Hamilton-Jacobi-Bellman approach for optimal control problems with discontinuous coefficients

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Contents

Abbreviations

DPP	Dynamical programming principle
HJ	Hamilton-Jacobi
HJB	Hamilton-Jacobi-Bellman
lsc	lower semicontinuous
usc	upper semicontinuous

Notations

$\mathbb{R}^d$	the Euclidean d-dimensional space
$\mathcal{K}$	a closed subset of $\mathbb{R}^d$ as state constraints
Ω	an open subset of $\mathbb{R}^d$
$\mathcal{M}$	an open C^2 embedded manifold in $\mathbb{R}^d$
$\mathcal{T}_{\mathcal{M}}(x)$	the Bouligand's tangent cone of $\mathcal{M}$ at x
$B(x,r)$	the open ball centered at x with the radius r
$V_a^b(f)$	the total variation of the function $f : [a,b] \to \mathbb{R}^d$
$BV([a,b];\mathbb{R}^d)$	the set of functions $f : [a,b] \to \mathbb{R}^d$ with bounded total variation
$AC([a,b];\mathbb{R}^d)$	the set of absolutely continuous functions $f : [a,b] \to \mathbb{R}^d$
$f(t^+)$	the right limit of f at t
$f(t^-)$	the left limit of f at t
$[f]_t$	$f(t^+) - f(t^-)$
$\mathcal{E}p(u)$	the epigraph of u
$\mathcal{H}p(u)$	the hypograph of u
$a \vee b$	$\max(a,b)$
co E	the convex hull of the set E
$\overline{co}$ E	the closed convex hull of the set E

Dedicated to my family

Chapter 1

General introduction

The purpose of the present thesis is to study the deterministic optimal control problems with discontinuous coefficients via the Hamilton-Jacobi-Bellman (HJB) approach concerned with first order partial differential equations.

The optimal control theory is a mathematical optimization problem consisting of the problem of finding a control strategy for a given controlled dynamical system such that a certain optimality criterion is achieved. More precisely, let us start by studying the optimal control problems of the following form: given $T > 0$ and a group of control functions $\mathcal{A}$, for any $t \in [0, T]$ and $x \in \mathbb{R}^d$, consider the optimization problem

$$\inf \left\{ \varphi(y^\alpha_{t,x}(T)) \ : \ \alpha \in \mathcal{A} \right\},$$

where $y^\alpha_{t,x}$ is the solution of the controlled dynamical system

$$\dot{y}(s) = f(s, y(s), \alpha(s)) \text{ for } s \in (t, T), \ y(t) = x.$$

The functions f and φ represent respectively the dynamics and the cost. The essential problem is to search an optimal control strategy α such that the final cost of the associated trajectory $y^\alpha_{t,x}$ is minimized.

Stimulated greatly by the aerospace engineering applications, the study of optimal control problems has systematically started from the late 1950s. Among such applications was the problem of optimal flight trajectories for aircraft and space vehicles. However, the potential of applications covers a much wider range of fields, including engineering, chemical processing, biomedicine, vehicles control, economics, etc. One prominent advance is Dynamical Programming and HJB approach developed by Bellman during the 1950's. The first step in this approach is to introduce the *value function*, denoted by $v(t, x)$, which is the optimal value of the optimization problem. Then the fundamental idea is that v satisfies a functional equation, often called the *Dynamical Programming Principle* (DPP). From this DPP, when v is smooth enough, we can derive an appropriate HJB equation for the value function:

$$\begin{cases} -\partial_t v(t,x) + H(t,x,Dv(t,x)) = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ v(T,x) = \varphi(x) & \text{in } \mathbb{R}^d, \end{cases}$$

where H is called the *Hamiltonian* with the Bellman form

$$H(t,x,p) = \sup_{a\in A}\{-p\cdot f(t,x,a)\}.$$

Here A is the set in which the control functions α take value. This HJB equation contains all the relevant information to compute the value function and to design the optimal control strategy.

However, the problems are generally nonlinear and therefore, do not have analytic solutions. Besides, the derived HJB equation of optimal control is usually a nonlinear partial differential equation for which the traditional notions of weak solutions, based on the theory of distributions, are not adequate.

Two important breakthroughs occurred in the 1970's and the early 1980's which allow to deal with the value function which usually lacks smoothness. One was the theory of viscosity solutions, initiated by the papers of Crandall and Lions [65, 66], Crandall, Evans and Lions [62] and Lions [109]. They introduced a weak formulation for the generalized solutions of Hamilton-Jacobi (HJ) equations, which are called *viscosity solutions*. This theory provides a framework for proving existence, uniqueness and stability of viscosity solutions to broad classes of nonlinear partial differential equations, including the HJ equations arising from optimal control. The paper of Crandall, Ishii and Lions [63] provides a survey of the development of the theory, and we also would like to refer to Barles [21], Bardi and Capuzzo-Dolcetta [19], and Fleming and Soner [88] among various books on this theory. Another breakthrough is the nonsmooth analysis based on Clarke's generalized gradients. It refers to differential analysis in the absence of differentiability, and provides another approach to the problems of nonsmooth calculus of variations, in particular optimal control problems. See Aubin and Cellina [15], Aubin and Frankowska [16], Frankowska [79, 80], Clarke [58, 59], Clarke et al. [60, 61], Rockafellar and Wets [121], Vinter [131] for the fundamental theory of nonsmooth analysis and its applications to optimization and control theory.

Both the theory of viscosity solutions and the tools of nonsmooth analysis clear the bottleneck of HJB approach dealing with the value function in absence of smoothness. The value function, usually being Lipschitz continuous, can be characterized as the unique viscosity solution of HJB equation with the Hamiltonian being Lipschitz continuous. The Lipschitz character of the Hamiltonian is very important to apply the viscosity theory and the nonsmooth analysis to optimal control problems.

The theory of viscosity solutions has been developed later including solutions that are not necessarily continuous. The definition of discontinuous viscosity solutions was first introduced in Ishii [103], as well as the connection with optimal control. The first uniqueness result for discontinuous viscosity solutions is given in Barles and Perthame [27]. An important development is the theory of *bilateral viscosity solutions* originating from Barron and Jensen [31, 32] and revisited by Barles [20]. A different approach to control problems with discontinuous value function was pursued in Frankowska [81] using nonsmooth analysis.

The HJB approach has been extended later for a more general class of optimal control problems. One important direction is for the problems with state constraints where the trajectories of the controlled dynamical system must verify a state-space constraint, that is they have to stay in a given set for all time. More precisely, given a closed set $\mathcal{K} \subset \mathbb{R}^d$, we consider only the admissible trajectories with $y^{\alpha}_{t,x}(s) \in \mathcal{K}$ for all $s \in [t,T]$. In this case, the value function that we are interested in is the minimal value of the final cost of the admissible trajectories, and the problem is to characterize the value function via the appropriate HJB equations and boundary conditions.

The theory of *constrained viscosity solutions* was developed in several directions in Soner [126, 127], Capuzzo-Dolcetta and Lions[64], Ishii and Koike [105], and Soravia [128]. It is known that in presence of state constraints, the continuity of the value function is no longer satisfied unless a special controllability assumption is satisfied by the dynamics on the boundary of state constraints. It is called "inward pointing qualification condition (IQ)" first introduced by Soner in [126]. It asks that at each point of $\mathcal{K}$, there exists a field of the system pointing inward $\mathcal{K}$. Under this assumption, the value function is the unique continuous constrained viscosity solution to an HJB equation, see the mentioned [105, 126, 127] and also Capuzzo-Dolcetta and Lions [64], Motta [112].

Unfortunately, in many control problems, the condition (IQ) is not satisfied and the value function could be discontinuous. In this framework, another controllability assumption, called "outward pointing qualification condition (OQ)", was introduced in Blanc [37], Frankowska and Plaskacz[84], Frankowska and Vinter [86]. It states that every point on the boundary of state constraints $\mathcal{K}$ can be reached by a trajectory coming from the interior of $\mathcal{K}$. Under this assumption, the value function can be characterized as the unique discontinuous bilateral viscosity solution of an HJB equation. There are some recent work on the problems under weaker conditions than (IQ) and (OQ), see Bokanowski, Forcadel and Zidani [40], Frankowska and Mazzola [82, 83].

An important setting for solving optimal control problems via HJB approach is the regularity setting of the dynamics f and the cost φ. The regularity of f and φ has a significant impact on the regularity of the value function v and the Hamiltonian H. It turns out that the classical HJB theory may not work if v and H lack some properties of continuity. When the value function v is not continuous, we have mentioned that the bilateral viscosity theory can be applied to deal with this problem. However, if the Hamiltonian H is not continuous, due to the lack of continuity of the dynamics f, the problem is much more complicated.

The field of dynamical systems and Hamilton-Jacobi equations with discontinuous coefficients is of growing interest both from theoretical point of view and from the potential applications. It appears in the modeling of problems in various domains, such as mechanical systems with impacts, Fadaray waves, synaptic activity in neuroscience, ray light propagation in an inhomogeneous medium with discontinuous refraction index, traffic flow problems, etc. In this area, the well-posedness of the problem is not evident owing to some discontinuous settings and the characterization of the value function by the corresponding HJB equation remains a difficult issue.

The leading theme of the thesis is to develop the HJB approach for a general class of optimal control problems in discontinuous settings. The study of the thesis involves essentially two types of difficulty: the problems in presence of time discontinuity and the problems in presence of state discontinuity. The project of the thesis contains three parts: HJB approach for problems with discontinuity in time and in presence of state constraints, HJB approach for problems with discontinuity in state, and homogenization problems with discontinuity in state. In part I and II, we establish the characterization results for the value function and the comparison principles for the appropriate HJB equations. In part III, we investigate the perturbation of the model studied in part II in the microscopic scale and search for the limit model in the macroscopic scale.

Part I: HJB approach for problems with discontinuity in time

The first part deals with the optimal control of impulsive systems under state constraints. The control problem based on impulsive systems and the control problem with state constraints have been separately studied. However, the subject of the mixed problem involving both impulsive systems and state constraints is brand new. In our study, we have developed the HJB approach to solve this problem. Another contribution of our study is the HJB approach for the problems with time-measurable dynamics under time-dependent state constraints.

Consider the following impulsive system:

$$dy(s) = g_0(s, y(s), \alpha(s))ds + g_1(s, y(s))d\mu, \text{ for } s \in (t, T), \; y(t) = x,$$

where g_0, g_1 are regular enough and μ is a vector of Radon measures containing singularities eventually. Denote by $S_{[t,T]}(x)$ the set of the trajectories satisfying the above impulsive system. Given a closed subset $\mathcal{K}$ of $\mathbb{R}^d$ the control problem is the following:

$$v(t, x) = \inf \left\{ \varphi(y(T)) \; : \; y(\cdot) \in S_{[t,T]}(x), \; y(s) \in \mathcal{K}, \; \forall\, s \in [t, T] \right\}.$$

The impulsive system, which is a measure-driven dynamical systems appear in the modeling of applications in many fields, including mechanical systems with impacts [42, 53, 98, 100], Faraday waves [67, 101], and several other applications in biomedicine or neuroscience, see [68] and the references therein. It contains an impulsive term which consists of the product of a state-dependent regular function and a measure of Radon type. The singularity character of Radon measure may force the trajectories to jump at certain time and the discontinuity of the trajectories occurs. This fact makes the magnitude of the jump quite complicated to be determined because the impulsive term is state-dependent, then the definition of solutions to our dynamical system with impulsive character is not clear. Theoretically, several studies have been devoted to the question of giving a precise notion of solution to this type of systems, see Bressan and Rampazzo [48, 49], Dal-Maso and Rampazzo [70], Raymond [119].

We follow the definition introduced in the mentioned papers of Bressan, Dal-Maso and Rampazzo for a general class of impulsive systems, where a concept of *graph completion* has been considered. This graph completion technique consists of a reparametrization in the time variable for the primitive function of the measure. The idea is that at the moment when the singularity of the Radon measure is involved, a fictive time interval will be created manually. During this fictive time interval, the graph of the primitive of the measure is completed so that the singularity is erased technically. This process then leads to a reparametrization in the time variable which turns the original impulsive dynamical system into an equivalent reparametrized dynamical system. The good news is that there is no more singular term in this new system, but the inconvenient point is that the reparametrized dynamics become time-measurable.

Then we turn our attention to the optimal control problem based on the reparametrized dynamical system. A natural problem to address is the equivalence between this problem and the original optimal control problem. In absence of state constraints, it has been studied in Briani and Zidani [52] and the desired equivalence holds. However, it is not clear in presence of state constraints. The problem lies in the branches of the reparametrized trajectories during the fictive time intervals. In general, the behavior of these branches is not controllable and they may violate the state constraints eventually.

The first study is to deal with the case where the fictive branches satisfy the state constraints by assuming a controllability condition as in Soner [126]. Under this controllability assumption, the fictive part of the reparametrized trajectories stays always in the constrained region and the whole part of the trajectories will satisfy the state constraints if it is the same case for the corresponding original trajectories. Thus, the original problem and the reparametrized problem are equivalent, and we can focus on the characterization of the value function of the reparametrized problem. Note that the reparametrized problem is a state-constrained optimal control problem with time-measurable dynamics. Thus the difficulty comes mainly from the presence of state constraints and the time-measurable character of the dynamics.

Recall that the controllability assumption introduced in [126] is the (IQ) as mentioned before. Under this assumption, several studies have been devoted to analyze the behavior of the trajectories near the boundary of the state constraints where the trajectories are driven by a time-measurable dynamical system, see Frankowska and Vinter [86], Bettiol, Bressan and Vinter [35], Bettiol, Frankowska and Vinter [36]. Following these studies, the continuity of the value function is ensured when this condition is assumed and the set of state constraints $\mathcal{K}$ is smooth enough.

Another difficult arises from the time-measurable character of the dynamics, which leads to a time-measurable Hamiltonian. The viscosity theory has been extended for HJB equations with time-measurable Hamiltonians by Ishii in [102] and Lions-Perthame in [110]. Here we have extended this theory for the state constrained case, and the main results are the following: the value function of the reparametrized problem solves a constrained HJB equation with time-measurable Hamiltonian, and a comparison principle result is proved to ensure the uniqueness of solution to the HJB equation.

The second study deals with the general case where the fictive branches may violate the state constraints. The idea is relax the state constraints for the fictive part of the reparametrized trajectories. It can happen that the fictive part will never satisfy the state constraints even under the controllability assumptions, for example when the constrained region is not connected. Note that the fictive part is created manually during the graph completion and is of no interest to calculate the value function of the original control problem, we can also relax the state constraints manually for the fictive part so that this part will never violate the relaxed state constraints. Then we obtain an equivalent reparametrized control problem with state constraints which are time-dependent. Now, we do not need any controllability assumption, but the time-dependent character of the state constraints produces new difficulty.

The problems with time-dependent state constraints without any controllability assumption are quite complicated to solve. Besides, the smoothness of the state constraints is not required in our framework. To overcome this difficulty, the main idea inspired by Altarovici, Bokanowski and Zidani [2] and also Bokanowski, Forcadel and Zidani [39] through a level set approach is to characterize the epigraph of the value function instead of characterizing the value function directly. Our main result is the following: the epigraph of the value function is characterized by a variational inequality. We have also studied the case when the value function is discontinuous and we have extended the bilateral viscosity theory to this constrained case.

To conclude this part, we have developed the HJB approach for impulsive optimal control problems with state constraints. Another contribution we have made is the HJB approach for a large class of optimal control problems where the state constraints can be time-dependent and no controllability assumption is needed.

Part II: HJB approach for problems with discontinuity in state

The second part of the thesis is concerned with the optimal control problems and HJB system on multi-domains. The structure of the multi-domains is composed of several disjoint subdomains Ω_i which are separated by several lower-dimensional interfaces:

$$\mathbb{R}^d = \bigcup_{i=1}^{m} \overline{\Omega}_i, \;\; \Omega_i \cap \Omega_j = \emptyset, \;\text{ for } i \neq j.$$

In each subdomain, an HJB equation is imposed with an Hamiltonian H_i which can be completely different from the ones defined in other subdomains:

$$-\partial_t u(t,x) + H_i(x, Du(t,x)) = 0, \;\text{ for } t \in (0,T), \;\; x \in \Omega_i.$$

Due to the singular geometric structure of multi-domains, we can not expect to find a continuous Hamiltonian defined on the whole space which coincides with each Hamiltonian in each subdomain. Thus, the discontinuity of the Hamiltonian in the state variable is involved in this subject.

The investigation of control problems and HJB equations with discontinuity in state is mainly motivated by the study of hybrid system. One example, given in van der Schaft and Schumacher [132], comes from the variable-structure system described by dynamical system of the following form: $\dot{x} = f_1(x)$ if $h(x) \geq 0$, $\dot{x} = f_2(x)$ if $h(x) \leq 0$. The precise interpretation is in principle ambiguous since there is no requirement that $f_1(x) = f_2(x)$ when $h(x) = 0$. Hence the standard theory for existence and uniqueness of solutions to differential equations does not apply. Another motivation lies in the modeling of network with application on traffic flow problems, see Achdou, Camilli, Cutri and Tchou [1], Imbert, Monneau and Zidani [106]). In [106], the network is modeled as a union of finite half-lines with a common junction point. On each half-line, representing a road for the application, an HJ equation is imposed to describe the density of the traffic flow. It is interesting to understand what can happen on the junction point. This subject appears also in the problem of ray light propagation in an inhomogeneous medium with discontinuous refraction index.

The setting of the problem leads us to deal with the HJB equations with state-discontinuous Hamiltonians. The subject of giving a precise notion of solutions and providing a comparison result remains a difficult issue. Recall that the viscosity notion has been extended to the discontinuous case by Ishii [104]. Later, the viscosity notion was extended to the case where the Hamiltonian is state-measurable by Camilli-Siconolfi [54]. A comparison principle has been proved under a so-called transversality assumption which is quite restrictive. Under this assumption, the interfaces whose measure is zero can be ignored in the framework of measurable setting. However, the interest of our study lies mainly in the interfaces as in the applications shown before. We would like also mention Soravia [129] where the Hamiltonians are discontinuous and take a special form with some assumptions of transversality type. We refer also to Garavello and Soravia [95, 96], De Zan and Soravia [72], Giga, Gòrka and Rybka [94] for problems with discontinuous coefficients, where the uniqueness results are given using the special structure of discontinuity.

The objective of our study is to derive some *junction conditions* that have to be considered on the interfaces in order to get a comparison principle between supersolutions and subsolutions. Three papers have been particularly influential for our work. We would like to mention Bressan and Hong [46], which has been, as far as we know, the first paper on the subject and where the relevance of HJB tangential equations, namely equations posed on the interfaces, is pointed out. The second work are [23, 24] which have studied both the infinite horizon problem and the finite horizon problem in two-domains. The controls are divided between regular and singular, according to the behavior of associated velocities on the interface, and correspondingly, two different value functions are analyzed mainly by the PDE tools. The work considers at first the Ishii's notion of solutions and looks for the properties satisfied by the value functions which allow to obtain the characterization results. The controllability is assumed in the whole space in [23], and then has been weakened in [24] where the controllability is only assumed in the normal directions on the interface. The convexity of the set of velocities/costs is also needed. The comparison results for super/sub-solutions and the stability results for both value functions have been established. This approach is certainly interesting and capable of promising developments. In our work, we are particularly interesting in the value function associated to all controls of the integrated system which corresponds to the regularization. Another main difference is that the notion of solutions is not based on the Ishii's notion because we are interested in the minimal requirements for the junction conditions. The third reference is Barnard and Wolenski [29], which has attracted our attention by introducing the concept of *essential dynamics* to deduce the proper equations on the interfaces.

The main idea is to introduce an optimal control problem and the associated value function well defined in the whole space, and then investigate the equations satisfied by the value function on the interfaces. In the theory of viscosity solutions, the comparison result is obtained through some PDE technique with viscosity test functions. This argument is difficult to be adapted in the framework of discontinuous Hamiltonians. The method considered in our study is of dynamics type, using essentially the tools of nonsmooth analysis, see [60, 61]. The theory of nonsmooth analysis provides some geometric relations, called invariance properties, between the dynamics and the epigraph/hypograph of the super/sub-solutions. These properties are then interpreted as some optimality principles for super/sub-solutions from which the comparison result can be deduced. On the other hand, these properties can be characterized by HJB inequations, which are considered as the candidate transmission conditions for the super/sub-solutions. In particular, we are interested in two types of transmission conditions: the weakest conditions for super/sub-solutions and the conditions in the form of HJB equations with the same Hamiltonian for both supersolutions and subsolutions.

We take for the supersolution part on the interfaces the Bellman Hamiltonian corresponding to all control in A, which turns out to be equal to $\max\{H_i\}$. This is the Hamiltonian for supersolutions indicated by Ishii's theory [104], the reference frame for discontinuous HJ equations. However the Hamiltonian provided by the same theory for subsolutions, namely $\min\{H_i\}$, does not seem well adapted to our setting since it does not take into any special account controls corresponding to tangential velocities.

We consider for subsolutions the Hamiltonian of Bellman type with controls associated to tangential velocities to the interfaces, accordingly the corresponding equation is restricted on the interfaces, which means that viscosity tests take place at local constrained maximizers, or test functions can be possibly just defined on the interfaces. Same Hamiltonian also appears in [23], the difference is that in our case to satisfy such a tangential equation is the unique condition we impose on subsolutions on Γ, and not an additional one.

Note that the weakest transmission conditions for supersolution and subsolutions presented above do not have the same Hamiltonian. An inspiring point is the *essential Hamiltonian* involving the essential dynamics introduced in [29]. The essential dynamics are identified as a exact selection of dynamics which are realized by the trajectories. The equation with the essential Hamiltonian are stronger for the characterization of supersolutions and subsolutions, but the value function satisfies this equation on the interfaces. Besides, we will see the convenience of this stronger transmission condition in the study of homogenization problems coming later.

The study consists of two parts. The first part, as the first step of study, is concerned with the class of Hamiltonians involving only the dynamics, i.e.

$$H_i(x,p) = \sup_{q\in F_i(x)} \{-p\cdot q\}, \text{ for } x\in\Omega_i,\ p\in\mathbb{R}^d,$$

where F_i represents the set of dynamics in Ω_i. A strong controllability hypothesis has been assumed which leads to the coercivity of Hamiltonians. The comparison result proved in this part is between the lsc supersolutions and the Lipschitz continuous subsolutions. It is also exploited that two properties are crucial for our study: the continuity of the value function on the interfaces and the Lipschitz continuity of the tangential dynamics along the interfaces.

For the second part, the controllability hypothesis is only assumed on the interfaces, which is a much weaker assumption, to ensure the two crucial properties. Another improvement in this part is that we consider a more general class of Hamiltonians involving not only the dynamics, but also the terms containing the running costs:

$$H_i(x,p) = \sup_{a\in A} \{-p\cdot f_i(x,a) - \ell_i(x,a)\},$$

where f_i and ℓ_i presents respectively the dynamic and the running cost in Ω_i, and A is the set of control. And the equation considered in this part is of infinite horizon, i.e. given $\lambda > 0$,

$$\lambda u(x) + H_i(x, Du(x)) = 0, \text{ for } x\in\Omega_i.$$

The study is in the framework of two-domains with one interface. The main result is a comparison principle between lsc supersolutions and usc subsolutions which, in addition, are continuous on the interface.

To conclude this part, we have developed the HJB approach for the finite horizon and infinite horizon problems on multi-domains with discontinuity in state. The transmission conditions have been investigated and the comparison principles have been obtained.

Part III: Singular perturbation problems with discontinuity in state

In this part, we make a further investigation of problems with discontinuous coefficients in state: singular perturbation of optimal control problem which is concerned with the homogenization problems of HJ equations in the framework of discontinuous Hamiltonians. The HJ equations are considered in the domains with a periodic structure, and our main interest lies in the limit behavior of the solutions to the HJ equations when the scale of periodicity tends to 0.

The homogenization of HJ equations with Lipschitz continuous Hamiltonians has been well studied, see Lions, Papanicolaou and Varadhan[111], Evans [74]. The main goal lies in finding the limit equation which is also of HJ type. The Hamiltonian in the limit HJ equation is called *effective Hamiltonian*. The classical idea to determine this effective Hamiltonian is to introduce the *cell problem* defined in each unit of the periodic domain. The well-posedness of the cell problem is usually obtained by considering a group of approximated problems.

The singular perturbation of optimal control problem is considered with two different time scales, i.e. the dynamical system on which the problem is based involving two variables: a slow variable and a fast variable. In addition, the dynamics for the fast variable are defined on a periodic multi-domains. We aim at the limit behavior of the value function when the velocity of the fast variable goes to infinity.

Singular perturbation problems for deterministic controlled systems have been studied by many authors; see e.g., the books by Kokotović, Khalil, and O'Reilly [107], and Bensoussan [34], as well as the articles by Gaitsgory [89, 90], Quincampoix and Zhang [118], Quincampoix and Watbled [117], Gaitsgory and Rossomakhine [93], Gaitsgory and Quincampoix [92], Bagagiolo and Bardi [17], Alvarez and Bardi [3, 4], Alvarez, Bardi and Marchi [5] and the references therein. In general, the value function of the perturbed problems solves an HJB equation. Then the limit behavior of this perturbed value function is studied through the homogenization of the associated HJB equation.

In our case, due to the structure of multi-domains, the value function of perturbed problem is not supposed to solve a classical HJB equation with Lipschitz continuous Hamiltonian. The study on HJB system on multi-domains is then applied here. Among the candidate transmission conditions on the interfaces obtained in the part II, the HJB equations with essential Hamiltonian introduced in [29] are adapted in this study since the Hamiltonians for supersolutions and subsolutions to be homogenized are the same. Then the study is turned to the homogenization of HJB equation with this discontinuous essential Hamiltonian. The difficulty arising from the discontinuity is significant, and rather few work is devoted to this subject in the literature. In Oberman, Takei and Vladimirsky [116], an algorithm has been introduced to solve the piecewise-periodic problems numerically where the Hamiltonians are not continuous, without giving general theoretical result for this method. In Camilli and Siconolfi [55], the authors have given the homogenization result for HJ equations in the framework of measurable setting. However, this result is obtained under the transversality assumption where the interfaces are considered meaningless since their measure is zero. Therefore, this assumption is not suitable for our study.

The main idea in our study is quite similar as in the classical case: we introduce the cell problem and obtain the effective Hamiltonian for the limit HJB equation. The main technique difficulty lies in the well-posedness of the cell problem which is concerned with an HJB equation with discontinuous Hamiltonian. An important hypothesis, which appears in almost all the work on homogenization and singular perturbed problems, is the controllability assumption for the dynamics of the fast variable. It leads to the coercivity of the Hamiltonian with respect to the fast variable, and allows the fast variable to be able to run through the whole space. Thanks to the coercivity of Hamiltonian, a stability result has been established in the framework of discontinuous Hamiltonian.

To solve the cell problem, we classically introduce an approximated cell problem as in [74, 111]. However, the essential Hamiltonian which appears in this approximating cell problem is not continuous. Thus, the construction of approximated corrector is a difficult issue. To solve this problem, we use the fact that the essential Hamiltonian is defined from an optimal control point of view and we show that approximated correctors can be constructed as the value functions of infinite horizon optimal control problems. Another difficulty is to prove that approximated correctors converge toward a corrector of the cell problem. This uses the stability result as mentioned before which is proved in the framework of discontinuous hamiltonian, but only for Lipschitz continuous solutions.

The main result is the following: the limit of the value function of the singular perturbation problem solves the HJB equation with the effective Hamiltonian given by the cell problem. Here the effective Hamiltonian depends only on the slow variable.

To conclude, we investigate the singular perturbation problem of optimal control and we have obtained the limit HJ equation describing the limit behavior of the perturbed value function.

Conclusions

In this thesis, we have studied the Mayer's optimal control problems and their extensions, including the problems with state constraints, the problems on multi-domains and the singular perturbation problems. The dynamical systems on which the problems are based include the regular dynamical system, time-measurable system, impulsive system and system on multi-domains.

Recall that the study of the problem over regular dynamical system has been studied by viscosity theory and nonsmooth analysis for both continuous and discontinuous solutions, see [19, 60, 66, 81]. The problem over time-measurable system has been investigated by viscosity theory for both continuous and discontinuous solutions, see [31, 52, 102]. The problem over impulsive systems has been treated in [52].

A rich literature can be found for the state constrained problem over regular and time-measurable dynamical systems, including continuous and discontinuous solutions under different types of controllability assumptions. The problem without controllability assumption has been recently treated in [2] for continuous solutions, and we have extended the idea to the problem with time-measurable dynamics and time-dependent state constraints for both continuous and discontinuous solutions. A brand new contribution in this thesis is the study of state constrained problem over impulsive systems. The problem has been investigated in the case with controllability assumptions and the case without controllability assumptions. The characterization results have been proved in both cases by extending the viscosity theory.

The subject of problem on multi-domains is a quite recent and active subject, see [23, 24, 29, 46]. It is based on dynamical system with a structure of multi-domains. We aim at investigating the transmission conditions on the singular parts of the multi-domains. The transmission conditions obtained in our study include both the minimal conditions and the conditions in the form of HJB equations with discontinuous Hamiltonians. A comparison principle which ensures the uniqueness of solution has been proved by the tools of nonsmooth analysis. We would like to mention that the transmission HJB equations are convenient for deeper study of this subject, including the numerical approaches and the homogenization problems.

The singular perturbation of optimal control problem has been widely studied, see [3] for example. However, the subject of perturbed problem on multi-domains is brand new. Based on the previous study on problems on multi-domains our contribution is a convergence result which has given the limit behavior of the solution of the singular perturbed problem.

Publications of the thesis

[122] (with A. Siconolfi and H. Zidani) *Transmission conditions on interfaces for Hamilton-Jacobi-Bellman equations*, submitted. http://hal.inria.fr/hal-00820273

[76] (with N. Forcadel) *Singular perturbation of optimal control problems on multi-domains*, submitted. http://hal.archives-ouvertes.fr/hal-00812846

[123] (with H. Zidani) *Hamilton-Jacobi-Bellman equations on multi-domains*, Control and Optimization with PDE Constraints, International Series of Numerical Mathematics, 164:93-116, 2013.

[78] (with N. Forcadel and H. Zidani) *State-Constrained Optimal Control Problems of Impulsive Differential Equations*, Applied Mathematics and Optimization, 68:1-19, 2013.

[77] (with N. Forcadel and H. Zidani) *Optimal control problems of BV trajectories with pointwise state constraints*, Proceedings of the 18th IFAC World Congress, Milan, 18:2583-2588, 2011.

Chapter 2

Background for Hamilton-Jacobi-Bellman approach

In this chapter, we present some classical results of HJB approach for Mayer's deterministic optimal control problems. The systematic study of optimal control problems dates from the late 1950s, and one important method is the Dynamical Programming and HJB approach. This approach reduces the study to investigating the analytic solution to a partial differential equation of the HJB type. However, the problems are usually nonlinear and the analytic solutions do not exist. To deal with the problems lacking of smoothness, two important tools have been developed: the theory of viscosity solutions and the nonsmooth analysis. The theory of viscosity solutions for nonlinear HJ equations, introduced in the early 1980s by Crandall-Lions [65, 66] and Crandall-Evans-Lions [62]. It allows to analyze the generalized solutions to broad classes of nonlinear partial differential equations, including the HJB equations of optimal control problems. We refer also to the books [19, 21] for a more complete introduction about this theory. Another important tool is the nonsmooth analysis which refers to differential analysis for nonsmooth functions. This field is launched by Clarke's theory of generalized gradients. It is of growing interest in a large class of domains, including optimization and control theory. We would like to refer to [15, 58, 60, 131] for the introduction of the theory and its applications.

The chapter is organized as follows. At first, we introduce the optimal control problem and the aimed value function in a standard setting. And then the Dynamical Programming approach is introduced and the HJB equation is derived. Then we recall the theory of viscosity solution and the nonsmooth analysis, and we will show how the characterization result of the value function via the HJB equation can be obtained by the two different theories.

2.1 Optimal control problems

Let $T > 0$ be a fixed finite time. Consider the set-valued multifunction $F : [0,T] \times \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ satisfying the following assumptions:

(HF1) For each $(t,x) \in [0,T] \times \mathbb{R}^d$, $F(t,x)$ is nonempty, compact and convex.

(HF2) F is upper semicontinuous, i.e. for any $t \in [0,T]$, $x \in \mathbb{R}^d$, $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(t',x') \subseteq F(t,x) + \varepsilon B(0,1), \ \forall\, (t',x') \in B((t,x),\delta).$$

(HF3) F has a linear growth, i.e. there exists $c(\cdot) \in L^1([0,T])$ such that

$$\forall\, t \in [0,T],\ x \in \mathbb{R}^d,\ \sup_{p \in F(t,x)} \|p\| \leq c(t)(\|x\| + 1).$$

For some results, we will need more regularity of F:

(HF4) F is Lipschitz continuous, i.e. for any $t, t' \in [0,T]$, $x, x' \in \mathbb{R}^d$, there exists $L > 0$ such that

$$F(t',x') \subseteq F(t,x) + LB(0,1).$$

Given $x \in \mathbb{R}^d$ and $t > 0$, we introduce the following control system:

$$\begin{cases} \dot{y}(s) \in F(s, y(s)), & \text{a.e. } s \in (t,T) \\ y(t) = x. \end{cases} \tag{2.1.1}$$

The solutions for the above differential inclusion are in the class of absolutely continuous functions $W^{1,1}([0,T])$. Consider the set of admissible trajectories which are absolutely continuous solutions of the system (2.1.1) defined on $[t,T]$ starting from x by:

$$S_{[t,T]}(x) := \big\{ y_{t,x} \text{ absolutely continuous solution of (2.1.1)} \big\}.$$

Remark 2.1.1. Let us recall that under the assumptions **(HF1)-(HF3)**, the differential equation (2.1.1) admits an absolutely continuous solution and that the set $S_{[t,T]}(x)$ is compact in $W^{1,1}$. In addition, if **(H4)** holds true, then the application $x \mapsto S_{[t,T]}(x)$ is Lipschitz continuous (see [15]).

Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ satisfy

(HC1) φ is Lipschitz continuous.

Consider the following optimal control problem of Mayer's type:

$$v(t,x) := \inf_{y_{t,x} \in S_{[t,T]}(x)} \{\varphi(y_{t,x}(T))\}, \tag{2.1.2}$$

where v is called the value function. By remark 2.1.1, the infimum is actually attained since $S_{[t,T]}(x)$ is compact in $W^{1,1}$. Then v can be rewritten as

$$v(t,x) = \min_{y_{t,x} \in S_{[t,T]}(x)} \{\varphi(y_{t,x}(T))\}.$$

Remark 2.1.2. A more general class of finite horizon optimal control problems is the Bolza's problems with a regular running cost function. The Bolza's problem can be turned into an equivalent Mayer's problem by adding a new state variable taking the running cost function as its dynamics. Here for the simplicity we studies the problems of Mayer's type, but the results can be generalized in the case of Bolza's problems.

2.2 Elements of nonsmooth analysis

We start this section by recalling some fundamental elements of nonsmooth analysis.

Let $Z : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ be a set-valued multifunction with nonempty, compact and convex images. We say that Z is usc if for any $x \in \mathbb{R}^p$, $\varepsilon > 0$, there exists $\delta > 0$ such that

$$Z(x') \subseteq Z(x) + \varepsilon B(0,1), \ \forall\, x' \in B(x,\delta).$$

We say that Z is locally Lipschitz continuous if for any compact $\mathcal{K} \subset \mathbb{R}^p$, $x_1, x_2 \in \mathcal{K}$, there exists $L_{\mathcal{K}} > 0$ such that

$$Z(x_1) \subseteq Z(x_2) + L_{\mathcal{K}} \|x_1 - x_2\| B(0,1).$$

Z is called Lipschitz continuous if there exists $L > 0$ such that $L_{\mathcal{K}} \leq L$ for any compact $\mathcal{K}$.

We say that Z has a linear growth if there exists $c > 0$ such that

$$\forall\, x \in \mathbb{R}^p, \ \sup_{\zeta \in Z(x)} \|\zeta\| \leq c(\|x\| + 1).$$

Given $\mathcal{K}$ a closed subset of $\mathbb{R}^p$, let $d_{\mathcal{K}}(\cdot)$ be the distance function to $\mathcal{K}$. Denote by $\mathcal{T}_{\mathcal{K}}(x)$ the tangent cone of $\mathcal{K}$ at some x defined as

$$\mathcal{T}_{\mathcal{K}}(x) = \{\zeta \in \mathbb{R}^p : \liminf_{h \to 0^+} \frac{d_{\mathcal{K}}(x + h\zeta)}{h} = 0\}. \tag{2.2.1}$$

The tangent cone considered here is called Bouligand's Contingent Cone, see [15, Definition 1, pp.176].

Given a closed subset $\mathcal{C} \subset \mathbb{R}^p$ and $x \in \partial\mathcal{C}$, we define $\mathcal{N}_{\mathcal{C}}(x)$ as the normal cone to $\mathcal{C}$ at x as

$$\{p \in \mathbb{R}^p : \ \exists\, \varepsilon > 0 \text{ such that } \mathrm{proj}_{\mathcal{C}}(x + \varepsilon\, p) = x\},$$

where $\text{proj}_{\mathcal{C}}$ stands for the projection on $\mathcal{C}$. This notion of normal cone is called proximal normal cone in [60]. Notice that the previous relation still holds for any positive quantity less than ε. Up to reducing ε, we can also suppose that x is the unique projection point of $x + \varepsilon\, p$. Notice that, given $x \in \partial\mathcal{C}$, the set of nonzero normal vectors can be empty.

In the sequel, for any function $w : \mathbb{R}^p \to \mathbb{R}$, $\mathcal{E}p(w)$ and $\mathcal{H}p(w)$ denote, respectively, the epigraph and hypograph of w, i.e.

$$\mathcal{E}p(w) := \{(x, z) \,|\, w(x) \leq z,\ x \in \mathbb{R}^p,\ z \in \mathbb{R}\}\,,\ \mathcal{H}p(w) := \{(x, z) \,|\, w(x) \geq z,\ x \in \mathbb{R}^p,\ z \in \mathbb{R}\}\,.$$

We recall some results of [60, 120] which are crucial for matching normal vectors to epi/hypographs and differentials of viscosity test functions.

Proposition 2.2.1. Let $w : \mathbb{R}^p \to \mathbb{R}$ be a lsc (resp. usc) function. Assume that $(p, -1)$ (resp. $(-p, 1)$) is a normal vector to $\mathcal{E}p(w)$ (resp. to $\mathcal{H}p(w)$) at some point $(x_0, w(x_0))$, then there exists $\phi \in C^1(\mathbb{R}^p)$ such that $w - \phi$ attains a local minimum (resp. maximum) at x_0 with $D\phi(x_0) = p$.

Proposition 2.2.2. Let w be a lsc (resp. usc) function. Assume that $(p, 0)$ is a normal vector to $\mathcal{E}p(w)$ (resp. to $\mathcal{H}p(w)$) at some point $(x_0, w(x_0))$, then there are sequences $(x_k, w(x_k))$, (p_k, s_k), with $s_k \neq 0$ and (p_k, s_k) is a normal vector to $\mathcal{E}p(w)$ (resp. $\mathcal{H}p(w)$) at $(x_k, w(x_k))$, such that $(x_k, w(x_k)) \to (x_0, w(x_0))$ and $(p_k, s_k) \to (p, 0)$.

2.2.1 Invariance properties

An essential notion from the tools developed via nonsmooth analysis is that of *invariance*. It concerns the flow invariance of the pair of a multifunction $Z : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ and a given set $\mathcal{K} \subset \mathbb{R}^p$. The main concepts of invariance are recalled as follows (see also [60, Definition 4.2.3]).

Definition 2.2.3. Let $Z : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ and $\mathcal{K} \subset \mathbb{R}^p$.

- The pair $(\mathcal{K}, Z)$ is called *weakly invariant* provided that for any $x \in \mathcal{K}$, there exists a trajectory $y(\cdot)$ such that
$$y(0) = x,\ \dot{y}(s) \in Z(y(s)) \text{ and } y(s) \in \mathcal{K},\ \forall\, s \geq 0.$$
- The pair $(\mathcal{K}, Z)$ is called *strongly invariant* provided that for any $x \in \mathcal{K}$, every trajectory $y(\cdot)$ satisfying
$$y(0) = x,\ \dot{y}(s) \in Z(y(s))\ \forall\, s \geq 0,$$
it holds that $y(s) \in \mathcal{K}$, for all $s \geq 0$.

The two theorems recalled below give the necessary and sufficient conditions for the weak/strong invariance properties (see also [60, Theorem 4.2.10, Theorem 4.3.8]).

Theorem 2.2.4. Assume that $Z : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ is a usc multifunction with nonempty, compact and convex images which has linear growth, and that $\mathcal{K}$ is a given nonempty closed subset of $\mathbb{R}^p$. The following are equivalent:

- $(\mathcal{K}, Z)$ is weakly invariant;
- $Z(x) \cap T_{\mathcal{K}}(x) \neq \emptyset, \ \forall\, x \in \mathcal{K}$;
- $\inf_{p \in Z(x), q \in \mathcal{N}_{\mathcal{K}}(x)} \{p \cdot q\} \leq 0$.

Theorem 2.2.5. Assume that $Z : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ is a Lipschitz continuous multifunction with nonempty, compact and convex images which has linear growth, and that $\mathcal{K}$ is a given nonempty closed subset of $\mathbb{R}^p$. The following are equivalent:

- $(\mathcal{K}, Z)$ is strongly invariant;
- $Z(x) \subseteq T_{\mathcal{K}}(x), \ \forall\, x \in \mathcal{K}$;
- $\sup_{p \in Z(x), q \in \mathcal{N}_{\mathcal{K}}(x)} \{p \cdot q\} \leq 0$.

We will see the applications of the invariance properties for control problems in Section 2.4.

2.2.2 Filippov Approximation Theorem

Another essential tool in our analysis will be Filippov Approximation Theorem, which provides an estimate of how far a given curve, say y, is from some integral trajectory of a Lipschitz multifunction Z in terms of the distance to $Z(y(t))$ of $\dot{y}(t)$. It is recalled as follows (see [58, Theorem 3.1.6]).

Theorem 2.2.6. Let $Z : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ be a L-Lipschitz continuous multifunction with nonempty compact images. For $\varepsilon > 0$, let y be a curve defined in some interval $[a, b]$ and $\mathcal{C}$ be an open neighborhood of $y([a, b])$ such that

$$y([a,b]) + \varepsilon B(0,1) \subset \mathcal{C}, \ \ y(a) \in \mathcal{C}, \ \text{ and } d(\dot{y}(s), Z(y(s))) \leq \varepsilon^{-L(b-a)}, \ \ \forall, s \in [a,b].$$

Then there exists a trajectory y_* driven by Z, contained in $\mathcal{C}$, and with $y_*(a) = y(a)$, such that

$$|y_*(s) - y(s)| \leq e^{L(s-a)} \int_a^b d(\dot{y}(s), Z(y(s)))ds \text{ for any } s \in [a,b].$$

The original formulation is local in time, we now present a modified formulation which is a global result. This result will be applied for the problems on multi-domains in Chapter 4.

We first introduce the *reachable set* $\mathcal{R}_Z(B,T)$ for a given multifunction Z in $\mathbb{R}^p$, $B \subset \mathbb{R}^p$, $T > 0$. We consider all points reached from some initial set not only in the prescribed time T, but in any time shorter than it, as well.

$$\mathcal{R}_Z(B,T) = \bigcup_{t \in [0,T]} \{x \in \mathbb{R}^n \mid \exists \text{ traj. } y \text{ of } Z \text{ with } y(0) \in B,\, y(t) = x\}. \tag{2.2.2}$$

If B reduces to a singleton, say $\{x_0\}$, we will simply write $\mathcal{R}_Z(x_0, T)$.

If Z has linear growth then it is an immediate consequence of Gronwall Lemma that $\mathcal{R}_Z(B,T)$ is bounded for any bounded subset B and any $T>0$.

Theorem 2.2.7. Let $\mathcal{C}$ be a closed subset of $\mathbb{R}^p$, and $\mathcal{C}_\natural$ an open neighborhood of $\mathcal{C}$. Let y be a curve defined in some interval $[0,T]$ such that $y(0)\in\mathcal{C}$ and $y([0,T])\subset\mathcal{C}_\natural$.

Let Z be a locally Lipschitz–continuous multifunction defined in $\mathcal{C}_\natural$. Assume that Z is compact valued and has linear growth, and $\mathcal{C}$ is strongly invariant for Z.

Then there exists a trajectory y_* of Z defined in $[0,T]$, contained in $\mathcal{C}$, and with $y_*(0)=y(0)$, such that

$$|y_*(t)-y(t)|\le e^{L\,t}\int_0^T d(\dot y, Z(y))\,ds \quad \text{for any } t\in[0,T],$$

where L is the Lipschitz constant of Z in some bounded open neighborhoods of $\mathcal{R}_Z(y(0),T)$ contained in $\mathcal{C}_\natural$. (note that $\mathcal{R}_Z(y(0),T)$ is indeed bounded, Z being with linear growth, and is in addition contained in $\mathcal{C}$ because of the invariance assumption of $\mathcal{C}$ for Z).

Proof. We denote by B a bounded open neighborhood of $\mathcal{R}_Z(y(0),T)$ in $\mathcal{C}_\natural$, and by ρ, P positive constants with

$$\mathcal{R}_Z(y(0),T)+B(0,\rho)\subset B \tag{2.2.3}$$

and $|q|<P$ for $q\in Z(x)$, $x\in B\cup y([0,T])$. All the curves starting at $y(0)$ with (a.e.) velocity less than P are contained in B for $t\in[0,t_0]$, where $t_0=\min\left\{T,\frac{\rho}{P}\right\}$. We construct by recurrence a sequence of curves of this type as follows: we set $y_0=y$ and for $k\ge 1$ define

$$Z_k(t)=\{q\in Z(y_{k-1}(t)) \mid |q-\dot y_{k-1}(t)|=d(\dot y_{k-1}(t), Z(y_{k-1}(t)))\} \quad \text{for a.e. } t\in[0,t_0].$$

Since this multifunction is measurable, see [58], we extract a measurable selection denoted by f_k. We then define y_k in $[0,t_0]$ as the curve determined by $\dot y(t)=f_k(t)$, for a.e. t and $y_k(0)=y(0)$. We set

$$d_Z=\int_0^{t_0} d(\dot y(s), Z(y(s))\,ds.$$

We have for a.e. $t\in[0,t_0]$, $|\dot y_1(t)-\dot y(t)|=d(\dot y(t),Z(y(t))$, $|y_1(t)-y(t)|\le d_Z$. Then for $k\ge 1$, $\dot y_{k+1}(t)\in Z(y_k(t))$ and

$$\begin{aligned}|\dot y_{k+1}(t)-\dot y_k(t)| &= d(\dot y_k(t),Z(y_k(t))\le L\,|y_k(t)-y_{k-1}(t)|\\ |y_{k+1}(t)-y_k(t)| &\le L\int_0^t |y_k(s)-y_{k-1}(s)|\,ds.\end{aligned}$$

We deduce for any $t\ge 0$:

$$|y_2(t)-y_1(t)|\le L\int_0^t |y_1(s)-y(s)|\,ds\le L\,d_Z\,t,$$

$$|y_{k+1}(t)-y_k(t)|\le L\int_a^t |y_k(s)-y_{k-1}(s)|\,ds\le d_Z\,\frac{L^k\,t^k}{k!}.$$

It is straightforward to deduce from this information, see [58], that y_k uniformly converge to a trajectory $\overline{y}$ of Z in $[0, t_0]$ satisfying the assertion with t_0 in place of T.

If $t_0 < T$ then using the same argument as above we show that $\overline{y}$ can be extended, still satisfying the assertion, in the interval $[0, t_1]$, where $t_1 = \min\left\{T, 2\,\frac{\rho}{P}\right\}]$. To do that, we exploit that any curve defined in $[t_0, t_1]$, taking the value $\overline{y}(t_0)$ at t_0 and with velocity less than P is contained in B. This is in turn true because of (2.2.3) and $\overline{y}(t_0) \in \mathcal{R}_Z(y([0,T],T)$. The proof is then concluded because we can iterate the argument till we reach T. □

Following [29], we deduce from the previous argument a property for Lipschitz continuous convexed-valued multifunctions.

Corollary 2.2.8. We assume Z to be defined in an open set B of $\mathbb{R}^n$ and to be locally Lipschitz–continuous, compact convex valued. For any $x_0 \in B$, $q_0 \in Z(x_0)$, there is a C^1 integral curve y_* of Z, defined in some interval $[0, T]$, with $y_*(0) = x_0$, $\dot{y}_*(0) = q_0$.

Proof. We set $y(t) = x_0 + q_0\, t$, $t \in [0, T]$, for T small enough. It comes from assumptions that the correspondence

$$t \mapsto \{q \in F(y(t)) \mid |q - \dot{y}(t)| = d(\dot{y}(t), Z(y(t)))\}$$

defined in $[0, T]$ is univalued and continuous, furthermore it takes the value q_0 at $t = 0$. It follows that the curve y_1, defined as in the proof of Theorem 2.2.7, is of class C^1 and satisfies $y_1(0) = x_0$, $\dot{y}_1(0) = q_0$, same properties hold true for any of the y_k. Following Theorem 2.2.7, we see that both y_k, $\dot{y}_k$ uniformly converge, up to a subsequence, as $k \to +\infty$. The limit curve satisfies the claim. □

2.3 Dynamic programming and Hamilton-Jacobi-Bellman equation

The fundamental idea of Dynamic Programming is that the value function v satisfies a functional equation, often called the *Dynamic Programming Principle* (see [19, Proposition III.3.2]). This principle provides two types of properties which are defined below.

Definition 2.3.1. For any function $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$,

(i) we say that u satisfies the *super-optimality principle* if for any $t \in [0, T]$, $x \in \mathbb{R}^d$, there exists $y_{t,x} \in S_{[t,T]}[x]$ such that

$$u(t, x) \;\geq\; u(t + h, y_{x,t}(t + h)), \;\forall\, h \in [0, T - t];$$

(ii) we say that u satisfies the *sub-optimality principle* if for any $t \in [0, T]$, $x \in \mathbb{R}^d$, and $y_{t,x} \in S_{[t,T]}[x]$,

$$u(t, x) \;\leq\; u(t + h, y_{x,t}(t + h)), \;\forall\, h \in [0, T - t].$$

Proposition 2.3.2. **(Dynamic Programming Principle)**
Assume **(HF1)-(HF3)** and **(HC1)**, the value function v satisfies both the super-optimality principle and the sub-optimality principle, i.e. for all $x \in \mathbb{R}^d$ and $t \in [0, T]$, we have:

$$v(t, x) = \min_{y_{t,x} \in S_{[t,T]}(x)} v(t + h, y_{t,x}(t + h)), \ \ h \in [0, T - t].$$

The Dynamic Programming Principle (DPP) allows to determinate the value function at the point (t, x) by splitting the trajectories at time $t + h$ and starting with the position of the trajectory $y_{t,x}$ at time $t + h$. Some numerical schemes can be developed based on this principle to compute the value function.

If the function v is differentiable, we can derive v to get its differential version, the *Hamilton-Jacobi-Bellman* (HJB) equation:

$$\begin{cases} -v_t(t, x) + H(t, x, Dv(t, x)) = 0 & \text{for } (t, x) \in (0, T) \times \mathbb{R}^d, \\ v(T, x) = \varphi(x) & \text{for } x \in \mathbb{R}^d, \end{cases} \tag{2.3.1}$$

where the Hamiltonian is given by

$$H(t, x, q) = \sup_{p \in F(t,x)} \big\{ - p \cdot q \big\}. \tag{2.3.2}$$

Then we look at the regularity result of v (see also [19, Proposition III.3.1]).

Proposition 2.3.3. Assume **(HF1)-(HF4)** and **(HC1)**, the value function v is Lipschitz continuous.

Unfortunately v is only Lipschitz continuous and usually not differentiable, then it is not expected that v is the analytic solution for (2.3.1). To deal with the value function lacking of smoothness, as mentioned before, the theory of viscosity solutions and the nonsmooth analysis will be applied.

2.4 Characterization result via the viscosity theory

The section is devoted to the characterization result of the value function by the viscosity theory. Recall firstly the definition of viscosity solution for HJB equations (see [19]).

Definition 2.4.1. (viscosity solution) Let $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$.

- We say that u is a viscosity supersolution if u is lower semicontinuous (lsc) and for any $\phi \in C^1((0, T) \times \mathbb{R}^d)$ and $(t_0, x_0) \in (0, T) \times \mathbb{R}^d$ local minimum point of $u - \phi$, we have

$$-\phi_t(t_0, x_0) + H(t_0, x_0, D\phi(t_0, x_0)) \geq 0.$$

- We say that u is a viscosity subsolution if u is upper semicontinuous (usc) and for any $\phi \in C^1((0,T) \times \mathbb{R}^d)$ and $(t_0, x_0) \in (0,T) \times \mathbb{R}^d$ local maximum point of $u - \phi$, we have

$$-\phi_t(t_0, x_0) + H(t_0, x_0, D\phi(t_0, x_0)) \leq 0.$$

- We say that u is a viscosity solution if it is both a viscosity supersolution and a viscosity subsolution and the final condition is satisfied:

$$u(T, x) = \varphi(x) \text{ in } \mathbb{R}^d.$$

Remark 2.4.2. There are also some equivalent definitions which are more local using the super and sub-differentials, which means that the differentials of the test functions can be replaced by some weak differentials of the viscosity solution. See [19, 21, 81] for the definition using the Dini-differentials and [60] for the definition using the proximal differentials.

Then the value function can be characterized as in the following result (see [19, Theorem III.3.7]).

Theorem 2.4.3. Suppose that **(HF1)-(HF4)** and **(HC1)** hold. Then the value function v is the unique viscosity solution of (2.3.1) in the sense of definition 2.4.1.

The difficult part in the proof is the uniqueness of the solution. The classical method is based on the doubling variable technique (see [19, 21]). It consists of establishing a comparison principle between any subsolution u_1 and any supersolution u_2. The main idea is to consider

$$\sup_{t,s,x,y} \left\{ u_1(t,x) - u_2(s,y) - \frac{|t-s|^2 + |x-y|^2}{\varepsilon^2} \right\},$$

where $\varepsilon > 0$. It is a regularization technique, called sup/inf-convolution for sub/super-solutions. Then the regularization of u_1 and u_2 can be considered as the viscosity test functions for u_2 and u_1 respectively, and the comparison result is deduced by the information obtained through the viscosity tests.

2.5 Characterization result via the nonsmooth analysis

This section is devoted to the characterization of the super-optimality principle and sub-optimality principle via HJB inequalities. The invariance properties, recalled as the elements of nonsmooth analysis, are applied here to describe the behavior of the trajectories which are strongly linked with the optimality principles. At the end, we will provide another proof for the characterization result (2.4.3) by using the super-/sub-optimality principles.

2.5.1 Characterization of the super-optimality principle

The characterization result is the following.

Theorem 2.5.1. Let $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$. Assume **(HF1)-(HF3)**. Then u is a lsc viscosity supersolution of (2.3.1) if and only if u satisfies the super-optimality principle.

Proof. Assume that u is a lsc viscosity supersolution. We proceed to show that u satisfies the super-optimality principle. Recall that $\mathcal{E}p(u)$ is setted as

$$\mathcal{E}p(u) = \{(t,x,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \,|\, u(t,x) \le z\}.$$

$\mathcal{E}p(u)$ is closed since u is lsc. Define the augmented multifunction $\hat{F} : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R} \rightsquigarrow \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}$ by

$$\hat{F}(t,x,z) := \begin{cases} \{1\} \times F(t,x) \times \{0\} & \text{for } t < T, \\ [0,1] \times \overline{co}\,(F(T,x) \cup \{0\}) \times \{0\} & \text{for } t \ge T, \end{cases}$$

where $\overline{co}$ signifies the convex hull. $\hat{F}$ is usc since F is usc. For any $(t,x) \in [0,T] \times \mathbb{R}^d$, consider the following differential inclusion:

$$\begin{cases} (\dot{\tau}(h), \dot{y}(h), \dot{\xi}(h)) \in \hat{F}(\tau(h), y(h), \xi(h)) & h \in (0,+\infty), \\ (\tau(0), y(0), \xi(0)) = (t, x, u(t,x)). \end{cases} \tag{2.5.1}$$

For any $(t,x,z) \in \mathcal{E}p(u)$, $u(t,x) \le z$. We claim that for any $(q_t, q_x, \sigma) \in \mathcal{N}_{\mathcal{E}p(u)}(t,x,u(t,x))$,

$$\inf_{\hat{p} \in \hat{F}(t,x,u(t,x))} \langle \hat{p}, (q_t, q_x, \sigma) \rangle \le 0. \tag{2.5.2}$$

Indeed, let $(q_t, q_x, \sigma) \in \mathcal{N}_{\mathcal{E}p(u)}(t,x,u(t,x))$. Since $(0,0,1) \in \mathcal{T}_{\mathcal{E}p(u)}(t,x,u(t,x))$, we have

$$\langle (0,0,1), (q_t, q_x, \sigma) \rangle \le 0,$$

i.e. $\sigma \le 0$. Based on this fact, consider the following three cases.

Case 1: $\sigma = -1$.

If $t = T$, then (2.5.2) holds true since $(0,0,0) \in \hat{F}(T,x,z)$.

Suppose now $t < T$. By Proposition 2.2.1 there exists $\phi \in C^1((0,T) \times \mathbb{R}^d)$ such that $u - \phi$ attains a local minimum on (t,x) with $(\partial_t \phi, D\phi)(t,x) = (q_t, q_x)$. Then

$$\begin{aligned} & \inf_{\hat{p} \in \hat{F}(t,x,u(t,x))} \langle \hat{p}, (q_t, q_x, \sigma) \rangle \\ = & \inf_{p \in F(t,x)} \langle (1,p,0), (\partial_t \phi(t,x), D\phi(t,x), \sigma) \rangle \\ = & \ \partial_t \phi(t,x) - H(t,x,D\phi(t,x)) \le 0, \end{aligned}$$

where the last inequality holds true because u is a viscosity supersolution of (2.3.1).

Case 2: $\sigma < 0$.

In this case, $(q_t/|\sigma|, q_x/|\sigma|, -1) \in \mathcal{N}_{\mathcal{E}p(u)}(t, x, u(t, x))$. We deduce from the previous case that

$$\inf_{\hat{p} \in \hat{F}(t,x,u(t,x))} \langle \hat{p}, (q_t/|\sigma|, q_x/|\sigma|, -1)\rangle \leq 0,$$

which implies

$$\inf_{\hat{p} \in \hat{F}(t,x,u(t,x))} \langle \hat{p}, (q_t, q_x, \sigma)\rangle \leq 0.$$

Case 3: $\sigma = 0$.

By Proposition 2.2.2, there exists $(t_n, x_n) \to (t, x)$ et $(q_t^n, q_x^n, \sigma^n) \to (q_t, q_x, \sigma)$ such that

$$(q_t^n, q_x^n, \sigma^n) \in \mathcal{N}_{\mathcal{E}p(u)}(t_n, x_n, u(t_n, x_n)), \; \sigma^n < 0.$$

By the previous case, we have

$$\inf_{\hat{p} \in \hat{F}(t_n,x_n,u(t_n,x_n))} \langle \hat{p}, (q_t^n, q_x^n, \sigma^n)\rangle \leq 0,$$

i.e.

$$\inf_{\hat{p} \in \hat{F}(t_n,x_n,u(t,x))} \langle \hat{p}, (q_t^n, q_x^n, \sigma^n)\rangle \leq 0.$$

Using the upper semicontinuity of $\hat{F}$, we deduce that

$$\inf_{\hat{p} \in \hat{F}(t,x,u(t,x))} \langle \hat{p}, (q_t, q_x, \sigma)\rangle \leq 0,$$

which ends the proof of claim (2.5.2).

Claim (2.5.2) holds true, then by Theorem 2.2.4 and the upper semicontinuity of $\hat{F}$, $(\mathcal{E}p(u), \hat{F})$ is weakly invariant, i.e. (2.5.3) has a solution $(\tilde{\tau}(\cdot), \tilde{y}(\cdot), \tilde{\xi}(\cdot))$ such that

$$(\tilde{\tau}(h), \tilde{y}(h), \tilde{\xi}(h)) \in \mathcal{E}p(u), \; \forall\, h \in [0, T-h],$$

i.e.

$$u(\tilde{\tau}(h), \tilde{y}(h)) \leq \tilde{\xi}(h), \; \forall\, h \in [0, T-h].$$

Note that $\tilde{\tau}(h) = t + h$ and $\tilde{\xi}(h) = u(t, x)$, we finally have

$$u(t+h, \tilde{y}(h)) \leq u(t, x), \; \forall\, h \in [0, T-h],$$

where $\tilde{y}(\cdot - t) \in S_{[t,T]}(x)$. Then u satisfies the super-optimality principle.

Now assume that u is lsc and satisfies the super-optimality principle. Let $\phi \in C^1((0,T) \times \mathbb{R}^d)$ and $(t_0, x_0) \in (0,T) \times \mathbb{R}^d$ such that $u - \phi$ attains a local minimum point at (t_0, x_0). The super-optimality principle of u implies that there exists $\tilde{y} \in S_{[t_0,T]}(x_0)$ such that

$$u(t_0, x_0) \geq u(t_0 + h, \tilde{y}(t_0 + h)), \; \forall\, h \in [0, T - t_0].$$

By the property of $u-\phi$ we have

$$\phi(t_0,x_0) \geq \phi(t_0+h,\tilde{y}(t_0+h)),\ \forall\, h \in [0,T-t_0],$$

which implies

$$\int_0^h \left[-\partial_t\phi(t_0+s,\tilde{y}(t_0+s)) - D\phi(t_0+s,\tilde{y}(t_0+s))\cdot\dot{\tilde{y}}(t_0+s)\right] ds \geq 0.$$

Since $\dot{\tilde{y}}(\cdot) \in F(\cdot,\tilde{y}(\cdot))$, we deduce that

$$\int_0^h \left[-\partial_t\phi(t_0+s,\tilde{y}(t_0+s)) + \sup_{p\in F(t_0+s,\tilde{y}(t_0+s))}\{-p\cdot D\phi(t_0+s,\tilde{y}(t_0+s))\}\right] ds \geq 0.$$

By the upper semicontinuity of F, $\forall\,\varepsilon>0$ and h being small enough,

$$F(t_0+s,\tilde{y}(t_0+s)) \subseteq F(t_0,x_0)+\varepsilon B(0,1).$$

Thus,

$$\int_0^h \left[-\partial_t\phi(t_0+s,\tilde{y}(t_0+s)) + \sup_{p\in F(t_0,x_0)+\varepsilon B(0,1)}\{-p\cdot D\phi(t_0+s,\tilde{y}(t_0+s))\}\right] ds \geq 0.$$

By taking $h\to 0^+$, we obtain

$$-\partial_t\phi(t_0,x_0) + \sup_{p\in F(t_0,x_0)+\varepsilon B(0,1)}\{-p\cdot D\phi(t_0,x_0)\} \geq 0,\ \forall\,\varepsilon>0,$$

where we deduce that

$$-\partial_t\phi(t_0,x_0) + H(t_0,x_0,D\phi(t_0,x_0)) \geq 0.$$

□

2.5.2 Characterization of the sub-optimality principle

The characterization result is the following.

Theorem 2.5.2. Let $u:[0,T]\times\mathbb{R}^d\to\mathbb{R}$. Assume **(HF1), (HF3)** and **(HF4)**. Then u is a usc viscosity subsolution of (2.3.1) if and only if u satisfies the sub-optimality principle.

Proof. Assume that u is a usc viscosity subsolution. We proceed to show that u satisfies the sub-optimality principle. We set $w:=-u$ and $\mathcal{E}p(w)$ as

$$\mathcal{E}p(w) := \{(t,x,z)\in[0,T]\times\mathbb{R}^d\times\mathbb{R} | w(t,x)\leq z\}.$$

Then w is lsc and $\mathcal{E}p(w)$ is closed. Define the augmented multifunction $\hat{F} : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R} \rightsquigarrow \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}$ by

$$\hat{F}(t, x, z) := \begin{cases} \{1\} \times F(t, x) \times \{0\} & \text{for } t < T, \\ [0, 1] \times \overline{co}\,(F(T, x) \cup \{0\}) \times \{0\} & \text{for } t \geq T. \end{cases}$$

$\hat{F}$ is usc since F is usc. For any $(t, x) \in [0, T] \times \mathbb{R}^d$, consider the following differential inclusion:

$$\begin{cases} (\dot{\tau}(h), \dot{y}(h), \dot{\xi}(h)) \in \hat{F}(\tau(h), y(h), \xi(h)) & h \in (0, +\infty), \\ (\tau(0), y(0), \xi(0)) = (t, x, w(t, x)). \end{cases} \tag{2.5.3}$$

For any $(t, x, z) \in \mathcal{E}p(w)$, $w(t, x) \leq z$. We claim that for any $(q_t, q_x, \sigma) \in \mathbb{N}_{\mathcal{E}p(w)}(t, x, w(t, x))$,

$$\sup_{\hat{p} \in \hat{F}(t,x,w(t,x))} \langle \hat{p}, (q_t, q_x, \sigma) \rangle \leq 0. \tag{2.5.4}$$

Indeed, let $(q_t, q_x, \sigma) \in \mathcal{N}_{\mathcal{E}p(w)}(t, x, w(t, x))$. Since $(0, 0, 1) \in \mathcal{T}_{\mathcal{E}p(w)}(t, x, w(t, x))$, we have

$$\langle (0, 0, 1), (q_t, q_x, \sigma) \rangle \leq 0,$$

i.e. $\sigma \leq 0$. Based on this fact, consider the following three cases.

Case 1: $\sigma = -1$.

If $t = T$, then (2.5.4) holds true since $(0, 0, 0) \in \hat{F}(T, x, z)$.

Suppose now $t < T$. By Proposition 2.2.1, there exists $\psi \in C^1((0, T) \times \mathbb{R}^d)$ such that $w - \psi$ attains a local minimum on (t, x) with $(\partial_t \psi, D\psi)(t, x) = (q_t, q_x)$. Then

$$\begin{aligned} & \sup_{\hat{p} \in \hat{F}(t,x,w(t,x))} \langle \hat{p}, (q_t, q_x, \sigma) \rangle \\ = & \sup_{p \in F(t,x)} \langle (1, p, 0), (\partial_t \psi(t, x), D\psi(t, x), \sigma) \rangle. \end{aligned}$$

By setting $\phi = -\psi$, we have that $u - \phi$ attains a local maximum on (t, x) and $(\partial_t \psi(t, x), D\psi(t, x)) = (-\partial_t \phi(t, x), -D\phi(t, x))$. Then

$$\begin{aligned} & \sup_{\hat{p} \in \hat{F}(t,x,w(t,x))} \langle \hat{p}, (q_t, q_x, \sigma) \rangle \\ = & \sup_{p \in F(t,x)} \langle (1, p, 0), (-\partial_t \phi(t, x), -D\phi(t, x), \sigma) \rangle \\ = & -\partial_t \phi(t, x) + H(t, x, D\phi(t, x)) \leq 0, \end{aligned}$$

where the last inequality holds true because u is a viscosity subsolution of (2.3.1).

Case 2: $\sigma < 0$.

In this case, $(q_t/|\sigma|, q_x/|\sigma|, -1) \in \mathcal{N}_{\mathcal{E}p(u)}(t, x, u(t, x))$. We deduce from the previous case that

$$\sup_{\hat{p} \in \hat{F}(t,x,w(t,x))} \langle \hat{p}, (q_t/|\sigma|, q_x/|\sigma|, -1) \rangle \leq 0,$$

which implies

$$\sup_{\hat{p} \in \hat{F}(t,x,w(t,x))} \langle \hat{p}, (q_t, q_x, \sigma) \rangle \leq 0.$$

Case 3: $\sigma = 0$.

By Proposition 2.2.2, there exists $(t_n, x_n) \to (t, x)$ et $(q_t^n, q_x^n, \sigma^n) \to (q_t, q_x, \sigma)$ such that

$$(q_t^n, q_x^n, \sigma^n) \in \mathcal{N}_{\mathcal{E}p(u)}(t_n, x_n, u(t_n, x_n)), \;\; \sigma^n < 0.$$

By the previous case, we have

$$\sup_{\hat{p} \in \hat{F}(t_n,x_n,w(t_n,x_n))} \langle \hat{p}, (q_t^n, q_x^n, \sigma^n) \rangle \leq 0,$$

i.e.

$$\sup_{\hat{p} \in \hat{F}(t_n,x_n,w(t,x))} \langle \hat{p}, (q_t^n, q_x^n, \sigma^n) \rangle \leq 0.$$

Using the Lipschitz continuity of $\hat{F}$, we deduce that

$$\sup_{\hat{p} \in \hat{F}(t,x,w(t,x))} \langle \hat{p}, (q_t, q_x, \sigma) \rangle \leq 0,$$

which ends the proof of claim (2.5.4).

(2.5.4) holds true, then by Theorem 2.2.5 and the Lipschitz continuity of $\hat{F}$ on $(0, T) \times \mathbb{R}^d \times \mathbb{R}$, $(\mathcal{E}p(w), \hat{F})$ is strongly invariant, which is equivalent to say that any solution $(\tau(\cdot), y(\cdot), \xi(\cdot))$ of (2.5.3) satisfies

$$(\tau(h), y(h), \xi(h)) \in \mathcal{E}p(w), \;\; \forall\, h \in [0, T - t),$$

i.e.

$$w(\tau(h), y(h)) \leq \tilde{\xi}(h), \;\; \forall\, h \in [0, T - t).$$

Note that $\tau(h) = t + h$ and $\xi(h) = w(t, x)$, we finally have

$$w(t + h, y(h)) \leq w(t, x), \;\; \forall\, h \in [0, T - t),$$

where $y(\cdot - t) \in S_{[t,T]}(x)$. Then $u = -w$ satisfies

$$u(t, x) \leq u(t + h, y(t + h)), \;\; \forall, h \in [0, T - t), \;\; y(\cdot) \in S_{[t,T]}(x).$$

By the upper semi-continuity of u, we deduce that

$$u(t,x) \leq u(t+h, y(t+h)),\ \forall, h \in [0, T-t],\ y(\cdot) \in S_{[t,T]}(x),$$

which is the desired sub-optimality principle for u.

Now assume that u is usc and satisfies the sub-optimality principle. Let $\phi \in C^1((0,T) \times \mathbb{R}^d)$ and $(t_0, x_0) \in (0,T) \times \mathbb{R}^d$ such that $u - \phi$ attains a local maximum point at (t_0, x_0). For any $p \in F(t_0, x_0)$, the Lipschitz continuity of $F(t_0, \cdot)$ implies that there exists $y(\cdot) \in S_{[t_0,T]}(x_0)$ such that $y(\cdot) \in C^1([t_0, \tau))$ for some $\tau > t_0$ and

$$\dot{y}(t_0) = p.$$

The sub-optimality principle of u implies that

$$u(t_0, x_0) \leq u(t_0 + h, y(t_0 + h)),\ \forall\, h \in [0, T - t_0].$$

By the property of $u - \phi$ we have

$$\phi(t_0, x_0) \leq \phi(t_0 + h, y(t_0 + h)),\ \forall\, h \in [0, T - t_0],$$

which implies

$$\int_0^h \left[-\partial_t \phi(t_0 + s, y(t_0 + s)) - D\phi(t_0 + s, y(t_0 + s)) \cdot \dot{y}(t_0 + s)\right] ds \leq 0.$$

By taking $h \to 0^+$, we obtain

$$-\partial_t \phi(t_0, x_0) + \{-p \cdot D\phi(t_0, x_0)\} \leq 0,\ \forall\, p \in F(t_0, x_0),$$

where we deduce that

$$-\partial_t \phi(t_0, x_0) + H(t_0, x_0, D\phi(t_0, x_0)) \leq 0.$$

□

2.5.3 Proof of Theorem 2.4.3

The value function v satisfies the dynamical programming principle, i.e. v satisfies both the super- and sup-optimality principle. Then by Theorem 2.5.1 and Theorem 2.5.2, v is a solution of (2.3.1). The uniqueness result is based on the following comparison principle.

Theorem 2.5.3. Let u_1 be a subsolution of (2.3.1) and u_2 be a supersolution of (2.3.1) with $u_1(T,x) \leq \varphi(x) \leq u_2(T,x)$ for $x \in \mathbb{R}^d$. Then for any $t \in [0,T]$, $x \in \mathbb{R}^d$,

$$u_1(t,x) \leq u_2(t,x).$$

Proof. For any $t \in [0, T]$, $x \in \mathbb{R}^d$, let $y_1 \in S_{[t,T]}(x)$ such that

$$v(t, x) = \varphi(y_1(T)).$$

By Theorem 2.5.2, u_1 is a subsolution of (2.3.1) implies that u_1 satisfies the sub-optimality principle, then

$$u_1(t, x) \leq u_1(T, y_1(T)) \leq \varphi(y_1(T)) = v(t, x).$$

By Theorem 2.5.1, u_2 is a supersolution of (2.3.1) implies that u_2 satisfies the super-optimality principle, then there exists $y_2 \in S_{[t,T]}(x)$ such that

$$u_2(t, x) \geq u_2(T, y_2(T)) \geq \varphi(y_2(T)).$$

v satisfies the sub-optimality principle, then

$$v(t, x) \leq v(T, y_2(T)) = \varphi(y_2(T)) \leq u_2(t, x).$$

Finally we have

$$u_1(t, x) \leq v(t, x) \leq u_2(t, x).$$

□

Remark

As we have mentioned before, the theory of viscosity solutions and nonsmooth analysis have developed to solve optimal control problems. In the study of this thesis, we do not have privileged tools between the two theories. We shall choose the suitable tools in our convenience.

A crucial property to apply these two theories to obtain the characterization result is the Lipschitz continuity of the dynamics $F(t, x)$. If this property is no longer satisfied, it is not clear how to obtain the characterization result since both theories can not be applied directly. In our study, we are interested in the optimal control problems where the dynamics F are not Lipschitz continuous: F is measuralbe on the time variable in Chapter 3 and F is discontinuous on the state variable in Chapter 4, 5 and 6. In the first case where the dynamics are discontinuous on time, the characterization result is obtained by extending the theory of viscosity solutions for time-measurable HJB equations. While in the second case, the difficulty arising from the discontinuity of dynamics on the state variable is more significant. In Chapter 4 and 5, the problem is considered to be set on a structure of multi-domains where the dynamics are Lipschitz continuous in each subdomain. In this case, however, neither the viscosity theory nor the nonsmooth analysis can be applied here. Fortunately, some properties can be exploited in each subdomain through the tools of nonsmooth analysis, then the desired characterization result is obtained by gluing together these properties.

Chapter 3

State constrained problems of impulsive control systems

Publications of this chapter

(with N. Forcadel and H. Zidani) *State-Constrained Optimal Control Problems of Impulsive Differential Equations*, Applied Mathematics and Optimization, 68:1-19, 2013.

(with N. Forcadel and H. Zidani) *Optimal control problems of BV trajectories with pointwise state constraints*, Proceedings of the 18th IFAC World Congress, Milan, 18:2583-2588, 2011.

3.1 Introduction

This chapter deals with an optimal control problem of *measure-driven* dynamical systems of the form:

$$\begin{cases} dy(t) = g_0(t, y(t), \alpha(t))dt + g_1(t, y(t))d\mu \text{ for } t \in (\tau, T], \\ y(\tau^-) = x, \end{cases} \tag{3.1.1}$$

where g_0 and g_1 are continuous functions whose values, respectively, are in $\mathbf{R}^d$ and $\mathcal{M}_{d\times p}$ (the space of $d \times p$ matrices), and μ is a given vector-valued measure with values in $\mathbf{R}^p$ (see section 2 for precise assumptions). The input α is a measurable function belonging to the set of admissible controls $\mathcal{A}$, that is:

$$\mathcal{A} := \{\alpha : (0, T) \to \mathbf{R}^m \ \text{ measurable function}, \ \ \alpha(t) \in A \text{ a.e. in } (0, T)\},$$

with A a compact set of $\mathbf{R}^m$.

For a given closed subset $\mathcal{K} \subset \mathbf{R}^d$, and a final cost function $\varphi : \mathbf{R}^d \to \mathbf{R}$, the Mayer control problem is:

$$\begin{aligned} v(\tau, x) \quad := \quad & \inf \big\{ \varphi(y^{\alpha}_{\tau,x}(T)), \\ & y^{\alpha}_{\tau,x} \text{ satisfies } (3.1.1), \text{ and } y^{\alpha}_{\tau,x}(t) \in \mathcal{K} \text{ for } t \in [\tau, T] \big\}. \end{aligned} \tag{3.1.2}$$

Measure-driven dynamical systems arise in many physical or economic applications that undergo forces whose actions have instantaneous effects. These systems are also called impulsive, they include mechanical systems with impacts [42, 53, 98, 100], Faraday waves [67, 101], and several other applications in biomedicine or neuroscience, see [68] and the references therein.

The impulsive character of the dynamical system (3.1.1) forces the trajectories to be discontinuous with *implicit* jumps. The magnitude of this jump should be first clarified in order to well define the behavior of the trajectory at the times of jump and then to have a precise notion of solution. To see this point, consider an example of impulsive ODEs in $4d$ studied in [68]:

$$\begin{cases} dx/dt &= \frac{z}{\tau_{rec}} - \delta(t - t_*)xu, \\ dy/dt &= -\frac{y}{\tau_{in}} + \delta(t - t_*)xu, \\ dz/dt &= \frac{y}{\tau_{in}} - \frac{z}{\tau_{rec}}, \\ du/dt &= -\frac{u}{\tau_{facil}} + \delta(t - t_*)k(1 - u), \end{cases}$$

where $\delta(\cdot)$ is the Dirac delta function and t_* is a fixed instant. This is a model describing the transit of electrochemical signals between two neurons at a synapse. The signals are passed via neurotransmitters which are stored in vesicles. In this example, (x, y, z, u) represent the quantity of vesicles in different states and $\tau_{rec}, \tau_{in}, \tau_{facil}$ are fixed parameters. The trajectory $Y(\cdot) := (x(\cdot), y(\cdot), z(\cdot), u(\cdot))$ jumps at the time t_*, then to determine the magnitude of this jump, there are different choices such as $Y(t_*^-)$, $Y(t_*^+)$ and any intermediate value between those two. Besides, since the dimension of this system is larger than 1, more ambiguity is created when the trajectory jumps in several directions at the same time t_* (see [68] for more details).

Several studies have been devoted to the question of giving a precise meaning to the notion of solution of impulsive systems like (3.1.1) and more generally to defining the product of a measure by a discontinuous function.

An illuminating point of view was introduced and analyzed in a series of papers [48, 49, 70], where the authors used the concept of *graph completion* to define the multiplication of a point-mass measure with a discontinuous state-dependent term. Basically, we introduce a function $\mathcal{W} : (0, T) \rightsquigarrow (0, 1)$ to reparametrize the time variable for the primitive function B of the measure μ. $\mathcal{W}$ is uniquely determined at each continuity point of B, while at the discontinuity points t_i, $\mathcal{W}$ is discontinuous and $[\mathcal{W}(t_i^-), \mathcal{W}(t_i^+)]$ corresponds to a "fictive" time interval (see Figure 3.1). Then we consider a graph completion $(\phi^0, \phi^1) : [0, 1] \to [0, T] \times \mathbf{R}^p$ which consists of an absolutely continuous map, where ϕ^0 is nondecreasing mapping onto $[0, T]$, and ϕ^1 is an extension to the graph of B. When

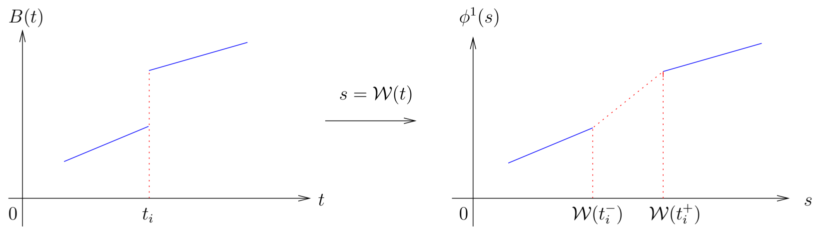


FIGURE 3.1: Reparametrization

$t \neq t_i$, $\mathcal{W}$ is continuous and

$$\phi^0(s) = t, \ \phi^1(s) = B(t) \text{ for } s = \mathcal{W}(t).$$

During the fictive time interval $[\mathcal{W}(t_i^-), \mathcal{W}(t_i^+)]$, we have

$$\phi^0(s) = t_i \Longleftrightarrow s \in [\mathcal{W}(t_i^-), \mathcal{W}(t_i^+)],$$

and the extension part of ϕ^1 prescribes an arc that connects the left and right hand limits of B at the points of discontinuity t_i.

In the sequel, the set of discontinuities of B will be denoted $\mathcal{T}$. In [70], the solution of (3.1.1) is defined as solution of an auxiliary differential system reparametrized in time. More precisely,

$$y(t) = z(\mathcal{W}(t)) \quad \text{for } t \in [\tau, T], \tag{3.1.3a}$$

where z is solution of

$$\begin{cases} \dot{z}(s) = \mathcal{F}(s, z(s), \alpha(s)), & s \in (\sigma, 1) \\ z(\sigma) = x, \end{cases} \tag{3.1.3b}$$

with $\sigma = \mathcal{W}(\tau^-)$, and $\mathcal{F}$ is a measurable function which depends on g_0, g_1, μ and on the graph completion (ϕ^0, ϕ^1) (the precise expression of $\mathcal{F}$ will be given in Section 2). The reparametrized solution z of (3.1.3b) is continuous and is well defined on the reparametrized time interval. In this way the multiplication of $g_1(y(t))$ by μ in the jump points is unambiguously defined.

In [70], a natural graph completion is introduced and analyzed. It consists on connecting the endpoints of the jumps of B by a straight line. This graph completion is said to be in the *canonical form* and it has been proved to lead to the same measure-solution given by the integral form:

$$y(t) = x + \int_\tau^t g_0(s, y(s), \alpha(s))\, ds + \int_{[\tau,t]} g_1(s, y(s))\, d\mu. \tag{3.1.4}$$

Of course, the above integral form has also to be well defined. It is known that each graph completion may lead to a different solution [44]. Further properties of the Graph completion concept and generalization to measure driven differential inclusions can be found in [119, 124, 133, 134].

In the present work, the solution of (3.1.1) will be defined by using the canonical graph completion. For the convenience of the reader, the concept of canonical graph completion and the notion of measure-solution are briefly recalled in Section 2.

With a precise definition of trajectories in hand, we can study the control problem (3.1.2). Let us mention that several works have been carried out on the necessary optimality conditions for problem (3.1.2) [13, 125]. The present chapter focuses mainly on the characterization of the value function v using the HJB approach. The main difficulties lie in the presence of the measure μ and of the state constraints.

It is easy to see that the value function v satisfies a Dynamic Programming Principle (DPP) which formally yields the following HJB equation:

$$\begin{cases} -v_t(t,x) + \sup_{a\in\mathcal{A}} \big\{ - Dv(t,x)\cdot(g_0(t,x,a) + g_1(t,x)d\mu)\big\} = 0, \\ v(T,x) = \varphi(x). \end{cases} \tag{3.1.5}$$

However, it is not clear in what sense the term "$Dv \cdot d\mu$" should be understood since there is no viscosity notion for this HJB equation with the measure term. In order to overcome this problem, using the concept of graph completion, one can consider a reparameterized optimal control problem where the new value function $\bar{v}_1$ is defined by:

$$\begin{aligned} \bar{v}_1(\sigma,x) \quad = \quad & \inf_{\alpha\in\mathcal{A}} \{\varphi(z^\alpha_{\sigma,x}(1)), \\ & z^\alpha_{\sigma,x} \text{ satisfies (3.1.3), and } \ z^\alpha_{\sigma,x}(s) \in \mathcal{K} \text{ in } (\sigma,1)\}. \end{aligned} \tag{3.1.6}$$

This problem is now classical and the characterization of $\bar{v}_1$ by a HJB equation falls into the already known theory if $\mathcal{K}$ satisfies some qualification conditions. Moreover, when $\mathcal{K}$ is the hole space $\mathbf{R}^d$ (no state constraints), it has been shown in [52] that the value function of the original problem (4.3.11) can be obtained by:

$$v(\tau,x) = \bar{v}_1(\mathcal{W}(\tau),x).$$

This relation is no more true when the control problem is in presence of state constraints (when $\mathcal{K} \neq \mathbf{R}^d$). Actually, as said before, by the graph-completion technics, to each trajectory y of the problem (3.1.1) correspond a trajectory z solution to the reparametrized system (3.1.3). However, it may happen that the trajectory y satisfies the state constraints while the trajectory z does not. Indeed, y and z coincide only on the branches of continuity of y. On these branches the state constraints should be satisfied for both y and z. However, z has also other branches corresponding to the fictive time intervals and it may happen that the state constraints fail to be satisfied on these intervals.

The first study is to deal with the case where the fictive branches of z satisfy the state constraints. To ensure this property, we make some controllability assumptions, then the relation of v and $\bar{v}$ holds true. And the control problem for $\bar{v}$ with state constraints $\mathcal{K}$ can be solved similarly as in [126].

The second study deals with the general case where the fictive branches of z may violate the state constraints eventually. In absence of any controllability assumptions, the idea is to relax the state constraints for the fictive branches. In this general case, it is more natural to consider the auxiliary control problem in the form of

$$\begin{aligned}\bar{v}_2(\sigma, x) \quad &= \quad \inf_{\alpha\in\mathcal{A}}\{\varphi(z^{\alpha}_{\sigma,x}(1), \\ &\quad z^{\alpha}_{\sigma,x} \text{ satisfies (3.1.3), and } z^{\alpha}_{\sigma,x}(s) \in \mathbb{K}_s \text{ in } (\sigma^-, 1)\},\end{aligned} \tag{3.1.7}$$

where $\mathbb{K}_s = \mathcal{K}$ for $s = \mathcal{W}(t)$ with $t \in [0, T] \setminus \mathcal{T}$ and $\mathbb{K}_s$ is any other big set containing all the trajectories for $s \in \cup_{t_i\in\mathcal{T}}[t_i^-, t_i^+]$. It is expected to define a $\mathbb{K}$ which is continuous, at least usc, in the time variable.

Hamilton-Jacobi approach for state-constrained control problems have been extensively studied in the literature [40, 84, 126**?** , 127]. When the state constraints are time-dependent, the characterization of the value function becomes more complicated [85].
The main idea to treat the time-dependent state constraints is to characterize the epigraph of the value function instead of characterizing the value function directly. Here, we extend the ideas developed in [2] to the case of time-dependent state constraints, and prove that the epigraph of ϑ can be characterized by means of a Lipschitz continuous viscosity solution of a time-measurable HJB equation (this notion of viscosity notion will be made precise in Section 4).

As mentioned in Chapter 2, to solve optimal control problems via HJB approach with Lipschitz continuous dynamics/Hamiltonians, the basic tools will be the theory of viscosity solutions and nonsmooth analysis. In the present work, we have firstly dealt with optimal control problems with time-measurable dynamics and viable state constraints, where the compatible theory is the theory of viscosity solutions under state constraints. Then the same problem with general state constraints (not necessarily viable) has been studied, and we apply the theory of viscosity solutions to the epigraph of the value function instead of the value function itself. Finally, we consider a more general case with discontinuous final costs, and the compatible theory is the theory of bilateral viscosity solutions.

3.2 Definition by graph completion and the control problem

In this section, we formulate a state-constrained control problem with discontinuous trajectories. Then, we recall the graph completion technics and the definition of solution for the state equation introduced in [47, 70].

3.2.1 The state equation and the graph completion technique

Let T be a fixed final time, $x \in \mathcal{K}$ be an initial position. Given a Radon measure μ and a control variable $\alpha \in \mathcal{A}$, we consider the controlled trajectory $y^{\alpha}_{x,\tau}(t) : \mathbf{R}^+ \to \mathbf{R}^d$ solution of

$$\begin{cases} dy(t) = g_0(t, y(t), \alpha(t))dt + g_1(t, y(t))d\mu \text{ for } t \in (\tau, T] \\ y(\tau^-) = x. \end{cases} \tag{3.2.1}$$

where α belongs to the set $\mathcal{A}$ of admissible controls, given by:

$$\mathcal{A} := \{\alpha : (0, T) \to \mathbf{R}^m \ \text{ measurable function}, \ \ \alpha(t) \in A \text{ a.e. in } (0, T)\},$$

with A a compact set of $\mathbf{R}^m$. The functions g_0 and g_1 will be assumed to satisfy:

(Hg1) $g_0 : (0, T) \times \mathbf{R}^d \times A \to \mathbf{R}^d$ and $g_1 : (0, T) \times \mathbf{R}^d \to \mathcal{M}_{d\times p}$ are measurable functions with respect to the time variable and are continuous with respect to the other variables. Moreover, for any $y \in \mathbf{R}^d$ and any $a \in A$, $g_0(\cdot, y, a) \in L^1(0, T)$ and $g_1(\cdot, y) \in L^1_\mu(0, T)$.

(Hg2) $\exists k_0 > 0$ such that $\forall y, z \in \mathbf{R}^d, a \in A, t \in \mathbf{R}^+$, we have:

$$|g_0(t, y, a) - g_0(t, z, a)| + |g_1(t, y) - g_1(t, z)| \leq k_0|y - z|.$$

$$|g_0(t, y, a)| \leq L_g \text{ and } |g_1(t, y)| \leq L_g, \ \forall\ y \ \in \mathbf{R}^d, \ a \in A, \text{ and a.e. } t \in \mathbf{R}^+.$$

Moreover, for a.e $t \in (0, T)$ and for every $x \in \mathbf{R}^d$, $g_0(t, x, A)$ is a convex set.

The state equation (3.1.1) is described by a driven-measure differential system, and as mentioned in the introduction, the jumps of the solution should be well described in order to define unambiguous notion of solution. Here we adapt the definition introduced in [47, 70]. Let B be the left continuous primitive of μ, i.e.

$$B(t) = \mu([0, t)), \tag{3.2.2}$$

then $B \in BV([0, T]; \mathbf{R}^p)$ and its distributional derivative $\dot{B}$ coincides with μ on $[0, T)$. Consider also $\mathcal{T} := \{t_i, i \in \mathcal{I}\}$ the set of all the discontinuity points of B, where $\mathcal{I}$ is the at most countable index of these discontinuity points.

Furthermore, let $\{\psi_t\}_{t\in\mathcal{T}}$ be a family of linear maps from $[0, 1]$ into $\mathbf{R}^M$ such that

$$\psi_{t_i}(t) := B(t_i^-) + t(B(t_i^+) - B(t_i^-)), \text{ for } t \in [0, 1], \ i \in \mathcal{I}. \tag{3.2.3}$$

Each ψ_{t_i} joins $B(t_i^-)$ to $B(t_i^+)$. We will denote by ξ the solution of:

$$\frac{d\xi(t)}{dt} = g_1(t, \xi(t))\frac{d\psi_{t_i}(t)}{dt}, \text{ for } \sigma \in (0, 1], \ \xi(0) = \bar{\xi}, \tag{3.2.4}$$

and we set $\xi(\bar{\xi}, \psi_{t_i}) := \xi(1) - \bar{\xi}$. Now, we are ready to state the definition of solution introduced by Dal Maso and Rampazzo in [70].

Definition 3.2.1. Fix initial position and time (τ, x) and a control variable $\alpha \in \mathcal{A}$, the function $y^{\alpha}_{\tau,x} \in BV([\tau, T]; \mathbf{R}^d)$ is a solution to (3.2.1) if for each Borel subset $\mathcal{B}$ of $]\tau, T[$ we have

$$\int_{\mathcal{B}} dy(t) = \int_{\mathcal{B}} g_0(t, y(t), \alpha(t))dt + \int_{\mathcal{B}\setminus\mathcal{T}} g_1(t, y(t))d\mu + \sum_{t_i \in \mathcal{T}\cap\mathcal{B}} \xi(y(t_i^-), \psi_{t_i}) \tag{3.2.5}$$

and $y(\tau^-) = x$. Moreover, if $\tau \in \mathcal{T}$ we have $y(\tau^+) = \xi(x, \psi_\tau)$.

Remark 3.2.2. Here for simplicity, we have considered $\{\psi_t\}_{t\in\mathcal{T}}$ as linear maps. In fact, $\{\psi_t\}_{t\in\mathcal{T}}$ can be any family of Lipschitz continuous maps from $[0, 1]$ into $\mathbf{R}^M$ with each ψ_t joining $B(t)$ to $B(t^+)$. But we also point out that a different choice of $\{\psi_t\}_{t\in\mathcal{T}}$ leads to a different definition of solution for (3.2.1).

This definition gives a precise notion for the solution of the equation (3.1.1). Recall now another definition based on the graph completion technique and which leads to a characterization of the solution through the unique absolutely continuous solution of a reparametrized system. In order to do that, we define $\mathcal{W} : [0, T] \to [0, 1]$ as follows:

$$\mathcal{W}(t) = \frac{t + V_0^t(B)}{T + V_0^T(B)}, \text{ for } t \in [0, T], \tag{3.2.6}$$

then $\mathcal{W}$ is continuous on $[0, T]\setminus\mathcal{T}$. The canonical graph completion of B corresponding to the family of linear functions $(\psi_t)_{t\in\mathcal{T}}$ is then defined by:

$$\begin{aligned} \Phi(s) &= (\phi^0; \phi^1)(s) \\ &= \begin{cases} (t; B(t)) & \text{if } s = \mathcal{W}(t), t \in [0, T]\setminus\{t_1, \ldots, t_M\} \\ \left(t_i; \psi_{t_i}\left(\frac{s-\mathcal{W}(t_i)}{[\mathcal{W}]_{t_i}}\right)\right) & \text{if } s \in [\mathcal{W}(t_i), \mathcal{W}(t_i^+)],\ t_i \in \mathcal{T}, \end{cases} \end{aligned} \tag{3.2.7}$$

where

$$\psi_{t_i}\left(\frac{s - \mathcal{W}(t_i)}{[\mathcal{W}]_{t_i}}\right) = B(t_i) + \frac{[B]_{t_i}}{[\mathcal{W}]_{t_i}}(s - \mathcal{W}(t_i)), \tag{3.2.8}$$

Following [70], we introduce the reparametrized system defined by:

$$\begin{cases} \frac{dz}{ds}(s) &= g_0(\phi^0(s), z(s), \alpha(\phi^0(s)))\frac{d\phi^0}{ds}(s) + \\ & \quad g_1(\phi^0(s), z(s))\left(\mu^a(\phi^0(s))\frac{d\phi^0}{ds}(s) + \frac{d\phi^1}{ds}(s)\right) \qquad \text{for } s \in (\sigma, 1], \\ z(\sigma) &= x. \end{cases} \tag{3.2.9}$$

where $\sigma := \mathcal{W}(\tau)$, μ^a is the absolutely continuous part of the measure μ with respect to the Lebesgue measure, i.e. $\mu(t) = \mu^a(t)dt + \mu^s$. We note that the derivatives of ϕ^0, ϕ^1 are measurable functions. Therefore, under assumptions (Hg1)-(Hg2), the Caratheodory system (3.2.9) has a unique solution and according to [52, Theorem 2.2]), the following holds.

Proposition 3.2.3. Assume **(Hg1)-(Hg2)**, then $y^{\alpha}_{\tau,x} \in BV([\tau,T];\mathbf{R}^d)$ is a solution of (3.2.1) (in the sense of Definition 3.2.1) if and only if there exists a solution $z^{\alpha}_{\sigma,x} \in AC([\sigma,1];\mathbf{R}^d)$ of (3.2.9) such that

$$z^{\alpha}_{\sigma,x}(\mathcal{W}(t)) = y^{\alpha}_{\tau,x}(t), \quad \forall\, t \in [\tau,T]. \tag{3.2.10}$$

The proof uses the same arguments introduced in [70, Theorem 2.2] for the Lipschitz continuous trajectories. The main difference here is to deal with the absolutely continuous trajectories which are less regular than Lipschitz arcs. To overcome this difficulty, we use a generalized chain rule for the composition of absolutely continuous functions and BV functions (presented in [50]).

The statement of proposition 3.2.3 links each BV trajectory solution of (3.2.1) with an absolutely continuous function satisfying the parametrized equation (3.2.9).

3.2.2 State constrained control problems

For a given measure μ and a given corresponding graph completion (ϕ^0,ϕ^1), consider the set of BV trajectories satisfying (3.2.1):

$$S_{[\tau,T]}(x) := \{y = y^{\alpha}_{\tau,x}, y \text{ satisfies (3.2.1) } \text{ in the sense of Definition 3.2.1 and } \alpha \in \mathcal{A}\},$$

and the set of reparametrized trajectories:

$$S^P_{[\sigma,1]}(x) := \{z = z^{\alpha}_{\sigma,x} \text{ satisfies (3.2.9) } \text{ and } \alpha \in \mathcal{A}\}.$$

Given a closed subset $\mathcal{K} \subset \mathbf{R}^d$ and a final cost function $\varphi : \mathbf{R}^d \to \mathbf{R}$, the Mayer control problem governed by the impulse systems is:

$$v(\tau,x) := \inf\left\{\varphi(y(T)),\ y \in S_{[\tau,T]}(x),\ \text{and } y(t) \in \mathcal{K} \text{ for } t \in [\tau,T]\right\}. \tag{3.2.11}$$

We assume in the sequel that:
(HC1) $\varphi : \mathbf{R}^d \to \mathbf{R}$ is a bounded Lipschitz continuous function.

It is easy to prove that the value function satisfies a classic Dynamic Programming Principle (see [113] for a general DPP). For each $\tau \in]0,T[$, and every $h \in [0, T-\tau]$, we have

$$\begin{aligned} &v(\tau,x) = \inf_{a\in\mathcal{A}} v(\tau+h, y^{\alpha}_{\tau,x}(\tau+h)) \quad \text{for } x \in \mathcal{K},\\ &v(\tau,x) = +\infty, \quad \text{for } x \notin \mathcal{K}. \end{aligned}$$

According to this DPP, we can formally derive the HJB equation:

$$\begin{cases} -v_t(t,x) + H(t,x,Dv(t,x)) = 0 & \text{for } (t,x) \in (0,T)\times\mathcal{K},\\ v(T,x) = \varphi(x) & \text{for } x \in \mathcal{K} \end{cases} \tag{3.2.12}$$

where the Hamiltonian is

$$H(t,x,p) = \sup_{a\in\mathcal{A}} \big\{ -p\cdot(g_0(t,x,a)+g_1(t,x)\mu)\big\}. \tag{3.2.13}$$

However, this equation is just formal and several difficulties arise when characterizing the value function by a HJB equation. The main difficulty comes from the fact that in general the value function is not C^1 and it is not clear in which sense the $Dv\cdot\mu$ should be understood. The second difficulty comes from the fact that the control problem is in presence of state constraints.

To deal with these difficulties, the idea would be to consider the reparametrized control problem instead of (3.2.12) (for $\sigma = \mathcal{W}(\tau)$):

$$\bar{v}_1(\sigma,x) := \inf\{\varphi(z(1)),\ z\in S^P_{[\sigma,1]}, z(s)\in\mathcal{K} \text{ for } s\in[\sigma,1]\}. \tag{3.2.14}$$

When the control problem is without state constraints (ie, when $\mathcal{K}\neq\emptyset$), we have (see [52]):

$$v(\tau,x) = \bar{v}_1(\mathcal{W}(\tau),x) \qquad \text{for any } x\in\mathbf{R}^d,\ \tau\in(0,T). \tag{3.2.15}$$

However, this relation may not be valid when the problem is in presence of state constraints (ie, when $\mathcal{K}\neq\mathbf{R}^d$). The reason is that even if an admissible trajectory y stays in $\mathcal{K}$ in $[\tau,T]$, it may happen that the reparametrized trajectory leave $\mathcal{K}$ during the "fictive" time intervalles $s\in[\mathcal{W}(t_i),\mathcal{W}(t_i^+)]$, where t_i is a discontinuous point of $\mathcal{W}$. The study is then divided into two parts with or without extra controllability assumptions for g_i.

3.3 Problems with pointwise state constraints

In this section, we consider the set of state constraints $\mathcal{K}$ as:

$$\mathcal{K} = \{x\ :\ h(x)\le 0\} \tag{3.3.1}$$

where $h\in C^{1,1}(\mathbf{R}^d)$.

Here in order to make sure that the "fictive" part of the trajectories of reparameterized system satisfies the state constraints, we need to consider the following viability condition: $\forall\ t\ge 0,\ x\in\partial\mathcal{K}, \forall\, i=1,\dots,p$,

$$g_1^i(t,x)\cdot\nabla_x h(x)\le 0, \tag{3.3.2}$$

where g_1^i is the i-th column of g_1. In view of Proposition 3.2.3, it is then natural to consider the auxiliary control problem governed by trajectories $Z^a_{X,\sigma}$ solutions of the reparameterized system 3.2.9. Then the corresponding value function is defined as follows:

$$\bar{v}_1(\sigma,x) = \inf_{\alpha\in\mathcal{A}} \big\{\varphi(z^\alpha_{\sigma,x}(1)),\ z^\alpha_{\sigma,x}(s)\in\mathcal{K} \text{ on } [\sigma,1]\big\}. \tag{3.3.3}$$

Theorem 3.3.1. Let v and $\bar{v}$ be defined respectively by (3.2.11) and (3.3.3). For each $x \in \mathcal{K}$ and $\tau \in [0, T]$, we have

$$v(\tau, x) = \bar{v}_1(\mathcal{W}(\tau), x) \tag{3.3.4}$$

where $\mathcal{W}$ is given by (3.2.6).

Proof. By Theorem 3.2.3 we have

$$y^{\alpha}_{\tau,x}(T) = z^{\alpha}_{\mathcal{W}(\tau),x}(\mathcal{W}(T)) = z^{\alpha}_{\sigma,x}(1),$$

then (3.3.4) holds by the definition of v and $\bar{v}_1$. □

According to this theorem, we can turn our attention to the HJB equation for the function $\bar{v}_1$ to avoid dealing with the Radon measures in the dynamics.

The dynamic programming principle satisfied by $\bar{v}_1$ leads to the following HJB equation:

$$\begin{cases} -\partial_s \bar{v}_1(s, x) + H(s, x, D\bar{v}_1(s, x)) = 0 & \text{for } (s, x) \in (0, 1) \times \mathcal{K}, \\ \bar{v}_1(1, x) = \varphi(x) & \text{for } x \in \mathcal{K}, \end{cases} \tag{3.3.5}$$

where the Hamiltonian is

$$\begin{aligned} H(s, x, p) \quad = \quad & \sup_{\alpha \in \mathcal{A}} \Big\{ -p \cdot \big(g_0(\phi^0(s), z^{\alpha}_{\sigma,x}(s), a(\phi^0(s)))\dot{\phi}^0(s) \\ & + g_1(\phi^0(s), z^{\alpha}_{\sigma,x}(s))(\mu^a(\phi^0(s))\dot{\phi}^0(s) + \dot{\phi}^1(s))\big) \Big\}. \end{aligned} \tag{3.3.6}$$

Note that $\mathcal{K}$ is a closed set. Moreover, the derivatives of ϕ^0 and ϕ^1 are just measurable functions, we should first make precise the definition of the constrained L^1-viscosity solution of (3.3.5).

3.3.1 State constrained optimal control problems with measurable time-dependent dynamics

In this section, we introduce the definition of viscosity solution for the HJB equation with a time measurable Hamiltonian and state constraints. To simplify the presentation, we state now the problem in a more general setting. Given $x \in \mathbf{R}^d$, $\tau > 0$ and a control $\alpha \in \mathcal{A}$, we consider the trajectory $y^a_{\tau,x}$ as the solution of the following system:

$$\begin{cases} \dot{y}(t) = f(t, y(t), \alpha(t)), & \text{for } t \in (\tau, 1) \\ y(\tau) = x, \end{cases} \tag{3.3.7}$$

where

$$f(t, y, \alpha) = g_0(\phi^0(t), y(t), \alpha(\phi^0(t)))\dot{\phi}^0(t) + g_1(\dot{\phi}^0(t), y(t))(\mu^a(\phi^0(t))\dot{\phi}^0(t) + \dot{\phi}^1(t))$$

which is measurable in t, Lipschitz continuous in y and continuous in α. Let $\mathcal{K}$ be the closed subset of $\mathbf{R}^d$ defined in (3.3.1) which is a smooth manifold. For each initial time and position $(\tau, x) \in [0, T) \times \mathcal{K}$, we define the set of admissible trajectories by

$$S^{\mathcal{K}}_{[\tau,1]}(x) := \left\{ y^{\alpha}_{\tau,x} \text{ solution of (3.3.7)}, y^{\alpha}_{\tau,x}(t) \in \mathcal{K} \text{ for } t \in [\tau, 1] \right\}.$$

Let $\varphi : \mathbf{R}^d \to \mathbf{R}$ be a given function satisfying:
(Hid) The function $\varphi : \mathbf{R}^d \to \mathbb{R}$ is Lipschitz continuous and bounded.
The optimal value function $\vartheta : \mathbf{R}^+ \times \mathbf{R}^d \to \mathbf{R}$ associated to this problem is defined by:

$$\vartheta(\tau, x) := \inf \left\{ \varphi(y^{\alpha}_{\tau,x}(1)),\ y^{\alpha}_{\tau,x} \in S^{\mathcal{K}}_{[\tau,1]}(x) \right\}. \tag{3.3.8}$$

Remark 3.3.2. We adopt the convention $\vartheta(\tau, x) = \|\varphi\|_{L^\infty(\mathcal{K})} + 1$, when the set of admissible trajectories is empty: $S^{\mathcal{K}}_{[\tau,1]}(x) = \emptyset$. Of course this value can be replaced by any other constant bigger than $\|\varphi\|_{L^\infty(\mathcal{K})}$, and eventually by $+\infty$. But we need to take a finite constant in order to deal with finite valued functions.

Remark 3.3.3. Let us recall that under assumptions **(Hco)**, **(Hg1)-(Hg2)**, for every $a \in \mathcal{A}$, the differential equation (3.4.4) admits an absolutely continuous solution.

(HK1) Soner's inward pointing qualification condition: $\exists\, \beta > 0,\ \forall\, t \in [0, 1],\ y \in \partial\mathcal{K}$,

$$\exists\, a \in A \text{ s.t. } f(t, y, a) \cdot \nabla_x h(y) < -\beta. \tag{3.3.9}$$

Our first aim is to characterize the function ϑ in (3.3.8) as the unique L^1 viscosity solution (see the definition below) of the following HJB equation:

$$\begin{cases} -u_t(t, x) + H(t, x, Du(t, x)) = 0 & \text{for } (t, x) \in (0, 1) \times \mathcal{K}, \\ u(1, x) = \varphi(x) & x \in \mathcal{K} \end{cases} \tag{3.3.10}$$

where the Hamiltonian is

$$H(t, x, p) = \sup_{a \in A} \{-p \cdot f(t, x, a)\}. \tag{3.3.11}$$

3.3.2 Uniform continuity of the value function

We recall the dynamic programming principle for $\vartheta(\tau, x)$:

Proposition 3.3.4. Assume **(Hco)**, **(Hg1)-(Hg2)**, **(HC1)** and **(HK1)**. Then the value function ϑ satisfies the following:

i) for all $x \in \mathcal{K}$,

$$\vartheta(1, x) = \varphi(x).$$

ii) Dynamic programming principle: for all $x \in \mathcal{K}$, $\tau \in [0,1]$ and $h \in [0, 1-\tau]$, we have:

$$\vartheta(\tau, x) = \inf_{y^\alpha_{\tau,x} \in S^{\mathcal{K}}_{[\tau,1]}(x)} \vartheta(\tau + h, y^\alpha_{\tau,x}(\tau + h)), \tag{3.3.12}$$

We will prove the continuity of the value function on $(0,1) \times \mathcal{K}$. At first, let us recall the following result: **(Neighbouring feasible trajectories theorem)** in [35, Theorem 2.1]. It gives $W^{1,1}$ estimates for the trajectories under state constraints, which are important for the continuity of the value function on the boundary

Lemma 3.3.5. Assume **(Hco), (Hg1)-(Hg2)** and **(HK1)**. Take any $r_0 > 0$, given $t_0 \in [0,1]$ and an absolutely continuous $\hat{y}(\cdot)$ driven by f, there exists a constant C and an absolutely continuous trajectory $y(\cdot)$ such that

$$x(t) \in \mathcal{K}, \ \forall\, t \in [t_0, 1], \ \|y - \hat{y}\|_{W^{1,1}([t_0,1];\mathbf{R}^d)} \leq C h^+(\hat{y}(\cdot)),$$

where

$$h^+(\hat{y}(\cdot)) = \max_{t \in [t_0,1]} \{h(\hat{y}(t)) \vee 0\}.$$

According to this lemma, we note that for any $x_0 \in \overset{\circ}{\mathcal{K}}$, there exists an admissible trajectory x on $[t_0, 1]$ such that

$$x(t_0) = x_0, \ x(t) \in \overset{\circ}{\mathcal{K}}, \ \forall\, t \in [t_0, 1].$$

In fact, there exists a small enough $\epsilon > 0$ such that

$$x_0 \in \mathcal{K}_\epsilon := \{x \, : \, h(x) + \epsilon \leq 0\},$$

$$\min_{\nu \in f(t,x,A)} \nabla h(x) \cdot \nu < -\frac{\alpha}{2}, x \in \partial \mathcal{K}_\epsilon, \ t \in [0,1],$$

by the continuity of ∇h and $x \to f(t, x, A)$. Then by this theorem there exists an admissible trajectory contained in $\mathcal{K}_\epsilon$ which is in $\overset{\circ}{\mathcal{K}}$.

Proposition 3.3.6. Assume **(Hco)**, **(Hg1)-(Hg2)**, **(HC1)** and **(HK1)**, the value function $\vartheta(\cdot,\cdot)$ is continuous on $(0,1) \times \overset{\circ}{\mathcal{K}}$.

Proof. Fix $\tau \in [0,1]$, let us first prove that $\vartheta(\tau, \cdot)$ is continuous on $\overset{\circ}{\mathcal{K}}$. $\forall x \in \overset{\circ}{\mathcal{K}}$, let $\alpha \in \mathcal{A}$ and $y^\alpha(\cdot)$ be the solution of $\dot{y} = f(t, y, a), y(\tau) = x$ such that $y(\cdot) \in \overset{\circ}{\mathcal{K}}$ (by lemma 3.3.5). Suppose that $x_n \in \overset{\circ}{\mathcal{K}}$ and $x_n \to x$ when $n \to +\infty$. Let $y^\alpha_n(\cdot)$ and $y^\alpha(\cdot)$ be the solutions of $\dot{y}^\alpha_n = f(t, y^\alpha_n, \alpha), y^\alpha_n(\tau) = x_n$. By **(Hco)** and Gronwall, we get

$$|y^\alpha_n(t) - y^\alpha(t)| \leq \exp\left(\int_\tau^t k_0(s) ds\right) |x_n - x|, \tag{3.3.13}$$

so $y^\alpha_n(\cdot)$ converge to $y^\alpha(\cdot)$ uniformly on $[0,1]$, and as $\overset{\circ}{\mathcal{K}}$ is open, $y^\alpha_n(\cdot) \in \overset{\circ}{\mathcal{K}}$ when n is big enough. As φ is continuous, we get $\varphi(y^\alpha_n(1)) \to \varphi(y^\alpha(1))$ uniformly on α. Then we have $\vartheta(\tau, x_n) \to \vartheta(\tau, x)$, and we get the continuity of ϑ in x.

Now we fix $x \in \overset{\circ}{\mathcal{K}}$ and we will prove that $\vartheta(\cdot, x)$ is continuous on $(0,1)$. $\forall \tau \in (0,1)$ and $\forall \epsilon > 0$, let $a \in \mathcal{A}$ and y^a the associated solution such that

$$\varphi(y^a(1)) \leq \vartheta(\tau, x) + \epsilon. \tag{3.3.14}$$

For each $x \in \mathcal{K}$, let $\tau_n \in (0,T)$ and $\tau_n \to \tau$. Without loss of generality, we suppose that $\tau_n > \tau$. Then we have

$$\vartheta(\tau, x) \leq \vartheta(\tau_n, y^\alpha(\tau_n)) \leq \varphi(y^\alpha(1)) \leq \vartheta(\tau, x) + \epsilon,$$

then we have

$$|\vartheta(\tau_n, y^\alpha(\tau_n)) - \vartheta(\tau, x)| \leq \epsilon,$$

and

$$|\vartheta(\tau_n, x) - \vartheta(\tau, x)| \leq |\vartheta(\tau_n, x) - \vartheta(\tau_n, y^\alpha(\tau_n))| + \epsilon.$$

By assumption **(Hg2)**, we have

$$y^\alpha(\tau_n) - x = \int_\tau^{\tau_n} f(t, y^\alpha, \alpha) \leq K|\tau_n - \tau|.$$

So when $\tau_n \to \tau$, $y^\alpha(\tau_n) \to x$, and by the continuity of $\vartheta(\tau_n, \cdot)$, we get $\vartheta(\tau_n, y^\alpha(\tau_n)) \to \vartheta(\tau_n, x)$. Finally we have

$$\vartheta(\tau_n, x) \to \vartheta(\tau, x),$$

where we prove the continuity of ϑ in τ. □

In order to prove the continuity of the value function on the boundary, we use the relaxation method. The following proposition is a result of relaxation of state constraints:

Proposition 3.3.7. Assume **(Hco)**, **(Hg1)-(Hg2)**, **(HC1)** and **(HK1)**. Consider $(\mathcal{K}^\epsilon)_{\epsilon>0}$ a sequence of subsets of $\mathbf{R}^d$ such that

$$\mathcal{K}^\epsilon = \{x \,:\, h(x) - \epsilon \leq 0\},$$

and we denote by ϑ_ϵ the value function associated to the control problem (3.3.8) with state constraints in $\mathcal{K}^\epsilon$ (instead of $\mathcal{K}$). Then

$$\lim_{\epsilon \to 0} \vartheta_\epsilon(t, x) = \vartheta(t, x) \text{ uniformly on } (0,T) \times \mathcal{K}.$$

Proof. By the definition of $\mathcal{K}^\epsilon$, for every $x \in \mathcal{K}$, every $\epsilon > 0$ and $\eta \in (0, \epsilon)$ we have

$$\mathcal{K} \subset \mathcal{K}^\eta \subset \overset{\circ}{\mathcal{K}^\epsilon}, \ \lim_{\epsilon \to 0} d(x, \mathbf{R}^d \backslash \mathcal{K}^\epsilon) = 0, \tag{3.3.15}$$

then, for $t \in (0,1)$ and $x \in \mathcal{K}$ given, we have

$$\vartheta_\epsilon(t, x) \leq \vartheta(t, x). \tag{3.3.16}$$

Let us set $l := \liminf_{\epsilon\to 0} \vartheta_\epsilon(t,x)$. For $k \in \mathbb{N}$ large enough, $\exists\, \epsilon_k > 0$ such that $\epsilon_k \to 0$ and

$$\vartheta_{\epsilon_k}(t,x) \le l + \frac{1}{2k},$$

then by the definition of the value function ϑ_{ϵ_k}, there exists a trajectory $y^{\epsilon_k}_{x,t} \in S^{\mathcal{K}^{\epsilon_k}}_{[t,1]}(x)$ such that

$$\varphi(y^{\epsilon_k}_{x,t}(1)) \le \vartheta_{\epsilon_k}(t,x) + \frac{1}{2k} \le l + \frac{1}{k}. \tag{3.3.17}$$

By the compactness of $S^{\mathcal{K}^{\epsilon_k}}_{[t,1]}(x)$, we can extract from $y^{\epsilon_k}_{x,t}$ a convergent subsequence towards some trajectory $y_{x,t} \in S^{\mathcal{K}^{\epsilon_k}}_{[t,1]}(x)$ for every $k > 0$. We then obtain that $y_{x,t} \in S^{\mathcal{K}}_{[t,1]}(x)$ by using (3.3.15). Let k tend to $+\infty$ in (3.3.17) and use the fact that φ is continuous, we prove that

$$\varphi(y_{x,t}(1)) \le l.$$

Then we have

$$\vartheta(t,x) \le \varphi(y_{x,t}(1)) \le l = \lim\inf_{\epsilon\to 0} \vartheta_\epsilon(t,x). \tag{3.3.18}$$

Combining (3.3.16) and (3.3.18), we get that

$$\vartheta(t,x) \le \lim\inf_{\epsilon\to 0} \vartheta_\epsilon(t,x) \le \lim\sup_{\epsilon\to 0} \vartheta_\epsilon(t,x) \le \vartheta(t,x),$$

which implies $\lim_{\epsilon\to 0} \vartheta_\epsilon(t,x) = \vartheta(t,v)$.
For each $\epsilon > 0$, let $y^\epsilon_{x,t} \in S^{\mathcal{K}^\epsilon}_{[t,1]}(x)$ such that $\vartheta_\epsilon(t,x) = \varphi(y^\epsilon_{x,t}(1))$. By lemma 3.3.5, there exist a $\hat{y}_{x,t} \in S^{\mathcal{K}}_{[t,1]}(x)$ and a constant K such that

$$\|\hat{y}_{x,t} - y^\epsilon_{x,t}\|_{W^{1,1}([t,1];\mathbf{R}^d)} \le K\epsilon.$$

Then we have

$$\begin{aligned} 0 \le \vartheta(t,x) - \vartheta_\epsilon(t,x) &\le \varphi(\hat{y}_{x,t}(1)) - \varphi(y^\epsilon_{x,t}(1)) \\ &\le m_\varphi \|\hat{y}_{x,t} - y^\epsilon_{x,t}\|_{W^{1,1}} \le m_\varphi K\epsilon, \end{aligned}$$

where $m_\varphi(\cdot)$ is the Lipshitz constant of φ. Let $\epsilon \to 0$, we get that $\vartheta_\epsilon \to \vartheta$ uniformly on $(0,1)\times\mathcal{K}$.
□

And finally the theorem:

Theorem 3.3.8. Assume **(Hco) (Hg1)-(Hg2) (HC1)** and **(HK1)**, the value function $\vartheta(\cdot,\cdot)$ is uniformly continuous and bounded on $(0,1)\times\mathcal{K}$.

Proof. We only need to prove that $\forall\,\tau \in (0,1)$, $\vartheta(\tau,\cdot)$ is continuous on $\mathcal{K}$. For every $\tau \in (0,1)$ and every $x \in \mathcal{K}$, define $\mathcal{K}^\epsilon$ and ϑ_ϵ as in Proposition 3.3.7. Then we have

$$\vartheta(\tau,x) = \lim_{\epsilon\to 0} \vartheta_\epsilon(\tau,x) \text{ uniformly on } (0,1)\times\mathcal{K}.$$

According to **(HK1)**, by the continuity of $\nabla h(x)$ and $f(t,x,p)$ on x, for small ϵ, we have $\forall\ t \in [0,1],\ y \in \partial\mathcal{K}^\epsilon$,

$$\exists\, a \in \mathcal{A} \text{ s.t. } f(t,y,a)\cdot\nabla_x h(y) < -\frac{\beta}{2}. \tag{3.3.19}$$

Using (3.3.15) and Proposition 3.3.6, we get that $\vartheta_\epsilon(\tau,\cdot)$ is continuous on $\mathcal{K} \subset \mathring{\mathcal{K}}^\epsilon$ and bounded, then by the uniform convergence of ϑ_ϵ, the limit $\vartheta(\tau,\cdot)$ is continuous on $\mathcal{K}$.
Finally, since φ is Lipschitz continuous and bounded, we obtain that ϑ is uniformly continuous and bounded on $(0,1)\times\mathcal{K}$. □

3.3.3 Definition of L^1-viscosity solutions of HJB equations

This section is devoted to the definition of the L^1-viscosity solutions of the HJB equation (3.3.10) and the characterization of the value function ϑ. The following definition can be seen as the combination of the definition of L^1-viscosity solutions for the HJB equations with a time measurable Hamiltonian introduced in [52, 102, 110] and the definition of constrained viscosity solutions introduced in Soner [126].

Definition 3.3.9. (L^1-viscosity solution) Let $u:(0,T]\times\mathcal{K}\to\mathbf{R}$ be a bounded Lipschitz continuous function.

- We say that u is a L^1-viscosity super-solution if $\forall\, b \in L^1(0,1)$, $\phi \in C^1(\mathbf{R}^d)$ and $(t_0,x_0) \in (0,1)\times\mathcal{K}$ local minimum point of $u(t,x) - \int_0^t b(s)ds - \phi(x)$, we have

$$\lim_{\delta\to 0^+}\ \operatorname*{ess\,sup}_{|t-t_0|\le\delta}\ \sup_{x\in B(x_0,\delta)\cap\mathcal{K},\, p\in B(D\phi(x_0),\delta)} \{H(t,x,p) - b(t)\} \ge 0.$$

- We say that u is a L^1-viscosity sub-solution if $\forall\, b \in L^1(0,1)$, $\phi \in C^1(\mathbf{R}^d)$ and $(t_0,x_0) \in (0,1)\times\mathring{\mathcal{K}}$ local maximum point of $u(t,x) - \int_0^t b(s)ds - \phi(x)$, we have

$$\lim_{\delta\to 0^+}\ \operatorname*{ess\,inf}_{|t-t_0|\le\delta}\ \inf_{x\in B(x_0,\delta),\, p\in B(D\phi(x_0),\delta)} \{H(t,x,p) - b(t)\} \le 0.$$

- We say that u is a L^1-viscosity solution if it is both a L^1-viscosity super-solution and a L^1-viscosity sub-solution and the final condition is satisfied:

$$u(1,x) = \varphi(x) \text{ in } \mathcal{K}.$$

Remark 3.3.10. In fact, there are many other formulations. For example we may replace $\phi \in C^1$ by $\phi \in C^2$, C^∞, ... We may also replace local maximum by global, or local strict, or global strict. We can also give another equivalent formulation of definition by generalizing the definition introduced by Ishii [102] to a closed subset $\mathcal{K}$. For more details, see Lions and Perthame [110].

Theorem 3.3.11. Suppose **(Hco)**, **(Hg1)-(Hg2)**, **(HC1)**, **(HK1)** hold. Then the value function ϑ is a L^1-viscosity solution of (3.3.10).

Proof. We first prove that ϑ is a L^1-viscosity super-solution. Let $b \in L^1(0,1)$, $\phi \in C^1(\mathbf{R}^d)$ and $(t_0, x_0) \in (0,1) \times \mathcal{K}$ local minimum point of $\vartheta(t,x) - \int_0^t b(s)ds - \phi(x)$. Without loss of generality, we suppose that

$$\vartheta(t_0, x_0) - \int_0^{t_0} b(s)ds - \phi(x_0) = 0, \tag{3.3.20}$$

then we have $\exists\, \delta > 0$ small enough such that for $t \in [t_0 - \delta, t_0 + \delta],\ x \in B(x_0, \delta) \cap \mathcal{K}$,

$$\vartheta(t,x) - \int_0^t b(s)ds - \phi(x) \geq 0. \tag{3.3.21}$$

By the DPP, $\forall \epsilon > 0$, $\exists\, \alpha \infty \mathcal{A}$ s.t. $\forall\, h \in [0, 1 - t_0]$,

$$\vartheta(t_0 + h, y^a_{t_0,x_0}(t_0 + h)) \leq \vartheta(t_0, x_0) + \epsilon. \tag{3.3.22}$$

Let h be small enough $(h \leq \delta)$. By (3.3.21) we get

$$\vartheta(t_0 + h, y^\alpha_{t_0,x_0}(t_0 + h)) \geq \int_0^{t_0+h} b(s)ds + \phi(y^\alpha_{t_0,x_0}(t_0 + h)). \tag{3.3.23}$$

By (3.3.20), (3.3.22) and (3.3.23), we have

$$\int_0^{t_0+h} b(s)ds + \phi(y^\alpha_{t_0,x_0}(t_0 + h)) \leq \int_0^{t_0} b(s)ds + \phi(x_0) + \epsilon.$$

Then

$$\phi(x_0) - \phi(y^\alpha_{t_0,x_0}(t_0 + h)) - \int_{t_0}^{t_0+h} b(s)ds + \epsilon \geq 0,$$

i.e.

$$-\int_{t_0}^{t_0+h} \big[D\phi(y^\alpha_{t_0,x_0}(s)) \cdot f(s, y^\alpha_{t_0,x_0}(s), a) + b(s)\big] ds + \epsilon \geq 0.$$

Then by the definition of the Hamiltonian, we have $\forall\, \epsilon > 0$

$$\int_{t_0}^{t_0+h} \big[H(s, y^\alpha_{t_0,x_0}(s), D\phi(y^\alpha_{t_0,x_0}(s))) - b(s)\big] ds + \epsilon \geq 0,$$

and we deduce that

$$\int_{t_0}^{t_0+h} \big[H(s, y^\alpha_{t_0,x_0}(s), D\phi(y^\alpha_{t_0,x_0}(s))) - b(s)\big] ds \geq 0. \tag{3.3.24}$$

By contradiction, if

$$\lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|t-t_0| \leq \delta} \sup_{x \in B(x_0,\delta) \cap \mathcal{K}, p \in B(D\phi(x_0),\delta)} \{H(t,x,p) - b(t)\} < 0,$$

then $\exists\, \delta_1 > 0,\ E \subset [t_0 - \delta_1, t_0 + \delta_1]$ with $m(E) = 0$ such that $\forall\, t \in [t_0 - \delta_1, t_0 + \delta_1] \backslash E$, $x \in B(x_0, \delta_1)$ and $p \in B(D\phi(x_0), \delta_1)$, we have $H(t,x,p) - b(t) < 0$. By the continuity of $Y^a_{x_0,t_0}(\cdot)$, $D\phi(\cdot)$ and

$H(t,\cdot,\cdot)$, if h is small enough, we get that for $s \in [t_0, t_0+h]\backslash E$,

$$H(s, y^{\alpha}_{t_0,x_0}(s), D\phi(y^{\alpha}_{t_0,x_0}(s))) - b(s) < 0, \tag{3.3.25}$$

which is a contradiction with (3.3.24).

Now we start to prove that ϑ is a L^1-viscosity sub-solution. Let $b \in L^1(0,1)$, $\phi \in C^1(\mathbf{R}^d)$ and $(t_0, x_0) \in (0,1) \times \mathring{\mathcal{K}}$ local maximum point of $\vartheta(t,x) - \int_0^t b(s)ds - \phi(x)$. Without loss of generality, we suppose that

$$\vartheta(t_0, x_0) - \int_0^{t_0} b(s)ds - \phi(x_0) = 0. \tag{3.3.26}$$

Then, $\exists\, \delta > 0$ small enough such that for $t \in [t_0 - \delta, t_0 + \delta],\ x \in B(x_0, \delta) \cap \mathcal{K}$,

$$\vartheta(t,x) - \int_0^{t} b(s)ds - \phi(x) \leq 0. \tag{3.3.27}$$

By the DPP, $\forall\, \alpha \in \mathcal{A}$ and $y^{\alpha}_{t_0,x_0} \in \mathcal{K}$, we have $\forall\, h \in [0, 1-t_0]$,

$$\vartheta(t_0, x_0) \leq \vartheta(t_0 + h, y^{\alpha}_{t_0,x_0}(t_0+h)). \tag{3.3.28}$$

Let h small enough ($h \leq \delta$), by (3.3.27) we get

$$\vartheta(t_0 + h, y^{\alpha}_{t_0,x_0}(t_0+h)) \leq \int_0^{t_0+h} b(s)ds + \phi(y^{\alpha}_{t_0,x_0}(t_0+h)). \tag{3.3.29}$$

By (3.3.26), (3.3.28) and (3.3.29), we have

$$\int_0^{t_0} b(s)ds + \phi(x_0) \leq \int_0^{t_0+h} b(s)ds + \phi(y^{\alpha}_{t_0,x_0}(t_0+h))$$

then

$$\phi(x_0) - \phi(y^{\alpha}_{t_0,x_0}(t_0+h)) - \int_{t_0}^{t_0+h} b(s)ds \leq 0,$$

i.e.

$$-\int_{t_0}^{t_0+h} \big[D\phi(y^{\alpha}_{t_0,x_0}(s)) \cdot f(s, y^{\alpha}_{t_0,x_0}(s), a) + b(s)\big] ds \leq 0,$$

then by the definition of the Hamiltonian, we have

$$\int_{t_0}^{t_0+h} \big[H(s, y^{\alpha}_{t_0,x_0}(s), D\phi(y^{\alpha}_{t_0,x_0}(s))) - b(s)\big] ds \leq 0.$$

By the same argument as above, we get

$$\lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t-t_0|\leq\delta} \inf_{x\in B(x_0,\delta), p\in B(D\phi(x_0),\delta)} \{H(t,x,p) - b(t)\} \leq 0.$$

Finally, by the definition of ϑ, we have $\vartheta(1,x) = \varphi(x)$ because $y^{\alpha}_{1,x}(1) = x$. □

3.3.4 Uniqueness of the L^1 constrained viscosity solutions of HJB equations

This section is devoted to the main properties of the L^1-viscosity solutions we have defined in Definition 3.3.9. The uniqueness and stability results are given later.

Consider the following Hamilton-Jacobi-Bellman equation

$$\begin{cases} -u_t(t,x) + H(t,x,Du(t,x)) = 0 & \text{for } (t,x) \in (0,1) \times \mathcal{K}, \\ u(1,x) = \varphi(x) & x \in \mathcal{K}. \end{cases} \tag{3.3.30}$$

We prove the comparison principle from which we can deduce the uniqueness of L^1-viscosity solution of (3.3.10).

Theorem 3.3.12. **(Comparison Principle)**
Assume that **(Hco)**, **(Hg1)-(Hg2)**, and **(HC1)** hold, let u_1, u_2 be two bounded uniformly continuous functions. Suppose that u_1 is a L^1-viscosity sub-solution of the HJB equation (3.3.10), u_2 is a L^1-viscosity super-solution of (3.3.10), and u_1, u_2 satisfy the final condition $u_1(1,x) \le \varphi(x) \le u_2(1,x)$ for every $x \in \mathcal{K}$. Then for every $t \in [0,1]$ and $x \in \mathcal{K}$, we have

$$u_1(t,x) \le u_2(t,x).$$

Before we give the proof, we state the following lemma:

Lemma 3.3.13. Assume that u is a L^1-viscosity sub-solution (resp. super-solution) of (3.3.10), then $\forall \gamma \in \mathbf{R}$, the function $v = ue^{\gamma t}$ is a L^1-viscosity sub-solution (resp. super-solution) of

$$\begin{cases} -v_t(t,x) + \gamma v + H(t,x,Dv(t,x)) = 0 & \text{for } (t,x) \in (0,T) \times \mathcal{K}, \\ v(T,x) = \varphi(x)e^{\gamma T} & x \in \mathcal{K}, \end{cases} \tag{3.3.31}$$

in the following sense:
$\forall\, b \in L^1(0,T)$, $\phi \in C^1(K)$ and $(t_0,x_0) \in (0,T) \times \overset{\circ}{\mathcal{K}}$ local maximum point of $v(t,x) - \int_0^t b(s)ds - \phi(x)$, we have

$$\lim_{\delta \to 0^+} \operatorname*{ess\,inf}_{|t-t_0| \le \delta} \inf_{x \in B(x_0,\delta), p \in B(D\phi(x_0),\delta)} \{H(t,x,p) + \gamma v(t,x) - b(t)\} \le 0,$$

and respectively
$\forall\, b \in L^1(0,T)$, $\phi \in C^1(K)$ and $(t_0,x_0) \in (0,T) \times \mathcal{K}$ local minimum point of $v(t,x) - \int_0^t b(s)ds - \phi(x)$, we have

$$\lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|t-t_0| \le \delta} \sup_{x \in B(x_0,\delta) \cap \mathcal{K}, p \in B(D\phi(x_0),\delta)} \{H(t,x,p) + \gamma v(t,x) - b(t)\} \ge 0,$$

Proof. Let $b \in L^1(0,T)$, $\phi \in C^1(K)$ and $(t_0,x_0) \in (0,T) \times \overset{\circ}{\mathcal{K}}$ be a local maximum point of $v(t,x) - \int_0^t b(s)ds - \phi(x)$, using the fact that $v = ue^{\gamma t}$ and $e^{\gamma t} = 1 + \int_0^t \gamma e^{\gamma s} ds$, we deduce that (t_0,x_0) is a local maximum point of $u(t,x) + \int_0^t \gamma e^{\gamma s} u(t,x) ds - \int_0^t b(s)ds - \phi(x)$, then by the definition of u, we have

$$\lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t-t_0|\le\delta} \inf_{x\in B(x_0,\delta),p\in B(D\phi(x_0),\delta)} \{H(t,x,p)+\gamma e^{\gamma t}u(t,x)-b(t)\}\le 0,$$

which is the result. And the proof for the case of super-solutions is exactly similar. □

Proof. Let $v_1 = u_1 e^t$ and $v_2 = u_2 e^t$, then by Lemma 3.3.13 with $\gamma = 1$, v_1, v_2 are respectively L^1-viscosity sub-solution on $\overset{\circ}{\mathcal{K}}$ and L^1-viscosity super-solution on K to the HJB equation

$$-v_t(t,x)+v(t,x)+H(t,x,Dv(t,x))=0,\ \text{for } t\in(0,T),\ x\in\mathcal{K}.$$

Since $\mathcal{K}$ has a $C^{1,1}$ boundary, then $\mathcal{K}$ satisfies the following property:

(K1) There exists positive constants h, r and an $\mathbf{R}^{d+1}$-value bounded, uniformly continuous map η of $\mathcal{K}$ such that

$$B(y+t\eta(y),rt)\subseteq\mathcal{K},\ \forall\, y\in\mathcal{K} \text{ and } t\in(0,h].$$

This is actually the assumption **(A1)** in [126]. Let η, r, h be as in **(K1)**, pick $\rho > 0$ such that

$$|\eta(x)-\eta(y)|\le\frac{r}{2},\ \forall x,y\in\mathcal{K} \text{ and } |x-y|<\rho. \tag{3.3.32}$$

Suppose that

$$M:=\sup_{(t,x)\in[0,T]\times\mathcal{K}}\{v_1(t,x)-v_2(t,x)\}>0. \tag{3.3.33}$$

For all $\sigma\in(0,M)$, let $t_\sigma\in(0,M)$ and $z_\sigma\in\mathcal{K}$ such that

$$v_1(t_\sigma,z_\sigma)-v_2(t_\sigma,z_\sigma)\ge M-\sigma>0.$$

Define $\Phi^\varepsilon:[0,T]\times[0,T]\times\mathcal{K}\times\mathcal{K}\to\mathbf{R}$ as follows:

$$\begin{aligned}\Phi^\varepsilon(t',s',x',y') &= v_1(t',x')-v_2(s',y')-\left|\frac{x'-y'}{\varepsilon}-\frac{2}{r}\eta(z_\sigma)\right|^2-\left|\frac{y'-z_\sigma}{\rho}\right|^2\\ &-\left|\frac{t'-t_\sigma}{\nu}\right|^2-\left|\frac{t'-s'}{\alpha}\right|^2+\int_0^{t'}b_\epsilon(\tau)d\tau+\int_0^{s'}b_\epsilon(\tau)d\tau,\end{aligned} \tag{3.3.34}$$

where $b_\epsilon(\cdot)\in L^1(\mathbf{R}^+)$ is positive and $b_\epsilon\to 0$ in $L^1(\mathbf{R}^+)$ when $\epsilon\to 0$. Note that $z_\sigma+(2\varepsilon/r)\eta(z_\sigma)$ is in $\mathcal{K}$ for small ε. We have

$$\begin{aligned}&\Phi^\varepsilon(t_\sigma,t_\sigma,z_\sigma+\frac{2\varepsilon}{r}\eta(z_\sigma),z_\sigma)\\ =\ & u_1(t_\sigma,z_\sigma+\frac{2\varepsilon}{r}\eta(z_\sigma))-u_2(t_\sigma,z_\sigma)+2\int_0^{t_\sigma}b_\epsilon(\tau)d\tau\\ \ge\ & u_1(t_\sigma,z_\sigma)-u_2(t_\sigma,z_\sigma)-\omega_1(c_1\varepsilon)+2\int_0^{t_\sigma}b_\epsilon(\tau)d\tau\\ \ge\ & M-\sigma-\omega_1(c_1\varepsilon),\end{aligned} \tag{3.3.35}$$

where $c_1 = 2/r \max\{\eta(\cdot)\}$ and $\omega_1(\cdot)$ is the modulus of continuity of $u_1(t_0, \cdot)$. Let $\sigma, \varepsilon, \eta$ be small enough such that

$$\sigma + \omega_1(c_1\varepsilon) < M,$$

then we have

$$\max_{[0,T]\times[0,T]\times\mathcal{K}\times\mathcal{K}} \Phi^\varepsilon(t,s,x,y) \geq \Phi^\varepsilon(t_\sigma, t_\sigma, z_\sigma + \frac{2\varepsilon}{r}\eta(z_\sigma), z_\sigma) > 0. \tag{3.3.36}$$

Suppose that Φ^ε achieves its maximum at (t,s,x,y), then by (3.3.35) and (3.3.36) we have

$$\begin{aligned}
& \left|\frac{x-y}{\varepsilon} - \frac{2}{r}\eta(z_\sigma)\right|^2 + \left|\frac{y-z_\sigma}{\rho}\right|^2 + \left|\frac{t-t_\sigma}{\nu}\right|^2 + \left|\frac{t-s}{\alpha}\right|^2 \\
\leq\ & v_1(t,x) - v_2(s,y) - (v_1(t_\sigma, z_\sigma) - v_2(t_\sigma, z_\sigma)) \\
& + \omega_1(c_1\varepsilon) + \int_{t_\sigma}^t b_\epsilon(\tau)d\tau + \int_{t_\sigma}^s b_\epsilon(\tau)d\tau \\
\leq\ & v_1(s,y) - v_2(s,y) - (v_1(t_\sigma, z_\sigma) - v_2(t_\sigma, z_\sigma)) \\
& + \omega_1(|t-s| + |x-y|) + \omega_1(c_1\varepsilon) + \int_{t_\sigma}^t b_\epsilon(\tau)d\tau + \int_{t_\sigma}^s b_\epsilon(\tau)d\tau \\
\leq\ & \sigma + \omega_1(|t-s| + |x-y|) + \omega_1(c_1\varepsilon) + 2\int_0^T b_\epsilon(\tau)d\tau.
\end{aligned} \tag{3.3.37}$$

Since ω_1 is bounded and $b_\epsilon \in L^1(\mathbf{R}^+)$, there exists an $M_0 > 0$ such that

$$\sigma + \omega_1(|t-s| + |x-y|) + \omega_1(c_1\varepsilon) + 2\int_0^T b_\epsilon(\tau)d\tau \leq M_0^2.$$

Together with (3.3.37) we obtain that

$$|x-y| \leq c_0\varepsilon, \ \ |t-s| \leq M_0\alpha,$$

where $c_0 = c_1 + M_0$ is a positive constant. Then we have

$$\left|\frac{x-y}{\varepsilon} - \frac{2}{r}\eta(z_\sigma)\right|^2 + \left|\frac{y-z_\sigma}{\rho}\right|^2 + \left|\frac{t-t_\sigma}{\nu}\right|^2 + \left|\frac{t-s}{\alpha}\right|^2 \leq h(\sigma, \varepsilon, \alpha, \epsilon), \tag{3.3.38}$$

where h is a continuous function and $h \to 0$ when $(\sigma, \varepsilon, \alpha, \epsilon) \to 0$. So let $\sigma, \varepsilon, \alpha, \epsilon$ small enough such that $h(\sigma, \varepsilon, \alpha, \epsilon) < 1$, and by (3.3.38) we get

$$\left|\frac{x-y}{\varepsilon} - \frac{2}{r}\eta(z_\sigma)\right|^2 < 1, \ \ |y - z_\sigma| < \rho, |t - t_\sigma| < \nu, \ \ |t-s| < \alpha.$$

Then by (3.3.32) and the fact that $0 < t_\sigma < T$, with small ν, α we have

$$|\eta(y_0) - \eta(z_\sigma)| \leq \frac{r}{2}, 0 < t, s < T. \tag{3.3.39}$$

Combining these yields

$$x \in B(y + \frac{2\varepsilon}{r}\eta(z_\sigma), \varepsilon) \subset B(y + \frac{2\varepsilon}{r}\eta(y), \frac{2\varepsilon}{r}r). \tag{3.3.40}$$

Thus, **(K1)** implies $x \in \overset{\circ}{\mathcal{K}}$ for small ε. Now consider the maps:

$$\begin{aligned}
\phi_1(x') &= v_2(s,y) + \left|\frac{x'-y}{\varepsilon} - \frac{2}{r}\eta(z_\sigma)\right|^2 + \left|\frac{y-z_\sigma}{\rho}\right|^2 \\
&\quad + \frac{t_\sigma^2}{\nu^2} + \frac{s^2}{\alpha^2} - \int_0^s b_\varepsilon(\tau)d\tau, \\
b_1(t') &= \frac{2}{\nu^2}(t'-t_\sigma) + \frac{2}{\alpha^2}(t'-s) - b_\varepsilon(t), \\
\phi_2(y') &= v_1(t,x) - |\frac{x-y'}{\varepsilon} - \frac{2}{r}\eta(z_\sigma)|^2 - |\frac{y'-z_\sigma}{\rho}|^2 \\
&\quad - |\frac{t-t_\sigma}{\nu}|^2 - \frac{t^2}{\alpha^2} + \int_0^t b_\varepsilon(\tau)d\tau, \\
b_2(s') &= \frac{2}{\alpha^2}(t-s') + b_\varepsilon(s).
\end{aligned}$$

Then $v_1(t',x') - \int_0^{t'} b_1(\tau)d\tau - \phi_1(x')$ has a maximum at $(t,x) \in (0,T)\times\overset{\circ}{\mathcal{K}}$, and $v_2(s',y') - \int_0^{s'} b_2(\tau)d\tau - \phi_2(y')$ has a minimum at $(s,y) \in (0,T)\times\mathcal{K}$, by the definition of constrained L^1-viscosity solutions we have

$$\lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t'-t|\le\delta} \inf_{x'\in B(x,\delta), p\in B(p_\varepsilon,\delta)} \left\{H(t',x',p) + v_1(t',x') - b_1(t')\right\} \le 0, \tag{3.3.41}$$

$$\lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|s'-s|\le\delta} \sup_{y'\in B(y,\delta)\cap\mathcal{K}, q\in B(p_\varepsilon+q_\varepsilon,\delta)} \left\{H(s',y',q) + v_2(s',y') - b_2(s')\right\} \ge 0, \tag{3.3.42}$$

where $D\phi_1(x) = p_\varepsilon, D\phi_2(y) = p_\varepsilon + q_\varepsilon$ with

$$p_\varepsilon = \frac{2}{\varepsilon}\left(\frac{x-y}{\varepsilon} - \frac{2}{r}\eta(z_\sigma)\right), \quad q_\varepsilon = -2\frac{y-z_\sigma}{\rho^2}.$$

For any $\epsilon > 0$, let $\zeta_\epsilon \in C_c^\infty(\mathbf{R})$ be a standard mollifier, and we define H_ϵ by $H_\epsilon(\cdot,x',p) = \zeta_\epsilon \star H(\cdot,x',p)$ for $(x',p) \in \mathcal{K}\times\mathbf{R}^d$. For small ϵ, ρ, σ, setting

$$b_\epsilon(t') = \sup_{x\in B(z_\sigma,1), p\in B(p_\varepsilon,1)} |H_\epsilon(t',x,p) - H(t',x,p)| \tag{3.3.43}$$

for $t' \in (0,T)$, and we check that $b_\epsilon \to 0$ in $L^1(0,T)$ as $\epsilon \to 0$. Thus, we deduce from (3.3.41) and (3.3.42) that

$$\lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t'-t|\le\delta} \inf_{x'\in B(x,\delta), p\in B(p_\varepsilon,\delta)} \left\{H_\epsilon(t',x',p) + v_1(t',x') - \frac{2}{\nu^2}(t'-t_\sigma) - \frac{2}{\alpha^2}(t'-s)\right\} \le 0, \tag{3.3.44}$$

$$\lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|s'-s|\le\delta} \sup_{y'\in B(y,\delta)\cap\mathcal{K}, q\in B(p_\varepsilon+q_\varepsilon,\delta)} \left\{H_\epsilon(s',y',q) + v_2(s',y') - \frac{2}{\alpha^2}(t-s')\right\} \ge 0. \tag{3.3.45}$$

Then by the continuity of H_ϵ, v_1 and v_2, (3.3.44) and (3.3.45) are equivalent to

$$H_\epsilon(t,x,p_\varepsilon) + v_1(t,x) - \frac{2}{\nu^2}(t-t_\sigma) - \frac{2}{\alpha^2}(t-s) \le 0, \tag{3.3.46}$$

$$H_\epsilon(s,y,p_\varepsilon+q_\varepsilon)+v_2(s,y)-\frac{2}{\alpha^2}(t-s)\geq 0. \tag{3.3.47}$$

Subtract (3.3.47) from (3.3.46),

$$\begin{aligned}
v_1(t,x)-v_2(s,y) &\leq \frac{2}{\nu^2}(t-t_\sigma)+H_\epsilon(s,y,p_\varepsilon+q_\varepsilon)-H_\epsilon(t,x,p_\varepsilon)\\
&\leq \frac{2}{\nu^2}(t-t_\sigma)+\omega_\epsilon(|t-s|)+H_\epsilon(t,y,p_\varepsilon+q_\varepsilon)-H_\epsilon(t,x,p_\varepsilon)\\
&\leq \frac{2}{\nu^2}(t-t_\sigma)+\omega_\epsilon(|t-s|)+\zeta_\epsilon\star m(t,|x-y|+|q_\varepsilon|)\\
&\leq \frac{2}{\nu^2}(t-t_\sigma)+\omega_\epsilon(\alpha)+\zeta_\epsilon\star m\left(t,c_0\varepsilon+\frac{2}{\rho^2}|y-z_\sigma|\right),
\end{aligned} \tag{3.3.48}$$

where ω_ϵ is the modulus of continuity of H_ϵ and $m(\cdot,\cdot)$ is defined in **(H1)**. By (3.3.38), let $\sigma,\varepsilon,\alpha,\epsilon$ be small enough such that $h(\sigma,\varepsilon,\alpha,\epsilon)\leq\max\{\nu^4,\rho^4\}$, and we obtain that

$$\frac{|t-t_\sigma|}{\nu^2}\leq\nu,\ \frac{|y-z_\sigma|}{\rho^2}\leq\rho. \tag{3.3.49}$$

Using (3.3.48) and (3.3.49) we have

$$\begin{aligned}
v_1(t,x)-v_2(s,y) &\leq 2\nu+\omega_\epsilon(\alpha)+\zeta_\epsilon\star m(t,c_0\varepsilon+2\rho)\\
&\leq 2\nu+\omega_\epsilon(\alpha)+\zeta_\epsilon\star m(t,c_0\varepsilon+2\rho).
\end{aligned} \tag{3.3.50}$$

then we get that

$$\begin{aligned}
M &\leq \sigma+v_1(t_\sigma,z_\sigma)-v_2(t_\sigma,z_\sigma)\\
&\leq \sigma+v_1(t_\sigma,z_\sigma)-v_1(t,x)+v_2(s,y)-v_2(t_\sigma,z_\sigma)+v_1(t,x)-v_2(s,y)\\
&\leq \sigma+\omega_1(|t_\sigma-t|+|z_\sigma-y|+|y-x|)+\omega_2(|s-t|+|t-t_\sigma|+|y-z_\sigma|)\\
&\quad +2\nu+\omega_\epsilon(\alpha)+\zeta_\epsilon\star m(t,c_0\varepsilon+2\rho)\\
&\leq \sigma+\omega_1(\nu+\rho+c_0\varepsilon)+\omega_2(\alpha+\nu+\rho)+2\nu+\omega_\epsilon(\alpha)+\zeta_\epsilon\star m(t,c_0\varepsilon+2\rho),
\end{aligned}$$

where ω_2 is the modulus of continuity of v_2. Now send first α then $\sigma,\varepsilon,\epsilon$ and finally ν,ρ to zero, we get that

$$M\leq 0,$$

which is a contradiction with (3.3.33). So we have

$$\sup_{(t,x)\in[0,T]\times\mathcal{K}}\{v_1(t,x)-v_2(t,x)\}\leq 0.$$

As $u_1(t,x)=e^t v_1(t,x)$ and $u_2(t,x)=e^t v_2(t,x)$, we obtain

$$\sup_{(t,x)\in[0,T]\times\mathcal{K}}\{u_1(t,x)-u_2(t,x)\}\leq 0,$$

which ends the proof. □

3.4 Problems with relaxed state constraints

In this section, we deal with the general case where no controllability assumptions such as **(HK1)** are considered and no additional condition is made on the vector field g_1 on the boundary of $\mathcal{K}$. The first aim would be to find a more convenient auxiliary control problem for which the value function will coincide with the original function v. From the discussion of the previous section, it turns out that the state constraints should be somehow relaxed for the reparametrized trajectories during the "fictive" time intervals $[\mathcal{W}(t_i^-), \mathcal{W}(t_i^+)]$. For this, time-dependent state constraints in the form of $z(s) \in \mathbb{K}(s)$ should be considered with $\mathbb{K}(s)$ equal to $\mathcal{K}$ when $s = \mathcal{W}(t)$ for any $t \in [0,T] \setminus \mathcal{T}$, and $\mathbb{K}(s)$ is large enough for any $s \in \bigcup_{t_i \in \mathcal{T}}]\mathcal{W}(t_i^-), \mathcal{W}(t_i^+)[$ so that the constraints are satisfied by any reparametrized trajectory without assuming any viability conditions like (3.3.2).

In the sequel, we will use the notation:

$$\bar{s}_i^{\pm} = \mathcal{W}(t_i^{\pm}) \text{ for every } t_i \in \mathcal{T}, \tag{3.4.1}$$

where $\mathcal{T}$ is the set of discontinuity points of B. We have the following theorem:

Theorem 3.4.1. Assume **(Hg1)-(Hg2)** and assume $\mathcal{K}$ to be a closed subset of $\mathbf{R}^d$. Consider the set-valued map $x \rightsquigarrow \mathbb{K}(x)$ defined by:

$$\mathbb{K}(s) = \begin{cases} \mathcal{K} & \text{if } \ s = \mathcal{W}(t), t \in [0,T] \backslash \mathcal{T}, \\ \mathcal{K} + B(0, L_g \delta(s - \bar{s}_i^-)) & \text{if } \ s \in \left[\bar{s}_i^-, \frac{\bar{s}_i^- + \bar{s}_i^+}{2}\right], \\ \mathcal{K} + B(0, L_g \delta(\bar{s}_i^+ - s)) & \text{if } \ s \in \left[\frac{\bar{s}_i^- + \bar{s}_i^+}{2}, \bar{s}_i^+\right], \end{cases}$$

with L_g defined in (Hg2) and $\delta = T + V_0^T(B)$. Then the multi-application $\mathbb{K}$ is upper semicontinuous (usc, in short)
(ii) Moreover, if we define

$$\bar{v}(\sigma, x) = \inf_{a \in \mathcal{A}} \left\{ \varphi(z_{\sigma,x}^{\alpha}(1)), \ z_{x,\sigma}^{\alpha} \ \text{ solution of (3.2.9) and } z_{\sigma,x}^{\alpha}(s) \in \mathbb{K}(s), \ \forall \, s \in [\sigma, 1] \right\}, \tag{3.4.2}$$

then we have:

$$v(\tau, x) = \bar{v}(\mathcal{W}(\tau), x) \qquad \text{for every } \tau \in [0,T]. \tag{3.4.3}$$

Proof. To prove assertion (i), we claim that for any $t \in \left[\bar{s}_i^-, \frac{\bar{s}_i^- + \bar{s}_i^+}{2}\right]$ (resp. $t \in \left[\frac{\bar{s}_i^- + \bar{s}_i^+}{2}, \bar{s}_i^+\right]$), and $s \in [\bar{s}_i^-, t]$ (resp. $s \in [t, \bar{s}_i^+]$), then we have

$$dist(\mathbb{K}(s), \mathbb{K}(t)^c) \leq L_g \delta |t - s|.$$

Consider for example the case when $t \in \left[\bar{s}_i^-, \frac{\bar{s}_i^- + \bar{s}_i^+}{2}\right]$ and assume that for any $x \in \partial \mathbb{K}(s)$ and $y \in \partial \mathbb{K}(t)$, the following holds:

$$\|x - y\| > L_g \delta (t - s).$$

Let $y_0 \in \partial\mathbb{K}(t)$ and set $z_0 \in P_{\mathcal{K}_i}(y_0)$. By the definition of $\mathbb{K}$, we deduce that

$$\|y_0 - z_0\| = L_g\delta(t - \bar{s}_i^-).$$

Let $x_0 \in [y_0, z_0] \cap \partial\mathbb{K}(s)$. We then have

$$\begin{aligned}\|x_0 - z_0\| =& \|y_0 - z_0\| - \|y_0 - x_0\| \\ <& L_g\delta(t - \bar{s}_i^-) - L_g\delta(t - s) \\ =& L_g\delta(s - \bar{s}_i^-)\end{aligned}$$

which contradicts the fact that $x_0 \in \partial\mathbb{K}(s)$.

To prove assertion (ii), let us consider some $z^\alpha_{\sigma,x}$ solution of (3.2.9) and satisfying that $z^\alpha_{\sigma,x}(s) \in \mathbb{K}(s), \forall\, s \in [\sigma, 1]$. For any $t \in [\tau, T]$ and $s = \mathcal{W}(t)$, we have $\mathbb{K}(s) = \mathcal{K}$ which implies

$$y^\alpha_{\tau,x}(t) = z^\alpha_{\sigma,x}(\mathcal{W}(t)) \in \mathcal{K}.$$

On the other side, consider some $z^\alpha_{\sigma,x}$ solution of (3.2.9) and satisfying that $z^\alpha_{\sigma,x}(\mathcal{W}(t)) \in \mathcal{K}, \forall\, t \in [\tau, T]$. We want to prove that $z^\alpha_{\sigma,x}(s) \in \mathbb{K}(s), \forall\, s \in [\sigma, 1]$. If $s = \mathcal{W}(t)$ for some $t \in [\tau, T]\backslash\mathcal{T}$, $\mathbb{K}(s) = \mathcal{K}$ and the result is obvious. If $s \in [\bar{s}_i^-, \frac{\bar{s}_i^- + \bar{s}_i^+}{2}]$, since the dynamic of $z^\alpha_{\sigma,x}$ is bounded by $L_g\delta$, we have

$$|z^\alpha_{\sigma,x}(s) - z^\alpha_{\sigma,x}(\bar{s}_i^-)| < L_g\delta(s - \bar{s}_i^-),$$

and we know that $z^\alpha_{\sigma,x}(\bar{s}_i^-) \in \mathcal{K}$, then

$$dist(z^\alpha_{\sigma,x}(s), \mathcal{K}) < L_g\delta(s - \bar{s}_i^-),$$

which implies that

$$z^\alpha_{\sigma,x}(s) \in \mathbb{K}(s).$$

In the case of $s \in [\frac{\bar{s}_i^- + \bar{s}_i^+}{2}, \bar{s}_i^+]$, the argument is quite similar by considering a backward dynamical system and using the fact that $z^\alpha_{\sigma,x}(\bar{s}_i^+) \in \mathcal{K}$. We conclude that

$$y^\alpha_{\tau,x}(t) \in \mathcal{K} \text{ for } t \in [\tau, T] \Leftrightarrow z^\alpha_{\sigma,x}(s) \in \mathbb{K}(s) \text{ for } s \in [\sigma, 1].$$

Then (3.4.3) follows by the fact that $y^\alpha_{\tau,x}(T) = z^\alpha_{\mathcal{W}(\tau),x}(\mathcal{W}(T)) = z^\alpha_{\sigma,x}(1)$ and the definitions of v and $\bar{v}$. □

Theorem 3.4.1 suggests to compute first the new auxiliary value function and then deduce the original value function v by the formula (3.4.3). The auxiliary reparametrized control problem is in presence of time-dependent state constraints. Recall that several papers have been devoted to study the characterization of the value function for state constrained control problems. Under some controllability assumption and when the set of state-constraints is not time-dependent, the value function can be shown to be the unique constrained-viscosity solution to an adequate HJB equation, see in [84, 126, 127]. We refer also to [2, 40] for a discussion on the general case where the control problem is lacking controllability properties.

Here the control problem (3.4.2) is in presence of time-dependent state constraints and no controllability assumption is assumed. The characterization of $\bar{v}$ by an HJB equation on a tube $\mathbb{K}_.$ is not a simple task because the evolution of $\bar{v}$ depends also on the evolution of the map $\mathbb{K}$. Here we extend to time-dependent state-constrained control problems an idea developed recently in [2] which allows to compute all the epigraph of the value function $\bar{v}$ by solving an appropriate variational HJB equation.

3.4.1 Optimal control problems with time-dependent state constraints

In this section, the main result concerns optimal control problems with time-dependent state constraints and time-measurable Hamiltonians. Introduce the function $\mathcal{F}$ defined by:

$$f(s,z,a) = g_0(\phi^0(s),z,a)\dot{\phi}^0(s) + g_1(\phi^0(s),z)\left(\mu^a(\phi^0(s))\dot{\phi}^0(s) + \dot{\phi}^1(s)\right),$$

where ϕ^0 and ϕ^1 are given in (3.2.7).

Remark 3.4.2. All the results of this section hold in a more general setting, where the following time-dependent state constrained Mayer's control problem is considered:

$$\begin{aligned}\vartheta(\sigma,x) \quad &= \quad \inf_{a\in\mathcal{A}}\Big\{\varphi(z^{\alpha}_{x,\sigma}(1)),\\ &\qquad z^{\alpha}_{x,\sigma} \;\text{ solution of (3.4.4) and } z^{\alpha}_{x,\sigma}(\theta)\in\mathbb{K}(\theta),\ \forall\,\theta\in[\sigma,1]\Big\}.\end{aligned}$$

with the convention that $\inf\emptyset = +\infty$, where the state equation is given by:

$$\begin{cases} \dot{z}(s) = \mathcal{F}(s,z(s),\alpha(s)), & \text{for } s\in(\sigma,1)\\ z(\sigma) = x, \end{cases} \tag{3.4.4}$$

and with $\mathcal{F}$ and $\mathbb{K}$ satisfying:

(HF1) $\mathcal{F} : (0,1)\times\mathbf{R}^d\times A \to \mathbf{R}^d$ is measurable with respect to the time variable, and is continuous with respect to the last two variables z and a. Moreover, for each $(z,a)\in\mathbf{R}^d\times A$, we have $\mathcal{F}(\cdot,z,a)\in L^1(0,1)$, and $\mathcal{F}(t,z,\mathcal{A})$ is nonempty compact and convex set, for every $x\in\mathbf{R}^d$ and for almost every $t\in(0,1)$.

(HF2) There exists $k_0>0$ such that $\forall\,s\in(0,1),\ x,z\in\mathbf{R}^d,\ a\in A$, we have

$$|\mathcal{F}(s,x,a)-\mathcal{F}(s,z,a)| \le k_0|x-z|, \qquad |\mathcal{F}(t,z,a)|\le k_0.$$

(HK2) the set-valued application $\theta \rightsquigarrow \mathbb{K}(\theta)$ is upper semicontinuous on $[0,1]$.

Our goal is to characterize the new value function ϑ. It is easy to check that the corresponding control problem does not satisfy any controllability condition. Indeed, the field $\mathcal{F}$ can never be inward pointing (resp. outward pointing) on $\bigcup_{s\in[0,1]} \times\mathbb{K}(s)$. Then the characterization of ϑ as constrained viscosity solution of an HJB equation does not hold in the general case [40].

3.4.2 Epigraph of ϑ

First of all, to deal with the state constraints, we introduce a Lipschitz continuous function $\Psi : [0,1] \times \mathbf{R}^d \to \mathbf{R}$ such that

$$\Psi(\theta, x) \le 0 \Leftrightarrow x \in \mathbb{K}(\theta), \ \forall\theta \in [0,1], x \in \mathbf{R}^d.$$

(Note that this is always possible to find such a function Ψ. In particular, according to theorem 3.4.1, the distance function to the set $\bigcup_{\theta\in[0,1]}\{\theta\} \times \mathbb{K}(\theta)$ fulfilled the conditions).
By using an idea introduced in [2], an equivalent way to characterize the epigraph of ϑ consists of considering the control problem

$$w(\sigma, x, \xi) = \inf_{\substack{a\in\mathcal{A} \\ \dot\zeta=0,\zeta(\sigma)=\xi}} \left\{ \left(\varphi(z^\alpha_{x,\sigma}(1)) - \zeta(1)\right) \bigvee \max_{\theta\in[\sigma,1]} \Psi(\theta, y^\alpha_{x,\sigma}(\theta))\right\} \tag{3.4.5}$$

where now, the state constraints are *included* in the cost function to be minimized. In the above expression the notation $a \vee b$ means the $\max(a, b)$. The following result shows the relation between the 0-level set of w and the epigraph of ϑ:

Proposition 3.4.3. Assume that **(Hg1)-(Hg2)** and **(HC1)** hold true, then we have

$$\begin{aligned} & \{(\sigma, x, z) \in [0,1] \times \mathbf{R}^d \times \mathbf{R} \mid w(\sigma, x, z) \le 0\} \\ = \ & \{(\sigma, x, z) \in [0,1] \times \mathbf{R}^d \times \mathbf{R} | \vartheta(\sigma, x) \le z\} =: \mathcal{E}p\ \vartheta, \end{aligned}$$

and $\vartheta(\sigma, x) = \min\{z \mid w(\sigma, x, z) \le 0\}$.

Proof. First, let us point out that under assumptions (Hg1)-(Hg2), by Filippov theorem, the set of trajectories $S^P_{[\sigma,1]}(x)$ is compact in $C([\sigma, 1])$, then the infimum in the definition of w is achieved. Moreover, when ϑ is finite, the infimum in the definition of ϑ is achieved too. Let (σ, x, ξ) be in $[0,1] \times \mathbf{R}^d \times \mathbf{R}$, it comes that:

$$\begin{aligned} w(\sigma, x, \zeta) \le 0 \ & \Leftrightarrow \ \exists a \in \mathcal{A} \text{ s.t. } \varphi(z^\alpha_{x,\sigma}(1)) \le \zeta, \ \Psi(\theta, z^\alpha_{x,\sigma}(\theta)) \le 0, \forall\,\theta \in [\sigma, 1] \\ & \Leftrightarrow \ \exists a \in \mathcal{A} \text{ s.t. } \varphi(z^\alpha_{x,\sigma}(1)) \le \zeta, \ z^\alpha_{x,\sigma}(\theta) \in \mathbb{K}(\theta), \forall\,\theta \in [\sigma, 1] \\ & \Leftrightarrow \ \vartheta(\sigma, x) \le \zeta. \end{aligned}$$

□

Proposition 3.4.3 shows that once the auxiliary function w is computed, the epigraph of ϑ can be deduced as the 0-level set of w.

3.4.3 Characterization of w

Hence, the goal now is to characterize the auxiliary function w. As in the classical case, the function w can be characterized as the unique solution of a Hamilton-Jacobi equation. More precisely, considering the Hamiltonian

$$\mathcal{H}(\sigma, x, p) := \sup_{a \in A} \big(-\mathcal{F}(\sigma, x, a) \cdot p \big),$$

we have

Theorem 3.4.4. Assume that **(Hg1)-(Hg2)**, **(HC1)** hold and that $\mathcal{K}$ is a closed nonempty set of $\mathbf{R}^d$. Then w is the unique continuous viscosity solution of the variational inequality

$$\min\Big(-\partial_s w(\sigma, x, \xi) + \mathcal{H}(\sigma, x, Dw),\ w(\sigma, x, \xi) - \Psi(\sigma, x) \Big) = 0, \tag{3.4.6a}$$

for $s \in (0,1)$, $(x, \xi) \in \mathbf{R}^{d+1}$, and

$$w(1, x, \xi) = \max(\varphi(x) - \xi, \Psi(1, x)),\ x, \xi \in \mathbf{R}^{d+1}. \tag{3.4.6b}$$

As usual, the proof of Theorem 3.4.4 is based on the dynamic programming principle (DPP) satisfied by w, and that can be stated here as follows:

Lemma 3.4.5. The function w is characterized by

1. for all $t \in [0,1]$ and $\tau \in [0, 1-t]$, for all $x, \xi \in \mathbf{R}^{d+1}$,

$$w(t, x, \xi) = \inf_{a \in \mathcal{A}} \left\{ w(t+\tau, z^{\alpha}_{x,t}(t+\tau), \xi) \bigvee \max_{\theta \in [t, t+\tau]} \Psi(\theta, z^{\alpha}_{x,t}(\theta)) \right\}, \tag{3.4.7}$$

2. $w(1, x, \xi) = \max(\varphi(x) - \xi, \Psi(1, x))$, $(x, \xi) \in \mathbf{R}^{d+1}$.

The first consequence of the above lemma is the continuity of the value function w:

Proposition 3.4.6. Assume **(Hg1)-(Hg2)** and **(HC1)** hold, and $\mathcal{K}$ is a closed set of $\mathbf{R}^d$. Then w is Lipschitz continuous on $[0,1] \times \mathbf{R}^{d+1}$.

Proof. Let $t \in [0,1]$ and $(x, \xi), (x', \xi') \in \mathbf{R}^{n+1}$. By using the definition of w and the simple inequalities:

$$\max(A, B) - \max(C, D) \leq \max(A - C, B - D),$$
$$\inf A_\alpha - \inf B_\alpha \leq \sup(A_\alpha - B_\alpha),$$

we get:

$$\begin{aligned}
& |w(t,x,\xi) - w(t,x',\xi')| \\
\leq\ & \sup_{a\in\mathcal{A}} \max\Big(|\varphi(z^{\alpha}_{x,t}(1)) - \varphi(z^{\alpha}_{x',t}(1))| + |\xi-\xi'|,\ \max_{\theta\in[t,1]} |\Psi(\theta, z^{\alpha}_{x,t}(\theta)) - \Psi(\theta, z^{\alpha}_{x',t}(\theta))| \Big) \\
\leq\ & \sup_{a\in\mathcal{A}} \Big(m_{\varphi} \|z^{\alpha}_{x,t}(1) - z^{\alpha}_{x',t}(1)\| + |\xi-\xi'|,\ \max_{\theta\in[t,1]} \|z^{\alpha}_{x,t}(\theta) - z^{\alpha}_{x',t}(\theta)\| \Big),
\end{aligned}$$

where m_{Φ} is the Lipschitz constant of φ. By assumption **(Hg1)-(Hg2)**, we know that $\|z^{\alpha}_{x,t}(\theta) - z^{\alpha}_{x',t}(\theta)\| \leq e^{k_0}\|x-x'\|$ for all $a\in\mathcal{A}, \theta\in[t,1]$, then we conclude that:

$$|w(t,x,\xi) - w(t,x',\xi')| \leq \max\Big(m_{\Phi} e^{k_0}\|x-x'\| + \|\xi-\xi'|, e^{k_0}\|x-x'\| \Big), \tag{3.4.8}$$

and we deduce that $w(t,\cdot,\cdot)$ is Lipschitz continuous in $\mathbf{R}^d\times\mathbf{R}$. Now let $(x,\xi)\in\mathbf{R}^{d+1}$ and $t\in[0,1], \tau\in[0,1-t]$. Remarking that $w(t+\tau,x,\xi)\geq\Psi(t+\tau,x,\xi)$ and by using the DPP, it follows that:

$$\begin{aligned}
& |w(t,x,\xi) - w(t+\tau,x,\xi)| \\
=\ & \Big| \inf_{a\in\mathcal{A}} \max(w(t+\tau, z^{\alpha}_{x,t}(t+\tau),\xi), \max_{\theta\in[t,t+\tau]} \Psi(\theta, z^{\alpha}_{x,t}(\theta)))\ \ - \max(w(t+\tau,x,\xi), g(t+\tau,x)) \Big| \\
\leq\ & \sup_{a\in\mathcal{A}} \max\Big(\Big|w(t+\tau, z^{\alpha}_{x,t}(t+\tau),\xi) - w(t+\tau,x,\xi)\Big|,\ \Big| \max_{\theta\in[t,t+\tau]} \Psi(\theta, z^{\alpha}_{x,t}(\theta)) - \Psi(t+\tau,x) \Big| \Big) \\
\leq\ & \max\Big(m_{\varphi}(e^{k_0}k_0\tau), e^{k_0}k_0\tau, (1+k_0)\tau \Big)
\end{aligned}$$

where we have used (3.4.8) and assumptions **(Hg1)-(Hg2)**. This completes the proof. □

Before proving Theorem 3.4.4, once need first to make more precise the notion of L^1-viscosity solution for (3.4.6). Here we extend the L^1-viscosity notion introduced by Ishii in[102].

Definition 3.4.7. A lower semi-continuous (resp. upper semi-continuous) function $u:(0,1)\times\mathbf{R}^d\times\mathbf{R}\to\mathbf{R}$ is a L^1-viscosity supersolution (resp. subsolution) of (3.4.6) if

1. $u(1,x,\xi)\geq(\varphi(x)-\xi)\bigvee\Psi(1,x)$ (resp. $u(1,x,\xi)\leq(\varphi(x)-\xi)\bigvee\Psi(1,x)$);
2. For any test function $b\in L^1(0,1)$, $\phi\in C^1(\mathbf{R}^{d+1})$ such that $u(t,x,\xi)-\int_0^1 b(s)ds-\phi(x,\xi)$ achieves a local minimum (resp. maximum) on $(t_0,x_0,\xi_0)\in(0,1)\times\mathbf{R}^{n+1}$, we have

$$\min\left(\lim_{\delta\to0^+} \operatorname*{ess\,sup}_{|t-t_0|\leq\delta} \sup_{\substack{(x,\xi)\in B_\delta(x_0,\xi_0),\\ p\in B_\delta(D\phi(x_0,\xi_0))}} \{\mathcal{H}(t,x,p)-b(t)\}, (u-\Psi)(t_0,x_0) \right) \geq 0$$

$$(\text{resp. } \min\Big(\lim_{\delta\to0^+} \operatorname*{ess\,inf}_{|t-t_0|\leq\delta} \inf_{\substack{(x,\xi)\in B_\delta(x_0,\xi_0),\\ p\in B_\delta(D\phi(x_0,\xi_0))}} \{\mathcal{H}(t,x_0,p)-b(t)\}, (u-\Psi)(t_0,x_0) \Big) \leq 0).$$

A continuous function u is a L^1-viscosity solution of (3.4.6) if u is both a supersolution and a subsolution of (3.4.6).

Now, we can give the proof of Theorem 3.4.4.

Proof of Theorem 3.4.4. We first show that w is a solution of (3.4.6). The fact that w satisfies the initial condition is a direct consequence of Lemma 5.3.2(ii).
Let us check the L^1-supersolution property of w. By the definition of w, for every $(\sigma, x, \xi) \in [0,1] \times \mathbf{R}^d \times \mathbf{R}$, we have

$$w(\sigma, x, \xi) \geq \inf_{a \in \mathcal{A}} \max_{\theta \in [\sigma,1]} \Psi(\theta, z^{\alpha}_{x,\sigma}(\theta)) \geq \Psi(\sigma, x). \tag{3.4.9}$$

Let $b \in L^1(0,1)$, $\phi \in C^1(\mathbf{R}^{n+1})$ and $(\sigma_0, x_0, \xi_0) \in (0,1) \times \mathbf{R}^{n+1}$ be a local minimum point of $w(\sigma, x, \xi) - \int_0^1 b(s)ds - \phi(x, \xi)$, then there exists $\delta > 0$ such that

$$w(\sigma, x, \xi) - \int_0^{\sigma} b(s)ds - \phi(x, \xi) \geq w(\sigma_0, x_0, \xi_0) - \int_0^{\sigma_0} b(s)ds - \phi(x_0, \xi_0), \tag{3.4.10}$$

for any $\sigma \in [\sigma_0 - \delta, \sigma_0 + \delta], (x, \xi) \in B_\delta(x_0, \xi_0)$. By Lemma 5.3.2(i), for all $\varepsilon > 0$, there exists $\alpha_0 \in \mathcal{A}$ such that

$$w(\sigma_0, x_0, \xi_0) \geq w(\sigma_0 + \tau, z^{\alpha_0}_{x_0,\sigma_0}(\sigma_0 + \tau), \xi_0) - \varepsilon, \ \forall \tau \in [0, 1 - \sigma_0]. \tag{3.4.11}$$

Consider some $\tau \leq \delta$, then by (3.4.10) and (3.4.11), we have

$$\int_0^{\sigma_0} b(s)ds + \phi(x_0, \xi_0) \geq \int_0^{\sigma_0+\tau} b(s)ds + \phi(z^{\alpha_0}_{x_0,\sigma_0}(\sigma_0 + \tau)) - \varepsilon,$$

i.e.

$$-\int_{\sigma_0}^{\sigma_0+\tau} [D\phi(z^{\alpha_0}_{x_0,\sigma_0}(s)) \cdot \mathcal{F}(s, z^{\alpha_0}_{x_0,\sigma_0}(s), a_0(s)) + b(s)]ds \geq -\varepsilon$$

then by the definition of the Hamiltonian, we have

$$\int_{\sigma_0}^{\sigma_0+\tau} [\mathcal{H}(s, z^{\alpha_0}_{z_0,\sigma_0}(s), D\phi(z^{\alpha_0}_{x_0,\sigma_0}(s))) - b(s)]ds \geq -\varepsilon, \ \forall \varepsilon > 0,$$

and we deduce that

$$\int_{\sigma_0}^{\sigma_0+\tau} [\mathcal{H}(s, z^{\alpha_0}_{x_0,\sigma_0}(s), D\phi(z^{\alpha_0}_{x_0,\sigma_0}(s))) - b(s)]ds \geq 0. \tag{3.4.12}$$

By contradiction, we assume that

$$\lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|\sigma - \sigma_0| \leq \delta} \sup_{\substack{(x,\xi) \in B_\delta(x_0,\xi_0), \\ p \in B_\delta(D\phi(x_0,\xi_0))}} \{\mathcal{H}(\sigma, x, \xi, p) - b(t)\} < 0,$$

then there exists $\delta_1 > 0$, $E \subset [\sigma_0 - \delta_1, \sigma_0 + \delta_1]$ with $m(E) = 0$ such that $\forall\, s \in [\sigma_0 - \delta_1, \sigma_0 + \delta_1] \backslash E$, $(x, \xi) \in B_{\delta_1}(x_0, \xi_0)$ and $p \in B_{\delta_1}(D\phi(x_0, \xi_0))$, we have $\mathcal{H}(s, x, p) - b(s) < 0$. By the continuity of

$z^{\alpha}_{x_0,\sigma_0}(\cdot)$, $D\phi(\cdot)$ and $\mathcal{H}(t,\cdot,\cdot)$, for τ small enough, we get

$$H(s, z^{\alpha}_{x_0,\sigma_0}(s), D\phi(z^{\alpha}_{x_0,t_0}(s), \xi_0)) - b(s) < 0, \text{ for } s \in [t_0, t_0+\tau]\backslash E, \tag{3.4.13}$$

which contradicts (3.4.12). Combined with (3.4.9), we get

$$\min\Big(\lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|\sigma-\sigma_0|\le\delta} \sup_{(x,\xi)\in B_\delta(x_0,\xi),p\in B_\delta(D\phi(x_0,\xi_0))} \{\mathcal{H}(\sigma,x,p) - b(\sigma)\}, \\ w(\sigma_0,x_0,\xi_0) - \Psi(\sigma_0,x_0)\Big) \ge 0.$$

Let us now prove that w is a L^1-subsolution. Let $(\sigma_0,x_0,\xi_0) \in (0,1)\times \mathbf{R}^{d+1}$. If $w(\sigma_0,x_0,\xi_0) \le \Psi(\sigma_0,x_0)$, it is obvious that w satisfies

$$\min\big(-\partial_t w(\sigma_0,x_0,\xi_0) + \mathcal{H}(\sigma_0,x_0,Dw(t_0,x_0,\xi_0)),\ w(\sigma_0,x_0,\xi_0) - \Psi(t_0,x_0,\xi_0)\big) \le 0$$

in the L^1-viscosity sense. Now, assume that $w(t_0,x_0,\xi_0) > \Psi(t_0,x_0,\xi_0)$. By continuity of w and Ψ, there exists some $\tau > 0$ such that $w(\sigma_0+\tau, z^{\alpha}_{x_0,\sigma_0}(\sigma_0+\tau)) > \Psi(\theta, z^{\alpha}_{x_0,t_0}(\theta))$ for all $\theta \in [\sigma_0,\sigma_0+\tau]$ (since $z^{\alpha}_{x_0,\sigma_0}(\theta)$ will stay in a neighborhood of x_0 which is controlled uniformly with respect to a). Hence, by using Lemma 5.3.2(i), we get that

$$w(\sigma_0,x_0,\xi_0) = \inf_{a\in\mathcal{A}} w(\sigma_0+h, z^{\alpha}_{x_0,t_0}(t_0+h)),\ \forall\, h\in[0,\tau].$$

We then deduce by the same argument as for the supersolution property that

$$-\partial_t w(\sigma_0,x_0,\xi_0) + \mathcal{H}(\sigma_0,x_0,Dw(\sigma_0,x_0,\xi_0)) \le 0$$

in the L^1-viscosity sense. Therefore, w is a L^1-viscosity subsolution.

The uniqueness follows from the following comparison principle result. □

Proposition 3.4.8 (Comparison principle). If u is a L^1-viscosity subsolution and v is a L^1-viscosity supersolution of (3.4.6), then we have

$$u \le v, \text{ on } (0,1)\times\mathbf{R}^{d+1}.$$

Proof. By Definition 3.4.7, for any $(t,x,\xi)\in(0,1)\times\mathbf{R}^{d+1}$ we have that

$$\min(-\partial_t u(t,x,\xi) + H(t,x,Du),\ u(t,x,\xi) - \Psi(t,x)) \le 0$$
$$\min(-\partial_t v(t,x,\xi) + H(t,x,Dv),\ v(t,x,\xi) - \Psi(t,x)) \ge 0$$

in the L^1-viscosity sense. If $u(t,x,\xi) - \Psi(t,x) \le 0$, we get

$$u(t,x,\xi) \le \Psi(t,x) \le v(t,x,\xi).$$

If $u(t,x,\xi) - \Psi(t,x) > 0$, then we have

$$-\partial_t u(t,x,\xi) + H(t,x,Du) \le 0, \qquad -\partial_t v(t,x,\xi) + H(t,x,Dv) \ge 0,$$

where we get $u(t,x,\xi) \le v(t,x,\xi)$ from a classical comparison principle (see Theorem 8.1 in Ishii[102]). □

3.4.4 Problems with discontinuous final cost

This subsection is devoted to the characterization of the value function of the same optimal control problem with discontinuous final cost. More precisely, we consider the final cost function $\varphi : \mathbf{R}^d \to \mathbf{R}$ satisfying the following:

(HC2) φ is a bounded lower semi-continuous function.

The optimal control problem and the value function v are defined as in (3.2.11). We follow the same idea as in the above problem where φ is Lipschitz continuous, and what we need to investigate is to characterize the auxiliary function w defined in (3.4.5).

In the sequel, for the convenience of notations, we set $X := (x,\xi) \in \mathbf{R}^{d+1}$ and $w(t,x,\xi)$ is rewritten as $w(t,X)$.

We begin by some definitions and preliminary results. The first point is that, since the cost function is only lsc, the value function w is also lsc:

Proposition 3.4.9. Assume that **(Hg1)-(Hg2) (HC2)** hold and that $\mathcal{K}$ is a closed nonempty set of $\mathbf{R}^d$. Then w is lower semi-continuous on $[0,1] \times \mathbf{R}^{d+1}$.

At this point, the classical good notion of viscosity solutions is lsc viscosity solutions which is also called bilateral viscosity solutions. We now give the precise definition of L^1-bilateral viscosity solutions for our problem which is equivalent to the classical bilateral viscosity solutions when the Hamiltonian is continuous.

Definition 3.4.10. Let $u : (0,1) \times \mathbf{R}^{n+1}$ be a bounded lsc function. We say that u is a L^1-bilateral viscosity solution of (3.4.6) if:
for any $b \in L^1(0,1)$, $\phi \in C^1(\mathbf{R}^{d+1})$ and $(t_0, X_0) \in (0,1) \times \mathbf{R}^{d+1}$ local minimum point of $u(t,X) - \int_0^1 b(s)ds - \phi(X)$, then we have

$$\min\left(\lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|t-t_0|\le\delta} \sup_{X \in B_\delta(X_0), p \in B_\delta(D\phi(X_0))} \{\mathcal{H}(t,X,p) - b(t)\}, (u-\Phi)(t_0,X_0) \right) \ge 0$$

and

$$\min\left(\lim_{\delta \to 0^+} \operatorname*{ess\,inf}_{|t-t_0|\le\delta} \inf_{X \in B_\delta(X_0), p \in B_\delta(D\phi(X_0))} \{H(t,X,p) - b(t)\}, (u-\Phi)(t_0,X_0) \right) \le 0.$$

Moreover, the final condition is satisfied in the following sense:

$$\phi(X) = \inf \left\{ \liminf_{n\to+\infty} u(t_n, X_n) : \ t_n \uparrow 1, X_n \to X \right\}.$$

Then we have the following characterization result for w.

Theorem 3.4.11. Assume that **(Hg1)-(Hg2) (HC2)** hold and that $\mathcal{K}$ is a closed nonempty set of $\mathbf{R}^d$. Then w is the unique lsc L^1-bilateral viscosity solution of (3.4.6) on $[0,1]\times\mathbf{R}^{d+1}$.

The main idea of the proof is to approximate the lsc (lower semi-continuous) final cost function φ by a sequence of continuous function. Then the proof is based on some stability and consistency results.

Lemma 3.4.12. Assume that **(Hg1)-(Hg2) (HC2)** hold and that $\mathcal{K}$ is a closed nonempty set of $\mathbf{R}^d$. If the final cost function φ is a continuous function, u is a continuous L^1-viscosity solution of (3.4.6) implies that u is also a L^1-bilateral viscosity solution of (3.4.6).

The key tool to prove this lemma of consistency is a lemma introduced in Barron and Jensen [30]. We recall here this result.

Lemma 3.4.13. [[30], Theorem 15] Let W be a continuous function on $[0,+\infty)\times\mathbf{R}^{d+1}$ such that W has a zero maximum (minimum) at (τ,ξ). Let $\varepsilon>0$. Then there is a smooth function ψ, a finite set of numbers $\alpha_k \geq 0$ summing to one, and a finite collection of points (t_k, x_k) such that

1. $W-\psi$ has a zero minimum (maximum) at (t_k,x_k);
2. $(t_k,x_k)\in B((s,y),o(\varepsilon\sqrt{\varepsilon}))$ for some $(s,y)\in B((\tau,\xi),o(\varepsilon))$;
3. $|D_{t,x}\psi(t_k,x_k)| = o(\sqrt{\varepsilon})$;
4. $\sum_k \alpha_k D_{t,x}\psi(t_k,x_k) = 0$.

Proof of Lemma 3.4.12. According to the Definition 3.4.7 and Definition 3.4.10, we only have to show that fix $b\in L^1(0,1)$, $\phi\in C^1(\mathbf{R}^{d+1})$ and $(t_0,X_0)\in(0,1)\times\mathbf{R}^{d+1}$ local minimum point of $u(t,X)-\int_0^1 b(s)ds-\phi(X)$ we have

$$\lim_{\delta\to0^+}\ \operatorname*{ess\,inf}_{|t-t_0|\leq\delta}\ \inf_{X\in B_\delta(X_0),p\in B_\delta(D\phi(X_0))}\{\mathcal{H}(t,X,p)-b(t)\}\leq 0, \tag{3.4.14}$$

when $u(t_0,X_0)>\Phi(t_0,X_0)$. For each $\delta>0$, we choose $\varepsilon,\eta>0$ small enough such that

$$o(\varepsilon)+o(\varepsilon\sqrt{\varepsilon})+\eta\leq\delta,\ \ m_{D\phi}(o(\varepsilon)+o(\varepsilon\sqrt{\varepsilon}))+o(\sqrt{\varepsilon})+\eta\leq\delta,$$

where $m_{D\phi}$ is the continuity modulus of $D\phi$. We now apply Lemma 3.4.13 with $W(t,X)=u(t,X)-\int_0^1 b(s)ds-\phi(X)$, there exists a smooth function ψ and a finite collection of points (t_k,X_k) such that $u(t,X)-\int_0^1 b(s)ds-\phi(X)-\psi(t,X)$ has a maximum at (t_k,X_k) and for each k

$$B_\eta(t_k,X_k)\subset B_\delta(t_0,X_0),\ \ B_\eta(D\phi(X_k)+D\psi(t_k,X_k))\subset B_\delta(D\phi(X_0)).$$

Thus

$$
\begin{aligned}
& \operatorname*{ess\,inf}_{|t-t_0|\le\delta} \inf_{X\in B_\delta(X_0),p\in B_\delta(D\phi(X_0))} \{\mathcal{H}(t,X,p)-b(t)\} \\
\le\ & \operatorname*{ess\,inf}_{|t-t_k|\le\eta} \inf_{X\in B_\eta(X_k),p\in B_\eta(D\phi(X_k)+D\psi(t_k,X_k))} \{\mathcal{H}(t,X,p)-b(t)\}. \qquad (3.4.15)
\end{aligned}
$$

Moreover, since u is a L^1-viscosity subsolution and $u(t_k,X_k) > \Phi(t_k,X_k)$ when ε is small enough, we have

$$
\begin{aligned}
\lim_{\eta\to0^+} \operatorname*{ess\,inf}_{|t-t_k|\le\eta} \inf_{X\in B_\eta(X_k),p\in B_\eta(D\phi(X_k)+D\psi(t_k,X_k))} \{ \quad & H(t,X,p)-b(t) \\
& -\partial_t\psi(t,X)\} \le 0. \qquad (3.4.16)
\end{aligned}
$$

Using (3.4.16) and the fact that $|\partial_t\psi(t_k,X_k)| \le o(\sqrt{\varepsilon}) < \delta$, letting $\delta\to0^+$ ($\Rightarrow \eta\to0^+$) in (3.4.15), we obtain (3.4.14) and conclude the proof. □

The result below is a stability property w.r. to φ.

Lemma 3.4.14. For each $k\in\mathbf{N}$, let u_k be a L^1-bilateral viscosity solution of (3.4.6) with the final condition

$$u_k(1,X) = \varphi_k(X) \text{ in } \mathbf{R}^{d+1},$$

where the function $\varphi_k \in C(\mathbf{R}^{d+1})$ is bounded. Moreover, assume that the sequence $(\varphi_k)_{k\in\mathbf{N}}$ is monotone increasing and that

$$\lim_{k\to+\infty} \varphi_k(X) = \varphi(X),\ \forall\, X\in\mathbf{R}^{d+1}.$$

Suppose that the sequence u_k converges and we set

$$u(t,X) := \lim_{k\to\infty} u_k(t,X),$$

then u is a L^1-bilateral viscosity solution of equation (3.4.6) with final condition

$$u(1,X) = \varphi(X) \text{ in } \mathbf{R}^{d+1}.$$

Proof. We first prove the final condition. Since $\varphi_k\in C(\mathbf{R}^{d+1})$, then u_k is continuous on $[0,1]\times\mathbf{R}^{d+1}$. Therefore, for each sequence $(t_k,X_k)\to(1,X)$, we have

$$\varphi(X) = \lim_{k\to\infty}\varphi_k(X) = \lim_{k\to\infty} u_k(1,X) = \lim_{k\to\infty} u_k(t_k,X_k) = \lim_{k\to\infty} u(t_k,X_k).$$

Now let $b\in L^1(0,1)$, $\phi\in C^1(\mathbf{R}^{d+1})$ and $(t_0,X_0)\in(0,1)\times\mathbf{R}^{d+1}$ be a local minimum point of $u(t,X) - \int_0^1 b(s)ds - \phi(X)$. Without loss of generality, we suppose that (t_0,X_0) is a strict local minimum. It is easy to see that u_k converges increasing to u. Therefore, when k is big enough, there exists a local minimum point (t_k,X_k) of $u_k(t,X) - \int_0^1 b(s)ds - \phi(X)$ such that $(t_k,X_k)\to(t_0,X_0)$.

Then since u_k is a L^1-bilateral viscosity solution, we get

$$\min\left(\lim_{\eta\to 0^+}\operatorname*{ess\,sup}_{|t-t_k|\le\eta}\sup_{X\in B_\eta(X_k),p\in B_\eta(D\phi(X_k))}\{\mathcal{H}(t,X,p)-b(t)\},(u_k-\Phi)(t_k,X_k)\right)\ge 0 \quad (3.4.17)$$

and

$$\min\left(\lim_{\eta\to 0^+}\operatorname*{ess\,inf}_{|t-t_k|\le\eta}\inf_{X\in B_\eta(X_k),p\in B_\eta(D\phi(X_k))}\{\mathcal{H}(t,X,p)-b(t)\},\ (u_k-\Phi)(t_k,X_k)\right)\le 0. \quad (3.4.18)$$

For every $\delta>0$, we choose k big enough and η small enough such that

$$B_\eta(t_k,X_k)\subset B_\delta(t_0,X_0),\ \ B_\eta(D\phi(X_k))\subset B_\delta(D\phi(X_0)).$$

Then by (3.4.17) we obtain

$$\begin{aligned}\lim_{\delta\to 0^+}\operatorname*{ess\,sup}_{|t-t_0|\le\delta}\sup_{X\in B_\delta(X_0),p\in B_\delta(D\phi(X_0))}\{\mathcal{H}(t,X,p)-b(t)\} &\ge \\ \lim_{\eta\to 0^+}\operatorname*{ess\,sup}_{|t-t_k|\le\eta}\sup_{X\in B_\eta(X_k),p\in B_\eta(D\phi(X_k))}\{\mathcal{H}(t,X,p)-b(t)\} &\ge 0\end{aligned}$$

and

$$u(t_0,X_0)=\lim_{k\to\infty}u_k(t_k,X_k)\ge\lim_{k\to\infty}\Phi(t_k,X_k)=\Phi(t_0,X_0).$$

These two inequalities imply that

$$\min\left(\lim_{\delta\to 0^+}\operatorname*{ess\,sup}_{|t-t_0|\le\delta}\sup_{X\in B_\delta(X_0),p\in B_\delta(D\phi(X_0))}\{\mathcal{H}(t,X,p)-b(t)\},(u-\Phi)(t_0,X_0)\right)\ge 0.$$

On the other side, suppose that $(u-\Phi)(t_0,X_0)>0$, using the fact that $u_k\to u$ and Φ,u_k are continuous, we have $(u_k-\Phi)(t_k,X_k)>0$ when k is big enough. Then by (3.4.18) we obtain

$$\begin{aligned}\lim_{\delta\to 0^+}\operatorname*{ess\,inf}_{|t-t_0|\le\delta}\inf_{X\in B_\delta(X_0),p\in B_\delta(D\phi(X_0))}\{\mathcal{H}(t,X,p)-b(t)\} &\le \\ \lim_{\eta\to 0^+}\operatorname*{ess\,inf}_{|t-t_k|\le\eta}\inf_{X\in B_\eta(X_k),p\in B_\eta(D\phi(X_k))}\{\mathcal{H}(t,X,p)-b(t)\} &\le 0.\end{aligned}$$

We then deduce that

$$\min\left(\lim_{\delta\to 0^+}\operatorname*{ess\,inf}_{|t-t_0|\le\delta}\inf_{X\in B_\delta(X_0),p\in B_\delta(D\phi(X_0))}\{\mathcal{H}(t,X,p)-b(t)\},(u-\Phi)(t_0,X_0)\right)\le 0.$$

□

We now give the proof of Theorem 3.4.11.

Proof of Theorem 3.4.11. First it is easy to verify that w fulfils the final condition $w(1,X)=\varphi(X)$ in the sense given by Definition 3.4.10. Moreover, since φ is lsc, consider $(\varphi_k)_{k\in\mathbf{N}}$ a monotone increasing sequence of continuous functions, from $\mathbf{R}^{d+1}$ to $\mathbf{R}$, converging pointwise to φ. We set

$$w_k(t,X) := \inf_{a\in\mathcal{A}}\{\max(\varphi_k(Y^a_{X,t}(1)),\ \max_{\theta\in[t,1]}\Phi(\theta,Y^a_{X,t}(\theta)))\},$$

then by Theorem 3.4.4 w_k is L^1-viscosity solution of (3.4.6) with final condition $w_k(1,X)=\varphi_k(X)$. Then by Lemma 3.4.12, we get that w_k is also a L^1-bilateral viscosity solution of (3.4.6) with final condition $w_k(1,X)=\varphi_k(X)$.

Next we show that w_k converges increasingly pointwise to w. Since $\varphi_k\le\varphi$, we immediately have that $w_k\le w$. By comparison principle for continuous L^1-viscosity solutions, we also know that $w_k\le w_{k+1}, k\in\mathbf{N}$. Therefore,

$$\lim_{k\to\infty} w_k \le w. \tag{3.4.19}$$

On the other hand, there exists for each $k\in\mathbf{N}$ an optimal control a_k such that

$$w_k(t,X)=\max(\Phi_k(Y^{a_k}_{X,t}(1)),\ \max_{\theta\in[t,1]}\Phi(\theta,Y^{a_k}_{X,t}(\theta))).$$

By the compactness of the set of trajectories with the initial data (t,X), there exists a $a_0\in\mathcal{A}$ such that (up to a subsequence):

$$Y^{a_k}_{X,t}\to Y^{a_0}_{X,t}\text{ uniformly on }[t,1]$$

as $k\to\infty$. Then

$$\begin{aligned}
w_k(t,X) &= \max(\varphi_k(Y^{a_k}_{X,t}(1)),\ \max_{\theta\in[t,1]}\Phi(\theta,Y^{a_k}_{X,t}(\theta)))\\
&\ge \max(\varphi_k(Y^{a_k}_{X,t}(1)),\ \max_{\theta\in[t,1]}\Phi(\theta,Y^{a_k}_{X,t}(\theta)))\\
&\quad -\max(\varphi(Y^{a_0}_{X,t}(1)),\ \max_{\theta\in[t,1]}\Phi(\theta,Y^{a_0}_{X,t}(\theta)))+w(t,X)\\
&\ge \min\big(\varphi_k(Y^{a_k}_{X,t}(1))-\varphi(Y^{a_0}_{X,t}(1)),\\
&\quad \max_{\theta\in[t,1]}(\Phi(\theta,Y^{a_k}_{X,t}(\theta))-\Phi(\theta,Y^{a_0}_{X,t}(\theta)))\big)+w(t,X),
\end{aligned} \tag{3.4.20}$$

where we have used the definition of w and the simple inequality:

$$\max(A,B)-\max(C,D)\ge\min(A-C,B-D).$$

Given $\varepsilon>0$ and fix $k_0>0$ such that

$$0\le\varphi(Y^{a_0}_{X,t}(1))-\varphi_{k_0}(Y^{a_0}_{X,t}(1))<\tfrac{\varepsilon}{2},$$
$$\max_{\theta\in[t,1]}|\Phi(\theta,Y^{a_{k_0}}_{X,t}(\theta))-\Phi(\theta,Y^{a_0}_{X,t}(\theta))|\le\max_{\theta\in[t,1]}\|Y^{a_{k_0}}_{X,t}(\theta)-Y^{a_0}_{X,t}(\theta)\|<\tfrac{\varepsilon}{2}.$$

Since Φ_k is increasing, then for any $k\ge k_0$

$$\begin{aligned}
\liminf_{k\to\infty}\varphi_k(Y^{a_k}_{X,t}(1)) &\ge \liminf_{k\to\infty,k\ge k_0}\varphi_{k_0}(Y^{a_k}_{X,t}(1))\\
&=\varphi_{k_0}(Y^{a_0}_{X,t}(1))>\varphi(Y^{a_0}_{X,t}(1))-\varepsilon/2.
\end{aligned} \tag{3.4.21}$$

Then combining (3.4.19)-(3.4.21), we prove that w_k converges increasingly pointwise to w. Therefore, the conclusion follows from the stability result with respect to the final condition as in Lemma 3.4.14, and finally the uniqueness follows by the following theorem. □

Theorem 3.4.15. Assume that **(Hg1)-(Hg2) (HC2)** hold and that $\mathcal{K}$ is a closed nonempty set of $\mathbf{R}^d$. Then there exists at most one L^1-bilateral viscosity solution of (3.4.6).

Proof. The proof follows the idea of [21, Theorem 5.14]. Suppose that there exist u and w two L^1-bilateral viscosity solutions of (3.4.6). We fix $(t', X') \in (0,1) \times \mathbf{R}^{n+1}$. For any $\beta > 0$, let $\zeta_\beta \in C_c^\infty(\mathbf{R})$ be a standard mollifier, and we define $\mathcal{H}_\beta$ and b_ϵ by

$$\mathcal{H}_\beta(\cdot, X, p) = \zeta_\beta \star \mathcal{H}(\cdot, X, p),$$

$$b_\epsilon(t) = \sup_{(X,p)\in B_{2M\varepsilon}(X')\times B_{4Me^K/\varepsilon}(0)} |\mathcal{H}_\beta(t, X, p) - \mathcal{H}(t, X, p)|.$$

Let $u_{\varepsilon,\epsilon}$ and $u^\alpha_{\varepsilon,\epsilon}$ be defined by

$$u_{\varepsilon,\epsilon}(t, X) := \inf_{Y\in\mathbf{R}^{n+1}} \{u(t, Y) + e^{Kt}\tfrac{|X-Y|^2}{\varepsilon^2}\} + \textstyle\int_0^t b_\epsilon(\tau)d\tau,$$
$$u^\alpha_{\varepsilon,\epsilon}(t, X) := \inf_{(s,Y)\in[0,1]\times\mathbf{R}^{n+1}} \{u(s, Y) + e^{Kt}\tfrac{|X-Y|^2}{\varepsilon^2} + \tfrac{(t-s)^2}{\alpha^2} + \textstyle\int_0^s b_\epsilon(\tau)d\tau\},$$

where K is a positive constant which will be defined later. We know that $\|b_\epsilon\|_{L^1} \to 0$ when $\beta \to 0$, then we have $\|b_\epsilon\|_{L^1} < 1$ for a small enough β. We note that

$$u^\alpha_{\varepsilon,\epsilon} \le u_{\varepsilon,\epsilon} \le u + \int_0^t b_\epsilon(\tau)d\tau \le M^2,$$

where $M = \sqrt{2\|u\|_\infty + 1}$. Since these functions are bounded, we can prove that $u_{\varepsilon,\epsilon}$ is locally Lipschitz continuous on X and $u^\alpha_{\varepsilon,\epsilon}$ is locally Lipschitz continuous on t and X by the classical arguments (see for example [21] Theorem 5.14). We will prove that $u^\alpha_{\varepsilon,\epsilon}$ is a sub-solution of an HJB equation in $O_\alpha = (M\alpha, 1 - M\alpha) \times \mathbf{R}^{n+1}$ then we will let $\alpha \to 0$. Let $\phi \in C^1(O_\alpha)$ and $(t', X') \in O_\alpha$ be a local minimum point of $u^\alpha_{\varepsilon,\epsilon} - \phi$. Let (s', Y') such that

$$u^\alpha_{\varepsilon,\epsilon}(t', X') = u(s', Y') + e^{Kt'}\frac{|X' - Y'|^2}{\varepsilon^2} + \frac{(t' - s')^2}{\alpha^2} + \int_0^{s'} b_\epsilon(\tau)d\tau,$$

then (t', s', X', Y') is a local minimum point of the function:

$$(t, s, X, Y) \mapsto u(s, Y) + e^{Kt}\frac{|X - Y|^2}{\varepsilon^2} + \frac{(t - s)^2}{\alpha^2} + \int_0^s b_\epsilon(\tau)d\tau - \phi(t, X),$$

and satisfies

$$e^{Kt'}\frac{|X' - Y'|^2}{\varepsilon^2} + \frac{(t' - s')^2}{\alpha^2} \le M^2. \tag{3.4.22}$$

Consider the case where $u^\alpha_{\varepsilon,\epsilon}(t', X') > \Phi(t', X')$. By the continuity of both functions, we have $u^\alpha_{\varepsilon,\epsilon}(s', Y') > \Phi(s', Y')$ when ε, α are small enough. We first fix $(t, X) = (t', X')$ and consider this function on (s, Y), since u is a L^1-bilateral viscosity solution and $u(s', Y') > \Phi(s', Y')$ for β small enough, we have

$$\lim_{\delta \to 0^+} \operatorname*{ess\,inf}_{|s-s'| \le \delta} \inf_{Y \in B_\delta(Y'), p \in B_\delta(p')} \{\mathcal{H}(s, Y, p) + b_\epsilon(s) + \frac{2(s'-t')}{\alpha^2}\} \le 0,$$

where $p' = e^{Kt'} \frac{2(X'-Y')}{\varepsilon^2}$. By the definition of b_ϵ, we note that $\mathcal{H}_\beta \le H + b_\epsilon$ and $\mathcal{H}_\beta$ is continuous, then we have

$$\mathcal{H}_\beta(s', Y', p') + \frac{2(s'-t')}{\alpha^2} \le 0. \tag{3.4.23}$$

Then we fix $(s, Y) = (s', Y')$ and consider the function on (t, X), then we have

$$\frac{\partial \phi}{\partial t}(t', X') = \frac{2(t'-s')}{\alpha^2} + K e^{Kt'} \frac{|X'-Y'|^2}{\varepsilon^2},$$

and

$$D\phi(t', X') = p'.$$

By using these two equalities, we deduce that

$$-\frac{\partial \phi}{\partial t}(t', X') + \mathcal{H}_\beta(s', Y', D\phi(t', X')) + K e^{Kt'} \frac{|X'-Y'|^2}{\varepsilon^2} \le 0. \tag{3.4.24}$$

By the inequality (3.4.22), we get that $|D\phi(t', X')| \le \frac{2Me^K}{\varepsilon}$, then we introduce the continuity modulus m_β of H_β on t for $|p| \le \frac{2Me^K}{\varepsilon}$. We obtain by using this continuity modulus that

$$-\frac{\partial \phi}{\partial t}(t', X') + \mathcal{H}_\beta(t', X', D\phi(t', X')) \tag{3.4.25}$$

$$\le \quad m_\beta(|t'-s'|) + \|k_0\|_\infty e^{Kt'} \frac{2|X'-Y'|^2}{\varepsilon^2} - K e^{Kt'} \frac{|X'-Y'|^2}{\varepsilon^2}, \tag{3.4.26}$$

then we can choose a K big enough such that

$$-\frac{\partial \phi}{\partial t}(t', X') + \mathcal{H}_\beta(t', X', D\phi(t', X')) \le m_\beta(M\alpha). \tag{3.4.27}$$

Combining with the case $u^\alpha_{\varepsilon,\epsilon}(t', X') \le \Phi(t', X')$, the local Lipschitz continuous function $u^\alpha_{\varepsilon,\epsilon}$ is a L^1-viscosity sub-solution of

$$\min\big(-\frac{\partial u}{\partial t} + \mathcal{H}_\beta(t, X, Du) - m_\beta(M\alpha), u - \Phi\big) \le 0, \tag{3.4.28}$$

in O_α. Then we set $\alpha \to 0$ then $\beta \to 0$, by a stability result in [22], we obtain that $(u_\varepsilon)^* := \limsup^* u^\alpha_{\varepsilon,\epsilon}$ is a L^1-viscosity sub-solution of (3.4.6). Therefore, by the comparison result for L^1-viscosity solutions, we have

$$(u_\varepsilon)^*(t, X) \le w(t, x), \ \forall (t, X) \in (0, 1) \times \mathbf{R}^{d+1}.$$

Let $\varepsilon \to 0$, we have

$$u(t, X) \le w(t, X), \ \forall (t, X) \in (0, 1) \times \mathbf{R}^{d+1},$$

then by reversing the roles of u and w, the uniqueness follows. □

3.5 Numerical tests

Given $T > 0$, $\tau \in [0, T]$, $x \in \mathbf{R}^2$, consider the following controlled system:

$$\begin{cases} dy(t) = u(t) \begin{pmatrix} cos(\alpha(t)) \\ sin(\alpha(t)) \end{pmatrix} dt + V d\delta_{t=1}, \\ y(\tau) = x, \end{cases}$$

where the control variables $u : (0, T) \to U$ and $\alpha : (0, T) \to A$. At time $t = 1$, the trajectories jump with the magnitude V. Let $\mathcal{C} \subset \mathbf{R}^2$ be the target, and $\mathcal{K} \subset \mathbf{R}^2$ be the set of state constraints. Set that φ and ψ is respectively the signed distance function to $\mathcal{C}$ and $\mathcal{K}$.

Consider the value function of the Rendez-vous problem:

$$v(\tau, x) := \inf\{\varphi(y(T)),\ y(t) \in \mathcal{K},\ \forall t \in [0, T]\}.$$

The reparametrized function can be computed:

$$\Phi(s) := \begin{cases} (3s, 0) & 0 \le s \le \frac{1}{3}, \\ (1, 3s - 1) & \frac{1}{3} \le s \le \frac{2}{3}, \\ (3s - 1, 1) & \frac{2}{3} \le s \le 1. \end{cases}$$

Then the reparametrized dynamics are

$$\mathcal{F}(s, x, u, \alpha) := \begin{cases} 3u(cos(\alpha), sin(\alpha))^T & 0 \le s \le \frac{1}{3}, \\ 3V & \frac{1}{3} < s < \frac{2}{3}, \\ 3u(cos(\alpha), sin(\alpha))^T & \frac{2}{3} \le s \le 1. \end{cases}$$

The time-dependent state constraints $\mathbb{K}$ for the reparametrized problem are defined as in Theorem 3.4.1 and let $\Psi : [0, 1] \times \mathbf{R}^d \to \mathbf{R}$ such the 0-sublevel of $\Psi(s, \cdot)$ represents $\mathbb{K}(s)$ for $s \in [0, 1]$.

Let ϑ be the value function of the reparametrized problem, i.e.

$$\vartheta(\sigma, x) = \inf\{\varphi(z(1)) \,:\, \dot{z}(s) = \mathcal{F}(s, z(s), u(s), \alpha(s)),\ z(\sigma) = x,\ \Psi(s, z(s)) \le 0\}.$$

We set

$$w(\sigma, x, \xi) = \inf\{\varphi(z(1)) - \xi \,:\, \dot{z}(s) = \mathcal{F}(s, z(s), u(s), \alpha(s)),\ z(\sigma) = x,\ \Psi(s, z(s)) \le 0\}.\}$$

We need to solve the equation

$$\begin{cases} \min\big(- \partial_s w(s, x, \xi) + \mathcal{H}(s, x, Dw), w(s, x, \xi) - \Psi(s, x)\big) = 0, \\ w(1, x, \xi) = \max(\varphi(x) - \xi, \Psi(1, x)). \end{cases}$$

Then the ξ-sublevel of ϑ is obtained by

$$\{(s,x)\,|\,\vartheta(s,x)\le\xi\}=\{(s,x)\,|\,w(s,x,\xi)\le 0\},$$

and

$$v(t,x)=\begin{cases}\vartheta(\frac{t}{3},x) & 0\le t\le 1,\\ \vartheta(\frac{t+1}{3},x) & 1<t\le 2.\end{cases}$$

We show two numerical tests for this problem using the software ROC-HJ solver [38]. For both tests, we show at first the value function of the reparametrized problem ϑ, in particular the 0-level set of ϑ. Then the value function of the original problem with impulsive system can be computed easily since v is the restriction of ϑ on $[0,\frac{1}{3}]\cup[\frac{2}{3},1]$. The optimal trajectories have also been computed for both problems. The optimal trajectory is continuous for the reparametrized problem since the state constraints are relaxed when the trajectory crosses the square obstacle, while the optimal trajectory is discontinuous for the original problem.

For both tests, we set $T=2$, $U=[0,1]$ and $A=[0,2\pi]$ be the control sets, and $\mathbb{C}=B(0,1)$ as the target with $\varphi(x)=\sqrt{x_1^2+x_2^2}-1$. The state constraints are taken as $\mathcal{K}=\mathbf{R}^2\backslash\mathcal{O}$ where $\mathcal{O}$ is an obstacle avoided by the trajectories.

- **Test 1.**

We take $V=(-1,0)$ and $\mathcal{O}$ is a square obstacle defined by

$$\mathcal{O}=\left\{(x_1,x_2)\,|\,\left|x_1-\frac{5}{2}\right|\le\frac{1}{2},\ |x_2|\le\frac{1}{2}\right\}.$$

Then $\Psi(s,x)$ is taken as

$$\begin{cases}-\max\left\{|x_1-\frac{5}{2}|-\frac{1}{2},|x_2|-\frac{1}{2}\right\} & 0\le s\le\frac{1}{3},\\ -\max\left\{|x_1-\frac{5}{2}|-\max\left\{\frac{1}{2}-M\left(s-\frac{1}{3}\right),0\right\},|x_2|-\max\left\{\frac{1}{2}-M\left(s-\frac{1}{3}\right),0\right\}\right\} & \frac{1}{3}\le s\le\frac{1}{2},\\ -\max\left\{|x_1-\frac{5}{2}|-\max\left\{\frac{1}{2}-M\left(\frac{2}{3}-s\right),0\right\},|x_2|-\max\left\{\frac{1}{2}-M\left(\frac{2}{3}-s\right),0\right\}\right\} & \frac{1}{2}\le s\le\frac{2}{3},\\ -\max\left\{|x_1-\frac{5}{2}|-\frac{1}{2},|x_2|-\frac{1}{2}\right\} & \frac{2}{3}\le s\le 1.\end{cases}$$

The value functions are computed by using the second-order ENO scheme in the domain $[-3,5]\times[-4,4]$ with 200^2 mesh points. The 0-level sets of ϑ and v are shown in Figure 3.2(a) and Figure 3.2(b) respectively. We observe the discontinuity of the value of v and that the optimal trajectory, taking the starting point at $(\frac{7}{2},0)$, jumps over the obstacle at time $t=1$.

- **Test 2.**

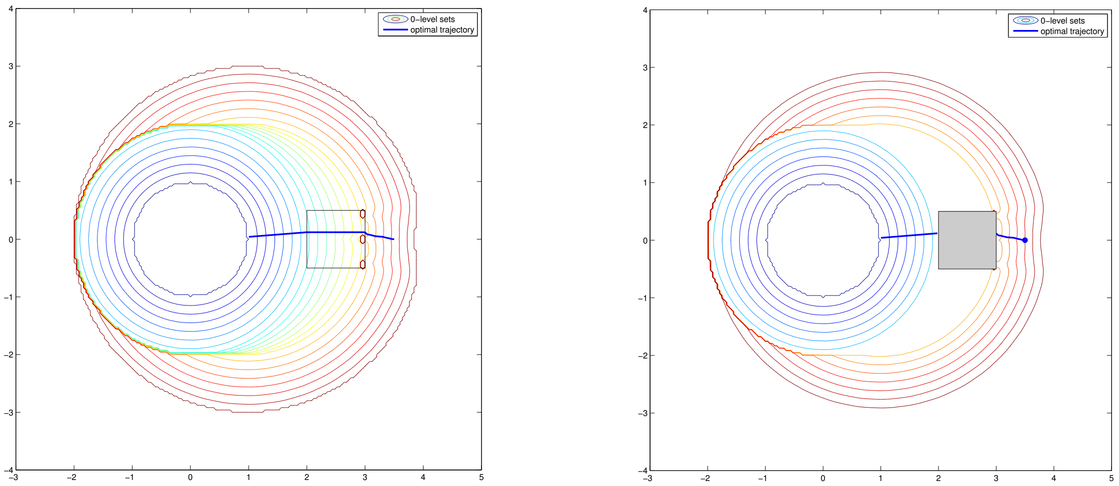
We take $V=(-2,-1)$ and $\mathcal{O}$ is an obstacle defined by

$$\mathcal{O}=\{(x_1,x_2)\,|\,2\le x_1\le 3\}.$$

In this case, the set of state constraints $\mathcal{K}$ is not connected. Then $\Psi(s,x)$ is taken as

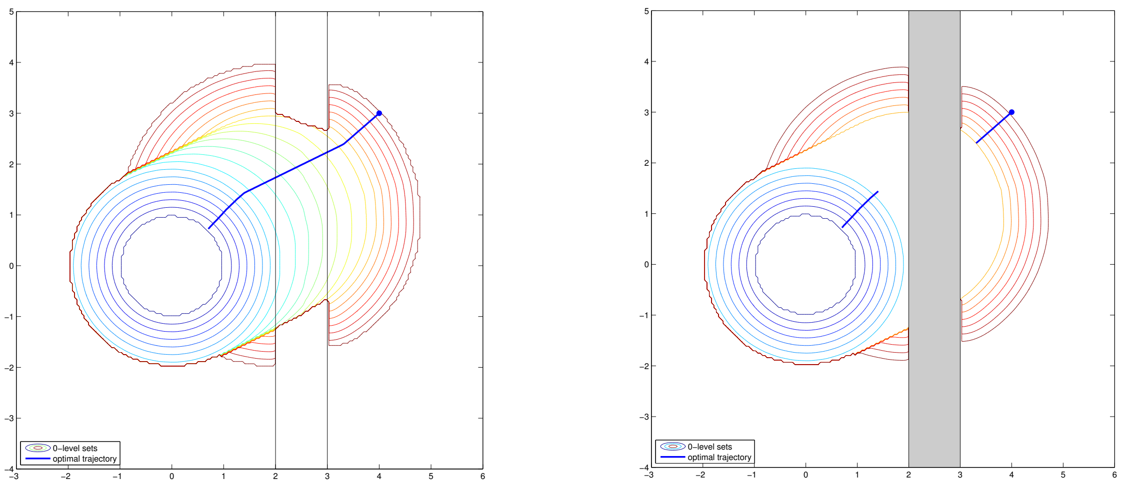
$$\begin{cases} -\max\left\{2-x_1, x_1-3\right\} & 0 \le s \le \frac{1}{3}, \\ -\max\left\{2-x_1+\min\left\{M\left(s-\frac{1}{3}\right),\frac{1}{2}\right\}, x_1-3+\min\left\{M\left(s-\frac{1}{3}\right),\frac{1}{2}\right\}\right\} & \frac{1}{3} \le s \le \frac{1}{2}, \\ -\max\left\{2-x_1+\min\left\{M\left(\frac{2}{3}-s\right),\frac{1}{2}\right\}, x_1-3+\min\left\{M\left(\frac{2}{3}-s\right),\frac{1}{2}\right\}\right\} & \frac{1}{2} \le s \le \frac{2}{3}, \\ -\max\left\{2-x_1, x_1-3\right\} & \frac{2}{3} \le s \le 1. \end{cases}$$

The value functions are computed by using the second-order ENO scheme in the domain $[-3,6]\times[-4,5]$ with 200^2 mesh points. The 0-level sets of ϑ and v are shown in Figure 3.3(a) and Figure 3.3(b) respectively. We compute the optimal trajectory with the starting point at $(4,3)$. We observe that, although the set of state constraints is not connected, the optimal trajectory can jump from one connected component to the other at time $t=1$.



(a) 0-level sets of ϑ (b) 0-level sets of v

FIGURE 3.2: Test 1



(a) 0-level sets of ϑ (b) 0-level sets of v

FIGURE 3.3: Test 2

Chapter 4

Transmission conditions for Hamilton-Jacobi-Bellman system on multi-domains

Publications of this chapter

(with A. Siconolfi and H. Zidani) *Transmission conditions on interfaces for Hamilton-Jacobi-Bellman equations*, submitted. http://hal.inria.fr/hal-00820273

(with H. Zidani) *Hamilton-Jacobi-Bellman equations on multi-domains*, Control and Optimization with PDE Constraints, International Series of Numerical Mathematics, 164:93-116, 2013.

4.1 Introduction

The present work aims at investigating a system of Hamilton-Jacobi-Bellman equations on a so-called structure of multi-domains. This form was introduced by Bressan-Hong [46] and Barnard-Wolenski [29]. It is concerned with the repartition of $\mathbf{R}^d$ by disjoint subdomains $(\Omega_i)_{i=1,\dots,m}$ with

$$\mathbf{R}^d = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_m, \quad \Omega_i \cap \Omega_j = \emptyset \ \text{ for } i \neq j.$$

Consider a collection of Hamilton-Jacobi-Bellman (HJB) equations

$$\begin{cases} -\partial_t u(t,x) + H_i(x, Du(t,x)) = 0, & \text{for } t \in (0,T), x \in \Omega_i, \\ u(T,x) = \varphi(x), & \text{for } x \in \Omega_i, \end{cases} \tag{4.1.1}$$

with the different Hamiltonians H_i satisfying standard assumptions, and where $\phi : \mathbf{R}^d \to \mathbf{R}$ is a Lipschitz continuous function. We address the question to know what condition should be considered on the interfaces (i.e., the intersections of the sets $\overline{\Omega}_i$) in order to get the existence and uniqueness of solution, and also what should be the precise notion of solution.

In order to identify a global solution satisfying (4.1.1) on each subdomain Ω_i, one can define a global HJB equation with the Hamiltonian H defined on the whole $\mathbf{R}^d$ with $H(x,p) = H_i(x,p)$ whenever $x \in \Omega_i$. However, H can not be expected to be continuous and the definition of H on the interfaces between the subdomains Ω_i is not clear.

The investigation of HJ equations with discontinuous coefficients is of growing interest both from theoretical viewpoint and for applications. It appears in the modeling of several physical problems, such as the problem of ray light propagation in an inhomogeneous medium with discontinuous refraction index. It is also motivated by some recent research [1, 106] on network, modeled as a union of finite half-lines with a single common point, with the applications on traffic flow problems. Another important motivation comes from the applications in Hybrid Control Theory.

Recall that the viscosity notion has been introduced by Crandall-Lions to give a precise meaning to the HJ equations with continuous Hamiltonians. This notion has been extended to the discontinuous case by Ishii (see [102]) where the lsc supersolutions satisfy the HJB inequality with the usc envelop of the Hamiltonian and the usc subsolutions satisfy the HJB inequality with the lsc envelop of the Hamiltonian. More precisely, in our case, the HJB equations on the interfaces will be

$$\begin{cases} -\partial_t u(t,x) + \max_{i=1,\dots,m} \{H_i(x, Du(t,x))\} \geq 0, \\ -\partial_t u(t,x) + \min_{i=1,\dots,m} \{H_i(x, Du(t,x))\} \leq 0. \end{cases}$$

This notion provides the first vision on this subject, and we then look for stronger conditions to establish the comparison principle result. Later, the viscosity notion was extended to the case where the Hamiltonian is measurable with respect to the space variable by Camilli-Siconolfi (see [54]). A comparison principle is obtained under a restrictive assumption, called transversality assumption, that prevents the trajectories related to the behavior of the solutions from complex interactions with interfaces. Therefore, the interfaces can be ignored under this assumption. It can be also found in the area of hybrid control theory. We consider the present study as the first step in the direction of hybrid systems without transversality conditions.

In [129], a class of stationary HJ equations with discontinuous Lagrangian has been studied where the Hamiltonian is the type of $H(x,p)+g(x)$ with continuous H and discontinuous g. A uniqueness result is proved under some assumptions on g, while we are interested in the more general case with weaker assumptions.

The objective of our study is to derive some *junction* conditions that have to be considered on the interfaces in order to guarantee the existence and uniqueness of the viscosity solution of (4.1.1). Three papers have been particularly influential for our work. We would like to mention [46], which has been, as far as we know, the first paper on the subject and where the relevance of HJB tangential equations, namely equations posed on the interfaces, is pointed out. The second work are [23, 24] which have studied both the infinite horizon problem and the finite horizon problem in two-domains.

The controls are divided between regular and singular, according to the behavior of associated velocities on the interface, and correspondingly, two different value functions are analyzed mainly by the PDE tools. The work considers at first the Ishii's notion of solutions and looks for the properties satisfied by the value functions which allow to obtain the characterization results. The controllability is assumed in the whole space in [23], and then has been weakened in [24] where the controllability is only assumed in the normal directions on the interface. The convexity of the set of velocities/costs is also needed. The comparison results for super/sub-solutions and the stability results for both value functions have been established. This approach is certainly interesting and capable of promising developments. In our work, we are particularly interesting in the value function associated to all controls of the integrated system which corresponds to the regularization. Another main difference is that the notion of solutions is not based on the Ishii's notion because we are interested in the minimal requirements for the junction conditions. The third reference is [29], which has attracted our attention on the fact that admissible curves of integrated system are actually integral trajectory of an essential, somehow hidden, dynamics and have showed the effectiveness of Filippov Approximation Theorem in this context. Our topic is also related, at least for difficulty to be tackled, with studies of Hamilton-Jacobi equations in domains with junctions or on networks, see [1, 106].

As mentioned before, the main difficulty in our study is the non Lipschitz continuity of the Hamiltonians in HJB equations. We take two steps of our study. At first, we consider the coercive Hamiltonians taking a relatively simple form without running costs. After we have obtained the junction conditions in this relatively more restrictive setting and have understood the essential properties needed for the comparison principle, the running costs are involved and the coercivity setting of Hamiltonians is erased for the second step. Then a stronger comparison principle result is obtained in this general setting with only some necessary assumptions.

The first step of our study is to consider the Hamiltonian of the following form

$$H_i(x,p) = \sup_{q\in F_i(x)} \{-p\cdot q\}.$$

Then the associated optimal control problems belong to the class of finite horizon problems. F_i are interpreted as the dynamics in each Ω_i, and we focus on the difficulty arising from the switch of dynamics around the interfaces. For simplicity, a strong controllability assumption is considered which leads to the coercivity of the Hamiltonians. We detect the possible HJB inequalities for super and subsolutions, especially the HJB inequalities with the same Hamiltonian for both supersolutions and subsolutions. The main idea developed here follows the concept of *Essential Hamiltonian* introduced in [29], and provides a new viscosity notion that is quite different from the notion of Ishii [102]. This new definition gives a precise meaning to the transmission conditions between Ω_i and provides the uniqueness of viscosity solution. It is discovered that two properties are crucial for our study: the continuity of the value function on the interfaces and the Lipschitz continuity of the tangential dynamics along the interfaces. Both of them are consequences of the controllability assumption. The main result is the following: we are able to compare lsc supersolutions and Lipschitz continuous subsolutions.

For the second step of our study, we take the Hamiltonians of the following form involving running cost which is more general:

$$H_i(x, p) := \sup_{a_i \in A_i} \{-p \cdot f_i(x, a_i) - \ell_i(x, a_i)\},$$

where A_i is the set of control on Ω_i. Here we assume the state variable space $\mathbf{R}^d$ to be partitioned in two disjoint open sets Ω_1, Ω_2 plus their common boundary, the interface, that we denote by Γ and take of class C^2 without requiring any connectedness condition. Then the difficulty arising from the switch of running costs ℓ_i is also involved. As mentioned above, two crucial properties are important for our study: the continuity of the value function on the interface and the Lipschitz continuity of the tangential dynamics. Regarding our hypotheses, the strong controllability assumption in the first step of our study is replaced by a much weaker controllability condition. More precisely, we assume a sort of permeability of the interface, namely the possibility to go from the interface to any of the two open regions following admissible trajectories. This is unavoidable if we want the value function to be continuous on Γ. Moreover, some controllability of tangential type on Γ are required to imply that the subsolutions are Lipschitz continuous when restricted to the interfaces.

We emphasize that no coercivity requirements on the Bellman Hamiltonians related to systems in Ω_i are assumed. Theses are actually quite onerous from a control theoretic viewpoint and implies Lipschitz continuity of subsolutions on the whole space, which simplifies to some extent the analysis. To the best of our knowledge, all comparison results holding for HJB equations in presence of some sort of interface, junctions or posed on networks have been established to date assuming coercivity of corresponding Hamiltonians.

The last necessary assumption in this step of study is the convexity, at any point of the interface, of the set of all admissible velocities/costs. In our understanding, this is actually the less satisfactory and more technical requirement, we crucially exploit it to prove a regularity result for an augmented dynamics on Γ. Same assumption appears in [23] and [29]. The use of relaxed controls will hopefully allow to weaken it or at least clarify its meaning in relation to the model.

The main result of this step is the following: we show that the value function is a bounded continuous solution of the HJB system, and we are able to compare lsc supersolutions with usc subsolutions, which are in addition continuous on Γ. It is deduced that, as a consequence of the previous properties, the value function is the unique solution of the HJB system.

The main idea to look for the transmission conditions on the interface is to analyze the behavior of the trajectories near the interface. To obtain the comparison principle, we are required to be able to compare the supersolutions and the subsolutions. It is known that the properties of the super/sub-solutions are strongly related to some invariance properties of the trajectories. These properties have been analyzed and then characterized by HJB inequalities, which are considered as the candidate transmission conditions for the super/sub-solutions. In particular, we are interested in two types of transmission conditions: the weakest conditions for super/sub-solutions and the conditions in the form of HJB equations with the same Hamiltonian for both super- and sub-solutions.

The methodology we use is of dynamical type, see [60], [61]. Namely, instead of directly working with viscosity test functions, we get the comparison by first establishing optimality properties for sub/supersolution, or equivalently invariance of the hypograph of any subsolution and epigraph of any supersolutions with respect to an augmented controlled dynamics defined in $\mathbf{R}^d \times \mathbf{R}$.

We take for the supersolution part on Γ the Bellman Hamiltonian corresponding to all control in A, which turns out to be equal to $\max\{H_1, H_2\}$. This is the Hamiltonian for supersolutions indicated by Ishii's theory, the reference frame for discontinuous HJ equations. However the Hamiltonian provided by the same theory for subsolutions, namely $\min\{H_1, H_2\}$, does not seem well adapted to our setting since it does not take into any special account controls corresponding to tangential velocities.

We consider for subsolutions the Hamiltonian of Bellman type with controls associated to tangential velocities, accordingly the corresponding equation is restricted on the interface, which means that viscosity tests take place at local constrained maximizers with constraint Γ, or test functions can be possibly just defined on Γ. Same Hamiltonian also appears in [23], the difference is that in our case to satisfy such a tangential equation is the unique condition we impose on subsolutions on Γ, and not an additional one.

This is in our opinion the most relevant new point in this work. It deeply changes the nature of the system because now equations pertaining to subsolutions are completely separated in the three regions of the partition. This requires, first, some compatibility conditions, otherwise there is no hope to get comparison results. Secondly, comparison must be based not on semicontinuity property of the Hamiltonian, that we do not have, at least for the subsolution part, but on a separation principle of the controlled dynamics of the integrated system, related to the partition, we will explain later on.

The transmission conditions in the form of HJB inequations presented as above are actually the weakest conditions in our study: $\max\{H_1, H_2\}$ for supersolution and the Hamiltonian related to the tangent directions for subsolutions. Both of them can be replaced by the stronger transmission condition as HJB equation with the essential Hamiltonian, and the existence and uniqueness of solution still holds. The interest of considering this stronger condition lies in the advantage that the Hamiltonian is the same for super/sub-solutions, which we will see the convenience in the study of numerical approaches and homogenization problems coming after the present study.

4.2 The finite horizon problem under a strong controllability assumption

4.2.1 Setting of the problem

Consider the following structure on $\mathbf{R}^d$: given $m \in \mathbf{N}$, let $\{\Omega_1, \dots, \Omega_m\}$ be a finite collection of C^2 open d-manifolds embedded in $\mathbf{R}^d$. For each $i = 1, \dots, m$, the closure of Ω_i is denoted as $\overline{\Omega}_i$. Assume that this collection of manifolds satisfies the following:

$$\textbf{(H1)} \begin{cases} \text{(i)} & \mathbf{R}^d = \bigcup_{i=1}^m \overline{\Omega}_i \text{ and } \Omega_i \cap \Omega_j = \emptyset \text{ when } i \neq j,\ i,j \in \{1, \dots, m\}; \\ \text{(ii)} & \text{Each } \overline{\Omega}_i \text{ is proximally smooth and wedged.} \end{cases}$$

The concepts of proximally smooth and wedged are introduced in [60]. For any set $\Omega \subseteq \mathbf{R}^d$, we recall that $\overline{\Omega}$ is proximally smooth means that the signed distance function to $\overline{\Omega}$ is differentiable on a tube neighborhood of $\overline{\Omega}$. $\overline{\Omega}$ is said to be wedged means that the interior of the tangent cone of $\overline{\Omega}$ at each point of $\overline{\Omega}$ is nonempty.

Let $\varphi : \mathbf{R}^d \to \mathbf{R}$ be a given function satisfying:

(H2) φ is a bounded Lipschitz continuous function.

Let $T > 0$ be a given final time, for $i = 1, \dots, m$, consider the following system of Hamilton-Jacobi (HJ) equations:

$$\begin{cases} -\partial_t u(t,x) + H_i(x, Du(t,x)) = 0, & \text{for } t \in (0,T), x \in \Omega_i, \\ u(T,x) = \varphi(x), & \text{for } x \in \Omega_i. \end{cases} \tag{4.2.1}$$

The system above implies that on each d-manifold Ω_i, a classical HJ equation is considered. However, there is no information on the boundaries of the d-manifolds which are the junctions between Ω_i. We then address the question to know what condition should be considered on the boundaries in order to get the existence and uniqueness of solution to all the equations.

In the sequel, we call the singular subdomains contained in the boundaries of the d-manifolds the interfaces. Let $\ell \in \mathbf{N}$ be the number of the interfaces and we denote $\Gamma_j,\ j = 1, \dots, \ell$ the interfaces which are also open embedded manifolds with dimensions strictly smaller than d. Assume that the interfaces satisfy the following:

$$\textbf{(H3)} \begin{cases} \text{(i)} & \mathbf{R}^d = \left(\bigcup_{i=1}^m \Omega_i\right) \cup \left(\bigcup_{j=1}^\ell \Gamma_j\right),\ \Gamma_j \cap \Gamma_k = \emptyset,\ j \neq k,\ j,k = 1, \dots, \ell; \\ \text{(ii)} & \text{If } \Gamma_j \cap \overline{\Omega}_i \neq \emptyset, \text{ then } \Gamma_j \subseteq \overline{\Omega}_i, \text{ for } i = 1, \dots, m, j = 1, \dots, \ell; \\ \text{(iii)} & \text{If } \Gamma_k \cap \overline{\Gamma}_j \neq \emptyset, \text{ then } \Gamma_k \subseteq \overline{\Gamma}_j, \text{ for } j,k \in \{1, \dots, \ell\}; \\ \text{(iv)} & \text{Each } \overline{\Gamma}_j \text{ is proximally smooth and relatively wedged.} \end{cases}$$

For any open embedded manifold Γ with dimension $p < d$, $\overline{\Gamma}$ is said to be relatively wedged if the relative interior (in $\mathbf{R}^p$) of the tangent cone of $\overline{\Gamma}$ at each point of $\overline{\Gamma}$ is nonempty.

Example 4.2.1. A simple example is shown in Figure 4.1 with $d = 1, m = 2$ and $\ell = 1$. Here $\mathbf{R} = \Omega_1 \cup \Gamma_1 \cup \Omega_2$ with

$$\Omega_1 = \{x : x < 0\},\ \Omega_2 = \{x : x > 0\},\ \Gamma_1 = \{0\}.$$

Note that Ω_1, Ω_2 are two one dimensional manifolds, and the only interface is the zero dimensional manifold Γ_1.

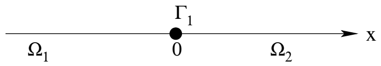
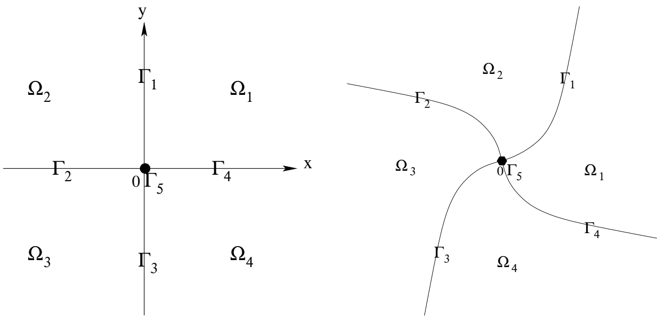


FIGURE 4.1: A multi-domain in 1d.

Other possible examples in $\mathbf{R}^2$ are depicted in the following figure.



We are interested particularly in the HJ equations with the Hamiltonians $H_i : \overline{\Omega}_i \times \mathbf{R}^d \to \mathbf{R},\ i = 1, \ldots, m$ of the following Bellman form: for $(x, q) \in \overline{\Omega}_i \times \mathbf{R}^d$,

$$H_i(x, q) = \sup_{p \in F_i(x)} \{-p \cdot q\},$$

where $F_i : \overline{\Omega}_i \rightsquigarrow \mathbf{R}^d$ are multifunctions defined on $\overline{\Omega}_i$ and satisfy the following assumptions

$$(\mathbf{H4}) \begin{cases} \text{(i)} & \forall\, x \in \overline{\Omega}_i,\ F_i(x) \text{ is a nonempty, convex, and compact set;} \\ \text{(ii)} & F_i \text{ is Lipschitz continuous on } \Omega_i \text{ with respect to the Hausdorff metric;} \\ \text{(iii)} & \exists\, \mu > 0 \text{ so that } \max\{|p| : p \in F_i(x)\} \le \mu(1 + \|x\|)\ \forall x \in \overline{\Omega}_i; \\ \text{(iv)} & \exists\, \delta > 0 \text{ so that } \forall\, x \in \overline{\Omega}_i,\ \delta \overline{B(0,1)} \subseteq F_i(x). \end{cases}$$

The hypothesis **(H4)(i)-(iii)** are classical for the study of HJB equations, whereas **(H4)(iv)** is a strong controllability assumption. Although this controllability assumption is restrictive, we use it here in order to ensure the continuity of solutions for the system (4.2.1). The continuity property plays an important role in our analysis, but it can be obtained under weaker assumption than **(H4iv)**, see [122].

Remark 4.2.2. For the simplicity, we define the multifunction F_i on $\overline{\Omega}_i$. In fact, if F_i is only defined on Ω_i and satisfies **(H4)**, it can be extended to the whole $\overline{\Omega}_i$ by its local Lipschitz continuity.

4.2.2 Essential Hamiltonian

The main goal of this work is to identify the junction conditions that ensure the uniqueness of the solution for the HJ system (4.2.1). In [54], the uniqueness of the solution of space-measurable HJ equations has been studied under some special conditions, called "*transversality*" conditions. Roughly speaking, this transversality condition would mean, in the case of problem (4.2.1), that the interfaces can be ignored and the behavior of the solution on the interfaces is not relevant. Here we consider the case when no transversality condition is assumed and we analyze the behavior of the solution on the interfaces.

First of all, in order to define a multifunction on the whole $\mathbf{R}^d$, an immediate idea is to consider the approach of Filippov regularization [75] of $(F_i)_{i=1,\dots,m}$. For this consider the multifunction $G : \mathbf{R}^d \rightsquigarrow \mathbf{R}^d$ given by:

$$\forall\, x \in \mathbf{R}^d,\ G(x) := \mathrm{co}\,\{F_i(x) :\ i \in \{1,\dots,m\},\, x \in \overline{\Omega}_i\}.$$

G is the smallest upper semi-continuous (usc) envelope of $(F_i)_{i=1,\dots,m}$ such that $G(x) = F_i(x)$ for $x \in \Omega_i$. Consider the Hamiltonian associated to G:

$$H_G(x,q) = \sup_{p\in G(x)} \{-p\cdot q\}.$$

If $H_G(\cdot,q)$ is Lipschitz continuous, then one could define the HJB equations on the interfaces with the Hamiltonian H_G and the uniqueness result would follow from the classical theory. However, G is not necessarily Lipschitz continuous and the characterization by means of HJB equations is not valid, see [69].

The next step is to define the multifunctions on the interfaces Γ_j. We first recall the notion of tangent cone which is defined as in (2.2.1). For any C^2 smooth $\mathcal{C} \subseteq \mathbf{R}^p$ with $1 \le p \le d$, the tangent cone $\mathcal{T}_{\mathcal{C}}(x)$ at $x \in \mathcal{C}$ is defined as

$$\mathcal{T}_{\mathcal{C}}(x) = \{v \in \mathbf{R}^p : \liminf_{t\to 0^+} \frac{d_{\mathcal{C}}(x+tv)}{t} = 0\},$$

where $d_{\mathcal{C}}(\cdot)$ is the distance function to $\mathcal{C}$. For $j = 1,\dots,\ell$, we define the multifunction $\widetilde{G}_j : \Gamma_j \rightsquigarrow \mathbf{R}^d$ on the interface Γ_j by

$$\forall\, x \in \Gamma_j,\ \widetilde{G}_j(x) := G(x) \cap \mathcal{T}_{\Gamma_j}(x).$$

Note that $\mathcal{T}_{\Gamma_j}(x)$ agrees with the tangent space of Γ_j at x, and the dimension of $\mathcal{T}_{\Gamma_j}(x)$ is strictly smaller than d. On $\widetilde{G}_j$ we have the following regularity result for which the proof is postponed to Appendix 4.2.8.

Lemma 4.2.3. Under the assumptions **(H1)** and **(H4)**, $\widetilde{G}_j(\cdot) : \Gamma_j \rightsquigarrow \mathbf{R}^d$ is locally Lipschitz continuous on Γ_j.

Through this section, and for the sake of simplicity of the notations, for $k = 1, \ldots, m+\ell$ we set

$$\mathcal{M}_k = \begin{cases} \Omega_k, & \text{for } k = 1, \ldots, m; \\ \Gamma_{k-m}, & \text{for } k = m+1, \ldots, m+\ell, \end{cases}$$

and we define a new multifunction $F_k^{new} : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ by

$$F_k^{new}(x) := \begin{cases} F_k(x) & \text{for } x \in \mathcal{M}_k,\ k = 1, \ldots, m; \\ \widetilde{G}_{k-m}, & \text{for } x \in \mathcal{M}_k,\ k = m+1, \ldots, m+\ell. \end{cases}$$

In all the sequel, we will also need the "essential multifunction" F^E which will be used in the junction conditions:

Definition 4.2.4. **(The essential multifunction.)**
The essential multifunction $F^E : \mathbf{R}^d \rightsquigarrow \mathbf{R}^d$ is defined by

$$F^E(x) := \bigcup_{k \in \{1, \ldots, m+\ell\}} \left\{ F_k^E(x) :\ x \in \overline{\mathcal{M}}_k \right\},\ \forall\, x \in \mathbf{R}^d,$$

where $F_k^E : \overline{\mathcal{M}}_k \rightsquigarrow \mathbf{R}^d$ is defined by

$$F_k^E(x) = F_k^{new}(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_k}(x),\ \text{for } x \in \overline{\mathcal{M}}_k.$$

F^E is called essential velocity multifunction in [29]. According to the definition, $F^E(x)$ is the union of the corresponding inward and tangent directions to each subdomain near x. We note that

$$F^E|_{\mathcal{M}_i} = F_i,\ \text{for } i = 1, \ldots, m,\ \text{and } F^E(x) \subseteq G(x),\ \text{for } x \in \mathbf{R}^d.$$

Example 4.2.5. Suppose the following dynamic data for the domain in Example 1:

$$F_1(x) = [-\frac{1}{2}, 1],\ \forall\, x \in \Omega_1,\ \text{and } F_2(x) = [-1, \frac{1}{2}],\ \forall\, x \in \Omega_2.$$

On this simple example, one can easily see that G and F^E are different on the interface $\{0\}$:

$$G(0) = [-1, 1],\ F^E(0) = [-\frac{1}{2}, \frac{1}{2}].$$

Now, define the "essential" Hamiltonian $H^E : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ by:

$$H^E(x, q) = \sup_{p \in F^E(x)} \{-p \cdot q\}, \ \forall \, (x, q) \in \mathbf{R}^d \times \mathbf{R}^d.$$

We point out that on each d-manifold Ω_i, for each $q \in \mathbf{R}^d$

$$H^E(x, q) = H_i(x, q), \text{ whenever } x \in \Omega_i.$$

In general, H^E is not Lipschitz continuous with respect to the first variable. Some properties of H^E will be discussed in Section 3.

4.2.3 Main results

We now state the main existence and uniqueness result.

Theorem 4.2.6. Assume that **(H1)-(H4)** hold. The following system:

$$-\partial_t u(t, x) + H_i(x, Du(t, x)) = 0, \text{ for } t \in (0, T), x \in \Omega_i \quad i = 1, \ldots, m; \tag{4.2.2a}$$

$$-\partial_t u(t, x) + H^E(x, Du(t, x)) = 0, \text{ for } t \in (0, T), x \in \Gamma_j \quad j = 1, \ldots, \ell; \tag{4.2.2b}$$

$$u(T, x) = \varphi(x), \text{ for } x \in \mathbf{R}^d, \tag{4.2.2c}$$

has a unique viscosity solution in the sense of Definition 4.2.8.

Note that the system (4.2.2) can be rewritten as

$$\begin{cases} -\partial_t u(t, x) + H^E(x, Du(t, x)) = 0, & \text{for } t \in (0, T), x \in \mathbf{R}^d \\ u(T, x) = \varphi(x), & \text{for } x \in \mathbf{R}^d, \end{cases}$$

which is an HJB equation on the whole space with a discontinuous Hamiltonian H^E.

Before giving the definition of viscosity solution, we need the following notion of extended differentials.

Definition 4.2.7. **(Extended differential)**
Let $\phi : (0, T) \times \mathbf{R}^d \to \mathbf{R}$ be a continuous function, and let $\mathcal{M} \subseteq \mathbf{R}^d$ be an open C^2 embedded manifold in $\mathbf{R}^d$. Suppose that $\phi \in C^1((0, T) \times \overline{\mathcal{M}})$. Then we define the differential of ϕ on any $(t, x) \in (0, T) \times \overline{\mathcal{M}}$ by

$$\nabla_{\overline{\mathcal{M}}} \phi(t, x) := \lim_{x_n \to x, x_n \in \mathcal{M}} \left(\phi_t(t, x_n), D\phi(t, x_n)\right).$$

Note that $\nabla \phi$ is continuous on $(0, T) \times \mathcal{M}$, the differential defined above is nothing but the extension of $\nabla \phi$ to the whole $\overline{\mathcal{M}}$.

Definition 4.2.8. **(Viscosity solution)**
Let $u : (0, T] \times \mathbf{R}^d \to \mathbf{R}$ be a bounded local Lipschitz continuous function. For any $x \in \mathbf{R}^d$, let $\mathbb{I}(x) := \{i, x \in \overline{\mathcal{M}_i}\}$ be the index set.

(i) We say that u is a supersolution of (4.2.2a)-(4.2.2b) if for any $(t_0, x_0) \in (0, T) \times \mathbf{R}^d$, $\phi \in C^1((0, T) \times \mathbf{R}^d)$ such that $u - \phi$ attains a local minimum on (t_0, x_0), we have

$$-\phi_t(t_0, x_0) + H^E(x_0, D\phi(t_0, x_0)) \geq 0.$$

(ii) We say that u is a subsolution of (4.2.2a)-(4.2.2b) if for any $(t_0, x_0) \in (0, T) \times \mathbf{R}^d$, any $\phi : (0, T) \times \mathbf{R}^d \to \mathbf{R}$ with $\phi|_{(0,T)\times\overline{\mathcal{M}_k}}$ being C^1 for any $k \in \mathbb{I}(x)$ such that $u - \phi$ attains a local maximum at (t_0, x_0) on $(0, T) \times \overline{\mathcal{M}_k}$, we have

$$-q_t + \sup_{p \in F_k^E(x_0)} \{-p \cdot q_x\} \leq 0, \text{ with } (q_t, q_x) = \nabla_{\overline{\mathcal{M}_k}} \phi(t_0, x_0).$$

(iii) We say that u is a viscosity solution of (4.2.2) if u is both a supersolution and a subsolution, and u satisfies the final condition

$$u(T, x) = \varphi(x), \ \forall x \in \mathbf{R}^d.$$

4.2.4 Finite horizon optimal control problems

Recall that for the classical optimal control problems of the Mayer's type, the value function can be characterized as the unique viscosity solution of the equations of the type (4.2.1) with Lipschitz continuous Hamiltonians. In our settings of problem, the multifunctions F_i are defined separately on $\overline{\Omega}_i$. A first idea would be to consider the "regularization" of F_i. However, the regularized multifunction G is only usc in general, and this is not enough to guarantee the existence and uniqueness of solution for (4.2.1). So in our framework, in order to link the Hamilton-Jacobi equation with a Mayer's optimal control problem, we need to well define the global trajectories driven by the dynamics $(F_i)_{i=1,\ldots,m}$. Consider the following differential inclusion

$$\begin{cases} \dot{y}(s) \in G(y(s)), & \text{for } s \in (t, T) \\ y(t) = x. \end{cases} \tag{4.2.3}$$

Since G is usc, (4.2.3) admits an absolutely continuous solution defined on $[\tau, T]$. For any $(t, x) \in [0, T] \times \mathbf{R}^d$, we denote the set of absolutely continuous trajectories by

$$S_{[t,T]}(x) := \{y_{t,x}, \ y_{t,x} \text{ satisfies } (4.2.3)\}.$$

Now consider the following Mayer's problem

$$v(t, x) := \min\{\varphi(y(T)), \ y(\cdot) \in S_{[t,T]}(x)\}. \tag{4.2.4}$$

Since G is usc and convex, the set $S_{[t,T]}(x)$ of absolutely continuous arcs is compact in $C(t,T;\mathbf{R}^d)$ (See Theorem 1, [15] pp. 60). And then the problem (4.2.4) has an optimal solution for any $t \in [0,T], x \in \mathbf{R}^d$.

As in the classical case, v satisfies a Dynamical programming principle (DPP) as in Definition 2.3.1.

Proposition 4.2.9. Assume that **(H1)-(H3)** hold. Then for any $(t,x) \in [0,T] \times \mathbf{R}^d$ the following holds.

(i) **The super-optimality principle.** $\exists\, \bar{y}_{t,x} \in S_{[t,T]}(x)$ such that

$$v(t,x) \geq v(t+h, \bar{y}_{t,x}(t+h)), \text{ for } h \in [0, T-t].$$

(ii) **The sub-optimality principle.** $\forall\, y_{t,x} \in S_{[t,T]}(x)$ such that

$$v(t,x) \leq v(t+h, y_{t,x}(t+h)), \text{ for } h \in [0, T-t].$$

An important fact resulting from the assumptions **(H2)** and **(H4)(iv)** is the local Lipschitz continuity of the value function v.

Proposition 4.2.10. Assume that **(H1)-(H4)** hold. Then the value function v is locally Lipschitz continuous on $[0,T] \times \mathbf{R}^d$.

Proof. For any $t \in [0,T]$, we first prove that $v(t,\cdot)$ is locally Lipschitz continuous on $\mathbf{R}^d$. Let $x, z \in \mathbf{R}^d$, without loss of generality, suppose that

$$v(t,x) \geq v(t,z)$$

There exists $\overline{y}_{t,z} \in S_{[t,T]}(z)$ such that

$$v(t,z) = \varphi(\overline{y}_{t,z}(T)).$$

We set

$$h = \frac{\|x-z\|}{\delta}, \; \xi(s) = x + \delta \frac{z-x}{\|z-x\|}(s-t) \text{ for } s \in [t, t+h].$$

Note that $\xi(t) = x, \; \xi(t+h) = z$. By the controllability assumption **(H4)(iv)**, we can define the following trajectory

$$\widetilde{y}_{t,x}(s) = \begin{cases} \xi(s) & \text{for } s \in [t, t+h], \\ \overline{y}_{t,z}(s-h) & \text{for } s \in [t+h, T]. \end{cases}$$

By denoting $L_\varphi > 0$ the Lipschitz constant of φ, we have

$$\begin{aligned} v(t,x) - v(t,z) &\leq \varphi(\widetilde{y}_{t,x}(T)) - \varphi(\overline{y}_{t,z}(T)) \\ &\leq L_\varphi \|\widetilde{y}_{t,x}(T) - \overline{y}_{t,z}(T)\| \\ &\leq L_\varphi \|\overline{y}_{t,z}(T-h) - \overline{y}_{t,z}(T)\| \\ &\leq L_\varphi \|G\| h = \frac{L_\varphi \|G\|}{\delta} \|x - z\|, \end{aligned}$$

where we deduce the local Lipschitz continuity of $v(t,\cdot)$.
Then for $x \in \mathbf{R}^d$, we prove the Lipschitz continuity of $v(\cdot,x)$ on $[0,T]$. For any $t,s \in [0,T]$, without loss of generality suppose that $t < s$. By the super-optimality principle, there exists $y^{op} \in S_{[t,T]}(x)$ such that

$$v(t,x) = v(s, y^{op}(s)).$$

Then

$$|v(t,x) - v(s,x)| = |v(s, y^{op}(s)) - v(s,x)| \leq L_v \|G\| (s-t),$$

where L_v is the local Lipschitz constant of $v(s,\cdot)$. And the proof is complete. □

Remark 4.2.11. Assumption **(H4)(iv)** plays an important role in our proof for the Lipschitz continuity of the value function. However, it is worth mentioning that the Lipschitz continuity can also be satisfied in some cases where **(H4)(iv)** is not satisfied. In Example 4.2.1, if one take $F_1 = F_2$ Lipschitz continuous dynamics, then the value function will be Lipschitz continuous without assuming any controllability property. For multi-domains problems, some weaker assumptions of controllability are analyzed in [122].

The following result analyses the structure of the dynamics and makes clear the behavior of the trajectories.

Proposition 4.2.12. Suppose $y(\cdot) : [t,T] \to \mathbf{R}^d$ is an absolutely continuous arc. Then the following are equivalent.

(i) $y(\cdot)$ satisfies (4.2.3);

(ii) For each $k = 1, \ldots, m + \ell$, $y(\cdot)$ satisfies $y(t) = x$ and

$$\dot{y}(s) \in F_k^{new}(y(s)), \text{ a.e. whenever } y(s) \in \mathcal{M}_k,$$

(iii) $y(\cdot)$ satisfies

$$\begin{cases} \dot{y}(s) \in F^E(y(s)) & \text{for } s \in (t,T), \\ y(t) = x. \end{cases}$$

Proof. It is clear that (ii) implies (i) since $F_k^{new}(x) \subseteq G(x)$ whenever $x \in \mathcal{M}_k$. So assume that (i) holds, and let us show that (ii) holds as well.

The proof is essentially the same as in Proposition 2.1 of [29]. For any $k = 1, \dots, m + \ell$, let $J_k := \{s \in [t, T] : y(s) \in \mathcal{M}_k\}$. Without loss of generality, suppose that the Lebesgue measure $mes(J_k) \neq 0$. We set

$$\tilde{J}_k := \{s \in J_k : \dot{y}(s) \text{ exists in } G(y(s)) \text{ and } s \text{ is a Lebesgue point of } J_k\}.$$

It is clear that $\tilde{J}_k$ has full measure in J_k. For any $s \in \tilde{J}_k$, then being a Lebesgue point implies that there exists a sequence $\{s_n\}$ such that $s_n \to s$ as $n \to \infty$ with $s \neq s_n \in \tilde{J}_k$ for all n. Since $y(s_n) \in \mathcal{M}_k$, we have

$$\dot{y}(s) = \lim_{n \to \infty} \frac{y(s_n) - y(s)}{s_n - s} \in \mathcal{T}_{\mathcal{M}_k}(y(s)).$$

Then by the definition of F_k^{new}, we have

$$\dot{y}(s) \in G(y(s)) \cap \mathcal{T}_{\mathcal{M}_k}(y(s)) = F_k^{new}(y(s)), \ \forall\, s \in \tilde{J}_k,$$

which proves (ii).
It is clear that (ii)⇒(iii)⇒(i) since $F_k^{new}(\cdot) \subseteq F^E(\cdot) \subseteq G(\cdot)$, which ends the proof. □

Proposition 4.2.12 will be very useful in the characterization of the super-optimality principle and the sub-optimality principle by HJ equations involving the essential Hamiltonian H^E.

4.2.5 Supersolutions and super-optimality principle

The following proposition shows the characterization of the super-optimality principle by the supersolutions of HJ equations. This is a classical result since G is usc.

Proposition 4.2.13. Suppose $u : [0, T] \times \mathbf{R}^d \to \mathbf{R}$ is continuous. Then the following are equivalent.

(i) u satisfies the super-optimality principle;

(ii) u satisfies the following inequality in the viscosity sense

$$-u_t(t_0, x_0) + H_G(x_0, Du(t_0, x_0)) \geq 0; \tag{4.2.5}$$

(iii) u satisfies the following inequality in the viscosity sense

$$-u_t(t_0, x_0) + \max_{i : x_0 \in \overline{\Omega}_i} \{H_i(x_0, Du(t_0, x_0))\} \geq 0. \tag{4.2.6}$$

Proof. The part "(i)⇔(ii)" a straightforward consequence of Theorem 3.2 & Lemma 4.3 in [81] (See also [19]).

The part "(ii)⇔(iii)" is easy to prove since G is the convexification of all the F_i around x_0. □

Due to the structure of the dynamics G illustrated in Proposition 4.2.12, it is possible to replace G by F^E to get a more precise HJB inequality since the set of trajectories driven by G or F^E is the same. But the difficulty here is that in general F^E is not usc.

At first, we have the following result concerning the dynamics of the optimal trajectories.

Lemma 4.2.14. Let $y(\cdot) \in S_{[t,T]}(x)$ be an absolutely continuous arc along which the value function v satisfies the super-optimality principle. For any $p \in \mathbf{R}^d$ such that there exists $t_n \to 0^+$ with $\frac{y(t_n)-x}{t_n} \to p$, by denoting co $F^E(x)$ the convex hull of $F^E(x)$ we have

$$p \in \text{co } F^E(x).$$

The proof of Lemma 4.2.14 is presented in Appendix 4.2.8. In the next theorem, we will use the statement of Lemma 4.2.14 to show that the functions satisfying the super-optimality principle condition is also a solution to a more precise HJB equation with H^E than the HJB equation (4.2.6) with the Hamiltonian H_G even if F^E is not usc.

Theorem 4.2.15. Suppose $u : [0,T] \times \mathbf{R}^d \to \mathbf{R}$ is continuous and $u(T,x) = \varphi(x)$ for all $x \in \mathbf{R}^d$. u satisfies the super-optimality principle if and only if u is a supersolution of (4.2.2).

Proof. **($\Rightarrow$)** for any $(t_0, x_0) \in (0,T) \times \mathbf{R}^d$, let $\bar{y}_{t_0,x_0}$ be the optimal trajectory along which u satisfies the super-optimality principle. Then for any $\phi \in C^1((0,T) \times \mathbf{R}^d)$ such that $u - \phi$ attains a local minimum on (t_0, x_0), by the same argument in Proposition 4.2.13, we obtain

$$\frac{1}{h}(\phi(t_0, x_0) - \phi(t_0 + h, \bar{y}_{t_0,x_0}(t_0 + h))) \geq 0,$$

i.e.

$$\frac{1}{h}\int_0^h [-\phi_t(t_0 + s, \bar{y}_{t_0,x_0}(t_0 + s)) - D\phi(t_0 + s, \bar{y}_{t_0,x_0}(t_0 + s)) \cdot \dot{\bar{y}}_{t_0,x_0}(t_0 + s)]\, ds \geq 0.$$

Up to a subsequence, let $h_n \to 0^+$ so that $x_n := \bar{y}_{t_0,x_0}(t_0 + h_n)$ satisfies $\frac{x_n - x}{h_n} \to p$ for some $p \in \mathbf{R}^d$. We then get

$$-\phi_t(t_0, x_0) - p \cdot D\phi(t_0, x_0) \geq 0.$$

Lemma 4.2.14 leads to

$$p \in \text{co } F^E(x_0). \tag{4.2.7}$$

Then we deduce that

$$-\phi_t(t_0, x_0) + \sup_{p \in \text{co } F^E(x_0)} \{-p \cdot D\phi(t_0, x_0)\} \geq 0.$$

By the separation theorem

$$-\phi_t(t_0, x_0) + \sup_{p \in F^E(x_0)} \{-p \cdot D\phi(t_0, x_0)\} \geq 0.$$

(⇐) For any $(t_0, x_0) \in (0,T) \times \mathbf{R}^d$, $\phi \in C^1((0,T) \times \mathbf{R}^d)$ such that $u - \phi$ attains a local minimum on (t_0, x_0), since u is a supersolution, we have

$$-\phi_t(t_0, x_0) + \sup_{p \in F^E(x_0)} \{-p \cdot D\phi(t_0, x_0)\} \geq 0.$$

Note that $F^E(x_0) \subseteq G(x_0)$, then we deduce that

$$-\phi_t(t_0, x_0) + \sup_{p \in G(x_0)} \{-p \cdot D\phi(t_0, x_0)\} \geq 0.$$

Then we deduce the desired result by Proposition 4.2.13. □

4.2.6 Subsolutions and sub-optimality principle

As mentioned before, if G is Lipschitz continuous, one can characterize the sub-optimality principle by the opposite HJB inequalities:

$$-u_t(t_0, x_0) + H_G(x_0, Du(t_0, x_0)) \leq 0$$

in the viscosity sense. However, G is only usc on the interfaces. And the characterization using H_G fails because there are dynamics in G which are not "*essential*", which means for some $p \in G(x)$, there does not exist any trajectory coming from x using the dynamic p. For instance in Example 4.2.5, at the point 0, $G(0) = [-1, 1]$. Consider the dynamic $p = 1 \in G(0)$, if there exists a trajectory y starting from 0 using the dynamic 1, y goes immediately into Ω_2 and y is not admissible since 1 is not contained in the dynamics F_2.

In the sequel, we consider the essential dynamic multifunction F^E to replace G by eliminating the useless nonessential dynamics. Note that F^E in general is not Lipschitz either. The significative role of F^E is shown in the following result.

Lemma 4.2.16. For any $p \in F^E(x)$, there exists $\tau > t$ and a solution $y(\cdot)$ of (4.2.3) which is C^1 on $[t, \tau]$ with $\dot{y}(t) = p$.

Proof. This is a partial result of in [29, Proposition 5.1]. For the convenience of reader, a sketch of the proof is given in the appendix. □

More precisely, Lemma 4.2.16 can be rewritten as:

Lemma 4.2.17. Let $k \in \{1, \ldots, m + \ell\}$, $x \in \overline{\mathcal{M}}_k$. Then for any $p \in F_k^E(x)$, there exist $\tau > t$ and a trajectory of (4.2.3) $y(\cdot)$ which is C^1 on $[t, \tau]$ with $\dot{y}(t) = p$ and $y(s) \in \overline{\mathcal{M}}_k$ for $s \in [t, \tau]$.

The following two results give the characterization of sub-optimality principle by HJB inequations.

Proposition 4.2.18. Let $u : [0,T] \times \mathbf{R}^d \to \mathbf{R}$ be locally Lipschitz continuous and $u(T, x) = \varphi(x)$ for all $x \in \mathbf{R}^d$. Suppose that u satisfies the sub-optimality principle, then u is a subsolution of (4.2.2) in the sense of Definition 4.2.8.

Proof. Given $(t_0, x_0) \in [0,T] \times \mathbf{R}^d$, for any $k \in \mathbb{I}(x_0), p \in F_k^E(x_0)$, by Lemma 4.2.17, there exists $h > 0$ and a solution $y(\cdot)$ of (4.2.3) C^1 on $[t_0, t_0+h]$ with $\dot{y}(t_0) = p, y(t_0) = x_0$ and $y(s) \in \overline{\mathcal{M}}_k, \forall\, s \in [t_0, t_0+h]$. By the sub-optimality principle of u

$$u(t_0, x_0) \leq u(t_0 + h, y(t_0 + h)).$$

For any $\phi \in C^0((0,T) \times \mathbf{R}^d) \cap C^1((0,T) \times \mathcal{M}_k)$ such that $u - \phi$ attains a local maximum at (t_0, x_0) on $(0,T) \times \overline{\mathcal{M}}_k$, we have

$$u(t_0 + h, y(t_0 + h)) - \phi(t_0 + h, y(t_0 + h)) \leq u(t_0, x_0) - \phi(t_0, x_0).$$

Then we deduce that

$$\frac{1}{h}(\phi(t_0, x_0) - \phi(t_0, y(t_0 + h))) \leq 0.$$

By taking $h \to 0$ we have

$$-q_t - p \cdot q_x \leq 0, \text{ where } p \in F_k^E(x_0), \ (q_t, q_x) \in \nabla_{\overline{\mathcal{M}}_k} \phi(t_0, x_0),$$

i.e.

$$-q_t + \sup_{p \in F_k^E(x_0)} \{-p \cdot q_x\} \leq 0.$$

□

We present a precise example to illustrate that H^E is the proper Hamiltonian for the subsolution characterization of the value function.

Example 4.2.19. Consider again the same 1d structure as in Example 4.2.1 and Example 4.2.5, i.e. $\mathbf{R} = \Omega_1 \cup \Omega_2 \cup \Gamma_1$ with

$$\Omega_1 = (-\infty, 0), \ \Omega_2 = (0, +\infty), \ \Gamma_1 = \{0\},$$

and the dynamics

$$F_1(x) = [-\frac{1}{2}, 1], \ \forall\, x \in \Omega_1, \text{ and } F_2(x) = [-1, \frac{1}{2}], \ \forall\, x \in \Omega_2.$$

At the point 0, the convexified dynamics $G(0) = [-1, 1]$ and the essential dynamics $F^E(0) = [-\frac{1}{2}, \frac{1}{2}]$. Let $T > 0$ be a given final time and the final cost function $\varphi_2(x) = x$. Then from any initial data $(t,x) \in [0,T] \times \mathbf{R}$, the optimal strategy is to go on the left as far as possible. Thus the value function is given by

$$v_2(t,x) := \min\{\varphi_2(y_{t,x}(T))\} = \begin{cases} x - \frac{1}{2}(T-t) & x \leq 0, \\ -\frac{1}{2}(T-t-x) & 0 \leq x \leq T, \\ x - (T-t) & x \geq T - t. \end{cases}$$

At the point $(t,x) = (0,0)$, $\partial_t v_2(0,0) = \frac{1}{2}$, $Dv_2(0,0^-) = 1$, $Dv_2(0,0^+) = \frac{1}{2}$, $D^+ v_2(0,0) = [\frac{1}{2}, 1]$. Then we have

$$-\partial_t v_2(0,0) + \max_{p \in F^E(0)} \{-p \cdot D^+ v_2(0,0)\} = 0 \leq 0,$$

while

$$-\partial_t v_2(0,0) + \max_{p\in G(0)} \{-p \cdot D^+ v_2(0,0)\} = \frac{1}{2} > 0.$$

We see that the subsolution property fails if we replace F^E by G which is larger.

Proposition 4.2.15 indicates that any function satisfying the sub-optimality principle is a subsolution of (4.2.2). The inverse result needs more elaborated arguments. The difficulty arises mainly from handling the trajectories oscillating near the interfaces, i.e. the trajectories cross the interfaces infinitely in finite time which exhibit a type of "Zeno" effect. The proofs of Theorem 4.2.22 and of Proposition 5.3.7 contain details on how to construct the "nice" approximate trajectories to deal with Zeno-type trajectories.

At first, we give the following result containing the key fact of Zeno-type trajectories. The idea is that one can accordingly divide the trajectories into disjoint pieces when the trajectories lie in different regions of the partition. Some partial regularities can easily deduced for each piece, and they could be glued together to get the desired property as in the usual case without interface.

Proposition 4.2.20. Let u be a Lipschitz continuous subsolution of (4.2.2). Suppose $\mathcal{M}_k$ is a sub-domain and $\mathcal{M}$ is a union of subdomains with $\mathcal{M}_k \subseteq \overline{\mathcal{M}}$. Assume $\mathcal{M}$ has the following property: for every trajectory $y(\cdot)$ of (4.2.3) defined on $[a,b] \subseteq [t,t+h]$ with $y(\cdot) \subseteq \mathcal{M}$, we have

$$u(a, y(a)) \leq u(b, y(b)). \tag{4.2.8}$$

Then for any trajectory $y(\cdot)$ of (4.2.3) defined on $[a,b] \subseteq [t,t+h]$ lying totally within $\mathcal{M}_k \cup \mathcal{M}$, we have

$$u(a, y(a)) \leq u(b, y(b)).$$

Proof. Here we adapt an idea introduced in [29] in a context of stratified control problems. Let $y(\cdot)$ be a trajectory of (4.2.3) with $y(\cdot) \subseteq \mathcal{M}_k \cup \mathcal{M}$ satisfying (4.2.8). Without loss of generality, suppose that $y(a) \in \mathcal{M}_k$ and $y(b) \in \mathcal{M}_k$. By **(H3)**, we have $\mathcal{M}_k \cap \mathcal{M} = \emptyset$. Let $J := \{s \in [a,b] : y(s) \notin \mathcal{M}_k\}$, which is an open set and so can be written as

$$J = \bigcup_{n=1}^{\infty} (a_n, b_n)$$

where the intervals are pairwise disjoint. For a fixed p, we set

$$J_p := \bigcup_{n=1}^{p} (a_n, b_n),$$

which after reindexing can be assumed to satisfy

$$b_0 := a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_p < b_p \leq a_{p+1} := b.$$

Choose p sufficiently large so that

$$meas(J\backslash J_p) < \frac{r}{2\, e^{LT}\, \|G\|},$$

where $\|G\|$ is an upper bound of the norm of any velocity that may appear, and $r > 0$ is given by

$$r := \inf_{\substack{s\in[b_0,b] \\ w\in\overline{\mathcal{M}}_k\backslash\mathcal{M}_k}} \|y(s) - w\|.$$

For $n = 1,\dots,p$, $y(s) \in \mathcal{M}$ for $s \in (a_n, b_n)$. Let $\varepsilon > 0$ small enough such that $[a_n + \varepsilon, b_n - \varepsilon] \subseteq (a_n, b_n)$, then by (4.2.8)

$$u(a_n + \varepsilon, y(a_n + \varepsilon)) \leq u(b_n - \varepsilon, y(b_n - \varepsilon)).$$

Taking $\varepsilon \to 0$ and by the continuity of u and $y(\cdot)$, we deduce that

$$u(a_n, y(a_n)) \leq u(b_n, y(b_n)).$$

Next we need to deal with $y(\cdot)$ restricted to $[b_n, a_{n+1}]$. For $n = 0,\dots,p$, by Proposition 4.2.12 $\dot{y}(s) \in F_k^{new}\big(y(s)\big)$ for almost all $s \in [b_n, a_{n+1}]\backslash J$. For $n = 0,\dots,p$, set $\varepsilon_n := meas\big([b_n, a_{n+1}] \cap J\big)$, and note that $\sum_{n=0}^{p} \varepsilon_n = meas(J\backslash J_p)$. We calculate how far $y(\cdot)$ is from a trajectory lying in $\mathcal{M}_k$ with dynamics F_k^{new} by

$$\xi_n := \int_{b_n}^{a_{n+1}} \mathrm{dist}\Big(\dot{y}(s), F_k^{new}\big(y(s)\big)\Big)\, ds \leq 2\, \|G\| \varepsilon_n.$$

By the Filippov approximation theorem (see [58, Theorem 3.1.6] and also [60, Proposition 3.2]), there exists a trajectory $z_n(\cdot)$ of F_k^{new} defined on the interval $[b_n, a_{n+1}]$ that lies in $\mathcal{M}_k$ with $z_n(b_n) = y(b_n)$ and satisfies

$$\big\| z_n(a_{n+1}) - y(a_{n+1}) \big\| \leq e^{L(a_{n+1}-b_n)} \xi_n \leq 2\|G\| e^{L(a_{n+1}-b_n)} \varepsilon_n. \tag{4.2.9}$$

Since u is subsolution of (4.2.2), then for any $x \in \mathcal{M}_k$, note that $F_k^{new}(x) \subseteq \mathcal{T}_{\mathcal{M}_k}(x)$ and $\mathcal{T}_{\mathcal{M}_k}(x) = \mathcal{T}_{\overline{\mathcal{M}}_k}(x)$ by Definition 4.2.8

$$-\partial_t \phi(t,x) + \sup_{p\in F_k^{new}(x)} \{-p \cdot D\phi(t,x)\} \leq 0 \tag{4.2.10}$$

with $\phi \in C^0((0,T) \times \mathbf{R}^d) \cap C^1((0,T) \times \mathcal{M}_k)$ and $u - \phi$ attains a local maximum at (t,x) on $(0,T) \times \mathcal{M}_k$. Since $z_n(\cdot)$ lies in $\mathcal{M}_k$ on $[b_n, a_{n+1}]$ driven by the Lipschitz dynamics F_k^{new}, then (4.2.10) implies that the sub-optimality principle of u is satisfied on $z_n(\cdot)|_{[b_n,a_{n+1}]}$, i.e.

$$u(b_n, z_n(b_n)) \leq u(a_{n+1}, z_n(a_{n+1})).$$

Then by (5.3.10) we have

$$\begin{aligned} u(b_n, y(b_n)) &= u(b_n, z_n(b_n)) \leq u(a_{n+1}, z_n(a_{n+1})) \\ &\leq u(a_{n+1}, y(a_{n+1})) + 2L_u\|G\|e^{L(a_{n+1}-b_n)}\varepsilon_n. \end{aligned}$$

We set $\varepsilon^p := meas(J \backslash J_p)$, and we deduce that

$$\begin{aligned} u(a, y(a)) &\leq u(a_1, y(a_1)) + 2L_u\|G\|e^{L(a_1-b_0)}\varepsilon_1 \\ &\leq u(a_2, y(a_2)) + 2L_u\|G\|e^{L(a_2-b_0)}(\varepsilon_1 + \varepsilon_2) \\ &\dots \\ &\leq u(a_{p+1}, y(a_{p+1})) + 2L_u\|G\|e^{L(a_{p+1}-b_0)}\varepsilon^p \\ &= u(b, y(b)) + 2L_u\|G\|e^{L(b-a)}\varepsilon^p. \end{aligned}$$

By taking $p \to +\infty$, we have $\varepsilon^p \to 0$ and the desired result is obtained. □

Remark 4.2.21. Note that the partition of the trajectory $y(\cdot)$ in the previous proof is not unique. The crucial step hidden in this partition is to cut off a set of accumulation points of J with the measure ε^p. In this proof, as the first step of study on the subject, we has not given all the details on the partition and we would like to focus on showing how to deal with different pieces of the trajectory and how the properties deduced on each piece can be glued together. In Section 4.2, we will give the complete details on the partition of the trajectories and we will consider only the optimal partitions.

Theorem 4.2.22. Suppose u is a locally Lipschitz continuous subsolution of (4.2.2). Then u satisfies the sub-optimality principle, i.e. for any trajectory $y(\cdot) \in S_{[t,T]}(x)$, one has

$$u(t, x) \leq u(t + h, y(t + h)), \ \forall h \in [0, T - t].$$

Proof. Let $\mathcal{M}$ be a union of subdomains (manifolds or interfaces). Let $\bar{d}_{\mathcal{M}} \in \{0, \dots, d\}$ be the minimal dimension of the subdomains in $\mathcal{M}$. We claim that for any $h \in [0, T-t]$ and any trajectory $y(\cdot)$ of (4.2.3) lying totally within $\mathcal{M}$, we have

$$u(a, y(a)) \leq u(b, y(b)), \text{ for any } [a, b] \subseteq [t, t + h]. \tag{4.2.11}$$

The proof of (4.2.11) is based on an induction argument with regard to the minimal dimension $\bar{d}_{\mathcal{M}}$:

(HR) for $\tilde{d} \in \{1, \dots, d\}$, suppose that for any $\mathcal{M}$ with $\bar{d}_{\mathcal{M}} \geq \tilde{d}$ and for any trajectory $y(\cdot)$ that lies within $\mathcal{M}$, (4.2.11) holds.

Step (1): let us first check the case when $\tilde{d} = d$. In this case, $\bar{d}_{\mathcal{M}} = d$, then $\mathcal{M}$ is a union of d-manifolds which are disjoint by **(H1)**. For any trajectory $y(\cdot)$ of (4.2.3) lying within $\mathcal{M}$, since $y(\cdot)$ is continuous, $y(\cdot)$ lies entirely in one of the d-manifolds, denoted by Ω_i. The subsolution property of u implies that

$$-\partial_t u(t,x) + \sup_{p\in F_i(x)} \{-p\cdot Du(t,x)\} \le 0$$

holds in the viscosity sense. Since the dynamics on Ω_i is F_i which is Lipschitz continuous, then by the classical theory u satisfies the sub-optimality principle along $y(\cdot)$ and (4.2.11) holds true.

Step (2): now assume that **(HR)** is true for $\tilde{d} \in \{1,\dots,d\}$, and let us prove that **(HR)** is true for $\tilde{d}-1$. In this case, the minimal dimension of subdomains in $\mathcal{M}$ is $\bar{d}_{\mathcal{M}} = \tilde{d}-1$, $\tilde{d} \in \{1,\dots,d\}$. As an induction hypothesis, assume that for any trajectory that lies within a union of subdomains each with dimension greater than $\tilde{d}$, then (4.2.11) holds. Three cases can occur.

- If $\mathcal{M}$ contains only one subdomain, i.e. $\mathcal{M} = \mathcal{M}_k$ with dimension $\bar{d}_{\mathcal{M}}$ for some $k \in \{1,\dots,m+\ell\}$, then for any trajectory $y(\cdot)$ lying within $\mathcal{M}_k$, the subsolution property of u implies that u satisfies the sub-optimality principle along $y(\cdot)$ since the dynamics F_k^{new} is Lipschitz continuous on $\mathcal{M}_k$.
- If $\mathcal{M}$ contains more than one subdomain and $\mathcal{M}$ is connected, let $\mathcal{M}'_1,\dots,\mathcal{M}'_p$ be all the subdomains contained in $\mathcal{M}$ with dimension $\bar{d}_{\mathcal{M}}$. Then $\widetilde{\mathcal{M}} := \mathcal{M}\backslash \left(\cup_{k=1}^{p}\mathcal{M}'_k\right)$ is a union of subdomains with dimension greater than $\tilde{d}$. We note that $\mathcal{M}'_k \subseteq \overline{\widetilde{\mathcal{M}}}$ for each $k=1,\dots,p$. Then by the induction hypothesis and Proposition 5.3.7, (4.2.11) holds true for any trajectory lying entirely within $\widetilde{\mathcal{M}} \cup \mathcal{M}'_1$. Then by applying Proposition 5.3.7 for $\widetilde{\mathcal{M}} \cup \mathcal{M}'_1$ and $\mathcal{M}'_2$, (4.2.11) holds true for any trajectory lying entirely within $\widetilde{\mathcal{M}} \cup \mathcal{M}'_1 \cup \mathcal{M}'_2$. We continue this process and finally we have (4.2.11) holds true for any trajectory lying entirely within $\mathcal{M} = \widetilde{\mathcal{M}} \bigcup \left(\cup_{k=1}^{p}\mathcal{M}'_k\right)$.
- If $\mathcal{M}$ is not connected, for any trajectory $y(\cdot)$ lying within $\mathcal{M}$, since $y(\cdot)$ is continuous, then $y(\cdot)$ lies within one connected component of $\mathcal{M}$. Then by the same argument as above, (4.2.11) holds true for $y(\cdot)$. And the induction step is complete.

Finally, to complete the proof of the theorem, we remark that for any trajectory $y(\cdot)$ of (4.2.3), by considering $\mathcal{M} = \mathbf{R}^d$ with $\bar{d}_{\mathcal{M}} = 0$, taking $a=t, b=t+h$ in (4.2.11) we have

$$u(t,x) \le u(t+h, y(t+h)),$$

which ends the proof. □

We have proved the link between the subsolutions of (4.2.2) and the sub-optimality principle. Moreover, note that in the proof of proposition 5.3.7, we only need the HJB inequations (4.2.10) on each $\mathcal{M}_k$. Then the suboptimality can also be characterized by these weaker HJB inequations as follows.

Theorem 4.2.23. Let $u : [0,T]\times\mathbf{R}^d \to \mathbf{R}$ be a Lipschitz continuous function. Then the following are equivalent.

(i) u satisfies the sub-optimality principle;

(ii) u is the subsolution of (4.2.2);

(iii) u satisfies the following inequalities in the viscosity sense: for any $t \in (0,T)$, $x \in \mathbf{R}^n$, $\mathcal{M}_k$ such that $x \in \mathcal{M}_k$,

$$-\partial_t u(t,x) + \sup_{p \in F_k^{new}(x)} \{-p \cdot Du(t,x)\} \leq 0. \tag{4.2.12}$$

Proof. (i) ⇒ (ii) is proved by proposition 4.2.18.

(ii) ⇒ (iii) since $F_k^{new}(x) \subseteq F^E(x)$ for $x \in \mathcal{M}_k$.

(iii) ⇒ (i) by following the same proof for theorem 4.2.22. □

4.2.7 Proof of the main result Theorem 4.2.6

Since v satisfies the super-optimality principle and sub-optimality principle, by Theorem 4.2.15 and Theorem 4.2.18 v is a viscosity solution of (4.2.2).

The uniqueness result is obtained by the following result of comparison principle.

Proposition 4.2.24. Suppose that $u : [0,T] \times \mathbf{R}^d \to \mathbf{R}$ is Lipschitz continuous and $u(T,x) = \varphi(x)$ for any $x \in \mathbf{R}^d$.

(i) If u satisfies the super-optimality principle, then $v(t,x) \leq u(t,x)$ for all $(t,x) \in [0,T] \times \mathbf{R}^d$;

(ii) If u satisfies the sub-optimality principle, then $v(t,x) \geq u(t,x)$ for all $(t,x) \in [0,T] \times \mathbf{R}^d$.

Proof. **(i)** For any $(t,x) \in [0,T] \times \mathbf{R}^d$, by the super-optimality principle of u, there exists a trajectory $\overline{y}_{t,x}$ such that

$$u(t,x) \geq u(T, \overline{y}_{t,x}(T)) = \varphi(\overline{y}_{t,x}(T)).$$

By the sub-optimality principle of v, we have

$$v(t,x) \leq v(T, \overline{y}_{t,x}(T)) = \varphi(\overline{y}_{t,x}(T)).$$

Then we deduce that

$$v(t,x) \leq u(t,x).$$

(ii) The proof is completed by the same argument by considering the super-optimality principle of v and the sub-optimality principle of u. □

4.2.8 Proof of technical lemmas

Proof. *(Proof of Lemma 4.2.3).*
Note that although G is only usc on $\mathbf{R}^d$, G is Lipschitz continuous on Γ_j since G is the convexification of a finite group of Lipschitz continuous multifunctions on Γ_j. For any $x \in \Gamma_j$, there exists $\alpha > 0$ and a diffeomorphism $g \in C^{1,1}(\mathbf{R}^d)$ such that

$$\overline{B(x,\alpha) \cap \Gamma_j} = \{x : g(x) = 0\} \text{ and } \nabla g(y) \neq 0, \ \forall y \in \overline{B(x,\alpha)}.$$

We can take g as the signed distance function to Γ_j for instance. Then there exists $\beta > 0$ such that

$$\|\nabla g(y)\| \geq \beta, \ \forall y \in B(x,\alpha) \cap \Gamma_j.$$

For any $w \in G(x) \cap \mathcal{T}_{\Gamma_j}(x)$, by the Lipschitz continuity of G there exists $v \in G(y)$ such that

$$\|w - v\| \leq L_G \|x - y\|,$$

where L_G is the Lipschitz constant of $G(\cdot)$. Since $w \in \mathcal{T}_{\Gamma_j}(x)$, we have

$$w \cdot \nabla g(x) = 0.$$

Then

$$\|v \cdot \nabla g(x)\| = \|(v - w) \cdot \nabla g(x)\| \leq L_G \|\nabla g\| \|x - y\|.$$

Thus,

$$\begin{aligned} \|v \cdot \nabla g(y)\| &\leq \|v \cdot \nabla g(x)\| + \|v \cdot (\nabla g(y) - \nabla g(x))\| \\ &\leq (L_G \|\nabla g\| + \|G\| L'_g) \|x - y\|, \end{aligned}$$

where L'_g is the Lipschitz constant of $\nabla g(\cdot)$. We consider the following three cases:
if $v \cdot \nabla g(y) = 0$, then $v \in \mathcal{T}_{\Gamma_j}(y)$ and we deduce that

$$w \in G(y) \cap \mathcal{T}_{\Gamma_j}(y) + L_G \|x - y\| B(0,1).$$

If $v \cdot \nabla g(y) := -\gamma < 0$, let $p := \delta \nabla g(y) / \|\nabla g(y)\|$, then by **(H4)(iv)**

$$p \in G(y) \text{ and } p \cdot \nabla g(y) := \tilde{\beta} \geq \delta\beta > 0.$$

We set

$$q := \frac{\tilde{\beta}}{\tilde{\beta} + \gamma} v + \frac{\gamma}{\tilde{\beta} + \gamma} p,$$

then $q \cdot \nabla g(y) = 0$, i.e. $q \in \mathcal{T}_{\Gamma_j}(y)$. And since $G(y)$ is convex, we have $q \in G(y)$. Then we obtain

$$\begin{aligned}
\|w - q\| &\leq \|w - v\| + \|v - q\| \\
&\leq L_G\|x - y\| + \frac{\gamma}{\tilde{\beta} + \gamma}\|v - p\| \\
&\leq \left(L_G + \frac{L_G\|\nabla g\| + \|G\|L'_g}{\delta\beta} 2\|G\|\right)\|x - y\|,
\end{aligned}$$

where we deduce that

$$w \in G(y) \cap \mathcal{T}_{\Gamma_j}(y) + L\|x - y\|B(0,1), \tag{4.2.13}$$

with $L := L_G + 2\|G\|(L_G\|\nabla g\| + \|G\|L'_g)/\delta\beta$.
If $v \cdot \nabla g(y) > 0$, then by the same argument taking $p = -\delta\nabla g(y)/\|\nabla g(y)\|$, (4.2.13) holds true as well.
Finally, (4.2.13) implies the local Lipschitz continuity of $G(\cdot) \cap \mathcal{T}_{\Gamma_j}(\cdot)$ on Γ_j with the local constant L. □

Proof. (Proof of Lemma 4.2.14).
For $k = 1, \dots, m + \ell$, we set

$$J_k^n := \{t \in [0, t_n] : y(t) \in \mathcal{M}_k\}, \ \mu_k^n := meas(J_k^n), \ \mathbb{K}(x) := \{k : \mu_k^n > 0, \forall n \in \mathbf{N}\}.$$

For each $k \in \mathbb{K}(x)$, we have $x \in \mathcal{M}_k$. Up to a subsequence, there exists $0 \leq \lambda_k \leq 1$ and $p_k \in \mathbf{R}^d$ so that

$$\frac{\mu_k^n}{t_n} \to \lambda_k, \ \sum_{k \in \mathbb{K}(x)} \lambda_k = 1, \ \frac{1}{\mu_k^n}\int_{J_k^n} \dot{y}(s)ds \to p_k$$

as $n \to +\infty$. By Proposition 4.2.12 and the Lipschitz continuity of F_k^{new}, we have

$$\begin{aligned}
p_k &= \lim_{n\to\infty} \frac{1}{\mu_k^n}\int_{J_k^n} \dot{y}(s)ds \\
&\in \lim_{n\to\infty} \frac{1}{\mu_k^n}\int_{J_k^n} F_k^{new}(y(s))ds \\
&\subseteq \lim_{n\to\infty} \left[\frac{1}{\mu_k^n}\int_{J_k^n} F_k^{new}(x)ds + \frac{1}{\mu_k^n}\int_{J_k^n} L_k\|y(s) - x\|B(0,1)ds\right] \\
&\subseteq \lim_{n\to\infty} \left[F_k^{new}(x) + L_k\|F\|\left[\frac{1}{\mu_k^n}\int_{J_k^n} sds\right]B(0,1)\right] = F_k^{new}(x).
\end{aligned}$$

We then have

$$\begin{aligned} p &= \lim_{n\to\infty} \frac{y(t_n)-x}{t_n} = \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} \dot{y}(s)ds \\ &= \sum_{k\in\mathbb{K}(x)} \lim_{n\to\infty} \frac{\mu_k^n}{t_n}\left[\frac{1}{\mu_k^n}\int_{J_k^n} \dot{y}(s)ds\right] \\ &= \sum_{k\in\mathbb{K}(x)} \lambda_k p_k \in \sum_{k\in\mathbb{K}(x)} \lambda_k F_k^{new}(x) \subseteq \text{co} \bigcup_{k\in\mathbb{K}(x)} F_k^{new}(x). \end{aligned}$$

Now set $\mathcal{M} := \cup_{k\in\mathbb{K}(x)}\mathcal{M}_k$, and since $y(t_n) \in \mathcal{M}$ for all large n, we have $p \in \mathcal{T}_{\overline{\mathcal{M}}}(x)$. Then we obtain

$$p \in \left(\text{co} \bigcup_{k\in\mathbb{K}(x)} F_k^{new}(x)\right) \bigcap \mathcal{T}_{\overline{\mathcal{M}}}(x).$$

The fact that $F_k^{new}(z) \subseteq \mathcal{T}_{\mathcal{M}_k}(z)$ whenever $z \in \mathcal{M}_k$ implies

$$F_k^{new}(x) \bigcap \mathcal{T}_{\overline{\mathcal{M}}}(x) = F_k^{new}(x) \bigcap \mathcal{T}_{\overline{\mathcal{M}}_k}(x)$$

whenever $x \in \overline{\mathcal{M}}_k$. Hence

$$p \in \text{co} \bigcup_{k\in\mathbb{K}(x)} \left(F_k^{new}(x) \bigcap \mathcal{T}_{\overline{\mathcal{M}}_k}(x)\right) = \text{co } F^E(x).$$

□

4.3 The infinite horizon problem under a weaker controllability assumption

In this part, we partition $\mathbf{R}^d$ as

$$\mathbf{R}^d = \Omega_1 \cup \Omega_2 \cup \Gamma,$$

where Ω_1, Ω_2 are two nonempty open disjoint subsets and

$$\Gamma = \partial\Omega_1 = \partial\Omega_2$$

is a C^2 hypersurface (not necessarily connected), namely an imbedded submanifold of $\mathbf{R}^d$ of codimension 1, here *embedded* simply means that the submanifold topology is the relative topology inherited by $\mathbf{R}^d$. We will refer to it throughout the section as *the interface.*

Consider two separate different dynamics together with cost functions $f_1,\ \ell_1,\ f_2,\ \ell_2$ respectively defined in the open regions Ω_1 and Ω_2 are considered. Assume that the discount factor is the same, say $\lambda > 0$. For $i = 1, 2$, let A_i be the control sets which are compact subsets of $\mathbf{R}^m$ for some $m \in \mathbf{N}$. We assume that

(H1) $f_i : \overline{\Omega}_i \times A_i \to \mathbf{R}^d$ is continuous and L-Lipschitz continuous in the first variable with a positive constant L, uniformly with respect to the second one.

(H2) $\ell_i : \overline{\Omega}_i \times A_i \to \mathbf{R}^d$ is continuous and L-Lipschitz continuous in the first variable with a positive constant L, uniformly with respect to the second one, and bounded by a positive constant M.

For $i = 1, 2$, we introduce the Hamiltonians $H_i : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ defined by for any $(x, p) \in \overline{\Omega}_i \times \mathbf{R}^d$

$$H_i(x, p) = \max\{-p \cdot f_i(x, \mathrm{A}_i) - \ell_i(x, \mathrm{A}_i) \mid \mathrm{A}_i \in A_i\}.$$

Then consider the system of HJB equation: for $i = 1, 2$,

$$\lambda u(x) + H_i(x, Du(x)) = 0 \text{ for } x \in \Omega_i. \tag{4.3.1}$$

We address the question to know which are the transmission conditions on the interface Γ to get the unique solution for (4.3.1).

4.3.1 Preliminaries and definitions

We make the arbitrary choice of defining the signed distance from Γ looking at it as boundary of Ω_2. Namely:

$$g(x) = d(x, \Gamma)\mathbf{1}_{\{x \in \overline{\Omega}_1\}} - d(x, \Gamma)\mathbf{1}_{\{x \in \overline{\Omega}_2\}}. \tag{4.3.2}$$

It is clear that, at any $x \in \Gamma$, $Dg(x)$ is the unit normal vector of Γ pointing outside Ω_2 and inside Ω_1. We denote by $\mathcal{T}_\Gamma(x)$, $\mathcal{T}_\Gamma^*(x)$ tangent and cotangent space, respectively, at any $x \in \Gamma$, the *cotangent bundle* $\mathcal{T}^*\Gamma$ is made up by all the pairs (x, p) with $x \in \Gamma$ and $p \in \mathcal{T}_\Gamma^*(x)$. We indicate by $d_\Gamma(\cdot)$ the Riemannian distance on Γ induced by the Euclidean metric of $\mathbf{R}^d$, which is given by any pair x, z of Γ by

$$d_\Gamma(x, z) = \inf\left\{ \int_0^1 |\dot{y}|\, ds \mid y : [0, 1] \to \Gamma,\ y(0) = x,\ y(1) = z \right\}.$$

It is clearly finite in each connected component of Γ. We will use the following well known facts:

(i) Γ has countably many connected component.

(ii) There is an open neighborhood $\Gamma_\natural$ of Γ in $\mathbf{R}^d$ where the projection on Γ is of class C^1.

(iii) The signed distance g is of class C^2 in $\Gamma_\natural$.

(iv) Given a connected component Γ_0 of Γ and $x \in \Gamma_0$, the function $d_\Gamma(x, \cdot)$ is of class C^2 in $\Gamma_0 \backslash \{x\}$. Moreover d_Γ is locally equivalent in Γ_0 to Euclidean distance. Namely for any compact subset Θ of Γ_0 there is $N > 1$ with

$$|x - z| \leq d_\Gamma(x, z) \leq N\,|x - z| \quad \text{for any } x,\ z \text{ in } \Theta.$$

(v) For any pair of points belonging to the same connected component of Γ, say Γ_0, there is a minimal geodesic for d_Γ of class C^1 linking them, namely such curve lies in Γ_0 and its Euclidean length realizes the Riemannian distance.

Item (i) directly comes from paracompactness of Γ, second item is a consequence of ε–neighborhood Theorem, see [99]. The third comes from the fact that proj_Γ appears in the derivative of distance, see [108] and [69, Remark 5.6]. Item(iv) basically depends on the fact that for any point of Γ the differential of the exponential map is the identity at 0. For the last one we exploit that any connected component of Γ is complete because is closed in $\mathbf{R}^d$ and invoke Hopf–Rinow Theorem.

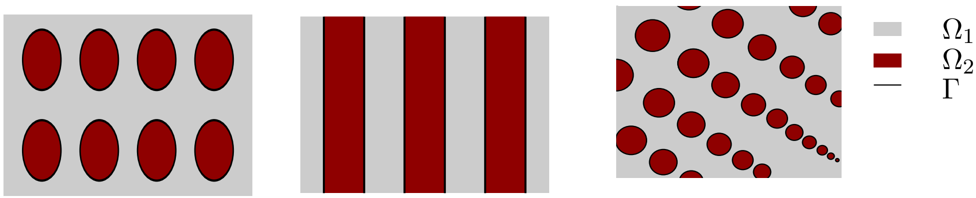


FIGURE 4.2: Some examples of partitions that be considered within the framework of this work.

Some examples of partitions are given in Figure 4.2. In fig.4.2(a), Ω_1 is the union of spheres with the same radius and located at a same distance each to another, the interface Γ is the union of the spheres' boundaries. In fig.4.2(b), the interface is the union of vertical lines. In fig.4.2(c), the domain Ω_2 is union of spheres that are disjoint but closer and closer when going to infinity. In this example, the interface is the union of the boundaries of the spheres.

Remark 4.3.1. Being Γ an embedded submanifold of $\mathbf{R}^d$, any point of it belonging, say, to the connected component Γ_0, must have a neighborhood U in $\mathbf{R}^d$ with $U \cap \Gamma \subset \Gamma_0$. This implies: first, that only a finite number of connected component of Γ can intersect a given compact subset of $\mathbf{R}^d$ and, second, that for any connected component Γ_0 of Γ the set $\Gamma_0^\natural := \{x \in \Gamma_\natural \mid \text{proj}_\Gamma(x) \in \Gamma_0\}$ is a connected component of $\Gamma_\natural$.

Following [23], we introduce the control set

$$A := A_1 \times A_2 \times [0,1]. \tag{4.3.3}$$

A_1, A_2 can be considered as subsets of A identified with $A_1 \times A_2 \times \{0\}$ and $A_1 \times A_2 \times \{1\}$, respectively. We set

$$A(x) = \begin{cases} A_i & \text{for } x \in \Omega_i,\ i = 1,2, \\ A & \text{for } x \in \Gamma. \end{cases}$$

The three components representation (4.3.3) allows to univocally associate, to any control, cost and dynamics by performing convex combinations. More precisely, let us define velocities and costs for the integrated system, for $x \in \mathbf{R}^d$, $(\text{A}_1, \text{A}_2, \mu) \in A(x)$ by

$$f(x, \text{A}_1, \text{A}_2, \mu) = \mu f_1(x, \text{A}_1) + (1 - \mu) f_2(x, \text{A}_2), \tag{4.3.4}$$

$$\ell(x, \text{A}_1, \text{A}_2, \mu) = \mu \ell_1(x, \text{A}_1) + (1 - \mu) \ell_2(x, \text{A}_2). \tag{4.3.5}$$

Note that f and ℓ restricted to $\Omega_i \times A_i$ gives back f_i, ℓ_i.

We proceed introducing the transmission conditions of dynamics and costs on the interface on which our analysis are based. The first is a controllability condition which, loosely speaking, is divided in a tangential and normal parts with respect to Γ.

(H3)(i) For $i = 1, 2$, any $x \in \Gamma$, there is A, B in A_i with $Dg(x) \cdot f_i(x, \text{A}) > 0$ and $Dg(x) \cdot f_i(x, \text{B}) < 0$, where g is defined as in (4.3.2).

(ii) There exists $R > 0$ such that for any $x \in \Gamma$

$$\big\{ f(x, \text{A}) \mid \text{A} \in A \big\} \supset B_R \cap \mathcal{T}_\Gamma(x).$$

Secondly, we require convexity of costs and admissible velocities. It will be specifically used in the proof of Theorem 4.3.12.

(H4) For any $x \in \Gamma$ the set $\big\{ (f(x, \text{A}), \ell(x, \text{A})) \mid \text{A} \in A \big\}$ is convex.

Remark 4.3.2. Condition **(H3)(i)** can be equivalently expressed saying that for any point of the interface there are admissible displacements of the two systems pointing strictly inward and outward Ω_1 and Ω_2.

Unless differently stated, **(H1)-(H4)** will be in place throughout the work. Dynamics of the integrated system is given by the multivalued vector field

$$F(x) = \{ f(x, \text{A}) \mid \text{A} \in A(x) \} \quad \text{for any } x \in \mathbf{R}^d.$$

Clearly F is Lipschitz–continuous in Ω_1 and Ω_2, but just usc on the whole of $\mathbf{R}^d$, in addition it has linear growth and possess compact, but in general non convex values, therefore existence of integral trajectories for any positive times is not in principle guaranteed. However it can be deduced from transmission conditions **(H3)**, for instance by (ii) any integral curve reaching the interface can be extended on $[0, +\infty)$ in a sliding mode along it.

It is convenient to single out controls and dynamics corresponding to tangential displacements on Γ putting

$$A_\Gamma(x) = \{ a \in A \mid f(x, \text{A}) \in \mathcal{T}_\Gamma(x) \} \quad \text{for any } x \in \Gamma,$$

$$F_\Gamma(x) = \{ f(x, \text{A}) \mid \text{A} \in A_\Gamma(x) \} = F(x) \cap \mathcal{T}_\Gamma(x) \quad \text{for any } x \in \Gamma.$$

It is a consequence of assumption **(H3)** that $A_\Gamma(x)$ and $F_\Gamma(x)$ are nonempty for any $x \in \Gamma$.

Now we define the controls and dynamics corresponding to the essential directions by

$$A_E(x) = \begin{cases} A_i & \text{for } x \in \Omega_i,\ i = 1, 2, \\ A_\Gamma(x) \cup A_E^1(x) \cup A_E^2(x) & \text{for } x \in \Gamma, \end{cases}$$

where

$$A_E^i(x) = \{f_i(x, \mathrm{A}_i) \,|\, \mathrm{A}_i \in A_i\} \cap \mathcal{T}_{\overline{\Omega}_i}(x),\ i = 1, 2.$$

$$F_E(x) = \{f(x, \mathrm{A}) \mid \mathrm{A} \in A_E(x)\}.$$

For any $(x, p) \in \Gamma \times \mathbf{R}^d$, we set the tangent Hamiltonian

$$H_\Gamma(x, p) = \max\{-p \cdot f(x, \mathrm{A}) - \ell(x, \mathrm{A}) \mid \mathrm{A} \in A_\Gamma(x)\}.$$

For any $(x, p) \in \mathbf{R}^d \times \mathbf{R}^d$, we set the essential Hamiltonian

$$H_E(x, p) = \max\{-p \cdot f(x, \mathrm{A}) - \ell(x, \mathrm{A}) \mid \mathrm{A} \in A_E(x)\}.$$

4.3.2 Main results

Let us state the main results. We start by presenting all the possible transmission conditions on the interface Γ:

$$\begin{cases} \lambda u(x) + \max\{H_1(x, Du(x)), H_2(x, Du(x))\} \geq 0, & \forall\, x \in \Gamma, \\ \lambda u(x) + H_\Gamma(x, Du(x)) \leq 0, & \forall\, x \in \Gamma, \end{cases} \tag{4.3.6}$$

$$\lambda u(x) + H_E(x, Du(x)) = 0,\ \forall\, x \in \Gamma. \tag{4.3.7}$$

Before we give the viscosity sense of the solutions for the above inequations/equations, we consider the following notion of differential.

Definition 4.3.3. Let $\mathcal{M}$ be a C^2 embedded manifold of $\mathbf{R}^d$ and $\phi \in C^1(\overline{\mathcal{M}})$. For any $x \in \overline{\mathcal{M}}$, we define

$$D_{\overline{\mathcal{M}}}\phi(x) := \lim_{z \to x,\, z \in \overline{\mathcal{M}},\, z \neq x} \frac{\phi(z) - \phi(x)}{z - x}.$$

The precise viscosity notions are given in the following definition.

Definition 4.3.4. Let $u : \mathbf{R}^d \to \mathbf{R}$ be a bounded function.

- u is a supersolution of (4.3.6) ((4.3.7) resp.) if u is lsc and for any $x_0 \in \Gamma$, $\phi \in C^1(\mathbf{R}^d)$ such that $u - \phi$ attains a local minimum at x_0,

$$\lambda u(x_0) + \max\{H_1(x_0, D\phi(x_0)), H_2(x_0, D\phi(x_0))\} \geq 0$$

(
$$\lambda u(x_0) + H_E(x_0, D\phi(x_0)) \geq 0$$
resp.).

- u is a subsolution of (4.3.6) if u is usc and for any $x_0 \in \Gamma$, $\phi \in C^1(\Gamma)$ such that $u|_\Gamma - \phi$ attains a local maximum at x_0,
$$\lambda u(x_0) + H_\Gamma(x_0, Du_\Gamma(x_0)) \leq 0.$$

- u is a subsolution of (4.3.7) if u is usc and for any $\mathcal{M} \in \{\Omega_1, \Omega_2, \Gamma\}$, $x_0 \in \Gamma$, $\phi \in C^1(\overline{\mathcal{M}})$ such that $u|_{\overline{\mathcal{M}}} - \phi$ attains a local maximum at x_0 in $\mathcal{M}$,
$$\lambda u(x_0) + \sup_{\mathrm{A}_i \in A^i_E(x_0)} \left\{ -D_{\overline{\mathcal{M}}}\phi(x_0) \cdot f_i(x_0, \mathrm{A}_i) - \ell_i(x_0, \mathrm{A}_i) \right\} \leq 0, \text{ if } \mathcal{M} = \Omega_i, \ i = 1, 2,$$
or
$$\lambda u(x_0) + \sup_{\mathrm{A} \in A_\Gamma(x_0)} \{ -D_\Gamma \phi(x_0) \cdot f(x_0, \mathrm{A}) - \ell(x_0, \mathrm{A}) \} \leq 0, \text{ if } \mathcal{M} = \Gamma.$$

- u is a viscosity solution of (4.3.6) ((4.3.7) resp.) if u is continuous and u is both a supersolution and subsolution of (4.3.6) ((4.3.7) resp.).

Here is the comparison principle which is a uniqueness result.

Theorem 4.3.5. Assume **(H1)-(H4)**. Let $u : \mathbf{R}^d \to \mathbf{R}$ be a bounded usc function and $w : \mathbf{R}^d \to \mathbf{R}$ be a bounded lsc function. Assume, in addition, that u is continuous at any point of Γ. If u is a subsolution of (4.3.1)-(4.3.6), and w is a supersolution of (4.3.1)-(4.3.6), then $u \leq w$ in $\mathbf{R}^d$.

Corollary 4.3.6. Assume **(H1)-(H4)**. Let $u : \mathbf{R}^d \to \mathbf{R}$ be a bounded usc function and $w : \mathbf{R}^d \to \mathbf{R}$ be a bounded lsc function. Assume, in addition, that u is continuous at any point of Γ. If u is a subsolution of (4.3.1)-(4.3.6) or (4.3.1)-(4.3.7), and w is a supersolution of (4.3.1)-(4.3.6) or (4.3.1)-(4.3.7), then $u \leq w$ in $\mathbf{R}^d$.

Theorem 4.3.7. Assume **(H1)-(H4)**. Both (4.3.1)-(4.3.6) and (4.3.1)-(4.3.7) have a unique solution in $\mathbf{R}^d$.

4.3.3 Infinite optimal control problems

Recall that the unique solution of the HJB equation of the type (4.3.1) with Lipschitz Hamiltonian is the value function of the associated infinite optimal control problem. The idea in our study is to introduce an optimal control problem then investigate the transmission conditions satisfied by the value function on the interface.

Consider the integral curves driving by F. Since F is usc, from Filippov Implicit Function Lemma ([114]) for any trajectory y defined in $[0, +\infty))$ of F, there is a measurable selection $\alpha(t)$ of $t \mapsto A(y(t))$ with
$$\dot{y}(t) = f(y(t), \alpha(t)) \quad \text{for a.e. } t. \tag{4.3.8}$$

The pairs trajectory/control (α, y) related as in (4.3.8) will be called *admissible*. Consider the value function

$$v(x) := \inf\left\{\int_0^{+\infty} e^{-\lambda s}\ell(y(s),\alpha(s))\,ds \mid (\alpha,y) \text{ satisfies (4.3.8) with } y(0)=x\right\}. \tag{4.3.9}$$

Then straightforwardly the value function satisfies the dynamical programming principle which is the combination of two notions of optimality principles. Recall that we have seen these notions in Definition 2.3.1 in the finite horizon case.

Definition 4.3.8. A lsc (resp. usc) function u satisfies the ***super-optimality*** (resp. ***sub-optimality***) principle if for any $x \in \mathbf{R}^d$, $t \in [0,+\infty)$,

$$u(x) \geq (\text{ resp. } \leq) \inf\left\{e^{-\lambda t}u(y(t)) + \int_0^t e^{-\lambda s}\ell\left(y(s),\alpha(s)\right)ds \mid \right.$$
$$\left. (\alpha,y) \text{ satisfies (4.3.8)}, \ y(0)=x\right\}.$$

An important fact is that the assumption **(H3(i))** leads to the continuity of the value function. To show this result, we first prove a lemma on the behavior of controlled dynamics around the interface, which is direct consequence of the controllability conditions **(H3)(i)**.

Lemma 4.3.9. Given any compact subset of Γ, say Θ, there exist in correspondence positive constants r and S such that if $x \in \Omega_i \cap (\Theta + B(0,r))$, $i = 1, 2$, we can find two trajectories $\overline{y}$, $\underline{y}$ of F and $\overline{T}$,$\underline{T}$ less than $S\,|g(x)|$ with

$$\overline{y}(0) = x, \quad \overline{y}(\overline{T}) \in \Gamma, \quad \overline{y}([0,\overline{T})) \subset \Omega_i, \tag{4.3.10}$$

$$\underline{y}(0) \in \Gamma, \quad \underline{y}(\underline{T}) = x, \quad \underline{y}((0,\underline{T}]) \subset \Omega_i. \tag{4.3.11}$$

A remark is preliminary to the proof.

Remark 4.3.10. Controlled vector fields f_i can be extended to $(\Omega_i \cup \Gamma_\natural) \times A_i$ by setting

$$f_i(x,a) = f_i(\text{proj}_\Gamma(x), a).$$

The extended f_i are continuous in both arguments and locally Lipschitz–continuous when first variable varies in $\Gamma_\natural$. Accordingly, the related multivalued maps $x \mapsto f_i(x, A_i)$ are locally Lipschitz–continuous in $\Gamma_\natural$.

Proof. (*of Lemma 4.3.9*) Let us first prove that the assertion for $i = 1$. The functions

$$x \mapsto \min\{Dg(x)\cdot f_1(x,\text{A}) \mid \text{A} \in A_1\}, \ \ x \mapsto \max\{Dg(x)\cdot f_1(x,\text{A}) \mid \text{A} \in A_1\}$$

are continuous in $\Gamma_\natural$ and, in force of assumption **(H3)(i)**, the first is moreover strictly negative and the latter strictly positive; they consequently keep same sign in $\Theta + B(0,\rho) \subset \Gamma_\natural$ for a suitable

$\rho > 0$. Then for an appropriate choice of $C > 0$, the set–valued functions

$$\underline{F}(x) = \{f_1(x, \mathrm{A}) \mid Dg(x) \cdot f_i(x, \mathrm{A}) \leq -C\}, \; \overline{F}(x) = \{f_1(x, \mathrm{A}) \mid Dg(x) \cdot f_i(x, \mathrm{A}) \geq C\}$$

take nonempty compact values in $\Theta + B(0, \rho)$. They are, in addition, usc. However, since in general they do not possess better continuity properties and are not convex–valued, the existence of solutions to the corresponding differential inclusions is not guaranteed. For this reason, we pass to relaxed problems and apply later Relaxation Theorem. The differential inclusions

$$\dot{y} \in \mathrm{co} \; \underline{F}(y), \; \dot{y} \in -\mathrm{co} \; \overline{F}(y)$$

posed in $\Theta + B(0, \rho)$, admit in fact solutions for any initial point, being the right hand–side multi-functions upper semicontinuous with convex compact nonempty values. Further, if y is one of these solutions and $[0, T)$ its maximal interval of definition, with $T < +\infty$, then

$$\lim_{t \to T} y(t) \in \partial(\Theta + B(0, \rho)). \tag{4.3.12}$$

We set $S = \frac{2}{C}$ and $r > 0$ with

$$r < \min \left\{ \frac{\rho \, C}{4 \, M_0}, \frac{\rho}{3} \right\}, \tag{4.3.13}$$

where M_0 is a constant estimating from above the norm of any element of $f_1(x, A_1)$, for x varying in $\Theta + B(0, \rho)$.

Given $x \in (\Theta + B(0, r)) \cap \Omega_1$, let y be an integral curve of co $\underline{F}$ starting at x, we denote by $[0, T)$ its maximal interval of definition. If $T \leq S \, g(x)$ then, taking into account (4.3.13) and that $g(x) \leq r$, it gives for any $t \in [0, T)$

$$d(y, \Theta) \leq |y(t) - x| + r < t \, M_0 + \frac{\rho}{3} < \frac{\rho}{2} + \frac{\rho}{3},$$

which is in contrast with (4.3.12). Consequently $T > S \, g(x)$ must hold, then

$$g(y(S \, g(x))) = g(x) + \int_0^{S \, g(x)} Dg(y) \cdot \dot{y} \, ds \leq g(x) - C \, S \, g(x) < 0,$$

so that $y(S \, g(x)) \in \Omega_2$. y is also a trajectory of the relaxed dynamics co $f_1(x, A_1)$, and, being $f_1(x, A_1)$ Lipschitz–continuous in $\Theta + B(0, \rho)$ (Remark 4.3.10), it can uniformly approximated in $[0, S \, g(x)]$ by integral curves of $f_1(x, A_1)$ with same initial point, thanks to [16, Theorem 10.4.4]. There thus exists one such trajectory, say $\overline{y}$, satisfying $\overline{y}(0) = x$, $\overline{y}(S \, g(x)) \in \Omega_2$, so that the first exit time of it from Ω_1, say $\overline{T}$, is less than $S \, g(x)$. The curve $\overline{y}$ in $[0, \overline{T}]$ satisfies (4.3.10). Same argument, with slight adaptations, shows the existence of an integral curve $\underline{y}$ of $-F_1$ and $\underline{T} < S \, g(x)$ with

$$\underline{y}(0) = x, \quad \underline{y}(\underline{T}) \in \Gamma, \quad \underline{y}([0, \underline{T})) \subset \Omega_1.$$

We then prove (4.3.11) by considering $t \mapsto \underline{y}(\underline{T} - t)$ in $[0, \underline{T}]$.

The proof for $i = 2$ is the same, up to obvious adjustments. □

Theorem 4.3.11. Under assumptions **(H1)**–**(H4)** the value function v is bounded and continuous in $\mathbf{R}^d$. It is moreover locally Lipschitz continuous on Γ.

Proof. The proof is divided into three steps:

(i) Local Lipschitz–continuity on Γ. This property is easily obtained using suboptimality of v plus assumption **(H3)(ii)** and local equivalence of Riemannian and Euclidean distance in any connected component of the interface.

(ii) Continuity at any point of Γ. Taking into account that v, restricted on the interface, is continuous, according to previous step and Remark 4.3.1, it is enough to show

$$v(x_n) \to v(x_0) \quad \text{for any } x_0 \in \Gamma,\ x_n \to x_0,\ x_n \in \Omega_i \text{ for any } n,\ i = 1 \text{ or } 2.$$

By applying Lemma 4.3.9 with $\Theta = \{x_0\}$, for a suitable $S > 0$ and n large enough there exist positive sequences T_n, $\hat{T}_n$ satisfying $T_n \le S|g(x_n)|$, $\hat{T}_n \le S|g(x_n)|$ for any n, and admissible trajectories y_n, $\hat{y}_n$, defined in $[0, T_n]$, $[0, \hat{T}_n]$, respectively, with $y_n([0, T_n)) \subset \Omega_i$, $\hat{y}_n([0, \hat{T}_n)) \subset \Omega_i$, corresponding to controls α_n, $\hat{\alpha}_n$ respectively, such that

$$y_n(0) = x_n, \quad y_n(T_n) =: z_n \in \Gamma,\ \hat{y}_n(0) =: \hat{z}_n \in \Gamma, \quad \hat{y}_n(\hat{T}_n) = x_n.$$

Since all supports of such curves is contained in some compact set, their velocities are equibounded, so that

$$z_n \to x_0 \quad \text{and} \quad \hat{z}_n \to x_0 \quad \text{as } n \to +\infty. \tag{4.3.14}$$

By suboptimality and boundedness condition on ℓ_i we have

$$v(x_n) \;\le\; \int_0^{T_n} e^{-\lambda s}\, \ell_i(y_n, \alpha_n)\, ds + e^{-\lambda T_n}\, v(z_n) \le M\, S\, |g(x_n)| + v(z_n) \tag{4.3.15}$$

$$v(\hat{z}_n) \;\le\; \int_0^{\hat{T}_n} e^{-\lambda s}\, \ell_i(\hat{y}_n, \hat{\alpha}_n)\, ds + e^{-\lambda \hat{T}_n}\, v(x_n) \le M\, S\, |g(x_n)| + v(x_n) \tag{4.3.16}$$

where M is defined as in **(H2)**. Putting together (4.3.14), (4.3.15), (4.3.16), we derive

$$\limsup v(x_n) \le \lim v(z_n) = v(x_0),\ \liminf v(x_n) \ge \lim v(\hat{z}_n) = v(x_0),$$

which shows the assertion.

(3) Final part: continuity of v in $\mathbf{R}^d$. We consider a bounded subset B of Ω_i. We will prove that, given any $\varepsilon > 0$, a $\delta > 0$ can be determined with

$$v(x_1) - v(x_0) < 4\,\varepsilon \quad \text{for any pair of elements } x_0,\ x_1 \text{ of } B \text{ with } |x_0 - x_1| < \delta. \tag{4.3.17}$$

This fact, combined with previous steps, will fully give the assertion. We then fix ε and in correspondence some entities we need in the proof. We select $T_\varepsilon > 1$ such that

$$\int_{T_\varepsilon}^{+\infty} e^{-\lambda s} |\ell(y,\alpha)| \, ds < \varepsilon. \tag{4.3.18}$$

for any admissible pair (α, y). We denote by K a compact set containing the support of any integral curve of F, starting at B, and defined in $[0, T_\varepsilon]$, and by $\nu(\cdot)$ an uniform continuity modulus for both ℓ_i in $K \times A_i$ and v in $\Gamma \cap K$. We assume, to simplify notation, that M, besides bounding cost, also bounds the velocities in $F(x)$, when x varies in K. Finally, we denote by r, S the constants provided by Lemma 4.3.9 with $\Gamma \cap K$ in place of Θ. We take δ with

$$\delta \, e^{L T_\varepsilon} \leq \min\left\{r, \frac{\varepsilon}{M\,S}\right\}, \quad \nu\left(\delta \,(1 + M\,S)\, e^{L T_\varepsilon}\right) \leq \frac{\varepsilon}{T_\varepsilon} < \varepsilon. \tag{4.3.19}$$

Let x_0, x_1 be a pair of elements of B with $|x_0 - x_1| < \delta$. Let α_0 be an ε–optimal control for $v(x_0)$ and y_0 the corresponding trajectory starting at x_0. We denote by T_0 its first exit time from Ω_i. We consider the problem

$$\dot{y}_1 = f_i(y_1, \alpha_0), \;\; y_1(0) = x_1.$$

in $\Omega_i \times (0, T_0)$. Let $[0, T_1)$ be the maximal interval of definition of the solution. If $T_1 < T_0$ then such solution can be extended in $[0, T_1]$ and $y_1(T_1) \in \Gamma$. We set $T = \min\{T_0, T_1\}$. We clearly have

$$|y_1(t) - y_0(t)| \leq \delta \, e^{L\,t} \, |x_1 - x_0| \quad \text{for any } t \in [0, T].$$

If $T \geq T_\varepsilon$ then the interface does not enter in the deduction of the estimate (4.3.17), which goes as in the usual case.

If instead $T < T_\varepsilon$, we have $|y_1(T) - y_0(T)| < r$ by (4.3.19), and at least one between $y_1(T)$ and $y_0(T)$ belongs to Γ, say $y_0(T) \in \Gamma$ to fix our ideas. By Lemma 4.3.9, there is an integral curve of the controlled dynamics f_i joining $y_1(T)$ to a point $z \in \Gamma$ in a time less or equal $S\,g(y_1(T))$. We deduce

$$\begin{aligned} v(y_1(T)) &\leq S\,g(y_1(T))\,M + v(z) \leq S\,M\,\delta\,e^{L\,T} + v(z) \\ |y_0(T) - z| &\leq |y_0(T) - y_1(T)| + |y_1(T) - z| \leq \delta\,e^{L\,T} + S\,M\,\delta\,e^{L\,T}. \end{aligned}$$

We deduce from this estimate and (4.3.19)

$$v(y_1(T)) \leq \varepsilon + v(z), \;\; |v(y_0(T)) - v(z)| \leq \varepsilon.$$

Therefore

$$\begin{aligned} v(x_1) - v(x_0) &\leq \int_0^T |\ell_1(y_1(s)) - \ell_1(y_0(s))| \, ds + e^{-L\,T} \big(v(y_1(T)) - v(y_0(T))\big) + \varepsilon \\ &\leq \varepsilon + \big(v(y_1(T)) - v(z)\big) + |v(y_0(T)) - v(z)| + \varepsilon \leq 4\,\varepsilon \end{aligned}$$

as desired. If instead $y_1(T) \in \Gamma$ then we apply Lemma 4.3.9 considering an admissible trajectory from some point of Γ to $y_0(T)$ to get the same conclusion. □

4.3.4 Augmented dynamics

In this section, let us show some technique results concerning the augmented dynamics which will be useful for the characterization of super and sub-optimality principles in the next sections. The inequalities in the super and sub-optimality principles 4.3.8 can be rewritten as

$$u(y(t)) \le (\ge \text{ resp. }) \, e^{\lambda t}\left(u(x) - \int_0^t e^{-\lambda s}\ell(y(s),\alpha(s))\right), \ \forall\, t \ge 0.$$

We set

$$\eta(t) := e^{\lambda t}\left(u(x) - \int_0^t e^{-\lambda s}\ell(y(s),\alpha(s))\right),$$

$$\mathcal{E}p(u) := \{(x,z) \,|\, u(x) \le z, \ x \in \mathbf{R}^d, \ z \in \mathbf{R}\}, \ \mathcal{H}p(u) := \{(x,z) \,|\, u(x) \ge z, \ x \in \mathbf{R}^d\}.$$

Then the super and sub-optimality principles are equivalent to

$$(y(t),\eta(t)) \in \mathcal{E}p(u) \ (\mathcal{H}p(u) \text{ resp. }), \ \forall\, t \ge 0.$$

Note that

$$\dot\eta(t) = \lambda\eta(t) - \ell(y(t),\alpha(t)), \ \text{a.e. } t > 0,$$

we then define the augmented dynamics for $(y(\cdot),\eta(\cdot))$:

$$G(x,\xi) = \{(f(x,\text{A}),\lambda\xi - \ell(x,\text{A})) \mid \text{A} \in A(x)\} \quad (x,\xi) \in \mathbf{R}^d \times \mathbf{R}, \tag{4.3.20}$$

$$G_\Gamma(x,\xi) = \{(f(x,\text{A}),\lambda\xi - \ell(x,\text{A})) \mid \text{A} \in A_\Gamma(x)\} \quad (x,\xi) \in \Gamma \times \mathbf{R}, \tag{4.3.21}$$

$$G_E(x,\xi) = \{(f(x,\text{A}),\lambda\xi - \ell(x,\text{A})) \mid \text{A} \in A_E(x)\} \quad (x,\xi) \in \Gamma \times \mathbf{R}. \tag{4.3.22}$$

The multifunction G is usc and possess linear growth, in addition we see from its very definition that the diameter of G is locally bounded in $\mathbf{R}^d \times \mathbf{R}$.

The rest of this section is devoted to establish a Lipschitz–continuity property for G_Γ with tangential controls.

We will use that estimate

$$d_H\big(G(x,\xi),G(z,\eta)\big) \le (2\,L+\lambda)(|x-z|+|\xi-\eta|) \quad \text{for any } (x,\xi),\ (z,\eta) \text{ in } \mathbf{R}^d \times \mathbf{R}, \tag{4.3.23}$$

where d_H stands for the Hausdorff distance corresponding to the norm of $\mathbf{R}^d \times \mathbf{R}$ appearing in the right hand–side.

Proposition 4.3.12. The multifunction G_Γ is locally Lipschitz continuous on Γ.

Proof. We fix a compact subset K of $\mathbf{R}^d$ and set

$$C = \min \left\{ - \max_{x \in K \cap \Gamma} \min_{q \in F(x)} Dg(x) \cdot q, \min_{x \in K \cap \Gamma} \max_{q \in F(x)} Dg(x) \cdot q \right\}.$$

We assume, without clearly loosing any generality, that $M \geq 1$, so that

$$|D(g(x))| \leq M \quad \text{for any } x \in \Gamma, \tag{4.3.24}$$

and that the constant L appearing in **(H1)-(H2)** is also the Lipschitz constant for Dg in $\Gamma \cap K$. Note that C is strictly positive because of assumption **(H3)(i)**. We pick (x, ξ), (z, η) in $(\Gamma \cap K) \times \mathbf{R}$ with

$$|x - z| < \frac{C}{3\, L\, M}. \tag{4.3.25}$$

Let $\big((f(x,a), \lambda\xi - \ell(x,a)\big)$ be in $G_\Gamma(x,\xi)$. In force of Lipschitz continuity in the state variable of f_i, ℓ_i, $i = 1, 2$ on Γ, we have

$$|f(x,a) - f(z,a)| + |\lambda\xi + \ell(x,a) - \lambda\eta - \ell(z,a)| < (2\,L + \lambda)\big(|x - z| + |\xi - \eta|\big). \tag{4.3.26}$$

We first assume $Dg(z) \cdot f(z,a)$ strictly positive. By the very definition of C and assumptions **(H3)(i)**, **(H3)(ii)** there is $b \in A$ with

$$Dg(x) \cdot f(x,b) = -C, \tag{4.3.27}$$

being $-3\,M\,L|x - z| > -C$ by (4.3.25), we can take $c \in A$ such that $\big(f(x,c), \ell(x,c)\big)$ lies in the segment joining $\big(f(x,a), \ell(x,a)\big)$ to $\big(f(x,b), \ell(x,b)\big)$ and satisfies

$$Dg(x) \cdot f(x,c) = -3\,M\,L|x - z|. \tag{4.3.28}$$

We have

$$\big(f(x,a) - f(x,c), \ell(x,a) - \ell(x,c)\big) = \rho\,\big(f(x,a) - f(x,b), \ell(x,a) - \ell(x,b)\big),$$

for some ρ positive, and, because of (4.3.27), (4.3.28), $\rho = \frac{3\,M\,L}{C}\,|x - z|$, which, in turn, implies

$$|f(x,a) - f(x,c)| = \frac{3\,M\,L}{C}\,|x - z|\,|f(x,a) - f(x,b)| \tag{4.3.29}$$

$$|\ell(x,a) - \ell(x,c)| = \frac{3\,M\,L}{C}\,|x - z|\,|\ell(x,a) - ell(x,b)|. \tag{4.3.30}$$

Since $\big|f(x,a) - f(x,b)\big| \leq 2\,M$, $\quad \big|\ell(x,a) - \ell(x,b)\big)\big| \leq 2\,M$, we derive from (4.3.29), (4.3.30)

$$\big|f(x,a) - f(x,c)\big| + \big|\ell(x,a) - \ell(x,c)\big| < \frac{12\,M^2\,L}{C}\,|x - z|,$$

exploiting this estimate and the inequality

$$\begin{aligned}\big|f(x,a)-f(z,c)\big|+\big|\ell(x,a)-\ell(z,c)\big| &\leq \big|f(x,a)-f(z,a)\big|+\big|f(z,a)-f(z,c)\big| \\ &+ \big|\ell(x,a)-\ell(z,a)\big|+\big|\ell(z,a)-\ell(z,c)\big|\end{aligned}$$

we finally yield

$$\big|f(x,a)-f(z,c)\big|+\big|\ell(x,a)-\ell(z,c)\big| < \left(\frac{12\,M^2\,L}{C}+2\,L\right)|x-z| \tag{4.3.31}$$

We proceed to determine the sign of $Dg(z)\cdot f(z,c)$ by writing it, with the usual trick of adding–subtracting the same quantity, as

$$\big(Dg(z)-Dg(x)\big)\cdot f(z,c)+Dg(x)\cdot(f(z,c)-f(x,c))+Dg(x)\cdot f(x,c).$$

Taking into account (4.3.28), (4.3.24), the boundedness of $|Dg|$, F and estimating term by term, we get

$$Dg(y)\cdot f(z,c)\leq (M\,L+M\,L-3\,M\,L)\,|x-y|<0.$$

Since $Dg(z)\cdot f(z,a)$ and $Dg(z)\cdot f(z,c)$ have opposite sign, there is $d\in A$ such that $f(z,d)\in T_\Gamma(z)$ and $\big(f(z,d),\ell(z,d)\big)$ lies in the segment joining $\big(f(z,a),\ell(z,a)\big)$ to $\big(f(z,c),\ell(z,c)\big)$. The function

$$(p,\sigma)\mapsto |p-f(x,a)|+|\sigma-\ell(x,a)|$$

is convex, and so

$$|f(z,d)-f(x,a)|+|\ell(z,d)-\ell(x,a)|\leq \max\big\{|f(z,a)-f(x,a)|+|\ell(z,a)-\ell(x,a)|,|f(z,c)-f(x,a)|+|\ell(z,c)-\ell(x,a)|\big\}$$

Taking into account (4.3.31) and that L is a Lipschitz constant for both ℓ and f we conclude

$$\big|f(x,a)-f(z,d)\big|+\big|\ell(x,a)-\ell(z,d)\big| < \left(\frac{12\,M^2\,L}{C}+2\,L\right)|x-z|)$$

and consequently

$$\big|f(x,a)-f(z,c)\big|+\big|+\lambda\,\xi-\ell(x,a)-\lambda\,\eta+\ell(z,c)\big| < \left(\frac{12\,M^2\,L}{C}+2\,L+\lambda\right)\big(|x-z|+|\xi-\eta|\big)$$

The same estimate is obtained, using the same argument with obvious change, if $Dg(z)\cdot f(z,a)<0$. If instead $Dg(z)\cdot f(z,a)=0$, then $\big(f(z,a),\lambda\,\eta-\ell(z,a)\big)\in G_\Gamma(z,\eta)$ and

$$\begin{aligned}\big|f(x,a)-f(z,a)\big|+\big|+\lambda\,\xi-\ell(x,a)-\lambda\,\eta+\ell(z,a)\big| &< (2\,L+\lambda)\,\big(|x-z|+|\xi-\eta|\big) \\ &< \left(\frac{12\,M^2\,L}{C}+2\,L+\lambda\right)\big(|x-z|+|\xi-\eta|\big).\end{aligned}$$

This ends the proof. □

We extend G_Γ in $\Gamma_\natural \times \mathbf{R}$ by setting

$$G_\Gamma(x,\xi) = G_\Gamma(\mathrm{proj}_\Gamma(x),\xi). \tag{4.3.32}$$

Exploiting the Lipschitz–continuity of the projection on the interface for points in $\Gamma_\natural$, we deduce from the previous theorem:

Corollary 4.3.13. The multifunction G_Γ, extended as in (4.3.32), is Lipschitz–continuous in $B \times \mathbf{R}$, for any bounded subset B of $\Gamma_\natural$.

Proof. We denote by L_0 a positive quantity which is at the same time Lipschitz constant for proj_Γ in B and for G_Γ in $\mathrm{proj}_\Gamma(B) \times \mathbf{R}$. Given (x_1,ξ_1), (x_2,ξ_2) in $B \times \mathbf{R}$, $(q_1,\sigma_1) \in G_\Gamma(x_1,\xi_1) = G_\Gamma(\mathrm{proj}_\Gamma(x_1),\xi_1)$ there exists $(q_2,\sigma_2) \in G_\Gamma(\mathrm{proj}_\Gamma(x_2,\xi_2) = G_\Gamma(x_2,\xi_2)$ with

$$|(q_1,\sigma_1) - (q_2,\sigma_2)| \leq L_0\, |\mathrm{proj}_\Gamma(x_1),\xi_1) - \mathrm{proj}_\Gamma(x_2),\xi_2)| \leq L_0^2\, |(x_1,\xi_1) - (x_2,\xi_2)|.$$

This proves the assertion. □

4.3.5 Supersolutions and super-optimality principle

In this section, we aim at showing the relation between super-optimality principle and supersolutions. However, we do not have a characterization result for the super-optimality principle via HJB inequations as in the finite horizon case. Recall that the characterization result for the super-optimality principle can be established either when the dynamics are Lipschitz continuous or when the dynamics are usc and convex-valued. In our case, because of the presence of the running cost ℓ_i and the assumption **(H4)**, we have considered the augmented dynamics G which is usc but not convex-valued everywhere. So the result we have obtained here is an approximate super-optimality principle for supersolutions. Although this is not a characterization result, it will be enough to prove the comparison principle result Theorem 4.3.5.

The main result of this section is stated as follows.

Theorem 4.3.14. Assume **(H1)-(H4)**. Let $u : \mathbf{R}^d \to \mathbf{R}$ be a lsc function. We have the following.

(i) If u satisfies the super-optimality, then u is a supersolution of (4.3.1)-(4.3.7);

(ii) If u is a supersolution of (4.3.1)-(4.3.7), then u is a supersolution of (4.3.1)-(4.3.6);

(iii) If u is a supersolution of (4.3.1)-(4.3.6) and $M_w > 0$ with $|w| < M_w$ in $\mathbf{R}^d$. Given $x_0 \in \mathbf{R}^d$ and positive constants T_0 and δ, there exists (y,α) admissible with $y(0) = x_0$ such that

$$w(x_0) \geq \int_0^T e^{-\lambda s}\, \ell(y,\alpha)\, ds - e^{-\lambda T}\, M_w - \delta \quad \text{for some } T \in (T_0, 4\,T_0 + 1).$$

The proof is split into several parts. We start by proving the following result.

Theorem 4.3.15. Assume **(H1)-(H4)**. Let $u : \mathbf{R}^d \to \mathbf{R}$ be a lsc function satisfying the super-optimality principle. Then u is a supersolution of (4.3.1)-(4.3.7).

Proof. For any $x_0 \in \mathbf{R}^d$, $\phi \in C^1(\mathbf{R}^d)$ such that $u - \phi$ attains a local minimum at x_0. If x_0 lies in Ω_1 or Ω_2, the result is classical since G is Lipschitz continuous in Ω_1 and Ω_2.

Suppose now $x_0 \in \Gamma$. u satisfies the super-optimality, then there exists $(y(\cdot), \eta(\cdot))$ driven by G with $(y, \eta)(0) = (x_0, u(x_0))$ such that

$$u(y(h)) \leq \eta(h), \ \forall\, h \geq 0.$$

By the very definition of ϕ, we deduce that

$$u(y(h)) - \phi(y(h)) \geq u(x_0) - \phi(x_0).$$

Then we obtain that

$$\eta(h) - \eta(0) \geq u(y(h)) - u(x_0) \geq \phi(y(h)) - \phi(x_0),$$

i.e.

$$\frac{1}{h}\int_0^h [\dot{\eta}(s) - D\phi(y(s)) \cdot \dot{y}(s)]\, ds.$$

Up to a subsequence, let $h_n \to 0^+$ so that $(x_n, \xi_n) := (y(h_n), \eta(h_n))$ satisfies

$$\frac{1}{h_n}(y(h_n) - x_0, \eta(h_n) - \eta(0)) \to (p, q), \text{ for some } (p, q) \in \mathbf{R}^d \times \mathbf{R}.$$

Lemma 4.2.14 leads to

$$(p, q) \in \text{co } G_E(x_0, u(x_0)).$$

Then we deduce that

$$\sup_{(p,q)\in \text{co } G_E(x_0,u(x_0))} \{-D\phi(x_0) \cdot p + q\} \geq 0,$$

which is equivalent to

$$\sup_{(p,q)\in G_E(x_0,u(x_0))} \{-D\phi(x_0) \cdot p + q\} \geq 0.$$

By the definition of G_E, we deduce that

$$\sup_{a\in A_E(x_0)} \{-D\phi(x_0) \cdot f(x_0, a) + \lambda u(x_0) - \ell(x_0, a)\} \geq 0,$$

i.e.

$$\lambda u(x_0) + H_E(x_0, D\phi(x_0)) \geq 0.$$

□

For technical reasons, we define a function $Q : \mathbf{R} \times (0, +\infty) \to \mathbf{R}$ via

$$Q(\xi, T) = \lambda\, e^{\lambda T} \left(|\xi| + M \left(T + \frac{1}{\lambda}\right) \right), \tag{4.3.33}$$

where M is defined as in **(H2)**. The following fact justifies the introduction of Q.

Lemma 4.3.16. Let (y, η) be an integral curve of G defined in some interval $[a, b]$, then

$$|\dot\eta(t)| \le Q(|\eta(a)|, b-a) \quad \text{for a.e. } t \in (a, b).$$

Proof. For a.e. t and a suitable $\mathrm{A} \in A$ we have:

$$\begin{aligned} |\dot\eta(t)| &= |\lambda\, \eta(t) - \ell(x, \mathrm{A})| \le \lambda\, \exp(\lambda\,(t-a)) \left(|\eta(a)| + M\,(t-a)\right) + M \\ &\le \lambda\, e^{\lambda\,(b-a)} \left(|\eta(a)| + M \left(b - a + \frac{1}{\lambda}\right) \right) = Q(\eta(a), b-a). \end{aligned}$$

□

We record a couple of elementary properties of function Q for which the proof can be done by direct calculation and it is skipped.

Lemma 4.3.17.

- **(i)** $Q(\xi, T_1) < Q(\xi, T_2)$ for any ξ and $T_1 < T_2$.
- **(ii)** Let (y, η) is a trajectory of G defined in some interval $[a, b]$, then

$$Q(\eta(t), b-t) \le Q(\eta(a), b-a) \quad \text{for any } t \in (a, b).$$

Theorem 4.3.18. Assume **(H1)-(H4)**. Let w be a bounded lsc supersolution of (4.3.1)-(4.3.6) and $M_w > 0$ with $|w| < M_w$ in $\mathbf{R}^d$. Given $x_0 \in \mathbf{R}^d$ and positive constants T_0 and δ, there exists (y, α) admissible with $y(0) = x_0$ such that

$$w(x_0) \ge \int_0^T e^{-\lambda\, s}\, \ell(y, \alpha)\, ds - e^{-\lambda\, T}\, M_w - \delta \quad \text{for some } T \in (T_0, 4\, T_0 + 1).$$

We will use the following property of co G for the epigraphs of supersolutions, which can be straightforwardly obtained as in the usual non partitioned case:

Proposition 4.3.19. Let w be a lsc supersolution to (4.3.1)-(4.3.6), and $(y(\cdot), \eta(\cdot)) : [0, +\infty) \to \mathbf{R}^d \times \mathbf{R}$ be driven by co G with $(y(0), \eta(0)) \in \mathcal{E}p(w)$. Then $(y(t), \eta(t)) \in \mathcal{E}p(w)$ for all $t \ge 0$.

The difficulty in deducing Theorem 4.3.18 from Proposition 4.3.19 in presence of an interface is that, as usual, we do not have Lipschitz–continuity of the multivalued vector field on the whole $\mathbf{R}^d \times \mathbf{R}$, and this prevents us from directly applying Relaxation Theorem to approximate curves of the relaxed dynamics, see [15, Theorem 2, pp. 124]. We break the arguments in two parts and use Relaxation Theorem for the portions of curves far from the interface and Filippov Approximation Theorem 2.2.7 for those closer to Γ. The two parts will be glued together by exploiting the controllability conditions of **(H3)**.

Proof. (*of Theorem 4.3.18*) By Proposition 4.3.19 there is an integral curve (y_0, η_0) of co G taking the value $(x_0, w(x_0))$ at $t = 0$, defined in $[0, 2\,T_0]$, and lying in $\mathcal{E}p(w)$. We select a compact set $K_0 \subset \mathbf{R}^d$ containing in its interior

$$\mathcal{R}_{F_\Gamma}(y_0([0, 2\,T_0]) \cap \Gamma, 2\,T_0).$$

Recall (Remark 4.3.1) that there is only a finite number of connected components of Γ intersecting K_0. Introducing some quantities we will use in the forthcoming estimates:

- P estimates from above the diameter of $G(x, \xi)$ on $y_0([0, 2\,T_0]) \times \eta_0([0, 2\,T_0])$.
- $N > 1$ express the equivalence of Euclidean distance and d_Γ in $K_0 \cap \Gamma_i$, $i = 1, \cdots, n$, namely $|x - z| \leq d_\Gamma(x, z) \leq N\,|x - z|$ for x, z in $K_0 \cap \Gamma_i$.
- L_G is a Lipschitz constant for G_Γ in $(K_0 \cap \Gamma_\natural) \times \mathbf{R}$.

We finally recall that R is the constant related to the controllability condition on the interface, stipulated in **(H3)(ii)**. We define

$$C = \frac{N}{R}\,\exp(2\,L_G\,T_0)\,P \tag{4.3.34}$$

Consider ε small enough such that any integral curve of F defined in some compact interval and with support contained in K_0 and any ε–partition related to it satisfy the weak separation principle stated in Proposition 4.4.3. We claim the following property that will be proved by induction.

($\mathbf{P_k}$) Given an interval $[a_0, b_0] \subset [0, 2\,T_0]$ such that $\jmath_\varepsilon(y_0; a_0, b_0) = k$, there exists, for any $\xi_0 \in \mathbf{R}$, a trajectory of G defined in some interval $[a, b]$ with

- $(y(a), \eta(a)) = (y_0(a), \xi_0)$.
- $C\,\varepsilon + 2\,(b_0 - a_0) > b - a > \frac{b_0 - a_0}{2}$.
- $\eta(b) + \exp(\lambda\,(b - a))\,[\eta_0(a_0) - \xi_0]^+ + P\,\left(1 + \frac{N}{R}\,Q(\xi_0, b - a)\right)\,\exp((L_G + \lambda)\,(b - a))\,\varepsilon \geq \eta_0(b_0)$.
- $y(b) = y_0(b_0)$ whenever $y_0(b_0) \in \Gamma$.

The function $Q(\cdot, \cdot)$ has been defined in (4.3.33).

The proof is divided into several steps.

Step (1): Proving $(\mathbf{P_2})$ when $y_0([a_0, b_0]) \cap \Gamma = \emptyset$

We first show $(\mathbf{P_2})$ assuming $y_0([a_0, b_0])$ contained in one of the two open region of the partition, say Ω_i. Since G is locally Lipschitz–continuous in $\Omega_i \times \mathbf{R}$, and y is at a positive distance from the interface, we find in this case by Relaxation Theorem, for any given ρ, an integral curve $(y, \overline{\eta})$ of G, defined in $[a_0, b_0]$, with $(y(a_0), \overline{\eta}(a_0)) = (y_0(a_0), \eta_0(a_0))$ and

$$|y(t) - y_0(t)| + |\overline{\eta}(t) - \eta_0(t)| < \rho \quad \text{for } t \in [a_0, b_0]. \tag{4.3.35}$$

By Filippov Implicit Function Lemma (see [114]), y is an integral trajectory in $[a_0, b_0]$ of $f_i(y, \alpha)$ for some admissible control α. Denote by η satisfying $\dot{\eta} = \lambda \eta - \ell_i(y, \alpha)$ with $\eta(a_0) = \xi_0$, then $\overline{\eta}(b_0) - \eta(b_0) \leq \exp(\lambda (b_0 - a_0)) (\eta_0(a_0) - \xi_0)$, and consequently

$$\begin{aligned} \eta_0(b_0) &\leq |\overline{\eta}(b_0) - \eta_0(b_0)| + (\overline{\eta}(b_0) - \eta(b_0)) + \eta(b_0) \\ &\leq \rho + \exp(\lambda (b_0 - a_0)) [\eta_0(a_0) - \xi_0]^+ + \eta(b_0), \end{aligned}$$

which proves the assertion with $[a, b] = [a_0, b_0]$, being ρ arbitrary.

Step (2): Proving $(\mathbf{P_2})$ when $y_0((a_0, b_0)) \cap \Gamma = \emptyset$ and $y_0(a_0)$, $y_0(b_0)$ possibly in Γ

Now assume $y_0((a_0, b_0)) \subset \Omega_i$, and both $y_0(a_0)$ and $y_0(b_0)$ to be in Γ. We again apply Relaxation Theorem in a slightly reduced interval to stay away from Γ. We find, for any $\rho > 0$ sufficiently small, an integral curve $(y, \overline{\eta})$ of G in $[a_0 + \rho, b_0 - \rho]$ with $(y(a_0 + \rho), \overline{\eta}(a_0 + \rho)) = (y_0(a_0 + \rho), \eta_0(a_0 + \rho))$ and

$$|y(t) - y_0(t)| + |\overline{\eta}(t) - \eta_0(t)| < \rho \quad \text{for } t \in [a_0 + \rho, b_0 - \rho]. \tag{4.3.36}$$

We have

$$\begin{aligned} |y(a_0 + \rho) - y_0(a_0)| &\leq |y(a_0 + \rho) - y_0(a_0 + \rho)| + |y_0(a_0) - y_0(a_0 + \rho)| \\ &\leq \rho + O(\rho) = O(\rho) \end{aligned} \tag{4.3.37}$$

and the same inequality holds for $|y(b_0 - \rho) - y_0(b_0)|$, therefore, bearing in mind that $y_0(a_0)$ and $y_0(b_0)$ are on the interface, we have

$$|g(y(a_0 + \rho))| = O(\rho) \quad \text{and} \quad |g(y(b_0 - \rho))| = O(\rho).$$

We can thus apply Lemma 4.3.9 to continuously extend y in $[a_0 + \rho - t_1, b_0 - \rho + t_2]$, for some t_1, t_2 positive, through concatenation with other trajectories of F such that

$$\begin{aligned} & t_1 = O(\rho),\ t_2 = O(\rho) \\ & y(a_0 + \rho - t_1) \text{ and } y_0(a_0) \text{ belong to the same connected component of } \Gamma \\ & y(b_0 - \rho + t_2) \text{ and } y_0(b_0) \text{ belong to the same connected component of } \Gamma. \end{aligned} \tag{4.3.38}$$

We proceed considering a geodesics on Γ linking $y_0(a_0)$ to $y(a_0+\rho-t_1)$ and $y(b_0-\rho+t_2)$ to $y_0(b_0)$. We parametrize it with constant velocity R in intervals $[a_0+\rho-t_1-t_1', t_1+\rho-t_1]$, $[b_0-\rho+t_2, t_2-\rho+t_2+t_2']$, respectively, for appropriate $t_1' \geq 0$, $t_2' \geq 0$. By assumption **(H3)(ii)** these curves are admissible for the controlled dynamics, and we employ it to further extend y by concatenation in $[a_0+\rho-t_1-t_1', b_0-\rho+t_2+t_2']$.

The next step is to estimate t_1', t_2'. We actually make explicit calculations just for t_1', being those for t_2' identical. We preliminarily calculate using (4.3.37), (4.3.38)

$$\begin{aligned} |y(a_0+\rho-t_1)-y_0(a_0)| &\leq |y(a_0+\rho-t_1)-y(a_0+\rho)|+|y(a_0+\rho)-y_0(a_0)| \\ &\leq O(\rho)+O(\rho)=O(\rho). \end{aligned}$$

Being d_Γ locally equivalent to the Euclidean distance, this implies

$$d_\Gamma(y(a_0+\rho-t_1), y_0(a_0))) \leq O(\rho)$$

and, taking into account that the geodesics have been parametrized with velocity R,

$$t_1' \leq \frac{O(\rho)}{R} = O(\rho), \quad t_2' = O(\rho). \tag{4.3.39}$$

We set $a = a_0+\rho-t_1-t_1'$, $\; b = b_0-\rho+t_2+t_2'$. The curve y in $[a,b]$ is altogether an integral curve of F and so it is in correspondence with an admissible control α. By construction we have

$$y(a) = y_0(a_0) \quad \text{and} \quad y(b) = y_0(b_0). \tag{4.3.40}$$

Denote by η, for $t \in [a,b]$, the solution of $\dot\eta = \lambda\,\eta - \ell(y,\alpha)$ with $\eta(a) = \xi_0$. Then, bearing in mind (4.3.38), (4.3.39)

$$\begin{aligned} &\eta_0(a_0+\rho)-\eta(a_0+\rho) \leq |\eta_0(a_0+\rho)-\eta_0(a_0)| + \eta_0(a_0) - \eta(a_0+\rho) \\ \leq\;& O(\rho)+\eta_0(a_0)-\exp(\lambda\,(t_1+t_1'))\left(\xi_0 - \int_a^{a_0+\rho} \exp(-\lambda\,(t-a))\,\ell(y,\alpha)\,dt\right) \\ \leq\;& O(\rho)+\eta_0(a_0)+\exp(\lambda\,(t_1+t_1'))\big(-\xi_0+M\,(t_1+t_1')\big) \\ \leq\;& O(\rho)+(1-\exp(\lambda\,(t_1+t_1'))\,\xi_0+[\eta_0(a_0)-\xi_0]^+ = O(\rho)+[\eta_0(a_0)-\xi_0]^+. \end{aligned}$$

This implies, taking into account that $\overline{\eta}(a_0+\rho) = \eta_0(a_0+\rho)$ and $b_0-a_0-2\,\rho \leq b-a$

$$\begin{aligned} \overline{\eta}(b_0-\rho)-\eta(b_0-\rho) &\leq \exp(\lambda\,(b_0-a_0-2\,\rho))\,(\overline{\eta}(a_0+\rho)-\eta(a_0+\rho)) \\ &\leq O(\rho)+\exp(\lambda\,(b-a))\,[\eta_0(a_0)-\xi_0]^+. \end{aligned}$$

By this last inequality, (4.3.36), (4.3.38), (4.3.39), we get

$$\begin{aligned}\eta_0(b_0) &\leq |\eta_0(b_0-\rho)-\eta_0(b_0)| + |\overline{\eta}(b_0-\rho)-\eta_0(b_0-\rho)| + (\overline{\eta}(b_0-\rho)-\eta(b_0-\rho)) \\ &\quad +|\eta(b)-\eta(b_0-\rho)| + \eta(b) \\ &\leq O(\rho)+\rho+O(\rho)+\exp(\lambda\,(b-a))\,[\eta_0(a_0)-\xi_0]^+ + Q(\xi_0,b-a)\,O(\rho)+\eta(b) \\ &= O(\rho)+\exp(\lambda\,(b-a))\,[\eta_0(a_0)-\xi_0]^+ + Q(\xi_0,b-a)\,O(\rho)+\eta(b).\end{aligned}$$

We recall that the function $Q(\cdot,\cdot)$ is defined in (4.3.33). Taking into account the above formula, the fact that ρ can be chosen arbitrarily small and (4.3.40), $\eta(a)=\xi_0$, the assertion is proved. The above argument can be easily adapted to the case where just one of the two extremal points $y_0(a_0)$, $y_0(b_0)$ belongs to the interface. Notice that if $y_0(a_0)\notin\Gamma$ then a can be taken equal to a_0 and similarly $b=b_0$ whenever $y_0(b_0)\notin\Gamma$. The proof of this part is therefore concluded.

Step (3): Proving $(\mathbf{P_2})$ when $y_0((a_0,b_0))\cap\Gamma\neq\emptyset$. Since $\jmath_\varepsilon(y_0;a_0,b_0)=2$

$$|\{t\in[a_0,b_0]\mid y_0(t)\notin\Gamma\}|<\varepsilon. \tag{4.3.41}$$

Along the same lines in Theorem 4.3.26, it gives

$$\int_{a_0}^{b_0} d((\dot{y}_0(s),\dot{\eta}_0(s)),G_\Gamma((y_0(s)),\eta(s)))\,ds\leq\varepsilon\,P.$$

By the assumption on ε, we conclude that $y_0([a_0,b_0])$ is contained in $\Gamma_\natural$. We apply Theorem 2.2.7 with $\mathcal{C}=\Gamma\times\mathbf{R}$, $\mathcal{C}_\natural=\Gamma_\natural\times\mathbf{R}$, and the multifunction $Z=G_\Gamma$, taking into account that L_G is a Lipschitz constant of G_Γ in $(K_0\cap\Gamma_\natural)\times\mathbf{R}$ which contains a bounded open neighborhood of $\mathcal{R}_{G_\Gamma}((y_0(a_0),\eta_0(a_0)),b_0-a_0))$, as prescribed in that theorem. We get the existence of an integral curve $(y,\overline{\eta})$ of G_Γ defined in $[a_0,b_0]$ and contained in the interface with

$$(y(a_0),\overline{\eta}(a_0))=(y_0(a_0),\eta_0(a_0)) \tag{4.3.42}$$

and

$$|y(b_0)-y_0(b_0)| \leq \exp\big(L_G\,(b_0-a_0)\big)\,\varepsilon\,P \tag{4.3.43}$$

$$|\overline{\eta}(b_0)-\eta_0(b_0)| \leq \exp\big(L_G\,(b_0-a_0)\big)\,\varepsilon\,P. \tag{4.3.44}$$

We extend y in some interval $[a_0,b_0+t_2]$, for a suitable $t_2\geq 0$ by concatenation with a geodesics in Γ_j joining $y(b_0)$ to $y_0(b_0)$, parametrized with constant velocity R. Since $y(b_0)\,,y_0(b_0)\in K_0\cap\Gamma_j$

$$d_\Gamma(y(b_0),y_0(b_0))<N\,|y(b_0)-y_0(b_0)|<N\,\exp\big(L_G\,(b_0-a_0)\big)\,\varepsilon\,P,$$

which, in turn, implies $t_2<\frac{N}{R}\exp\big(L_G\,(b_0-a_0)\big)\,\varepsilon\,P$.

We set $a = a_0$ and $b = b_0 + t_2$. Recalling the definition of C given in (4.3.34) and the above estimate of t_2, we have

$$C\,\varepsilon + (b_0 - a_0) \geq b - a \geq b_0 - a_0. \tag{4.3.45}$$

The curve y so extended in $[a, b]$ is an integral curve of F_Γ and so it is in correspondence with an admissible control α, in addition it satisfies

$$y(b) = y(b_0 + t_2) = y_0(b_0). \tag{4.3.46}$$

We denote by η, for $t \in [a, b]$, the curve identified by $\dot{\eta} = \lambda\,\eta - \ell(y, \alpha)$ and $\eta(a) = \xi_0$. Together with (4.3.42) we have

$$\overline{\eta}(b_0) - \eta(b_0) \leq \exp(\lambda\,(b_0 - a_0))\,[\eta_0(a_0) - \xi_0]^+.$$

We finally gather information from (4.3.44) and the above formula to get

$$\begin{aligned} \eta_0(b_0) &\leq |\eta_0(b_0) - \overline{\eta}(b_0)| + \overline{\eta}(b_0) - \eta(b_0) + |\eta(b_0) - \eta(b_0 + t_2)| + \eta(b_0 + t_2) \\ &\leq \exp\left(L_G\,(b_0 - a_0)\right)\varepsilon\,P + \exp(\lambda\,(b_0 - a_0))\,[\eta_0(a_0) - \xi_0]^+ \\ &+ \frac{N}{R}\exp\left(L_G\,(b_0 - a_0)\right)P\,Q(\xi_0, b - a)\,\varepsilon + \eta(b). \end{aligned}$$

Therefore, using (4.3.45)

$$\eta_0(b_0) \leq P\left(1 + \frac{N}{R}\,Q(\xi_0, b - a)\right)e^{L_G\,(b-a)}\,\varepsilon + e^{\lambda\,(b-a)}\,[\eta_0(a_0) - \xi_0]^+ + \eta(b). \tag{4.3.47}$$

We claim that (y, η) satisfies all the properties in $(\mathbf{P_2})$. In fact, such curve is continuous in $[a, b]$ because of (4.3.46), it is an integral curve of G by construction, and satisfies the basic estimate because of (4.3.47). Moreover $y(b) = y(b_0 + t_2) = \overline{y}(b_0 + t_2) = y_0(b_0)$ by (4.3.46), the condition at $t = a = a_0$ is also satisfied thanks to (4.3.42). Finally (4.3.45) gives the desired estimate on $b - a$ in terms of $b_0 - a_0$, C and ε.

Step (4): Proving $(\mathbf{P_{k+1}})$ We assume $(\mathbf{P_2}), \cdots, (\mathbf{P_k})$ to hold and $\jmath(y_0; a_0, b_0) = k + 1$. The idea is to exploit Proposition 4.4.5, we denote by

$$\{t_1 = a_0, \cdots, t_{k+1} = b_0\}$$

a minimal ε–partition of $[a_0, b_0]$ related to y_0, then there are two positive constant ε_1, ε_2 with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ satisfying

$$\jmath_{\varepsilon_1}(y_0; a, t_k) = k \quad \text{and} \quad \jmath_{\varepsilon_2}(y_0; t_k, b) = 2.$$

By inductive step there are two integral curves y_1 ad y_2 of G, defined in intervals $[a_1, b_1]$, $[a_2, b_2]$, respectively, enjoying the following properties:

(i) $(y_1(a_1), \eta_1(a_1)) = (y_0(a_0), \xi_0)$ and $(y_2(a_2), \eta_2(a_2)) = (y_0(t_k), \eta_1(b_1))$.

(ii) $C\,\varepsilon_1 + 2\,(t_k - a_0) > b_1 - a_1 > \frac{t_k - a_0}{2}$ and $C\,\varepsilon_2 + 2\,(b_0 - t_k) > b_2 - a_2 > \frac{b_0 - t_k}{2}$.

(iii) $\eta_1(b_1) + \exp(\lambda\,(b_1 - a_1))\,[\eta_0(a_0) - \xi_0]^+ + P\,\left(1 + \frac{N}{R}\,Q(\xi_0, b_1 - a_1)\right)\,\exp((L_G + \lambda)\,(b_1 - a_1))\,\varepsilon_1 \geq \eta_0(t_k)$.

(iv) $\eta_2(b_2) + \exp(\lambda\,(b_2 - a_2))\,[\eta_0(t_k) - \eta_1(b_1)]^+ + P\,\left(1 + \frac{N}{R}\,Q(\eta_1(b_1), b_2 - a_2)\right)\,\exp((L_G + \lambda)\,(b_2 - a_2))\,\varepsilon_2 \geq \eta_0(b_0)$.

(v) $y_1(b_1) = y_0(t_k)$ because $y_0(t_k) \in \Gamma$, see the definition of ε–partition.

(vi) $y_2(b_2) = y_0(b_0)$ if $y_0(b_0) \in \Gamma$.

We set $a = a_1$, $b = b_1 + b_2 - a_2$ and define a curve in $[a, b]$ by setting

$$\begin{cases} (y(t), \eta(t)) = (y_1(t), \eta_1(t)) & \text{for } t \in [a_1, b_1] \\ (y(t), \eta(t)) = (y_2(t + a_2 - b_1), \eta_2(t + a_2 - b_1)) & \text{for } t \in [b_1, b_1 + b_2 - a_2]. \end{cases}$$

Notice that (y, η) is continuous because of items (i), (v), and it is an integral curve of G being the concatenation of two of such curves. It attains the value $(y_0(a_0), \xi_0)$ at a thanks to (i), inequalities

$$C\,\varepsilon + 2\,(b_0 - a_0) > b - a > \frac{b_0 - a_0}{2}$$

hold by (ii), and the condition at $t = b$, in case $y_0(b_0)$ is on Γ, is satisfied by (vi). Finally we combine estimates in (iii) and (iv) to get

$$\begin{aligned} \eta_0(b_0) \leq\ & \eta_2(b_2) + \exp(\lambda\,(b_2 - a_2))\,[\eta_0(t_k) - \eta_1(b_1)]^+ \\ & + P\left(1 + \frac{N}{R}\,Q(\eta_1(b_1), b_2 - a_2)\right)\,\exp((L_G + \lambda)\,(b_2 - a_2))\,\varepsilon_2 \\ \leq\ & \eta_2(b_2) + \exp(\lambda\,(b_2 - a_2))\,\big\{\exp(\lambda\,(b_1 - a_1))\,[\eta_0(a_0) - \xi_0]^+ \\ & + P\left(1 + \frac{N}{R}\,Q(\xi_0, b_1 - a_1)\right)\,\exp((L_G + \lambda)\,(b_1 - a_1))\,\varepsilon_1\big\} \\ & + P\left(1 + \frac{N}{R}\,Q(\eta_1(b_1), b_2 - a_2)\right)\,\exp((L_G + \lambda)\,(b_2 - a_2))\,\varepsilon_2. \end{aligned}$$

By Lemma 4.3.17, $Q(\eta_1(b_1), b_2 - a_2) \leq Q(\xi_0, b - a)$ and $Q(\xi_0, b_1 - a_1) \leq Q(\xi_0, b - a)$. Plugging these relations in the previous estimates, it gives

$$\begin{aligned} \eta_0(b_0) \leq\ & \eta(b) + e^{\lambda\,(b-a)}[\eta_0(a_0) - \xi_0]^+ + P\left(1 + \frac{N}{R}\,Q(\xi_0, b - a)\right)\,e^{(\lambda + L_G)\,(b-a)}\,\varepsilon_1 \\ & + P\left(1 + \frac{N}{R}\,Q(\xi_0, b - a)\right)\,e^{(L_G + \lambda)\,(b_2 - a_2)}\,\varepsilon_2 \\ \leq\ & \eta(b) + e^{\lambda\,(b-a)}[\eta_0(a_0) - \xi_0]^+ + P\left(1 + \frac{N}{R}\,Q(\xi_0, b - a)\right)\,e^{(L_G + \lambda)\,(b-a)}\,\varepsilon. \end{aligned}$$

This segment of the proof is then complete.

Step (5): Final part. We fix $\delta > 0$ and ε with $P\left(1 + \frac{N}{R}Q(w(x_0), 4T_0 + 1)\right)\exp((L_G + \lambda)T)\varepsilon < \delta$ and $C\varepsilon < 1$. Owing to the above part of the proof, we find a trajectory (y, η) of G defined in some interval $[a,b]$ of length $b-a =: T \in (T_0, 4T_0+1)$ such that $(y(a),\eta(a)) = (x_0, w(x_0))$ and

$$\eta(b)+\delta > \eta(b) + P\left(1+\frac{N}{R}\,Q(w(x_0),4T_0+C\,\varepsilon)\right)\exp((L_G+\lambda)T)\,\varepsilon \geq \eta_0(2T_0).$$

It is not restrictive to assume $[a,b]=[0,T]$. Taking into account that (y_0,η_0) is contained in $\mathcal{E}p(w)$ we further obtain

$$\eta(T)+\delta \geq \eta_0(2T_0) \geq w(y_0(2T_0)) \geq -M_w. \tag{4.3.48}$$

Since y is an integral curve of F, there exists an admissible control α such that

$$\eta(T) = e^{\lambda T}\left(w(x_0) - \int_0^T e^{-\lambda t}\,\ell(y,\alpha)\,dt\right).$$

Plugging this relation in (4.3.48) we find

$$e^{\lambda T}\left(w(x_0) - \int_0^T e^{-\lambda t}\,\ell(y,\alpha)\,dt\right) \geq -\delta - M_w,$$

and finally

$$w(x_0) \geq \int_0^T e^{-\lambda t}\,\ell(y,\alpha)\,dt - e^{-\lambda T}(M+\delta).$$

□

Finally, we state the proof of Theorem 4.3.14.

Proof. (i) holds true thanks to Theorem 4.3.15 and (iii) holds true thanks to Theorem 4.3.18.

We proceed to prove (ii). It is sufficient to prove this result on Γ. Note that for $x \in \Gamma$, $p \in \mathbf{R}^d$,

$$\begin{aligned} \max\{H_1(x,p), H_2(x,p)\} &= \max\{-p\cdot f(x,\text{A}) - \ell(x,\text{A}) \mid \text{A} \in A\} \\ &\geq \max\{-p\cdot f(x,\text{A}) - \ell(x,\text{A}) \mid \text{A} \in A_E\} \\ &= H_E(x,p), \end{aligned}$$

thus, it is clear to see that if u is a supersolution of (4.3.7) on Γ, then u is a supersolution of (4.3.6) on Γ. □

4.3.6 Subsolutions and sub-optimality principle

In this section, we aim at the characterization of the sub-optimality principle. The main result is the following.

Theorem 4.3.20. Assume **(H1)-(H3)**. Let $u : \mathbf{R}^d \to \mathbf{R}$ be a usc function. The following are equivalent.

(i) u satisfies the sub-optimality principle;

(ii) u is a subsolution of (4.3.1)-(4.3.6);

(iii) u is a subsolution of (4.3.1)-(4.3.7).

At first, we exploit some regularity result of the subsolutions of (4.3.6) as a consequence of the assumption **(H3)(ii)**.

Proposition 4.3.21. Any bounded usc subsolution to (4.3.6) in Γ is locally Lipschitz-continuous on Γ.

Proof. This is the usual argument which holds for subsolutions of equations with coercive Hamiltonians. Some adaptation is just required since the problem is posed in an hypersurface. By **(H3)(ii)**

$$\lim_{\substack{|p|\to+\infty \\ p\in\mathcal{T}^*_\Gamma(x)}} H_\Gamma(x,p) = +\infty \quad \text{uniformly in } \Gamma.$$

Being our subsolution, say u, bounded we deduce

$$|Du| \le C \quad \text{on } \Gamma \text{ for a suitable } C \tag{4.3.49}$$

again, this must be understood in the viscosity sense on Γ, we will consider test functions defined on Γ, with differentials in the cotangent bundle of Γ. Now fix a connected component Γ_0 of Γ, $z \in \Gamma_0$ and $C' > C$. The function

$$u(x) - u(z) - C'\, d_\Gamma(z,x)$$

attains maximum in Γ_0. If it is strictly positive then corresponding maximizers are different from z and $C'\, d_\Gamma(z,\cdot)$ is an admissible test function for (4.3.49) at any of them, which is impossible because

$$C'\, |Dd_\Gamma(z,x)| \ge C' > C \quad \text{for all } x \in \Gamma_0.$$

Therefore maximum in object must be zero, then

$$|u(x) - u(z)| \le C\, d_\Gamma(x,z) \quad \text{for any } x,\ z \text{ in } \Gamma_0,$$

which in turns implies that u is locally Lipschitz–continuous in Γ_0, being d_Γ and the Euclidean distance are locally equivalent in Γ_0. The full assertion, namely local Lipschitz continuity in Γ and not just on connected components, just comes from the fact that any compact subset of $\mathbf{R}^d$ intersects only a finite number of connected components of Γ (Remark 4.3.1), and they are at a positive distance apart. □

Theorem 4.3.22. Assume **(H1)**–**(H4)**. If $u : \mathbf{R}^d \to \mathbf{R}$ is a bounded usc function satisfying the suboptimality property then it is a subsolution to (4.3.1)-(4.3.6).

Proof. Outside the interface there is nothing new, so we focus on $x_0 \in \Gamma$ where u admits a C^1 viscosity test function from above, say ϕ, with x_0 local constrained maximizer of $u - \phi$ on Γ, we also assume $\phi(x_0) = u(x_0)$. We aim at proving

$$\lambda u(x_0) + \max\{-D\phi(x_0) \cdot f(x_0, \mathrm{A}) - \ell(x_0, \mathrm{A}) \mid \mathrm{A} \in A_\Gamma(x_0)\} \leq 0. \tag{4.3.50}$$

By Theorem 4.3.12 and Corollary 4.3.13, the multifunction G_Γ, suitably extended outside the interface, is locally Lipschitz–continuous in $\Gamma_\natural$. Therefore, given $\mathrm{A}_0 \in A_\Gamma(x_0)$, we can apply Corollary 2.2.8 to find a C^1 integral curve of G_Γ, say (y, η), in $[0, T]$, for some $T > 0$, with $(y(0), \eta(0)) = (x_0, u(x_0))$, $(\dot{y}(0), \dot{\eta}(0)) = (f(x_0, \mathrm{A}_0), \lambda\, u(x_0) - \ell(x_0, \mathrm{A}_0))$. Clearly $y(t) \in \Gamma$ for any t and there is an admissible control α such that for all $t \in [0, T]$

$$\dot{y}(t) = f(y(t), \alpha(t))), \;\; \dot{\eta}(t) = \lambda\, \eta(t) - \ell(y(t), \alpha(t))),$$

in addition $t \mapsto \ell(y(t), \alpha(t)))$ is continuous and its limit, as $t \to 0$, is $\ell(x_0, \mathrm{A}_0)$. Because of the suboptimality of u, $\phi(x_0) = u(x_0)$ and $y(t) \in \Gamma$ for any t, we have for $t \in [0, T]$

$$u(x_0) \leq e^{-\lambda\, t}\, \phi(y(t)) + \int_0^t e^{-\lambda\, s}\, \ell(y, \alpha)\, ds$$

and consequently

$$\frac{\phi(x_0) - e^{-\lambda\, t}\, \phi(y(t))}{t} \leq \frac{1}{t} \int_0^t e^{-\lambda\, s}\, \ell(y, \alpha)\, ds.$$

This implies, passing at the limit for $t \to 0$ and exploiting the aforementioned continuity properties of cost in t

$$\lambda\, u(x_0) - D\phi(x_0) \cdot f(x_0, \mathrm{A}_0) \leq \ell(x_0, \mathrm{A}_0).$$

This concludes the proof because A_0 has been selected arbitrarily in $A_\Gamma(x_0)$. □

Theorem 4.3.23. Assume **(H1)**–**(H4)**. If $u : \mathbf{R}^d \to \mathbf{R}$ is a bounded usc function satisfying the suboptimality property then it is a subsolution to (4.3.1)-(4.3.7).

Proof. The result is classical outside the interface. Then for any $x_0 \in \Gamma$, we need to check three types of viscosity tests in Γ, $\overline{\Omega}_1$ and Ω_2 separately. At first, for any $\phi \in C^1(\Gamma)$ such that $u|_\Gamma - \phi$ attains a local maximum at x_0 in Γ, Theorem 4.3.22 implies that

$$\lambda u(x_0) + \sup_{\mathrm{A} \in A_\Gamma(x_0)} \{-D_\Gamma \phi(x_0) \cdot f(x_0, \mathrm{A}) - \ell(x_0, \mathrm{A})\} \leq 0.$$

Then for second type of viscosity tests, consider $\phi \in C^1(\overline{\Omega}_1)$ such that $u|_{\overline{\Omega}_1} - \phi$ attains a local maximum at x_0 in $\overline{\Omega}_1$, given $\mathrm{A}_1 \in A^1_E(x_0)$, then $f_1(x_0, \mathrm{A}_1) \in T_{\overline{\Omega}_1}(x_0)$, i.e.

$$Dg(x_0) \cdot f_1(x_0, \mathrm{A}_1) \leq 0.$$

If $Dg(x_0) \cdot f_1(x_0, \text{A}_1) = 0$, then $\text{A}_1 \in A_\Gamma(x_0)$. Note that $\phi|_\Gamma \in C^1(\Gamma)$ and $D_{\overline{\Omega}_1}\phi(x_0) = D_\Gamma \phi|_\Gamma(x_0)$, by applying Theorem 4.3.22,

$$\lambda u(x_0) - D_{\overline{\Omega}_1}\phi(x_0) \cdot f(x_0, \text{A}_1) - \ell(x_0, \text{A}_1) \leq 0,$$

where A_1 is chosen arbitrarily. Then we deduce that

$$\lambda u(x_0) + \sup_{\text{A}_1 \in A^1_E(x_0)} \{-D_{\overline{\Omega}_1}\phi(x_0) \cdot f(x_0, \text{A}_1) - \ell(x_0, \text{A}_1)\} \leq 0.$$

Now if $Dg(x_0) \cdot f_1(x_0, \text{A}_1) < 0$, consider the trajectory $y_1 : (0, +\infty) \to \mathbf{R}^d$ satisfying

$$\dot{y}_1(s) = f_1(y_1(s), \text{A}_1), \ \forall\, s > 0, \text{ with } y_1(0) = x_0.$$

By the continuity of $f_1(\cdot, \text{A}_1)$ and $y_1(\cdot)$, there exists $\tau > 0$ such that

$$Dg(y_1(s)) \cdot f_1(y_1(s), \text{A}_1) < 0, \text{ for } s \in [0, \tau).$$

Then $y_1(s) \in \Omega_1$ for $s \in (0, \tau)$. The sub-optimality principle satisfied by u leads to

$$u(y_1(h)) \geq \eta_1(h), \ \forall\, h \in [0, \tau),$$

where

$$\eta_1(h) = e^{\lambda h}\left(u(x_0) - \int_0^h e^{-\lambda s}\ell(y_1(s), \text{A}_1)ds\right).$$

Note that

$$u(y_1(h)) - \phi(y_1(h)) \leq u(x_0) - \phi(x_0), \ \forall\, h \in [0, \tau),$$

we then deduce that

$$\eta(h) - \eta(0) \geq \phi(y_1(h)) - \phi(x_0), \ \forall\, h \in [0, \tau).$$

Thus,

$$\frac{1}{h}\int_0^h \left[\dot{\eta}(s) - D_{\overline{\Omega}_1}\phi(y_1(s)) \cdot \dot{y}_1(s)\right] ds \leq 0.$$

Let $h \to 0$, we obtain that

$$\lambda u(x_0) - \ell(x_0, a_1) - D_{\overline{\Omega}_1}\phi(x_0) \cdot f_1(x_0, a_1) \leq 0,$$

which concludes the proof of this part.

Finally, the arguments for viscosity tests in Ω_2 are the same as in Ω_1. □

For the converse implication some preliminary material is needed. We derive a first invariance result for the hypograph of u on Γ through the Filippov Approximation Theorem and the local Lipschitz–continuous character of G_Γ.

Proposition 4.3.24. Let u be an usc subsolution to (4.3.1)-(4.3.6), then for any $x_0 \in \Gamma$, $\xi \in \mathbf{R}$, any $(y(\cdot), \eta(\cdot))$ driven by G_Γ with $(y(0), \eta(0)) = (x_0, \xi) \in \mathcal{H}p(u)$, we have $(y(t), \eta(t)) \in \mathcal{H}p(u)$ for $t \geq 0$.

Proof. In view of Corollary 4.3.13, we have just to check that G_Γ satisfies the strong tangential condition on $\mathcal{H}p(u) \cap \big(\Gamma \times \mathbf{R}\big)$. Being the interior of such set empty, this condition must be satisfied at any of its points. If $(x_0, \xi_0) \in (\mathrm{int}\, \mathcal{H}p(u)) \cap \big(\Gamma \times \mathbf{R}\big)$ then any nonzero normal vector at it has the form $(p, 0)$ with p normal to Γ at x_0, then the strong tangential condition comes from the fact that $F_\Gamma(x_0) \subset \mathcal{T}_\Gamma(x_0)$.

If instead $(x_0, \xi_0) \in (\partial \mathcal{H}p(u)) \cap \big(\Gamma \times \mathbf{R}\big)$ then $\xi_0 = u(x_0)$ since u is continuous in Γ. We consider (p, s) as a normal vector to $\mathcal{H}p(u) \cap \big(\Gamma \times \mathbf{R}\big)$ at $(x_0, u(x_0))$ and pick $\varepsilon > 0$ such that

$$(x_0 + \varepsilon\, p, u(x_0) + \varepsilon\, s) \text{ has } (x_0, u(x_0)) \text{ as unique projection on } \mathcal{H}p(u) \cap \big(\Gamma \times \mathbf{R}\big). \tag{4.3.51}$$

The argument can be divided according to whether s is vanishing or strictly positive. In the first instance, we reach the sought conclusion arguing as in the first step provided that p is normal to Γ at x_0. We show by contradiction that $s = 0$ and p not normal is impossible because of the Lipschitz–continuity of u on Γ. Take $q \in \mathcal{T}_\Gamma(x_0)$ with $c := p \cdot q > 0$, and consider a regular curve y defined in some small interval $[0, T]$ and lying on Γ with $y(0) = x_0$ and $\dot{y}(0) = q$.

On the other hand, denote by L_u a Lipschitz constant for u in a bounded subset of Γ containing the support of y. We have for t small enough

$$\begin{aligned} & |y(t) - (x_0 + \varepsilon\, p)|^2 + |u(y(t) - u(x_0)|^2 \\ \leq\; & (1 + L_u^2)\, |y(t) - x_0|^2 - 2\,\varepsilon\, (y(t) - x_0) \cdot p + \varepsilon^2\, |p|^2 \leq o(t) - c\,\varepsilon\, t + \varepsilon^2\, |p|^2, \end{aligned}$$

in contrast with (4.3.51), recall that $s = 0$. The case $s > 0$ is left, we can assume $s = 1$. The ball of $\mathbf{R}^d \times \mathbf{R}$ centered at $(x_0 + \varepsilon\, p, u(x_0) + \varepsilon)$ and with radius $\varepsilon\, \sqrt{|p|^2 + 1}$ is locally at $(x_0, u(x_0))$ the graph of a smooth function, say ϕ, with $-D\phi(x_0) = p$, which is viscosity test function from above to u at x_0 with Γ as constraint. This implies, being u subsolution to (4.3.1)-(4.3.6)

$$(p, 1) \cdot (f(x, a), \lambda w(x_0) - \ell(x_0, a)) = \lambda w(x_0) - D\phi(x_0) \cdot f(x, a) - \ell(x_0, a)) \leq 0$$

for any $a \in A_\Gamma(x_0)$, concluding the proof. □

Next result is about an invariance property for G outside Γ. For this we essentially exploit the continuity condition of u on the interface. This is actually the unique point where such a condition enters into play.

Proposition 4.3.25. Let u, (y, η) be an usc subsolution to (4.3.1)-(4.3.6), which is, in addition, continuous at any point of Γ, and an integral curve of G defined in an interval $[a, b]$, respectively. Assume that $(y(a), \eta(a)) \in \mathcal{H}p(u)$, and $y(t) \notin \Gamma$ for $t \in (a, b)$.

Then, $(y(t), \eta(t)) \in \mathcal{H}p(u)$ for $t \in [a, b]$.

Proof. Given $\rho > 0$, consider a Lipschitz–continuous cutoff function $\phi_\rho : [0, +\infty) \to [0, 1]$ with $\phi_\rho(0) = 0$ for $s \in [0, \frac{\rho}{2}]$, $\phi_\rho(s) = 1$ for $s \geq \rho$ and define

$$G_\rho(x, \xi) = \phi_\rho(|g(x)|)\, G(x, \xi) \text{ for any } (x, \xi) \in \mathbf{R}^d \times \mathbf{R}.$$

The multifunction G_ρ is locally Lipschitz–continuous in the whole $\mathbf{R}^d \times \mathbf{R}$ and reduces to $\{0\}$ in a suitable neighborhood of $\Gamma \times \mathbf{R}$.

We claim that $\mathcal{H}p(u)$ is strongly invariant for G_ρ. It is enough to check strong tangential condition for $\mathcal{H}p(u)$ with respect to G_ρ, and, in addition to check it for $(x_0, \xi_0) \in \partial\mathcal{H}p(u)$ with x_0 outside Γ or even far enough from it, where the images of G are different from $\{0\}$. We then consider $(x_0, \xi_0) \in \partial\mathcal{H}p(u)$ and $x_0 \in \Omega_i$, $i = 1$ or 2, with (p, s) being a normal vector to $\mathcal{H}p(u)$ at it. The argument is well known, we sketch it for reader's convenience. If $\xi_0 = u(x_0)$ and $s > 0$, so that we can assume $s = 1$, then we find a smooth viscosity test function from above ϕ to u at x_0 with $D\phi(x_0) = -p$. Given $(f_1(x_0, a), \lambda\, u(x_0) - \ell_1(x_0, a))$, we exploit that u is subsolution of (4.3.1)-(4.3.6) to get

$$\begin{aligned} & (p, 1) \cdot \phi_\rho(g(x_0))\, (f(x, a), \lambda w(x_0) - \ell(x_0, a)) \\ = \;& \phi_\rho(g(x_0))\big(\lambda w(x_0) - D\phi(x_0) \cdot f_1(x, a) - \ell_1(x_0, a))\big) \leq 0. \end{aligned}$$

In the case where $s = 0$ or $\xi_0 > u(x_0)$, Proposition can be used to get similar estimate. The claim is in the end proved.

Now consider a curve y as in the statement, with $y((a, b)) \subset \Omega_i$. If $y(a) \notin \Gamma$ then

$$(y, \eta)([a, b - \varepsilon]) \cap \Gamma = \emptyset \quad \text{for any } \varepsilon > 0$$

then $\min\{g(y(t)) \mid t \in [a, b - \varepsilon]\} = \rho$, for some $\rho = \rho(\varepsilon) > 0$, so that (y, η) is a trajectory of G_ρ and by the first part of the proof,

$$(y(t), \eta(t)) \in \mathcal{H}p(u) \quad \text{for } t \in [a, b - \varepsilon]\ .$$

Taking into account that $\mathcal{H}p(u)$ is closed, we get the assertion sending ε to 0. If, on the contrary, $y(a) \in \Gamma$, we exploit that u is continuous at $y(a)$ and $(y(a), \eta(a)) \in \mathcal{H}p(u)$ to find for any $\varepsilon > 0$ small a $\delta_\varepsilon > 0$ satisfying

$$(y(a + \varepsilon), \eta(a + \varepsilon) - \delta_\varepsilon) \in \mathcal{H}p(u), \tag{4.3.52}$$

and $\delta_\varepsilon \to 0$ as ε goes to 0. Being the support of (y, η), for $t \in [a + \varepsilon, b - \varepsilon]$, compact and disjoint from Γ, we can argue as above to deduce from (4.3.52)

$$(y(t), \eta(t) - \delta_\varepsilon\, e^{\lambda\,(t - a - \varepsilon)}) \in \mathcal{H}p(u) \quad \text{for } t \in [a + \varepsilon, b - \varepsilon]\ .$$

Then the assertion is obtained passing at the limit for $\varepsilon \to 0$. □

In the forthcoming proof it is couched the crucial induction argument on the index $\jmath_\varepsilon$, see the notion of ε–partition introduced in Appendix 4.4. It will be also employed, with suitable adaptations, to prove the results on superoptimality in the next section.

Theorem 4.3.26. Assume **(H1)-(H4)**. Let u be a bounded usc subsolution to (4.3.1)-(4.3.6), which is, in addition, continuous at any point of Γ, then for any (y, η) driven by G with $(y(0), \eta(0)) \in \mathcal{H}p(u)$,

$$(y(t), \eta(t)) \in \mathcal{H}p(u), \ \forall, t \geq 0.$$

Proof. We consider a trajectory (y, η) of G with $(y(0), \eta(0)) \in \mathcal{H}p(u)$ in the interval $[0, T]$, for $T > 0$. We select a compact set $K_0 \subset \mathbf{R}^d$ containing in its interior the reachable set (see (2.2.2) for the definition)

$$\mathcal{R}_{F_\Gamma}(y([0, T]) \cap \Gamma, T).$$

We introduce some constants that will appear in the forthcoming estimates.

- M_u, L_u is an upper bound for $|u|$ in $\mathbf{R}^d$ and a Lipschitz constant for u in $(K_0 \cap \Gamma) \times \mathbf{R}$, respectively, see Proposition 4.3.21.
- P estimates from above the diameter of $G(x, \xi)$ for $(x, \xi) \in \mathcal{R}_G(y[0, T] \times (\eta([0, T]) \cup [-M_u, M_u])), T)$.
- L_G is a Lipschitz constant for G_Γ (suitably extended outside the interface, see (4.3.32)) in $(K_0 \cap \Gamma_\natural) \times \mathbf{R}$.

The argument will be broken down into slices depending on a positive integer index and prove the result by induction. Consider ε small enough so that any integral curve of F defined in some compact interval and with support contained in K_0 and any ε–partition related to it satisfy the weak separation principle stated in Proposition 4.4.3. Consider the statement of the sequence of properties that will be proved by induction:

$(\mathbf{P_k})$ For any interval $[a, b] \subset [0, T]$ such that $\jmath_\varepsilon(y; a, b) \leq k$, one has

$$\eta(b) - \exp(\lambda\,(b-a))\,[\eta(a) - u(y(a))]^+ - (1 + L_u)\,\exp((L_G + \lambda)\,(b-a))\,P\,\varepsilon \leq u(y(b)),$$

where $[\cdot]^+$ stands for the positive part.

We first show $(\mathbf{P_2})$. Fix $[a, b] \subset [0, T]$ with $\jmath_\varepsilon(y; a, b) = 2$, and modify the component $\eta(t)$ of our curve in $[a, b]$ setting

$$\zeta(t)) := \eta(t) - [\eta(a) - u(y(a))]^+\, e^{\lambda\,(t-a)}. \tag{4.3.53}$$

(y, ζ) is still a trajectory of G in $[a, b]$, but now the initial datum at $t = a$ satisfies

$$\zeta(a) = \eta(a) - [\eta(a) - u(y(a))]^+ \in \mathcal{H}p(u). \tag{4.3.54}$$

Since $\zeta(a)$ is either equal to $\eta(a)$ or to $u(y(a))$, then

$$y([a,b]) \times \zeta([a,b]) \subset \mathcal{R}_G(y[0,T] \times (\eta([0,T]) \cup [-M_u, M_u]), T). \tag{4.3.55}$$

We divide the proof according on whether $y((a,b)) \cap \Gamma$ is empty or not. In the first instance by Proposition 4.3.25, and (4.3.54) the modified curve is contained in $\mathcal{H}p(u)$, and so $\zeta(b) = \eta(b) - e^{\lambda\,(b-a)}\,[\eta(a) - u(y(a))]^+ \leq u(y(b))$, which implies the claimed inequality. In the second case $y(a)$ and $y(b)$ belong to the interface and

$$|\{t \in [a,b] \mid y(t) \notin \Gamma\}| < \varepsilon, \tag{4.3.56}$$

in addition

$$(\dot{y}(t), \dot{\zeta}(t)) \in G_\Gamma(y(t), \zeta(t)) \qquad \text{for a.e. } t \in (a,b) \setminus J, \tag{4.3.57}$$

where J the time set appearing in (4.3.56). On the other side, bearing in mind (4.3.55) and that (y, ζ) is an integral curve of G, we deduce from the very definition of P

$$d((\dot{y}(t), \dot{\zeta}(t)), G_\Gamma(y(t), \zeta(t))) < P \qquad \text{for a.e. } t \in J. \tag{4.3.58}$$

Combining (4.3.56), (4.3.57), (4.3.58), we finally obtain

$$\int_a^b d((\dot{y}(s), \dot{\zeta}(s)), G_\Gamma((y(s)), \zeta(s)))\, ds \leq \varepsilon\, P. \tag{4.3.59}$$

By the assumption on ε, $y([a,b])$ is contained in $\Gamma_\natural$. We can then apply Theorem 2.2.7 with $\mathcal{C} = \Gamma \times \mathbf{R}$, $\mathcal{C}_\natural = \Gamma_\natural \times \mathbf{R}$, and the multifunction $Z = G_\Gamma$, taking into account that L_G is a Lipschitz constant of G_Γ in $(K_0 \cap \Gamma_\natural) \times \mathbf{R}$, and this set clearly contains a bounded open neighborhood of $\mathcal{R}_{G_\Gamma}((y(a), \zeta_0(a)), b-a))$, as prescribed in that theorem. We get the existence of an integral curve (z_0, ζ_0) of G_Γ, defined in $[a,b]$, with $(z_0(a), \zeta_0(a)) = (y(a), \zeta(a))$, satisfying by (4.3.59)

$$|z_0(b) - y(b)| \leq \exp\big(L_G\,(b-a)\big)\,\varepsilon\, P \tag{4.3.60}$$

$$|\zeta_0(b) - \zeta(b)| \leq \exp\big(L_G\,(b-a)\big)\,\varepsilon\, P. \tag{4.3.61}$$

Since $(z_0(a), \zeta_0(a)) \in \Gamma \times \mathbf{R}$ then by Proposition 4.3.24 and (4.3.54)

$$\zeta_0(b) \leq u(z_0(b)). \tag{4.3.62}$$

By Lipschitz–continuity on Γ_j of subsolution u, we derive from (4.3.60)

$$u(y(b)) + L_u \exp\big(L_G\,(b-a)\big)\,\varepsilon\, P \geq u(z_0(b)), \tag{4.3.63}$$

and taking also into account (4.3.61), (4.3.62), we get

$$\zeta(b) - \exp\big(L_G\,(b-a)\big)\,\varepsilon\, P \leq u(y(b)) + L_u \exp\big(L_G\,(b-a)\big)\,\varepsilon\, P.$$

Recalling the definition of $\zeta(t)$ given in (4.3.53) we further obtain

$$\eta(b) - e^{\lambda\,(b-a)}\,[\eta(a) - u(y(a))]^+ - (1 + L_u)\,\exp\big(L_G\,(b-a)\big)\,\varepsilon\,P \le u(y(b)),$$

and, replacing L_G in the second exponential by $L_G + \lambda$, which is larger, we reach the sought inequality, ending the proof of $(\mathbf{P_2})$.

Given $k \in \mathbf{N}$, $k \ge 2$, we now assume $(\mathbf{P_2}), \cdots, (\mathbf{P_k})$ to hold and prove $(\mathbf{P_{k+1}})$. Taking $\jmath(y; a, b) = k + 1$, we denote by $\{t_1 = a, \cdots, t_{k+1} = b\}$ a minimal ε–partition of $[a, b]$ related to y, by Proposition 4.4.5 there are two positive constant ε_1, ε_2 with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ satisfying $\jmath_{\varepsilon_1}(y; a, t_k) = k$ and $\jmath_{\varepsilon_2}(y; t_k, b) = 2$. By inductive step

$$\begin{aligned} u(y(t_k)) &\ge \eta(t_k) - e^{\lambda\,(t_k-a)}\,[\eta(a) - u(y(a))]^+ - (1 + L_u)\,e^{(L_G+\lambda)\,(t_k-a)}\,P\,\varepsilon_1 \\ u(y(b)) &\ge \eta(b) - e^{\lambda\,(b-t_k)}\,[\eta(t_k) - u(y(t_k))]^+ - (1 + L_u)\,e^{(L_G+\lambda)\,(b-t_k)}\,P\,\varepsilon_2. \end{aligned}$$

Replacing in the second inequality of above the estimate of $[\eta(t_k) - u(y(t_k))]^+$ provided in the first one, we get

$$\begin{aligned} u(y(b)) &\ge \eta(b) - e^{\lambda\,(b-t_k)}\,\big(e^{\lambda\,(t_k-a)}\,[\eta(a) - u(y(a))]^+ \\ &\quad -(1 + L_u)\,e^{(L_G+\lambda)\,(t_k-a)}\,P\,\varepsilon_1\big) - (1 + L_u)\,e^{(L_G+\lambda)\,(b-t_k)}\,P\,\varepsilon_2 \\ &\ge \eta(b) - e^{\lambda\,(b-a)}\,[\eta(a) - u(y(a))]^+ - (1 + L_u)\,e^{(L_G+\lambda)\,(b-a)}\,P\,(\varepsilon_1 + \varepsilon_2). \end{aligned}$$

This finishes the proof by induction. We apply the property so far established to (y, η) in the whole of $[0, T]$. Taking into account that $[\eta(0) - u(y(0)]^+ = 0$ by assumption, that ε can be arbitrarily small and the error in $(\mathbf{P_k})$ goes to 0 as $\varepsilon \to 0$, we deduce $(y(T), \eta(T)) \in \mathcal{H}p(u)$. This completes the argument, being T arbitrary. □

Proof. (*Proof of Theorem 4.3.20*) By putting together Theorems 4.3.22 and 4.3.26 we get the equivalence between (i) and (ii).

(iii) $\Rightarrow$ (ii) holds true since $H_E(\cdot,\cdot) \ge H_\Gamma(\cdot,\cdot)$.

(i) $\Rightarrow$ (iii) is obtained by Theorem 4.3.23. □

4.3.7 Proof of the main results

Proof. (*of Theorem 4.3.5*) let w, u, x_0 be a a bounded lower semicontinuous supersolution, a bounded upper semicontinuous subsolution continuous at any point of Γ, and a point of $\mathbf{R}^d$, respectively. We take a common upper bound M_0 for $|w|$, $|u|$, $|v|$ in $\mathbf{R}^d$. We aim at proving

$$w(x_0) \ge v(x_0) \ge u(x_0), \tag{4.3.64}$$

which gives the assertion with arbitrary x_0 in $\mathbf{R}^d$. Fix $\varepsilon > 0$ and thereafter δ, T_0 with

$$2\, e^{-\lambda\, T_0}\, M_0 + \delta < \varepsilon. \tag{4.3.65}$$

We recall that v satisfies the dynamical programming principle, and invoke Theorem 4.3.14 for w, to get for a suitable pair (y, α) admissible with $y(0) = x_0$ and $T > T_0$

$$v(x_0) \leq \int_0^T \ell(y, \alpha)\, ds + e^{-\lambda\, T}\, v(y(T)), \;\; w(x_0) \geq \int_0^T \ell(y, \alpha)\, ds - e^{-\lambda\, T}\, M_0 - \delta.$$

We deduce $w(x_0) \geq v(x_0) - 2\, e^{-\lambda\, T}\, M_0 - \delta$, and taking into account (4.3.65)

$$w(x_0) \geq v(x_0) - \varepsilon. \tag{4.3.66}$$

Similarly, we invoke Theorem 4.3.20 for u and again dynamical programming principle for v to get for a suitable pair (y, α) admissible with $y(0) = x_0$

$$v(x_0) \geq \int_0^{T_0} \ell(y, \alpha)\, ds + e^{-\lambda\, T}\, v(y(T)) - \delta, \;\; u(x_0) \leq \int_0^{T_0} \ell(y, \alpha)\, ds + e^{-\lambda\, T}\, u(y(T)).$$

We deduce $v(x_0) \geq u(x_0) - 2\, e^{-\lambda\, T_0}\, M_0 - \delta$, and taking into account (4.3.65)

$$v(x_0) \geq u(x_0) - \varepsilon. \tag{4.3.67}$$

Relations (4.3.66) and (4.3.67) imply (4.3.64) since ε is arbitrary. □

Proof. (*of Theorem 4.3.7*) Since the value function satisfies both the super-optimality principle and the sub-optimality principe, it is the unique solution which is a direct consequence of Theorem 4.3.20, 4.3.14 and 4.3.5. □

4.4 ε–partitions

Given a curve y defined in some compact interval $[a, b]$, we define the *event set* as

$$E_y = \partial\, \{t \in [a, b] \mid y(t) \in \Omega_1 \cup \Omega_2\};$$

this terminology, we have adapted from hybrid control theory, reflects the fact that at such times something memorable happens, namely the possible passage from one basic phase of the life of the curve to another, theses are the times when y lies in one of the open sets Ω_1, Ω_2, or it is sliding along the interface.

In the special case where E_y is made of isolated points, and so it is finite being the interval of definition compact, then such phases follow one another in a well ordered and separated way, there is in fact a finite partition of $[a, b]$ with points of E_y such that in the interior of any interval the curve is in Ω_1 or Ω_2 or Γ.

This nice frame could be messed up in presence of accumulation points of E_y. Around these times the curve may wildly oscillates among the regions of partition. However, we point out in this section that for any $\varepsilon > 0$ a partition of $[a, b]$ keeping some separation property among different phases can be defined also if the Zeno set is nonempty, up to time sets of 1–dimensional measure less than ε. These are the ε–partitions mentioned in the title of the section. We adopt the following terminology:

A *partition* of $[a, b]$ is any finite strictly increasing sequence of times $\{t_1, \cdots, t_k\}$ with $t_1 = a$, $t_k = b$.

An *interval of the partition* is any interval with two subsequent elements of the partition as endpoints.

Definition 4.4.1. **(ε–partition)** Given $\varepsilon > 0$, and a curve y defined in $[a, b]$, a partition of $[a, b]$ will be called ε– *partition related to* y provided the following conditions hold:

(i) All points of it, except possibly a and b, belong to E_y.

(ii) Given the (possibly empty) family

$$\mathcal{I} = \{\text{open intervals } I \text{ of the partition with } y(I) \cap \Gamma \neq \emptyset\} \tag{4.4.1}$$

then all the endpoints of intervals in $\mathcal{I}$ belong to E_y.

(iii) $\sum_{I \in \mathcal{I}} |I \setminus \{t \in [a, b] \mid y(t) \in \Gamma\}| < \varepsilon$.

Notice that item (ii) of the previous definition is about the status of endpoints a and b. It is equivalent of requiring

$$y(t) \notin \Gamma \quad \text{for } t \in (a, t_2) \text{ whenever } y(a) \notin \Gamma$$

and same property, *mutatis mutandis*, for b.

Proposition 4.4.2. Given a curve y in $\mathbf{R}^d$ defined in some compact interval $[a, b]$ and $\varepsilon > 0$, there exists an ε–partition of $[a, b]$ related to y.

Proof. We set

$$J = \{t \in (a, b) \mid y(t) \in \Omega_1 \cup \Omega_2\}. \tag{4.4.2}$$

If $J = (a, b)$ or $J = \emptyset$, then we simply take the partition $\{a, b\}$ to prove the assertion. In the other cases, J being open is the disjoint union of a countable family of open intervals. Being its measure finite we can find a finite subfamily $\{J'_1, \cdots, J'_h\}$ for some $h \in \mathbf{N}$ with

$$\left| \bigcup_{l=1}^{h} J'_l \right| = \sum_{l=1}^{h} |J'_l| > |J| - \varepsilon. \tag{4.4.3}$$

We set for $l = 1 \dots h$

$$a_l = \begin{cases} \max\{t \le \inf J'_l \mid y(t) \in \Gamma\} & \text{if the set under the max is nonempty} \\ a & \text{otherwise} \end{cases}$$

and

$$b_l = \begin{cases} \min\{t \ge \sup J'_l \mid y(t) \in \Gamma\} & \text{if the set under the min is nonempty} \\ b & \text{otherwise} \end{cases}$$

We define new open intervals by $J_l = (a_l, b_l)$ for $l = 1, \cdots, h$. We further set

$$\begin{aligned} J_{00} &= \big(a, \min\{t \in [a,b] \mid y(t) \in \Gamma\}\big) \quad (J_{00} = \emptyset \text{ if } y(a) \in \Gamma) \\ J_0 &= \big(\max\{t \in [a,b] \mid y(t) \in \Gamma\}, b\big) \quad (J_0 = \emptyset \text{ if } y(b) \in \Gamma) \end{aligned}$$

By construction $\bigcup_{l=1}^h J'_l \subset \bigcup_{l=1}^h J_l \subset J$, therefore by (4.4.3)

$$\left| \bigcup_{l=1}^h J_l \cup J_{00} \cup J_0 \right| > |J| - \varepsilon. \tag{4.4.4}$$

Consider the family of enlarged intervals plus J_{00}, J_0. We claim that two of such intervals either coincide or are disjoint. Take first J_m, J_n for some $1 \le m \ne n \le h$, assume, to fix our ideas

$$\sup J'_n \le \inf J'_m \tag{4.4.5}$$

(recall that $J'_m \cap J'_n = \emptyset$), if $J_m \cap J_n \ne \emptyset$ then $b_n > a_m$ but this implies, by the very definition of a_m and taking into account that $y(J_m) \cap \Gamma = \emptyset$, that $b_n \ge \sup J'_m$ which in turn gives $b_n \ge b_m$; being the opposite inequality direct consequence of (4.4.5), we finally get $b_n = b_m$. Arguing similarly we also prove equality of right endpoints, under the assumption of nonempty intersection, and show the claim for J_m, J_n.

Now, assume $J_{00} \ne \emptyset$ and take any $m \in \{1, \cdots, h\}$, if $a_m > a$, then $a_m \ge \min\{t \in [a,b] \mid y(t) \in \Gamma\}$, and the quantity in the right hand–side is the right endpoint of J_{00}. This shows $J_{00} \cap J_m = \emptyset$. If, on the contrary, $a_m = a$, then since $J_m \subset J$ then $b_m = \min\{t \in [a,b] \mid y(t) \in \Gamma\}$, which shows $J_{00} = J_m$.

Similarly, if $J_0 \ne \emptyset$ one proves that either it coincides with J_l, for some $l = 1, \cdots, h$ or it is disjoint with any of them. Finally, J_0, J_{00} are disjoint by their very definition. The claim is then fully proved.

Therefore, up to removing copies, and possibly empty intervals, and reindexing, we end up with a family $\{J_1, \cdots, J_k\}$, for some $k \in \mathbf{N}$, of disjoint open intervals all contained in J , satisfying by (4.4.4)

$$\left| \bigcup_{l=1}^k J_l \right| = \sum_{l=1}^h |J_l| > |J| - \varepsilon. \tag{4.4.6}$$

and enjoying conditions (i), (ii) of the definition of ε–partition, which actually justifies the previous construction. Consider the partition given by all their endpoints, suitably indexed, plus a and b, and take $\mathcal{I}$ as defined in (4.4.1). If $I \in \mathcal{I}$ then $I \cap \cup_l J_l = \emptyset$ and so

$$\left(\bigcup_{I \in \mathcal{I}} I \right) \cap J \subset J \setminus \left(\bigcup_{l=1}^{k} J_l \right).$$

From this we derive

$$\sum_{I \in \mathcal{I}} |I \cap J| = \left| \left(\bigcup_{I \in \mathcal{I}} I \right) \cap J \right| \leq \left| J \setminus \bigcup_{l=1}^{k} J_l \right| < \varepsilon$$

which gives the assertion. □

We proceed deducing that when ε is small with the respect to the velocity of the curve in object, then a sort of *weak separation principle* holds for any ε–partition. We emphasize that the size of such an ε does not depend on the length of intervals but just on velocities. In next proposition we state this property just for integral trajectories of F, since these are the curves we are interested on.

Proposition 4.4.3. Given a compact subset K_0 of $\mathbf{R}^d$, there is $\varepsilon_0 > 0$ such that for any integral curve y of F defined in some compact interval $[a, b]$, with $y([a, b]) \subset K_0$ one has: If I is a closed interval of an ε–partition of $[a, b]$ related to y with $\varepsilon < \varepsilon_0$, then the two (mutually non–exclusive) possibilities hold

$$\text{either} \quad y(I) \subset \Omega_i, \ i = 1, 2, \quad \text{or} \quad y(I) \subset \Gamma_\natural.$$

Proof. We denote by M_0 a constant estimating from above $|f|$ in $\cup_i((K_0 \cap \Omega_i) \times A_i) \cup ((K_0 \cap \Gamma) \times A)$.

If $y(I) \cap \Gamma = \emptyset$ then $y(I) \subset \Omega_i$ for a suitable choice of i. If instead $y(I) \cap \Gamma \neq \emptyset$, we take $t_0 \in I$, and consider a time neighborhood I_0 of t_0 of radius ε and so of measure $2\,\varepsilon$. if I_0 contains a endpoint of I then $y(I_0) \cap \Gamma \neq \emptyset$ by item (ii) in the definition of ε–partition, same conclusion is reached in force of item (iii), if instead $I_0 \subset I$. Summing up: there is $t_1 \in [a, b]$ with $y(t_1) \in \Gamma$, $|t_1 - t_0| < \varepsilon$, therefore

$$|y(t_1) - y(t_0)| \leq M_0\, |t_1 - t_0| < M_0\, \varepsilon. \tag{4.4.7}$$

Being $\Gamma_\natural$ open and K_0 compact there is $\delta > 0$ with $(\Gamma \cap K_0) + B(0, \delta) \subset \Gamma_\natural$. Taking into account (4.4.7) and that the support of y is contained in K_0, it is enough, for proving the assertion, to take $\varepsilon_0 < \frac{\delta}{M}$.

□

We attach to any curve defined in a compact interval a natural number, namely the smallest cardinality of an ε–partition related to the curve. Loosely speaking, its size captures, when ε varies, how complicated is the behavior of the curve around the interface. Results in Sections 4.3.5 and

4.3.6, on which, in turn, the main comparison theorem is based, are obtained by means of an inductive argument on this index.

Definition 4.4.4. **(minimal ε–partition)** We say that an ε–partition is *minimal* if there are no ε–partitions of $[a,b]$ for y with less elements. We denote the cardinality of any such ε–minimal partition by $\jmath_\varepsilon(y;a,b)$.

We point out for later use a sort of additive property of the index $\jmath_\varepsilon$.

Proposition 4.4.5. Given $\varepsilon > 0$, consider an ε–minimal partition $\{t_1 = a, t_2, \cdots t_k = b\}$ with $k = \jmath_\varepsilon(y;a,b) > 2$. For any $1 < h < k$, there exist two positive constants ε_1, ε_2 with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ such that

$$\jmath_{\varepsilon_1}(y;a,t_h) = h \quad \text{and} \quad \jmath_{\varepsilon_2}(y;t_h,b) = k - h + 1.$$

Proof. Basically there is nothing to prove, we just exploit the very definition of ε–minimal partition and additivity of measure. We define $\mathcal{I}$ as in (4.4.1) and set

$$\mathcal{I}_1 = \{I \in \mathcal{I} \mid I \subset [a,t_h]\},\ \mathcal{I}_2 = \{I \in \mathcal{I} \mid I \subset [t_h,b]\},$$

clearly

$$\sum_{I\in\mathcal{I}_1} |I \setminus \{t \in [a,b] \mid y(t) \in \Gamma\}| + \sum_{I\in\mathcal{I}_2} |I \setminus \{t \in [a,b] \mid y(t) \in \Gamma\}| < \varepsilon,$$

and we can thus find ε_1, ε_2 with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ such that

$$\sum_{I\in\mathcal{I}_i} |I \setminus \{t \in [a,b] \mid y(t) \in \Gamma\}| < \varepsilon_i \quad \text{for } i = 1,2.$$

This shows that $\{t_1 \cdots, t_h\}$ is an ε_1–partition for y in $[a,t_k]$ and $\{t_h \cdots, t_k\}$ an ε_2–partition in $[t_h,b]$. We claim that both these partitions are minimal. In fact, if there were an ε_1–minimal partition of $[a,t_k]$ with less than h elements that the union of it with $\{t_{h+1}, \cdots, t_k\}$ should yield an $(\varepsilon_1 + \varepsilon_2 = \varepsilon)$–partition of the whole of $[a,b]$ with less then k element, which is contrast with $\jmath(y;a,b) = k$. Same conclusion is reached denying $\jmath_{\varepsilon_2}(y;t_h,b) = k - h + 1$. This proves the claim, which, in turn, immediately implies the assertion. □

4.5 Perspective: numerical approaches for HJB equations on multi-domains

We complete the study on HJB equations on multi-domains by investigating the numerical approaches. To simplify, consider the multi-domains where the whole space $\mathbf{R}^d$ is separated by a hyperplane Γ into two disjoint open subsets Ω_1 and Ω_2:

$$\Gamma = \{0\} \times \mathbf{R}^{d-1},\ \Omega_1 = (-\infty,0) \times \mathbf{R}^{d-1},\ \Omega_2 = (0,+\infty) \times \mathbf{R}^{d-1}.$$

For $i = 1, 2$, let $F_i : \mathbf{R}^d \rightsquigarrow \mathbf{R}^d$ be multifunctions satisfying the following:

- F_i is L-Lipschitz continuous with nonempty, compact and convex images. There exists $M > 0$, $\delta > 0$ such that for any $x \in \mathbf{R}^d$, $p \in F_i(x)$, we have $\|p\| \leq M$, $\overline{B(0,\delta)} \subseteq F_i(x)$.

Given $T > 0$, consider the HJB system defined by

$$\begin{cases} \partial_t u(t,x) + H_i(x, Du(t,x)) = 0, & \text{for } t \in (0,T), x \in \Omega_i,\ i = 1,2, \\ u(0,x) = \varphi(x), & \text{for } x \in \mathbf{R}^d, \end{cases} \tag{4.5.1}$$

where φ is Lipschitz continuous and $H_i(x,p) := \sup_{q \in F_i(x)}\{-p \cdot q\}$.

For given mesh sizes $\Delta t > 0$, $\Delta x > 0$, we define

$$\mathcal{G} := \{I\Delta x,\ I \in \mathbf{Z}^d\}.$$

Let N_T be the integer part of $T/\Delta t$. The discrete running point is (t^n, x_I) with $t_n = n\Delta t$, $x_I = I\Delta x$. The approximation of the solution u at the node (t^n, x_I) is written as U_I^n.

In general, a numerical scheme for this equation is given by

$$S(t_n, x_I, U_I^{n+1}, U^n) = 0,\ \forall\, n = 0, \ldots, N_T - 1,\ I \in \mathbf{Z}^d;\ U_I^0 = \varphi(x_I),\ \forall\, I \in \mathbf{Z}^d. \tag{4.5.2}$$

Assume the following on $S : (0,T) \times \mathbf{R}^d \times \mathbf{R} \times L^\infty(\mathbf{R}^d)$.

(i) **Monotonicity.** $S(t,x,r,u) \leq S(t,x,r,w)$ if $u \geq w$.

(ii) **Stability.** If u^Δ is a solution of (4.5.2), then u^Δ is bounded uniformly on Δt, Δx.

(iii) **Consistency.** There exists $K > 0$ such that for any $\phi \in C^{n,1}((0,T) \times \mathbf{R}^d)$, $t \in (0,T)$, $x \in \mathbf{R}^d$, we have

$$|\partial_t \phi(t,x) + H^E(x, D\phi(t,x)) - S(t,x,\phi(t+\Delta t, x), \phi(t,\cdot))| \leq K\|\phi\|_{n,1}(\Delta t + \Delta x).$$

where $C^{n,1}(\mathbf{R}^d)$ is denoted as the space of n times continuously differentiable functions $u : \mathbf{R}^d \to \mathbf{R}$ with the finite norm

$$\|u\|_{n,1} = \sum_{i=0}^{n} \sup_{x \in \mathbf{R}^d} |D^i u| + \sup_{x,y \in \mathbf{R}^d, x \neq y} \frac{|D^n u(x) - D^n u(y)|}{|x-y|}.$$

In [28], it is proved that the numerical solution U^Δ of any schemes satisfying (i)-(iii) converges to the continuous viscosity solution of HJB equations with Lipschitz continuous Hamiltonians. The main idea to prove the infimum limit of U^Δ is a lsc supersolution and the superum limit of U^Δ is a usc subsolution. Then by the comparison principle for lsc supersolutions and usc subsolutions, the convergence result is deduced. However, in our case, the comparison principle is only established for lsc supersolutions and Lipschitz continuous subsolutions! Then the significant difficulty is to prove the Lipschitz continuity of the numerical solution U^Δ in the framework of discontinuous Hamiltonian.

To solve the classical HJB equations with Lipschitz continuous Hamiltonians, one can find a rich literature on the numerical approaches. We would like to refer to [28] for a general convergence result and [32] for error estimate theory. The fundamental schemes for HJB equations include finite difference schemes and semi-Lagrangian schemes. To develop the numerical schemes for (4.5.3), the difficulty remains the discontinuity of H^E. In the following, we will discuss the extensions of classical schemes in the case of multi-domains, mainly finite different schemes and semi-Lagrangian schemes.

4.5.1 Finite difference schemes

Based on the study on the transmission conditions, the transmission HJB equations with the essential Hamiltonian H^E takes our attention. Then the HJB system defined in $\mathbf{R}^d$ which is going to be discretized will be

$$\begin{cases} \partial_t u(t,x) + H^E(x, Du(t,x)) = 0, & \text{for } t \in (0,T), x \in \mathbf{R}^d, \\ u(0,x) = \varphi(x), & \text{for } x \in \mathbf{R}^d, \end{cases} \tag{4.5.3}$$

Let us recall the definition of H^E step by step as follows. For $\mathcal{M} \in \{\Omega_1, \Omega_2, \Gamma\}$, we set

$$F_{\mathcal{M}} = \begin{cases} F_i & \text{for } \mathcal{M} = \Omega_i, \ i = 1,2, \\ co(F_1, F_2) \cap \mathcal{T}_\Gamma & \text{for } \mathcal{M} = \Gamma. \end{cases}$$

The *essential dynamics* $F^E : \mathbf{R}^d \rightsquigarrow \mathbf{R}^2$ is defined as follows:

$$F^E(x) = \bigcup_{x \in \overline{\mathcal{M}}, \ \mathcal{M} \in \{\Omega_1, \Omega_2, \Gamma\}} \{F_{\mathcal{M}}(x) \cap \mathcal{T}_{\overline{\mathcal{M}}}(x)\}.$$

Then the *essential Hamiltonian* $H^E : \mathbf{R}^d \times \mathbf{R}^d$ is defined by

$$H^E(x,p) = \sup_{q \in F^E(x)} \{-p \cdot q\}.$$

An example of scheme of finite difference type fulfilling the assumptions **(i)-(iii)** is the following

$$S(t_n, x_I, U_I^{n+1}, U^n) = \frac{U_I^{n+1} - U_I^n}{\Delta t} + h^E(x_I, U^n), \tag{4.5.4}$$

where

$$h^E(x_I, U^n) := \begin{cases} h_1(x_I, U^n) & \text{if } x_I \in \Omega_1, \\ h_2(x_I, U^n) & \text{if } x_I \in \Omega_2, \\ \max\{h_1^-(x_I, U^n), \ h_2^+(x_I, U^n), \ h_3(x_I, U^n)\} & \text{if } x_I \in \Gamma. \end{cases}$$

More precisely, we denote by $a^+ = \max\{a, 0\}$, $a^- = \min\{a, 0\}$, $D_j^+(U_I^n) = (U_{I+e_j}^n - U_I^n)/\Delta x$, $D_j^-(U_I^n) = (U_I^n - U_{I-e_j}^n)/\Delta x$ for $j = 1, \dots, d$,

$$h_i(x_I, U^n) = \max_{p\in F_i(x_I)} \left\{ \sum_{j=1}^{d} (-p_j^+ D_j^+(U_I^n) - p_j^- D_j^-(U_I^n)) \right\}, \text{ for } i = 1, 2,$$

$$h_1^-(x_I, U^n) = \max_{p\in F_1(x_I)} \left\{ -p_1^- D_1^-(U_I^n) + \sum_{j=2}^{d} (-p_j^+ D_j^+(U_I^n) - p_j^- D_j^-(U_I^n)) \right\},$$

$$h_2^+(x_I, U^n) = \max_{p\in F_2(x_I)} \left\{ -p_1^+ D_1^+(U_I^n) + \sum_{j=2}^{d} (-p_j^+ D_j^+(U_I^n) - p_j^- D_j^-(U_I^n)) \right\},$$

$$h_3(x_I, U^n) = \max_{(0,q)\in F^E(x_I)} \left\{ \sum_{j=2}^{d} (-q_j^+ D_j^+(U_I^n) - q_j^- D_j^-(U_I^n)) \right\}.$$

As discussed before, the bottleneck to prove the convergence of the above scheme lies in the theoretical result of comparison principle.

4.5.2 Semi-Lagragian schemes

The idea of construct the Semi-Lagrangian schemes is based on the dynamical programming principle. Let v be the solution of (4.5.3), recall that v satisfies the following DPP:

$$v(t, x) = \min_{y_x\in S_{[0,T]}(x)} \{v(t-h, y_x(h))\}, \ \forall\, h \in [0, t], \tag{4.5.5}$$

where $S_{[0,t]}(x)$ is the set of absolutely continuous trajectories satisfying the following differential inclusion:

$$\begin{cases} \dot{y}(s) \in F(y(s)) & \text{for } s \in (0, t), \\ y(0) = x. \end{cases} \tag{4.5.6}$$

If we take $t = t_{n+1}$, $h = \Delta t$ and $x = x_I$ in (4.5.5), the formal semi-Lagrangian scheme is constructed as follows:

$$\begin{cases} v_I^{n+1} = \min_{p\in\mathcal{S}(x_I)} [v^n](x_I + p\Delta t), \\ v_I^0 = \varphi(x_I), \end{cases} \tag{4.5.7}$$

where $[v^n]$ represents the interpolation value of v^n on the discrete mesh, and $\mathcal{S}(x_I)$ is a sort of strategy set which needs to be determined. In the classical case where the dynamics set F is Lipschitz continuous, $\mathcal{S}(x_I)$ is nothing but $F(x_I)$. For any $p \in F(x_I)$, it is expected that any trajectory $y_x(h)$ can be expanded as

$$y_x(h) = x + hp + O(h^2), \text{ for } h > 0. \tag{4.5.8}$$

The expected error estimate for semi-Lagrangian scheme is $O(\frac{\Delta x}{\Delta t} + \Delta t)$. Usually, one can take $\Delta t = \sqrt{\Delta x}$ so that the error estimate will be $O(\sqrt{\Delta x})$.

In our case, it is natural to try $\mathcal{S}(x_I) = F^E(x_I)$ since F^E represents the proper dynamics used by the trajectories. However, it is not clear if we can get the approximation result (4.5.8) since F^E is not Lipschitz continuous. For any x_I close to the interface Γ and given $p \in F^E(x_I)$, (4.5.8) may hold true for a small time. However, it can happen that y_{x_I} cross the interface and p is no longer suitable for y_x, then the approximate discretization (4.5.8) fails. This situation is totally possible since the time scale Δt $(= \sqrt{\Delta x})$ is much bigger than the space scale Δx.

Consequently, $\mathcal{S}(x_I)$ should be a set of strategy depending on the position of x_I. It can be F^{x_I} for x_I far away from the interface, but for those x_I close to Γ, the strategy should involve the possible switch of dynamics for y_{x_I}.

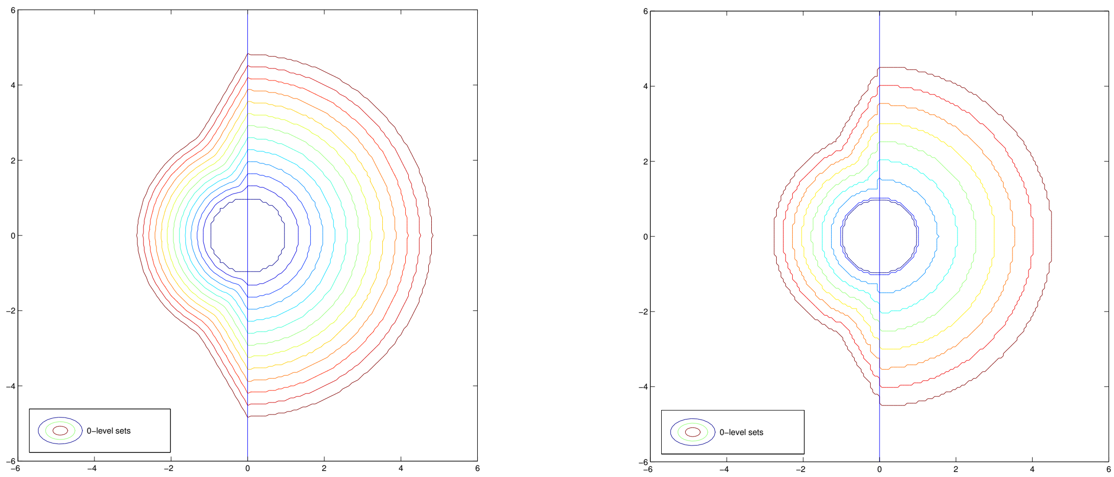
4.5.3 A numerical test

Although no convergence result has been proved at the moment, a numerical test is provided to show how the schemes discussed in the previous subsections work.

Consider the final time $T = 2$, the dynamics $F_1 = B(0, 1)$ and $F_2 = B(0, 2)$, and the initial condition $\varphi(x_1, x_2) = x_1^2 + x_2^2 - 1$. This problem can be interpreted as the propagation of the front whose initial position is the unit ball. The velocity of the front in Ω_1 and Ω_2 is 1 and 2 respectively.

In Figure 4.3(a), we use the first-order finite difference scheme (4.5.4) to compute the solution u to (4.5.3) in the domain $[-6, 6]^2$ with 200^2 mesh points. The evolution of the front during $[0, T]$ given by the 0-level set of u at each time step. The scheme works well in this case even in absence of convergence result.

In Figure 4.3(b), the solution u to (4.5.3) is computed through the semi-Lagrangian scheme (4.5.7) where we take $\mathcal{S} = F^E$ as the first attempt. For this scheme, we take less time steps than the finite difference scheme with $\Delta t \approx \sqrt{\Delta x}$. It is observed that the numerical result is almost the same as in Figure 4.3(a).



(a) Finite Difference scheme (b) Semi-Lagrangian scheme

FIGURE 4.3: Evolution of the front.

Chapter 5

Singular perturbation of optimal control problems on multi-domains

Publications of this chapter

(with N. Forcadel) *Singular perturbation of optimal control problems on multi-domains*, submitted. http://hal.archives-ouvertes.fr/hal-00812846

5.1 Introduction

In the present work, we investigate a class of singular perturbation problems for Hamilton-Jacobi-Bellman equations motivated by optimal control systems with different time scales on multi-domains. The multi-domains considered here is the following repartition of $\mathbf{R}^2$ by two disjoint open subsets Ω_1, Ω_2 with

$$\mathbf{R}^2 = \overline{\Omega}_1 \cup \overline{\Omega}_2,\ \Omega_1 \cap \Omega_2 = \emptyset.$$

Consider the nonlinear controlled systems of the following form: given the final time $T > 0$ and the initial data $t \geq 0, x \in \mathbf{R}^d, y \in \mathbf{R}^2$,

$$\begin{cases} \dot{X}(s) = f(X(s), Y(s), \alpha(s)) & \text{for } \alpha(s) \in A,\ s \in (t,T), \\ \dot{Y}(s) = \frac{1}{\varepsilon} g_i(X(s), Y(s), \alpha(s)) & \text{for } Y(s) \in \Omega_i, i = 1,2,\ \alpha(s) \in A,\ s \in (t,T), \\ (X(t), Y(t)) = (x,y), \end{cases} \tag{5.1.1}$$

where $\varepsilon > 0$, A is compact, f and g_i are Lipschitz continuous in the state variables and continuous. The optimal control problem that we are interested in is of Mayer's type:

$$v^\varepsilon(t,x,y) := \inf_{\alpha(\cdot)} \{\varphi(X(T), Y(T))\},$$

where φ is Lipschitz continuous.

The goal of this work is to obtain a characterization of the limit of v^ε as ε goes to zero. Singular perturbation problems for deterministic controlled systems have been studied by many authors; see e.g., the books by Kokotović, Khalil, and O'Reilly [107], and Bensoussan [34], as well as the articles by Gaitsgory [89, 90], Bagagiolo and Bardi [17], Alvarez and Bardi [3, 4], and the references therein.

However, up to our knowledge, there is no result for this kind of problem on multi-domains. In our setting, the dynamics of the fast state variable $Y(\cdot)$ switch to g_i when $Y(\cdot)$ goes into Ω_i. Then the definitions for the dynamical system (5.1.1) and the optimal control problem are not clear since the dynamics of $Y(\cdot)$ is not continuous on $\mathbf{R}^2$. The subject of optimal control problems on multi-domains is quite recent and we would like to refer to [1, 23, 29, 106, 122, 123]. The main difficulty lies in finding out the proper junction condition between Ω_1 and Ω_2 to characterize the value function of optimal control problems. Thanks to the recent work [29] on optimal control problems on stratified domains and [123] on the HJB equations on multi-domains, optimal control problems on multi-domains can be associated to HJB equations with discontinuity by introducing the concept of *Essential Hamiltonians*. The existence and uniqueness result for the solution of HJB equations with essential Hamiltonians has been established in [123]. Roughly speaking, the idea of this essential Hamiltonians consists in selecting the useful dynamics on the interfaces between Ω_1 and Ω_2 that drive the trajectories either to go into the interior of Ω_i or to travel on the interfaces between them. The value function v^ε is then characterized as the unique solution of

$$-\partial_t v^\varepsilon(t,x,y) + H^E(x,y,D_x v^\varepsilon(t,x,y),\frac{1}{\varepsilon}D_y v^\varepsilon(t,x,y)) = 0 \text{ on } (0,T)\times\mathbf{R}^d\times\mathbf{R}^2,$$

where H^E is the essential Hamiltonian (see Definition 5.2.1 below), with the final condition

$$v^\varepsilon(T,x,y) = \varphi(x,y) \text{ on } \mathbf{R}^d\times\mathbf{R}^2.$$

We are interested in the limit behavior as $\varepsilon \to 0$ of the solution of the above HJB equation. However, this essential Hamiltonian H^E is not necessarily Lipschitz continuous, which is a significant difficulty. There are some works [6, 116] dealing with the homogenization of metric Hamilton-Jacobi equations where the Hamiltonians are continuous and coercive. But when the Hamiltonians become discontinuous, this problem remains a difficult issue. In [116], an algorithm has been introduced to solve the piecewise-periodic problems numerically where the Hamiltonians are not continuous, but there is no general theoretical result for this method.

In this work, we consider coercive Hamiltonians by assuming a controllability condition on the fast variable $Y(\cdot)$: $\exists\, r_0 > 0$,

$$B_{\mathbf{R}^2}(0,r_0) \subseteq \{g_i(x,y,a), a\in A\},\ \forall\, x\in\mathbf{R}^d,\ y\in\mathbf{R}^2,\ i=1,2.$$

We also assume that the multi-domains have a periodic structure so that the dynamics for $Y(\cdot)$ is bounded. Our main result states that the limit $v(t,x)$, as $\varepsilon\to 0$, of the value function $v^\varepsilon(t,x,y)$ is the unique solution of

$$-\partial_t v(t,x) + \overline{H}(x, D_x v(t,x)) = 0 \text{ on } (0,T)\times \mathbf{R}^d, \text{ and } v(T,x) = \inf_{y\in\mathbf{R}^2} \varphi(x,y) \text{ on } \mathbf{R}^d.$$

The Hamiltonian $\overline{H}$ is called the *effective Hamiltonian* and is classically determined by the following cell problem: for each fixed $x \in \mathbf{R}^d$, $P \in \mathbf{R}^d$, there exists a unique constant $\overline{H}(x,P)$ such that the cell problem

$$H^E(x,y,P,D_y w(y)) = \overline{H}(x,P)$$

has a periodic viscosity solution w.

To solve the cell problem, we classically introduce an approximated cell problem (see [74, 111]). However, the essential Hamiltonian H^E which appears in this approximating cell problem is not continuous. Thus, the construction of approximated corrector is a difficult issue. To solve this problem, we use the fact that the essential Hamiltonian is defined from an optimal control point of view and we show that approximated correctors can be constructed as the value functions of infinite horizon optimal control problems.

Another difficulty is to prove that approximated correctors converge toward a corrector of the cell problem. This uses a stability result which we prove in the framework of discontinuous hamiltonian (but only for Lipschitz continuous solutions).

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5.1.1 Setting of the problem

We are interested in the limit value of the optimal control problems of Mayer's type. Let $T > 0$ be a fixed final time and $\mathcal{A}$ be the set of controls given by

$$\mathcal{A} := \{\alpha : (0,T) \to \mathbf{R}^m \text{ measurable functions}, \ \alpha(t) \in A \text{ a.e. in } (0,T)\}$$

with A being a compact subset of $\mathbf{R}^m$. In the sequel, all the periodic functions we consider have the period

$$S = (-1,1)^2,$$

then "f is S-periodic" means:

$$\forall\, k \in \mathbf{Z}^2,\ \forall\, x \in \mathbf{R}^2,\ f(x+2k) = f(x).$$

We assume that the function $f : \mathbf{R}^d \times \mathbf{R}^2 \times A \to \mathbf{R}^d$ satisfies the following:

$$
\textbf{(H1)} \begin{cases} \text{(i)} & \forall\, x \in \mathbf{R}^d, y \in \mathbf{R}^2,\ \{f(x,y,a) : a \in A\} \text{ is nonempty, convex, and compact;} \\ \text{(ii)} & f(x,y,a) \text{ is } L\text{-Lipschitz continuous w.r.t } x,\ y,\ \text{and continuous w.r.t } a; \\ \text{(iii)} & \exists\, M > 0 \text{ so that } \|f(x,y,a)\| \le M,\ \forall, (x,y) \in \mathbf{R}^d \times \mathbf{R}^2, a \in A. \end{cases}
$$

For $i = 1,2$, we assume that the functions $g_i : \mathbf{R}^d \times \mathbf{R}^2 \times A \to \mathbf{R}^2$ satisfies the following assumption

$$
\textbf{(H2)} \begin{cases} \text{(i)} & \forall\, x \in \mathbf{R}^d, y \in \mathbf{R}^2,\ \{g_i(x,y,a) : a \in A\} \text{ is nonempty, convex, and compact;} \\ \text{(ii)} & g_i(x,y,a) \text{ is } L\text{-Lipschitz continuous w.r.t } x,\ y,\ \text{and continuous w.r.t } a; \\ \text{(iii)} & \exists r_0 > 0 \text{ so that } \forall\, (x,y) \in \mathbf{R}^d \times \mathbf{R}^2,\ B(0,r_0) \subseteq \{g_i(x,y,a) : a \in A\}; \\ \text{(iv)} & \forall\, x \in \mathbf{R}^d,\ a \in A,\ g_i(x,\cdot,a) \text{ is } S\text{-periodic.} \end{cases}
$$

The last requirement needed in our study is the convexity of velocities. For $i = 1,2$, $(x,y) \in \mathbf{R}^d \times \mathbf{R}^2$, we set

$$
\Phi_i(x,y) := \left\{ \begin{pmatrix} f(x,y,a) \\ g_i(x,y,a) \end{pmatrix}, a \in A \right\}.
$$

The convexity assumption is the following:

(H3) $\forall\, x \in \mathbf{R}^d, y \in \mathbf{R}^2$, $\Phi_i(x,y)$ is convex.

We consider the following periodic chessboard structure (see also Figure 5.1)

$$
S_1 := \{(0,1) \times (0,1) + kS,\ k \in \mathbf{Z}^2\} \cup \{(-1,0) \times (-1,0) + kS,\ k \in \mathbf{Z}^2\},\ \Omega_1 := \bigcup_{\mathcal{M} \in S_1} \mathcal{M}.
$$

$$
S_2 := \{(-1,0) \times (0,1) + kS,\ k \in \mathbf{Z}^2\} \cup \{(0,1) \times (-1,0) + kS,\ k \in \mathbf{Z}^2\},\ \Omega_2 := \bigcup_{\mathcal{M} \in S_2} \mathcal{M}.
$$

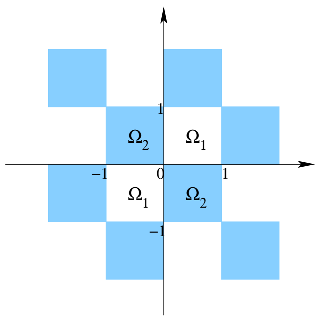

FIGURE 5.1: The periodic chessboard structure.

Remark 5.1.1. The structure of multi-domains we considered here is the type of chessboard structure. In fact, due to the work [29, 123] our results can be generalized on any periodic structure of multi-domains $(\mathcal{M}_i)_{i=1,\ldots,n}, n \in \mathbf{N}$ satisfying the following: each $\mathcal{M}_i$ is a C^2 open embedded 2-manifold in $\mathbf{R}^2$, each $\overline{\mathcal{M}}_i$ is proximally smooth and wedged, and

$$S = \bigcup_{i=1}^{n} \overline{\mathcal{M}}_i, \ \mathcal{M}_i \cap \mathcal{M}_j = \emptyset \text{ for } i \neq j, \ i,j = 1, \dots, n.$$

The concepts of proximally smooth and wedged are introduced in [60]. For any set $\mathcal{M} \subseteq \mathbf{R}^d$, we recall that $\overline{\mathcal{M}}$ is proximally smooth means that the signed distance function to $\overline{\mathcal{M}}$ is differentiable on a tubular neighborhood of $\overline{\mathcal{M}}$. $\overline{\mathcal{M}}$ is said to be wedged means that the interior of $\mathcal{T}_{\overline{\mathcal{M}}}(x)$ is nonempty for each $x \in \overline{\mathcal{M}}$. Here $\mathcal{T}_{\overline{\mathcal{M}}}(x)$ is the tangent cone of $\overline{\mathcal{M}}$ at x defined by

$$\mathcal{T}_{\overline{\mathcal{M}}}(x) = \{\zeta \in \mathbf{R}^2 : \liminf_{t \to 0^+} \frac{d_{\overline{\mathcal{M}}}(x + t\zeta)}{t} = 0\},$$

where $d_{\overline{\mathcal{M}}}(\cdot)$ is the distance function to $\overline{\mathcal{M}}$.

Now in order to well define a dynamical system on the whole $\mathbf{R}^2$ for $Y(\cdot)$, we need to determine the dynamics on the interfaces between the sets of S_1 and S_2. The idea is to consider the approach of Filippov regularization of the dynamics around the interfaces, i.e. consider the multifunction $\Phi : \mathbf{R}^d \times \mathbf{R}^2 \rightsquigarrow \mathbf{R}^2$ defined by

$$\Phi(x,y) := \begin{cases} \Phi_i(x,y) & \text{if } y \in \Omega_i, \\ \overline{co}(\Phi_1(x,y), \Phi_2(x,y)) & \text{otherwise,} \end{cases}$$

where $\overline{co}(\Phi_1(x,y), \Phi_2(x,y))$ is defined as the set

$$\left\{ (1-\theta) \begin{pmatrix} f(x,y,a_1) \\ g_1(x,y,a_1) \end{pmatrix} + \theta \begin{pmatrix} f(x,y,a_2) \\ g_2(x,y,a_2) \end{pmatrix} \mid \theta \in [0,1], \ a_1, a_2 \in A \right\}.$$

Now we are ready to introduce the optimal control problem. Given the initial time $t \in [0,T]$ and the initial state $(x,y) \in \mathbf{R}^d \times \mathbf{R}^2$, we consider the controlled trajectories $(X,Y)(\cdot) : [0,T] \to \mathbf{R}^d \times \mathbf{R}^2$ satisfying

$$\begin{cases} \begin{pmatrix} \dot{X}(s) \\ \varepsilon \dot{Y}(s) \end{pmatrix} \in \Phi(X(s), Y(s)) \quad \text{for } s \in (t,T), \\ \qquad X(t) = x, \ Y(t) = y. \end{cases} \tag{5.1.2}$$

We denote by $S^{\varepsilon}_{[t,T]}(x,y)$ the set of absolutely continuous trajectories satisfying (5.1.2). Let $\varphi : \mathbf{R}^d \times \mathbf{R}^2 \to \mathbf{R}$ be a bounded Lipschitz continuous function. Consider the following Mayer's problem: for any $\varepsilon > 0$,

$$v^{\varepsilon}(t,x,y) := \inf \left\{ \varphi(X(T), Y(T)) : (X(\cdot), Y(\cdot)) \in S^{\varepsilon}_{[t,T]}(x,y) \right\}. \tag{5.1.3}$$

Note that Φ is upper semi-continuous and convex valued, but Φ is not necessarily Lipschitz continuous. The characterization of the value function via the Hamilton-Jacobi-Bellman approach is a difficult issue and we refer to [123] in order to prove that v^ε is the unique solution of

$$\begin{cases} -\partial_t v^\varepsilon(t,x,y) + H^E(x,y,D_x v^\varepsilon(t,x,y), \frac{1}{\varepsilon} D_y v^\varepsilon(t,x,y)) = 0 & \text{on } (0,T)\times \mathbf{R}^d \times \mathbf{R}^2, \\ v^\varepsilon(T,x,y) = \varphi(x,y) & \text{on } \mathbf{R}^d \times \mathbf{R}^2, \end{cases} \tag{5.1.4}$$

where H^E is the essential Hamiltonian which is discontinuous in general and will be defined in Section 5.2.

5.1.2 Main results

We now want to characterize the limit v of v^ε as the velocity of the fast variable goes to infinity (i.e. $\varepsilon \to 0$).

The main results are the following.

Theorem 5.1.2 (Definition of the effective Hamiltonian). For each fixed $x \in \mathbf{R}^d$, $P \in \mathbf{R}^d$, there exists a unique $\lambda := \overline{H}(x,P) \in \mathbf{R}$ such that the cell problem

$$H^E(x,y,P,D_y w(y)) = \lambda \tag{5.1.5}$$

has a periodic viscosity solution w. Moreover, seen as a function of x and P, $\overline{H}$ is Lipschitz continuous.

Theorem 5.1.3 (Convergence result). Assume **(H1)-(H3)**. The value function v^ε defined in (5.1.3) converges uniformly on $[0,T]\times \mathbf{R}^d \times \mathbf{R}^2$ to the unique viscosity solution v of

$$\begin{cases} -\partial_t v(t,x) + \overline{H}(x, D_x v(t,x)) = 0 & \text{for } t\in(0,T),\ x\in \mathbf{R}^d, \\ v(T,x) = \inf_{y\in\mathbf{R}^2} \varphi(x,y) & \text{for } x \in \mathbf{R}^d. \end{cases} \tag{5.1.6}$$

Note the fact that the limiting equation does not depend on the fast variable, (5.1.6) can be understood by looking at the controllability assumptions which implies that at the limit, the fast variable can travel over all the space $\mathbf{R}^2$ with infinite velocity (this also explains the terminal condition).

We also want to point out that the effective Hamiltonian $\overline{H}$ is Lipschitz continuous in x and so the perturbed test function (introduced by Evans [74]) can be adapted to our case.

This chapter is organized as follows. In section 5.2, we give some preliminary results including the notion of essential Hamiltonians. Section 5.3 discusses the cell problem while Section 5.4 is devoted to the properties of the effective Hamiltonian $\overline{H}$. The proof of the convergence result is given in Section 5.5.

5.2 Preliminary results

We now state the definition of the essential Hamiltonian. Note that we have two types of interfaces according to their dimensions, we set

$$I := \{(k,k+1)\times\{m\},\ (k,m)\in \mathbf{Z}^2\}\cup\{\{k\}\times(m,m+1),\ (k,m)\in\mathbf{Z}^2\}\cup\mathbf{Z}^2$$

as the union of all the 1-dimensional interfaces and 0-dimensional interfaces.

For any $\mathcal{M}\in S_1\cup S_2\cup I$, we denote by $\Phi_{\mathcal{M}}:\mathbf{R}^d\times\mathbf{R}^2\rightsquigarrow\mathbf{R}^d\times\mathbf{R}^2$ defined by

$$\Phi_{\mathcal{M}}(x,y) := \begin{cases} \Phi_i(x,y) & \text{if } \mathcal{M}\in S_i,\ i=1,2,\\ \Phi(x,y) & \text{if } \mathcal{M}\in I.\end{cases}$$

Consider the *essential multifunction* Φ^E (introduced in [29, 123]) defined as follows.

Definition 5.2.1. [Essential dynamics and essential Hamiltonian] Let $\Phi^E:\mathbf{R}^d\times\mathbf{R}^2\rightsquigarrow\mathbf{R}^d\times\mathbf{R}^2$ be a multifunction defined for any $(x,y)\in\mathbf{R}^d\times\mathbf{R}^2$ by

$$\Phi^E(x,y) := \bigcup_{\mathcal{M}\in S_1\cup S_2\cup I,\ y\in\overline{\mathcal{M}}}\big(\Phi_{\mathcal{M}}(x,y)\cap(\mathbf{R}^d\times\mathcal{T}_{\overline{\mathcal{M}}}(y))\big).$$

We also denote by $H^E:\mathbf{R}^d\times\mathbf{R}^2\times\mathbf{R}^d\times\mathbf{R}^2\to\mathbf{R}$ the essential Hamiltonian defined by

$$H^E(x,y,\xi,\zeta) := \sup_{(p,q)\in\Phi^E(x,y)}\{-p\cdot\xi-q\cdot\zeta\}.$$

Example 5.2.2. Here we give a precise example to see more clearly the elements in Φ^E. We ignore the variable X since there is no singularity in the structure of the dynamics of X. Consider $g_1\equiv(1,1)$ and $g_2\equiv(-1,1)$, Figure 5.2 shows the differences between Φ and Φ^E on the interfaces (elements in I). In fact, on the interfaces Φ contains all the possible directions (the whole triangles) in which

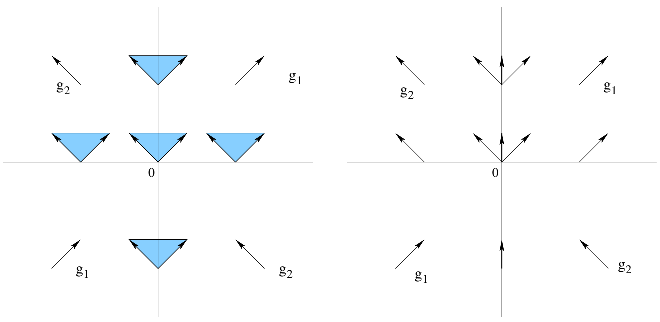

FIGURE 5.2: Φ and Φ^E.

some of them may be useless. While the definition of Φ^E allows to select only the useful dynamics for the trajectories in $S^{\varepsilon}_{[t,T]}(x,y)$: the directions g_i which are inward for $\overline{\Omega}_i$ and the tangent directions for the interfaces. We refer to [29, 123] for more details.

Remark 5.2.3. $\Phi^E(x,y)$ is Lipschitz continuous in x since $\Phi(\cdot,y)$ is Lipschitz continuous. However, $\Phi^E(x,y)$ is not necessarily continuous in y because of the geometrical singularity of the dynamical structure for the variable y. Therefore, the essential Hamiltonian $H^E(x,y,\xi,\zeta)$ is Lipschitz continuous in x, but not necessarily continuous in y.

Then here is the characterization result ([123, Theorem 2.4]) for the value function.

Lemma 5.2.4 (Characterization of the value function). The value function v^ε is the unique Lipschitz continuous viscosity solution of (5.1.4) in the sense of Definition 5.2.6.

Before giving the definition of viscosity solution, we need the following notion of extended differentials.

Definition 5.2.5 (Extended differential). Let $\phi : (0,T)\times \mathbf{R}^d \times \mathbf{R}^2 \to \mathbf{R}$ be a continuous function and $\mathcal{M} \in S_1 \cup S_2 \cup I$. Suppose that $\phi \in C^1((0,T)\times \mathbf{R}^d \times \overline{\mathcal{M}})$, then for any $t \in (0,T), x \in \mathbf{R}^d, y \in \overline{\mathcal{M}}$, the extended differential of ϕ on (t,x,y) is defined by

$$D_{\overline{\mathcal{M}}}\phi(t,x,y) := \lim_{z\to y, z\in\mathcal{M}} D\phi(t,x,z).$$

Note that since $D\phi(t,x,\cdot)$ is continuous on $\overline{\mathcal{M}}$, the extended differential is nothing but the extension of $D\phi(t,x,\cdot)$ to the whole $\overline{\mathcal{M}}$.

We now state the definition of viscosity solution for (5.1.4).

Definition 5.2.6 (Viscosity solution for (5.1.4)). Let $u : (0,T]\times\mathbf{R}^d\times\mathbf{R}^2 \to \mathbf{R}$ be a bounded Lipschitz continuous function.

(i) We say that u is a supersolution of (5.1.4) if for any $(t_0,x_0,y_0) \in (0,T)\times\mathbf{R}^d\times\mathbf{R}^2$, $\phi \in C^1((0,T)\times\mathbf{R}^d\times\mathbf{R}^2)$ such that $u-\phi$ attains a local minimum on (t_0,x_0,y_0), we have

$$-\phi_t(t_0,x_0,y_0) + H^E(x_0,y_0,D_x\phi(t_0,x_0,y_0), \frac{1}{\varepsilon}D_y\phi(t_0,x_0,y_0)) \geq 0.$$

(ii) We say that u is a subsolution of (5.1.4) if for any $(t_0,x_0,y_0)\in(0,T)\times\mathbf{R}^d\times\mathbf{R}^2$, any continuous $\phi:(0,T)\times\mathbf{R}^d\times\mathbf{R}^2\to\mathbf{R}$ with $\phi|_{(0,T)\times\mathbf{R}^d\times\overline{\mathcal{M}}}$ being C^1 for each $\mathcal{M}\in S_1\cup S_2\cup I$ with $y_0\in\overline{\mathcal{M}}$ such that $u-\phi$ attains a local maximum at (t_0,x_0,y_0), we have

$$\begin{aligned}-\phi_t(t_0,x_0,y_0) + \sup_{(p,q)\in\Phi_{\mathcal{M}}(x_0,y_0)\cap(\mathbf{R}^d\times\mathcal{T}_{\overline{\mathcal{M}}}(y_0))} \{ & -p\cdot D_x\phi(t_0,x_0,y_0) \\ & -\frac{1}{\varepsilon}q\cdot D_{\overline{\mathcal{M}}}\phi(t_0,x_0,y_0)\} \leq 0.\end{aligned}$$

(iii) We say that u is a viscosity solution of (5.1.4) if u is both a supersolution and a subsolution, and u satisfies the final condition

$$u(T,x,y) = \varphi(x,y),\ \forall\,(x,y)\in\mathbf{R}^d\times\mathbf{R}^2.$$

In the following, we will also use different equations (in particular for the cell problem and for the approximated cell problem). We then give the definition of viscosity solution for a more general equation of the form

$$H_1(u(y)) + H^E(x, y, P, Du(y)) = 0. \tag{5.2.1}$$

Definition 5.2.7 (Viscosity solution for (5.2.1)). Let $u : \mathbf{R}^2 \to \mathbf{R}$ be a bounded Lipschitz continuous function.

(i) We say that u is a supersolution of (5.2.1) if for any $y_0 \in \mathbf{R}^2$, $\phi \in C^1(\mathbf{R}^2)$ such that $u - \phi$ attains a local minimum on y_0, we have

$$H_1(u(y_0)) + H^E(x, y_0, P, D\phi(y_0)) \geq 0.$$

(ii) We say that u is a subsolution of (5.2.1) if for any $y_0 \in \mathbf{R}^2$, any continuous $\phi : \mathbf{R}^2 \to \mathbf{R}$ with $\phi|_{\overline{\mathcal{M}}}$ being C^1 for each $\mathcal{M} \in S_1 \cup S_2 \cup I$ with $y_0 \in \overline{\mathcal{M}}$ such that $u - \phi$ attains a local maximum at y_0, we have

$$H_1(u(y_0)) + \sup_{(p,q) \in \Phi_{\mathcal{M}}(x, y_0) \cap (\mathbf{R}^d \times \mathcal{T}_{\overline{\mathcal{M}}}(y_0))} \{-p \cdot P - q \cdot D_{\overline{\mathcal{M}}}\phi(y_0))\} \leq 0.$$

(iii) We say that u is a viscosity solution of (5.2.1) if u is both a supersolution and a subsolution.

We now state a comparison principle for the equation (5.1.4) on bounded domain

Theorem 5.2.8 (Comparison principle in bounded domain). For any open bounded $\Omega \subseteq (0, T) \times \mathbf{R}^d$, let $u_1, u_2 : (0, T) \times \mathbf{R}^d \times \mathbf{R}^2 \to \mathbf{R}$ be Lipschitz continuous. If u_1 is a subsolution of (5.1.4) and u_2 is a supersolution of (5.1.4) in $\Omega \times \mathbf{R}^2$, then we have

$$\sup_{(t,x,y) \in \Omega \times \mathbf{R}^2} \{u_1(t, x, y) - u_2(t, x, y)\} \leq \sup_{(t,x,y) \in \partial\Omega \times \mathbf{R}^2} \{u_1(t, x, y) - u_2(t, x, y)\}.$$

Before we start the proof, we have the following lemma which is a direct consequence of [123, Theorem 3.7, Theorem 3.11].

Lemma 5.2.9 (Dynamics programming principle). Let $u : (0, T) \times \mathbf{R}^d \times \mathbf{R}^2$ be Lipschitz continuous.

- If u is a supersolution of (5.1.4), then for any $(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^2$ there exists $(\overline{X}, \overline{Y}) \in S^{\varepsilon}_{[t,T]}(x, y)$ such that

$$u(t, x, y) \geq u(t + h, \overline{X}(t + h), \overline{Y}(t + h)), \text{ for } 0 \leq h \leq T - t.$$

- If u is a subsolution of (5.1.4), then for any $(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^2$ and any $(X, Y) \in S^{\varepsilon}_{[t,T]}(x, y)$

$$u(t, x, y) \leq u(t + h, X(t + h), Y(t + h)), \text{ for } 0 \leq h \leq T - t.$$

Proof of Theorem 5.2.8. For any $(t_0, x_0, y_0) \in \Omega \times \mathbf{R}^2$, u_2 is a supersolution on Ω implies that there exists an absolutely continuous function $(\overline{X}, \overline{Y}) \in S^{\varepsilon}_{[t_0,T]}(x_0, y_0)$ such that

$$u_2(t_0, x_0, y_0) \geq u_2(t_0 + h, \overline{X}(t_0 + h), \overline{Y}(t_0 + h)), \text{ for } 0 \leq h \leq h_0,$$

where

$$h_0 := \inf\{h > 0 : (t_0 + h, \overline{X}(t_0 + h)) \notin \Omega\}.$$

Similarly, u_1 is a subsolution on Ω implies that

$$u_1(t_0, x_0, y_0) \leq u_1(t_0 + h, \overline{X}(t_0 + h), \overline{Y}(t_0 + h)), \text{ for } 0 \leq h \leq h_0.$$

We then deduce that

$$(u_1 - u_2)(t_0, x_0, y_0) \leq (u_1 - u_2)(t_0 + h_0, \overline{X}(t_0 + h), \overline{Y}(t_0 + h)).$$

The definition of h_0 implies that $(t_0 + h_0, \overline{X}(t_0 + h)) \in \partial\Omega$, then we obtain

$$u_1(t_0, x_0, y_0) - u_2(t_0, x_0, y_0) \leq \max_{(t,x,y)\in\partial\Omega\times\mathbf{R}^2} \{u_1(t, x, y) - u_2(t, x, y)\},$$

which leads to the desired result. □

5.3 The cell problem

In this section, we focus on the the cell problem: given $x \in \mathbf{R}^d$, $P \in \mathbf{R}^d$, find $\lambda \in \mathbf{R}$ such that the equation (5.1.5) has a viscosity solution.

5.3.1 Approximating problem

To solve the cell problem, we classically introduce an approximated cell problem. Given $x \in \mathbf{R}^d$, $P \in \mathbf{R}^d$ and $\beta > 0$, we consider the problem

$$\beta v^{\beta}(y) + H^E(x, y, P, Dv^{\beta}(y)) = 0, \; y \in \mathbf{R}^2. \tag{5.3.1}$$

Then we investigate the limit of the approximating equation (5.3.1) as $\beta \to 0$ by proving that $v^{\beta} \to v$ and $\beta v^{\beta} \to -\lambda$ with v solution of (5.1.5)

Since H^E is not Lipschitz continuous in y, the existence and uniqueness of the solution for (5.3.1) need to be carrefully studied. A simple idea is to link the HJB equation (5.3.1) with an optimal control problem. For any $y \in \mathbf{R}^2$, we denote the set of absolutely continuous trajectories by

$$S[x, y] := \{(X, Y), \; (\dot{X}(s), \dot{Y}(s)) \in \Phi(x, Y(s)), \; X(0) = x, \; Y(0) = y\}.$$

Given $P \in \mathbf{R}^2$, consider the value function w^β of the following infinite horizon optimal control problem:

$$w^\beta(y) := \min_{(X,Y)\in S[x,y]} \int_0^{+\infty} e^{-\beta s} P \cdot \dot{X}(s) ds.$$

The main result of this subsection is the following characterization of the value function w^β:

Theorem 5.3.1 (Characterization of the value function w^β). The value function w^β is the unique viscosity solution of (5.3.1) in the sense of Definition 5.2.7.

We begin by the existence part. As in the classical case (see [19, Proposition III.2.5]), w^β satisfies a Dynamical programming principle (DPP).

Proposition 5.3.2 (Dynamic programming principle). Assume that **(H1)-(H3)** hold. Then for any $y \in \mathbf{R}^2$, $h \geq 0$, the following holds.

(i) **The super-optimality.** $\exists\ (\overline{X}, \overline{Y}) \in S[x, y]$ such that

$$w^\beta(y) \geq \int_0^h e^{-\beta s} P \cdot \dot{\overline{X}}(s) ds + e^{-\beta h} w^\beta(\overline{Y}(h));$$

(ii) **The sub-optimality.** $\forall\ (X, Y) \in S[x, y]$ we have

$$w^\beta(y) \leq \int_0^h e^{-\beta s} P \cdot \dot{X}(s) ds + e^{-\beta h} w^\beta(Y(h)).$$

The value function w^β satisfies the following properties.

Proposition 5.3.3 (Regularity of w^β). Assume that **(H1)-(H3)** hold. Then w^β is bounded and Lipschitz continuous. Moreover, the Lipschitz constant is uniform in β.

Proof. By the definition of w^β, for any $y \in \mathbf{R}^2$,

$$|w^\beta(y)| \leq \int_0^{+\infty} e^{-\beta s} \|P\| M ds = \frac{\|P\| M}{\beta}. \tag{5.3.2}$$

Now we prove the Lipschitz continuity. For any $y, z \in \mathbf{R}^2$, consider the following trajectory:

$$Y(s) := y + r_0 \frac{z - y}{\|y - z\|} s, \text{ for } s \geq 0.$$

We set $h = \|y - z\|/r_0$, then we have $Y(0) = y$, $Y(h) = z$. Note that $\|\dot{y}_x(s)\| = r_0$, so by **(H2)(iii)** there exists X such that $(X, Y) \in S[x, y]$. Since w^β satisfies the sub-optimality along (X, Y), we obtain

$$\begin{aligned} w^\beta(y) &\le \int_0^h e^{-\beta s} P\cdot \dot X(s)ds + e^{-\beta h} w^\beta(z) \\ &\le w^\beta(z) + \int_0^h e^{-\beta s} P\cdot \dot X(s)ds + (e^{-\beta h}-1)w^\beta(z) \\ &\le w^\beta(z) + \int_0^h e^{-\beta s}\|P\| M ds + (1-e^{-\beta h})|w^\beta(z)| \\ &\le w^\beta(z) + 2(1-e^{-\beta h})\frac{\|P\|M}{\beta} \\ &\le w^\beta(z) + 2h\|P\|M = w^\beta(z) + \frac{2\|P\|M}{r_0}\|y-z\|, \end{aligned}$$

which implies the Lipschitz continuity of w^β (the Lipschitz constant is independent on β). □

Then we have that w^β is solution of the equation (5.3.1).

Proposition 5.3.4 (w^β satisfies (5.3.1)). The value function w^β is a viscosity solution of (5.3.1).

Proof. We first prove that w^β is a supersolution. For any $y_0 \in \mathbf{R}^2$, let $\phi \in C^1(\mathbf{R}^2)$ such that $u-\phi$ attains a local minimum on y_0. By the super-optimality satisfied by w^β, $\exists\ (\overline{X}, \overline{Y}) \in S[y_0]$ such that

$$w^\beta(y_0) \ge \int_0^h e^{-\beta s} P\cdot \dot{\overline{X}}(s)ds + e^{-\beta h} w^\beta(\overline{Y}(h)). \tag{5.3.3}$$

By definition of ϕ, we have

$$w^\beta(y_0) - \phi(y_0) \le w^\beta(\overline{Y}(h)) - \phi(\overline{Y}(h)),\ \forall\, h>0. \tag{5.3.4}$$

Then, (5.3.3) and (5.3.4) imply that

$$w^\beta(y_0) \ge \int_0^h e^{-\beta s} P\cdot \dot{\overline{X}}(s)ds + e^{-\beta h}(w^\beta(y_0) + \phi(\overline{Y}(h)) - \phi(y_0)), \tag{5.3.5}$$

i.e.

$$\frac{1-e^{-\beta h}}{h} w^\beta(y_0) - \frac{1}{h}\int_0^h e^{-\beta s} P\cdot \dot{\overline{X}}(s)ds - \frac{e^{-\beta h}}{h}\int_0^h D\phi(\overline{Y}(s))\cdot \dot{\overline{Y}}(s)ds \ge 0. \tag{5.3.6}$$

By [123, Lemma 3.6], there exists $h_n \to 0$ such that $\frac{(\overline{X}(h_n),\overline{Y}(h_n))-(x,y_0)}{h_n} \to (p_0,q_0)$ for some $(p_0,q_0) \in \mathrm{co}\left(\Phi^E(x,y_0)\right)$ where $\mathrm{co}\left(\Phi^E(x,y_0)\right)$ is the convex hull of $\Phi^E(x,y_0)$. We then get

$$\beta w^\beta(y_0) - p_0\cdot P - q_0 \cdot D\phi(y_0) \ge 0,$$

which leads to

$$\beta w^\beta(y_0) + \sup_{(p,q)\in \mathrm{co}\,(\Phi^E(x,y_0))} \{-p\cdot P - q\cdot D\phi(y_0)\} \ge 0.$$

Since $(p,q) \mapsto -p \cdot P - q \cdot D\phi(y_0)$ is linear, we have

$$\sup_{(p,q)\in \text{co}\,(\Phi^E(x,y_0))} \{-p \cdot P - q \cdot D\phi(y_0)\} = \sup_{(p,q)\in \Phi^E(x,y_0)} \{-p \cdot P - q \cdot D\phi(y_0)\}.$$

Thus

$$\beta w^\beta(y_0) + \sup_{(p,q)\in \Phi^E(x,y_0)} \{-p \cdot P - q \cdot D\phi(y_0)\} \geq 0,$$

which ends the proof for the supersolution property.

Now we prove that w^β is a subsolution. Let $\phi \in C(\mathbf{R}^2)$ such that $u - \phi$ attains a local maximum at y_0 with $\phi \in C^1(\overline{\mathcal{M}})$ for every $\mathcal{M} \in S_1 \cup S_2 \cup I$ such that $y_0 \in \overline{\mathcal{M}}$. If $y_0 \in \mathcal{M}$ with $\mathcal{M} \in S_1 \cup S_2$, since g_1 and g_2 are Lipschitz continuous, then the proof is classical (see [19]) and we skip it. We then assume that y_0 lies in an element of I. For each $\mathcal{M} \in S_1 \cup S_2 \cup I$ with $y_0 \in \overline{\mathcal{M}}$, any $(p,q) \in \Phi_{\mathcal{M}}(x,y_0) \cap (\mathbf{R}^d \times \mathcal{T}_{\overline{\mathcal{M}}}(y_0))$, by [123, Lemma 3.9] there exists $h > 0$ and a solution $(X,Y) \in S[x,y_0]$ which is C^1 on $[0,h]$ with $(\dot{X}(0), \dot{Y}(0)) = (p,q)$ and $Y(s) \in \overline{\mathcal{M}}, \forall\, s \in [0,h]$. By the sub-optimality of w^β,

$$w^\beta(y_0) \leq \int_0^h e^{-\beta s} P \cdot \dot{X}(s) ds + e^{-\beta h} w^\beta(Y(h)).$$

We have also

$$w^\beta(y_0) - \phi(y_0) \geq w^\beta(Y(h)) - \phi(Y(h)), \ \forall\, h > 0.$$

By a similar argument as in the supersolution property case, we can deduce that

$$\frac{1 - e^{-\beta h}}{h} w^\beta(y_0) - \frac{1}{h}\int_0^h e^{-\beta s} P \cdot \dot{Y}(s) ds - \frac{e^{-\beta h}}{h} \int_0^h D\phi(Y(s))\dot{Y}(s) ds \leq 0.$$

Taking $h \to 0$ leads to

$$\beta w^\beta(y_0) - (p \cdot P + q \cdot D_{\overline{\mathcal{M}}}\phi(y_0)) \leq 0.$$

The point (p,q) being arbitrary in $\Phi_{\mathcal{M}}(x,y_0) \cap (\mathbf{R}^d \times \mathcal{T}_{\overline{\mathcal{M}}}(y_0))$, we deduce that

$$\beta w^\beta(y_0) + \sup_{(p,q)\in \Phi_{\mathcal{M}}(x,y_0) \cap (\mathbf{R}^d \times \mathcal{T}_{\overline{\mathcal{M}}}(x,y_0))} \left\{-p \cdot P - q \cdot D_{\overline{\mathcal{M}}}\phi(x_0)\right\} \leq 0,$$

which ends the proof. □

Before we prove the uniqueness result, we state the following results dealing with the relation between supersolution (resp. subsolution) and super-optimality (resp. sub-optimality).

Theorem 5.3.5 (Supersolution implies super-optimality). Let $u : \mathbf{R}^2 \to \mathbf{R}$ be a supersolution of (5.3.1), then u satisfies the super-optimality.

Proof. We want to prove that there exists $(\overline{X}, \overline{Y}) \in S[x,y]$ such that

$$u(y) \geq \int_0^h e^{-\beta s} P \cdot \dot{\overline{X}}(s) ds + e^{-\beta h} u(\overline{Y}(h)), \ \text{for } h > 0,$$

i.e.

$$u(\overline{Y}(h)) \leq \overline{\xi}(h),\ \overline{\xi}(h) := e^{\beta h}\left(u(y) - \int_0^h e^{-\beta s} P \cdot \dot{\overline{X}}(s)ds\right),\ h > 0.$$

For any $y \in \mathbf{R}^2$, consider the following viability problem:

$$\begin{cases} (\dot{X}(h), \dot{Y}(h) \in \Phi(x, Y(h)) & \text{for } h \in (0,\infty), \\ \dot{\xi}(h) = \beta\xi(h) - P \cdot \dot{X}(h) & \text{for } h \in (0,\infty), \\ (X(0), Y(0), \xi(0)) = (x, y, u(y)), & \\ (Y(h), \xi(h)) \in \text{epi }(u). & \end{cases} \tag{5.3.7}$$

For any $(y, \xi) \in$ epi (u), we have $u(y) \leq \xi$. We claim that for any $(\zeta, \sigma) \in [\mathcal{T}_{\text{epi }(u)}(y, u(y))]^-$[1],

$$\inf_{(p,q)\in\Phi(x,y)} \langle (q, \beta\xi - P \cdot p), (\zeta, \sigma)\rangle \leq 0. \tag{5.3.8}$$

Indeed, let $(\zeta, \sigma) \in [\mathcal{T}_{\text{epi }(u)}(y, u(y))]^-$. Since $(0, 1) \in \mathcal{T}_{\text{epi }(u)}(y, u(y))$, by the definition of $[\mathcal{T}_{\text{epi }(u)}(y, u(y))]^-$ we have

$$\langle (\zeta, \sigma), (0, 1)\rangle \leq 0,$$

i.e. $\sigma \leq 0$. Based on this fact, we consider the following three cases.

Case 1: $\sigma = -1$

By [81, Proposition 4.1] there exists $\phi \in C^1(\mathbf{R}^d)$ such that $u - \phi$ attains a local minimum on y with $D\phi(y) = \zeta$. Then

$$\begin{aligned} \inf_{(p,q)\in\Phi(x,y)} \langle (q, \beta\xi - P \cdot p), (\zeta, -1)\rangle &= -\beta\xi + \inf_{(p,q)\in\Phi(x,y)} \{D\phi(y) \cdot q + P \cdot p\} \\ &\leq -\beta u(y) + \inf_{(p,q)\in\Phi(x,y)} \{D\phi(y) \cdot q + P \cdot p\} \\ &\leq -\beta u(y) + \inf_{(p,q)\in\Phi^E(x,y)} \{D\phi(y) \cdot q + P \cdot p\} \leq 0. \end{aligned}$$

Case 2 : $\sigma < 0$

In that case, $(\zeta/|\sigma|, -1) \in [\mathcal{T}_{\text{epi }(u)}(y, u(y))]^-$. We deduce using the previous case, that

$$\inf_{(p,q)\in\Phi(x,y)} \langle (q, \beta\xi - P \cdot p), (\frac{\zeta}{|\sigma|}, -1)\rangle \leq 0,$$

which implies

$$\inf_{(p,q)\in\Phi(x,y)} \langle (q, \beta\xi - P \cdot p), (\zeta, \sigma)\rangle \leq 0.$$

[1] $[\mathcal{T}_{\text{epi }(u)}(y, u(y))]^-$ is the negative polar cone of $\mathcal{T}_{\text{epi }(u)}(y, u(y))$, i.e. $p \in [\mathcal{T}_{\text{epi }(u)}(y, u(y))]^-$ if and only if $\langle p, q\rangle \leq 0$ for any $q \in \mathcal{T}_{\text{epi }(u)}(y, u(y))$.

Case 3 : $\sigma = 0$
By [81, Lemma 4.2] there exists $y_n \to y$, $(\zeta_n, \sigma_n) \to (\zeta, 0)$ such that

$$(\zeta_n, \sigma_n) \in [\mathcal{T}_{\text{epi }(u)}(y_n, u(y_n))]^-, \ \sigma_n < 0.$$

Using Case 2, we get that

$$\inf_{(p,q)\in\Phi(x,y_n)} \langle (q, \beta\xi - P\cdot p), (\zeta_n, \sigma_n)\rangle \leq 0.$$

Since Φ is upper semicontinuous, we deduce that

$$\inf_{(p,q)\in\Phi(x,y)} \langle (q, \beta\xi - P\cdot p), (\zeta, 0)\rangle \leq 0.$$

which ends the proof of (5.3.8).

Note that

$$\begin{pmatrix} \dot{Y}(h) \\ \dot{\xi}(h) \end{pmatrix} \in \begin{pmatrix} 0 \\ \beta\xi(h) \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -P & 0 \end{pmatrix} \Phi(x, Y(h)) := \Psi(Y(h), \xi(h)),$$

where Ψ is upper semicontinuous since Φ is upper semicontinuous. Equation (5.3.8) can be rewritten as

$$\inf_{(p,q)\in\Phi(x,y)} \left\langle \begin{pmatrix} 0 \\ \beta\xi \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -P & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} \zeta \\ \sigma \end{pmatrix} \right\rangle \leq 0,$$

which, by the definition of Ψ, is equivalent to

$$\inf_{(p',q')\in\Psi(x,y)} \langle (p', q'), (\zeta, \sigma)\rangle \leq 0.$$

Then we deduce that

$$\Psi(y,\xi) \cap \mathcal{T}_{\text{epi }(u)}(y, u(y)) \neq \emptyset, \ \text{for } (y,\xi) \in \text{epi }(u).$$

For any $(y,\xi) \in \text{epi }(u)$, if $\xi \neq u(y)$, i.e. $\xi > u(y)$, then $(y,\xi) \in$ *int* epi (u), we have

$$\mathcal{T}_{\text{epi }(u)}(y,\xi) = \mathbf{R}^3 \supseteq \mathcal{T}_{\text{epi }(u)}(y, u(y)).$$

Thus,

$$\Psi(y,\xi) \cap \mathcal{T}_{\text{epi }(u)}(y, \xi) \neq \emptyset, \ \text{for } (y,\xi) \in \text{epi }(u).$$

Since $(y, u(y)) \in \text{epi }(u)$ and Ψ are usc, the viability theorem [15, pp. 180] yields that problem (5.3.7) has a viable solution $(\overline{X}(\cdot), \overline{Y}(\cdot), \overline{\xi}(\cdot))$, i.e.

$$(\overline{Y}(h), \overline{\xi}(h)) \in \text{epi }(u), \ \forall\, h \geq 0,$$

which leads to $u(\overline{Y}(h)) \leq \overline{\xi}(h), \ \forall\, h \geq 0.$ □

Theorem 5.3.6 (Subsolution implies sub-optimality). Let $u : \mathbf{R}^2 \to \mathbf{R}$ be a subsolution of (5.3.1), then u satisfies the sub-optimality.

To do the proof, we need the following result

Proposition 5.3.7. Let u be a subsolution of (5.3.1). Suppose that $\mathcal{M} \in S_1 \cup S_2 \cup I$ and Ω is a finite union of sets contained in $S_1 \cup S_2 \cup I$ with $\mathcal{M} \subseteq \overline{\Omega}$. Assume that Ω has the following property: for any $0 \leq a \leq b$ and any trajectory $(X, Y) \in S[x, y]$ with $Y(\cdot) \subset \Omega$, we have

$$u(Y(a)) \leq \int_a^b e^{-\beta(s-a)} P \cdot \dot{X}(s) ds + e^{-\beta(b-a)} u(Y(b)). \tag{5.3.9}$$

Then for any trajectory $(X, Y) \in S[x, y]$ with $Y(\cdot) \subseteq \Omega \cup \mathcal{M}$, we still have

$$u(Y(a)) \leq \int_a^b e^{-\beta(s-a)} P \cdot \dot{X}(s) ds + e^{-\beta(b-a)} u(Y(b)).$$

Proof. Let $(X, Y) \in S[x, y]$ with $Y(\cdot) \subseteq \Omega \cup \mathcal{M}$ satisfying the property (5.3.9). Without loss of generality, suppose that $Y(a) \in \mathcal{M}, Y(b) \in \mathcal{M}$ (otherwise we consider the first arrival time and the last exit time of Y for $\mathcal{M}$). Let $J := \{s \in [a, b] : Y(s) \notin \mathcal{M}\}$, which is an open set and so can be written as

$$J = \bigcup_{n=1}^{\infty} (a_n, b_n)$$

where the intervals are disjoint. For a fixed $p \in \mathbf{N}$, we set

$$J_p := \bigcup_{n=1}^{p} (a_n, b_n)$$

as the union of the first p intervals which, without loss of generality, after reindexing can be assumed to satisfy

$$a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_p < b_p.$$

We set $b_0 := a$ and $a_{p+1} := b$. Then $a \leq a_1$ and $b_p \leq b$. For $n = 1, \ldots, p$, $Y(s) \in \Omega$ for $s \in (a_n, b_n)$. Let $\eta > 0$ small enough such that $[a_n + \eta, b_n - \eta] \subset (a_n, b_n)$, then by (5.3.9)

$$u(Y(a_n + \eta)) \leq \int_{a_n+\eta}^{b_n-\eta} e^{-\beta(s-a_n-\eta)} P \cdot \dot{X}(s) ds + e^{-\beta(b_n-a_n-2\eta)} u(Y(b_n - \eta)).$$

Taking $\eta \to 0$ and by the continuity of u, $Y(\cdot)$ and the integral, we deduce that

$$u(Y(a_n)) \leq \int_{a_n}^{b_n} e^{-\beta(s-a_n)} P \cdot \dot{X}(s) ds + e^{-\beta(b_n-a_n)} u(Y(b_n)).$$

Next we need to deal with $Y(\cdot)$ restricted to $[b_n, a_{n+1}]$. For $n = 0, \ldots, p$, we note that $Y(s) \in \mathcal{M}$ for all $s \in [b_n, a_{n+1}] \backslash J$, then $(\dot{X}(s), \dot{Y}(s)) \in \Phi(x, Y(s)) \cap (\mathbf{R}^d \times \mathcal{T}_{\mathcal{M}}(Y(s)))$ for almost all $s \in [b_n, a_{n+1}] \backslash J$. For $n = 0, \ldots, p$, set $\eta_n := meas\big([b_n, a_{n+1}] \cap J\big)$, and note that $\sum_{n=0}^{p} \eta_n = meas(J \backslash J_p)$.

Then we have

$$\left|\int_{b_n}^{a_{n+1}} e^{-\beta(s-b_n)} P \cdot \dot{X}(s) ds\right| \leq M|P|\eta_n.$$

We now calculate how far $(X(\cdot), Y(\cdot))$ is from a trajectory lying in $\mathbf{R}^d \times \mathcal{M}$ with dynamics $\Phi(X(\cdot), Y(\cdot)) \cap (\mathbf{R}^d \times \mathcal{T}_{\mathcal{M}}(Y(\cdot)))$ by

$$\delta_n := \int_{b_n}^{a_{n+1}} \operatorname{dist}\Big((\dot{X}(s), \dot{Y}(s)), \Phi(X(s), Y(s)) \cap (\mathbf{R}^d \times \mathcal{T}_{\mathcal{M}}(Y(s)))\Big)\, ds \leq \frac{2M}{\varepsilon}\eta_n,$$

where ε is given in (5.1.2). By the Filippov approximation theorem (see [58, Theorem 3.1.6]) and also [60, Proposition 3.2]), there exists a trajectory $(X_n, Z_n)(\cdot)$ of $\Phi(x, Z_n(\cdot)) \cap (\mathbf{R}^d \times \mathcal{T}_{\mathcal{M}}(Z_n(\cdot)))$ defined on the interval $[b_n, a_{n+1}]$ that lies in $\mathbf{R}^d \times \mathcal{M}$ with $Z_n(b_n) = Y(b_n)$ and satisfies for any $, s \in [b_n, a_{n+1}]$

$$\big\|(X_n, Z_n)(s) - (X, Y)(s)\big\| \leq e^{L(s-b_n)/\varepsilon}\delta_n \leq \frac{2M}{\varepsilon} e^{L(s-b_n)/\varepsilon}\eta_n \leq \frac{2M}{\varepsilon} e^{L(a_{n+1}-b_n)/\varepsilon}\eta_n. \tag{5.3.10}$$

Since $(X_n, Z_n)(\cdot)$ lies in $\mathbf{R}^d \times \mathcal{M}$ and is driven by $\Phi(x, Z_n(\cdot)) \cap (\mathbf{R}^d \times \mathcal{T}_{\mathcal{M}}(Z_n(\cdot)))$ which is Lipschitz continuous, the subsolution property of u implies that

$$u(Z_n(b_n)) \leq \int_{b_n}^{a_{n+1}} e^{-\beta(s-b_n)} P \cdot \dot{X}_n(s) ds + e^{-\beta(a_{n+1}-b_n)} u(Z_n(a_{n+1})).$$

Then by (5.3.10) we have

$$\begin{aligned}
& u(Y(b_n)) = u(Z_n(b_n)) \\
\leq\ & \int_{b_n}^{a_{n+1}} e^{-\beta(s-b_n)} P \cdot \dot{X}(s) ds + \left(\int_{b_n}^{a_{n+1}} e^{-\beta(s-b_n)} ds \|P\| L \frac{2M}{\varepsilon} e^{L(a_{n+1}-b_n)/\varepsilon}\eta_n\right) \\
& + e^{-\beta(a_{n+1}-b_n)} u(Y(a_{n+1})) + e^{-\beta(a_{n+1}-b_n)} L_u \cdot \frac{2M}{\varepsilon} e^{L(a_{n+1}-b_n)/\varepsilon}\eta_n \\
\leq\ & \int_{b_n}^{a_{n+1}} e^{-\beta(s-b_n)} P \cdot \dot{X}(s) ds + e^{-\beta(a_{n+1}-b_n)} u(Y(a_{n+1})) \\
& + (\|P\| L + L_u) \frac{2M}{\varepsilon} e^{L(a_{n+1}-b_n)/\varepsilon}\eta_n,
\end{aligned} \tag{5.3.11}$$

where L_u is the Lipschitz constant of u. Then we deduce that

$$\begin{aligned}
u(Y(a_n)) \ \leq\ & \int_{a_n}^{b_n} e^{-\beta(s-a_n)} P \cdot \dot{X}(s) ds + \int_{b_n}^{a_{n+1}} e^{-\beta(s-a_n)} P \cdot \dot{X}(s) ds \\
& + e^{-\beta(a_{n+1}-a_n)} u(Y(a_{n+1})) + e^{-\beta(b_n-a_n)} \cdot (\|P\| L + L_u) \frac{2M}{\varepsilon} e^{L(a_{n+1}-b_n)/\varepsilon}\eta_n \\
\leq\ & \int_{a_n}^{a_{n+1}} e^{-\beta(s-a_n)} P \cdot \dot{X}(s) ds + e^{-\beta(a_{n+1}-a_n)} u(Y(a_{n+1})) \\
& + (\|P\| L + L_u) \frac{2M}{\varepsilon} e^{L(a_{n+1}-b_n)/\varepsilon}\eta_n.
\end{aligned} \tag{5.3.12}$$

By using (5.3.11) for $n=0$ and (5.3.12) for $n=1,\dots,p$, we obtain

$$
\begin{aligned}
u(Y(a)) &\le \int_a^{a_1} e^{-\beta(s-a)}P\cdot\dot{X}(s)ds + e^{-\beta(a_1-a)}u(Y(a_1)) + (\|P\|L+L_u)\frac{2M}{\varepsilon}e^{L(a_1-b_0)/\varepsilon}\eta_0\\
&\le \int_a^{a_2} e^{-\beta(s-a)}P\cdot\dot{X}(s)ds + e^{-\beta(a_2-a)}u(Y(a_2)) + (\|P\|L+L_u)\frac{2M}{\varepsilon}e^{L(a_2-b_0)/\varepsilon}(\eta_0+\eta_1)\\
&\cdots\\
&\le \int_a^{a_{p+1}} e^{-\beta(s-a)}P\cdot\dot{X}(s)ds + e^{-\beta(a_{p+1}-a)}u(Y(a_{p+1})) + (\|P\|L+L_u)\frac{2M}{\varepsilon}e^{L(a_{p+1}-b_0)/\varepsilon}\sum_{n=0}^{p}\eta_n\\
&= \int_a^{b} e^{-\beta(s-a)}P\cdot\dot{X}(s)ds + e^{-\beta(b-a)}u(Y(b)) + (\|P\|L+L_u)\frac{2M}{\varepsilon}e^{L(b-a)/\varepsilon}\sum_{n=0}^{p}\eta_n.
\end{aligned}
$$

By taking $p\to+\infty$, we have $\sum_{n=0}^{p}\eta_n = meas(J\backslash J_p)\to 0$ and the desired result is obtained. □

Now we state the proof of Theorem 5.3.6.

Proof of Theorem 5.3.6. Let u be a subsolution of (5.3.1). For any trajectory $(X,Y)(\cdot)\in S[x,y]$, any $[a,b]\subset[0,+\infty)$, we want to prove that (5.3.9) is true, i.e.

$$
u(Y(a)) \le \int_a^{b} e^{-\beta(s-a)}P\cdot\dot{X}(s)ds + e^{-\beta(b-a)}u(Y(b)).
$$

We set

$$
\Omega = \{\mathcal{M}\in S_1\cup S_2\cup I \,|\, \exists\, s\in[a,b] \text{ such that } Y(s)\in\mathcal{M}\}\,.
$$

Note that Ω is connected since $Y(\cdot)$ is continuous.

Let d_Ω be the minimal dimension of the manifolds contained in Ω.

Case 1: $d_\Omega = 2$**.**
Then $\Omega\subset\Omega_1\cup\Omega_2$. Since $Y(\cdot)$ is continuous, then $Y|_{[a,b]}$ lies entirely in Ω_1 or Ω_2. Since the dynamics g_i of $Y(\cdot)$ is Lipschitz continuous, then the subsolution property of u implies that u satisfies the sub-optimality along $(X,Y)|_{[a,b]}$, i.e. (5.3.9) holds true.

Case 2: $d_\Omega = 1$**.**
Two cases can happen.

Case 2.1: Ω contains only one manifold
In that case, $\Omega\in I$ with dimension 1, then the subsolution property of u implies (5.3.9) since the dynamics $\Phi\cap(\mathbf{R}^d\times\mathcal{T}_\Omega)$ is Lipschitz continuous on Ω.

Case 2.2: Ω contains more than one manifold
Let $\mathcal{M}'_1,\dots,\mathcal{M}'_p$ be all the manifolds contained in Ω with dimension 1. Then $\Omega' := \Omega\backslash\left(\cup_{k=1}^{p}\mathcal{M}'_k\right)$ contains only manifolds of dimension 2. For any $(\overline{X},\overline{Y})\in S[x,y]$ with $\overline{Y}(\cdot)\subset\Omega'$, (5.3.9) is satisfied

(see Case 1). Then using Proposition 5.3.7, we get that (5.3.9) holds true for every trajectory $(\overline{X}, \overline{Y}) \in S[x, y]$ with $\overline{Y}(\cdot) \subset \Omega' \cup \mathcal{M}'_1$ because $\mathcal{M}'_1 \subset \overline{\Omega'}$. By induction, (5.3.9) holds true for every $(\overline{X}, \overline{Y}) \in S[x, y]$ with $\overline{Y}(\cdot) \subset \Omega' \cup \mathcal{M}'_1 \cup \cdots \cup \mathcal{M}'_p = \Omega$.

Case 3: $d_\Omega = 0$**.**
The arguments are quite similar to the ones of Case 2 and we skip it.

Finally, to complete the proof, we remark that the sub-optimality of u is proved by taking $a = 0, b = h$ in (5.3.9). □

We are now ready to prove the following comparison principle

Lemma 5.3.8 (Comparison principle for (5.3.1)). Let $u, w : \mathbf{R}^2 \to \mathbf{R}$ be Lipschitz continuous functions. Suppose that u is a subsolution of (5.3.1) and w is a supersolution of (5.3.1). Then we have

$$u(y) \leq w(y), \;\forall\, y \in \mathbf{R}^2.$$

Proof. By contradiction, suppose that

$$\sup_{y \in \mathbf{R}^2} \{u(y) - w(y)\} := M > 0. \tag{5.3.13}$$

Then there exists $y_0 \in \mathbf{R}^2$ such that

$$u(y_0) - w(y_0) > \frac{M}{2}. \tag{5.3.14}$$

Since w is a supersolution, by Theorem 5.3.5, w satisfies the super-optimality, i.e. $\exists (\bar{X}, \bar{Y}) \in S[y_0]$ such that

$$w(y_0) \geq e^{-\beta h} w(\bar{Y}(h)) + \int_0^h e^{-\beta s} P \cdot \dot{\bar{X}}(s) ds, \;\forall\, h \geq 0. \tag{5.3.15}$$

Since u is a subsolution, by Theorem 5.3.6, u satisfies the sub-optimality, i.e.

$$u(y_0) \leq e^{-\beta h} u(\bar{Y}(h)) + \int_0^h e^{-\beta s} P \cdot \dot{\bar{X}}(s) ds, \;\forall\, h \geq 0. \tag{5.3.16}$$

Equations (5.3.15) and (5.3.16) leads to

$$u(y_0) - w(y_0) \leq e^{-\beta h}(u(\bar{Y}(h)) - w(\bar{Y}(h))), \;\forall\, h \geq 0.$$

If there exists $h_0 > 0$ such that $\bar{Y}(h) = y_0$, then we deduce that

$$u(y_0) - w(y_0) \leq 0,$$

which contradicts (5.3.14). Otherwise, we set $z_h = \bar{Y}(h)$ with $z_h \neq y_0$ and $h = \log 2/\beta$. We then have

$$u(z_h) - w(z_h) \geq e^{\beta h}(u(y_0) - w(y_0)) > e^{\beta h}\frac{M}{2} = M,$$

which is a contradiction to (5.3.13). Thus $M \leq 0$ and the desired result holds. □

We now give the proof of Theorem 5.3.1.

Proof of Theorem 5.3.1. The fact that w^β is a viscosity solution of (5.3.1) is a consequence of Proposition 5.3.4. The uniqueness is deduced from Lemma 5.3.8. □

5.3.2 Proof of Theorem 5.1.2

Before we start the proof, we need the following stability result.

Lemma 5.3.9. Let v^β be the viscosity solution of

$$\beta v^\beta(y) + a^\beta + H^E(x, y, P, Dv^\beta(y)) = 0 \tag{5.3.17}$$

with $a^\beta \in \mathbf{R}$. Assume that there exist $\lambda \in \mathbf{R}$ and $v : \mathbf{R}^2 \to \mathbf{R}$ such that

$$\beta v^\beta + a^\beta \to -\lambda \text{ uniformly and } v^\beta \to v \text{ uniformly when } \beta \to 0.$$

Then v is a viscosity solution of (5.1.5).

Proof. We first prove that v is a subsolution. Let $y_0 \in \mathbf{R}^2$, $\phi \in C(\mathbf{R}^2)$ and $\phi \in C^1(\overline{\mathcal{M}})$ for each $\mathcal{M} \in S_1 \cup S_2 \cup I$ with $y_0 \in \overline{\mathcal{M}}$ such that $v(x) - \phi(x)$ attains a strict maximum at y_0. We want to prove that

$$-\lambda + \sup_{(p,q)\in\Phi_{\mathcal{M}}(x,y_0)\cap(\mathbf{R}^d\times\mathcal{T}_{\overline{\mathcal{M}}}(y_0))} \{-p \cdot P - q \cdot D_{\overline{\mathcal{M}}}\phi(y_0)\} \leq 0.$$

Let $\mathcal{M} \in S_1 \cup S_2 \cup I$ such that $y_0 \in \overline{\mathcal{M}}$.

For any $y \in \mathbf{R}^2$, let $P_{\overline{\mathcal{M}}}(y)$ be the projection of y on $\overline{\mathcal{M}}$, and $\text{dist}(y, \overline{\mathcal{M}})$ be the distance function to $\overline{\mathcal{M}}$. Consider the penalized function $\Psi(y) := v(y) - \phi(y) - C\text{dist}(y, \overline{\mathcal{M}})$ with

$$C > \|Dv^\beta - D\phi\|.$$

We have

$$v(y) - \phi(y) - C\text{dist}(y, \overline{\mathcal{M}}) \leq v(y) - \phi(y) < v(y_0) - \phi(y_0), \ \forall\, y \neq y_0,$$

which implies that $v(y) - \phi(y) - C\text{dist}(y, \overline{\mathcal{M}})$ attains a strict maximum at y_0. Since $v^\beta \to v$ uniformly, $v^\beta - \phi + C\text{dist}(y, \overline{\mathcal{M}})$ attains a local maximum at some y_β with $y_\beta \to y_0$. For any

$y \notin \overline{\mathcal{M}}$, we have

$$\begin{aligned} & v^\beta(y) - \phi(y) - C\text{dist}(y, \overline{\mathcal{M}}) \\ \leq \; & v^\beta(P_{\overline{\mathcal{M}}}(y)) - \phi(P_{\overline{\mathcal{M}}}(y)) + \|Dv^\beta - D\phi\| \cdot \|y - P_{\overline{\mathcal{M}}}(y)\| - C\text{dist}(y, \overline{\mathcal{M}}) \\ < \; & v^\beta(P_{\overline{\mathcal{M}}}(y)) - \phi(P_{\overline{\mathcal{M}}}(y)). \end{aligned}$$

Then we deduce that the maximum $y_\beta \in \overline{\mathcal{M}}$.

v^β is the subsolution of (5.3.17), thus

$$\beta v^\beta(y_\beta) + a_\beta + \sup_{(p,q)\in\Phi_{\mathcal{M}}(x,y_\beta)\cap(\mathbf{R}^d\times\mathcal{T}_{\overline{\mathcal{M}}}(y_\beta))} \{-p\cdot P - q\cdot D\phi(y_\beta)\} \leq 0. \tag{5.3.18}$$

We claim that

$$\mathcal{T}_{\overline{\mathcal{M}}}(y_0) \subseteq \mathcal{T}_{\overline{\mathcal{M}}}(y_\beta). \tag{5.3.19}$$

If $y_0 \in$ *r-int* $\overline{\mathcal{M}}$ (the relative interior of $\overline{\mathcal{M}}$), then $y_\beta \in$ *r-int* $\overline{\mathcal{M}}$ for β small enough. Therefore,

$$\mathcal{T}_{\overline{\mathcal{M}}}(y_\beta) = \mathcal{T}_{\mathcal{M}}(y_\beta) = \mathcal{T}_{\mathcal{M}}(y_0) = \mathcal{T}_{\overline{\mathcal{M}}}(y_0).$$

If $y_0 \in$ *r-bdry* $\overline{\mathcal{M}}$ (the relative boundary of $\overline{\mathcal{M}}$), note that $y_\beta \to y_0$ and $y_\beta \in \overline{\mathcal{M}}$, then $y_\beta \in$ *r-bdry* $\overline{\mathcal{M}}$ or $y_\beta \in$ *r-int* $\overline{\mathcal{M}}$. If $y_\beta \in$ *r-bdry* $\overline{\mathcal{M}}$, then

$$\mathcal{T}_{\overline{\mathcal{M}}}(y_\beta) = \mathcal{T}_{\overline{\mathcal{M}}}(y_0).$$

Otherwise $y_\beta \in$ *r-int* $\overline{\mathcal{M}}$, then

$$\mathcal{T}_{\overline{\mathcal{M}}}(y_0) \subset \mathcal{T}_{\mathcal{M}}(y_\beta) = \mathcal{T}_{\overline{\mathcal{M}}}(y_\beta).$$

Finally, we conclude that (5.3.19) holds true.

Equations (5.3.18) and (5.3.19) implies that

$$\beta v^\beta(y_\beta) + a^\beta + \sup_{(p,q)\in\Phi_{\mathcal{M}}(x,y_\beta)\cap(\mathbf{R}^d\times\mathcal{T}_{\overline{\mathcal{M}}}(y_0))} \{-p\cdot P - q\cdot D\phi(y_\beta)\} \leq 0.$$

By letting $\beta \to 0$, we obtain

$$-\lambda + \sup_{(p,q)\in\Phi_{\mathcal{M}}(x,y_0)\cap(\mathbf{R}^d\times\mathcal{T}_{\overline{\mathcal{M}}}(y_0))} \{-p\cdot P - q\cdot D_{\overline{\mathcal{M}}}\phi(y_0))\} \leq 0.$$

Now we prove that v is a supersolution. Let $\phi \in C^1(\mathbf{R}^2)$ such that $v - \phi$ attains a strict minimum at y_0. Since $v^\beta \to v$ uniformly, $v^\beta - \phi$ attains a minimum at some y_β such that $y_\beta \to y_0$. Then we have

$$\beta v^\beta(y_\beta) + a^\beta + \sup_{(p,q)\in\Phi^E(x,y_\beta)} \{-p\cdot P - q\cdot D\phi(y_\beta))\} \geq 0.$$

Since $\Phi^E(\cdot) \subseteq \Phi(\cdot)$, we have

$$\beta v^\beta(y_\beta) + a^\beta + \sup_{(p,q)\in\Phi(x,y_\beta)} \{-p \cdot P - q \cdot D\phi(y_\beta))\} \geq 0.$$

By sending $\beta \to 0$ and the upper semi-continuity of Φ, we get

$$-\lambda + \sup_{(p,q)\in\Phi(x,y_0)} \{-p \cdot P - q \cdot D\phi(y_0))\} \geq 0,$$

which, by [123, Proposition 3.5, Theorem 3.7], is equivalent to

$$-\lambda + \sup_{(p,q)\in\Phi^E(x,y_0)} \{-p \cdot P - q \cdot D\phi(y_0))\} \geq 0.$$

□

Now we state the proof of Theorem 5.1.2.

Proof of Theorem 5.1.2. By Theorem 5.3.1, given $x \in \mathbf{R}^d$, $P \in \mathbf{R}^d$, for each $\beta > 0$, we know that the approximating problem

$$\beta w^\beta(y) + H^E(x, y, P, Dw^\beta(y)) = 0, \text{ for } y \in \mathbf{R}^2$$

has a unique bounded Lipschitz continuous viscosity solution w^β.

Step 1: Estimate on w^β.
We now prove that w^β is S-periodic. For $k \in \mathbf{Z}^2$, we set $\widetilde{w}^\beta(y) := w^\beta(y + k)$. It is then easy to check that $\widetilde{w}^\beta$ is still a solution of (5.3.1). Thus, by uniqueness, we get

$$\widetilde{w}^\beta = w^\beta,$$

which implies that w^β is S-periodic.

Since w^β is uniformly Lipschitz continuous (see Proposition 5.3.3), then w^β is differentiable almost everywhere and

$$\sup_{0<\beta<1} \|Dw^\beta\| \leq C_1.$$

Moreover, by (5.3.2), we get that

$$\|\beta w^\beta\| \leq \|P\| M. \tag{5.3.20}$$

Let $v^\beta = w^\beta - \min_S w^\beta$. Since w^β is continuous and periodic, there exists $y_0 \in S$ such that $v^\beta = w^\beta - w^\beta(x_0)$. Then

$$\|v^\beta\| \leq 2\sqrt{2}\|Dw^\beta\| \leq 2\sqrt{2}C_1, \quad Dv^\beta = Dw^\beta. \tag{5.3.21}$$

Using the fact that w^β is a viscosity solution of (5.3.1), we get that v^β is a viscosity solution of

$$\beta v^\beta(y) + H^E(x, y, P, Dv^\beta(y)) = -\min_S(\beta w^\beta), \ \forall\, y \in \mathbf{R}^2.$$

Step 2: Passing to the limit
Using (5.3.20), (5.3.21) and Arzela-Ascoli Theorem, up to a subsequence, we get

$$v^\beta \to v \text{ uniformly on } \mathbf{R}^2 \quad \text{and} \quad \min_S(\beta w^\beta) \to -\lambda$$

for some v Lipschitz continuous and S-periodic and $\lambda \in \mathbf{R}$. Moreover, since v^β is uniformly bounded (see (5.3.21)), we get

$$\beta v^\beta \to 0 \text{ uniformly on } \mathbf{R}^2.$$

Then by Lemma 5.3.9, we deduce that

$$H^E(x, y, P, Dv(y)) = \lambda.$$

Step 3: Uniqueness of λ
Suppose that there exists (v_1, λ_1) and (v_2, λ_2) solutions of the cell problem (5.1.5) with $\lambda_1 \neq \lambda_2$. Assume without lost of generality that $\lambda_1 < \lambda_2$. Note that v_1, v_2 are both continuous and periodic, thus they are bounded. By adding a suitable constant to v_1, we may assume that $v_1 > v_2$.

Since $\lambda_1 < \frac{\lambda_1+\lambda_2}{2} < \lambda_2$, v_1, v_2 are bounded, we deduce that for ε small enough, v_1, v_2 are respectively subsolution and supersolution of

$$\varepsilon v + H^E(x, y, P, Dv) = \frac{\lambda_1 + \lambda_2}{2}.$$

Using the comparison principle for the equation (5.3.1), we obtain $v_1 \leq v_2$ which is a contradiction. $\square$

5.4 Properties of the effective Hamiltonian

For every $x \in \mathbf{R}^d, P \in \mathbf{R}^d$, we denote by $\overline{H}(x, P)$ the unique constant such that there exists a periodic solution of (5.1.5).

Proposition 5.4.1. $\overline{H}(\cdot, \cdot)$ is Lipschitz continuous.

Proof. Let $x_1, x_2 \in \mathbf{R}^d$ and $P_1, P_2 \in \mathbf{R}^2$. For each $\beta > 0$, suppose that w_i^β, $i = 1, 2$ is a solution of

$$\beta w_i^\beta(y) + H^E(x_i, y, P_i, Dw_i^\beta) = 0,$$

For any $y \in \mathbf{R}^2$, $q \in \mathbf{R}^2$, by the Lipschitz continuity of $H^E(\cdot, y, \cdot, q)$, there exists $C > 0$ such that

$$H^E(x_2, y, P_2, q) \leq H^E(x_1, y, P_1, q) + C(\|x_1 - x_2\| + \|P_1 - P_2\|).$$

Then we deduce that $w_1^\beta - \frac{C}{\beta}(\|x_1 - x_2\| + \|P_1 - P_2\|)$ is a subsolution of

$$\beta w^\beta + H^E(x_2, y, P_2, Dw^\beta) = 0.$$

By the comparison principle for (5.3.1), we get

$$w_1^\beta - \frac{C}{\beta}(\|x_1 - x_2\| + \|P_1 - P_2\|) \leq w_2^\beta,$$

i.e.

$$\beta w_1^\beta - \beta w_2^\beta \leq C(\|x_1 - x_2\| + \|P_1 - P_2\|).$$

Letting $\beta \to 0$ leads to

$$\overline{H}(x_1, P_1) - \overline{H}(x_2, P_2) \leq C(\|x_1 - x_2\| + \|P_1 - P_2\|).$$

Exchanging the role of (x_1, P_1) and (x_2, P_2), we conclude that

$$|\overline{H}(x_1, P_1) - \overline{H}(x_2, P_2)| \leq C(\|x_1 - x_2\| + \|P_1 - P_2\|),$$

which implies the Lipschitz continuity of $\overline{H}(\cdot, \cdot)$. □

As studied in [3, 19], the effective Hamiltonian $\overline{H}$ can be evaluated as the optimal average cost of an ergodic control problem in the y variable.

Proposition 5.4.2. Given $x \in \mathbf{R}^d$, $P \in \mathbf{R}^d$,

$$\overline{H}(x, P) = \lim_{t \to +\infty} \sup_{(X,Y) \in S[x,y]} \left\{ -\frac{1}{t} P \cdot (X(t) - x) \right\}, \tag{5.4.1}$$

for any $y \in \mathbf{R}^2$.

Proof. This result is quite similar to the formula (10) obtained in [3]. Here we give a sketch of the proof. Consider the value function

$$v(t, y) = \inf_{(X,Y) \in S[x,y]} \{P \cdot (X(t) - x)\}.$$

Then v solves the HJB equation

$$\partial_t v(t, y) + H^E(x, y, P, D_y v(t, y)) = 0 \text{ on } (0, +\infty) \times \mathbf{R}^2,$$

where x, P are fixed, and the initial condition $v(0, \cdot) \equiv 0$. Let $w(\cdot)$ be a solution of the cell problem (5.1.5) with $\lambda = \overline{H}(x, P)$, then $w(y) - t\overline{H}(x, P)$ is a solution of the same Cauchy problem but with a different initial condition. Note that the HJB equation above is the same type as (5.1.4), the comparison result Theorem 5.2.8 implies that $v(t, y) - w(y) + t\overline{H}(x, P)$ is bounded by $\|w\|_\infty$. Since w is bounded, $-v(t, y)/t \to \overline{H}(x, P)$ as $t \to +\infty$, uniformly in y. □

Remark 5.4.3. If we consider the same case as in [17] where the controls acting on the slow variable X and and fast variable Y are separated, more precisely given A, B two independent control sets,

$$f = f(x, y, a),\ a \in A,\ g_i = g_i(x, y, b),\ b \in B,\ i = 1, 2.$$

Let $H_1 : \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^d \to \mathbf{R}$ defined by

$$H_1(x, y, P) = \sup_{a \in A}\{-P \cdot f(x, y, a)\}.$$

Then the effective Hamiltonian satisfies the following formula:

$$\overline{H}(x, P) = \max_{y \in \mathbf{R}^2} H_1(x, y, P),\ \forall x \in \mathbf{R}^d,\ P \in \mathbf{R}^d.$$

This is the same formula (12) obtained in [3]. It is proved through the formula (5.4.1) and the controllability assumption on the fast variable Y.

5.5 Proof of Theorem 5.1.3

We define

$$\overline{u}(t, x) = \limsup_{t,x,\varepsilon}\ \sup_{y \in \mathbf{R}^2} u^\varepsilon(t, x, y) \quad \text{and} \quad \underline{u}(t, x) = \liminf_{t,x,\varepsilon}\ \inf_{y \in \mathbf{R}^2} u^\varepsilon(t, x, y).$$

The proof is divided into several steps.

Step 1 : $\overline{u}$ is subsolution of (5.1.6).
Let $\phi \in C^1((0, T) \times \mathbf{R}^d)$ such that $\overline{u} - \phi$ has a strict local maximum at (t_0, x_0). We want to prove that

$$-\phi_t(t_0, x_0) + \overline{H}(x_0, D\phi(t_0, x_0)) \le 0.$$

We assume by contradiction that

$$-\phi_t(t_0, x_0) + \overline{H}(x_0, D\phi(t_0, x_0)) = \theta > 0. \tag{5.5.1}$$

We set $P := D\phi(t_0, x_0)$ and let v be a periodic Lipschitz continuous viscosity solution of the cell problem

$$-\overline{H}(x_0, P) + H^E(x_0, y, P, Dv(y)) = 0.$$

We use the perturbed test function introduced by Evans. For any $\varepsilon > 0$, we define $\phi^\varepsilon(t, x, y) = \phi(t, x) + \varepsilon v(y)$. We want to prove that ϕ^ε is a supersolution of (5.1.4) in $B((t_0, x_0), r) \times \mathbf{R}^2$ for

$r > 0$ small enough. Let $\psi \in C^1((0,T)\times \mathbf{R}^d \times \mathbf{R}^2)$ such that $\phi^\varepsilon - \psi$ attains a minimum at $(t_1, x_1, y_1) \in B((t_0, x_0), r) \times \mathbf{R}^2$. Then

$$\phi^\varepsilon(t_1, x_1, y_1) - \psi(t_1, x_1, y_1) \le \phi^\varepsilon(t, x, y) - \psi(t, x, y).$$

This implies that

$$v(y_1) - \Gamma(y_1) \le v(y) - \Gamma(y)$$

where $\Gamma(y) = \frac{1}{\varepsilon}[\psi(t_1, x_1, y) - \phi(t_1, x_1)]$. We deduce that $v(y) - \Gamma(y)$ attains a minimum at y_1, then

$$-\overline{H}(x_0, P) + H^E(x_0, y_1, P, D\Gamma(y_1)) \ge 0,$$

i.e.

$$-\phi_t(t_0, x_0) - \theta + H^E(x_0, y_1, D\phi(t_0, x_0), \frac{1}{\varepsilon} D_y\psi(t_1, x_1, y_1)) \ge 0.$$

We then deduce that

$$-\phi_t(t_1, x_1) + H^E(x_1, y_1, D\phi(t_1, x_1), \frac{1}{\varepsilon} D_y\psi(t_1, x_1, y_1)) \ge \phi_t(t_0, x_0) - \phi_t(t_1, x_1) - \theta$$
$$+H^E(x_1, y_1, D\phi(t_1, x_1), \frac{1}{\varepsilon} D_y\psi(t_1, x_1, y_1)) - H^E(x_0, y_1, D\phi(t_0, x_0), \frac{1}{\varepsilon} D_y\psi(t_1, x_1, y_1)).$$

Since $\phi \in C^1((0,T)\times \mathbf{R}^d)$ and $H^E(\cdot, y, \cdot, q)$ is continuous, we have for $r > 0$ small enough

$$-\phi_t(t_1, x_1) + H^E(x_1, y_1, D\phi(t_1, x_1), \frac{1}{\varepsilon} D_y\psi(t_1, x_1, y_1)) \ge \frac{\theta}{2}.$$

Note that $v(\cdot)$ is independent on t and x, the application $t \mapsto \phi(t, x_1) - \psi(t, x_1, y_1)$ is C^1 and attains a minimum at t_1 and the application $x \mapsto \phi(t_1, x) - \psi(t_1, x, y_1)$ is C^1 and attains a minimum at x_1, we obtain

$$\phi_t(t_1, x_1) = \psi_t(t_1, x_1, y_1), \;\; D\phi(t_1, x_1) = D_x\psi(t_1, x_1, y_1).$$

We conclude that

$$-\psi_t(t_1, x_1, y_1) + H^E(x_1, y_1, D_x\psi(t_1, x_1, y_1), \frac{1}{\varepsilon} D_y\psi(t_1, x_1, y_1)) \ge \frac{\theta}{2},$$

which implies that ϕ^ε is a supersolution of (5.1.4). Then by Theorem 5.2.8, we have

$$\sup_{B((t_0,x_0),r)\times\mathbf{R}^2} \{u^\varepsilon(t,x,y) - \phi^\varepsilon(t,x,y)\} \le \sup_{\partial B((t_0,x_0),r)\times\mathbf{R}^2} \{u^\varepsilon(t,x,y) - \phi^\varepsilon(t,x,y)\}.$$

Then we deduce that

$$\sup_{(t,x)\in B((t_0,x_0),r)} \{\overline{u}(t,x) - \phi(t,x)\} \le \sup_{(t,x)\in \partial B((t_0,x_0),r)} \{\overline{u}(t,x) - \phi(t,x)\},$$

which contradicts the fact that (t_0, x_0) is a local strict maximum of $\overline{u} - \phi$.

Step 2 : $\underline{u}$ is a supersolution of (5.1.6).

The proof is very similar. The main difference is to check that ϕ^ε is a subsolution. By contradiction, assume that there exists $\phi \in C^1((0,T)\times \mathbf{R}^d)$ such that $\underline{u}-\phi$ has a strict local minimum at (t_0,x_0) and such that

$$-\phi_t(t_0,x_0)+\overline{H}(x_0,D\phi(t_0,x_0))=-\theta<0. \tag{5.5.2}$$

We set $P := D\phi(t_0,x_0)$ and let v be a periodic Lipschitz continuous viscosity solution of the cell problem

$$-\overline{H}(x_0,P)+H^E(x_0,y,P,Dv(y))=0.$$

For any $\varepsilon>0$, we define $\phi^\varepsilon(t,x,y)=\phi(t,x)+\varepsilon v(y)$. We want to prove that ϕ^ε is a supersolution of (5.1.4) in $B((t_0,x_0),r)\times\mathbf{R}^2$ for $r>0$ small enough. Let $\psi:(0,T)\times\mathbf{R}^d\times\mathbf{R}^2\to\mathbf{R}$ be continuous with $\psi|_{(0,T)\times\mathbf{R}^d\times\overline{\mathcal{M}}}$ being C^1 for each $\mathcal{M}\in S_1\cup S_2\cup I$ with $y_1\in\overline{\mathcal{M}}$ such that $\phi^\varepsilon-\psi$ attains a local maximum at $(t_1,x_1,y_1)\in B((t_0,x_0),r)\times\mathbf{R}^2$. As in the previous step, we deduce that $v-\Gamma$ reaches a maximum at y_1 where

$$\Gamma(y)=\frac{1}{\varepsilon}[\psi(t_1,x_1,y)-\phi(t_1,x_1)].$$

Then

$$-\overline{H}(x_0,P)+\sup_{(p,q)\in\Phi_{\mathcal{M}}(x_0,y_1)\cap(\mathbf{R}^d\times\mathcal{T}_{\overline{\mathcal{M}}}(y_1))}\{-p\cdot P-q\cdot D_{\overline{\mathcal{M}}}\psi(t_1,x_1,y_1))\}\le 0.$$

i.e.

$$-\phi_t(t_0,x_0)+\theta+\sup_{(p,q)\in\Phi_{\mathcal{M}}(x_0,y_1)\cap(\mathbf{R}^d\times\mathcal{T}_{\overline{\mathcal{M}}}(y_1))}\{-p\cdot D\phi(t_0,x_0)-q\cdot D_{\overline{\mathcal{M}}}\psi(t_1,x_1,y_1))\}\le 0.$$

Then we deduce that

$$\begin{aligned}
&-\phi_t(t_1,x_1)+\sup_{(p,q)\in\Phi_{\mathcal{M}}(x_1,y_1)\cap(\mathbf{R}^d\times\mathcal{T}_{\overline{\mathcal{M}}}(y_1))}\{-p\cdot D\phi(t_1,x_1)-q\cdot D_{\overline{\mathcal{M}}}\psi(t_1,x_1,y_1))\}\\
\le\ &\phi_t(t_0,x_0)-\phi_t(t_1,x_1)\\
&+\sup_{(p,q)\in\Phi_{\mathcal{M}}(x_1,y_1)\cap(\mathbf{R}^d\times\mathcal{T}_{\overline{\mathcal{M}}}(y_1))}\{-p\cdot D\phi(t_1,x_1)-q\cdot D_{\overline{\mathcal{M}}}\psi(t_1,x_1,y_1))\}\\
&-\sup_{(p,q)\in\Phi_{\mathcal{M}}(x_0,y_1)\cap(\mathbf{R}^d\times\mathcal{T}_{\overline{\mathcal{M}}}(y_1))}\{-p\cdot D\phi(t_0,x_0)-q\cdot D_{\overline{\mathcal{M}}}\psi(t_1,x_1,y_1))\}-\theta.
\end{aligned}$$

Since $\phi\in C^1((0,T)\times\mathbf{R}^d)$ and $\Phi_{\mathcal{M}}(\cdot,y_1)\cap(\mathbf{R}^d\times\mathcal{T}_{\overline{\mathcal{M}}}(y_1))$ are continuous, we have for $r>0$ small enough

$$-\phi_t(t_1,x_1)+\sup_{(p,q)\in\Phi_{\mathcal{M}}(x_1,y_1)\cap(\mathbf{R}^d\times\mathcal{T}_{\overline{\mathcal{M}}}(y_1))}\{-p\cdot D\phi(t_1,x_1)-q\cdot D_{\overline{\mathcal{M}}}\psi(t_1,x_1,y_1))\}\le-\frac{\theta}{2}.$$

Using that

$$\phi_t(t_1,x_1)=\psi_t(t_1,x_1,y_1),\ \ D\phi(t_1,x_1)=D_x\psi(t_1,x_1,y_1),$$

we conclude that

$$-\psi_t(t_1,x_1,y_1)+\sup_{(p,q)\in\Phi_{\mathcal{M}}(x_1,y_1)\cap(\mathbf{R}^d\times\mathcal{T}_{\overline{\mathcal{M}}}(y_1))}\{-p\cdot D\phi(t_1,x_1)-q\cdot D_{\overline{\mathcal{M}}}\psi(t_1,x_1,y_1))\}\le-\frac{\theta}{2},$$

which implies that ϕ^ε is a subsolution of (5.1.4). We then get a contradiction as in the previous step.

Step 3: Terminal condition

Now we check the terminal condition. We set

$$\underline{\varphi}(x):=\inf_{y\in\mathbf{R}^2}\varphi(x,y).$$

The Lipschitz continuity of φ implies that $\underline{\varphi}$ is Lipschitz continuous. Since $u^\varepsilon(T,x,y)=\varphi(x,y)$, we have

$$\inf_{y\in\mathbf{R}^2}u^\varepsilon(T,x,y)=\underline{\varphi}(x).$$

Then we deduce that $\underline{u}(T,x)=\underline{\varphi}(x)$.

On the other hand, for any $t\in[0,T]$, $x\in\mathbf{R}^d$ and $y\in S(=(-1,1)^2)$,

$$\begin{aligned}u^\varepsilon(t,x,y) &= \inf_{(X,Y)\in S^\varepsilon_{[t,T]}(x,y)}\{\varphi(X(T),Y(T))\}\\ &\le \inf_{(X,Y)\in S^\varepsilon_{[t,T]}(x,y)}\{\varphi(x,Y(T))+L_\varphi\|x-X(T)\|\}\\ &\le \inf_{(X,Y)\in S^\varepsilon_{[t,T]}(x,y)}\varphi(x,Y(T))+ML_\varphi(T-t).\end{aligned}$$

By the controllability assumption **(H2)(iii)** for g_i, we set that for any $x'\in\mathbf{R}^d$, $Y(\cdot)$ such that $(x',Y)\in S[x',y]$,

$$\inf\varphi(x',Y(T))=\inf_{y\in S}\varphi(x',y)=\underline{\varphi}(x'),\ \text{for } T\ge t+\frac{2\sqrt{2}\varepsilon}{r_0},$$

where we have used that $S\subset B(0,\sqrt{2})$.

Then for any $t<T$ we can restrict $\varepsilon<r_0(T-t)/(2\sqrt{2})$ and get

$$\limsup_{\varepsilon\to0,t\to T^-,x'\to x}\ \sup_{y\in\mathbf{R}^2}\ \inf_{(X',Y)\in S^\varepsilon_{[t,T]}(x',y)}\varphi(x',Y(T))=\limsup_{x'\to x}\underline{\varphi}(x')=\underline{\varphi}(x).$$

Therefore,

$$\overline{u}(T,x)\le\underline{\varphi}(x)+\limsup_{t\to T^-}ML_\varphi(T-t)=\underline{\varphi}(x).$$

We conclude that

$$\overline{u}(T,x)\le\underline{\varphi}(x)=\underline{u}(T,x). \tag{5.5.3}$$

Step 4 : Conclusion
Since $\overline{u}$ is a subsolution of (5.1.6) and $\underline{u}$ is a supersolution of (5.1.6), by (5.5.3) and the comparison principle for (5.1.6) we have

$$\overline{u}(t,x) \leq \underline{u}(t,x), \text{ for } (t,x) \in (0,T) \times \mathbf{R}^d,$$

which gives

$$\overline{u} = \underline{u} = u \text{ in } (0,T) \times \mathbf{R}^d,$$

and implies the convergence of u^ε to u which is the viscosity solution of (5.1.6).

Chapter 6

Perspectives

The problems studied in the present thesis provide a rich spectrum of further research issues, and we would like to name a few among them.

In chapter 3, the study of the control problem is based on the impulsive dynamical system where the dynamics are $g_0(t, y, \alpha) + g_1(t, y)d\mu$. Here g_1 depends on the time t and the state y, and it is independent from the control α. In this study, we have focused on the proper definition of the magnitude of the jumps of the trajectories using the graph completion method. Once the graph completion is given, the magnitude of the jumps is determined. For further development on this issue, the impulsive system can be more complicated. The first further study is to consider the case when g_1 depends on the control variable α such that the jumps can be controlled. Another issue can be generalized lies in the measure μ. In our study, μ is given and the moments when the jumps take place are fixed. Then the second further study is to consider the case where the moments of the jumps can be controlled. These two further topics both require more investigation for the controlled impulsive system, and provide more potential for applications.

In chapter 4, we have studied the finite horizon problems on general multi-domains and the infinite horizon problems on two-domains. The first interesting topic is the study of infinite horizon problems on multi-domains which is the general case. The second further topic is the relevant study of hybrid control problems which allows also some interesting applications. Another issue related to this study is the numerical approach. The framework of our study is well adapted for the domain decomposition method, which is one of the motivations of this work.

The results for the homogenization problem on multi-domains have been given in dimension 2 for the fast variable. The first further study will be the general case with any dimension for the fast variable. Another issue lies in some restrictive assumptions for the couple of slow variable and fast variable taking the same control parameter. The assumptions can be weaken eventually in the future study. Finally, the numerical approach for this problem is also interesting to exploit.

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Title: Hamilton-Jacobi-Bellman approach for optimal control problems with discontinuous coefficients.

Abstract: This thesis deals with the Dynamical Programming and Hamilton-Jacobi-Bellman approach for a general class of deterministic optimal control problems with discontinuous coefficients. The tools essentially used in this work are based on the control theory, the viscosity theory for Partial Differential Equations, the nonsmooth analysis and the dynamical systems.

The first part of the thesis is concerned with the state constrained problem of discontinuous trajectories driven by impulsive dynamical systems. A characterization result of the value function of this problem has been obtained. Another contribution of this part consists of the extension of the HJB approach for the problems with time-measurable dynamical systems and in presence of time-dependent state constraints.

The second part is devoted to the problem on stratified domain, which consists of a union of subdomains separated by several interfaces. One of the motivations of this work comes from the hybrid control problems. Here new transmission conditions on the interfaces have been obtained to ensure the uniqueness and the characterization of the value function.

The third part investigates the homogenization of Hamilton-Jacobi equations in the framework of state-discontinuous Hamiltonians. This work considers the singular perturbation of optimal control problem on a periodic stratified structure. The limit problem has been analyzed and the associated Hamilton-Jacobi equation has been established. This equation describes the limit behavior of the value function of the perturbed problem when the scale of periodicity tends to 0.

Keywords: optimal control problems, Hamilton-Jacobi-Bellman equations, viscosity solutions, nonsmooth analysis, impulsive differential equations, state constraints, multi-domains, stratified dynamical system, transmission conditions, singular perturbation.

Titre: L'approche Hamilton-Jacobi-Bellman pour des problèmes de contrôle optimal avec des coefficients discontinus.

Résumé: Cette thèse porte sur l'approche de Programmation dynamique et Hamilton-Jacobi-Bellman pour une classe générale de problèmes déterministes de contrôle optimal avec des coefficients discontinus. Les outils utilisés dans ce travail se basent essentiellement sur la théorie de contrôle, la théorie de viscosité pour les équations aux dérivées partielles, l'analyse nonlisse et les systèmes dynamiques.

La première partie de la thèse concerne le problème des trajectoires discontinues sous contraintes sur l'état, où les trajectoires sont solutions de systèmes dynamiques impulsionnels. Un résultat de caractérisation de la fonction de valeur pour de tels problème a été obtenu. Une autre contribution issue de cette partie consiste en l'extension de l'approche HJB pour des problèmes gouvernés par des systèmes dynamiques mesurables en temps et en présence de contraintes sur l'état dépendantes du temps.

La deuxième partie est consacrée au problème de contrôle optimal sur domaine stratifié, qui consiste en une réunion de sous-domaines séparés par plusieurs interfaces. Une de motivations de ce travail vient du problème de contrôle hybride. Ici on obtient de nouvelles conditions de transmission sur les interfaces qui garantissent l'unicité et la caractérisation de la fonction de valeur.

La troisième partie consiste à étudier l'homogénéisation des équations d'Hamilton-Jacobi dans le cadre d'Hamiltonians discontinus en état. Ce travail considère la perturbation singulière des problèmes de contrôle optimal sur une structure périodique stratifié. Le problème limite est analysé et une équation d'Hamilton-Jacobi associée est établie. Cette équation décrit le comportement limite de la fonction de valeur du problème perturbé lorsque l'échelle de périodicité tend vers 0.

Mots-clés: problèmes de contrôle optimal, équations d'Hamilton-Jacobi-Bellman, solutions de viscosité, analyse nonlisse, équations différentielles impulsionnelles, contraintes sur létat, multi-domaines, système dynamique stratifié, conditions de transmissions, perturbation singulière.