Nonlinear Vibrations of Thin Rectangular Plates

A Numerical Investigation with Application to Wave Turbulence and Sound Synthesis

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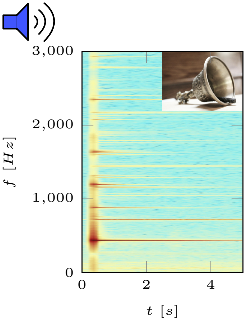
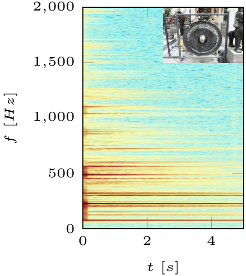
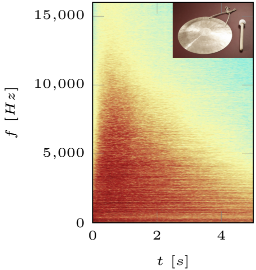
introduction: motivation for this work

An *idiophone* is any musical instrument which creates sound primarily by way of the instrument's vibrating, without the use of strings or membranes. Listen to the sound produced by the following common instruments belonging to the group: bells, gongs, cymbals.







Linear

- Few frequencies (eigenmodes)
- No harmonic relations
- Small amplitude of vibrations ($|w| \ll h$)

Weakly Nonlinear

- Coupled frequencies
- Amplitude-dependent frequencies (pitch glides)
- Moderate amplitudes of vibrations ($|w| \sim h$)

Strongly Nonlinear

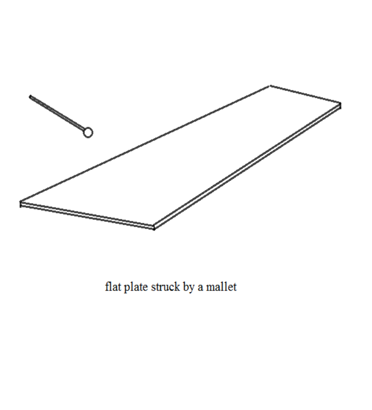
- Cascade of energy
- Continuum Spectrum
- Large amplitudes of vibrations ($|w| > h$)

Mechanical properties of idiophones

- Linear elastic material
- Shape is, in many cases, a curved shell
- Type of coupling depends on the amplitude of vibration (*geometrical nonlinearity*)

Two types of complexity

Geometrical Complexity	Dynamical Complexity



A flat plate displays the same dynamical complexity as the curved idiophones, but it has a straightforward geometry.
Dynamical equations: von Kármán equations.

[von Kármán , Enk. Mat. Wiss. 1910, Thomas *et al.*, JSV 2008; Book by Nayfeh and Pai, 2004]

Which numerical scheme?

- ▷ Finite Difference Scheme: [Bilbao, NMPDE 2008]
- ▷ Modal method: To be developed during PhD!

Big challenge: many interacting modes, never done before for nonlinear synthesis of plates

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plate equations and modes

Von Kármán equations

- Dimensions L_x, L_y
- Thickness h ($h \ll L_x, L_y$)
- Density ρ, Young's modulus E, Poisson's ratio ν
- Rigidity $D = Eh^3/12(1-\nu^2)$

The dynamics is described the flexural wave field $w(\mathbf{x}, t)$, which is the unknown of the equations. The function $F(\mathbf{x}, t)$ is called Airy's stress function and quantifies the movement in the plane directions.

$$\overbrace{\underbrace{\rho h \ddot{w} =}_{\text{inertial term}} \underbrace{-D\Delta\Delta w}_{\text{elastic force}} \underbrace{-c\dot{w}}_{\text{damping}} \underbrace{+P}_{\text{external loads}}}^{\text{Kirchhoff linear plate equation}} \underbrace{+\mathcal{L}(w, F)}_{\text{nonlinear term}}$$

$$\underbrace{\Delta\Delta F = -\frac{Eh}{2}\mathcal{L}(w, w)}_{\text{equation for stress function } F}$$

Modal Equations

Need to choose a suitable basis: associated linear system

Flexural waves

$$w = S_w \sum_{i=1}^{N_w} \frac{\Phi_i(\mathbf{x})}{\|\Phi_i\|} q_i(t);$$

$$\Delta\Delta\Phi_i(\mathbf{x}) = \frac{\rho h}{D}\omega_i^2 \Phi_i(\mathbf{x}).$$

Airy stress function

$$F = S_F \sum_{i=1}^{N_F} \frac{\Psi_i(\mathbf{x})}{\|\Psi_i\|} \eta_i(t);$$

$$\Delta\Delta\Psi_i(\mathbf{x}) = \zeta_i^4 \Psi_i(\mathbf{x}).$$

Orthogonality and projection

- Modes are orthogonal: $\langle \Phi_i, \Phi_j \rangle_S = \int_S d\mathbf{x}\ \Phi_i\, \Phi_j = \|\Phi_i\|^2 \delta_{ij}$
- Use orthogonality to project onto the coordinate s

$$\ddot{q}_s(t) + 2\chi_s \omega_s \dot{q}_s(t) + \omega_s^2 q_s(t) =$$

$$-\frac{E}{2\rho} \sum_{n,p,q,r=1}^{\infty} \frac{H^n_{q,r} E^s_{p,n}}{\zeta_n^4} q_p(t) q_q(t) q_r(t)$$

$$+\frac{\langle \Phi_s, P(\mathbf{x},t) \rangle_S}{\|\Phi_s\| \rho h}$$

Coupling coefficients

Two third order tensors appear. These are:

$$H^n_{q,r} = \frac{\langle \Psi_n, L(\Phi_q, \Phi_r) \rangle_S}{\|\Psi_n\| \|\Phi_q\| \|\Phi_r\|}$$

$$E^s_{p,n} = \frac{\langle \Phi_s, L(\Phi_p, \Psi_n) \rangle_S}{\|\Phi_p\| \|\Phi_s\| \|\Psi_n\|}$$

The two tensors can be combined to give the tensor of nonlinear coupling coefficients

$$\Gamma^s_{p,r,q} \equiv \sum_{n=1}^{N_F} \frac{H^n_{q,r} E^s_{p,n}}{\zeta_n^4}$$

Boundary Conditions

In-plane direction

- Movable

$$\Psi_{,nt} = \Psi_{,tt} = 0$$

- Immovable (with $w = 0$)

$$\Psi_{,nn} - \nu\Psi_{,tt} =$$
$$\Psi_{,nnn} + (2+\nu)\Psi_{,ntt} = 0$$

Edge Rotation

- Rotationally Free

$$\Phi_{,nn} + \nu\Phi_{,tt} = 0$$

- Rotationally Immovable

$$\Phi_{,n} = 0$$

Edge Vertical Translation

- Transversely Movable

$$\Phi_{,nn} + (2-\nu)\Phi_{,ntt}$$
$$-\frac{1}{D}(\Psi_{,tt}\Phi_{,n} - \Psi_{,nt}\Phi_{,t}) = 0$$

$\Phi_{,nt} = 0$ at corners.

- Transversely Immovable

$$\Phi = 0$$

Boundary Conditions

In-plane direction

- Movable

$$\Psi_{,nt} = \Psi_{,tt} = 0$$

Edge Rotation

- Rotationally Free

$$\Phi_{,nn} + \nu\Phi_{,tt} = 0$$

Edge Vertical Translation

- Transversely Immovable

$$\Phi = 0$$

The selected boundary conditions can be reduced to

- $\Phi = \Phi_{,nn} = 0$
- $\Psi = \Psi_{,n} = 0$

Equation for flexural modes

$$\Delta\Delta\Phi = \frac{\rho h}{D}\omega^2\Phi$$
$$\Phi = \Phi_{,nn} = 0$$

Simply Supported Kirchhoff Plate Equation

$\Phi \propto \sin\frac{m_1 \pi x}{L_x} \sin\frac{m_2 \pi y}{L_y}$
[Leissa, 1993; Hagedorn and DasGupta, 2007]

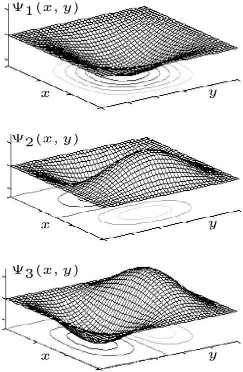
Equation for Airy modes

$$\Delta\Delta\Psi = \zeta^4\Psi$$
$$\Psi = \Psi_{,n} = 0$$

Clamped Kirchhoff Plate Equation

Ψ?

Modes of a Clamped Plate



Eigenfunctions of a clamped plate of aspect ratio 2/3

The Clamped Plate Problem

Requirements for this work

- ▷ Many (hundreds) of modes
- ▷ Fast convergence
- ▷ Stable algorithm

Brief literature review

The literature reveals that the clamped plate problem has been treated by many in the course of history.

Ad-hoc methods (only clamped plate)

- ▷ Leissa's book (a collection of them) [Leissa, 1993]
- ▷ Gorman's superposition method [Gorman *et al.*, Comp. & Struct. 2012]

Not enough frequencies! (∼ 10)

General methods (all boundary conditions)

- ▷ Li (use of flexural and rotational springs to simulate different boundary conditions) [Li, JSV 2004]

Easier to implement but does not meet all requirements! (number of modes...)

NONE OF THE METHODS SATISFIES ALL THE REQUIREMENTS!

Idea

Modification of general Li's method in order to

- ▷ Create an ad-hoc, stable solution for the clamped plate
- ▷ Create an ad-hoc, stable solution for the free plate (impossible to treat in Li's general method!)

Implementation

- ▷ Rayleigh-Ritz method
- ▷ Modified cosine Fourier series
- ▷ Expansion function must satisfy *a priori* the geometrical boundary conditions of the problem

Rayleigh-Ritz Method

Write eigenfunction as

$$\Psi(\mathbf{x}) = \sum_{i=1}^{N_\Psi} a_i \Lambda_i(\mathbf{x})$$

Insert into energy functionals

$$T[\Psi] = \mathbf{a}^T \mathbf{M} \mathbf{a}$$
$$U[\Psi] = \mathbf{a}^T \mathbf{K} \mathbf{a}$$

Use stiffness and mass matrices to define the algebraic eigenvalue problem

$$\mathbf{K}\mathbf{a} = \zeta^4 \mathbf{M}\mathbf{a}$$

Clamped Plate Problem

Expansion functions are chosen as

$$\Lambda_n(x,y) = X_{n_1}(x) Y_{n_2}(y)$$

where

$$X_{n_1}(x) = \cos\left(\frac{n_1 \pi x}{L_x}\right) + \frac{15(1+(-1)^{n_1})}{L_x^4} x^4 - \frac{4(8+7(-1)^{n_1})}{L_x^3} x^3 + \frac{6(3+2(-1)^{n_1})}{L_x^2} x^2 - 1$$

and similarly for $Y_{n_2}(y)$

Table: Convergence of clamped plate frequencies, $\zeta_k^2 L_x L_y$, $\xi = 1$ (square plate)

k	N_Ψ			
	12	112	312	392
1	35.986	35.985	35.985	35.985
10	218.67	210.52	210.52	210.52
50	-	805.35	805.34	805.34
100	-	1546.2	1546.1	1546.1
200	-	-	2848.0	2847.6
300	-	-	4191.6	4188.0

Observations

- Stability
- Hundreds of modes calculated VERY accurately (accuracy out of reach with other methods like FD or FEM)

Table: Comparison of clamped plate frequencies, $\zeta_k^2 L_x L_y$, $\xi = 1$ (square plate)

	Source		
k	R.R method ($N_\Psi = 400$)	Leissa	FD (161×161)
1	35.98	35.99	35.54
2	73.39	73.41	72.49
3	73.39	73.41	72.49
4	108.2	108.3	106.9
20	371.3	-	366.7

Results

- ▷ Discretised von Kármán system
- ▷ Clamped plate modes and frequencies calculated with great precision using a stable algorithm capable of calculating hundreds of modes (not available before!)
- ▷ Coupling coefficients calculated with great precision

What's next?

- ▷ Discretised system of ODEs needs to time-integrator
- ▷ Are standard integration routines ok for highly nonlinear dynamics?
- ▷ Can a stable integrator be constructed for this problem?

Contents

time integration schemes

Overview

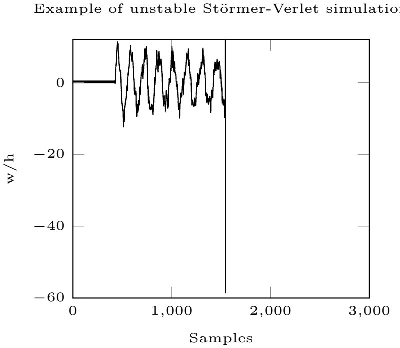
- A scheme is needed to find an approximate solution to the differential equation
$$\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, t)$$
- Introduce timestep k and mapping λ_k to push the solution from the step n to the step $n+1$, such that
$$\lambda_k : \mathbf{q}(n) \to \mathbf{q}(n+1)$$

Selecting an appropriate scheme

- Störmer-Verlet
- Newmark
- Runge-Kutta
- ....

Example of Instability

- ▷ Störmer-Verlet is usually fine, but not ok for high amplitudes of vibrations
- ▷ One must construct a stable scheme



Stability issue *must* be addressed for sound synthesis

- ▷ Nonlinearity
- ▷ Large amplitudes of vibrations
- ▷ Large number of modes (*i.e.* large frequency range)

Störmer-Verlet Scheme

Choice 1: Störmer-Verlet

Störmer-Verlet

- second-order
- symmetric
- symplectic

Implementation

$$\delta_{tt}\mathbf{q}(n) = \mathbf{f}(\mathbf{q}(n))$$

where

$$\delta_{tt}\mathbf{q}(n) = \frac{\mathbf{q}(n+1) - 2\mathbf{q}(n) + \mathbf{q}(n-1)}{k^2}$$

(second order accurate derivative operator)

- explicit scheme
- conserves energy when **f** is the linear plate system

Energy conservation reads

$$\delta_{t+}\left\{\sum_{s=1}^{N_\Phi} S_w^2 \frac{\rho h}{2}\left[(\delta_{t-}q_s(n))^2 + \omega_s^2 q_s(n)\left(e_{t-}q_s(n)\right)\right]\right\} = 0$$

or

$$\delta_{t+}\sum_{s=1}^{N_\Phi}\Big(\underbrace{\tau_s(n)}_{\text{K.E. of mode } s \text{ at time } n} + \underbrace{v_s^l(n)}_{\text{P.E. of mode } s \text{ at time } n}\Big) = 0$$

Energy-conserving, Stable Scheme

Choice 2: Energy conserving, Stable Scheme

Construction of an Energy Conserving Scheme

The scheme is constructed as follows

$$\delta_{tt} q_s(n) + K_{s,s} q_s(n) = \frac{S_F}{\rho h} \sum_{k=1}^{N_\Phi} \sum_{l=1}^{N_\Psi} E^s_{k,l} q_k(n) [\mu_{t\cdot} \eta_l(n)]$$

$$\mu_{t-} \eta_l(n) = -\frac{Eh}{2\zeta_l^4} \frac{S_w^2}{S_F} \sum_{i,j=1}^{N_\Phi} H^l_{i,j} q_i(n) [e_{t-} q_j(n)]$$

$$\mu_{t\cdot} \eta_l(n) = \frac{1}{2} (\eta_l(n+1) + \eta_l(n-1))$$

$$\mu_{t-} \eta_l(n) = \frac{1}{2} (\eta_l(n) + \eta_l(n-1))$$

$$e_{t-} q_j(n) = q_j(n+1)$$

Construction of an Energy Conserving Scheme

After some manipulations, one can show that

$$\delta_{t+}\left\{\sum_{s=1}^{N_\Phi} S_w^2 \frac{\rho h}{2}\left[\left(\delta_{t-}q_s(n)\right)^2+\omega_s^2 q_s(n)\left(e_{t-}q_s(n)\right)\right]+\frac{1}{2Eh}\sum_{l=1}^{N_\Psi}\left(\mu_{t-}\left(\eta_l(n)\eta_l(n)\right)\right)\zeta_l^4\right\}=0$$

or

$$\delta_{t+}\sum_{s=1}^{N_\Phi}\left(\tau_s(n)+\upsilon_s^l(n)\right)+\delta_{t+}\sum_{l=1}^{N_\Psi}\underbrace{\upsilon_l^{nl}(n)}_{\text{P.E. of Airy mode } l \text{ at time } n}=0$$

- Given $x = q_s(n)$ and $y = q_s(n-1)$ one has

$$x^2 + y^2 + 2\alpha xy = g(\epsilon_s^l(n)) \qquad \left(\alpha = \frac{k^2\omega_s^2}{2} - 1\right)$$

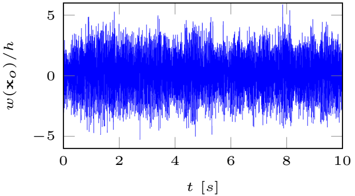
where $g(\epsilon_s^l(n))$ is a function of the linear energy of the mode s at time n.

- A closed conic (ellipse or circle) is obtained when $|\alpha| < 1$. This gives a bound on the solution size

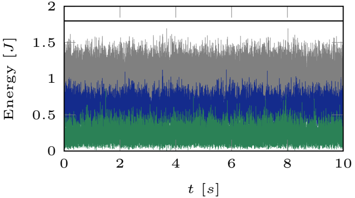
$$|x|, |y| \leq \sqrt{\frac{2k^2\epsilon_s^l(n)}{\rho h(1-\alpha^2)S_w^2}}$$

- Stability condition is $|\alpha| < 1$, or

$$k < \frac{2}{\omega_s}.$$



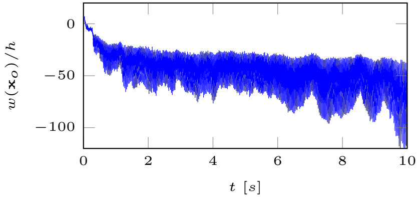
(a)



(b)

Stability Condition: OK

Time simulations of a steel plate of dimensions $L_x \times L_y = 0.4 \times 0.6\text{m}^2$ and thickness $h = 1$mm.
(a) Time series sampled at 10kHz;
(b) total energy (black thick line), kinetic (grey), linear potential (navy), nonlinear potential (dark green).



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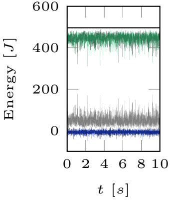
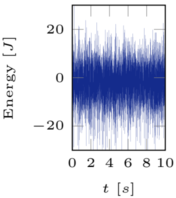


(b)



(c)

Stability Condition: NOT OK

Time simulations of a steel plate of dimensions $L_x \times L_y = 0.4 \times 0.6\text{m}^2$ and thickness $h = 1\text{mm}$.
(a) Time series;
(b) total energy (black thick line), kinetic (grey), linear potential (navy), nonlinear potential (dark green);
(c) linear potential energy showing non physical behaviour (it is not positive definite).

Results

- Constructed energy-conserving scheme
- Energy conservation leads to stability condition
- Nonlinear von Kármán system is now fully solved in terms of modes (not available before!)

What's next?

- Application: weakly nonlinear vibration analysis using the very precise modal scheme (not presented)
- Application: compare FD and modes for sound synthesis
- Application: use FD to create thousands of interacting modes and analyse the turbulent system

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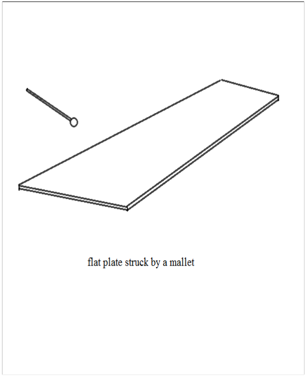
sound synthesis of idiophones



State of the Art

Linear Models

- MOSAIC and Modalys: sound synthesis using modal approach
  [Morrison *et al.*, Comp. Mus. Jour. 1993; Eckel *et al.*, Proceedings of ISMA 1995]
- Implementation of different damping laws using Finite Differences
  [Chaigne *et al.*, JASA 2001]
- Plate reverberation using Finite Differences
  [Arkas, PhD thesis 2009]

Nonlinear Models

- Von Kármán equations solved using energy-conserving Finite Differences
  [Bilbao, NMPDE 2008]
- Propagation model added
  [Torin *et al.*, Proceedings of DAFX 2013]

In this work

Extend the modal synthesis to nonlinear plate vibrations.

First: validate model by comparing with Finite Differences.

Second: Experiment with modal parameters and do synthesis

Note that an efficient modal scheme could open up possibilities that cannot be implemented in Finite Differences: DAMPING RATIOS.

Set a plate into motion: Strikes

- ▷ Impulsive forcing in time
- ▷ Dirac's delta in space

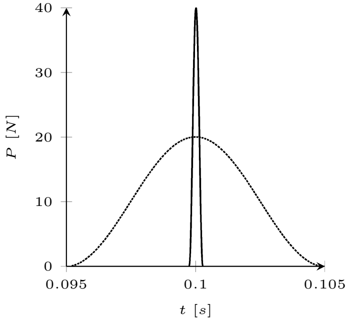
Raised cosine function

$$P(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}_0)p(t),$$

where

$$p(t) = \frac{p_0}{2}(1 + \cos(\pi(t - t_0)/\Delta t)),$$

for $|t - t_0| \leq \Delta t$, and zero otherwise.



Dashed line: mallet-like configuration, $\Delta t = 5$ms, $p_0 = 20$N. Thick line: drumstick-like configuration, $\Delta t = 0.3$ms, $p_0 = 40$N comparison with Finite Differences

Damping laws implemented in FD

Here two damping laws are considered

$$R_0(\mathbf{x},t) = 2\sigma_0 \dot{w}; \qquad R_1(\mathbf{x},t) = -2\sigma_1 \Delta \dot{w}.$$

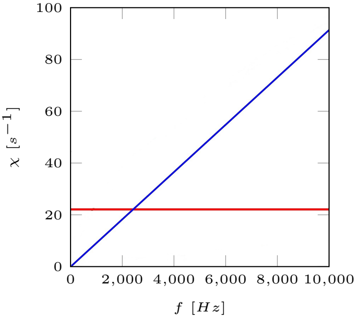
Taking Fourier transforms

$$\tilde{R}_0(\mathbf{k},t) = \gamma_0(f)\tilde{\dot{w}}(\mathbf{k},t)$$
$$\tilde{R}_1(\mathbf{k},t) = \gamma_1(f)\tilde{\dot{w}}(\mathbf{k},t)$$

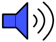
where

$$\gamma_0(f) = 2\sigma_0; \qquad \gamma_1(f) = 2\sigma_1 \frac{2\pi}{hc} f.$$

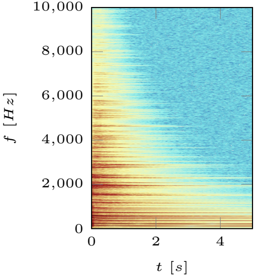
Damping laws implemented in FD code



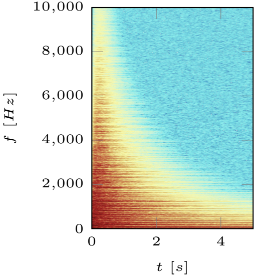
▷ LIMITED POSSIBILITIES IN FD!



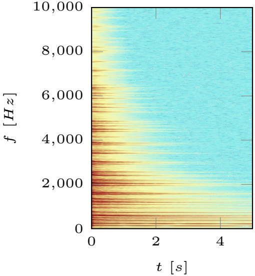
Weak FD



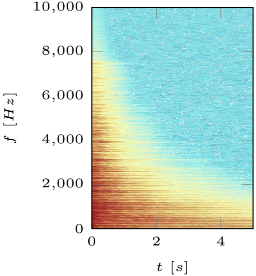
Strong FD



Weak Modes



Strong Modes

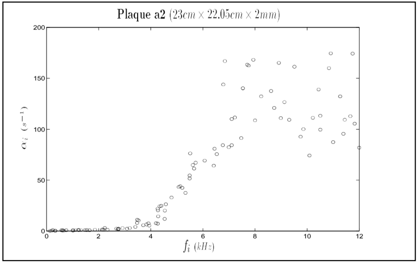


improved modal samples

Modal Damping

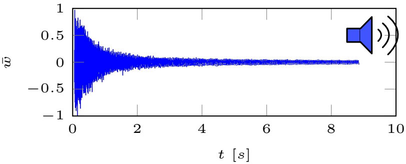
- ▷ Using the modal code, the damping ratios can be inserted directly in the code

Measured damping factors



(figure from [Lambourg, PhD thesis, 1997])

Example Time Series



- ▷ $N_\Phi \sim 600$ modes suffices for gong-like sounds!
- ▷ Very natural sounding synthesis thanks to natural decay ratios
- ▷ Very fast computations for weakly nonlinear dynamics (almost real time in Matlab,$N_\Phi \sim 100$)

Results

- Compared efficiently modes with FD
- Modal approach CAN reproduce the sound of strongly nonlinear vibrations (crashes)
- Possibility of adding realistic decaying rates for each one of the modes
- Modal synthesis CAN be extended to nonlinear dynamics for sound synthesis (NEW!!)

What's next?

- Ran simulations with $\sim 10^5$ modes using FD
- Analyse statistical properties of the cascade

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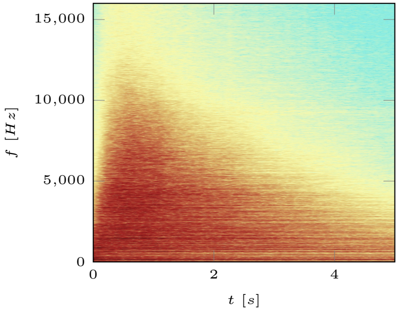
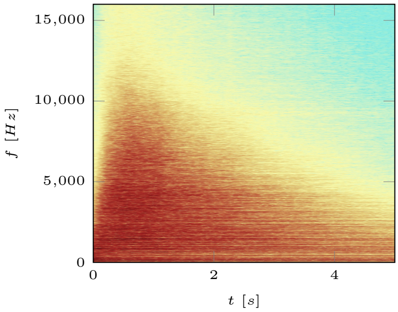
plates in a strongly nonlinear regime: wave turbulence



Observations

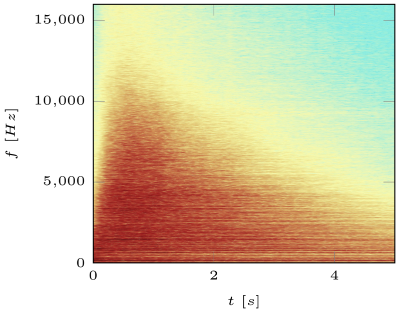
- ▷ Many (thousands) of modes are activated
- ▷ Production of a "cascade" to higher frequencies
- ▷ Modal interaction away from an equilibrium condition

This scenario is referred to as "turbulent". The interacting partners are the modes of the system (Fourier components), hence the name "wave turbulence" (WT)



State of the Art (Theory/Experiments)

- Plate equations are solved in terms of WT formalism and power spectrum formula given (KZ spectrum) [Düring *et al.*, PRL 2006]
- Experiments show deviations from theory [Boudaoud *et al.*, PRL 2008; Mordant, PRL 2008]
- WT assumptions (separation of time scales) are checked and verified [Miquel *et al.*, PRE 2011]
- Forcing introduces anisotropy in space; removing forcing gives spectra closer to theory [Miquel *et al.*, PRL 2011]
- Damping heavily responsible for slopes in the spectra [Humbert *et al.*, EPJ 2013]



State of the Art (Numerics)

- Spectral methods: when numerical experiments are set-up following the hypothesis of the theory, KZ spectra are recovered [Düring *et al.*, PRL 2006; Yokoyama *et al.*, PRL 2013]

All the numerical experiments are set up in *Fourier space*. In real experiments, however, one works in physical space.

In this work

Use FD scheme to produce a cascade of energy including thousands of modes

Create close-to-reality numerical conditions (physical boundary conditions, pointwise forcing, conservation of energy, ...)

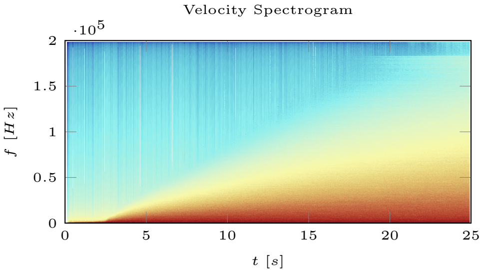
Study aspects of the turbulent regime that have not been considered

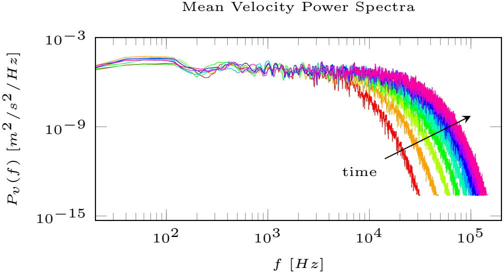
- ▷ Nonstationary turbulence
- ▷ Effects of geometrical imperfections

Note that the absence of damping will necessarily create a state of NONSTATIONARY turbulence: this is NOT the system described by the theoretical framework of [Düring *et al.*, PRL 2006] . Hence comparison with K-Z spectrum is not appropriate.
Framework in this case given by [Falkovich *et al.*, JNS 1991; Connaughton *et al.*Physica D 2003] .

Nonstationary Turbulence 1: Steady Forcing

Steady Sinusoidal Forcing in Conservative Flat Plates



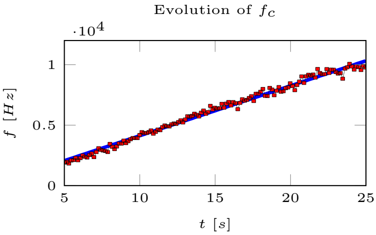


Comments

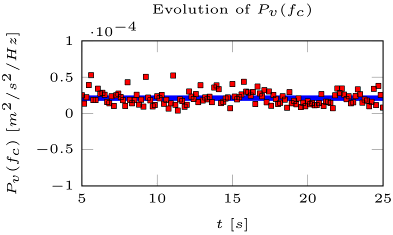
- Presence of an nonstationary turbulence
- Cascade developing in an infinite box limit (up to Nyquist frequency)

Analysis

- Try to quantify the front of the cascade in terms of a characteristic frequency f_c
- Study the evolution of f_c and $P_v(f_c)$
- Characterise the injection in terms of the flux ε (the injected power)



(a)



(b)

Definitions

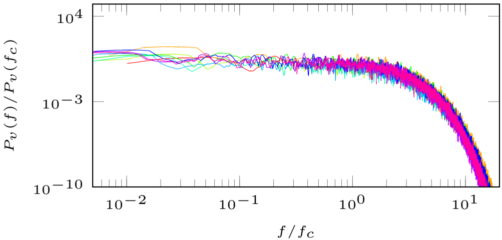
- ▷ Characteristic frequency: $f_c = \frac{\int_0^\infty P_v(f)\, f \, df}{\int_0^\infty P_v(f) \, df}$
- ▷ Spectral amplitude: $P_v(f_c)$

Results

- ▷ $f_c = c_f \cdot t$
- ▷ $P_v(f_c) \sim \text{cnst}$

Analysis

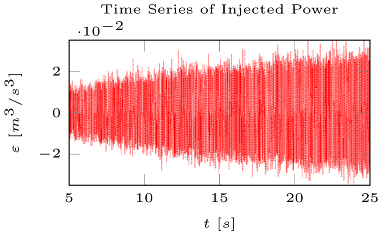
Scaling of Spectra: divide each of the spectra by $P_v(f_c)$ and plot against f/f_c



Observations

- Self-similar dynamics

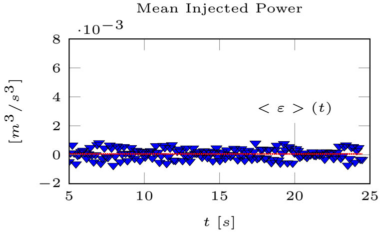
$$P_v(f) = P_v(f_c)\phi\left(\frac{f}{f_c}\right)$$



(a)

Injected Power

$$\varepsilon(t) = \frac{P(t) \cdot \dot{w}(\mathbf{x}_i, t)}{\rho S}$$



(b)

Observations

- ▷ $< \varepsilon > \sim \text{cnst} = \bar{\varepsilon}$
- ▷ Self-similar dynamics is linked to the constant injection $\bar{\varepsilon}$

To do: run more simulations

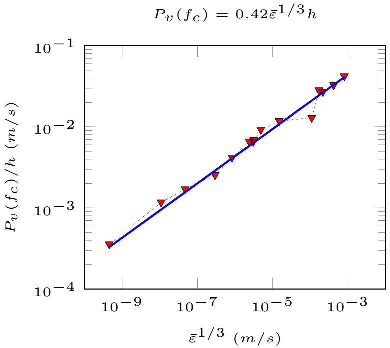
Change parameters of the system

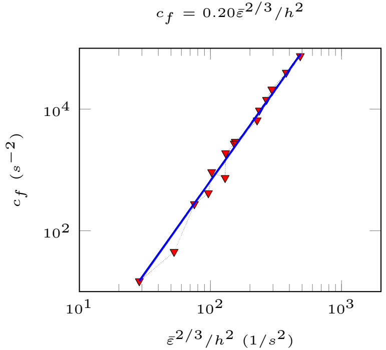
- Forcing amplitudes (large range! [0.005 – 70] N)
- Thickness ([0.1 – 1] mm)

Look for scaling laws

The constant injection $\bar{\varepsilon}$ and the thickness h are the defining parameters of the turbulent state.
IDEA: look for power law dependence of the spectral amplitude $P_v(f_c)$ and the cascade velocity c_f with appropriate combinations of $\bar{\varepsilon}$ and h using dimensional arguments. In other words

- $P_v(f_c) \propto \bar{\varepsilon}^{1/3} h$
- $c_f \propto \bar{\varepsilon}^{2/3} / h^2$


$P_v(f_c) = 0.42\bar{\varepsilon}^{1/3}h$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$P_v(f_c)/h$ (m/s)
10^{-9}
10^{-7}
10^{-5}
10^{-3}
$\bar{\varepsilon}^{1/3}$ (m/s)


$c_f = 0.20\bar{\varepsilon}^{2/3}/h^2$
10^4
10^2
c_f (s^{-2})
10^1
10^2
10^3
$\bar{\varepsilon}^{2/3}/h^2$ $(1/s^2)$

Final Remarks

- The dynamics of nondissipative plates under steady forcing is self-similar and nonstationary
- The self-similar function is

$$P_v(f) = P_v(f_c)\phi\left(\frac{f}{f_c}\right),$$

where $P_v(f_c)$ and c_f can be given in terms of $\bar{\varepsilon}$ and h, as in

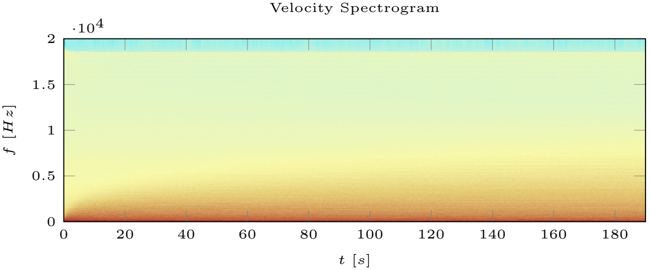
$$P_v(f_c) = 0.42\bar{\varepsilon}^{1/3}h$$
$$f_c = 0.20\bar{\varepsilon}^{2/3}/h^2 \cdot t$$

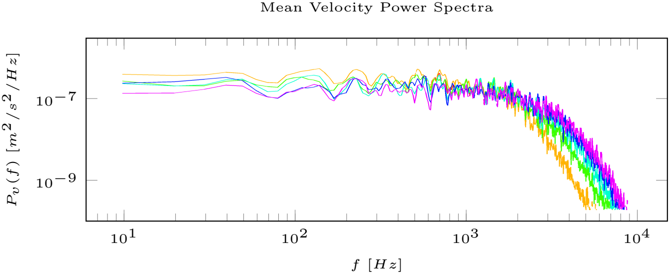
- NOTE that

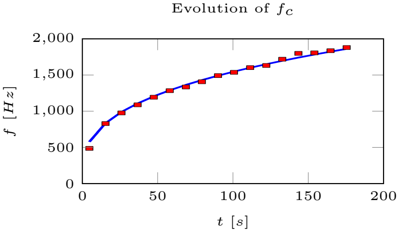
$$\phi\left(\frac{f}{f_c}\right) = \phi\left(\frac{f}{t}\right)$$

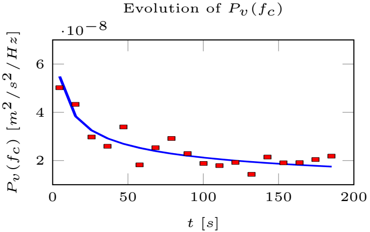
Nonstationary Turbulence 2: Impulsive Forcing

results for impulsively forced, undamped flat plates


Velocity Spectrogram
$\cdot 10^4$
2
1.5
1
0.5
0
$f\ [Hz]$
0 20 40 60 80 100 120 140 160 180
$t\ [s]$

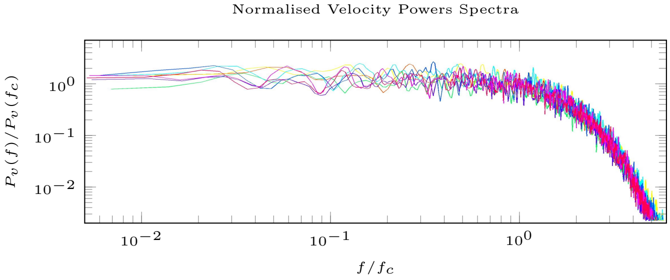

Mean Velocity Power Spectra
10^{-7}
10^{-9}
$P_v(f)\ [m^2/s^2/Hz]$
10^1 10^2 10^3 10^4
$f\ [Hz]$





Results

- $f_c \sim t^{1/3}$
- $P_v(f_c) \sim t^{-1/3}$



Comments

▷ Self-Similar Dynamics

Final Remarks

- The dynamics of nondissipative plates in free turbulence is self-similar and nonstationary.
  NOTE particularly (because of previous argument)

$$P_v(f) \propto t^{-1/3}\phi\left(\frac{f}{t^{1/3}}\right)$$

Theoretical Framework of Nonstationary Turbulence

theoretical framework for nonstationary turbulence

Consider the kinetic equation relating the wave action $n(k,t)$ to the collision integral $I(k)$

$$\frac{\partial n(k,t)}{\partial t} = I(k)$$

- ▷ Ansatz (self-similar dynamics)

$$n(k,t) = t^{-q} z(kt^{-p}) = t^{-q} z(\eta)$$

- ▷ Plug ansatz into kinetic equation and get

$$-t^{-q-1}\left[qz(\eta) + p\xi z'(\eta)\right] = I(\eta)t^{-3q+2p}$$

The last equality is derived considering the expression of $I(k)$ provided by [Düring *et al.*, PRL 2006] . The equality gives $2(q-p) = 1$.

Consider now the following easily derived relations

- ▷ $P_v(f,t) \propto fn(f,t)$ (definition of power spectral density)
- ▷ $\xi(t) = \int_0^\infty f\, P_v(f,t)\ df$ (definition of total energy)

In the two cases considered before, it was found that

- ▷ $\xi(t) \propto t$ (steadily forced turbulence)
- ▷ $\xi(t) \propto t^0$ (impulsively forced turbulence)

Putting all together gives

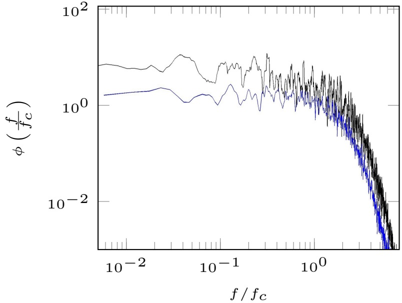
- $4p - q = 1$ (steadily forced turbulence)
- $4p - q = 0$ (impulsively forced turbulence)

Hence, this gives the following equations for the spectral density

- $P_v(f) \propto \phi_1\left(\frac{f}{t}\right)$ (steadily forced turbulence)
- $P_v(f) \propto t^{-1/3}\phi_2\left(\frac{f}{t^{1/3}}\right)$ (impulsively forced turbulence)

These laws are exactly the same as those found numerically!

The theory does not give a form for the functions ϕ_1, ϕ_2. Numerically the forms are



Imperfections

results for continuously forced, undamped imperfect plates

Why Deformations?

- Deformations are always present in experimental plates
- They introduce quadratic nonlinearities, e.g. 3-wave processes that might affect the dynamics

Deformations introduced as raised cosines in x and y directions. Deformation amplitudes up to 10 times the thickness.

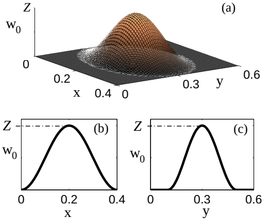


Plate with imperfection in the form of a raised cosine.

THE SELECTED DEFORMATIONS DO NOT CHANGE THE SCALING PROPERTIES OF THE SYSTEM

Results

- Successfully reproduced nonstationary turbulence in continuous and impulsive forcing
- The numerical scaling laws of the spectra is consistent with theory of nonsationary 4-wave processess
- Numerics gives form of self-similar functions that are not predicted in the theory
- Numerics gives also coefficients for evolution of the front of cascade and spectral amplitude
- Imperfections (3-wave processess) do not change the scaling properties of the system

Contents

Major Results

Successful reproduction and analysis of the nonlinear vibrations of plates using appropriate numerical schemes.
Most important results from today's discussion

- Difficult problems such as the clamped and free plate problems solved in terms of Rayleigh-Ritz method with high precision and stability for hundreds of modes
- Previously unavailable modal code developed for sound synthesis of plates. Damping can be now tuned at will
- Nonstationary wave turbulence of 4-wave processes analysed and self-similar function shape proposed. Precise coefficients given for scaling laws

Extensions

- Translate modal code from Matlab to C (improved memory management and speed of calculation)
- Use modal approach for circular plates/shells (difficult to treat in FD)
- Use modal code for wave turbulence (possibility of adding damping at arbitrary frequency)