Phi_1(T), \Phi_2(T), \left(\sqrt{1-\delta^2} + i\beta\delta\right)\Phi_3(T)\right), \quad \forall \delta > 0, \forall \nu \in \mathbb{R}.$$

L'hypothèse (1.37) est une adaptation de l'hypothèse (1.34) discutée en Section 1.2.4.

Au regard des Théorèmes 1.11 et 1.12, pour pouvoir gérer la multiplicité triple de la fréquence 0 dans le problème de moments, on introduit deux degrés de liberté supplémentaires : la phase globale et le retard global. En effet, en adaptant les trajectoires de référence de ces deux théorèmes, on prouve pour $N = 3$, la contrôlabilité au voisinage de (Φ_1, Φ_2, Φ_3) à phase globale près et à retard global près, sous des hypothèses génériques sur μ (voir le Théorème 3.5 page 91).

Théorème 1.14. *On suppose que* $N = 3$ *et* $\mu \in H^3((0,1), \mathbb{R})$ *vérifie (1.28) et* $5\langle \mu\varphi_1, \varphi_1\rangle - 8\langle \mu\varphi_2, \varphi_2\rangle + 3\langle \mu\varphi_3, \varphi_3\rangle \neq 0$*. Il existe* $\theta \in \mathbb{R}$*,* $T^* > 0$ *tels que, pour tout* $T \geq 0$*, il existe* $\varepsilon_0 > 0$ *et une application* C^1

$$\Gamma : \mathcal{O}_{\varepsilon_0,T} \to L^2((0, T^* + T), \mathbb{R})$$

où

$$\mathcal{O}_{\varepsilon_0,T} := \left\{ \left(\psi_f^1, \psi_f^2, \psi_f^3\right) \in \left(H^3_{(0)}\right)^3 ; \langle \psi_f^j, \psi_f^k \rangle = \delta_{j=k} \text{ et } \sum_{j=1}^{3} ||\psi_f^j - e^{i\theta}\Phi_j(T)||_{H^3_{(0)}} < \varepsilon_0 \right\},$$

telle que pour tout $\left(\psi_f^1, \psi_f^2, \psi_f^3\right) \in \mathcal{O}_{\varepsilon_0,T}$, *la solution de (1.26) avec* $u = \Gamma\left(\psi_f^1, \psi_f^2, \psi_f^3\right)$ *satisfasse*

$$\left(\psi^1(T^*+T), \psi^2(T^*+T), \psi^3(T^*+T)\right) = \left(\psi_f^1, \psi_f^2, \psi_f^3\right).$$

1.2.3.3 Contrôle exact global de N équations

La stratégie développée pour prouver le Théorème 1.14 ne s'étend pas directement pour $N \geq 4$ équations. Non seulement il semble que la résolution du problème de moments trigonométriques nécessite de nouveaux degrés de liberté, pour gérer la multiplicité N de la fréquence 0, mais on voit aussi apparaitre des fréquences résonantes (par exemple $\lambda_7 - \lambda_1 = \lambda_8 - \lambda_4$). Un moyen de résoudre ce problème de fréquences résonantes consiste à considérer un potentiel V non nul i.e.

$$\begin{cases} i\partial_t \psi^j = \left(-\partial_{xx}^2 + V(x)\right)\psi^j - u(t)\mu(x)\psi^j, & (t,x) \in (0,T)\times(0,1), \\ \psi^j(t,0) = \psi^j(t,1) = 0, & j \in \{1,\dots,N\}, \\ \psi^j(0,x) = \psi_0^j(x). \end{cases} \tag{1.38}$$

En tirant parti de ce potentiel V, on étend au Chapitre 4 les idées du Théorème 1.14 pour montrer la contrôlabilité exacte globale de (1.38). La prépublication [110], dont s'inspire le Chapitre 4, a été écrite en collaboration avec V. Nersesyan.

Pour simplifier les notations, on considère en symboles gras les vecteurs i.e. on note $\boldsymbol{\psi}$ pour le vecteur $(\psi^1, \dots, \psi^N)$. De même, pour un espace de Hilbert H, on désigne H^N par $\boldsymbol{H}$. Le principal résultat du Chapitre 4 est le suivant (voir le Théorème 4.1 page 122).

Théorème 1.15. *Soit* $N \in \mathbb{N}^*$. *Pour tout* $V \in H^4((0,1),\mathbb{R})$, *on a contrôlabilité exacte globale du système (1.38) dans* $\boldsymbol{H}^4_{(V)}$, *génériquement par rapport à* μ *dans* $H^4((0,1),\mathbb{R})$. *Plus précisément, pour tout* $V \in H^4((0,1),\mathbb{R})$, *il existe un ensemble résiduel* $\mathcal{Q}_V$ *de* $H^4((0,1),\mathbb{R})$ *tel que, pour tout* $\mu \in \mathcal{Q}_V$, *pour tout vecteurs* $\boldsymbol{\psi}_0, \boldsymbol{\psi}_f \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$ *unitairement équivalents, il existe un temps* $T > 0$ *et un contrôle* $u \in L^2((0,T),\mathbb{R})$ *tels que la solution de (1.38) satisfasse*

$$\boldsymbol{\psi}(T) = \boldsymbol{\psi}_f.$$

Deux vecteurs $\boldsymbol{\psi}_0, \boldsymbol{\psi}_f \in \boldsymbol{\mathcal{S}}$ sont dits unitairement équivalents s'il existe une application unitaire $\mathcal{U}$ de L^2 telle que $\boldsymbol{\psi}_f = \mathcal{U}\boldsymbol{\psi}_0$ i.e.

$$\psi_f^j = \mathcal{U}\psi_0^j, \quad \forall j \in \{1,\dots,N\}.$$

Au vu de l'évolution unitaire de l'équation de Schrödinger considérée, cette hypothèse d'équivalence unitaire entre la cible et la condition initiale n'est pas restrictive.
Un argument de perturbation, développé ultérieurement (voir le système (1.40)), permet de considérer une partie du moment dipolaire comme un potentiel supplémentaire. Ainsi, même pour V quelconque, on se ramène à étudier le système (1.38) sous des hypothèses

favorables sur les éléments propres de l'opérateur libre. Du fait de l'absence de restrictions sur le potentiel, même dans le cas d'une seule équation, le Théorème 1.15 constitue une généralisation de la littérature existante concernant la contrôlabilité des systèmes quantiques bilinéaires unidimensionnels. La preuve du Théorème 1.15 repose sur les idées suivantes.

- On montre, en utilisant la méthode de Lyapunov, la contrôlabilité globale approchée vers des sommes finies d'états propres.
- En adaptant la trajectoire de référence construite pour la preuve du Théorème 1.14, la méthode du retour de J.-M. Coron permet de conclure à la contrôlabilité exacte locale au voisinage de certaines sommes finies d'états propres.
- Grâce à des arguments de connexité et de compacité, on étend cette propriété de contrôlabilité exacte au voisinage de vecteurs (dont les composantes sont des sommes finies d'états propres) différents pour les conditions initiales et finales.
- On conclut en combinant les résultats précédents et l'argument de réversibilité en temps (1.36).

On termine cette section en donnant les énoncés et des éléments de preuve de chacune de ces étapes.

Contrôle global approché : fonction de Lyapunov. Si l'on définit pour $M \in \mathbb{N}^*$,

$$\mathcal{C}_M := \mathrm{Vect}\{\varphi_{1,V}, \dots, \varphi_{M,V}\}, \tag{1.39}$$

on prouve le résultat de contrôle approché suivant (voir le Théorème 4.2 page 126).

Théorème 1.16. *Soit $N \in \mathbb{N}^*$. On suppose que $V, \mu \in H^4((0,1),\mathbb{R})$ vérifient*

- $\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle \neq 0$ *pour tout* $j \in \{1,\dots,N\}$, $k \in \mathbb{N}^*$.
- $\lambda_{j,V} - \lambda_{k,V} \neq \lambda_{p,V} - \lambda_{q,V}$ *pour tout* $j \in \{1,\dots,N\}$, $k,p,q \in \mathbb{N}^*$ *tels que* $\{j,k\} \neq \{p,q\}$ *et* $k \neq j$.

Alors, pour tout $\boldsymbol{\psi}_0 \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$ vérifiant $\langle \psi_0^j, \varphi_{j,V}\rangle \neq 0$ pour tout $j \in \{1,\dots,N\}$, il existe $M \in \mathbb{N}^*$, $\boldsymbol{\psi}_f \in \boldsymbol{\mathcal{C}}_M$, des suites $T_n > 0$ et $u_n \in C_0^\infty((0,T_n),\mathbb{R})$ tels que la solution de (1.38) associée satisfasse

$$\boldsymbol{\psi}(T_n) \underset{n\to\infty}{\longrightarrow} \boldsymbol{\psi}_f \quad \text{dans } \boldsymbol{H}^3.$$

Ce résultat est l'adaptation au cas $N > 1$ du résultat démontré par V. Nersesyan avec $N = M = 1$ dans [112]. L'idée principale consiste à construire une fonction de Lyapunov contrôlée, $\mathcal{V}(\boldsymbol{\psi})$, minimale en un point de $\boldsymbol{\mathcal{C}}_M$ (en l'occurrence $(c_1\varphi_{1,V},\dots,c_N\varphi_{N,V})$ avec $c_j \in \mathbb{C}$ tels que $|c_j| = 1$) et qui assure une borne des solutions dans $\boldsymbol{H}^4$. Le contrôle, qui garantit la décroissance de cette fonctionnelle le long des trajectoires, n'est pas une loi de rétroaction explicite, mais un contrôle en boucle ouverte, dont on montre l'existence par un argument variationnel comme dans [112]. L'analyse développée ne nous permet pas de démontrer que la solution $\boldsymbol{\psi}$ de (1.38) approche $(c_1\varphi_{1,V},\dots,c_N\varphi_{N,V})$ (qui est le minimum de $\mathcal{V}$) mais seulement un vecteur composé de sommes finies d'états propres (qui est un point critique de $\mathcal{V}$).

Comme souligné en page 10, ce résultat n'est pas couvert par le Théorème 1.3 et la notion de système faiblement couplé.

Contrôlabilité exacte locale au voisinage de sommes finies d'états propres : méthode du retour. Au vu du Théorème 1.16, une étape clé pour conclure à la contrôlabilité exacte globale est de montrer la contrôlabilité exacte localement au voisinage de sommes finies d'états propres. Ce résultat est l'objet du théorème suivant (voir le Théorème 4.3 page 130).

Théorème 1.17. *Soit $N \in \mathbb{N}^*$. On suppose que $V, \mu \in H^3((0,1),\mathbb{R})$ vérifient*

- *il existe $C > 0$ tel que*

$$|\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle| \geq \frac{C}{k^3}, \quad \forall j \in \{1,\dots,N\}, \forall k \in \mathbb{N}^*,$$

- $\lambda_{j,V} - \lambda_{k,V} \neq \lambda_{p,V} - \lambda_{n,V}$ *pour tout $j, p \in \{1,\dots,N\}$, $k \geq j+1$, $p \geq n+1$ tels que $\{j,k\} \neq \{p,n\}$,*
- $1, \lambda_{1,V}, \dots, \lambda_{N,V}$ *sont rationnellement indépendants.*

Soient C_0, C_f des matrices unitaires $N \times N$ et $\boldsymbol{z}_0 := C_0 \boldsymbol{\varphi}_V$, $\boldsymbol{z}_f := C_f \boldsymbol{\varphi}_V$. Alors, il existe $\delta > 0$ et $T > 0$ tels que, si l'on pose

$$\mathcal{O}_{\delta,C_0} := \Big\{ \boldsymbol{\phi} \in \boldsymbol{H}^3_{(V)}\,;\, \langle \phi^j, \phi^k \rangle = \delta_{j=k} \text{ et } \sum_{j=1}^N \|\phi^j - z_0^j\|_{H^3_{(V)}} < \delta \Big\},$$

$$\mathcal{O}_{\delta,C_f} := \Big\{ \boldsymbol{\phi} \in \boldsymbol{H}^3_{(V)}\,;\, \langle \phi^j, \phi^k \rangle = \delta_{j=k} \text{ et } \sum_{j=1}^N \|\phi^j - z_f^j\|_{H^3_{(V)}} < \delta \Big\},$$

pour tout $\boldsymbol{\psi}_0 \in \mathcal{O}_{\delta,C_0}$ et $\boldsymbol{\psi}_f \in \mathcal{O}_{\delta,C_f}$, il existe un contrôle $u \in L^2((0,T),\mathbb{R})$ tel que la solution associée de (1.38) issue de $\boldsymbol{\psi}_0$ satisfasse $\boldsymbol{\psi}(T) = \boldsymbol{\psi}_f$.

La norme $\|\cdot\|_{H^3_{(V)}}$ est définie par

$$\|\psi\|_{H^3_{(V)}} := \left(\sum_{k=1}^{\infty} |k^3 \langle \psi, \varphi_{k,V}\rangle|^2 \right)^{\frac{1}{2}}.$$

De façon usuelle, pour C matrice de taille $N \times N$, les composants du vecteur $C\boldsymbol{\psi}$ sont donnés par

$$(C\boldsymbol{\psi})^k = \sum_{j=1}^N C_{k,j}\psi^j.$$

Tout comme pour le Théorème 1.16, les hypothèses du Théorème 1.17 semblent trop fortes pour conclure à la contrôlabilité globale exacte pour un potentiel arbitraire. Cette question est discutée au paragraphe suivant. Pour $C_0 = I_N$ la matrice identité, $\boldsymbol{\psi}_0 = \boldsymbol{\varphi}_V$, et $\boldsymbol{\psi}_f = C\boldsymbol{\varphi}_V$ pour toute matrice C unitaire de taille N, ce théorème montre la réalisation exacte de n'importe quelle porte logique quantique, en temps grand. Pour de plus amples détails sur ces portes logiques et une preuve de réalisation approchée, on renvoie à [32] par N. Boussaïd, M. Caponigro et T. Chambrion.

On conclut ce paragraphe avec la stratégie de preuve du Théorème 1.17. Ce théorème contient principalement deux aspects : la contrôlabilité exacte locale au voisinage d'un vecteur dont les composantes sont des sommes finies d'états propres et le fait que les vecteurs $\boldsymbol{z}_0$ et $\boldsymbol{z}_f$ au voisinage desquels sont considérées les conditions initiales et finales puissent être différents. Pour traiter ce deuxième aspect, on utilise la connexité dans le groupe unitaire, qui induit l'existence d'un chemin entre C_0 et C_f. Par un argument de compacité, on est ramené à étudier la contrôlabilité exacte locale en chaque point de ce chemin i.e. le Théorème 1.17 dans le cas où $C_0 = C_f$ est une matrice unitaire arbitraire. Par linéarité du système (1.38) vis à vis de l'état, il est alors suffisant de démontrer le Théorème 1.17 dans le cas où $C_0 = C_f = I_N$ est la matrice identité.

Ce résultat est technique et nécessite plusieurs étapes. La première étape est une généralisation de la méthode du retour utilisée pour démontrer le Théorème 1.11, dans le cas de deux équations : on montre la contrôlabilité exacte pour des conditions initiales proches de $\boldsymbol{\varphi}_V$ et des conditions finales proches de $(e^{i\theta_1}\varphi_{1,V}, \dots, e^{i\theta_N}\varphi_{N,V})$ où les θ_j sont des termes de phases inconnus. Plus précisément, on prouve le résultat suivant (voir la Proposition 4.3 page 131).

Proposition 1.1. *Supposons que $V, \mu \in H^3((0,1),\mathbb{R})$ vérifient les conditions du Théorème 1.17. Pour tout $T > 0$, il existe $\theta_1, \dots, \theta_N \in \mathbb{R}$, $\delta > 0$ et une application C^1*

$$\Gamma : \mathcal{O}^0_\delta \times \mathcal{O}^f_\delta \to L^2((0,T),\mathbb{R}),$$

où

$$\mathcal{O}^0_\delta := \Big\{ \boldsymbol{\phi} \in \boldsymbol{H}^3_{(V)} \,;\, \langle \phi^j, \phi^k \rangle = \delta_{j=k} \ \ et \ \sum_{j=1}^N \|\phi^j - \varphi_{j,V}\|_{H^3_{(V)}} < \delta \Big\},$$

$$\mathcal{O}^f_\delta := \Big\{ \boldsymbol{\phi} \in \boldsymbol{H}^3_{(V)} \,;\, \langle \phi^j, \phi^k \rangle = \delta_{j=k} \ \ et \ \sum_{j=1}^N \|\phi^j - e^{i\theta_j}\varphi_{j,V}\|_{H^3_{(V)}} < \delta \Big\},$$

telle que pour tout $\boldsymbol{\psi}_0 \in \mathcal{O}^0_\delta$ et $\boldsymbol{\psi}_f \in \mathcal{O}^f_\delta$, la solution de (1.38) avec contrôle $u := \Gamma(\boldsymbol{\psi}_0, \boldsymbol{\psi}_f)$ issue $\boldsymbol{\psi}_0$ satisfasse $\boldsymbol{\psi}(T) = \boldsymbol{\psi}_f$.

La preuve de cette proposition est une variation de celle du Théorème 1.14. Les hypothèses sur V assurent que le spectre de l'opérateur libre est non résonant. Les valeurs propres $\lambda_{k,V}$ vérifiant la même asymptotique que les λ_k, les fréquences apparaissant dans les différents problèmes de moments trigonométriques satisfont une condition de saut suffisante pour la résolution de ces problèmes de moments en temps arbitraire. On prouve alors la Proposition 1.1 pour un nombre N d'équations quelconque en appliquant la méthode du retour de J.-M. Coron. Là encore, la construction de la trajectoire de référence est basée sur des résultats de contrôle partiel et l'utilisation des invariants (1.27). Comme l'on n'impose aucune condition sur les termes de phase θ_j, la fréquence 0 (dont la multiplicité était un obstacle à l'extension du Théorème 1.14 pour $N \geq 4$) n'apparait plus dans le problème de moments associé à la construction de la trajectoire de référence.

Conjointement à l'hypothèse d'indépendance rationnelle du spectre de l'opérateur libre, la Proposition 1.1 permet de conclure la preuve du Théorème 1.17. En effet, l'indépendance rationnelle des réels $\{1, \lambda_{1,V}, \dots, \lambda_{N,V}\}$ et le théorème d'approximation diophantienne simultanée de Kronecker impliquent

$$\text{Adh}\,\big\{ (e^{i\lambda_{1,V}t}, \dots, e^{i\lambda_{N,V}t}) \,;\, t \in (0,+\infty) \big\} = \{ e^{i\theta} \,;\, \theta \in \mathbb{R} \}^N.$$

Ainsi, il existe un temps de rotation $T_r > 0$, tel que $\overline{\boldsymbol{\zeta}}$ appartienne à $\mathcal{O}^f_\delta$ où

$$\boldsymbol{\zeta} := (e^{i(\theta_1 - \lambda_{1,V} T_r)}\varphi_{1,V}, \dots, e^{i(\theta_N - \lambda_{N,V} T_r)}\varphi_{N,V}).$$

Ce vecteur $\boldsymbol{\zeta}$ est la solution au temps T_r du système (1.38) libre (contrôle nul) issue de $(e^{i\theta_1}\varphi_{1,V}, \dots, e^{i\theta_N}\varphi_{N,V})$. En utilisant la Proposition 1.1, on obtient l'existence de contrôles tels que les solutions associées de (1.38) issues de $\boldsymbol{\psi}_0$ et de $\overline{\boldsymbol{\psi}_f}$ soient égales au temps T respectivement à $(e^{i\theta_1}\varphi_{1,V}, \dots, e^{i\theta_N}\varphi_{N,V})$ et à $\overline{\boldsymbol{\zeta}}$. L'argument de réversibilité en temps permet alors d'atteindre $\boldsymbol{\psi}_f$ à partir de $\boldsymbol{\psi}_0$ en temps $2T + T_r$, ce qui conclut l'heuristique de la preuve du Théorème 1.17.

Contrôle exact global : perturbations. L'utilisation conjointe des Théorèmes 1.16 et 1.17 conduit à la contrôlabilité exacte globale sous des hypothèses favorables sur V et sur μ (voir le Théorème 4.4 page 140). Par un argument de perturbation, on se ramène au système (1.38) où les fonctions V et μ sont remplacées par $V + \mu$ et μ. En effet, le système (1.7) avec contrôle $u(t) = \tilde{u}(t) - 1$ s'écrit

$$\begin{cases} i\partial_t \psi = \left(-\partial^2_{xx} + V(x) + \mu(x)\right)\psi - \tilde{u}(t)\mu(x)\psi, & (t,x) \in (0,T) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0, \\ \psi(0,x) = \psi_0(x). \end{cases} \tag{1.40}$$

Le potentiel de ce système est alors $V+\mu$ et les éléments propres $\lambda_{k,V+\mu}$ et $\varphi_{k,V+\mu}$, peuvent alors satisfaire des hypothèses favorables, même si V est arbitraire. Ceci justifie le fait qu'on puisse supposer aux Théorèmes 1.16 et 1.17 des hypothèses restrictives sur les éléments propres de l'opérateur libre et conclure au Théorème 1.15 à la contrôlabilité globale exacte pour un potentiel V arbitraire. La généricité de telles hypothèses sur μ est prouvée en Section 4.A.2 page 145 en étendant la stratégie utilisée par V. Nersesyan [111, 112]. Pour des résultats de généricité similaires, incluant aussi la généricité de certaines propriétés spectrales vis à vis du domaine considéré, on renvoie à [118, 102] par Y. Privat, M. Sigalotti et P. Mason.

1.2.4 Perspectives

Au vu des résultats présentés dans la Section 1.2.3 et détaillés dans la Partie I, certaines questions et extensions apparaissent naturellement.

Non contrôlabilité en temps petit. Dans le Théorème 1.6 nous avons mis en évidence la non contrôlabilité en temps petit, avec des contrôles petits dans L^2. Ceci donne lieu à l'existence d'un temps minimal strictement positif pour la contrôlabilité locale autour de l'état fondamental au Théorème 1.9. Une question intéressante est alors la caractérisation de ce temps minimal. Cette question, encore ouverte, est au cœur de la Section 2.6 page 80 où l'on propose un encadrement du temps minimal dans le cas où uniquement la première direction est perdue ainsi qu'une conjecture dans le cas général. Le fait que le résultat négatif se base sur le système auxiliaire (1.23) alors que le résultat positif est obtenu directement sur le système (1.7) introduit une différence de cadres fonctionnels. Pour cette raison on obtient, même dans le cadre favorable présenté, un encadrement et non une caractérisation du temps minimal. L'estimation du temps minimal serait aussi intéressante pour une meilleure compréhension des Théorèmes 1.10 et 1.13 basés sur les mêmes arguments.

Un autre point commun à ces trois théorèmes est que leur validité repose sur une hypothèse supplémentaire sur μ (par exemple $A_K \neq 0$ dans (1.18)). Analysons cette hypothèse. La non contrôlabilité en temps petit est impliquée par le fait que la forme quadratique définie par (1.21) vérifie une inégalité de coercivité de la forme (1.22). D'après le Lemme 2.1 page 56, cette forme quadratique se réécrit

$$\mathcal{Q}_{K,T}(s) := -A_K \int_0^T s(t)^2 \cos[(\lambda_K - \lambda_1)(t-T)]dt + \int_0^T s(t) \int_0^t s(\tau) k_{K,T}(t,\tau) d\tau dt,$$

où $k_{K,T} \in C^0(\mathbb{R} \times \mathbb{R})$ et $s(t) := \int_0^t v(\tau) \mathrm{d}\tau$. Lorsque le coefficient A_K est non nul et que le temps T est petit, le premier terme de la forme quadratique $\mathcal{Q}_{K,T}$ domine le second. La forme quadratique $\mathcal{Q}_{K,T}$ a donc un signe (celui de $-A_K$) et vérifie une inégalité de coercivité. Dans le cas où $A_K = 0$, l'existence d'un signe strict pour la forme quadratique $\mathcal{Q}_{K,T}$ est un problème ouvert. Notons qu'une intégration par parties supplémentaire (en primitivant $s(t)$) n'est pas envisageable, pour des raisons de sommabilité. Lorsque la forme quadratique $\mathcal{Q}_{K,T}$ est identiquement nulle, on montre, au cours de la preuve du Théorème 2.6, que la direction associée (perdue à l'ordre un et à l'ordre deux) est automatiquement récupérée à l'ordre trois, et ce, en tout temps $T > 0$. Un temps minimal strictement positif se détecte donc uniquement sur les ordres pairs.

Dans cette direction, on peut donner une autre interprétation du coefficient A_K. On réécrit (1.7) sous la forme abstraite

$$\partial_t \psi = f_0(\psi) + u(t) f_1(\psi), \tag{1.41}$$

où $f_0(\psi) := i\partial_{xx}^2 \psi$ avec domaine $D(f_0) := H^2 \cap H_0^1((0,1), \mathbb{C})$ et $f_1(\psi) := iu(t)\mu(x)\psi$ avec domaine $D(f_1) := L^2((0,1), \mathbb{C})$.

On remarque alors que, pour $\psi \in D(f_0)$, les crochets de Lie itérés suivants sont bien définis

$$\begin{aligned} &[f_1, f_0](\psi) = \mu''(x)\psi + 2\mu'(x)\partial_x \psi, \\ &\big[f_1, [f_1, f_0]\big](\psi) = -2i\mu'(x)^2 \psi, \\ &\big[f_1, \big[f_1, [f_1, f_0]\big]\big](\psi) = 0. \end{aligned}$$

Ainsi,

$$A_K \neq 0 \Longleftrightarrow \langle \big[f_1, [f_1, f_0]\big](\varphi_1), \varphi_K \rangle \neq 0.$$

La preuve du Théorème 1.6 se basant sur des arguments à l'ordre deux, il est cohérent de trouver une hypothèse sur les crochets itérés de longueur deux. Bien que l'utilisation des crochets de Lie pour les systèmes de dimension infinie soit à l'heure actuelle relativement mal comprise, cette interprétation pourrait donner de nouvelles idées pour gérer le cas où $A_K = 0$.

Les résultats positifs de contrôlabilité en temps grand, obtenus au Théorème 1.8, reposent en partie sur les idées utilisées par E. Cerpa et E. Crépeau, dans [44], pour le système de Korteweg-de Vries

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, & (t,x) \in (0,T) \times (0,L) \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = \kappa(t), \end{cases} \tag{1.42}$$

pour des longueurs critiques $L \in \left\{ 2\pi \sqrt{\frac{k^2+kl+l^2}{3}}\ ;\ k,l \in \mathbb{N}^* \right\}$. Pour ce système, E. Cerpa et E. Crépeau ont montré la contrôlabilité exacte locale autour de $y=0$ en temps grand. La contrôlabilité en temps arbitraire étant ouverte, les outils développés pour prouver le Théorème 1.6 pourraient aider à apporter une réponse à cette question. Néanmoins la différence de structure entre ces deux systèmes (contrôle au bord et équation non linéaire pour Korteweg-de Vries, contrôle bilinéaire et équation linéaire pour Schrödinger) ne permet pas d'appliquer directement la stratégie développée ici.

Une autre question d'intérêt est de savoir si le temps minimal pour la contrôlabilité exacte locale subsiste sans la condition de petitesse sur les contrôles. L'étude présentée ici consiste en l'analyse des différents ordres dans le développement en puissances de ϵ du contrôle. Il est évident que, pour des contrôles grands, les systèmes d'ordre un ou deux ne sont pas une bonne approximation de la dynamique du système de contrôle non linéaire. Cette question nécessite donc l'introduction de nouveaux outils et de nouvelles méthodes.

Contrôlabilité simultanée. Là encore, le temps pour la contrôlabilité simultanée est au cœur des préoccupations. Dans les Théorèmes 1.12 et 1.14, on montre la contrôlabilité exacte (resp. à phase globale près et) à retard global près localement pour deux équations (resp. trois équations). D'après la construction de ce retard global en Section 3.3.3 page 98, on voit que T^* tend vers $\frac{2}{\pi}$, qui est la période commune des états propres, quand la trajectoire de référence tend vers $(\Phi_1, \dots, \Phi_N)$. Si le retard global était égal à $\frac{2}{\pi}$ on obtiendrait alors la contrôlabilité en temps suffisamment grand en lieu et place de la contrôlabilité à retard global près.

Le Théorème 1.15 prouve la contrôlabilité exacte globale d'un nombre quelconque d'équations de Schrödinger bilinéaires unidimensionnelles en temps suffisamment grand. Dans la preuve, la contrôlabilité approchée vers des sommes finies d'états propres du Théorème 1.16 et la contrôlabilité exacte locale du Théorème 1.17 nécessitent un temps suffisamment grand. La propriété de contrôle approché étant basée sur la méthode de Lyapunov, elle nécessite fondamentalement des temps suffisamment grands. L'adaptation du Théorème 1.17 en temps arbitraire est une question ouverte. Dans cette preuve, deux étapes utilisent un temps suffisamment grand : la rotation pendant un temps adéquat (donné par le théorème d'approximation diophantienne de Kronecker) et l'utilisation d'un argument de compacité (pour passer du cas $C_0 = C_f$ au cas où C_0 et C_f sont des matrices unitaires arbitraires).

Une autre possibilité d'amélioration du Théorème 1.15 réside dans le cadre fonctionnel. Du fait de la convergence faible donnée par la méthode de Lyapunov, on a considéré des conditions initiales et des cibles plus régulières que $\boldsymbol{H}^3_{(V)}$ qui est le cadre optimal pour la contrôlabilité exacte locale. Avec une fonction de Lyapunov (non stricte) de type distance H^s à la cible, si l'on souhaite démontrer la stabilisation pour la topologie forte de H^s, le principe d'invariance de LaSalle nécessite la compacité H^s des trajectoires du système bouclé. Cette propriété de compacité est difficile à établir. Le problème de stabilisation forte reste donc relativement mal compris (voir [55] par J.-M. Coron et B. D'Andréa-Novel pour un exemple de mise en œuvre de cette méthode). L'approche développée ici consiste à utiliser la compacité faible H^4 des bornés de H^4. Ceci justifie la différence entre la régularité H^3 des résultats de contrôlabilité exacte locale (voir Théorème 1.17 et Proposition 1.1) et la régularité H^4 pour la contrôlabilité exacte globale (voir Théorème 1.15).

Si la stabilisation forte dans H^3 était possible, cela conduirait à la contrôlabilité exacte globale dans H^3, qui est l'espace fonctionnel optimal. Ce résultat de stabilisation forte est ouvert.

La robustesse des différentes techniques présentées pour d'autres équations est abordée dans la section suivante pour des modèles avec un terme quadratique en le contrôle en plus du terme bilinéaire.

1.3 Contrôle d'équations de Schrödinger avec un terme de polarisabilité

1.3.1 Modèle

Nous avons vu en Section 1.2.1, que la dérivation du modèle bilinéaire (1.7), étudié par de nombreux auteurs, et au cœur de la Partie I, est obtenue par une approximation d'ordre un sur l'hamiltonien d'interaction. Pour des champs électriques de faible amplitude, le modèle bilinéaire produit une assez bonne description des interactions. Cependant, dans certains cas (voir les travaux de C.M. Dion *et al.* [64, 65]) il est nécessaire de poursuivre le développement de l'hamiltonien à un ordre supérieur. Le terme suivant dans ce développement est $-u(t)^2\mu_2(x)$ où μ_2 est le moment de polarisabilité. On considère donc le système suivant.

$$\begin{cases} i\partial_t\psi = \left(-\Delta + V(x)\right)\psi - u(t)\mu_1(x)\psi - u(t)^2\mu_2(x)\psi, \quad x \in D, \\ \psi_{|\partial D} = 0, \\ \psi(0,x) = \psi_0(x). \end{cases} \tag{1.43}$$

Du point de vue de la contrôlabilité, on a donc un seul contrôle $u(t)$ intervenant à la fois de manière linéaire (moment dipolaire) et de manière quadratique (moment de polarisabilité). Si l'introduction de ce terme est justifiée physiquement, on peut aussi chercher à en tirer profit mathématiquement : si ce terme n'est pas négligé (i.e. μ_2 non nul) peut-on obtenir la contrôlabilité dans des cas où le terme bilinéaire n'est pas suffisant ? Cette question est le point central de la Partie II dont les principaux résultats sont présentés dans les sous-sections suivantes.

1.3.2 Résultats précédents

Les résultats de contrôlabilité pour un système quantique avec polarisabilité de la forme (1.43) antérieurs à ceux de ce mémoire ont été obtenus pour le système de dimension finie

$$\begin{cases} i\dfrac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \left(H_0 + u(t)H_1 + u(t)^2H_2\right)\psi(t), \\ \psi(0) = \psi_0, \end{cases} \tag{1.44}$$

où H_0, H_1 et H_2 sont des matrices hermitiennes de taille $n \times n$. Pour $k \in \{1, \dots, n\}$, on note λ_k les valeurs propres de H_0 et φ_k les vecteurs propres associés.

Les propriétés de contrôle de ce système de dimension finie ont été étudiées dans [135] par G. Turinici. En utilisant les critères de contrôlabilité donnés par l'algèbre de Lie engendrée par H_0, H_1 et H_2, G. Turinici démontre la contrôlabilité de (1.44) sous les mêmes hypothèses que pour le système

$$\begin{cases} i\dfrac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \big(H_0 + u(t)H_1 + v(t)H_2\big)\psi(t), \\ \psi(0) = \psi_0, \end{cases}$$

avec deux contrôles indépendants. La stabilisation de (1.44) par boucle de rétroaction a aussi été étudiée. Dans [80], A. Grigoriu, C. Lefter et G. Turinici ont montré, en utilisant des arguments de Lyapunov, la stabilisation du premier étape propre φ_1 si le spectre de H_0 est non dégénéré et que tous les autres vecteurs propres φ_k sont couplés à φ_1 via H_1 i.e.

$$\langle H_1\varphi_1, \varphi_k\rangle \neq 0, \quad \forall k \in \{2, \dots, n\}.$$

La stratégie utilisée est une adaptation de [105] par M. Mirrahimi, P. Rouchon et G. Turinici, dans le cadre $H_2 = 0$, sous les mêmes hypothèses sur les matrices H_0 et H_1. Afin de tirer profit de l'introduction du terme de polarisabilité, J.-M. Coron, A. Grigoriu, C. Lefter et G. Turinici ont étudié, dans [57], la stabilisation de φ_1 dans le cas où chaque vecteur propre φ_k est couplé à φ_1 via H_1 ou via H_2 i.e.

$$\forall k \in \{2, \dots, n\},\ \langle H_1\varphi_1, \varphi_k\rangle \neq 0 \text{ ou } \langle H_2\varphi_1, \varphi_k\rangle \neq 0. \tag{1.45}$$

Dans cet article, les auteurs ont proposé des lois de rétroaction discontinues et des lois de rétroaction dynamiques périodiques en temps. Une analyse de stabilité pour le cas de la loi de rétroaction discontinue est faite dans [79] par A. Grigoriu. On détaille maintenant la stratégie basée sur des lois de rétroaction périodiques, qui sera appliquée au système infini dimensionnel (1.43) dans le Chapitre 5. Pour une introduction aux lois de rétroaction dynamiques on renvoie à [54, Chapitre 11] par J.-M. Coron. L'idée développée dans [57] consiste à chercher u sous la forme

$$u(t, \psi) = \alpha(\psi) + \beta(\psi)\sin\left(\frac{t}{\varepsilon}\right). \tag{1.46}$$

On introduit ce contrôle périodique en temps dans (1.44), et on considère le système moyen

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi_{av} = \left(H_0 + \alpha H_1 + \left(\alpha^2 + \frac{1}{2}\beta^2\right)H_2\right)\psi_{av}. \tag{1.47}$$

Le système moyen de $\dot{x} = f(t, x)$, avec $f(\cdot, x)$ T-périodique, est défini par $\dot{x}_{av} = f_{av}(x_{av})$ où $f_{av}(x) := \frac{1}{T}\int_0^T f(t, x)\mathrm{d}t$. Suivant l'idée développée dans [105], la phase globale de la fonction d'onde n'ayant pas de signification physique, on peut introduire un contrôle supplémentaire, dit fictif, ω à valeurs réelles et considérer plutôt le système

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi_{av} = \left(H_0 + \alpha H_1 + \left(\alpha^2 + \frac{1}{2}\beta^2\right)H_2 + \omega\right)\psi_{av}. \tag{1.48}$$

Les solutions des deux systèmes (1.47) et (1.48) diffèrent uniquement du terme de phase $e^{i\int_0^t \omega(\tau)\mathrm{d}\tau}$: il est donc équivalent d'introduire ce contrôle fictif ou de travailler à phase globale près.

Les lois de rétroactions α et β sont construites pour assurer la stabilisation du système moyen (1.48) vers φ_1 en supposant que le spectre de H_0 est non dégénéré et que (1.45) est vérifié. Des résultats classiques sur les systèmes dynamiques en dimension finie impliquent que le système moyen (1.48) est une bonne approximation du système (1.44) avec loi de rétroaction dynamique (1.46) si ce contrôle est suffisamment oscillant i.e. si ε est suffisamment petit. Le couplage de ces deux résultats conduit alors à la stabilisation approchée du système bouclé (1.44), (1.46).

1.3.3 Principaux résultats

1.3.3.1 Contrôlabilité approchée par des contrôles explicites

Le premier résultat présenté dans ce mémoire sur le système (1.43) est l'adaptation de la stratégie de [57] basée sur des lois de rétroaction périodiques fortement oscillantes au cas de la dimension infinie. Ces résultats sont détaillés au Chapitre 5. L'article [108], dont s'inspire le Chapitre 5, est publié dans le journal *Mathematics of Control, Signals, and Systems*.

La stratégie mise en place est la même que pour le système de dimension finie. On cherche une loi de rétroaction périodique oscillante de la forme (1.46). On peut alors réécrire le système (1.43) sous la forme

$$\begin{cases} \partial_t \psi(t) = -iA_V\psi(t) + F\left(\dfrac{t}{\varepsilon}, \psi(t)\right), \\ \psi_{|\partial D} = 0, \end{cases} \tag{1.49}$$

où A_V est défini par (1.4) et

$$F(s,z) := i\left(\alpha(z) + \beta(z)\sin(s)\right)\mu_1 z + i\left(\alpha(z) + \beta(z)\sin(s)\right)^2 \mu_2 z.$$

Stabilisation du système moyen : méthode de Lyapunov. En adaptant les techniques de dimension finie, on pose $F^0(z) := \frac{1}{T}\int_0^T F(t,z)\mathrm{d}t$ et on étudie le système moyen

$$\begin{cases} \partial_t \psi_{av} = -iA_V\psi_{av} + F^0(\psi_{av}), \\ \psi_{av_{|\partial D}} = 0. \end{cases} \tag{1.50}$$

Le calcul de F^0 conduit alors à expliciter le système moyen (1.50) sous la forme

$$\begin{cases} i\partial_t\psi_{av} = \big(-\Delta + V(x)\big)\psi_{av} - \alpha(\psi_{av})\mu_1(x)\psi_{av} - \left(\alpha(\psi_{av})^2 + \dfrac{1}{2}\beta(\psi_{av})^2\right)\mu_2(x)\psi_{av}, \\ \psi_{av_{|\partial D}} = 0, \\ \psi_{av}(0,\cdot) = \psi_0. \end{cases} \tag{1.51}$$

Soient $\mathcal{P}$ la projection sur l'espace engendré par $\{\varphi_{k,V}\,;\,k \geq 2\}$ et

$$\mathcal{L}(z) := \gamma\|(-\Delta + V)\mathcal{P}z\|_{L^2}^2 + 1 - |\langle z, \varphi_{1,V}\rangle|^2, \quad \forall z \in \mathcal{S}\cap H^2 \cap H_0^1, \tag{1.52}$$

où $\gamma > 0$ est une constante à déterminer. On pose

$$\alpha(z) := -\kappa I_1(z), \qquad \beta(z) := g(I_2(z)), \quad \forall z \in H^2 \tag{1.53}$$

avec $\kappa > 0$ suffisamment petit (fixé uniformément pour toute condition initiale dans une boule de H^2) et

$$g \in C^2(\mathbb{R}, \mathbb{R}^+) \text{ telle que } g' \text{ bornée, } g(x) = 0 \text{ si et seulement si } x \geq 0,$$

et pour $j \in \{1, 2\}$,

$$I_j(z) = \mathrm{Im}\Big[-\gamma\langle(-\Delta + V)\mathcal{P}\,\mu_j z, (-\Delta + V)\mathcal{P}z\rangle + \langle\mu_j z, \phi\rangle\langle\phi, z\rangle\Big].$$

Soient $J_{\neq 0} := \Big\{k \geq 2\,;\, \langle\mu_1\varphi_{1,V}, \varphi_{k,V}\rangle \neq 0\Big\}$ et $J_0 := \Big\{k \geq 2\,;\, \langle\mu_1\varphi_{1,V}, \varphi_{k,V}\rangle = 0\Big\}$. On obtient alors le théorème suivant prouvant la stabilisation du système moyen vers $\varphi_{1,V}$ (voir le Théorème 5.2 page 158).

Théorème 1.18. *Supposons que $V, \mu_1, \mu_2 \in C^\infty(\overline{D}, \mathbb{R})$ vérifient*

- $\forall k \in J_0, \langle\mu_2\varphi_{1,V}, \varphi_{k,V}\rangle \neq 0$ *i.e. tous les vecteurs propres $\varphi_{k,V}$ sont couplés à $\varphi_{1,V}$ via μ_1 ou μ_2,*
- $Card(J_0) < \infty$ *i.e. le nombre de directions perdues par μ_1 est fini,*
- $\lambda_{1,V} - \lambda_{k,V} \neq \lambda_{p,V} - \lambda_{q,V}$ *pour $k, p, q \geq 1$ tels que $\{1, k\} \neq \{p, q\}$.*

Soit $\psi_0 \in \mathcal{S} \cap H^2 \cap H^1_0$ avec $0 < \mathcal{L}(\psi_0) < 1$. La solution ψ_{av} du système bouclé (1.51), (1.53) vérifie

$$\psi_{av}(t) \underset{t\to\infty}{\rightharpoonup} \mathcal{C} := \{c\varphi_{1,V}\,;\, c \in \mathbb{C} \text{ et } |c| = 1\} \quad \text{dans } H^2.$$

La condition $\mathcal{L}(\psi_0) < 1$ n'est pas restrictive : elle est assurée par le choix de la constante γ dans (1.52). Dans le cadre unidimensionnel, nous avons vu en (1.25) que l'hypothèse $Card(J_0) < \infty$ est vérifiée si $\mu_1'(0) \pm \mu_1'(1) \neq 0$. Ce théorème est obtenu par la méthode de Lyapunov et le principe d'invariance de LaSalle pour la fonction de Lyapunov $\mathcal{L}$ définie par (1.52). Cette fonction de Lyapunov est celle utilisée dans [111, 19] par K. Beauchard et V. Nersesyan dans le cadre $\mu_2 = 0$. La dérivation de cette fonction de Lyapunov le long des trajectoires conduit à

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi_{av}(t)) = 2\alpha I_1(\psi_{av}(t)) + 2\left(\alpha^2 + \frac{1}{2}\beta^2\right) I_2(\psi_{av}(t)).$$

L'expression des lois de rétroaction α et β en fonction des I_j est inspiré du cadre fini dimensionnel de [57].

Approximation du système oscillant par le système moyen. Afin de pouvoir utiliser le Théorème 1.18 de stabilisation du système moyen (1.51), (1.53) dans l'étude du système (1.43), on étend au cas de la dimension infinie les propriétés d'approximation d'un système oscillant par le système moyen. Pour $\psi_0 \in \mathcal{S} \cap H^2_{(0)}$, on pose

$$u^\varepsilon(t) := \alpha(\psi_{av}(t)) + \beta(\psi_{av}(t)) \sin\left(\frac{t}{\varepsilon}\right), \tag{1.54}$$

où ψ_{av} est la solution du système bouclé (1.51), (1.53). On prouve le résultat d'approximation suivant (voir la Proposition 5.5 page 162).

Théorème 1.19. *Soit* $[s,L]$ *un intervalle de temps fixé et* $\psi_0 \in \mathcal{S} \cap H^4_{(0)}$ *avec* $0 < \mathcal{L}(\psi_0) < 1$. *Soit* ψ_{av} *la solution du système bouclé (1.51), (1.53) avec condition initiale* $\psi_{av}(s,\cdot) = \psi_0$. *Pour tout* $\delta > 0$, *il existe* $\varepsilon_0 > 0$ *tel que, si* ψ_ε *est la solution de (1.43) associée au contrôle* u^ε *défini par (1.54) avec* $\varepsilon \in (0, \varepsilon_0)$, *avec la même condition initiale* $\psi_\varepsilon(s,\cdot) = \psi_0$, *alors*

$$||\psi_\varepsilon(t,\cdot) - \psi_{av}(t,\cdot)||_{H^2} \leq \delta, \quad \forall t \in [s,L].$$

La démonstration utilise des bornes sur la norme H^2 de $\partial_t \psi_{av}$. Pour cette raison, les conditions initiales sont supposées plus régulières que le cadre H^2 dans lequel cette propriété d'approximation est prouvée.

Si ce résultat est classique en dimension finie, son extension à la dimension infinie n'est pas directe. On note ici une différence avec le cas de la dimension finie : le système moyen ne sert plus seulement à définir les lois de rétroaction α et β. Dans (1.54), ces lois sont calculées le long de la trajectoire associée au système moyen : le système (1.43) n'est pas considéré avec une loi de rétroaction, comme en dimension finie, mais avec le contrôle explicite u^ε défini par (1.54).

Contrôle approché vers l'état fondamental avec des contrôles explicites. La stabilisation du système moyen au Théorème 1.18 et l'approximation du système oscillant par le système moyen au Théorème 1.19 conduisent à la contrôlabilité approchée vers l'état fondamental en norme H^s pour $s < 2$ avec des contrôles explicites à phase globale près (on rappelle que l'ensemble $\mathcal{C}$ est défini au Théorème 1.18). Plus précisément, on obtient le théorème suivant (voir le Théorème 5.1 page 154).

Théorème 1.20. *On suppose que* V, μ_1 *et* μ_2 *vérifient les hypothèses du Théorème 1.18. Pour tout* $s < 2$ *et pour tout* $\psi_0 \in \mathcal{S} \cap H^4_{(0)}$ *avec* $0 < \mathcal{L}(\psi_0) < 1$, *il existe une suite croissante de temps* $(T_n)_{n\in\mathbb{N}}$ *dans* $\mathbb{R}^*_+$ *tendant vers* $+\infty$ *et une suite décroissante* $(\varepsilon_n)_{n\in\mathbb{N}}$ *dans* $\mathbb{R}^*_+$ *tels que* ψ_ε *la solution de (1.43) associée au contrôle* u^ε *défini par (1.54) avec* $\varepsilon \in (0, \varepsilon_n)$ *vérifie pour tout* $n \in \mathbb{N}$,

$$dist_{H^s}\left(\psi_\varepsilon(t,\cdot), \mathcal{C}\right) \leq \frac{1}{2^n}, \quad \forall t \in [T_n, T_{n+1}].$$

En temps suffisamment grand, si le contrôle (1.54) est suffisamment oscillant, on approche arbitrairement près le premier état propre $\varphi_{1,V}$, à phase globale près, avec des contrôles explicites. Les contrôles utilisés étant explicites, on réalise en Section 5.5 page 167 des simulations de convergence comme présenté en Figure 1.1. Il est à noter que l'hypothèse sur μ_1 faite au Théorème 1.20 est plus faible que l'hypothèse $\langle \mu_1 \varphi_{1,V}, \varphi_{k,V} \rangle \neq 0$ pour tout $k \geq 2$ utilisée dans [19]. La prise en compte du terme de polarisabilité a donc permis le contrôle avec des hypothèses sur μ_1, sous lesquelles la littérature existante (au moment de la production de ce résultat) ne concluait pas.

Résultats postérieurs. Toujours sur le modèle avec polarisabilité citons les travaux de N. Boussaïd, M. Caponigro et T. Chambrion [31] postérieurs à cette étude. En utilisant la méthode de contrôle géométrique des approximations de Galerkin (décrite en Section 1.2.2.1) et la notion de système faiblement couplé, ils ont montré la contrôlabilité globale approchée du système

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi = (A + u(t)B + u(t)^2 C)\psi,$$

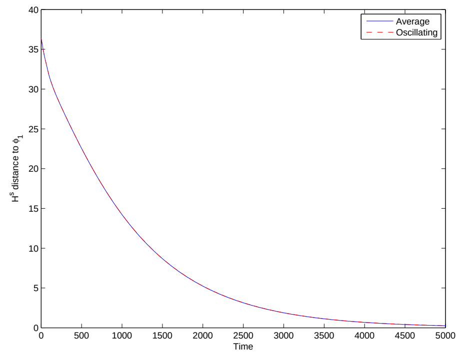


FIGURE 1.1: Convergence $H^{1.8}$ du système moyen (trait plein) et du système oscillant (pointillé) vers $\varphi_{1,V}$ avec $V(x) = (x - \frac{1}{2})^2$, $\mu_1(x) = x^2$, $\mu_2(x) = x$ et $\psi_0 = \frac{1}{\sqrt{3}}\varphi_{1,V} + \frac{1}{\sqrt{3}}\varphi_{2,V} + \frac{i}{\sqrt{3}}\varphi_{3,V}$.

sous l'hypothèse de l'existence d'une chaîne de connexité pour $B + \alpha C$ avec $\alpha \in \mathbb{R}$. Ce résultat est plus général que celui du Théorème 1.20, puisqu'il démontre la contrôlabilité globale approchée, et non la contrôlabilité vers l'état fondamental. De plus, les hypothèses sur le modèle sont moins restrictives. Ainsi, pour un système de la forme (1.43) en dimension un, leur résultat est appliqué dans [31] à un modèle d'alignement de molécules HCN,

$$i\partial_t \psi = -\Delta\psi + u(t)\cos(x)\psi + u(t)^2\cos(2x)\psi.$$

Le potentiel considéré dans ce système étant nul, les hypothèses sur les valeurs propres de l'opérateur libre du Théorème 1.18 ne sont pas vérifiées. De plus, l'hypothèse d'existence d'une chaîne de connexité est moins restrictive pour les couplages entre états propres que les hypothèses du Théorème 1.18. En effet, dans notre cadre, l'hypothèse

$$\langle \mu_2 \varphi_{1,V}, \varphi_{k,V} \rangle \neq 0, \quad \forall k \in J_0,$$

est suffisante pour obtenir l'existence d'une chaîne de connexité et appliquer les résultats de [31], sans supposer $\mathrm{Card}(J_0) < \infty$.

Cependant, l'un des intérêts d'un résultat de contrôle approché de la forme du Théorème 1.20 est de pouvoir être couplé avec un résultat de contrôle exact local au voisinage de l'état fondamental, afin d'obtenir le contrôle exact global. Or, comme mentionné en page 10, les espaces fonctionnels pour lesquels le résultat de contrôle global approché de [31] est vérifié ne sont pas compatibles avec la contrôlabilité exacte globale. La stratégie développée pour prouver le Théorème 1.20 ne souffre, a priori, d'aucune obstruction à l'extension pour des espaces plus réguliers.

La section suivante est consacrée à la contrôlabilité exacte globale du modèle avec polarisabilité (1.43) unidimensionnel sans restrictions sur le potentiel ni sur le moment dipolaire.

1.3.3.2 Contrôle exact global du modèle unidimensionnel

La stratégie précédente est spécifique au modèle avec polarisabilité. Bien que le système de contrôle (1.43) ne soit pas bilinéaire, certains des outils développés pour les systèmes de contrôle bilinéaires présentés en Section 1.2 peuvent s'étendre à ce modèle dans le cadre unidimensionnel

$$\begin{cases} i\partial_t \psi = \left(-\partial^2_{xx} + V(x)\right)\psi - u(t)\mu_1(x)\psi - u(t)^2\mu_2(x)\psi, & (t,x) \in (0,T)\times(0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T), \\ \psi(0,x) = \psi_0(x), & x \in (0,1). \end{cases} \tag{1.55}$$

Ces résultats sont détaillés au Chapitre 6.

Dans un premier temps, on s'intéresse à la contrôlabilité exacte locale dans le cas $V = 0$. Le principal argument du Théorème 1.5 de K. Beauchard et C. Laurent est la contrôlabilité du linéarisé autour de la trajectoire $(\Phi_1, u \equiv 0)$, dans $H^3_{(0)}$, avec des contrôles $L^2((0,T),\mathbb{R})$. Formellement, du fait du terme quadratique, le système (1.55) admet le même linéarisé que (1.7) autour de la trajectoire $(\Phi_1, u \equiv 0)$ qui est donc contrôlable dans $H^3_{(0)}$ avec des contrôles $L^2((0,T),\mathbb{R})$ sous des hypothèses favorables sur μ_1. Cependant, cette stratégie présente deux inconvénients :

- Cette stratégie conduirait à la contrôlabilité du modèle avec polarisabilité sous les mêmes hypothèses que lorsque $\mu_2 = 0$. On ne tire alors aucun profit de l'ajout du terme de polarisabilité.
- Le contrôle du linéarisé se fait via des contrôles L^2. Le terme u^2 dans le modèle de polarisabilité possède donc seulement une régularité L^1 insuffisante pour appliquer les résultats de [16] et obtenir le caractère bien posé dans $H^3_{(0)}$.

Pour résoudre le second problème, une possibilité est d'adapter un autre résultat : la contrôlabilité exacte locale, dans $H^5_{(0)}$, autour de φ_1, avec des contrôles $H^1_0((0,T),\mathbb{R})$ (voir [16, Théorème 2]). En effet, si $u \in H^1_0((0,T),\mathbb{R})$, alors $u^2 \in H^1_0((0,T),\mathbb{R})$ et le contrôle exact local dans $H^5_{(0)}$ est obtenu de manière identique que μ_2 soit nul ou non.

Pour résoudre le premier problème, on considère un argument de perturbation similaire à celui présenté en (1.40) pour démontrer la contrôlabilité globale exacte simultanée du Théorème 1.15. En effet, le système (1.55) avec le contrôle $u(t) = \tilde{u}(t) + 2$ s'écrit

$$\begin{cases} i\partial_t \psi = \left(-\partial^2_{xx} + V(x) - 2\mu_1(x) - 4\mu_2(x)\right)\psi - \tilde{u}(t)(\mu_1 + 4\mu_2)(x)\psi - \tilde{u}(t)^2\mu_2(x)\psi, \\ \psi(t,0) = \psi(t,1) = 0, \\ \psi(0,x) = \psi_0(x). \end{cases} \tag{1.56}$$

Ce système est donc le même que (1.55) où le potentiel V est remplacé par $V - 2\mu_1 - 4\mu_2$ et le moment dipolaire μ_1 par $\mu_1 + 4\mu_2$. Ainsi, pour V et μ_1 quelconques, on se ramène à étudier le système (1.56) avec des hypothèses suffisantes, sur le potentiel et le moment dipolaire, pour pouvoir conclure à la contrôlabilité exacte locale au voisinage du premier état propre $\varphi_{1,V-2\mu_1-4\mu_2}$ de l'opérateur $(-\partial^2_{xx} + V - 2\mu_1 - 4\mu_2)$.

La contrôlabilité approchée vers l'état fondamental dans des espaces réguliers obtenue par V. Nersesyan dans [112] (et adaptée au Théorème 1.16 pour N équations bilinéaires) est essentiellement basée sur les propriétés du linéarisé de (1.7) autour de trajectoires associées au contrôle $u \equiv 0$. Ce linéarisé est le même pour le système avec polarisabilité. Ainsi, sous des hypothèses favorables sur le potentiel et le moment dipolaire de (1.56), on obtient la contrôlabilité approchée vers $\varphi_{1,V-2\mu_1-4\mu_2}$.

Finalement, en regroupant ces deux résultats, on obtient le théorème suivant (voir le Théorème 6.1 page 174).

Théorème 1.21. *Pour tout* $V, \mu_1 \in H^6((0,1),\mathbb{R})$ *le système (1.55) est globalement exactement contrôlable dans* $H^6_{(V)}$*, génériquement par rapport à* $\mu_2 \in H^6((0,1),\mathbb{R})$*. Plus précisément, pour tout* $V, \mu_1 \in H^6((0,1),\mathbb{R})$*, il existe un ensemble* $\mathcal{Q}_{V,\mu_1}$ *résiduel dans* $H^6((0,1),\mathbb{R})$ *tel que, si* $\mu_2 \in \mathcal{Q}_{V,\mu_1}$*, pour tout* $\psi_0, \psi_f \in \mathcal{S} \cap H^6_{(V)}$*, il existe* $T > 0$ *et* $u \in H^1_0((0,T),\mathbb{R})$ *tels que la solution associée de (1.55) satisfasse* $\psi(T) = \psi_f$*.*

Ainsi, la prise en compte du terme de polarisabilité permet d'obtenir des résultats de contrôlabilité dans des cas non couverts par le modèle bilinéaire (1.7) (e.g. $\mu_1 = 0$ ou V quelconque et $\mu_1 \notin \mathcal{Q}_V$ comme défini au Théorème 1.15).

1.3.4 Perspectives

Au vu des résultats présentés dans la Section 1.3.3, certaines questions et extensions apparaissent naturellement.

Approximation par le système moyen en dimension infinie. Le Théorème 1.19 montre que sur un intervalle de temps donné, si le système (1.43) avec contrôle (1.54) est suffisamment oscillant alors la solution associée est arbitrairement proche de la solution du système couplé (1.51), (1.53) .

Une première direction d'amélioration de ce résultat consisterait à affaiblir les hypothèses de régularité supplémentaire sur les conditions initiales. Le schéma de la preuve utilisant fortement une borne H^2 sur $\partial_t \psi_{av}$, ce problème est ouvert.

Une seconde direction consiste à analyser l'intervalle de temps sur lequel cette propriété est valable. L'énoncé du Théorème 1.19 laisse à penser que, si l'on souhaite agrandir l'intervalle de temps $[s, L]$ considéré en $[s, \tilde{L}]$ avec $\tilde{L} > L$, alors le paramètre d'oscillation ε doit être réduit. Les simulations effectuées pour différents choix de V, μ_1 et μ_2 semblent indiquer qu'il n' en est rien (voir la Section 5.5 page 167). La Figure 1.2 représente l'évolution de la différence des deux solutions en norme H^2.

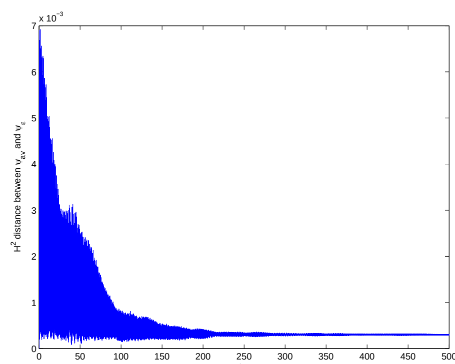


FIGURE 1.2: Ecart H^2 entre le système moyen et le système oscillant avec $V(x) = (x - \frac{1}{2})^2$, $\mu_1(x) = x^2$, $\mu_2(x) = x$ et $\psi_0 = \frac{1}{\sqrt{2}}\varphi_{1,V} + \frac{i}{\sqrt{2}}\varphi_{2,V}$.

Ces simulations induisent l'idée que la propriété d'approximation par le système moyen puisse être valable sur $[0,+\infty)$, ce qui permettrait de remplacer le Théorème 1.20 par un résultat de stabilisation approchée.

Cadre fonctionnel pour la contrôlabilité exacte globale Le Théorème 1.21 prouve la contrôlabilité exacte globale dans $H^6_{(V)}$ de (1.55) en temps grand. Le temps grand et la régularité supplémentaire (condition initiale $H^6_{(V)}$ pour un résultat de contrôle exact local dans $H^5_{(V)}$) sont inhérents à la stratégie de Lyapunov employée pour le résultat de contrôle approché du premier état propre. Cependant, l'espace $H^5_{(V)}$ dans lequel est prouvé la contrôlabilité exacte locale n'est pas optimal : le système linéarisé est contrôlable dans $H^3_{(V)}$ avec des contrôles $L^2((0,T),\mathbb{R})$. En utilisant les résultats de [16], le système (1.55) est bien posé dans $H^3_{(V)}$ avec des contrôles $L^4((0,T),\mathbb{R})$. Pour obtenir la contrôlabilité exacte locale dans $H^3_{(V)}$, une piste envisageable serait donc de montrer la régularité $L^4((0,T),\mathbb{R})$ du contrôle réalisant la contrôlabilité du linéarisé (obtenu comme solution d'un problème de moments trigonométriques). Cette question est ouverte.

1.4 Contrôle d'équations de Grushin singulières

1.4.1 Modèle

Après avoir étudié des problèmes de contrôle bilinéaire pour un système quantique en Partie I, puis l'ajout d'un terme quadratique en le contrôle en Partie II, ce mémoire présente en Partie III un problème de contrôle linéaire pour l'équation de Grushin singulière

$$\partial_t f - \partial^2_{xx} f - |x|^{2\gamma}\partial^2_{yy} f + \frac{c}{x^2} f = u(t,x,y)\chi_\omega(x,y), \tag{1.57}$$

pour $(t,x,y) \in (0,T)\times\Omega$ où $\Omega = (-1,1)\times(0,1)$ et le contrôle u est localisé en espace sur le sous-domaine $\omega \subset \Omega$.

Ce modèle est inspiré de l'équation de la chaleur pour l'opérateur de Laplace-Beltrami associé à la métrique de Grushin généralisée $g(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & |x|^{-2\gamma} \end{pmatrix}$, étudié par U. Boscain et C. Laurent dans [28] i.e.

$$Lu = \partial^2_{xx} u + |x|^{2\gamma}\partial^2_{yy} u - \frac{\gamma}{x}\partial_x u. \tag{1.58}$$

Le changement de variables $u = |x|^{\frac{\gamma}{2}} v$ unitaire de $L^2\left(\mathbb{R}\times\mathbb{T}, \frac{1}{|x|^\gamma}\mathrm{d}x\mathrm{d}y\right)$ vers $L^2(\mathbb{R}\times\mathbb{T}, \mathrm{d}x\mathrm{d}y)$ conduit à l'opérateur

$$\tilde{L}v = \partial^2_{xx} v + |x|^{2\gamma}\partial^2_{yy} v - \frac{\gamma}{2}\left(\frac{\gamma}{2}+1\right)\frac{v}{x^2}. \tag{1.59}$$

Afin de découpler et d'analyser séparément les effets de la dégénérescence et de la singularité on choisit dans (1.59) pour le potentiel singulier un coefficient c indépendant de γ.

L'équation (1.57) est un problème de contrôle linéaire pour un opérateur parabolique dégénéré et singulier sur l'ensemble $\{x = 0\}$, à l'intérieur du domaine. On rappelle, dans la section suivante, la stratégie utilisée pour les problèmes de contrôle linéaire en dimension infinie et on mentionne quelques résultats précédents dont les outils ou les résultats sont semblables à ceux utilisés pour l'étude de (1.57).

1.4.2 Problèmes de contrôle linéaires et dualité

On considère un système de contrôle linéaire abstrait avec un contrôle interne

$$\begin{cases} y' = \mathcal{A}y + h\chi_\omega, \\ y(0) = y_0, \end{cases} \tag{1.60}$$

où $\mathcal{A}$ génère un semigroupe continu d'opérateurs sur l'espace de Hilbert $\mathcal{H}$. La théorie générale de la méthode HUM est valable pour des opérateurs de contrôle plus généraux que celui du contrôle interne (voir [67, 98] par S. Dolecki, D.L. Russell et J.-L. Lions pour les résultats historiques et [133] par M. Tucsnak et G. Weiss pour une présentation complète). Ce cadre étant celui étudié dans ce mémoire, on se limite dans cette présentation au contrôle interne. On associe à (1.60) le système adjoint en temps rétrograde

$$\begin{cases} z' = -\mathcal{A}^* z, \\ z(T) = z_T. \end{cases} \tag{1.61}$$

Le système (1.60) est approximativement contrôlable en temps $T > 0$ si pour tout $y_0, y_1 \in \mathcal{H}$, pour tout $\epsilon > 0$, il existe $h \in L^2((0,T), \mathcal{H})$ tel que la solution de (1.60) satisfasse

$$\|y(T) - y_1\|_{\mathcal{H}} \leq \epsilon.$$

Par linéarité, il est suffisant de considérer le cas $y_0 = 0$. Le contrôle approché en temps T est alors équivalent à la densité de l'image de l'application

$$\mathcal{R}_T : h \in L^2((0,T), \mathcal{H}) \mapsto y(T) \in \mathcal{H},$$

où y est la solution de (1.60). Par un argument de dualité, la contrôlabilité approchée est alors équivalente à l'injectivité de $\mathcal{R}_T^*$ i.e. à la continuation unique du système adjoint : si z solution de (1.61) vérifie z nulle sur $(0,T) \times \omega$ alors $z_T = 0$. Cette question peut être traitée par des résultats d'unicité du type du Théorème d'Holmgren (voir aussi [84, Chapitre 28] par L. Hörmander et [121, 130] par L. Robbiano, C. Zuily et D. Tataru pour d'autres hypothèses sur la régularité des coefficients de l'opérateur différentiel $\mathcal{A}$).

Selon les propriétés de l'opérateur $\mathcal{A}$, la contrôlabilité exacte peut être inenvisageable, par exemple, dans le cas où le système (1.60) possède une propriété de régularisation. On considère alors la notion de contrôlabilité aux trajectoires : pour toute trajectoire $(\overline{y}, \overline{h})$ de (1.60), pour tout $y_0 \in \mathcal{H}$, il existe $h \in L^2((0,T), \mathcal{H})$ tel que la solution de (1.60) satisfasse $y(T) = \overline{y}(T)$. Par linéarité, cette propriété est équivalente à la contrôlabilité à zéro : pour tout $y_0 \in \mathcal{H}$, il existe $h \in L^2((0,T), \mathcal{H})$ tel que la solution de (1.60) satisfasse $y(T) = 0$. Par dualité, la méthode HUM montre l'équivalence entre la contrôlabilité à zéro en temps T et l'observabilité du système adjoint i.e. l'existence de $C > 0$ telle que

$$\|z(0)\|_{\mathcal{H}} \leq C \int_0^T \|\chi_\omega z(t)\|_{\mathcal{H}}^2 \mathrm{d}t.$$

La stratégie pour prouver la contrôlabilité à zéro ou la contrôlabilité approchée de systèmes de la forme (1.60) est donc plus balisée que pour les systèmes de contrôle bilinéaire précédents. Cependant, la preuve d'inégalités d'observabilité est parfois loin d'être aisée. Les outils et méthodes sont variées selon les propriétés de l'opérateur $\mathcal{A}$. En préambule aux équations paraboliques dégénérées ou singulières, on mentionne la contrôlabilité à zéro en tout temps $T > 0$ de l'équation de la chaleur

$$\begin{cases} \partial_t y - \Delta y = \chi_\omega h, & (t,x) \in [0,T] \times \Omega, \\ y = 0, & (t,x) \in [0,T] \times \partial\Omega, \\ y(0) = y_0, & x \in \Omega, \end{cases} \tag{1.62}$$

avec des contrôles $L^2((0,T) \times \omega)$ par A.V. Fursikov et O.Y. Imanuvilov [72] et G. Lebeau et L. Robbiano [97] où Ω est un domaine borné régulier de $\mathbb{R}^d$ et $\omega \subset \Omega$ un ouvert non vide.

Pour des résultats de contrôlabilité à zéro pour des opérateurs uniformément paraboliques avec des coefficients discontinus, on mentionne les résultats [68, 123, 21, 22] de A. Doubova, A. Osses, J.-P. Puel, A. Benabdallah, Y. Dermenjian et J. Le Rousseau.

1.4.3 Résultats précédents

Solutions d'équations de la chaleur singulières. La continuation unique (et donc la contrôlabilité approchée) pour l'opérateur parabolique dégénéré avec un potentiel singulier défini en (1.57) est au cœur de la Partie III.

La première difficulté pour l'étude d'une équation parabolique avec un potentiel singulier de la forme $\frac{1}{|x|^2}$ réside dans le caractère bien posé de l'équation. Un outil essentiel pour cette étude est l'inégalité de Hardy suivante. Soit $\Omega \subset \mathbb{R}^d$ un domaine borné régulier. On a

$$\lambda^*(d) \int_\Omega \frac{u^2}{|x|^2} \mathrm{d}x \leq \int_\Omega |\nabla u|^2 \mathrm{d}x, \quad \forall u \in H_0^1(\Omega), \tag{1.63}$$

où $\lambda^*(d) = \frac{(d-2)^2}{4}$ est la meilleure constante dans cette inégalité (qui n'est pas atteinte). Dans le cas où $d = 2$, cette inégalité est triviale. Dans le cas où $d = 1$, on suppose que 0 est au bord du domaine : en effet si $u(0) \neq 0$, cette inégalité n'est plus valable.

Pour une equation de la chaleur avec potentiel singulier $a \in L^1_{loc}(\Omega)$, X. Cabré et Y. Martel ont montré dans [35] que l'existence de solutions faibles locales en temps pour des conditions initiales positives est conditionnée à la validité d'une inégalité de Hardy avec poids $a(x)$. On retrouve alors le cas particulier de P. Baras et J.A. Goldstein [6] suivant.

Théorème 1.22. *On considère le système*

$$\begin{cases} u_t - \Delta u - \dfrac{\lambda}{|x|^2} u = 0, & (t,x) \in (0,T) \times \Omega, \\ u = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{1.64}$$

avec Ω satisfaisant les conditions précédentes d'application de l'inégalité de Hardy.
Si $\lambda \leq \lambda^*(d)$, alors (1.64) a une solution faible globale pour toute condition initiale $u_0 \in L^2(\Omega)$ positive ou nulle.
Si $\lambda > \lambda^*(d)$, alors pour tout $T > 0$ et tout $u_0 \in L^1_{loc}(\Omega)$ avec $u_0 \geq 0$ non nulle, il n'existe pas de solution faible de (1.64), même localement en temps.

La plupart des études s'intéressant à une équation de la chaleur avec potentiel singulier en $\frac{1}{|x|^2}$ se focalisent donc sur la gamme de constantes pour laquelle l'inégalité de Hardy (1.63) est vérifiée. Dans ce cadre, le caractère bien posé sans restriction de signe sur la condition initiale est dû à J.L. Vazquez et E. Zuazua [139].

Contrôlabilité à zéro d'équations paraboliques dégénérées et singulières Les outils utilisés pour montrer la contrôlabilité à zéro (via l'observabilité) d'équations paraboliques avec un tel potentiel singulier ont d'abord été développés pour l'étude d'équations paraboliques dégénérées. Dans [38, 39] P. Cannarsa, P. Martinez et J. Vancostenoble ont montré la contrôlabilité à zéro de

$$u_t - (x^\alpha u_x)_x = h\chi_\omega, \quad (t,x) \in (0,T) \times (0,1), \tag{1.65}$$

pour $\alpha \in [0,2)$ avec des conditions au bord (de type Dirichlet ou Neumann en 0) adaptées à la dégénérescence, en tout temps $T > 0$, pour toute condition initiale $u_0 \in L^2(0,1)$, avec des contrôles $h \in L^2((0,T) \times \omega)$. L'observabilité du système adjoint repose sur la preuve d'inégalités de Carleman adaptées. Ces résultats ont ensuite été étendus pour des dégénérescences plus générales puis pour des opérateurs de dimension deux dégénérant au bord dans [101, 40]. Pour le cas du contrôle frontière sur le bord dégénéré i.e.

$$\begin{cases} u_t - (x^\alpha u_x)_x = 0, & (t,x) \in (0,T) \times (0,1), \\ u(t,0) = \gamma(t),\ u(t,1) = 0, & t \in (0,T), \end{cases} \tag{1.66}$$

avec $\alpha \in [0,1)$, la contrôlabilité approchée est due à P. Cannarsa, J. Tort et M. Yamamoto [41] en utilisant une inégalité de Carleman, et la contrôlabilité à zéro à M. Gueye [81] en utilisant une décomposition dans une base de Fourier non harmonique pour le problème hyperbolique associé et la méthode de transmutation pour revenir au problème parabolique.

La contrôlabilité de certains opérateurs présentant une dégénérescence interne a déjà étudiée. Dans cette direction, on mentionne la contrôlabilité régionale obtenue par P. Martinez, J.-P. Raymond et J. Vancostenoble [100] pour un opérateur issu d'une équation de Crocco linéarisée

$$u_t + u_x - u_{yy} = \chi_\omega(x,y)h(t,x,y), \quad (t,x,y) \in (0,T) \times (0,\ell) \times (0,1), \tag{1.67}$$

et l'étude de la contrôlabilité à zéro menée par K. Beauchard [13] pour des opérateurs de type Kolmogorov

$$u_t + v^\gamma u_x - u_{vv} = \chi_\omega(x,v)h(t,x,v), \quad (t,x,v) \in (0,T) \times \mathbb{T} \times (-1,1), \tag{1.68}$$

généralisant un résultat précédent de K. Beauchard et E. Zuazua [20]. La contrôlabilité à zéro de l'équation de Grushin (1.57) dégénérée mais non singulière (i.e. $c = 0$) par K. Beauchard, P. Cannarsa et R. Guglielmi [14] est détaillée au paragraphe suivant.

Concernant la contrôlabilité de l'équation parabolique singulière (1.64) avec un contrôle distribué sur un sous-domaine ω, J. Vancostenoble et E. Zuazua ont montré dans [138] le résultat suivant.

Théorème 1.23. *Soient $d \geq 3$ et $\Omega \subset \mathbb{R}^d$ un domaine régulier borné tel que $0 \in \Omega$. Soit $\omega \subset \Omega$ un ouvert contenant une couronne centrée sur la singularité. On suppose que $\lambda \leq \lambda^*(d)$. Alors, pour tout temps $T > 0$, le système*

$$\begin{cases} u_t - \Delta u - \dfrac{\lambda}{|x|^2} u = h\chi_\omega, & (t,x) \in (0,T) \times \Omega, \\ u = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{1.69}$$

est contrôlable à zéro pour toute condition initiale $u_0 \in L^2(\Omega)$ avec des contrôles $h \in L^2((0,T) \times \Omega)$.

La preuve de l'observabilité repose sur une décomposition en harmoniques sphériques et sur l'observabilité des équations unidimensionnelles avec potentiel singulier qui en découlent. Les poids utilisés dans l'inégalité de Carleman 1D sont ceux développés dans [39] pour le problème dégénéré (1.65).

Les restrictions géométriques sur le domaine de contrôle ω ont été supprimées par S. Ervedoza [69] conduisant à la contrôlabilité à zéro de (1.69) pour tout sous-domaine $\omega \subset \Omega \subset \mathbb{R}^d$ pour $d \geq 3$ si $\lambda \leq \lambda^*(d)$.

Finalement, pour la contrôlabilité à zéro d'un problème 1D, à la fois singulier et dégénéré au bord de l'intervalle, on mentionne l'article [137] de J. Vancostenoble qui utilise les idées des inégalités de Carleman précédentes, conjointement à certaines améliorations de l'inégalité de Hardy.

Equations de type Grushin. Les singularités étudiées dans ce manuscrit (voir (1.57)) sont de nature différentes de celles étudiées par J. Vancostenoble, E. Zuazua et S. Ervedoza (voir (1.69)). En effet, ces auteurs considèrent des singularités soit au bord, soit en un point intérieur en dimension supérieure à trois. Le modèle 2D étudié dans ce manuscrit est lui simultanément dégénéré et singulier sur le segment $\{x = 0\}$, qui sépare le domaine $\Omega = (-1,1) \times (0,1)$ en deux composantes. Les propriétés de modèles similaires à (1.57) ont déjà été étudiées sous plusieurs angles.

Dans [28], U. Boscain et C. Laurent ont montré dans le cadre $\gamma > 0$ que l'opérateur $\tilde{L}$ défini par (1.59) avec domaine $C_0^\infty((\mathbb{R}\backslash\{0\}) \times \mathbb{T})$ est essentiellement autoadjoint sur $L^2(\mathbb{R} \times \mathbb{T})$ si et seulement si $\gamma \in [1, +\infty)$. Ainsi, si $\gamma \in [1, +\infty)$, aucune information ne peut traverser la singularité pour l'équation de la chaleur, des ondes ou de Schrödinger associée : la solution issue d'une condition initiale supportée dans $\Omega^+ := \{(x,y) \in \mathbb{R} \times \mathbb{T}; x > 0\}$ reste à support dans Ω^+. Ces résultats s'adaptent directement à l'opérateur (1.57) : pour $\gamma > 0$, l'opérateur associé avec domaine $C_0^\infty(\Omega\backslash\{x = 0\})$ est essentiellement autoadjoint dans $L^2(\Omega)$ si $c \geq \frac{3}{4}$. Il est donc vain de chercher des propriétés de contrôlabilité avec un contrôle localisé d'un côté de la singularité pour $c \geq \frac{3}{4}$.

L'étude des propriétés de l'opérateur de Laplace-Beltrami L défini par (1.58) a été poursuivie par U. Boscain et D. Prandi [29] dans le cadre $\gamma \in \mathbb{R}$. Parmi d'autres résultats ils ont montré, pour $\gamma \in (-1,1)$, l'existence d'une extension autoadjointe pour laquelle l'équation de la chaleur associée est bien posée et vérifie des conditions de continuité de part et d'autre de la singularité.

Du point de vue de la contrôlabilité, les propriétés de (1.57) ont été étudiées, dans [14], par K. Beauchard, P. Cannarsa et R. Guglielmi dans le cadre non singulier i.e. $c = 0$. Ils ont montré le résultat suivant concernant la contrôlabilité à zéro.

Théorème 1.24. *Soient* $\Omega = (-1, 1) \times (0, 1)$ *et* ω *un ouvert de* $(0, 1) \times (0, 1)$.

- *Si* $\gamma \in (0, 1)$, *le système*

$$\begin{cases} \partial_t f - \partial_{xx}^2 f - |x|^{2\gamma} \partial_{yy}^2 f = u(t, x, y)\chi_\omega(x, y), & (t, x, y) \in (0, \infty) \times \Omega, \\ f(t, x, y) = 0, & (t, x, y) \in (0, \infty) \times \partial\Omega, \\ f(0, x, y) = f_0(x, y), & (x, y) \in \Omega, \end{cases} \tag{1.70}$$

  est contrôlable à zéro en tout temps $T > 0$.

- *Si* $\gamma = 1$ *et* $\omega = (a, b) \times (0, 1)$ *avec* $0 < a < b \leq 1$, *il existe* $T^* \geq \frac{a^2}{2}$ *tel que le système (1.70) soit contrôlable à zéro en temps* T *pour* $T > T^*$ *et non contrôlable à zéro en temps* T *pour* $T < T^*$.

- *Si* $\gamma > 1$, *le système (1.70) est non contrôlable à zéro pour tout temps* $T > 0$.

La preuve repose sur la décomposition de Fourier en la variable y des solutions du système adjoint de (1.70). Le système est contrôlable à zéro si et seulement si les systèmes unidimensionnels, satisfaits par les coefficients de Fourier, sont observables uniformément par rapport aux fréquences de Fourier. Les inégalités de Carleman développées par K. Beauchard, P. Cannarsa et R. Guglielmi portent donc une attention toute particulière au suivi des fréquences de Fourier. Un corollaire de cette stratégie est la continuation unique du système adjoint de (1.70) pour tout $\gamma > 0$.

Ces idées ont été adaptées partiellement au cas de (1.57) avec $c \neq 0$ dans le cas où la singularité et la dégénérescence sont au bord du domaine par P. Cannarsa et R. Guglielmi [37]. Ils ont montré la contrôlabilité approchée en tout temps de

$$\begin{cases} \partial_t f - \partial_{xx}^2 f - |x|^{2\gamma} \partial_{yy}^2 f + \dfrac{c}{x^2} f = u(t, x, y)\chi_\omega(x, y), & (t, x, y) \in (0, \infty) \times \Omega, \\ f(t, x, y) = 0, & (t, x, y) \in (0, \infty) \times \partial\Omega, \\ f(0, x, y) = f_0(x, y), & (x, y) \in \Omega, \end{cases} \tag{1.71}$$

pour $\Omega = (0, 1) \times (0, 1)$, $\omega \subset \Omega$, $\gamma > 0$ et $c > -\frac{1}{4}$ et la contrôlabilité à zéro en temps suffisamment grand pour $\Omega = (0, 1) \times (0, 1)$, $\omega = (a, b) \times (0, 1)$ avec $0 < a < b \leq 1$, $\gamma = 1$ et $c > -\frac{1}{4}$.

1.4.4 Principaux résultats

Dans ce manuscrit, on donne une condition nécessaire et suffisante sur le potentiel singulier pour obtenir la contrôlabilité approchée de (1.57) via la continuation unique du système adjoint. Ce résultat, ainsi que deux corollaires, sont détaillés au Chapitre 7. Le Chapitre 7 s'inspire de la prépublication [107].

Dans un premier temps, on considère le problème de contrôle (1.57) muni de conditions au bord de Dirichlet en x et périodiques en y i.e.

$$\begin{cases} \partial_t f - \partial_{xx}^2 f - |x|^{2\gamma}\partial_{yy}^2 f + \dfrac{c_\nu}{x^2} f = u(t,x,y)\chi_\omega(x,y), & (t,x,y) \in (0,T)\times\Omega, \\ f(t,-1,y) = f(t,1,y) = 0, & (t,y)\in(0,T)\times(0,1), \\ f(t,x,0) = f(t,x,1), & (t,x)\in(0,T)\times(-1,1), \\ \partial_y f(t,x,0) = \partial_y f(t,x,1), & (t,x)\in(0,T)\times(-1,1), \\ f(0,x,y) = f_0(x,y), & (x,y)\in\Omega, \end{cases} \tag{1.72}$$

avec $\omega \subset \Omega = (-1,1)\times(0,1)$, $\gamma > 0$ et $c_\nu = \nu^2 - \frac{1}{4}$ pour $\nu \in (0,1)$. La borne inférieure $-\frac{1}{4}$ pour la constante du potentiel singulier permet la validité de l'inégalité de Hardy

$$\int_{-1}^{1} f'(x)^2 + \frac{c_\nu}{x^2} f(x)^2 \mathrm{d}x \geq 0, \quad \text{pour tout } f \in H^1(-1,1) \text{ tel que } f(0) = 0. \tag{1.73}$$

La borne supérieure $\frac{3}{4}$ pour la constante du potentiel singulier permet d'assurer que l'opérateur étudié n'est pas essentiellement autoadjoint, conformément aux résultats de U. Boscain et C. Laurent [28].

On rappelle que le système (1.72) est approximativement contrôlable en temps $T > 0$ si pour tout $\epsilon > 0$, pour tout $f_0, f_T \in L^2(\Omega)$, il existe $u \in L^2((0,T)\times\Omega)$ tel que la solution de (1.72) associée satisfasse

$$\|f(T) - f_T\|_{L^2(\Omega)} \leq \epsilon. \tag{1.74}$$

Le principal résultat du Chapitre 7 est le suivant (voir le Théorème 7.1 page 187).

Théorème 1.25. *Soient $T > 0$, $\gamma > 0$ et $\nu \in (0,1)$. Si ω est un sous-domaine de l'une des deux composantes de $\Omega\backslash\{x = 0\}$ alors (1.72) est approximativement contrôlable en temps T si et seulement si $\nu \in \left(0, \frac{1}{2}\right]$ i.e. si et seulement si $c_\nu \in \left(-\frac{1}{4}, 0\right]$.*

Comme nous le verrons plus tard, si ω intersecte les deux composantes de $\Omega\backslash\{x = 0\}$, alors la contrôlabilité approchée a lieu pour tout $\gamma > 0$ et tout $\nu \in (0,1)$. Ceci est une conséquence de la continuation unique des opérateurs uniformément paraboliques. Ainsi, le fait d'avoir une singularité interne, et non au bord (comme considéré dans [37]), affecte profondément les propriétés de contrôlabilité.

La première difficulté de ce théorème est de donner un sens aux solutions de (1.72). En effet, à cause de la singularité interne $\frac{1}{x^2}$ les techniques usuelles d'étude du caractère bien posé sont mises en défaut. Dans ce contexte, la régularité n'est pas un facteur limitant. Ainsi, on peut construire des fonctions $f \in C_0^\infty(\Omega)$ telles que $(x,y) \mapsto \frac{1}{x^2} f(x,y) \notin L^2(\Omega)$. Le facteur limitant est ici le comportement de la fonction considérée au voisinage de la singularité. L'inégalité de Hardy (1.73) n'est plus valable si la fonction f n'est pas nulle sur $\{x = 0\}$. Pour les problèmes 1D avec un potentiel singulier au bord, comme par exemple [138, 137], la condition de nullité en $x = 0$ est assurée par les conditions au bord de Dirichlet homogène considérées. La singularité étant ici interne, le cadre fonctionnel doit contenir des informations sur le comportement des fonctions au voisinage de la singularité.

D'après les travaux de U. Boscain et C. Laurent [28], on sait que pour $\nu \in (0,1)$ l'opérateur $-\partial_{xx}^2 - |x|^{2\gamma}\partial_{yy}^2 + \frac{c_\nu}{x^2}$ avec domaine $C_0^\infty(\Omega\backslash\{x = 0\})$ admet plusieurs extensions autoadjointes sur $L^2(\Omega)$. On construit $(\mathcal{A}, D(\mathcal{A}))$ une telle extension positive. Ainsi, pour tout $f^0 \in L^2(\Omega)$ et $u \in L^2((0,T)\times\Omega)$, le système

$$\begin{cases} f'(t) = \mathcal{A}f(t) + v(t), \quad t \in [0,T], \\ f(0) = f^0, \end{cases} \tag{1.75}$$

où $v(t) : (x,y) \in \Omega \mapsto u(t,x,y)\chi_\omega(x,y)$ admet une unique solution faible. La solution de (1.72) est alors entendue au sens de la solution de (1.75). Il en va de même pour le système adjoint de (1.72). La construction de cet opérateur $\mathcal{A}$ est réalisée en Section 7.2. On présente ici la stratégie de cette construction.

Inspirés par les idées du Théorème 1.24, on considère les systèmes 1D obtenus de manière formelle par la décomposition de la solution du système (1.72) homogène dans la base de Fourier périodique en la variable y i.e.

$$\partial_t f_n - \partial_{xx}^2 f_n + \left(\frac{c_\nu}{x^2} + (2n\pi)^2|x|^{2\gamma}\right) f_n = 0, \quad (t,x) \in (0,T)\times(-1,1). \tag{1.76}$$

On construit un domaine $D(A_n)$ tel que l'opérateur

$$A_n := -\partial_{xx}^2 + \left(\frac{c_\nu}{x^2} + (2n\pi)^2|x|^{2\gamma}\right) \tag{1.77}$$

soit autoadjoint positif. On obtient alors la solution de (1.76) par le semigroupe généré par $-A_n$. Les éléments de $D(A_n)$ sont obtenus comme somme d'une partie régulière H^2 nulle en $x = 0$ et à dérivée nulle en $x = 0$ et d'une partie singulière solution de part et d'autre de la singularité de

$$-\partial_{xx}^2 f + \frac{c_\nu}{x^2} f = 0. \tag{1.78}$$

On impose des conditions de transmission adaptées à travers la singularité pour la partie singulière. Le caractère autoadjoint est obtenu grâce aux conditions de transmission. Le caractère positif est obtenu grâce aux conditions de transmission et à l'inégalité de Hardy (1.73) pour la partie régulière. Finalement, on construit un semigroupe continu de $L^2(\Omega)$ par la procédure suivante

- on décompose $f^0 \in L^2(\Omega)$ dans la base de Fourier périodique en la variable y,
- on résout (1.76) avec condition initiale f_n^0 donnée par la décomposition précédente,
- on somme dans la base de Fourier précédente les solutions ainsi obtenues.

Le générateur infinitésimal $\mathcal{A}$ du semigroupe $S(t)$ ainsi construit est alors une extension de l'opérateur $-\partial_{xx}^2 - |x|^{2\gamma}\partial_{yy}^2 + \frac{c_\nu}{x^2}$ avec domaine $C_0^\infty(\Omega\backslash\{x=0\})$. Conformément au caractère essentiellement autoadjoint de cet opérateur, cette construction n'est pas licite pour $\nu \geq 1$: les fonctions singulières considérées dans le domaine de l'opérateur unidimensionnel ne sont plus des fonctions de L^2.

Le Théorème 1.25 est alors prouvé en montrant la continuation unique du système adjoint de (1.76) dont les solutions sont données par le semigroupe $S(t)$ construit précédemment. La singularité interne empêche d'appliquer les résultats de continuation unique de type Holmgren mentionnés précédemment. On obtient le résultat suivant ramenant la continuation unique du problème en dimension deux à celle d'un problème unidimensionnel avec singularité au bord (voir la Proposition 7.6 page 195).

Proposition 1.2. *Soit $T > 0$, $\gamma > 0$, $\nu \in (0,1)$ et ω un ouvert de $(-1,0) \times (0,1)$. Si $g^0 \in L^2(\Omega)$ vérifie $\chi_\omega S(t) g^0 \equiv 0$ pour presque tout $t \in [0,T]$, alors $S(t)g^0$ est identiquement nul sur $(-1,0) \times (0,1)$. Pour $n \in \mathbb{Z}$, la partie singulière du n^e coefficient de Fourier est identiquement nulle sur $(-1,1)$.*

Ce résultat utilise les propriétés d'ellipticité de l'opérateur $\mathcal{A}$ sur $(-1,-\varepsilon) \times (0,1)$ pour tout $\varepsilon \in (0,1)$ et les résultats classiques de continuation unique pour les équations paraboliques non singulières. La nullité de la partie singulière sur $(-1,1)$ est assurée par sa nullité sur $(-1,0)$ et les conditions de transmission à travers la singularité. Ce résultat prouve la continuation unique pour tout $\nu \in (0,1)$ et tout $\gamma > 0$ si ω intersecte les deux composantes de $\Omega \backslash \{x = 0\}$.

On est alors ramenés à étudier la continuation unique du problème unidimensionnel avec singularité au bord suivant

$$\begin{cases} \partial_t g_n - \partial_{xx}^2 g_n + \left(\dfrac{c_\nu}{x^2} + (2n\pi)^2 x^{2\gamma}\right) g_n = 0, & (t,x) \in (0,T) \times (0,1), \\ g_n(t,0) = g_n(t,1) = 0, & t \in (0,T), \\ \partial_x g_n(t,0) = 0, & t \in (0,T). \end{cases} \tag{1.79}$$

Notons que, les problèmes unidimensionnels singuliers au bord mentionnés en Section 1.4.3 se focalisant sur des contrôles distribués sur un sous-domaine, la continuation unique pour ce problème n'est pas traitée dans la littérature.

Pour $\nu \in \left(0, \frac{1}{2}\right]$, on prouve une inégalité de Carleman qui entraine pour tout $n \in \mathbb{Z}$, la nullité de g_n solution de (1.79). Cette inégalité repose principalement sur la condition de Neumann supplémentaire en $x = 0$ et sur le fait que $c_\nu \leq 0$ pour $\nu \in \left(0, \frac{1}{2}\right]$.

Pour $\nu \in \left(\frac{1}{2}, 1\right)$, on construit en utilisant des fonctions de Bessel une solution explicite non nulle de (1.79) pour $n = 0$, ce qui prouve le Théorème 1.25.

En corollaire, on déduit la condition nécessaire et suffisante suivante de contrôlabilité approchée de l'équation de la chaleur unidimensionnelle avec potentiel singulier $\frac{1}{x^2}$ (voir le Théorème 7.3 page 188).

Théorème 1.26. *Soient $T > 0$ et $\nu \in (0,1)$. Si ω est un ouvert de $(-1,0)$ ou $(0,1)$, alors le système*

$$\begin{cases} \partial_t f - \partial_{xx}^2 f + \dfrac{c_\nu}{x^2} f = u(t,x)\chi_\omega(x), & (t,x) \in (0,T) \times (-1,1), \\ f(t,-1) = f(t,1) = 0, & t \in (0,T), \end{cases} \tag{1.80}$$

est approximativement contrôlable en temps T si et seulement si $\nu \in \left(0, \frac{1}{2}\right]$ i.e. si et seulement si $c_\nu \in \left(-\frac{1}{4}, 0\right]$.

Les solutions du système (1.80) sont définies via le semigroupe généré par l'opérateur $(-A_0, D(A_0))$ mentionné en (1.77).

Un deuxième corollaire consiste en l'adaptation du Théorème 1.25 (voir le Théorème 7.2 page 187) pour des conditions au bord de Dirichlet homogène.

Théorème 1.27. *Soient $T>0$, $\gamma>0$ et $\nu\in(0,1)$. Pour $\ell>0$, on pose $\Omega^\ell:=(-1,1)\times(0,\ell)$. On considère l'opérateur (1.57) avec conditions de Dirichlet homogènes.*

$$\begin{cases} \partial_t f-\partial_{xx}^2 f-|x|^{2\gamma}\partial_{yy}^2 f+\frac{c_\nu}{x^2}f=u(t,x,y)\chi_\omega(x,y), & (t,x,y)\in(0,T)\times\Omega^\ell,\\ f(t,x,y)=0, & (t,x,y)\in(0,T)\times\partial\Omega^\ell,\\ f(0,x,y)=f^0(x,y), & (x,y)\in\Omega^\ell. \end{cases} \tag{1.81}$$

Si $\nu\in\left(0,\frac{1}{2}\right]$, alors le système (1.81) est approximativement contrôlable en temps T, pour tout $\ell>0$.
Si $\nu\in\left(\frac{1}{2},1\right)$ et $\gamma=1$, il existe des valeurs de $\ell>0$ telles que pour tout $T>0$ le système (1.81) n'est pas approximativement contrôlable en temps T

La construction des solutions de (1.81) est similaire à celle des solutions de (1.72). Le résultat positif est identique à celui du Théorème 1.27. Le résultat négatif du Théorème 1.27 étant basé sur un contre-exemple explicite pour le coefficient de Fourier associé à la fréquence nulle, on adapte ce résultat pour des fréquences non nulles. C'est cette adaptation qui utilise des longueurs spécifiques du domaine dans la direction y et la restriction $\gamma=1$.

1.4.5 Perspectives

Au vu des résultats présentés dans la Section 1.4.4, certaines questions et extensions apparaissent naturellement.

Contrôlabilité à zéro. Lorsque le système (1.72) est approximativement contrôlable, on peut s'interroger sur la contrôlabilité à zéro de ce système et donc sur l'observabilité du système adjoint. La validité d'inégalités de Carleman pour des opérateurs 2D de type Grushin est un problème ouvert. Vu la construction du semigroupe $S(t)$ et la stratégie utilisée dans [14], une direction d'étude serait de prouver l'observabilité, par un intervalle $(a,b)\subset(-1,0)$, des systèmes adjoints unidimensionnels, uniformément par rapport à la fréquence de Fourier. Le système 2D (1.72) serait alors observable par une bande verticale $\omega=(a,b)\times(0,1)$, grâce à l'égalité de Bessel-Parseval. Dans cette direction, il est peu probable que l'inégalité de Carleman démontrée au Chapitre 7 soit utilisable. En effet, la condition de Neumann supplémentaire en $x=0$ dans (1.79) est fondamentale pour la validité de cette inégalité. De plus, la présence de fonctions singulières compromet la stratégie d'inégalités de Carleman basée sur des intégrations par parties. Déjà, pour l'équation de la chaleur unidimensionnelle singulière (1.80) la contrôlabilité à zéro est un problème ouvert.

Cas des conditions de Dirichlet On peut aussi s'interroger sur la non contrôlabilité du système (1.81) avec conditions de bord de Dirichlet pour $\gamma>0$ et ℓ quelconque. Les conditions imposées sur γ et ℓ au Théorème 1.27 sont purement techniques et liées à l'utilisation de solutions explicites. On conjecture que le système (1.81) ne vérifie pas la propriété de contrôle approché pour $\gamma>0$ et ℓ quelconque. La preuve de ce résultat nécessite probablement des outils différents.

Première partie

Contrôle bilinéaire d'équations de Schrödinger

Chapitre 2

Temps minimal pour la contrôlabilité exacte locale

Ce chapitre est inspiré de l'article [18] écrit en collaboration avec K. Beauchard et accepté pour publication dans le journal *Mathematical Control and Related Fields.*

Contents

2.1 Introduction

2.1.1 The problem

Let us consider the 1D Schrödinger equation

$$\begin{cases} i\partial_t\psi(t,x) = -\partial_x^2\psi(t,x) - u(t)\mu(x)\psi(t,x), & (t,x) \in \mathbb{R}\times(0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t\in\mathbb{R}. \end{cases} \tag{2.1}$$

Such an equation arises in the modelization of a quantum particle, in an infinite square potential well, in a uniform electric field with amplitude $u(t)$. The function $\mu : (0,1) \to \mathbb{R}$ is the dipolar moment of the particle. The system (2.1) is a bilinear control system in which the state is the wave function ψ, with $\|\psi(t)\|_{L^2(0,1)} = 1, \forall t \in \mathbb{R}$ and the control is the real valued function u.

In this article, we study the minimal time required for the local controllability of (2.1) around the ground state. Before going into details, let us introduce several notations. The operator A is defined by

$$D(A) := H^2 \cap H^1_0((0,1),\mathbb{C}), \quad A\varphi := -\frac{d^2\varphi}{dx^2}. \tag{2.2}$$

Its eigenvalues and eigenvectors are

$$\lambda_k := (k\pi)^2, \quad \varphi_k(x) := \sqrt{2}\sin(k\pi x), \forall k \in \mathbb{N}^*. \tag{2.3}$$

The family $(\varphi_k)_{k\in\mathbb{N}^*}$ is an orthonormal basis of $L^2((0,1),\mathbb{C})$ and

$$\Phi_k(t,x) := \varphi_k(x)e^{-i\lambda_k t}, \forall k \in \mathbb{N}^*$$

is a solution of (2.1) with $u \equiv 0$ called eigenstate, or ground state, when $k = 1$. We denote by $\mathcal{S}$ the unit $L^2((0,1),\mathbb{C})$-sphere. In this article, we consider two types of initial conditions for (2.1): the ground state

$$\psi(0,x) = \varphi_1(x), \quad x \in (0,1), \tag{2.4}$$

or an arbitrary one

$$\psi(0,x) = \psi_0(x), \quad x \in (0,1). \tag{2.5}$$

Now, let us define the concept of local controllability used in this article.

Definition 2.1. Let $T>0$, X and Y be normed spaces such that $X\subset L^2((0,1),\mathbb{C})$ and $Y\subset L^2((0,T),\mathbb{R})$. The system (2.1) is **controllable in X, locally around the ground state, with controls in Y, in time** T, if, for every $\epsilon>0$, there exists $\delta>0$ such that, for every $\psi_f\in\mathcal{S}\cap X$ with $\|\psi_f-\Phi_1(T)\|_X<\delta$, there exists $u\in Y$ with $\|u\|_Y<\epsilon$ such that the solution of the Cauchy problem (2.1)-(2.4) satisfies $\psi(T)=\psi_f$.

In particular, this definition requires that arbitrarily small motions may be done with arbitrarily small controls.
In this introduction, we first recall two previous results concerning local controllability of systems similar to (2.1). We present a positive result in arbitrary time and a setting for which there exists a positive minimal time. Then, we present the main results of this article i.e. we give a precise setting where local controllability hold in time larger than a minimal time and fails otherwise. We end by a short bibliography and by setting some notations.

2.1.2 A first previous result

First, let us introduce the normed spaces

$$H^s_{(0)}((0,1),\mathbb{C}):=D(A^{s/2}),\quad \|\psi\|_{H^s_{(0)}}:=\left(\sum_{k=1}^{\infty}|k^s\langle\psi,\varphi_k\rangle|^2\right)^{1/2},\quad \forall s>0. \tag{2.6}$$

The following result, proved in [16], emphasizes that the local controllability holds in any positive time when the dipolar moment μ satisfies an appropriate non-degeneracy assumption.

Theorem 2.1. *Let $T>0$ and $\mu\in H^3((0,1),\mathbb{R})$ be such that*

$$\exists c>0 \text{ such that } \frac{c}{k^3}\leqslant|\langle\mu\varphi_1,\varphi_k\rangle|,\quad \forall k\in\mathbb{N}^*. \tag{2.7}$$

There exists $\delta>0$ and a C^1 map $\Gamma:\Omega_T\to L^2((0,T),\mathbb{R})$ where

$$\Omega_T:=\{\psi_f\in\mathcal{S}\cap H^3_{(0)}((0,1),\mathbb{C})\,;\,\|\psi_f-\Phi_1(T)\|_{H^3}<\delta\},$$

such that, $\Gamma(\Phi_1(T))=0$ and for every $\psi_f\in\Omega_T$, the solution of the Cauchy problem (2.1)-(2.4) with control $u:=\Gamma(\psi_f)$ satisfies $\psi(T)=\psi_f$.

First, let us remark that the assumption (2.7) holds for example with $\mu(x)=x^2$. Actually, it holds generically in $H^3((0,1),\mathbb{R})$ (see [16, Proposition 16]). Indeed, for $\mu\in H^3((0,1),\mathbb{R})$, three integrations by part and the Riemann-Lebesgue Lemma prove that

$$\langle\mu\varphi_1,\varphi_k\rangle=2\int_0^1\mu(x)\sin(\pi x)\sin(k\pi x)dx=\frac{4[(-1)^{k+1}\mu'(1)-\mu'(0)]}{k^3\pi^2}+\underset{k\to+\infty}{o}\left(\frac{1}{k^3}\right). \tag{2.8}$$

In particular, a necessary (but not sufficient) condition on μ for (2.7) to be satisfied is $\mu'(1)\pm\mu'(0)\neq0$.

Note that the function spaces in Theorem 2.1 are optimal. Indeed, they are the same as for the well posedness of the Cauchy problem (2.1)(2.4) (see Proposition 2.1).

Finally, let us summarize the proof of Theorem 2.1 in [16]. This proof relies on the linear test (see [54, Chapter 3.1]), the inverse mapping theorem and a regularizing effect. In particular, the assumption (2.7) is necessary for the linearized system to be controllable in $H^3_{(0)}((0,1),\mathbb{C})$ with controls in $L^2((0,T),\mathbb{R})$. When one of the coefficients $\langle \mu\varphi_1, \varphi_k\rangle$ vanishes, then the linearized system is not controllable anymore and the strategy of [16] fails.

2.1.3 A second previous result

The first article in which a positive minimal time is proved, for the local controllability of systems similar to (2.1), is [53]. In this reference, Coron considers the control system

$$\begin{cases} i\partial_t\psi(t,x) = -\partial_x^2\psi(t,x) - u(t)(x-1/2)\psi(t,x), & (t,x)\in\mathbb{R}\times(0,1), \\ \psi(t,0)=\psi(t,1)=0, & t\in\mathbb{R}, \\ s'(t)=u(t), \quad d'(t)=s(t), & t\in\mathbb{R}, \end{cases} \tag{2.9}$$

where the state is (ψ, s, d) and the control is the real valued function u. This system represents a quantum particle in a moving box: u, s, d are the acceleration, the speed and the position of the box.

Note that, here, the relation (2.7) is not satisfied:

$$\langle (x-1/2)\varphi_1, \varphi_k\rangle = \begin{cases} 0 \text{ if } k \text{ is odd}, \\ \frac{8k}{\pi^2(k^2-1)^2} \text{ if } k \text{ is even}, \end{cases}$$

thus Theorem 2.1 does not apply.

On one hand, it is proved in [15] that this system is controllable in $H^7_{(0)}((0,1),\mathbb{C})\times\mathbb{R}\times\mathbb{R}$, locally around the ground state $(\psi=\Phi_1, s=0, d=0)$, with controls $u\in L^\infty((0,T),\mathbb{R})$, in time T large enough.

On the other hand, Coron proved in [53] that this local controllability does not hold in arbitrary time: contrary to Theorem 2.1, a positive minimal time is required for the local controllability. Precisely, Coron proved the following statement.

Theorem 2.2. *There exists $\epsilon>0$ such that, for every $\overline{d}\neq 0$ and $u\in L^2((0,\epsilon),\mathbb{R})$ satisfying $|u(t)|<\epsilon, \forall t\in(0,\epsilon)$, the solution $(\psi,s,d)\in C^0([0,\epsilon],H^1_0((0,1),\mathbb{C}))\times C^0([0,\epsilon],\mathbb{R})\times C^1([0,\epsilon],\mathbb{R})$ of (2.9) with initial condition $(\psi,s,d)(0)=(\Phi_1(0),0,0)$ satisfies $(\psi,s,d)(\epsilon)\neq(\Phi_1(\epsilon),0,\overline{d})$.*

The goal of this article is to go further in this analysis:

- we propose a general context for the minimal time to be positive (in particular, the variables s and d are not required anymore in the state),
- we propose a sufficient condition for the local controllability to hold in large time; this assumption is compatible with the previous context and weaker than (2.7),
- we work in an optimal functional frame, for instance, our non controllability result requires u small in L^2-norm, not in L^∞-norm as in Theorem 2.2,
- we perform a first step toward the characterization of the minimal time.

2.1.4 Main results of this article

The first result of this article is the following one.

Theorem 2.3. *Let $K \in \mathbb{N}^*$, $\mu \in H^3((0,1),\mathbb{R})$ be such that*

$$\langle \mu\varphi_1, \varphi_K \rangle = 0 \quad \text{and} \quad A_K := \langle (\mu')^2 \varphi_1, \varphi_K \rangle \neq 0, \tag{2.10}$$

and $\alpha_K \in \{-1,+1\}$ be defined by

$$\alpha_K := sign(A_K). \tag{2.11}$$

There exists $T_K^* > 0$ such that, for every $T < T_K^*$, there exists $\epsilon > 0$ such that, for every $u \in L^2((0,T),\mathbb{R})$ with

$$||u||_{L^2(0,T)} < \epsilon \tag{2.12}$$

the solution of (2.1)(2.4) satisfies $\psi(T) \neq [\sqrt{1-\delta^2}\varphi_1 + i\alpha_K \delta \varphi_K] e^{-i\lambda_1 T}$ for every $\delta > 0$.

First, we remark that the assumption (2.10) holds, for example, with $\mu(x) = (x - 1/2)$ and $K = 1$. In particular, Theorem 2.3 applies to the particular case studied by Coron in [53]. Thus, the variables (s, d) are not required in the state for the minimal time to be positive. Moreover, the control u does not need to be small in $L^\infty(0,T)$ as in Theorem 2.2: a smallness assumption in $L^2(0,T)$ is sufficient.

Note that the validity of the same result without the assumption '$A_K \neq 0$' is an open problem (see remark 2.2 for technical reasons). A possible (but not optimal) value of T_K^* is given in (2.37). The proof of Theorem 2.3 relies on an expansion of the solution to the second order.

The second result of this article is the following one.

Theorem 2.4. *Let $\mu \in H^3((0,1),\mathbb{R})$ be such that*

$$\mu'(0) \pm \mu'(1) \neq 0. \tag{2.13}$$

Then, the system (2.1) is controllable in $H^3_{(0)}((0,1),\mathbb{C})$, locally around the ground state, with controls $u \in L^2((0,T),\mathbb{R})$, in large enough time T.

A direct consequence of Theorems 2.3 and 2.4 is the following result.

Theorem 2.5. *Let $\mu \in H^3((0,1),\mathbb{R})$ be such that (2.10) and (2.13) hold for some $K \in \mathbb{N}^*$. Then, there exists $T_{min} > 0$ such that the controllability of (2.1) in $H^3_{(0)}((0,1),\mathbb{C})$, locally around the ground state, with controls in $L^2((0,T),\mathbb{R})$ does not hold when $T < T_{min}$, and holds when $T > T_{min}$.*

First, we remark that the assumption (2.13) is weaker than (2.7) and that the assumptions (2.10) and (2.13) are compatible: consider, for instance $\mu(x) := x^2 - \langle x^2\varphi_1, \varphi_2\rangle \varphi_2/\varphi_1$.

Note that an explicit upper bound $T_\sharp$ for the minimal time T_{min} is proposed in the proof (see (2.71)).

We emphasize that, when $\mu'(0)=\mu'(1)=0$, then, the appropriate functional frame stops to be $(\psi \in H^3_{(0)}, u \in L^2)$. For instance, with the tools developed in this article, one may prove: if $L \in \mathbb{N}$, $\mu \in H^{2L+3}((0,1),\mathbb{R})$ are such that $\mu^{(2k+1)}(0)=\mu^{(2k+1)}(1)=0$ for $k=0,...,L-1$ and $\mu^{(2L+1)}(0) \pm \mu^{(2L+1)}(1) \neq 0$, then, the system (2.1) is controllable in $H^{2L+3}_{(0)}((0,1),\mathbb{C})$, locally around the ground state, with controls in $L^2((0,T),\mathbb{R})$, in large enough time T.

Finally, we summarize the proof of Theorem 2.4. Under assumption (2.13), only a finite number of the coefficients $\langle \mu\varphi_1, \varphi_k\rangle$ vanish (see (2.8)). Thus, the linearized system around the ground state is not controllable along a finite number of directions. We will see that all of these directions are recovered at the second order. Moreover, all these directions excepted one, present a rotation phenomena in the complex plane, for the null input solution. This idea of using a power series expansion and exploiting a rotation phenomena was first used on a Korteweg-de Vries equation by Cerpa and Crépeau in [44]. However, their strategy has to be adapted in our situation, because one lost direction does not exhibit a rotation phenomenon (see Remark 2.3).

Under a weaker assumption than (2.13) and still in the framework $(\psi \in H^3_{(0)}, u \in L^2)$, we prove the following result.

Theorem 2.6. *Let $\mu \in H^3((0,1),\mathbb{R})$ be such that*

$$\mu'(0)=\mu'(1)\neq 0 \quad (\textit{resp. } \mu'(0)=-\mu'(1)\neq 0). \tag{2.14}$$

For $N \in \mathbb{N}^*$, we define $\mathcal{N}_N := \{k \in \mathbb{N}^*\,;\, k$ is odd and $k \leq N$ or k is even $\}$ (resp. $\mathcal{N}_N := \{k \in \mathbb{N}^*\,;\, k$ is even and $k \leq N$ or k is odd $\}$). Let $\mathbb{P}_N$ be the orthogonal projection from $L^2((0,1),\mathbb{C})$ to $\mathbb{V}_N := Span\{\varphi_k; k \in \mathcal{N}_N\}$. Then, for every $\epsilon>0$, there exists $T>0$ and $\delta>0$ such that, for every $\widetilde{\psi}_f \in \mathbb{V}_N \cap H^3_{(0)}(0,1)$ with $\|\widetilde{\psi}_f - \mathbb{P}_N\Phi_1(T)\| < \delta$, there exists $u \in L^2(0,T)$ with $\|u\|_{L^2} < \epsilon$ such that the solution of (2.1)-(2.4) satisfies $\mathbb{P}_N\psi(T)=\widetilde{\psi}_f$.

The sketch of the proof is the following. Under assumption (2.14), we prove that

- an infinite number of directions are controlled at the first order, in any positive time,
- all the lost directions are recovered either at the second order, or at the third order,
- any direction corresponding to vanishing first and second orders, are recovered at the third order in arbitrary time.

Note that even if $\mu'(0)=\mu'(1)\neq 0$ (resp. $\mu'(0)=-\mu'(1)\neq 0$), one may sometimes control the whole wave function ψ in large time. For instance in [10], the local controllability in $H^7_{(0)}((0,1),\mathbb{C})$, with controls in $H^1_0((0,T),\mathbb{R})$, in large time T, is proved for $\mu(x)=(x-1/2)$, with the return method.

2.1.5 A review about control of bilinear systems

The first controllability result for bilinear Schrödinger equations such as (2.1) is negative and proved by Turinici [134], as a corollary of a more general result by Ball, Marsden and Slemrod [5]. Then, it has been adapted to nonlinear Schrödinger equations in [86] by Ilner, Lange and Teismann. Because of such noncontrollability results, these equations have been considered as non controllable for a long time. However, progress have been made and this question is now better understood.

Concerning exact controllability issues, local results for 1D models have been proved in [10, 11] by Beauchard; almost global results have been proved in [15], by Coron and Beauchard. In [16], Beauchard and Laurent proposed an important simplification of the above proofs. In [53], Coron proved that a positive minimal time may be required for the local controllability of the 1D model. In [12], Beauchard studied the minimal time for the local controllability of 1D wave equations with bilinear controls. In this reference, the origin of the minimal time is the linearized system, whereas in the present article, the minimal time is related to the nonlinearity of the system. Exact controllability has also been studied in infinite time by Nersesyan and Nersisyan in [113, 114].

Now, we quote some approximate controllability results. Mirrahimi and Beauchard proved in [17] the global approximate controllability, in infinite time, for a 1D model and in [103] Mirrahimi proved a similar result for equations involving a continuous spectrum. Approximate controllability, in finite time, has been proved for particular models by Boscain and Adami in [1], by using adiabatic theory and intersection of the eigenvalues in the space of controls. Approximate controllability, in finite time, for more general models, have been studied by three teams, with different tools: by Boscain, Chambrion, Mason, Sigalotti [45, 25, 30], with geometric control methods; by Nersesyan [111, 112] with feedback controls and variational methods; and by Ervedoza and Puel [70] thanks to a simplified model.

Optimal control techniques have also been investigated for Schrödinger equations with a non linearity of Hartee type in [7, 8] by Baudouin, Kavian, Puel and in [36] by Cances, Le Bris, Pilot. An algorithm for the computation of such optimal controls is studied in [9] by Baudouin and Salomon.

Finally, we quote some references concerning bilinear wave equations. In [91, 90, 89], Khapalov considers nonlinear wave equations with bilinear controls. He proves the global approximate controllability to nonnegative equilibrium states.

2.1.6 Notations

We introduce some conventions and notations valid in all this article. Unless otherwise specified, the functions considered are complex valued and, for example, we write $H^1_0(0,1)$ for $H^1_0((0,1),\mathbb{C})$. When the functions considered are real valued, we specify it and we write, for example, $L^2((0,T),\mathbb{R})$. The same letter C denotes a positive constant, that can change from one line to another one. If $(X,\|.\|)$ is a normed vector space, $x\in X$ and $R>0$, $B_X(x,R)$ denotes the open ball $\{y\in X; \|x-y\|<R\}$ and $\overline{B}_X(x,R)$ denotes the closed ball $\{y\in X; \|x-y\|\leqslant R\}$. We denote by $\langle\cdot,\cdot\rangle$ the $L^2(0,1)$ hermitian inner product

$$\langle f,g\rangle=\int_0^1 f(x)\overline{g(x)}dx,$$

and by $T_{\mathcal{S}}\varphi:=\{\xi\in L^2(0,1)\,;\,\mathrm{Re}\langle\varphi,\xi\rangle=0\}$ the tangent space to $\mathcal{S}$ at any point $\varphi\in\mathcal{S}$. We also introduce for any $s>0$, the spaces

$$h^s(\mathbb{N}^*,\mathbb{C}):=\left\{a=(a_k)_{k\in\mathbb{N}^*}\in\mathbb{C}^{\mathbb{N}^*}\,;\,\sum_{k=1}^{+\infty}|k^s a_k|^2<+\infty\right\}$$

equipped with the norm

$$||a||_{h^s} := \left(\sum_{k=1}^{+\infty} |k^s a_k|^2 \right)^{1/2}.$$

2.1.7 Structure of this article

In Section 2.2, we recall a well posedness result concerning system (2.1). In Section 2.3, we prove Theorem 2.3. In Section 2.4, we prove Theorem 2.4 thanks to power series expansions to the second order as in [44] (see also ([54, Chapter 8])). In Section 2.5, we prove Theorem 2.6 thanks to power series expansions to the order 2 and 3. In Section 2.6, we perform a first step toward the characterization of the minimal time, in a favorable situation. Finally, in Section 2.7, we gather several concluding remarks and perspectives.

2.2 Well posedness

This section is dedicated to the well posedness of the Cauchy problem

$$\begin{cases} i\partial_t \psi = -\partial_x^2 \psi - u(t)\mu(x)\psi - f(t,x), & (t,x) \in (0,T) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T), \\ \psi(0,x) = \psi_0(x), & x \in (0,1). \end{cases} \tag{2.15}$$

proved in [16, Proposition 3].

Proposition 2.1. *Let $\mu \in H^3((0,1),\mathbb{R})$, $T > 0$, $\psi_0 \in H^3_{(0)}(0,1)$, $f \in L^2((0,T), H^3 \cap H^1_0)$ and $u \in L^2((0,T),\mathbb{R})$. There exists a unique weak solution of (2.15), i.e. a function $\psi \in C^0([0,T], H^3_{(0)})$ such that the following equality holds in $H^3_{(0)}(0,1)$ for every $t \in [0,T]$,*

$$\psi(t) = e^{-iAt}\psi_0 + i \int_0^t e^{-iA(t-\tau)}[u(\tau)\mu\psi(\tau) + f(\tau)]d\tau. \tag{2.16}$$

Moreover, for every $R > 0$, there exists $C = C(T,\mu,R) > 0$ such that, if $\|u\|_{L^2(0,T)} < R$, then this weak solution satisfies

$$\|\psi\|_{C^0([0,T],H^3_{(0)})} \leqslant C\Big(\|\psi_0\|_{H^3_{(0)}} + \|f\|_{L^2((0,T),H^3\cap H^1_0(0,1))} \Big). \tag{2.17}$$

If $f \equiv 0$ then

$$\|\psi(t)\|_{L^2(0,1)} = \|\psi_0\|_{L^2(0,1)}, \forall t \in [0,T]. \tag{2.18}$$

2.3 Examples of impossible motions in small time

The goal of this section is to prove Theorem 2.3.

2.3.1 Heuristic

Since we are interested in small motions around the trajectory ($\psi = \Phi_1, u = 0$), with small controls, it is natural to try to do them, in a first step, with the first and the second order terms. We consider a control u of the form $u = 0 + \epsilon v + \epsilon^2 w$. Then, formally, the solution ψ of (2.1)(2.4) writes $\psi = \Phi_1 + \epsilon\Psi + \epsilon^2\xi + o(\epsilon^2)$ where

$$\begin{cases} i\partial_t \Psi = -\partial_x^2 \Psi - v(t)\mu(x)\Phi_1, & (t,x) \in (0,T)\times(0,1), \\ \Psi(t,0) = \Psi(t,1) = 0, & t \in (0,T), \\ \Psi(0,x) = 0, & x \in (0,1), \end{cases} \tag{2.19}$$

$$\begin{cases} i\partial_t \xi = -\partial_x^2 \xi - v(t)\mu(x)\Psi - w(t)\mu(x)\Phi_1, & (t,x) \in (0,T)\times(0,1), \\ \xi(t,0) = \xi(t,1) = 0, & t \in (0,T), \\ \xi(0,x) = 0, & x \in (0,1). \end{cases} \tag{2.20}$$

From the property $\|\psi(t)\|_{L^2} \equiv 1$, we deduce that $\mathrm{Re}\langle \Psi(t), \Phi_1(t)\rangle = 0$ (i.e. $\Psi(t) \in T_{\mathcal{S}}\Phi_1(t)$, $\forall t$) and

$$\|\Psi(t)\|_{L^2}^2 + 2\,\mathrm{Re}\langle \xi(t), \Phi_1(t)\rangle \equiv 0. \tag{2.21}$$

We have

$$\Psi(T,x) = i\sum_{j=1}^{\infty} \langle \mu\varphi_1, \varphi_j\rangle \int_0^T v(t) e^{i\omega_j t} dt\, \Phi_j(T,x) \tag{2.22}$$

where

$$\omega_j := \lambda_j - \lambda_1, \quad \forall j \in \mathbb{N}^*. \tag{2.23}$$

We assume that (2.10) holds for some $K \in \mathbb{N}^*$. By adapting the choice of the control $v \in L^2((0,T),\mathbb{R})$, $\Psi(T)$ can reach any target in the closed subspace $\mathrm{Adh}_{H^3_{(0)}(0,1)}[\mathrm{Span}\{\varphi_k\,;\, k \in \mathcal{J}\}]$ where

$$\mathcal{J} := \{j \in \mathbb{N}^*\,;\, \langle \mu\varphi_1, \varphi_j\rangle \neq 0\} \tag{2.24}$$

(see Proposition 2.19 in Appendix); but the complex direction $\langle \Psi(T), \Phi_K(T)\rangle$ is lost. Let us show that, when T is small, the second order term imposes a sign on the component along this lost direction, preventing the local exact controllability around the ground state.

Using (2.20) and (2.22), we get

$$\langle \xi(T), \Phi_K(T)\rangle = Q_{K,T}^2(v), \tag{2.25}$$

where

$$Q_{K,T}^2(v) := \int_0^T v(t) \int_0^t v(\tau) h_K^2(t,\tau) d\tau dt, \tag{2.26}$$

$$h_K^2(t,\tau) := -\sum_{j=1}^{\infty} \langle \mu\varphi_K, \varphi_j\rangle \langle \mu\varphi_j, \varphi_1\rangle e^{i[(\lambda_K-\lambda_j)t + (\lambda_j - \lambda_1)\tau]}. \tag{2.27}$$

The index 2 in $Q_{K,T}^2$ and h_K^2 is related to the fact that ξ is the second order of the power series expansion. Integrations by part show that

$$|\langle \mu\varphi_K, \varphi_j\rangle| \text{ and } |\langle \mu\varphi_1, \varphi_j\rangle| \leqslant \frac{C}{j^3}, \quad \forall j \in \mathbb{N}^*, \tag{2.28}$$

for some constant $C = C(\mu) > 0$, thus $h_K^2 \in C^0(\mathbb{R}^2, \mathbb{C})$ and the quadratic form $Q_{K,T}^2$ is well defined on $L^2((0,T),\mathbb{R})$. In particular,

$$\mathrm{Im}[\langle \xi(T), \varphi_K e^{-i\lambda_1 T}\rangle] = \widetilde{Q}_{K,T}^2(v) \tag{2.29}$$

where

$$\widetilde{Q}^2_{K,T}(v) := \int_0^T v(t) \int_0^t v(\tau)\widetilde{h}^2_{K,T}(t,\tau)d\tau dt, \tag{2.30}$$

$$\widetilde{h}^2_{K,T}(t,\tau) := \sum_{j=1}^{\infty}\langle \mu\varphi_K, \varphi_j\rangle\langle \mu\varphi_j, \varphi_1\rangle \sin[(\lambda_j - \lambda_K)t - \omega_j\tau + (\lambda_K - \lambda_1)T]. \tag{2.31}$$

Now, we try to move $\epsilon\Psi(T) + \epsilon^2\xi(T)$ in the direction of $+i\alpha_K\varphi_K e^{-i\lambda_1 T}$ (see (2.11) for the definition of α_K). Since $\Psi(T)$ lives in $\mathrm{Adh}_{H^3_{(0)}(0,1)}[\mathrm{Span}\{\varphi_k\,;\, k \neq K\}]$, then, necessarily $\Psi(T) = 0$, i.e. v belongs to

$$V_T := \left\{ v \in L^2((0,T),\mathbb{R})\,;\, \int_0^T v(t)e^{i\omega_j t}dt = 0, \forall j \in \mathcal{J} \right\} \tag{2.32}$$

and $\xi(T) = i\delta\alpha_K\varphi_K e^{-i\lambda_1 T}$ for some $\delta > 0$. Thus the sign of $\widetilde{Q}^2_{K,T}(v)$ has to be α_K. The following two lemmas show that this is not possible when T is small.

Lemma 2.1. *For every $v \in V_T$, we have $\widetilde{Q}^2_{K,T}(v) = \mathcal{Q}_{K,T}(S)$ where $S(t) := \int_0^t v(\tau)d\tau$ and*

$$\mathcal{Q}_{K,T}(S) := -A_K \int_0^T S(t)^2 \cos[(\lambda_K - \lambda_1)(t-T)]dt + \int_0^T S(t) \int_0^t S(\tau)k_{K,T}(t,\tau)d\tau dt, \tag{2.33}$$

$$k_{K,T}(t,\tau) := \sum_{j=1}^{\infty}(\lambda_j - \lambda_K)\omega_j\langle \mu\varphi_1, \varphi_j\rangle\langle \mu\varphi_K, \varphi_j\rangle \sin[(\lambda_j - \lambda_K)t - \omega_j\tau + (\lambda_K - \lambda_1)T]. \tag{2.34}$$

Remark 2.1. Note that $\mathcal{Q}_{K,T}$ is well defined on $L^2(0,T)$ because $k_{K,T} \in L^\infty(\mathbb{R}\times\mathbb{R})$ (see (2.28)).

Proof of Lemma 2.1. Let $T > 0$ and $v \in V_T - \{0\}$. Integrations by parts show that, for every $j \in \mathcal{J}$,

$$\begin{aligned}
&\int_0^T v(t)\int_0^t v(\tau)e^{i[(\lambda_j-\lambda_K)t-\omega_j\tau]}d\tau dt \\
&= -\int_0^T S(t)\left(v(t)e^{i(\lambda_1-\lambda_K)t} + i(\lambda_j - \lambda_K)\int_0^t v(\tau)e^{i[(\lambda_j-\lambda_K)t-\omega_j\tau]}d\tau \right) dt \\
&= -\frac{1}{2}S(T)^2 e^{i(\lambda_1-\lambda_K)T} + \frac{i(\lambda_1-\lambda_K)}{2}\int_0^T S(t)^2 e^{i(\lambda_1-\lambda_K)t}dt \\
&\quad - i(\lambda_j-\lambda_K)\int_0^T S(t)\left(S(t)e^{i(\lambda_1-\lambda_K)t} + i\omega_j\int_0^t S(\tau)e^{i[(\lambda_j-\lambda_K)t-\omega_j\tau]}d\tau \right) dt \\
&= -\frac{1}{2}S(T)^2 e^{i(\lambda_1-\lambda_K)T} - i\left(\lambda_j - \frac{\lambda_1+\lambda_K}{2}\right)\int_0^T S(t)^2 e^{i(\lambda_1-\lambda_K)t}dt \\
&\quad + (\lambda_j-\lambda_K)\omega_j\int_0^T S(t)\int_0^t S(\tau)e^{i[(\lambda_j-\lambda_K)t-\omega_j\tau]}d\tau dt.
\end{aligned}$$

The relations

$$\sum_{j=1}^{\infty}\langle\mu\varphi_1,\varphi_j\rangle\langle\mu\varphi_K,\varphi_j\rangle=\langle\mu\varphi_1,\mu\varphi_K\rangle,$$

$$\sum_{j=1}^{\infty}\left(\lambda_j-\frac{\lambda_1+\lambda_K}{2}\right)\langle\mu\varphi_1,\varphi_j\rangle\langle\mu\varphi_K,\varphi_j\rangle=\langle(\mu')^2\varphi_1,\varphi_K\rangle=A_K, \tag{2.35}$$

give the conclusion.

□

Lemma 2.2. *Let $\mu\in H^3((0,1),\mathbb{R})$ be such that (2.10) holds for some $K\in\mathbb{N}^*$. There exists $T_K^*>0$ such that, for every $T<T_K^*$*

$$\mathcal{Q}_{K,T}(S)\left\{\begin{array}{l}\leqslant -\dfrac{A_K}{4}\displaystyle\int_0^T S(t)^2dt \text{ if } A_K>0,\\ \geqslant -\dfrac{A_K}{4}\displaystyle\int_0^T S(t)^2dt \text{ if } A_K<0,\end{array}\right\}, \forall S\in L^2((0,T),\mathbb{R}). \tag{2.36}$$

Remark 2.2. This statement enlightens the importance of the assumption $A_K\neq 0$ in Theorem 2.3. Indeed, if A_K vanishes then we do not know whether the quadratic form $\widetilde{Q}^2_{K,T}$ has a sign on V_T in small time T. Note that another integration by parts (leading to a quadratic form in $\sigma(t):=\int_0^t S$) is not possible, because of problems of divergence in infinite sums.

Proof of Lemma 2.2. One may assume that $A_K>0$, $\alpha_K=1$. We define the quantity

$$C_K:=\sum_{j=1}^{\infty}|(\lambda_j-\lambda_K)\omega_j\langle\mu\varphi_1,\varphi_j\rangle\langle\mu\varphi_K,\varphi_j\rangle|.$$

By (2.35) and (2.10), there exists $j\in\mathbb{N}^*-\{1,K\}$ such that $\langle\mu\varphi_1,\varphi_j\rangle\langle\mu\varphi_K,\varphi_j\rangle\neq 0$. Thus, $C_K>0$. We introduce

$$T_K^*:=\left\{\begin{array}{l}\frac{|A_1|}{2C_1}\text{ if } K=1,\\ \min\left\{\frac{|A_K|}{2C_K};\frac{\pi}{3(\lambda_K-\lambda_1)}\right\}\text{ if } K\geqslant 2.\end{array}\right. \tag{2.37}$$

Let $T\in(0,T_K^*)$. Using the inequality

$$\cos[(\lambda_K-\lambda_1)(t-T)]\geqslant\frac{1}{2},\quad\forall t\in(0,T),$$

(2.33), (2.34) and Cauchy-Schwarz inequality we get, for every $S\in L^2((0,T),\mathbb{R})$,

$$\begin{aligned}\mathcal{Q}_{K,T}(S)&\leqslant -\frac{A_K}{2}\int_0^T S(t)^2dt+C_K\int_0^T|S(t)|\int_0^t|S(\tau)|d\tau dt\\&\leqslant -\frac{1}{2}\left[A_K-TC_K\right]\int_0^T S(t)^2dt.\end{aligned}$$

□

With additional arguments, one may prove that, for $T < T_K^*$,

$$\sup\{\widetilde{Q}^2_{K,T}(v)\,;\, v \in V_T, \|v\|_{L^2} = 1\} = 0.$$

The non existence of a positive constant $c(T) > 0$ such that

$$\widetilde{Q}^2_{K,T}(v) \leqslant -c(T)\|v\|^2_{L^2}, \forall v \in V_T, \forall T < T_K^*$$

prevents from proving the non controllability in a simple way. Our solution relies on the fact that, for T small, the quadratic form $\widetilde{Q}^2_{K,T}$ is coercive in $S(t) := \int_0^t v(\tau)d\tau$ (see (2.36)). This justifies several technical developments and the use of an auxiliary system in the next section.

2.3.2 Auxiliary system

We consider the function $\widetilde{\psi}(t,x)$ defined by

$$\psi(t,x) = \widetilde{\psi}(t,x)e^{is(t)\mu(x)} \text{ where } s(t) := \int_0^t u(\tau)d\tau, \tag{2.38}$$

which is a weak solution of

$$\begin{cases} i\partial_t\widetilde{\psi} = -\partial_x^2\widetilde{\psi} - is(t)[2\mu'(x)\partial_x\widetilde{\psi} + \mu''(x)\widetilde{\psi}] + s(t)^2\mu'(x)^2\widetilde{\psi}, & (t,x) \in (0,T) \times (0,1), \\ \widetilde{\psi}(t,0) = \widetilde{\psi}(t,1) = 0, & t \in (0,T), \\ \widetilde{\psi}(0,x) = \varphi_1(x), & x \in (0,1). \end{cases} \tag{2.39}$$

We deduce from (2.38) and Proposition 2.1 (applied to (2.1)(2.4)) the following well posedness result for (2.39).

Proposition 2.2. *Let $\mu \in H^3((0,1),\mathbb{R})$, $T > 0$, $s \in H^1((0,T),\mathbb{R})$ with $s(0) = 0$. There exists a unique weak solution $\widetilde{\psi} \in C^0([0,T], H^3 \cap H^1_0(0,1))$ of (2.39). Moreover, for every $R > 0$, there exists $C = C(T,\mu,R) > 0$ such that, if $\|\dot{s}\|_{L^2(0,T)} < R$, then this weak solution satisfies*

$$\|\widetilde{\psi}\|_{L^\infty((0,T),H^3\cap H^1_0)} \leqslant C. \tag{2.40}$$

The proof of Theorem 2.3 is a direct consequence of the following result.

Theorem 2.7. *Let $K \in \mathbb{N}^*$, $\mu \in H^3((0,1),\mathbb{R})$ be such that (2.10) holds and T_K^* be as in Lemma 2.2. For every $T < T_K^*$, there exists $\epsilon > 0$ such that for every $s \in H^1((0,T),\mathbb{R})$ with $s(0) = 0$ and $\|\dot{s}\|_{L^2} < \epsilon$, the solution of the Cauchy problem (2.39) satisfies*

$$\widetilde{\psi}(T,.) \neq (\sqrt{1-\delta^2}\varphi_1 + i\alpha_K\delta\varphi_K)e^{-i\lambda_1 T}e^{i\theta\mu}, \quad \forall \delta > 0, \forall \theta \in \mathbb{R}. \tag{2.41}$$

The proof of Theorem 2.7 requires several steps, thus, it is developed in Section 2.3.4.

2.3.3 Proof of Theorem 2.3 thanks to Theorem 2.7

Let $T < T_K^*$. Let $\epsilon > 0$ be as in Theorem 2.7. Let $u \in L^2((0,T),\mathbb{R})$ be such that $||u||_{L^2} < \varepsilon$. We assume that the solution of the Cauchy problem (2.1)(2.4) satisfies $\psi(T) = (\sqrt{1-\delta^2}\varphi_1 + i\alpha_K\delta\varphi_K)e^{-i\lambda_1 T}$ for some $\delta > 0$. Then, the function $\widetilde{\psi}$ defined by (2.38) solves (2.39) and satisfies $\widetilde{\psi}(T) = (\sqrt{1-\delta^2}\varphi_1 + i\alpha_K\delta\varphi_K)e^{-i\lambda_1 T}e^{-is(T)\mu}$. By Theorem 2.7, this is impossible.

2.3.4 Proof of Theorem 2.7

The proof of Theorem 2.7 requires the following preliminary result.

Proposition 2.3. *Let $T>0$, $K\in\mathbb{N}^*$, $\mu\in H^3((0,1),\mathbb{R})$ be such that $\langle\mu\varphi_1,\varphi_K\rangle=0$. When $||u||_{L^2}\to 0$,*

$$\left|\operatorname{Im}\langle\widetilde{\psi}(T),\varphi_K e^{-i\lambda_1 T}\rangle-\mathcal{Q}_{K,T}(s)\right|=o(||s||_{L^2}^2), \tag{2.42}$$

$$|\operatorname{Im}\langle\widetilde{\psi}(T),\Phi_1(T)\rangle|=o(||s||_{L^2}), \tag{2.43}$$

$$\left\|\left(\langle\widetilde{\psi}(T),\Phi_j(T)\rangle-\omega_j\langle\mu\varphi_1,\varphi_j\rangle\int_0^T s(t)e^{i\omega_j t}dt\right)_{j\in\mathcal{J}-\{1\}}\right\|_{h^1}=o(||s||_{L^2}). \tag{2.44}$$

The proof will use the following lemma which is a straightforward adaptation of [16, Lemma 1]. Its proof is postponed to Appendix 2.B.

Lemma 2.3. *Let $T>0$ and $f\in L^2((0,T),H^1)$. The function defined by $F(t):=\int_0^t e^{iA\tau}f(\tau)d\tau$ belongs to $C^0([0,T],H_0^1)$ and satisfies*

$$||F||_{L^\infty((0,T),H_0^1)}\leqslant c_1(T)||f||_{L^2((0,T),H^1)}$$

where $c_1(T)>0$.

Proof of Proposition 2.3. Let $T>0$. We work with functions $u\in L^2((0,T),\mathbb{R})$ such that $||u||_{L^2}<1$.

First step : We prove that $||\widetilde{\psi}-\Phi_1||_{L^\infty((0,T),H_0^1)}=O(||s||_{L^2})$ when $||u||_{L^2}\to 0$.
From Proposition 2.1, we know that $\psi\in C^0([0,T],H^3_{(0)})$ and

$$||\psi||_{L^\infty((0,T),H^3_{(0)})}\leqslant C. \tag{2.45}$$

We deduce from (2.38) that $\widetilde{\psi}\in C^0([0,T],H^3\cap H_0^1)$ and

$$||\widetilde{\psi}||_{L^\infty((0,T),H^3\cap H_0^1)}\leqslant\tilde{C}. \tag{2.46}$$

By Lemma 2.3 the following equality holds in $H_0^1(0,1)$, for every $t\in[0,T]$

$$\widetilde{\psi}(t)=\Phi_1(t)-\int_0^t e^{iA(t-\tau)}\big[s(\tau)(2\mu'\partial_x\widetilde{\psi}(\tau)+\mu''\widetilde{\psi}(\tau))+is(\tau)^2\mu'^2\widetilde{\psi}(\tau)\big]d\tau, \tag{2.47}$$

and

$$\begin{aligned}||\widetilde{\psi}-\Phi_1||_{L^\infty((0,T),H_0^1)}\leqslant C(T)\Big[&||s||_{L^2(0,T)}||2\mu'\partial_x\widetilde{\psi}+\mu''\widetilde{\psi}||_{L^\infty((0,T),H^1)}\\&+||s||^2_{L^2(0,T)}||\mu'^2\widetilde{\psi}||_{L^\infty((0,T),H_0^1)}\Big].\end{aligned}$$

This inequality, together with (2.46) ends the first step.

Second step : We prove that $||\widetilde{\psi}-\Phi_1-\widetilde{\Psi}||_{L^\infty((0,T),H^1_0)}=o(||s||_{L^2})$ *when* $||u||_{L^2}\to 0$ where $\widetilde{\Psi}(t,x)$ is defined by

$$\Psi(t,x)=\widetilde{\Psi}(t,x)+is(t)\mu(x)\Phi_1(t,x) \tag{2.48}$$

and Ψ is the solution of (2.19). From Proposition 2.1 (applied to system (2.19)), we know that $\Psi\in C^0([0,T],H^3_{(0)})$. We deduce from (2.48) that $\widetilde{\Psi}\in C^0([0,T],H^3\cap H^1_0)$. Note that $\widetilde{\Psi}$ is a weak solution of

$$\begin{cases} i\partial_t\widetilde{\Psi}=-\partial_x^2\widetilde{\Psi}-is(t)\big[2\mu'\partial_x\Phi_1+\mu''\Phi_1\big],\\ \widetilde{\Psi}(t,0)=\widetilde{\Psi}(t,1)=0,\\ \widetilde{\Psi}(0,x)=0.\end{cases} \tag{2.49}$$

By Lemma 2.3, the following equality holds in $H^1_0(0,1)$, for every $t\in[0,T]$

$$\widetilde{\Psi}(t)=-\int_0^t e^{iA(t-\tau)}s(\tau)\Big[2\mu'\partial_x\Phi_1(\tau)+\mu''\Phi_1(\tau)\Big]d\tau. \tag{2.50}$$

Subtracting this relation to (2.47) and applying Lemma 2.3, we get

$$\begin{aligned}||\widetilde{\psi}-\Phi_1-\widetilde{\Psi}||_{L^\infty((0,T),H^1_0)}\leqslant\ & C(T)\big(||s||^2_{L^2(0,T)}||\mu'^2\widetilde{\psi}||_{L^\infty((0,T),H^1_0)}\\ &+||s||_{L^2(0,T)}||2\mu'\partial_x(\Phi_1-\widetilde{\psi})+\mu''(\Phi_1-\widetilde{\psi})||_{L^\infty((0,T),H^1)}\big).\end{aligned}$$

We deduce from (2.46) the existence of a constant $C>0$ (independent of u) such that

$$||\widetilde{\psi}-\Phi_1-\widetilde{\Psi}||_{L^\infty((0,T),H^1_0)}\leqslant C\big(||s||_{L^2}||\widetilde{\psi}-\Phi_1||_{L^\infty((0,T),H^2)}+||s||^2_{L^2}\big).$$

Thus, to end the proof of the second step, we only need to prove that

$$||\widetilde{\psi}-\Phi_1||_{L^\infty((0,T),H^2)}\to 0 \text{ when } ||u||_{L^2}\to 0. \tag{2.51}$$

Using (2.38) and (2.45), we get

$$\begin{aligned}||\widetilde{\psi}-\Phi_1||_{L^\infty((0,T),H^2)}&\leqslant||(e^{is(t)\mu}-1)\psi||_{L^\infty((0,T),H^2)}+||\psi-\Phi_1||_{L^\infty((0,T),H^2)}\\ &\leqslant C||s||_{L^\infty(0,T)}+||\psi-\Phi_1||_{L^\infty((0,T),H^2)}.\end{aligned}$$

Thus (2.51) is a consequence of Proposition 2.1 (applied to (2.1)(2.4)).

Third step : We prove that $||\widetilde{\psi}-\Phi_1-\widetilde{\Psi}-\widetilde{\xi}||_{L^\infty((0,T),L^2)}=o(||s||^2_{L^2})$ *when* $||u||_{L^2}\to 0$ where $\widetilde{\xi}(t,x)$ is defined by

$$\xi(t,x)=\widetilde{\xi}(t,x)+is(t)\mu(x)\widetilde{\Psi}(t,x)-\frac{s(t)^2}{2}\mu(x)^2\Phi_1(t,x) \tag{2.52}$$

and ξ is the solution of (2.20). Note that $\widetilde{\xi}$ is a weak solution of

$$\begin{cases} i\partial_t\widetilde{\xi}=-\partial_x^2\widetilde{\xi}-is(t)\big[2\mu'(x)\partial_x\widetilde{\Psi}+\mu''(x)\widetilde{\Psi}\big]+s(t)^2\mu'(x)^2\Phi_1,\\ \widetilde{\xi}(t,0)=\widetilde{\xi}(t,1)=0,\\ \widetilde{\xi}(0,x)=0.\end{cases} \tag{2.53}$$

Thus, the following equation holds in $L^2(0,1)$ for every $t \in [0,T]$

$$\widetilde{\xi}(t) = -\int_0^t e^{iA(t-\tau)} \Big[s(\tau)(2\mu' \partial_x \widetilde{\Psi}(\tau) + \mu'' \widetilde{\Psi}(\tau)) + is(\tau)^2 \mu'^2 \Phi_1(\tau) \Big] d\tau. \tag{2.54}$$

Using (2.47) and (2.50) we deduce that

$$\begin{aligned}(\widetilde{\psi} - \Phi_1 - \widetilde{\Psi} - \widetilde{\xi})(t) = -\int_0^t e^{iA(t-\tau)} \Big[s(\tau) \Big(2\mu' \partial_x (\widetilde{\psi} - \Phi_1 - \widetilde{\Psi})(\tau) \\ + \mu'' (\widetilde{\psi} - \Phi_1 - \widetilde{\Psi})(\tau) \Big) + is(\tau)^2 \mu'^2 (\widetilde{\psi} - \Phi_1)(\tau) \Big] d\tau\end{aligned}$$

in $L^2(0,1)$ for every $t \in [0,T]$. Thus,

$$||(\widetilde{\psi} - \Phi_1 - \widetilde{\Psi} - \widetilde{\xi})(t)||_{L^2} \leqslant C \int_0^t |s(\tau)| \, ||(\widetilde{\psi} - \Phi_1 - \widetilde{\Psi})(\tau)||_{H^1} + |s(\tau)|^2 ||(\widetilde{\psi} - \Phi_1)(\tau)||_{L^2} d\tau$$

Taking into account the first and second step, we get the conclusion of the third step.

Fourth step : Proof of (2.42). We deduce from (2.50) and (2.54) that

$$\text{Im}\langle \widetilde{\Psi}(T), \varphi_K e^{-i\lambda_1 T} \rangle = 0, \quad \text{Im}\langle \widetilde{\xi}(T), \varphi_K e^{-i\lambda_1 T} \rangle = \mathcal{Q}_{K,T}(s).$$

Using the third step, we get

$$\begin{aligned}\Big| \text{Im}\langle \widetilde{\psi}(T), \varphi_K e^{-i\lambda_1 T} \rangle - \mathcal{Q}_{K,T}(s) \Big| &= \Big| \text{Im}\langle (\widetilde{\psi} - \Phi_1 - \widetilde{\Psi} - \widetilde{\xi})(T), \varphi_K e^{-i\lambda_1 T} \rangle \Big| \\ &\leqslant ||(\widetilde{\psi} - \Phi_1 - \widetilde{\Psi} - \widetilde{\xi})(T)||_{L^2} \\ &= o(||s||_{L^2}^2) \text{ when } ||u||_{L^2} \to 0.\end{aligned}$$

Fifth step : Proof of (2.43). We deduce from (2.50) and the relation $\langle 2\mu' \varphi_1' + \mu'' \varphi_1, \varphi_1 \rangle = 0$ that $\text{Im}\langle \widetilde{\Psi}(T), \Phi_1(T) \rangle = 0$. Thus, the second step gives

$$\begin{aligned}\Big| \text{Im}\langle \widetilde{\psi}(T), \Phi_1(T) \rangle \Big| &= \Big| \text{Im}\langle (\widetilde{\psi} - \Phi_1 - \widetilde{\Psi})(T), \Phi_1(T) \rangle \Big| \\ &= o(||s||_{L^2}) \text{ when } ||u||_{L^2} \to 0.\end{aligned}$$

Sixth step : Proof of (2.44). We deduce from (2.50) that

$$\langle \widetilde{\Psi}(T), \Phi_j(T) \rangle = \omega_j \langle \mu \varphi_1, \varphi_j \rangle \int_0^T s(t) e^{i\omega_j t} dt, \quad \forall j \in \mathbb{N}^* - \{1\}.$$

Using the second step, we get

$$\begin{aligned}&\Big|\Big| \Big(\langle \widetilde{\psi}(T), \Phi_j(T) \rangle - \omega_j \langle \mu \varphi_1, \varphi_j \rangle \int_0^T s(t) e^{i\omega_j t} dt \Big)_{j \in \mathcal{J} - \{1\}} \Big|\Big|_{h^1} \\ &= \Big|\Big| \big(\langle (\widetilde{\psi} - \Phi_1 - \widetilde{\Psi})(T), \Phi_j(T) \big)_{j \in \mathcal{J} - \{1\}} \Big|\Big|_{h^1} \\ &\leqslant C ||(\widetilde{\psi} - \Phi_1 - \widetilde{\Psi})(T)||_{H_0^1} \\ &= o(||s||_{L^2}) \text{ when } ||u||_{L^2} \to 0.\end{aligned}$$

This ends the proof of Proposition 2.3.

□

Proof of Theorem 2.7. One may assume that $A_K > 0$ i.e. $\alpha_K = 1$. Let $T < T_K^*$. Working by contradiction, we assume that, for every $\epsilon > 0$, there exists $s_\epsilon \in H^1((0,T),\mathbb{R})$ with $s_\epsilon(0) = 0$ and $\|\dot{s}_\epsilon\|_{L^2} < \epsilon$ such that the solution $\widetilde{\psi}_\epsilon$ of (2.39) satisfies

$$\widetilde{\psi}_\epsilon(T,.) = (\sqrt{1-\delta_\epsilon^2}\varphi_1 + i\delta_\epsilon\varphi_K)e^{i\theta_\epsilon\mu(.)}e^{-i\lambda_1 T} \tag{2.55}$$

for some $\delta_\epsilon > 0$ and $\theta_\epsilon \in \mathbb{R}$. Then $\theta_\epsilon, \delta_\epsilon \to 0$ when $\epsilon \to 0$.

First step: We prove that

$$|\theta_\epsilon| + |\delta_\epsilon| = \underset{\epsilon\to 0}{O}(\|s_\epsilon\|_{L^2}). \tag{2.56}$$

Using (2.55) and the assumption $\langle \mu\varphi_1, \varphi_K\rangle = 0$, we have

$$\begin{aligned}
&\frac{1}{2}\|(\widetilde{\psi}_\epsilon - \Phi_1)(T)\|^2_{L^2(0,1)} \\
=&1 - \mathrm{Re}\int_0^1 \widetilde{\psi}_\epsilon(T,x)\overline{\Phi_1(T,x)}dx \\
=&1 - \int_0^1 \left(\sqrt{1-\delta_\epsilon^2}\varphi_1(x)^2\cos[\theta_\epsilon\mu(x)] - \delta_\epsilon\varphi_1(x)\varphi_K(x)\sin[\theta_\epsilon\mu(x)]\right)dx \\
=&1 - \left(1 - \frac{\delta_\epsilon^2}{2} + O(\delta_\epsilon^4)\right)\left(1 - \frac{\theta_\epsilon^2}{2}\|\mu\varphi_1\|^2 + O(\theta_\epsilon^4)\right) + \underset{\epsilon\to 0}{O}(\delta_\epsilon\theta_\epsilon^3) \\
=&\frac{\delta_\epsilon^2}{2} + \frac{\theta_\epsilon^2}{2}\|\mu\varphi_1\|^2 + \underset{\epsilon\to 0}{O}(\delta_\epsilon^4 + \theta_\epsilon^4 + \delta_\epsilon\theta_\epsilon^3).
\end{aligned}$$

As proved, in Proposition 2.3,

$$||\widetilde{\psi} - \Phi_1||_{L^\infty((0,T),H_0^1)} = O(||s||_{L^2}) \text{ when } ||u||_{L^2} \to 0.$$

This concludes the first step.

Second step: We prove that

$$\mathrm{Im}\langle\widetilde{\psi}_\epsilon(T), \varphi_K e^{-i\lambda_1 T}\rangle = \delta_\epsilon + \underset{\epsilon\to 0}{o}(\|s_\epsilon\|^2_{L^2}). \tag{2.57}$$

Using (2.55) and the assumption $\langle \mu\varphi_1, \varphi_K\rangle = 0$, we get

$$\begin{aligned}
&\mathrm{Im}\langle\widetilde{\psi}_\epsilon(T), \varphi_K e^{-i\lambda_1 T}\rangle \\
&= \int_0^1 \left(\sqrt{1-\delta_\epsilon^2}\varphi_1(x)\varphi_K(x)\sin[\theta_\epsilon\mu(x)] + \delta_\epsilon\varphi_K(x)^2\cos[\theta_\epsilon\mu(x)]\right)dx \\
&= \left(1 + \underset{\epsilon\to 0}{O}(\delta_\epsilon^2)\right)\underset{\epsilon\to 0}{O}(\theta_\epsilon^3) + \delta_\epsilon\left(1 + \underset{\epsilon\to 0}{O}(\theta_\epsilon^2)\right) \\
&= \delta_\epsilon + \underset{\epsilon\to 0}{O}(\theta_\epsilon^3 + \delta_\epsilon\theta_\epsilon^2).
\end{aligned}$$

Thus, (2.57) is a consequence of (2.56).

Third step: Conclusion. Using (2.57), (2.42) and (2.36), we get

$$\begin{aligned}0 < \delta_\epsilon &= \mathrm{Im}\langle \widetilde{\psi}_\epsilon(T), \varphi_K e^{-i\lambda_1 T}\rangle + \underset{\epsilon\to 0}{o}(\|s_\epsilon\|^2_{L^2}) \\ &= \mathcal{Q}_{K,T}(s_\epsilon) + \underset{\epsilon\to 0}{o}(\|s_\epsilon\|^2_{L^2}) \\ &\leqslant -\frac{A_K}{4}\|s_\epsilon\|^2_{L^2} + \underset{\epsilon\to 0}{o}(\|s_\epsilon\|^2_{L^2}),\end{aligned}$$

which is impossible when ϵ is small.

□

2.4 Local controllability in large time

The goal of this section is to prove Theorem 2.4.

2.4.1 Preliminary

The goal of this section is the proof of the following result.

Proposition 2.4. *Let $\mu \in H^3((0,1),\mathbb{R})$ be such that $\mu'(0) \pm \mu'(1) \neq 0$.*

- *Then, $N := \sharp\{k \in \mathbb{N}^* \,;\, \langle \mu\varphi_1, \varphi_k\rangle = 0\}$ is finite.*
- *Let $K_1 < ... < K_N \in \mathbb{N}^*$ be such that $\langle \mu\varphi_1, \varphi_{K_j}\rangle = 0$ for $j = 1, ..., N$. Then, for every $j \in \{1, ..., N\}$ and $T > 0$ there exists $v \in V_T$ such that $Q^2_{K_j,T}(v) \neq 0$.*
- *There exists $c > 0$ such that*

$$|\langle \mu\varphi_1, \varphi_k\rangle| \geqslant \frac{c}{k^3}, \forall k \in \mathbb{N}^* - \{K_1, ..., K_N\}. \tag{2.58}$$

We recall that $Q^2_{K_j,T}$ is defined in (2.26)(2.27), and V_T in (2.32). For the proof of Proposition 2.4, we need the following preliminary result.

Proposition 2.5. *Let $\mu \in H^3((0,1),\mathbb{R})$ and $K \in \mathbb{N}^*$ be such that for some $n \in \mathbb{N}^*$ $\langle \mu\varphi_K, \varphi_n\rangle\langle \mu\varphi_n, \varphi_1\rangle \neq 0$. The following statements are equivalent.*

- *There exists $T^* > 0$ such that, for every $T < T^*$, $Q^2_{K,T} \equiv 0$ on V_T.*
- *The support of the sequence $(\langle \mu\varphi_K, \varphi_j\rangle\langle \mu\varphi_j, \varphi_1\rangle)_{j\in\mathbb{N}^*}$ is contained in the finite set*

$$\{j_* \in \mathcal{J} \cap [1, K]\,;\, \exists k_* \in \mathcal{J} \cap [1, K], \lambda_{j_*} - \lambda_1 = \lambda_K - \lambda_{k_*}\},$$

and for every $j_*, k_* \in \mathcal{J} \cap [1, K]$ which satisfy $\lambda_{j_*} - \lambda_1 = \lambda_K - \lambda_{k_*}$, we have $\langle \mu\varphi_K, \varphi_{j_*}\rangle\langle \mu\varphi_{j_*}, \varphi_1\rangle = \langle \mu\varphi_K, \varphi_{k_*}\rangle\langle \mu\varphi_{k_*}, \varphi_1\rangle$.

Proof of Proposition 2.5. To simplify the notation of this proof, we write Q_T and h, instead of $Q^2_{K,T}$ and h^2_K. Let us assume that $Q_T \equiv 0$ on V_T, for every $T < T^*$. Then $\nabla Q_T(v) \perp V_T$, for every $v \in V_T$ and $T < T^*$. Easy computations show that, for $v \in V_T$,

$$\nabla Q_T(v) : t \mapsto \int_0^t v(\tau)h(t,\tau)d\tau + \int_t^T v(\tau)h(\tau,t)d\tau = \int_t^T v(\tau)[h(\tau,t) - h(t,\tau)]d\tau. \tag{2.59}$$

First step: We prove that $\nabla Q_T(v)=0, \forall v \in V_T$. Let T, T_1 be such that $0<T<T_1<T^*$ and $v \in V_{T_1}$ supported on $(0,T)$. Since $\nabla Q_{T_1}(v) \perp V_{T_1}$, there exists a unique sequence $(\alpha_k)_{k\in\mathbb{Z}-\{0\}} \in l^2$ such that

$$\nabla Q_{T_1}(v)=\sum_{k=1}^{\infty}\alpha_k e^{i(\lambda_k-\lambda_1)t}+\sum_{k=2}^{\infty}\alpha_{-k}e^{-i(\lambda_k-\lambda_1)t} \quad \text{ in } L^2((0,T_1),\mathbb{R})$$

(decomposition on a Riesz-basis). We have $\nabla Q_{T_1}(v)\equiv 0$ on (T,T_1) because v is supported on $(0,T)$ (see(2.59)). Using Ingham inequality on (T,T_1) we get $\alpha_k=0, \forall k$ (see Proposition 2.19 in Appendix).

Second step: We prove that $V_T|_{(t,T)}=L^2(t,T)$. Let $T\in(0,T^*)$ and $t\in(0,T)$ be fixed. Let $v\in L^2((t,T),\mathbb{R})$. We define $d_j:=0$, for $j\in\mathbb{N}^*-\mathcal{J}$ and

$$d_j:=-\int_t^T v(\tau)e^{i\omega_j\tau}d\tau, \quad \text{for } j\in\mathcal{J}.$$

Thus, $d=(d_j)_{j\in\mathbb{N}^*}\in\ell_r^2(\mathbb{N}^*,\mathbb{C})$ and Proposition 2.19 imply that there exists $\tilde{v}\in L^2((0,t),\mathbb{R})$ such that

$$\int_0^t \tilde{v}(\tau)e^{i\omega_j\tau}d\tau=d_j=-\int_t^T v(\tau)e^{i\omega_j\tau}d\tau, \quad \forall j\in\mathcal{J}.$$

Then if we extend $\tilde{v}$ on (t,T) by setting $\tilde{v}_{|(t,T)}=v$, it comes that $\tilde{v}\in V_T$. Thus, $V_T|_{(t,T)}=L^2(t,T)$.

Third step: We prove that $h(\tau,t)=h(t,\tau), \forall t,\tau\in[0,T^*]$. Using the first step, we get

$$\int_t^T v(\tau)[h(\tau,t)-h(t,\tau)]d\tau=0, \quad \forall 0<t<T<T^*, \forall v\in V_T. \tag{2.60}$$

Using the second step, we deduce from (2.60) that $\tau\mapsto h(t,\tau)-h(\tau,t)$ vanishes in $L^2(t,T)$, for every $0<t<T<T^*$. This gives the conclusion because the function $(t,\tau)\mapsto h(\tau,t)-h(t,\tau)$ is continuous.

Fourth step: Conclusion. Let $k^*\in\mathbb{N}^*$ be such that $b_{k^*}:=\langle\mu\varphi_K,\varphi_{k^*}\rangle\langle\mu\varphi_{k^*},\varphi_1\rangle\neq 0$. The equality $h(t,\tau)-h(\tau,t)=0$ with $\tau=0$ gives

$$b_{k^*}e^{i(\lambda_K-\lambda_{k^*})t}=\sum_{j\in\mathcal{J}}b_je^{i(\lambda_j-\lambda_1)t}-\sum_{k\in\mathcal{J}-\{k^*\}}b_ke^{i(\lambda_K-\lambda_k)t}. \tag{2.61}$$

The equality $\frac{d}{d\tau}[h(t,\tau)-h(\tau,t)]=0$ with $\tau=0$ gives

$$(\lambda_{k^*}-\lambda_1)b_{k^*}e^{i(\lambda_K-\lambda_{k^*})t}=\sum_{j\in\mathcal{J}}(\lambda_K-\lambda_j)b_je^{i(\lambda_j-\lambda_1)t}-\sum_{k\in\mathcal{J}-\{k^*\}}(\lambda_k-\lambda_1)b_ke^{i(\lambda_K-\lambda_k)t}. \tag{2.62}$$

Thus, an obvious linear combination of (2.61) and (2.62) leads to

$$0=\sum_{j\in\mathcal{J}}\big((\lambda_K-\lambda_j)-(\lambda_{k^*}-\lambda_1)\big)b_je^{i(\lambda_j-\lambda_1)t}-\sum_{k\in\mathcal{J}-\{k^*\}}\big((\lambda_k-\lambda_1)-(\lambda_{k^*}-\lambda_1)\big)b_ke^{i(\lambda_K-\lambda_k)t}.$$

In the right hand side of the previous equality, the frequencies $(\lambda_j - \lambda_1)$ are non-negative for every $j \in \mathcal{J}$, while the frequencies $(\lambda_K - \lambda_k)$ are negative for every $k > K$. Thus, for every $k > K$ the frequency $(\lambda_K - \lambda_k)$ appears only one time in the right hand side of the previous equality. The uniqueness of the decomposition on a Riesz basis gives

$$(\lambda_{k^*} - \lambda_1) b_k = (\lambda_k - \lambda_1) b_k, \quad \forall k > K \text{ with } k \neq k^*.$$

Thus, $b_k = 0$, $\forall k \in \mathcal{J} - \{k^*\}$ with $k > K$. Coming back to (2.61), we only have a finite sum in the right hand side, over $j \in \mathcal{J}$ with $j \leqslant K$ and over $k \in \mathcal{J} - \{k^*\}$ with $k \leqslant K$. We deduce the existence of a unique $j^* \in \mathcal{J}$ with $j^* \leqslant K$ such that $\lambda_K - \lambda_{k^*} = \lambda_{j^*} - \lambda_1$ and $b_{k^*} = b_{j^*}$.

Reciprocally, let $\alpha := \lambda_K - \lambda_{k^*} = \lambda_{j^*} - \lambda_1$, $\beta := \lambda_K - \lambda_{j^*} = \lambda_{k^*} - \lambda_1$. Then $h(t,\tau) := b_{k^*}[e^{i[\alpha t + \beta \tau]} + e^{i[\beta t + \alpha \tau]}]$, satisfies $h(t,\tau) = h(\tau, t)$ and $\nabla Q_T \equiv 0$ on V_T, for every $T > 0$. By linearity, the same conclusion holds when h is a finite sum of such terms.

□

Proof of Proposition 2.4. Performing three integrations by part and using the Riemann-Lebesgue Lemma, we get for every K and n in $\mathbb{N}^*$,

$$\langle \mu\varphi_K, \varphi_n \rangle = \frac{4K[(-1)^{K+n}\mu'(1) - \mu'(0)]}{n^3\pi^2} + \underset{n\to+\infty}{o}\left(\frac{1}{n^3}\right). \tag{2.63}$$

Thus, for n large enough $\langle \mu\varphi_1, \varphi_n \rangle \neq 0$. This proves the first and third statements of Proposition 2.4.

Let $j \in \{1, ..., N\}$. Using (2.63), we have simultaneously $\langle \mu\varphi_1, \varphi_n \rangle \neq 0$ and $\langle \mu\varphi_{K_j}, \varphi_n \rangle \neq 0$ for arbitrarily large values of n. Thus, Proposition 2.5 gives the conclusion.

□

2.4.2 Strategy for the proof of Theorem 2.4

Until the end of Section 2.4, we fix $\mu \in H^3((0,1),\mathbb{R})$ such that $\mu'(1) \pm \mu'(0) \neq 0$, $N \in \mathbb{N}$ and $K_1, ..., K_N \in \mathbb{N}^*$ as in Proposition 2.4. To simplify the notations, we assume that $K_1 = 1$. We define the space

$$\mathcal{H} := \text{Span}_{\mathbb{C}}\Big(\Phi_k(T), k \in \mathbb{N}^* - \{K_1, ..., K_N\}\Big), \tag{2.64}$$

and, for $j = 1, ..., N$ the space

$$M^j := \begin{cases} \text{Span}_{\mathbb{C}}\left(\Phi_{K_j}(T)\right) \text{ if } K_j \neq 1, \\ i\text{Span}_{\mathbb{R}}(\Phi_1(T)) \text{ if } K_j = 1. \end{cases} \tag{2.65}$$

Let

$$M := \bigoplus_{j=1}^{N} M^j. \tag{2.66}$$

The global strategy relies on power series expansion of the solutions to the second order as in [44] (see also [54]). In Section 2.4.3, we prove the local exact controllability 'in $\mathcal{H}$', with a first order strategy. Then, in Section 2.4.4, we prove that any direction in M is reached with the second order term. Finally, in Section 2.4.5, we conclude with a fixed point argument.

2.4.3 Controllability in $\mathcal{H}$ in arbitrarily small time

We introduce the orthogonal projection

$$\left| \begin{array}{llll} \mathcal{P}_T : & L^2(0,1) & \to & \mathcal{H} \\ & \psi & \mapsto & \psi - \sum_{j=1}^{N} \langle \psi, \Phi_{K_j}(T) \rangle \Phi_{K_j}(T) \end{array} \right. \tag{2.67}$$

The goal of this section is the proof of the following result.

Theorem 2.8. *Let $T_1, T > 0$ be such that $T_1 < T$. There exists $\delta_1 > 0$ and a C^1-map $\Gamma_{[T_1,T]} : \Omega_{T_1} \times \Omega_T \to L^2((T_1,T),\mathbb{R})$ where*

$$\Omega_{T_1} := \left\{ \psi_0 \in \mathcal{S} \cap H^3_{(0)}(0,1)\,;\, \|\psi_0 - \Phi_1(T_1)\|_{H^3_{(0)}} < \delta_1 \right\},$$

$$\Omega_T := \left\{ \widetilde{\psi}_f \in \mathcal{H} \cap H^3_{(0)}(0,1)\,;\, \|\widetilde{\psi}_f - \mathcal{P}_T[\Phi_1(T)]\|_{H^3_{(0)}} < \delta_1 \right\},$$

such that $\Gamma_{[T_1,T]}(\Phi_1(T_1), \mathcal{P}_T[\Phi_1(T)]) = 0$ and for every $(\psi_0, \widetilde{\psi}_f) \in \Omega_{T_1} \times \Omega_T$, the solution of (2.1) with initial condition $\psi(T_1) = \psi_0$ and control $u := \Gamma_{[T_1,T]}(\psi_0, \widetilde{\psi}_f)$ satisfies $\mathcal{P}_T[\psi(T)] = \widetilde{\psi}_f$.

This theorem may be proved exactly as Theorem 2.1 in [16]. We recall the main steps of the proof because several intermediate results will also be used in the end of this article. To simplify the notations, we take $T_1 = 0$.

By Proposition 2.1, we can consider the map

$$\left| \begin{array}{llll} \Theta_T : & [\mathcal{S} \cap H^3_{(0)}(0,1)] \times L^2((0,T),\mathbb{R}) & \to & [\mathcal{S} \cap H^3_{(0)}(0,1)] \times [\mathcal{H} \cap H^3_{(0)}(0,1)] \\ & (\psi_0, u) & \mapsto & (\psi_0, \mathcal{P}_T[\psi(T)]) \end{array} \right. \tag{2.68}$$

where ψ is the solution of (2.1)(2.5). Then Theorem 2.8 corresponds to the local surjectivity of the nonlinear map Θ_T around the point $(\varphi_1, 0)$, that will be proved thanks to the inverse mapping theorem. Thus, the first property required is the C^1-regularity of Θ_T, which is a consequence of [16, Proposition 3].

Proposition 2.6. *Let $T > 0$ and $\mu \in H^3((0,1),\mathbb{R})$. The map Θ_T defined by (2.68) is C^1. Moreover, for every $\psi_0, \Psi_0 \in H^3_{(0)}(0,1)$, $u, v \in L^2((0,T),\mathbb{R})$, we have*

$$d\Theta_T(\psi_0, u).(\Psi_0, v) = (\Psi_0, P_T[\Psi(T)]) \tag{2.69}$$

where Ψ is the weak solution of the linearized system

$$\left\{ \begin{array}{ll} i\partial_t \Psi = -\partial_x^2 \Psi - u(t)\mu(x)\Psi - v(t)\mu(x)\psi, & (t,x) \in (0,T) \times (0,1), \\ \Psi(t,0) = \Psi(t,1) = 0, & t \in (0,T), \\ \Psi(0,x) = \Psi_0, & x \in (0,1) \end{array} \right. \tag{2.70}$$

and ψ is the solution of (2.1)(2.5).

The second property required for the application of the inverse mapping theorem is the existence of a continuous right inverse for $d\Theta_T(\varphi_1, 0)$, that may be proved exactly as [16, Proposition 4] (it is a consequence of Proposition 2.19 in Appendix).

Proposition 2.7. *Let $T>0$ and $\mu \in H^3((0,1),\mathbb{R})$ be such that (2.58) holds. The linear map*

$$d\Theta_T(\varphi_1,0):[T_{\mathcal{S}}\varphi_1\cap H^3_{(0)}]\times L^2((0,T),\mathbb{R})\to[T_{\mathcal{S}}\varphi_1\cap H^3_{(0)}]\times[\mathcal{H}\cap H^3_{(0)}]$$

has a continuous right inverse

$$d\Theta_T(\varphi_1,0)^{-1}:[T_{\mathcal{S}}\varphi_1\cap H^3_{(0)}]\times[\mathcal{H}\cap H^3_{(0)}]\to[T_{\mathcal{S}}\varphi_1\cap H^3_{(0)}]\times L^2((0,T),\mathbb{R}).$$

Thus, Propositions 2.6 and 2.7 allow to apply the inverse mapping theorem to Θ_T at the point $(\varphi_1,0)$ and thus to prove Theorem 2.8.

2.4.4 Reaching the missed directions, at the second order, in large time.

The goal of this section is the proof of the following result.

Proposition 2.8. *Let $T>T_\sharp$ where*

$$T_\sharp:=\begin{cases}2^{N-1}T^2_{min}+\displaystyle\sum_{k=2}^{N}((k-1)+2^{k-2})\frac{\pi}{\lambda_{K_k}-\lambda_1} & \text{if } K_1=1,\\ \displaystyle\sum_{k=1}^{N}\frac{k\pi}{\lambda_{K_k}-\lambda_1} & \text{if } K_1\neq 1.\end{cases}\tag{2.71}$$

There exists a continuous map

$$\Lambda_T:\ \begin{array}{ccc} M & \to & L^2((0,T),\mathbb{R})^2\\ z & \mapsto & (v,w)\end{array}$$

such that, for every $z\in M$, the solutions Ψ and ξ of (2.19) and (2.20) satisfy $\Psi(T)=0$ and $\xi(T)=z$.

In this statement, the quantity T^2_{min} is defined as follows.

Lemma 2.4. *The quantity*

$$T^2_{min}:=\inf\{T>0\,;\,\exists v_\pm\in V_T \text{ such that } \widetilde{Q}^2_{1,T}(v_\pm)=\pm 1\}$$

is well defined and belongs to $(0,2/\pi]$.

Let us recall that $K_1=1$ and $\widetilde{Q}^2_{1,T}$ and V_T are defined in (2.30), (2.32).

Proof of Lemma 2.4. Let $T\geqslant 2/\pi$. Let T_1^* be defined as in Lemma 2.2. If $v_-\in V_T-\{0\}$ is supported on $(0,T_1^*)$, then Lemma 2.2 implies that $\widetilde{Q}^2_{1,T}(v)<0$. Let $v_+(t):=\cos(\pi^2 t)1_{[0,2/\pi]}(t)$. Then, explicit computations prove that $v_+\in V_T$ and

$$\widetilde{Q}^2_{1,T}(v_+)=\sum_{j=2}^{\infty}\langle\mu\varphi_1,\varphi_j\rangle^2\frac{(j^2-1)}{\pi^3 j^2(j^2-2)}>0.$$

□

2.4.4.1 Preliminary

Our proof of Proposition 2.8 requires three preliminary results. The first one consists in proving the existence of controls such that the projections of the second order term on the lost directions are non zero.

Proposition 2.9. *Let $T > 0$. For every $j \in \{1, ..., N\}$, there exists $v_j, w_j \in L^2((0,T),\mathbb{R})$ such that the associated solutions Ψ^j and ξ^j of (2.19) and (2.20) satisfy*

$$\begin{aligned} &\Psi^j(T,\cdot) = 0, \\ &\langle \xi^j(T,\cdot), \Phi_{K_j}(T)\rangle \neq 0, \\ &\langle \xi^j(T,\cdot), \Phi_k(T)\rangle = 0, \quad \forall k \in \mathbb{N}^* - \{K_1, ..., K_N\}. \end{aligned} \tag{2.72}$$

Proof of Proposition 2.9. Let $j \in \{1, ..., N\}$. By Proposition 2.4, there exists $v_j \in V_T$ such that $Q^2_{K_j,T}(v_j) \neq 0$. Using (2.25) we get $\langle \xi^j(T), \Phi_{K_j}(T)\rangle = Q^2_{K_j,T}(v_j) \neq 0$. As $v_j \in V_T$, (2.22) and (2.32) imply $\Psi^j(T) = 0$. The equality (2.72) is equivalent to the following trigonometric moment problem on w_j,

$$\int_0^T w_j(t)e^{i\omega_k t}dt = \frac{1}{\langle \mu\varphi_1, \varphi_k\rangle} \int_0^T v_j(t)\langle \mu\Psi^j(t), \varphi_k\rangle e^{i\lambda_k t}dt, \forall k \in \mathbb{N}^* - \{K_1, ..., K_N\}. \tag{2.73}$$

By (2.58) and [16, Lemma 1], the right hand side belongs to l^2. Thus, Proposition 2.19 ensures the existence of a solution $w_j \in L^2((0,T),\mathbb{R})$.

□

The second preliminary result for the proof of Proposition 2.8 is a measure of the rotation of the null input solution, precised in the next statement.

Lemma 2.5. *Let $T, \tilde{T}, \theta > 0$ be such that $0 < T < T + \theta \leqslant \tilde{T}$, $v, w \in L^2((0,T),\mathbb{R})$ and $v_\theta, w_\theta \in L^2((0,\tilde{T}),\mathbb{R})$ be defined by*

$$(v_\theta, w_\theta)(t) := \begin{cases} (0,0) & \text{if } t \in (0,\theta), \\ (v,w)(t-\theta) & \text{if } t \in (\theta, \theta+T), \\ (0,0) & \text{if } t \in (\theta+T, \tilde{T}). \end{cases}$$

We denote by (Ψ,ξ) and $(\Psi_\theta, \xi_\theta)$ the associated solutions of (2.19) and (2.20). Then, for every $k \in \mathbb{N}^*$

$$\begin{aligned} \langle \Psi_\theta(\tilde{T}), \Phi_k(\tilde{T})\rangle = e^{i(\lambda_k-\lambda_1)\theta}\langle \Psi(T), \Phi_k(T)\rangle, \\ \langle \xi_\theta(\tilde{T}), \Phi_k(\tilde{T})\rangle = e^{i(\lambda_k-\lambda_1)\theta}\langle \xi(T), \Phi_k(T)\rangle. \end{aligned}$$

Remark 2.3. Note that, for $k = 1$, there is no rotation phenomenon.

Proof of Lemma 2.5. We have

$$\Psi_\theta(t) = \begin{cases} 0 & \text{for } 0 < t < \theta, \\ \Psi(t-\theta)e^{-i\lambda_1\theta} & \text{for } \theta < t < \theta + T, \\ e^{-iA(t-\theta-T)}\Psi_\theta(\theta+T) & \text{for } \theta + T < t \leqslant \tilde{T}, \end{cases}$$

thus

$$\begin{aligned}\Psi_\theta(\tilde{T}) &= \sum_{k=1}^{\infty} \langle \Psi(T), \varphi_k \rangle e^{-i\lambda_1 \theta} e^{-i\lambda_k(\tilde{T}-\theta-T)} \varphi_k \\ &= \sum_{k=1}^{\infty} \langle \Psi(T), \Phi_k(T) \rangle e^{i(\lambda_k-\lambda_1)\theta} \Phi_k(\tilde{T}).\end{aligned}$$

The same relations hold for ξ_θ.

□

The third preliminary result for the proof of Proposition 2.8 is the non overlapping principle.

Proposition 2.10. *Let $T > 0$ and $T_1 \in (0,T)$. Let $v_j \in V_T$, $w_j \in L^2((0,T),\mathbb{R})$, Ψ_j and ξ_j be the associated solutions of (2.19) and (2.20) for $j = 1, 2$. We assume that v_1 is supported on $(0,T_1)$ and v_2 is supported on (T_1,T). Let $v := v_1 + v_2$, $w := w_1 + w_2$, Ψ and ξ be the associated solutions of (2.19) and (2.20). Then $\Psi(T) = 0$ and $\xi(T) = \xi_1(T) + \xi_2(T)$.*

Proof of Proposition 2.10. We have $\Psi(T) = 0$ because $v \in V_T$ (see (2.22) and (2.32)). The control v_1 is supported on $(0,T_1)$ and belongs to V_{T_1} thus Ψ_1 is supported on $(0,T_1)\times(0,1)$ (see (2.22) and (2.32)). The function v_2 is supported on (T_1,T) thus Ψ_2 is supported on $(T_1,T)\times(0,1)$. Therefore

$$(v_1 + v_2)\mu(\Psi_1 + \Psi_2) = v_1\mu\Psi_1 + v_2\mu\Psi_2 \text{ on } (0,T)\times(0,1),$$

i.e. $\xi_1 + \xi_2$ and ξ solve the same Cauchy problem, thus $\xi = \xi_1 + \xi_2$.

□

2.4.4.2 Proof of Proposition 2.8 in a simplified case

The strategy for the proof of Proposition 2.8 is the same as in [44]. It relies strongly on the rotation of the lost directions, emphasized in Lemma 2.5. However, the strategy of [44] needs to be adapted because there is no rotation phenomenon on our first lost direction. In order to simplify the notations, we prove Proposition 2.8 in the case

$$N = 2, \quad K_1 = 1, \quad K_2 = 2, \quad T_\sharp = 2T_{min}^2 + \frac{3\pi}{\lambda_2 - \lambda_1},$$

where T_{min}^2 is defined in Lemma 2.4. We will explain in Section 2.4.4.3 how it can be adapted for $N \geqslant 3$ and $K_1, ..., K_N$ arbitrary.

Let T, T_1, T_θ, T_c, $T_c^1 > 0$ be such that

$$T > T_\sharp := 2T_{min}^2 + \frac{3\pi}{\lambda_2 - \lambda_1}, \tag{2.74}$$

$$\frac{\pi}{\lambda_2 - \lambda_1} < T_1 < T - \frac{2\pi}{\lambda_2 - \lambda_1} - 2T_{min}^2, \tag{2.75}$$

$$T_c < T_\theta, \quad T_c + T_\theta < \min\left\{ \frac{\pi}{\lambda_2 - \lambda_1} \,;\, T_1 - \frac{\pi}{\lambda_2 - \lambda_1} \right\}, \tag{2.76}$$

$$T_{min}^2 < T_c^1 < \frac{1}{2}\left(T - T_1 - \frac{2\pi}{\lambda_2 - \lambda_1} \right). \tag{2.77}$$

Recall that T^2_{min} is defined in Lemma 2.4. Since $T^1_c > T^2_{min}$, there exists controls $(v_\pm, w_\pm) \in L^2((0,T^1_c),\mathbb{R})^2$, such that the associated solutions $\Psi^\pm$ and $\xi^\pm$ of (2.19) and (2.20) satisfy

$$\begin{aligned}&\Psi^\pm(T^1_c) = 0,\\ &\langle \xi^\pm(T^1_c), \Phi_1(T^1_c)\rangle = \pm i,\\ &\langle \xi^\pm(T^1_c), \Phi_k(T^1_c)\rangle = 0, \quad \forall k \geq 3.\end{aligned} \tag{2.78}$$

Indeed Lemma 2.4 implies the existence of $v_\pm \in V_{T^1_c}$ such that $\widetilde{Q}^2_{1,T^1_c}(v_\pm) = \pm 1$. Then, (2.21) implies $\langle \xi^\pm(T^1_c), \Phi_1(T^1_c)\rangle = \pm i$. Defining $w_\pm$ as the solution of an adequate moment problem as in the proof of Proposition 2.9 proves (2.78).
According to Proposition 2.9, there exists controls $(v^2, w^2) \in L^2((0,T_c),\mathbb{R})^2$ such that the associated solutions (Ψ^2, ξ^2) of (2.19) and (2.20) satisfy

$$\begin{aligned}&\Psi^2(T_c) = 0,\\ &\langle \xi^2(T_c), \Phi_2(T_c)\rangle \neq 0,\\ &\langle \xi^2(T_c), \Phi_k(T_c)\rangle = 0, \quad \forall k \geq 3.\end{aligned} \tag{2.79}$$

First step: Construction of a basis for $M^2 = Span_{\mathbb{C}}(\Phi_2(T))$, with nonoverlapping controls. Let

$$\theta_1 := T - T_1, \qquad \theta_2 := T - T_1 + T_\theta,$$
$$\theta_3 := T - T_1 + \frac{\pi}{\lambda_2-\lambda_1}, \qquad \theta_4 := T - T_1 + T_\theta + \frac{\pi}{\lambda_2-\lambda_1}$$

and $(v^2_j, w^2_j) := (v^2_{\theta_j}, w^2_{\theta_j})$ for $j = 1, ..., 4$ with the notations of Lemma 2.5 (in which $(T, \tilde{T})$ is replaced by (T_c, T)). Then $\mathrm{supp}(v^2_j) \subset (\theta_j, \theta_j + T_c)$ for $j = 1, ..., 4$ and

$$T - T_1 = \theta_1 < \theta_1 + T_c < \theta_2 < \theta_2 + T_c < \theta_3 < \theta_3 + T_c < \theta_4 < \theta_4 + T_c < T$$

(see (2.76)), thus the supports do not overlapp:

$$\forall j_1, j_2 \in \{1,2,3,4\} \text{ with } j_1 \neq j_2 \text{ then } \mathrm{Supp}(v^2_{j_1}) \cap \mathrm{Supp}(v^2_{j_2}) = \emptyset. \tag{2.80}$$

We denote by (Ψ^2_j, ξ^2_j) the associated solutions of (2.19) and (2.20). Then, $\Psi^2_j(T) = 0$ and $\xi^2_j(T) = \tilde{f}^2_j + f^2_j$ for $j = 1, ..., 4$ where (see Lemma 2.5)

$$\begin{aligned}&\tilde{f}^2_1 = \langle \xi^2(T_c), \Phi_1(T_c)\rangle \Phi_1(T), && f^2_1 = e^{i(\lambda_2-\lambda_1)(T-T_1)}\langle \xi^2(T_c), \Phi_2(T_c)\rangle \Phi_2(T) \neq 0,\\ &\tilde{f}^2_2 = \tilde{f}^2_1, && f^2_2 = e^{i(\lambda_2-\lambda_1)T_\theta} f^2_1,\\ &\tilde{f}^2_3 = \tilde{f}^2_1, && f^2_3 = e^{i(\lambda_2-\lambda_1)\frac{\pi}{\lambda_2-\lambda_1}} f^2_1 = -f^2_1,\\ &\tilde{f}^2_4 = \tilde{f}^2_1, && f^2_4 = e^{i(\lambda_2-\lambda_1)(\frac{\pi}{\lambda_2-\lambda_1}+T_\theta)} f^2_1 = -f^2_2.\end{aligned}$$

Moreover, (2.21) imply that

$$\mathrm{Re}\langle \tilde{f}^2_j, \Phi_1(T)\rangle = \mathrm{Re}\langle \xi^2(T_c), \Phi_1(T_c)\rangle = -\|\Psi^2(T_c)\|^2 = 0, \quad \forall j = 1, \dots, 4. \tag{2.81}$$

Note that $(\lambda_2-\lambda_1)T_\theta \in (0,\pi)$, thus (f^2_1, f^2_2) is a $\mathbb{R}$-basis of M^2. This leads to $M^2 = \bigcup_{j=1}^4 M^2_j$ where

$$\begin{aligned}&M^2_1 = \{d^2_1 f^2_1 + d^2_2 f^2_2\,;\, d^2_1 \geq 0, d^2_2 \geq 0\},\\ &M^2_2 = \{d^2_1 f^2_2 + d^2_2 f^2_3\,;\, d^2_1 > 0, d^2_2 \geq 0\},\\ &M^2_3 = \{d^2_1 f^2_3 + d^2_2 f^2_4\,;\, d^2_1 \geq 0, d^2_2 \geq 0\},\\ &M^2_4 = \{d^2_1 f^2_4 + d^2_2 f^2_1\,;\, d^2_1 > 0, d^2_2 \geq 0\}.\end{aligned} \tag{2.82}$$

Second step : Construction of a basis for M^1, with non overlapping controls. The time interval $(T_c^1, T-T_1-T_c^1)$ has length $(T-T_1-2T_c^1) > 2\pi/(\lambda_2-\lambda_1)$ (see (2.77)), thus there exists an odd integer k such that

$$\mathcal{T} := \frac{k\pi}{\lambda_2-\lambda_1} \in (T_c^1, T-T_1-T_c^1). \tag{2.83}$$

Let us consider the following controls

$$(V_\pm, W_\pm)(t) := \begin{cases} (v_\pm, w_\pm)(t) & \text{if } t \in (0, T_c^1), \\ (0,0) & \text{if } t \in (T_c^1, \mathcal{T}), \\ (v_\pm, w_\pm)(t-\mathcal{T}) & \text{if } t \in (\mathcal{T}, \mathcal{T}+T_c^1), \\ (0,0) & \text{if } t \in (\mathcal{T}+T_c^1, T), \end{cases}$$

We denote by $(\Psi_\pm^1, \xi_\pm^1)$ the associated solutions of (2.19) and (2.20). Then $\text{supp}(V_\pm) \subset [0, T-T_1)$ (see (2.83)), thus

$$\forall j \in \{1,...,4\}, \text{Supp}(V_\pm) \cap \text{Supp}(v_j^2) = \emptyset. \tag{2.84}$$

Then, $\Psi_\pm^1(T) = 0$ and

$$\xi_\pm^1(T) = \pm 2i\Phi_1(T) + \langle \xi^\pm(T_c^1), \Phi_2(T_c^1)\rangle [1 + e^{i\mathcal{T}(\lambda_2-\lambda_1)}]\Phi_2(T) = \pm 2i\Phi_1(T)$$

by Proposition 2.10, Lemma 2.5 and (2.78).
As $M^1 = i\text{Span}_{\mathbb{R}}(\Phi_1(T))$, we can thus reach a $\mathbb{R}$-basis of M^1 with non-negative coefficients.

Third step : Conclusion. Let $z \in M$. We construct controls $(v,w) \in L^2((0,T),\mathbb{R})^2$ such that the associated solutions (Ψ, ξ) of (2.19) and (2.20) satisfy $\Psi(T) = 0$ and $\xi(T) = z$. The proof relies on the two following facts:

- $\pm 2i\Phi_1(T)$ and $f_j^2 + \tilde{f}_j^2$ for $j = 1,2,3,4$ are reachable states, with controls such that their supports do not overlap (see (2.80) and (2.84)),
- any vector in M is a linear combination of three of theses vectors, with only non negative coefficients before $f_j^2 + \tilde{f}_j^2$.

There exists a unique $j \in \{1,2,3,4\}$ such that $z \in M^1 + M_j^2$ (see (2.82)). Then,

$$z = ix\Phi_1(T) + d_1 f_j^2 + d_2 f_{j+1}^2 \text{ for some } d_1, d_2 \geqslant 0, x \in \mathbb{R}$$

with the convention $f_5^2 = f_1^2$. We have

$$z = \left(ix - d_1\tilde{f}_j^2 - d_2\tilde{f}_{j+1}^2\right) + d_1\left(\tilde{f}_j^2 + f_j^2\right) + d_2\left(\tilde{f}_{j+1}^2 + f_{j+1}^2\right).$$

As $\text{Re}(\langle \tilde{f}_j^2, \Phi_1(T)\rangle) = 0$, for all $j = 1, \dots, 4$ (see (2.81)), there exists $\kappa \in \{+,-\}$ and $c \geq 0$ such that

$$ix - d_1\tilde{f}_j^2 - d_2\tilde{f}_{j+1}^2 = \kappa 2ic\Phi_1(T).$$

Then,

$$z = \kappa 2ic\Phi_1(T) + d_1(f_j^2 + \tilde{f}_j^2) + d_2(f_{j+1}^2 + \tilde{f}_{j+1}^2),$$

i.e. z is a linear combination of three states that are reachable with non overlapping controls. Hence the map

$$\Lambda_T(z) := (v,w) := \left(\sqrt{c}V_\kappa + \sqrt{d_1}v_j^2 + \sqrt{d_2}v_{j+1}^2, cW_\kappa + d_1 w_j^2 + d_2 w_{j+1}^2\right),$$

gives the conclusion.

2.4.4.3 Proof of Proposition 2.8 in the general case

Let us explain the adaptation of the strategy developed in Section 2.4.4.2 for $N \geq 3$. As previously, we denote by $K_1 < \cdots < K_N$ the directions missed at the first order and we explain how to reach a basis of missed directions on the second order (2.20), iteratively. In this proof, the term 'projection on M^j' denotes $\text{Im}\langle \xi(T), \Phi_1(T)\rangle$ if $j = 1$ and $K_1 = 1$ and $\langle \xi(T), \Phi_{K_j}(T)\rangle$ otherwise.

• The first step consists in reaching a $\mathbb{R}^+$ basis of M^N, the projections on $M^1, \dots, M^{N-1}$ being possibly non zero. This is done as in the first step of the proof of Proposition 2.8, by designing four controls with non overlapping supports. It is done in any time $T_1 > \frac{\pi}{\lambda_{K_N} - \lambda_1}$.

• The $(k+1)^{th}$ step consists in reaching a $\mathbb{R}^+$ basis of M^{N-k} while driving to zero the projections on M^j, for $j = N-k+1, \dots, N$. This can be done iteratively in the following way. Let $(v^{(0)}, w^{(0)})$ be as in Proposition 2.9 for a sufficiently small time and for $j = N-k$. Then, the controls

$$\big(v^{(1)}, w^{(1)}\big) := \big(v^{(0)}, w^{(0)}\big) + \big(v_\theta^{(0)}, w_\theta^{(0)}\big), \quad \text{with } \theta = \frac{\pi}{\lambda_{K_N} - \lambda_1}$$

drive the projection on M^N to zero while the projection on M^{N-k} is still non zero. This is ensured by Lemma 2.5. This is the same strategy as the second step of Section 2.4.4.2 where we drove the projection on M^2 to zero while the projection on M^1 was non zero. We iterate this construction

$$\big(v^{(j+1)}, w^{(j+1)}\big) := \big(v^{(j)}, w^{(j)}\big) + \big(v_\theta^{(j)}, w_\theta^{(j)}\big), \quad \text{with } \theta = \frac{\pi}{\lambda_{K_{N-j}} - \lambda_1},$$

for $j = 0, ..., k-1$. Then the controls $\big(v, w\big) := \big(v^{(k)}, w^{(k)}\big)$ drive the projection on $M^N, \dots, M^{N-k+1}$ to zero while the projection on M^{N-k} is still non zero. Finally, we can find T_θ sufficiently small such that $\big(v, w\big)$ and $\big(v_{T_\theta}, w_{T_\theta}\big)$ have non overlapping supports and the four pairs of control $\big(v, w\big)$, $\big(v_{T_\theta}, w_{T_\theta}\big)$, $\big(v_p, w_p\big)$ and $\big(v_{p+T_\theta}, w_{p+T_\theta}\big)$ with $p = \frac{\pi}{\lambda_{K_{N-k}} - \lambda_1}$ allow to conclude the $(k+1)^{th}$ step. This can be done in any time $T > \frac{\pi}{\lambda_{K_{N-k}} - \lambda_1} + \cdots + \frac{\pi}{\lambda_{K_N} - \lambda_1}$.

• The final step depends on the value of K_1. If $K_1 \geq 2$, we end with the same strategy. This step can be done in any time $T > \frac{\pi}{\lambda_{K_1} - \lambda_1} + \cdots + \frac{\pi}{\lambda_{K_N} - \lambda_1}$ and leads to the expression (2.71) of $T_\sharp$.
If $K_1 = 1$, the elementary brick of control cannot be designed in arbitrary small time (it was already the case in the second step of Section 2.4.4.2 where $(V_\pm, W_\pm)$ were constructed). In this case, the controls $(v^{(0)}, w^{(0)})$ have a time support greater than T_{min}^2. The iterative process then gives that this step can be done in any time $T > 2^{N-1} T_{min}^2 + \sum_{k=2}^N \frac{2^{k-2}\pi}{\lambda_{K_k} - \lambda_1}$ and leads to the expression (2.71) of $T_\sharp$.

Figure 2.1 illustrates the support of controls during the fourth step with $p_j := \frac{\pi}{\lambda_{K_{N-j}} - \lambda_1}$. The small rectangles indicates that the control is active. The phases of control associated with the same index define one of the four pair of controls $\big(v, w\big)$, $\big(v_{T_\theta}, w_{T_\theta}\big)$, $\big(v_p, w_p\big)$ and $\big(v_{p+T_\theta}, w_{p+T_\theta}\big)$. The first rectangle (at the left) stands for the support of $(v^{(0)}, w^{(0)})$.

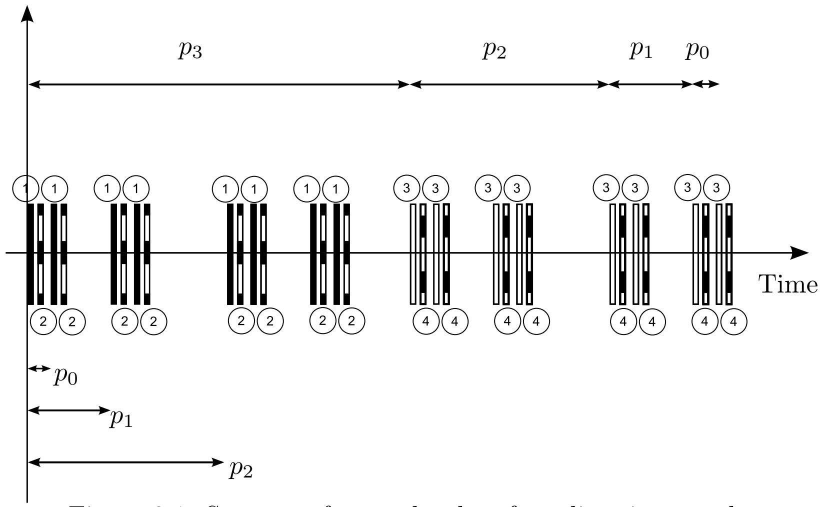


Figure 2.1: Support of controls when four directions are lost

2.4.5 Proof of Theorem 2.4

Let $T > T_\sharp$ and $\psi_f \in H^3_{(0)}(0,1)$ be close enough to $\Phi_1(T)$ (this will be precised later on). The goal of this section is the construction of $u \in L^2((0,T),\mathbb{R})$ such that

- the solution of (2.1)-(2.4) satisfies $\psi(T) = \psi_f$,
- u tends to 0 in $L^2((0,T),\mathbb{R})$ when $\psi_f \to \Phi_1(T)$ in $H^3_{(0)}(0,1)$.

To simplify the notations, we assume $K_1 = 1$.

Let $T_1 \in (T_\sharp, T)$ and $\delta_1 > 0$ associated to the map $\Gamma_{[T_1,T]}$ of Theorem 2.8. From now on, we assume that

$$\|\psi_f - \Phi_1(T)\|_{H^3_{(0)}} < \delta_1. \tag{2.85}$$

One may assume that δ_1 is small enough so that condition (2.85) implies $\mathrm{Re}\langle\psi_f, \Phi_1(T)\rangle > 0$. We introduce the map

$$\left| \begin{array}{cccc} F_{\psi_f}: & M \cap B_{L^2(0,1)}(0,\rho) & \to & M \\ & z & \mapsto & \mathcal{P}_M[\psi_z(T)] \end{array} \right. \tag{2.86}$$

where

- $\rho \in (0,1)$ will be chosen later on,
- $\mathcal{P}_M : L^2(0,1) \to M$ is the L^2-orthogonal projection on M

$$\mathcal{P}_M(\zeta) := i\,\mathrm{Im}(\langle\zeta, \Phi_1(T)\rangle)\Phi_1(T) + \sum_{j=2}^{N} \langle\zeta, \Phi_{K_j}(T)\rangle\Phi_{K_j}(T).$$

- ψ_z is the solution of (2.1)-(2.4) associated to the control u_z defined by

$$u_z := \left\{ \begin{array}{ll} \sqrt{\|z\|}v_z + \|z\|w_z & \text{on } (0,T_1), \\ \Gamma_{[T_1,T]}(\psi_z(T_1), \mathcal{P}_T[\psi_f]) & \text{on } (T_1,T). \end{array} \right.$$

where $\|.\|$ is the $L^2(0,1)$-norm, $\Gamma_{[T_1,T]}$ is defined in Theorem 2.8, $\mathcal{P}_T$ is defined by (2.67) and

$$(v_z, w_z) := \Lambda_{T_1}\left(\frac{e^{iA(T-T_1)}z}{\|z\|}\right),$$

with Λ_{T_1} defined in Proposition 2.8.

Note that, for every z, $\mathcal{P}_T[\psi_z(T)] = \mathcal{P}_T[\psi_f]$. Thus, our goal is to find z_* such that $F_{\psi_f}(z_*) = \mathcal{P}_M[\psi_f]$.

First, we check that the map F_{ψ_f} is well defined when ρ is small enough.

Proposition 2.11. *There exists $\rho > 0$ such that, for every $\psi_f \in H^3_{(0)}(0,1)$ with (2.85), the map F_{ψ_f} defined by (2.86) is well defined and continuous on $M \cap B_{L^2(0,1)}(0,\rho)$.*

Proof of Proposition 2.11. In order to prove that F_{ψ_f} is well defined, it is sufficient to find $\rho > 0$ such that

$$\|z\| < \rho \quad \Rightarrow \quad \|\psi_z(T_1) - \Phi_1(T_1)\|_{H^3_{(0)}} < \delta_1. \tag{2.87}$$

By Proposition 2.1, there exists $C_1, C_1' > 0$ such that, for every $z \in M$,

$$\|\psi_z(T_1) - \Phi_1(T_1)\|_{H^3_{(0)}} \leqslant C_1 \|u_z\|_{L^2(0,T_1)} \leqslant C_1' \sqrt{\|z\|}.$$

Thus, (2.87) holds with $\rho := \min\{1; (\delta_1/C_1')^2\}$. The continuity of F_{ψ_f} is a consequence of the continuity of $\Gamma_{[T_1,T]}$ and the continuity of the solutions of (2.1)(2.5) with respect to the control u and the initial condition ψ_0 (see (2.17)).

□

One may assume ρ small enough so that

$$\|z\| < \rho \quad \Rightarrow \quad \mathrm{Re}\langle \psi_z(T), \Phi_1(T)\rangle > 0.$$

The goal of this section is the proof of the following result, which proves Theorem 2.4.

Proposition 2.12. *There exists $\delta \in (0, \delta_1]$ such that, for every $\psi_f \in H^3_{(0)}(0,1)$ with*

$$\|\psi_f - \Phi_1(T)\|_{H^3_{(0)}} < \delta \tag{2.88}$$

there exists $z_* = z_*(\psi_f) \in M \cap B_{L^2}(0,\rho)$ such that $F_{\psi_f}(z_*) = \mathcal{P}_M[\psi_f]$. Moreover, $z_*(\psi_f) \to 0$ when $\psi_f \to \Phi_1(T)$ in $H^3_{(0)}(0,1)$.

Combining $\mathcal{P}_T[\psi_z(T)] = \mathcal{P}_T[\psi_f]$, Proposition 2.12 and $||\psi_z(T)||_{L^2} = ||\psi_f||_{L^2}$ ends the proof of Theorem 2.4. The proof of Proposition 2.12 requires the following preliminary result.

Proposition 2.13. *There exists $\mathcal{C} > 0$ such that, for every $\psi_f \in H^3_{(0)}(0,1)$ with (2.85) and $z \in M \cap B_{L^2(0,1)}(0,\rho)$, we have*

$$\|F_{\psi_f}(z) - z\| \leqslant \mathcal{C}[\|\psi_f - \Phi_1(T)\|^2_{H^3_{(0)}} + \|z\|^{3/2}].$$

Proof of Proposition 2.13. First step: Existence of $C_1 > 0$ *such that*

$$\|\psi_z(T_1) - \Phi_1(T_1) - e^{iA(T-T_1)}z\|_{H^3_{(0)}} \leqslant C_1\|z\|^{3/2}, \quad \forall z \in M \cap B_{L^2(0,1)}(0,\rho). \tag{2.89}$$

Let Ψ_z, ξ_z be the solution of (2.19) and (2.20) associated to the controls v_z and w_z. Then, $\Psi_z(T_1) = 0$ and $\xi(T_1) = e^{iA(T-T_1)}z/\|z\|$. Explicit computations show that $\psi_z - \Phi_1 - \sqrt{\|z\|}\Psi_z - \|z\|\xi_z$ is solution of (2.15) with control u_z, null initial condition and the following source term

$$(t,x) \mapsto ||z||^{3/2} w_z(t)\mu(x)\Psi_z(t,x) + ||z||u_z(t)\mu(x)\xi_z(t,x).$$

In Proposition 2.11, ρ was assumed to be smaller than 1, thus Proposition 2.1 implies that there exists $C > 0$ such that

$$\|\psi_z - \Phi_1 - \sqrt{\|z\|}\Psi_z - \|z\|\xi_z\|_{L^\infty((0,T),H^3_{(0)})} \leqslant C\|z\|^{3/2}, \forall z \in M \cap B_{L^2(0,1)}(0,\rho).$$

which gives (2.89).

Second step: Existence of $C_2 > 0$ *such that*

$$\|u_z\|_{L^2(T_1,T)} \leqslant C_2[\|\psi_f - \Phi_1(T)\|_{H^3_{(0)}} + \|z\|], \quad \forall z \in M \cap B_{L^2(0,1)}(0,\rho). \tag{2.90}$$

The map $\Gamma_{[T_1,T]}$ is C^1 and $\Gamma_{[T_1,T]}(\Phi_1(T_1),0) = 0$, thus there exists $C > 0$ such that, for every $z \in M \cap B_{L^2(0,1)}(0,\rho)$,

$$\begin{aligned}\|u_z\|_{L^2(T_1,T)} &= \|\Gamma_{[T_1,T]}(\psi_z(T_1), \mathcal{P}_T[\psi_f])\|_{L^2(T_1,T)} \\ &\leqslant C[\|\psi_z(T_1) - \Phi_1(T_1)\|_{H^3_{(0)}} + \|\mathcal{P}_T[\psi_f]\|_{H^3_{(0)}}].\end{aligned}$$

Explicit computations show that $\psi_z - \Phi_1 - \sqrt{\|z\|}\Psi_z$ is solution of (2.15) with control u_z, null initial condition and the following source term

$$(t,x) \mapsto ||z||w_z(t)\mu(x)\Phi_1(t,x) + \sqrt{||z||}u_z(t)\mu(x)\Psi_z(t,x).$$

Thus Proposition 2.1 implies that there exists $C > 0$ such that

$$\|\psi_z(T_1) - \Phi_1(T_1)\|_{H^3_{(0)}} \leqslant \|\psi_z - \Phi_1 - \sqrt{\|z\|}\Psi_z\|_{L^\infty((0,T),H^3_{(0)})} \leqslant C\|z\|. \tag{2.91}$$

Then, $\mathcal{P}_T[\psi_f] = \mathcal{P}_T[\psi_f - \Phi_1(T)]$ implies (2.90).

Third step: Existence of $C_3 > 0$ *such that*

$$\|\psi_z - \Phi_1\|_{L^\infty((T_1,T),H^3_{(0)})} \leqslant C_3[\|\psi_f - \Phi_1(T)\|_{H^3_{(0)}} + \|z\|], \quad \forall z \in M \cap B_{L^2(0,1)}(0,\rho). \tag{2.92}$$

Explicit computations show that $\psi_z - \Phi_1$ is solution of (2.15) on (T_1,T) with control u_z, initial condition $\psi_z(T_1) - \Phi_1(T_1)$ and the following source term

$$(t,x) \mapsto u_z(t)\mu(x)\Phi_1(t,x).$$

Using Proposition 2.1 and (2.91), we get a constant $C > 0$ such that

$$\|\psi_z - \Phi_1\|_{L^\infty((T_1,T),H^3_{(0)})} \leqslant C[\|z\| + \|u_z\|_{L^2(T_1,T)}], \forall z \in M \cap B_{L^2(0,1)}(0,\rho),$$

which, together with (2.90), gives (2.92).

Fourth step: Conclusion. Using the Duhamel formula, the commutativity between e^{iAt} and $\mathcal{P}_M$ and the isometry on $L^2(0,1)$ of e^{iAt}, we get for every $z \in M \cap B_{L^2(0,1)}(0,\rho)$

$$\begin{aligned}\|F_{\psi_f}(z) - z\| &= \|\mathcal{P}_M[\psi_z(T)] - z\| \\ &\leqslant \|\mathcal{P}_M[e^{-iA(T-T_1)}\psi_z(T_1)] - z\| + \int_{T_1}^{T} |u_z(\tau)|\, \|\mathcal{P}_M[\mu\psi_z(\tau)]\| d\tau\end{aligned}$$

Then, using the relation $\mathcal{P}_M[\mu\Phi_1(t)] \equiv 0$ (that holds because $\langle \mu\varphi_1, \varphi_{K_j}\rangle = 0$ for $j = 1, ..., N$), Cauchy-Schwarz inequality and estimates (2.89),(2.90),(2.92) it comes that

$$\begin{aligned}&\|F_{\psi_f}(z) - z\| \\ &\leqslant \|\mathcal{P}_M[\psi_z(T_1) - \Phi_1(T_1) - e^{iA(T-T_1)}z]\| + \int_{T_1}^{T} |u_z(\tau)|\, \|\mathcal{P}_M[\mu(\psi_z - \Phi_1)(\tau)]\| d\tau \\ &\leqslant \|\psi_z(T_1) - \Phi_1(T_1) - e^{iA(T-T_1)}z\| + \sqrt{T-T_1}\|u_z\|_{L^2(T_1,T)}\|\psi_z - \Phi_1\|_{L^\infty((T_1,T),L^2)} \\ &\leqslant C_1\|z\|^{3/2} + \sqrt{T-T_1}C_2C_3[\|\psi_f - \Phi_1(T)\|_{H^3_{(0)}} + \|z\|]^2 \\ &\leqslant C(\rho)[\|z\|^{3/2} + \|\psi_f - \Phi_1(T)\|^2_{H^3_{(0)}}].\end{aligned}$$

This proves Proposition 2.13.

□

Proof of Proposition 2.12. We introduce the map

$$\left| \begin{array}{rrcl} G_{\psi_f} : & M \cap B_{L^2(0,1)}(0,\rho) & \to & M \\ & z & \mapsto & z + \mathcal{P}_M[\psi_f] - F_{\psi_f}(z). \end{array}\right.$$

Our goal is to prove the existence of a fixed point $z_* = z_*(\psi_f)$ to the map G_{ψ_f}. By Proposition 2.13, there exists $\mathcal{C} > 0$ (independent of ψ_f) such that, for every $z \in M \cap B_{L^2(0,1)}(0,\rho)$,

$$\begin{aligned}\|G_{\psi_f}(z)\| &\leqslant \|z - F_{\psi_f}(z)\| + \|\mathcal{P}_M[\psi_f]\| \\ &\leqslant \mathcal{C}[\|\psi_f - \Phi_1(T)\|^2_{H^3_{(0)}} + \|z\|^{3/2}] + \|\psi_f - \Phi_1(T)\|_{H^3_{(0)}}.\end{aligned} \tag{2.93}$$

Let $\rho' \in (0,\rho)$ be such that

$$\mathcal{C}\sqrt{\rho'} < \frac{1}{2} \tag{2.94}$$

and $\delta \in (0,\delta_1)$ be such that $\mathcal{C}\delta^2 + \delta < \rho'/2$. If ψ_f satisfies (2.88), then G_{ψ_f} maps continuously $M \cap B_{L^2(0,1)}(0,\rho')$ into itself. The Brouwer fixed point theorem implies the existence of a fixed point $z_* = z_*(\psi_f)$ of G_{ψ_f} in $M \cap B_{L^2(0,1)}(0,\rho')$. We deduce from (2.93) and (2.94) that

$$\|z_*(\psi_f)\| \leqslant 2[\mathcal{C}\|\psi_f - \Phi_1(T)\|^2_{H^3_{(0)}} + \|\psi_f - \Phi_1(T)\|_{H^3_{(0)}}],$$

thus $z_*(\psi_f) \to 0$ when $\psi_f \to \Phi_1(T)$ in $H^3_{(0)}(0,1)$.

□

2.5 Proof of Theorem 2.6

In this section, we prove Theorem 2.6 when $\mu'(0) = \mu'(1) \neq 0$. The case $\mu'(0) = -\mu'(1) \neq 0$ may be proved similarly. The strategy is similar to the one of the previous section, excepted that, for some lost directions, the second order may vanish and thus, we need to go to a higher order. We prove that the third order is sufficient.

2.5.1 Heuristic

We consider a control u of the form $u = \epsilon v + \epsilon^2 w + \epsilon^3 \nu$, then, formally $\psi = \Phi_1 + \epsilon\Psi + \epsilon^2\xi + \epsilon^3\zeta + o(\epsilon^3)$, where Ψ and ξ solve (2.19) and (2.20) and

$$\begin{cases} i\partial_t\zeta = -\partial_x^2\zeta - v(t)\mu(x)\xi - w(t)\mu(x)\Psi - \nu(t)\mu(x)\Phi_1, & (t,x) \in (0,T)\times(0,1), \\ \zeta(t,0) = \zeta(t,1) = 0, & t \in (0,T), \\ \zeta(0,x) = 0. & x \in (0,1). \end{cases} \tag{2.95}$$

We assume that $K \in \mathbb{N}^*$ satisfies $\langle \mu\varphi_1, \varphi_K\rangle = 0$ and that the quadratic form $Q^2_{K,T}$ vanishes on V_T (see Proposition 2.5). Then, one may prove that for any $v \in V_T$, $\langle \zeta(T), \Phi_K(T)\rangle = Q^3_{K,T}(v)$ where $Q^3_{K,T}$ is the cubic form (the index 3 is related to the fact that ζ is the third order of the power series expansion)

$$Q^3_{K,T}(v) := \int_0^T v(t_1)\int_0^{t_1} v(t_2)\int_0^{t_2} v(t_3) h^3_{K,T}(t_1,t_2,t_3)dt_3dt_2dt_1,$$

$$h^3_{K,T}(t_1,t_2,t_3) := -i\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} B_{j_1,j_2} e^{i[(\lambda_K-\lambda_{j_1})t_1+(\lambda_{j_1}-\lambda_{j_2})t_2+(\lambda_{j_2}-\lambda_1)t_3]},$$

$$B_{j_1,j_2} := \langle \mu\varphi_K, \varphi_{j_1}\rangle\langle \mu\varphi_{j_1}, \varphi_{j_2}\rangle\langle \mu\varphi_{j_2}, \varphi_1\rangle.$$

Proposition 2.14. *Let $\mu \in H^3((0,1),\mathbb{R})$ and $K \in \mathbb{N}^*$ be such that $\mu'(0) = \mu'(1) \neq 0$ and $\langle \mu\varphi_1, \varphi_K\rangle = 0$. Then,*

- *either, for every $N^* > 0$, there exists $n \geqslant N^*$ such that*

$$\langle \mu\varphi_K, \varphi_n\rangle\langle \mu\varphi_n, \varphi_1\rangle \neq 0 \tag{2.96}$$

- *or, for every $N^* > 0$, there exists $n_1, n_2 \geqslant N^*$ such that*

$$\langle \mu\varphi_K, \varphi_{n_1}\rangle\langle \mu\varphi_{n_1}, \varphi_{n_2}\rangle\langle \mu\varphi_{n_2}, \varphi_1\rangle \neq 0. \tag{2.97}$$

Proof of Proposition 2.14. The proof relies on the equality (2.63). If K is odd, then (2.96) holds with n odd and large enough. If K is even and (2.96) does not hold, then (2.97) holds with n_1 odd, n_2 even, both large enough.

□

The previous and next propositions show that any lost direction (at the first order) is recovered either at the second order, or at the third order.

Proposition 2.15. *Let $\mu \in H^3((0,1),\mathbb{R})$, $K \in \mathbb{N}^*$ be such that*

$$\langle \mu\varphi_K, \varphi_{n_1}\rangle\langle \mu\varphi_{n_1}, \varphi_{n_2}\rangle\langle \mu\varphi_{n_2}, \varphi_1\rangle \neq 0 \text{ for some } n_1, n_2 > K.$$

Then, $Q^3_{K,T} \neq 0$ on V_T, $\forall T > 0$.

Proof of Proposition 2.15. To simplify the notations, we write Q_T and h instead of $Q^3_{K,T}$ and h^3_K. Working by contradiction, we assume that $Q_T \equiv 0$ on V_T, for every $T < T^*$. Then $\nabla Q_T(v) \perp V_T$, for every $v \in V_T$ and $T < T^*$. Easy computations show that, for $v \in V_T$,

$$\nabla Q_T(v) : t_3 \mapsto \int_{(0,T)^2} v(t_1)v(t_2)[\tilde{h}(t_1,t_2,t_3) + \tilde{h}(t_1,t_3,t_2) + \tilde{h}(t_3,t_2,t_1)]dt_1dt_2$$

where $\tilde{h}(t_1,t_2,t_3) := h(t_1,t_2,t_3)1_{t_1>t_2>t_3}$. Let $v \in V_T$ with a compact support $(a,b) \subset (0,T)$. For $t_3 \in (0,a)$, we have

$$\nabla Q_T(v)(t_3) = \sum_{k_2=1}^{\infty} \alpha_{k_2}(v) e^{i(\lambda_{k_2}-\lambda_1)t_3} \tag{2.98}$$

where

$$\alpha_{k_2}(v) := -i \int_{(a,b)^2} v(t_1)v(t_2) \sum_{k_1=1}^{\infty} B_{k_1,k_2} e^{i[(\lambda_K-\lambda_{k_1})t_1+(\lambda_{k_1}-\lambda_{k_2})t_2]} dt_2 dt_1.$$

We know that $\nabla Q_T(v)$ belongs to $\mathrm{Adh}_{L^2(0,T)}(\mathrm{Span}\{e^{\pm i(\lambda_j-\lambda_1)t}\,;\, j \in \mathcal{J}\})$ because $\nabla Q_T(v) \perp V_T$. The uniqueness of the decomposition on a Riesz basis ensures that (2.98) holds for all $t_3 \in (0,T)$. For $t_3 \in (b,T)$, we have

$$\nabla Q_T(v)(t_3) = i \sum_{k_1=1}^{\infty} \langle \mu\varphi_K, \varphi_{k_1}\rangle Q^2_{k_1,T}(v) e^{i(\lambda_K-\lambda_{k_1})t_3},$$

where $Q^2_{k_1,T}$ is defined in (2.26)-(2.27). Thus,

$$i \sum_{k_1=1}^{\infty} \langle \mu\varphi_K, \varphi_{k_1}\rangle Q^2_{k_1,T}(v) e^{i(\lambda_K-\lambda_{k_1})t_3} = \sum_{k_2=1}^{\infty} \alpha_{k_2}(v) e^{i(\lambda_{k_2}-\lambda_1)t_3}, \ \forall b < t_3 < T.$$

Notice that the frequencies $(\lambda_K - \lambda_{k_1})$ in the left hand side are negative when $k_1 > K$, and the frequencies $(\lambda_{k_2} - \lambda_1)$ in the right hand side are non-negative. Thus,

$$\langle \mu\varphi_K, \varphi_{k_1}\rangle Q^2_{k_1,T}(v) = 0, \forall k_1 > K.$$

But $C^0_c(0,T) \cap V_T$ is dense in V_T, thus

$$\langle \mu\varphi_K, \varphi_{k_1}\rangle Q^2_{k_1,T} \equiv 0 \text{ on } V_T, \forall k_1 > K. \tag{2.99}$$

Let $n_1, n_2 > K$ be such that $\langle \mu\varphi_K, \varphi_{n_1}\rangle\langle \mu\varphi_{n_1}, \varphi_{n_2}\rangle\langle \mu\varphi_{n_2}, \varphi_1\rangle \neq 0$. In particular, $\langle \mu\varphi_K, \varphi_{n_1}\rangle \neq 0$ and $Q^2_{n_1,T} \neq 0$ on V_T, for every $T > 0$ by Proposition 2.5. This is in contradiction with (2.99).

□

Remark 2.4. Note that the third order may be necessary. For example, with $\mu(x) := x - \langle x\varphi_1, \varphi_K\rangle\varphi_K/\varphi_1$, where $K \in \mathbb{N}$ is even, we have $\langle \mu\varphi_1, \varphi_n\rangle\langle \mu\varphi_n, \varphi_K\rangle = 0$, $\forall n \in \mathbb{N}^*$, thus $Q^2_{K,T} \equiv 0$.

2.5.2 Reaching the missed directions at the second or third order

We here only detail the changes with respect to the proof of Section 2.4. Using, (2.63) and the fact that $\mu'(0) = \mu'(1) \neq 0$, it comes that $\{K \in \mathcal{N}_N \,;\, \langle \mu\varphi_1, \varphi_K\rangle = 0\}$ is finite. Thus, there exists $p \in \mathbb{N}^*$ and $K_1 < \cdots < K_p \in \mathbb{N}^*$ such that for any $j \in \{1, \ldots, p\}$, $\langle \mu\varphi_1, \varphi_{K_j}\rangle = 0$. The estimate (2.63) also implies the existence of $C > 0$ such that

$$|\langle \mu\varphi_1, \varphi_k\rangle| \geq \frac{C}{k^3}, \quad \forall k \in \mathcal{N}_N - \{K_1, \ldots, K_p\}.$$

For any $T > 0$, Propositions 2.14 and 2.15 imply that for any $j \in \{1, \ldots, p\}$, if $Q^2_{K_j,T}$ vanishes on V_T, then $Q^3_{K_j,T} \not\equiv 0$ on V_T.

Let $\mathcal{K}_{(2)} := \left\{ j \in \{1, \ldots, p\} \,;\, Q^2_{K_j,T} \not\equiv 0 \text{ on } V_T \right\}$ and $\mathcal{K}_{(3)} := \{1, \ldots, p\} - \mathcal{K}^{(2)}$. The spaces M^j and M are defined as in (2.65), (2.66). Let us define

$$M_{(2)} := \bigoplus_{j \in \mathcal{K}^{(2)}} M^j, \quad M_{(3)} := \bigoplus_{j \in \mathcal{K}^{(3)}} M^j.$$

Thus, $M = M_{(2)} \oplus M_{(3)}$. Proposition 2.8 holds with M replaced by $M_{(2)}$. The cubic form $Q^3_{K_j,T}$ satisfies $Q^3_{K_j,T}(-v) = -Q^3_{K_j,T}(v)$. Thus, one does not have to exploit the rotation phenomenon as in Proposition 2.8 and we can reach a basis with real non negative coefficients of $M^{(3)}$ on the third order in arbitrary time. More precisely, the following proposition holds.

Proposition 2.16. *Let $T > 0$. There exists a continuous map*

$$\left| \begin{array}{rccc} \tilde{\Lambda}_T : & M_{(3)} & \to & L^2((0,T), \mathbb{R})^2 \\ & z & \mapsto & (v, w, \nu) \end{array} \right.$$

such that, for every $z \in M_{(3)}$, the solutions Ψ, ξ and ζ of (2.19), (2.20) and (2.95) satisfy $\Psi(T) = 0$, $\xi(T) = 0$ and $\zeta(T) = z$.

Finally, let us define the control u_z by

$$u_z := \begin{cases} \sqrt{\|z_2\|} v_{z_2} + \|z_2\| w_{z_2} + \|z_3\|^{1/3} \tilde{v}_{z_3} + \|z_3\|^{2/3} \tilde{w}_{z_3} + \|z_3\| \tilde{\nu}_{z_3} & \text{on } (0, T_1), \\ \Gamma_{[T_1,T]}(\psi_z(T_1), \mathcal{P}_T[\psi_f]) & \text{on } (T_1, T). \end{cases}$$

where $z_2 + z_3 = z$ with $(z_2, z_3) \in M_{(2)} \times M_{(3)}$ and

$$(v_{z_2}, w_{z_2}) := \Lambda_{T_1}\left(\frac{e^{iA(T-T_1)} z_2}{\|z_2\|} \right), \quad (\tilde{v}_{z_3}, \tilde{w}_{z_3}, \tilde{\nu}_{z_3}) := \tilde{\Lambda}_{T_1}\left(\frac{e^{iA(T-T_1)} z_3}{\|z_3\|} \right).$$

Theorem 2.6 is then proved, as Theorem 2.4 in Section 2.4.5, using a fixed point argument.

Remark 2.5. As Proposition 2.8 is the only step requiring a minimal time, it has to be noticed that if $\mathcal{K}^{(2)} = \emptyset$, Theorem 2.6 holds in arbitrary time

2.6 A first step to the characterization of the minimal time

In this section, we focus on the system

$$\begin{cases} i\partial_t\psi(t,x) = -\partial_x^2\psi(t,x) - u(t)\mu(x)\psi(t,x), & (t,x) \in \mathbb{R}\times(0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in \mathbb{R}, \\ s'(t) = u(t), & t \in \mathbb{R}, \end{cases} \tag{2.100}$$

associated to the initial conditions

$$(\psi, s)(0) = (\varphi_1, 0). \tag{2.101}$$

We consider a dipolar moment $\mu \in H^3((0,1),\mathbb{R})$ such that $\langle \mu\varphi_1, \varphi_1\rangle = 0$ (for instance $\mu(x) = (x - 1/2)$). We use the notation $\mathcal{Q}_T$ instead of $\mathcal{Q}_{1,T}$ (see (2.33)-(2.34)), Q_T instead of $\widetilde{Q}^2_{1,T}$ (see (2.26)-(2.27)), $k(t,\tau)$ instead of $k_{1,T}(t,\tau)$ and the spaces

$$V_T^1 := \left\{ v \in L^2(0,T)\,;\, \int_0^T v(t)e^{i\omega_j t}dt = 0, \forall j \in \mathcal{J}\cup\{1\} \right\}$$

$$\mathcal{V}_T := \left\{ S \in L^2((0,T),\mathbb{R})\,;\, \int_0^T S(t)e^{i\omega_j t}dt = 0, \forall j \in \mathcal{J} \right\}$$

where $\mathcal{J}$ is defined by (2.24). We introduce the quantities

$$\tilde{T}^1_{min} := \sup\{T \geqslant 0\,;\, \mathcal{Q}_T \leqslant 0 \text{ on } \mathcal{V}_T\},$$

$$\tilde{T}^2_{min} := \inf\{T \geqslant 0\,;\, \exists S_\pm \in \mathcal{V}_T \cap H^1_0(0,T) \text{ such that } \mathcal{Q}_T(S_\pm) = \pm 1\}. \tag{2.102}$$

Lemma 2.2 ensures that $\tilde{T}^1_{min} > 0$ and the following proposition justifies the existence of $\tilde{T}^2_{min}$.

Proposition 2.17. *Let $\mu \in H^3((0,1),\mathbb{R})$ be such that $\langle \mu\varphi_1,\varphi_1\rangle = 0$. For every $T > 2/\pi$, there exists $S_\pm \in \mathcal{V}_T \cap H^1_0(0,T)$ such that $\mathcal{Q}_T(S_\pm) = \pm 1$; or, equivalently, there exists $v_\pm \in V^1_T$ such that $Q_T(v_\pm) = \pm 1$. Thus,*

$$0 < T_1^* < \tilde{T}^1_{min} \leqslant \tilde{T}^2_{min} \leqslant \frac{2}{\pi},$$

where T_1^* was defined in Lemma 2.2.

This proposition may be proved as Lemma 2.4. The goal of this section is the proof of the following theorem.

Theorem 2.9. *Let $\mu \in H^3((0,1),\mathbb{R})$ be such that*

$$\langle \mu\varphi_1, \varphi_1\rangle = 0 \quad \text{and} \quad \exists c > 0 \text{ such that } \frac{c}{k^3} \leqslant |\langle \mu\varphi_1, \varphi_k\rangle|, \forall k \in \mathcal{J}. \tag{2.103}$$

- *For every $T < \tilde{T}^1_{min}$, there exists $\epsilon > 0$ such that, for every $u \in L^2((0,T),\mathbb{R})$ with(2.12), the solution of (2.100)-(2.101) satisfies*

$$(\psi, s)(T) \neq \left([\sqrt{1-\delta^2} + i\delta]\Phi_1(T), 0\right), \quad \forall \delta > 0.$$

- *If, moreover $\mathcal{J} = \mathbb{N}^* - \{1\}$, then, for every $T > \tilde{T}^2_{min}$, the system (2.100) is controllable in $H^3_{(0)}(0,1) \times \mathbb{R}$, locally around the ground state $(\psi = \Phi_1, s \equiv 0)$, in time T, with controls $u \in L^2((0,T),\mathbb{R})$.*

In particular, when $\mathcal{J} = \mathbb{N}^* - \{1\}$, the minimal time T_{min} required for the local controllability satisfies $T_{min} \in [\tilde{T}^1_{min}, \tilde{T}^2_{min}]$.

Remark 2.6. The equality between $\tilde{T}^1_{min}$ and $\tilde{T}^2_{min}$ is an open problem, equivalent to the question addressed in the next paragraph.

Let P_T be the orthogonal projection from $L^2((0,T),\mathbb{R})$ to the closed subspace $\mathcal{V}_T$ and $\mathcal{K}_T$ be the compact self adjoint operator on $L^2((0,T),\mathbb{R})$ defined by

$$\mathcal{K}_T := P_T \left[t \mapsto \int_0^t k(t,\tau) S(\tau) d\tau \right].$$

Recall that A_1 is defined by (2.10). We know that

- for any $T < \tilde{T}^1_{min}$ all the eigenvalues of $\mathcal{K}_T$ are $< A_1$ (see the first statement of Theorem 2.9),
- for any $T > \tilde{T}^1_{min}$, the largest eigenvalue of $\mathcal{K}_T$ is $> A_1$. (by definition of $\tilde{T}^1_{min}$).

For $T > \tilde{T}^1_{min}$, does the associated eigenvector belong to $H^1_0((0,T),\mathbb{R})$?

The proof of the second statement of Theorem 2.9 may be done exactly as the proof of Theorem 2.4 in Section 2.4. Indeed, when $\mathcal{J} = \mathbb{N}^* - \{1\}$, then

- the vector space M of lost directions (at the first order) is $i\mathbb{R}\Phi_1(T)$,
- for any $T_1 \in (\tilde{T}^2_{min}, T)$, the controls $S_\pm \in \mathcal{V}_{T_1} \cap H^1_0(0,T_1)$ allow to reach the states $\pm i\Phi_1(T_1)$ with the second order term; moreover, $(i\Phi_1(T_1), -i\Phi_1(T_1))$ is an '$\mathbb{R}^+$-basis' of M.

Thus, in this section, we focus only on the proof of the first statement of Theorem 2.9, which is a direct consequence of the following result.

Theorem 2.10. *Let $\mu \in H^3((0,1),\mathbb{R})$ that satisfies (2.103). For every $T < \tilde{T}^1_{min}$, there exists $\epsilon > 0$ such that for every $s \in H^1((0,T),\mathbb{R})$ with $s(0) = 0$ and $\|\dot{s}\|_{L^2} < \epsilon$, the solution of the Cauchy problem (2.39) satisfies $\widetilde{\psi}(T) \neq (\sqrt{1-\delta^2} + i\delta)\Phi_1(T)$, $\forall \delta > 0$.*

In section 2.6.1, we state a preliminary result for the proof of Theorem 2.10, which is detailled in section 2.6.2.

2.6.1 Preliminaries

For $T > 0$ and $\eta > 0$, we introduce the sets

$$\mathcal{V}_{T,\eta} := \left\{ S \in L^2(0,T)\,; \left\| \left(\int_0^T S(t) e^{i\omega_j t} dt \right)_{j \in \mathcal{J}} \right\|_{l^2} \leqslant \eta \|S\|_{L^2(0,T)} \right\} \tag{2.104}$$

where $\mathcal{J}$ is defined in (2.24).

Proposition 2.18. *For every $T < \tilde{T}^1_{min}$, there exists $\lambda = \lambda(T), \eta = \eta(T) > 0$ such that*

$$\mathcal{Q}_T(S) \leqslant -\lambda(T) \|S\|^2_{L^2(0,T)}, \quad \forall S \in \mathcal{V}_T, \tag{2.105}$$

$$\mathcal{Q}_T(S) \leqslant -\frac{\lambda(T)}{2} \|S\|^2_{L^2(0,T)}, \quad \forall S \in \mathcal{V}_{T,\eta}. \tag{2.106}$$

This proposition may be proved with the formalism of Legendre quadratic forms (see [24]). For this article to be self contained, we propose an elementary proof in Appendix 2.C.

2.6.2 Proof of Theorem 2.10

Let $T < \tilde{T}^1_{min}$. We proceed as in the proof of Theorem 2.7. Working by contradiction, we assume that, for every $\epsilon > 0$, there exists $s_\epsilon \in H^1(0,T)$ with $s_\epsilon(0) = 0$ and $\|\dot{s}_\epsilon\|_{L^2} < \epsilon$ such that the solution $\widetilde{\psi}_\epsilon$ of (2.39) satisfies

$$\widetilde{\psi}_\epsilon(T) = (\sqrt{1-\delta_\epsilon^2} + i\delta_\epsilon)\Phi_1(T), \tag{2.107}$$

for some $\delta_\epsilon > 0$.

First step : For $\epsilon > 0$ small enough, $s_\epsilon \in \mathcal{V}_{T,\eta}$ (with $\eta = \eta(T)$ as in Proposition 2.18). Using (2.103), Proposition 2.3 and (2.107) we have

$$\begin{aligned} \left\| \left(\int_0^T s_\epsilon(t) e^{i\omega_j t} dt \right)_{j\in\mathcal{J}} \right\|_{l^2} &\leqslant C \left\| \left(\omega_j \langle \mu\varphi_1, \varphi_j \rangle \int_0^T s_\epsilon(t) e^{i\omega_j t} dt \right)_{j\in\mathcal{J}} \right\|_{h^1} \\ &\leqslant C \left\| \left(\langle \widetilde{\psi}_\epsilon(T), \Phi_j(T) \rangle \right)_{j\in\mathcal{J}} \right\|_{h^1} + \underset{\epsilon\to 0}{o}(\|s_\epsilon\|_{L^2}) \\ &= \underset{\epsilon\to 0}{o}(\|s_\epsilon\|_{L^2}) \end{aligned} \tag{2.108}$$

which gives the conclusion.

Second step : Conclusion. Using (2.42) with $K = 1$, the first step and (2.106) it comes that

$$\begin{aligned} 0 < \delta_\epsilon &= \operatorname{Im}\langle \widetilde{\psi}_\epsilon(T), \Phi_1(T) \rangle \\ &= \mathcal{Q}_T(s_\epsilon) + \underset{\epsilon\to 0}{o}(\|s_\epsilon\|^2_{L^2}) \\ &\leqslant -\frac{\lambda(T)}{2} \|s_\epsilon\|^2_{L^2} + \underset{\epsilon\to 0}{o}(\|s_\epsilon\|^2_{L^2}), \end{aligned}$$

which is impossible for ϵ small enough. This ends the proof of Theorem 2.10.

2.6.3 Comments about generalizations

Let us consider a situation in which the first order misses exactly $N \geq 2$ directions associated to the indexes $K_1, \dots, K_N$. Let $Q^2_{K_1,T}, \dots, Q^2_{K_N,T}$ be the associated complex-valued quadratic forms. A natural candidate for the minimal time T_{min} could be the minimal time $\tilde{T}_{min}$ for the image of

$$\left(Q^2_{K_1,T}, \dots, Q^2_{K_N,T}\right) : V_T \to \mathbb{C}^N$$

to cover $\mathbb{C}^N$. The positive controllability result in time $T > \tilde{T}_{min}$ could be proved with the technics of this article. The negative controllability result in time $T < \tilde{T}_{min}$ is more difficult. To transfer an impossible motion from the second order to the nonlinear system we need a coercivity property which is not obvious in this case.

2.7 Conclusion, open problems, perspectives

In Theorem 2.3, we have proposed a general context for the local controllability of the system (2.1) to require a positive minimal time. This statement extends Coron's previous result in [53] because:

- it does not use the variables (s, d) in the state,
- the control u has to be small in L^2 (not in L^∞),
- $\mu(x)$ is not necessarily $(x - 1/2)$.

The validity of the conclusion without the assumption $A_K \neq 0$ is an open problem.

In Theorem 2.4, we have proposed a sufficient condition for the system (2.1) to be controllable around the ground state in large time. This sufficient condition is compatible with the general context of Theorem 2.3, thus there exists a large class of functions μ for which local controllability holds in large time, but not in small time.

The existence of a positive minimal time for the controllability is closely related to a second order approximation of the solution. When a direction is not controllable neither at the first order, nor at the second one, then it is recovered at the third one, and no minimal time is required.

The characterization of the minimal time for the local controllability around the ground state is essentially an open problem. A first step has been done in this article, when only the first direction is lost.

In [44], Crépeau and Cerpa prove the local controllability of the KdV equation, with boundary control. When the length of the domain is critical, the linearized system is not controllable along a finite number of directions, but all of them are recovered at the second or third order. The existence of a positive minimal time, required for the local controllability is an open problem. The technics developed in this article may be helpful for this question.

2.A Trigonometric moment problems

In this article, we use several times the following result (see, for instance [16, Corollary 1 in Appendix B] for a proof).

Proposition 2.19. *Let $(\omega_k)_{k\in\mathbb{N}^*}$ be an increasing sequence of $[0,+\infty)$ such that $\omega_{k+1}-\omega_k \to +\infty$ when $k\to+\infty$ and $\omega_1=0$. Let $l^2_r(\mathbb{N}^*,\mathbb{C}) := \{d=(d_k)_{k\in\mathbb{N}^*}\in l^2(\mathbb{N}^*,\mathbb{C})\,;\, d_1\in\mathbb{R}\}$.*

- *For every $T>0$, there exists a continuous linear map*

$$\begin{array}{rccc} L_T: & l^2_r(\mathbb{N}^*,\mathbb{C}) & \to & L^2((0,T),\mathbb{R}) \\ & d & \mapsto & L_T(d) \end{array}$$

  such that, for every $d=(d_k)_{k\in\mathbb{N}^*}\in l^2(\mathbb{N}^*,\mathbb{C})$, the function $v:=L_T(d)$ solves

$$\int_0^T v(t)e^{i\omega_k t}dt = d_k, \quad \forall k\in\mathbb{N}^*.$$

- *For every $T>0$ there exists a constant $C=C(T)$ such that (Ingham inequality)*

$$\sum_{k=1}^{\infty}|a_k|^2 \leqslant C\int_0^T\left|\sum_{k=1}^{\infty}a_k e^{i\omega_k t}\right|^2 dt, \quad \forall (a_k)_{k\in\mathbb{N}^*}\in l^2(\mathbb{N}^*,\mathbb{C}).$$

2.B Proof of Lemma 2.3

This appendix is devoted to the proof of Lemma 2.3. It is a straightforward adaptation of [16, Lemma 1]. By definition,

$$F(t) = \sum_{k=1}^{\infty}\Big(\int_0^t \langle f(\tau),\varphi_k\rangle e^{i\lambda_k\tau}d\tau\Big)\varphi_k, \quad \text{in } L^2(0,1).$$

For almost every $\tau\in(0,T)$, $f(\tau)\in H^1$ and

$$\begin{aligned}\langle f(\tau),\varphi_k\rangle &= \sqrt{2}\int_0^1 f(\tau,x)\sin(k\pi x)dx \\ &= \frac{-\sqrt{2}}{k\pi}\big((-1)^k f(\tau,1)-f(\tau,0)\big)+\frac{\sqrt{2}}{k\pi}\int_0^1 f'(\tau,x)\cos(k\pi x)dx.\end{aligned}$$

Thus,

$$\begin{aligned}||F(t)||_{H^1_0} &= \Big|\Big|\int_0^t\langle f(\tau),\varphi_k\rangle e^{i\lambda_k\tau}d\tau\Big|\Big|_{h^1} \\ &\leqslant \frac{\sqrt{2}}{\pi}\Big(\Big|\Big|\int_0^t f(\tau,1)e^{i\lambda_k\tau}d\tau\Big|\Big|_{\ell^2}+\Big|\Big|\int_0^t f(\tau,0)e^{i\lambda_k\tau}d\tau\Big|\Big|_{\ell^2}\Big) \\ &\quad+\frac{1}{\pi}\Big|\Big|\int_0^t\langle f'(\tau),\sqrt{2}\cos(k\pi x)\rangle e^{i\lambda_k\tau}d\tau\Big|\Big|_{\ell^2}.\end{aligned}$$

As $(\sqrt{2}\cos(k\pi x))_{k\in\mathbb{N}^*}$ is orthonormal in $L^2(0,1)$,

$$\begin{aligned}\Big|\Big|\int_0^t \langle f'(\tau),\sqrt{2}\cos(k\pi x)\rangle e^{i\lambda_k\tau}d\tau\Big|\Big|_{\ell^2} &= \left(\sum_{k=1}^{\infty}\Big|\int_0^t \langle f'(\tau),\sqrt{2}\cos(k\pi x)\rangle e^{i\lambda_k\tau}d\tau\Big|^2\right)^{1/2}\\ &\leqslant \left(\sum_{k=1}^{\infty} t\int_0^t \left|\langle f'(\tau),\sqrt{2}\cos(k\pi x)\rangle\right|^2 d\tau\right)^{1/2}\\ &\leqslant \sqrt{t}\left(\int_0^t ||f'(\tau)||^2_{L^2}d\tau\right)^{1/2}\\ &\leqslant \sqrt{t}||f||_{L^2((0,t),H^1)}.\end{aligned}$$

Finally [16, Appendix B, Corollary 4] imply

$$\begin{aligned}||F(t)||_{H^1_0} &\leqslant \frac{\sqrt{2}C(t)}{\pi}\big(||f'(\cdot,1)||_{L^2(0,t)}+||f'(\cdot,0)||_{L^2(0,t)}\big)+\frac{\sqrt{2}}{\pi}||f||_{L^2((0,t),H^1)}\\ &\leqslant c_1(t)||f||_{L^2((0,t),H^1)}\end{aligned}$$

where $c_1(t)$ is bounded for t lying in bounded intervals. This proves that $F(t)\in H^1_0(0,1)$ for every $t\in[0,T]$ and that $t\mapsto F(t)\in H^1_0$ is continuous at $t=0$. The continuity at any $t\in[0,T]$ may be proved similarly.

2.C Proof of Proposition 2.18

This appendix is devoted to the proof of Proposition 2.18. The proof is divided in two steps. First, using a maximizing sequence we prove (2.105). Then, solving an adequate moment problem, we prove (2.106).

First step: Proof of (2.105). For $T\in(0,\tilde{T}^1_{min})$, we define the quantity $\lambda(T)\geqslant 0$ by

$$-\lambda(T) := \sup\{\mathcal{Q}_T(S)\,;\, S\in\mathcal{V}_T, ||S||_{L^2(0,T)}=1\}. \tag{2.109}$$

First, let us emphasize that, if $\lambda(T)\leqslant 0$, then, there exists $S\in\mathcal{V}_T$ such that $||S||_{L^2(0,T)}=1$ and $\mathcal{Q}_T(S)=\lambda(T)$ (consider a weak $L^2(0,T)$-limit, of a maximizing sequence and use the compactness of the operator $K:L^2(0,T)\to L^2(0,T)$ defined by $KS:t\mapsto\int_0^t S(\tau)k(t,\tau)d\tau$).

Let us assume that there exists $T\in(0,\tilde{T}^1_{min})$ such that $\lambda(T)=0$. Let $T_1\in(T,\tilde{T}^1_{min})$. Let $S_*\in\mathcal{V}_T$ such that $\|S_*\|_{L^2(0,T)}=1$ and $\mathcal{Q}_T(S_*)=0$. We extend S_* on (T,T_1) by zero. Then, $S_*\in\mathcal{V}_{T_1}$ and $\mathcal{Q}_{T_1}(S_*)=\max\{\mathcal{Q}_{T_1}(S)\,;\,S\in\mathcal{V}_T\}=0$ thus (Euler equation) $\nabla\mathcal{Q}_{T_1}(S_*)\perp\mathcal{V}_{T_1}$, i.e. there exists a unique sequence $(a_j)_{j\in\mathcal{J}-\{1\}}\in l^2$ such that

$$\nabla\mathcal{Q}_{T_1}S_*(t)=\sum_{j\in\mathcal{J}-\{1\}}a_je^{i\omega_jt}\text{ in }L^2(0,T_1).$$

However, we have

$$\nabla\mathcal{Q}_{T_1}(S_*)(t)=-A_1S_*(t)+\int_0^t S_*(\tau)k(t,\tau)d\tau,\forall t\in(0,T_1).$$

In particular, $\nabla \mathcal{Q}_{T_1}(S_*) \equiv 0$ on (T, T_1) thus (Ingham inequality, see Proposition 2.19) $a_j \equiv 0$. We have proved that

$$S_*(t) = \frac{1}{A_1} \int_0^t S_*(\tau) k(t,\tau) d\tau, \forall t \in (0,T).$$

Thus, $S_*(0) = 0$, $S_* \in H^1((0,T),\mathbb{R})$ and S_*' satisfies the same relation. Iterating this result, we get $S_*^{(n)}(0) = 0$ and $S_*^{(n)} \in \text{Ker}(-A_1 Id + K)$ for every $n \in \mathbb{N}$. But K is compact, so $\dim[\text{Ker}(-A_1 Id + K)] < +\infty$. Thus there exists $N \in \mathbb{N}^*$ and $a_0, \cdots, a_{N-1} \in \mathbb{R}$ such that

$$\begin{cases} S_*^{(N)} = a_0 S_* + a_1 S_*' + \cdots + a_{N-1} S_*^{(N-1)} \\ S_*(0) = 0, \cdots, S_*^{(N-1)}(0) = 0 \end{cases}$$

Therefore $S_* = 0$, which is a contradiction.

Second step: Proof of (2.106): Let $\eta > 0$ and $S \in \mathcal{V}_{T,\eta}$ with $\|S\|_{L^2} = 1$. Let $d := (d_k)_{k \geqslant 2}$ be defined by

$$d_k := \int_0^T S(t) e^{i\omega_k t} dt, \forall k \geqslant 2.$$

Then $\|d\|_{l^2} \leqslant \eta$. Let $\tilde{S} := L_T(d)$ and $S_0 := S - \tilde{S}$, where L_T is as in Proposition 2.19. Let $C(T) := \|L_T\|$. We have

$$\|\tilde{S}\|_{L^2} \leqslant C(T)\eta \quad \text{and} \quad 1 - C(T)\eta \leqslant \|S_0\|_{L^2} \leqslant 1 + C(T)\eta. \tag{2.110}$$

Using the first step and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\mathcal{Q}_T(S) = \mathcal{Q}_T(S_0 + \tilde{S}) \\
&= \mathcal{Q}_T(S_0) + \mathcal{Q}_T(\tilde{S}) + \int_0^T S_0(t) \int_0^t \tilde{S}(s) k(t,s) ds dt + \int_0^T \tilde{S}(t) \int_0^t S_0(s) k(t,s) ds dt \\
&\leqslant -\lambda(T) \|S_0\|_{L^2}^2 + \frac{T}{2} \|k\|_\infty \|\tilde{S}\|_{L^2}^2 + 2T \|k\|_\infty \|S_0\|_{L^2} \|\tilde{S}\|_{L^2} \\
&\leqslant -\lambda(T)[1 - C(T)\eta]^2 + \frac{T}{2} \|k\|_\infty C(T)^2 \eta^2 + 2T \|k\|_\infty [1 + C(T)\eta] C(T)\eta.
\end{aligned}$$

Thus, for η small enough, we get $\mathcal{Q}_T(S) \leqslant -\dfrac{\lambda(T)}{2} < 0$.

Chapitre 3

Contrôlabilité simultanée de deux et trois équations

Ce chapitre est inspiré de l'article [109] publié dans le journal *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire.*

Contents

3.1 Introduction

3.1.1 Main results

We consider a quantum particle in a one dimensional infinite square potential well coupled to an external laser field. The evolution of the wave function ψ is given by the following Schrödinger equation

$$\begin{cases} i\partial_t\psi = -\partial_{xx}^2\psi - u(t)\mu(x)\psi, & (t,x)\in(0,T)\times(0,1), \\ \psi(t,0)=\psi(t,1)=0, & t\in(0,T), \end{cases} \tag{3.1}$$

where $\mu \in H^3((0,1),\mathbb{R})$ is the dipolar moment and $u : t \in (0,T) \mapsto \mathbb{R}$ is the amplitude of the laser field. This is a bilinear control system in which the state ψ lives on a sphere of $L^2((0,1),\mathbb{C})$. Similar systems have been studied by various authors (see e.g. [16, 103, 122, 46]).
We are interested in simultaneous controllability of system (3.1) and thus we consider, for $N \in \mathbb{N}^*$, the system

$$\begin{cases} i\partial_t\psi^j = -\partial_{xx}^2\psi^j - u(t)\mu(x)\psi^j, & (t,x)\in(0,T)\times(0,1),\ j\in\{1,\dots,N\}, \\ \psi^j(t,0)=\psi^j(t,1)=0, & t\in(0,T),\ j\in\{1,\dots,N\}. \end{cases} \tag{3.2}$$

It is a simplified model for the evolution of N identical and independent particles submitted to a single external laser field where entanglement has been neglected. This can be seen as a first step towards more sophisticated models.
Before going into details, let us set some notations. In this article, $\langle\cdot,\cdot\rangle$ denotes the usual scalar product on $L^2((0,1),\mathbb{C})$ i.e.

$$\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}\mathrm{d}x$$

and $\mathcal{S}$ denotes the unit sphere of $L^2((0,1),\mathbb{C})$. We consider the operator A defined by

$$\mathcal{D}(A) := H^2\cap H^1_0((0,1),\mathbb{C}), \quad A\varphi := -\partial^2_{xx}\varphi.$$

Its eigenvalues and eigenvectors are

$$\lambda_k := (k\pi)^2, \quad \varphi_k(x) := \sqrt{2}\sin(k\pi x), \quad \forall k\in\mathbb{N}^*.$$

The family $(\varphi_k)_{k\in\mathbb{N}^*}$ is an Hilbert basis of $L^2((0,1),\mathbb{C})$. The eigenstates are defined by

$$\Phi_k(t,x) := \varphi_k(x)e^{-i\lambda_k t}, \quad (t,x)\in\mathbb{R}^+\times(0,1),\ k\in\mathbb{N}^*.$$

Any N-tuple of eigenstates is solution of system (3.2) with control $u\equiv 0$. Finally, we define the spaces

$$H^s_{(0)}((0,1),\mathbb{C}) := \mathcal{D}(A^{s/2}),\ \forall s>0,$$

endowed with the norm

$$||\cdot||_{H^s_{(0)}} := \left(\sum_{k=1}^{+\infty}|k^s\langle\,\cdot\,,\varphi_k\rangle|^2\right)^{1/2}$$

and

$$h^s(\mathbb{N}^*,\mathbb{C}) := \left\{ a = (a_k)_{k\in\mathbb{N}^*} \in \mathbb{C}^{\mathbb{N}^*} \; ; \; \sum_{k=1}^{+\infty} |k^s a_k|^2 < +\infty \right\}$$

endowed with the norm

$$||a||_{h^s} := \left(\sum_{k=1}^{+\infty} |k^s a_k|^2 \right)^{1/2}.$$

Our goal is to control simultaneously the particles modelled by (3.2) with initial conditions

$$\psi^j(0,x) = \varphi_j(x), \quad x \in (0,1), \, j \in \{1,\dots,N\}, \tag{3.3}$$

locally around $\big(\Phi_1,\dots,\Phi_N\big)$ using a single control.

Remark 3.1. Before getting to controllability results, it has to be noticed that for any control $v \in L^2((0,T),\mathbb{R})$, the associated solution of (3.2) satisfies

$$\langle \psi^j(t), \psi^k(t)\rangle = \langle \psi^j(0), \psi^k(0)\rangle, \quad \forall t \in [0,T].$$

This invariant has to be taken into account since it imposes compatibility conditions between targets and initial conditions.

The case $N = 1$ of a single equation was studied, in this setting, in [16, Theorem 1] by Beauchard and Laurent. They proved exact controllability, in $H^3_{(0)}$, in arbitrary time, locally around Φ_1. Their proof relies on the linear test, the inverse mapping theorem and a regularizing effect. We prove that this result cannot be extended to the case $N = 2$.
In the spirit of [16], we assume the following hypothesis.

Hypothesis 3.1. The dipolar moment $\mu \in H^3((0,1),\mathbb{R})$ is such that there exists $c > 0$ satisfying

$$|\langle \mu\varphi_j, \varphi_k\rangle| \geq \frac{c}{k^3}, \quad \forall k \in \mathbb{N}^*, \, \forall j \in \{1,\dots,N\}.$$

Remark 3.2. In the same way as in [16, Proposition 16], one may prove that Hypothesis 3.1 holds generically in $H^3((0,1),\mathbb{R})$.

Using [16, Theorem 1], Hypothesis 3.1 implies that the j^{th} equation of system (3.2) is locally controllable in $H^3_{(0)}$ around Φ_j.

Hypothesis 3.2. The dipolar moment $\mu \in H^3((0,1),\mathbb{R})$ is such that

$$\mathcal{A} := \langle \mu\varphi_1, \varphi_1\rangle \langle (\mu')^2\varphi_2, \varphi_2\rangle - \langle \mu\varphi_2, \varphi_2\rangle \langle (\mu')^2\varphi_1, \varphi_1\rangle \neq 0.$$

Remark 3.3. For example, $\mu(x) := x^3$ satisfies both Hypothesis 3.1 and 3.2. Unfortunately, the case $\mu(x) := x$ studied in [122] does not satisfy these hypotheses. But, as in [16, Proposition 16], one may prove that Hypotheses 3.1 and 3.2 hold simultaneously generically in $H^3((0,1),\mathbb{R})$.

Remark 3.4. Hypothesis 3.2 implies that there exists $j \in \{1,2\}$ such that $\langle \mu\varphi_j, \varphi_j\rangle \neq 0$. Without loss of generality, when Hypothesis 3.2 is assumed to hold, one should consider that $\langle \mu\varphi_1, \varphi_1\rangle \neq 0$.

Theorem 3.1. *Let $N = 2$ and $\mu \in H^3((0,1),\mathbb{R})$ be such that Hypothesis 3.2 hold. Let $\alpha \in \{-1,1\}$ be defined by $\alpha := sign(\mathcal{A}\langle\mu\varphi_1,\varphi_1\rangle)$. There exists $T_* > 0$ and $\varepsilon > 0$ such that for any $T < T_*$, for every $u \in L^2((0,T),\mathbb{R})$ with $||u||_{L^2(0,T)} < \varepsilon$, the solution of system (3.2)-(3.3) satisfies*

$$\left(\psi^1(T),\psi^2(T)\right) \neq \Big(\Phi_1(T),\Big(\sqrt{1-\delta^2}+i\alpha\delta\Big)\,\Phi_2(T)\Big),\quad \forall\delta>0.$$

Thus, under Hypothesis 3.2, simultaneous controllability does not hold for (ψ^1,ψ^2) around (Φ_1,Φ_2) in small time with small controls. The smallness assumption on the control is in L^2 norm. This prevents from extending [16, Theorem 1] to the case $N \geq 2$. Notice that the proposed target that cannot be reached satisfies the compatibility conditions of Remark 3.1. However, when modelling a quantum particle, the global phase is physically meaningless. Thus for any $\theta \in \mathbb{R}$ and $\psi^1,\psi^2 \in L^2((0,1),\mathbb{C})$, the states $e^{i\theta}(\psi^1,\psi^2)$ and (ψ^1,ψ^2) are physically equivalent. Working up to a global phase, we prove the following theorem.

Theorem 3.2. *Let $N = 2$. Let $T > 0$. Let $\mu \in H^3((0,1),\mathbb{R})$ satisfy Hypothesis 3.1 and $\langle\mu\varphi_1,\varphi_1\rangle \neq \langle\mu\varphi_2,\varphi_2\rangle$. There exists $\theta\in\mathbb{R}$, $\varepsilon_0 > 0$ and a C^1 map*

$$\Gamma : \mathcal{O}_{\varepsilon_0} \to L^2((0,T),\mathbb{R})$$

where

$$\mathcal{O}_{\varepsilon_0} := \Big\{\left(\psi_f^1,\psi_f^2\right) \in H^3_{(0)}((0,1),\mathbb{C})^2\,;\,\langle\psi_f^j,\psi_f^k\rangle = \delta_{j=k} \text{ and } \sum_{j=1}^{2}||\psi_f^j - e^{i\theta}\Phi_j(T)||_{H^3_{(0)}} < \varepsilon_0\Big\},$$

such that for any $\left(\psi_f^1,\psi_f^2\right) \in \mathcal{O}_{\varepsilon_0}$, the solution of system (3.2) with initial condition (3.3) and control $u = \Gamma\left(\psi_f^1,\psi_f^2\right)$ satisfies

$$(\psi^1(T),\psi^2(T)) = (\psi_f^1,\psi_f^2).$$

Remark 3.5. Notice that, using Remark 3.1, the condition $\langle\psi_f^j,\psi_f^k\rangle = \delta_{j=k}$ is not restrictive. Indeed, as $\psi^j(0) = \varphi_j$, we can only reach targets satisfying such an orthonormality condition.

Remark 3.6. The same theorem holds with initial conditions (ψ_0^1,ψ_0^2) close enough to (φ_1,φ_2) in $H^3_{(0)}$ satisfying the constraints $\langle\psi_0^1,\psi_0^2\rangle = \langle\psi_f^1,\psi_f^2\rangle$ (see Remark 3.14 in Section 3.4.2).

Working in time large enough we can drop the global phase and prove the following theorem.

Theorem 3.3. *Let $N = 2$. Let $\mu \in H^3((0,1),\mathbb{R})$ satisfy Hypothesis 3.1 and $4\langle\mu\varphi_1,\varphi_1\rangle - \langle\mu\varphi_2,\varphi_2\rangle \neq 0$. There exists $T^* > 0$ such that, for any $T \geq 0$, there exists $\varepsilon_0 > 0$ and a C^1 map*

$$\Gamma : \mathcal{O}_{\varepsilon_0,T} \to L^2((0,T^*+T),\mathbb{R})$$

where

$$\mathcal{O}_{\varepsilon_0,T} := \Big\{\left(\psi_f^1,\psi_f^2\right) \in H^3_{(0)}((0,1),\mathbb{C})^2\,;\,\langle\psi_f^j,\psi_f^k\rangle = \delta_{j=k} \text{ and } \sum_{j=1}^{2}||\psi_f^j - \Phi_j(T)||_{H^3_{(0)}} < \varepsilon_0\Big\},$$

such that for any $\big(\psi^1_f,\psi^2_f\big)\in\mathcal{O}_{\varepsilon_0,T}$*, the solution of system (3.2) with initial condition (3.3) and control* $u=\Gamma\big(\psi^1_f,\psi^2_f\big)$ *satisfies*

$$\big(\psi^1(T^*+T),\psi^2(T^*+T)\big)=\big(\psi^1_f,\psi^2_f\big).$$

Remark 3.7. Remark 3.6 is still valid in this case.

We now turn to the case $N=3$. We prove that under an extra generic assumption, Theorem 3.2 cannot be extended to three particles. Assume the following hypothesis.

Hypothesis 3.3. The dipolar moment $\mu\in H^3((0,1),\mathbb{R})$ is such that

$$\begin{aligned}\mathcal{B}:=&\big(\langle\mu\varphi_3,\varphi_3\rangle-\langle\mu\varphi_2,\varphi_2\rangle\big)\langle(\mu')^2\varphi_1,\varphi_1\rangle\\&+\big(\langle\mu\varphi_1,\varphi_1\rangle-\langle\mu\varphi_3,\varphi_3\rangle\big)\langle(\mu')^2\varphi_2,\varphi_2\rangle\\&+\big(\langle\mu\varphi_2,\varphi_2\rangle-\langle\mu\varphi_1,\varphi_1\rangle\big)\langle(\mu')^2\varphi_3,\varphi_3\rangle\neq 0.\end{aligned}$$

Remark 3.8. Hypothesis 3.3 implies that there exist $j,k\in\{1,2,3\}$ such that $\langle\mu\varphi_j,\varphi_j\rangle\neq\langle\mu\varphi_k,\varphi_k\rangle$. Without loss of generality, when Hypothesis 3.3 is assumed to hold, one should consider that $\langle\mu\varphi_1,\varphi_1\rangle\neq\langle\mu\varphi_2,\varphi_2\rangle$.

Remark 3.9. Again, one gets that Hypotheses 3.1 and 3.3 hold simultaneously generically in $H^3((0,1),\mathbb{R})$.

We prove the following theorem.

Theorem 3.4. *Let* $N=3$ *and* $\mu\in H^3((0,1),\mathbb{R})$ *be such that Hypothesis 3.3 hold. Let* $\beta\in\{-1,1\}$ *be defined by* $\beta:=sign\big(\mathcal{B}(\langle\mu\varphi_2,\varphi_2\rangle-\langle\mu\varphi_1,\varphi_1\rangle)\big)$*. There exists* $T_*>0$ *and* $\varepsilon>0$ *such that, for any* $T<T_*$*, for every* $u\in L^2((0,T),\mathbb{R})$ *with* $||u||_{L^2(0,T)}<\varepsilon$*, the solution of system (3.2)-(3.3) satisfies*

$$\big(\psi^1(T),\psi^2(T),\psi^3(T)\big)\neq e^{i\nu}\left(\Phi_1(T),\Phi_2(T),\left(\sqrt{1-\delta^2}+i\beta\delta\right)\Phi_3(T)\right),\quad\forall\delta>0,\forall\nu\in\mathbb{R}.$$

Thus, in small time, local exact controllability with small controls does not hold for $N\geq 3$, even up to a global phase. The next statement ensures that it holds up to a global phase and a global delay.

Theorem 3.5. *Let* $N=3$*. Let* $\mu\in H^3((0,1),\mathbb{R})$ *satisfy Hypothesis 3.1 and* $5\langle\mu\varphi_1,\varphi_1\rangle-8\langle\mu\varphi_2,\varphi_2\rangle+3\langle\mu\varphi_3,\varphi_3\rangle\neq 0$*. There exists* $\theta\in\mathbb{R}$*,* $T^*>0$ *such that, for any* $T\geq 0$*, there exists* $\varepsilon_0>0$ *and a* C^1 *map*

$$\Gamma:\mathcal{O}_{\varepsilon_0,T}\to L^2((0,T^*+T),\mathbb{R})$$

where

$$\mathcal{O}_{\varepsilon_0,T}:=\Big\{\big(\psi^1_f,\psi^2_f,\psi^3_f\big)\in H^3_{(0)}((0,1),\mathbb{C})^3\,;\,\langle\psi^j_f,\psi^k_f\rangle=\delta_{j=k}\text{ and }\sum_{j=1}^{3}||\psi^j_f-e^{i\theta}\Phi_j(T)||_{H^3_{(0)}}<\varepsilon_0\Big\},$$

such that for any $\big(\psi^1_f,\psi^2_f,\psi^3_f\big)\in\mathcal{O}_{\varepsilon_0,T}$*, the solution of system (3.2) with initial condition (3.3) and control* $u=\Gamma\big(\psi^1_f,\psi^2_f,\psi^3_f\big)$ *satisfies*

$$\big(\psi^1(T^*+T),\psi^2(T^*+T),\psi^3(T^*+T)\big)=\big(\psi^1_f,\psi^2_f,\psi^3_f\big).$$

Remark 3.10. Remark 3.6 is still valid in this case.

3.1.2 Heuristic

Contrarily to the case $N=1$, the linearized system around a N-tuple of eigenstates is not controllable when $N \geq 2$. Let us consider, for $N=2$, the linearization of system (3.2) around (Φ_1, Φ_2)

$$\begin{cases} i\partial_t \Psi^j = -\partial^2_{xx}\Psi^j - v(t)\mu(x)\Phi_j, & (t,x) \in (0,T)\times(0,1),\ j\in\{1,2\}, \\ \Psi^j(t,0) = \Psi^j(t,1) = 0, & t \in (0,T), \\ \Psi^j(0,x) = 0, & x \in (0,1). \end{cases} \tag{3.4}$$

For $j=1,2$, straightforward computations lead to

$$\Psi^j(T) = i\sum_{k=1}^{+\infty}\langle \mu\varphi_j, \varphi_k\rangle \int_0^T v(t)e^{i(\lambda_k-\lambda_j)t}\mathrm{d}t\Phi_k(T). \tag{3.5}$$

Thus, thanks to Hypothesis 3.1, we could, by solving a suitable moment problem, control any direction $\langle \Psi^j(T), \Phi_k(T)\rangle$, for $k \geq 2$ (with a slight abuse of notation for the direction Φ_k of the j^{th} equation). Straightforward computations using (3.5) lead to

$$\langle \Psi^1(T), \Phi_2(T)\rangle + \overline{\langle \Psi^2(T), \Phi_1(T)\rangle} = 0.$$

This comes from the linearization of the invariant (see Remark 3.1)

$$\langle \psi^1(t), \psi^2(t)\rangle = \langle \psi^1_0, \psi^2_0\rangle, \quad \forall t \in (0,T),$$

and can be overcome (see Subsection 3.4.2). However, (3.5) also implies that

$$\langle \mu\varphi_2, \varphi_2\rangle\langle \Psi^1(T), \Phi_1(T)\rangle = \langle \mu\varphi_1, \varphi_1\rangle\langle \Psi^2(T), \Phi_2(T)\rangle,$$

for any $v \in L^2((0,T),\mathbb{R})$. This is a strong obstacle to controllability and leads to Theorem 3.1 (see Section 3.6).
In this situation, where a direction is lost at the first order, one can try to recover it at the second order. This strategy was used for example by Cerpa and Crépeau in [44] on a Korteweg De Vries equation and adapted on the considered bilinear Schrödinger equation (3.1) by Beauchard and the author in [18]. Let, for $j \in \{1,2\}$,

$$\begin{cases} i\partial_t \xi^j = -\partial^2_{xx}\xi^j - v(t)\mu(x)\Psi^j - w(t)\mu(x)\Phi_j, & (t,x) \in (0,T)\times(0,1), \\ \xi^j(t,0) = \xi^j(t,1) = 0, & t \in (0,T), \\ \xi^j(0,x) = 0, & x \in (0,1). \end{cases}$$

The main idea of this strategy is to exploit a rotation phenomenon when the control is turned off. However, as proved in [18, Lemma 4], there is no rotation phenomenon on the diagonal directions $\langle \xi^j(T), \Phi_j(T)\rangle$ and this power series expansion strategy cannot be applied to this situation.
Thus, the local exact controllability results in this article are proved using Coron's return method. This strategy, detailed in [54, Chapter 6], relies on finding a reference trajectory of the non linear control system with suitable origin and final positions such that the linearized

system around this reference trajectory is controllable. Then, the inverse mapping theorem allows to prove local exact controllability.
As the Schrödinger equation is not time reversible, the design of the reference trajectory $(\psi^1_{ref}, \dots, \psi^N_{ref}, u_{ref})$ is not straightforward. The reference control u_{ref} is designed in two steps. The first step is to impose restrictive conditions on u_{ref} on an arbitrary time interval $(0, \varepsilon)$ in order to ensure the controllability of the linearized system. Then, u_{ref} is designed on (ε, T^*) such that the reference trajectory at the final time coincides with the target. For example, to prove Theorem 3.5, the reference trajectory is designed such that

$$\big(\psi^1_{ref}(T^*), \psi^2_{ref}(T^*), \psi^3_{ref}(T^*)\big) = e^{i\theta}\big(\varphi_1, \varphi_2, \varphi_3\big). \tag{3.6}$$

3.1.3 Structure of the article

This article is organized as follows. We recall, in Section 3.2, well posedness results.
To emphasize the ideas developed in this article, we start by proving Theorem 3.5. Section 3.3 is devoted to the construction of the reference trajectory. In Subsection 3.4.1, we prove the controllability of the linearized system around the reference trajectory. In Subsection 3.4.2, we conclude the return method thanks to an inverse mapping argument.
In Section 3.5, we adapt the construction of the reference trajectory for two equations leading to Theorems 3.2 and 3.3.
Finally, Section 3.6 is devoted to non controllability results and the proofs of Theorems 3.1 and 3.4.

3.1.4 A review of previous results

Let us recall some previous results about the controllability of Schrödinger equations. In [5], Ball, Marsden and Slemrod proved a negative result for infinite dimensional bilinear control systems. The adaptation of this result to Schrödinger equations, by Turinici [134], proves that the reachable set with L^2 controls has an empty interior in $\mathcal{S} \cap H^2_{(0)}((0,1), \mathbb{C})$. Although this is a negative result it does not prevent controllability in more regular spaces.

Actually, in [10], Beauchard proved local exact controllability in H^7 using Nash-Moser theorem for a one dimensional model. The proof of this result was simplified, by Beauchard and Laurent in [16], by exhibiting a regularizing effect allowing to apply the classical inverse mapping theorem. In [15], Beauchard and Coron also proved exact controllability between eigenstates for a particle in a moving potential well.

Using stabilization techniques and Lyapunov functions, Nersesyan proved in [112] that Beauchard and Laurent's result holds globally in $H^{3+\varepsilon}$. Other stabilization results on similar models were obtained in [17, 103, 111, 19, 108] by Mirrahimi, Beauchard, Nersesyan and the author.

Unlike exact controllability, approximate controllability results have been obtained for Schrödinger equations on multidimensional domains. In [45], Chambrion, Mason, Sigalotti and Boscain proved approximate controllability in L^2, thanks to geometric technics on the Galerkin approximation both for the wave function and density matrices. These results were extended to stronger norms in [30] by Boussaid, Caponigro and Chambrion. Approximate controllability in more regular spaces (containing H^3) were obtained by Nersesyan and Nersisyan [114] using exact controllability in infinite time. Approximate controllability has also been obtained by Ervedoza and Puel in [70] on a model of trapped ions.

Simultaneous exact controllability of quantum particles has been obtained on a finite dimensional model in [136] by Turinici and Rabitz. Their model uses specific orientation of the molecules and their proof relies on iterated Lie brackets. In addition to the results of [45], simultaneous approximate controllability was also studied in [46] by Chambrion and Sigalotti. They used controllability of the Galerkin approximations for a model of different particles with the same control operator and a model of identical particles with different control operators. These simultaneous approximate controllability results are valid regardless of the number of particles considered.

Finally, let us give some details about the return method. This idea of designing a reference trajectory such that the linearized system is controllable was developed by Coron in [49] for a stabilization problem. It was then successfully used to prove exact controllability for various systems : Euler equations in [50, 74, 76] by Coron and Glass, Navier-Stokes equations in [51, 73, 48, 58] by Coron, Fursikov, Imanuvilov, Chapouly and Guerrero, Bürgers equations in [85, 78, 47] by Horsin, Glass, Guerrero and Chapouly and many other models such as [52, 75, 77, 59]. This method was also used for a bilinear Schrödinger equation in [10] by Beauchard.

The question of simultaneous exact controllability for linear PDE is already present in the book [98] by Lions. He considered the case of two wave equations with different boundary controls. This was later extended to other systems by Avdonin, Tucsnak, Moran and Kapitonov in [4, 3, 88].

To conclude, the question of impossibility of certain motions in small time, at stake in this article, for bilinear Schrödinger equations was studied in [53, 18] by Coron, Beauchard and the author.

3.2 Well posedness

First, we recall the well posedness of the considered Schrödinger equation with a source term which proof is in [16, Proposition 2]. Consider

$$\begin{cases} i\partial_t\psi(t,x) = -\partial^2_{xx}\psi(t,x) - u(t)\mu(x)\psi(t,x) - f(t,x), & (t,x)\in(0,T)\times(0,1), \\ \psi(t,0)=\psi(t,1)=0, & t\in(0,T), \\ \psi(0,x)=\psi_0(x), & x\in(0,1). \end{cases} \tag{3.7}$$

Proposition 3.1. *Let $\mu\in H^3((0,1),\mathbb{R})$, $T>0$, $\psi_0\in H^3_{(0)}(0,1)$, $u\in L^2((0,T),\mathbb{R})$ and $f\in L^2((0,T),H^3\cap H^1_0)$. There exists a unique weak solution of (3.7), i.e. a function $\psi\in C^0([0,T],H^3_{(0)})$ such that the following equality holds in $H^3_{(0)}((0,1),\mathbb{C})$ for every $t\in[0,T]$,*

$$\psi(t) = e^{-iAt}\psi_0 + i\int_0^t e^{-iA(t-\tau)}[u(\tau)\mu\psi(\tau)+f(\tau)]d\tau.$$

Moreover, for every $R>0$, there exists $C=C(T,\mu,R)>0$ such that, if $\|u\|_{L^2(0,T)}<R$, then this weak solution satisfies

$$\|\psi\|_{C^0([0,T],H^3_{(0)})} \leqslant C\Big(\|\psi_0\|_{H^3_{(0)}} + \|f\|_{L^2((0,T),H^3\cap H^1_0)}\Big).$$

If $f\equiv 0$, then

$$\|\psi(t)\|_{L^2(0,1)} = \|\psi_0\|_{L^2(0,1)}, \quad \forall t\in[0,T].$$

3.3 Construction of the reference trajectory for three equations

The goal of this section is the design of the following family of reference trajectories to prove Theorem 3.5.

Theorem 3.6. *Let $N = 3$. Let $\mu \in H^3((0,1),\mathbb{R})$ satisfy Hypothesis 3.1 and $5\langle \mu\varphi_1, \varphi_1\rangle - 8\langle \mu\varphi_2, \varphi_2\rangle + 3\langle \mu\varphi_3, \varphi_3\rangle \neq 0$. Let $T_1 > 0$ be arbitrary, $\varepsilon \in (0, T_1)$ and $\varepsilon_1 \in (\frac{\varepsilon}{2}, \varepsilon)$. There exist $\overline{\eta} > 0$, $C > 0$ such that for every $\eta \in (0, \overline{\eta})$, there exist $T^\eta > T_1$, $\theta^\eta \in \mathbb{R}$ and $u^\eta_{ref} \in L^2((0,T^\eta),\mathbb{R})$ with*

$$||u^\eta_{ref}||_{L^2(0,T^\eta)} \leq C\eta \tag{3.8}$$

such that the associated solution $\left(\psi^{1,\eta}_{ref}, \psi^{2,\eta}_{ref}, \psi^{3,\eta}_{ref}\right)$ of (3.2)-(3.3) satisfies

$$\begin{aligned}
&\langle \mu\psi^{1,\eta}_{ref}(\varepsilon_1), \psi^{1,\eta}_{ref}(\varepsilon_1)\rangle = \langle \mu\varphi_1, \varphi_1\rangle + \eta,\\
&\langle \mu\psi^{2,\eta}_{ref}(\varepsilon_1), \psi^{2,\eta}_{ref}(\varepsilon_1)\rangle = \langle \mu\varphi_2, \varphi_2\rangle,\\
&\langle \mu\psi^{3,\eta}_{ref}(\varepsilon_1), \psi^{3,\eta}_{ref}(\varepsilon_1)\rangle = \langle \mu\varphi_3, \varphi_3\rangle,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\langle \mu\psi^{1,\eta}_{ref}(\varepsilon), \psi^{1,\eta}_{ref}(\varepsilon)\rangle = \langle \mu\varphi_1, \varphi_1\rangle,\\
&\langle \mu\psi^{2,\eta}_{ref}(\varepsilon), \psi^{2,\eta}_{ref}(\varepsilon)\rangle = \langle \mu\varphi_2, \varphi_2\rangle + \eta,\\
&\langle \mu\psi^{3,\eta}_{ref}(\varepsilon), \psi^{3,\eta}_{ref}(\varepsilon)\rangle = \langle \mu\varphi_3, \varphi_3\rangle,
\end{aligned} \tag{3.10}$$

and

$$\left(\psi^{1,\eta}_{ref}(T^\eta), \psi^{2,\eta}_{ref}(T^\eta), \psi^{3,\eta}_{ref}(T^\eta)\right) = e^{i\theta^\eta}\left(\varphi_1, \varphi_2, \varphi_3\right). \tag{3.11}$$

Remark 3.11. For any $T \geq 0$, u^η_{ref} is extended by zero on $(T^\eta, T^\eta + T)$. Thus, there exists $C > 0$ such that, $||u^\eta_{ref}||_{L^2(0,T^\eta+T)} \leq C\eta$, (3.9), (3.10) are satisfied and

$$\left(\psi^{1,\eta}_{ref}(T^\eta + T), \psi^{2,\eta}_{ref}(T^\eta + T), \psi^{3,\eta}_{ref}(T^\eta + T)\right) = e^{i\theta^\eta}\left(\Phi_1(T), \Phi_2(T), \Phi_3(T)\right).$$

Remark 3.12. The choice of a parameter η sufficiently small together with conditions (3.9) and (3.10) will be used in Section 3.4.1 to prove the controllability of the linearized system around the reference trajectory. The control u^η_{ref} will be designed on $(0, T_1)$ and extended by zero on (T_1, T^η).

The proof of Theorem 3.6 is divided in two steps : the construction of u^η_{ref} on $(0, \varepsilon)$ to prove (3.9) and (3.10) and then, the construction on (ε, T_1) to prove (3.11). This is what is detailed in the next subsections.

3.3.1 Construction on $(0, \varepsilon)$

Let $u^\eta_{ref} \equiv 0$ on $[0, \frac{\varepsilon}{2})$. We prove the following proposition.

Proposition 3.2. *Let $\mu \in H^3((0,1),\mathbb{R})$ satisfy Hypothesis 3.1. There exists $\eta^* > 0$ and a C^1 map*

$$\hat{\Gamma} : (0, \eta^*) \to L^2\left(\left(\frac{\varepsilon}{2}, \varepsilon\right), \mathbb{R}\right),$$

such that $\hat{\Gamma}(0) = 0$ and for any $\eta \in (0, \eta^*)$, the solution $\left(\psi^{1,\eta}_{ref}, \psi^{2,\eta}_{ref}, \psi^{3,\eta}_{ref}\right)$ of system (3.2) with control $u^\eta_{ref} := \hat{\Gamma}(\eta)$ and initial conditions $\psi^{j,\eta}_{ref}(\frac{\varepsilon}{2}) = \Phi_j(\frac{\varepsilon}{2})$, for $j = 1, 2, 3$, satisfies (3.9) and (3.10).

Proof of Proposition 3.2. Using Proposition 3.1, it comes that the map

$$\begin{array}{cccc} \tilde{\Theta}: & L^2((\frac{\varepsilon}{2},\varepsilon),\mathbb{R}) & \to & \mathbb{R}^3\times\mathbb{R}^3 \\ & u & \mapsto & \left(\tilde{\Theta}_1(u),\tilde{\Theta}_2(u)\right) \end{array}$$

where

$$\tilde{\Theta}_1(u) := \left(\langle\mu\psi^j(\varepsilon_1),\psi^j(\varepsilon_1)\rangle - \langle\mu\varphi_j,\varphi_j\rangle\right)_{j=1,2,3},$$

and

$$\tilde{\Theta}_2(u) := \left(\langle\mu\psi^j(\varepsilon),\psi^j(\varepsilon)\rangle - \langle\mu\varphi_j,\varphi_j\rangle\right)_{j=1,2,3},$$

is well defined, C^1, satisfies $\tilde{\Theta}(0)=0$ and

$$\mathrm{d}\tilde{\Theta}(0).v = \left(\left(2\,\mathrm{Re}(\langle\mu\Psi^j(\varepsilon_1),\Phi_j(\varepsilon_1)\rangle)\right)_{1\leq j\leq 3}, \left(2\,\mathrm{Re}(\langle\mu\Psi^j(\varepsilon),\Phi_j(\varepsilon)\rangle)\right)_{1\leq j\leq 3}\right), \tag{3.12}$$

where $\left(\Psi^1,\Psi^2,\Psi^3\right)$ is the solution of (3.4) on the time interval $\left(\frac{\varepsilon}{2},\varepsilon\right)$ with control v and initial conditions $\Psi^j\left(\frac{\varepsilon}{2},\cdot\right)=0$. Let us prove that $\mathrm{d}\tilde{\Theta}(0)$ is surjective; then the inverse mapping theorem will give the conclusion.

Let $\gamma=(\gamma_j)_{1\leq j\leq 6}\in\mathbb{R}^6$ and $K\geq 4$. By Proposition 3.11 (see the appendix), there exist $v_1\in L^2((\frac{\varepsilon}{2},\varepsilon_1),\mathbb{R})$ and $v_2\in L^2((\varepsilon_1,\varepsilon),\mathbb{R})$ such that

$$\begin{aligned}
&\int_{\frac{\varepsilon}{2}}^{\varepsilon_1} v_1(t)e^{i(\lambda_k-\lambda_j)t}\mathrm{d}t = 0, \quad \forall k\in\mathbb{N}^*\backslash\{K\}, \forall 1\leq j\leq 3,\\
&\int_{\frac{\varepsilon}{2}}^{\varepsilon_1} v_1(t)e^{i(\lambda_K-\lambda_j)t}\mathrm{d}t = \frac{e^{i(\lambda_K-\lambda_j)\varepsilon_1}\gamma_j}{2i\langle\mu\varphi_j,\varphi_K\rangle^2}, \quad \forall 1\leq j\leq 3,\\
&\int_{\varepsilon_1}^{\varepsilon} v_2(t)e^{i(\lambda_k-\lambda_j)t}\mathrm{d}t = 0, \quad \forall k\in\mathbb{N}^*\backslash\{K\}, \forall 1\leq j\leq 3,\\
&\int_{\varepsilon_1}^{\varepsilon} v_2(t)e^{i(\lambda_K-\lambda_j)t}\mathrm{d}t = \frac{e^{i(\lambda_K-\lambda_j)\varepsilon}\gamma_{3+j}}{2i\langle\mu\varphi_j,\varphi_K\rangle^2} - \frac{e^{i(\lambda_K-\lambda_j)\varepsilon_1}\gamma_j}{2i\langle\mu\varphi_j,\varphi_K\rangle^2}, \quad \forall 1\leq j\leq 3.
\end{aligned}$$

Notice that the moments associated to redundant frequencies in the previous moment problem are all set to the same value and, as $K\geq 4$, the frequencies $\lambda_K-\lambda_j$ for $1\leq j\leq 3$ are distinct. Let $v\in L^2\left(\frac{\varepsilon}{2},\varepsilon\right)$ be defined by v_1 on $\left(\frac{\varepsilon}{2},\varepsilon_1\right)$ and by v_2 on $(\varepsilon_1,\varepsilon)$. Straightforward computations lead to $\mathrm{d}\tilde{\Theta}(0).v=\gamma$.

□

3.3.2 Construction on (ε, T_1)

For any $j\in\mathbb{N}^*$, let $\mathcal{P}_j$ be the orthogonal projection of $L^2((0,1),\mathbb{C})$ onto $\mathrm{Span}_{\mathbb{C}}(\varphi_k, k\geq j+1)$ i.e.

$$\mathcal{P}_j(\psi) := \sum_{k=j+1}^{+\infty}\langle\psi,\varphi_k\rangle\varphi_k.$$

The goal of this subsection is the proof of the following proposition.

Proposition 3.3. *Let $0 < T_0 < T_f$. Let $\mu \in H^3((0,1),\mathbb{R})$ satisfy Hypothesis 3.1 and $5\langle \mu\varphi_1,\varphi_1\rangle - 8\langle \mu\varphi_2,\varphi_2\rangle + 3\langle \mu\varphi_3,\varphi_3\rangle \neq 0$. There exist $\delta > 0$ and a C^1-map*

$$\tilde{\Gamma}_{T_0,T_f} : \tilde{\mathcal{O}}_{\delta,T_0} \to L^2((T_0,T_f),\mathbb{R})$$

with

$$\tilde{\mathcal{O}}_{\delta,T_0} := \left\{ \left(\psi_0^1,\psi_0^2,\psi_0^3\right) \in (\mathcal{S}\cap H^3_{(0)}(0,1))^3 \; ; \; \sum_{j=1}^{3} ||\psi_0^j - \Phi_j(T_0)||_{H^3_{(0)}} < \delta \right\},$$

such that $\tilde{\Gamma}_{T_0,T_f}\big(\Phi_1(T_0),\Phi_2(T_0),\Phi_3(T_0)\big) = 0$ and, if $(\psi_0^1,\psi_0^2,\psi_0^3) \in \tilde{\mathcal{O}}_{\delta,T_0}$, the solution $\big(\psi^1,\psi^2,\psi^3\big)$ of system (3.2) with initial conditions $\psi^j(T_0,\cdot) = \psi_0^j$, for $j = 1,2,3$, and control $u := \tilde{\Gamma}_{T_0,T_f}\big(\psi_0^1,\psi_0^2,\psi_0^3\big)$ satisfies

$$\mathcal{P}_1\big(\psi^1(T_f)\big) = \mathcal{P}_2\big(\psi^2(T_f)\big) = \mathcal{P}_3\big(\psi^3(T_f)\big) = 0, \tag{3.13}$$

$$\operatorname{Im}\Big(\langle\psi^1(T_f),\Phi_1(T_f)\rangle^5\overline{\langle\psi^2(T_f),\Phi_2(T_f)\rangle^8}\langle\psi^3(T_f),\Phi_3(T_f)\rangle^3\Big) = 0. \tag{3.14}$$

Remark 3.13. The conditions (3.13) and (3.14) will be used in the next subsection to prove (3.11). Equation (3.14) will be used to define the global phase θ^η.

Proof of Proposition 3.3. Let us define the following space

$$X_1 := \Big\{(\phi_1,\phi_2,\phi_3) \in H^3_{(0)}((0,1),\mathbb{C})^3 \; ; \; \langle\phi_j,\varphi_k\rangle = 0, \text{ for } 1 \leq k \leq j \leq 3\Big\}.$$

We consider the following end-point map

$$\Theta_{T_0,T_f} : \quad L^2((T_0,T_f),\mathbb{R}) \times H^3_{(0)}(0,1)^3 \quad \to \quad H^3_{(0)}(0,1)^3 \times X_1 \times \mathbb{R},$$

defined by

$$\Theta_{T_0,T_f}\big(u,\psi_0^1,\psi_0^2,\psi_0^3\big) := \Big(\psi_0^1,\psi_0^2,\psi_0^3,\, \mathcal{P}_1\big(\psi^1(T_f)\big),\mathcal{P}_2\big(\psi^2(T_f)\big),\mathcal{P}_3\big(\psi^3(T_f)\big),$$
$$\operatorname{Im}\big(\,\langle\psi^1(T_f),\Phi_1(T_f)\rangle^5\overline{\langle\psi^2(T_f),\Phi_2(T_f)\rangle^8}\langle\psi^3(T_f),\Phi_3(T_f)\rangle^3\,\big)\Big)$$

where (ψ^1,ψ^2,ψ^3) is the solution of (3.2) with initial condition $\psi^j(T_0,\cdot) = \psi_0^j$ and control u. Thus, we have

$$\Theta_{T_0,T_f}\big(0,\Phi_1(T_0),\Phi_2(T_0),\Phi_3(T_0)\big) = \big(\Phi_1(T_0),\Phi_2(T_0),\Phi_3(T_0),0,0,0,0\big).$$

Proposition 3.3 is proved by application of the inverse mapping theorem to Θ_{T_0,T_f} at the point $\big(0,\Phi_1(T_0),\Phi_2(T_0),\Phi_3(T_0)\big)$.
Using the same arguments as in [16, Proposition 3], it comes that Θ_{T_0,T_f} is a C^1 map and that

$$\begin{aligned}
&\mathrm{d}\Theta_{T_0,T_f}\big(0,\Phi_1(T_0),\Phi_2(T_0),\Phi_3(T_0)\big).(v,\Psi_0^1,\Psi_0^2,\Psi_0^3)\\
=&\Big(\Psi_0^1,\Psi_0^2,\Psi_0^3,\,\mathcal{P}_1\big(\Psi^1(T_f)\big),\mathcal{P}_2\big(\Psi^2(T_f)\big),\mathcal{P}_3\big(\Psi^3(T_f)\big),\\
&5\operatorname{Im}(\langle\Psi^1(T_f),\Phi_1(T_f)\rangle) - 8\operatorname{Im}(\langle\Psi^2(T_f),\Phi_2(T_f)\rangle) + 3\operatorname{Im}(\langle\Psi^3(T_f),\Phi_3(T_f)\rangle)\Big),
\end{aligned}$$

where (Ψ^1, Ψ^2, Ψ^3) is the solution of (3.4) on the time interval (T_0, T_f) with control v and initial conditions $\Psi^j(T_0, \cdot) = \Psi^j_0$.
It remains to prove that $\mathrm{d}\Theta_{T_0,T_f}\big(0, \Phi_1(T_0), \Phi_2(T_0), \Phi_3(T_0)\big) : L^2((T_0,T_f),\mathbb{R}) \times H^3_{(0)}(0,1)^3 \to H^3_{(0)}(0,1)^3 \times X_1 \times \mathbb{R}$ admits a continuous right inverse.
Let $(\Psi^1_0, \Psi^2_0, \Psi^3_0) \in H^3_{(0)}(0,1)^3$, $(\psi^1_f, \psi^2_f, \psi^3_f) \in X_1$ and $r \in \mathbb{R}$. Straightforward computations lead to

$$\Psi^j(T_f) = \sum_{k=1}^{+\infty} \Big(\langle \Psi^j_0, \Phi_k(T_0)\rangle + i\langle \mu\varphi_j, \varphi_k\rangle \int_{T_0}^{T_f} v(t) e^{i(\lambda_k - \lambda_j)t} \mathrm{d}t \Big) \Phi_k(T_f).$$

Finding $v \in L^2((T_0, T_f), \mathbb{R})$ such that

$$\begin{aligned}
&\mathcal{P}_j(\Psi^j(T_f)) = \psi^j_f, \quad \forall j \in \{1,2,3\},\\
&\mathrm{Im}\,\big(5\langle \Psi^1(T_f), \Phi_1(T_f)\rangle - 8\langle \Psi^2(T_f), \Phi_2(T_f)\rangle + 3\langle \Psi^3(T_f), \Phi_3(T_f)\rangle\big) = r,
\end{aligned}$$

is equivalent to solving the following trigonometric moment, $\forall j = 1,2,3$, $\forall k \geq j+1$

$$\begin{aligned}
&\int_{T_0}^{T_f} v(t) e^{i(\lambda_k - \lambda_j)t} \mathrm{d}t = \frac{1}{i\langle \mu\varphi_j, \varphi_k\rangle} \big(\langle \psi^j_f, \Phi_k(T_f)\rangle - \langle \Psi^j_0, \Phi_k(T_0)\rangle \big),\\
&\int_{T_0}^{T_f} v(t)\mathrm{d}t = \frac{r - \mathrm{Im}\,\big(5\langle \Psi^1_0, \Phi_1(T_0)\rangle - 8\langle \Psi^2_0, \Phi_2(T_0)\rangle + 3\langle \Psi^3_0, \Phi_3(T_0)\rangle\big)}{5\langle \mu\varphi_1, \varphi_1\rangle - 8\langle \mu\varphi_2, \varphi_2\rangle + 3\langle \mu\varphi_3, \varphi_3\rangle}.
\end{aligned} \tag{3.15}$$

Using Proposition 3.11 and the hypotheses on μ, this ends the proof of Proposition 3.3.

□

3.3.3 Proof of Theorem 3.6

Let $\delta > 0$ be the radius defined in Proposition 3.3 with $T_0 = \varepsilon$ and $T_f = T_1$. For $\eta > 0$ we define the following control

$$u^\eta_{ref}(t) := \begin{cases} 0 & \text{for } t \in (0, \frac{\varepsilon}{2}),\\ \hat{\Gamma}(\eta) & \text{for } t \in (\frac{\varepsilon}{2}, \varepsilon),\\ \tilde{\Gamma}_{\varepsilon,T_1}(\psi^{1,\eta}_{ref}(\varepsilon), \psi^{2,\eta}_{ref}(\varepsilon), \psi^{3,\eta}_{ref}(\varepsilon)) & \text{for } t \in (\varepsilon, T_1), \end{cases} \tag{3.16}$$

where $\hat{\Gamma}$ and $\tilde{\Gamma}$ are defined respectively in Proposition 3.2 and 3.3. We prove that, for η small enough, this control satisfies the conditions of Theorem 3.6.

Proof of Theorem 3.6. The proof is decomposed into two parts. First, we prove that there exists $\overline{\eta} > 0$ such that for $\eta \in (0, \overline{\eta})$, u^η_{ref} is well defined, satisfies $||u^\eta_{ref}||_{L^2(0,T_1)} \leq C\eta$ and the conditions (3.9), (3.10) are satisfied. Then, we prove the existence of $T^\eta > 0$ and $\theta^\eta \in \mathbb{R}$ such that if u^η_{ref} is extended by 0 on (T_1, T^η), the condition (3.11) is satisfied.

First step : u^η_{ref} is well defined.

Using Proposition 3.2, the control u^η_{ref} is well defined on $(0,\varepsilon)$ as soon as $\eta \in (0,\eta^*)$. Moreover, using Lipschitz property of $\hat{\Gamma}$, there exists $C(\eta^*) > 0$ such that

$$||u^\eta_{ref}||_{L^2(\frac{\varepsilon}{2},\varepsilon)} = ||\hat{\Gamma}(\eta) - \hat{\Gamma}(0)||_{L^2(\frac{\varepsilon}{2},\varepsilon)} \leq C(\eta^*)\eta.$$

Thanks to Proposition 3.1, there exists $C(\varepsilon) > 0$ such that if $||u||_{L^2(0,\varepsilon)} < 1$, the associated solution of (3.2)-(3.3) satisfies

$$||(\psi^j - \Phi_j)(\varepsilon)||_{H^3_{(0)}} \leq C(\varepsilon)||u||_{L^2(0,\varepsilon)}, \quad \text{for } j = 1,2,3.$$

Thus, using Proposition 3.3, if $C(\varepsilon)C(\eta^*)\eta < \frac{\delta}{3}$, we get that for $j = 1,2,3$,

$$||(\psi^{j,\eta}_{ref} - \Phi_j)(\varepsilon)||_{H^3_{(0)}} < \frac{\delta}{3}.$$

Thus, u^η_{ref} is well defined on $(0,T_1)$. Moreover, there exists $C(\delta) > 0$ such that

$$\begin{aligned}||u^\eta_{ref}||_{L^2(\varepsilon,T_1)} &= ||\tilde{\Gamma}_{\varepsilon,T_1}\left(\psi^{1,\eta}_{ref}(\varepsilon), \psi^{2,\eta}_{ref}(\varepsilon), \psi^{3,\eta}_{ref}(\varepsilon)\right) - \tilde{\Gamma}_{\varepsilon,T_1}\left(\Phi_1(\varepsilon), \Phi_2(\varepsilon), \Phi_3(\varepsilon)\right)||_{L^2(\varepsilon,T_1)} \\ &\leq C(\delta)\sum_{j=1}^{3} ||(\psi^{j,\eta}_{ref} - \Phi_j)(\varepsilon)||_{H^3_{(0)}} \\ &\leq 3C(\delta)C(\varepsilon)C(\eta^*)\eta.\end{aligned}$$

Finally, choosing

$$\overline{\eta} < \min\left(\eta^*, \frac{\delta}{3C(\varepsilon)C(\eta^*)}, \frac{1}{C(\eta^*)}\right),$$

implies that $||u^\eta_{ref}||_{L^2(0,T_1)} \leq C\eta$. Here and throughout this paper C denotes a positive constant that may vary each time it appears. Thanks to Proposition 3.2, it comes that (3.9) and (3.10) hold.

Second step : We prove the existence of a final time $T^\eta > 0$ and a global phase $\theta^\eta \in \mathbb{R}$ such that (3.11) holds.
Proposition 3.3, implies

$$\psi^{j,\eta}_{ref}(T_1) = \sum_{k=1}^{j} \langle \psi^{j,\eta}_{ref}(T_1), \Phi_k(T_1)\rangle \Phi_k(T_1), \quad \forall j = 1,2,3, \tag{3.17}$$

$$\mathrm{Im}\left(\langle \psi^{1,\eta}_{ref}(T_1), \Phi_1(T_1)\rangle^5 \overline{\langle \psi^{2,\eta}_{ref}(T_1), \Phi_2(T_1)\rangle^8} \langle \psi^{3,\eta}_{ref}(T_1), \Phi_3(T_1)\rangle^3\right) = 0. \tag{3.18}$$

Using the invariant of the system, $\langle \psi^{j,\eta}_{ref}, \psi^{k,\eta}_{ref}\rangle \equiv \delta_{j=k}$, for $j,k \in \{1,2,3\}$, this leads to the existence of θ^η_1, $\theta^\eta_2, \theta^\eta_3 \in (-\pi,\pi]$ such that

$$\psi^{j,\eta}_{ref}(T_1) = e^{-i\theta^\eta_j}\Phi_j(T_1), \quad \forall j = 1,2,3.$$

Using (3.18), it comes that

$$\sin\left(5\theta^\eta_1 - 8\theta^\eta_2 + 3\theta^\eta_3\right) = 0.$$

Using Proposition 3.1, it comes that, up to a choice of a smaller $\overline{\eta}$,

$$5\theta^\eta_1 - 8\theta^\eta_2 + 3\theta^\eta_3 = 0. \tag{3.19}$$

Recall that $\lambda_k = k^2\pi^2$. Let T^η and θ^η be such that $T^\eta > T_1$ and

$$\begin{cases} T^\eta \equiv \dfrac{\theta_1^\eta - \theta_2^\eta}{\lambda_2 - \lambda_1} \left[\dfrac{2}{\pi}\right], \\ \theta^\eta \equiv \dfrac{\lambda_2}{\lambda_2 - \lambda_1}\theta_1^\eta - \dfrac{\lambda_1}{\lambda_2 - \lambda_1}\theta_2^\eta \, [2\pi]. \end{cases}$$

This choice leads to

$$\begin{cases} \theta_1^\eta + \lambda_1 T^\eta - \theta^\eta \equiv 0 \, [2\pi], \\ \theta_2^\eta + \lambda_2 T^\eta - \theta^\eta \equiv 0 \, [2\pi]. \end{cases}$$

Then, using the definitions of T^η and θ^η together with (3.19) we get

$$\begin{aligned} \theta_3^\eta + \lambda_3 T^\eta - \theta^\eta &\equiv \theta_3^\eta + \frac{\lambda_3}{\lambda_2 - \lambda_1}(\theta_1^\eta - \theta_2^\eta) - \frac{\lambda_2}{\lambda_2 - \lambda_1}\theta_1^\eta + \frac{\lambda_1}{\lambda_2 - \lambda_1}\theta_2^\eta \, [2\pi] \\ &\equiv \frac{1}{3}\big(5\theta_1^\eta - 8\theta_2^\eta + 3\theta_3^\eta\big) \, [2\pi] \\ &\equiv 0 \, [2\pi]. \end{aligned}$$

Finally, if we extend u_{ref}^η by 0 on (T_1, T^η), we have that $\big(\psi_{ref}^{1,\eta}, \psi_{ref}^{2,\eta}, \psi_{ref}^{3,\eta}\big)$ is solution of (3.2)-(3.3) with control u_{ref}^η and satisfies for $j \in \{1,2,3\}$

$$\psi_{ref}^{j,\eta}(T^\eta) = e^{-i(\theta_j^\eta + \lambda_j T^\eta)}\varphi_j = e^{-i\theta^\eta}\varphi_j.$$

This ends the proof of Theorem 3.6.

□

3.4 Proof of Theorem 3.5

This section is dedicated to the proof of Theorem 3.5 which is done in the case $T = 0$, the extension to the general case being straightforward. The proof is divided in two parts. In Subsection 3.4.1, the functional setting is specified and we prove the controllability of the linearized system around $(\psi_{ref}^{1,\eta}, \psi_{ref}^{2,\eta}, \psi_{ref}^{3,\eta}, u_{ref}^\eta)$,

$$\begin{cases} i\partial_t \Psi^{j,\eta} = -\partial_{xx}^2 \Psi^{j,\eta} - u_{ref}^\eta(t)\mu(x)\Psi^{j,\eta} - v(t)\mu(x)\psi_{ref}^{j,\eta}, & (t,x) \in (0,T^\eta) \times (0,1), \\ \Psi^{j,\eta}(t,0) = \Psi^{j,\eta}(t,1) = 0, & t \in (0,T^\eta), \\ \Psi^{j,\eta}(0,x) = 0, & x \in (0,1), \end{cases} \tag{3.20}$$

when η is small enough. In Subsection 3.4.2, we conclude the proof of Theorem 3.5 using the inverse mapping theorem.

3.4.1 Controllability of the linearized system

For any $t > 0$, let

$$\begin{aligned} X_t^f := \Big\{ &(\phi^1, \phi^2, \phi^3) \in H^3_{(0)}((0,1),\mathbb{C})^3 \, ; \, \mathrm{Re}(\langle \phi^j, \psi_{ref}^{j,\eta}(t)\rangle) = 0, \text{ for } j = 1,2,3 \\ &\text{and } \langle \phi^j, \psi_{ref}^{k,\eta}(t)\rangle = -\overline{\langle \phi^k, \psi_{ref}^{j,\eta}(t)\rangle}, \text{ for } (j,k) = (2,1),(3,1),(3,2) \Big\}. \end{aligned} \tag{3.21}$$

The following proposition holds.

Proposition 3.4. *There exists $\hat{\eta} \in (0, \overline{\eta})$ such that, for any $\eta \in (0, \hat{\eta})$, if T^η, u^η_{ref} and $(\psi^{1,\eta}_{ref}, \psi^{2,\eta}_{ref}, \psi^{3,\eta}_{ref})$ are defined as in Theorem 3.6, there exists a continuous linear map*

$$L^\eta : \begin{array}{ccc} X^f_{T^\eta} & \to & L^2((0,T^\eta),\mathbb{R}) \\ (\psi^1_f, \psi^2_f, \psi^3_f) & \mapsto & v \end{array}$$

such that for any $(\psi^1_f, \psi^2_f, \psi^3_f) \in X^f_{T^\eta}$, the solution $(\Psi^{1,\eta}, \Psi^{2,\eta}, \Psi^{3,\eta})$ of system (3.20) with control $v = L^\eta(\psi^1_f, \psi^2_f, \psi^3_f)$ satisfies

$$\big(\Psi^{1,\eta}(T^\eta), \Psi^{2,\eta}(T^\eta), \Psi^{3,\eta}(T^\eta)\big) = \big(\psi^1_f, \psi^2_f, \psi^3_f\big).$$

Before proving Proposition 3.4 we set some notations. For any $\eta \in (0, \overline{\eta})$, for any $t \in (0, T^\eta)$, let $U^\eta(t)$ be the propagator of the following system

$$\begin{cases} i\partial_t \psi = -\partial^2_{xx}\psi - u^\eta_{ref}(t)\mu(x)\psi, & (t,x) \in (0,T^\eta) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T^\eta), \\ \psi(0,x) = \psi^0(0,x), & x \in (0,1), \end{cases} \tag{3.22}$$

i.e. $U^\eta(t)\psi^0 = \psi(t)$. We will work in the Hilbert basis $(\Phi^\eta_k(t) := U^\eta(t)\varphi_k)_{k\in\mathbb{N}^*}$ of $L^2((0,1),\mathbb{C})$. Notice that for $j = 1,2,3$, $\Phi^\eta_j = \psi^{j,\eta}_{ref}$. As the proof of Proposition 3.4 is quite long and technical, let us detail the different steps. Let

$$\mathcal{I} := \big\{(j,k) \in \{1,2,3\} \times \mathbb{N}^* \,;\, k \geq j+1\big\} \cup \{(3,3)\}\,.$$

The first step consists in proving the controllability of the components $\langle \Psi^{j,\eta}(T_f), \Phi^\eta_k(T_f)\rangle$ for $(j,k) \in \mathcal{I}$, for any $T_f > 0$ and η sufficiently small, as stated in Lemma 3.1. First, we prove that these components are controllable when $\eta = 0$: it corresponds to solving a trigonometric moment problem with an infinite asymptotic gap between successive frequencies. Then, we extend the controllability of these components to small values of η, by an argument of close linear maps.

In the second step (Lemmas 3.2 and 3.3), using Riesz basis and biorthogonal family arguments, we prove that we can also control the two diagonal directions $\langle \Psi^{j,\eta}(T_f), \Phi^\eta_j(T_f)\rangle$ for $j = 1,2$. This would not have been possible directly in the first step. Indeed for $\eta = 0$, the three directions $\langle \Psi^{j,\eta}(T_f), \Phi^\eta_j(T_f)\rangle$ for $j = 1,2,3$ are associated to the same frequency in the moment problem. But for $\eta > 0$, the construction of the reference trajectory (and more precisely conditions (3.9) and (3.10)) will allow to control those two directions.

Finally, in the third step, due to the conditions imposed in the definition of X^f_t (in (3.21)) the remaining directions $\langle \Psi^{j,\eta}, \Phi^\eta_k\rangle$ for $1 \leq k < j$ are automatically controlled.

Proof of Proposition 3.4. The map L^η will be designed on $(0, T_1)$ and extended by 0 on (T_1, T^η), where T_1 is as in Theorem 3.6. Let

$$\mathcal{V}_0 := \Big\{(d^1, d^2, d^3) \in h^3(\mathbb{N}^*, \mathbb{C})^3 \,;\, d^j_k = 0, \text{ if } (j,k) \notin \mathcal{I} \text{ and } \operatorname{Re}(d^3_3) = 0\Big\}\,.$$

Let $R : \mathcal{I} \to \mathbb{N}$ be the rearrangement such that, if $\omega_n := \lambda_k - \lambda_j$ with $n = R(j,k)$, the sequence $(\omega_n)_{n\in\mathbb{N}}$ is increasing. Notice that $0 = R(3,3)$.

First step of the proof of Proposition 3.4 : we prove that for $(j,k) \in \mathcal{I}$ the directions $\langle \Psi^{j,\eta}(T_f), \Phi_k^\eta(T_f)\rangle$ are controllable in any positive time T_f for η small enough.
Let

$$d_{T_f}^\eta : \psi = (\psi^1, \psi^2, \psi^3) \in X_{T_f}^f \mapsto \left(d_{T_f}^{1,\eta}(\psi), d_{T_f}^{2,\eta}(\psi), d_{T_f}^{3,\eta}(\psi)\right) \in \mathcal{V}_0,$$

where for $j = 1,2,3$,

$$d_{T_f,k}^{j,\eta}(\psi) := \langle \psi^j, \Phi_k^\eta(T_f)\rangle, \quad \text{if } (j,k) \in \mathcal{I},$$
$$d_{T_f,k}^{j,\eta}(\psi) := 0, \quad \text{if } (j,k) \notin \mathcal{I}.$$

The next lemma ensures the controllability of the directions $\langle \Psi^{j,\eta}(T_f), \Phi_k^\eta(T_f)\rangle$ for $(j,k) \in \mathcal{I}$.

Lemma 3.1. *Let $T_f > 0$ and*

$$\begin{array}{rccc} F^\eta : & L^2((0,T_f),\mathbb{R}) & \to & \mathcal{V}_0 \\ & v & \mapsto & d_{T_f}^\eta(\Psi(T_f)) \end{array}$$

where $\Psi := \left(\Psi^1, \Psi^2, \Psi^3\right)$ is the solution of (3.20) with control v. There exists $\hat{\eta} = \hat{\eta}(T_f) \in (0, \overline{\eta})$ such that, for any $\eta \in (0, \hat{\eta})$, the map F^η has a continuous right inverse

$$F^{\eta^{-1}} : \mathcal{V}_0 \to L^2((0,T_f),\mathbb{R}).$$

Proof of Lemma 3.1. Straightforward computations lead to

$$\langle \Psi^{j,\eta}(T_f), \Phi_k^\eta(T_f)\rangle = i\int_0^{T_f} v(t)\langle \mu\psi_{ref}^{j,\eta}(t), \Phi_k^\eta(t)\rangle \mathrm{d}t, \quad \text{for } (j,k) \in \mathcal{I}. \tag{3.23}$$

Let us define

$$f_n^\eta(t) := \frac{\langle \mu\psi_{ref}^{j,\eta}(t), \Phi_k^\eta(t)\rangle}{\langle \mu\varphi_j, \varphi_k\rangle}, \quad \text{for } (j,k) \in \mathcal{I} \text{ and } n = R(j,k), \tag{3.24}$$

and $f_{-n}^\eta(t) := \overline{f_n^\eta(t)}$, for $n \in \mathbb{N}^*$. We consider the following map

$$\begin{array}{rccc} J^\eta : & L^2((0,T_f),\mathbb{C}) & \to & \ell^2(\mathbb{Z},\mathbb{C}) \\ & v & \mapsto & \left(\int_0^{T_f} v(t) f_n^\eta(t)\mathrm{d}t\right)_{n\in\mathbb{Z}} \end{array}.$$

Notice that $f_n^0(t) = e^{i\omega_n t}$ with $\omega_n = \lambda_k - \lambda_j$ for any $n = R(j,k) \in \mathbb{N}$. Thus (see 3.A), J^0 is continuous with values in $\ell^2(\mathbb{Z},\mathbb{C})$. Moreover, J^0 is an isomorphism from $H_0 := \mathrm{Adh}_{L^2(0,T_f)}\left(\mathrm{Span}\{f_n^0\,;\, n \in \mathbb{Z}\}\right)$ to $\ell^2(\mathbb{Z},\mathbb{C})$.

First step : we prove the existence of $\tilde{C} > 0$ such that

$$||(J^\eta - J^0)(v)||_{\ell^2} \leq \tilde{C}\eta||v||_{L^2(0,T_f)}, \quad \forall v \in L^2((0,T_f),\mathbb{C}). \tag{3.25}$$

Let $(j,k) \in \mathcal{I}$, $n = R(j,k) \in \mathbb{N}$ and $v \in L^2((0,T_f),\mathbb{C})$. Using (3.23) and (3.24), the triangular inequality and Hypothesis 3.1, we get

$$\left|\int_0^{T_f} v(t)(f_n^0 - f_n^\eta)(t)\mathrm{d}t\right| = \left|\frac{\langle \Psi^{j,0}(T_f), \Phi_k(T_f)\rangle}{\langle \mu\varphi_j, \varphi_k\rangle} - \frac{\langle \Psi^{j,\eta}(T_f), \Phi_k^\eta(T_f)\rangle}{\langle \mu\varphi_j, \varphi_k\rangle}\right|$$
$$\leq Ck^3 \left(\left|\langle (\Psi^{j,0} - \Psi^{j,\eta})(T_f), \Phi_k(T_f)\rangle\right| + \left|\langle (U^\eta(T_f) - U^0(T_f))^* \Psi^{j,\eta}(T_f), \varphi_k\rangle\right|\right)$$

because $(\Phi_k^\eta - \Phi_k)(t) = (U^\eta(t) - U^0(t))\varphi_k$ (we denoted by $*$ the $L^2((0,1),\mathbb{C})$ adjoint operator). Thus,

$$||(J^0 - J^\eta)(v)||_{\ell^2} \leq C \sum_{j=1}^{3} \Big(||(\Psi^{j,0} - \Psi^{j,\eta})(T_f)||_{H^3_{(0)}} + ||(U^\eta(T_f) - U^0(T_f))^* \Psi^{j,\eta}(T_f)||_{H^3_{(0)}} \Big). \tag{3.26}$$

Proposition 3.1 implies that

$$\begin{aligned} ||(\Psi^{j,0} - \Psi^{j,\eta})(T_f)||_{H^3_{(0)}} &\leq C||u_{ref}^\eta(t)\mu\Psi^{j,0}(t) + v(t)\mu(\psi_{ref}^{j,\eta} - \Phi_j)(t)||_{L^2((0,T_f),H^3\cap H^1_0)} \\ &\leq C||u_{ref}^\eta||_{L^2(0,T_f)}||v||_{L^2(0,T_f)}. \end{aligned} \tag{3.27}$$

Using unitarity, it comes that $U^\eta(T_f)^*$ is the propagator at time T_f of system

$$\begin{cases} i\partial_t\psi = \partial_{xx}^2\psi + u_{ref}^\eta(T_f - t)\mu(x)\psi, & (t,x) \in (0,T_f) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T_f). \end{cases}$$

Thus Proposition 3.1 may be applied again leading to

$$\begin{aligned} \left|\left|(U^\eta(T_f) - U^0(T_f))^* \Psi^{j,\eta}(T_f)\right|\right|_{H^3_{(0)}} &\leq C\,||u_{ref}^\eta(t)\mu U^0(t)^* \Psi^{j,\eta}(T_f)||_{L^2((0,T_f),H^3\cap H^1_0)} \\ &\leq C\,||u_{ref}^\eta||_{L^2(0,T_f)}||v||_{L^2(0,T_f)}. \end{aligned} \tag{3.28}$$

From inequalities (3.26), (3.27), (3.28) above and (3.8) we get the conclusion of the first step.

Second step : conclusion.
Let $\hat{\eta}(T_f) := \min\left\{\overline{\eta}, \tilde{C}^{-1}||(J^0)^{-1}||^{-1}_{\mathcal{L}(H_0,\ell^2)}\right\}$ where $\tilde{C}$ is defined by (3.25) and let $\eta \in (0,\hat{\eta}(T_f))$. We deduce from the first step that J^η is an isomorphism from H_0 to $\ell^2(\mathbb{Z},\mathbb{C})$. Let $(d^1,d^2,d^3) \in \mathcal{V}_0$. We define $\tilde{d}_n := \dfrac{d_k^j}{i\langle \mu\varphi_j, \varphi_k\rangle}$, for $(j,k) \in \mathcal{I}$ and $n = R(j,k) \in \mathbb{N}$, and $\tilde{d}_{-n} := \overline{\tilde{d}_n}$, for $n \in \mathbb{N}^*$. Then,

$$F^{\eta^{-1}}(d^1,d^2,d^3) := (J^\eta_{|H_0})^{-1}(\tilde{d})$$

is the unique solution v in H_0 of the equation $F^\eta(v) = (d^1,d^2,d^3)$. The uniqueness implies that v is real valued. This ends the proof of Lemma 3.1. □

Second step of the proof of Proposition 3.4 : Riesz basis and minimality.
To prove that we can also control the directions $\langle \Psi^{j,\eta}(T_f), \Phi_j^\eta(T_f)\rangle$, for $j = 1,2$, we will use the following lemmas.

Lemma 3.2. *Let $T_f > 0$ and $H^\eta := Adh_{L^2(0,T_f)}\big(Span\{f_n^\eta, n \in \mathbb{Z}\}\big)$. If $\eta < \hat{\eta}(T_f)$, then $(f_n^\eta)_{n\in\mathbb{Z}}$ is a Riesz basis of H^η.*

Proof of Lemma 3.2. Using [16, Proposition 19], it comes that $(f_n^\eta)_{n\in\mathbb{Z}}$ is a Riesz basis of H^η if and only if there exists $C_1, C_2 > 0$ such that for any complex sequence $(a_n)_{n\in\mathbb{Z}}$ with finite support

$$C_1\left(\sum_n |a_n|^2\right)^{1/2} \leq \left|\left|\sum_n a_n f_n^\eta\right|\right|_{L^2(0,T_f)} \leq C_2\left(\sum_n |a_n|^2\right)^{1/2}. \tag{3.29}$$

Lemma 3.1 together with [23, Theorem 1] imply the first inequality of (3.29). Using again [23, Theorem 1], we get that the second inequality of (3.29) holds if and only if, for any $g \in L^2((0,T_f),\mathbb{C})$

$$\left(\sum_{n\in\mathbb{Z}}\Big|\int_0^{T_f} g(t) f_n^\eta(t)\mathrm{d}t\Big|^2\right)^{1/2} \leq C_2||g||_{L^2}.$$

This is implied by the continuity of J^0, the triangular inequality and (3.25). This ends the proof of Lemma 3.2. ☐

From now on, we consider $\hat{\eta} < \min\big(\hat{\eta}(\frac{\varepsilon}{2}), \hat{\eta}(T_1)\big)$ and $\eta \in (0,\hat{\eta})$ fixed for all what follows.

Lemma 3.3. *Let $f_{j,j}^\eta := \dfrac{\langle \mu\psi_{ref}^{j,\eta}, \psi_{ref}^{j,\eta}\rangle}{\langle \mu\varphi_j, \varphi_j\rangle}$, for $j \in \{1,2\}$. The family $\Xi := \big(f_n^\eta\big)_{n\in\mathbb{Z}} \cup \{f_{1,1}^\eta, f_{2,2}^\eta\}$ is minimal in $L^2((0,T_1),\mathbb{C})$.*

Proof of Lemma 3.3. Assume that there exist $(c_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$ and $c_{1,1}$, $c_{2,2} \in \mathbb{C}$, not all being zero, such that

$$c_{1,1}f_{1,1}^\eta + c_{2,2}f_{2,2}^\eta + \sum_{n\in\mathbb{Z}} c_n f_n^\eta = 0, \quad \text{in } L^2((0,T_1),\mathbb{C}). \tag{3.30}$$

Thus,

$$c_{1,1}f_{1,1}^\eta + c_{2,2}f_{2,2}^\eta + \sum_{n\in\mathbb{Z}} c_n f_n^\eta = 0, \quad \text{in } L^2((0,\frac{\varepsilon}{2}),\mathbb{C}).$$

As $f_0^\eta = f_{1,1}^\eta = f_{2,2}^\eta = 1$ on $(0,\frac{\varepsilon}{2})$, then

$$c_{1,1}f_{1,1}^\eta + c_{2,2}f_{2,2}^\eta + c_0 f_0^\eta = c f_0^\eta, \quad \text{on } (0,\frac{\varepsilon}{2}),$$

where $c := c_{1,1} + c_{2,2} + c_0$. Thus,

$$c f_0^\eta + \sum_{n\in\mathbb{Z}^*} c_n f_n^\eta = 0, \quad \text{in } L^2((0,\frac{\varepsilon}{2}),\mathbb{C}).$$

As $\eta < \hat{\eta}(\varepsilon/2)$, Lemma 3.2 with $T_f = \varepsilon/2$ implies minimality of $(f_n^\eta)_{n\in\mathbb{Z}}$ in $L^2((0,\frac{\varepsilon}{2}),\mathbb{C})$. Thus,

$$c = 0 \quad \text{and} \quad c_n = 0, \quad \forall n \in \mathbb{Z}^*.$$

Then, equation (3.30) implies that,

$$c_{1,1}f^{\eta}_{1,1} + c_{2,2}f^{\eta}_{2,2} + c_0 f^{\eta}_0 = 0, \text{ on } (0,T_1). \tag{3.31}$$

Finally, as $c = 0$, conditions (3.9) and (3.10) in (3.31) lead to $c_{1,1} = c_{2,2} = 0$ and then $c_0 = 0$. This is a contradiction, thus the family Ξ is proved to be minimal in $L^2((0,T_1),\mathbb{C})$. □

The proof of Lemma 3.3 makes important use of the conditions (3.9) and (3.10) from the construction of the reference trajectory. This is the main interest of the construction of the reference trajectory : for $\eta = 0$, one gets $f^0_{1,1} = f^0_{2,2} = f^0_0$. Thus, one could not control simultaneously $\langle \Psi^{j,0}(T_1), \Phi_j(T_1)\rangle$ for $j = 1,2,3$. In our setting, the minimal family property allows together with Lemma 3.1 to conclude the proof of Proposition 3.4.

Third step of the proof of Proposition 3.4 : conclusion.
Using [16, Proposition 18], Lemma 3.3 implies that there exists a unique biorthogonal family associated to Ξ in $\text{Adh}_{L^2(0,T_1)}\big(\text{Span}(\Xi)\big)$ denoted by $\big\{g^{\eta}_{1,1}, g^{\eta}_{2,2}, (g^{\eta}_n)_{n\in\mathbb{Z}}\big\}$. This construction ensures that $g^{\eta}_{1,1}$ and $g^{\eta}_{2,2}$ are real valued.
Let $\psi_f \in X^f_{T^\eta}$ and $\tilde{\psi}_f := \big(e^{iA(T^\eta - T_1)}\psi^1_f,\, e^{iA(T^\eta - T_1)}\psi^2_f,\, e^{iA(T^\eta - T_1)}\psi^3_f\big)$. As u^{η}_{ref} is identically equal to 0 on (T_1, T^η), it comes that $\tilde{\psi}_f \in X^f_{T_1}$. The map L^η is defined by

$$L^\eta : \psi_f \in X^f_{T^\eta} \mapsto v \in L^2((0,T^\eta),\mathbb{R}),$$

where v is defined on $(0,T_1)$ by

$$v := v_0 + \sum_{j=1}^{2}\left(\frac{\text{Im}\left(\langle \tilde{\psi}^j_f, \psi^{j,\eta}_{ref}(T_1)\rangle\right)}{\langle \mu\varphi_j, \varphi_j\rangle} - \int_0^{T_1} v_0(t) f^{\eta}_{j,j}(t)\mathrm{d}t\right) g^{\eta}_{j,j},$$

with $v_0 := F^{\eta^{-1}}\big(d_{T_1}(\tilde{\psi}_f)\big)$ and extended by 0 on (T_1, T^η). The map $F^{\eta^{-1}}$ is given by Lemma 3.1 with $T_f = T_1$. Notice that L^η is linear and continuous and that as v_0, $g^{\eta}_{1,1}$ and $g^{\eta}_{2,2}$ are real valued so is v.
Let (Ψ^1, Ψ^2, Ψ^3) be the solution of (3.20) with control v. Using the biorthogonal properties, the definition of v_0 and Lemma 3.1 we get that

$$\langle \Psi^j(T_1), \Phi^{\eta}_k(T_1)\rangle = \langle \tilde{\psi}^j_f, \Phi^{\eta}_k(T_1)\rangle, \quad \forall (j,k) \in \mathcal{I} \cup \{(1,1),(2,2)\}.$$

We check that v also controls the remaining extra-diagonal terms. Straightforward computations give

$$\langle \Psi^2(T_1), \Phi^{\eta}_1(T_1)\rangle = -\overline{\langle \Psi^1(T_1), \Phi^{\eta}_2(T_1)\rangle}.$$

Yet, by definition of v and $X^f_{T_1}$,

$$\langle \Psi^1(T_1), \psi^{2,\eta}_{ref}(T_1)\rangle = \langle \tilde{\psi}^1_f, \Phi^{\eta}_2(T_1)\rangle = -\overline{\langle \tilde{\psi}^2_f, \Phi^{\eta}_1(T_1)\rangle}.$$

This leads to

$$\langle \Psi^2(T_1), \Phi^{\eta}_1(T_1)\rangle = \langle \tilde{\psi}^2_f, \Phi^{\eta}_1(T_1)\rangle.$$

The same computations hold for $\langle \Psi^3(T_1), \Phi^{\eta}_1(T_1)\rangle$ and $\langle \Psi^3(T_1), \Phi^{\eta}_2(T_1)\rangle$.
Thus, as $(\Phi^{\eta}_k(T_1))_{k\in\mathbb{N}^*}$ is a Hilbert basis of $L^2((0,T_1),\mathbb{C})$, it comes that

$$\big(\Psi^1(T_1), \Psi^2(T_1), \Psi^3(T_1)\big) = \big(\tilde{\psi}^1_f, \tilde{\psi}^2_f, \tilde{\psi}^3_f\big).$$

As v is set to zero on (T_1, T^η), this ends the proof of Proposition 3.4.

□

3.4.2 Controllability of the nonlinear system

In this subsection, we end the proof Theorem 3.5. First, using the inverse mapping theorem and Proposition 3.4, we prove in Proposition 3.5 that we can control the projections associated to the space $X^f_{T^\eta}$ (see below for precise statements and notations). Then, using the invariants of the system (see Remark 3.1) we prove that it is sufficient to control those projections.
We define

$$\begin{array}{rccl} \Lambda : & L^2((0,T^\eta),\mathbb{R}) & \to & X^f_{T^\eta} \\ & u & \mapsto & \left(\tilde{\mathcal{P}}_j(\psi^j(T^\eta))_{j=1,2,3}\right) \end{array}$$

where (ψ^1,ψ^2,ψ^3) is the solution of (3.2)-(3.3) with control u and $\tilde{\mathcal{P}}$ is defined by

$$\begin{aligned} \tilde{\mathcal{P}}_j(\phi^j) :&= \phi^j - \mathrm{Re}\left(\langle \phi^j, \psi^{j,\eta}_{ref}(T^\eta)\rangle\right) \psi^{j,\eta}_{ref}(T^\eta) \\ &- \sum_{k=1}^{j-1} \left(\langle \phi^j, \psi^{k,\eta}_{ref}(T^\eta)\rangle + \langle \psi^{j,\eta}_{ref}(T^\eta), \phi^k\rangle\right) \psi^{k,\eta}_{ref}(T^\eta). \end{aligned}$$

Thanks to this definition, Λ takes value in $X^f_{T^\eta}$ (defined in (3.21)) and $\Lambda(u^\eta_{ref}) = (0,0,0)$. As announced, we prove that we can control the projections $\tilde{\mathcal{P}}_j$. More precisely, we prove the following proposition.

Proposition 3.5. *There exists $\delta > 0$ and a C^1-map*

$$\Upsilon : \Omega_\delta \to L^2((0,T^\eta),\mathbb{R}),$$

with

$$\Omega_\delta := \left\{ \left(\tilde{\psi}^1_f, \tilde{\psi}^2_f, \tilde{\psi}^3_f\right) \in X^f_{T^\eta} \; ; \; \sum_{j=1}^{3} ||\tilde{\psi}^j_f||_{H^3_{(0)}} < \delta \right\}$$

such that $\Upsilon\big(0,0,0\big) = u^\eta_{ref}$ and for any $\left(\tilde{\psi}^1_f, \tilde{\psi}^2_f, \tilde{\psi}^3_f\right) \in \Omega_\delta$, the solution of system (3.2)-(3.3) with control $u := \Upsilon\left(\tilde{\psi}^1_f, \tilde{\psi}^2_f, \tilde{\psi}^3_f\right)$ satisfies

$$\left(\tilde{\mathcal{P}}_1(\psi^1(T^\eta)), \tilde{\mathcal{P}}_2(\psi^2(T^\eta)), \tilde{\mathcal{P}}_3(\psi^3(T^\eta))\right) = \left(\tilde{\psi}^1_f, \tilde{\psi}^2_f, \tilde{\psi}^3_f\right).$$

Proof of Proposition 3.5. This proposition is proved by application of the inverse mapping theorem to Λ at the point u^η_{ref}. Using the same arguments as in [16, Proposition 3], it comes that Λ is C^1 and for any $v \in L^2((0,T^\eta),\mathbb{R})$,

$$\mathrm{d}\Lambda(u^\eta_{ref}).v = \left(\tilde{\mathcal{P}}_1(\Psi^1(T^\eta)), \tilde{\mathcal{P}}_2(\Psi^2(T^\eta)), \tilde{\mathcal{P}}_3(\Psi^3(T^\eta))\right),$$

where $(\Psi^j)_{j=1,2,3}$ is the solution of system (3.20) with control v. Straightforward computations lead to $\tilde{\mathcal{P}}_j(\Psi^j(T^\eta)) = \Psi^j(T^\eta)$ and thus

$$\mathrm{d}\Lambda(u^\eta_{ref}).v = \left(\Psi^1(T^\eta), \Psi^2(T^\eta), \Psi^3(T^\eta)\right).$$

Proposition 3.4 proves that $\mathrm{d}\Lambda(u^\eta_{ref}) : L^2((0,T^\eta),\mathbb{R}) \to X^f_{T^\eta}$ admits a continuous right inverse. This ends the proof of Proposition 3.5.

□

Proof of Theorem 3.5. Let $\tilde{\varepsilon} > 0$ and $\big(\psi_f^1, \psi_f^2, \psi_f^3\big) \in H^3_{(0)}((0,1),\mathbb{C})^3$ be such that

$$\langle \psi_f^j, \psi_f^k \rangle = \delta_{j=k} \text{ and } \sum_{j=1}^{3} ||\psi_f^j - \psi_{ref}^{j,\eta}(T^\eta)||_{H^3_{(0)}} < \tilde{\varepsilon}.$$

Let

$$\big(\tilde{\psi}_f^1, \tilde{\psi}_f^2, \tilde{\psi}_f^3\big) := \big(\tilde{\mathcal{P}}_1(\psi_f^1), \tilde{\mathcal{P}}_2(\psi_f^2), \tilde{\mathcal{P}}_3(\psi_f^3)\big).$$

Let δ be the radius defined in Proposition 3.5. There exists $\varepsilon_0 > 0$ such that for any $\tilde{\varepsilon} \in (0, \varepsilon_0)$, $\big(\tilde{\psi}_f^1, \tilde{\psi}_f^2, \tilde{\psi}_f^3\big) \in \Omega_\delta$ and

$$\operatorname{Re}\big(\langle \psi_f^j, \psi_{ref}^{j,\eta}(T^\eta) \rangle\big) > 0, \quad \forall j \in \{1,2,3\}. \tag{3.32}$$

Let $u := \Upsilon\big(\tilde{\psi}_f^1, \tilde{\psi}_f^2, \tilde{\psi}_f^3\big)$. Let (ψ^1, ψ^2, ψ^3) be the solution of system (3.2)-(3.3) with control u. We prove that

$$\big(\psi^1(T^\eta), \psi^2(T^\eta), \psi^3(T^\eta)\big) = \big(\psi_f^1, \psi_f^2, \psi_f^3\big).$$

Up to a reduction of ε_0, we can assume that

$$\operatorname{Re}\big(\langle \psi^j(T^\eta), \psi_{ref}^{j,\eta}(T^\eta) \rangle\big) > 0, \quad \forall j \in \{1,2,3\}. \tag{3.33}$$

By definition of Υ and $\tilde{\mathcal{P}}_1$ it comes that

$$\psi^1(T^\eta) - \operatorname{Re}(\langle \psi^1(T^\eta), \psi_{ref}^{1,\eta}(T^\eta)\rangle)\psi_{ref}^{1,\eta}(T^\eta) = \psi_f^1 - \operatorname{Re}(\langle \psi_f^1, \psi_{ref}^{1,\eta}(T^\eta)\rangle)\psi_{ref}^{1,\eta}(T^\eta).$$

Thanks to (3.32)-(3.33) and the fact that $||\psi^1(T^\eta)||_{L^2} = ||\psi_f^1||_{L^2}$, we get

$$\psi^1(T^\eta) = \psi_f^1. \tag{3.34}$$

The equality $\tilde{\mathcal{P}}_2(\psi^2(T^\eta)) = \tilde{\psi}_f^2$ gives

$$\begin{aligned} &\psi^2(T^\eta) - \langle \psi^2(T^\eta), \psi_{ref}^{1,\eta}(T^\eta)\rangle \psi_{ref}^{1,\eta}(T^\eta) - \operatorname{Re}(\langle \psi^2(T^\eta), \psi_{ref}^{2,\eta}(T^\eta)\rangle)\psi_{ref}^{2,\eta}(T^\eta) \\ &= \psi_f^2 - \langle \psi_f^2, \psi_{ref}^{1,\eta}(T^\eta)\rangle \psi_{ref}^{1,\eta}(T^\eta) - \operatorname{Re}(\langle \psi_f^2, \psi_{ref}^{2,\eta}(T^\eta)\rangle)\psi_{ref}^{2,\eta}(T^\eta). \end{aligned} \tag{3.35}$$

Taking the scalar product of (3.35) with ψ_f^1, using (3.34) and the constraints $\langle \psi_f^2, \psi_f^1 \rangle = \langle \psi^2(T^\eta), \psi^1(T^\eta)\rangle = 0$, it comes that

$$\begin{aligned} &\langle \psi^2(T^\eta), \psi_{ref}^{1,\eta}(T^\eta)\rangle \langle \psi_{ref}^{1,\eta}(T^\eta), \psi_f^1\rangle + \operatorname{Re}\big(\langle \psi^2(T^\eta), \psi_{ref}^{2,\eta}(T^\eta)\rangle\big)\langle \psi_{ref}^{2,\eta}(T^\eta), \psi_f^1\rangle \\ &= \langle \psi_f^2, \psi_{ref}^{1,\eta}(T^\eta)\rangle \langle \psi_{ref}^{1,\eta}(T^\eta), \psi_f^1\rangle + \operatorname{Re}\big(\langle \psi_f^2, \psi_{ref}^{2,\eta}(T^\eta)\rangle\big)\langle \psi_{ref}^{2,\eta}(T^\eta), \psi_f^1\rangle. \end{aligned} \tag{3.36}$$

As $||\psi^2(T^\eta)||_{L^2} = ||\psi_f^2||_{L^2}$, we also get

$$\begin{aligned} &|\langle \psi^2(T^\eta), \psi_{ref}^{1,\eta}(T^\eta)\rangle|^2 + \operatorname{Re}\big(\langle \psi^2(T^\eta), \psi_{ref}^{2,\eta}(T^\eta)\rangle\big)^2 \\ &\qquad = |\langle \psi_f^2, \psi_{ref}^{1,\eta}(T^\eta)\rangle|^2 + \operatorname{Re}\big(\langle \psi_f^2, \psi_{ref}^{2,\eta}(T^\eta)\rangle\big)^2. \end{aligned} \tag{3.37}$$

Straightforward computations prove that, up to an a priori reduction of ε_0, equalities (3.36) and (3.37) imply

$$\operatorname{Re}\big(\langle \psi^2(T^\eta), \psi_{ref}^{2,\eta}(T^\eta)\rangle\big) = \operatorname{Re}\big(\langle \psi_f^2, \psi_{ref}^{2,\eta}(T^\eta)\rangle\big) \tag{3.38}$$

Then, (3.36) imply $\langle \psi^2(T^\eta), \psi^{1,\eta}_{ref}(T^\eta)\rangle = \langle \psi^2_f, \psi^{1,\eta}_{ref}(T^\eta)\rangle$. Finally, using these two last equalities in (3.35), we obtain

$$\psi^2(T^\eta) = \psi^2_f. \tag{3.39}$$

Using $\tilde{\mathcal{P}}_3(\psi^3(T^\eta)) = \tilde{\psi}^3_f$ and the exact same strategy we also get

$$\psi^3(T^\eta) = \psi^3_f. \tag{3.40}$$

Thus equalities (3.34), (3.39) and (3.40) end the proof of Theorem 3.5 with $T^* := T^\eta$ and

$$\Gamma : \big(\psi^1_f, \psi^2_f, \psi^3_f\big) \mapsto \Upsilon\big(\tilde{\mathcal{P}}_1(\psi^1_f), \tilde{\mathcal{P}}_2(\psi^2_f), \tilde{\mathcal{P}}_3(\psi^3_f)\big).$$

□

Remark 3.14. As mentioned in Remark 3.6, a slight change in the proof allows to prove Theorem 3.5 for initial conditions $(\psi^1_0, \psi^2_0, \psi^3_0)$ close enough to $(\varphi_1, \varphi_2, \varphi_3)$ satisfying

$$\langle \psi^j_0, \psi^k_0\rangle = \langle \psi^j_f, \psi^k_f\rangle, \quad \forall j,k \in \{1,2,3\}. \tag{3.41}$$

To this aim, the inverse mapping theorem is applied at the point $(u^\eta_{ref}, \varphi_1, \varphi_2, \varphi_3)$ to the map

$$\Lambda : L^2((0,T^\eta),\mathbb{R}) \times (\mathcal{S} \cap H^3_{(0)}(0,1))^3 \to (\mathcal{S} \cap H^3_{(0)}(0,1))^3 \times X^f_{T^\eta}$$

defined by

$$\Lambda\big(u, \psi^1_0, \psi^2_0, \psi^3_0\big) = \big((\psi^j_0)_{j=1,2,3}, \tilde{\mathcal{P}}_j(\psi^j(T^\eta))_{j=1,2,3}\big).$$

The compatibility condition (3.41) will then lead to (3.36), the conclusion being unchanged.

3.5 Controllability results for two equations

Theorem 3.5 leads to local exact controllability up to a global phase and a global delay in the case $N = 2$. Actually the strategy we developed can be improved in this case to obtain less restrictive results, namely Theorems 3.2 and 3.3. Here, we only detail the construction of the reference trajectory, the application of the return method being very similar to Section 3.4. Subsection 3.5.1 will imply Theorem 3.2 and Subsection 3.5.2 will imply Theorem 3.3. In all this section, we consider $N = 2$. Let $T_1 > 0$ and $\varepsilon \in (0, T_1)$. As in Theorem 3.6, the reference control is designed in two steps.
Let $u \equiv 0$ on $[0, \frac{\varepsilon}{2})$. Proposition 3.2 is replaced by the following proposition.

Proposition 3.6. *There exists $\eta^* > 0$ and a C^1 map*

$$\hat{\Gamma} : (0,\eta^*) \to L^2\left(\left(\frac{\varepsilon}{2}, \varepsilon\right), \mathbb{R}\right),$$

satisfying $\hat{\Gamma}(0) = 0$ such that for any $\eta \in (0,\eta^*)$, the solution $(\psi^{1,\eta}_{ref}, \psi^{2,\eta}_{ref})$ of system (3.2) with control $u := \hat{\Gamma}(\eta)$ and initial conditions $\psi^{j,\eta}_{ref}(\frac{\varepsilon}{2}) = \Phi_j(\frac{\varepsilon}{2})$ for $j = 1, 2$ satisfies

$$\langle \mu\psi^{1,\eta}_{ref}(\varepsilon), \psi^{1,\eta}_{ref}(\varepsilon)\rangle = \langle \mu\varphi_1, \varphi_1\rangle + \eta,$$
$$\langle \mu\psi^{2,\eta}_{ref}(\varepsilon), \psi^{2,\eta}_{ref}(\varepsilon)\rangle = \langle \mu\varphi_2, \varphi_2\rangle.$$

As previously, this proposition will ensure controllability of the linearized system around the reference trajectory. The proof is a simple adaptation of Proposition 3.2 and is not detailed.
We now turn to two different constructions of reference trajectories on (ε, T_1), to replace Proposition 3.3.

3.5.1 Controllability up to a global phase in arbitrary time : Theorem 3.2

Let $T > 0$ be arbitrary. Up to a reduction of ε, we assume that $T = T_1$. We prove that there exists a global phase $\theta^\eta > 0$ and a control u^η_{ref} on (ε, T) such that the associated trajectory $(\psi^{1,\eta}_{ref}, \psi^{2,\eta}_{ref})$ of (3.2)-(3.3) satisfies Proposition 3.6,

$$(\psi^{1,\eta}_{ref}(T), \psi^{2,\eta}_{ref}(T)) = e^{i\theta^\eta}(\Phi_1(T), \Phi_2(T)), \tag{3.42}$$

and $||u^\eta_{ref}||_{L^2(0,T)} \leq C\eta$.
Proposition 3.3 is replaced by the following proposition which proof is a simple adaptation of the one of Proposition 3.3 and is not detailed.

Proposition 3.7. *There exists $\delta > 0$ and a C^1-map*

$$\tilde{\Gamma} : \tilde{\mathcal{O}}_\delta \to L^2((\varepsilon, T), \mathbb{R})$$

with

$$\tilde{\mathcal{O}}_\delta := \left\{ \big(\psi^1_0, \psi^2_0\big) \in (\mathcal{S} \cap H^3_{(0)}(0,1))^2 \, ; \, \sum_{j=1}^{2} ||\psi^j_0 - \Phi_j(\varepsilon)||_{H^3_{(0)}} < \delta \right\},$$

such that $\tilde{\Gamma}\big(\Phi_1(\varepsilon), \Phi_2(\varepsilon)\big) = 0$ and, if $(\psi^1_0, \psi^2_0) \in \tilde{\mathcal{O}}_\delta$, the solution $\big(\psi^1, \psi^2\big)$ of system (3.2) with initial conditions $\psi^j(\varepsilon, \cdot) = \psi^j_0$, for $j = 1, 2$, and control $u := \tilde{\Gamma}\big(\psi^1_0, \psi^2_0\big)$ satisfies

$$\mathcal{P}_1\big(\psi^1(T)\big) = \mathcal{P}_2\big(\psi^2(T)\big) = 0, \tag{3.43}$$

$$\mathrm{Im}\left(\langle \psi^1(T), \Phi_1(T)\rangle \overline{\langle \psi^2(T), \Phi_2(T)\rangle}\right) = 0. \tag{3.44}$$

There exists $\overline{\eta} > 0$ such that for $\eta \in (0, \overline{\eta})$, the control

$$u^\eta_{ref}(t) := \begin{cases} 0 & \text{for } t \in (0, \frac{\varepsilon}{2}), \\ \hat{\Gamma}(\eta) & \text{for } t \in (\frac{\varepsilon}{2}, \varepsilon), \\ \tilde{\Gamma}(\psi^{1,\eta}_{ref}(\varepsilon), \psi^{2,\eta}_{ref}(\varepsilon)) & \text{for } t \in (\varepsilon, T), \end{cases} \tag{3.45}$$

is well defined and satisfies $||u^\eta_{ref}||_{L^2(0,T)} \leq C\eta$, where $\hat{\Gamma}$ and $\tilde{\Gamma}$ are defined respectively in Proposition 3.6 and 3.7. Proposition 3.7 implies that

$$\begin{aligned}
&\psi^{1,\eta}_{ref}(T) = \langle \psi^{1,\eta}_{ref}(T), \Phi_1(T)\rangle \Phi_1(T), \\
&\psi^{2,\eta}_{ref}(T) = \langle \psi^{2,\eta}_{ref}(T), \Phi_1(T)\rangle \Phi_1(T) + \langle \psi^{2,\eta}_{ref}(T), \Phi_2(T)\rangle \Phi_2(T), \\
&\mathrm{Im}\left(\langle \psi^{1,\eta}_{ref}(T), \Phi_1(T)\rangle \overline{\langle \psi^{2,\eta}_{ref}(T), \Phi_2(T)\rangle}\right) = 0.
\end{aligned}$$

Thus, using the invariant of the system, it comes that there exist θ_1^η, $\theta_2^\eta \in [0, 2\pi)$ such that

$$(\psi_{ref}^{1,\eta}(T), \psi_{ref}^{2,\eta}(T)) = \big(e^{-i\theta_1^\eta}\Phi_1(T), e^{-i\theta_2^\eta}\Phi_2(T)\big),$$

and

$$\theta_1^\eta - \theta_2^\eta \equiv 0\,[2\pi].$$

Finally, this implies that there exists $\theta^\eta \in \mathbb{R}$ such that

$$(\psi_{ref}^{1,\eta}(T), \psi_{ref}^{2,\eta}(T)) = e^{i\theta^\eta}(\Phi_1(T), \Phi_2(T)).$$

Then, application of the return method along this trajectory as in Section 3.4 implies Theorem 3.2.

Remark 3.15. To investigate controllability properties up to a global phase, as proposed in [105], one can introduce a fictitious control ω in the following way

$$\begin{cases} i\partial_t\psi^j = -\partial_{xx}^2\psi^j - u(t)\mu(x)\psi^j - \omega(t)\psi^j, & (t,x) \in (0,T)\times(0,1),\ j \in \{1,2\}, \\ \psi^j(t,0) = \psi^j(t,1) = 0, & t \in (0,T),\ j \in \{1,2\}. \end{cases}$$

Adapting the strategy of [16, Theorem 1], one can prove local controllability of this system by linearization around the trajectory $(\Phi_1, \Phi_2, u \equiv 0, \omega \equiv 0)$. This would lead to local controllability up to a global phase. However, in this case, one would obtain for each target (ψ_f^1, ψ_f^2) close enough to (Φ_1, Φ_2) a global phase $\theta = \theta(\psi_f^1, \psi_f^2)$ such that there exists a control driving the solution of (3.2) from (3.3) to $e^{i\theta}(\psi_f^1, \psi_f^2)$.

3.5.2 Exact controllability up to a global delay : Theorem 3.3

We prove that there exists $T^\eta > 0$ and a control u_{ref}^η on (ε, T_1) such that if u_{ref}^η is extended by 0 on (T_1, T^η), the associated trajectory $(\psi_{ref}^{1,\eta}, \psi_{ref}^{2,\eta})$ of (3.2)-(3.3) satisfies Proposition 3.6,

$$(\psi_{ref}^{1,\eta}(T^\eta), \psi_{ref}^{2,\eta}(T^\eta)) = (\varphi_1, \varphi_2), \tag{3.46}$$

and $||u_{ref}^\eta||_{L^2(0,T^\eta)} \leq C\eta$.
Proposition 3.3 is replaced by the following proposition which proof is a simple adaptation of the one of Proposition 3.3 and is not detailed.

Proposition 3.8. *There exists $\delta > 0$ and a C^1-map*

$$\tilde{\Gamma} : \tilde{\mathcal{O}}_\delta \to L^2((\varepsilon, T_1), \mathbb{R})$$

with

$$\tilde{\mathcal{O}}_\delta := \left\{ \big(\psi_0^1, \psi_0^2\big) \in (\mathcal{S} \cap H^3_{(0)}(0,1))^2 \,;\, \sum_{j=1}^2 ||\psi_0^j - \Phi_j(\varepsilon)||_{H^3_{(0)}} < \delta \right\},$$

such that $\tilde{\Gamma}\big(\Phi_1(\varepsilon), \Phi_2(\varepsilon)\big) = 0$ and, if $(\psi_0^1, \psi_0^2) \in \tilde{\mathcal{O}}_\delta$, the solution $\big(\psi^1, \psi^2\big)$ of system (3.2) with initial conditions $\psi^j(\varepsilon, \cdot) = \psi_0^j$, for $j = 1, 2$, and control $u := \tilde{\Gamma}\big(\psi_0^1, \psi_0^2\big)$ satisfies

$$\mathcal{P}_1\big(\psi^1(T_1)\big) = \mathcal{P}_2\big(\psi^2(T_1)\big) = 0, \tag{3.47}$$

$$\operatorname{Im}\Big(\langle\psi^1(T_1), \Phi_1(T_1)\rangle^4\, \overline{\langle\psi^2(T_1), \Phi_2(T_1)\rangle}\Big) = 0. \tag{3.48}$$

There exists $\overline{\eta} > 0$ such that for $\eta \in (0, \overline{\eta})$, the control

$$u^{\eta}_{ref}(t) := \begin{cases} 0 & \text{for } t \in (0, \frac{\varepsilon}{2}), \\ \hat{\Gamma}(\eta) & \text{for } t \in (\frac{\varepsilon}{2}, \varepsilon), \\ \tilde{\Gamma}(\psi^{1,\eta}_{ref}(\varepsilon), \psi^{2,\eta}_{ref}(\varepsilon)) & \text{for } t \in (\varepsilon, T_1), \end{cases} \tag{3.49}$$

is well defined and satisfies $||u^{\eta}_{ref}||_{L^2(0,T_1)} \leq C\eta$, where $\hat{\Gamma}$ and $\tilde{\Gamma}$ are defined respectively in Proposition 3.6 and 3.8. Proposition 3.8 implies the existence of θ^{η}_1, $\theta^{\eta}_2 \in [0, 2\pi)$ such that

$$\begin{aligned} (\psi^{1,\eta}_{ref}(T_1), \psi^{2,\eta}_{ref}(T_1)) &= \big(e^{-i\theta^{\eta}_1}\Phi_1(T_1), e^{-i\theta^{\eta}_2}\Phi_2(T_1)\big), \\ 4\theta^{\eta}_1 - \theta^{\eta}_2 &\equiv 0\,[2\pi]. \end{aligned}$$

Let $T^{\eta} > T_1$ be such that

$$\theta^{\eta}_1 + \lambda_1 T^{\eta} \equiv 0\,[2\pi]$$

Thus,

$$\theta^{\eta}_2 + \lambda_2 T^{\eta} \equiv 4\big(\theta^{\eta}_1 + \lambda_1 T^{\eta}\big) \equiv 0\,[2\pi].$$

Finally, if we extend u^{η}_{ref} by 0 on (T_1, T^{η}), we have that $(\psi^{1,\eta}_{ref}, \psi^{2,\eta}_{ref})$ is solution of (3.2)-(3.3) with control u^{η}_{ref} and satisfies

$$\psi^{j,\eta}_{ref}(T^{\eta}) = e^{-i(\theta^{\eta}_j + \lambda_j T^{\eta})}\varphi_j = \varphi_j.$$

Then, application of the return method along this trajectory as in Section 3.4 implies Theorem 3.3.

3.6 Non controllability results in small time

The goal of this section is the proof of Theorems 3.1 and 3.4.

3.6.1 Heuristic of non controllability

We adapt the strategy developed in [18] by Beauchard and the author in the case $N = 1$. Using power series expansion, we consider

$$\begin{aligned} u &= 0 + \varepsilon v, \\ \psi^j &= \Phi_j + \varepsilon \Psi^j + \varepsilon^2 \xi^j + o(\varepsilon^2), \quad \forall j \in \{1, \dots, N\}. \end{aligned} \tag{3.50}$$

Here and in the following, we use the classical Landau notations. We say that $f = \underset{x\to a}{O}(g)$ if there exist $C > 0$ and a neighbourhood $\mathcal{V}(a)$ of a such that $||f(x)|| \leq C||g(x)||$ for $x \in \mathcal{V}(a)$. We say that $f = \underset{x\to a}{o}(g)$ if for any $\delta > 0$ there exists a neighbourhood $\mathcal{V}(a)$ of a such that $||f(x)|| \leq \delta||g(x)||$ for $x \in \mathcal{V}(a)$.
Considering (3.50), we define the following systems for $j \in \{1, \dots, N\}$,

$$\begin{cases} i\partial_t \Psi^j = -\partial^2_{xx}\Psi^j - v(t)\mu(x)\Phi_j, & (t,x) \in (0,T) \times (0,1), \\ \Psi^j(t,0) = \Psi^j(t,1) = 0, & t \in (0,T), \\ \Psi^j(0,x) = 0, & x \in (0,1), \end{cases} \tag{3.51}$$

and

$$\begin{cases} i\partial_t \xi^j = -\partial^2_{xx}\xi^j - v(t)\mu(x)\Psi^j, & (t,x)\in(0,T)\times(0,1),\\ \xi^j(t,0)=\xi^j(t,1)=0, & t\in(0,T),\\ \xi^j(0,x)=0, & x\in(0,1). \end{cases} \tag{3.52}$$

We focus in this heuristic on the case $N=2$. Let us try to reach

$$\big(\psi^1(T),\psi^2(T)\big) = \Big(\Phi_1(T), (\sqrt{1-\delta^2}+i\alpha\delta)\Phi_2(T)\Big), \tag{3.53}$$

with $\delta>0$ and α defined in Theorem 3.1 from $(\psi^1(0),\psi^2(0))=(\varphi_1,\varphi_2)$. Condition (3.53) imposes $\Psi^1(T)=0$ i.e.

$$v\in V_T := \left\{ v\in L^2((0,T),\mathbb{R})\,;\ \int_0^T v(t)e^{i(\lambda_k-\lambda_1)t}\mathrm{d}t = 0,\ \forall k\in\mathbb{N}^* \right\}.$$

Let us define the following quadratic forms, for $j\in\{1,2\}$, associated to the second order

$$\begin{aligned} Q_{T,j}(v) :&= \mathrm{Im}\left(\langle \xi^j(T),\Phi_j(T)\rangle\right)\\ &= \int_0^T v(t)\int_0^t v(\tau)\left(\sum_{k=1}^{+\infty}\langle\mu\varphi_j,\varphi_k\rangle^2\sin((\lambda_k-\lambda_j)(t-\tau))\right)\mathrm{d}\tau\mathrm{d}t, \end{aligned}$$

and

$$Q_T(v) := \langle\mu\varphi_1,\varphi_1\rangle Q_{T,2}(v) - \langle\mu\varphi_2,\varphi_2\rangle Q_{T,1}(v). \tag{3.54}$$

The following proposition states that in time small enough, the quadratic form Q_T has a sign on V_T.

Proposition 3.9. *Assume that μ satisfies Hypothesis 3.2. Then, there exists $T_*>0$ such that for any $T\in(0,T_*)$, for any $v\in V_T\backslash\{0\}$,*

$$\mathcal{A}Q_T(v)<0,$$

where $\mathcal{A}\in\mathbb{R}^*$ is defined in Hypothesis 3.2.

Proof of Proposition 3.9. Let $v\in V_T$ and $s: t\in(0,T)\mapsto\int_0^t v(\tau)\mathrm{d}\tau$. Performing integrations by part, we define a new quadratic form

$$\mathcal{Q}_{T,j}(s) := -\langle(\mu')^2\varphi_j,\varphi_j\rangle\int_0^T s(t)^2\mathrm{d}t + \int_0^T s(t)\int_0^t s(\tau)h_j(t-\tau)\mathrm{d}\tau\mathrm{d}t = Q_{T,j}(v), \tag{3.55}$$

where $h_j: t\mapsto\sum_{k=1}^{+\infty}(\lambda_k-\lambda_j)^2\langle\mu\varphi_j,\varphi_k\rangle^2\sin((\lambda_k-\lambda_j)t)$. As $\mu\in H^3((0,1),\mathbb{R})$, it comes that $h_j\in C^0(\mathbb{R},\mathbb{R})$. Thus, if we define

$$\mathcal{Q}_T(s) := \langle\mu\varphi_1,\varphi_1\rangle\mathcal{Q}_{T,2}(s) - \langle\mu\varphi_2,\varphi_2\rangle\mathcal{Q}_{T,1}(s), \tag{3.56}$$

we get that

$$Q_T(v) = \mathcal{Q}_T(s) = -\mathcal{A}||s||^2_{L^2} + \int_0^T s(t)\int_0^t s(\tau)h(t-\tau)\mathrm{d}\tau\mathrm{d}t,$$

with

$$h := \langle \mu\varphi_1, \varphi_1\rangle h_2 - \langle \mu\varphi_2, \varphi_2\rangle h_1 \in C^0(\mathbb{R}, \mathbb{R}).$$

We can assume, without loss of generality, that $\mathcal{A} > 0$. Thus, there exists $C = C(\mu) > 0$ such that

$$Q_T(v) \leq \big(-\mathcal{A} + CT\big)||s||^2_{L^2}. \tag{3.57}$$

We conclude the proof by choosing $T_* < \frac{\mathcal{A}}{C}$.

□

Remark 3.16. This Proposition indicates that, in small time, there are targets that cannot be reached. However, using the theory of Legendre form (see e.g. [83, 24]), we can prove that Q_T lacks coercivity in $L^2((0,T),\mathbb{R})$. This is why we work directly with the quadratic form $\mathcal{Q}_T$ adapted to the auxiliary system defined in Subsection 3.6.2 where the control is s and not v.

Remark 3.17. This strategy is only valid for small time and we do not know if this quadratic form changes sign in time large enough on V_T. Following the strategy of [18], this would imply local exact controllability in large time but it is an open question.

3.6.2 Auxiliary system

For $j \in \{1,\dots,N\}$, we consider the function $\tilde{\psi}^j$ defined by

$$\psi^j(t,x) = \tilde{\psi}^j(t,x)e^{is(t)\mu(x)} \text{ with } s(t) := \int_0^t u(\tau)\mathrm{d}\tau. \tag{3.58}$$

It is a weak solution of

$$\begin{cases} i\partial_t\tilde{\psi}^j = -\partial^2_{xx}\tilde{\psi}^j - is(t)\big(2\mu'(x)\partial_x\tilde{\psi}^j + \mu''(x)\tilde{\psi}^j\big) + s(t)^2\mu'(x)^2\tilde{\psi}^j, \\ \tilde{\psi}^j(t,0) = \tilde{\psi}^j(t,1) = 0, \\ \tilde{\psi}^j(0,\cdot) = \varphi_j. \end{cases} \tag{3.59}$$

Using Proposition 3.1 on (3.2) and (3.58), it follows that the following well posedness result holds. In the following, the time derivative of s will be denoted by $\dot{s}$.

Proposition 3.10. *Let $\mu \in H^3((0,1),\mathbb{R})$, $T > 0$, $s \in H^1((0,T),\mathbb{R})$ with $s(0) = 0$. There exists a unique weak solution $(\tilde{\psi}^1,\dots,\tilde{\psi}^N) \in C^0([0,T], H^3 \cap H^1_0)^N$ of system (3.59). Moreover, for every $R > 0$, there exists $C = C(T,\mu,R) > 0$ such that, if $\|\dot{s}\|_{L^2(0,T)} < R$, then this weak solution satisfies for any $j \in \{1,\dots,N\}$,*

$$\|\tilde{\psi}^j\|_{L^\infty((0,T),H^3\cap H^1_0)} \leqslant C.$$

3.6.3 Non exact controllability in arbitrary time with $N = 2$.

In this subsection, we consider system (3.2) with $N = 2$ and prove Theorem 3.1. This result is a corollary of the following theorem for the auxiliary system.

Theorem 3.7. *Let $\mu \in H^3((0,1),\mathbb{R})$ be such that Hypothesis 3.2 hold. Let $T_* > 0$ be as in Proposition 3.9 and $\alpha \in \{-1,1\}$ as in Theorem 3.1. For any $T < T_*$, there exists $\varepsilon > 0$ such that for every $s \in H^1((0,T),\mathbb{R})$ with $s(0)=0$ and $||\dot{s}||_{L^2} < \varepsilon$, the solution of system (3.59) satisfies*

$$\left(\tilde{\psi}^1(T), \tilde{\psi}^2(T)\right) \neq \left(\Phi_1(T)e^{i\theta\mu}, \left(\sqrt{1-\delta^2}+i\alpha\delta\right)\Phi_2(T)e^{i\theta\mu}\right), \quad \forall \delta > 0, \forall \theta \in \mathbb{R}.$$

Before getting into the proof of Theorem 3.7, we prove that it implies Theorem 3.1.

Proof of Theorem 3.1. Let $T < T_*$ and $\varepsilon > 0$ defined by Theorem 3.7. Let $u \in L^2((0,T),\mathbb{R})$ be such that $||u||_{L^2(0,T)} < \varepsilon$. Assume by contradiction that

$$\left(\psi^1(T), \psi^2(T)\right) = \left(\Phi_1(T), \left(\sqrt{1-\delta^2}+i\alpha\delta\right)\Phi_2(T)\right),$$

for some $\delta > 0$. Let s and $\tilde{\psi}^j$ be defined by (3.58). Then $s(0)=0$, $||\dot{s}||_{L^2} < \varepsilon$ and $\tilde{\psi}^j$ is solution of (3.59) and satisfies

$$\left(\tilde{\psi}^1(T), \tilde{\psi}^2(T)\right) = \left(\Phi_1(T)e^{-is(T)\mu}, \left(\sqrt{1-\delta^2}+i\alpha\delta\right)\Phi_2(T)e^{-is(T)\mu}\right).$$

Thanks to Theorem 3.7, this is impossible.

□

Proof of Theorem 3.7. Without loss of generality, we assume that $\mathcal{A} > 0$.

First step : we prove that $-\mathcal{Q}_T$ is coercive for $T < T_*$.
Using the same estimates as in (3.57) and the fact that $T_* < \frac{\mathcal{A}}{\mathcal{C}}$, we get that there exists $C_* > 0$ such that for $T < T_*$

$$\mathcal{Q}_T(s) \leq -C_*||s||^2_{L^2}, \quad \forall s \in L^2((0,T),\mathbb{R}). \tag{3.60}$$

Second step : approximation of first and second order.
Using the first and second order approximation of (3.59), the following lemma holds.

Lemma 3.4. *Let $T > 0$ and $\mu \in H^3((0,1),\mathbb{R})$. For all $j \in \{1,\dots,N\}$*

$$\left|\mathrm{Im}(\langle\tilde{\psi}^j(T), \Phi_j(T)\rangle) - \mathcal{Q}_{T,j}(s)\right| = o(||s||^2_{L^2}) \text{ when } ||\dot{s}||_{L^2} \to 0,$$
$$\left|\mathrm{Im}(\langle\tilde{\psi}^j(T), \Phi_j(T)\rangle)\right| = o(||s||_{L^2}) \text{ when } ||\dot{s}||_{L^2} \to 0.$$

Proof of Lemma 3.4. Let $j \in \{1,\dots,N\}$. As proved in [18, Proposition 3], if we define the first and second order approximations, $\tilde{\Psi}^j$ and $\tilde{\xi}^j$, by

$$\Psi^j(t,x) = \tilde{\Psi}^j(t,x) + is(t)\mu(x)\Phi_j(t,x), \tag{3.61}$$

and

$$\xi^j(t,x) = \tilde{\xi}^j(t,x) + is(t)\mu(x)\tilde{\Psi}^j(t,x) - \frac{s(t)^2}{2}\mu(x)^2\Phi_j(t,x), \tag{3.62}$$

it comes that, when $||\dot{s}||_{L^2} \to 0$

$$||\tilde{\psi}^j - \Phi_j - \tilde{\Psi}^j||_{L^\infty((0,T),H^1_0)} = o(||s||_{L^2}) \tag{3.63}$$

and

$$||\tilde{\psi}^j - \Phi_j - \tilde{\Psi}^j - \tilde{\xi}^j||_{L^\infty((0,T),L^2)} = o(||s||^2_{L^2}). \tag{3.64}$$

Straightforward computations using (3.61) imply $\text{Im}(\langle \tilde{\Psi}^j(T), \Phi_j(T)\rangle) = 0$. Thus, from (3.63) we deduce

$$\Big|\text{Im}(\langle \tilde{\psi}^j(T), \Phi_j(T)\rangle)\Big| = \Big|\text{Im}(\langle (\tilde{\psi}^j - \Phi_j - \tilde{\Psi}^j)(T), \Phi_j(T)\rangle)\Big| = \underset{||\dot{s}||_{L^2}\to 0}{o}(||s||_{L^2}).$$

Straightforward computations using (3.62) imply $\text{Im}(\langle \tilde{\xi}^j(T), \Phi_j(T)\rangle) = \mathcal{Q}_{T,j}(s)$. Thus, from (3.64) we deduce

$$\begin{aligned}\Big|\text{Im}(\langle \tilde{\psi}^j(T), \Phi_j(T)\rangle) - \mathcal{Q}_{T,j}(s)\Big| &= \Big|\text{Im}(\langle (\tilde{\psi}^j - \Phi_j - \tilde{\Psi}^j - \tilde{\xi}^j)(T), \Phi_j(T)\rangle)\Big| \\ &= \underset{||\dot{s}||_{L^2}\to 0}{o}(||s||^2_{L^2}).\end{aligned}$$

This ends the proof of Lemma 3.4. □

Third step : conclusion.
Let $T < T_*$. Assume by contradiction, that $\forall \varepsilon > 0$, $\exists s_\varepsilon \in H^1((0,T),\mathbb{R})$ with $s_\varepsilon(0) = 0$ and $||\dot{s}_\varepsilon||_{L^2} < \varepsilon$ such that the associated solution of (3.59) satisfies

$$\left(\tilde{\psi}^1_\varepsilon(T), \tilde{\psi}^2_\varepsilon(T)\right) = \left(\Phi_1(T)e^{i\theta_\varepsilon \mu}, \left(\sqrt{1-\delta_\varepsilon^2} + i\alpha\delta_\varepsilon\right)\Phi_2(T)e^{i\theta_\varepsilon\mu}\right),$$

with $\delta_\varepsilon > 0$ and $\theta_\varepsilon \in \mathbb{R}$. Notice that

$$\delta_\varepsilon \underset{\varepsilon\to 0}{\to} 0, \quad \theta_\varepsilon \underset{\varepsilon\to 0}{\to} 0.$$

Explicit computations lead to

$$\text{Im}\left(\langle \tilde{\psi}^1_\varepsilon(T), \Phi_1(T)\rangle\right) = \langle \mu\varphi_1, \varphi_1\rangle \theta_\varepsilon + \underset{\varepsilon\to 0}{O}(\theta_\varepsilon^3),$$

and

$$\text{Im}\left(\langle \tilde{\psi}^2_\varepsilon(T), \Phi_2(T)\rangle\right) = \alpha\delta_\varepsilon + \sqrt{1-\delta_\varepsilon^2}\langle \mu\varphi_2, \varphi_2\rangle\theta_\varepsilon + \underset{\varepsilon\to 0}{O}(\theta_\varepsilon^2).$$

Thus, it comes that

$$\begin{aligned}&\langle \mu\varphi_1,\varphi_1\rangle \,\text{Im}(\langle \tilde{\psi}^2_\varepsilon(T), \Phi_2(T)\rangle) - \langle \mu\varphi_2,\varphi_2\rangle\,\text{Im}(\langle \tilde{\psi}^1_\varepsilon(T), \Phi_1(T)\rangle) \\ &= \alpha\langle\mu\varphi_1,\varphi_1\rangle\delta_\varepsilon - \langle\mu\varphi_1,\varphi_1\rangle\langle\mu\varphi_2,\varphi_2\rangle \frac{\delta_\varepsilon^2}{\sqrt{1-\delta_\varepsilon^2}+1}\theta_\varepsilon + \underset{\varepsilon\to 0}{O}(\theta_\varepsilon^2).\end{aligned}$$

Using Lemma 3.4 to estimate $\text{Im}\left(\langle \tilde{\psi}^1_\varepsilon(T), \Phi_1(T)\rangle\right)$ and $\text{Im}\left(\langle \tilde{\psi}^2_\varepsilon(T), \Phi_2(T)\rangle\right)$ it comes that

$$\theta_\varepsilon = \underset{\varepsilon\to 0}{o}(||s_\varepsilon||_{L^2}), \quad \delta_\varepsilon = \underset{\varepsilon\to 0}{o}(||s_\varepsilon||_{L^2}).$$

Thus,

$$\begin{aligned}&\langle\mu\varphi_1,\varphi_1\rangle\,\mathrm{Im}(\langle\tilde{\psi}^2_\varepsilon(T),\Phi_2(T)\rangle)-\langle\mu\varphi_2,\varphi_2\rangle\,\mathrm{Im}(\langle\tilde{\psi}^1_\varepsilon(T),\Phi_1(T)\rangle)\\&=\langle\mu\varphi_1,\varphi_1\rangle\alpha\delta_\varepsilon+\underset{\varepsilon\to0}{o}(||s_\varepsilon||^2_{L^2}).\end{aligned}$$

Finally, combining this with Lemma 3.4 and (3.60), we obtain

$$\begin{aligned}0&<\alpha\langle\mu\varphi_1,\varphi_1\rangle\delta_\varepsilon\\&=\langle\mu\varphi_1,\varphi_1\rangle\,\mathrm{Im}(\langle\tilde{\psi}^2_\varepsilon(T),\Phi_2(T)\rangle)-\langle\mu\varphi_2,\varphi_2\rangle\,\mathrm{Im}(\langle\tilde{\psi}^1_\varepsilon(T),\Phi_1(T)\rangle)+\underset{\varepsilon\to0}{o}(||s_\varepsilon||^2_{L^2})\\&=\mathcal{Q}_T(s_\varepsilon)+\underset{\varepsilon\to0}{o}(||s_\varepsilon||^2_{L^2})\\&\leq -C_*||s_\varepsilon||^2_{L^2}+\underset{\varepsilon\to0}{o}(||s_\varepsilon||^2_{L^2}).\end{aligned}$$

This is impossible for ε sufficiently small. This ends the proof of Theorem 3.7.

□

3.6.4 Non exact controllability up to a global phase in arbitrary time with $N = 3$.

In this subsection, we consider system (3.2) with $N = 3$ and prove Theorem 3.4. As previously, this result is a corollary of the following theorem for the auxiliary system.

Theorem 3.8. *Let $\mu \in H^3((0,1),\mathbb{R})$ be such that Hypothesis 3.3 hold. Let $\beta \in \{-1,1\}$ be defined as in Theorem 3.4. There exists $T_* > 0$ and $\varepsilon > 0$ such that for any $T < T_*$, for every $s \in H^1((0,T),\mathbb{R})$ with $s(0) = 0$ and $||\dot{s}||_{L^2} < \varepsilon$, the solution of system (3.59) satisfies*

$$\left(\tilde{\psi}^1(T),\tilde{\psi}^2(T),\tilde{\psi}^3(T)\right)\neq e^{i\nu}\left(\Phi_1(T)e^{i\theta\mu},\Phi_2(T)e^{i\theta\mu},\left(\sqrt{1-\delta^2}+i\beta\delta\right)\Phi_3(T)e^{i\theta\mu}\right),$$

for all $\delta > 0$, for all $\nu, \theta \in \mathbb{R}$.

The proof is very close to the one of Theorem 3.7.

Proof of Theorem 3.8. Without loss of generality, we can assume $\mathcal{B} > 0$. We consider the following quadratic form

$$\begin{aligned}\mathcal{Q}_T(s):=&\big(\langle\mu\varphi_3,\varphi_3\rangle-\langle\mu\varphi_2,\varphi_2\rangle\big)\mathcal{Q}_{T,1}(s)+\big(\langle\mu\varphi_1,\varphi_1\rangle-\langle\mu\varphi_3,\varphi_3\rangle\big)\mathcal{Q}_{T,2}(s)\\&+\big(\langle\mu\varphi_2,\varphi_2\rangle-\langle\mu\varphi_1,\varphi_1\rangle\big)\mathcal{Q}_{T,3}(s),\end{aligned}$$

where $\mathcal{Q}_{T,j}$ is defined as in (3.55). This is rewritten as

$$\mathcal{Q}_T(s)=-\mathcal{B}||s||^2_{L^2}+\int_0^T s(t)\int_0^t s(\tau)h(t-\tau)\mathrm{d}\tau\mathrm{d}t,$$

with $h \in C^0(\mathbb{R},\mathbb{R})$. Thus, there exists $T_* > 0$, $C_* > 0$ such that for all $T < T_*$,

$$\mathcal{Q}_T(s)\leq -C_*||s||^2_{L^2},\quad \forall s\in L^2((0,T),\mathbb{R}).$$

Let $T < T_*$ and assume, by contradiction, that $\forall \varepsilon > 0$, $\exists s_\varepsilon \in H^1((0,T),\mathbb{R})$ with $s_\varepsilon(0) = 0$ and $||\dot{s}_\varepsilon||_{L^2} < \varepsilon$ such that the associated solution of (3.59) satisfies

$$\big(\tilde{\psi}^1_\varepsilon(T), \tilde{\psi}^2_\varepsilon(T), \tilde{\psi}^3_\varepsilon(T)\big) = e^{i\nu_\varepsilon}\big(\Phi_1(T)e^{i\theta_\varepsilon \mu}, \Phi_2(T)e^{i\theta_\varepsilon \mu}, (\sqrt{1-\delta_\varepsilon^2} + i\beta\delta_\varepsilon)\Phi_3(T)e^{i\theta_\varepsilon \mu}\big),$$

with $\nu_\varepsilon, \theta_\varepsilon \in \mathbb{R}$ and $\delta_\varepsilon > 0$. Notice that,

$$\delta_\varepsilon \underset{\varepsilon\to 0}{\to} 0, \quad \theta_\varepsilon \underset{\varepsilon\to 0}{\to} 0, \quad e^{i\nu_\varepsilon} \underset{\varepsilon\to 0}{\to} 1.$$

Straightforward computations and Lemma 3.4 to estimate the terms $\operatorname{Im}\big(\langle \tilde{\psi}^1_\varepsilon(T), \Phi_1(T)\rangle\big) - \operatorname{Im}\big(\langle \tilde{\psi}^2_\varepsilon(T), \Phi_2(T)\rangle\big)$, $\operatorname{Im}\big(\langle \tilde{\psi}^1_\varepsilon(T), \Phi_1(T)\rangle\big)$ and $\operatorname{Im}\big(\langle \tilde{\psi}^3_\varepsilon(T), \Phi_3(T)\rangle\big)$ lead to

$$\theta_\varepsilon = \underset{\varepsilon\to 0}{o}(||s_\varepsilon||_{L^2}), \quad \sin(\nu_\varepsilon) = \underset{\varepsilon\to 0}{o}(||s_\varepsilon||_{L^2}), \quad \delta_\varepsilon = \underset{\varepsilon\to 0}{o}(||s_\varepsilon||_{L^2}). \tag{3.65}$$

For the sake of clarity, let us denote

$$\begin{aligned}\mathcal{T}(s_\varepsilon) := &\big(\langle \mu\varphi_3, \varphi_3\rangle - \langle \mu\varphi_2, \varphi_2\rangle\big) \operatorname{Im}(\langle \tilde{\psi}^1_\varepsilon(T), \Phi_1(T)\rangle) \\ &+ \big(\langle \mu\varphi_1, \varphi_1\rangle - \langle \mu\varphi_3, \varphi_3\rangle\big) \operatorname{Im}(\langle \tilde{\psi}^2_\varepsilon(T), \Phi_2(T)\rangle) \\ &+ \big(\langle \mu\varphi_2, \varphi_2\rangle - \langle \mu\varphi_1, \varphi_1\rangle\big) \operatorname{Im}(\langle \tilde{\psi}^3_\varepsilon(T), \Phi_3(T)\rangle).\end{aligned}$$

Using estimates (3.65), straightforward computations lead to

$$\mathcal{T}(s_\varepsilon) = \beta\big(\langle \mu\varphi_2, \varphi_2\rangle - \langle \mu\varphi_1, \varphi_1\rangle\big)\cos(\nu_\varepsilon)\delta_\varepsilon + \underset{\varepsilon\to 0}{o}(||s_\varepsilon||^2_{L^2}).$$

Finally, for ε sufficiently small,

$$\begin{aligned}0 &< \beta\big(\langle \mu\varphi_2, \varphi_2\rangle - \langle \mu\varphi_1, \varphi_1\rangle\big)\cos(\nu_\varepsilon)\delta_\varepsilon \\ &= \mathcal{T}(s_\varepsilon) + \underset{\varepsilon\to 0}{o}(||s_\varepsilon||^2_{L^2}) \\ &= \mathcal{Q}_T(s_\varepsilon) + \underset{\varepsilon\to 0}{o}(||s_\varepsilon||^2_{L^2}) \\ &\leq -C_*||s_\varepsilon||^2_{L^2} + \underset{\varepsilon\to 0}{o}(||s_\varepsilon||^2_{L^2}).\end{aligned}$$

This is impossible and ends the proof of Theorem 3.8.

□

3.7 Conclusion, open problems and perspectives.

In this article, we have proved that the local exact controllability result of Beauchard and Laurent for a single bilinear Schrödinger equation cannot be adapted to a system of such equations with a single control. Thus, we developed a strategy based on Coron's return method to obtain controllability in arbitrary time up to a global phase or exactly up to a global delay for two equations. For three equations local controllability up to a global phase does not even hold in small time with small controls. Thus, in this setting and under generic assumptions no local controllability result can be proved in small time if $N \geq 3$. Finally, the main result of this article is the construction of a reference trajectory and application

of the return method to prove local exact controllability up to a global phase and a global delay around (Φ_1, Φ_2, Φ_3).
However our non controllability strategy is only valid for small time and we do not know if local exact controllability around the eigenstates (Φ_1, Φ_2) hold in time large enough (for two equations or more). This would be the case if one manages to prove that the global delay T^* can be designed to be the common period of the eigenstates Φ_k i.e. $T^* = \frac{2}{\pi}$. This is an open problem. Moreover, when Hypothesis 3.2 or 3.3 are not satisfied, we do not know if the considered quadratic forms still have a sign. Thus, the question of non controllability when these hypotheses do not hold is an open problem. The question of non controllability with large controls has not been addressed here since our strategy relies on a second order approximation valid for small controls.
The question of controllability of four equations or more is also open. In fact, each time we add an equation there is another diagonal coefficient $\langle \Psi^j, \Phi_j \rangle$ which is lost. We proved that we can recover this lost direction using either a global phase or a global delay for $N = 2$ and both a global phase and a global delay in the case $N = 3$. It seems that there is no other degree of freedom to use to obtain controllability for $N \geq 4$. Moreover, there are other directions than the diagonal ones with the same gap frequencies (e.g. $\lambda_7 - \lambda_1 = \lambda_8 - \lambda_4$). Thus, for $N \geq 4$ one should consider a model with a potential that prevents such resonances.

3.A Moment problems

We define the following space

$$\ell^2_r(\mathbb{N}, \mathbb{C}) := \left\{ (d_k)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{C}) \, ; \, d_0 \in \mathbb{R} \right\}.$$

In this article, we use several times the following moment problem result.

Proposition 3.11. *Let $T>0$. Let $(\omega_n)_{n\in\mathbb{N}}$ be the increasing sequence defined by*

$$\{\omega_n \, ; \, n \in \mathbb{N}\} = \{\lambda_k - \lambda_j \, ; \, j \in \{1,2,3\}, \, k \geq j+1 \text{ and } k = j = 3\}.$$

There exists a continuous linear map

$$\mathcal{L} \, : \, \ell^2_r(\mathbb{N}, \mathbb{C}) \to L^2((0,T), \mathbb{R}),$$

such that for all $d := (d_n)_{n\in\mathbb{N}} \in \ell^2_r(\mathbb{N}, \mathbb{C})$,

$$\int_0^T \mathcal{L}(d)(t) e^{i\omega_n t} \mathrm{d}t = d_n, \quad \forall n \in \mathbb{N}.$$

Proof of Proposition 3.11. For $n \in \mathbb{N}^*$, let $\omega_{-n} := -\omega_n$. Using [92, Theorems 9.1, 9.2], it comes that for any finite interval I, there exists $C_1, C_2 > 0$, such that all finite sums

$$f(t) := \sum_n c_n e^{i\omega_n t}, \quad c_n \in \mathbb{C},$$

satisfy

$$C_1 \sum_n |c_n|^2 \leq \int_I |f(t)|^2 \mathrm{d}t \leq C_2 \sum_n |c_n|^2.$$

This relies on Ingham inequality which holds true for any finite interval as $\lambda_k = k^2\pi^2$. Let $T > 0$ and $H_0 := \text{Adh}_{L^2(0,T)}\big(\text{Span}\{e^{i\omega_n \cdot}\,;\, n \in \mathbb{Z}\}\big)$. Thus, $(e^{i\omega_n \cdot})_{n\in\mathbb{Z}}$ is a Riesz basis of H_0 i.e.

$$\begin{array}{cccc} J_0 : & H_0 & \to & \ell^2(\mathbb{Z},\mathbb{C}) \\ & f & \mapsto & \left(\int_0^T f(t)e^{i\omega_n t}\mathrm{d}t\right)_{n\in\mathbb{Z}} \end{array}$$

is an isomorphism (see e.g. [16, Propositions 19, 20]). Let $d \in \ell^2_r(\mathbb{N},\mathbb{C})$. We define $\tilde{d} := (\tilde{d}_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$ by $\tilde{d}_n := d_n$, for $n \geq 0$ and $\tilde{d}_n := \overline{d_{-n}}$, for $n < 0$. The map $\mathcal{L}$ is defined by $\mathcal{L}(d) := J_0^{-1}(\tilde{d})$. The construction of $\tilde{d}$ and the isomorphism property ensure that $\mathcal{L}(d)$ is real valued.

□

Chapitre 4

Contrôle exact global simultané de N équations

Ce chapitre est inspiré de la prépublication [110] écrite en collaboration avec V. Nersesyan.

Contents

4.1 Introduction

The evolution of a 1D quantum particle submitted to an external laser field is described by the following linear Schrödinger equation

$$\begin{cases} i\partial_t\psi = \left(-\partial^2_{xx} + V(x)\right)\psi - u(t)\mu(x)\psi, & (t,x) \in (0,T)\times(0,1), \\ \psi(t,0) = \psi(t,1) = 0, \end{cases} \tag{4.1}$$

where $V(x)$ is the potential, $\mu(x)$ is the dipole moment of the particle, $\psi(t,x)$ its wave function, and $u(t)$ is the amplitude of the laser. In this setting, we consider N identical and independent particles. Then neglecting entanglement effects, the system will be described by the following equations

$$\begin{cases} i\partial_t \psi^j = \left(-\partial_{xx}^2 + V(x)\right)\psi^j - u(t)\mu(x)\psi^j, & (t,x) \in (0,T)\times(0,1), \\ \psi^j(t,0) = \psi^j(t,1) = 0, & j \in \{1,\dots,N\}, \\ \psi^j(0,x) = \psi_0^j(x). \end{cases} \tag{4.2}$$

This can be seen as a step towards more sophisticated and realistic models. From the point of view of controllability, this is a bilinear control system where the state is the N-tuple of wave functions $(\psi^1,\dots,\psi^N)$ and the control is the real-valued function u. The main result of this article is the global exact controllability of (4.2) for an arbitrary number N of particles, arbitrary potential V, and a generic dipole moment μ.

Before stating our main result, let us introduce some notations. We denote by $\mathcal{S}$ the unit sphere in $L^2((0,1),\mathbb{C})$ and $\boldsymbol{\mathcal{S}} := \mathcal{S}^N$. Since the functions V,μ and the control u are real-valued, for any initial condition $\boldsymbol{\psi}_0 := (\psi_0^1,\dots,\psi_0^N)$ in $\boldsymbol{\mathcal{S}}$, the solution $\boldsymbol{\psi}(t) := (\psi^1(t),\dots,\psi^N(t))$ belongs to $\boldsymbol{\mathcal{S}}$. We say that the vectors $\boldsymbol{\psi}_0, \boldsymbol{\psi}_f \in \boldsymbol{\mathcal{S}}$ are unitarily equivalent, if there is a unitary operator $\mathcal{U}$ in L^2 such that $\boldsymbol{\psi}_f = \mathcal{U}\boldsymbol{\psi}_0$, i.e. $\psi_f^j = \mathcal{U}\psi_0^j$ for all $j = 1,\dots,N$. Finally, we define the operator A_V by

$$\mathcal{D}(A_V) := H^2 \cap H_0^1((0,1),\mathbb{C}), \quad A_V\varphi := \left(-\partial_{xx}^2 + V(x)\right)\varphi$$

and, for $s > 0$, we set $H^s_{(V)} := \mathcal{D}\big(A_V^{s/2}\big)$ and write $\boldsymbol{H}^s_{(V)}$ instead of $(H^s_{(V)})^N$.

Theorem 4.1. *For any given $V \in H^4((0,1),\mathbb{R})$, problem (4.2) is globally exactly controllable in $\boldsymbol{H}^4_{(V)}$ generically with respect to μ in $H^4((0,1),\mathbb{R})$. More precisely, there is a residual set $\mathcal{Q}_V$ in $H^4((0,1),\mathbb{R})$ such that for any $\mu \in \mathcal{Q}_V$ and for any unitarily equivalent vectors $\boldsymbol{\psi}_0, \boldsymbol{\psi}_f \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$ there is a time $T > 0$ and a control $u \in L^2((0,T),\mathbb{R})$ such that the solution of (4.2) satisfies*

$$\boldsymbol{\psi}(T) = \boldsymbol{\psi}_f.$$

First of all, notice that the unitary equivalence assumption on the initial condition and the target is not restrictive. Indeed, the evolution of the considered Schrödinger equation (4.1) is unitary, hence the system can be controlled from a given initial state only to a unitarily equivalent target.

The problem of controllability for the bilinear Schrödinger equation has been widely studied in the literature. A negative controllability result for bilinear quantum systems is proved by Turinici [134] as a corollary of a general result by Ball, Marsden, and Slemrod [5]. It states that the complement of the reachable set with L^2 controls from any initial condition in $\mathcal{S} \cap H^2_{(0)}$ is dense in $\mathcal{S} \cap H^2_{(0)}$. Thus, these equations have been considered to be non-controllable.

This negative result is actually only due to the choice of the functional setting. For a single particle, Beauchard proved in [10] local exact controllability in large time in $H^7_{(0)}$ in the case $\mu(x) = x$, $V(x) = 0$, using Coron's return method, quasi-static deformations, and Nash–Moser theorem. Exhibiting a regularizing effect, this result was extended to the space $H^3_{(0)}$ for generic dipole moment μ, still in the case $V = 0$, by Beauchard and Laurent [16]. Thus, as we are dealing with an arbitrary potential V and a generic dipole moment μ, Theorem 4.1 with $N = 1$ is already an improvement of the previous literature. In [15], Beauchard and Coron proved exact controllability between eigenstates for a particle in a moving potential well as studied by Rouchon in [122].

Different methods have been developed to study approximate controllability. A first strategy of the proof of approximate controllability is due to Chambrion, Mason, Sigalotti, and Boscain [45] and relies on the geometric techniques based on the controllability of the Galerkin approximations. The hypotheses of this result were refined by Boscain, Caponigro, Chambrion, and Sigalotti in [25]. In a more recent paper [26] of this team, in particular, it is proved a simultaneous approximate controllability property in Sobolev spaces for an arbitrary number of equations. For more details and more references about the geometric techniques, we refer the reader to the recent survey [27]. Although the results presented in these papers cover an important class of models, the functional setting used there is always incompatible with the one which is necessary for the exact controllability. More precisely, approximate controllability is proved in less regular spaces than the one needed for exact controllability.

The second method which is used in the literature to prove approximate controllability for the bilinear Schrödinger equation is the Lyapunov strategy. This method was used by Mirrahimi in [103] in the case of a mixed spectrum and by Beauchard and Mirrahimi in [17] in the case $V = 0$ and $\mu(x) = x$. Both results prove approximate stabilization in L^2. Global approximate controllability with generic assumptions both on the potential and the dipole moment is obtained by the second author in [111] and extended to higher norms leading to the first global exact controllability result for a bilinear quantum system in [112]. For a model involving also a quadratic control, we refer to [108]. Approximate controllability in regular spaces (containing H^3) can also be deduced from the exact controllability results in infinite time [113, 114] by Nersisyan and the second author. The novelty of Theorem 4.1 with respect to the above papers is the fact that N particles are controlled simultaneously in a regular space for an arbitrary fixed potential V.

Simultaneous exact controllability of quantum particles has been obtained for a finite dimensional model in [136] by Turinici and Rabitz. Their model uses specific orientation of the molecules and their proof relies on iterated Lie brackets. To our best knowledge, the only exact simultaneous controllability results for infinite dimensional bilinear quantum systems were obtained in [109] by the first author locally around eigenstates in the case $V = 0$ for $N = 2$ or $N = 3$. This is proved either up to a global phase in arbitrary time or exactly up to a global delay in the case $N = 2$ and up to a global phase and a global delay in the case $N = 3$. In that paper, it is also proved that, under generic assumptions on the dipole moment, local exact controllability (resp. local controllability up to a global phase) with controls small in L^2 does not hold in small time for $N \geq 2$ (resp. $N \geq 3$). A key issue for the positive results of this paper is the construction of a suitable reference trajectory which coincides (up to global phase and/or a global delay) at the final time with the vector of eigenstates. Extending directly this result to the case $N \geq 4$ presents two difficulties: in the trigonometric moment problem solved for the construction of the reference trajectory resonant frequencies appear (e.g. $\lambda_7 - \lambda_1 = \lambda_8 - \lambda_4$) and the frequency 0 appears with multiplicity N. The use of a global phase and/or a global delay, by adding new degrees of freedom, allowed to deal with the frequency 0 having multiplicity two or three. In our setting, we do not impose any conditions on the phase terms of the reference trajectory (see Proposition 4.3). Thus, the frequency 0 does not appear in the associated trigonometric moment problems. Taking advantage of the assumptions on the spectrum of the free operator, we prove local exact controllability around $(\varphi_{1,V}, \dots, \varphi_{N,V})$ (see the First step of the proof of Theorem 4.3). The price to pay is that we lose track of the time of control.

Structure of the article. Theorem 4.1 is proved in three steps. First, under favourable hypotheses on V and μ, we prove that any initial condition can be driven arbitrarily close to some finite sum of eigenfunctions. This is done in Section 4.3 using a Lyapunov strategy inspired by [112]. Then, adapting the ideas of [109], using favourable assumptions on the spectrum of A_V and a compactness argument, we prove in Section 4.4 exact controllability locally around specific finite sums of eigenfunctions. Finally, for any potential V, using a perturbation argument, leading to the potential $V+\mu$ instead of V, we gather in Section 4.5 the two previous results to prove Theorem 4.1. Let us mention that, essentially with the same proof, one can prove global exact controllability in $\boldsymbol{H}^{3+\epsilon}_{(V)}$, for any $\epsilon > 0$.

Notations

The space $L^2((0,1),\mathbb{C})$ is endowed with the usual scalar product

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\mathrm{d}x,$$

and we denote by $\|\cdot\|$ the associated norm. For any $s > 0$, we denote by $\|\cdot\|_s$ the classical norm on the Sobolev space $H^s((0,1),\mathbb{C})$. The eigenvalues and eigenvectors of the operator A_V are denoted respectively by $\lambda_{k,V}$ and $\varphi_{k,V}$. The eigenstates are defined by

$$\Phi_{k,V}(t,x) := \varphi_{k,V}(x)e^{-i\lambda_{k,V}t}, \quad (t,x) \in \mathbb{R}^+ \times (0,1),\ k \in \mathbb{N}^*.$$

Any N-tuple of eigenstates is a solution of system (4.2) with control $u \equiv 0$. Notice that

$$H^3_{(V)} = \left\{\varphi \in H^3((0,1),\mathbb{C})\,;\, \varphi|_{x=0,1} = \varphi''|_{x=0,1} = 0\right\} = H^3_{(0)}$$

for any $V \in H^3((0,1),\mathbb{R})$. We endow this space with the norm

$$\|\psi\|_{H^3_{(V)}} := \left(\sum_{k=1}^{\infty} |k^3\langle\psi,\varphi_{k,V}\rangle|^2\right)^{\frac{1}{2}}.$$

We use bold characters to denote vector functions or product spaces. For instance, we denote by $\boldsymbol{\psi}(t)$ the vector $(\psi^1(t),\dots,\psi^N(t))$ of solutions of (4.2) and by $\boldsymbol{H}^s_{(V)}$ the space $(H^s_{(V)})^N$. With coherent notations, $\boldsymbol{\varphi}_V$ denotes the vector $(\varphi_{1,V},\dots,\varphi_{N,V})$.

Let us denote by $U(H)$ the set of unitary operators from a Hilbert space H into itself, and by U_N the set of $N\times N$ unitary matrices. Any $N\times M$ matrix $C = (c_{ij})$ defines a linear map from H^M to H^N (denoted again by C) which associates to the vector $(z^1,\dots,z^M)$ the vector $(\sum_{j=1}^M c_{1j}z^j,\dots,\sum_{j=1}^M c_{Nj}z^j)$.

For a Banach space X, let $B_X(a,d)$ be the closed ball of radius $d>0$ centred on $a \in X$. A subset of X is said to be residual if it contains a countable intersection of open and dense sets.
The symbol $\delta_{j=k}$ is the classical Kronecker symbol, i.e., $\delta_{j=k}=1$ if $j=k$ and $\delta_{j=k}=0$ otherwise.

Finally, we define the space

$$\ell^2_r(\mathbb{N},\mathbb{C}) := \big\{ d \in \ell^2(\mathbb{N},\mathbb{C})\,;\, d_0 \in \mathbb{R} \big\}$$

which is endowed with the natural metric.

4.2 Well-posedness

In the following proposition, we recall a well-posedness result of the Cauchy problem for the Schrödinger equation

$$\begin{cases} i\partial_t \psi = \big(-\partial^2_{xx} + V(x)\big)\,\psi - u(t)\mu(x)\psi - v(t)\mu(x)\zeta, & (t,x)\in(0,T)\times(0,1),\\ \psi(t,0)=\psi(t,1)=0, \\ \psi(0,x)=\psi_0(x), \end{cases} \tag{4.3}$$

and list properties of the solution that will be used in the proofs of the main results in the subsequent sections.

Proposition 4.1. *Let us assume that $V,\mu \in H^3((0,1),\mathbb{R})$ and $T>0$. Then, for any $\psi_0 \in H^3_{(0)}$, $\zeta \in C^0([0,T],H^3_{(0)})$ and $u,v \in L^2((0,T),\mathbb{R})$ there is a unique weak solution of (4.3), i.e., a function $\psi \in C([0,T],H^3_{(0)})$ such that the following equality holds in $H^3_{(0)}$ for every $t\in[0,T]$*

$$\psi(t) = e^{-iA_V t}\psi_0 + i\int_0^t e^{-iA_V(t-\tau)}\big(u(\tau)\mu\psi(\tau)+v(\tau)\mu\zeta(\tau)\big)\mathrm{d}\tau.$$

For every $R>0$, there exists $C=C(T,V,\mu,R)>0$ such that, if $\|u\|_{L^2(0,T)}<R$, this weak solution satisfies

$$\|\psi\|_{C^0([0,T],H^3_{(V)})} \le C\left(\|\psi_0\|_{H^3_{(V)}} + \|v\|_{L^2(0,T)}\|\zeta\|_{L^\infty((0,T),H^3_{(V)})}\right).$$

Moreover, if $v\equiv 0$ the solution satisfies

$$\|\psi(t)\| = \|\psi_0\| \quad \textit{for all } t\in[0,T],$$

and the following properties hold in the case $v\equiv 0$.

Differentiability. *Let us denote by $\psi(t,\psi_0,u)$ the solution of (4.3) corresponding to $\psi_0 \in H^3_{(0)}$, $u\in L^2((0,T),\mathbb{R})$ and $v=0$. The mapping*

$$\begin{array}{rccc} \psi(T,\psi_0,\cdot): & L^2((0,T),\mathbb{R}) & \to & H^3_{(0)}, \\ & u & \mapsto & \psi(T,\psi_0,u) \end{array} \tag{4.4}$$

is C^1, and for any $u, v \in L^2((0,T),\mathbb{R})$, we have $\partial_u \psi(T,\psi_0,u)v = \Psi(T)$, where Ψ is the weak solution of the linearized system

$$\begin{cases} i\partial_t \Psi = \left(-\partial^2_{xx} + V(x)\right)\Psi - u(t)\mu(x)\Psi - v(t)\mu(x)\psi, & (t,x) \in (0,T)\times(0,1), \\ \Psi(t,0) = \Psi(t,1) = 0, \\ \Psi(0,x) = 0, \end{cases}$$

with $\psi = \psi(\cdot,\psi_0,u)$.

Regularity. *Assume that $V, \mu \in H^4((0,1),\mathbb{R})$. For any $u \in W^{1,1}((0,T),\mathbb{R})$ and $\psi_0 \in H^4_{(V-u(0)\mu)}$, we have $\psi(t) \in H^4_{(V-u(t)\mu)}$ for all $t \in [0,T]$.*

Time reversibility. *Suppose that $\psi(T,\overline{\psi_f},u) = \overline{\psi_0}$ for some $\psi_0, \psi_f \in H^3_{(0)}$, $T > 0$ and $u \in L^2((0,T),\mathbb{R})$. Then $\psi(T,\psi_0,w) = \psi_f$, where $w(t) = u(T-t)$.*

See [16, Propositions 2 and 3] for the proof of the well-posedness in $H^3_{(0)}$ and for the differentiability property. The property of regularity is established in [10, Proposition 47]. In these references, the case of $V = 0$ is considered, but the case of a non-zero V is proved by literally the same arguments (see [113]). The time reversibility property is obvious. Proposition 4.1 implies that similar properties hold for the solutions of system (4.2). We denote by $\boldsymbol{\psi}(t,\boldsymbol{\psi}_0,u)$ the solution of (4.2) corresponding to $\boldsymbol{\psi}_0 \in \boldsymbol{H}^3_{(0)}$ and $u \in L^2((0,T),\mathbb{R})$.

4.3 Approximate controllability

4.3.1 Approximate controllability towards finite sums of eigenvectors

In this section, we assume that the following conditions are satisfied for the functions $V, \mu \in H^4((0,1),\mathbb{R})$

($\mathbf{C_1}$) $\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle \neq 0$ for all $j \in \{1,\dots,N\}$, $k \in \mathbb{N}^*$.

($\mathbf{C_2}$) $\lambda_{j,V} - \lambda_{k,V} \neq \lambda_{p,V} - \lambda_{q,V}$ for all $j \in \{1,\dots,N\}$, $k,p,q \in \mathbb{N}^*$ such that $\{j,k\} \neq \{p,q\}$ and $k \neq j$.

For any $M \in \mathbb{N}^*$, let us define the sets

$$\mathcal{C}_M := \operatorname{Span}\{\varphi_{1,V},\dots,\varphi_{M,V}\}, \quad \boldsymbol{\mathcal{C}}_M := (\mathcal{C}_M)^N, \tag{4.5}$$

$$\boldsymbol{E} := \left\{ \boldsymbol{\psi} \in \boldsymbol{L}^2 \,;\, \prod_{j=1}^{N} \langle \psi^j, \varphi_{j,V}\rangle \neq 0 \right\}. \tag{4.6}$$

The following theorem is the main result of this section.

Theorem 4.2. *Assume that Conditions* **($\mathbf{C_1}$)** *and* **($\mathbf{C_2}$)** *are satisfied for the functions $V, \mu \in H^4((0,1),\mathbb{R})$. Then, for any $\boldsymbol{\psi}_0 \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)} \cap \boldsymbol{E}$, there are $M \in \mathbb{N}^*$, $\boldsymbol{\psi}_f \in \boldsymbol{\mathcal{C}}_M$, sequences $T_n > 0$ and $u_n \in C_0^\infty((0,T_n),\mathbb{R})$ such that*

$$\boldsymbol{\psi}(T_n,\boldsymbol{\psi}_0,u_n) \underset{n\to\infty}{\longrightarrow} \boldsymbol{\psi}_f \quad \text{in} \quad \boldsymbol{H}^3. \tag{4.7}$$

Proof. See [112, Theorem 2.3] for the proof of a similar result in the case $N = 1$ (in that case one gets $M = 1$). To simplify notations, we shall write λ_k, φ_k instead of $\lambda_{k,V}, \varphi_{k,V}$. For any $\boldsymbol{z} = (z^1, \dots, z^N) \in \boldsymbol{H}^4_{(V)}$, let us define the following Lyapunov function

$$\mathcal{V}(\boldsymbol{z}) = \alpha \sum_{j=1}^{N} \|(-\partial^2_{xx} + V)^2 \mathcal{P}_N z^j\|^2 + 1 - \prod_{j=1}^{N} |\langle z^j, \varphi_j \rangle|^2, \tag{4.8}$$

where $\alpha > 0$ is a constant that will be chosen later and $\mathcal{P}_N$ is the orthogonal projection in L^2 onto the closure of the vector span of $\{\varphi_k\}_{k \geq N+1}$, i.e.,

$$\mathcal{P}_N(z) := \sum_{k \geq N+1} \langle z, \varphi_k \rangle \varphi_k. \tag{4.9}$$

Clearly, we have that $\mathcal{V}(\boldsymbol{z}) \geq 0$ for any $\boldsymbol{z} \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$ and $\mathcal{V}(\boldsymbol{z}) = 0$ if and only if $\boldsymbol{z} = (c_1\varphi_1, \dots, c_N\varphi_N)$ for some $c_i \in \mathbb{C}$ such that $|c_i| = 1, i = 1, \dots, N$. Furthermore, for any $\boldsymbol{z} \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$, we have

$$\mathcal{V}(\boldsymbol{z}) \geq \alpha \sum_{j=1}^{N} \|(-\partial^2_{xx} + V)^2 \mathcal{P}_N z^j\|^2 \geq C_1 \sum_{j=1}^{N} \|z^j\|_4^2 - C_2.$$

Thus

$$C(1 + \mathcal{V}(\boldsymbol{z})) \geq \|\boldsymbol{z}\|_4^2 \tag{4.10}$$

for some constant $C > 0$. We need the following result which is a generalization of [112, Proposition 2.6].

Proposition 4.2. *Under the conditions of Theorem 4.2, for any initial condition* $\boldsymbol{\psi}_0 \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)} \cap \boldsymbol{E} \setminus (\cup_{M=1}^{\infty} \boldsymbol{\mathcal{C}}_M)$ *there is a time* $T > 0$ *and a control* $u \in C_0^\infty((0,T), \mathbb{R})$ *such that*

$$\mathcal{V}(\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, u)) < \mathcal{V}(\boldsymbol{\psi}_0).$$

See Section 4.3.2 for the proof of this result.

Let us choose $\alpha > 0$ in (4.8) so small that $\mathcal{V}(\boldsymbol{\psi}_0) < 1$ and define the set

$$\mathcal{K} := \Big\{ \boldsymbol{\psi} \in \boldsymbol{H}^4_{(V)} \,;\, \boldsymbol{\psi}(T_n, \boldsymbol{\psi}_0, u_n) \underset{n \to \infty}{\longrightarrow} \boldsymbol{\psi} \text{ in } \boldsymbol{H}^3 \text{ for some } T_n \geq 0,\, u_n \in C_0^\infty((0,T_n), \mathbb{R}) \Big\}.$$

Then the infimum $m := \inf_{\boldsymbol{\psi} \in \mathcal{K}} \mathcal{V}(\boldsymbol{\psi})$ is attained, there is $\boldsymbol{e} \in \mathcal{K}$ such that

$$\mathcal{V}(\boldsymbol{e}) = \inf_{\boldsymbol{\psi} \in \mathcal{K}} \mathcal{V}(\boldsymbol{\psi}). \tag{4.11}$$

Indeed, any minimizing sequence $\boldsymbol{\psi}_n \in \mathcal{K}$, $\mathcal{V}(\boldsymbol{\psi}_n) \to m$ is bounded in $\boldsymbol{H}^4$, by (4.10). Extracting a subsequence if necessary, we may assume that $\boldsymbol{\psi}_n \rightharpoonup \boldsymbol{e}$ in $\boldsymbol{H}^4$ for some $\boldsymbol{e} \in \boldsymbol{H}^4_{(V)}$. This implies that $\mathcal{V}(\boldsymbol{e}) \leq \liminf_{n \to \infty} \mathcal{V}(\boldsymbol{\psi}_n) = m$. Let us show that $\boldsymbol{e} \in \mathcal{K}$. As $\boldsymbol{\psi}_n \in \mathcal{K}$, there are sequences $T_n > 0$ and $u_n \in C_0^\infty((0,T_n), \mathbb{R})$ such that

$$\|\boldsymbol{\psi}(T_n, \boldsymbol{\psi}_0, u_n) - \boldsymbol{\psi}_n\|_{H^3_{(V)}} \leq \frac{1}{n}. \tag{4.12}$$

On the other hand, $\boldsymbol{\psi}_n \to \boldsymbol{e}$ in $\boldsymbol{H}^3$, and (4.12) implies that $\boldsymbol{\psi}(T_n, \boldsymbol{\psi}_0, u_n) \to \boldsymbol{e}$ in $\boldsymbol{H}^3$. Thus $\boldsymbol{e} \in \mathcal{K}$ and $\mathcal{V}(\boldsymbol{e}) = m$.

Let us prove that $\boldsymbol{e} \in \boldsymbol{\mathcal{C}}_M$ for some $M \in \mathbb{N}^*$. Suppose, by contradiction, that $\boldsymbol{e} \notin \cup_{M=1}^{\infty} \boldsymbol{\mathcal{C}}_M$. It follows from (4.11) and from the choice of α that $\mathcal{V}(\boldsymbol{e}) \leq \mathcal{V}(\boldsymbol{\psi}_0) < 1$. This shows that $\boldsymbol{e} \in \boldsymbol{E}$. Proposition 4.2 implies that there are $T > 0$ and $u \in C_0^\infty((0,T), \mathbb{R})$ such that

$$\mathcal{V}(\boldsymbol{\psi}(T, \boldsymbol{e}, u)) < \mathcal{V}(\boldsymbol{e}). \tag{4.13}$$

Define $\tilde{u}_n(t) = u_n(t)$, $t \in [0, T_n]$ and $\tilde{u}_n(t) = u(t - T_n)$, $t \in [T_n, T_n + T]$. Then $\tilde{u}_n \in C_0^\infty((0, T_n + T), \mathbb{R})$ and, by the continuity in H^3 of the resolving operator for (4.2), we get

$$\boldsymbol{\psi}(T_n + T, \boldsymbol{\psi}_0, \tilde{u}_n) \underset{n}{\longrightarrow} \boldsymbol{\psi}(T, \boldsymbol{e}, u) \quad \text{in } \boldsymbol{H}^3,$$

hence $\boldsymbol{\psi}(T, \boldsymbol{e}, u) \in \mathcal{K}$. Together with (4.13), this contradicts (4.11). Thus $\boldsymbol{e} \in \boldsymbol{\mathcal{C}}_M$, and we get (4.7) with $\boldsymbol{\psi}_f = \boldsymbol{e}$.

□

4.3.2 Proof of Proposition 4.2

Let us take any vector $\boldsymbol{\psi}_0 \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)} \cap \boldsymbol{E} \backslash (\cup_{M=1}^{\infty} \boldsymbol{\mathcal{C}}_M)$, any time $T > 0$, any control $w \in C_0^\infty((0,T), \mathbb{R})$, and consider the mapping

$$\begin{array}{rccc} \mathcal{V}(\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, (\cdot) w)) : & \mathbb{R} & \to & \mathbb{R}, \\ & \sigma & \mapsto & \mathcal{V}(\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, \sigma w)). \end{array}$$

It suffices to show that, for an appropriate choice of T and w, we have

$$\left.\frac{\mathrm{d}\mathcal{V}(\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, \sigma w))}{\mathrm{d}\sigma}\right|_{\sigma=0} \neq 0. \tag{4.14}$$

Indeed, (4.14) implies that there is $\sigma_0 \in \mathbb{R}$ close to zero such that

$$\mathcal{V}(\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, \sigma_0 w)) < \mathcal{V}(\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, 0)) = \mathcal{V}(\boldsymbol{\psi}_0),$$

which completes the proof of Proposition 4.2.

To prove (4.14), notice that

$$\begin{aligned} \left.\frac{\mathrm{d}\mathcal{V}(\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, \sigma w))}{\mathrm{d}\sigma}\right|_{\sigma=0} = 2 \sum_{j=1}^{N} \operatorname{Re} \Big(& \alpha \langle (-\partial_{xx}^2 + V)^2 \mathcal{P}_N \psi^j(T), (-\partial_{xx}^2 + V)^2 \mathcal{P}_N \Psi^j(T) \rangle \\ & - \langle \psi^j(T), \varphi_j \rangle \langle \varphi_j, \Psi^j(T) \rangle \prod_{q=1, q \neq j}^{N} |\langle \psi_0^q, \varphi_q \rangle|^2 \Big), \end{aligned} \tag{4.15}$$

where

$$\psi^j(t) = \psi(t, \psi_0^j, 0) = \sum_{k=1}^{\infty} e^{-i\lambda_k t} \langle \psi_0^j, \varphi_k \rangle \varphi_k, \tag{4.16}$$

and Ψ^j is the solution of the linearized problem

$$\begin{cases} i\partial_t \Psi^j = \big(-\partial^2_{xx} + V(x)\big)\Psi^j - w(t)\mu(x)\psi^j, & (t,x)\in(0,T)\times(0,1),\\ \Psi^j(t,0) = \Psi^j(t,1) = 0, & j\in\{1,\dots,N\},\\ \Psi^j(0,x) = 0. \end{cases}$$

Rewriting this in the Duhamel form

$$\Psi^j(t) = i\int_0^t e^{-iA_V(t-\tau)} w(\tau)\mu\psi(\tau)\mathrm{d}\tau$$

and using (4.16), we get that

$$\langle \Psi^j(T), \varphi_p\rangle = ie^{-i\lambda_p T}\sum_{k=1}^{\infty}\langle \psi_0^j, \varphi_k\rangle\langle \mu\varphi_k, \varphi_p\rangle \int_0^T e^{-i(\lambda_k-\lambda_p)\tau} w(\tau)\mathrm{d}\tau. \tag{4.17}$$

Replacing (4.16) and (4.17) into (4.15), we obtain

$$\frac{\mathrm{d}\mathcal{V}(\boldsymbol{\psi}(T,\boldsymbol{\psi}_0,\sigma w))}{\mathrm{d}\sigma}\Big|_{\sigma=0} = \int_0^T \Phi(\tau)w(\tau)\mathrm{d}\tau,$$

where

$$\begin{aligned} i\Phi(\tau) :=& \sum_{j=1}^{N}\Bigg(\sum_{p=N+1,k=1}^{\infty}\alpha\lambda_p^4\langle\psi_0^j,\varphi_p\rangle\langle\varphi_k,\psi_0^j\rangle\langle\mu\varphi_k,\varphi_p\rangle e^{i(\lambda_k-\lambda_p)\tau}\\ &-\sum_{p=N+1,k=1}^{\infty}\alpha\lambda_p^4\langle\varphi_p,\psi_0^j\rangle\langle\psi_0^j,\varphi_k\rangle\langle\mu\varphi_k,\varphi_p\rangle e^{-i(\lambda_k-\lambda_p)\tau}\\ &-\Big(\prod_{q=1,q\neq j}^{N}|\langle\psi_0^q,\varphi_q\rangle|^2\Big)\sum_{k=1}^{\infty}\langle\psi_0^j,\varphi_j\rangle\langle\varphi_k,\psi_0^j\rangle\langle\mu\varphi_k,\varphi_j\rangle e^{i(\lambda_k-\lambda_j)\tau}\\ &+\Big(\prod_{q=1,q\neq j}^{N}|\langle\psi_0^q,\varphi_q\rangle|^2\Big)\sum_{k=1}^{\infty}\langle\varphi_j,\psi_0^j\rangle\langle\psi_0^j,\varphi_k\rangle\langle\mu\varphi_k,\varphi_j\rangle e^{-i(\lambda_k-\lambda_j)\tau}\Bigg)\\ =:& \sum_{1\le k<p<\infty}\Big(P(k,p)e^{i(\lambda_k-\lambda_p)\tau} + \tilde{P}(k,p)e^{-i(\lambda_k-\lambda_p)\tau}\Big), \end{aligned} \tag{4.18}$$

where $P(k,p)$ and $\tilde{P}(k,p)$ are constants. To prove (4.14), it suffices to show that $\Phi(\tau)\neq 0$ for some $\tau\geq 0$. Suppose, by contradiction, that $\Phi(\tau) = 0$ for all $\tau\geq 0$. Then Condition $\mathbf{(C_2)}$ and [111, Lemma 3.10] imply that $P(k,p) = 0$ for all $1\leq k\leq N<p<\infty$. Together with $\mathbf{(C_1)}$, this leads to

$$\left(\alpha\lambda_p^4 + \prod_{q=1,q\neq k}^{N}|\langle\psi_0^q,\varphi_q\rangle|^2\right)\langle\psi_0^k,\varphi_p\rangle\langle\varphi_k,\psi_0^k\rangle + \sum_{j=1,j\neq k}^{N}\alpha\lambda_p^4\langle\psi_0^j,\varphi_p\rangle\langle\varphi_k,\psi_0^j\rangle = 0.$$

Assume that for some integer $p>N$ we have

$$\sum_{j=1}^{N}|\langle\psi_0^j,\varphi_p\rangle| > 0. \tag{4.19}$$

Let us set $a_k(\lambda) := \lambda + \prod_{q=1,q\neq k}^{N} |\langle \psi_0^q, \varphi_q\rangle|^2$ and consider the determinant

$$\Lambda(\lambda) = \begin{vmatrix} a_1(\lambda)\langle \psi_0^1, \varphi_1\rangle & \lambda\langle \psi_0^2, \varphi_1\rangle & \cdots & \lambda\langle \psi_0^N, \varphi_1\rangle \\ \lambda\langle \psi_0^1, \varphi_2\rangle & a_2(\lambda)\langle \psi_0^2, \varphi_2\rangle & \cdots & \lambda\langle \psi_0^N, \varphi_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \lambda\langle \psi_0^1, \varphi_N\rangle & \lambda\langle \psi_0^2, \varphi_N\rangle & \cdots & a_N(\lambda)\langle \psi_0^N, \varphi_N\rangle \end{vmatrix}.$$

Then $\Lambda(\lambda)$ is a polynomial of degree less or equal to N which vanishes at $\lambda = \alpha\lambda_p^4$. The free term in $\Lambda(\lambda)$ is $\prod_{k=1}^{N} a_k(0)\langle \psi_0^k, \varphi_k\rangle$ which is non-zero by the assumption $\boldsymbol{\psi}_0 \in \boldsymbol{E}$. Thus $\Lambda(\lambda)$ has at most N roots and the number of indices p such that (4.19) holds is finite. This gives the existence of $M \in \mathbb{N}^*$ such that $\boldsymbol{\psi}_0 \in \boldsymbol{\mathcal{C}}_M$ and completes the proof of Proposition 4.2.

□

4.4 Local exact controllability

4.4.1 Local exact controllability around finite sums of eigenstates

In this section, we assume that the following conditions are satisfied for the functions $V, \mu \in H^3((0,1),\mathbb{R})$.

(C₃) There exists $C > 0$ such that

$$|\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle| \geq \frac{C}{k^3}, \quad \forall j \in \{1,\dots,N\}, \forall k \in \mathbb{N}^*.$$

($\mathbf{C_4}$) $\lambda_{k,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{n,V}$ for all $j,n \in \{1,\dots,N\}$, $k \geq j+1$, $p \geq n+1$ with $\{j,k\} \neq \{p,n\}$.

($\mathbf{C_5}$) $1, \lambda_{1,V}, \dots, \lambda_{N,V}$ are rationally independent.

The goal of this section is the proof of the following theorem.

Theorem 4.3. *Assume that Conditions* $(\mathbf{C_3}) - (\mathbf{C_5})$ *are satisfied for* $V, \mu \in H^3((0,1),\mathbb{R})$. *Let us take any* $C_0, C_f \in U_N$ *and set* $\boldsymbol{z}_0 := C_0\boldsymbol{\varphi}_V$, $\boldsymbol{z}_f := C_f\boldsymbol{\varphi}_V$. *Then there exist* $\delta > 0$ *and* $T > 0$ *such that if we define*

$$\mathcal{O}_{\delta,C_0} := \Big\{\boldsymbol{\phi} \in \boldsymbol{H}^3_{(0)}\,;\, \langle \phi^j, \phi^k\rangle = \delta_{j=k} \text{ and } \sum_{j=1}^{N} \|\phi^j - z_0^j\|_{H^3_{(V)}} < \delta\Big\},$$

$$\mathcal{O}_{\delta,C_f} := \Big\{\boldsymbol{\phi} \in \boldsymbol{H}^3_{(0)}\,;\, \langle \phi^j, \phi^k\rangle = \delta_{j=k} \text{ and } \sum_{j=1}^{N} \|\phi^j - z_f^j\|_{H^3_{(V)}} < \delta\Big\},$$

then for any $\boldsymbol{\psi}_0 \in \mathcal{O}_{\delta,C_0}$ *and* $\boldsymbol{\psi}_f \in \mathcal{O}_{\delta,C_f}$, *there is a control* $u \in L^2((0,T),\mathbb{R})$ *such that the associated solution of (4.2) with initial condition* $\boldsymbol{\psi}(0) = \boldsymbol{\psi}_0$ *satisfies* $\boldsymbol{\psi}(T) = \boldsymbol{\psi}_f$.

Remark 4.1. Notice that the condition

$$\langle \phi^j, \phi^k\rangle = \delta_{j=k}, \quad \forall j,k \in \{1,\dots,N\}$$

is equivalent to the fact that $\boldsymbol{\phi}$ is unitarily equivalent to $\boldsymbol{\varphi}_V$. In this section, we will always consider such initial conditions. Thus, the associated trajectories will satisfy the following invariants

$$\langle \psi^j(t), \psi^k(t)\rangle \equiv \delta_{j=k}, \quad \forall j,k \in \{1,\dots,N\}. \tag{4.20}$$

Remark 4.2. A quantum logical gate is a unitary operator $\hat{\mathcal{U}}$ in $L^2((0,1),\mathbb{C})$ such that for some $n \in \mathbb{N}^*$, the space $\operatorname{Span}\{\varphi_{1,V},\dots,\varphi_{n,V}\}$ is stable for $\hat{\mathcal{U}}$. Designing such a quantum gate means finding a control $u \in L^2((0,T),\mathbb{R})$ such that the associated solution of (4.2) with initial condition $(\varphi_{1,V},\dots,\varphi_{n,V})$ satisfies

$$\left(\psi^1(T),\dots,\psi^n(T)\right) = \left(\hat{\mathcal{U}}\varphi_{1,V},\dots,\hat{\mathcal{U}}\varphi_{n,V}\right).$$

See [32] for L^2–approximate realization of such quantum logical gates with error estimates and numerical simulations on two classical examples. Theorem 4.3 thus proves exact realization of quantum logical gates in large time under Conditions $\mathbf{(C_3)}$ – $\mathbf{(C_5)}$ of size n. Applying directly Theorem 4.1 leads to exact realization of any quantum gate, for an arbitrary potential with a generic dipole moment.

The proof of Theorem 4.3 is based on the following proposition which is an adaptation of [109, Theorem 1.5].

Proposition 4.3. *Assume that Conditions* $\mathbf{(C_3)}$–$\mathbf{(C_4)}$ *are satisfied for* $V,\mu \in H^3((0,1),\mathbb{R})$. *For any* $T>0$*, there exist* $\theta_1,\dots,\theta_N \in \mathbb{R}$*,* $\delta>0$*, and a* C^1 *map*

$$\Gamma : \mathcal{O}^0_\delta \times \mathcal{O}^f_\delta \to L^2((0,T),\mathbb{R}),$$

where

$$\mathcal{O}^0_\delta := \Big\{\boldsymbol{\phi} \in \boldsymbol{H}^3_{(0)}\,;\, \langle \phi^j,\phi^k\rangle = \delta_{j=k} \text{ and } \sum_{j=1}^N \|\phi^j - \varphi_{j,V}\|_{H^3_{(V)}} < \delta\Big\},$$

$$\mathcal{O}^f_\delta := \Big\{\boldsymbol{\phi} \in \boldsymbol{H}^3_{(0)}\,;\, \langle \phi^j,\phi^k\rangle = \delta_{j=k} \text{ and } \sum_{j=1}^N \|\phi^j - e^{i\theta_j}\varphi_{j,V}\|_{H^3_{(V)}} < \delta\Big\},$$

such that for any initial condition $\boldsymbol{\psi}_0 \in \mathcal{O}^0_\delta$ *and for any target* $\boldsymbol{\psi}_f \in \mathcal{O}^f_\delta$*, the solution of system (4.2) associated to the control* $u := \Gamma(\boldsymbol{\psi}_0,\boldsymbol{\psi}_f)$ *satisfies* $\boldsymbol{\psi}(T) = \boldsymbol{\psi}_f$.

In the case $N=2$ and $V=0$, the previous proposition is exactly [109, Theorem 1.2] with $\theta_j = \theta - \lambda_{j,V}T$. As here we do not impose any condition on the phase terms θ_j, the proof of Proposition 4.3 does not introduce new ideas with respect to [109]. Anyway, dealing with an arbitrary number of equations (instead of two or three equations in [109]) needs some adaptations that are described in Sections 4.4.2, 4.4.3, and 4.4.4. Dealing with a potential V instead of $V=0$ is done with literally the same arguments.

To highlight the novelties of this work, we postpone the proof of Proposition 4.3 to Section 4.4.2 and first prove how this proposition implies Theorem 4.3. We start with the proof of Theorem 4.3 in the particular case $C_0 = C_f = I_N$, where I_N is the $N\times N$ identity matrix. This is done using Proposition 4.3, a rotation phenomenon for the solution corresponding to the null control on a suitable time interval, and a time reversibility argument. Then, for any $C \in U_N$, using a linearity argument, we prove Theorem 4.3 in the case $C_0 = C_f = C$. We end the proof using connectedness of the set of unitary matrices and a compactness argument.

Proof of Theorem 4.3. To simplify notations, until the end of Section 4.4, we shall write λ_k, φ_k instead of $\lambda_{k,V}$, $\varphi_{k,V}$.

First step : proof in the case $C_0 = C_f = I_N$.
Let us take any $T > 0$. Let $\delta > 0$ and $\theta_1, \dots, \theta_N$ be the constants given in Proposition 4.3. Let $\boldsymbol{\psi}_0, \boldsymbol{\psi}_f \in \mathcal{O}_{\delta, I_N}$. As $\mathcal{O}_{\delta, I_N} = \mathcal{O}_\delta^0$, there exists $u \in L^2((0,T),\mathbb{R})$ such that the associated solution of (4.2) with initial condition $\boldsymbol{\psi}_0$ satisfies

$$\boldsymbol{\psi}(T) = (e^{i\theta_1}\varphi_1, \dots, e^{i\theta_N}\varphi_N). \tag{4.21}$$

Using Condition $\mathbf{(C_5)}$ and the Kronecker theorem on simultaneous diophantine approximation (see e.g. [128, Corollary 10]), there exists a rotation time $T_r > 0$ such that

$$|\lambda_j|^{3/2} \left| e^{i(2\theta_j - \lambda_j T_r)} - 1 \right| < \frac{\delta}{N}, \quad \forall j \in \{1, \dots, N\}.$$

Thus, it comes that $\displaystyle\sum_{j=1}^N \| e^{i(\theta_j - \lambda_j T_r)}\varphi_j - e^{-i\theta_j}\varphi_j\|_{H^3_{(V)}} < \delta$. Together with (4.21), this implies that if we extend u by zero on $(T, T+T_r)$ then

$$\sum_{j=1}^N \|\psi^j(T+T_r) - e^{-i\theta_j}\varphi_j\|_{H^3_{(V)}} < \delta.$$

Thus,

$$\overline{\boldsymbol{\psi}}(T+T_r) \in \mathcal{O}_\delta^f. \tag{4.22}$$

As $\boldsymbol{\psi}_f \in \mathcal{O}_{\delta, I_N} = \mathcal{O}_\delta^0$ and the eigenvectors φ_j being real-valued, we have $\overline{\boldsymbol{\psi}}_f \in \mathcal{O}_\delta^0$. Then, Proposition 4.3 implies the existence of $v \in L^2((0,T),\mathbb{R})$ such that the associated solution of (4.2) with initial condition $\overline{\boldsymbol{\psi}}_f$ equals to $\overline{\boldsymbol{\psi}}(T+T_r)$ at time T. Finally, the time reversibility property proves that if u is defined by $u(T+T_r+t) = v(T-t)$ for $t \in (0,T)$, then the associated solution of (4.2) with initial condition $\boldsymbol{\psi}_0$ satisfies

$$\boldsymbol{\psi}(T+T_r+T) = \boldsymbol{\psi}_f. \tag{4.23}$$

This ends the proof of Theorem 4.3 in the case $C_0 = C_f = I_N$ in time $T^* := 2T + T_r$.

Second step : proof in the case $C_0 = C_f = C$.
Let $\delta > 0$ be as in the first step, $C \in U_N$, and $\boldsymbol{z} := C\boldsymbol{\varphi}$. Let $\delta_{\boldsymbol{z}} > 0$ be sufficiently small to satisfy

$$C^*\Big(B_{\boldsymbol{H}^3_{(V)}}(\boldsymbol{z}, \delta_{\boldsymbol{z}})\Big) \subset B_{\boldsymbol{H}^3_{(V)}}(\boldsymbol{\varphi}, \delta).$$

Let us take any $\boldsymbol{\psi}_0, \boldsymbol{\psi}_f \in \mathcal{O}_{\delta_{\boldsymbol{z}}, C}$ and define

$$\tilde{\boldsymbol{\psi}}_0 := C^*\boldsymbol{\psi}_0, \quad \tilde{\boldsymbol{\psi}}_f := C^*\boldsymbol{\psi}_f. \tag{4.24}$$

The unitarity of C implies that $\langle \tilde{\psi}_0^j, \tilde{\psi}_0^k \rangle = \delta_{j=k}$ and $\langle \tilde{\psi}_f^j, \tilde{\psi}_f^k \rangle = \delta_{j=k}$. Thus, from the definition of $\delta_{\boldsymbol{z}}$ it follows that $\tilde{\boldsymbol{\psi}}_0, \tilde{\boldsymbol{\psi}}_f \in \mathcal{O}_{\delta, I_N}$. Then, by the first step, there is a control $u \in L^2((0,T^*),\mathbb{R})$ such that

$$\boldsymbol{\psi}(T^*, \tilde{\boldsymbol{\psi}}_0, u) = \tilde{\boldsymbol{\psi}}_f.$$

Since system (4.2) is linear with respect to the state, the resolving operator commutes with C. Thus, in view of (4.24), we have

$$\boldsymbol{\psi}(T^*, \boldsymbol{\psi}_0, u) = \boldsymbol{\psi}(T^*, C\tilde{\boldsymbol{\psi}}_0, u) = C\boldsymbol{\psi}(T^*, \tilde{\boldsymbol{\psi}}_0, u) = C\tilde{\boldsymbol{\psi}}_f = \boldsymbol{\psi}_f. \tag{4.25}$$

This ends the proof the second step.

Third step : conclusion.
Since U_N is connected, there is a continuous mapping $t \in [0,1] \mapsto C(t) \in U_N$ with $C(0) = C_0$ and $C(1) = C_f$. By the previous step, for any $\boldsymbol{z} \in F := \{C(t)\boldsymbol{\varphi}\,;\, t \in [0,1]\}$, there is $\delta_{\boldsymbol{z}} > 0$ such that (4.2) is exactly controllable in $B_{\boldsymbol{H}^3_{(V)}}(\boldsymbol{z}, \delta_{\boldsymbol{z}})$ in time T^*. Using the compactness of the set F, we get the existence of $\boldsymbol{z}_j \in F$, $j = 1, \ldots, L$ with $L \in \mathbb{N}^*$ such that

$$F \subset \bigcup_{j=0}^{L} B_{\boldsymbol{H}^3_{(V)}}(\boldsymbol{z}_j, \delta_{\boldsymbol{z}_j}).$$

Without loss of generality, we can assume that $\boldsymbol{z}_L = \boldsymbol{z}_f$. Finally, setting $T := (L+1)T^*$ and $\delta := \min\{\delta_{\boldsymbol{z}_0}, \delta_{\boldsymbol{z}_f}\}$, we see that for any $\boldsymbol{\psi}_0 \in \mathcal{O}_{\delta, C_0}$ and $\boldsymbol{\psi}_f \in \mathcal{O}_{\delta, C_f}$ there is a control $u \in L^2((0,T),\mathbb{R})$ such that

$$\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, u) = \boldsymbol{\psi}_f$$

This completes the proof of Theorem 4.3. □

The rest of this section is dedicated to the proof of Proposition 4.3.

4.4.2 Construction of the reference trajectory

The proof of Proposition 4.3 relies on the return method introduced by Coron (see [54, Chapter 6] for a comprehensive introduction). The natural strategy to obtain local exact controllability around $\boldsymbol{\varphi}$ is to prove controllability for the linearized system

$$\begin{cases} i\partial_t \Psi^j = \left(-\partial^2_{xx} + V(x)\right)\Psi^j - v(t)\mu(x)\Phi_j, & (t,x) \in (0,T) \times (0,1), \\ \Psi^j(t,0) = \Psi^j(t,1) = 0, & j \in \{1, \ldots, N\}, \\ \Psi^j(0,x) = 0. \end{cases} \tag{4.26}$$

However, straightforward computations lead to

$$\langle \mu\varphi_k, \varphi_k \rangle \langle \Psi^j(T), \Phi_j(T) \rangle = \langle \mu\varphi_j, \varphi_j \rangle \langle \Psi^k(T), \Phi_k(T) \rangle, \quad \forall j,k \in \{1, \ldots, N\}. \tag{4.27}$$

Thus, the linearized system (4.26) is not controllable and we use the return method. In our setting, the main idea of this method is to design a reference control u_{ref} such that the associated solution $\boldsymbol{\psi}_{ref}$ of system (4.2) with initial condition $\boldsymbol{\varphi}$ satisfies

$$\boldsymbol{\psi}_{ref}(T) = (e^{i\theta_1}\varphi_1, \ldots, e^{i\theta_N}\varphi_N),$$

for some $\theta_1,\dots,\theta_N \in \mathbb{R}$ and the linearized system around this trajectory is controllable. Then, an application of the inverse mapping theorem leads to local controllability of (4.2) around the trajectory $(u_{ref}, \boldsymbol{\psi}_{ref})$ and proves Proposition 4.3. The main ideas of this proof are adapted from [109, Theorem 1.5]. For the sake of completeness, we precise the adaptations that have been made and give a sketch of the proofs. The reference trajectory is designed in the following proposition.

Proposition 4.4. *Assume that Conditions* $\mathbf{(C_3)}-\mathbf{(C_4)}$ *are satisfied for* $V, \mu \in H^3((0,1),\mathbb{R})$*. Let* $T>0$ *and* $0<\varepsilon_0<\dots<\varepsilon_{N-1}=:\varepsilon<T$*. There exist* $\overline{\eta}>0$ *and* $C>0$ *such that for every* $\eta\in(0,\overline{\eta})$*, there are* $\theta_1^\eta,\dots,\theta_N^\eta\in\mathbb{R}$ *and a control* $u_{ref}^\eta \in L^2((0,T),\mathbb{R})$ *with*

$$\|u_{ref}^\eta\|_{L^2(0,T)} \leq C\eta \tag{4.28}$$

such that the associated solution $\boldsymbol{\psi}_{ref}^\eta$ *of (4.2) with initial condition* $\boldsymbol{\varphi}$ *satisfies for* $j \in \{1,\dots,N\}$ *and* $k\in\{1,\dots,N-1\}$

$$\langle \mu\psi_{ref}^{j,\eta}(\varepsilon_k), \psi_{ref}^{j,\eta}(\varepsilon_k)\rangle = \langle \mu\varphi_j,\varphi_j\rangle + \eta\delta_{j=k}, \tag{4.29}$$

and

$$\boldsymbol{\psi}_{ref}^\eta(T) = \left(e^{i\theta_1^\eta}\varphi_1,\dots,e^{i\theta_N^\eta}\varphi_N\right). \tag{4.30}$$

Remark 4.3. As in [109], the conditions (4.29), together with an appropriate choice of the parameter η, will imply the controllability of the linearized system around this reference trajectory (see Section 4.4.3).

Sketch of the proof of Proposition 4.4. We split the proof in two steps. In the first step, we construct u_{ref}^η on $(0,\varepsilon)$ such that (4.29) is satisfied. Then in the second step, we extend u_{ref}^η to (ε,T) in such a way that (4.30) is verified.

First step : Let us take $u_{ref}^\eta\equiv 0$ on $[0,\varepsilon_0)$. Following the proof of [109, Proposition 3.1], we construct a control u_{ref}^η such that condition (4.29) is satisfied and

$$\|u_{ref}^\eta\|_{L^2(\varepsilon_0,\varepsilon)} \leq C\eta, \tag{4.31}$$

by an application of the inverse mapping theorem to the map

$$\begin{array}{rccc}\tilde{\Theta}: & L^2((\varepsilon_0,\varepsilon),\mathbb{R}) & \to & \mathbb{R}^N\times\dots\times\mathbb{R}^N \\ & u & \mapsto & \left(\tilde{\Theta}_1(u),\dots,\tilde{\Theta}_{N-1}(u)\right)\end{array}$$

at the point $u=0$, where

$$\tilde{\Theta}_k(u) := \left(\langle \mu\psi^j(\varepsilon_k),\psi^j(\varepsilon_k)\rangle - \langle\mu\varphi_j,\varphi_j\rangle\right)_{1\leq j\leq N}.$$

The C^1 regularity of $\tilde{\Theta}$ follows from the differentiability property in Proposition 4.1. A continuous right-inverse of $\mathrm{d}\tilde{\Theta}(0)$ is constructed by the resolution of a suitable trigonometric moment problem using Proposition 4.7.

Second step : For any $j\in\mathbb{N}^*$, let $\mathcal{P}_j$ be the orthogonal projection defined by (4.9). We prove that for any initial condition at time ε close enough to $\big(\Phi_1,\dots,\Phi_N\big)(\varepsilon)$, the projections

$\big(\mathcal{P}_1(\psi^1(T)),\dots,\mathcal{P}_N(\psi^N(T))\big)$ can be brought to 0 by a small control $u \in L^2((\varepsilon,T),\mathbb{R})$. This is sufficient to prove Proposition 4.4. Indeed, if

$$\mathcal{P}_1\big(\psi^{1,\eta}_{ref}(T)\big) = \cdots = \mathcal{P}_N\big(\psi^{N,\eta}_{ref}(T)\big) = 0, \tag{4.32}$$

using the invariants (4.20), it comes that there exist $\theta_1^\eta,\dots,\theta_N^\eta \in \mathbb{R}$ such that (4.30) holds. As in [109, Proposition 3.2], the condition (4.32) with a control satisfying

$$\|u^\eta_{ref}\|_{L^2(\varepsilon,T)} \leq C\eta \tag{4.33}$$

is obtained by an application of the inverse mapping theorem to the map

$$\Theta : L^2((\varepsilon,T),\mathbb{R}) \times \boldsymbol{H}^3_{(0)} \to \boldsymbol{H}^3_{(0)} \times \boldsymbol{X},$$

at the point $\big(0,\Phi_1(\varepsilon),\dots,\Phi_N(\varepsilon)\big)$, where

$$\Theta\big(u,\boldsymbol{\psi}_0\big) := \Big(\boldsymbol{\psi}_0,\, \mathcal{P}_1\big(\psi^1(T)\big),\dots,\mathcal{P}_N\big(\psi^N(T)\big)\Big)$$

and

$$\boldsymbol{X} := \Big\{\boldsymbol{\phi} \in \boldsymbol{H}^3_{(0)}\,;\, \langle \phi^j,\varphi_k\rangle = 0 \text{ for all } 1 \leq k \leq j \leq N\Big\}. \tag{4.34}$$

Again, the C^1 regularity of Θ is obtained thanks to Proposition 4.1. The continuous right-inverse of $\mathrm{d}\Theta\big(0,\Phi_1(\varepsilon),\dots,\Phi_N(\varepsilon)\big)$ is given by the resolution of a suitable trigonometric moment problem with frequencies

$$\{\lambda_k - \lambda_j\,;\, j \in \{1,\dots,N\},\, k \geq j+1\}.$$

The solution of that moment problem is given by Proposition 4.7.

□

4.4.3 Controllability of the linearized system

This section is dedicated to the proof of controllability of the following system which is the linearization of (4.2) around the reference trajectory $\boldsymbol{\psi}^\eta_{ref}$:

$$\begin{cases} i\partial_t \Psi^j = \big(-\partial^2_{xx} + V(x)\big)\,\Psi^j - u^\eta_{ref}(t)\mu(x)\Psi^j - v(t)\mu(x)\psi^{j,\eta}_{ref}, & (t,x) \in (0,T)\times(0,1), \\ \Psi^j(t,0) = \Psi^j(t,1) = 0, & j \in \{1,\dots,N\}, \\ \Psi^j(0,x) = \Psi^j_0(x). \end{cases} \tag{4.35}$$

For any $t \in [0,T]$, let us define the following space

$$\begin{aligned} \boldsymbol{X}_t : = \Big\{\boldsymbol{\phi} \in \boldsymbol{H}^3_{(0)}\,;\, &\mathrm{Re}(\langle \phi^j, \psi^{j,\eta}_{ref}(t)\rangle) = 0 \text{ for } j = 1,\dots,N \\ &\text{and } \langle \phi^j, \psi^{k,\eta}_{ref}(t)\rangle = -\overline{\langle \phi^k, \psi^{j,\eta}_{ref}(t)\rangle} \text{ for } j = 2,\dots,N,\, k<j\Big\}. \end{aligned}$$

This space is given by the linearization of the invariants (4.20) around the reference trajectory.

We prove the following controllability result.

Proposition 4.5. *There exists* $\hat{\eta} \in (0, \overline{\eta})$ *such that for any* $\eta \in (0, \hat{\eta})$*, there exists a continuous linear map*

$$\begin{array}{rccc} L^\eta : & \boldsymbol{X}_0 \times \boldsymbol{X}_T & \to & L^2((0,T),\mathbb{R}) \\ & (\boldsymbol{\Psi}_0, \boldsymbol{\Psi}_f) & \mapsto & v \end{array}$$

such that for any $\boldsymbol{\Psi}_0 \in \boldsymbol{X}_0$ *and* $\boldsymbol{\Psi}_f \in \boldsymbol{X}_T$*, the solution* $\boldsymbol{\Psi}$ *of system (4.35) with initial condition* $\boldsymbol{\Psi}_0$ *and control* $v := L^\eta(\boldsymbol{\Psi}_0, \boldsymbol{\Psi}_f)$ *satisfies* $\boldsymbol{\Psi}(T) = \boldsymbol{\Psi}_f$*.*

The proof of Proposition 4.5 is adapted from [109, Proposition 4.1]. As the proof is quite long and technical, we recall the main steps and arguments. Let us set some notations that will be used throughout this proof. For any $\eta \in (0, \overline{\eta})$ and $k \in \mathbb{N}^*$, let $\Phi_k^\eta = \psi(\cdot, \varphi_k, u_{ref}^\eta)$ as defined by (4.4). Notice that for $j \in \{1, \dots, N\}$, $\Phi_j^\eta = \psi_{ref}^{j,\eta}$ and for any $t \in [0,T]$, $\{\Phi_k^\eta(t)\}_{k \in \mathbb{N}^*}$ is a Hilbert basis of $L^2((0,1),\mathbb{C})$, as an image of a Hilbert basis by a unitary operator. Let

$$\mathcal{I} := \left\{(j,k) \in \{1, \dots, N\} \times \mathbb{N}^* \,;\, k \geq j+1\right\} \cup \{(N,N)\}\,.$$

In the first step we prove the controllability of the directions $\langle \Psi^j(T), \Phi_k^\eta(T) \rangle$ for $(j,k) \in \mathcal{I}$ for η small enough. This comes from the solvability of the trigonometric moment problem associated to the case $\eta = 0$ and a close linear maps argument. Then, we exhibit a minimal family that allows to control, simultaneously to the previous direction, the remaining diagonal directions $\langle \Psi^j(T), \Phi_j^\eta(T) \rangle$ for $j \in \{1, \dots, N-1\}$. This is the main feature of the design of the reference trajectory. Indeed, we enlightened in (4.27) that those diagonal directions were the ones leading to non controllability of the linearized system in the case $\eta = 0$. Finally, due to the definition of $\boldsymbol{X}_T$, the remaining directions $\langle \Psi^j(T), \Phi_k^\eta(T) \rangle$ for $1 \leq k < j$ are automatically controlled.

Sketch of the proof of Proposition 4.5. Let $R : \mathcal{I} \to \mathbb{N}$ be the rearrangement such that, if $\omega_n := \lambda_k - \lambda_j$ with $n = R(j,k)$, the sequence $(\omega_n)_{n \in \mathbb{N}}$ is increasing. Notice that $0 = R(N,N)$.

First step. Let us take any $T_f \in (0,T]$ and prove that there is $\hat{\eta} = \hat{\eta}(T_f) \in (0, \overline{\eta})$ such that for any $\eta \in (0, \hat{\eta})$ there exists a continuous linear map

$$G_{T_f}^\eta : \boldsymbol{X}_0 \times \ell_r^2(\mathbb{N},\mathbb{C}) \to L^2((0,T_f),\mathbb{R})$$

such that for any $\boldsymbol{\Psi}_0 \in \boldsymbol{X}_0$, $d = (d_n)_{n \in \mathbb{Z}} \in \ell_r^2(\mathbb{N},\mathbb{C})$, the solution $\boldsymbol{\Psi}$ of system (4.35) with initial condition $\boldsymbol{\Psi}_0$ and control $v = G_{T_f}^\eta(\boldsymbol{\Psi}_0, d)$ satisfies

$$\frac{\langle \Psi^j(T_f), \Phi_k^\eta(T_f) \rangle}{i \langle \mu \varphi_j, \varphi_k \rangle} = d_n, \quad \forall (j,k) \in \mathcal{I},\, n = R(j,k).$$

Let

$$f_n^\eta : t \in [0,T] \mapsto \frac{\langle \mu \psi_{ref}^{j,\eta}(t), \Phi_k^\eta(t) \rangle}{\langle \mu \varphi_j, \varphi_k \rangle} \quad \text{ for } (j,k) \in \mathcal{I} \text{ and } n = R(j,k),$$

$f_{-n}^\eta := \overline{f_n^\eta}$ for $n \in \mathbb{N}^*$ and $H_0 := \mathrm{Adh}_{L^2(0,T_f)}\left(\mathrm{Span}\{e^{i\omega_n \cdot}, n \in \mathbb{Z}\}\right)$. As in [109, Lemma 4.1], the construction of $G_{T_f}^\eta$ relies on the fact that the map

$$\begin{array}{rccc} J^\eta : & L^2((0,T_f),\mathbb{C}) & \to & \ell^2(\mathbb{Z},\mathbb{C}) \\ & v & \mapsto & \left(\int_0^{T_f} v(t) f_n^\eta(t) \mathrm{d}t\right)_{n \in \mathbb{Z}} \end{array}$$

is an isomorphism from H_0 to $\ell^2(\mathbb{Z},\mathbb{C})$. Indeed, for any $(j,k)\in\mathcal{I}$ and $n=R(j,k)$, straightforward computations lead to

$$\langle \Psi^j(T_f), \Phi_k^\eta(T_f)\rangle = \langle \Psi_0^j, \varphi_k\rangle + i\langle \mu\varphi_j,\varphi_k\rangle \int_0^{T_f} v(t) f_n^\eta(t)\mathrm{d}t.$$

The isomorphism property of J^η comes from the estimate

$$\|J^\eta - J^0\|_{\mathcal{L}(L^2(0,T_f),\ell^2)} \le C\|u_{ref}^\eta\|_{L^2(0,T_f)} \le C\eta,$$

(see [109, Proof of Lemma 4.1] for the proof of this estimate) and the fact that, due to Proposition 4.7, J^0 is an isomorphism from H_0 to $\ell^2(\mathbb{Z},\mathbb{C})$.

Second step. Let $\hat\eta < \min(\hat\eta(T), \hat\eta(\varepsilon_0))$ with ε_0 as in Proposition 4.4. In all what follows we assume $\eta\in(0,\hat\eta)$. Let

$$f_{j,j}^\eta : t\in[0,T] \mapsto \frac{\langle \mu\psi_{ref}^{j,\eta}(t), \psi_{ref}^{j,\eta}(t)\rangle}{\langle\mu\varphi_j,\varphi_j\rangle} \quad \text{for } j\in\{1,\dots,N-1\}. \tag{4.36}$$

Then, the family $\Xi := (f_n^\eta)_{n\in\mathbb{Z}} \cup \{f_{1,1}^\eta,\dots,f_{N-1,N-1}^\eta\}$ is minimal in $L^2((0,T),\mathbb{C})$. The proof of this is a straightforward extension of [109, Lemma 4.3] and is not detailed. It relies on the fact that $(f_n^\eta)_{n\in\mathbb{Z}}$ is a Riesz basis of $\mathrm{Adh}_{L^2(0,T)}\big(\mathrm{Span}\{f_n^\eta, n\in\mathbb{Z}\}\big)$ and conditions (4.29).

Third step : conclusion. From the second step, we get the existence of a biorthogonal family associated to Ξ in $\mathrm{Adh}_{L^2(0,T)}\big(\mathrm{Span}\{\Xi\}\big)$ denoted by

$$\big\{g_{1,1}^\eta,\dots,g_{N-1,N-1}^\eta,(g_n^\eta)_{n\in\mathbb{Z}}\big\}, \tag{4.37}$$

with $g_{j,j}^\eta$ being real-valued for $j\in\{1,\dots,N\}$. The map L^η is defined by

$$L^\eta : \big(\boldsymbol{\Psi}_0,\boldsymbol{\Psi}_f\big)\in \boldsymbol{X}_0\times\boldsymbol{X}_T \mapsto v\in L^2((0,T),\mathbb{R}),$$

where

$$v := v_0 + \sum_{j=1}^{N-1}\Big(\frac{\mathrm{Im}(\langle\Psi_f^j,\psi_{ref}^{j,\eta}(T)\rangle) - \mathrm{Im}(\langle \Psi_0^j,\varphi_j\rangle)}{\langle\mu\varphi_j,\varphi_j\rangle} - \int_0^T v_0(t) f_{j,j}^\eta(t)\mathrm{d}t\Big) g_{j,j}^\eta,$$

and $v_0 := G_T^\eta(\boldsymbol{\Psi}_0, d(\boldsymbol{\Psi}_f))$ with $d(\boldsymbol{\Psi}_f)_n := \dfrac{\langle\Psi_f^j,\Phi_k^\eta(T)\rangle}{i\langle\mu\varphi_j,\varphi_k\rangle}$, for $(j,k)\in\mathcal{I}$ and $n=R(j,k)$. The biorthogonality properties and the first step imply

$$\langle\Psi^j(T),\Phi_k^\eta(T)\rangle = \langle\Psi_f^j,\Phi_k^\eta(T)\rangle, \quad \forall (j,k)\in\mathcal{I}\cup\{(1,1),\dots,(N-1,N-1)\}.$$

Finally, for $j\in\{2,\dots,N\}$ and $k<j$ explicit computations lead to

$$\langle\Psi^j(T),\psi_{ref}^{k,\eta}(T)\rangle = -\overline{\langle\Psi^k(T),\psi_{ref}^{j,\eta}(T)\rangle}.$$

As $\boldsymbol{\Psi}_f\in\boldsymbol{X}_T$, this ends the proof of Proposition 4.5.

□

4.4.4 Controllability of the nonlinear system

In this subsection, we end the proof of Proposition 4.3. We consider the reference trajectory designed in Proposition 4.4. Let $\hat{\eta}$ be given by Proposition 4.5. We assume in all what follows that $\eta \in (0, \hat{\eta})$ is fixed. Using the inverse mapping theorem and Proposition 4.5, we prove in Proposition 4.6 that the projections onto the space $\boldsymbol{X}_T$ (see (4.39) for a precise definition) are exactly controlled. Then, using the invariants (4.20) of the system, we prove that controlling these projections is sufficient to control the full trajectory. Let us set

$$\boldsymbol{\Omega} := \left\{ \boldsymbol{\phi} \in \boldsymbol{H}^3_{(0)} \, ; \, \langle \phi^j, \phi^k \rangle = \delta_{j=k}, \ \forall j,k \in \{1,\dots,N\} \right\} \tag{4.38}$$

and define

$$\begin{array}{rccc} \Lambda : & \boldsymbol{\Omega} \times L^2((0,T),\mathbb{R}) & \to & \boldsymbol{\Omega} \times \boldsymbol{X}_T \\ & (\boldsymbol{\psi}_0, u) & \mapsto & \big(\boldsymbol{\psi}_0, \, \tilde{\mathcal{P}}_1(\psi^1(T)), \dots, \tilde{\mathcal{P}}_N(\psi^N(T))\big), \end{array}$$

where $\boldsymbol{\psi} := \boldsymbol{\psi}(\cdot, \boldsymbol{\psi}_0, u)$ and

$$\begin{aligned} \tilde{\mathcal{P}}_j(\phi^j) :&= \phi^j - \mathrm{Re}\left(\langle \phi^j, \psi^{j,\eta}_{ref}(T) \rangle\right) \psi^{j,\eta}_{ref}(T) \\ &\quad - \sum_{k=1}^{j-1} \left(\langle \phi^j, \psi^{k,\eta}_{ref}(T) \rangle + \langle \psi^{j,\eta}_{ref}(T), \phi^k \rangle \right) \psi^k_{ref}(T). \end{aligned} \tag{4.39}$$

Thus, Λ takes values in $\boldsymbol{\Omega} \times \boldsymbol{X}_T$ and $\Lambda(\boldsymbol{\varphi}, u^\eta_{ref}) = (\boldsymbol{\varphi}, 0)$. The following proposition holds.

Proposition 4.6. *There exist $\tilde{\delta} > 0$ and a C^1 map*

$$\Upsilon : \mathcal{O}^0_{\tilde{\delta}} \times \tilde{\mathcal{O}}_{T,\tilde{\delta}} \to L^2((0,T),\mathbb{R}),$$

where $\mathcal{O}^0_{\tilde{\delta}}$ is defined in Proposition 4.3 and

$$\tilde{\mathcal{O}}_{T,\tilde{\delta}} := \left\{ \tilde{\boldsymbol{\psi}}_f \in \boldsymbol{X}_T \, ; \, \sum_{j=1}^N \|\tilde{\psi}^j_f\|_{H^3_{(V)}} < \tilde{\delta} \right\},$$

such that $\Upsilon\big(\boldsymbol{\varphi}, \boldsymbol{0}\big) = u^\eta_{ref}$ and for any $\boldsymbol{\psi}_0 \in \mathcal{O}^0_{\tilde{\delta}}$ and $\tilde{\boldsymbol{\psi}}_f \in \tilde{\mathcal{O}}_{T,\tilde{\delta}}$ the solution $\boldsymbol{\psi}$ of system (4.2) with initial condition $\boldsymbol{\psi}_0$ and control $u = \Upsilon\big(\boldsymbol{\psi}_0, \tilde{\boldsymbol{\psi}}_f\big)$ satisfies

$$\big(\tilde{\mathcal{P}}_1(\psi^1(T)), \dots, \tilde{\mathcal{P}}_N(\psi^N(T))\big) = \tilde{\boldsymbol{\psi}}_f.$$

Sketch of proof. As [109, Proposition 4.2], this proposition is proved by an application of the inverse mapping theorem to the map Λ at the point $(\boldsymbol{\varphi}, u^\eta_{ref})$. This map is C^1 by Proposition 4.7, and a continuous right inverse of the map

$$\mathrm{d}\Lambda\big(\boldsymbol{\varphi}, u^\eta_{ref}\big) : \boldsymbol{X}_0 \times L^2((0,T),\mathbb{R}) \to \boldsymbol{X}_0 \times \boldsymbol{X}_T$$

is given by Proposition 4.5.

□

Finally, we prove Proposition 4.3.

Proof of Proposition 4.3. Let us take any $\boldsymbol{\psi}_0 \in \mathcal{O}_\delta^0$ and $\boldsymbol{\psi}_f \in \mathcal{O}_\delta^f$, where the sets $\mathcal{O}_\delta^0$ and $\mathcal{O}_\delta^f$ are defined in Proposition 4.3 and $\delta > 0$ will be specified later on. Let $\tilde{\delta}$ be the constant in Proposition 4.6 and

$$\tilde{\boldsymbol{\psi}}_f := \big(\tilde{\mathcal{P}}_1(\psi_f^1), \dots, \tilde{\mathcal{P}}_N(\psi_f^N)\big).$$

For sufficiently small $\delta \in (0, \tilde{\delta})$, we have $\tilde{\boldsymbol{\psi}}_f \in \tilde{\mathcal{O}}_{T,\tilde{\delta}}$ and

$$\operatorname{Re}(\langle \psi_f^j, \psi_{ref}^{j,\eta}(T)\rangle) > 0, \quad \forall j \in \{1, \dots, N\}, \tag{4.40}$$

for any $\boldsymbol{\psi}_f \in \mathcal{O}_\delta^f$. Let $u := \Upsilon\big(\boldsymbol{\psi}_0, \tilde{\boldsymbol{\psi}}_f\big)$ and let $\boldsymbol{\psi}$ be the associated solution of (4.2) with initial condition $\boldsymbol{\psi}_0$. We prove that (up to an a priori reduction of δ)

$$\boldsymbol{\psi}(T) = \boldsymbol{\psi}_f. \tag{4.41}$$

Thanks to the regularity of Υ and Proposition 4.1, it comes that, up to a reduction of δ, one can assume that

$$\operatorname{Re}(\langle \psi^j(T), \psi_{ref}^{j,\eta}(T)\rangle) > 0, \quad \forall j \in \{1, \dots, N\}. \tag{4.42}$$

By Proposition 4.6, we get

$$\psi^1(T) - \operatorname{Re}(\langle \psi^1(T), \psi_{ref}^{1,\eta}(T)\rangle)\psi_{ref}^{1,\eta}(T) = \psi_f^1 - \operatorname{Re}(\langle \psi_f^1, \psi_{ref}^{1,\eta}(T)\rangle)\psi_{ref}^{1,\eta}(T).$$

Thus, using the fact that $\|\psi^1(T)\| = \|\psi_f^1\|$ and (4.40), (4.42), we get $\psi^1(T) = \psi_f^1$. Assume that

$$\big(\psi^1, \dots, \psi^{j-1}\big)(T) = \big(\psi_f^1, \dots, \psi_f^{j-1}\big) \quad \text{for } j \in \{2, \dots, N\}.$$

Then the equality $\tilde{\mathcal{P}}_j(\psi^j(T)) = \tilde{\psi}_f^j$ gives

$$\begin{aligned} &\psi^j(T) - \operatorname{Re}(\langle \psi^j(T), \psi_{ref}^{j,\eta}(T)\rangle)\psi_{ref}^{j,\eta}(T) - \sum_{k=1}^{j-1}\langle \psi^j(T), \psi_{ref}^{k,\eta}(T)\rangle \psi_{ref}^{k,\eta}(T) \\ &= \psi_f^j - \operatorname{Re}(\langle \psi_f^j, \psi_{ref}^{j,\eta}(T)\rangle)\psi_{ref}^{j,\eta}(T) - \sum_{k=1}^{j-1}\langle \psi_f^j, \psi_{ref}^{k,\eta}(T)\rangle \psi_{ref}^{k,\eta}(T). \end{aligned} \tag{4.43}$$

Taking the scalar product of (4.43) with $\psi^n(T) (= \psi_f^n)$ for $n \in \{1, \dots, j-1\}$ and using the constraints $\langle \psi^j(T), \psi^k(T)\rangle = \langle \psi_f^j, \psi_f^k\rangle = \delta_{j=k}$, we get

$$\begin{aligned} &\operatorname{Re}(\langle \psi^j(T), \psi_{ref}^{j,\eta}(T)\rangle)\, \langle \psi_{ref}^{j,\eta}(T), \psi_f^n\rangle + \sum_{k=1}^{j-1}\langle \psi^j(T), \psi_{ref}^{k,\eta}(T)\rangle\, \langle \psi_{ref}^{k,\eta}(T), \psi_f^n\rangle \\ &= \operatorname{Re}(\langle \psi_f^j, \psi_{ref}^{j,\eta}(T)\rangle)\, \langle \psi_{ref}^{j,\eta}(T), \psi_f^n\rangle + \sum_{k=1}^{j-1}\langle \psi_f^j, \psi_{ref}^{k,\eta}(T)\rangle\, \langle \psi_{ref}^{k,\eta}(T), \psi_f^n\rangle. \end{aligned}$$

Straightforward algebraic manipulations of these equations lead to the existence of constants $\gamma_1, \dots, \gamma_{j-1} \in \mathbb{C}$ that are proved to be arbitrarily small (up to an a priori reduction of δ) such that for $k \in \{1, \dots, j-1\}$

$$\langle \psi^j(T), \psi_{ref}^{k,\eta}(T)\rangle = \langle \psi_f^j, \psi_{ref}^{k,\eta}(T)\rangle + \gamma_k\Big(\operatorname{Re}(\langle \psi^j(T), \psi_{ref}^{j,\eta}(T)\rangle) - \operatorname{Re}(\langle \psi_f^j, \psi_{ref}^{j,\eta}(T)\rangle)\Big). \tag{4.44}$$

If the γ_j's are small enough this is consistent with $\|\psi^j(T)\| = \|\psi^j_f\|$ only if

$$\mathrm{Re}(\langle \psi^j(T), \psi^{j,\eta}_{ref}(T)\rangle) = \mathrm{Re}(\langle \psi^j_f, \psi^{j,\eta}_{ref}(T)\rangle).$$

Together with (4.44), this implies $\psi^j(T) = \psi^j_f$ and ends the proof of Proposition 4.3. □

4.5 Global exact controllability

4.5.1 Global exact controllability under favourable hypothesis

In this section, combining the properties of approximate controllability proved in Theorem 4.2 and local exact controllability proved in Theorem 4.3, we establish global exact controllability for (4.2), under the following hypotheses on the functions $V, \mu \in H^4((0,1),\mathbb{R})$

$(\mathbf{C_6})$ For any $j \in \mathbb{N}^*$, there exists $C_j > 0$ such that

$$|\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle| \geq \frac{C_j}{k^3} \quad \text{for all } k \in \mathbb{N}^*.$$

$(\mathbf{C_7})$ The numbers $\{1, \lambda_{j,V}\}_{j\in\mathbb{N}^*}$ are rationally independent, i.e., for any $M \in \mathbb{N}^*$ and $\boldsymbol{r} \in \mathbb{Q}^{M+1}\backslash\{\mathbf{0}\}$, we have

$$r_0 + \sum_{j=1}^{M} r_j \lambda_{j,V} \neq 0.$$

Notice that these conditions imply Conditions $(\mathbf{C_1}) - (\mathbf{C_5})$.

Theorem 4.4. *Assume that Conditions* $(\mathbf{C_6})$ *and* $(\mathbf{C_7})$ *are satisfied for the functions* $V, \mu \in H^4((0,1),\mathbb{R})$. *Then, for any unitarily equivalent vectors* $\boldsymbol{\psi}_0, \boldsymbol{\psi}_f \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$, *there is a time* $T > 0$ *and a control* $u \in L^2((0,T),\mathbb{R})$ *such that the solution of (4.2) satisfies*

$$\boldsymbol{\psi}(T) = \boldsymbol{\psi}_f.$$

Proof. In this proof, we use vectors of different size. In bold characters we denote only the vectors of size N.

First step. Let us take any $M \in \mathbb{N}^*$ and $\boldsymbol{z} \in \boldsymbol{\mathcal{C}}_M$ and prove that there is a time $T > 0$ and a constant $\delta > 0$ such that for any $\boldsymbol{\psi}_0, \boldsymbol{\psi}_f \in B_{\boldsymbol{H}^3_{(V)}}(\boldsymbol{z}, \delta)$ which are unitarily equivalent to $\boldsymbol{z}$, there is a control $u \in L^2((0,T),\mathbb{R})$ satisfying $\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, u) = \boldsymbol{\psi}_f$. Here we use the following technical lemma whose proof is postponed to the end of this subsection.

Lemma 4.1. *For any* $\boldsymbol{z} \in \boldsymbol{\mathcal{C}}_M$ *and* $\epsilon > 0$, *there is* $\delta > 0$ *such that for any* $\boldsymbol{\phi} \in B_{\boldsymbol{H}^3_{(V)}}(\boldsymbol{z}, \delta)$, *which is unitarily equivalent to* $\boldsymbol{z}$, *there exists* $\mathcal{U}_\phi \in U(L^2)$ *satisfying* $\mathcal{U}_\phi \boldsymbol{z} = \boldsymbol{\phi}$ *and* $\|\mathcal{U}_\phi \varphi_{j,V} - \varphi_{j,V}\|_{H^3_{(V)}} < \epsilon$ *for* $j = 1, \ldots, M$.

Notice that under Conditions $(\mathbf{C_6})$ and $(\mathbf{C_7})$, we can apply Theorem 4.3 in the case of M equations and $C_0 = C_f = I_M$. We denote by δ_* and T_* the corresponding radius and time given in Theorem 4.3. Let δ be the constant in Lemma 4.1 corresponding to $\epsilon = \frac{\delta_*}{M}$. Then for any $\boldsymbol{\psi}_f \in B_{\boldsymbol{H}^3_{(V)}}(\boldsymbol{z}, \delta)$, which is unitarily equivalent to $\boldsymbol{z}$, we have $\sum_{j=1}^{M} \|\mathcal{U}_{\psi_f} \varphi_{j,V} -$

$\varphi_{j,V}\|_{H^3_{(V)}} < \delta_*$. Thus Theorem 4.3 implies the existence of a control $u_f \in L^2((0,T_*),\mathbb{R})$ driving the solution of (4.2) of size M from $(\varphi_{1,V},\dots,\varphi_{M,V})$ to $\mathcal{U}_{\psi_f}(\varphi_{1,V},\dots,\varphi_{M,V})$. As $\boldsymbol{z} \in \boldsymbol{\mathcal{C}}_M$, there exists a matrix $C \in \mathbb{C}^{N\times M}$ such that $\boldsymbol{z} = C(\varphi_{1,V},\dots,\varphi_{M,V})$. Then we have

$$C\mathcal{U}_{\psi_f}(\varphi_{1,V},\dots,\varphi_{M,V}) = \mathcal{U}_{\psi_f}C(\varphi_{1,V},\dots,\varphi_{M,V}) = \mathcal{U}_{\psi_f}\boldsymbol{z} = \boldsymbol{\psi}_f.$$

Combining this with the fact that (4.2) is linear with respect to the state, we get that the control u_f also drives the solution of (4.2) of size N from $\boldsymbol{z}$ to $\boldsymbol{\psi}_f$ (cf. (4.25)).
The same strategy leads to the existence of a control $u_0 \in L^2((0,T_*),\mathbb{R})$ driving the solution of (4.2) of size N from $\overline{\boldsymbol{z}}$ to $\overline{\boldsymbol{\psi}_0}$. Thus, using the time reversibility property and setting $T = 2T_*$, $u(t) = u_0(T_* - t)$ on $(0,T_*)$ and $u(t) = u_f(t - T_*)$ on (T_*,T), we end the proof of the first step.

Second step. Let $M \in \mathbb{N}^*$ and $\boldsymbol{z}_0, \boldsymbol{z}_f \in \boldsymbol{\mathcal{C}}_M$ be unitarily equivalent. In this step, we prove that there is a constant $\delta > 0$ and a time $T > 0$ such that for any $\boldsymbol{\psi}_0 \in B_{\boldsymbol{H}^3_{(V)}}(\boldsymbol{z}_0,\delta)$ and $\boldsymbol{\psi}_f \in B_{\boldsymbol{H}^3_{(V)}}(\boldsymbol{z}_f,\delta)$, which are unitarily equivalent to $\boldsymbol{z}_0$, there is a control $u \in L^2((0,T),\mathbb{R})$ such that $\boldsymbol{\psi}(T,\boldsymbol{\psi}_0,u) = \boldsymbol{\psi}_f$.
As $\boldsymbol{z}_0, \boldsymbol{z}_f \in \boldsymbol{\mathcal{C}}_M$, there exists $\mathcal{U} \in U(\mathcal{C}_M)$ such that $\boldsymbol{z}_f = \mathcal{U}\boldsymbol{z}_0$. Since $U(\mathcal{C}_M)$ is connected, we can choose a continuous mapping $t \in [0,1] \mapsto \mathcal{U}(t) \in U(\mathcal{C}_M)$ such that $\mathcal{U}(0) = I_M$ and $\mathcal{U}(1) = \mathcal{U}$. Then using the exact controllability result proved in the first step for the vectors $\mathcal{U}(t)\boldsymbol{z}_0$, $t \in [0,1]$ and an argument of compactness, as in the third step of the proof of Theorem 4.3, we get the required property.

Third step. Let us take any unitarily equivalent $\boldsymbol{\psi}_0, \boldsymbol{\psi}_f \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)} \cap \boldsymbol{E}$ and prove that there is a time $T > 0$ and a control $u \in L^2((0,T),\mathbb{R})$ such that $\boldsymbol{\psi}(T,\boldsymbol{\psi}_0,u) = \boldsymbol{\psi}_f$. Applying Theorem 4.2 to $\boldsymbol{\psi}_0$ and $\overline{\boldsymbol{\psi}}_f$, we find sequences T_{0n}, T_{fn} and $u_{0n} \in L^2((0,T_{0n}),\mathbb{R}), u_{fn} \in L^2((0,T_{fn}),\mathbb{R})$ such that

$$\|\boldsymbol{\psi}(T_{0n},\boldsymbol{\psi}_0,u_{0n}) - \boldsymbol{\psi}_{01}\|_{H^3_{(V)}} + \|\boldsymbol{\psi}(T_{fn},\overline{\boldsymbol{\psi}}_f,u_{fn}) - \overline{\boldsymbol{\psi}}_{f1}\|_{H^3_{(V)}} \underset{n\to\infty}{\longrightarrow} 0$$

for some $\boldsymbol{\psi}_{01}, \boldsymbol{\psi}_{f1} \in \boldsymbol{\mathcal{C}}_M$. By the second step, we have exact controllability between some δ-neighbourhoods of $\boldsymbol{\psi}_{01}$ and $\boldsymbol{\psi}_{f1}$ (notice that these vectors are unitarily equivalent). Choosing n so large that

$$\|\boldsymbol{\psi}(T_{0n},\boldsymbol{\psi}_0,u_{0n}) - \boldsymbol{\psi}_{01}\|_{H^3_{(V)}} + \|\overline{\boldsymbol{\psi}(T_{fn},\overline{\boldsymbol{\psi}}_f,u_{fn})} - \boldsymbol{\psi}_{f1}\|_{H^3_{(V)}} < \delta,$$

we find a time $\tilde{T}$ and a control $\tilde{u} \in L^2((0,\tilde{T}),\mathbb{R})$ such that

$$\boldsymbol{\psi}(\tilde{T},\boldsymbol{\psi}(T_{0n},\boldsymbol{\psi}_0,u_{0n}),\tilde{u}) = \overline{\boldsymbol{\psi}(T_{fn},\overline{\boldsymbol{\psi}}_f,u_{fn})}.$$

Taking $T = T_{0n} + \tilde{T} + T_{fn}$ and $u(t) = u_{0n}(t)$ for $t \in (0,T_{0n})$, $u(t) = \tilde{u}(t - T_{0n})$ for $t \in (T_{0n}, T_{0n}+\tilde{T})$, and $u(t) = u_{fn}(T-t)$ for $t \in (T_{0n}+\tilde{T},T)$, and using the time reversibility property, we get $\boldsymbol{\psi}(T,\boldsymbol{\psi}_0,u) = \boldsymbol{\psi}_f$.

Fourth step. By the time reversibility property, to complete the proof of the theorem, it remains to show that for any $\boldsymbol{\psi}_0 \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$ we have $\boldsymbol{\psi}(T,\boldsymbol{\psi}_0,u) \in \boldsymbol{H}^4_{(V)} \cap \boldsymbol{E}$ for some $T > 0$ and $u \in L^2((0,T),\mathbb{R})$.

Let us take any $\boldsymbol{\psi}_{0n}, \boldsymbol{\psi}_f \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)} \cap \boldsymbol{E}$ such that $\boldsymbol{\psi}_{0n} \underset{n\to\infty}{\longrightarrow} \boldsymbol{\psi}_0$ in $\boldsymbol{L}^2$. From the previous step, there are sequences T_n and $u_n \in L^2((0,T_n),\mathbb{R})$ such that $\boldsymbol{\psi}(T_n, \boldsymbol{\psi}_{0n}, u_n) = \boldsymbol{\psi}_f$. Then

$$\|\boldsymbol{\psi}(T_n, \psi_0^j, u_n) - \boldsymbol{\psi}_f\| = \|\boldsymbol{\psi}_0 - \boldsymbol{\psi}_{0n}\| \underset{n\to\infty}{\longrightarrow} 0,$$

therefore

$$\prod_{j=1}^N |\langle \psi(T_n, \psi_0^j, u_n), \varphi_{j,V}\rangle|^2 \underset{n\to\infty}{\longrightarrow} \prod_{j=1}^N |\langle \psi_f^j, \varphi_{j,V}\rangle|^2 \neq 0.$$

Thus $\boldsymbol{\psi}(T_n, \boldsymbol{\psi}_0, u_n) \in \boldsymbol{E}$ for sufficiently large n. Finally, taking a control $u \in C_0^\infty((0,T_n),\mathbb{R})$ sufficiently close to u_n in $L^2((0,T_n),\mathbb{R})$, we get $\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, u) \in \boldsymbol{H}^4_{(V)} \cap \boldsymbol{E}$. This completes the proof of Theorem 4.4.

□

We end this section by the proof of Lemma 4.1.

Proof of Lemma 4.1. Let $\mathcal{A}_{\boldsymbol{\phi}} := \mathrm{Span}\{\phi_i\,;\, i = 1, \dots, N\}$. As $\boldsymbol{\phi}$ and $\boldsymbol{z}$ are unitarily equivalent, there exists a linear map $L_{\boldsymbol{\phi}} : \mathcal{A}_{\boldsymbol{z}} \to \mathcal{A}_{\boldsymbol{\phi}}$ such that $L_{\boldsymbol{\phi}} \boldsymbol{z} = \boldsymbol{\phi}$ and

$$\langle L_{\boldsymbol{\phi}}\xi, L_{\boldsymbol{\phi}}\zeta\rangle = \langle \xi, \zeta\rangle, \quad \forall \xi, \zeta \in \mathcal{A}_{\boldsymbol{z}}.$$

Let $\{\psi_k^z\}_{1\le k\le M}$ be an orthonormal basis in $\mathcal{C}_M$ (with respect to the L^2 scalar product) such that $\{\psi_k^z\}_{1\le k\le n}$ is a basis in $\mathcal{A}_{\boldsymbol{z}}$. If we define $\psi_j^\phi := L_{\boldsymbol{\phi}}\psi_j^z$ for $j = 1, \dots, n$, then $\{\psi_k^\phi\}_{1\le k\le n}$ will be an orthonormal basis in $\mathcal{A}_{\boldsymbol{\phi}}$ and $\psi_j^\phi \underset{\boldsymbol{\phi}\to\boldsymbol{z}}{\longrightarrow} \psi_j^z$ in $H^3_{(V)}$ for $j = 1, \dots, n$. Let

$$\begin{aligned} \tilde{\psi}_k^\phi &:= \psi_k^\phi, \quad \forall k \in \{1, \dots, n\}, \\ \tilde{\psi}_k^\phi &:= \psi_k^z - \sum_{j=1}^n \langle \psi_k^z, \psi_j^\phi\rangle \psi_j^\phi, \quad \forall k \in \{n+1, \dots, M\}. \end{aligned}$$

It is easy to see that $\tilde{\psi}_k^\phi \underset{\boldsymbol{\phi}\to\boldsymbol{z}}{\longrightarrow} \psi_k^z$ in $H^3_{(V)}$ for $k = 1, \dots, M$. Thus if $\boldsymbol{\phi}$ is sufficiently close to $\boldsymbol{z}$ in $\boldsymbol{H}^3_{(V)}$, then $\{\tilde{\psi}_k^\phi\}_{1\le k\le M}$ is linearly independent. We denote by $\{\hat{\psi}_k^\phi\}_{1\le k\le M}$ the associated orthonormal family given by the Gram-Schmidt process. Notice that $\hat{\psi}_k^\phi = \psi_k^\phi$ for $k \in \{1, \dots, n\}$ and $\hat{\psi}_k^\phi \underset{\boldsymbol{\phi}\to\boldsymbol{z}}{\longrightarrow} \psi_k^z$ in $H^3_{(V)}$ for $k = 1, \dots, M$. Let $\mathcal{U}_\phi \in U(L^2)$ be any operator such that $\mathcal{U}_\phi \psi_j^z = \hat{\psi}_j^\phi$ for every $j \in \{1, \dots, M\}$. By construction we have that $\mathcal{U}_\phi \boldsymbol{z} = L_\phi \boldsymbol{z} = \boldsymbol{\phi}$ and $\|\mathcal{U}_\phi \varphi_{j,V} - \varphi_{j,V}\|_{H^3_{(V)}} \underset{\boldsymbol{\phi}\to\boldsymbol{z}}{\longrightarrow} 0$ for any $j \in \{1, \dots, M\}$. This ends the proof of Lemma 4.1.

□

4.5.2 Proof of Theorem 4.1

Let us fix an arbitrary $V \in H^4$, and let $\mathcal{Q}_V$ be the set of all functions $\mu \in H^4$ such that Conditions $\mathbf{(C_6)}$ and $\mathbf{(C_7)}$ are satisfied with the functions V and μ replaced by the functions $V+\mu$ and μ. Let us prove that (4.2) is exactly controllable in $\boldsymbol{H}^4_{(V)}$ for any $\mu \in \mathcal{Q}_V$. Along with (4.2), let us consider the system

$$\begin{cases} i\partial_t \psi^j = \big(-\partial^2_{xx} + V(x) + \mu(x)\big)\psi^j - u(t)\mu(x)\psi^j, & (t,x) \in (0,T)\times(0,1), \\ \psi^j(t,0) = \psi^j(t,1) = 0, & j \in \{1,\dots,N\}, \\ \psi^j(0,x) = \psi^j_0(x), \end{cases} \tag{4.45}$$

and denote by $\tilde{\boldsymbol{\psi}}$ its resolving operator. Clearly, we have

$$\tilde{\boldsymbol{\psi}}(t,\boldsymbol{\psi}_0,u) = \boldsymbol{\psi}(t,\boldsymbol{\psi}_0,u-1) \tag{4.46}$$

for any $\boldsymbol{\psi}_0 \in \boldsymbol{H}^3_{(0)}$, $t\in[0,T]$, and $u \in L^2((0,T),\mathbb{R})$. By Theorem 4.4, system (4.45) is exactly controllable in $\boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V+\mu)}$ for any $\mu \in \mathcal{Q}_V$.

Let us take any $\boldsymbol{\psi}_0, \boldsymbol{\psi}_f \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$ and any control $u_1 \in W^{1,1}((0,1),\mathbb{R})$ such that $u_1(0) = 0$ and $u_1(1) = -1$. By Proposition 4.1, $\boldsymbol{\psi}(1,\boldsymbol{\psi}_0,u_1) =: \boldsymbol{\psi}_{01} \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V+\mu)}$ and $\boldsymbol{\psi}(1,\overline{\boldsymbol{\psi}_f},u_1) =: \overline{\boldsymbol{\psi}_{f1}} \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V+\mu)}$. The time reversibility property implies that $\boldsymbol{\psi}(1,\boldsymbol{\psi}_{f1},u_2) = \boldsymbol{\psi}_f$, where $u_2(t) = u_1(1-t), t\in[0,1]$. Since (4.45) is exactly controllable, there is a time $\tilde{T}$ and a control $\tilde{u} \in L^2((0,T),\mathbb{R})$ such that $\tilde{\boldsymbol{\psi}}(\tilde{T},\boldsymbol{\psi}_{01},\tilde{u}) = \boldsymbol{\psi}_{f1}$. Finally, choosing $T = \tilde{T}+2$ and $u(t) = u_1(t)$ for $t\in(0,1)$, $u(t) = \tilde{u}(t-\tilde{T}) - 1$ for $t \in (1, 1+\tilde{T})$, and $u(t) = u_2(t-1-\tilde{T})$ for $t \in (1+\tilde{T},T)$, we get $\boldsymbol{\psi}(T,\boldsymbol{\psi}_0,u) = \boldsymbol{\psi}_f$. This proves the global exact controllability of (4.2) in $\boldsymbol{H}^4_{(V)}$ for any $\mu \in \mathcal{Q}_V$.

It remains to show that the set $\mathcal{Q}_V$ is residual in H^4. Let us write $\mathcal{Q}_V = \mathcal{Q}^6_V \cap \mathcal{Q}^7_V$, where $\mathcal{Q}^j_V$ is the set of all functions $\mu \in H^4$ such that Condition $(\mathbf{C_j})$ is satisfied with V and μ replaced by $V+\mu$ and μ, $j = 6,7$. Since the intersection of two residual sets is residual, the proof of Theorem 4.1 follows from the following result.

Lemma 4.2. *For any $V \in H^s, s \geq 4$, the sets $\mathcal{Q}^6_V$ and $\mathcal{Q}^7_V$ are residual in H^s.*

This lemma is proved in Section 4.A.2. See [102] for the proof of the fact that $\mathcal{Q}^7_V$ is residual in a much more general case. Nevertheless, we give its proof in the Appendix, since it is simpler in our setting.

Conclusion and open problems

In this article, we have proved simultaneous global exact controllability between any unitarily equivalent N-tuples of functions in $\mathcal{S} \cap H^4_{(V)}$. Our result is valid in large time, for an arbitrary number of equations, and for an arbitrary potential. Hence, the spectrum of the free operator can be extremely resonant. Thus, not only we extend previous results on exact controllability for a single particle to simultaneous controllability of N particles, but we also improve the existing literature in 1D for $N = 1$.

Our proof combines several ideas. Using a Lyapunov strategy, we proved that any initial condition can be driven arbitrarily close to some finite sum of eigenfunctions. Then, designing a reference trajectory and using a rotation phenomenon on a suitable time interval we proved local exact controllability in $\boldsymbol{H}^3_{(V)}$ around $\boldsymbol{\varphi}_V$. Finally combining linearity of the equation with respect to the state and a compactness argument, we obtained global exact controllability under favourable hypotheses. The case of an arbitrary potential is dealt with a perturbation argument.

We mention here two possible ways to improve this result. The optimal functional setting for exact controllability is $H^3_{(V)}$. While using our Lyapunov function, we have dealt with more regular initial and final conditions to get convergence in H^3 from the boundedness in H^4. This issue of strong stabilization in infinite dimension is not specific to bilinear quantum system and is an open problem. The other possible improvement concerns the time of control. In our strategy, there are three steps requiring a time large enough : the approximate controllability, the rotation argument in local exact controllability, and the compactness argument.

4.A Appendix

4.A.1 Moment problem

In this article, we use several times the following result about the trigonometric moment problem.

Proposition 4.7. *Assume that Condition* $(\mathbf{C_4})$ *is satisfied. Let* $(\omega_n)_{n\in\mathbb{N}}$ *be the increasing sequence defined by*

$$\{\omega_n\,;\, n\in\mathbb{N}\} = \{\lambda_{k,V}-\lambda_{j,V}\,;\, j\in\{1,\dots,N\},\, k\geq j+1 \text{ and } k=j=N\}.$$

Then, for any $T>0$*, there exists a continuous linear map*

$$\mathcal{L} : \ell^2_r(\mathbb{N},\mathbb{C}) \to L^2((0,T),\mathbb{R})$$

such that for every $d=(d_n)_{n\in\mathbb{N}}\in\ell^2_r(\mathbb{N},\mathbb{C})$*, we have*

$$\int_0^T \mathcal{L}(d)(t)e^{i\omega_n t}\mathrm{d}t = d_n, \quad \forall n\in\mathbb{N}.$$

Proof. Let us set $\omega_{-n} := -\omega_n$ for $n\in\mathbb{N}$, and let D^+ be the upper density of the sequence $(\omega_n)_{n\in\mathbb{Z}}$, i.e.,

$$D^+ := \lim_{r\to\infty}\frac{n^+(r)}{r},$$

where $n^+(r)$ is the largest number of elements of the sequence $(\omega_n)_{n\in\mathbb{Z}}$ in an interval of length r. By the Beurling theorem (e.g., see [92, Theorem 9.2]), if the uniform gap condition

$$\omega_{n+1}-\omega_n \geq \gamma, \quad \forall n\in\mathbb{N} \tag{4.47}$$

is satisfied for some $\gamma>0$, then for any $T>2\pi D^+$, the family $(e^{i\omega_n\cdot})_{n\in\mathbb{Z}}$ is a Riesz basis of $H_0 := \mathrm{Adh}_{L^2(0,T)}\big(\mathrm{Span}\{e^{i\omega_n\cdot}\,;\, n\in\mathbb{Z}\}\big)$. Let us show that, under Condition $(\mathbf{C_4})$, the sequence $(\omega_n)_{n\in\mathbb{Z}}$ has a uniform gap and $D^+=0$.

Indeed, by the well-known asymptotic formula for the eigenvalues (e.g., see [117, Theorem 4]),

$$\lambda_{k,V} = k^2\pi^2 + \int_0^1 V(x)\mathrm{d}x + r_k, \quad \text{with } \sum_{k=1}^{\infty} r_k^2 < +\infty. \tag{4.48}$$

This implies that for some sufficiently large integers n_0 and k_0, we have

$$\omega_{n_0+n} = \lambda_{k_0+p,V} - \lambda_{j,V}, \quad \text{where } n = pN + j,\ 1 \le j \le N, p \in \mathbb{N}.$$

Thus, the frequencies $(\omega_n)_{n \ge n_0}$ can be gathered as successive packets of N frequencies such that the minimal gap inside each packet is

$$\tilde{\gamma} := \min_{1 \le q < m \le N} (\lambda_{m,V} - \lambda_{q,V}).$$

Using Condition $\mathbf{(C_4)}$, we obtain $\tilde{\gamma} > 0$. The gap between the $(\ell+1)^{th}$ packet and the ℓ^{th} packet is

$$\lambda_{\ell+1,V} - \lambda_{\ell,V} + \lambda_{1,V} - \lambda_{N,V}$$

which goes to infinity as $\ell \to \infty$, by (4.48). On the other hand, $\omega_n \neq \omega_k$ for $n \neq k$, by Condition $\mathbf{(C_4)}$. Hence we get the uniform gap condition (4.47). From (4.48) it follows immediately that $D^+ = 0$. Thus the family $(e^{i\omega_n \cdot})_{n \in \mathbb{Z}}$ is a Riesz basis of H_0. This implies that the map

$$\begin{array}{rccc} J_0 : & H_0 & \to & \ell^2(\mathbb{Z}, \mathbb{C}) \\ & f & \mapsto & \left(\int_0^T f(t) e^{i\omega_n t} \mathrm{d}t \right)_{n \in \mathbb{Z}} \end{array}$$

is an isomorphism. Then, the map $\mathcal{L} : d \in \ell^2_r(\mathbb{N}, \mathbb{C}) \mapsto J_0^{-1}(\tilde{d})$, where $\tilde{d}_n := d_n$ and $\tilde{d}_{-n} := \overline{d_n}$ for $n \in \mathbb{N}$, satisfies the required properties.

□

4.A.2 Proof of Lemma 4.2

First step. Let us show that $\mathcal{Q}_V^7$ is residual in H^s. It suffices to show that the set $\mathcal{Q}_0^7$ of all functions $W \in H^s$, such that the numbers $\{1, \lambda_{j,W}\}_{j \in \mathbb{N}^*}$ are rationally independent, is residual in H^s. Let us take any $M \in \mathbb{N}^*$ and $\boldsymbol{r} \in \mathbb{Q}^{M+1} \setminus \{\mathbf{0}\}$ and denote by $\mathcal{Q}_{M,\boldsymbol{r}}$ the set of all functions $W \in H^s$ such that

$$r_0 + \sum_{j=1}^{M} r_j \lambda_{j,W} \neq 0.$$

Then we have $\mathcal{Q}_0^7 = \bigcap_{M \in \mathbb{N}^*, \boldsymbol{r} \in \mathbb{Q}^M \setminus \{\mathbf{0}\}} \mathcal{Q}_{M,\boldsymbol{r}}$. Thus it is sufficient to prove that $\mathcal{Q}_{M,\boldsymbol{r}}$ is open and dense in H^s. Continuity of the eigenvalues[1] $\lambda_{k,W}$ from L^2 to $\mathbb{R}$ implies that $\mathcal{Q}_{M,\boldsymbol{r}}$ is open in H^s. Let us show that $\mathcal{Q}_{M,\boldsymbol{r}}$ is dense in H^s. For any $W, P \in H^s$ and $\sigma \in \mathbb{R}$, differentiating the identity

$$(-\partial_{xx}^2 + W + \sigma P - \lambda_{j,W+\sigma P}) \varphi_{j,W+\sigma P} = 0$$

with respect to σ at $\sigma = 0$, we get

$$(-\partial_{xx}^2 + W - \lambda_{j,W}) \frac{\mathrm{d}\varphi_{j,W+\sigma P}}{\mathrm{d}\sigma}\Big|_{\sigma=0} + \Big(P - \frac{\mathrm{d}\lambda_{j,W+\sigma P}}{\mathrm{d}\sigma}\Big|_{\sigma=0} \Big) \varphi_{j,W} = 0.$$

1. By [117, Theorem 3], the eigenvalues $\lambda_{k,W}$ and eigenfunctions $\varphi_{k,W}$ are real-analytic functions with respect to $W \in L^2$.

Taking the scalar product of this identity with $\varphi_{j,W}$, we obtain

$$\frac{\mathrm{d}\lambda_{j,W+\sigma P}}{\mathrm{d}\sigma}\Big|_{\sigma=0} = \langle P, \varphi_{j,W}^2\rangle.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\Big(r_0 + \sum_{j=1}^{M} r_j\lambda_{j,W+\sigma P}\Big)\Big|_{\sigma=0} = \langle P, \sum_{j=1}^{M} r_j\varphi_{j,W}^2\rangle. \tag{4.49}$$

By [117, Theorem 9], for any $W \in L^2$, the functions $\{\varphi_{j,W}^2\}_{j=1}^{\infty}$ are linearly independent. Hence we can find $P \in H^s$ such that

$$\langle P, \sum_{j=1}^{M} r_j\varphi_{j,W}^2\rangle \neq 0.$$

Then (4.49) implies that $W + \sigma P \in \mathcal{Q}_{M,\boldsymbol{r}}$ for any σ sufficiently close to 0. This shows that $\mathcal{Q}_{M,\boldsymbol{r}}$ is dense in H^s. Thus $\mathcal{Q}_V^7$ is residual in H^s.

Second step. Recall that $\mathcal{Q}_V^6$ is the set of all functions $\mu \in H^s$ such that for any $j \in \mathbb{N}^*$ there exists $C_j > 0$ verifying

$$|\langle \mu\varphi_{j,V+\mu}, \varphi_{k,V+\mu}\rangle| \geq \frac{C_j}{k^3} \quad \text{for all } k \in \mathbb{N}^*.$$

We will use the following well known estimates for any $W \in L^2$

$$\|\varphi_{k,W} - \varphi_{k,0}\|_{L^\infty} \leq \frac{C}{k}, \tag{4.50}$$

$$\|\varphi'_{k,W} - \varphi'_{k,0}\|_{L^\infty} \leq C, \tag{4.51}$$

(e.g., see [117, Theorem 4]). Integrating by parts, we get for any $W \in H^s$

$$\begin{aligned}\langle \mu\varphi_{j,W}, \varphi_{k,W}\rangle =& \frac{1}{\lambda_{k,W}}\langle(-\partial_{xx}^2 + W)(\mu\varphi_{j,W}), \varphi_{k,W}\rangle\\ =& \frac{1}{\lambda_{k,W}}\left(\langle -\mu''\varphi_{j,W}, \varphi_{k,W}\rangle + 2\langle -\mu'\varphi'_{j,W}, \varphi_{k,W}\rangle + \lambda_{j,W}\langle \mu\varphi_{j,W}, \varphi_{k,W}\rangle\right).\end{aligned}$$

This implies that for $k \neq j$, we have

$$\langle \mu\varphi_{j,W}, \varphi_{k,W}\rangle = \frac{1}{\lambda_{j,W} - \lambda_{k,W}}(\langle \mu''\varphi_{j,W}, \varphi_{k,W}\rangle + 2\langle \mu'\varphi'_{j,W}, \varphi_{k,W}\rangle). \tag{4.52}$$

Again integrating by parts, we obtain

$$\begin{aligned}\langle \mu'\varphi'_{j,W}, \varphi_{k,W}\rangle =& \frac{1}{\lambda_{k,W}}\langle \mu'\varphi'_{j,W}, (-\partial_{xx}^2 + W)\varphi_{k,W}\rangle\\ =& -\frac{1}{\lambda_{k,W}}\mu'\varphi'_{j,W}\varphi'_{k,W}\Big|_{x=0}^{x=1} + \frac{1}{\lambda_{k,W}}\langle(-\partial_{xx}^2 + W)(\mu'\varphi'_{j,W}), \varphi_{k,W}\rangle.\end{aligned} \tag{4.53}$$

Using (4.52) with μ replaced by μ'', we get

$$\langle \mu''\varphi_{j,W}, \varphi_{k,W}\rangle = \frac{1}{\lambda_{j,W} - \lambda_{k,W}}(\langle \mu^{(4)}\varphi_{j,W}, \varphi_{k,W}\rangle + 2\langle \mu^{(3)}\varphi'_{j,W}, \varphi_{k,W}\rangle).$$

Combination of this last equality with (4.48), (4.50)-(4.53) and the explicit expression $\varphi_{k,0}(x) = \sqrt{2}\sin(k\pi x)$, yields that

$$\begin{aligned} k^3\langle \mu\varphi_{j,W}, \varphi_{k,W}\rangle &= -4j\pi^{-1}\mu'\cos(j\pi x)\cos(k\pi x)\Big|_{x=0}^{x=1} + c_{k,j}k^{-1} \\ &= -4j\pi^{-1}((-1)^{j+k}\mu'(1) - \mu'(0)) + c_{k,j}k^{-1}, \end{aligned}$$

where for any $j \in \mathbb{N}^*$ the sequence $c_{j,k}, k > j$ is bounded in $\mathbb{R}$. Thus for any μ from the set

$$\mathcal{B} = \{\mu \in H^s\,;\, \mu'(1) \pm \mu'(0) \neq 0\}$$

and for any $W \in H^s$, there is $K_j \in \mathbb{N}^*$ such that

$$|\langle \mu\varphi_{j,W}, \varphi_{k,W}\rangle| \geq \frac{C_j}{k^3}$$

for all $k \geq K_j$. In particular, this is true for $W = V + \mu$. Combining this with the following result, we complete the proof.

Lemma 4.3. *For any $V \in H^s$, the set $\mathcal{Q}_V^1$ of all functions $\mu \in H^s$ such that*

$$\langle \mu\varphi_{j,V+\mu}, \varphi_{k,V+\mu}\rangle \neq 0 \tag{4.54}$$

for all $j, k \in \mathbb{N}^*$, is residual in H^s.

Indeed, $\mathcal{B}$ is open and dense in H^s and $\mathcal{B} \cap \mathcal{Q}_V^1 \subset \mathcal{Q}_V^6$. Then $\mathcal{B} \cap \mathcal{Q}_V^1$ is residual as an intersection of two residual sets. Hence $\mathcal{Q}_V^6$ is a residual set in H^s.

Proof of Lemma 4.3. For any $j, k \in \mathbb{N}^*$, let $\mathcal{Q}_{V,j,k}^1$ be the set of functions $\mu \in H^s$ such that (4.54) holds. Then $\mathcal{Q}_V^1 = \cap_{j,k\in\mathbb{N}^*}\mathcal{Q}_{V,j,k}^1$ and it suffices to show that $\mathcal{Q}_{V,j,k}^1$ is open and dense in H^s. As above, the fact that $\mathcal{Q}_{V,j,k}^1$ is open follows immediately from the continuous dependence of the eigenfunction $\varphi_{k,V+\mu}$ on μ. Let us show that $\mathcal{Q}_{V,j,k}^1$ is dense in H^s. Since $\varphi_{j,V}(x)\varphi_{k,V}(x)$ is not identically equal to zero, the set of functions μ such that $\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle \neq 0$ is dense in H^s. For any μ_0 from that set, the function $\langle \mu_0\varphi_{j,V+s\mu_0}, \varphi_{k,V+s\mu_0}\rangle$ is non-zero real-analytic function with respect to $s \in \mathbb{R}$. Thus $s\mu_0 \in \mathcal{Q}_{V,j,k}^1$ almost surely for any $s \in \mathbb{R}$. This proves that $\mathcal{Q}_{V,j,k}^1$ is dense in H^s.

□

Deuxième partie

Contrôle d'équations de Schrödinger avec un terme de polarisabilité

Chapitre 5

Contrôle approché explicite par moyennisation

Ce chapitre est inspiré de l'article [108] publié dans le journal *Mathematics of Control, Signals, and Systems*.

Contents

5.1 Introduction

5.1.1 Main result

Following [122] we consider a quantum particle in a potential $V(x)$ and an electric field of amplitude $u(t)$. We assume that the dipolar approximation is not valid (see [64, 65]). Then, the particle is represented by its wave function $\psi(t,x)$ solution of the following Schrödinger equation

$$\begin{cases} i\partial_t\psi = (-\Delta + V(x))\psi - u(t)\mu_1(x)\psi - u(t)^2\mu_2(x)\psi, & x \in D, \\ \psi_{|\partial D} = 0, \end{cases} \tag{5.1}$$

with initial condition

$$\psi(0,x) = \psi^0(x), \quad x \in D, \tag{5.2}$$

where $D \subset \mathbb{R}^m$ is a bounded domain with smooth boundary. The functions $V, \mu_1, \mu_2 \in C^\infty(\overline{D}, \mathbb{R})$ are given, μ_1 is the dipolar moment and μ_2 the polarizability moment. For the sake of simplicity, we denote by L^2, H_0^1 and H^2 respectively the usual Lebesgue and Sobolev spaces $L^2(D,\mathbb{C})$, $H_0^1(D,\mathbb{C})$ and $H^2(D,\mathbb{C})$. The following well-posedness result holds (see [42]) by application of the Banach fixed point theorem.

Proposition 5.1. *For any $\psi^0 \in H_0^1 \cap H^2$ and $u \in L^2_{loc}([0,+\infty),\mathbb{R})$, the system (5.1)-(5.2) has a unique weak solution $\psi \in C^0([0,+\infty), H_0^1 \cap H^2)$. Moreover, for all $t > 0$, $||\psi(t,.)||_{L^2} = ||\psi^0||_{L^2}$ and there exists $C = C(\mu_1,\mu_2) > 0$ such that for any $t > 0$,*

$$||\psi(t,\cdot)||_{H^2} \leq ||\psi^0||_{H^2} e^{C\int_0^t |u(\tau)|+|u(\tau)|^2 \mathrm{d}\tau}.$$

Let $\mathcal{S} := \left\{\psi \in L^2(D,\mathbb{C}); ||\psi||_{L^2} = 1\right\}$ and $\langle\cdot,\cdot\rangle$ be the usual scalar product on $L^2(D,\mathbb{C})$

$$\langle f, g\rangle = \int_D f(x)\overline{g(x)}\mathrm{d}x, \quad \text{for } f,g \in L^2(D,\mathbb{C}).$$

We consider the operator A_V defined by

$$A_V\psi := (-\Delta + V(x))\,\psi, \quad D\,(A_V) := H_0^1 \cap H^2, \tag{5.3}$$

and denote by $(\lambda_{k,V})_{k\in\mathbb{N}^*}$ the non-decreasing sequence of its eigenvalues and by $(\varphi_{k,V})_{k\in\mathbb{N}^*}$ the associated eigenvectors in $\mathcal{S}$. The family $(\varphi_{k,V})_{k\in\mathbb{N}^*}$ is a Hilbert basis of L^2. We also define the space $H^4_{(V)} := D(A_V^2)$. As V is fixed, the eigenelements $\varphi_{k,V}$ and $\lambda_{k,V}$ will be denoted by φ_k and λ_k. The operator A_V will be denoted by A.

Our goal is to stabilize the ground state. As the global phase of the wave function is physically meaningless, our target set is

$$\mathcal{C} := \{c\varphi\,;\, c \in \mathbb{C} \text{ and } |c| = 1\}, \tag{5.4}$$

where $\varphi := \varphi_1$.

Let $J_{\neq 0} := \Big\{k \geq 2; \langle\mu_1\varphi,\varphi_k\rangle \neq 0\Big\}$ and $J_0 := \Big\{k \geq 2; \langle\mu_1\varphi,\varphi_k\rangle = 0\Big\}$. We assume that the following hypotheses hold.

Hypothesis 5.1.

i) $\forall k \in J_0, \langle\mu_2\varphi,\varphi_k\rangle \neq 0$ i.e. all coupling are realized either by μ_1 or μ_2,

ii) $Card(J_0) < \infty$ i.e. only a finite number of coupling is missed by μ_1,

iii) $\lambda_1 - \lambda_k \neq \lambda_p - \lambda_q$ for $k,p,q \geq 1$ such that $\{1,k\} \neq \{p,q\}$ and $k \neq 1$.

Remark 5.1. The hypothesis *i)* is weaker than the one in [19] (i.e. $J_0 = \emptyset$). As proved in [111, Section 3.4], we get that generically with respect to μ_1 and μ_2 in $C^\infty(\overline{D}, \mathbb{R})$, the scalar products $\langle \mu_1\varphi, \varphi_k\rangle$ and $\langle \mu_2\varphi, \varphi_k\rangle$ are all non-zero. The spectral assumption *iii)* does not hold in every physical situation. For example, it is not satisfied in 1D if $V = 0$. However, it is proved in [111, Lemma 3.12] that if D is the rectangle $[0,1]^n$, Hypothesis 1.1 *iii)* hold generically with respect to V in the set $\mathcal{G} := \big\{V \in C^\infty(D,\mathbb{R})\,;\, V(x_1,\dots,x_n) = V_1(x_1) + \dots + V_n(x_n),\ \text{with } V_k \in C^\infty([0,1],\mathbb{R})\big\}$.

As in [57], we use a time-periodic oscillating control of the form

$$u(t,\psi) := \alpha(\psi) + \beta(\psi)\sin\left(\frac{t}{\varepsilon}\right). \tag{5.5}$$

Following classical techniques (see e.g. [124]) of dynamical systems in finite dimension let us introduce the averaged system

$$\begin{cases} i\partial_t\psi_{av} = \big(-\Delta + V(x)\big)\psi_{av} - \alpha(\psi_{av})\mu_1(x)\psi_{av} - \left(\alpha(\psi_{av})^2 + \dfrac{1}{2}\beta(\psi_{av})^2\right)\mu_2(x)\psi_{av}, \\ \psi_{av_{|\partial D}} = 0, \end{cases} \tag{5.6}$$

with initial condition

$$\psi_{av}(0,\cdot) = \psi^0. \tag{5.7}$$

Let $\mathcal{P}$ be the orthogonal projection in L^2 onto the closure of Span $\{\varphi_k; k \geq 2\}$ and γ be a positive constant (to be determined later).

Our stabilization strategy relies on the following Lyapunov function (used in [19]) defined on $\mathcal{S} \cap H^1_0 \cap H^2$ by

$$\mathcal{L}(\psi) := \gamma||(-\Delta + V)\mathcal{P}\psi||^2_{L^2} + 1 - |\langle\psi,\varphi\rangle|^2. \tag{5.8}$$

This leads to feedback laws given by

$$\alpha(\psi_{av}(t,\cdot)) := -kI_1(\psi_{av}(t,\cdot)), \qquad \beta(\psi_{av}(t,\cdot)) := g(I_2(\psi_{av}(t,\cdot)), \tag{5.9}$$

with $k > 0$ small enough and

$$g \in C^2(\mathbb{R},\mathbb{R}^+) \text{ satisfying } g(x) = 0 \text{ if and only if } x \geq 0,\ g' \text{ bounded}, \tag{5.10}$$

and for $j \in \{1,2\}$, for $z \in H^2$,

$$I_j(z) = \operatorname{Im}\Big[-\gamma\langle(-\Delta+V)\mathcal{P}\,\mu_j z, (-\Delta+V)\mathcal{P}z\rangle + \langle\mu_j z,\varphi\rangle\langle\varphi,z\rangle\Big]. \tag{5.11}$$

We can now state the well-posedness of the averaged closed loop system (5.6).

Proposition 5.2. *Let $R > 0$. There exists $k_0 = k_0(V,\mu_2,R) > 0$ such that for any $\psi^0 \in H^2 \cap H^1_0 \cap \mathcal{S}$ with $\mathcal{L}(\psi^0) < R$ and $k \in (0,k_0)$, the closed-loop system (5.6)-(5.7)-(5.9) has a unique solution $\psi_{av} \in C^0([0,+\infty), H^2 \cap H^1_0)$. There exists $M > 0$ such that*

$$||\psi_{av}(t)||_{H^2} \leq M, \quad \forall t \geq 0. \tag{5.12}$$

Moreover, if $\Delta\psi^0 \in H^1_0 \cap H^2$, then $\Delta\psi_{av} \in C^0([0,+\infty), H^1_0 \cap H^2)$.

For an initial condition $\psi^0 \in \mathcal{S} \cap H^4_{(0)}$, we define the control

$$u^\varepsilon(t) := \alpha(\psi_{av}(t)) + \beta(\psi_{av}(t)) \sin\left(\frac{t}{\varepsilon}\right), \tag{5.13}$$

where ψ_{av} is the solution of (5.6)-(5.7)-(5.9).

The main result of this article is the following one.

Theorem 5.1. *Assume that Hypotheses 5.1 hold. Let $\mathcal{C}$, the target set, be defined by (5.4). There exists $k_0 = k_0(V, \mu_2) > 0$ such that for any $k \in k_0$, for any $s < 2$ and for any $\psi^0 \in \mathcal{S} \cap H^4_{(0)}$ with $0 < \mathcal{L}(\psi^0) < 1$, there exist an increasing time sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+^*$ tending to $+\infty$ and a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+^*$ such that if ψ_ε is the solution of (5.1)-(5.2) associated to the control u^ε defined by (5.13) then for all $n \in \mathbb{N}$, if $\varepsilon \in (0, \varepsilon_n)$,*

$$dist_{H^s}\left(\psi_\varepsilon(t,\cdot), \mathcal{C}\right) \leq \frac{1}{2^n}, \quad \forall t \in [T_n, T_{n+1}].$$

Remark 5.2. Theorem 5.1 gives the semi-global approximate controllability with explicit controls of system (5.1). Hypotheses 5.1 are needed to ensure that the invariant set coincides with the target set. The semi-global aspect comes from the hypothesis $0 < \mathcal{L}(\psi^0) < 1$: by reducing γ (in a way dependant of ψ^0), this condition can be fulfilled as soon as $\psi^0 \notin \mathcal{C}$.

In Theorem 5.1, there is a gap between the H^4 regularity of the initial condition and the approximate controllability in H^s with $s < 2$. The extra-regularity is used in this article to prove an approximation property in H^2 between the oscillating system and the averaged one (see Section 5.3). Weakening this regularity assumption is an open problem for which an alternative strategy is required. The last loss of regularity comes from the application of a weak LaSalle principle instead of a strong one due to lack of compactness in infinite dimension.

5.1.2 A review of previous results

In this section, we recall previous results about quantum systems with bilinear controls. The model (5.1) of an infinite potential well was proposed by Rouchon [122] in the dipolar approximation ($\mu_2 = 0$). A classical negative result was obtained in [5] by Ball, Marsden and Slemrod for infinite dimensional bilinear control systems. This result implies, for system (5.1) with $\mu_2 = 0$, that the set of reachable states from any initial data in $H^2 \cap H^1_0 \cap \mathcal{S}$ with control in $L^2(0,T)$ has a dense complement in $H^2 \cap H^1_0 \cap \mathcal{S}$. However, exact controllability was proved in 1D by Beauchard [10] for $V = 0$ and $\mu_1(x) = x$ in more regular spaces (H^7). This result was then refined in [16] by Beauchard and Laurent for more general μ_1 and a regularity H^3.

The question of stabilization is addressed in [19] where Beauchard and Nersesyan extended previous results from Nersesyan [111]. They proved, under appropriate assumptions on μ_1, the semi-global weak H^2 stabilization of the wave function towards the ground state using explicit feedback control and Lyapunov techniques in infinite dimension.

However sometimes, for example in the case of higher laser intensities, this model is not efficient (see e.g. [64, 65]) and we need to add a polarizability term $u(t)^2\mu_2(x)\psi$ in the model. This term, if not neglected, can also be helpful in mathematical proofs. Indeed the result of [19] only holds if μ_1 couples the ground state to any other eigenstate and then the use of the polarizability enables us to weaken this assumption. Mathematical use of the expansion of the Hamiltonian beyond the dipolar approximation was used by Grigoriu, Lefter and Turinici in [80, 135]. A finite dimension approximation of this model was studied in [57] by Coron, Grigoriu, Lefter and Turinici. The authors proposed discontinuous feedback laws and periodic highly oscillating feedback laws to stabilize the ground state. In this article, we extend in our infinite dimensional framework their idea of using (time-dependent) periodic feedback laws. We also refer to the book by Coron [54] for a comprehensive presentation of the feedback strategy and the use of time-varying feedback laws.

How to adapt the Lyapunov or LaSalle strategy in an infinite dimensional framework is not clear because closed bounded sets are not compact so the trajectories may lack compactness in the considered topology. In this direction we should cite some related works of Mirrahimi and Beauchard [17, 103] where the idea was to prove approximate convergence results. In this article, we will use an adaptation of the LaSalle invariance principle for weak convergence which was used for example in [19] by Beauchard and Nersesyan. There are other strategies to show a strong stabilization property. Coron and d'Andréa-Novel proved in [55] the compactness of the trajectories by a direct method for a beam equation and thus the strong stabilization. Couchouron [61, 62] gave sufficient conditions to obtain the compactness in favorable cases where the control acts diagonally on the state. Another strategy to obtain strong results is to look for a strict Lyapunov function, which is an even trickier question, and was done for example by Coron, d'Andréa-Novel and Bastin [56] for a system of conservation laws.

The question of approximate controllability has been addressed by various authors using various techniques. In [112], Nersesyan uses a Lyapunov strategy to obtain approximate controllability in large time in regular spaces. In [45], Chambrion, Mason, Sigalotti and Boscain proved approximate controllability in L^2 for a wider class of systems using geometric control tools for the Galerkin approximations. The hypotheses needed were weakened in [25] and the approximate controllability was extended to some H^s spaces in [30].

Explicit approximate controllability in large time has also been obtained by Ervedoza and Puel in [70] on a model of trapped ion, using different tools.

5.1.3 Structure of this article

As announced in Section 5.1.1, we study the system (5.1) by introducing a highly oscillating time-periodic control and the corresponding averaged system. Section 5.2 is devoted to the introduction of this averaged system and its weak stabilization using Lyapunov techniques and an adaptation of the LaSalle invariance principle in infinite dimension.

In Section 5.3 we study the approximation property between the solution of the averaged system and the solution of (5.1) with the same initial condition. We prove that on every finite time interval these two solutions remain arbitrarily close provided that the control is oscillating enough. This is an extension of classical averaging results for finite dimension dynamical systems.

Finally gathering the stabilization result of Section 5.2 and the approximation property of Section 5.3, we prove Theorem 5.1 in Section 5.4.

Section 5.5 is devoted to numerical simulations illustrating several aspects of Theorem 5.1 and of the averaging strategy.

5.2 Stabilization of the averaged system

5.2.1 Definition of the averaged system

System (5.1) with feedback law u defined by (5.5) can be rewritten as

$$\begin{cases} \partial_t \psi(t) = -iA\psi(t) + F\left(\frac{t}{\varepsilon}, \psi(t)\right), \\ \psi_{|\partial D} = 0, \end{cases} \tag{5.14}$$

where the operator A is defined by (5.3) and

$$F(s,z) := i\left(\alpha(z) + \beta(z)\sin(s)\right)\mu_1 z + i\left(\alpha(z) + \beta(z)\sin(s)\right)^2 \mu_2 z. \tag{5.15}$$

For any z, $F(.,z)$ is T-periodic (with here $T = 2\pi$). Following classical techniques of averaging, we introduce $F^0(z) := \frac{1}{T}\int_0^T F(t,z)\mathrm{d}t$. We can define the averaged system associated to (5.14) by

$$\begin{cases} \partial_t \psi_{av} = -iA\psi_{av} + F^0(\psi_{av}), \\ \psi_{av_{|\partial D}} = 0. \end{cases} \tag{5.16}$$

Straightforward computations of F^0 show that the system (5.16) can be rewritten as (5.6).

We show by Lyapunov techniques that we can choose α and β such that the solution of the averaged system (5.16) is weakly convergent in H^2 towards our target set $\mathcal{C}$.

5.2.2 Control Lyapunov function and damping feedback laws

Our candidate for the Lyapunov function, $\mathcal{L}$, is defined in (5.8). It is clear that $\mathcal{L}(\psi) \geq 0$ whenever $\psi \in \mathcal{S} \cap H_0^1 \cap H^2$ and that $\mathcal{L}(\psi) = 0$ if and only if $\psi \in \mathcal{C}$.

The main advantage of this Lyapunov function is that it can be used to bound the H^2 norm. In fact, for any $\psi \in \mathcal{S} \cap H_0^1 \cap H^2$,

$$\mathcal{L}(\psi) \geq \gamma ||(-\Delta + V)\mathcal{P}\psi||_{L^2}^2 \geq \frac{\gamma}{2}||\Delta(\mathcal{P}\psi)||_{L^2}^2 - C \geq \frac{\gamma}{4}||\Delta\psi||_{L^2}^2 - C,$$

where here, as in all this article, C is a positive constant possibly different each time it appears. This leads to the existence of $\tilde{C} > 0$ satisfying

$$||\psi||_{H^2}^2 \leq \tilde{C}(1 + \mathcal{L}(\psi)), \quad \forall \psi \in \mathcal{S} \cap H_0^1 \cap H^2. \tag{5.17}$$

Remark 5.3. Although the idea of using a feedback of the form (5.5) is inspired by [57], the construction of the Lyapunov function and of the controls is here different because we are dealing with an infinite dimensional framework. We follow the strategy used in [111, 19].

Choice of the feedbacks. We would like to choose the feedbacks α and β such that for all $t \geq 0$, $\dfrac{d}{dt}\mathcal{L}(\psi_{av}(t)) \leq 0$ where ψ_{av} is the solution of (5.6),(5.7).

If $\Delta\psi_{av}(t) \in H_0^1 \cap H^2$ for all $t \geq 0$ then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi_{av}(t)) = 2\gamma \mathrm{Re}\big[\langle(-\Delta+V)\mathcal{P}\partial_t\psi_{av}, (-\Delta+V)\mathcal{P}\psi_{av}\rangle\big] - 2\mathrm{Re}\big[\langle\partial_t\psi_{av}, \varphi\rangle\langle\varphi, \psi_{av}\rangle\big]\\
&= 2\gamma\mathrm{Re}\Big[\langle(-\Delta+V)\mathcal{P}\big(i\Delta\psi_{av} - iV\psi_{av} + i\alpha\mu_1\psi_{av} + i(\alpha^2+\frac{1}{2}\beta^2)\mu_2\psi_{av}\big), (-\Delta+V)\mathcal{P}\psi_{av}\rangle\Big]\\
&- 2\mathrm{Re}\Big[\langle i\Delta\psi_{av} - iV\psi_{av} + i\alpha\mu_1\psi_{av} + i(\alpha^2+\frac{1}{2}\beta^2)\mu_2\psi_{av}, \varphi\rangle\langle\varphi, \psi_{av}\rangle\Big].
\end{aligned}$$

Then, we perform integration by parts. As $\mathcal{P}$ commutes with $(-\Delta+V)$, V is real and thanks to the following boundary conditions

$$(-\Delta+V)\mathcal{P}\psi_{av|\partial D} = \psi_{av|\partial D} = \varphi_{|\partial D} = 0,$$

we have

$$\begin{aligned}
&2\gamma\mathrm{Re}\Big[\langle -i(-\Delta+V)^2\mathcal{P}\psi_{av}, (-\Delta+V)\mathcal{P}\psi_{av}\rangle\Big] - 2\mathrm{Re}\Big[\langle(i\Delta - iV)\psi_{av}, \varphi\rangle\langle\varphi, \psi_{av}\rangle\Big]\\
&= 2\gamma\mathrm{Re}\Big[\langle -i\nabla(-\Delta+V)\mathcal{P}\psi_{av}, \nabla(-\Delta+V)\mathcal{P}\psi_{av}\rangle\Big]\\
&+ 2\gamma\mathrm{Re}\Big[\langle -iV(-\Delta+V)\mathcal{P}\psi_{av}, (-\Delta+V)\mathcal{P}\psi_{av}\rangle\Big] + 2\lambda_1\mathrm{Re}\Big[\langle i\psi_{av}, \varphi\rangle\langle\varphi, \psi_{av}\rangle\Big]\\
&= 0.
\end{aligned}$$

This leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi_{av}(t)) = 2\alpha I_1(\psi_{av}(t)) + 2\left(\alpha^2 + \frac{1}{2}\beta^2\right) I_2(\psi_{av}(t)), \tag{5.18}$$

where I_j is defined in (5.11).
In order to have a decreasing Lyapunov function we define the feedback laws α and β as in (5.9). Thus (5.18) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi_{av}(t)) = -2\left(kI_1^2(1-kI_2) - \frac{1}{2}I_2 g^2(I_2)\right). \tag{5.19}$$

If we assume that we can choose the constant k such that $(1-kI_2) > 0$ for all $t \geq 0$ and if $\Delta\psi_{av}(t) \in H_0^1 \cap H^2$ then the feedbacks (5.9) in system (5.6) lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi_{av}(t)) \leq 0, \quad \forall t \geq 0. \tag{5.20}$$

Well-posedness and boundedness proofs. Using the previous heuristic on the Lyapunov function, we can state and prove the well-posedness of the closed loop system (5.6)-(5.9) globally in time and derive a uniform bound on the H^2 norm of the solution. Namely, we prove Proposition 5.2.

Proof of Proposition 5.2. By the explicit expression (5.11) of I_2, we get for any $z \in H^2$, $|I_2(z)| \leq f\big(||z||_{H^2}\big)$ where

$$f(x) := ||\mu_2||_{L^\infty} + \gamma(x + ||V||_{L^\infty} + \lambda_1)(||\mu_2||_{\mathcal{C}^2}x + ||V||_{L^\infty}||\mu_2||_{L^\infty} + \lambda_1||\mu_2||_{L^\infty}).$$

Notice that f is increasing on $\mathbb{R}^+$. Let $K := 2f\Big(\sqrt{\tilde{C}(1+R)}\Big)$ where $\tilde{C}$ is defined by (5.17), $k_0 := \frac{1}{K}$ and $k \in (0, k_0)$.

The local existence and regularity is obtained by a classical fixed point argument : there exists $T^* > 0$ such that the closed loop system (5.6) with initial condition (5.7) and feedback laws (5.9) admits a unique solution defined on $(0, T^*)$ and satisfying either $T^* = +\infty$ or $T^* < +\infty$ and

$$\limsup_{t \to T^*} ||\psi_{av}(t)||_{H^2} = +\infty.$$

We have

$$|I_2(\psi_{av}(0))| \leq f(||\psi^0||_{H^2}) \leq f\left(\sqrt{\tilde{C}(1+\mathcal{L}(\psi^0))}\right) \leq \frac{K}{2},$$

thus, by continuity, $|I_2(\psi_{av}(t))| \leq K$ for t small enough.
Let

$$T_{max} := \sup\left\{t \in (0, T^*); |I_2(\psi_{av}(\tau))| \leq K, \forall \tau \in (0,t)\right\}.$$

We want to prove that $T_{max} = T^* = +\infty$.
For all $t \in [0, T_{max})$, we have $\big(1 - kI_2(\psi_{av}(t))\big) > 0$, which implies (by (5.19)), $\mathcal{L}(\psi_{av}(\cdot))$ is decreasing on $[0, T_{max})$. Estimate (5.17) leads to

$$||\psi_{av}(t)||_{H^2} \leq \sqrt{\tilde{C}(1+\mathcal{L}(\psi_{av}(t))} \leq \sqrt{\tilde{C}(1+\mathcal{L}(\psi^0))}, \quad \forall t \in [0, T_{max}). \tag{5.21}$$

Let us proceed by contradiction and assume that $T_{max} < T^*$. Thus $|I_2(\psi_{av}(T_{max}))| = K$. By definition of K,

$$|I_2(\psi_{av}(t))| \leq f\left(\sqrt{\tilde{C}(1+\mathcal{L}(\psi^0)}\right) \leq \frac{K}{2} \quad \forall t \in [0, T_{max}).$$

This is inconsistent with $|I_2(\psi_{av}(T_{max}))| = K$ so $T_{max} = T^*$ and the solution is bounded in H^2 when it is defined. As no blow-up is possible thanks to (5.21) we obtain that $T_{max} = T^* = +\infty$ and thus the solution is global in time and bounded.

Finally, taking the time derivative of the equation we obtain the announced regularity.

□

5.2.3 Convergence Analysis

In all this section we assume that $k \in (0, k_0)$ where k_0 is defined in Proposition 5.2 with $R = 1$. The closed-loop stabilization for the averaged system (5.6) is given by the next statement.

Theorem 5.2. *Assume that Hypotheses 5.1 hold. If $\psi^0 \in \mathcal{S} \cap H^4_{(0)}$ with $0 < \mathcal{L}(\psi^0) < 1$, then the solution ψ_{av} of the closed-loop system (5.6)-(5.9) with initial condition (5.7) satisfies*

$$\psi_{av}(t) \underset{t \to \infty}{\rightharpoonup} \mathcal{C} \quad \text{in } H^2.$$

We prove this theorem by adapting the LaSalle invariance principle to infinite dimension in the same spirit as in [19]. This is done in two steps. First we prove that the invariant set, relatively to the closed-loop system (5.6)-(5.9) and the Lyapunov function $\mathcal{L}$, is $\mathcal{C}$. Here, Hypotheses 5.1 are crucial. Then we prove that every adherent point for the weak H^2 topology of the solution of this closed-loop system is contained in $\mathcal{C}$. This is due to the continuity of the propagator of the closed-loop system for the weak H^2 topology.

5.2.3.1 Invariant set

Proposition 5.3. *Assume that Hypotheses 5.1 hold. Assume that ψ^0 belongs to $\mathcal{S}\cap H_0^1\cap H^2$ and satisfies $\langle\psi^0,\varphi\rangle \neq 0$. If the function $t\mapsto \mathcal{L}(\psi_{av}(t))$ is constant, then $\psi^0\in\mathcal{C}$.*

Proof. Thanks to (5.19), the fact that $\big(1-kI_2(\psi_{av}(t))\big)>0$ for all $t\geq 0$ and (5.10) we get

$$I_1[\psi_{av}(\cdot)]\equiv 0, \quad I_2(\psi_{av}(\cdot))g^2\left(I_2(\psi_{av}(\cdot))\right)\equiv 0 \text{ i.e. } I_2(\psi_{av}(t))\geq 0, \quad \forall t\geq 0.$$

By (5.9) this implies that $\alpha(\psi_{av}(\cdot))\equiv\beta(\psi_{av}(\cdot))\equiv 0$ and then ψ_{av} is solution of the uncontrolled Schrödinger equation. So,

$$\psi_{av}(t)=\sum_{j=1}^{\infty}e^{-i\lambda_j t}\langle\psi^0,\varphi_j\rangle\varphi_j.$$

Recall that $\varphi:=\varphi_1$ is the ground state. Following the idea of [111], we obtain after computations and gathering the terms with different exponential term

$$\begin{aligned}I_1(\psi_{av}(t))=&\sum_{j,k\geq 2}\tilde{P}(\psi^0,j,k,\mu_1)e^{-i(\lambda_j-\lambda_k)t}+\sum_{j\in J_{\neq 0}}\tilde{\tilde{P}}(\psi^0,j,\mu_1)e^{i(\lambda_j-\lambda_1)t}\\&-\sum_{j\in J_{\neq 0}}\langle\psi^0,\varphi_j\rangle\langle\varphi,\psi^0\rangle\langle\mu_1\varphi_j,\varphi\rangle(1+\gamma\lambda_j^2)e^{-i(\lambda_j-\lambda_1)t},\end{aligned}$$

where $\tilde{P}(\psi^0,j,k,\mu_1)$ and $\tilde{\tilde{P}}(\psi^0,j,\mu_1)$ are constants. Then, by [111, Lemma 3.10],

$$\langle\psi^0,\varphi_j\rangle\langle\varphi,\psi^0\rangle\langle\mu_1\varphi_j,\varphi\rangle(1+\gamma\lambda_j^2)=0, \quad \forall j\in J_{\neq 0}.$$

Using the assumption $\langle\varphi,\psi^0\rangle\neq 0$ and Hypotheses 5.1 it comes that for all $j\in J_{\neq 0}$, $\langle\psi^0,\varphi_j\rangle=0$. This leads to

$$\psi_{av}(t)=e^{-i\lambda_1 t}\langle\psi^0,\varphi\rangle\varphi+\sum_{j\in J_0}e^{-i\lambda_j t}\langle\psi^0,\varphi_j\rangle\varphi_j,$$

where by Hypotheses 5.1, J_0 is a finite set. By simple computations we obtain,

$$\begin{aligned}I_2(\psi_{av}(t))=\mathrm{Im}\Big(&-\sum_{k,j\in J_0}\gamma\lambda_j\langle\varphi_j,\psi^0\rangle\langle\psi^0,\varphi_k\rangle\langle(-\Delta+V)\mathcal{P}(\mu_2\varphi_k),\varphi_j\rangle e^{i(\lambda_j-\lambda_k)t}\\&-\sum_{j\in J_0}\gamma\lambda_j\langle\varphi_j,\psi^0\rangle\langle\psi^0,\varphi\rangle\langle(-\Delta+V)\mathcal{P}(\mu_2\varphi),\varphi_j\rangle e^{i(\lambda_j-\lambda_1)t}\\&+\sum_{j\in J_0}\langle\psi^0,\varphi_j\rangle\langle\varphi,\psi^0\rangle\langle\mu_2\varphi_j,\varphi\rangle e^{-i(\lambda_j-\lambda_1)t}+|\langle\psi^0,\varphi\rangle|^2\langle\mu_2\varphi,\varphi\rangle\Big)\geq 0.\end{aligned}\tag{5.22}$$

There exists $N_0\in\mathbb{N}^*$ and $(\omega_n)_{n\in\{0,\dots,N_0\}}$ such that

$$\{\omega_n\,;\,n\in\{0,\dots,N_0\}\}=\{\pm(\lambda_k-\lambda_j)\,;\,(k,j)\in J_0\times(J_0\cup\{1\})\},$$

with $\omega_0 = 0$ and $\omega_j \neq \omega_k$ if $j \neq k$. Thus, (5.22) implies that for any $n \in \{0, \dots, N_0\}$, there exists $\Lambda_n = \Lambda_n(\psi^0, \mu_2) \in \mathbb{C}$ such that

$$\mathrm{Im}\Big(\sum_{j=0}^{N_0} \Lambda_j e^{i\omega_j t}\Big) \geq 0, \quad \forall t \geq 0. \tag{5.23}$$

Straightforward computations give

$$\Lambda_0 = |\langle \psi^0, \varphi\rangle|^2 \langle \mu_2 \varphi, \varphi\rangle - \sum_{j \in J_0} \gamma \lambda_j^2 |\langle \varphi_j, \psi^0\rangle|^2 \langle \mu_2 \varphi_j, \varphi_j\rangle.$$

Thus, $\mathrm{Im}\big(\Lambda_0\big) = 0$ and our inequality (5.23) can be rewritten as

$$\mathrm{Im}\Big(\sum_{j=1}^{N_0} \Lambda_j e^{i\omega_j t}\Big) \geq 0, \quad \forall t \geq 0,$$

with the ω_j being all different and non-zero. Then using the same argument as in [57, Proof of Theorem 3.1], we get that $\Lambda_j = 0$ for $j \geq 1$ and then using (5.22) in particular that the coefficient of $e^{-i(\lambda_j - \lambda_1)t}$ vanishes. It implies $\langle \psi^0, \varphi_j\rangle = 0$ for all $j \in J_0$. Consequently, $\psi^0 = \langle \psi^0, \varphi\rangle \varphi$. As $\psi^0, \varphi \in \mathcal{S}$, we obtain $\psi^0 \in \mathcal{C}$.

□

5.2.3.2 Weak H^2 continuity of the propagator

We denote by $\mathcal{U}_t(\psi^0)$ the propagator of the closed-loop system (5.6)-(5.9). We detail here the continuity property of this propagator and of the feedback laws we need to apply the LaSalle invariance principle.

Proposition 5.4. *Let $z_n \in \mathcal{S} \cap H_0^1 \cap H^2$ be a sequence such that $z_n \rightharpoonup z_\infty$ in H^2. For every $T > 0$, there exists $N \subset (0, T)$ of zero Lebesgue measure verifying for all $t \in (0,T) \backslash N$,*

i) $\mathcal{U}_t(z_n) \underset{n\to\infty}{\rightharpoonup} \mathcal{U}_t(z_\infty)$ in H^2,

ii) $\alpha\big(\mathcal{U}_t(z_n)\big) \underset{n\to\infty}{\to} \alpha\big(\mathcal{U}_t(z_\infty)\big)$ and $\beta\big(\mathcal{U}_t(z_n)\big) \underset{n\to\infty}{\to} \beta\big(\mathcal{U}_t(z_\infty)\big)$.

Proof. Proof of ii). We start by proving that if $(z_n)_{n\in\mathbb{N}} \in H_0^1 \cap H^2$ satisfy

$z_n \underset{n\to\infty}{\rightharpoonup} z_\infty$ in H^2 then $\alpha(z_n) \underset{n\to\infty}{\to} \alpha(z_\infty)$ and $\beta(z_n) \underset{n\to\infty}{\to} \beta(z_\infty)$. Thus *ii)* will be a simple consequence of *i)*. As proved in [19, Proposition 2.2], using the fact that the regularity $H^{3/2}$ is sufficient to define the feedback, we get

$$I_j(z_n) \underset{n\to+\infty}{\longrightarrow} I_j(z_\infty), \quad \text{for } j = 1, 2.$$

So by the design of our feedback,

$$\alpha(z_n) \underset{n\to+\infty}{\longrightarrow} \alpha(z_\infty), \quad \beta(z_n) \underset{n\to+\infty}{\longrightarrow} \beta(z_\infty).$$

Proof of i). The exact same proof as in [19, Proposition 2.2] based on extraction in less regular spaces, uniqueness property of the closed loop system and taking into account the polarizability term leads to the announced result.

□

5.2.3.3 LaSalle invariance principle

We now have all the needed tools to prove Theorem 5.2.

Proof of Theorem 5.2. Consider $\psi^0 \in \mathcal{S} \cap H^4_{(0)}$ with $0 < \mathcal{L}(\psi^0) < 1$. Thanks to the bound (5.17), $\mathcal{U}_t(\psi^0)$ is bounded in H^2. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of times tending to $+\infty$ and $\psi_\infty \in H^2$ be such that $\mathcal{U}_{t_n}(\psi^0) \underset{n\to\infty}{\rightharpoonup} \psi_\infty$ in H^2. We want to show that $\psi_\infty \in \mathcal{C}$.

We prove that $\alpha(\mathcal{U}_t(\psi_\infty)) = 0$ and $\beta(\mathcal{U}_t(\psi_\infty)) = 0$. Indeed, the function $t \mapsto \alpha\big(\mathcal{U}_t(\psi^0)\big)$ belongs to $L^2(0,+\infty)$ (because of (5.19) and (5.9)) so the sequence of functions $(t \in (0,+\infty) \mapsto \alpha\big(\mathcal{U}_{t_n+t}(\psi^0)\big)_n$ tends to zero in $L^2(0,+\infty)$. Then by the Lebesgue reciprocal theorem there exists a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ and $N_1 \subset (0,+\infty)$ of zero Lebesgue measure such that

$$\alpha\big(\mathcal{U}_{t+t_{n_k}}(\psi^0)\big) \underset{k\to\infty}{\rightarrow} 0, \quad \forall t \in (0,+\infty)\backslash N_1.$$

Let $T \in (0,+\infty)$. Using Proposition 5.4, there exists $N \subset (0,T)$ of zero Lebesgue measure such that

$$\alpha\big(\mathcal{U}_{t+t_{n_k}}(\psi^0)\big) \underset{k\to\infty}{\rightarrow} \alpha\big(\mathcal{U}_t(\psi_\infty)\big), \quad \forall t \in (0,T)\backslash N.$$

Hence, $\alpha\big(\mathcal{U}_t(\psi_\infty)\big) = 0$ for all $t \in (0,T)\backslash(N_1 \cup N)$. The function $t \mapsto \alpha\big(\mathcal{U}_t(\psi_\infty)\big)$ being continuous we get $\alpha\big(\mathcal{U}_t(\psi_\infty)\big) = 0$ for all $t \in [0,T]$, and this for all $T > 0$. Finally $\alpha\big(\mathcal{U}_t(\psi_\infty)\big) = 0$ for all $t \geq 0$.

The same argument holds for β as $\tilde{g} : t \mapsto I_2\big(\mathcal{U}_t(\psi^0)\big)g^2\left(I_2\big(\mathcal{U}_t(\psi^0)\big)\right)$ belongs to $L^1(0,+\infty)$. Then by the proof of Proposition 5.4,

$$\tilde{g}\left(\mathcal{U}_{t+t_{n_k}}(\psi^0)\right) \underset{k\to\infty}{\rightarrow} \tilde{g}\left(\mathcal{U}_t(\psi_\infty)\right), \quad \forall t \in (0,T)\backslash N,$$

and $\tilde{g}\big(\mathcal{U}_t(\psi_\infty)\big) = 0$ implies $\beta\big(\mathcal{U}_t(\psi_\infty)\big) \equiv 0$.

These two results lead to the fact that $\mathcal{L}\big(\mathcal{U}_t(\psi_\infty)\big)$ is constant.

By (5.20), $\mathcal{L}(\psi_\infty) \leq \mathcal{L}(\psi^0) < 1$ so $\langle \psi_\infty, \varphi \rangle \neq 0$. All assumptions of Proposition 5.3 are satisfied then $\psi_\infty \in \mathcal{C}$.

This concludes the proof of Theorem 5.2 and the convergence analysis of (5.6).

□

5.3 Approximation by averaging

The method of averaging was mostly used for finite-dimensional dynamical systems (see e.g. [124]). The concept of averaging in quantum control theory has already produced interesting results. For example, in [106] the authors make important use of these averaging properties in finite dimension through what is called in quantum physics the rotating wave approximation. The main idea of using a highly oscillating control is that if it is oscillating enough the initial system behaves like the averaged system. We extend this concept in our infinite dimensional framework : we prove an approximation result on every finite time interval. More precisely we have the following result.

Proposition 5.5. *Let $[s, L]$ be a fixed interval and $\psi^0 \in \mathcal{S} \cap H^4_{(0)}$ with $0 < \mathcal{L}(\psi^0) < 1$. Let ψ_{av} be the solution of the closed loop system (5.6),(5.9) with initial condition $\psi_{av}(s, \cdot) = \psi^0$. For any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, if ψ_ε is the solution of (5.1) associated to the same initial condition $\psi_\varepsilon(s, \cdot) = \psi^0$ and control $u^\varepsilon(t)$ defined by (5.13) with $\varepsilon \in (0, \varepsilon_0)$ then*

$$||\psi_\varepsilon(t, \cdot) - \psi_{av}(t, \cdot)||_{H^2} \leq \delta, \quad \forall t \in [s, L].$$

Remark 5.4. Notice that the controls α and β were defined using the averaged system in a feedback form but the control u^ε used for the system (5.1) is explicit and is not defined as a feedback control.

Remark 5.5. Due to the infinite dimensional framework, we are facing regularity issues and cannot adapt directly the strategy of [124].

Proof. We define for $(t, z, \tilde{z}) \in \mathbb{R} \times H^2 \times H^2$,

$$\tilde{F}(t, z, \tilde{z}) := i\left(\alpha(\tilde{z}) + \beta(\tilde{z})\sin(t)\right)\mu_1 z + i\left(\alpha(\tilde{z}) + \beta(\tilde{z})\sin(t)\right)^2 \mu_2 z. \tag{5.24}$$

Notice that thanks to (5.15) for any $(t, z) \in \mathbb{R} \times H^2$,

$$\tilde{F}(t, z, z) = F(t, z). \tag{5.25}$$

With these notations the considered system (5.1) with control (5.13) and initial condition $\psi_\varepsilon(s, \cdot) = \psi^0$ can be rewritten as

$$\begin{cases} \partial_t \psi_\varepsilon(t) = -iA\psi_\varepsilon(t) + \tilde{F}\left(\dfrac{t}{\varepsilon}, \psi_\varepsilon(t), \psi_{av}(t)\right), \\ \psi_{\varepsilon|\partial D} = 0, \end{cases}$$

where ψ_{av} is the solution of the closed-loop system (5.6) with initial condition $\psi_{av}(s, \cdot) = \psi^0$.

Denoting by T_A the semigroup generated by $-iA$, we have for any $t \geq s$,

$$\psi_\varepsilon(t) = T_A(t-s)\psi^0 + \int_s^t T_A(t-\tau)\tilde{F}\left(\frac{\tau}{\varepsilon}, \psi_\varepsilon(\tau), \psi_{av}(\tau)\right) \mathrm{d}\tau,$$
$$\psi_{av}(t) = T_A(t-s)\psi^0 + \int_s^t T_A(t-\tau)F^0\big(\psi_{av}(\tau)\big)\mathrm{d}\tau.$$

This implies for any $t \geq s$,

$$\begin{aligned} ||\psi_\varepsilon(t) - \psi_{av}(t)||_{H^2} \leq & \left|\left|\int_s^t T_A(t-\tau)\left[F\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right) - F^0\left(\psi_{av}(\tau)\right)\right]\mathrm{d}\tau\right|\right|_{H^2} \\ & + \left|\left|\int_s^t T_A(t-\tau)\Big[\tilde{F}\left(\frac{\tau}{\varepsilon}, \psi_\varepsilon(\tau), \psi_{av}(\tau)\right) - F\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right)\Big]\mathrm{d}\tau\right|\right|_{H^2}. \end{aligned} \tag{5.26}$$

We study separately the two terms of the right-hand side of (5.26).

First step : We show the existence of $C > 0$ such that for any $t \geq s$, for any $\varepsilon > 0$,

$$\begin{aligned} & \left|\left|\int_s^t T_A(t-\tau)\left[\tilde{F}\left(\frac{\tau}{\varepsilon}, \psi_\varepsilon(\tau), \psi_{av}(\tau)\right) - F\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right)\right]\mathrm{d}\tau\right|\right|_{H^2} \\ & \qquad \leq C\int_s^t \big|\big|\psi_\varepsilon(\tau) - \psi_{av}(\tau)\big|\big|_{H^2}\mathrm{d}\tau. \end{aligned} \tag{5.27}$$

By (5.15),(5.24), it comes that for any $\tau \geq s$, for any $\varepsilon > 0$,

$$\begin{aligned}
&\tilde{F}\left(\frac{\tau}{\varepsilon}, \psi_\varepsilon(\tau), \psi_{av}(\tau)\right) - F\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right) \\
&\quad = i\left(\alpha(\psi_{av}(\tau)) + \beta(\psi_{av}(\tau))\sin\left(\frac{\tau}{\varepsilon}\right)\right)\mu_1\left[\psi_\varepsilon(\tau) - \psi_{av}(\tau)\right] \\
&\quad + i\left(\alpha(\psi_{av}(\tau)) + \beta(\psi_{av}(\tau))\sin\left(\frac{\tau}{\varepsilon}\right)\right)^2\mu_2\left[\psi_\varepsilon(\tau) - \psi_{av}(\tau)\right].
\end{aligned}$$

As ψ_{av} is bounded in H^2, using (5.9) and (5.11) we get the existence of $M_1 > 0$ such that for all $\tau \geq s$,

$$|\alpha(\psi_{av}(\tau))| + |\beta(\psi_{av}(\tau))| \leq M_1. \tag{5.28}$$

As $|\sin\left(\frac{\tau}{\varepsilon}\right)| \leq 1$, we get the existence of $C > 0$ independent of ε such that for any $\tau \geq s$, for any $\varepsilon > 0$,

$$\left|\left|\tilde{F}\left(\frac{\tau}{\varepsilon}, \psi_\varepsilon(\tau), \psi_{av}(\tau)\right) - F\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right)\right|\right|_{H^2} \leq C\left|\left|\psi_\varepsilon(\tau) - \psi_{av}(\tau)\right|\right|_{H^2}. \tag{5.29}$$

Then the contraction property of T_A implies (5.27).

Second step : We show that there exists $C > 0$ satisfying for all $t \in [s, L]$, for any $\varepsilon > 0$,

$$\left|\left|\int_s^t T_A(t-\tau)\left[F\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right) - F^0(\psi_{av}(\tau))\right]\mathrm{d}\tau\right|\right|_{H^2} \leq C\varepsilon. \tag{5.30}$$

We follow computations on the semigroup T_A done in [82]. For $(t, v) \in \mathbb{R}^+ \times C^1([s, L], H^2)$, we define U and H by

$$\begin{aligned}
U(t, v(\cdot)) &:= \int_0^t \left(F(\tau, v(\cdot)) - F^0(v(\cdot))\right)\mathrm{d}\tau, \\
H(t, v) &:= \mathrm{d}_v U(t, v)\dot{v},
\end{aligned}$$

where $\dot{v}$ is the time derivative of v.

Notice that the T-periodicity of $F(\cdot, v)$ and the definition of F^0 imply that $U(\cdot, v)$ is also T-periodic.

Lemma 5.1. *As $\psi_{av} \in C^1([s, L], H_0^1 \cap H^2)$, we have for any $t \in [s, L]$, for any $\varepsilon > 0$,*

$$\begin{aligned}
&\int_s^t T_A(t-\tau)\left[F\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right) - F^0(\psi_{av}(\tau))\right]\mathrm{d}\tau = \\
&\quad \varepsilon U\left(\frac{t}{\varepsilon}, \psi_{av}(t)\right) - \varepsilon T_A(t-s)U\left(\frac{s}{\varepsilon}, \psi_{av}(s)\right) \\
&\quad - i\varepsilon A\int_s^t T_A(t-\tau)U\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right)\mathrm{d}\tau - \varepsilon\int_s^t T_A(t-\tau)H\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right)\mathrm{d}\tau.
\end{aligned}$$

Proof. The proof is done in [82, Lemma 2.2] .

□

We study separately each term of the previous right-hand side.
• With $\kappa = \lfloor \frac{t}{\varepsilon T} \rfloor$, we have $\frac{t}{\varepsilon} - \kappa T \in [0, T]$ and by periodicity

$$U\left(\frac{t}{\varepsilon}, \psi_{av}(t)\right) = \int_0^{t/\varepsilon} \Big(F(\tau, \psi_{av}(t)) - F^0(\psi_{av}(t))\Big)\mathrm{d}\tau$$
$$= \int_0^{t/\varepsilon - \kappa T} \Big(F(\tau, \psi_{av}(t)) - F^0(\psi_{av}(t))\Big)\mathrm{d}\tau.$$

As ψ_{av} is bounded in H^2 and $\alpha(\psi_{av})$, $\beta(\psi_{av})$ are bounded there exists $M_2 > 0$ such that

$$||F(\tau, \psi_{av}(t))||_{H^2} \leq M_2, \quad ||F^0(\psi_{av}(t))||_{H^2} \leq M_2, \quad \forall \tau \geq 0, \forall t \geq s.$$

This leads to

$$\left|\left|U\left(\frac{t}{\varepsilon}, \psi_{av}(t)\right)\right|\right|_{H^2} \leq \int_0^{t/\varepsilon - \kappa T} 2M_2 \mathrm{d}\tau \leq 2M_2 T, \quad \forall t \geq s, \forall \varepsilon > 0.$$

The same computations lead to

$$\left|\left|T_A(t-s)U\left(\frac{s}{\varepsilon}, \psi_{av}(s)\right)\right|\right|_{H^2} \leq 2M_2 T, \quad \forall t \geq s, \forall \varepsilon > 0.$$

Then,

$$\left|\left|\varepsilon U\left(\frac{t}{\varepsilon}, \psi_{av}(t)\right) + \varepsilon T_A(t-s)U\left(\frac{s}{\varepsilon}, \psi_{av}(s)\right)\right|\right|_{H^2} \leq C\varepsilon, \quad \forall t \geq s, \forall \varepsilon > 0. \tag{5.31}$$

• By switching property,

$$A\int_s^t T_A(t-\tau)U\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right)\mathrm{d}\tau = \int_s^t T_A(t-\tau)AU\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right)\mathrm{d}\tau,$$

and for any $t \in [s, L]$, for any $\varepsilon > 0$

$$AU\left(\frac{t}{\varepsilon}, \psi_{av}(t)\right) = A\int_0^{t/\varepsilon - \kappa T} \left[F(\tau, \psi_{av}(t)) - F^0(\psi_{av}(t))\right]\mathrm{d}\tau$$
$$= \int_0^{t/\varepsilon - \kappa T} \left[AF(\tau, \psi_{av}(t)) - AF^0(\psi_{av}(t))\right]\mathrm{d}\tau.$$

By definition of F and F^0 we have

$$AF(t,z) = i(\alpha(z) + \beta(z)\sin(t))A(\mu_1 z) + i(\alpha(z) + \beta(z)\sin(t))^2 A(\mu_2 z),$$
$$AF^0(z) = i\alpha(z)A(\mu_1 z) + i\left(\alpha(z)^2 + \frac{1}{2}\beta(z)^2\right)A(\mu_2 z).$$

By regularity hypothesis on μ_1, μ_2 and V there exists $C > 0$ such that

$$||A(\mu_1 z)||_{H^2} \leq C||\Delta z||_{H^2}, \quad ||A(\mu_2 z)||_{H^2} \leq C||\Delta z||_{H^2}.$$

Thus thanks to Proposition 5.2 and the bound (5.28) on $\alpha(\psi_{av})$ and $\beta(\psi_{av})$, we get the existence of $M_3 > 0$ satisfying

$$||AF(\tau,\psi_{av}(t))||_{H^2} \leq M_3, \quad ||AF^0(\psi_{av}(t))||_{H^2} \leq M_3, \quad \forall \tau \geq 0, \forall t \in [s,L].$$

So, for any $t \in [s,L]$, for any $\varepsilon > 0$, $\left|\left|AU\left(\frac{t}{\varepsilon},\psi_{av}(t)\right)\right|\right|_{H^2} \leq 2M_3T$. Consequently , there exists $C > 0$ such that

$$\left|\left|i\varepsilon A\int_s^t T_A(t-\tau)U\left(\frac{\tau}{\varepsilon},\psi_{av}(\tau)\right)\mathrm{d}\tau\right|\right|_{H^2} \leq C\varepsilon, \quad \forall t \in [s,L], \forall \varepsilon > 0. \tag{5.32}$$

- For the last term we need to estimate $H\left(\frac{t}{\varepsilon},\psi_{av}(t)\right)$. We have

$$\begin{aligned} H\left(\frac{t}{\varepsilon},\psi_{av}(t)\right) &= \mathrm{d}_v U\left(\frac{t}{\varepsilon},\psi_{av}(t)\right).\partial_t\psi_{av}(t) \\ &= \int_0^{t/\varepsilon-\kappa T}\Big(\mathrm{d}_v F(\tau,\psi_{av}).\partial_t\psi_{av} - \mathrm{d}F^0(\psi_{av}).\partial_t\psi_{av}\Big)\mathrm{d}\tau. \end{aligned}$$

Using (5.9) and (5.11), we have for any $v,w \in C^0([s,L],H_0^1\cap H^2)$,

$$\mathrm{d}\alpha(v).w = -k\mathrm{d}I_1(v).w, \quad \mathrm{d}\beta(v).w = g'\big(I_2(v)\big)\mathrm{d}I_2(v).w, \tag{5.33}$$

where,

$$\begin{aligned} \mathrm{d}I_j(v).w = \mathrm{Im}\Big[&-\gamma\langle(-\Delta+V)\mathcal{P}(\mu_j w),(-\Delta+V)\mathcal{P}v\rangle \\ &-\gamma\langle(-\Delta+V)\mathcal{P}(\mu_j v),(-\Delta+V)\mathcal{P}w\rangle + \langle\mu_j w,\varphi\rangle\langle\varphi,v\rangle + \langle\mu_j v,\varphi\rangle\langle\varphi,w\rangle\Big]. \end{aligned}$$

Finally, we have

$$\begin{aligned} &\mathrm{d}_v F(t,v).w = i(\alpha(v)+\beta(v)\sin t)\mu_1 w + i(\mathrm{d}\alpha(v).w + \mathrm{d}\beta(v).w\sin t)\mu_1 v \\ &+i(\alpha(v)+\beta(v)\sin t)^2\mu_2 w + 2i(\alpha(v)+\beta(v)\sin t)(\mathrm{d}\alpha(v).w+\mathrm{d}\beta(v).w\sin t)\mu_2 v, \end{aligned} \tag{5.34}$$

and

$$\begin{aligned} \mathrm{d}F^0(v).w &= i\alpha(v)\mu_1 w + i\mathrm{d}\alpha(v).w\mu_1 v + i\left(\alpha(v)^2+\frac{1}{2}\beta(v)^2\right)\mu_2 w \\ &+i(2\alpha(v)\mathrm{d}\alpha(v).w + \beta(v)\mathrm{d}\beta(v).w\sin t)\mu_2 v. \end{aligned} \tag{5.35}$$

By Proposition 5.2, $\partial_t\psi_{av} \in C^0([0,+\infty),H_0^1\cap H^2)$ so there exists $M_4 > 0$ such that

$$||\partial_t\psi_{av}(t)||_{H^2} \leq M_4, \quad \forall t \in [s,L].$$

Hence, the same computations as previously lead to the existence of $C > 0$ satisfying

$$|\mathrm{d}\alpha(\psi_{av}(t)).\partial_t\psi_{av}(t)| + |\mathrm{d}\beta(\psi_{av}(t)).\partial_t\psi_{av}(t)| \leq C, \quad \forall t \in [s,L],$$

and thus by (5.34),(5.35), for any $t \in [s,L]$, for any $\tau \geq 0$,

$$||\mathrm{d}_v F(\tau,\psi_{av}(t)).\partial_t\psi_{av}(t)||_{H^2} + ||\mathrm{d}F^0(\psi_{av}(t)).\partial_t\psi_{av}(t)||_{H^2} \leq C.$$

As a consequence,

$$\left|\left|H\left(\frac{\tau}{\varepsilon},\psi_{av}(\tau)\right)\right|\right|_{H^2}\leq CT,\quad \forall\tau\in[s,L],\forall\varepsilon>0,$$

and then,

$$\left|\left|\varepsilon\int_s^t T_A(t-\tau)H\left(\frac{\tau}{\varepsilon},\psi_{av}(\tau)\right)\right|\right|_{H^2}\mathrm{d}\tau\leq(CLT)\varepsilon. \tag{5.36}$$

We are now able to deal with the remaining term of the right-hand side of (5.26). Gathering inequalities (5.31), (5.32) and (5.36) in Lemma 5.1 we obtain that there exists $C>0$ such that inequality (5.30) holds.

Third step : Putting together (5.26), (5.27) and (5.30) we obtain that there exists $C>0$ such that for any $t\in[s,L]$, for any $\varepsilon>0$,

$$||\psi_\varepsilon(t)-\psi_{av}(t)||_{H^2}\leq C\varepsilon+C\int_s^t||\psi_\varepsilon(\tau)-\psi_{av}(\tau)||_{H^2}\mathrm{d}\tau.$$

Hence Grönwall's lemma implies

$$||\psi_\varepsilon(t)-\psi_{av}(t)||_{H^2}\leq C\varepsilon e^{C(t-s)}\leq(Ce^{C(L-s)})\varepsilon,\quad\forall t\in[s,L],$$

and Proposition 5.5 is proved with $\varepsilon_0=\dfrac{\delta}{Ce^{C(L-s)}}$.

□

Remark 5.6. The proof we used is fundamentally based on the boundedness of $\Delta\psi_{av}(t)$ on $[s,L]$ and on Grönwall's lemma so it cannot be extended directly to an infinite time interval $[s,+\infty)$.

5.4 Explicit approximate controllability

The solution ψ_{av} of the averaged system (5.6),(5.9), can be driven in the H^2 weak topology to the target set $\mathcal{C}$. The solution ψ_ε of the system (5.1) associated to the same initial condition, with control u^ε, stays close to ψ_{av} on every finite time interval provided that the control is oscillating enough. Gathering these two results we prove Theorem 5.1.

Proof of Theorem 5.1. We consider $s<2$ fixed.

By Theorem 5.2, we can construct an increasing time sequence $(T_n)_{n\in\mathbb{N}}$ tending to $+\infty$ such that for any $n\in\mathbb{N}$,

$$\mathrm{dist}_{H^s}(\psi_{av}(t),\mathcal{C})\leq\frac{1}{2^{n+1}},\quad\forall t\geq T_n. \tag{5.37}$$

Using Proposition 5.5 on the time interval $[0,T_{n+1}]$ we then construct a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that for any $n\in\mathbb{N}$,

$$||\psi_\varepsilon(t)-\psi_{av}(t)||_{H^s}\leq\frac{1}{2^{n+1}},\quad\forall t\in[0,T_{n+1}],\forall\varepsilon\in(0,\varepsilon_n). \tag{5.38}$$

Then (5.37),(5.38) imply that

$$\forall n \in \mathbb{N}, \quad \operatorname{dist}_{H^s}(\psi_\varepsilon(t), \mathcal{C}) \leq \frac{1}{2^n}, \quad \forall t \in [T_n, T_{n+1}], \forall \varepsilon \in (0, \varepsilon_n),$$

which is the statement of Theorem 5.1.

□

5.5 Numerical simulations

This section is dedicated to numerical simulations of system (5.1). First, we detail how we approximate the solutions of (5.1) and (5.6). Then, we check the validity of the implemented code. Finally, we illustrate different aspects of Theorem 5.1 and of the averaging property, Proposition 5.5.

5.5.1 Settings

In all what follows, we set $D = [0,1]$. As the potential V will vary in this section, the eigenelements of $-\Delta + V$ are denoted $\varphi_{k,V}$ and $\lambda_{k,V}$. Any function $\psi \in L^2((0,1), \mathbb{C})$ is approximated by its first M modes

$$\psi(t) \approx \sum_{k=1}^{M} x_k(t) \varphi_{k,V.}$$

The unknown eigenvectors $\varphi_{k,V}$ are approximated in the following way

$$\varphi_{k,V} \approx \sum_{j=1}^{N} a_j^k \varphi_{k,0}.$$

The equality $(-\Delta + V)\varphi_{k,V} = \lambda_{k,V}\varphi_{k,V}$ leads to $Ba^k = \lambda_{k,V} a^k$ with

$$a^k = (a_1^k, \ldots, a_N^k)^t, \quad B = \operatorname{diag}(\lambda_{1,0}, \ldots, \lambda_{N,0}) + \big(\langle V\varphi_{i,0}, \varphi_{j,0}\rangle\big)_{1 \leq i,j \leq N}.$$

Notice that $\lambda_{k,0} = (k\pi)^2$ and $\varphi_{k,0} = \sqrt{2}\sin(k\pi\cdot)$ are explicit. The scalar products are approximated by the Matlab function `quadl`. The eigenelements a^k and $\lambda_{k,V}$ are then approximated by the Matlab function `eig`.

5.5.2 Approximation of ψ_ε and ψ_{av}

Let

$$H_0 := \operatorname{diag}(\lambda_{1,V}, \ldots, \lambda_{M,V}), \quad H_n := \big(\langle \mu_n \varphi_{i,V}, \varphi_{j,V}\rangle\big)_{1 \leq i,j \leq M}, \ n \in \{1,2\}.$$

It follows that the feedback laws (5.11) are approximated, for $X \in \mathbb{R}^M$ and $j \in \{1,2\}$, by

$$I_j(X) := \operatorname{Im}\Big(-\gamma\big(H_0(0, (H_jX)_2, \ldots, (H_jX)_M)^t\big)^t \left(H_0(0, \overline{x_2}, \ldots, \overline{x_M})\right) + (H_jX)_1 \overline{x_1}\Big),$$

leading to

$$\alpha(X) := -kI_1(X), \quad \beta(X) := -\min(I_2(X), 0).$$

Thus, if we define X_{av}, $X_\varepsilon \in \mathbb{R}^M$, systems (5.1) and (5.6) are approximated by

$$i\frac{\mathrm{d}X_\varepsilon}{\mathrm{d}t} = \big(H_0 - u_\varepsilon(t)H_1 - u_\varepsilon(t)^2 H_2\big)X_\varepsilon, \tag{5.39}$$

and

$$i\frac{\mathrm{d}X_{av}}{\mathrm{d}t} = \left(H_0 - \alpha(X_{av})H_1 - \left(\alpha^2(X_{av}) + \frac{1}{2}\beta(X_{av})^2\right)H_2\right)X_{av}, \tag{5.40}$$

where $u_\varepsilon(t) = \alpha(X_{av}(t)) + \beta(X_{av}(t))\sin(t/\varepsilon)$. Equations (5.39) and (5.40) are solved numerically (simultaneously) using Euler method with a time step $\mathrm{d}t$ and a Strang splitting method.

5.5.3 Validation

We now prove the validity of the implemented code. The eigenvectors $\varphi_{k,V}$ are approximated by $N = 50$ modes. We take, as a test case, $V(x) := (x-1/2)^2$, $\mu_1(x) := x^2$ and $\mu_2(x) := x$. The considered initial condition is $\psi^0 = \frac{1}{\sqrt{2}}\varphi_{1,V} + \frac{i}{\sqrt{2}}\varphi_{2,V}$. The value of the oscillating parameter is $\varepsilon = 10^{-3}$. The parameter γ is chosen such that $\mathcal{L}(\psi^0) = 3/4$. We compute the discrete Lyapunov function for the averaged system and the H^s norm (with $s = 1.8$) to the ground state for both the oscillating and the averaged system. The time scale is $[0, T]$ with $T = 1000$ and a time step $\mathrm{d}t = 10^{-3}$. For $M = 5$, we get the results presented in Figure 5.1.

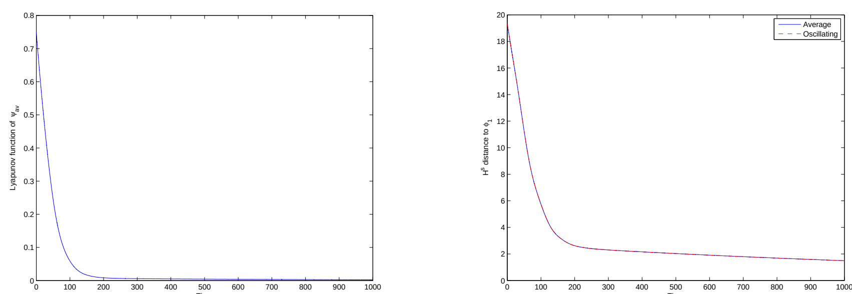


Figure 5.1: Lyapunov function of the averaged system (left). H^s norm to the ground state (right) for the averaged system (continuous line) and the oscillating system (dashed line).

As expected, we observe the convergence of the Lyapunov function to 0. The solutions of (5.39) and (5.40) are driven to the ground state (up to a global phase). To validate the simulations, we have also tested the code for $M = 10$ and $M = 20$. We obtained the same asymptotic behaviour and the same values for the Lyapunov function and the H^s distance to the target.

As the approximate controllability uses the fact that the controls are oscillating, the time step $\mathrm{d}t$ cannot be taken large with respect to the oscillating parameter ε. For $\varepsilon = 10^{-3}$, we obtain the same results with $\mathrm{d}t = 10^{-3}$ and $\mathrm{d}t = 10^{-4}$. However, instabilities appear on the oscillating system for $\mathrm{d}t = 10^{-2}$. Thus, in all what follows the time step will be chosen smaller than ε. We now present several simulations to illustrate various aspects of Theorem 5.1.

5.5.4 Influence of the initial condition

For every other initial condition tested, the asymptotic behaviour is the same. We present here the results for the same parameters as in Figure 5.1 but with the initial condition $\psi^0 = \frac{1}{\sqrt{3}}\varphi_{1,V} + \frac{1}{\sqrt{3}}\varphi_{2,V} + \frac{i}{\sqrt{3}}\varphi_{3,V}$. In this case, the stabilization of the averaged system is slower and we computed it for $T = 5000$.

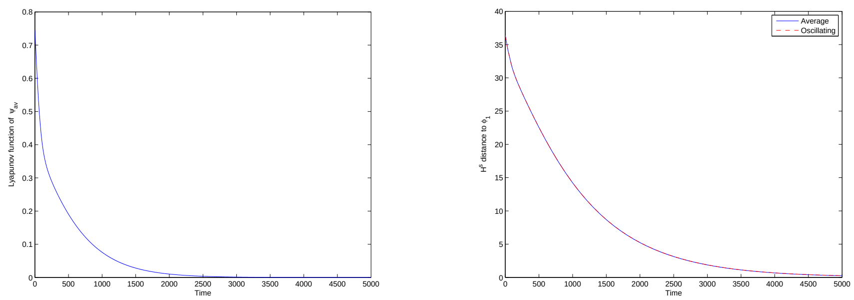


Figure 5.2: Lyapunov function of the averaged system (left). H^s norm to the ground state (right) for the averaged system (continuous line) and the oscillating system (dashed line).

We observe in Figure 5.2 the same asymptotic behaviour as in Figure 5.1.

5.5.5 Averaging strategy

We present numerically the influence of the oscillating parameter ε. First, we consider the same potential, dipolar and polarizability moments as in Figure 5.1. We compute the discrete H^s norm (for $s = 1.8$) to the ground state (up to a global phase) and the discrete H^2 norm of $X_{av} - X_\varepsilon$. Figure 5.3 is obtained with $\varepsilon = 10^{-3}$ while Figure 5.4 is obtained with $\varepsilon = 10^{-4}$. Both are computed with a time step $\mathrm{d}t = \varepsilon$ and final time $T = 500$. For a fixed parameter ε, we observe that the H^2 distance between the solution of (5.1) and the solution of (5.6) with the same initial condition does not increase as the time goes to infinity but rather tends to a limit value. This limit value is of the same order of magnitude as ε. We observe that

$$\frac{||X_{av}(T) - X_{10^{-3}}(T)||_{H^2}}{||X_{av}(T) - X_{10^{-4}}(T)||_{H^2}} \approx 30.$$

This validates numerically the results of Proposition 5.5 and indicates that this averaging property should be valid on an infinite time horizon.

The same behaviour has been obtained with other parameters. We present here the simulations with $\mu_1(x) := cos(x)$ and $\mu_2(x) := cos(2x)$, inspired by the physical situation of alignment dynamic of a HCN molecule as in [65]. Figure 5.5 is obtained with $\varepsilon = 10^{-3}$ while Figure 5.6 is obtained with $\varepsilon = 10^{-4}$. Both are computed with a time step $\mathrm{d}t = \varepsilon$ and final time $T = 1000$ (as the stabilization process seems slower in this case).

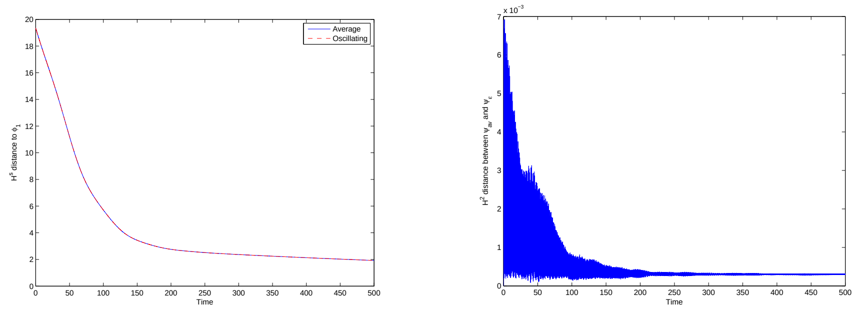


Figure 5.3: H^s norm to the ground state (left) for the averaged system (continuous line) and the oscillating system (dashed line). H^2 gap from the average (right).

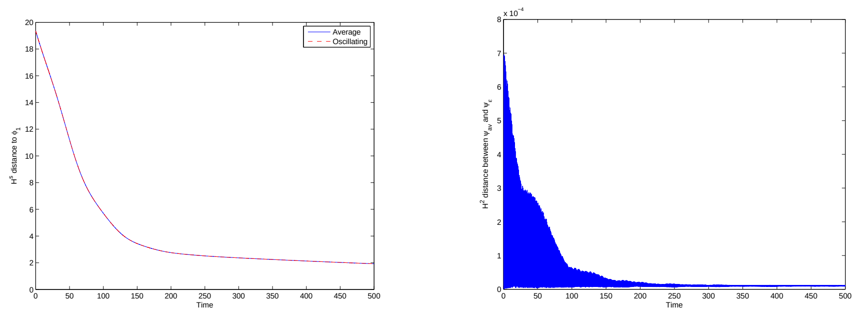


Figure 5.4: H^s norm to the ground state (left) for the averaged system (continuous line) and the oscillating system (dashed line). H^2 gap from the average (right).

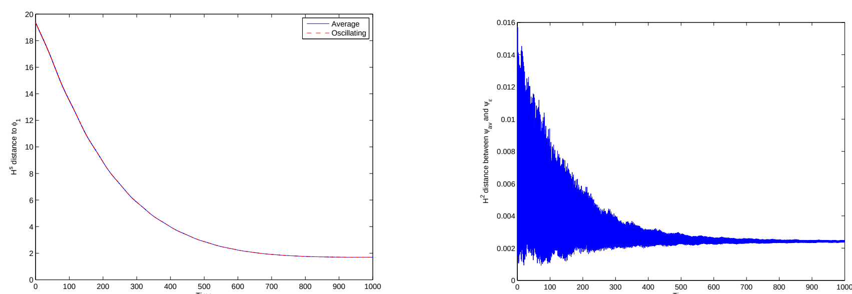
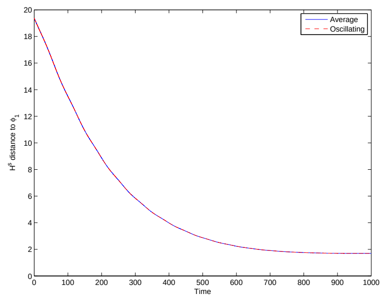


Figure 5.5: H^s norm to the ground state (left) for the averaged system (continuous line) and the oscillating system (dashed line). H^2 gap from the average (right).



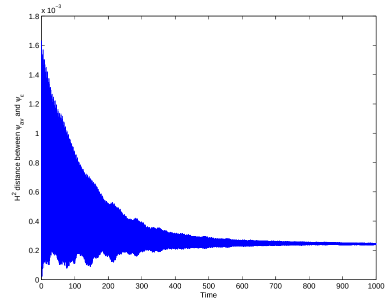


Figure 5.6: H^s norm to the ground state (left) for the averaged system (continuous line) and the oscillating system (dashed line). H^2 gap from the average (right).

Remark 5.7. Although the time scales at stake in these simulations can seem very large, one has to remember that the Schrödinger equation is considered in atomic unity.

5.6 Conclusion, open problems and perspectives

In this article we have defined explicit oscillating controls that drive the solution of our system arbitrarily close to the ground state provided that the control is oscillating enough and the time is large enough. To achieve this we have used and developed tools from the theory of finite dimension dynamical systems and applied them to the considered Schrödinger equation. We managed by adding a mathematically and physically meaningful term to weaken the previous assumptions on the coupling realized by this model. The assumptions that were made are proved to be generic with respect to the functions determining the system (potential, dipolar and polarizability moments). The results presented should be generalizable to a compact manifold with the Laplace-Beltrami operator. We performed numerical simulations to illustrate the approximate controllability. This gives numerical bounds on the time scale and on the values of the oscillating parameter needed to drive any initial condition arbitrarily close to the ground state.

A challenging question would be to prove an approximation property of the averaged system on an infinite time interval $[s, +\infty)$. This would lead to approximate stabilization to the ground state. Based on the numerical simulations, this result seems to hold. Unfortunately the tools developed here are really based on the finite time interval and cannot be extended directly. In [16], Beauchard and Laurent proved the local exact controllability in H^3 around the ground state for the system (5.1) in the dipolar approximation (i.e. $\mu_2 \equiv 0$) under some coupling assumptions in one dimension. If one manages to extend their result to the system (5.1) with suitable assumptions on μ_2, this may lead to a global exact controllability result around the ground state, at least for the one-dimensional case. The main difficulty would be to obtain the approximate convergence in the same functional setting as their local exact controllability result and with coherent assumptions on the polarizability moment.

Chapitre 6

Contrôle exact global

Sommaire

On considère une particule quantique dans un intervalle, modélisée par sa fonction d'onde ψ, dans un potentiel $V(x)$. On contrôle l'évolution de cette fonction d'onde par un champ extérieur d'amplitude $u(t)$ réelle. En prenant en compte le terme dipolaire et le terme de polarisabilité, on considère le système de Schrödinger

$$\begin{cases} i\partial_t\psi = \left(-\partial^2_{xx} + V(x)\right)\psi - u(t)\mu_1(x)\psi - u(t)^2\mu_2(x)\psi, & (t,x)\in(0,T)\times(0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t\in(0,T), \\ \psi(0,x) = \psi_0(x), & x\in(0,1), \end{cases} \tag{6.1}$$

présenté en Section 1.3 page 27. L'objectif de ce Chapitre est de montrer que certaines techniques développées pour le modèle bilinéaire

$$\begin{cases} i\partial_t\psi = \left(-\partial^2_{xx} + V(x)\right)\psi - u(t)\mu_1(x)\psi, & (t,x)\in(0,T)\times(0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t\in(0,T), \\ \psi(0,x) = \psi_0(x), & x\in(0,1), \end{cases} \tag{6.2}$$

présentées en Section 1.2 et détaillées en Partie I, s'appliquent au cas du modèle unidimensionnel avec un terme de polarisabilité (6.1).

On rappelle que, pour $V \in L^2((0,1),\mathbb{R})$, on note $\lambda_{k,V}$ et $\varphi_{k,V}$ les valeurs propres et vecteurs propres de l'opérateur A_V défini par $A_V\psi := \left(-\partial^2_{xx} + V(x)\right)\psi$ avec domaine $D(A_V) := H^2\cap H^1_0((0,1),\mathbb{C})$. Pour $s>0$, l'espace $H^s_{(V)} := D(A_V^{s/2})$ est muni de la norme

$$\|\psi\|_{H^s_{(V)}} := \left(\sum_{k=1}^{+\infty} \left|k^s\langle\psi,\varphi_{k,V}\rangle\right|^2\right)^{\frac{1}{2}}.$$

La sphère unité de $L^2((0,1),\mathbb{C})$ est notée $\mathcal{S}$. Le résultat principal de ce chapitre est le suivant.

Théorème 6.1. *Pour tout $V,\mu_1 \in H^6((0,1),\mathbb{R})$ le système (6.1) est globalement exactement contrôlable dans $H^6_{(V)}$, génériquement par rapport à $\mu_2 \in H^6((0,1),\mathbb{R})$. Plus précisément, pour tout $V,\mu_1 \in H^6((0,1),\mathbb{R})$, il existe un ensemble $\mathcal{Q}_{V,\mu_1}$ résiduel dans $H^6((0,1),\mathbb{R})$ tel que si $\mu_2 \in \mathcal{Q}_{V,\mu_1}$, pour tout $\psi_0,\psi_f \in \mathcal{S}\cap H^6_{(V)}$, il existe $T>0$ et $u \in H^1_0((0,T),\mathbb{R})$ tels que la solution associée de (6.1) satisfasse $\psi(T)=\psi_f$.*

Ce théorème montre que la prise en compte du terme de polarisabilité permet d'obtenir la contrôlabilité exacte globale en temps grand pour des modèles où la contrôlabilité sans ce terme de polarisabilité est fausse ou ouverte (par exemple $\mu_1 = 0$ ou V quelconque et $\mu_1 \notin \mathcal{Q}_V$ comme défini au Théorème 4.1 page 122). La preuve de ce théorème repose principalement sur trois arguments. En utilisant l'argument de perturbation développé en Section 4.5.2, si l'on considère le contrôle $u(t) = \tilde{u}(t) + 2$, alors le système (6.1) s'écrit

$$\begin{cases} i\partial_t\psi = \left(-\partial^2_{xx} + V(x) - 2\mu_1(x) - 4\mu_2(x)\right)\psi - \tilde{u}(t)(\mu_1+4\mu_2)(x)\psi - \tilde{u}(t)^2\mu_2(x)\psi, \\ \psi(t,0)=\psi(t,1)=0, \\ \psi(0,x)=\psi_0(x). \end{cases} \tag{6.3}$$

Ainsi, on peut distribuer une partie du moment de polarisabilité sur le potentiel et une partie sur le moment dipolaire. Même pour des fonctions V et μ_1 quelconques, on se ramène donc à étudier le système (6.1) avec des hypothèses favorables sur V et μ_1.

La deuxième étape de la preuve consiste à montrer le contrôle approché vers l'état fondamental $\varphi_{1,V}$ pour la norme H^5. Dans le cadre $\mu_2 = 0$, ce résultat est obtenu par V. Nersesyan [112] sous des hypothèses favorables sur V et μ_1. La preuve de ce résultat utilise la fonction de Lyapunov

$$\mathcal{L}(z) := \gamma \| \left(-\Delta+V\right)^3 \mathcal{P}z\|^2_{L^2} + 1 - |\langle z,\varphi_{1,V}\rangle|^2, \quad z \in \mathcal{S}\cap H^6_{(V)}, \tag{6.4}$$

où $\mathcal{P}$ est la projection orthogonale dans L^2 sur l'espace engendré par $\{\varphi_{k,V}\,;\,k\geq 2\}$ et $\gamma > 0$ une constante à déterminer. C'est cette fonction de Lyapunov (déjà utilisée par K. Beauchard et V. Nersesyan [111, 19]) qui a été adaptée au cadre de la contrôlabilité simultanée au Chapitre 4. Dans [112], la décroissance de la fonction de Lyapunov est assurée par un argument variationnel lié aux propriétés du linéarisé du système (6.2) au voisinage de trajectoires associées au contrôle $u \equiv 0$. Les systèmes (6.1) et (6.2) admettant le même linéarisé au voisinage de ces trajectoires, ce résultat est directement étendu au cas $\mu_2 \neq 0$.

Grâce à l'argument de réversibilité en temps classique, la dernière étape de la preuve du Théorème 6.1 consiste à prouver la contrôlabilité exacte locale dans $H^5_{(V)}$ autour de l'état fondamental $\varphi_{1,V}$. Dans le cadre $V=0$ et $\mu_2=0$, ce résultat est prouvé avec des contrôles $H^1_0((0,T),\mathbb{R})$ par K. Beauchard et C. Laurent [16]. Le cas d'un potentiel V non nul nécessite simplement quelques adaptations techniques pour obtenir le caractère bien posé. La preuve de contrôlabilité de [16] étant basée sur le contrôle du linéarisé au voisinage de la trajectoire $(\Phi_1, u\equiv 0)$ dans $H^5_{(0)}$ avec des contrôles $H^1_0((0,T),\mathbb{R})$, ce résultat s'étend directement au cadre μ_2 non nul. En effet, les systèmes (6.1) et (6.2) possèdent le même linéarisé au voisinage de la trajectoire $(\Phi_{1,V}, u\equiv 0)$ et la régularité $H^1_0((0,T),\mathbb{R})$ de u se transfère automatiquement à u^2. Cette dernière remarque justifie le cadre fonctionnel $H^5_{(V)}$

utilisé ici. En effet, une première idée serait d'adapter le résultat de contrôle du linéarisé dans $H^3_{(V)}$ avec des contrôles $L^2((0,T),\mathbb{R})$. Le terme u^2 aurait alors seulement une régularité $L^1((0,T),\mathbb{R})$ insuffisante pour appliquer les résultats de [16] et conclure au caractère bien posé de (6.1) dans $H^3_{(V)}$. Le contrôle du linéarisé dans $H^3_{(V)}$ avec des contrôles $L^4((0,T),\mathbb{R})$ est un problème ouvert.

Structure du chapitre. On commence par préciser en Section 6.1 les résultats de régularité pour les solutions du problème (6.1). En Section 6.2, on montre la contrôlabilité approchée vers l'état fondamental $\varphi_{1,V}$ sous des hypothèses favorables sur V et μ_1 en utilisant la fonction de Lyapunov $\mathcal{L}$ définie en (6.4). La Section 6.3 prouve la contrôlabilité exacte locale dans $H^5_{(V)}$ avec des contrôles $H^1_0((0,T),\mathbb{R})$ au voisinage de l'état fondamental sous des hypothèses favorables sur V et μ_1. On conclut la preuve du Théorème 6.1 en Section 6.4 en utilisant le système perturbé (6.3).

6.1 Régularité des solutions

Cette section est dédiée à l'étude du caractère bien posé et aux résultats de régularité du système (6.1). Le résultat suivant est une adaptation de [16, Proposition 5].

Proposition 6.1. *Soient $V, \mu_1, \mu_2 \in H^5((0,1),\mathbb{R})$, $T > 0$, $\psi_0 \in H^5_{(V)}$, $f \in H^1_0((0,T), H^3 \cap H^1_0)$ et $u \in H^1_0((0,T),\mathbb{R})$. Le système*

$$\begin{cases} i\partial_t \psi = \left(-\partial^2_{xx} + V(x)\right)\psi - u(t)\mu_1(x)\psi - u(t)^2\mu_2(x)\psi - f(t,x), & (t,x) \in (0,T)\times(0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T), \\ \psi(0,x) = \psi_0(x), & x \in (0,1), \end{cases} \tag{6.5}$$

admet une unique solution faible i.e. une fonction $\psi \in C^1([0,T], H^3_{(V)})$ telle que l'égalité

$$\psi(t) = e^{-iA_V t}\psi_0 + i\int_0^t e^{-iA_V(t-\tau)}\left(u(\tau)\mu_1\psi(\tau) + u(\tau)^2\mu_2\psi(\tau) + f(\tau)\right)\mathrm{d}\tau, \tag{6.6}$$

soit vérifiée dans $C^1([0,T], H^3_{(V)})$. Pour tout $R > 0$, il existe $C = C(T, \mu_1, \mu_2, R) > 0$ tel que si $\|u\|_{H^1_0(0,T)} < R$, la solution associée vérifie

$$\|\psi\|_{C^1([0,T],H^3_{(V)})} \leq C\left(\|\psi_0\|_{H^5_{(V)}} + \|f\|_{H^1((0,T),H^3\cap H^1_0)}\right).$$

Si $f \equiv 0$, alors

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \quad \text{pour tout } t \in [0,T],$$

et la solution vérifie $A_V\psi + u(t)\mu_1\psi + u(t)^2\mu_2\psi \in C^0([0,T], H^3_{(V)})$. En particulier, l'hypothèse $u(T) = 0$ implique $\psi(T) \in H^5_{(V)}$.

Dans la suite, la solution de (6.1) avec condition initiale ψ_0 et contrôle u sera notée $\psi(\cdot, \psi_0, u)$.

Démonstration. Comme $u \in H_0^1((0,T),\mathbb{R})$ implique $u^2 \in H_0^1((0,T),\mathbb{R})$, le passage du cas $\mu_2 = 0$ à $\mu_2 \in H^5((0,1),\mathbb{R})$ est immédiat. On précise donc simplement les adaptations nécessaires pour étendre [16, Proposition 5] au cadre V non nul. L'existence et l'unicité de la solution faible dans $C^0([0,T],H^3_{(V)})$ sont assurées par un théorème de point fixe et le lemme suivant.

Lemme 6.1. *Soient $T>0$ et $f \in L^2((0,T),H^3 \cap H_0^1)$. La fonction $G : t \mapsto \int_0^t e^{iA_V s} f(s)\mathrm{d}s$ appartient à $C^0([0,T],H^3_{(V)})$ et vérifie*

$$\|G\|_{L^\infty((0,T),H^3_{(V)})} \leq c_1(T)\|f\|_{L^2((0,T),H^3\cap H_0^1)}$$

où la constante $c_1(T)$ est uniformément bornée pour T dans un intervalle borné.

Pour $V=0$, ce lemme est [16, Lemme 1]. L'adaptation au cas V non nul en suivant la même stratégie est due à V. Nersesyan et H. Nersisyan [113, Proposition 3.1] en utilisant les estimations suivantes (voir par exemple [117, Théorème 4]) pour $V \in L^2((0,1),\mathbb{R})$

$$\lambda_{k,V} = k^2\pi^2 + \int_0^1 V(x)\mathrm{d}x + r_k, \quad \forall k \in \mathbb{N}^* \text{ avec } \sum_{k=1}^{\infty} r_k^2 < \infty, \tag{6.7}$$

$$\|\varphi_{k,V} - \varphi_k\|_{L^\infty} \leq \frac{C}{k}, \quad \forall k \in \mathbb{N}^*, \tag{6.8}$$

$$\|\varphi_{k,V}' - \varphi_k'\|_{L^\infty} \leq C, \quad \forall k \in \mathbb{N}^*. \tag{6.9}$$

Finalement, de même que pour la preuve de [16, Proposition 5], la régularité $C^1([0,T],H^3_{(V)})$ est obtenue en dérivant l'équation par rapport au temps et en utilisant le Lemme 6.1.

□

Pour des conditions initiales plus régulières, on utilise aussi le résultat de régularité suivant.

Proposition 6.2. *Soit $T>0$. Pour tout $u \in C^2([0,T],\mathbb{R})$ tel que $\dot{u}(0)=0$, pour tout $\psi_0 \in H^6_{(V-u(0)\mu_1-u(0)^2\mu_2)}$, (6.1) admet une unique solution faible $C^0([0,T],H^6)$. De plus, si $\dot{u}(T)=0$, alors $\psi(T) \in H^6_{(V-u(T)\mu_1-u(T)^2\mu_2)}$.*

Démonstration. Si $u \in C^2([0,T],\mathbb{R})$ avec $\dot{u}(0)=0$ alors u^2 vérifie les mêmes propriétés. Le résultat est donc impliqué par [112, Lemme 2.1] pour le cas $\mu_2=0$.

□

6.2 Contrôle approché vers l'état fondamental

Cette section est dédiée à la contrôlabilité approchée vers l'état fondamental $\varphi_{1,V}$. On suppose que les fonctions $V,\mu_1 \in H^6((0,1),\mathbb{R})$ vérifient

- **(C_1)** $\langle \mu_1\varphi_{1,V},\varphi_{k,V}\rangle \neq 0$, pour tout $k \in \mathbb{N}^*$,
- **(C_2)** $\lambda_{1,V}-\lambda_{j,V} \neq \lambda_{p,V}-\lambda_{q,V}$, pour tout $j,p,q \geq 1$ et $j \neq 1$.

Théorème 6.2. *Soient* $V, \mu_1, \mu_2 \in H^6((0,1),\mathbb{R})$ *tels que les Conditions* $(\mathbf{C_1})$ *et* $(\mathbf{C_2})$ *soient vérifiées. Soit* $\psi_0 \in \mathcal{S} \cap H^6_{(V)}$ *tel que* $\langle \psi_0, \varphi_{1,V} \rangle \neq 0$. *Pour tout* $\varepsilon > 0$, *il existe* $T > 0$ *et* $u \in C^2_0((0,T),\mathbb{R})$ *tels que*

$$\|\psi(T,\psi_0,u) - \varphi_{1,V}\|_{H^5} < \varepsilon. \tag{6.10}$$

Démonstration. Dans le cas $\mu_2 = 0$, la preuve est celle de [112, Théorème 2.3]. L'adaptation au cas $\mu_2 \in H^6((0,1),\mathbb{R})$ est immédiate, on rappelle simplement la trame de la preuve.

Pour tout $\psi_0 \in H^6_{(V)}$, les systèmes (6.1) et (6.2) ayant le même linéarisé au voisinage de la trajectoire $\psi(\cdot,\psi_0,0)$ on en déduit immédiatement le lemme suivant qui assure la décroissance de la fonction de Lyapunov (voir [112, Proposition 2.6]).

Lemme 6.2. *Soient* $V, \mu_1, \mu_2 \in H^6((0,1),\mathbb{R})$ *tels que les Conditions* $(\mathbf{C_1})$ *et* $(\mathbf{C_2})$ *soient vérifiées. Pour tout* $\psi_0 \in \mathcal{S} \cap H^6_{(V)}$ *vérifiant* $\langle \psi_0, \varphi_{1,V} \rangle \neq 0$ *et* $\mathcal{L}(\psi_0) > 0$, *il existe un temps* $T > 0$ *et un contrôle* $u \in C^2_0((0,T),\mathbb{R})$ *tels que*

$$\mathcal{L}(\psi(T,\psi_0,u)) < \mathcal{L}(\psi_0).$$

Soit $\psi_0 \in \mathcal{S} \cap H^6_{(V)}$ tel que $\langle \psi_0, \varphi_{1,V} \rangle \neq 0$. On fixe $\gamma > 0$ dans la définition (6.4) tel que $\mathcal{L}(\psi_0) < 1$. Si $\mathcal{L}(\psi_0) > 0$, on définit

$$\mathcal{K} := \left\{ \psi_f \in H^6_{(V)} \,;\, \psi(T_n,\psi_0,u_n) \underset{n\to\infty}{\longrightarrow} \psi_f, \text{ dans } H^5 \text{ où } T_n > 0, u_n \in C^2_0((0,T_n),\mathbb{R}) \right\}.$$

L'infimum de $\mathcal{L}$ sur $\mathcal{K}$ est atteint i.e. il existe $e \in \mathcal{K}$ tel que

$$\mathcal{L}(e) = \inf_{\psi \in \mathcal{K}} \mathcal{L}(\psi). \tag{6.11}$$

En particulier, $\mathcal{L}(e) \leq \mathcal{L}(\psi_0) < 1$ d'où $\langle e, \varphi_{1,V} \rangle \neq 0$. D'après le Lemme 6.2, on obtient que si $\mathcal{L}(e) > 0$ alors il existe $T > 0$ et $u \in C^2_0((0,T),\mathbb{R})$ tels que $\mathcal{L}(\psi(T,e,u)) < \mathcal{L}(e)$. Comme $\psi(T,e,u) \in \mathcal{K}$, on obtient une contradiction avec (6.11). D'où $\mathcal{L}(e) = 0$. Ceci implique $\varphi_{1,V} \in \mathcal{K}$ et conclut la preuve du Théorème 6.2.

□

6.3 Contrôle exact local autour de l'état fondamental

Cette section est dédiée à la contrôlabilité exacte locale autour de l'état fondamental $\varphi_{1,V}$ dans $H^5_{(V)}$ avec des contrôles $H^1_0((0,T),\mathbb{R})$. Dans cette section, on suppose que les fonctions $V, \mu_1 \in H^5((0,1),\mathbb{R})$ vérifient

$(\mathbf{C_3})$ il existe $C > 0$ tel que

$$|\langle \mu_1 \varphi_{1,V}, \varphi_{k,V} \rangle| \geq \frac{C}{k^3}, \text{ pour tout } k \in \mathbb{N}^*.$$

Théorème 6.3. *Soient* $V, \mu_1, \mu_2 \in H^5((0,1),\mathbb{R})$ *tels que la Condition* $(\mathbf{C_3})$ *soit vérifiée. Soit* $T > 0$. *Il existe* $\delta > 0$ *et une application* C^1

$$\Gamma : \mathcal{O}_T \longrightarrow H^1_0((0,T),\mathbb{R})$$

où

$$\mathcal{O}_T := \left\{ \psi_f \in \mathcal{S} \cap H^5_{(V)} \,;\, \|\psi_f - \Phi_{1,V}(T)\|_{H^5} < \delta \right\},$$

telle que $\Gamma(\Phi_{1,V}(T)) = 0$, *et pour tout* $\psi_f \in \mathcal{O}_T$, *la solution de (6.1) avec condition initiale* $\psi_0 = \varphi_{1,V}$ *et contrôle* $u = \Gamma(\psi_f)$ *satisfasse* $\psi(T) = \psi_f$.

Démonstration. Dans le cas $\mu_2 = 0$ et $V = 0$, la preuve est celle de [16, Théorème 2]. L'adaptation au cas $V, \mu_2 \in H^5((0,1),\mathbb{R})$ est directe, on rappelle simplement la trame de la preuve.

Soit $T > 0$. En utilisant la régularité obtenue en Proposition 6.1 et la stratégie de [16, Proposition 6] on obtient la régularité C^1 de l'application

$$\begin{array}{rccc} \Theta_T : & H^1_0((0,T),\mathbb{R}) & \to & \mathcal{H} \\ & u & \mapsto & \mathcal{P}_{\mathcal{H}}\big(\psi(T,\varphi_{1,V},u)\big) \end{array} \tag{6.12}$$

où

$$\mathcal{H} := \left\{ \psi \in H^5_{(V)} \,;\, \mathrm{Re}(\langle \psi, \varphi_{1,V} \rangle) = 0 \right\},$$

et $\mathcal{P}_{\mathcal{H}}$ est la projection orthogonale dans $L^2((0,1),\mathbb{C})$ sur $\mathcal{H}$ i.e.

$$\mathcal{P}_{\mathcal{H}}(\psi) = \psi - \mathrm{Re}(\langle \psi, \varphi_{1,V} \rangle)\varphi_{1,V}.$$

Sa différentielle en 0 est donnée par $\mathrm{d}\Theta(0).v = \Psi(T)$, où Ψ est la solution de

$$\begin{cases} i\partial_t \Psi = \left(-\partial^2_{xx} + V(x)\right)\Psi - v(t)\mu_1(x)\Phi_{1,V}, & (t,x) \in (0,T)\times(0,1), \\ \Psi(t,0) = \Psi(t,1) = 0, & t \in (0,T), \\ \Psi(0,x) = 0, & x \in (0,1). \end{cases} \tag{6.13}$$

En utilisant la Condition $(\mathbf{C_3})$, l'asymptotique (6.7) et [16, Corollaire 2], on obtient l'existence d'un application linéaire continue

$$\mathcal{M} : \mathcal{H} \mapsto L^2((0,T),\mathbb{R}),$$

telle que pour tout $\Psi_f \in \mathcal{H}$, $w := \mathcal{M}(\Psi_f)$ est solution du problème de moments

$$\begin{cases} \displaystyle\int_0^T w(t)\mathrm{d}t = 0, \\ \displaystyle\int_0^T w(t)(T-t)\mathrm{d}t = \frac{1}{\langle \mu_1\varphi_{1,V}, \varphi_{1,V}\rangle}\langle \Psi_f, \Phi_{1,V}(T)\rangle, \\ \displaystyle\int_0^T w(t)e^{i(\lambda_{k,V}-\lambda_{1,V})t}\mathrm{d}t = \frac{\lambda_{1,V}-\lambda_{k,V}}{\langle \mu_1\varphi_{1,V}, \varphi_{k,V}\rangle}\langle \Psi_f, \Phi_{k,V}(T)\rangle, \quad \forall k \geq 2. \end{cases} \tag{6.14}$$

Le choix $v := t \mapsto \int_0^t w(\tau)\mathrm{d}\tau \in H^1_0((0,T),\mathbb{R})$ fournit alors un inverse à droite continu de la différentielle $\mathrm{d}\Theta(0) : H^1_0((0,T),\mathbb{R}) \to \mathcal{H}$.

Finalement, l'application du théorème d'inversion locale à Θ_T au point $u = 0$, la conservation de la norme L^2 et l'hypothèse $\psi_f \in \mathcal{S}$ concluent la preuve du Théorème 6.3.

□

6.4 Contrôle exact global

En combinant la contrôlabilité globale approchée du Théorème 6.2 et la contrôlabilité exacte locale du Théorème 6.3 on obtient la contrôlabilité exacte globale de (6.1) sous des hypothèses favorables sur V et μ_1.

Théorème 6.4. *Soient $V, \mu_1, \mu_2 \in H^6((0,1),\mathbb{R})$ tels que les Conditions* **($\mathbf{C_2}$)** *et* **($\mathbf{C_3}$)** *soient vérifiées. Pour tout $\psi_0, \psi_f \in \mathcal{S} \cap H^6_{(V)}$, il existe $T > 0$ et $u \in H^1_0((0,T),\mathbb{R})$ tels que la solution de (6.1) associée satisfasse $\psi(T) = \psi_f$.*

Démonstration. Première étape. Soient $\psi_0, \psi_f \in \mathcal{S} \cap H^6_{(V)}$ tels que $\langle \psi_0, \varphi_{1,V} \rangle \neq 0$ et $\langle \psi_f, \varphi_{1,V} \rangle \neq 0$. Soient $T_* > 0$ et $\delta > 0$ le rayon de contrôlabilité exacte locale dans $H^5_{(V)}$ en temps T_* donné par le Théorème 6.3. D'après le Théorème 6.2, il existe $T_0, T_f > 0$, $u_0 \in C^2_0((0,T_0),\mathbb{R})$ et $u_f \in C^2_0((0,T_f),\mathbb{R})$ tels que

$$\|\psi(T_0, \psi_0, u_0) - \varphi_{1,V}\|_{H^5} + \|\psi(T_f, \overline{\psi_f}, u_f) - \varphi_{1,V}\|_{H^5} < \delta. \tag{6.15}$$

D'après le Théorème 6.3, il existe $u_* \in H^1_0((0,T_*),\mathbb{R})$ tel que

$$\psi(T_*, \varphi_{1,V}, u_*) = e^{-i\lambda_{1,V} T_*} \overline{\psi(T_0, \psi_0, u_0)}.$$

En utilisant la réversibilité en temps de (6.1), on obtient alors que si u est défini sur $[0, T_0 + T_*]$ par $u(t) = u_0(t)$ pour $t \in [0, T_0]$ et $u(t + T_0) = u(T_* - t)$ pour $t \in [0, T_*]$ alors $u \in H^1_0((0, T_0 + T_*),\mathbb{R})$ et

$$\psi(T_0 + T_*, \psi_0, u) = \Phi_{1,V}(T_*).$$

Ainsi il existe $T^* > 0$ tel que si l'on étend u par 0 sur $[T_0 + T_*, T_0 + T_* + T^*]$ alors,

$$\psi(T_0 + T_* + T^*, \psi_0, u) = \varphi_{1,V}. \tag{6.16}$$

Les mêmes arguments conduisent alors à l'existence de $\tilde{u} \in H^1_0((0, T_f + T_* + T^*),\mathbb{R})$ tel que

$$\psi(T_f + T_* + T^*, \overline{\psi_f}, \tilde{u}) = \varphi_{1,V}.$$

Conjointement à (6.16), la réversibilité en temps de (6.1) implique alors, pour $T := T_0 + T_f + 2T_* + 2T^*$, l'existence de $u \in H^1_0((0,T),\mathbb{R})$ tel que

$$\psi(T, \psi_0, u) = \psi_f.$$

Deuxième étape. On conclut la preuve du Théorème 6.4 en montrant que l'on peut se passer des hypothèses $\langle \psi_0, \varphi_{1,V} \rangle \neq 0$ et $\langle \psi_f, \varphi_{1,V} \rangle \neq 0$.

D'après la réversibilité en temps de (6.1), il est suffisant de montrer que pour tout $\psi_0 \in \mathcal{S} \cap H^6_{(V)}$, il existe $T > 0$ et $u \in C^2_0((0,T),\mathbb{R})$ tels que $\psi(T, \psi_0, u) \in \mathcal{S} \cap H^6_{(V)}$ et $\langle \psi(T, \psi_0, u), \varphi_{1,V} \rangle \neq 0$. Soit $\widehat{\psi_0} \in \mathcal{S} \cap H^6_{(V)}$ tel que $\langle \widehat{\psi_0}, \varphi_{1,V} \rangle \neq 0$ et $\|\psi_0 - \widehat{\psi_0}\|_{L^2} < \sqrt{2}$. D'après la première étape, il existe $T > 0$ et $\hat{u} \in H^1_0((0,T),\mathbb{R})$ tels que $\psi(T, \widehat{\psi_0}, \hat{u}) = \varphi_{1,V}$. La conservation de la norme L^2 implique

$$\|\psi(T, \psi_0, \hat{u}) - \varphi_{1,V}\|_{L^2} = \|\psi_0 - \widehat{\psi_0}\|_{L^2} < \sqrt{2}.$$

Finalement, le choix de $u \in C^2_0((0,T),\mathbb{R})$ suffisamment proche de $\hat{u}$ termine la preuve du Théorème 6.4.

□

En adaptant l'argument de perturbation utilisé en Section 4.5.2 et le Théorème 6.4 on prouve le Théorème 6.1.

Démonstration du Théorème 6.1. Soient $V, \mu_1 \in H^6((0,1),\mathbb{R})$. On définit $\mathcal{Q}_{V,\mu_1}$ l'ensemble des fonctions $\mu_2 \in H^6((0,1),\mathbb{R})$ telles que les Conditions $(\mathbf{C_2})$ et $(\mathbf{C_3})$ soient vérifiées avec les fonctions V et μ_1 remplacées respectivement par $V-2\mu_1-4\mu_2$ et $\mu_1+4\mu_2$ i.e. on définit

- $(\mathbf{C_2'})$ $\lambda_{1,V-2\mu_1-4\mu_2} - \lambda_{j,V-2\mu_1-4\mu_2} \neq \lambda_{p,V-2\mu_1-4\mu_2} - \lambda_{q,V-2\mu_1-4\mu_2}$, pour tout $j,p,q \geq 1$ et $j \neq 1$.
- $(\mathbf{C_3'})$ il existe $C>0$ tel que

$$|\langle(\mu_1+4\mu_2)\varphi_{1,V-2\mu_1-4\mu_2}, \varphi_{k,V-2\mu_1-4\mu_2}\rangle| \geq \frac{C}{k^3}, \text{ pour tout } k \in \mathbb{N}^*.$$

et

$$\mathcal{Q}_{V,\mu_1} := \left\{\mu_2 \in H^6((0,1),\mathbb{R})\,;\, \text{Conditions } (\mathbf{C_2'}) \text{ et } (\mathbf{C_3'})\right\}.$$

Première étape : contrôlabilité exacte globale pour $\mu_2 \in \mathcal{Q}_{V,\mu_1}$. On considère le système (6.1) avec les fonctions V et μ_1 remplacées respectivement par $V-2\mu_1-4\mu_2$ et $\mu_1+4\mu_2$ i.e. le système (6.3). On note $\widetilde{\psi}(T,\psi_0,u)$ le propagateur de (6.3) au temps T. On a

$$\widetilde{\psi}(t,\psi_0,u) = \psi(t,\psi_0,u+2), \quad \text{pour } t \in [0,T]. \tag{6.17}$$

Soient $\psi_0, \psi_f \in \mathcal{S} \cap H^6_{(V)}$ et $u_1 \in C^2([0,1],\mathbb{R})$ vérifiant $u_1(0) = \dot{u}_1(0) = \dot{u}_1(1) = 0$ et $u_1(1) = 2$. D'après la Proposition 6.2, on a

$$\widetilde{\psi}_0 := \psi(1,\psi_0,u_1) \in \mathcal{S} \cap H^6_{(V-2\mu_1-4\mu_2)},$$
$$\overline{\widetilde{\psi}_f} := \psi(1,\overline{\psi_f},u_1) \in \mathcal{S} \cap H^6_{(V-2\mu_1-4\mu_2)}.$$

Comme $\mu_2 \in \mathcal{Q}_{V,\mu_1}$, le Théorème 6.4 implique l'existence de $\tilde{T}>0$ et $\tilde{u} \in H^1_0((0,\tilde{T}),\mathbb{R})$ tels que

$$\widetilde{\psi}(\tilde{T},\widetilde{\psi}_0,\tilde{u}) = \widetilde{\psi}_f.$$

On pose $T = 2+\tilde{T}$ et

$$u(t) = \begin{cases} u_1(t) & \text{pour } t \in [0,1], \\ \tilde{u}(t-1)+2 & \text{pour } t \in [1,\tilde{T}+1], \\ u_1(1-(t-1-\tilde{T})) & \text{pour } t \in [\tilde{T}+1,T]. \end{cases}$$

Ainsi, la réversibilité en temps de (6.1) et (6.17) impliquent

$$\psi(T,\psi_0,u) = \psi_f,$$

avec $u \in H^1_0((0,T),\mathbb{R})$.

Deuxième étape. On conclut la preuve du Théorème 6.1 en montrant que $\mathcal{Q}_{V,\mu_1}$ est résiduel dans $H^6((0,1),\mathbb{R})$.

Pour tout $W \in H^6((0,1),\mathbb{R})$, d'après le Lemme 4.2 page 143 pour $s = 6$, l'ensemble $\mathcal{Q}_W$ des fonctions $\mu \in H^6((0,1),\mathbb{R})$ telles que $\{1, \lambda_{k,W+\mu}\}_{k\in\mathbb{N}^*}$ soient rationnellement indépendants et qu'il existe $C > 0$ tel que

$$|\langle \mu\varphi_{1,W+\mu}, \varphi_{k,W+\mu}\rangle| \geq \frac{C}{k^3}, \quad \forall k \in \mathbb{N}^*,$$

est résiduel dans $H^6((0,1),\mathbb{R})$. On pose $W := V - \mu_1 \in H^6((0,1),\mathbb{R})$. Pour tout $\mu \in \mathcal{Q}_W$, μ_2 défini par $\mu_2 := -\frac{1}{4}(\mu_1 + \mu)$ appartient à $\mathcal{Q}_{V,\mu_1}$. Ceci termine la preuve du Théorème 6.1. $\square$

Troisième partie

Contrôle d'équations de Grushin singulières

Chapitre 7

Continuation unique de l'équation de Grushin singulière 2D

Ce chapitre est inspiré de la prépublication [107].

Contents

7.1 Introduction

7.1.1 Main result

Let us consider for $\gamma > 0$ the following degenerate singular parabolic equation

$$\begin{cases} \partial_t f - \partial_{xx}^2 f - |x|^{2\gamma} \partial_{yy}^2 f + \dfrac{c_\nu}{x^2} f = u(t,x,y)\chi_\omega(x,y), & (t,x,y) \in (0,T) \times \Omega, \\ f(t,-1,y) = f(t,1,y) = 0, & (t,y) \in (0,T) \times (0,1), \\ f(t,x,0) = f(t,x,1), & (t,x) \in (0,T) \times (-1,1), \\ \partial_y f(t,x,0) = \partial_y f(t,x,1), & (t,x) \in (0,T) \times (-1,1), \end{cases} \tag{7.1}$$

with initial condition

$$f(0,x,y) = f^0(x,y), \quad (x,y) \in \Omega. \tag{7.2}$$

The domain is $\Omega := (-1,1) \times (0,1)$ and ω, the control domain, is an open subset of Ω and χ denotes the indicator function. The constant c_ν of the singular potential is defined by $c_\nu := \nu^2 - \frac{1}{4}$, for $\nu \in (0,1)$. The degeneracy set $\{x = 0\}$ coincides with the singularity set ; it separates the domain Ω in two connected components. Due to the singular potential, the first difficulty of this work is to give meaning to solutions of (7.1). Through the study of an associated 1D heat equation, we will design a suitable extension of the considered operator generating a continuous semigroup. The solutions considered in this article will be related to this semigroup. This is detailed in Section 7.2.

In [28], Boscain and Laurent studied the Laplace-Beltrami operator for the Grushin-like metric given by the orthonormal basis $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 \\ |x|^\gamma \end{pmatrix}$ on $\mathbb{R} \times \mathbb{T}$ with $\gamma > 0$ i.e.

$$Lu := \partial_{xx}^2 u + |x|^{2\gamma} \partial_{yy}^2 u - \frac{\gamma}{x} \partial_x u. \tag{7.3}$$

They proved that this operator with domain $C_0^\infty\left((\mathbb{R}\backslash\{0\}) \times \mathbb{T}\right)$ is essentially self-adjoint on $L^2(\mathbb{R} \times \mathbb{T})$ if and only if $\gamma > 1$. Thus, for the heat equation associated to this Laplace-Beltrami operator, no information passes through the singular set $\{x = 0\}$ when $\gamma > 1$. This prevents controllability from one side of the singularity.

Up to the change of variables $u = |x|^{\gamma/2} v$, the Laplace-Beltrami operator L is equal to

$$\Delta v = \partial_{xx}^2 v + |x|^{2\gamma} \partial_{yy}^2 v - \frac{\gamma}{2}\left(\frac{\gamma}{2} + 1\right) \frac{v}{x^2}.$$

The model (7.1) can then be seen as a heat equation for this Laplace-Beltrami operator. By choosing the coefficient c_ν instead of $\frac{\gamma}{2}\left(\frac{\gamma}{2} + 1\right)$ we authorize a wider class of singular potentials and decouple the effects of the degeneracy and the singularity for a better understanding of each one of these phenomena. Adapting the arguments of [28], one obtains that for any $\gamma > 0$, the operator $-\partial_{xx}^2 - |x|^{2\gamma}\partial_{yy}^2 + \frac{\lambda}{x^2}$ with domain $C_0^\infty(\Omega\backslash\{x = 0\})$ is essentially self-adjoint on $L^2(\Omega)$ if and only if $\lambda \geq \frac{3}{4}$. Thus, our study focuses on the range of constants $c_\nu < \frac{3}{4}$ i.e. $\nu < 1$.

The lower bound $c_\nu > -\frac{1}{4}$ for the range of constants considered comes from well posedness issues linked to the use of the following Hardy inequality (see e.g. [39] for a simple proof)

$$\int_0^1 \frac{z(x)^2}{x^2} \mathrm{d}x \leq 4 \int_0^1 z_x(x)^2 \mathrm{d}x, \quad \forall z \in H^1((0,1),\mathbb{R}) \text{ with } z(0) = 0. \tag{7.4}$$

The critical case $c_\nu = -\frac{1}{4}$ in this inequality is not covered by the technics of this article.

Recall that (7.1) is said to be approximately controllable in time $T > 0$ if for any $(f^0, f^T) \in L^2(\Omega)^2$, for any $\varepsilon > 0$, there exists $u \in L^2((0,T) \times \Omega)$ such that the solution of (7.1)-(7.2) satisfies

$$||f(T) - f^T||_{L^2(\Omega)} \leq \varepsilon.$$

The main result of this article is the following theorem.

Theorem 7.1. *Let $T > 0$, $\gamma > 0$ and $\nu \in (0,1)$. If ω is an open subset of one of the connected components of $\Omega \backslash \{x=0\}$ then (7.1) is approximately controllable in time T if and only if $\nu \in \left(0, \frac{1}{2}\right]$ i.e. if and only if $c_\nu \in \left(-\frac{1}{4}, 0\right]$.*

This theorem thus fills the gap, for the approximate controllability property, between validity of Hardy inequality ($c_\nu > -\frac{1}{4}$) and the essential self-adjointness property of [28] for $c_\nu \geq \frac{3}{4}$.

Remark 7.1. As it will be noticed during the proof (see Remark 7.4), if ω intersects both connected components of $\Omega \backslash \{x=0\}$, then approximate controllability holds for any $\nu \in (0,1)$ i.e. any $c_\nu \in \left(-\frac{1}{4}, \frac{3}{4}\right)$.

Theorem 7.1 can be partially adapted to the case of homogeneous Dirichlet boundary conditions in the following way.

Theorem 7.2. *Let $T > 0$, $\gamma > 0$ and $\nu \in (0,1)$. For $\ell > 0$, set $\Omega^\ell := (-1,1) \times (0,\ell)$. Consider the system with homogeneous Dirichlet boundary conditions*

$$\begin{cases} \partial_t f - \partial_{xx}^2 f - |x|^{2\gamma} \partial_{yy}^2 f + \dfrac{c_\nu}{x^2} f = u(t,x,y)\chi_\omega(x,y), & (t,x,y) \in (0,T) \times \Omega^\ell, \\ f(t,x,y) = 0, & (t,x,y) \in (0,T) \times \partial\Omega^\ell, \\ f(0,x,y) = f^0(x,y), & (x,y) \in \Omega^\ell. \end{cases} \tag{7.5}$$

If $\nu \in \left(0, \frac{1}{2}\right]$, then system (7.5) is approximately controllable in time T, for any $\ell > 0$.
If $\nu \in \left(\frac{1}{2}, 1\right)$ and $\gamma = 1$, there exist values of $\ell > 0$ such that for any $T > 0$ approximate controllability does not hold in time T for system (7.5).

Thus, the positive result of approximate controllability also holds for homogeneous Dirichlet boundary conditions. The negative result based on an explicit counterexample will necessitate special lengths in the y variable (i.e. particular values of ℓ) and only stands in the case $\gamma = 1$. These assumptions are technical and we conjecture that approximate controllability does not hold for system (7.5) for any $\gamma > 0$, any $\ell > 0$ if $\nu \in \left(\frac{1}{2}, 1\right)$. We will focus in the rest of the paper on Theorem 7.1 and detail only the modifications for Theorem 7.2 when necessary.

The model (7.1) can also be seen as an extension of [14] where Beauchard *et al.* studied the null controllability without the singular potential (i.e. in the case $\nu = \frac{1}{2}$). The authors proved that null controllability holds if $\gamma \in (0,1)$ and does not hold if $\gamma > 1$. In the case $\gamma = 1$, for ω a strip in the y direction, null controllability holds if and only if the time is large enough.

The inverse square potential for the Grushin equation has already been taken into account by Cannarsa and Guglielmi in [37] in the case where both degeneracy and singularity are at the boundary. With our notations, they proved null controllability in sufficiently large time for $\Omega = (0,1) \times (0,1)$, $\omega = (a,b) \times (0,1)$, $\gamma = 1$ and any $c_\nu > -\frac{1}{4}$. They also proved that approximate controllability holds for any control domain $\omega \subset \Omega$, any $\gamma > 0$ and any $c_\nu > -\frac{1}{4}$. Thus, the fact that our model presents an internal singularity instead of a boundary singularity deeply affects the approximate controllability property.
By a classical duality argument, Theorem 7.1 will be proved by unique continuation on the adjoint system. Following techniques used in [14] this problem will be studied trough the 1D equations satisfied by the coefficients of the solution in the expansion in Fourier series

in the y variable. As a corollary we will obtain the following approximate controllability result for the 1D heat equation with a singular inverse square potential.

Theorem 7.3. *Let $T > 0$ and $\nu \in (0,1)$. If ω is an open subset of $(-1,0)$ or $(0,1)$, then approximate controllability holds for*

$$\begin{cases} \partial_t f - \partial^2_{xx} f + \dfrac{c_\nu}{x^2} f = u(t,x)\chi_\omega(x), & (t,x) \in (0,T) \times (-1,1), \\ f(t,-1) = f(t,1) = 0, & t \in (0,T), \end{cases} \tag{7.6}$$

if and only if $\nu \in \left(0, \frac{1}{2}\right]$ i.e. if and only if $c_\nu \in \left(-\frac{1}{4}, 0\right]$.

The null controllability issue for the 1D heat equation with such an internal inverse square singularity remains an open question. Like (7.1), the solutions of (7.6) are related to the semigroup generated by a suitable extension of the Laplace operator with a singular potential.

7.1.2 Structure of this article

We end this introduction by a brief review of previous results concerning degenerate and/or singular parabolic equations.

Due to the internal singularity and the fact that the considered operators admit several self-adjoint extensions, the functional setting and the well posedness are crucial issues in this article. Section 7.2 is dedicated to these questions.

Section 7.3 is dedicated to the study of the unique continuation property. When it holds, unique continuation is proved using tools from the uniformly parabolic case and by adapting Carleman estimates to our setting. When $\nu \in \left(\frac{1}{2}, 1\right)$, explicit counterexamples will be constructed using Bessel functions.

7.1.3 A review of previous results

The first result for a heat equation with an inverse square potential $\frac{\lambda}{|x|^2}$ deals with well posedness issues. In [6], Baras and Goldstein proved complete instantaneous blow-up for positive initial conditions in space dimension N (the singularity being at the boundary of the domain in the one dimensional case) if $\lambda < \lambda^*(N) := -\frac{(N-2)^2}{4}$. Notice that this critical value is the best constant in the Hardy inequality. Cabré and Martel also studied in [35] the relation between blow-up of such equations and the existence of an Hardy inequality. Thus, most of the following studies focus on the range of constants $\lambda \geq \lambda^*(N)$. In this case, well posedness in $L^2(\Omega)$ has been proved in [139] by Vazquez and Zuazua.

The controllability issues were first studied for degenerate equations. In [38, 101, 39, 40], Cannarsa, Martinez and Vancostenoble proved null controllability with a distributed control for a one dimensional parabolic equation degenerating at the boundary. Then, they extended this result to more general degeneracies and in dimension two. These results are based on suitable Hardy inequalities and Carleman estimates. More recently, Gueye proved in [81] null controllability for a class of one dimensional hyperbolic equations degenerating at the boundary (and the corresponding parabolic degenerate equation via transmutation) with control on the degenerate boundary. Its proof relies on appropriate nonharmonic Fourier series.

Meanwhile, these Carleman estimates were adapted for heat equation with an inverse square potential $\frac{1}{|x|^2}$ in dimension $N \geq 3$. Notice that in this case the singularity is the point $\{0\}$. In [138], Vancostenoble and Zuazua proved null controllability in the case where the control domain ω contains an annulus centred on the singularity. Their proof relies on a decomposition in spherical harmonics reducing the problem to the study of a 1D heat equation with an inverse square potential which is singular at the boundary. The geometric assumptions on the control domain were then removed by Ervedoza in [69] using a direct Carleman strategy in dimension $N \geq 3$. Notice that although these results deal with internal singularity they cannot be adapted to our setting. Indeed, in [138] it is crucial that the singularity of the 1D problem obtained by decomposition in spherical harmonics is at the boundary. The Carleman strategy developed in [69] cannot be adapted in this article because our singularity is no longer a point but separates the domain in two connected components.
For null controllability for a one dimensional parabolic equation both degenerate and singular at the boundary we refer to [137] by Vancostenoble. The proof extends the previous Carleman strategy together with an improved Hardy inequality.
As the functional setting for this study is obtained through the design of a suitable self-adjoint extension of our Grushin-like operator, let us mention the work [29] conducted simultaneously to this study. In this paper, Boscain and Prandi studied some extensions of the Laplace-Beltrami operator (7.3) for $\gamma \in \mathbb{R}$. Among other things, they design for a suitable range of constants an extension called bridging extension that allows full communication through the singular set. Even if we authorize in this article a wider class of singular potentials, the approximate controllability from one side of the singularity given by Theorem 7.1 shows full agreement with the bridging extension of [29].

7.2 Well posedness

The previous results of the literature dealing with an inverse square potential were obtained thanks to some Hardy-type inequality. For a boundary inverse square singularity (as in [137]), the condition $z(0) = 0$ needed for (7.4) to hold is contained in the homogeneous Dirichlet boundary conditions considered. Thus, in [137], the appropriate functional setting to study the 1D operator $-\partial_{xx}^2 + \frac{\lambda}{x^2}$ with $\lambda > -\frac{1}{4}$ is

$$\left\{ f \in H^2_{loc}((0,1]) \cap H^1_0(0,1)\,;\, -\partial_{xx}^2 f + \frac{\lambda}{x^2} f \in L^2(0,1) \right\}.$$

For an internal inverse square singularity one still has

$$\int_{-1}^{1} \frac{z(x)^2}{x^2} \mathrm{d}x \leq 4 \int_{-1}^{1} z_x(x)^2 \mathrm{d}x, \quad \forall z \in H^1(-1,1) \text{ such that } z(0) = 0. \tag{7.7}$$

This inequality ceases to be true if $z(0) \neq 0$. Thus, the functional setting must contain some informations on the behaviour of the functions at the singularity.

As announced, we design a suitable self-adjoint extension of the operator $-\partial_{xx}^2 - |x|^{2\gamma} \partial_{yy}^2 + \frac{c_\nu}{x^2}$ on $C_0^\infty(\Omega \backslash \{x = 0\})$. The next subsection deals with an associated one dimensional equation. Subsection 7.2.2 will then relate this one dimensional problem to the original problem in dimension two.

7.2.1 Reduction to a 1D problem

For $n \in \mathbb{Z}$, $\gamma > 0$ and $\nu \in (0,1)$ let us consider the following homogeneous problem

$$\begin{cases} \partial_t f - \partial^2_{xx} f + \dfrac{c_\nu}{x^2} f + (2n\pi)^2 |x|^{2\gamma} f = 0, & (t,x) \in (0,T) \times (-1,1), \\ f(t,-1) = f(t,1) = 0, & t \in (0,T). \end{cases} \tag{7.8}$$

This equation is inspired by the equation satisfied by the coefficients of the Fourier expansion in the y variable done in [14] and will be linked to (7.1) in Subsection 7.2.2. From now on, we focus on the well posedness of (7.8).

Remark 7.2. A naive functional setting for this equation is the adaptation of [137]

$$\Big\{ f \in L^2(-1,1)\,;\, f_{|[0,1]} \in H^2_{loc}((0,1]) \cap H^1_0(0,1),\ f_{|[-1,0]} \in H^2_{loc}([-1,0)) \cap H^1_0(-1,0) \\ \text{and } -\partial^2_{xx} f + \frac{c_\nu}{x^2} f \in L^2(-1,1) \Big\}.$$

However, a functional setting where the two problems on $(-1,0)$ and $(0,1)$ are well posed is not pertinent for the control problem from one side of the singularity. It leads to decoupled dynamics on the connected components of $(-1,0) \cup (0,1)$.

We study the differential operator

$$A_n f := -\partial^2_{xx} f + \frac{c_\nu}{x^2} f(x) + (2n\pi)^2 |x|^{2\gamma} f(x).$$

As $\nu < 1$, the results of [28] imply that A_n defined on $C^\infty_0((-1,0) \cup (0,1))$ admits several self-adjoint extensions. Let us specify the self-adjoint extension that will be used. Let

$$\tilde{H}^2_0(-1,1) := \left\{ f \in H^2(-1,1)\,;\, f(0) = f'(0) = 0 \right\},$$

and

$$\mathcal{F}_s := \Big\{ f \in L^2(-1,1)\,; f = c_1^+ |x|^{\nu+1/2} + c_2^+ |x|^{-\nu+1/2} \text{ on } (0,1) \\ \text{and } f = c_1^- |x|^{\nu+1/2} + c_2^- |x|^{-\nu+1/2} \text{ on } (-1,0) \Big\}.$$

The domain of the operator is defined as

$$D(A_n) := \Big\{ f = f_r + f_s\,;\, f_r \in \tilde{H}^2_0(-1,1),\ f_s \in \mathcal{F}_s \text{ such that } f(-1) = f(1) = 0, \\ c_1^- + c_2^- + c_1^+ + c_2^+ = 0 \text{ and} \\ (\nu + 1/2) c_1^- + (-\nu + 1/2) c_2^- = (\nu + 1/2) c_1^+ + (-\nu + 1/2) c_2^+ \Big\}, \tag{7.9}$$

Notice that for $\nu \in (0,1)$, $D(A_n) \subset L^2(-1,1)$. As this domain is independent of n, it will be denoted by $D(A)$ in the rest of this article. The coefficients of the singular part will be denoted by c_1^+ if there is no ambiguity and $c_1^+(f)$ otherwise. The conditions imposed on these coefficients in (7.9) will be referred to as the "transmission conditions".

Using classical computations (see for instance [2, Proposition 3.1]) we get that for any $f_r \in \tilde{H}_0^2(-1,1)$, $x \mapsto \frac{1}{x^2} f_r(x) \in L^2(-1,1)$. Notice that for any $f_s \in \mathcal{F}_s$,

$$-\partial_{xx}^2 f_s + \frac{c_\nu}{x^2} f_s = 0. \tag{7.10}$$

Thus, for any $f \in D(A)$,

$$A_n f = \Big(-\partial_{xx}^2 f_r + \frac{c_\nu}{x^2} f_r\Big) + (2n\pi)^2 |x|^{2\gamma} f \in L^2(-1,1).$$

This operator satisfies the following properties

Proposition 7.1. *For any $n \in \mathbb{Z}$, the operator $(A_n, D(A))$ is self-adjoint. Moreover, for any $f \in D(A)$,*

$$\langle A_n f, f\rangle \geq m_\nu \int_{-1}^1 \partial_x f_r(x)^2 \mathrm{d}x + (2n\pi)^2 \int_{-1}^1 |x|^{2\gamma} f(x)^2 \mathrm{d}x \geq 0,$$

where $m_\nu := \min\{1, 4\nu^2\}$.

This proposition is proved in Appendix 7.A.

Remark 7.3. The construction of this self-adjoint extension of the minimal operator is inspired by [142, Theorem 13.3.1, Case 5] for general self-adjoint extensions of Sturm-Liouville differential operators and by [2] for the explicit characterization of the minimal and maximal domains. We have nevertheless detailed this proof for the sake of clarity. The reader interested in the link between this construction and the general theory of self adjoint extensions of Sturm-Liouville operator should refer to Appendix 7.B.

Using Proposition 7.1, the well posedness of the one dimensional system (7.8) follows from the Hille-Yosida theorem (see e.g. [43, Theorem 3.2.1]).

Proposition 7.2. *For any $n \in \mathbb{Z}$ and any $f^0 \in L^2(-1,1)$, problem (7.8) with initial condition $f(0,\cdot) = f^0$ has a unique solution*

$$f \in C^0([0,+\infty), L^2(-1,1)) \cap C^0((0,+\infty), D(A)) \cap C^1((0,+\infty), L^2(-1,1)).$$

This solution satisfies

$$||f(t)||_{L^2(-1,1)} \leq ||f^0||_{L^2(-1,1)}.$$

In all what follows, we denote by $e^{-A_n t}$ the semigroup generated by $-A_n$ i.e. for any $f^0 \in L^2(-1,1)$, the function $t \mapsto e^{-A_n t} f^0$ is the solution of (7.8) given by Proposition 7.2. We now turn back to the initial problem in dimension two.

7.2.2 Semigroup associated to the 2D problem

Let $f^0 \in L^2(\Omega)$. For almost every $x \in (-1,1)$, $f^0(x,\cdot) \in L^2(0,1)$ and thus can be expanded in Fourier series as follows

$$f^0(x,y) = \sum_{n\in\mathbb{Z}} f_n^0(x)\varphi_n(y), \tag{7.11}$$

where $(\varphi_n)_{n\in\mathbb{Z}}$ is the Hilbert basis of $L^2(0,1)$ of eigenvectors of the Laplace operator on $H^2(0,1)$ with periodic boundary conditions i.e.

$$\varphi_n(y) := \sqrt{2}\sin(2n\pi y),\ \forall n \in \mathbb{N}^*; \quad \varphi_{-n}(y) := \sqrt{2}\cos(2n\pi y),\ \forall n \in \mathbb{N}^*; \quad \varphi_0(y) := 1$$

and

$$f_n^0(x) := \int_{-1}^{1} f^0(x,y)\varphi_n(y)\mathrm{d}y.$$

For any $t \in (0,T)$, we define the following operator

$$(S(t)f^0)(x,y) := \sum_{n\in\mathbb{Z}} f_n(t,x)\varphi_n(y), \tag{7.12}$$

where for any $n \in \mathbb{Z}$, $f_n(t) := e^{-A_n t} f_n^0$. Then, the following proposition holds.

Proposition 7.3. *$S(t)$ defined by (7.12) is a continuous semigroup of contractions in $L^2(\Omega)$.*

Proof of Proposition 7.3. By Proposition 7.2, $S(t)$ is well defined, with value in $L^2(\Omega)$, it is a semigroup and satisfies the contraction property. For any $f^0 \in L^2(\Omega)$, we have

$$||S(t)f^0 - f^0||^2_{L^2(\Omega)} = \sum_{n\in\mathbb{Z}} ||f_n(t,\cdot) - f_n^0||^2_{L^2(-1,1)}.$$

By Proposition 7.2 it comes that

$$||f_n(t,\cdot) - f_n^0||_{L^2(-1,1)} \underset{t\to 0}{\longrightarrow} 0,$$
$$||f_n(t,\cdot) - f_n^0||_{L^2(-1,1)} \leq 2\,||f_n^0||_{L^2(-1,1)}.$$

Thus, by the dominated convergence theorem, $S(t)f^0 \underset{t\to 0}{\longrightarrow} f^0$ in $L^2(\Omega)$.

□

Recall that its infinitesimal generator $\mathcal{A}$ is defined on

$$D(\mathcal{A}) := \left\{ f \in L^2(\Omega)\,;\ \lim_{t\to 0} \frac{S(t)f - f}{t} \text{ exists } \right\},$$

by

$$\mathcal{A}f := \lim_{t\to 0} \frac{S(t)f - f}{t}.$$

The previous limits are related to the L^2 norm. Then, from [116, Theorems 1.3.1 and 1.4.3] it comes that $(\mathcal{A}, D(\mathcal{A}))$ is a closed dissipative densely defined operator and satisfies for any $\lambda > 0$, $R(\lambda I - \mathcal{A}) = L^2(\Omega)$. The following proposition links the system (7.1) and the semigroup $S(t)$.

Proposition 7.4. *The infinitesimal generator $\mathcal{A}$ of $S(t)$ is characterized by*

$$D(\mathcal{A}) = \Big\{ f \in L^2(\Omega)\,;\, f = \sum_{n\in\mathbb{Z}} f_n(x)\varphi_n(y) \text{ with } f_n \in D(A) \text{ and } \sum_{n\in\mathbb{Z}} ||A_n f_n||^2_{L^2(-1,1)} < +\infty \Big\}, \tag{7.13}$$

and

$$\mathcal{A}f = -\sum_{n\in\mathbb{Z}} (A_n f_n)(x)\varphi_n(y). \tag{7.14}$$

This operator extends the Grushin differential operator in the sense that

$$\mathcal{A}f = \partial^2_{xx} f + |x|^{2\gamma}\partial^2_{yy} f - \frac{c_\nu}{x^2} f, \quad \forall f \in C_0^\infty(\Omega\backslash\{x=0\}). \tag{7.15}$$

Proof of Proposition 7.4. Let $f^0 \in D(\mathcal{A})$. Then, $\mathcal{A}f^0 \in L^2(\Omega)$ and

$$\frac{S(t)f^0 - f^0}{t} \underset{t\to 0}{\longrightarrow} \mathcal{A}f^0, \quad \text{in } L^2(\Omega).$$

As $\mathcal{A}f^0 \in L^2(\Omega)$, it can be decomposed in Fourier series in the y variable i.e.

$$\mathcal{A}f^0(x,y) = \sum_{n\in\mathbb{Z}} (\mathcal{A}f^0)_n(x)\varphi_n(y).$$

Thus,

$$\left|\left|\frac{S(t)f^0 - f^0}{t} - \mathcal{A}f^0\right|\right|^2_{L^2(\Omega)} = \sum_{n\in\mathbb{Z}} \left|\left|\frac{f_n(t) - f_n^0}{t} - (\mathcal{A}f^0)_n\right|\right|^2_{L^2(-1,1)} \underset{t\to 0}{\longrightarrow} 0.$$

This implies that for any $n \in \mathbb{Z}$, $f_n^0 \in D(A)$ and

$$(\mathcal{A}f^0)_n = -A_n f_n^0.$$

We thus get

$$-\mathcal{A}f^0 = \sum_{n\in\mathbb{Z}} (A_n f_n^0)(x)\varphi_n(y).$$

Conversely, let $g \in L^2(\Omega)$ be such that for any $n \in \mathbb{Z}$, $g_n \in D(A)$ and $\sum_{n\in\mathbb{Z}} ||A_n g_n||^2_{L^2(-1,1)} < +\infty$. Let $f \in D(\mathcal{A})$. Then,

$$|\langle \mathcal{A}f, g\rangle| \le \sum_{n\in\mathbb{Z}} |\langle A_n f_n, g_n\rangle| \le \left(\sum_{n\in\mathbb{Z}} ||f_n||^2_{L^2(-1,1)}\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} ||A_n g_n||^2_{L^2(-1,1)}\right)^{\frac{1}{2}}.$$

This implies that $g \in D(\mathcal{A}^*)$. Finally, self-adjointness of $S(t)$ and thus of $\mathcal{A}$ ends the proof of (7.13). Straightforward computations lead to (7.15) and thus ends the proof of Proposition 7.4.

□

Using Proposition 7.4, we rewrite (7.1)-(7.2) in the form

$$\begin{cases} f'(t) = \mathcal{A}f(t) + v(t), & t \in [0,T], \\ f(0) = f^0, \end{cases} \tag{7.16}$$

where $v(t) : (x,y) \in \Omega \mapsto u(t,x,y)\chi_\omega(x,y)$. In the following a solution of (7.1) will mean a solution of (7.16). The following proposition is classical (see e.g. [116]) and ends this well posedness section

Proposition 7.5. *For any $f^0 \in L^2(\Omega)$, any $T > 0$ and $v \in L^2((0,T); L^2(\Omega))$, system (7.16) has a unique mild solution given by*

$$f(t) = S(t)f^0 + \int_0^t S(t-\tau)v(\tau)\mathrm{d}\tau, \quad t \in [0,T].$$

7.3 Unique continuation

Without loss of generality, we assume that $\omega \subset (-1,0) \times (0,1)$. Using the abstract formulation (7.16) we get that the adjoint system of (7.1) is given by

$$\begin{cases} \partial_t g - \partial_{xx}^2 g - |x|^{2\gamma}\partial_{yy}^2 g + \frac{c_\nu}{x^2} g = 0, & (t,x,y) \in (0,T) \times \Omega, \\ g(t,-1,y) = g(t,1,y) = 0, & (t,y) \in (0,T) \times (0,1), \\ g(t,x,0) = g(t,x,1), & (t,x) \in (0,T) \times (-1,1), \\ \partial_y g(t,x,0) = \partial_y g(t,x,1), & (t,x) \in (0,T) \times (-1,1), \\ g(0,x,y) = g^0(x,y), & (x,y) \in \Omega. \end{cases} \tag{7.17}$$

From Section 7.2, it comes that for any $g^0 \in L^2(\Omega)$, system (7.17) has a unique solution given by $S(t)g^0$. Thanks to a classical duality argument, Theorem 7.1 is proved by the following theorem dealing with unique continuation for the adjoint system (7.17).

Theorem 7.4. *Let $T > 0$, $\gamma > 0$ and $\nu \in (0,1)$. Assume that $g^0 \in L^2(\Omega)$ is such that $\chi_\omega S(t)g^0 \equiv 0$ for almost every $t \in [0,T]$. Then, g^0 must be identically zero on Ω if and only if $\nu \in \left(0, \frac{1}{2}\right]$ i.e. $c_\nu \in \left(-\frac{1}{4}, 0\right]$.*

The rest of this section is dedicated to the proof of Theorem 7.4. In Subsection 7.3.1, we prove that $S(t)g^0$ must be identically zero on the connected component of $\Omega \backslash \{x = 0\}$ containing ω using unique continuation for uniformly parabolic operators. This will imply that any Fourier component g_n has no singular part and is identically zero on one side of $[-1,1]\backslash\{0\}$. Then, we are left to study a one dimensional equation on the regular part with a boundary inverse square singularity. If $\nu \in \left(0, \frac{1}{2}\right]$, we prove in Subsection 7.3.2 that unique continuation holds thanks to a suitable Carleman-type estimate. Finally, if $\nu \in \left(\frac{1}{2}, 1\right)$, we construct explicit solutions that contradict unique continuation in Subsection 7.3.3.

7.3.1 Reduction to the case of a boundary singularity

The goal of this section is the proof of the following proposition

Proposition 7.6. *Let $T > 0$, $\gamma > 0$, $\nu \in (0,1)$ and ω be an open subset of $(-1,0) \times (0,1)$. Assume that $g^0 \in L^2(\Omega)$ is such that $\chi_\omega S(t)g^0 \equiv 0$ for almost every $t \in [0,T]$. Then $S(t)g^0$ is identically zero on $(-1,0) \times (0,1)$. For any $n \in \mathbb{Z}$, the singular part of the n^{th} Fourier component satisfies $g_{n,s}(t,x) = 0$ for every $(t,x) \in (0,T) \times (-1,1)$.*

Proof of Proposition 7.6. Let $\varepsilon > 0$ be such that

$$\omega \subset \Omega_\varepsilon^- := (-1,-\varepsilon) \times (0,1).$$

For every $t \in [0,T]$,

$$(S(t)g^0)(x,y) = \sum_{n\in\mathbb{Z}} g_n(t,x)\varphi_n(y),$$

where g_n is the solution of (7.8) with initial condition g_n^0.
The solution of (7.17) is defined through an abstract extension operator. We check that on Ω_ε^-, the operator $\mathcal{A}$ is uniformly elliptic. Let $h \in D(\mathcal{A})$ and $\phi \in C_0^\infty(\Omega_\varepsilon^-)$. Then,

$$\begin{aligned}
\langle \mathcal{A}h, \phi\rangle_{L^2(\Omega_\varepsilon^-)} &= \int_{-1}^{-\varepsilon}\int_0^1 \mathcal{A}h(x,y)\phi(x,y)\mathrm{d}y\mathrm{d}x \\
&= -\sum_{n\in\mathbb{Z}} \langle A_n h_n, \phi_n\rangle_{L^2(-1,-\varepsilon)} \\
&= -\sum_{n\in\mathbb{Z}} \langle h_n, A_n \phi_n\rangle_{L^2(-1,-\varepsilon)} \\
&= \langle h, \left(\partial_{xx}^2 + |x|^{2\gamma}\partial_{yy}^2 - \frac{c_\nu}{x^2}\right)\phi\rangle_{L^2(\Omega_\varepsilon^-)}.
\end{aligned}$$

Thus, $h \in D(\mathcal{A})$ implies that

$$\mathcal{A}h \overset{\mathcal{D}'(\Omega_\varepsilon^-)}{=} \left(\partial_{xx}^2 + |x|^{2\gamma}\partial_{yy}^2 - \frac{c_\nu}{x^2}\right) h.$$

As $h \in D(\mathcal{A})$, this equality also holds in $L^2(\Omega_\varepsilon^-)$. In particular, this implies that

$$\partial_{xx}^2 h + |x|^{2\gamma}\partial_{yy}^2 h \in L^2(\Omega_\varepsilon^-),$$

and also that $\mathcal{A}$ is uniformly elliptic on Ω_ε^-. Thus, using classical unique continuation results for uniformly parabolic operators with variable coefficients (see e.g. [125, Theorem 1.1]), it comes that $S(t)g^0 = 0$ for every $t \in (0,T]$ in $L^2(\Omega_\varepsilon^-)$. Then, it comes that $S(t)g^0 = 0$ for every $t \in (0,T]$ in $L^2(\Omega_0^-)$. If, for any $n \in \mathbb{Z}$, we decompose g_n in regular and singular part (respectively $g_{n,r}$ and $g_{n,s}$ as defined in (7.9)) we get

$$c_1^-(g_n(t)) = c_2^-(g_n(t)) = 0, \quad \forall t \in (0,T). \tag{7.18}$$

$$g_{n,r}(t,x) \equiv 0, \quad \forall (t,x) \in (0,T) \times (-1,0), \tag{7.19}$$

Using the transmission conditions in (7.9), it also comes that $c_1^+(g_n(t)) = c_2^+(g_n(t)) = 0$ and thus the singular part is identically zero on $(0,T) \times (-1,1)$. This ends the proof of Proposition 7.6.

□

Remark 7.4. Notice that Proposition 7.6 proves that if ω intersects both connected components of $\Omega\backslash\{x=0\}$, then unique continuation hold for any $\nu \in (0,1)$.

Proposition 7.6 implies that if $\chi_\omega S(t)g^0$ is identically zero then for any $n \in \mathbb{Z}$, $g_n \in C^0((0,T],H^2\cap H_0^1(0,1))\cap C^1((0,T],L^2(0,1))$ is solution of

$$\begin{cases} \partial_t g_n - \partial_{xx}^2 g_n + \left(\dfrac{c_\nu}{x^2} + (2n\pi)^2 x^{2\gamma}\right) g_n = 0, & (t,x)\in(0,T)\times(0,1),\\ g_n(t,0)=g_n(t,1)=0, & t\in(0,T),\\ \partial_x g_n(t,0)=0, & t\in(0,T). \end{cases} \tag{7.20}$$

For $\nu \in \left(0,\frac{1}{2}\right]$, we prove in Subsection 7.3.2 that $g_n \equiv 0$ using a suitable Carleman estimate. For $\nu \in \left(\frac{1}{2},1\right)$ we design in Subsection 7.3.3 explicit non trivial solutions.

7.3.2 Unique continuation for $\nu \in \left(0,\frac{1}{2}\right]$

In this subsection we assume that $\nu \in \left(0,\frac{1}{2}\right]$ and prove the Carleman type inequality stated in Proposition 7.7 below. Let us define the weights that will be used to prove this inequality (see Remark 7.6 for comments on the weights).
Let $\theta : t\in(0,T)\mapsto \frac{1}{t(T-t)}$. Let $p\in C^4([0,1])$ be such that there exist positive constants m_0,m_1,m_2 such that for any $x\in[0,1]$

$$p(x)\geq m_0>0,\quad p_x(x)\geq m_1>0,\quad -p_{xx}(x)\geq m_2>0. \tag{7.21}$$

We set $\sigma(t,x) := \theta(t)p(x)$. For any $n\in\mathbb{Z}$ and any $\gamma>0$, we introduce the following operator

$$\mathcal{P}_n := \partial_t - \partial_{xx}^2 + \left(\frac{c_\nu}{x^2} + (2n\pi)^2|x|^{2\gamma}\right).$$

Then, the following proposition holds.

Proposition 7.7. *Let $T>0$ and $Q_T := (0,T)\times(0,1)$. There exist $R_0, C_0>0$ such that for any $R\geq R_0$, any $g\in C^1((0,T],L^2(0,1))\cap C^0((0,T],H^2\cap H_0^1(0,1))$ with $\partial_x g(t,0)\equiv 0$ on $(0,T)$ satisfies*

$$C_0 \iint_{Q_T} \left(R^3\theta^3 g^2 + R\theta g_x^2\right) e^{-2R\sigma}\mathrm{d}x\mathrm{d}t \leq \iint_{Q_T} |\mathcal{P}_n g| e^{-2R\sigma}\mathrm{d}x\mathrm{d}t. \tag{7.22}$$

Before proving Proposition 7.7 let us point out that it ends the proof of the "if" assertion of Theorem 7.4. Let $g^0\in L^2(\Omega)$ be such that $\chi_\omega S(t)g^0\equiv 0$. Using Proposition 7.6 and the final comment of Subsection 7.3.1, it comes that for any $n\in\mathbb{Z}$, $g_n\in C^1((0,T],L^2(0,1))\cap C^0((0,T],H^2\cap H_0^1(0,1))$ with $\partial_x g_n(t,0)\equiv 0$ on $(0,T)$. As, g_n is solution of (7.20), it comes that $\mathcal{P}_n g_n\equiv 0$ on $(0,T)\times(0,1)$. Then, Proposition 7.7 implies that $g_n\equiv 0$ and thus, as $g\in C^0([0,T],L^2(0,1))$, we recover $g^0=0$.

Remark 7.5. Contrarily to Carleman estimates proved by Vancostenoble [137], there are no boundary terms in the right-hand side of the inequality. Actually, the homogeneous Neumann boundary condition at $x=0$ is crucial for inequality (7.22) to hold.

Proof of Proposition 7.7. We denote the partial derivative by subscripts: z_x stands for $\partial_x z$. We set for $R > 0$,

$$z(t,x) := e^{-R\sigma(t,x)} g(t,x). \tag{7.23}$$

Thus, for any $x \in (0,1)$, $z(0,x) = z(T,x) = z_t(0,x) = z_t(T,x) = 0$. The boundary conditions on g also imply that for any $t \in (0,T)$, $z(t,0) = z(t,1) = z_x(t,0) = 0$. Notice that these boundary conditions imply that $x \mapsto \frac{z(t,x)}{x^2} \in L^2(0,1)$ which justifies the following computations.
Straightforward computations lead to $e^{-R\sigma}\mathcal{P}_n g = P_R^+ z + P_R^- z$ where

$$\begin{aligned}
P_R^+ z &:= (R\sigma_t - R^2\sigma_x^2)z - z_{xx} + \left(\frac{c_\nu}{x^2} + (2n\pi)^2|x|^{2\gamma}\right) z,\\
P_R^- z &:= z_t - 2R\sigma_x z_x - R\sigma_{xx} z.
\end{aligned}$$

Then,

$$\iint_{Q_T} P_R^+ z P_R^- z \mathrm{d}x\mathrm{d}t \le \frac{1}{2}\iint_{Q_T} e^{-2R\sigma}|\mathcal{P}_n g|^2 \mathrm{d}x\mathrm{d}t. \tag{7.24}$$

The rest of the proof follows the classical Carleman strategy [87] (see [54] for a pedagogical presentation). We just pay attention to the singular terms.

First step : integrations by part lead to

$$\begin{aligned}
\iint_{Q_T} P_R^+ z P_R^- z \mathrm{d}x\mathrm{d}t &= R\int_0^T \sigma_x(t,1) z_x^2(t,1)\mathrm{d}t - 2R\iint_{Q_T} \sigma_{xx} z_x^2 \mathrm{d}x\mathrm{d}t\\
&+ \iint_{Q_T} \left(-\frac{R}{2}\sigma_{tt} + 2R^2\sigma_x\sigma_{xt} - 2R^3\sigma_x^2\sigma_{xx} + \frac{R}{2}\sigma_{xxxx}\right) z^2 \mathrm{d}x\mathrm{d}t\\
&+ R\iint_{Q_T} \left(-2\frac{c_\nu}{x^3} + 2\gamma(2n\pi)^2|x|^{2\gamma-1}\right)\sigma_x z^2 \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{7.25}$$

Performing integrations by parts, it is easily seen that $\langle P_R^+ z, P_R^- z\rangle = I_1 + \cdots + I_5$, where

$$\begin{aligned}
I_1 :=& \langle (R\sigma_t - R^2\sigma_x^2)z - z_{xx}, z_t\rangle = \iint_{Q_T} \Big(-\frac{R}{2}\sigma_{tt} + R^2\sigma_x\sigma_{xt}\Big) z^2 \mathrm{d}x\mathrm{d}t,\\
I_2 :=& -R^2\langle \sigma_t z, 2\sigma_x z_x + \sigma_{xx} z\rangle = R^2\iint_{Q_T} \sigma_{xt}\sigma_x z^2 \mathrm{d}x\mathrm{d}t,\\
I_3 :=& R^3\langle \sigma_x^2 z, 2\sigma_x z_x + \sigma_{xx} z\rangle = -R^3\iint_{Q_T} 2\sigma_x^2\sigma_{xx} z^2 \mathrm{d}x\mathrm{d}t,\\
I_4 :=& R\langle z_{xx}, 2\sigma_x z_x + \sigma_{xx} z\rangle\\
=& R\int_0^T \sigma_x(t,1) z_x^2(t,1)\mathrm{d}t - R\iint_{Q_T} 2\sigma_{xx} z_x^2 + \sigma_{xxx} z z_x \mathrm{d}x\mathrm{d}t\\
=& R\int_0^T \sigma_x(t,1) z_x^2(t,1)\mathrm{d}t - R\iint_{Q_T} 2\sigma_{xx} z_x^2 - \frac{1}{2}\sigma_{xxxx} z^2 \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

and

$$I_5 := \langle \Big(\frac{c_\nu}{x^2} + (2n\pi)^2|x|^{2\gamma}\Big) z, z_t - 2R\sigma_x z_x - R\sigma_{xx} z\rangle.$$

Integrations by parts with Lemma 7.4 to estimate the boundary terms lead to

$$\begin{aligned} I_5 &= -R\int_0^T \left[\left(\frac{c_\nu}{x^2}+(2n\pi)^2|x|^{2\gamma}\right)\sigma_x z^2\right]_0^1 \mathrm{d}t + R\iint_{Q_T}\left(\frac{c_\nu}{x^2}+(2n\pi)^2|x|^{2\gamma}\right)_x \sigma_x z^2 \mathrm{d}x\mathrm{d}t \\ &= R\iint_{Q_T}\left(-2\frac{c_\nu}{x^3}+2\gamma(2n\pi)^2|x|^{2\gamma-1}\right)\sigma_x z^2\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Summing these terms leads to (7.25). Combining with (7.24) it comes that

$$\begin{aligned} \frac{1}{2}\iint_{Q_T} e^{-2R\sigma}|\mathcal{P}_n g|^2\mathrm{d}x\mathrm{d}t &\geq R\int_0^T \sigma_x(t,1)z_x^2(t,1)\mathrm{d}t - 2R\iint_{Q_T}\sigma_{xx}z_x^2\mathrm{d}x\mathrm{d}t \\ &+ \iint_{Q_T}\left(-\frac{R}{2}\sigma_{tt}+2R^2\sigma_x\sigma_{xt}-2R^3\sigma_x^2\sigma_{xx}+\frac{R}{2}\sigma_{xxxx}\right)z^2\mathrm{d}x\mathrm{d}t \\ &+ R\iint_{Q_T}\left(-2\frac{c_\nu}{x^3}+2\gamma(2n\pi)^2|x|^{2\gamma-1}\right)\sigma_x z^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{7.26}$$

Second step : lower bounds on the right-hand side of (7.26). Recall that $\sigma(t,x) = \theta(t)p(x)$. Using (7.21) in inequality (7.26) leads to

$$\begin{aligned} \frac{1}{2}\iint_{Q_T} e^{-2R\sigma}|\mathcal{P}_n g|^2\mathrm{d}x\mathrm{d}t &\geq m_1\int_0^T R\theta(t)z_x^2(t,1)\mathrm{d}t + 2m_2R\iint_{Q_T}\theta z_x^2\mathrm{d}x\mathrm{d}t \\ &+ \iint_{Q_T}\left(-2R^3\theta^3p_x^2p_{xx}+2R^2\theta\theta_t p_x^2-\frac{R}{2}\theta_{tt}p+\frac{R}{2}\theta p_{xxxx}\right)z^2\mathrm{d}x\mathrm{d}t \\ &+ R\iint_{Q_T}\theta\left(-2\frac{c_\nu}{x^3}+2\gamma(2n\pi)^2|x|^{2\gamma-1}\right)p_x z^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{7.27}$$

We study these terms separately. As $p_x \geq m_1 > 0$ on $[0,1]$ and $c_\nu \leq 0$, it comes that

$$m_1\int_0^T R\theta(t)z_x^2(t,1)\mathrm{d}t + R\iint_{Q_T}\theta\left(-2\frac{c_\nu}{x^3}+2\gamma(2n\pi)^2|x|^{2\gamma-1}\right)p_x z^2\mathrm{d}x\mathrm{d}t \geq 0, \tag{7.28}$$

each one of these terms being nonnegative. The definition of θ implies the existence of $C > 0$ such that

$$|\theta\theta_t| + |\theta_{tt}| + \theta \leq C\theta^3, \quad \text{on } (0,T).$$

Together with (7.21), this leads to the existence of $\tilde{C} > 0$ such that for R large enough

$$\iint_{Q_T}\left(-2R^3\theta^3p_x^2p_{xx}+2R^2\theta\theta_t p_x^2-\frac{R}{2}\theta_{tt}p+\frac{R}{2}\theta p_{xxxx}\right)z^2\mathrm{d}x\mathrm{d}t \geq \tilde{C}R^3\iint_{Q_T}\theta^3 z^2\mathrm{d}x\mathrm{d}t. \tag{7.29}$$

Using (7.28) and (7.29) in (7.27) it comes that for R large enough

$$\frac{1}{2}\iint_{Q_T} e^{-2R\sigma}|\mathcal{P}_n g|^2\mathrm{d}x\mathrm{d}t \geq 2m_2R\iint_{Q_T}\theta z_x^2\mathrm{d}x\mathrm{d}t + \tilde{C}R^3\iint_{Q_T}\theta^3 z^2\mathrm{d}x\mathrm{d}t. \tag{7.30}$$

Thus, (7.23) leads to

$$\frac{1}{2}\iint_{Q_T} e^{-2R\sigma}|\mathcal{P}_n g|^2 \mathrm{d}x\mathrm{d}t \geq 2m_2 R\iint_{Q_T} \theta g_x^2 e^{-2R\sigma}\mathrm{d}x\mathrm{d}t + \iint_{Q_T} (\tilde{C}R^3\theta^3 - 2m_2R^2\theta^2 p_x^2)g^2 e^{-2R\sigma}\mathrm{d}x\mathrm{d}t.$$

The choice of R large enough ends the proof of Proposition 7.7.

□

Remark 7.6. Let us point some of the differences between Proposition 7.7 and the Carleman estimates established in the case of a boundary inverse square singularity in [138, 137]. In both estimates the singular potential appears as

$$\iint_{Q_T} \frac{\sigma_x}{x^3} z^2 \mathrm{d}x\mathrm{d}t.$$

In [138], the weight is defined by $p(x) = 1 - \frac{x^2}{2}$. Thus, the singular potential can be treated with some Hardy type inequalities. In our situation, we choose the weight as an increasing concave positive function(for example, let us take $p(x) = -x^2 + 4x + 1$). This allows to deal easily with the lower bounds for the boundary term and for the potential $x \mapsto x^{2\gamma}$. However the price to pay is that there is for the singular potential a remaining term of the form

$$\iint_{Q_T} \theta\frac{z^2}{x^3}\mathrm{d}x\mathrm{d}t.$$

This term is dealt with in (7.28) (and is finite) thanks to the boundary condition $\partial_x g(t,0) = 0$.

Adaptation to the case of Dirichlet boundary conditions. Let us explain how we can adapt this unique continuation result to prove the positive result of Theorem 7.2. The Fourier expansion is done in the Hilbert basis of eigenvectors of the Laplace operator on $H^2(0,1)$ with homogeneous Dirichlet boundary condition i.e. $\left(\varphi_n^D(y) := \sqrt{2}\sin(n\pi y)\right)_{n\in\mathbb{N}^*}$. Similarly, the semigroup associated is

$$(S^D(t)f^0)(x,y) := \sum_{n\in\mathbb{N}^*} f_n(t,x)\varphi_n^D(y). \tag{7.31}$$

As the previous results hold true for any coefficient, we recover the results of Propositions 7.5, 7.6 and 7.7 for system (7.5). This holds for $\ell = 1$ and then for any $\ell > 0$ by an obvious change of variables.

7.3.3 Non unique continuation for $\nu \in \left(\frac{1}{2}, 1\right)$

In all this subsection, we assume that $\nu \in \left(\frac{1}{2}, 1\right)$.

7.3.3.1 Periodic boundary conditions on y

The goal of this section is to prove the following proposition.

Proposition 7.8. *Let $\gamma > 0$ and ω be an open subset of $(-1,0)\times(0,1)$. There exists $g^0 \in L^2(\Omega)$ such that the associated solution of (7.17) is not identically zero on Ω and satisfies $\chi_\omega S(t)g^0 \equiv 0$.*

Let J_ν be the Bessel function of first kind of order ν. The following properties of Bessel functions are classical and can be found for example in [140]. The function J_ν is defined on $[0,+\infty)$ by

$$J_\nu(x) := \left(\frac{x}{2}\right)^\nu \sum_{k\in\mathbb{N}} \frac{(-1)^k}{2^{2k}k!\,\Gamma(k+\nu+1)} x^{2k},$$

and solves the following Bessel equation

$$x^2y''(x) + xy'(x) + (x^2-\nu^2)y = 0. \tag{7.32}$$

J_ν possesses an infinite number of positive zeros denoted $j_{\nu,n}$ for $n \in \mathbb{N}^*$. The construction of our explicit counterexample is based on the following lemma.

Lemma 7.1. *For any $\lambda \in \left\{j_{\nu,n}^2\,;\, n\in\mathbb{N}^*\right\}$, the function $b_\lambda(x) := x^{1/2}J_\nu(x\sqrt{\lambda})$ satisfies*

$$\begin{cases} -b_\lambda''(x) + \dfrac{c_\nu}{x^2}b_\lambda(x) = \lambda b_\lambda(x),\\ b_\lambda(0) = b_\lambda(1) = 0,\\ b_\lambda'(0) = 0.\end{cases}$$

Proof of Lemma 7.1. Using (7.32) we get

$$\begin{aligned} -b_\lambda''(x) + \frac{c_\nu}{x^2}b_\lambda(x) &= \frac{-1}{x^{3/2}}\left(\lambda x^2 J_\nu''(x\sqrt{\lambda}) + x\sqrt{\lambda}J_\nu'(x\sqrt{\lambda}) - \nu^2 J_\nu(x\sqrt{\lambda})\right)\\ &= \lambda x^{1/2}J_\nu(x\sqrt{\lambda}). \end{aligned}$$

As $\nu > 0$ it comes that $b_\lambda(0) = 0$. The fact that $\lambda \in \left\{j_{\nu,n}^2\,;\, n\in\mathbb{N}^*\right\}$ implies that $b_\lambda(1) = 0$. As,

$$b_\lambda(x) = \lambda^{\nu/2}\frac{x^{\nu+1/2}}{2^\nu}\sum_{k\in\mathbb{N}}\frac{(-1)^k\lambda^k}{2^{2k}k!\,\Gamma(k+\nu+1)}x^{2k},$$

and $\nu > \frac{1}{2}$ it comes that $b_\lambda'(0) = 0$. This ends the proof of Lemma 7.1.

□

We now prove Proposition 7.8. This ends the proof of Theorem 7.4.

Proof of Proposition 7.8. Let $\lambda \in \left\{j_{\nu,n}^2\,;\, n\in\mathbb{N}^*\right\}$ and b_λ be as in Lemma 7.1. We define

$$g^0 : (x,y)\in\Omega \mapsto b_\lambda(x)\chi_{x\geq 0}(x).$$

Then $g^0 \in L^2(\Omega)$ and for any $n \in \mathbb{Z}\backslash\{0\}$, $g_n^0 \equiv 0$. From Lemma 7.1, it comes that the associated solution of (7.17) is

$$g(t) = e^{-A_0 t}g_0^0 : (x,y) \mapsto e^{-\lambda t}b_\lambda(x)\chi_{x\geq 0}(x).$$

This construction ends the proof of Proposition 7.8.

□

Remark 7.7. Notice that for $\nu \in \left(0, \frac{1}{2}\right]$, the explicit solutions constructed in the previous lemma are still strong solutions but does not satisfy $b'_\lambda(0) = 0$. This enlightens the crucial importance of the functional setting for unique continuation to hold.

As this counterexample is fundamentally based on the coefficient $n = 0$, it does not extend to the case of homogeneous Dirichlet boundary conditions. We design for this case, in the next subsection, a similar counterexample for $\gamma = 1$ and specific values of the length ℓ in the y direction.

Adaptation to the 1D heat equation. Let us point out that the previous study proves Theorem 7.3. Proposition 7.1 for $n = 0$ implies the existence of a mild solution to (7.6) for any initial condition in $L^2(-1, 1)$ and any control $u \in L^2(0, T)$. Proposition 7.2 gives the well posedness of the adjoint system in $D(A)$ for any initial condition in $L^2(-1, 1)$. The arguments developed in Subsection 7.3.1 are automatically adapted to this one dimensional setting. Then, Proposition 7.7 with $n = 0$ gives unique continuation for $\nu \in \left(0, \frac{1}{2}\right]$. The counterexample designed in Proposition 7.8 being based on the one dimensional system for $n = 0$ ends the proof of Theorem 7.3.

7.3.3.2 Homogeneous Dirichlet boundary conditions on y

In all what follows, we assume that $\gamma = 1$. Recall that the semigroup S^D associated to Dirichlet boundary conditions is defined in (7.31). We end the proof of Theorem 7.2 with the following proposition

Proposition 7.9. *There exists $\ell > 0$ such that for any subset ω of $(-1, 0) \times (0, \ell)$, there exists $g^0 \in L^2(\Omega)$ such that $S^D(t)g^0$ is not identically zero on Ω and satisfies $\chi_\omega S^D(t)g^0 \equiv 0$.*

Following the study of Proposition 7.8, we prove that there exists $m > 0$ and $g \not\equiv 0$ such that

$$\begin{cases} \partial_t g - \partial^2_{xx} g + \dfrac{c_\nu}{x^2} g + m^2 x^2 g = 0, \\ g(t, 0) = g(t, 1) = 0, \\ \partial_x g(t, 0) = 0. \end{cases} \tag{7.33}$$

Let $M_{k,\mu}$ be the Whittaker function of first type with parameters $k \geq 1$ and $\mu > 0$. The properties of Whittaker functions can be found for example in [66, Sections 13.14 and 13.22]. The function $M_{k,\mu}$ is defined on $[0, +\infty)$ by

$$M_{k,\mu}(x) := x^{\mu+1/2} e^{-x/2} \frac{\Gamma(2\mu + 1)}{\Gamma(\mu - k + 1/2)} \sum_{n \in \mathbb{N}} \frac{\Gamma(n + \mu - k + 1/2)}{n!\, \Gamma(n + 2\mu + 1)} x^n,$$

and solves the following Whittaker equation

$$-y''(x) + \left(\frac{1}{4} - \frac{k}{x} + \frac{\mu^2 - 1/4}{x^2}\right) y(x) = 0. \tag{7.34}$$

Lemma 7.2. *Let $\lambda > 0$ be such that $M_{1,\frac{\nu}{2}}\left(\frac{\lambda}{4}\right) = 0$ and $m := \frac{\lambda}{4}$. Then, the function $w_\lambda(x) := x^{-1/2} M_{1,\frac{\nu}{2}}\left(\frac{\lambda}{4}x^2\right)$ satisfies*

$$\begin{cases} -w_\lambda''(x) + \dfrac{c_\nu}{x^2} w_\lambda(x) + m^2 x^2 w_\lambda(x) = \lambda w_\lambda(x), \\ w_\lambda(0) = w_\lambda(1) = 0, \\ w_\lambda'(0) = 0. \end{cases}$$

Thus, as Lemma 7.1 implied Proposition 7.8 it directly comes that if ℓ satisfies $\frac{n\pi}{\ell} = m$ for some $n \in \mathbb{N}^*$, Lemma 7.2 implies Proposition 7.9. Then, there is an infinite number of values of ℓ such that Proposition 7.9 holds.

Proof of Lemma 7.2. By [66, Section 13.22], if $\frac{1}{2} + \mu - k < 0$ and $1 + 2\mu > 0$, then $M_{k,\mu}$ admits a zero in $(0, +\infty)$. As $\nu \in \left(\frac{1}{2}, 1\right)$, there exists $\lambda > 0$ such that $M_{1,\frac{\nu}{2}}\left(\frac{\lambda}{4}\right) = 0$. Straightforward computations lead to

$$w_\lambda''(x) = \frac{3}{4x^{5/2}} M_{1,\frac{\nu}{2}}\left(\frac{\lambda}{4}x^2\right) + \frac{\lambda^2 x^{3/2}}{4} M_{1,\frac{\nu}{2}}''\left(\frac{\lambda}{4}x^2\right).$$

Thus, using (7.34), it comes that

$$-w_\lambda''(x) + \frac{c_\nu}{x^2} w_\lambda(x) + m^2 x^2 w_\lambda(x) = \lambda x^{-1/2} M_{1,\frac{\nu}{2}}\left(\frac{\lambda}{4}x^2\right).$$

Recall that

$$w_\lambda(x) = x^{\nu+\frac{1}{2}} \left(\frac{\lambda}{4}\right)^{\frac{\nu+1}{2}} e^{-\frac{\lambda}{2}x^2} \frac{\Gamma(\nu+1)}{\Gamma((\nu-1)/2)} \sum_{n\in\mathbb{N}} \frac{\Gamma(n+(\nu-1)/2)}{n!\,\Gamma(n+\nu+1)} \left(\frac{\lambda}{4}\right)^n x^{2n}.$$

Thus, as $\nu > \frac{1}{2}$, it comes that $w_\lambda(0) = w_\lambda'(0) = 0$. The choice of λ implies that $w_\lambda(1) = 0$. This ends the proof of Lemma 7.2.

□

Remark 7.8. Notice that for $\nu \in \left(0, \frac{1}{2}\right]$, the explicit solutions constructed in the previous proposition does not satisfy $w_\lambda'(0) = 0$.

As our strategy relies on explicit counterexamples, the restriction $\gamma = 1$ and particular values of ℓ seems only technical and we conjecture that for system (7.5), unique continuation does not hold for any $\gamma > 0$ and any value of $\ell > 0$.

7.4 Conclusion, open problems and perspectives

In this paper we have investigated the approximate controllability properties for a 2D Grushin-like equation which presents both a degeneracy and an inverse square singularity on the internal set $\{x = 0\}$. As the associated operator possesses several self-adjoint extensions, the functional setting in which we study the well posedness and unique continuation for the adjoint system is crucial. This functional setting relies on a precise study of the 1D associated operators.

We prove a necessary and sufficient condition on the coefficient c_ν of the potential $\frac{c_\nu}{x^2}$ for unique continuation to hold. The positive result is proved using classical unique continuation results for uniformly parabolic operators and a 1D Carleman type estimate that holds due to the construction of the functional setting. The negative result is proved by designing an explicit counterexample based on Bessel functions. These results have been extended to homogeneous Dirichlet boundary conditions in the y direction. The negative result in this setting for $\nu \in \left(\frac{1}{2}, 1\right)$, for any $\gamma > 0$ and any $\ell > 0$ remains an open problem.

An interesting open problem coming from this work is the question of null controllability in the case $\nu \in \left(0, \frac{1}{2}\right]$. The classical strategy would be to prove uniform observability for the 1D adjoint systems. This has been done in the case where there is no singular potential in [14] and with a singular potential for the one-side problem in [37]. The Carleman type estimate we proved in this paper might not be directly used as it holds true only for the regular part of the coefficient g_n. Dealing with the singular part in Carleman type estimates is quite tricky as we cannot perform integrations by part on the singular part. The other difficulty relies on the fact that we want these estimates to be uniform with respect to n.

7.A One dimensional operator

This appendix is dedicated to the proof of Proposition 7.1 where we investigate the self-adjointness and positivity properties of the operator associated to the one dimensional problem (7.8). The proof uses the following two lemmas.

Lemma 7.3. *For $f, g \in \tilde{H}_0^2(-1,1) \oplus \mathcal{F}_s$, if we define*

$$[f,g](x) := (fg' - f'g)(x), \quad \forall x \neq 0,$$

then

$$\int_{-1}^{1} \left(-\partial_{xx}^2 f + \frac{c_\nu}{x^2} f\right)(x) g(x) \mathrm{d}x = \int_{-1}^{1} f(x) \left(-\partial_{xx}^2 g + \frac{c_\nu}{x^2} g\right)(x) \mathrm{d}x + [f,g](1) - [f,g](0^+) + [f,g](0^-) - [f,g](-1).$$

Proof of Lemma 7.3. See [142, Lemma 9.2.3]. □

The following lemma characterizes the behaviour of the regular part at the singularity.

Lemma 7.4. *For $f \in \tilde{H}_0^2(-1,1)$,*

$$\lim_{x\to 0} \frac{f(x)}{|x|^{3/2}} = 0 \quad \textit{and} \quad \lim_{x\to 0} \frac{f'(x)}{|x|^{1/2}} = 0.$$

Proof of Lemma 7.4. As $f(0) = f'(0) = 0$, it comes that

$$f(x) = \int_0^x \int_0^t f''(s) \mathrm{d}s \mathrm{d}t.$$

Then, Cauchy-Schwarz inequality implies,

$$|f(x)| \leq \left| \int_0^x \sqrt{t} \left| \int_0^t |f''(s)|^2 \mathrm{d}s \right|^{1/2} \mathrm{d}t \right| \leq \frac{2}{3} \left| \int_0^x |f''(s)|^2 \mathrm{d}s \right|^{1/2} |x|^{3/2}.$$

The proof of the second limit is similar.

□

We now turn to the proof of Proposition 7.1.

Proof of Proposition 7.1. We start by proving that $(A_n, D(A))$ is a symmetric operator. Thus, A_n^* is an extension of A_n and self-adjointness will follow from the equality $D(A_n^*) = D(A_n)$.

First step : we prove that $(A_n, D(A))$ is a symmetric operator.
Let $f, g \in D(A)$. As $f(1) = g(1) = f(-1) = g(-1) = 0$, it comes that

$$[f, g](1) = [f, g](-1) = 0.$$

Lemma 7.4 imply that

$$[f, g](0^+) = [f_s, g_s](0^+) = \big(c_1^+(f)c_2^+(g) - c_2^+(f)c_1^+(g)\big)[|x|^{\nu+1/2}, |x|^{-\nu+1/2}](0^+),$$

and

$$\begin{aligned}[f, g](0^-) &= [f_s, g_s](0^-)\\ &= \big(c_1^-(f)c_2^-(g) - c_2^-(f)c_1^-(g)\big)[|x|^{\nu+1/2}, |x|^{-\nu+1/2}](0^-)\\ &= -\big(c_1^-(f)c_2^-(g) - c_2^-(f)c_1^-(g)\big)[|x|^{\nu+1/2}, |x|^{-\nu+1/2}](0^+).\end{aligned}$$

The transmission conditions on the coefficients of the singular part given in (7.9) can be rewritten as

$$\begin{pmatrix} c_1^+(f) \\ c_2^+(f) \end{pmatrix} = \frac{-1}{2\nu} \begin{pmatrix} -1 & 2\nu - 1 \\ 2\nu + 1 & 1 \end{pmatrix} \begin{pmatrix} c_1^-(f) \\ c_2^-(f) \end{pmatrix}, \quad \forall f \in D(A). \tag{7.35}$$

Thus, for any $f, g \in D(A)$

$$c_1^+(f)c_2^+(g) - c_2^+(f)c_1^+(g) = -\big(c_1^-(f)c_2^-(g) - c_2^-(f)c_1^-(g)\big).$$

This leads to

$$[f, g](0^+) = [f, g](0^-).$$

Finally, Lemma 7.3 imply that for any $f, g \in D(A)$, $\langle A_n f, g\rangle = \langle f, A_n g\rangle$.

Thus, to prove self-adjointness it remains to prove that $D(A_n^*) = D(A)$. As $D(A)$ is independent of n and $x \mapsto (2n\pi)^2|x|^{2\gamma} \in L^\infty(-1, 1)$ it comes that $D(A_n^*) = D(A_0^*)$.

Second step : minimal and maximal domains. First, we explicit the minimal and maximal domains in the case of a boundary singularity. Without loss of generality, we study the operator in $(0, 1)$.
Using [2, Proposition 3.1], the minimal and maximal domains associated to the differential expression A_0 in $L^2(0, 1)$ are respectively equal to

$$H_0^2([0, 1]) := \big\{y \in H^2([0, 1])\,;\, y(0) = y(1) = y'(0) = y'(1) = 0\big\}$$

and

$$\big\{y \in H^2([0, 1])\,;\, y(0) = y'(0) = 0\big\} \oplus \text{Span}\left\{x^{\nu+1/2},\, x^{-\nu+1/2}\right\}.$$

Then, [142, Lemma 13.3.1] imply that the minimal and maximal domains associated to A_0 on the interval $(-1,1)$ are given by

$$D_{min} := \left\{ f \in \tilde{H}_0^2(-1,1)\,;\, f(-1) = f(1) = f'(-1) = f'(1) = 0 \right\}, \tag{7.36}$$

and

$$D_{max} := \tilde{H}_0^2(-1,1) \oplus \mathcal{F}_s. \tag{7.37}$$

Besides, the minimal and maximal operators form an adjoint pair

Third step : self-adjointness. The operator A_0 being a symmetric extension of the minimal operator it comes that $D(A_0) \subset D(A_0^*) \subset D_{max}$. Let $g \in D(A_0^*)$ be decomposed as $g = g_r + g_s$ with $g_r \in \tilde{H}_0^2(-1,1)$ and $g_s \in \mathcal{F}_s$. We prove that g satisfy the boundary and transmission conditions. By the definition of $D(A_0^*)$, there exists $c > 0$ such that for any $f \in D(A)$,

$$|\langle A_0 f, g\rangle| \leq c||f||_{L^2}.$$

Let $f \in D(A) \cap \tilde{H}_0^2(-1,1)$ be such that $f \equiv 0$ in $(-1,0)$. Then, Lemma 7.3 implies that

$$\langle A_0 f, g\rangle = \langle f, A_0 g\rangle + [f,g](1) = \langle f, A_0 g\rangle + f'(1)g(1).$$

Thus, $g(1) = 0$. Symmetric arguments imply that $g(-1) = 0$.
We now turn to the transmission conditions. Let $f \in D(A)$ be such that its singular part is given by

$$c_1^+(f) := \frac{1}{2\nu}, \quad c_2^+(f) := -\frac{1}{2\nu}.$$

Then, the transmission conditions imply

$$c_1^-(f) = \frac{1}{2\nu}, \quad c_2^-(f) = -\frac{1}{2\nu}.$$

By Lemma 7.3

$$\langle A_0 f, g\rangle = \langle f, A_0 g\rangle + [f,g](0^-) - [f,g](0^+).$$

Using Lemma 7.4 it comes that $[f,g](0^-) = [f_s,g_s](0^-)$ and $[f,g](0^+) = [f_s,g_s](0^+)$. Straightforward computations lead to

$$[f,g](0^+) = -c_1^+(g) - c_2^+(g), \quad [f,g](0^-) = c_1^-(g) + c_2^-(g).$$

We thus recover the first transmission condition. The second transmission condition follow from the same computations with the choice of a particular $f \in D(A)$ satisfying

$$c_1^+(f) := -\frac{\nu - 1/2}{2\nu}, \quad c_2^+(f) := -\frac{\nu + 1/2}{2\nu}.$$

Finally, this proves that $(A_n, D(A))$ is a self-adjoint operator.

Fourth step : positivity. We end the proof of Proposition 7.1 by proving that for any $f \in D(A)$, $\langle A_n f, f\rangle \geq 0$. Let $f \in D(A)$.

Using Lemma 7.3 and integration by parts it comes that

$$\begin{aligned}\langle A_n f, f\rangle &= \int_{-1}^{1}\Big(-\partial_{xx}^2 f_r + \frac{c_\nu}{x^2} f_r\Big)(x) f(x)\mathrm{d}x + \int_{-1}^{1}(2n\pi)^2|x|^{2\gamma} f^2(x)\mathrm{d}x,\\ &= \int_{-1}^{1}(\partial_x f_r)^2(x) + \frac{c_\nu}{x^2} f_r^2(x)\mathrm{d}x + \int_{-1}^{1}(2n\pi)^2|x|^{2\gamma} f^2(x)\mathrm{d}x + (-\partial_x f_r)(1) f_r(1)\\ &\quad + \partial_x f_r(-1) f_r(-1) + [f_r, f_s](1) - [f_r, f_s](0^+) + [f_r, f_s](0^-) - [f_r, f_s](-1).\end{aligned}$$

Using Lemma 7.4, it comes that $[f_r, f_s](0^+) = [f_r, f_s](0^-) = 0$. Gathering the boundary terms and using $f(1) = f(-1) = 0$ it comes that

$$\begin{aligned}\langle A_n f, f\rangle = &\int_{-1}^{1}(\partial_x f_r)^2(x) + \frac{c_\nu}{x^2} f_r^2(x)\mathrm{d}x + \int_{-1}^{1}(2n\pi)^2|x|^{2\gamma} f^2(x)\mathrm{d}x\\ &+ f_r(1)\partial_x f_s(1) - f_r(-1)\partial_x f_s(-1).\end{aligned}$$

The transmission conditions on $c_1^+, c_2^+, c_1^-, c_2^-$ in $D(A)$ impose that

$$\begin{aligned}f_r(1)\partial_x f_s(1) &= -\big(c_1^+(f) + c_2^+(f)\big)\left(\left(\nu + \frac{1}{2}\right)c_1^+(f) + \left(-\nu + \frac{1}{2}\right)c_2^+(f)\right)\\ &= f_r(-1)\partial_x f_s(-1).\end{aligned}$$

Thus, using Hardy inequality (7.7)

$$\begin{aligned}\langle A f, f\rangle &\geq \int_{-1}^{1}(\partial_x f_r)^2(x) + \frac{c_\nu}{x^2} f_r^2(x)\mathrm{d}x + \int_{-1}^{1}(2n\pi)^2|x|^{2\gamma} f^2(x)\mathrm{d}x,\\ &\geq m_\nu \int_{-1}^{1}(\partial_x f_r)^2(x)\mathrm{d}x + \int_{-1}^{1}(2n\pi)^2|x|^{2\gamma} f^2(x)\mathrm{d}x, \qquad (7.38)\end{aligned}$$

where $m_\nu := \min\{1, 4\nu^2\}$. This ends the proof of Proposition 7.1.

□

7.B Abstract self adjoint extensions

This appendix is dedicated to enlighten the choices made in the construction of the functional setting leading to the definition (7.9) of $D(A)$.

The question of finding the self-adjoint extensions of a given closed symmetric operator is classical. In [120, Theorem X.2] such extensions are characterized by means of isometries between the deficiency subspaces. The particular case of Sturm-Liouville operators has been widely studied : most of these result are contained in [142]. The self-adjoint extensions are characterized by means of boundary conditions. In our case, we are concerned with the Sturm-Liouville operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{c_\nu}{x^2}$ on the interval $(-1, 1)$. This fits in the setting of [142, Chapter 13]. The number of boundary conditions to impose is given by the deficiency index. Following [2, Proposition 3.1], it comes that our operator on the interval $(0, 1)$ has deficiency index 2. This is closely related to the fact that $\nu \in (0, 1)$. Then, [142, Lemma 13.3.1] implies that the deficiency index for the interval $(-1, 1)$ is 4. We thus get the following proposition which is simply a rewriting of [142, Theorem 13.3.1 Case 5].

Proposition 7.10. *Let u and v in D_{max} be such that their restriction on $(0,1)$ (resp. $(-1,0)$) are linearly independent modulo $H_0^2(0,1)$ (resp. $H_0^2(-1,0)$) and*

$$[u,v](-1) = [u,v](0^-) = [u,v](0^+) = [u,v](1) = 1.$$

Let $M_1, \dots, M_4$ be 4×2 complex matrices. Then every self-adjoint extension of the minimal operator is given by the restriction of D_{max} to the functions f satisfying the boundary conditions

$$M_1 \begin{pmatrix} [f,u](-1) \\ [f,v](-1) \end{pmatrix} + M_2 \begin{pmatrix} [f,u](0^-) \\ [f,v](0^-) \end{pmatrix} + M_3 \begin{pmatrix} [f,u](0^+) \\ [f,v](0^+) \end{pmatrix} + M_4 \begin{pmatrix} [f,u](1) \\ [f,v](1) \end{pmatrix} = 0,$$

where the matrices satisfy $(M_1\, M_2\, M_3\, M_4)$ has full rank and

$$M_1 E M_1^* - M_2 E M_2^* + M_3 E M_3^* - M_4 E M_4^* = 0, \text{ with } E := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Conversely, every choice of such matrices defines a self-adjoint extension.

We end this appendix by giving the choices of such matrices that we made and give another functional setting that would lead to well posedness but that is not adapted to controllability issues. We define on $(0,1)$ u and v to be solutions of

$$-f''(x) + \frac{c_\nu}{x^2} f(x) = 0$$

with $(u(1) = 0, u'(1) = 1)$ and $(v(1) = -1, v'(1) = 0)$ i.e.

$$u(x) = \frac{1}{2\nu} x^{\nu+1/2} - \frac{1}{2\nu} x^{-\nu+1/2},$$
$$v(x) = -\frac{\nu - 1/2}{2\nu} x^{\nu+1/2} - \frac{\nu+1/2}{2\nu} x^{-\nu+1/2}.$$

Thus for any $f \in D_{max}$, $[f,u](1) = f(1)$ and $[f,v](1) = f'(1)$, and for any $x \in [0,1]$, $[u,v](x) \equiv 1$. We design u and v similarly on $(-1,0)$ i.e.

$$u(x) = -\frac{1}{2\nu} |x|^{\nu+1/2} + \frac{1}{2\nu} |x|^{-\nu+1/2},$$
$$v(x) = -\frac{\nu - 1/2}{2\nu} |x|^{\nu+1/2} - \frac{\nu+1/2}{2\nu} |x|^{-\nu+1/2}.$$

Due to the choice of functions u and v, the homogeneous Dirichlet conditions at ± 1 are implied by the choice

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \; M_2 = \begin{pmatrix} 0 \quad 0 \\ \tilde{M}_2 \\ 0 \quad 0 \end{pmatrix}, \; M_3 = \begin{pmatrix} 0 \quad 0 \\ \tilde{M}_3 \\ 0 \quad 0 \end{pmatrix}, \; M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then, the conditions of Proposition 7.10 are satisfied if and only if the matrix $(\tilde{M}_2\, \tilde{M}_3)$ has rank 2 and $\det(\tilde{M}_2) = \det(\tilde{M}_3)$. Straightforward computations lead to, for any $f \in D_{max}$

$$[f,u](0^+) = c_1^+ + c_2^+, \qquad [f,v](0^+) = \left(\nu + \frac{1}{2}\right) c_1^+ + \left(-\nu + \frac{1}{2}\right) c_2^+,$$
$$[f,u](0^-) = c_1^- + c_2^-, \qquad [f,v](0^-) = -\left(\nu + \frac{1}{2}\right) c_1^- - \left(-\nu + \frac{1}{2}\right) c_2^-.$$

Thus, the choice $\tilde{M}_2 = \tilde{M}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ lead to the definition of $D(A)$ in (7.9). The computations done in the fourth step of the proof of Proposition 7.1 (see (7.38)) prove the positivity and thus, Proposition 7.1 could also be seen as an application of Proposition 7.10.
At this stage, there is another choice that would lead to a self-adjoint positive extension. If, we set $\tilde{M}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\tilde{M}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then the domain with conditions

$$c_1^+ = -c_2^+, \quad c_1^- = -\frac{-\nu + 1/2}{\nu + 1/2} c_2^-,$$

give rise to a self-adjoint positive operator. However, from a point of view of controllability, this domain does not seem interesting as this conditions couple the coefficients on each side on the singularity and there is no transmission of information through the singular set. In particular, we cannot apply the results developed in this article to this functional setting.

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Résumé

Ce mémoire présente les travaux réalisés au cours de ma thèse sur le contrôle d'équations aux dérivées partielles.

La première partie est consacrée à l'étude d'équations de Schrödinger bilinéaires unidimensionnelles autour de deux axes : la non contrôlabilité en temps petit avec des contrôles petits et la contrôlabilité simultanée. On établit un cadre pour lequel, bien que la vitesse de propagation du système soit infinie, la contrôlabilité exacte locale avec des contrôles petits est vérifiée si et seulement si le temps est suffisamment grand. Ces résultats, basés sur la coercivité d'une forme quadratique associée au développement de l'état à l'ordre deux, sont étendus dans le contexte de la contrôlabilité simultanée. On montre alors, en utilisant la méthode du retour de J.-M. Coron, des résultats de contrôle exact local simultané pour deux ou trois équations, à phase globale et/ou à retard global près. La trajectoire de référence utilisée est construite via des résultats de contrôle partiel. En utilisant un argument de perturbation, on étend cette idée pour montrer la contrôlabilité exacte globale d'un nombre quelconque d'équations sans hypothèse sur le potentiel.

Dans la deuxième partie, on prend en compte dans le modèle un terme supplémentaire quadratique en le contrôle. Ce terme, dit de polarisabilité, généralement négligé, présente un intérêt physique dans la modélisation, mais aussi mathématique dans le cas où le terme bilinéaire est insuffisant pour conclure à la contrôlabilité. En dimension quelconque, on construit des contrôles explicites réalisant la contrôlabilité approchée de l'état fondamental. En adaptant conjointement l'argument de perturbation précédent et certains résultats du contrôle bilinéaire, on prouve la contrôlabilité globale exacte de l'équation de Schrödinger avec polarisabilité 1D.

La dernière partie de ce mémoire est consacrée à l'étude de la continuation unique pour un opérateur de type Grushin sur un rectangle 2D. Cet opérateur présente une singularité et une dégénérescence sur un segment séparant le domaine en deux composantes. On donne une condition nécessaire et suffisante sur le coefficient du potentiel singulier pour obtenir la continuation unique.

Abstract

This memoir presents the achievements of my thesis on the control of partial differential equations.

The first part mainly deals with two aspects of bilinear Schrödinger equations : negative controllability results in small time with small controls and simultaneous controllability. We propose a general setting for the existence of a positive minimal time for local exact controllability to hold with small controls. The negative result, based on the coercivity of a quadratic form associated to a second order power series expansion, is extended to simultaneous controllability. Using J.-M. Coron's return method, we prove simultaneous local exact controllability for two or three equations, up to a global phase and/or up to a global delay. The reference trajectory is designed using partial control results. Using a perturbation argument, this idea is extended to prove simultaneous global exact controllability of an arbitrary number of equations without restrictions on the potential.

In the second part, we add, in the model, a polarizability term which is quadratic with respect to the control. Taking into account this physically meaningfull term, usually neglected, is interesting when the dipole moment is not sufficient to control the particle. In any space dimension, we obtain approximate controllability towards the ground state with explicit controls. The previous perturbation argument together with tools from bilinear control lead to global exact controllability of the 1D Schrödinger equation with a polarizability term.

The last part deals with a Grushin-type operator on 2D rectangle. This operator is both degenerate and singular on a line that separates the domain in two components. A necessary and sufficient condition on the coefficient of the singular potential for unique continuation to hold.