Robust design of backstepping controllers for systems of linearhyperbolic PDEs

Jean Auriol

THÈSE DE DOCTORAT

de l'Université de recherche Paris Sciences et Lettres
PSL Research University

Préparée à MINES ParisTech

Contrôle robuste d'EDPs linéaires hyperboliques par méthodes de backstepping

École doctorale n°432

SCIENCES ET MÉTIERS DE L'INGÉNIEUR

Spécialité MATHÉMATIQUES ET AUTOMATIQUE

COMPOSITION DU JURY :

M. Yann Le Gorrec
FEMTO-ST, Président

M. Alexey Pavlov
Norwegian University of Science and Technology, Rapporteur

M. Rafael Vázquez
Universidad de Sevilla, Rapporteur

Mme. Kirsten Morris
University of Waterloo, Examinateur

M. Silviu Iulian Niculescu
L2S, Examinateur

M. Nicolas Petit
MINES ParisTech, Examinateur

M. Florent Di Meglio
MINES ParisTech, Examinateur

Soutenue par **Jean AURIOL**
le 4 Juillet 2018

Dirigée par **Florent DI MEGLIO**
Nicolas PETIT

MINES ParisTech
PSL RESEARCH UNIVERSITY PARIS

Remerciements

En tout premier lieu je souhaite remercier mes encadrants de thèse, Florent Di Meglio et Nicolas Petit pour leur confiance, leur enthousiasme et leur soutien permanents. Ce fut un grand honneur et un immense bonheur de travailler avec vous pendant ces trois années. J'ai énormément appris à votre contact, et je garderai de cette thèse un excellent souvenir, tant sur le plan humain que scientifique. Nicolas, véritable mentor bienveillant, vous m'avez accueilli au CAS il y a de cela cinq ans au moment de mon Semestre Recherche. Toujours disponible, source prodigue de conseils avisés (surtout après 18h30 le vendredi), pour toutes ces années, pour toutes ces discussions, soyez remercié. J'espère que vous ne serez pas trop désappointé par l'absence du théorème de Mercer dans ce manuscrit. Florent, vous fûtes le directeur de thèse idéal. D'une disponibilité rare, prenant toujours le temps de répondre à chacune de mes questions, enthousiaste face aux idées que je vous ai soumises, vous avez fait de ces trois années un véritable délice. Véritable chef d'orchestre de cette thèse, alliant à la fois fermeté et grande liberté d'interprétation pour l'instrumentiste. Si la carte de navigation que vous m'avez proposée dès le départ était extrêmement lisible et très bien détaillée, le fait d'avoir le guide avec soi est bien plus précieux pour cerner toutes les subtilités du monde des EDPs. Vous m'avez également fait confiance, ce dont je vous suis reconnaissant, en me laissant explorer de nouveaux chemins, en me permettant d'enseigner ou de co-encadrer des semestres recherche. Pour tout cela je vous remercie et vous suis reconnaissant.

Mes remerciements vont également à tous les membres du Centre Automatique et Systèmes de MINES ParisTech. J'ai passé au sein de ce laboratoire de très belles années. Cela est certes lié à l'exigeante atmosphère scientifique qui imprègne les murs mais également à la haute qualité du personnel qui y réside. Je remercie mes camarades doctorants et post-doctorants présents (Aurélien, Naveen, Matthieu, Gerardo, Pierre-Olivier), mais aussi ceux qui m'ont précédé au titre de docteur (Chucky, Rémi, Pauline, Pierre) pour les échanges fructueux que nous avons pu avoir. Je remercie également tous les membres permanents qui font de ce lieu un havre du savoir. Je remercie particulièrement Delphine Bresch-Pietri, qui du fait du fait de la proximité de nos bureaux s'est souvent retrouvée à répondre à de multiples questions à la pertinence variable, et Philippe Martin pour les nombreuses discussions que nous avons pu avoir.

Je tiens à remercier Alexey Pavlov et Rafael Vázquez, qui m'ont fait l'honneur d'accepter d'être rapporteurs de cette thèse. Merci également à Yann Le Gorrec, Kirsten Morris et Silviu-Iulian Niculescu d'avoir participé à mon jury de soutenance. Vos questions et précieux commentaires ont considérablement contribué à l'amélioration de ce manuscrit et m'ont fourni de nouvelles pistes de travail. Je dois, en outre, une reconnaissance particulière à Kirsten Morris, qui m'a accueilli pendant plus de deux mois au sein de son groupe de recherche à l'Université de Waterloo. J'y ai grandement bénéficié de son érudition et de ses commentaires avisés.

Je voudrais remercier les membres de ma famille au sein de laquelle je compte certains de mes plus grands admirateurs scientifiques (malheureusement ce ne sont pas eux qui relisent mes papiers). Même si pour vous le backstepping évoque plutôt une sorte de danse ésotérique qu'une véritable technique mathématique, même si pour vous la robustesse est plus une propriété du béton qu'autre chose, même si pour vous une EDP est plus proche d'une molécule ou d'un sigle cryptique que d'une quelconque équation, vous n'avez eu de cesse de m'accompagner et de me donner votre support moral et logistique à défaut d'un éventuel support technique. C'est grâce à vous que je peux aujourd'hui écrire ces lignes. Soyez en remerciés!

Enfin, je voudrais adresser mes remerciements à tous ceux qui, sans avoir directement pris part à ce travail m'ont soutenu au cours de ces trois années. J'ai une pensée particulière pour Jean Bernard ou François, amis doctorants avec lesquels il était possible de parler d'inversibilité d'opérateurs sans passer pour anormal mais je pense aussi à Maxime, Denis, Nicolas, Céline, Daphné, Marie-Emmanuelle, Romain, Mathieu, Père David, Marie-Laurence... Tous, je vous remerice chaleureusement. Je remercie également tous mes anciens professeurs qui m'ont donné le goût des mathématiques et tous ceux qui je n'ai pas eu la place de citer !

Contents

Chapter 1

Introduction

Context

In this thesis, we study the design of boundary control laws and observers for systems of coupled linear first-order hyperbolic Partial Differential Equations (LFOH-PDEs) on a one-dimensional bounded spatial domain. This class of systems naturally arises in industrial processes where the dynamics involve a transport phenomenon and for which the variation of the amount of some extensive quantity is balanced by its flux through the boundaries of the domain and its production or consumption inside the domain. Related applications include electric transmission lines [MWTA00], gas flow pipelines [RNM15]-[GD11], heat exchangers [XS02], traffic flow [AHB08], oil well drilling [Aam16], open channel flow [CANB99], [dHPC$^+$03], or multi-phase flow [DM11, DBVdHJ10, DBAR12]. In all these applications, the natural dynamics are three-dimensional. However, as the dominant phenomena evolve in one privileged coordinate dimension (while the phenomena in the other directions are negligible), the dynamics can be represented by one-dimensional hyperbolic balance laws. Until recently [KS08a], the literature focused on the existence of controllers and observers, i.e. on controllability and observability issues. Regarding an explicit design of such controllers, various approaches can be found in the literature. Among them, one can find flatness-based controllers [MZ04, SDMKR13], optimization controllers [Lio71] or Lyapunov-based controllers [CBdN08, Cor09, PGW12] that have for instance enabled the design of dissipative boundary conditions [CBdN08, Cor09]. This last approach uses only static output feedback (the output being the value of the state at the boundary) and even for systems of two equations, there are multiple examples for which it does not allow the synthesis of the control law (see [BC11]). In this context, the backstepping approach is a constructive method mainly used to design boundary controllers and observers. This method consists in performing an invertible change of variables that maps the original system to a so-called “target system" for which the control design is easier. These controllers are explicit, in the sense that they are expressed as a linear functional of the distributed state at each instant. The (distributed) gains can be computed offline. This method was originally introduced by Smyshlyaev and Krstic in [SK04] for the problem of boundary stabilization of a class of linear parabolic partial integro-differential equations and extended in [SK05] to design the corresponding observers (necessary to envision practical implementation and the design of an output-feedback law) using boundary measurements and exploiting duality concepts. Compared to anterior methods, this approach has some distinguishing features: it takes advantage of the structure of the system, resulting in a problem of solving of linear hyperbolic PDEs which is an object much easier than operator Riccatti equations arising in LQG approaches to boundary control, then the analysis is easy in contrast to standard abstract approaches as the problem is solved essentially by calculus. It has produced a number of results first on linear and non-linear parabolic equations [KMV08, KMV09] and then for the wave equation (see for instance [KGBS08, SCK10, SK09]) in presence of an internal spatially varying antidamping term (that puts the eigenvalues of the open-loop system in the right half complex plane). Moreover, the proposed designs often produce new Lyapunov functions. Concerning linear hyperbolic PDEs controlled at one boundary, the first results have been obtained for a single 1-D hyperbolic partial integro-differential equation [KS08a] (converting the potentially unstable PDE into a delay line system which converges to zero in finite time) and for the case of 2-state heterodirectional linear and quasilinear hyperbolic system [CVKB13, VKC11]. The proposed control laws yields exponential stability of the closed-loop system. Moreover, in the case of constant-coefficient case, it is possible to obtain explicit controllers expressed in terms of the Marcum Q-functions [VK14]. This has been done solving the control kernel equations and has required the development of a method that expresses their solution as the sum of solutions of an infinite set of explicitly solvable equations. These results have then been extended to the case of $n+1$ systems (i.e. n states convecting in one direction with one counter-convecting state that is controlled) in [DMVK13]. However, the authors only considered the case of an anti-collocated observer which is easier to derive than a collocated observer (which is of greater interest, e.g. in the case of multiphase flow control in oil production systems). The generalization of this result to an arbitrary number of equations has finally been solved in [HDMVK16] and [CHO17]. Once again, only anti-collocated observers were considered in these contributions. A complete history of the backstepping method for PDEs and of its extensions has recently been given in [VK17]. For these linear hyperbolic systems, one of the major by-product of the backstepping controllers is to (partially) solve the finite-time stabilization and observability problems stated in [Rus72, Rus78b] and generalized by Tatsien Li in [Li10]. More precisely, the considered class of LFOH systems can be stabilized or observed in finite time, the minimum stabilization (resp. observability) time reachable obviously depending on the number of actuators (resp. sensors) available. However, the results given in [Li10, Rus72, Rus78b] are only existence results and do not provide any explicit control law/observers. In that sense the backstepping approach, by giving an explicit expression of the corresponding control law, has opened the door for a large number of related problems to be solved, e.g., parameter identification [DMBPA14], output feedback adaptive control as in [BK14], and stabilization of quasilinear systems [HDMVK15].

By ensuring finite-time convergence these controllers thereby neglect the robustness aspects which are known to be the major limitation to envision practical applications. These robustness limitations may come from uncertainties on the parameters, disturbances acting on the system, noise on the measurements, neglected dynamics or delays acting on the actuators or on the sensors. For instance, it has been observed (see [DLP86, LRW96]) that for many feedback systems, the introduction of arbitrarily small time delays in the loop may cause instability for any feedback. To tackle these robustness problems, various concepts have been introduced such as the concepts of delay-robust stabilization [LRW96] or w-stability [CZ12]. Interestingly, similar robustness concepts and notions have been developed for neutral time-delay systems for which the community also usually focuses on uncertainties with respect to delays. This has led to the concept of *strong stability* [HVL93, HL02, MN07, MVZ$^+$09]. It has been long noticed that some linear hyperbolic systems of PDEs can be represented as neutral time-delay systems. The earliest link is made through D'Alembert's formula [D'A49] that transforms a wave equation into a difference equation. More generally, using the method of characteristics, systems of linear first-order *uncoupled* hyperbolic PDEs can be transformed into difference equations. In [Rus78a, Rus91], the existence of a mapping between the solutions of potentially coupled linear first order hyperbolic PDEs and zero-order neutral systems is proved using spectral methods. In [KK14], the mapping is proved to be unique, and is explicitly constructed for a single hyperbolic equation with a reaction term. These examples suggest the existence of an explicit mapping from the solutions of the general class of LFOH PDEs considered in this thesis and neutral systems. Furthermore, in [KK14], the authors have stated that the stability analysis is easier while converting the PDEs to a delay form. Indeed various methods have been proposed for the stability and robustness analysis of neutral systems such as necessary and sufficient stability conditions based on complex analysis [BC63, HL02]. Thus, one important consequence of the existence of such a mapping would be the adjustment of these methods to deal with the robustness analysis of the backstepping controllers.

Finally, considering application of such controllers to industrial problems, in most cases, only an approximation of the state is available for controller design and the controller needs to be approximated. This direct controller design approach is sometimes referred to as late lumping since the last step in the design is to approximate the controller by a finite-dimensional, or lumped parameter, system. The question of the convergence of late-lumping backstepping controllers has not been well-investigated, contrary to the approximation of the kernels themselves, e.g. in [JMK12] using a trapezoidal rule or in [AAP17] using a sum-of-squares approach. In [WRE17], a method for computing the bounded part of the control operator is proposed. It relies on a finite-dimensional approximation of the state and enables efficient computing of the feedback law. However, the unbounded part of the operator is not approximated and no guarantees of convergence are provided.

Problems addressed and thesis organization

The first part of the thesis provides some contributions in control theory. In Chapter 2, we properly introduce the problems of minimum finite-time stabilization and observability stated by Tatsien Li in [Li10]. In particular we distinguish the problem of one-sided controllability (resp. observability) for which the actuators (resp. sensors) are available at only one boundary of the problem of two-sided controllability (resp. observability) for which the actuators (resp. sensors) are available at both boundaries. When this thesis was started in 2015, the problem of one-sided finite stabilization had been partially solved in [HDMVK16] deriving a control law that ensures finite time convergence but not in the minimum-time. In Chapter 3, we adjusted the backstepping transformation derived in [HDMVK16] to solve the problem of one-sided boundary stabilization in minimum-time for $n+m$ equations. To obtain the minimum-time associated observer, we have developed a new technique based on the adjoint system. The two-sided problems have not received a lot of attention as far as we know, although they constitute an interesting source of performance improvements. Such a problem has been solved in [VK16] for reaction-diffusion PDEs and 2-states heterodirectional linear PDEs with *equal* transport velocities. In Chapter 4, we solve the problems of two-sided minimum-time boundary stabilization and observability for $n+m$ equations. Inspired by recent approaches (see [BAK15, CHO16] and recently [CHO17]), this is done introducing a Fredholm transformation (which can actually be rewritten as a Volterra transformation) that takes full advantages of the multiple actuators and sensors. Finally, in Chapter 5, we introduce a new tool for the stability analysis of systems of linear hyperbolic PDEs deriving an explicit mapping from the space generated by the solutions of such systems to the space generated by the solutions of a general class of neutral systems with distributed delays. This mapping can be used to derive new sufficient stability conditions.

The second part of the thesis addresses the robustness properties of the resulting closed-loop systems. These aspects are essential for practical applications. We start in Chapter 6 by analyzing the robustness properties of the minimum-time controllers and observers designed in the first part with respect to delays in the actuator. This is done using the previously introduced equivalence with neutral systems. We then prove that finite-time stabilization often yields vanishing delay margins, making it an impractical control objective. Consequently, we introduce a degree of freedom (by means of a tuning parameter) that makes possible a potential trade-off between performance and delay-robustness. An extension to the case of a system composed of two coupled PDEs and of an Ordinary Differential Equation is given in Chapter 7. Finally, in Chapter 8, we solve the problem of robust output regulation and Input-to-State Stability for a system of two coupled PDEs. The resulting output-feedback law introduces three degrees of freedom that can be tuned to enable a trade-off between performance and robustness but also between noise sensitivity and disturbance rejection.

Summing up, this thesis provides the three following contributions

Contribution 1: an explicit solution to the problems of minimum finite-time control and observer design addressed in [Li10]. This is done in Chapters 2-4.

Contribution 2: an explicit mapping from the space generated by the solutions of LFOH systems to the space generated by the solutions of a neutral system with distributed delays. This is done in Chapter 5.

Contribution 3: design of a robust output feedback law for a system of two coupled PDEs, introducing multiple degrees of freedom. This is done in Chapters 6-8

Publications

The work presented in this thesis has resulted in the following publications:

- Journals

  1. J. Auriol, F. Di Meglio, *minimum-time control of heterodirectional linear coupled hyperbolic PDEs*, Automatica, Vol. 71, p300-307, 2016
  2. J. Auriol, F. Di Meglio, *Two sided boundary stabilization of heterodirectional linear coupled hyperbolic PDEs*, IEEE Transactions on Automatic Control, Accepted.
  3. J. Auriol, U. J. F. Aarsnes, P. Martin, F. Di Meglio, *Delay-robust control design for two heterodirectional linear coupled hyperbolic PDEs*, IEEE Transactions on Automatic Control, Accepted.
  4. J. Auriol, F. Bribiesca-Argomedo, D. Bou Saba, M. Di Loreto, F. Di Meglio, *Delay-robust stabilization of a hyperbolic PDE-ODE system*, Automatica, Accepted.
  5. J. Auriol, K. A. Morris, F. Di Melgio, *Late-lumping backstepping control of partial differential equations*, Automatica, Submitted.
  6. J. Auriol, F. Di Meglio, *An explicit mapping from linear first order hyperbolic PDEs to difference systems*, Systems an Control Letters, Submitted.
  7. D. Bou Saba, F. Bribiesca-Argomedo, J. Auriol, M. Di Loreto, F. Di Meglio, *A sufficient stability condition for a class of linear 2 x 2 hyperbolic PDEs using a backstepping transform*, IEEE Transactions on Automatic Control, Submitted.

- Conferences

  1. J. Auriol, F. Di Meglio, *Two sided boundary stabilization of two linear hyperbolic PDEs in minimum-time*, Proc. of the 55th IEEE Conference on Decision and Control, Las Vegas, Nevada, USA, 2016.
  2. J. Auriol, F. Di Meglio, *Trajectory tracking for a system of two linear hyperbolic PDEs with uncertainties*, 20th World Congress of the International Federation of Automatic Control, Toulouse, France, 2017.
  3. P.-O. Lamare, J. Auriol, U. J. F. Aarsnes, F. Di Meglio, *Robust output regulation of 2×2 hyperbolic systems: Control law and Input-to-State Stability*, IEEE American Control Conference, Milwaukee, Wisconsin, USA, 2018.

Introduction

Dans cette thèse, nous étudions la synthèse de lois de commande frontières et d'observateurs pour des systèmes d'Équations aux Dérivées Partielles Hyperboliques Linéaires du Premier Ordre (EDPs-HLPO) définies sur un domaine spatial uni-dimensionel borné. Cette classe de systèmes apparaît naturellement lors de la modélisation de procédés industriels pour lesquels la dynamique fait intervenir un phénomène de transport ou pour lesquels la variation d'une certaine quantité extensive est compensée par son flux aux frontières et par sa production ou consommation à l'intérieur du domaine. Parmi les applications correspondantes à de tels systèmes, on trouve la modélisation de lignes de transmission électriques [MWTA00], de flux de gaz dans des conduites [RNM15]-[GD11], d'échangeurs de chaleur [XS02], d phénomènes de trafic routiers [AHB08], de forage pétroliers [Aam16], de canaux de navigation [CANB99], [dHPC$^+$03], ou d'écoulements multiphasiques [DM11, DBVdHJ10, DBAR12]. Pour toutes ces applications, la dynamique naturelle du système est tri-dimensionelle. Toutefois, comme le phénomène dominant privilégie une seule direction d'évolution (les phénomènes dans les autres directions sont négligeables), la dynamique peut être simplifiée en une loi d'équilibre hyperbolique uni-dimensionelle. Jusqu'à récemment [KS08a], la littérature s'intéressait principalement à l'existence de contrôleurs et d'observateurs pour de tels systèmes. En ce qui concerne la synthèse explicite de tels contrôleurs, différentes approches peuvent être considérées. Parmi elles, on peut citer les contrôleurs par platitude [MZ04, SDMKR13], par méthodes d'optimisation [Lio71] ou obtenus à partir de fonctionnelles de Lyapunov [CBdN08, Cor09, PGW12]. Cette dernière approche a ainsi permis l'obtention de conditions frontières dissipatives [CBdN08, Cor09]. Elle n'utilise néanmoins que des feedbacks pour retour de sortie statiques (la sortie étant la valeur de l'état aux frontières) et même dans le cas de systèmes de deux équations, il existe de nombreux exemples pour lesquels la synthèse de lois de commande stabilisantes est impossible (c.f. [BC11]). Dans ce contexte, l'approche par backstepping offre une méthode constructive permettant la synthèse explicite de telles lois de commande et d'observateurs. Cette méthode consiste à transformer de façon inversible le système originel en un système cible pour lequel la synthèse de contrôleurs est aisée. Les contrôleurs ainsi obtenus sont explicites, dans le sens où ils peuvent être exprimés à chaque instant comme une fonction linéaire de l'état distribué. Les gains (distribués) peuvant être calculés hors ligne. Cette méthode a été originellement introduite par Smyshlyaev et Krstic dans [SK04] pour le problème de stabilisation frontière d'une classe d'équations integro-différentielles linéaires paraboliques, et fut ensuite étendue dans [SK05] pour obtenir les observateurs associés (nécessaires pour envisager une quelconque implémentation pratique), en utilisant les mesures aux frontières et en se servant du principe de dualité. En comparaison avec les méthodes antérieures, cette approche présente certaines particularités: en utilisant la structure intrinsèque du système, le problème est ramené à la résolution d'un système d'EDPs linéaires hyperboliques (ce qui est un objet plus facile à manier que des opérateurs de Riccatti qui apparaissent lorsque sont utilisées des approches LQG), par ailleurs, l'analyse demeure simple en comparaison avec les approches conventionnelles abstraites puisque le problème à résoudre peut l'être par calculs. Cela a permis l'obtention de nombreux résultats, premièrement pour des systèmes linéaires et non-linéaires d'équations paraboliques [KMV08, KMV09] puis pour l'équation des ondes (c.f. [KGBS08, SCK10, SK09] par exemple) en présence de termes d'antidamping variables en espace (qui déplacent les valeurs propres du système en boucle ouverte vers le demi-plan complexe droit). En outre, les méthodes proposées permettent souvent l'obtention de nouvelles fonctionnelles de Lyapunov. En ce qui concerne les EDPs hyperboliques linéaires avec actionneurs situés à une seule frontière, les premiers résultats ont été d'abord obtenus pour un système d'une équation hyperbolique integro-différentielle [KS08a] (l'équation potentiellement instable est convertie en une équation à retard convergeant vers zéro en temps fini), puis pour le cas de systèmes hyperboliques linéaires et quasilinéaires de deux équations [CVKB13, VKC11]. Les lois de commande ainsi proposées permettent la stabilité exponentielle du système en boucle fermée. Par ailleurs, dans le cas de coefficients constants, il est possible d'obtenir une formulation explicite de ces contrôleurs sous la forme de fonctions de Marcum [VK14]. Ce résultat a été obtenu en résolvant les équations satisfaites par les noyaux de la loi de commande et a nécessité le développement d'une méthode permettant l'expression des solutions correspondantes comme étant la somme de solutions d'équations pouvant être résolues explicitement. Ces résultats ont ensuite été étendus dans [DMVK13] au cas de systèmes formés de $n+1$ équations (i.e, n états se propagent dans une direction tandis qu'un état se propage dans la direction opposée). Il est à noter que les auteurs ne considèrent cependant que le cas d'un observateur utilisant des mesures provenant de la frontière opposée à celle où se trouve l'actionneur. Un tel observateur est en effet plus facile à synthétiser par rapport à un observateur qui utiliserait des mesures provenant de la même frontière que l'actionneur (ce qui est en général d'un plus grand intérêt pratique, par exemple dans le cas de systèmes multiphasiques). La généralisation de ce résultat à un nombre arbitraire d'équations a finalement été proposé par [HDMVK16] et [CHO17]. Néanmoins, là encore, les observateurs proposées utilisent uniquement des mesures provenant de la frontière opposée à celle où se trouve l'actionneur. Un historique complet de l'utilisation de la méthode de backstepping pour le contrôle d'EDPs a été récemment proposé par [VK17]. Pour ces systèmes linéaires hyperboliques, un des avantages de la méthode de backstepping est de permettre une résolutions explicite des problèmes de stabilisation et d'observabilité temps finis tels que formulés dans [Rus72, Rus78b] et généralisés par Tatsien Li dans [Li10]. Plus précisément, la classe de systèmes HLPO considérée peut être stabilisée ou observée en temps fini, le temps de stabilisation minimal (resp. d'observabilité) dépendant du nombre d'actionneurs (resp. capters) disponibles. Toutefois, les résultats donnés par [Li10, Rus72, Rus78b] ne sont que des résultats d'existence et ne permettent pas une synthèse explicite de telles lois de commande/observateurs. En ce sens, la méthode de backstepping, en permettant l'obtention de contrôleurs explicites, a permis d'ouvrir la porte à l'étude d'une large gamme de problèmes, comme les problèmes d'identification de paramètres [DMBPA14], de contrôle adaptatif [BK14] ou de stabilisation de systèmes quasi-linéaires [HDMVK15].

En se focalisant sur la convergence en temps fini, ces contrôleurs négligent les questions de robustesse. Ces questions sont connues comme étant la principale limitation en vu d'implémentations réelles. Ces problèmes de robustesse peuvent être liés à la présence d'incertitudes concernant les différents paramètres du système, de perturbations agissant sur le système, de bruit affectant les mesures, de dynamiques négligées ou de retards agissant sur les actionneurs ou sur les capteurs. Il a été ainsi observé (c.f [DLP86, LRW96]) que pour de nombreux systèmes stabilisés par feedback, l'introduction d'un retard (arbitrairement petit) dans la boucle de rétroaction peut engendrer une instabilité, et ce quel que soit la loi de feedback considéré. Pour appréhender de tels problèmes, de nombreux concepts ont été introduits tels que ceux de stabilisation robuste au retards [LRW96] ou de w-stabilité [CZ12]. De façon similaire, des concepts de robustesse analogues ont été simulatanément développés pour des systèmes à retard de type neutre. Pour ces derniers, la communauté s'intéresse en particulier à la présence d'incertitudes agissant sur les différents retards intrinséques au système. Cela a induit l'introduction du concept de *stabilité forte* [HVL93, HL02, MN07, MVZ$^+$09]. Il est connu depuis longtemps que certains systèmes d'EDPs peuvent être réprésentées comme des systèmes à retard de type neutre. Le premier lien a été établi par la formule de D'Alembert [D'A49] qui permet de réécrire une équation des ondes sous la forme d'une équation aux différences. Plus généralement, en utilisant la méthode des caractéristiques, des systèmes d'EDPs hyperboliques linéaires du premier ordre sans couplage peuvent être transformées en des équations aux différences. Dans [Rus78a, Rus91], les auteurs prouvent à l'aide de méthodes spectrales l'existence d'une telle transformation entre les solutions de systèmes d'EDPs hyperboliques linéaires du premier ordre couplées et de systèmes neutres d'ordre zéro. Dans [KK14], cette transformation est prouvée comme étant unique, et est explicitée dans le cas d'une équation hyperbolique avec un terme de réaction. Ces exemples suggèrent l'existence d'une transformation explicite entre les solutions de la classe générale d'EDPs HLPO que nous considérons dans cette thèse et les systèmes neutres. En outre, dans [KK14], les auteurs ont montré que l'analyse de stabilité était plus simple en analysant la forme neutre. Diverses méthodes utilisant l'analyse complexe ont été introduites pour permettre l'analyse de stabilité et de robustesse de tels systèmes. Cela a permis l'obtention de conditions nécessaires et suffisantes de stabilité [BC63, HL02]. Ainsi une conséquence fondamentale de l'existence d'une telle transformation serait de permettre l'utilisation de méthodes classiquement utilisées pour les sytèmes neutres afin d'analyser les propriétés de robustesse des contrôleurs par backstepping.

Au final, lorsque l'on considère l'utilisation de tels contrôleurs pour des problèmes industriels, dans la plupart des cas seule une approximation de l'état est disponible pour réaliser la synthèse de la loi de commande. Par conséquent, seule une approximation du contrôleur est disponible pour assurer la rétroaction. Ce type d'approche est parfois appelée late-lumping puisque la dernière étape consiste à approximer la loi de commande par un système de dimension finie. La question de la convergence des contrôleurs late-lumping dérivés de contrôleurs backstepping n'a pas été considérablement étudiée à l'heure actuelle; contrairement à la question de l'approximation des noyaux, intervenant dans de telles lois de commande comme proposé en [JMK12] (en utilisant une méthode des trapèzes) ou en [AAP17] (en utilisant une approche par somme de carrés). Dans [WRE17], les auteurs proposent une méthode permettant l'implémentation (i.e. l'approximation) de la partie bornée de l'opérateur de contrôle. Cette méthode repose sur une approximation en dimension finie de l'état et permet une implémentation efficace (en termes de temps de calcul) de la loi de commmande. Néanmoins, la partie non bornée de l'opérateur n'est pas approximée et nulle garantie de convergence n'est assurée.

Problèmes considérés et organisation de la thèse

La première partie de cette thèse propose plusieurs contributions en théorie des systèmes. Dans le Chapitre 2, nous introduisons le problème de synthèse de lois de commande et d'observateurs temps-fini tels qu'introduits par Tatsien Li dans [Li10]. Nous distinguons en particulier les problèmes de contrôlabilité (resp. d'observabilité) unilatérale, pour lesquels les actionneurs (resp. capteurs) sont uniquement disponibles à une des frontières du système, des problèmes de contrôlabilité (resp. d'observabilité) bilatérale, pour lesquels les actionneurs (resp. capteurs) sont disponibles aux deux frontières. Lorsque cette thèse a débuté en 2015, le problème de stabilisation unilatérale n'avait été que partiellement résolu dans [HDMVK16], les auteurs proposant une loi de commande assurant une stabilisation en un temps fini mais non minimal. Dans le Chapitre 3, nous ajustons la transformation de backstepping introduite en [HDMVK16] pour résoudre le problème de stabilisation unilatérale en temps minimal dans le cas d'un système composé de $n+m$ équations. L'observateur en temps minimal associé est quant a lui obtenu grâce à une nouvelle technique basée sur l'utilisation du système adjoint. À notre connaissance, les problèmes bilatéraux, bien que constituant une perspective intéressante en termes d'améliorations de performances, n'ont été que peu considérés dans la littérature. De tels problèmes ont toutefois été résolus dans [VK16] pour un système d'EDPs réaction-diffusion et pour le cas de 2 EDPs linéaires hétérodirectionnelles avec des vitesses de transport *égales*. Dans le Chapitre 4, nous résolvons les problèmes de stabilisation et d'observabilité bilatérales pour un système composé de $n+m$ équations. Inspiré par les approches récentes (c.f. [BAK15, CHO16] et plus récemment [CHO17]), cela est fait en introduisant une transformation de Fredholm (pouvant en réalité se réécrire comme une transformation de Volterra) permettant de tirer pleinement avantage des multiples actionneurs et capteurs. Finalement, dans le Chapitre 5, nous introduisons un nouvel outil permettant l'analyse de stabilité de systèmes d'EDPs hyperboliques linéaires, en proposant une transformation explicite entre l'espace généré par les solutions de tels systèmes et l'espace généré par les solutions d'une classe de systèmes neutres à retards distribués. Cette transformation peut ainsi être utilisée pour obtenir de nouvelles conditions de stabilité.

La seconde partie de cette thèse est dédiée aux propriétés de robustesse des contrôleurs précédemment introduits. Ces aspects sont essentiels pour envisager une quelconque application industrielle. Dans le Chapitre 6, nous commençons par analyser les propriétés de robustesse aux retards. Nous utilisons pour cela l'équivalence entre systèmes d'EDPs et systèmes neutres. Nous montrons que la stabilisation en temps fini peut parfois conduire à des marges de robustesse nulles, ce qui rend de tels contrôleurs inapplicables. Par conséquent, nous introduisons un degré de liberté (à l'aide d'une paramètre ajustable), qui rend ainsi possible un potentiel compromis entre performance et robustesse aux retards. Une extension pour un système composé de deux EDPs couplées avec une Équation Différentielle Ordinaire est proposée dans le Chapitre 7. Le problème de régulation robuste de sortie et de Stabilité Entrée-État est finalement résolu dans le Chapitre 8 pour un système de deux EDPs couplées. La loi de commande par retour de sortie que nous proposons présente trois degrés de liberté qui peuvent être ajustés pour permettre un compromis entre robustesse et performance mais aussi entre sensibilité au bruit et rejet de perturbations.

Nous proposons dans cette thèse les contributions suivantes.

Contribution 1: Synthèse explicite de contrôleurs et d'observateurs temps-fini tels que définis dans [Li10]. Cette contribution est présentée dans les Chaptitres 2-4.

Contribution 2: Mise en relation explicite entre systèmes linéaires hyperboliques du premier ordre et systèmes neutres d'ordre zéro à retards distribués. Cette contribution est présentée au Chapitre 5.

Contribution 3: synthèse d'une loi de feedback par retour de sortie robuste à des retards et incertitudes dans le cas d'un système de deux EDPs couplées. La loi de commande proposée introduit de multiples degrés de liberté. Cette contribution est présentée dans les Chapitres 6-8.

Publications

Le travail présenté dans cette thèse a donné lieu aux publications suivantes:

- Journals

  1. J. Auriol, F. Di Meglio, *minimum-time control of heterodirectional linear coupled hyperbolic PDEs*, Automatica, Vol. 71, p300-307, 2016
  2. J. Auriol, F. Di Meglio, *Two sided boundary stabilization of heterodirectional linear coupled hyperbolic PDEs*, IEEE Transactions on Automatic Control, Accepté.
  3. J. Auriol, U. J. F. Aarsnes, P. Martin, F. Di Meglio, *Delay-robust control design for two heterodirectional linear coupled hyperbolic PDEs*, IEEE Transactions on Automatic Control, Accepté.
  4. J. Auriol, F. Bribiesca-Argomedo, D. Bou Saba, M. Di Loreto, F. Di Meglio, *Delay-robust stabilization of a hyperbolic PDE-ODE system*, Automatica, Accepté.
  5. J. Auriol, K. A. Morris, F. Di Melgio, *Late-lumping backstepping control of partial differential equations*, Automatica, Soumis.
  6. J. Auriol, F. Di Meglio, *An explicit mapping from linear first order hyperbolic PDEs to difference systems*, Systems an Control Letters, Soumis.

7. D. Bou Saba, F. Bribiesca-Argomedo, J. Auriol, M. Di Loreto, F. Di Meglio, *A sufficient stability condition for a class of linear 2 x 2 hyperbolic PDEs using a backstepping transform*, IEEE Transactions on Automatic Control, Soumis.

- Conferences

1. J. Auriol, F. Di Meglio, *Two sided boundary stabilization of two linear hyperbolic PDEs in minimum-time*, Proc. of the 55th IEEE Conference on Decision and Control, Las Vegas, Nevada, USA, 2016.
2. J. Auriol, F. Di Meglio, *Trajectory tracking for a system of two linear hyperbolic PDEs with uncertainties*, 20th World Congress of the International Federation of Automatic Control, Toulouse, France, 2017.
3. P.-O. Lamare, J. Auriol, U. J. F. Aarsnes, F. Di Meglio, *Robust output regulation of 2×2 hyperbolic systems: Control law and Input-to-State Stability*, IEEE American Control Conference, Milwaukee, Wisconsin, USA, 2018.

Notations

We present here some notations that are used throughout the thesis.

1. The underlying field will always be assumed to be real numbers and, for simplification, will often be omitted in the notations. For $p \in \mathbb{N}^*$ and for $X \in \mathbb{R}^p$, the norm of X is defined by
$$||X|| = \sqrt{\sum_{i=1}^{p}(X_i)^2}.$$

2. For $p \in \mathbb{N}^*$ and $q \in \mathbb{N}^*$, we denote by $\mathcal{M}_{p,q}(\mathbb{R})$ the set of $p \times q$ real matrices. The identity matrix of dimension q is denoted Id_q (or Id if no confusion arises).

3. For $p \in \mathbb{N}^*$ and $q \in \mathbb{N}^*$, for $K \in \mathcal{M}_{p,q}(\mathbb{R})$, we define
$$||K|| = \max\{||K\xi||;\ \xi \in \mathbb{R}^p,\ ||x|| = 1\}.$$

4. For $p \in \mathbb{N}^*$ and for $K \in \mathcal{M}_{p,p}$, we denote $\mathrm{Sp}(K)$ the spectral radius of the matrix K.

5. For $n \in \mathbb{N}$, and for $(p_1, \cdots, p_n) \in \mathbb{R}^n$, we denote $D = \mathsf{diag}\{p_1, \cdots, p_n\}$, the diagonal matrix that satisfies for all $1 \leq i \leq n$, $D_{ii} = p_i$.

6. We denote $\mathcal{C}^q([0,1])$ (with $q \in \mathbb{N} \cup \{\infty\}$) the space of functions defined in $[0,1]$ that are q times differentiable and whose the q^{th} differentiate is continuous. We often denote $\mathcal{C}^0([0,1])$ as $\mathcal{C}([0,1])$.

7. We denote by $L^1([0,1],\mathbb{R})$, or $L^1([0,1])$ if no confusion arises, the space of real-valued functions defined on $[0,1]$ whose absolute value is integrable. This space is equipped with the standard L^1 norm, that is, for any $f \in L^1([0,1])$
$$||f||_{L^1} = \int_0^1 |f(x)|\mathrm{d}x.$$

8. We denote $L^2([0,1],\mathbb{R})$ the space of the real functions defined on $[0,1]$ whose square is integrable. This space is equipped with the standard L^2 norm. For any $f \in L^2([0,1],\mathbb{R})$
$$||f||_{L^2} = \sqrt{\int_0^1 f^2(x)dx}.$$
The associated scalar product is denoted $\langle \cdot, \cdot \rangle$.

9. We denote $L^\infty([0,1],\mathbb{R})$ the space of real bounded functions defined on $[0,1]$ with the standard L^∞ norm, *i.e.*, for any $f \in L^\infty([0,1],\mathbb{R})$
$$||f||_{L^\infty} = \sup_{x\in[0,1]} |f(x)|.$$

10. We denote the Sobolev space $H^1([0,1],\mathbb{R}) = W^{1,2}([0,1],\mathbb{R})$ as the subset of functions f in $L^2([0,1],\mathbb{R})$ such that the function f and its weak derivative of order 1 have a finite L^2 norm. This space is equipped with the standard H^1 norm. For any $f \in H^1([0,1],\mathbb{R})$

$$||f||_{H^1} = \sqrt{\int_0^1 f^2(x)dx + \int_0^1 (f'(x))^2 dx}.$$

11. Given a set $\Omega \subseteq \mathbb{R}$, its characteristic function will be denoted by

$$\mathbb{1}_\Omega(\theta) = \begin{cases} 1 & \text{if } \theta \in \Omega \\ 0 & \text{otherwise.} \end{cases}$$

12. For an integer k_1 and two reals $a < b$, a real-value function f defined on $[a,b]$ is said to be k_1-Lipschitz if for all $(x,y) \in [a,b]^2$, it satisfies

$$|f(x) - f(y)| \leq k_1|x - y|.$$

13. For any $s \in \mathbb{C}$, we denote $\Re(s)$ its real part.

14. We denote $\mathbb{C}^+$ the complex Right Half Plane: $\mathbb{C}^+ = \{s \in \mathbb{C},\ \Re(s) \geq 0\}$.

15. Provided it is well defined, we denote $\hat{f}(s)$ the Laplace transform of a function $f(t)$.

16. For $(u,v) \in (L^2([0,1]))^2$, and $X \in \mathbb{R}^p$, we denote

$$||(u,v,X)||_2 = ||u||_{L^2} + ||v||_{L^2} + ||X|| \tag{1.1}$$

17. We denote $\mathcal{A}$ the space of BIBO-stable generalized functions [Vid72]: a function $g(\cdot)$ belongs to $\mathcal{A}$ if it can be expressed

$$g(t) = g_r(t) + \sum_{i=0}^{\infty} g_i \delta(t - t_i),$$

where $g_r \in L^1(\mathbb{R}^+,\mathbb{R})$, $\sum_{i \geq 0} |g_i| < \infty$, $0 = t_0 < t_1 < \ldots$ and δ is the Dirac distribution. The associated norm is

$$\|g\|_{\mathcal{A}} = \|g_r\|_{L^1} + \sum_{i \geq 0} |g_i|.$$

We denote $\hat{\mathcal{A}}$ the Banach algebra of Laplace transforms of functions of $\mathcal{A}$. The associated norm is defined by

$$\|\hat{g}\|_{\hat{\mathcal{A}}} = \|g\|_{\mathcal{A}}$$

18. We denote by $\mathcal{K}$ the set of continuous increasing functions $h : \mathbb{R}^+ \to \mathbb{R}^+$ with $h(0) = 0$.

19. We denote by $\mathcal{KL}$ the set of functions $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for each fixed $t \geq 0$ the function $g(\cdot, t)$ is a $\mathcal{K}$ function, and such that for each fixed $s \geq 0$, the function $g(s, \cdot)$ is decreasing.

20. For any $p \in \mathbb{N}^*$ and any $1 \leq i,j \leq p$, we denote $E^p_{i,j}$ the elementary matrix of order p. More precisely, the matrix $E^p_{i,j} \in \mathcal{M}_{p,p}(\mathbb{R})$, has all its components equal to zero except the one located at the intersection of the i^{th}-line and of the j^{th}-column whose value is one.

Part I

Control theory

Chapter 2

A primer on boundary controllability and observability for linear first order hyperbolic systems

Chapitre 2: Commandabilité et observabilié frontière pour des systèmes hyperboliques linéaires du premier ordre. Ce chapitre introductif présente le cadre mathématique général sur lequel est construit cette thèse. Nous considérons les problèmes généraux de stabilisation et d'observation en temps fini introduits par Tatsien Li dans [Li10]. Plus précisément, concernant la classe de systèmes considérés dans cette thèse, il est théoriquement possible de synthétiser une loi de commande agissant à la frontière du domaine (resp. un observateur utilisant les mesures aux frontières) qui assure la convergence à zero de l'état (resp. la convergence de l'observateur vers l'état réel) en temps fini. Ce temps de convergence minimal dépend du nombre d'actionneurs (resp. de capteurs) disponibles. L'objectif de cette partie est de proposer une synthèse explicite de ces controlleurs (resp. observateurs). L'approche retenue est basée sur la méthode de *backstepping*. Afin de permettre au lecteur de se familiariser avec les outils utilisés dans les chapitres suivants, nous présentons la synthèse explicite d'une telle loi de commande et d'un tel observateur pour le cas particulier d'un système composé de deux équations telle que réalisée dans [CVKB13, VCKB11].

Contents

In this introductory chapter we address the general problems of finite-time boundary stabilization and state estimation stated by David L. Russell in [Rus72, Rus78b] and generalized by Tatsien Li in [Li10]. More precisely for the class of systems considered in this thesis, it is theoretically possible to design a boundary feedback control law (resp. a boundary output observer) that ensures the convergence of the state to zero (resp. the convergence of the observer to the real state) in finite-time. This minimum convergence time depends on the number of actuators (resp. sensors) available. The purpose of this part is to explicitly design these minimum-time control laws and observers. The proposed approach is based on the backstepping methodology. To make the reader familiar with the tools used in the next chapters, we recall the explicit design of the control law and of the observer done in [CVKB13, VCKB11] for the specific case of two equations.

2.1 System under consideration and well-posedness

In this section, we alternatively consider a PDE formulation and an abstract formulation of the class of systems considered in this thesis. Although the first formulation is the most frequently used along this thesis, some proofs require the latter approach. The well-posedness of the system is for instance assessed using the operator formulation. Let us consider the following general LFOH PDE system

$$\partial_t u(t,x) + \Lambda^+ \partial_x u(t,x) = \Sigma^{++}(x)u(t,x) + \Sigma^{+-}(x)v(t,x), \tag{2.1}$$
$$\partial_t v(t,x) - \Lambda^- \partial_x v(t,x) = \Sigma^{-+}(x)u(t,x) + \Sigma^{--}(x)v(t,x), \tag{2.2}$$

with the following linear boundary conditions

$$u(t,0) = Q_0 v(t,0) + U(t), \quad v(t,1) = R_1 u(t,1) + V(t), \tag{2.3}$$

where the states $u = (u_1 \dots u_n)^T$, $v = (v_1 \dots v_m)^T$ are vector functions of (t,x) evolving in $\{(t,x)| \quad 0 < t < T, \quad x \in [0,1]\}$ with values in $\mathbb{R}^n$ and $\mathbb{R}^m$. The matrices Λ^+ and Λ^- are $n \times n$ and $m \times m$ diagonal matrices defined by

$$\Lambda^+ = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad \Lambda^- = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_m \end{pmatrix}.$$

Their eigenvalues satisfy

$$-\mu_m < \dots < -\mu_1 < 0 < \lambda_1 < \dots < \lambda_n.$$

The spatially-varying inside domain couplings matrices are defined as follows

$$\Sigma^{++}(x) = \{(\sigma^{++})_{ij}(x)\}_{1\le i\le n, 1\le j\le n}, \quad \Sigma^{+-}(x) = \{(\sigma^{+-})_{ij}(x)\}_{1\le i\le n, 1\le j\le m},$$
$$\Sigma^{-+}(x) = \{(\sigma^{-+})_{ij}(x)\}_{1\le i\le m, 1\le j\le n}, \quad \Sigma^{--}(x) = \{(\sigma^{--})_{ij}(x)\}_{1\le i\le m, 1\le j\le m}.$$

Their different entries are assumed to belong to $\mathcal{C}^0([0,1],\mathbb{R})$. The boundary coupling terms Q_0 and R_1 are assumed to be constant matrices of dimension $n \times m$ and $m \times n$ whose components are real. The functions U and V are input functions that respectively have values in $\mathbb{R}^n$ and $\mathbb{R}^m$. We assume that the initial conditions of the system (2.1)-(2.3), denoted

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \Big((u_1)_0(\cdot) \quad \dots \quad (u_n)_0(\cdot) \quad (v_1)_0(\cdot) \quad \dots \quad (v_m)_0(\cdot) \Big)^T, \tag{2.4}$$

belong to $(L^2([0,1]))^{n+m}$.

Remark 2.1.1 *Even if the matrices Λ^+ and Λ^- are assumed to be constant in this thesis, most of the presented results can be extended to the case of spatially varying matrices whose components are $C^1([0,1],\mathbb{R})$-functions.*

Let us now give the operator formulation of equations (2.1)-(2.2). Multiplying formally (2.1)-(2.2) by smooth test functions (ϕ, ψ) and integrating by parts, we are lead to the following definition of a solution:

Definition 2.1.1.

Let us consider a time $T > 0$, an initial condition $\begin{pmatrix} u_0 & v_0 \end{pmatrix} \in (L^2([0,1]))^{n+m}$, $U \in (L^2([0,T]))^n$ and $V \in (L^2([0,T]))^m$. We say that (u,v) is a (weak) solution to (2.1)-(2.3) if $(u,v) \in (\mathcal{C}^0([0,T]; L^2([0,1])))^{n+m}$ and

$$\begin{aligned}
0 = &\int_0^\tau \int_0^1 (-\partial_t \phi^T(t,x) - \partial_x \phi^T(t,x)\Lambda^+ - \phi^T(t,x)\Sigma^{++}(x) - \psi^T(t,x)\Sigma^{-+}(x))u(t,x) + (-\partial_t \psi^T(t,x) \\
&+ \partial_x \psi^T(t,x)\Lambda^- - \psi^T(t,x)\Sigma^{--}(x) - \phi^T(t,x)\Sigma^{++}(x))u(t,x)dxdt + \int_0^1 \phi^T(\tau,x)u(\tau,x) \\
&- \phi^T(0,x)u(0,x) + \psi^T(\tau,x)v(\tau,x) - \psi^T(0,x)v(0,x)dx - \int_0^\tau \phi^T(t,0)\Lambda^+ U(t) + \psi^T(t,1)\Lambda^- V(t)dt \\
&+ \int_0^\tau (\phi^T(t,1)\Lambda^+ - \psi^T(t,1)\Lambda^- R_1)u(t,1) + (\psi^T(t,0)\Lambda^- - \phi^T(t,0)\Lambda^+ Q_0)v(t,0)dt, \qquad (2.5)
\end{aligned}$$

for every $(\phi, \psi) \in \mathcal{C}^1([0,\tau]\times[0,1])^{n+m}$ such that $\phi(\cdot,1) = \psi(\cdot,0) = 0$, and for every $\tau \in [0,T]$.

The class of systems defined by (2.1)-(2.3) belongs to the general class of boundary control systems [Sal87]. These systems can be rewritten in an abstract state space form (although this is not always necessary [CM03]), generally using unbounded control operators; that is, a control operator bounded to some Hilbert space larger than the state space [Sal87]. There is an extensive literature dealing with systems having unbounded control operators; see for instance [CS86, CP78, DLS85, HR83, LT83a, PW78, Sal87, Was79]). Considering system (2.1)-(2.3), we have the following abstract formulation

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} + B \begin{pmatrix} U \\ V \end{pmatrix}, \qquad (2.6)$$

where we can identify the operators A and B through their adjoints taking formally the canonical scalar product of (2.6) with smooth test functions (ϕ, ψ) and comparing with (2.5). The operator A is thus defined by

$$\begin{aligned}
A : &D(A) \subset (L^2(0,1))^{n+m} \to (L^2(0,1))^{n+m} \\
&\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} -\Lambda^+\partial_x u + \Sigma^{++}(x)u + \Sigma^{+-}(x)v \\ \Lambda^-\partial_x v + \Sigma^{-+}(x)u + \Sigma^{--}(x)v \end{pmatrix},
\end{aligned} \qquad (2.7)$$

on the domain

$$D(A) = \{(u,v) \in (L^2(0,1))^{n+m} |\ u(0) = Q_0 v(0),\ v(1) = R_1 u(1)\}.$$

The operator A is densely defined. Its adjoint A^* is defined by

$$\begin{aligned}
A^* : &D(A^*) \subset (L^2(0,1))^{n+m} \to (L^2(0,1))^{n+m} \\
&\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} \Lambda^+\partial_x u + (\Sigma^{++}(x))^T u + (\Sigma^{-+}(x))^T v \\ -\Lambda^-\partial_x v + (\Sigma^{+-}(x))^T u + (\Sigma^{++}(x))^T v \end{pmatrix},
\end{aligned}$$

with

$$D(A^*) = \{(u,v) \in (L^2(0,1))^{n+m} |\ u(1) = (\Lambda^+)^{-1} R_1^T \Lambda^- v(t,1),\ v(0) = (\Lambda^-)^{-1} Q_0^T \Lambda^+ u(t,0)\}.$$

It has been proved in [Rus78b, Theorem 3.1] that A generates a $\mathcal{C}^0$-semigroup $(\mathcal{S}(t))_{t\geq 0}$.

Definition 2.1.2.

Let us consider a Hilbert space $\mathcal{Z}$. An operator $A : D(A) \subset \mathcal{Z} \to \mathcal{Z}$, is dissipative if there exists $\omega < 0$ such that for all $z \in D(A)$, $\Re(\langle Az, z\rangle) \leq \omega\langle z, z\rangle$.

Theorem 2.1.1. [LP61, Corrolary 3.1]

Let A be a linear operator defined on a dense linear subspace $D(A)$ of a reflexive Banach space. Then, A generates a strongly continuous contraction semigroup and there exists $\omega < 0$ such that $||\exp(At)|| \leq \exp(\omega t)$ if and only if A is closed[1], A is dissipative and its adjoint A^* is dissipative.

Since A^* is closed, its domain $D(A^*)$ is then a Hilbert space, equipped with the scalar product associated to the graph norm $||z||_{D(A^*)} = (||z||^2_{L^2} + ||A^*z||^2_{L^2})$, where $z \in D(A^*)$. Note that on $D(A^*)$ the norms $||\cdot||_{D(A^*)}$ and $||\cdot||_{H^1}$ are equivalent. The operator $B \in \mathcal{L}(\Re^{n+m}, D(A^*)')$ is defined by

$$< B \begin{pmatrix} U \\ V \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} >= z_1(0)^T\Lambda^+U + z_2(1)^T\Lambda^-V, \tag{2.8}$$

while its adjoint $B^* \in \mathcal{L}(D(A^*), \Re^{n+m})$ is defined by $B^* \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1(0)^T\Lambda^+ \\ z_2(1)^T\Lambda^- \end{pmatrix}$. The operator B is well-defined since, due to the trace theorem $B \begin{pmatrix} U \\ V \end{pmatrix}$ is continuous on $H^1([0, 1])$. One can prove that the operator B satisfies the following so-called admissibility condition (see [BDPDM07, LT83b, LT91, Wei89]):

$$\exists M > 0, \quad \int_0^T |B^*\mathcal{S}(T-t)^*z|^2 dt \leq M||z||^2_{L^2([0,1])}, \quad \forall z \in D(A^*) \tag{2.9}$$

where we recall that $\mathcal{S}$ is the semigroup generated by the operator A. This condition implies that for any t, the state $z(t)$ remains in $L^2([0, 1])$ and depends continuously on the input (U, V). Since A generates a $\mathcal{C}^0$-semigroup and since B is admissible, we can write the following well-posedness lemma

Lemma 2.1.1. [Cor09, Rus78b]

For every initial condition $\begin{pmatrix} u_0 & v_0 \end{pmatrix} \in (L^2([0, 1]))^{n+m}$, and every control laws $(U, V) \in (L^2([0, 1]))^{n+m}$, the problem (2.1)-(2.3) along with the boundary conditions (2.3) admits a unique solution $(u, v) \in \mathcal{C}^0([0, T]; (L^2([0, 1]))^{n+m})$ on the domain $\{(t, x)| \quad 0 < t < T, \quad x \in [0, 1]\}$. Moreover, there exists a constant $C_T > 0$ (which does not depend on the initial condition nor the control law) such that

$$||(u, v)||_{\mathcal{C}^0([0,T];L^2([0,1])^{n+m})} \leq C_T \left(||(u_0^T, v_0^T)||_{(L^2([0,1]))^{n+m}} + ||(U, V)||_{(L^2([0,T]))^{n+m}}\right)$$

According to the theory on the semi-global $\mathcal{C}^1$ solution [LJ01], we also have the following lemma, that requires stronger regularity hypothesis on the initial condition.

[1] We recall that an operator A defined on $D(A)$ is closed if and only if for all sequence $x_n \in D(A)$ such that $x_n \to x$ and $Ax_n \to y$, then we have $x \in D(a)$ and $Ax = y$.

Lemma 2.1.2. [LR10, Lemma 1.1]

Consider system (2.1)-(2.3) along with the initial condition $(u_0, v_0) \in (\mathcal{C}^1([0,1]))^{n+m}$ and the $\mathcal{C}^1$ control laws (U, V). Suppose furthermore that the conditions of $\mathcal{C}^1$ compatibility are satisfied at the points $(t, x) = (0, 0)$ and $(0, 1)$, respectively. For any given and possibly quite large $T_0 > 0$, if $||(u_0, v_0)||_{(\mathcal{C}^1([0,1]))^{n+m}}$ and $||(U, V)||_{(\mathcal{C}^1[0,T_0])^{n+m}}$ are sufficiently small (depending on T_0), then there exists a unique semi-global $\mathcal{C}^1$ solution $(u(t, x), v(t, x))$ on the domain $R(T_0) = \{(t, x)| \quad 0 < t < T_0, \quad x \in [0, 1]\}$. Moreover we have the existence of a positive constant C_0 (depending on T_0) such that

$$||(u, v)||_{(\mathcal{C}^1(R(T_0)))^{n+m}} \leq C_0 \left(||(u_0, v_0)||_{(\mathcal{C}^1([0,1]))^{n+m}} + ||(U, V)||_{(\mathcal{C}^1[0,T_0])^{n+m}} \right)$$

Remark 2.1.2 *We recall that the compatibility conditions are given by*

$$\begin{aligned}
&u_0(0) = Q_0 v_0(0), \quad v_0(1) = R_1 u_0(1), \\
&u_0'(0) = (\Lambda^+)^{-1}((\Sigma^{++}(0) - Q_0\Sigma^{-+}(0))u(0) + (\Sigma^{+-}(0) - Q_0\Sigma^{--}(0))v(0) - Q_0\Lambda^-) + U(0), \\
&v_0'(0) = (\Lambda^-)^{-1}((R_1\Sigma^{++}(1) - R_1\Sigma^{-+}(1))u(1) + (R_1\Sigma^{+-}(1) - \Sigma^{--}(1))v(1) - R_1\Lambda^+) + U'(0).
\end{aligned}$$

2.2 Stability analysis

In this section we recall some results on the stability properties of system (2.1)-(2.3) in open loop (i.e $U \equiv 0$ and $V \equiv 0$). The objective is to find conditions on the different coupling terms such that the system is dissipative (i.e exponentially stable). We give the following definition of exponential stability.

Definition 2.2.1. [BC16]

The hyperbolic system (2.1)-(2.3) is exponentially stable, if there exist $\nu > 0$, and $C > 0$ such that, for every initial condition $(u_0, v_0) \in (L^2([0,1]))^{n+m}$, the L^2-solution of the problem (2.1)-(2.3) satisfies

$$||(u, v)||_{(L^2([0,1]))^{n+m}} \leq C\mathrm{e}^{-\nu t} \left(||(u_0^T, v_0^T)||_{(L^2([0,1]))^{n+m}} \right). \tag{2.10}$$

In what follows, we present a quick review of existing stability results for LFOH PDEs.

2.2.1 Linear conservation laws equations

We first consider the case of linear conservation laws under static boundary conditions, i.e the in-domain couplings $\Sigma^{\cdot\cdot}$ are assumed to be equal to zero. System (2.1)-(2.3) rewrites

$$\partial_t u(t, x) + \Lambda^+ \partial_x u(t, x) = 0, \quad \partial_t v(t, x) - \Lambda^- \partial_x v(t, x) = 0, \tag{2.11}$$
$$u(t, 0) = Q_0 v(t, 0), \quad v(t, 1) = R_1 u(t, 1). \tag{2.12}$$

This problem has been considered in [Sle83, Tie85, Zha86] relying on a systematic use of direct estimates of the solutions and their derivatives along the characteristic curves. As system (2.11)-(2.12) is composed of transport equations coupled at the boundaries, it can be rewritten as a neutral system. More precisely, denoting $u_+(t, x) = u(t, 1-x)$, the system (2.11)-(2.12) rewrites

$$\partial_t u_+(t, x) - \Lambda^+ \partial_x u_+(t, x) = 0, \quad \partial_t v(t, x) - \Lambda^- \partial_x v(t, x) = 0, \tag{2.13}$$
$$u_+(t, 1) = Q_0 v(t, 0), \quad v(t, 1) = R_1 u_+(t, 0). \tag{2.14}$$

Denoting $\phi(t) = \begin{pmatrix} u_+(t,1) \\ v(t,1) \end{pmatrix}$ and using the method of characteristic we immediately have for all $t > \max\ \{\frac{1}{\mu_1}; \frac{1}{\lambda_1}\}$ and for all $1 \leq i \leq n+m$

$$\phi_i(t) = \sum_{k=1}^{n+m} \begin{pmatrix} 0 & Q_0 \\ R_1 & 0 \end{pmatrix}_{ik} \phi_k(t-\tau_k), \tag{2.15}$$

where

$$\tau_k = \begin{cases} \frac{1}{\lambda_k}, & \text{if } k \leq n \\ \frac{1}{\mu_k}, & \text{otherwise} \end{cases}.$$

This system can be rewritten in the vector form

$$\phi(t) = \sum_{k=1}^{n+m} A_k \phi(t-\tau_k), \tag{2.16}$$

for some matrices A_k that only depend on R_1 and Q_0. The stability properties of such systems are often assessed by writing the corresponding complex characteristic equation [HL02, HVL93, Nic01a]. More precisely, the location of the roots of the characteristic equation in the complex plane gives the stability properties of the system. It is proved in [HVL93], that system (2.16) is exponentially stable if and only if there exists $\delta > 0$ such that

$$\left(\det(Id_{n+m} - \sum_{k=1}^{n+m} A_k \mathrm{e}^{-\tau_k s}) = 0, s \in \mathbb{C}\right) \Rightarrow (\Re(s) \leq -\delta) \tag{2.17}$$

where $\Re(s)$ denotes the real part of the complex number s. The following theorem provides a numerical method to test this condition.

Theorem 2.2.1. [HVL93, Theorem 6.1], [HL02]

If the delays $(\tau_1, \cdots, \tau_{n+m})$ are rationally independent then, there exists $\epsilon > 0$ such that the holomorphic function $F(s) = \det(Id_{n+m} - \sum_{k=1}^{n+m} A_k \mathrm{e}^{-\tau_k s})$ has all its roots in the left half-plane $\{s \in \mathbb{C} \mid \Re(s) \leq -\epsilon\}$ if and only if

$$\sup_{\theta_k \in [0,2\pi]^{n+m}} \mathrm{Sp}\ (\sum_{k=1}^{n+m} A_k \exp(i\theta_k)) < 1. \tag{2.18}$$

Moreover, if

$$\sup_{\theta_k \in [0,2\pi]^{n+m}} \mathrm{Sp}\ (\sum_{k=1}^{n+m} A_k \exp(i\theta_k)) > 1, \tag{2.19}$$

the function $F(s)$ has an infinite number of roots in the right half plane $\{s \in \mathbb{C} \mid \Re(s) \geq 0\}$.

This condition can be rewritten in the scalar case as follows.

Theorem 2.2.2. [HVL93, Corollary 6.1],[HL02]

If each A_k is a scalar (denoted a_k) and if the delays $(\tau_1, \cdots, \tau_N)$ are rationally independent then, the function $F(s) = \det(Id_{n+m} - \sum_{k=1}^{n+m} a_k \mathrm{e}^{-\tau_k s})$ has all its roots in the left half-plane $\{s \in \mathbb{C} \mid \Re(s) < 0\}$ if and only if $\sum_{k=1}^{n+m} |a_k| < 1$. Moreover, if $\sum_{k=1}^{n+m} |a_k| > 1$ the function $F(s)$ has an infinite number of roots in the right half-plane $\{s \in \mathbb{C} \mid \Re(s) \geq 0\}$.

Note that if the delays are not rationally independent, the conditions (2.18) is only sufficient (see [HVL93] for details). Condition (2.18) requires iterative optimization methods to be tested and is not constructive, in general. For state-feedback synthesis, some numerically tractable sufficient conditions have been proposed using Lyapunov-Krasovskii theory [Car96, DDLM15, Fri02, GFF17, MA13, Nic01a, Nic01b, Pep05]. We present below some of these sufficient conditions that are easily implementable. An easy one is for instance given by the following theorem.

Theorem 2.2.3. [Li94, Theorem 1.3]

The system (2.11)-(2.12) is exponentially stable for the L^2-norm if Sp $\left(\begin{pmatrix} 0 & |Q_0| \\ |R_1| & 0 \end{pmatrix}\right) < 1$.

More recently, a better explicit sufficient condition has been derived in [CBdN08] using a Lyapunov approach. Let us define, for $H \in \mathcal{M}_{p,p}(\mathbb{R})$

$$\mathrm{Sp}_1(K) = \mathrm{Inf}\{||\Delta H \Delta^{-1}||;\ \Delta \in \mathcal{D}_{p,+}\}, \tag{2.20}$$

where $\mathcal{D}_{p,+}$ denotes the set of $p \times p$ real diagonal matrices with strictly positive diagonal elements. Then, we have

Theorem 2.2.4. [CBdN08, Theorem 2.3]

The system (2.11)-(2.12) is exponentially stable for the L^2-norm if $\mathrm{Sp}_1\left(\begin{pmatrix} 0 & Q_0 \\ R_1 & 0 \end{pmatrix}\right) < 1$.

It has been proved in [CBdN08] that this sufficient condition is weaker than the previous one, i.e for every $K \in \mathcal{M}_{p,p}(\mathbb{R})$, $\mathrm{Sp}_1(K) \leq \mathrm{Sp}(|K|)$. Comparing this sufficient condition with the necessary and sufficient one given in (2.18), we have the following lemma.

Lemma 2.2.1. [CBdN08, Proposition 3.7]

For every $p \in \mathbb{N}^*$ and for every $K \in \mathcal{M}_{p,p}(\mathbb{R})$, $\mathrm{Sp}(K) \leq \mathrm{Sp}_1(K)$. Moreover, if $p \leq 5$, then $\mathrm{Sp}(K) = \mathrm{Sp}_1(K)$ and if $p > 5, \mathrm{Sp}(K) < \mathrm{Sp}_1(K)$.

This lemma proves that for a small number of equations, the stability condition given in Theorem 2.2.4 is sufficient and necessary.

2.2.2 Case of coupled system (2.1)-(2.3)

To study the stability of the open-loop system (2.1)-(2.3), it is possible, using a Lyapunov approach, to derive sufficient conditions taking the form of Linear Matrices Inequalities (LMIs) that guarantee exponential stability [BC16, DBC12].

Theorem 2.2.5. [BC16, Proposition 5.2]

The solution of the open-loop Cauchy problem (2.1)-(2.3) in presence of the initial condition (u_0, v_0) exponentially converges to its zero-equilibrium for the L^2-norm if there exist some reals $\mu \neq 0$, $p_i > 0$, $i = 1, \cdots, n$ and $q_j > 0$, $j = 1, \cdots, m$ such that the following matrix inequalities hold:

1. the matrix

$$\begin{pmatrix} P^+\Lambda^+ e^{-\mu} & 0 \\ 0 & P^-\Lambda^- \end{pmatrix} - \begin{pmatrix} 0 & Q_0 \\ R_1 & 0 \end{pmatrix}^T \begin{pmatrix} P^+\Lambda^+ & 0 \\ 0 & P^-\Lambda^- e^{\mu} \end{pmatrix} \begin{pmatrix} 0 & Q_0 \\ R_1 & 0 \end{pmatrix},$$

is positive semi-definite;

2. the matrix

$$\mu\begin{pmatrix}\Lambda^+ & 0\\ 0 & \Lambda^-\end{pmatrix}\begin{pmatrix}P^+\mathrm{e}^{-\mu x} & 0\\ 0 & P^-\mathrm{e}^{\mu x}\end{pmatrix}-\begin{pmatrix}\Sigma^{++}(x) & \Sigma^{+-}(x)\\ \Sigma^{-+}(x) & \Sigma^{--}(x)\end{pmatrix}^T\begin{pmatrix}P^+\mathrm{e}^{-\mu x} & 0\\ 0 & P^-\mathrm{e}^{\mu x}\end{pmatrix}$$
$$-\begin{pmatrix}P^+\mathrm{e}^{-\mu x} & 0\\ 0 & P^-\mathrm{e}^{\mu x}\end{pmatrix}\begin{pmatrix}\Sigma^{++}(x) & \Sigma^{+-}(x)\\ \Sigma^{-+}(x) & \Sigma^{--}(x)\end{pmatrix},$$

is positive definite for all $x \in [0,1]$;

where $P^+ = \mathsf{diag}\{p_1, \cdots, p_n\}$ and $P^- = \mathsf{diag}\{q_1, \cdots, q_m\}$.

For general linear systems of the form (2.1)-(2.3), it is clear that more explicit conditions can only be derived on a case by case basis, when the specific structure or the numerical values of the matrices are specified. However, more precise conditions similar to the one stated in Theorem (2.2.4) can be obtained for constant parameters [BC16]. More precisely, we have the following theorem.

Theorem 2.2.6. [BC16, Theorem 5.4]

Let us denote

$$M = \begin{pmatrix}\Sigma^{++} & \Sigma^{+-}\\ \Sigma^{-+} & \Sigma^{--}\end{pmatrix},\; K = \begin{pmatrix}0 & Q_0\\ R_1 & 0\end{pmatrix}\; \Delta = \begin{pmatrix}P^+\Lambda^+ & 0\\ 0 & P^-\Lambda^-\end{pmatrix}.$$

If there exists $P \in \mathcal{D}^+_{n+m}$ such that

1. $M^T\begin{pmatrix}P^+ & 0\\ 0 & P^-\end{pmatrix} + \begin{pmatrix}P^+ & 0\\ 0 & P^-\end{pmatrix}M$ is negative semi-definite;
2. $||\sqrt{\Delta}K\sqrt{\Delta}^{-1}|| < 1$;

then the system (2.1)-(2.3) is exponentially stable in the sense of the L^2-norm

In the case of two coupled scalar equations with only constant anti-diagonal terms, this condition can be simplified. Then, let us consider the system

$$\partial_t u(t,x) + \lambda\partial_x u(t,x) = \sigma^{+-}v(t,x) \tag{2.21}$$
$$\partial_t v(t,x) - \mu\partial_x v(t,x) = \sigma^{-+}u(t,x) \tag{2.22}$$

along with the boundary conditions

$$u(t,0) = qv(t,0), \quad v(t,1) = \rho u(t,1). \tag{2.23}$$

If $q \neq 0$, Theorem 2.2.6 implies the following lemma

Lemma 2.2.2. [BC16, Corollary 5.5]

The system (2.21)-(2.23) is exponentially stable if $\sigma^{+-} = \sigma^{-+} = 0$ or if

$$\sigma^{+-}\sigma^{-+} < 0, \text{ and } \rho^2 < -\frac{\lambda\sigma^{-+}}{\mu\sigma^{+-}} < \frac{1}{q^2}. \tag{2.24}$$

This sufficient condition is conservative as if $\sigma^{+-}\sigma^{-+} > 0$ or only one of them is equal to zero, no conclusion can be made regarding the stability. In Chapter 5, the Lyapunov-based condition stated in Lemma 2.2.2 is compared to a new criterion obtained by rewriting system (2.21)-(2.23) as a neutral system with distributed delays.

2.3 Boundary controllability and observability problems

In this section we present the boundary controllability and observability problems that we solve in this part of the thesis. We recall the principal definitions and existing results that have, for instance, been stated in [Li10, LR10] for the more general framework of quasilinear hyperbolic systems.

2.3.1 Boundary controllability

Various concepts of controllability have been defined for systems of the form (2.1)-(2.3). The problem of weak exact boundary controllability addressed by Tatsien Li in [Li10, LR10] is the following

Definition 2.3.1. [LR10, Weak exact boundary controllability]

A system of the form (2.1)-(2.3) is said to be weakly exactly controllable if for any given initial condition (u_0, v_0) with small $\mathcal{C}^1$-norm, there exists a $0 < T$ and a control law (U, V) (or a part of this control law) with small $\mathcal{C}^1([0, T_0])$ norm, such that the corresponding mixed initial-boundary value problem (2.1)-(2.3) admits a unique semi-global $\mathcal{C}^1$ solution $(u(t,x), v(t,x))$ on the domain $\{(t,x)| \quad 0 < t < T, \quad x \in [0,1]\}$ with small $\mathcal{C}^1$ norm, which satisfies exactly the final condition

$$\text{For all } 0 \leq x \leq 1,\ u(T,x) = v(T,x) = 0. \tag{2.25}$$

This property is called weak controllability by opposition to the strong exact controllability property, for which any final condition $\Psi(x)$ (with small norm) should be reachable in finite time. There are a number of contributions concerned with the exact controllability for linear hyperbolic systems [Lio88a, Lio88b, Rus78b]. The exact boundary controllability of first order linear hyperbolic system has been established by the characteristic methods in [Rus78b]. The Hilbert Uniqueness Method (HUM) introduced by Lions [Lio88a, Lio88b] gives a more general and systematic framework for the study of exact boundary controllability for wave equations. Some extensions have been obtained in [Cir69, FI96, LR02, LY03, LR03, LR10] for first order quasilinear hyperbolic systems. Based on the existence result for the semi-global $\mathcal{C}^1$ solution to the mixed initial-boundary value problems for quasilinear hyperbolic systems, general controllability results have been obtained in [Li10, LR10]. We give here two properties concerning weak controllability for the system (2.1)-(2.3).

Theorem 2.3.1. [LR10, Two-sided Weak Boundary Controllability]

Let $T_0 > \max(\frac{1}{\lambda_1}, \frac{1}{\mu_1})$. There exist boundary controls (U, V) with small $\mathcal{C}^1([0, T_0])$ norm such that the boundary-control problem (2.1)-(2.3) admits a unique semi-global $\mathcal{C}^1$ solution with small $\mathcal{C}^1$ norm which satisfies exactly the final zero-condition (2.25).

The "minimum time" $\max(\frac{1}{\lambda_1}, \frac{1}{\mu_1})$ is the time needed for the slowest characteristic to travel the entire spatial domain. A similar result for the case where only actuation at one boundary is available (i.e, either U or V is equal to zero)has been obtained. This situation can occur for multiple industrial problems: channel regulation, heat exchangers, gas flow pipelines (see [BC16]).

Theorem 2.3.2. [LR10, One-sided Weak Boundary Controllability]

Let $T_1 > \frac{1}{\lambda_1} + \frac{1}{\mu_1}$. Assume that the control $U(t)$ (resp. $V(t)$) is set to zero. For any given initial condition (u_0, v_0) with small $\mathcal{C}^1([0,1])$ norm, such that the conditions of $\mathcal{C}^1$ compatibility are satisfied at the point $(t,x)=(0,0)$ (resp. $(t,x)=(0,1)$). There exist a boundary controls $V(t)$ (resp. $U(t)$) with small $\mathcal{C}^1([0,T_1])$ norm such that the boundary-control problem (2.1)-(2.3) admits a unique semi-global $\mathcal{C}^1$ solution with small $\mathcal{C}^1$ norm which satisfies exactly the final zero-condition (2.25).

In this case, the "minimum time" $\frac{1}{\lambda_1}+\frac{1}{\mu_1}$ is the sum of the times needed for the slowest characteristic in each direction to travel the entire spatial domain. It has been proved [Rus67b, Rus67a] that this minimum time is "critical" in the sense that it is in general impossible to satisfy the given initial and terminal conditions if less time is allowed. Moreover, for the specific case of two equations, it has been proved that the control law that ensures finite-time boundary stabilization in the minimum time is unique [Rus72]. Note that the minimum convergence time proposed in Theorem 2.3.1 is smaller than the one given in Theorem 2.3.2 due to the fact that one can simultaneously use the two boundary control laws to accelerate the convergence. The proofs of these two theorems are based on the explicit evolution of the Riemann invariants along the characteristics (see [Li10] for details). They are straightforward in the absence of in-domain couplings $\Sigma^{\cdot\cdot}$, as detailed in the next example.

Example 2.3.1 In the absence of in-domain couplings, the original system (2.1)-(2.3) rewrites

$$\partial_t u(t,x)+\Lambda^+\partial_x u(t,x)=0,$$
$$\partial_t v(t,x)-\Lambda^-\partial_x v(t,x)=0,$$

$$u(t,0)=Q_0v(t,0)+U(t),\quad v(t,1)=R_1u(t,1)+V(t)$$

Thus, if the two controls are available, choosing $U(t)=-Q_0v(t,0)$ and $V(t)=-R_1u(t,1)$ leads to $(n+m)$ independent transport equations that reach their zero equilibrium in finite-time $\max(\frac{1}{\lambda_1},\frac{1}{\mu_1})$ (the slowest transport time). Physically, it corresponds to the time needed for the system to "forget" its initial condition. If only one control is available ($U(t)\equiv 0$ for instance), then choosing $V(t)=-R_1u(t,1)$ leads to a cascade of two systems composed of transport equations. It converges to its zero equilibrium in time $\frac{1}{\lambda_1}+\frac{1}{\mu_1}$. The purpose of such a control law is to increase the convergence rate or to have finite-time convergence. This improvement of the controller performance is related to impedance matching methods (see [AAHP13, AGA$^+$14, EG02]). This method is used, for instance, to improve the control performance for the heave rejection problem in Managed Pressure Drilling([AGA$^+$14]), one can match the load impedance (the pressure to flow ratio in the frequency domain at the boundary) to the characteristic line impedance (the pressure to flow ratio in the frequency domain in the transmission line).

The controllability theorems (Theorem (2.3.1) and Theorem (2.3.2)) focus on the **existence** of a control law that stabilizes the original system (2.1)-(2.3) in finite time (which depends on the number of actuators available). Their proofs (that can be found in [Li10]), are based on explicit computation of the solution along the characteristic curves in the framework of the $\mathcal{C}^1$ norm [GT84], [Li94], [Qin85]. Due to the complexity of the underlying methods, these controllability analysis methods cannot be directly used to derive the corresponding stabilizing control laws.

For the case of **one-sided boundary control** (i.e. $U\equiv 0$) this problem has been overcome in [CVKB13] for two equations ($m=n=1$). A similar approach is used in [DMVK13] to design output feedback laws for the case of $m=1$ controlled negative velocity and n (uncontrolled) positive ones. The generalization of this result to an arbitrary number m of controlled negative velocities is presented in [HDMVK16]. The proposed control law yields finite-time convergence to zero, but the convergence time is larger than the minimum control time given in Theorem 2.3.2. Based on Theorem 2.3.1 and Theorem 2.3.2, the objective of Part I is to derive exact stabilizing control law in the framework presented in Section 2.1. More precisely, we have the following control objectives.

Objective A: Finite-time one-sided boundary stabilization. In the case where only one of the two control laws is available (e.g $U(t) \equiv 0$), we derive a state-feedback law $V \in (L^2([0,T]))^m$ that ensures, for any initial condition $(u_0^T, v_0^T) \in L^2([0,1])^{n+m}$, the stabilization in the sense of the L^2-norm of system (2.1)-(2.3) (i.e the convergence of $||(u,v)||_{L^2([0,1])}$ to zero) in finite-time $\frac{1}{\lambda_1} + \frac{1}{\mu_1}$. This is the purpose of Chapter 3.

Objective B: Finite-time two-sided boundary stabilization. In the case where the two control laws are available, we derive a state-feedback law $(U,V) \in L^2([0,T])^{n+m}$ that ensures for any initial condition (u_0, v_0) the stabilization in the sense of the L^2-norm of system (2.1)-(2.3) (i.e the convergence of $||(u,v)||_{L^2([0,1])}$ to zero) in finite-time $\max(\frac{1}{\lambda_1}; \frac{1}{\mu_1})$. This is done in Chapter 4.

Note that, since the two control laws are available, Objective B requires a smaller convergence time compared to Objective A (and in that sense is not included in Objective A). The control laws we derive in the next sections to fulfill Objective A and Objective B ensure the weak exact boundary controllability in the sense of Theorem 2.3.1 and Theorem 2.3.2 considering $\mathcal{C}^1$ initial conditions (along with compatible boundary conditions. In the next section, we define the dual estimation problems.

2.3.2 Boundary observability

We now discuss observability. Let us consider the following observability problem defined in [LR10].

Definition 2.3.2. [LR10, Weak exact boundary observability]

A system of the form (2.1)-(2.3) is said to be weakly exactly observable, if for any given initial condition (u_0, v_0) with small $\mathcal{C}^1$-norm, for any known control law (U,V) with small $\mathcal{C}^1$-norm such that the conditions of $\mathcal{C}^1$ compatibility are satisfied at the points $(t,x) = (0,0)$ and $(0,1)$ respectively, then boundary observations, $u(t,1)$ and $v(t,0)$ (or a part of these observations), can be used to uniquely determine the final data $(u(T,x), v(T,x))$ of the corresponding mixed initial-boundary value problem (2.1)-(2.3).

This "backward" problem, introduced by D. Russell in [Rus78b] is called weak observability problem since it only requires to uniquely determine the final data from boundary observations. Conversely, the strong exact observability problem consists in uniquely determining the initial condition (u_0, v_0) using the boundary measurements. Obviously, strong exact observability implies weak exact observability by forward integration. These two observability properties can be equivalent in some situations. Practical implementations often only require weak observability as the purpose of the observer is to be coupled with a state feedback law to obtain an output feedback control law.We recall here two results concerning weak observability developed in [Li08, Li10, LR10] which are the analogous of the controllability results introduced in Section 2.3.1.

Theorem 2.3.3. [LR10, Two-sided Weak Boundary Observability]

Let $T > \max(\frac{1}{\lambda_1}, \frac{1}{\mu_1})$. For any initial condition (u_0, v_0) with small $\mathcal{C}^1([0,1])$ norm, satisfying the conditions of $\mathcal{C}^1$ compatibility at the points $(t,x) = (0,0)$ and $(0,1)$, respectively, the boundary observations $u(t,1)$ and $v(t,0)$ can be used to uniquely determine the final data $(u(t,x), v(t,x))$ at $t = T$. Moreover, there exists a positive constant C_1 such that the following weak observability inequality holds

$$||(u(T,x),v(T,x))||_{\mathcal{C}^1([0,1])} \leq C_1(\sum_{k=1}^{n} ||u(\cdot,1)||_{\mathcal{C}^1([0,T_0])} + \sum_{k=1}^{m} ||v(\cdot,0)||_{\mathcal{C}^1([0,T_0])} + ||(U,V)||_{\mathcal{C}^1([0,T_0])}). \tag{2.26}$$

Similarly, we get the following theorem for the one-sided weak boundary observability.

Theorem 2.3.4. [LR10, One-sided Weak Boundary Observability]

Let $T > \frac{1}{\lambda_1} + \frac{1}{\mu_1}$. For any initial condition (u_0, v_0) with small $\mathcal{C}^1([0,1])$ norm, satisfying the conditions of $\mathcal{C}^1$ compatibility at the points $(t,x) = (0,0)$ and $(0,1)$, respectively, the boundary observation $u(t,1)$ can be used to uniquely determine the final data $(u(t,x), v(t,x))$ at $t = T$. Moreover, there exists a positive constant C_2 such that the following weak observability inequality holds

$$||(u(T,x),v(T,x))||_{\mathcal{C}^1([0,1])} \leq C_2 \left(\sum_{k=1}^{n} ||u(\cdot,1)||_{\mathcal{C}^1([0,T_0])} + ||(U,V)||_{\mathcal{C}^1([0,T_0])}\right). \tag{2.27}$$

Similarly, for any initial condition (u_0, v_0) with small $\mathcal{C}^1([0,1])$ norm, satisfying the conditions of $\mathcal{C}^1$ compatibility at the points $(t,x) = (0,0)$ and $(0,1)$, respectively, the boundary observation $v(t,0)$ can be used to uniquely determine the final data $(u(t,x), v(t,x))$ at $t = T$. Moreover, there exists a positive constant C_2 such that the following weak observability inequality holds

$$||(u(T,x),v(T,x))||_{\mathcal{C}^1([0,1])} \leq C_2 \left(\sum_{k=1}^{m} ||v(\cdot,0)||_{\mathcal{C}^1([0,T_0])} + ||(U,V)||_{\mathcal{C}^1([0,T_0])}\right). \tag{2.28}$$

Based on Theorem 2.3.3 and Theorem 2.3.4, the objective of Part I is to derive exact observers in the framework presented in section 2.1. More precisely, we have the following objectives.

Objective A': Finite-time one-sided boundary observability. In the case where the measurements are only available at one of the two boundaries, we design an observer whose trajectories $(\hat{u}, \hat{v})$ converge in the sense of the L^2-norm to the solutions of system (2.1)-(2.3) (i.e $||(u-\hat{u}, v-\hat{v})||_{L^2([0,1])}$ converges to zero) in finite-time $\frac{1}{\lambda_1} + \frac{1}{\mu_1}$, for any initial condition $(u_0, v_0) \in L^2([0,1])^{n+m}$. This is the done in Chapter 3.

Objective B': Finite-time two-sided boundary stabilization. In the case where the measurements are available at both boundaries, we design an observer whose trajectories $(\hat{u}, \hat{v})$ converge in the sense of the L^2-norm to the solutions of system (2.1)-(2.3) (i.e $||(u-\hat{u}, v-\hat{v})||_{L^2([0,1])}$ converges to zero) in finite-time $\max(\frac{1}{\lambda_1}, \frac{1}{\mu_1})$, for any initial condition $(u_0, v_0) \in L^2([0,1])^{n+m}$. This is done in Chapter 4.

Again, one must be aware that the convergence time required in Objective B' (for which sensors are available at both side) is smaller than the one required in Objective A'. We prove that given $\mathcal{C}^1$ initial conditions for the original system and for the observer (along with compatible boundary conditions), the resulting observers ensure the weak exact boundary observability in the sense of Theorem 2.3.3 and Theorem 2.3.4. We also prove that combining these observers with the corresponding stabilizing boundary state-feedback control law yields finite-time stabilizing boundary output-feedback control laws.

Comparing the statements of observability Theorem 2.3.3 (resp. Theorem 2.3.4) with the controllability Theorem 2.3.1 (resp. Theorem 2.3.2), one can notice the following

- The controllability time is equal to the observability time.
- The number of actuators is equal to the number of measurements.
- The resolution method (see [Li10]) to obtain the controllability is first to solve a forward mixed problem and a backward mixed problem and then to solve a leftward mixed problem and a rightward mixed problem, while, to get the observability, one needs to first solve a leftward Cauchy problem and a rightward Cauchy problem and then solve a backward mixed problem.

This suggests a relation of duality between the exact controllability problem and the exact observability problem. If this duality relation between observability and controllability is well-known [CZ12, Li10, Rus78b], we prove in this thesis that it can be used to explicitly solve observer design problems. More precisely, the problem of observer design for system (2.1)-(2.3) can actually be rewritten as a controllability problem for the adjoint system. Consequently, if one can solve the problem of one-sided (resp. two-sided) boundary stabilization for system (2.1)-(2.3), it becomes possible to easily solve the problem of one-sided (resp. two-sided) boundary observability associated to the same system. This is the technique used in this thesis.

2.4 One-sided controllabilty and observability: tutorial case of two equations

The results of this section are taken from [CVKB13, VCKB11] we reformulate them here for tutorial purposes. We consider here the example of two coupled equations with a single input. The original system (2.1)-(2.3) rewrites

$$\partial_t u(t,x) + \lambda \partial_x u(t,x) = \sigma^{++}(x)u(t,x) + \sigma^{+-}(x)v(t,x), \tag{2.29}$$

$$\partial_t v(t,x) - \mu \partial_x v(t,x) = \sigma^{-+}(x)u(t,x) + \sigma^{--}(x)v(t,x), \tag{2.30}$$

with the linear boundary conditions

$$u(t,0) = qv(t,0), \quad v(t,1) = \rho u(t,1) + V(t), \tag{2.31}$$

where the velocities λ and μ are assumed to be strictly positive, the boundary couplings q and ρ are constant and are respectively called **distal reflection** (reflection at the unactuated boundary) and **proximal reflection** (reflection at the actuated boundary). The in-domain couplings belong to $\mathcal{C}^0([0,1],\mathbb{R})$. The states u and v have values in $\mathbb{R}$ and the corresponding initial condition is denoted $(u_0,v_0) \in (L^2([0,1],\mathbb{R}))^2$. We assume that only the right actuator is available (i.e $U(t) \equiv 0$). This system is schematically pictured in Figure 2.1. The objective is to design an explicit full-state feedback control law $V(t)$ that stabilizes system (2.29)-(2.31) (in the sense given in Objective A) in finite time $t_f = \frac{1}{\lambda} + \frac{1}{\mu}$ and to derive an observer based on boundary measurements that converges (in the sense given in Objective A') to the real state in the same time t_f. Using the characteristic method, the controllability problem has been solved in [Rus72] designing an integral feedback control law. Recently, an other approach has been proposed in [VCKB11, CVKB13] using the backstepping method [KS08b]. The purpose of this example is to introduce the backstepping method to make the reader familiar with it in so far as most of the proof derived in this thesis are based on this technique.

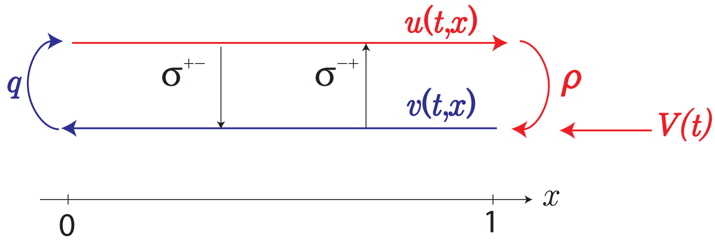

Figure 2.1: Schematic representation of the system (2.29)-(2.31). The black arrows represent coupling terms.

Remark 2.4.1 *The growth terms $\sigma^{++}(x)$ and $\sigma^{--}(x)$ in equations* (2.29)-(2.30) *can, without any loss of generality, be assumed equal to 0. More precisely, let us consider the invertible exponential transformation defined for all $t>0$ and all $x \in [0,1]$ by*

$$\bar{u}(t,x) = u(t,x)\mathrm{e}^{-\int_0^x \frac{\sigma^{++}(\nu)}{\lambda}d\nu}, \quad \bar{v}(t,x) = v(t,x)\mathrm{e}^{+\int_0^x \frac{\sigma^{--}(\nu)}{\mu}d\nu}, \tag{2.32}$$

which is well-defined since the functions $\sigma^{--}(\cdot)$ and $\sigma^{++}(\cdot)$ are integrable. This yields

$$\begin{aligned}\partial_t\bar{u}(t,x) + \lambda\partial_x\bar{u}(t,x) &= (\partial_t u(t,x) + \lambda\partial_x u(t,x) - \sigma^{++}(x)u(t,x))\mathrm{e}^{-\int_0^x \frac{\sigma^{++}(\nu)}{\lambda}d\nu} \\ &= \sigma^{+-}v(t,x)\mathrm{e}^{-\int_0^x \frac{\sigma^{++}(\nu)}{\lambda}d\nu} \\ &= \sigma^{+-}(x)\mathrm{e}^{-\int_0^x \frac{\sigma^{++}(\nu)}{\lambda}d\nu}\mathrm{e}^{-\int_0^x \frac{\sigma^{--}(\nu)}{\mu}d\nu}\bar{v}(t,x) = \bar{\sigma}^{+-}(x)\bar{v}(t,x),\end{aligned} \tag{2.33}$$

where we have denoted $\bar{\sigma}^{+-}(x) = \mathrm{e}^{-\int_0^x \frac{\sigma^{++}(\nu)}{\lambda}d\nu}\mathrm{e}^{-\int_0^x \frac{\sigma^{--}(\nu)}{\mu}d\nu}\sigma^{+-}(x)$. Similarly, denoting $\bar{\sigma}^{-+}(x) = \mathrm{e}^{+\int_0^x \frac{\sigma^{++}(\nu)}{\lambda}d\nu}\mathrm{e}^{+\int_0^x \frac{\sigma^{--}(\nu)}{\mu}d\nu}\sigma^{+-}(x)$, we obtain

$$\partial_t\bar{v}(t,x) - \mu\partial_x\bar{v}(t,x) = \bar{\sigma}^{-+}(x)\bar{u}(t,x). \tag{2.34}$$

The boundary conditions (2.31) *rewrites*

$$\bar{u}(t,0) = q\bar{v}(t,0), \quad \bar{v}(t,1) = \bar{\rho}u(t,1) + \bar{V}(t), \tag{2.35}$$

with $\bar{\rho} = \rho\mathrm{e}^{+\int_0^1 \frac{\sigma^{++}(\nu)}{\lambda}d\nu + \int_0^1 \frac{\sigma^{--}(\nu)}{\mu}d\nu}$ and $\bar{V}(t) = \mathrm{e}^{+\int_0^1 \frac{\sigma^{--}(\nu)}{\mu}d\nu}V(t)$. Consequently, adjusting the corresponding initial conditions, this proves that system (2.29)-(2.30) *and system* (2.33)-(2.35) *are equivalent.* ***In what follows, we always assume that the coupling terms*** $\sigma^{++}(x)$ ***and*** $\sigma^{--}(x)$ ***are equal to zero for the scalar equation***(2.29)-(2.31). *Similar techniques can be used to prove that in the case of the general system* (2.1)-(2.3) *the diagonal terms of the matrices Σ^{++} and Σ^{--} can be considered as equal to zero.*

2.4.1 Backstepping transformation and control design

Let us assume $q \neq 0$ [2]. The design introduced [VCKB11, CVKB13] to stabilize in finite time the system (2.29)-(2.31) is based on the backstepping approach. This method consists in performing an integral change of variables (usually using a Volterra or a Fredholm transformation) that maps the original system to a so-called "target system" for which the control design is easier. Provided that the transformation is invertible (which is always the case for a Volterra

[2] This assumption leads to a simple target system. However, the case $q = 0$ can be tackled in a similar way and is discussed bellow.

transformation [KS08b]), the original system and the corresponding target system have equivalent stability properties [KS08b]. The existence of such a transformation is strongly interwoven with the choice of the target system. More precisely, if the chosen target system has a structure much different from the one of the original system, it imposes conditions on the transformation that cannot be fulfilled. Consequently, while using the backstepping approach, one must at the same time find a suitable target system (i.e a system for which the control design is easier) and prove the existence of the integral transformation that maps the original system to this target system. In early works [KS08b, VCKB11, CVKB13], the target system has been chosen to be an autonomous, stable system. Its structure imposed specific conditions on the transformation and the feedback control law. Here, we propose a slightly different view of the approach. We fix the transformation a priori and several features of the target system (in particular, the structure of the in-domain couplings), but keep the degree of freedom of the control law. In other word, we look for a target system that is “more intuitive" to control, rather than stable. In the case of system (2.29)-(2.31), the difficulties for stabilizing are mostly due to the in-domain couplings $\sigma^{-+}(x)$ and $\sigma^{+-}(x)$ (see [BC11, BC16]). Thus, we choose to remove them in the target system:

$$\partial_t\alpha(t,x) + \lambda\partial_x\alpha(t,x) = 0, \quad \partial_t\beta(t,x) - \mu\partial_x\beta(t,x) = 0. \tag{2.36}$$

As explained above, the boundary conditions associated to (2.36) are not imposed for the moment. To map the original system to this target system, we consider the following Volterra change of coordinates

$$\alpha(t,x) = u(t,x) - \int_0^x K^{uu}(x,\xi)u(t,\xi) + K^{uv}(x,\xi)v(t,\xi)d\xi \tag{2.37}$$

$$\beta(t,x) = v(t,x) - \int_0^x K^{vu}(x,\xi)u(t,\xi) + K^{vv}(x,\xi)v(t,\xi)d\xi. \tag{2.38}$$

The functions K^{uu}, K^{uv}, K^{vu} and K^{vv} are referred as the kernels of the transformation and are all defined on the triangular domain $\mathcal{T} = \{(x,\xi) \in [0,1]^2,\ \xi \leq x\}$. The transformation (2.37)-(2.38) imposes the following boundary conditions

$$\alpha(t,0) = q\beta(t,0), \tag{2.39}$$

$$\beta(t,1) = \rho u(t,1) - \int_0^1 K^{vu}(1,\xi)u(t,\xi) + K^{vv}(1,\xi)v(t,\xi)d\xi + V(t). \tag{2.40}$$

Note that the second boundary condition depends on the original set of variables for the moment.

Well-posedness of the transformation (2.37)-(2.38)

The first objective consists in proving the existence of the transformation (2.37)-(2.38), i.e the existence of kernels K^{uu}, K^{uv}, K^{vu} and K^{vv} such that equations (2.37)-(2.38) are satisfied. Differentiating (2.37) with respect to time and integrating by parts, recalling that $\sigma^{++}(x)$ and $\sigma^{--}(x)$ are assumed to be equal to zero, one gets

$$\begin{aligned}
\partial_t\alpha(t,x) =& \partial_t u(t,x) - \int_0^x K^{uu}(x,\xi)\partial_t u(t,\xi) + K^{uv}(x,\xi)\partial_t v(t,\xi)d\xi \\
=& -\lambda\partial_x u(t,x) + \sigma^{+-}(x)v(t,x) - \int_0^x K^{uu}(x,\xi)\sigma^{+-}(\xi)v(t,x) - \lambda K^{uu}(x,\xi)\partial_x u(t,\xi) \\
&+ \mu K^{uv}(x,\xi)\partial_x v(t,\xi) + K^{uv}(x,\xi)\sigma^{-+}(\xi)u(t,\xi)d\xi \\
=& -\lambda\partial_x u(t,x) + \sigma^{+-}(x)v(t,x) + \lambda K^{uu}(x,x)u(t,x) - \lambda K^{uu}(x,0)u(t,0) \\
&- \mu K^{uv}(x,x)v(t,x) + \mu K^{uv}(x,0)v(t,0) - \int_0^x (K^{uu}(x,\xi)\sigma^{+-}(\xi) \\
&- \mu\partial_\xi K^{uv}(x,\xi))v(t,\xi) + (\lambda\partial_\xi K^{uu}(x,\xi) + K^{uv}(x,\xi)\sigma^{-+}(\xi))u(t,\xi)d\xi.
\end{aligned} \tag{2.41}$$

Similarly, differentiating (2.37) with respect to space, one obtains

$$\partial_x\alpha(t,x) = \partial_x u(t,x) - K^{uu}(x,x)u(t,x) - K^{uv}(x,x)v(t,x) - \int_0^x \partial_x K^{uu}(x,\xi)u(t,\xi) + \partial_x K^{uv}(x,\xi)v(t,\xi)d\xi. \tag{2.42}$$

This yields

$$\partial_t\alpha(t,x) + \lambda\partial_x\alpha(t,x) = (\sigma^{+-}(x) - \lambda K^{uv}(x,x) - \mu K^{uv}(x,x))v(t,x) + (\mu K^{uv}(x,0) - q\lambda K^{uu}(x,0))v(t,0) - \int_0^1 (\lambda\partial_x K^{uu}(x,\xi) + \lambda\partial_\xi K^{uv}(x,\xi) + \sigma^{-+}K^{uv}(x,\xi))u(t,\xi)d\xi - \int_0^1 (\lambda\partial_x K^{uv}(x,\xi) - \mu\partial_\xi K^{uv}(x,\xi) + \sigma^{+-}K^{uu}(x,\xi))v(t,\xi)d\xi.$$

Thus, to prove the existence of the first part of the transformation, it is sufficient to prove the existence of K^{uu} and K^{uv} that satisfy

$$\lambda\partial_x K^{uu}(x,\xi) + \lambda\partial_\xi K^{uu}(x,\xi) = -\sigma^{-+}(\xi)K^{uv}(x,\xi), \tag{2.43}$$
$$\lambda\partial_x K^{uv}(x,\xi) - \mu\partial_\xi K^{uv}(x,\xi) = -\sigma^{+-}(\xi)K^{uu}(x,\xi), \tag{2.44}$$

along with the boundary conditions

$$K^{uv}(x,x) = \frac{\sigma^{+-}(x)}{\lambda+\mu}, \quad K^{uu}(x,0) = \frac{\mu}{q\lambda}K^{uv}(x,0), \tag{2.45}$$

since $q \neq 0$. In a similar way, differentiating (2.38) with respect to space and time, to prove the existence of the second part of the transformation, it is sufficient to prove the existence of K^{vu} and K^{vv} that satisfy

$$\mu\partial_x K^{vu}(x,\xi) - \lambda\partial_\xi K^{vu}(x,\xi) = \sigma^{-+}(\xi)K^{vv}(x,\xi), \tag{2.46}$$
$$\mu\partial_x K^{vv}(x,\xi) + \mu\partial_\xi K^{vv}(x,\xi) = \sigma^{+-}(\xi)K^{vu}(x,\xi), \tag{2.47}$$

along with the boundary conditions

$$K^{vu}(x,x) = -\frac{\sigma^{-+}(x)}{\lambda+\mu}, \quad K^{vv}(x,0) = \frac{q\lambda}{\mu}K^{uv}(x,0). \tag{2.48}$$

We have the following theorem [CVKB13]

Theorem 2.4.1. [CVKB13, Theorem A.1]

The system (2.43)-(2.48) admits a unique continuous solution on $\mathcal{T}$.

Proof : The proof is quite classical (see [HDMVK16], [Joh60] and [Whi11]) and consists in writing the integral equations associated to equations (2.43)-(2.48) using the method of characteristics. These integral equations are then solved using the method of successive approximations. ∎

Thus, this theorem proves the existence of a unique solution to system (2.43)-(2.48) and consequently, the existence of the transformation (2.37)-(2.38). Note that for the case of constant coefficients, an explicit solution to equations (2.43)-(2.48) using Bessel and Marcum Q-functions can be found in [VK14].

Invertibility of the transformation (2.37)-(2.38)

To ensure that the original system and the target system have equivalent stability properties, the transformation (2.37)-(2.38) has to be invertible. Although this is granted by the fact that the transformation (2.37)-(2.38) is a Volterra transformation [KS08b], the explicit expression of the inverse transformation is sometimes required to obtain some of the results presented in this thesis. This inverse transformation takes the following form

$$u(t,x) = \alpha(t,x) + \int_0^x (L^{\alpha\alpha}(x,\xi)\alpha(t,\xi) + L^{\alpha\beta}(x,\xi)\beta(t,\xi))d\xi, \tag{2.49}$$

$$v(t,x) = \beta(t,x) + \int_0^x (L^{\beta\alpha}(x,\xi)\alpha(t,\xi) + L^{\beta\beta}(x,\xi)\beta(t,\xi))d\xi, \tag{2.50}$$

where the kernels $L^{\alpha\alpha}$, $L^{\alpha\beta}$, $L^{\beta\alpha}$ and $L^{\beta\beta}$ belong to $L^\infty(\mathcal{T})$ and are defined by the set of PDEs

$$\lambda\partial_x L^{\alpha\alpha}(x,\xi) + \lambda\partial_\xi L^{\alpha\alpha}(x,\xi) = \sigma^{+-}(\xi)L^{\beta\alpha}(x,\xi), \tag{2.51}$$

$$\lambda\partial_x L^{\alpha\beta}(x,\xi) - \mu\partial_\xi L^{\alpha\beta}(x,\xi) = \sigma^{+-}(\xi)L^{\beta\beta}(x,\xi), \tag{2.52}$$

$$\mu\partial_x L^{\beta\alpha}(x,\xi) - \lambda\partial_\xi L^{\beta\alpha}(x,\xi) = -\sigma^{-+}(\xi)L^{\alpha\alpha}(x,\xi), \tag{2.53}$$

$$\mu\partial_x L^{\beta\beta}(x,\xi) + \mu\partial_\xi L^{\beta\beta}(x,\xi) = -\sigma^{-+}(\xi)L^{\alpha\beta}(x,\xi), \tag{2.54}$$

with the boundary conditions

$$L^{\alpha\alpha}(x,0) = \frac{\mu}{q\lambda}L^{\alpha\beta}(x,0), \quad L^{\beta\beta}(x,0) = \frac{q\lambda}{\mu}L^{\beta\alpha}(x,0) \tag{2.55}$$

$$L^{\alpha\beta}(x,x) = \frac{\sigma^{+-}(x)}{\lambda+\mu}, \quad L^{\beta\alpha}(x,x) = -\frac{\sigma^{-+}(x)}{\lambda+\mu}. \tag{2.56}$$

Again by [CVKB13, Theorem A.1], one finds that there is a unique solution to equations (2.51)-(2.56), which is in $C(\mathcal{T})$.

Remark 2.4.2 *Using this inverse transformation, the right boundary* (2.38) *condition can be rewritten as*

$$\begin{aligned}\beta(t,1) = \rho\alpha(t,1) &+ \int_0^1 (\rho L^{\alpha\alpha}(1,\xi) - L^{\beta\alpha}(1,\xi))\alpha(t,\xi)d\xi \\ &+ \int_0^1 (\rho L^{\alpha\beta}(1,\xi) - L^{\beta\beta}(1,\xi))\beta(t,\xi)d\xi + V(t).\end{aligned} \tag{2.57}$$

Denoting

$$N^\alpha(\xi) = L^{\beta\alpha}(1,\xi) - \rho L^{\alpha\alpha}(1,\xi), \quad N^\beta(\xi) = L^{\beta\beta}(1,\xi) - \rho L^{\alpha\beta}(1,\xi), \tag{2.58}$$

we obtain

$$\beta(t,1) = \rho\alpha(t,1) - \int_0^1 N^\alpha(\xi)\alpha(t,\xi) + N^\beta(\xi)\beta(t,\xi)d\xi + V(t). \tag{2.59}$$

Control law and stabilization

We have now proved the existence of an invertible transformation that maps the original system (2.29)-(2.31) to the **decoupled** target system (2.36)-(2.40). Thus, these two systems have equivalent stability properties. In particular, if $||(\alpha,\beta)||_{L^2}$ converges to zero in finite time, then the same property holds for $||(u,v)||_{L^2}$. To obtain finite-time convergence for the target system (2.36)-(2.40), one can use the control law $V(t)$ to set to zero the right boundary condition of (2.40), i.e. choose $V(t)$ as

$$V(t) = -\rho u(t,1) + \int_0^1 K^{vu}(1,\xi)u(t,\xi) + K^{vv}(1,\xi)v(t,\xi)d\xi. \tag{2.60}$$

The target system (2.36)-(2.40) becomes a cascade of two transport equations with a zero-boundary condition. This system, pictured in Figure 2.2, converges, for any initial condition, to its equilibrium in finite time $t_F = \frac{1}{\lambda} + \frac{1}{\mu}$ (see [CVKB13]). Thus, the same property holds for the original system (2.29)-(2.31). This solves the problem of the problem of one-sided boundary stabilization given by Objective A.

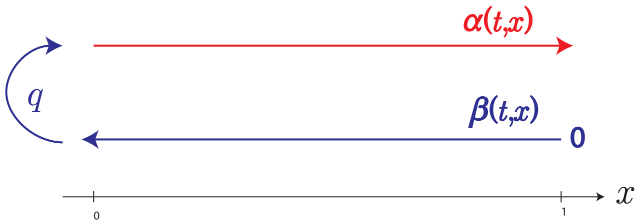

Figure 2.2: Schematic representation of the target system

The case $q = 0$

If the coefficient q is zero, the method presented above is not valid since the boundary condition (2.45) cannot be properly defined.However, the proposed method can be adjusted, slightly modifying the target system by

$$\partial_t \alpha(t,x) + \lambda \partial_x \alpha(t,x) = g(x)\beta(t,0), \quad \partial_t \beta(t,x) - \mu \partial_x \beta(t,x). \tag{2.61}$$

and choosing the control law $V(t)$ to still get the zero boundary condition.

$$\alpha(t,0) = q\beta(t,0), \quad \beta(t,1) = 0. \tag{2.62}$$

Again, the system (2.61)-(2.62) is a cascade system which is L^2 exponentially stable and converges in finite time to zero. The kernel equations resulting from the backstepping transformation are identical to (2.43)-(2.48) except that the boundary condition for K^{uu} has been changed by $K^{uu}(x,0) = h(x)$ where h can be chosen as desired. Note that this has no impact on the resulting feedback law since it does not change the expression of K^{vu} and K^{vv} used in (2.60).

2.4.2 Boundary observability

The control law (2.60) is a full-state feedback as it requires the value of the state across the domain. Full-state distributed measurements are almost never available in practice. Next, we assume that we can measure $u(t,x)$ at the boundary $x = 1$ (collocated measurements). The goal is to design an observer to estimate both infinite-dimensional states and fulfill Objective B', i.e we want the estimations to be equal (in the sense of the L^2-norm) to the real states in finite time $t_f = \frac{1}{\lambda} + \frac{1}{\mu}$. In [VCKB11], the dynamics of the estimates $(\hat{u}, \hat{v})$ are a copy of the original system with output injection terms. They rewrite as follows

$$\partial_t \hat{u}(t,x) + \lambda \partial_x \hat{u}(t,x) = \sigma^{+-}(x)\hat{v}(t,x) + P_1(x)(u(t,1) - \hat{u}(t,1)), \tag{2.63}$$
$$\partial_t \hat{v}(t,x) - \mu \partial_x \hat{v}(t,x) = \sigma^{-+}(x)\hat{u}(t,x) + P_2(x)(u(t,1) - \hat{u}(t,1)), \tag{2.64}$$

along with the boundary conditions

$$\hat{u}(t,0) = q\hat{v}(t,0), \quad \hat{v}(t,1) = \rho u(t,1) + V(t). \tag{2.65}$$

The functions $P_1(x)$ and $P_2(x)$ correspond to output error injection gains (observer gains) and have to be designed. Note the presence of the measurements in the right boundary condition (2.65) of the observer such that this boundary condition is identical to (2.31). Denoting the error estimates by $\tilde{u}(t,x) = u(t,x) - \hat{u}(t,x)$ and $\tilde{v}(t,x) = v(t,x) - \hat{v}(t,x)$, we get the following error system

$$\partial_t \tilde{u}(t,x) + \lambda \partial_x \tilde{u}(t,x) = \sigma^{+-}(x)\tilde{v}(t,x) + P_1(x)\tilde{u}(t,1), \tag{2.66}$$
$$\partial_t \tilde{v}(t,x) - \mu \partial_x \tilde{v}(t,x) = \sigma^{-+}(x)\tilde{u}(t,x) + P_2(x)\tilde{u}(t,1), \tag{2.67}$$

along with the boundary conditions

$$\tilde{u}(t,0) = q\tilde{v}(t,0), \quad \tilde{v}(t,1) = 0. \tag{2.68}$$

Similarly to the stabilization case, the objective is to find a backstepping transformation that removes the in-domain couplings. This transformation will impose the values of the observer gains P_1 and P_2. More precisely, consider the invertible backstepping transformation

$$\tilde{u}(t,x) = \tilde{\alpha}(t,x) - \int_x^1 P^{uu}(x,\xi)\tilde{\alpha}(t,\xi) - P^{uv}(x,\xi)\tilde{\beta}(t,\xi)d\xi, \tag{2.69}$$
$$\tilde{v}(t,x) = \tilde{\beta}(t,x) - \int_x^1 P^{vu}(x,\xi)\tilde{\alpha}(t,\xi) - P^{vv}(x,\xi)\tilde{\beta}(t,\xi)d\xi, \tag{2.70}$$

where the kernels P^{uu}, P^{uv}, P^{vu} and P^{vv} are continuous functions defined on $\mathcal{T}_1 = \{(x,\xi) \in [0,1]^2,\ \xi \geq x\}$ by the following set of PDEs

$$\lambda \partial_x P^{uu}(x,\xi) + \lambda \partial_\xi P^{uu}(x,\xi) = -\sigma^{+-}(x)P^{vu}(x,\xi), \tag{2.71}$$
$$\lambda \partial_x P^{uv}(x,\xi) - \mu \partial_\xi P^{uv}(x,\xi) = -\sigma^{+-}(x)P^{vv}(x,\xi), \tag{2.72}$$
$$\mu \partial_x P^{vu}(x,\xi) - \lambda \partial_\xi P^{vu}(x,\xi) = +\sigma^{-+}(x)P^{uu}(x,\xi), \tag{2.73}$$
$$\mu \partial_x P^{vv}(x,\xi) + \mu \partial_\xi P^{vv}(x,\xi) = +\sigma^{+-}(x)P^{uv}(x,\xi), \tag{2.74}$$

along with the boundary conditions

$$P^{uu}(0,\xi) = qP^{vu}(0,\xi),\ P^{uv}(x,x) = \frac{\sigma^{+-}(x)}{\lambda+\mu}, \tag{2.75}$$
$$P^{vv}(0,\xi) = \frac{1}{q}P^{uv}(0,\xi),\ P^{vu}(x,x) = -\frac{\sigma^{-+}(x)}{\lambda+\mu}. \tag{2.76}$$

We have implicitly assumed that $q \neq 0$ (the particular case $q = 0$ can be solved as above). The system (2.71)-(2.76) has a unique solution which is continuous on $\mathcal{T}_1$ [VCKB11, Theorem 4]. We now impose the following conditions on the output injection kernels:

$$P_1(x) = -\lambda P^{uu}(x,1), \quad P_2(x) = -\lambda P^{vu}(x,1). \tag{2.77}$$

It is proved in [VCKB11] that it maps the error system (2.66)-(2.68) to the following target system

$$\partial_t \tilde{\alpha}(t,x) + \lambda \partial_x \tilde{\alpha}(t,x) = 0, \quad \partial_t \tilde{\beta}(t,x) - \mu \partial_x \tilde{\beta}(t,x) = 0, \tag{2.78}$$

along with the boundary conditions

$$\tilde{\alpha}(t,0) = q\tilde{\beta}(t,0), \quad \tilde{\beta}(t,1) = 0. \tag{2.79}$$

This system consists of a cascade of transport equations with a zero-boundary conditions, the associated L^2 converges to zero in finite time $t_F = \frac{1}{\lambda} + \frac{1}{\mu}$. Since the transformation (2.69)-(2.70) is invertible, the original system (2.66)-(2.68) has the same properties. Thus $||(u-\hat{u}, v-\hat{v})||_{L^2([0,1])}$ converges to zero in finite time t_F and Objective B' is fulfilled.

Collocated output feedback control

Combining the full state feedback law and the observer estimates, we choose the output-feedback law

$$V(t) = -\rho\hat{u}(t,1) + \int_0^1 K^{vu}\hat{u}(t,\xi) + K^{vv}\hat{v}(t,\xi)d\xi, \tag{2.80}$$

where $\hat{u}$ and $\hat{v}$ are the solutions of (2.63)-(2.65) and where the kernels K and P are respectively the solutions of (2.43)-(2.48) and (2.71)-(2.76). We have the following theorem

Theorem 2.4.2. [VCKB11, Theorem 3]

Consider system (2.29)-(2.31), initial condition $u_0 \in L^2([0,1],\mathbb{R})$ and $v_0 \in L^2([0,1],\mathbb{R})$, and control law (2.80) and (2.63)-(2.65). The equilibrium $u \equiv v \equiv 0$ is exponentially stable in the L^2 sense. Moreover, the equilibrium is reached in finite time $t = 2t_F = 2(\frac{1}{\lambda} + \frac{1}{\mu})$.

It is important to remark that the time $2t_F$ in this theorem is due to the fact that one must have the convergence of the observer before being able to start stabilizing (as there is a cascade from the error system $(\tilde{u},\tilde{v})$ to the system (u,v)). The proof of this theorem is straightforward and can be found in [VCKB11]

2.5 Summary and organization of Part I

The present part of this thesis (in Chapters 3-4-5) contributes to some advances in system theory by solving the finite-time boundary stabilization problems A and A' and the finite-time boundary observability problems B and B' for general systems of the form (2.1)-(2.3). Considering $\mathcal{C}^1$ initial conditions (along with compatible boundary conditions) these problems are equivalent to the ones stated by Tatsien Li in [LR10]. The one-sided boundary problems A and B have received a lot of attention. The case of two equations presented above acts as precursor as it introduces the backstepping approach for this class of PDEs. The presented method has been successfully adjusted for the case of three equations ($n = 2$, $m = 1$) [DMVKP12] and systems featuring n rightward convecting transport PDEs and one leftward convecting transport [DMVK13]. The extension to the general class of systems described by (2.1)-(2.3) has been done in [HDMVK16] but the convergence time there is larger than the theoretical one given in the definition of Objectives A and A'. The two-sided problems have not received a lot of attention as far as we know, although they constitute an interesting problem for performance improvements. The two-sided control problem has been solved in [VK16] for reaction-diffusion PDEs and 2-states heterodirectional linear PDEs with *equal* transport velocities. The next Chapters are organized as follows.

Chapter 3 : One-sided boundary stabilization and observability. This chapter solves Objectives A and A'. The controller is designed through a backstepping approach adjusted from the one presented in [HDMVK16]. To obtain the associated observer, we develop a new technique based on the adjoint system.

Chapter 4: Two-sided boundary stabilization and observability. This chapter solves Objectives B and B'. The techniques introduced in the previous chapter cannot be simply adjusted and a new Fredholm-backstepping transformation has to be introduced to take full advantage of the multiple actuators and sensors.

Chapter 5: An explicit mapping from LFOH PDEs to neutral systems. We introduce a new tool for the stability analysis of system (2.1)-(2.3) by showing their equivalence, in a certain sense, to neutral systems with distributed delays. This opens the perspective of adapting stability analysis methods for time delay systems, such as those developed in [DDLM15, HL02, Nic01a, Nic01b].

Chapter 3

One-sided boundary stabilization and observability

Chapitre 3 Stabilisation et observabilité unilatérale. *Dans ce chapitre nous résolvons les problèmes de stabilisation et d'observabilité unilatérale pour la classe générale de systèmes d'EDPs* (2.1)-(2.3)*, tels que formulés par les Objectifs A et A'. À l'aide d'une transformation de Volterra, le système initial est transformé en un système cible ayant des propriétés de stabilité idoines. Ce système cible possède une structure similaire à celle du système initial à l'exception des termes sources qui présentent désormais une structure en cascade. Cette cascade permet d'assurer la convergence exponentielle du système cible vers son équilibre. Les noyaux de la transformation intégrale satisfont un système d'équations ayant également une structure cascade, proche de celle du système cible. Du fait de cette structure il est possible d'adapter la preuve de [HDMVK16] pour prouver de manière récursive l'existence de tels noyaux. À partir de cette transformation, il est aisé d'obtenir une loi de commande par retour d'état assurant la convergence exponentielle du système vers son équilibre au sens de la norme L^2. Le problème d'observabilité correspondant (exprimé par l'Objectif A') est résolu en se servant de l'observateur proposé en [HDMVK16]. Nous prouvons que le système d'erreur obtenu est l'adjoint du système que nous souhaitions initialement commander. Cette approche ouvre de nouvelles perspectives en termes de synthèse d'observateurs.*

Contents

In this chapter, we solve the problem of one-sided boundary stabilization and observability for the general class of hyperbolic PDEs (2.1)-(2.3) as stated by Objective A and Objective A'. Using a backstepping approach (with a Volterra transformation) the system is mapped to a *target* system with desirable stability properties. This target system is a copy of the original dynamics with a modified in-domain couplings structure. More precisely, the target system is designed as an exponentially stable cascade removing some of the in-domain coupling terms. The transformation kernels satisfy a system of equations with a cascade structure akin to that of the target system. This structure enables a recursive proof of existence of the transformation kernels using tools similar to [HDMVK16]. A full-state feedback law guaranteeing exponential stability of the zero equilibrium in the L^2-norm is then designed. To solve the corresponding observability problem (stated in Objective A'), we adjust the structure of the observer proposed in [HDMVK16] and prove that the resulting system is the adjoint of the previously considered controlled system. This opens new prospects for the design of boundary observers. The content of this chapter has been published in [ADM16a]. Let us recall the equations of system (2.1)-(2.3):

$$\partial_t u(t,x) + \Lambda^+ \partial_x u(t,x) = \Sigma^{++}(x)u(t,x) + \Sigma^{+-}(x)v(t,x), \tag{3.1}$$
$$\partial_t v(t,x) - \Lambda^- \partial_x v(t,x) = \Sigma^{-+}(x)u(t,x) + \Sigma^{--}(x)v(t,x), \tag{3.2}$$

with the following linear boundary conditions

$$u(t,0) = Q_0 v(t,0), \quad v(t,1) = R_1 u(t,1) + V(t), \tag{3.3}$$

The control law $U(t)$ has been set zero (as we consider one-sided problem). We recall that the matrices Λ^+ and Λ^- are diagonal matrices defined for all $1 \leq i \leq n$, for all $1 \leq j \leq m$ by $\Lambda^+_{ii} = \lambda_i$ and $\Lambda^-_{jj} = \mu_j$ whose eigenvalues satisfy

$$-\mu_m < \cdots < -\mu_1 < 0 < \lambda_1 < \cdots < \lambda_n.$$

Remark 3.0.1 *The case $V(t) \equiv 0$ can be treated in a similar way using the change of variable $\bar{x} = 1 - x$.*

In what follows we denote t_F the minimum time (defined in Objectives A and A') in which we want to stabilize the system (3.1)-(3.3):

$$t_F = \frac{1}{\mu_1} + \frac{1}{\lambda_1}. \tag{3.4}$$

3.1 One-sided finite time stabilization

3.1.1 Target system and Volterra transformation

It has been proved in [HDMVK16] that the original system (3.1)-(3.3) can be mapped to a target system that converges to its zero equilibrium in finite-time. However, this time was larger that the theoretical minimum time t_F due to the presence of non-local coupling terms. To reach the minimum convergence time we have chosen to target local coupling terms instead of non-local ones. This leads to the following target-system candidate:

$$\begin{aligned}\partial_t \alpha(t,x) + \Lambda^+ \partial_x \alpha(t,x) =& \Sigma^{++}(x)\alpha(t,x) + \Sigma^{+-}(x)\beta(t,x) \\ &+ \int_0^x C^+(x,\xi)\alpha(t,\xi)d\xi + \int_0^x C^-(x,\xi)\beta(t,\xi)d\xi \end{aligned} \tag{3.5}$$
$$\partial_t \beta(t,x) - \Lambda^- \partial_x \beta(t,x) = \Omega(x)\beta(t,x), \tag{3.6}$$

with the boundary conditions

$$\alpha(t,0) = Q_0 \beta(t,0) \quad \beta(t,1) = 0, \tag{3.7}$$

where C^+ and C^- are $\mathcal{C}^0$ matrix functions on the domain

$$\mathcal{T} = \{0 \leq \xi \leq x \leq 1\}, \tag{3.8}$$

while $\Omega \in L^\infty(0,1)$ is an upper triangular matrix with the following structure

$$\Omega(x) = \begin{pmatrix} \omega_{1,1}(x) & \omega_{1,2}(x) & \dots & \omega_{1,m}(x) \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \omega_{m-1,m-1}(x) & \omega_{m-1,m}(x) \\ 0 & \dots & 0 & \omega_{m,m}(x) \end{pmatrix}. \tag{3.9}$$

The integral couplings appearing in (3.5) do not have any impact on the stability of the target system: since all the velocities are strictly positive, the integral terms are feedforward terms. A schematic representation of this target system is given in Figure 3.1 for $n = 1$ and $m = 2$.

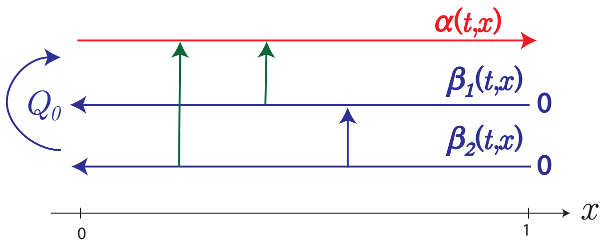

Figure 3.1: Schematic representation of the target system (3.5)-(3.7)

The following lemma assesses the finite-time stability of the target system.

Lemma 3.1.1.

For any initial condition $(\alpha(0,\cdot), \beta(0,\cdot)) \in (L^2([0,1]))^{n+m}$, the system (3.5), (3.6) reaches its zero equilibrium in finite-time t_F.

Proof : The proof of this lemma is straightforward using the proof of [HDMVK16, Lemma 3.1].The system is a cascade of β-system (that has zero input at the right boundary) into the α-system (that has zero input at the left boundary once β becomes equal to zero). ■

Remark 3.1.1 *The zero equilibrium of* (3.5)-(3.6) *with boundary conditions* (3.7) *and initial conditions* $(\alpha(0,\cdot), \beta(0,\cdot)) \in (L^2([0,1]))^{n+m}$ *is exponentially stable in the* L^2 *sense. This can be proved using the fact that for initial condition in* L^2*, the solution stays in* L^2*, and becomes identically zero in finite time. Moreover, if the initial condition* $(\alpha(0,\cdot), \beta(0,\cdot)) \in \mathcal{C}^1([0,1])$ *and satisfies the corresponding compatibility conditions, the solution stays in* $\mathcal{C}^1$*, and becomes identically zero in finite time. An alternative proof based on a Lyapunov approach can be adjusted from [HDMVK16].*

Volterra transformation

To map the original system (3.1)-(3.3) to the target system (3.5)-(3.7), we use the following Volterra transformation

$$\alpha(t,x) = u(t,x) \tag{3.10}$$

$$\beta(t,x) = v(t,x) - \int_0^x (K(x,\xi)u(\xi) + L(x,\xi)v(\xi))d\xi, \tag{3.11}$$

where the kernels matrices K and L, defined on $\mathcal{T} = \{(x,\xi) \in [0,1]^2 | \quad \xi \leq x\}$ have yet to be defined. Differentiating (3.11) with respect to space and using the Leibniz rule yields

$$\partial_x \beta(t,x) = \partial_x v(t,x) - K(x,x)u(t,x) - L(x,x)v(t,x) - \int_0^x \partial_x K(x,\xi)u(t,\xi) + \partial_x L(x,\xi)v(t,\xi)d\xi.$$

Differentiating (3.11) with respect to time, using (3.1), (3.2) and integrating by parts, we obtain

$$\partial_t \bar{\beta}(t,x) = \Lambda^- \partial_x v(t,x) + \Sigma^{-+}(x)u(t,x) + \Sigma^{--}(x)v(t,x) - \int_0^x \Bigg[K(x,\xi)\Sigma^{++}u(t,\xi) + K(x,\xi)\Sigma^{+-}$$

$$v(t,\xi) + L(x,\xi)\Sigma^{-+}(\xi)u(t,\xi) + L(x,\xi)\Sigma^{--}v(t,\xi)\Bigg]d\xi + K(x,x)\Lambda^+ u(t,x) - K(x,0)\Lambda^+ u(t,0)$$

$$- L(x,x)\Lambda^- v(t,x) + L(x,0)\Lambda^- v(t,0) - \int_0^x [\partial_\xi K(x,\xi)\Lambda^+ u(t,\xi) - \partial_\xi L(x,\xi)\Lambda^- v(t,\xi)]d\xi.$$

Plugging these expressions into the target system (3.5)-(3.7), taking $x = 0$ in (3.11) and using the corresponding boundary conditions (3.3), we get the following system of kernel equations

$$0 = \Lambda^- \partial_x K(x,\xi) - \partial_\xi K(x,\xi)\Lambda^+ - K(x,\xi)\Sigma^{++}(\xi) - L(x,\xi)\Sigma^{-+}(\xi) + \Omega(x)K(x,\xi), \tag{3.12}$$
$$0 = \Lambda^- \partial_x L(x,\xi) + \partial_\xi L(x,\xi)\Lambda^- - L(x,\xi)\Sigma^{--}(\xi) - K(x,\xi)\Sigma^{+-}(\xi) + \Omega(x)L(x,\xi), \tag{3.13}$$
$$0 = \Sigma^{-+} + K(x,x)\Lambda^+ + \Lambda^- K(x,x), \quad 0 = \Sigma^{--} + \Lambda^- L(x,x) - L(x,x)\Lambda^- - \Omega(x), \tag{3.14}$$
$$0 = K(x,0)\Lambda^+ Q_0 - L(x,0)\Lambda^-. \tag{3.15}$$

Moreover, the functions $C^-(x,\xi)$ and $C^+(x,\xi)$ satisfy the following equations

$$C^-(x,\xi) = \Sigma^{+-}L(x,\xi) + \int_\xi^x C^-(x,s)L(s,\xi)ds, \tag{3.16}$$
$$C^+(x,\xi) = \Sigma^{+-}K(x,\xi) + \int_\xi^x C^-(x,s)K(s,\xi)ds. \tag{3.17}$$

Remark 3.1.2 *Similarly to [HDMVK16, Remark 3], one can notice that for each $x \in [0,1]$, equation (3.16) is a Volterra equation on $[0,x]$ where $C^-(x,\cdot)$ is the unknown. Thus it is well-defined [Yos60]. Assuming that K and L are well defined and bounded, so is C^-. Using (3.17), it becomes possible to explicitly express C^+ as a function of C^- and K.*

Developing equations (3.12)-(3.15) we get (for $1 \leq i \leq m$, $1 \leq j \leq n$) the following set of kernel PDEs:

$$\begin{aligned}\mu_i \partial_x K_{ij}(x,\xi) - \lambda_j \partial_\xi K_{ij}(x,\xi) =& \sum_{k=1}^n \sigma_{kj}^{++}(\xi)K_{ik}(x,\xi) + \sum_{p=1}^m \sigma_{pj}^{-+}(\xi)L_{ip}(x,\xi) \\ &- \sum_{i\leq p\leq m} K_{pj}(x,\xi)\omega_{ip}(x),\end{aligned} \tag{3.18}$$

$$\begin{aligned}\mu_i \partial_x L_{ij}(x,\xi) + \mu_j \partial_\xi L_{ij}(x,\xi) =& \sum_{k=1}^m \sigma_{kj}^{--}(\xi)L_{ik}(x,\xi) + \sum_{p=1}^n \sigma_{pj}^{+-}(\xi)K_{ip}(x,\xi) \\ &- \sum_{i\leq p\leq m} L_{pj}(x,\xi)\omega_{ip}(x),\end{aligned} \tag{3.19}$$

along with the set of boundary conditions

$$\forall 1 \leq i \leq m, \forall j \leq n,\ K_{ij}(x,x) = -\frac{\sigma_{ij}^{-+}(x)}{\mu_i + \lambda_j} = k_{ij}(x), \tag{3.20}$$

$$\forall 1 \leq i, j \leq m, j < i\ L_{ij}(x,x) = \frac{-\sigma_{ij}^{--}(x)}{\mu_i - \mu_j}, \tag{3.21}$$

$$\forall 1 \leq i, j \leq m,\ \mu_j L_{ij}(x,0) = \sum_{k=1}^{n} \lambda_k K_{ik}(x,0) q_{kj}. \tag{3.22}$$

Besides, (3.14) imposes

$$\forall i \leq j \quad \omega_{ij}(x) = (\mu_i - \mu_j) L_{ij}(x,x) + \sigma_{ij}^{--}(x). \tag{3.23}$$

This induces a coupling between the kernels through equations (3.18) and (3.19) that could appear as non linear at first sight. However, as it will appear, this coupling has a linear cascade structure. More precisely, the well-posedness of the target system is assessed in the following theorem.

Theorem 3.1.1.

Consider system (3.18)-(3.22). There exists a unique solution K and L in $L^\infty(\mathcal{T})$.

The proof of this theorem is detailed in the next section and uses the cascade structure of the kernel equations (which is due to the upper-triangular shape of the matrix Ω).

3.1.2 Well-posedness of the kernel equations: Proof of Theorem 3.1.1

To prove the well-posedness of the kernel equations (and that consequently, transformation (3.10)-(3.11) exists), we classically (see [Joh60] and [Whi11]) transform the kernel equations into integral equations and use the method of successive approximations. By induction, let us consider the following property $P(s)$ defined for all $1 \leq s \leq m$: "$\forall\ 1 \leq j \leq n$, $\forall 1 \leq l \leq m$ and $\forall\ m+1-s \leq i \leq m$, the problem (3.18)-(3.22) where Ω is defined by (3.23) has a unique solution $K_{ij}(\cdot,\cdot)$, $L_{il}(\cdot,\cdot) \in L^\infty(\mathcal{T})$". This property means that we successively prove the well-posedness of each line of the matrices K and M starting from the last one.

The property $P(1)$ is a direct consequence of [HDMVK16, Theorem 3.3]. Let us now assume that the property $P(s-1)$ ($1 \leq s \leq m-1$) is true. We consequently have that $\forall\ m+2-s \leq p \leq m$, $\forall\ 1 \leq j \leq n$, $\forall\ 1 \leq l \leq m$ $K_{pj}(\cdot,\cdot)$ and $L_{pl}(\cdot,\cdot)$ are bounded. In the following we take $i = m+1-s$. We show that (3.18)-(3.22) is well-posed and that $K_{ij}(\cdot,\cdot)$ and $L_{il}(\cdot,\cdot) \in L^\infty(\mathcal{T})$.

Characteristics of the K kernels

For each $1 \leq j \leq n$ and $(x,\xi) \in \mathcal{T}$, we define the following characteristic lines $(x_{ij}(x,\xi,\cdot), \xi_{ij}(x,\xi,\cdot))$ corresponding to equation (3.18)

$$\begin{cases} \dfrac{dx_{ij}}{ds}(x,\xi,s) = -\mu_i \quad s \in [0, s_{ij}^F(x,\xi)] \\ x_{ij}(x,\xi,0) = x, \quad x_{ij}(x,\xi,s_{ij}^F(x,\xi)) = x_{ij}^F(x,\xi), \end{cases} \tag{3.24}$$

$$\begin{cases} \dfrac{d\xi_{ij}}{ds}(x,\xi,s) = \lambda_j \quad s \in [0, s_{ij}^F(x,\xi)] \\ \xi_{ij}(x,\xi,0) = \xi, \quad \xi_{ij}(x,\xi,s_{ij}^F(x,\xi)) = x_{ij}^F(x,\xi). \end{cases} \tag{3.25}$$

These lines originate at the point (x,ξ) and terminate at the point $(x_{ij}^F(x,\xi), x_{ij}^F(x,\xi))$ on the hypotenuse. Integrating (3.18) along these characteristics and using the boundary conditions (3.20) we get

$$
\begin{aligned}
&K_{ij}(x,\xi) = k_{ij}(x) + \int_0^{s_{ij}^F(x,\xi)} [\sum_{k=1}^{n} \sigma_{kj}^{++}(\xi_{ij}(x,\xi,s))K_{ik}(x_{ij}(x,\xi,s),\xi_{ij}(x,\xi,s)) \\
&+ \sum_{k=1}^{m} \sigma_{kj}^{-+}(\xi_{ij}(x,\xi,s))L_{ik}(x_{ij}(x,\xi,s),\xi_{ij}(x,\xi,s)) - \sum_{i\le p\le m} K_{pj}(x_{ij}(x,\xi,s),\xi_{ij}(x,\xi,s)) \\
&\cdot((\mu_i-\mu_p)L_{ip}(x_{ij}(x,\xi,s),x_{ij}(x,\xi,s)) + \sigma_{ip}^{--}(x_{ij}(x,\xi,s)))]ds. \qquad (3.26)
\end{aligned}
$$

We can notice that the last sum uses the expression of K_{pj} for $i \le p \le m$. This term is known and bounded for $p > i$ (induction assumption). For $p = i$, we have $\mu_i = \mu_p$ and the term $(\mu_i - \mu_p)L_{ip}(x_{ij}(x,\xi,s),x_{ij}(x,\xi,s))$ cancel. Therefore, equation (3.26) is linear in the unknowns $K_{ij},\ 1 \le j \le n$ and $L_{ij},\ 1 \le j \le n$.

Characteristics of the L kernels

For each $1 \le j \le n$ and $(x,\xi) \in \mathcal{T}$, we define the following characteristic lines $(\chi_{ij}(x,\xi,\cdot),\zeta_{ij}(x,\xi,\cdot))$ corresponding to equation (3.19)

$$
\begin{cases}
\dfrac{d\chi_{ij}}{d\nu}(x,\xi,\nu) = -\mu_i \quad \nu \in [0,\nu_{ij}^F(x,\xi)] \\
\chi_{ij}(x,\xi,0) = x, \quad \chi_{ij}(x,\xi,\nu_{ij}^F(x,\xi)) = \chi_{ij}^F(x,\xi),
\end{cases} \qquad (3.27)
$$

$$
\begin{cases}
\dfrac{d\zeta_{ij}}{d\nu}(x,\xi,\nu) = -\mu_j \quad \nu \in [0,\nu_{ij}^F(x,\xi)] \\
\zeta_{ij}(x,\xi,0) = \xi, \quad \zeta_{ij}(x,\xi,\nu_{ij}^F(x,\xi)) = \zeta_{ij}^F(x,\xi).
\end{cases} \qquad (3.28)
$$

These lines all originates from (x,ξ) and terminate at the point $(\chi_{ij}^F(x,\xi),\zeta_{ij}^F(x,\xi))$, i.e. either at $(\chi_{ij}^F(x,\xi),\chi_{ij}^F(x,\xi))$ or at $(\chi_{ij}^F(x,\xi),0)$. Integrating (3.19) along these characteristic and using the boundary conditions (3.21)-(3.22) yields

$$
\begin{aligned}
&L_{ij}(x,\xi) = -\delta_{ij}(x,\xi)\frac{\sigma_{ij}^{--}(x)}{\mu_i-\mu_j} + (1-\delta_{ij})\frac{1}{\mu_j}\sum_{k=1}^{n}\lambda_k q_{kj} K_{ik}(\chi_{ij}^F(x,\xi),0) + \int_0^{\nu_{ij}^F(x,\xi)} [\sum_{p=1}^{m}\sigma_{pj}^{--}(\zeta_{ij}(x,\xi,\nu)) \\
&L_{ip}(\chi_{ij}(x,\xi,\nu),\zeta_{ij}(x,\xi,\nu)) + \sum_{k=1}^{n}\sigma_{kj}^{+-}(\zeta_{ij}(x,\xi,\nu))K_{ik}(\chi_{ij}(x,\xi,\nu),\zeta_{ij}(x,\xi,\nu)) - \sum_{i\le p\le m} L_{pj}(\chi_{ij}(x, \\
&\xi,\nu),\zeta_{ij}(x,\xi,\nu))((\mu_i-\mu_p)L_{ip}(\chi_{ij}(x,\xi,\nu),\chi_{ij}(x,\xi,\nu)) + \sigma_{ip}^{--}(\chi_{ij}(x,\xi,\nu)))]d\nu, \qquad (3.29)
\end{aligned}
$$

where the coefficient $\delta_{ij}(x,\xi)$ is defined by

$$
\delta_{i,j}(x,\xi) = \begin{cases} 1 & \text{if } j < i \quad \text{and} \quad \mu_i\xi - \mu_j x \ge 0, \\ 0 & \text{else.} \end{cases} \qquad (3.30)
$$

This coefficient reflects the fact that, as mentioned above, some characteristics terminate on the hypotenuse and others on the axis $\xi = 0$. We can now plug (3.26) evaluated at $(\chi_{ij}^F(x,\xi),0)$

into (3.29). This yields

$$L_{ij}(x,\xi) = -\delta_{ij}(x,\xi)\frac{\sigma_{ij}^{--}(x)}{\mu_i-\mu_j} + (1-\delta_{ij}(x,\xi))\frac{1}{\mu_j}\sum_{k=1}^{n}\lambda_k q_{kj}k_{ik}(0) + (1-\delta_{ij}(x,\xi))\frac{1}{\mu_j}\sum_{r=1}^{n}\lambda_r q_{rj}$$
$$\int_0^{s_{ir}^F(\chi_{ij}^F(x,\xi),0)}[\sum_{k=1}^{n}\sigma_{kr}^{++}(0)K_{ik}(x_{ir}(\chi_{ij}^F(x,\xi),0,s),\xi_{ir}(\chi_{ij}^F(x,\xi),0,s)) + \sum_{k=1}^{m}\sigma_{kr}^{-+}(0)L_{ik}(x_{ir}(\chi_{ij}^F(x,\xi),$$
$$0,s),\xi_{ir}(\chi_{ij}^F(x,\xi),0,s)) - \sum_{i\le p\le m}K_{pr}(x_{ir}(\chi_{ij}^F(x,\xi),0,s),\xi_{ir}(\chi_{ij}^F(x,\xi),0,s))\cdot((\mu_i-\mu_p)$$
$$L_{ip}(x_{ir}(\chi_{ij}^F(x,\xi),0,s),x_{ir}(\chi_{ij}^F(x,\xi),0,s)) + \sigma_{ip}^{--}(0))]ds + \int_0^{\nu_{ij}^F(x,\xi)}[\sum_{p=1}^{m}\sigma_{pj}^{--}(\zeta_{ij}(x,\xi,\nu))$$
$$L_{ip}(\chi_{ij}(x,\xi,\nu),\zeta_{ij}(x,\xi,\nu)) + \sum_{k=1}^{n}\sigma_{kj}^{+-}(\zeta_{ij}(x,\xi,\nu))K_{ik}(\chi_{ij}(x,\xi,\nu),\zeta_{ij}(x,\xi,\nu)) - \sum_{i\le p\le m}L_{pj}(\chi_{ij}(x$$
$$,\xi,\nu),\zeta_{ij}(x,\xi,\nu))\cdot((\mu_i-\mu_p)L_{ip}(\chi_{ij}(x,\xi,\nu),\chi_{ij}(x,\xi,\nu)) + \sigma_{ip}^{--}(\chi_{ij}(x,\xi,\nu)))]d\nu. \tag{3.31}$$

Again, equation (3.31) is linear.

Method of successive approximations

To solve the integral equations (3.26)-(3.31) we use the method of successive approximations. We define

$$\begin{aligned}
\forall 1\le j\le n \quad & \phi_j^1(x,\xi) = k_{ij}(x) - \int_0^{s_{ij}^F(x,\xi)}\sum_{i<p\le m}K_{pj}(x_{ij}(x,\xi,s),\xi_{ij}(x,\xi,s))\sigma_{ip}^{--}(\xi_{ij}(x,\xi,s))ds,\\
\forall 1\le j\le m \quad & \phi_j^2(x,\xi) = -\delta_{ij}(x,\xi)\frac{\sigma_{ij}^{--}(x)}{\mu_i-\mu_j} + (1-\delta_{ij}(x,\xi))\frac{1}{\mu_j}\sum_{k=1}^{n}\lambda_k q_{kj}k_{ik} - (1-\delta_{ij}(x,\xi))\frac{1}{\mu_j}\\
& \sum_{r=1}^{n}\lambda_r q_{rj}\int_0^{s_{ir}^F(\chi_{ij}^F(x,\xi),0)}\sum_{i<p\le m}K_{pr}(x_{ir}(\chi_{ij}^F(x,\xi),0,s),\xi_{ir}(\chi_{ij}^F(x,\xi),0,s))\sigma_{ip}^{--}(0)\\
& -\int_0^{\nu_{ij}^F(x,\xi)}\sum_{i<p\le m}L_{pj}(\chi_{ij}(x,\xi,\nu),\zeta_{ij}(x,\xi,\nu))\sigma_{ip}^{--}(0)d\nu.
\end{aligned}$$

Besides we denote H and Ψ as

$$H = \begin{pmatrix}K_{i1} & \dots & K_{in} & L_{i1} & \cdots & L_{im}\end{pmatrix}^\top, \quad \Psi = \begin{pmatrix}\phi_1^1 & \dots & \phi_n^1 & \phi_1^2 & \dots & \phi_m^2\end{pmatrix}^\top, \tag{3.32}$$

We now consider the following operators, defined for all $1\le j\le n$ by

$$\begin{aligned}
\Phi_j^1(H)(x,\xi) = & \int_0^{s_{ij}^F(x,\xi)}[\sum_{k=1}^{n}\sigma_{kj}^{++}(\xi_{ij}(x,\xi,s))K_{ik}(x_{ij}(x,\xi,s),\xi_{ij}(x,\xi,s))\\
& + \sum_{k=1}^{m}\sigma_{kj}^{-+}(\xi_{ij}(x,\xi,s))L_{ik}(x_{ij}(x,\xi,s),\xi_{ij}(x,\xi,s)) - \sum_{i<p\le m}K_{pj}(x_{ij}(x,\xi,s),\xi_{ij}(x,\xi,s))\\
& \cdot((\mu_i-\mu_p)L_{ip}(x_{ij}(x,\xi,s),x_{ij}(x,\xi,s))) + \sigma_{ii}^{--}(\xi_{ij}(x,\xi,s))K_{ij}(x_{ij}(x,\xi,s),\xi_{ij}(x,\xi,s))]ds,
\end{aligned}$$

$$\Phi^2_j(H)(x,\xi) = (1-\delta_{ij}(x,\xi))\frac{1}{\mu_j}\sum_{r=1}^{n}\lambda_r q_{rj}\int_0^{s^F_{ir}(\chi^F_{ij}(x,\xi),0)}[\sum_{k=1}^{n}\sigma^{++}_{kr}(0)K_{ik}(x_{ir}(\chi^F_{ij}(x,\xi),0,s),$$
$$\xi_{ir}(\chi^F_{ij}(x,\xi),0,s)) + \sum_{k=1}^{m}\sigma^{-+}_{kr}(0)L_{ik}(x_{ir}(\chi^F_{ij}(x,\xi),0,s),\xi_{ir}(\chi^F_{ij}(x,\xi),0,s)) - \sum_{i<p\le m}K_{pr}(x_{ir}(\chi^F_{ij}(x,\xi)$$
$$,0,s),\xi_{ir}(\chi^F_{ij}(x,\xi),0,s))\cdot((\mu_i-\mu_p)L_{ip}(x_{ir}(\chi^F_{ij}(x,\xi),0,s),x_{ir}(\chi^F_{ij}(x,\xi),0,s))) - K_{ir}(x_{ir}(\chi^F_{ij}(x,\xi)$$
$$,0,s),\xi_{ir}(\chi^F_{ij}(x,\xi),0,s))\sigma^{--}_{ii}(0)]ds + \int_0^{\nu^F_{ij}(x,\xi)}[\sum_{p=1}^{m}\sigma^{--}_{pj}(\zeta_{ij}(x,\xi,\nu))L_{ip}(\chi_{ij}(x,\xi,\nu),\zeta_{ij}(x,\xi,\nu))$$
$$+\sum_{k=1}^{n}\sigma^{+-}_{kj}\zeta_{ij}(x,\xi,\nu)K_{ik}(\chi_{ij}(x,\xi,\nu),\zeta_{ij}(x,\xi,\nu)) - \sum_{i<p\le m}L_{pj}(\chi_{ij}(x,\xi,\nu),\zeta_{ij}(x,\xi,\nu))$$
$$\cdot((\mu_i-\mu_p)L_{ip}(\chi_{ij}(x,\xi,\nu),\chi_{ij}(x,\xi,\nu))) - L_{ij}(\chi_{ij}(x,\xi,\nu),\zeta_{ij}(x,\xi,\nu))\sigma^{--}_{ii}\chi_{ij}(x,\xi,\nu)]d\nu.$$

We set $\Phi[H](x,\xi) = [\Phi^1[H](x,\xi)^T, \Phi^2[H](x,\xi)^\top]^\top$. We define the following sequence

$$H^0(x,\xi)=0,\quad H^q(x,\xi) = \Psi(x,\xi) + \Phi(H^{q-1})(x,\xi). \tag{3.33}$$

Consequently, if the sequence H^q has a limit, then this limit is a solution of the integral equation and therefore of the original system. We define the increment $\Delta H^q = H^q - H^{q-1}$ (with $\Delta H^0 = \Psi$). Provided the limit exists one has

$$H(x,\xi) = \lim_{q\to+\infty} H^q(x,\xi) = \sum_{q=0}^{+\infty}\Delta H^q(x,\xi). \tag{3.34}$$

We now prove the convergence of the series.

Convergence of the successive approximation series

Similarly to [DMVK13], [HDMVK16] we want to find a recursive upper bound in order to prove the convergence of the series. We first define

$$\bar\Phi = \max_j \max_{(x,\xi)\in\mathcal{T}}\{|\phi^1_{i,j}(x,\xi)|,|\phi^2_{ij}(x,\xi)|\},\quad \bar\sigma = \max_{k,j}\{\sigma^{++}_{kj},\sigma^{+-}_{kj},\sigma^{-+}_{kj},\sigma^{--}_{kj}\},\quad \bar q = \max_{k,j}\{q_{kj}\},$$
$$\bar\mu = \max_p\{|\mu_i-\mu_p|\},\quad \bar\lambda = \max\{\lambda_n,\mu_n\},\quad \tilde\lambda = \max\{\frac{1}{\lambda_1},\frac{1}{\mu_1}\},\quad M_\lambda = \max_{j=1,\dots,m}\{\frac{1}{\mu_j}\}.$$

We then define $\bar S = \max_{p>i,1\le j\le n}\{||K_{pj}||,||L_{pj}||\}$ which is well defined according to the hypothesis $P(s-1)$. Moreover we set

$$M = (n\bar\lambda\tilde\lambda\bar q+1)[(n+m+1)\bar\sigma + m\bar\mu\bar S]M_\lambda. \tag{3.35}$$

Lemma 3.1.2.

Assume that for some $1\le q$, one has, for all $(x,\xi)\in\mathcal{T}$

$$\forall j=1,...m+n \quad |\Delta H^q_j(x,\xi)| \le \bar\Phi\frac{M^q x^q}{q!}, \tag{3.36}$$

where $\Delta H^q_j(x,\xi)$ is the j-th component of $\Delta H^q(x,\xi)$.
Then, one has

$$\forall j=1,...m+n \quad |\Delta H^{q+1}_j(x,\xi)| \le \bar\Phi\frac{M^{q+1}x^{q+1}}{(q+1)!}. \tag{3.37}$$

The proof of this lemma is direct since all the characteristic lines have the same direction along the x-axis, i.e. $\frac{d\xi_{ij}}{ds} < 0$ and $\frac{d\chi_{ij}}{d\nu}(x,\xi,\nu) < 0$. Consequently, using similar methods as the ones presented in [DMVK13, VCKB11], we get that the successive approximation series (3.34) is bounded and converges uniformly. This proves the existence part of the property $P(s)$. The uniqueness of the kernels and their continuity can be proved in similar way as the one proposed in [CVKB13]. Thus the property $P(s)$ is true. This concludes the proof by induction of Theorem 3.1.1.

3.1.3 Control law and finite time boundary stabilization

In the previous section, we have proved that the set of equations (3.18)-(3.22) admits an unique solution and that consequently, the transformation (3.10)-(3.11) does exist. In order to get the right boundary condition (3.7) and reach finite-time convergence, we choose the following control law.

$$V(t) = -R_1 u(t,1) + \int_0^1 [K(1,\xi)u(t,\xi) + L(1,\xi)v(t,\xi)]d\xi. \tag{3.38}$$

We now prove that this control law fulfills Objective A. We first have the following result,

Lemma 3.1.3.

There exists an invertible bounded linear map $\mathcal{F} : (L^2[0,1])^{n+m} \to (L^2[0,1])^{n+m}$ such that, in presence of the control law (3.38), for every initial condition $(u_0,v_0) \in (L^2([0,1]))^{n+m}$, if $(\alpha_0,\beta_0) \in (L^2([0,1]))^{n+m}$ denotes the solution to (3.5)-(3.7) satisfying the initial data $(\alpha_0(0,\cdot),\beta_0(0,\cdot)) = \mathcal{F}(u_0,v_0)$, then $(u(t),v(t)) = \mathcal{F}^{-1}(\alpha(t),\beta(t))$.

Proof : Let us consider the invertible bounded linear operator,

$$\begin{aligned}\mathcal{F} : (L^2[0,1])^{n+m} &\to (L^2[0,1])^{n+m}\\ \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} &\mapsto \begin{pmatrix} u(t,x) \\ v(t,x) - \int_0^x (K(x,\xi)u(\xi) + L(x,\xi)v(\xi))d\xi \end{pmatrix},\end{aligned} \tag{3.39}$$

which is well-defined due to Theorem 3.1.1. Let $(\alpha,\beta) \in \mathcal{C}^0([0,T],(L^2([0,1]))^{n+m})$ be the solution to (3.5)-(3.7) associated with the initial data $(\alpha(0,\cdot),\beta(0,\cdot)) = \mathcal{F}(u_0,v_0)$. Note that the invertibility of the operator is a direct consequence of the use of a Volterra integral in (3.39). Then, $(u(t),v(t)) = \mathcal{F}^{-1}(\alpha(t),\beta(t))$ is the solution to the Cauchy problem (3.1)-(3.3) along with the initial conditions (u_0,v_0). ■

Finally, we have the following theorem (published in [ADM16a]).

Theorem 3.1.2.

Consider system (3.1)-(3.2) along with boundary conditions (3.3) and the feedback control law (3.38). Then, for any initial condition $(u_0,v_0) \in (L^2([0,1]))^{n+m}$, it reaches its zero equilibrium in the minimum finite time $t_F = \frac{1}{\lambda_1} + \frac{1}{\mu_1}$. Moreover, if the initial conditions belong to $(\mathcal{C}^1([0,1]))^{n+m}$ (and satisfy the corresponding compatibility conditions), the control law (3.38) ensures the weak exact boundary controllability in the sense of Theorem (2.3.2).

Proof : The proof is a direct consequence of Lemma 3.1.3 and 3.1.1, since for any initial condition in $(L^2([0,1]))^{n+m}$, the system (3.5)-(3.7) reaches its zero equilibrium in finite time t_F. Additionally, if the initial conditions (u_0,v_0) belong to $(\mathcal{C}^1([0,1]))^{n+m}$ (and satisfy the corresponding compatibility conditions), then so do the initial data $(\alpha(0,\cdot),\beta(0,\cdot)) = \mathcal{F}(u_0,v_0)$. Thus, due to Lemma 2.1.2, the solution of (3.5)-(3.7) remains in $(\mathcal{C}^1([0,1]))^{n+m}$ and so do (u,v). ■

3.2 One-sided boundary observability

In this section we design an observer for the original system (3.1)-(3.3) relying on the measurements of u at the right boundary (collocated case), i.e. we measure

$$y(t) = u(t, 1).$$

As stated by Objective A', we want our estimation to converge (in the sense of the L^2-norm) to the real state in finite time t_F. The chosen structure for the observer is adjusted from the one presented in [HDMVK16]. We then prove that the corresponding error system is the dual of a system that has a similar structure that the closed-loop system (3.1)-(3.3). Designing a stabilizing full-state feedback law for this new system, one can obtain the corresponding observer gains.

3.2.1 Observer design

The observer equations read as follows

$$\partial_t \hat{u}(t,x) + \Lambda^+ \partial_x \hat{u}(t,x) = \Sigma^{++}\hat{u}(t,x) + \Sigma^{+-}\hat{v}(t,x) - P^+(x)(\hat{u}(t,1) - u(t,1)), \tag{3.40}$$
$$\partial_t \hat{v}(t,x) - \Lambda^- \partial_x \hat{v}(t,x) = \Sigma^{-+}\hat{u}(t,x) + \Sigma^{--}\hat{v}(t,x) - P^-(x)(\hat{u}(t,1) - u(t,1)), \tag{3.41}$$

with the boundary conditions

$$\hat{u}(t,0) = Q_0 \hat{v}(t,0), \quad \hat{v}(t,1) = R_1 u(t,1) + V(t), \tag{3.42}$$

and with arbitrary initial condition $(\hat{u}(0,\cdot), \hat{v}(0,\cdot) \in (L^2([0,1]))^{n+m}$. The observer gains $P^+(\cdot)$ and $P^-(\cdot)$ have yet to be designed. Defining the error estimates $\tilde{u}(t,x) = u(t,x) - \hat{u}(t,x)$ and $\tilde{v}(t,x) = u(v,x) - \hat{v}(t,x)$, this yields the following error system

$$\partial_t \tilde{u}(t,x) + \Lambda^+ \partial_x \tilde{u}(t,x) = \Sigma^{++}\tilde{u}(t,x) + \Sigma^{+-}\tilde{v}(t,x) - P^+(x)\tilde{u}(t,1), \tag{3.43}$$
$$\partial_t \tilde{v}(t,x) - \Lambda^- \partial_x \tilde{v}(t,x) = \Sigma^{-+}\tilde{u}(t,x) + \Sigma^{--}\tilde{v}(t,x) - P^-(x)\tilde{u}(t,1), \tag{3.44}$$

with the boundary conditions

$$\tilde{u}(t,0) = Q_0 \tilde{v}(t,0), \quad \tilde{v}(t,1) = 0. \tag{3.45}$$

This system evolves in $[0,T] \times [0,1]$ and its initial condition $(\tilde{u}(0,x), \tilde{v}(0,x)) = (\tilde{u}_0(x), \tilde{v}_0(x))$ belongs to $L^2([0,1])^{(n+m)}$. We define the operator $\bar{A}$ by

$$\begin{aligned} \bar{A} : & D(\bar{A}) \subset (L^2(0,1))^{n+m} \to (L^2(0,1))^{n+m} \\ & \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \longmapsto \begin{pmatrix} -\Lambda^+ \partial_x \bar{u} + \Sigma^{++}\bar{u} + \Sigma^{+-}\bar{v} - P^+(x)\bar{u}(t,1) \\ \Lambda^- \partial_x \bar{v} + \Sigma^{-+}\bar{u} + \Sigma^{--}\bar{v} - P^-(x)\bar{u}(t,1) \end{pmatrix}, \end{aligned} \tag{3.46}$$

with

$$D(\bar{A}) = \{(u,v) \in (H^1(0,1))^{n+m} | \ \bar{u}(0) = Q_0 \bar{v}(t,0), \quad \bar{v}(1) = 0\}.$$

We now define the following (adjoint) system (which is derived from (3.43)-(3.45) changing t into $(T-t)$, evolving in $[0,T] \times [0,1]$:

$$-\frac{d}{dt}\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \bar{A}\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}. \tag{3.47}$$

Its (arbitrary) initial conditions are defined by

$$\begin{pmatrix} \bar{u}(T,x) \\ \bar{v}(T,x) \end{pmatrix} = \begin{pmatrix} \bar{u}_T(x) \\ \bar{v}_T(x) \end{pmatrix} \in L^2([0,1])^{(n+m)}. \tag{3.48}$$

Note that system (3.47) does not include any control operator since it has been canceled (see equation (3.45)). To find the observer gains we define a control problem that is the dual of the observer problem. The observer gains will then be defined by the gains of the *dual controller*.

3.2.2 A new control problem: use of the adjoint

The design of the observer we propose is inspired by the adjoint lemma [Kai80, p. 627]. Let us consider the following system

$$\partial_t \phi(t,x) - \Lambda^+ \partial_x \phi(t,x) = (\Sigma^{-+})^T \psi(t,x) + (\Sigma^{++})^T \phi(t,x), \tag{3.49}$$

$$\partial_t \psi(t,x) + \Lambda^- \partial_x \psi(t,x) = (\Sigma^{--})^T \psi(t,x) + (\Sigma^{+-})^T \phi(t,x), \tag{3.50}$$

evolving in $\{(t,x)| \quad 0 < t < T, \quad x \in [0,1]\}$, with the following linear boundary conditions

$$\psi(t,0) = (\Lambda^-)^{-1} Q_0^T \Lambda^+, \quad \phi(t,1) = V_0(t), \tag{3.51}$$

and the arbitrary initial conditions (belonging to $(L^2([0,1]))^{(n+m)}$)

$$\phi(0,x) = \phi_0(x), \quad \psi(0,x) = \psi_0(x).$$

Using Theorem 3.1.2, there exist L^∞ kernels K_1, L_1 defined on $\mathcal{T}$ such that system (3.49)-(3.50) with the following feedback law

$$V_0(t) = \int_0^1 (K_1(1,\xi)\phi(t,\xi) + L_1(1,\xi)\psi(t,\xi))d\xi, \tag{3.52}$$

reaches its zero equilibrium (in the sense of the L^2-norm) in time t_F. We now prove that this closed loop system is the adjoint of the error system (3.43)-(3.45) as long as the observer gains P^+ and P^- satisfy some conditions that will be explicited. Thus, due to the adjoint lemma, as system (3.49)-(3.51) converges to zero in finite time, so does the error system (3.43)-(3.45). The proof is based on the abstract formulation. System (3.49)-(3.51) can be rewritten in the abstract form as

$$\frac{d}{dt}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = A_1 \begin{pmatrix}\phi\\ \psi\end{pmatrix} + B_1 V_0, \tag{3.53}$$

with the initial condition

$$\begin{pmatrix}\phi(0,x)\\ \psi(0,x)\end{pmatrix} = \begin{pmatrix}\phi_0(x)\\ \psi_0(x)\end{pmatrix}, \tag{3.54}$$

where the operators A_1 and B_1 are defined in a similar form as the ones presented in equation (2.6):

$$\begin{aligned} A_1 :& D(A_1) \subset (L^2(0,1))^{n+m} \to (L^2(0,1))^{n+m} \\ & \begin{pmatrix}\phi\\ \psi\end{pmatrix} \longmapsto \begin{pmatrix}\Lambda^+\partial_x \phi + (\Sigma^{-+}(x))^T\psi + (\Sigma^{++}(x))^T v\phi \\ -\Lambda^-\partial_x v + (\Sigma^{--}(x))^T\psi + (\Sigma^{+-}(x))^T\phi\end{pmatrix}, \end{aligned} \tag{3.55}$$

with

$$D(A_1) = \{(\phi,\psi) \in (H^1(0,1))^{n+m} | \psi(0) = (\Lambda^-)^{-1} Q_0^T \Lambda^+, \ \phi(1) = 0\}.$$

Its adjoint A_1^* is defined by

$$\begin{aligned} A_1^* :& D(A_1^*) \subset (L^2(0,1))^{n+m} \to (L^2(0,1))^{n+m} \\ & \begin{pmatrix}\phi\\ \psi\end{pmatrix} \longmapsto \begin{pmatrix}-\Lambda^+\partial_x \phi + \Sigma^{++}(x)\phi + \Sigma^{+-}(x)\psi \\ \Lambda^-\partial_x v + \Sigma^{-+}(x)\phi + \Sigma^{--}(x)\psi\end{pmatrix}, \end{aligned}$$

with

$$D(A_1^*) = \{(\phi,\psi) \in (H^1(0,1))^{n+m} | \psi(1) = 0, \ \phi(0) = Q_0\psi(0)\}.$$

Note that the operator A_1^* corresponds to the original operator A defined in (2.7) but has a different domain of definition. The operator $B_1 \in \mathcal{L}(\Re^m, D(A^*)')$ is defined by

$$< B_1 V, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} >= z_1(1)^T \Lambda^- V, \tag{3.56}$$

while its adjoint $B_1^* \in \mathcal{L}(D(A^*), \Re^m)$ is defined by $B_1^* \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1(1)^T \Lambda^-$. In the same time the control law can be rewritten as

$$V_0(t) = \Gamma_0 \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

where

$$\begin{aligned} \Gamma_0 :& D(A_1) \subset (L^2(0,1))^{n+m} \to \mathbb{R}^m \\ &\begin{pmatrix} \phi \\ \psi \end{pmatrix} \longmapsto \int_0^1 (K_1(1,\xi)\phi(t,\xi) + L_1(1,\xi)\psi(t,\xi))d\xi. \end{aligned} \tag{3.57}$$

We now define the gains of the observer as

$$P^+(x) = -K_1(1,x)^T \Lambda^+, \quad P^-(x) = -L_1(1,x)^T \Lambda^+. \tag{3.58}$$

With these observer gains, we have that the operator $\bar{A}$ corresponds to the operator $A_1^* + \Gamma^* B_1^*$. Consequently the two systems have similar properties and the observer (3.40)-(3.42) fulfills Objective A'. More precisely, we have the following theorem whose proof is similar

Theorem 3.2.1.

Consider system (3.1)-(3.2) along with boundary conditions (3.3) and the feedback control law (3.38). Consider the error system (3.43)-(3.45) (where P^+ and P^- are defined by (3.58)). Then, for any initial condition $(u_0, v_0) \in (L^2([0,1]))^{n+m}$, for any observer initial condition $(\hat{u}_0, \hat{v}_0) \in (L^2([0,1]))^{n+m}$ the observer state $(\hat{u}, \hat{v})$ is equal (in the sense of the L^2-norm) to the real state in the minimum finite time t_F. Moreover, if the initial conditions (u_0, v_0) and $(\hat{u}_0, \hat{v}_0)$ belong to $(\mathcal{C}^1([0,1]))^{n+m}$ (and satisfy the corresponding compatibility conditions), the observer system (3.43)-(3.45) ensures the weak exact boundary observability in the sense of Theorem (2.3.4).

Proof : We recall that we denote $\langle \cdot, \cdot \rangle$ the scalar product associated to the L^2-norm. For every solution $(\bar{u}, \bar{v})$ of (3.47)-(3.48) and every solution (ϕ, ψ) of (3.53)-(3.54) (with any initial conditions) we have

$$\langle \frac{d}{dt} \begin{pmatrix} \phi \\ \psi \end{pmatrix} - A_1 \begin{pmatrix} \phi \\ \psi \end{pmatrix} - B_1 V_0, \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \rangle = 0, \tag{3.59}$$

since the left part of the scalar product is zero (due to (3.53)). This yields

$$\begin{aligned} 0 = \int_0^\tau \int_0^1 & \bar{u}^T(t,x)(\partial_t \phi(t,x) - \Lambda^+ \partial_x \phi(t,x) - (\Sigma^{++}(x))^T \phi(t,x) - (\Sigma^{+-})^T \psi(t,x)) + \bar{v}(t,x)(\partial_t \psi(t,x) \\ &+ \Lambda^- \partial_x \psi(t,x) - (\Sigma^{-+}(x))^T \phi(t,x) - (\Sigma^{--}(x))^T \psi(t,x)) dx dt - \int_0^\tau \bar{u}(t,1) \Lambda^+ V(t) dt. \end{aligned} \tag{3.60}$$

This formulation corresponds to the adjoint lemma given in [Kai80, p. 627]. Integrating (3.60) by parts, we obtain

$$\begin{aligned}0 = &\int_0^\tau \int_0^1 (-\partial_t \bar{u}^T(t,x) + \partial_x \bar{u}^T(t,x)\Lambda^+ - \bar{u}^T(t,x)(\Sigma^{++}(x))^T - \bar{v}(t,x)^T(\Sigma^{+-}(x))^T)\phi(t,x) + (-\partial_t \bar{v}^T(t,x) \\ &- \partial_x \bar{v}^T(t,x)\Lambda^- - \bar{u}(t,x)(\Sigma^{-+}(x))^T - \bar{v}(t,x)(\Sigma^{--}(x))^T)\psi(t,x)dxdt + \int_0^1 (\bar{v}^T(\tau,x)\psi(\tau,x) - \bar{v}^T(0,x)\psi(0,x) \\ &+ \bar{u}^T(\tau,x)\phi(\tau,x) - \bar{u}^T(0,x)\phi(0,x))dx + \int_0^\tau -\bar{u}^T(t,0)\Lambda^+\phi(t,0) + \bar{u}^T(t,1)\Lambda^+\phi(t,1) \\ &+ \bar{v}^T(t,1)\Lambda^-\psi(t,1) - \bar{v}^T(t,0)\Lambda^-\psi(t,0)dt + \int_0^\tau \int_0^1 \bar{u}(t,1)^T\Lambda^+(K_1(1,x)\phi(t,x) + L_1(1,x)\psi(t,x))dx)dt.\end{aligned}$$

Using the definition of $D(A_1)$ and $D(\bar{A})$, we have that $\int_0^\tau -\bar{u}^T(t,0)\Lambda^+\phi(t,0) + \bar{u}^T(t,1)\Lambda^+\phi(t,1) + \bar{v}^T(t,1)\Lambda^-\psi(t,1) - \bar{v}^T(t,0)\Lambda^-\psi(t,0)dt = 0$ Computing $\langle \begin{pmatrix}\phi\\ \psi\end{pmatrix}, -\frac{d}{dt}\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix} - \bar{A}\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix}\rangle$, one easily obtains

$$\begin{aligned}0 =&\langle \begin{pmatrix}\phi\\ \psi\end{pmatrix}, -(\frac{d}{dt} + \bar{A})\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix}\rangle = \langle (\frac{d}{dt} - A_1)\begin{pmatrix}\phi\\ \psi\end{pmatrix} - B_1 V, \begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix}\rangle \\ &- \int_0^1 (\bar{v}^T(\tau,x)\psi(\tau,x) - \bar{v}^T(0,x)\psi(0,x)dx - \int_0^1 \bar{u}^T(\tau,x)\phi(\tau,x) - \bar{u}^T(0,x)\phi(0,x))dx.\end{aligned}$$

Both scalar products are equal to zero due to (3.47) and (3.53). Choosing $\tau = t_F$, and using the fact that $\phi(t_F,\cdot)$ and $\psi(t_F,\cdot)$ are equal to zero almost everywhere, one can cancel some of the remaining integrals and finally obtain:

$$0 = \int_0^1 \bar{u}^T(0,x)\phi(0,x) + \bar{v}^T(0,x)\psi(0,x)dx.$$

This has to be true for any initial condition $\phi(0,x)$ and $\psi(0,x)$. It implies that $\bar{u}(0)$ and $\bar{v}(0)$ are equal to zero almost everywhere. Consequently, using the change of variable $r = t_F - t$ we obtain $\tilde{u}(t_F)$ and $\tilde{v}(t_F)$ are equal to zero almost everywhere. Moreover, if the functions are $\mathcal{C}^1$ (which is the case in presence of $\mathcal{C}^1$ initial conditions that satisfy the compatibility conditions due to Lemma 2.1.2), this holds everywhere. This concludes the proof. ■

3.2.3 Output feedback control law

The estimates can be used in a observer-controller to derive an output feedback law yielding finite-time stability of the zero equilibrium.

Theorem 3.2.2.

Consider the system composed of (3.1)-(3.3) and of the observer system (3.40)-(3.42) along with the control law

$$V(t) = \int_0^1 [K(1,\xi)\hat{u}(t,\xi) + L(1,\xi)\hat{v}(t,\xi)]d\xi - R_1 u(t,1) \tag{3.61}$$

where K and L are defined by (3.18)-(3.23). Then, for any initial condition $(u_0, v_0) \in (L^2([0,1]))^{n+m}$, for any observer initial condition $(\hat{u}_0, \hat{v}_0) \in (L^2([0,1]))^{n+m}$, its solutions $(u, v, \hat{u}, \hat{v})$ converge (in the sense of the L^2-norm) in finite time to zero.

Proof : The convergence of the observer error states $\tilde{u}, \tilde{v}$ to zero for $t_F \leq t$ is ensured by Theorem 3.2.1, along with the existence of the backstepping transformation. Thus, once $t_F \leq t$, $v(t,0) = \hat{v}(t,0)$ and one can use Theorem 3.1.2. Therefore for $2t_F \leq t$, one has $(\tilde{u}, \tilde{v}, \hat{u}, \hat{v}) \equiv 0$ which yields $(u, v) \equiv 0$. ■

3.3 Numerical example

In this section we illustrate the presented results with simulations on a toy problem. The numerical values of the parameters are as follows.

$$n = 2, \quad m = 2, \quad \Lambda^+ = \begin{pmatrix} 10 & 0 \\ 0 & 12 \end{pmatrix}, \quad \Lambda^- = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}, \tag{3.62}$$

$$\Sigma^{++} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma^{+-} = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \quad \Sigma^{-+} = \begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix}, \tag{3.63}$$

$$\Sigma^{--} = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 0.4 & 0.4 \\ 0.08 & 0 \end{pmatrix}, \quad R_1 = 0. \tag{3.64}$$

The first step consists in computing the kernels defined by (3.18)-(3.22) and which are required in the minimum-time control law (3.38). The algorithm we use follows the proof of Theorem 3.1.1. The objective is to find the fixed point of Φ as defined by (3.33). This fixed point is the solution of the kernel equations. The algorithm starts defining a mesh on the triangular domain $\mathcal{T}$. The number of points of this mesh is defined by a parameter p (the greater is p the greater is the precision of the algorithm). An arbitrary value (namely 0) is given for the kernels K_{ij} and L_{ij} on each point of the mesh. Then, for each point (x, ξ) of the mesh we compute the characteristic lines (3.24)-(3.25) and (3.27)-(3.28) that originate at this point and terminate at the point $(x_{ij}^F(x,\xi), \xi_{ij}^F(x,\xi))$ (as seen above this final point depends on the slopes of the characteristic and on the initial point (x, ξ)). This is done using the Matlab ODE solver ODE45. This parametrization of the characteristic lines can be used to compute the right part of the integral equations (3.26)-(3.29). The values of the kernels K_{ij} and L_{ij} are then updated for each point of the mesh. Once the fixed point is reached (namely if the difference of the norm of the kernels between two iterations is smaller than a tolerance ϵ), the algorithm stops. Finally, these kernels can be used to compute the control law (3.38). The simulations of the closed-loop system are done using a classical finite volume method based on a Godunov scheme. The integral of the control law is computed using a trapezoidal method with a precision that corresponds to the size of the mesh used previously to compute the kernels.

The results are compared to the one obtained in [HDMVK16]. The parameters values are chosen such that there is a large benefit in using the presented result compared to [HDMVK16] since the minimum reachable time $t_F = \dfrac{1}{\lambda_1} + \dfrac{1}{\mu_1} = 0.35$ is almost half of the convergence time $t_{F_1} = \dfrac{1}{\lambda_1} + \dfrac{1}{\mu_1} + \dfrac{1}{\mu_2} = 0.55$ that could be obtained in [HDMVK16]. However, one could think that the control effort required to achieve convergence in time t_F is greater than the control effort required to achieve convergence in time t_{F_1}. This intuition is further reenforced by (3.19) and (3.21) that suggest that

- close transport velocities yields larger control gains
- the magnitude of the gains increases with the number of leftward convective states (due to the recursive dependance of the kernels from one line to the next) contrary to [HDMVK16].

However, as it appears the simulation results are contrary to the intuition. Figure 3.2 pictures the $\mathcal{L}^2-$norm of the state (u, v) and the total control effort $V^2(t)$ defined by $V^2(t) = V_1^2(t) + V_2^2(t)$ in open loop, using the control law presented in [HDMVK16] and then using the control law (3.38) presented in this paper. The convergence times are consistent with the theory. The control effort is significantly lower for the minimum-time control. This surprising result may be explained as follows: the control gains depicted in absolute value on Figure 3.3 are of comparable magnitude on a large part of the spatial domain. Since the control law takes the form of a spatial integral, the two controllers are expected to yield similar magnitude of control action for a given

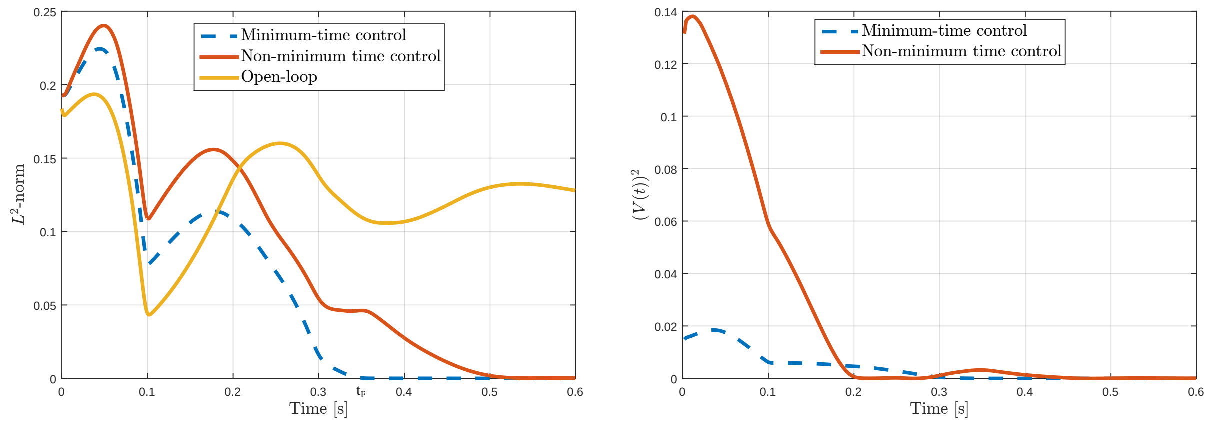

Figure 3.2: Left: Time evolution of the L^2-norm in open loop and using two different controllers. Right: Time evolution of the control effort.

norm of the states. Due to the cascade structure of tits corresponding target system, the non-minimum time control "waits" for fast states to converge before stabilizing slower states, which exponentially grow in the meantime. This result combined with a larger overshoot entails a larger control effort. The observer gains have not been pictured as the have the properties as the controller gains. Finally, we have pictured on Figure 3.4 the time evolution of the L^2-norm using the output-feedback law (3.61). As expected by the theory, it converges to zero in finite-time $2t_F = 0.7$.

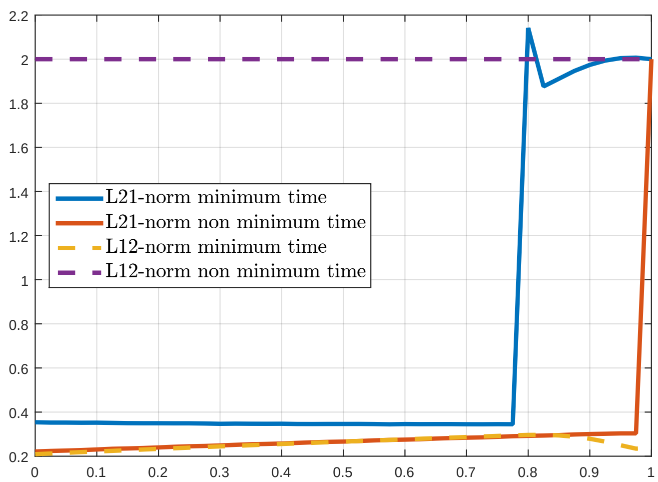

Figure 3.3: Backstepping controller gains

For practical implementation not only performance (or robustness) of the controller is important, but also the computational burden. Here, the principal computational expense is due to the computation of the kernels K_{ij} and L_{ij} as it is required to solve $(n+m)$ hyperbolic PDEs. The method we propose may be quite slow (although it is reliable) and simpler Euler methods could be used to reduce the computational burden. However, one must be aware that for a given set of parameters, the computation of the kernels has to be done only once. Moreover, this can be done offline.

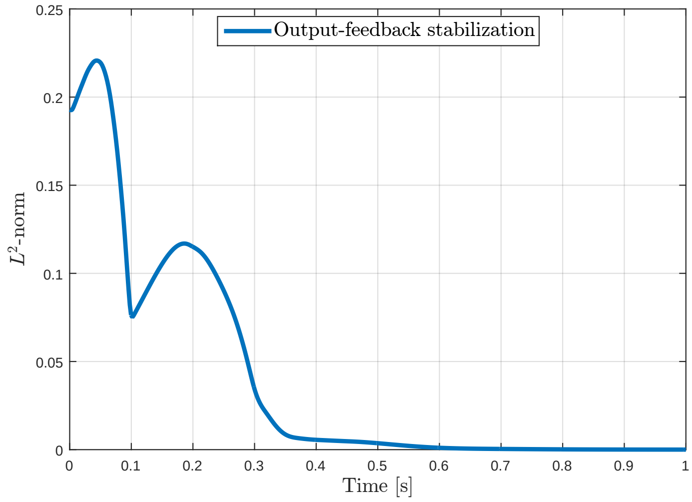

Figure 3.4: Time evolution of the L^2-norm using the output feedback law (3.61)

3.4 Comments on the observer design

In the previous section, we have introduced an observer-design method in which the observer gains are seen as controller gains for a dual problem. Although this approach is standard for finite dimensional system it is original in a backstepping framework. The proposed method skirts one of the most important difficulties of the backstepping method for observer design aka the target system design. More precisely, there are some observability problems (e.g. the two sided-boundary weak observability problem) for which the observer design methods introduced in [VKC11] (and detailed in Section 2.4.2) are hard to adjust due to the difficulty to either find a suitable target system or prove its stability. Thus, for these complex systems, designing the observer using the proposed method can be extremely useful (even more if one knows a procedure to design the corresponding controller).

Chapter 4

Two-sided boundary stabilization and observability

Chapitre 4 Stabilisation et observabilité bilatérale. *Dans ce chapitre nous résolvons les problèmes de stabilisation et d'observabilité bilatèrale pour la classe générale de systèmes d'EDPs* (2.1)-(2.3)*, tels que formulés par les Objectifs B et B'. Pour stabiliser de tels systèmes, la littérature propose en général d'utiliser des conditions aux frontières dissipatives. Cela ne permet pas de garantir la stabilisation en temps minimal et est possible uniquement pour de faibles termes de couplage entre les EDPs mais peut en général* être *réalisé par retour de sortie statique et est numériquement peu couteux. Récemment, le problème de stabilisation bilatérale a été résolu pour un système d'EDPs de réaction-diffusion et pour deux EDPs hyperboliques couplées mais ayant des vitesses de propagation identiques. Les techniques utilisées dans le chapitre précédent ne peuvent pas directement s'étendre pour ce nouveau problème car elles ne permettent pas de tirer le plein potentiel d'une action simultanée aux deux frontières de l'équation. Nous proposons donc dans ce chapitre l'emploi d'une transformation de Fredholm inversible transformant le système initial en un système cible aux propriétés idoines. À partir de cette transformation, il est aisé d'obtenir une loi de commande par retour d'état assurant la convergence exponentielle du système vers son équilibre au sens de la norme L^2. Cette loi de commande satisfait l'Objectif B. L'observateur correspondant est obtenu à l'aide de la technique de l'adjoint proposée au Chapitre 3. Des applications pour lesquelles les actionneurs et/ou les capteurs sont situés aux deux frontières sont décrites par [BC11, DMBPA14] pour des systèmes de canaux ou pour des problèmes d'estimation de paramètres survenant lors de forages pétroliers [ADMEA14].*

Contents

In this chapter we solve the problem of two-sided boundary stabilization for the general class of hyperbolic PDEs (2.1)-(2.3) as stated by Objective B and Objective B'. In this situation, the literature usually focuses on design of dissipative boundary conditions to stabilize the system. This does not guarantee stabilization in the minimum theoretical time, and is only possible for small coupling terms between PDEs, but can generally be achieved using static boundary output feedback, which is not computationally intensive. Recently the problem of stabilizing a system of two coupled PDEs with control at both sides in minimum time has been solved in [VK16] in the case of reaction-diffusion PDEs and for 2-state heterodirectional linear PDEs with equal transport velocities. As the techniques presented in the previous chapter for the case of one-sided controlled systems, based on a Volterra transformation cannot be straightforwardly extended when actuation is applied at both boundaries, we propose in this chapter to use an invertible Fredholm transformation to map the system to a *target system* with desirable stability properties. This target-system is designed as an exponentially stable cascade. The well-posedness of the Fredholm transformation is a consequence of a clever choice of the domain on which the kernels are defined and of the cascade structure of the target system. A full-state feedback law guaranteeing exponential stability of the zero equilibrium in the L^2-norm is then designed. This feedback law satisfies Objective B. The corresponding boundary observer is designed by adjusting the technique of the adjoint introduced in Chapter 3. Applications where controls and/or sensors are located at the two boundaries include control of open channel flow [BC11] and state and parameter estimation for oil drilling [DMBPA14]. In this chapter, we investigate the benefits of using sensors at both boundaries by conducting simulations on a distributed model for two-phase flow [ADMEA14]. The content of this chapter has been published in [ADM16b] for the case of two equations and in [ADM17] for the general case of a $n+m$ system .

4.1 Tutorial case of two equations

In this section we consider the tutorial case of two coupled equations ($n=m=1$). The original system (2.1)-(2.3) rewrites

$$\partial_t u(t,x) + \lambda \partial_x u(t,x) = \sigma^{+-}(x) v(t,x), \tag{4.1}$$
$$\partial_t v(t,x) - \mu \partial_x v(t,x) = \sigma^{-+}(x) u(t,x), \tag{4.2}$$

with the following linear boundary conditions

$$u(t,0) = qv(t,0) + U(t), \quad v(t,1) = \rho u(t,1) + V(t), \tag{4.3}$$

The corresponding initial condition is denoted (u_0, v_0) and belongs to $(L^2([0,1],\mathbb{R}))^2$. Note that as explained in Remark 2.4.1, the coefficients $\sigma^{++}(x)$ and $\sigma^{--}(x)$ are considered as equal to zero, without any loss of generality. The goal is to design feedback control inputs $U(t)$ and $V(t)$ such that the zero equilibrium is reached in minimum time t_F,

$$t_F = \max\{\frac{1}{\mu}, \frac{1}{\lambda}\}. \tag{4.4}$$

This “minimum time" is the time needed for the slowest characteristic to travel the entire length of the spatial domain. We do not solve in this section the observer design problem B' associated to this system (this is only done in the general case in Section 4.2). The control design is based on a modified backstepping approach: using a specific transformation, we map the system (4.1)-(4.3) to a target system with desirable properties of stability. However, unlike the classical backstepping approach where a Volterra transformation is used, we use a Fredholm transformation here.

Target system design

Without any loss of generality, we assume that $\lambda \leq \mu$. We map the system (4.1)-(4.3) to the following system

$$\partial_t \alpha(t,x) + \lambda \partial_x \alpha(t,x) = \Omega(x)\beta(t,x)\mathbb{1}_{[\frac{\lambda}{\lambda+\mu},1]}(x), \tag{4.5}$$

$$\partial_t \beta(t,x) - \mu \partial_x \beta(t,x) = 0, \tag{4.6}$$

with the following boundary conditions

$$\alpha(t,0) = 0 \quad \beta(t,1) = 0, \tag{4.7}$$

while $\Omega \in \mathcal{C}^0(0,1)$ is a function that will be defined later. This system is designed as a copy of the original dynamics, from which most of the coupling terms are removed: only couplings from the β equation to the α equation acting on the segment $[\frac{\lambda}{\lambda+\mu},1]$ are preserved. Note that as explained in Chapter 2 some of the coupling terms must be preserved in the target system to be able to prove the existence of a backstepping transformation mapping the original system to this target system. This explains the presence of the terms $\Omega(\cdot)$. This target system is pictured in Figure 4.1.

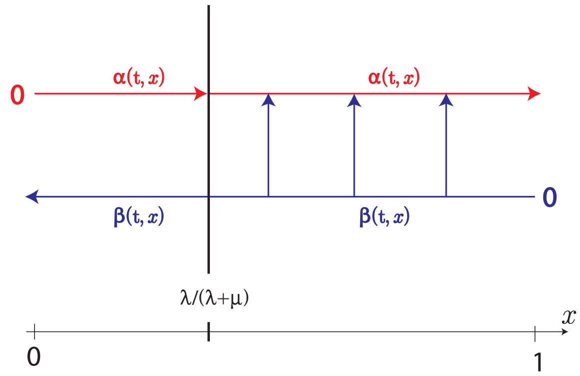

Figure 4.1: Schematic representation of the target system.

Remark 4.1.1 *One must be aware that although the coupling term* $\Omega(x)\beta(t,x)\mathbb{1}_{[\frac{\lambda}{\lambda+\mu},1]}(x)$ *is equal to zero on*$[0,\frac{\lambda}{\lambda+\mu}[$*, there is no discontinuity in the system.*

Lemma 4.1.1.

The zero equilibrium of (4.5)-(4.6) with boundary conditions (4.7) and initial conditions $(\alpha_0,\beta_0) \in L^2([0,1])$ is exponentially stable in the L^2 sense. Moreover, this equilibrium is reached in finite-time $t_F = \max\{\frac{1}{\lambda},\frac{1}{\mu}\} = \frac{1}{\lambda}$.

Proof : Using the same arguments than the ones presented in [HDMVK16, Lemma 3.1] (i.e the characteristic method), we can easily prove that for $t \geq \frac{1}{\lambda+\mu}$

$$\beta(t,x) = 0 \quad \text{if} \quad x \geq \frac{\lambda}{\lambda+\mu}, \qquad \alpha(t,x) = 0 \quad \text{if} \quad x \leq \frac{\lambda}{\lambda+\mu}. \tag{4.8}$$

Consequently, for $t \geq \frac{1}{\lambda+\mu}$, the system (4.5)-(4.6) can be rewritten

$$\partial_t \alpha(t,x) + \lambda \partial_x \alpha(t,x) = 0, \quad \partial_t \beta(t,x) - \mu \partial_x \beta(t,x) = 0, \tag{4.9}$$

with the additional conditions

$$\alpha(t, \frac{\lambda}{\lambda+\mu}) = 0 \quad \beta(t, \frac{\lambda}{\lambda+\mu}) = 0. \tag{4.10}$$

Once again, using the method of characteristics, we can prove that $\forall x \in [0,1]$, $\alpha(t,x) = 0$ for $t \geq \frac{1}{\lambda+\mu} + \frac{1-\frac{\lambda}{\lambda+\mu}}{\lambda} = \frac{1}{\lambda}$ and that $\beta(t,x) = 0$ for $t \geq \frac{\lambda}{\mu(\lambda+\mu)} + \frac{\frac{\lambda}{\lambda+\mu}}{\mu} \geq \frac{1}{\lambda}$.
Therefore (4.5)-(4.6) reaches its zero equilibrium in finite-time $t_F = \frac{1}{\lambda}$. ■

Fredholm transformation

To map the original system (4.1)-(4.3) to a target system of the form (4.5)-(4.7), we use the following transformation

$$\begin{aligned}\alpha(t,x) = u(t,x) &+ \mathbb{1}_{[0,\frac{\lambda}{\mu+\lambda}[}(x)\int_x^{-\frac{\mu}{\lambda}x+1}(K(x,\xi)u(t,\xi) + L(x,\xi)v(t,\xi))d\xi \\ &+ \mathbb{1}_{[\frac{\lambda}{\mu+\lambda},1]}(x)\int_{\frac{\lambda}{\mu}(1-x)}^{x}(M(x,\xi)u(t,\xi) + N(x,\xi)v(t,\xi))d\xi,\end{aligned} \tag{4.11}$$

$$\begin{aligned}\beta(t,x) = v(t,x) &+ \mathbb{1}_{[0,\frac{\lambda}{\mu+\lambda}[}(x)\int_x^{\frac{\lambda}{\mu}(1-x)}(\bar{K}(x,\xi)u(t,\xi) + \bar{L}(x,\xi)v(t,\xi))d\xi \\ &+ \mathbb{1}_{[\frac{\lambda}{\mu+\lambda},1]}(x)\int_{\frac{\lambda}{\mu}(1-x)}^{x}(\bar{M}(x,\xi)u(t,\xi) + \bar{N}(x,\xi)v(t,\xi))d\xi.\end{aligned} \tag{4.12}$$

We define the following triangular domains, depicted in Figure 4.2:

$$\mathcal{T}_0 = \{(x,\xi)| \quad x \in [0, \frac{\lambda}{\lambda+\mu}], \quad x \leq \xi \leq -\frac{\mu}{\lambda}x + 1\}, \tag{4.13}$$

$$\mathcal{T}_2 = \{(x,\xi)| \quad x \in [0, \frac{\lambda}{\lambda+\mu}], \quad x \leq \xi \leq \frac{\lambda}{\mu}(1-x)\}, \tag{4.14}$$

$$\tilde{\mathcal{T}}_0 = \{(x,\xi)| \quad x \in]\frac{\lambda}{\lambda+\mu}, 1], \quad \frac{\lambda}{\mu}(1-x) < \xi \leq x\}. \tag{4.15}$$

The kernels K, L are defined on $\mathcal{T}_0$, M, N are defined on $\tilde{\mathcal{T}}_0$. The kernels $\bar{K}, \bar{L}$ are defined on $\mathcal{T}_2$ and $\bar{M}, \bar{N}$ are defined on $\tilde{\mathcal{T}}_0$. They are continuous in their domains of assumed definition. They all have yet to be defined.

Remark 4.1.2 *One may think that due to the use of the characteristic-functions, the transformation presents a discontinuity in $x = \frac{\lambda}{\mu+\lambda}$. Nevertheless, one can check that the right and left limits are equal since the integral vanishes and that consequently we do not have any discontinuity. Similarly, as it appears in the computations, the derivatives do not have any discontinuity in $x = \frac{\lambda}{\mu+\lambda}$.*

Remark 4.1.3 *A careful reader could notice that the transformation* (4.11)-(4.12) *is actually a Volterra transformation. This can be proved using the folding procedure introduced for a much simpler case in [VK16]. To simplify, let us assume that $\lambda = 1$. We have*

$$\alpha(t,x) = u(t,x) + \left\{\begin{array}{ll} \int_x^{1-\mu x}(K(x,\xi)u(t,\xi) + L(x,\xi)v(t,\xi))\,d\xi & x \in [0, \frac{1}{\mu+1}[\\ \int_{\frac{1-x}{\mu}}^{x}(M(x,\xi)u(t,\xi) + N(x,\xi)v(t,\xi))\,d\xi & x \in [\frac{1}{\mu+1}, 1]. \end{array}\right.$$

Denote now

$$\alpha(t,x) = \left\{\begin{array}{ll} \alpha_1(t,x) & x \in [0, \frac{1}{\mu+1}[\\ \alpha_2(t,x) & x \in [\frac{1}{\mu+1}, 1]. \end{array}\right. \quad u(t,x) = \left\{\begin{array}{ll} u_1(t,x) & x \in [0, \frac{1}{\mu+1}[\\ u_2(t,x) & x \in [\frac{1}{\mu+1}, 1]. \end{array}\right. \quad v(t,x) = \left\{\begin{array}{ll} v_1(t,x) & x \in [0, \frac{1}{\mu+1}[\\ v_2(t,x) & x \in [\frac{1}{\mu+1}, 1]. \end{array}\right.$$

Thus,

$$\begin{aligned}\alpha_1(t,x) = u_1(t,x) + &\int_x^{\frac{1}{1+\mu}} \left(K(x,\xi)u_1(t,\xi) + L(x,\xi)v_1(t,\xi)\right) d\xi \\ &+ \int_{\frac{1}{1+\mu}}^{1-\mu x} \left(K(x,\xi)u_2(t,\xi) + L(x,\xi)v_2(t,\xi)\right) d\xi, \\ \alpha_2(t,x) = u_2(t,x) + &\int_{\frac{1-x}{\mu}}^{\frac{1}{1+\mu}} \left(M(x,\xi)u_1(t,\xi) + N(x,\xi)v_1(t,\xi)\right) d\xi \\ &+ \int_{\frac{1}{1+\mu}}^{x} \left(M(x,\xi)u_2(t,\xi) + N(x,\xi)v_2(t,\xi)\right) d\xi.\end{aligned}$$

Now redefine

$$\begin{aligned}\bar{\alpha}_1(x) &= \alpha_1\left(\frac{1-x}{1+\mu}\right), \quad \bar{u}_1(x) = u_1\left(\frac{1-x}{1+\mu}\right), \quad v_1(x) = v_1\left(\frac{1-x}{1+\mu}\right), \\ \bar{\alpha}_2(x) &= \alpha_2\left(\frac{1+\mu x}{1+\mu}\right), \quad \bar{u}_2(x) = u_2\left(\frac{1+\mu x}{1+\mu}\right), \quad v_2(x) = v_2\left(\frac{1+\mu x}{1+\mu}\right).\end{aligned}$$

Thus:

$$\begin{aligned}\bar{\alpha}_1(t,x) = \bar{u}_1(t,x) + &\int_{\frac{1-x}{1+\mu}}^{\frac{1}{1+\mu}} \left(K(\frac{1-x}{1+\mu},\xi)u_1(t,\xi) + L(\frac{1-x}{1+\mu},\xi)v_1(t,\xi)\right) d\xi \\ &+ \int_{\frac{1}{1+\mu}}^{\frac{1+\mu x}{1+\mu}} \left(K(\frac{1-x}{1+\mu},\xi)u_2(t,\xi) + L(\frac{1-x}{1+\mu},\xi)v_2(t,\xi)\right) d\xi, \\ \bar{\alpha}_2(t,x) = \bar{u}_2(t,x) + &\int_{\frac{1-x}{1+\mu}}^{\frac{1}{1+\mu}} \left(M(\frac{1+\mu x}{1+\mu},\xi)u_1(t,\xi) + N(\frac{1+\mu x}{1+\mu},\xi)v_1(t,\xi)\right) d\xi \\ &+ \int_{\frac{1}{1+\mu}}^{\frac{1+\mu x}{1+\mu}} \left(M(\frac{1+\mu x}{1+\mu},\xi)u_2(t,\xi) + N(\frac{1+\mu x}{1+\mu},\xi)v_2(t,\xi)\right) d\xi.\end{aligned}$$

Do now the change of variables $\xi = \frac{1-s}{\mu+1}$ in the first integral and $\xi = \frac{1+\mu s}{1+\mu}$ in the second. We obtain

$$\begin{aligned}\bar{\alpha}_1(t,x) = \bar{u}_1(t,x) + &\int_0^x \left(K(\frac{1-x}{1+\mu},\frac{1-s}{1+\mu})\bar{u}_1(t,\xi) + L(\frac{1-x}{1+\mu},\frac{1-s}{1+\mu})\bar{v}_1(t,\xi)\right) ds \\ &+ \mu\int_0^x \left(K(\frac{1-x}{1+\mu},\frac{1+\mu s}{1+\mu})\bar{u}_2(t,\xi) + L(\frac{1-x}{1+\mu},\frac{1+\mu s}{1+\mu})\bar{v}_2(t,\xi)\right) ds, \\ \bar{\alpha}_2(t,x) = \bar{u}_2(t,x) + &\int_0^x \left(M(\frac{1+\mu x}{1+\mu},\frac{1-s}{1+\mu})\bar{u}_1(t,\xi) + N(\frac{1+\mu x}{1+\mu},\frac{1-s}{1+\mu})\bar{v}_1(t,\xi)\right) ds \\ &+ \mu\int_0^x \left(M(\frac{1+\mu x}{1+\mu},\frac{1+\mu s}{1+\mu})\bar{u}_2(t,\xi) + N(\frac{1+\mu x}{1+\mu},\frac{1+\mu s}{1+\mu})\bar{v}_2(t,\xi)\right) ds.\end{aligned}$$

which is an clearly a Volterra form (after some re-definition of the kernels). Consequently, all the properties of Volterra transformations still hold (in particular its invertibility).

We now differentiate the Fredholm transformation (4.11)-(4.12) with respect to time and space to compute the equations satisfied by the kernels. We start with the β-transformation (4.12). If $x \geq \frac{\lambda}{\mu+\lambda}$: differentiating (4.12) with respect to space and using the Leibniz rule yields

$$\begin{aligned}\partial_x\beta(t,x) =& \partial_x v(t,x) + \bar{M}(x,x)u(t,x) + \bar{N}(x,x)v(t,x) + \frac{\lambda}{\mu}\bar{M}(x,\frac{\lambda}{\mu}(1-x))u(t,\frac{\lambda}{\mu}(1-x)) \\ &+ \frac{\lambda}{\mu}\bar{N}(x,\frac{\lambda}{\mu}(1-x))v(t,\frac{\lambda}{\mu}(1-x)) + \int_{\frac{\lambda}{\mu}(1-x)}^x \partial_x\bar{M}(x,\xi)u(t,\xi) + \partial_x\bar{N}(x,\xi)v(t,\xi)d\xi.\end{aligned}$$

Differentiating (4.12) with respect to time, using (4.1), (4.2) and integrating by parts yields

$$\partial_t\beta(t,x) = \mu\partial_x v(t,x) + \sigma^{-+}u(t,x) + \mu\bar{N}(x,x)v(t,x) - \mu\bar{N}(x,\frac{\lambda}{\mu}(1-x))v(t,\frac{\lambda}{\mu}(1-x))$$
$$-\lambda\bar{M}(x,x)u(t,x) + \lambda\bar{M}(x,\frac{\lambda}{\mu}(1-x))u(t,\frac{\lambda}{\mu}(1-x)) + \int_{\frac{\lambda}{\mu}(1-x)}^{x}\lambda\partial_\xi\bar{M}(x,\xi)u(t,\xi)d\xi$$
$$+\int_{\frac{\lambda}{\mu}(1-x)}^{x} -\mu\partial_\xi\bar{N}(x,\xi)v(t,\xi) + \sigma^{-+}(\xi)\bar{N}(x,\xi)u(t,\xi) + \sigma^{+-}(\xi)\bar{M}(x,\xi)v(t,\xi)d\xi.$$

Plugging these expressions into the target system (4.5)-(4.6) yields the following system of kernel equations

$$0 = -\mu\partial_x\bar{M}(x,\xi) + \lambda\partial_\xi\bar{M}(x,\xi) + \sigma^{-+}(\xi)\bar{N}(x,\xi), \tag{4.16}$$
$$0 = -\mu\partial_x\bar{N}(x,\xi) - \mu\partial_\xi\bar{N}(x,\xi) + \sigma^{+-}(\xi)\bar{M}(x,\xi), \tag{4.17}$$
$$0 = \bar{M}(x,x) - \frac{\sigma^{-+}}{\lambda+\mu}, \quad 0 = \bar{N}(x,\frac{\lambda}{\mu}(1-x)). \tag{4.18}$$

If $x < \frac{\lambda}{\mu+\lambda}$: similarly we get

$$0 = -\mu\partial_x\bar{K}(x,\xi) + \lambda\partial_\xi\bar{K}(x,\xi) + \sigma^{-+}(\xi)\bar{L}(x,\xi), \tag{4.19}$$
$$0 = -\mu\partial_x\bar{L}(x,\xi) - \mu\partial_\xi\bar{L}(x,\xi) + \sigma^{+-}(\xi)\bar{K}(x,\xi), \tag{4.20}$$
$$0 = \bar{K}(x,x) + \frac{\sigma^{-+}(x)}{\lambda+\mu}, \quad 0 = \bar{L}(x,\frac{\lambda}{\mu}(1-x)). \tag{4.21}$$

The corresponding domains, characteristic lines and boundary conditions in Figure 4.2. We now focus on the α-transformation.
If $x \leq \frac{\lambda}{\mu+\lambda}$: as above, differentiating (4.11) with respect to space and time and then plugging into the target system (4.5)-(4.6) yields the following system of kernel equations

$$0 = \lambda\partial_x L(x,\xi) - \mu\partial_\xi L(x,\xi) + \sigma^{+-}(\xi)K(x,\xi), \tag{4.22}$$
$$0 = \lambda\partial_x K(x,\xi) + \lambda\partial_\xi K(x,\xi) + \sigma^{-+}(\xi)L(x,\xi), \tag{4.23}$$
$$0 = L(x,x) - \frac{\sigma^{+-}(x)}{\lambda+\mu}, \quad 0 = K(x,\frac{\mu}{\lambda}(1-x)). \tag{4.24}$$

If $x > \frac{\lambda}{\mu+\lambda}$: similarly we get

$$\begin{aligned}0 =& \lambda\partial_x M(x,\xi) + \lambda\partial_\xi M(x,\xi) + \sigma^{-+}(\xi)N(x,\xi) - (\lambda+\mu)\bar{M}(x,\xi)N(x,x)\\ &- \sigma^{+-}(\xi)\bar{M}(x,\xi),\end{aligned} \tag{4.25}$$
$$\begin{aligned}0 =& \lambda\partial_x N(x,\xi) - \mu\partial_\xi N(x,\xi) + \sigma^{+-}(\xi)M(x,\xi) - (\lambda+\mu)\bar{N}(x,\xi)N(x,x)\\ &- \sigma^{+-}(\xi)\bar{N}(x,\xi),\end{aligned} \tag{4.26}$$
$$0 = N(x,\frac{\lambda}{\mu}(1-x)), \quad 0 = M(x,\frac{\lambda}{\mu}(1-x)). \tag{4.27}$$

To have a well-posed system, we add the artificial boundary condition $N(1,\xi) = 0$. The function $\Omega(x)$ is defined by

$$\Omega(x) = \sigma^{+-}(x) + (\mu+\lambda)N(x,x). \tag{4.28}$$

The corresponding domains, characteristic lines and boundary conditions are pictured in Figure 4.2.

Remark 4.1.4 *The artificial boundary condition we add for the kernel N is not a degree of freedom since it has no impact on the control law and on the stability of the target system. It was just chosen to be equal to zero for convenience. However, it could obviously be defined in a different way.*

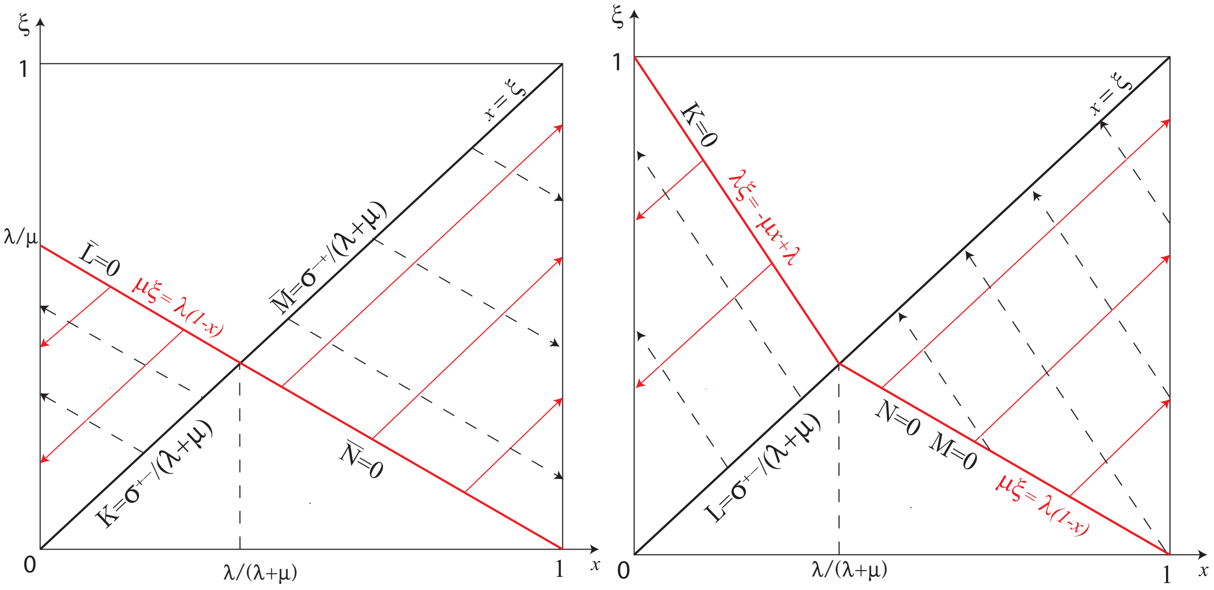

Figure 4.2: Representation of the β-kernels and of the α-kernels

Well-posedness of the kernel equations

Theorem 4.1.1 *Consider the systems* (4.16)-(4.18), (4.19)-(4.21), (4.22)-(4.24), (4.25)-(4.28). *There exists a unique solution K, L (defined on $L^\infty(\mathcal{T})$), M, N (defined on $L^\infty(\tilde{\mathcal{T}}_0)$), $\bar{K}, \bar{L}$ (defined on $L^\infty(\mathcal{T}_2)$), $\bar{M}, \bar{N}$ (defined on $L^\infty(\tilde{\mathcal{T}}_0)$).*

Proof : The proof uses the same tools as the one of Theorem 2.4.1. We start to prove the result for the systems (4.16)-(4.18), (4.19)-(4.21),(4.22)-(4.24) and finish with the system (4.25)-(4.28) since for this last one we need to use the fact that $\bar{M}(x, \xi)$ and $\bar{N}(x, \xi)$ are bounded. ■

Invertibility of the Fredholm transformation

Unlike the Volterra transformation (2.37)-(2.38), the Fredholm transformation (4.11)-(4.12) is not always invertible. However, it has been proved in Remark 4.1.3 that the transformation (4.11)-(4.12) is actually a Volterra transformation. Thus, its invertibility is granted.

Finite-time boundary stabilization

We now state the main stabilization result. The resulting control laws fulfill Objective B.

Theorem 4.1.2.

Consider system (4.1)-(4.3) and the following feedback control laws

$$U(t) = -qu(t,0) - \int_0^1 (K(0,\xi)u(t,\xi) + L(0,\xi)v(t,\xi))d\xi, \tag{4.29}$$

$$V(t) = -\rho v(t,1) - \int_0^1 (\bar{M}(1,\xi)u(t,\xi) + \bar{N}(1,\xi)v(t,\xi))d\xi, \tag{4.30}$$

where K, L and $\bar{M}, \bar{N}$ are defined by (4.22)-(4.24) and (4.16)-(4.18). Then, for any initial condition $(u_0, v_0) \in (L^2([0,1]))^2$, it reaches its zero equilibrium in the minimum finite time $t_F = \frac{1}{\lambda_1} + \frac{1}{\mu_1}$. Moreover, if the initial conditions belong to $(\mathcal{C}^1([0,1]))^2$ (and satisfy the corresponding compatibility conditions), the control laws (4.29)-(4.30) ensure the weak exact boundary controllability in the sense of Theorem (2.3.1).

Proof : Notice that evaluating (4.11) at $x = 0$ yields (4.29) and evaluating (4.12) at $x = 1$ yields (4.30). Applying Lemma 4.1.1 and using the invertibility of the transformation (4.11)-(4.12) imply that (α, β) go to zero in finite time t_F, therefore (u, v) converge to zero in finite time t_F ■

The purpose of this example was to introduce the new class of target system used to solve Objective B. Similar ideas based on the proposed partial decoupling can be used in the general case. The question of the observer design is not considered in this tutorial example since, using the adjoint approach introduced in Chapter 3 it can easily be solved for the general case.

4.2 Two-sided finite time stabilization of a $(n + m)$ system

We now consider the general system (2.1)-(2.3) whose equations are rewritten bellow:

$$\partial_t u(t, x) + \Lambda^+ \partial_x u(t, x) = \Sigma^{++}(x)u(t, x) + \Sigma^{+-}(x)v(t, x), \tag{4.31}$$

$$\partial_t v(t, x) - \Lambda^- \partial_x v(t, x) = \Sigma^{-+}(x)u(t, x) + \Sigma^{--}(x)v(t, x), \tag{4.32}$$

along with the linear boundary conditions

$$u(t, 0) = U(t), \quad v(t, 1) = V(t). \tag{4.33}$$

The control problem stated in Objective B consists in designing feedback control inputs $U(t) = (U_1(t), \ldots, U_n(t))^T$ and $V(t) = (V_1(t), \ldots, V_m(t))^T$ such that the zero equilibrium is reached in minimum time $t = t_F$, where

$$t_F = \max\left\{\frac{1}{\mu_1}, \frac{1}{\lambda_1}\right\}, \tag{4.34}$$

where we recall that the matrices Λ^+ and Λ^- are diagonal matrices defined for all $1 \leq i \leq n$, for all $1 \leq j \leq m$ by $\Lambda^+_{ii} = \lambda_i$ and $\Lambda^-_{jj} = \mu_j$ whose eigenvalues satisfy

$$-\mu_m < \cdots < -\mu_1 < 0 < \lambda_1 < \cdots < \lambda_n.$$

The proposed approach is similar to one presented above for two equations and consists in introducing an invertible Fredholm transformation that maps the system to a target system with desirable properties of stability. In the following we denote

$$\forall\ i \in [1, n],\ j \in [1, m] \quad a_{ij} = \frac{\lambda_i}{\lambda_i + \mu_j}. \tag{4.35}$$

It is straightforward to show that for $i \in [1, n],\ j \in [1, m]$

$$\forall\ k \geq i\ a_{ij} \leq a_{kj}, \quad \forall\ l \leq j\ a_{ij} \leq a_{il}. \tag{4.36}$$

We also define $\bar{a}$ such that

$$|\bar{a} - \frac{1}{2}| = \min_{1 \leq i \leq n, 1 \leq j \leq m} |a_{ij} - \frac{1}{2}|. \tag{4.37}$$

The a_{ij} and $\bar{a}$ play an important role in the design of the target system. The parameter $\bar{a}$ can be written as a_{kl} with $k \in [0, n]$ and $l \in [0, m]$ (the uniqueness is not guaranteed). For this particular choice we denote $\bar{\lambda}$ and $\bar{\mu}$ the corresponding transport velocities.

Remark 4.2.1 *For sake of simplicity, we have removed the terms $Q_0 v(t, 0)$ and $R_1 u(t, 1)$ from* (4.33) *since these terms can be directly compensating in the corresponding control law (regarding output feedback stabilization, they correspond to known quantities (see Section 4.3 for details)).*

Remark 4.2.2 *Without any loss of generality we can assume that $\bar{a} \geq \frac{1}{2}$ (if this is not the case we make the change of variables $\bar{x} = 1 - x$).*

Similarly to the tutorial case of two equations of Section 4.1, we use a Fredholm transformation to map the system (4.31)-(4.33) to a target system with desirable properties of stability.

4.2.1 Target system design

We map the system (4.31)-(4.33) to the following system

$$\alpha_t(t,x) + \Lambda^+ \alpha_x(t,x) = \Omega(x)\alpha(t,x) + \Gamma(x)\beta(t,x), \tag{4.38}$$
$$\beta_t(t,x) - \Lambda^- \beta_x(t,x) = \bar{\Omega}(x)\beta(t,x) + \bar{\Gamma}(x)\alpha(t,x), \tag{4.39}$$

with the following boundary conditions

$$\alpha(t,0) = 0 \quad \beta(t,1) = 0, \tag{4.40}$$

while Ω and $\bar{\Omega} \in L^\infty(0,1)$ are upper triangular matrices with the following structure

$$\Omega(x) = \begin{pmatrix} \omega_{1,1}(x) & \omega_{1,2}(x) & \dots & \omega_{1,n}(x) \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \omega_{n-1,n-1}(x) & \omega_{n-1,n}(x) \\ 0 & \dots & 0 & \omega_{n,n}(x) \end{pmatrix},$$
$$\bar{\Omega}(x) = \begin{pmatrix} \bar{\omega}_{1,1}(x) & \bar{\omega}_{1,2}(x) & \dots & \bar{\omega}_{1,m}(x) \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \bar{\omega}_{m-1,m-1}(x) & \bar{\omega}_{m-1,m}(x) \\ 0 & \dots & 0 & \bar{\omega}_{m,m}(x) \end{pmatrix}.$$

The coefficients of the matrices $\Gamma(x)$ and $\bar{\Gamma}(x)$ are defined by

$$\forall 1 \le i \le n \quad \forall 1 \le j \le m, \quad \Gamma_{ij}(x) = \begin{cases} 0 & \text{if} \quad a_{ij} \ge \bar{a} \quad \text{or} \quad x < a_{ij}, \\ \gamma_{ij}(x) & \text{otherwise.} \end{cases}$$
$$\forall 1 \le i \le m \quad \forall 1 \le j \le n, \quad \bar{\Gamma}_{ij}(x) = \begin{cases} 0 & \text{if} \quad a_{ji} < 1-\bar{a} \quad \text{or} \quad x > a_{ji}, \\ \bar{\gamma}_{ij}(x) & \text{otherwise.} \end{cases}$$

Remark 4.2.3 *The a_{ij} coefficients correspond to the spatial position where the characteristic leaving $x=0$ with velocity λ_i and the one leaving $x=1$ with velocity μ_j intersect.*

Remark 4.2.4 *As it will appear in the proof, the matrices Γ and $\bar{\Gamma}$ have no destabilizing effect due to their particular cascade structure. Their presence is necessary to prove the well-posedness of the backstepping transformation presented below. The following example illustrates this particular structure in a simple case.*

Example 4.2.1 We consider the following coefficients:

$$n=3,\ m=2,\ \Lambda^+ = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},\ \Lambda^- = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}.$$

We define the matrix A such that $A_{ij} = a_{ij} = \frac{\lambda_i}{\lambda_i + \mu_j}$:

$$A = \begin{pmatrix} \frac{1}{7} & \frac{1}{11} \\ \frac{2}{5} & \frac{2}{7} \\ \frac{4}{7} & \frac{4}{9} \end{pmatrix}^T.$$

One obtains $\bar{a} = a_{32} = \frac{4}{9}$. Consequently the matrices Γ and $\bar{\Gamma}$ have the following structure

$$\Gamma(x) = \begin{pmatrix} * & * \\ * & * \\ 0 & 0 \end{pmatrix} \quad \bar{\Gamma}(x) = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \end{pmatrix},$$

where the potential non-zero terms are represented by $*$. We can notice that the last line of Γ is equal to zero. Moreover, if $\Gamma(x)_{ij} = 0$ then, $\forall k > i, \Gamma(x)_{kj} = 0$ and $\forall p < j, \Gamma(x)_{ip} = 0$. The same holds for $\bar{\Gamma}$. These structural properties still hold in the general case (under a more general form). They are deeply analyzed in Section 4.2.1.

Besides, the following lemma assesses the finite-time convergence of the target system.

Lemma 4.2.1.

The system (4.38)-(4.39) reaches its zero equilibrium in finite-time $t_F = \max\{\frac{1}{\lambda_1}, \frac{1}{\mu_1}\} = \frac{1}{\lambda_1}$.

Proof : $\forall 1 \leq i \leq n$, $\forall 1 \leq j \leq n$, system (4.38)-(4.39) can be rewritten as

$$\alpha_t^i(t,x) + \lambda_i \alpha_x^i(t,x) = \sum_{p=i}^{n} \omega_{ip}(x)\alpha^p(t,x) + \sum_{p=1}^{m} \gamma_{ip}(x)\mathbb{1}_{[a_{ip},1]}(x)\beta^p(t,x), \tag{4.41}$$

$$\beta_t^j(t,x) - \mu_j \beta_x^j(t,x) = \sum_{p=j}^{m} \bar{\omega}_{jp}(x)\beta^p(t,x) + \sum_{p=1}^{n} \bar{\gamma}_{jp}(x)\mathbb{1}_{[0,a_{pj}]}(x)\alpha^p(t,x), \tag{4.42}$$

with

$$\begin{aligned} \gamma_{ij}(x) = 0 \quad \text{if} \quad a_{ij} > \bar{a}, \\ \bar{\gamma}_{ij}(x) = 0 \quad \text{if} \quad a_{ji} \leq 1 - \bar{a}. \end{aligned}$$

By induction, let us consider the following property $P(s)$ defined for all $1 \leq s \leq n$

$$P(s): \quad \forall p \geq n+1-s, \quad \text{if} \quad t \geq \frac{x}{\lambda_p} \quad \text{then} \quad \alpha^p(t,x) = 0.$$

The proof of this property follows three steps:

- Step (A). First, for the considered component α^r of the state α, we give the integral formulation of equation (4.41) using the method of characteristics.
- Step (B). Then, we prove that due to the induction hypothesis and due to the triangular structure of the matrix Ω, after some time, this integral equation only depends on α^r itself and on the function β.
- Step (C). We prove by induction that due to the presence of the characteristic functions, the β-terms cancel after some time.
- Step (D). We prove that this time corresponds to the time stated in the induction hypothesis.

Initialization: The initialization can be proved using a similar technique than the one presented below in the induction and is not detailed here.

Induction: Let us assume that the property $P(s-1)$ $(1 < s \leq n)$ is true. We denote $r = n+1-s$.
Step (A): integrating the r^{th} line of (4.41) along its characteristic lines and using the boundary condition $\alpha^r(t,0) = 0$, yields:

$$\begin{aligned} \alpha^r(t,x) = \int_0^{\frac{x}{\lambda_r}} & \sum_{p=r}^{n} \omega_{rp}(x-\lambda_r\nu)\alpha^p(t-\nu, x-\lambda_r\nu) \\ & + \sum_{p=1}^{m} \gamma_{rp}(x-\lambda_r\nu)\mathbb{1}_{[a_{rp},1]}(x-\lambda_r\nu)\beta^p(t-\nu, x-\lambda_r\nu)d\nu, \end{aligned} \tag{4.43}$$

with $x \in [0,1]$ and $t \geq \frac{x}{\lambda_r}$.
Step (B): consequently, $\forall p > r$:

$$
\begin{aligned}
t \geq \frac{x}{\lambda_r} \Rightarrow (1-\frac{\lambda_r}{\lambda_p})\frac{x}{\lambda_r} \leq t - \frac{x}{\lambda_p} \Rightarrow (1-\frac{\lambda_r}{\lambda_p})\nu \leq t - \frac{x}{\lambda_p} \quad \forall \nu \in [0, \frac{x}{\lambda_r}] \\
\Rightarrow t - \nu \geq \frac{x - \lambda_r \nu}{\lambda_p} \Rightarrow \alpha^p(t-\nu, x - \lambda_r \nu) = 0.
\end{aligned} \tag{4.44}
$$

The last implication uses the fact that $P(s-1)$ is true (this step is useless considering the initialization due to the triangular structure of the matrix Ω).
Step (C): let us now consider the following property $P_1(q)$ defined for all $1 \leq q \leq m$ by

$$P_1(q) : \forall x > a_{rq}, \quad \forall t \geq \frac{1-x}{\mu_q}, \quad \beta^q(t,x) = 0.$$

Doing the recursion, the proof follows three steps

- •Step (A'). For the considered component β^q of β, we give the integral formulation of equation (4.42) using the method of characteristics.
- •Step (B'). Then, we prove that, due to the hypothesis $P(s)$, after some time, the terms in α present in this equation vanish.
- •Step (C'). Finally, using the triangular structure of the matrix $\bar{\Omega}$ and the induction hypothesis, we prove the property P_1.

Initialization: The initialization can be proved using a similar technique than the one presented below in the induction and is not detailed here.

Induction: Let us assume that the property $P_1(q-1)$ $(1 < q \leq m)$ is true.
Step (A'): integrating the q^{th} line of (4.42) along its characteristic lines and using the boundary condition $\beta_q(t,1) = 0$, yields:

$$
\begin{aligned}
\beta^q(t,x) = \int_0^{\frac{1-x}{\mu_q}} \sum_{p=q}^{n} \bar{\omega}_{qp}(\mu_q\nu + x)\beta^p(t-\nu, \mu_q\nu + x) \\
+ \sum_{p>r}^{n} \bar{\gamma}_{qp}(\mu_q\nu + x)\mathbb{1}_{[0,a_{pq}]}(\mu_q\nu + x)\alpha^p(t-\nu, \mu_q\nu + x)d\nu,
\end{aligned} \tag{4.45}
$$

with $1 \geq x > a_{rq}$, and $t \geq \frac{1-x}{\mu_q}$ (this explains why the last sum starts at $p > r$).
Step (B'): consequently, $\forall p \geq q$ and $\forall \nu \in [0, \frac{1-x}{\mu_q}]$ such that $\mu_q\nu + x \leq a_{pq}$ (in order to have $\mathbb{1}_{[0,a_{pq}]}(\mu_q\nu + x) \neq 0$):

$$
\begin{aligned}
t \geq \frac{1-x}{\mu_q} \Rightarrow & t - \frac{x}{\lambda_p} \geq (\frac{\lambda_p}{\mu_q(\lambda_p + \mu_q)} - \frac{x}{\mu_q})(1 + \frac{\mu_q}{\lambda_p}) \Rightarrow t - \frac{x}{\lambda_p} \geq (\frac{a_{pq}}{\mu_q} - \frac{x}{\mu_q})(1 + \frac{\mu_q}{\lambda_p}) \\
\Rightarrow & t - \frac{x}{\lambda_p} \geq (1 + \frac{\mu_q}{\lambda_p})\nu \Rightarrow t - \nu \geq \frac{x + \mu_q\nu}{\lambda_p}.
\end{aligned}
$$

Consequently, using the fact that $P(s-1)$ holds, one obtains that $\forall \nu \in [0, \frac{1-x}{\mu_q}]$ such that $\mu_q\nu + x \leq a_{pq}$

$$\alpha^p(t-\nu, \mu_q\nu + x) = 0.$$

Consequently the second sum in (4.45) is always zero for $t \geq \frac{1-x}{\mu_q}$.
Step (C'): moreover, using the fact that $P_1(q-1)$ is true, we can simplify the first sum removing most of the terms (in the case of the initialization, this is a direct result of the triangular structure of the matrix $\bar{\Omega}$. We can rewrite (4.45) as

$$\beta^q(t,x) = \int_0^{\frac{1-x}{\mu_q}} \bar{\omega}_{qq}(\mu_q\nu + x)\beta^q(t-\nu, \mu_q\nu + x), d\nu$$

with $1 \geq x > a_{rq}$, and $t \geq \frac{1-x}{\mu_q}$. Consequently $\beta^q(t,x) = 0$ and $P_1(q)$ is true. This achieves the proof of $P_1(q)$ for all $1 \leq q \leq m$.
Step (D): for a given p we now focus on the term $\kappa = \gamma_{rp}(x - \lambda_r\nu)\mathbb{1}_{[a_{rp},1]}(x - \lambda_r\nu)\beta^p(t-\nu, x - \lambda_r\nu)$ of (4.43). This term is not equal to zero only if

$$x - \lambda_r\nu \geq a_{rp} \quad \forall \nu \in [0, \frac{x}{\lambda_r}].$$

This yields

$$\frac{\mu_q + \lambda_r}{\lambda_r} x - (\mu_q + \lambda_r)\nu \geq 1 \Leftrightarrow \frac{\mu_q}{\lambda_r} x + x - 1 \geq (\mu_q + \lambda_r)\nu.$$

Since $t \geq \frac{x}{\lambda_r}$, one obtains

$$\mu_q t + x - 1 \geq (\mu_q + \lambda_r)\nu \Rightarrow \mu_q(t - \nu) \geq 1 - x + \lambda_r \nu.$$

Using P_1, we can deduce that κ is always equal to zero. Consequently, combining this result with (4.44), we can rewrite (4.43) as

$$\alpha^s(t,x) = \int_0^{\frac{x}{\lambda_r}} \omega_{rr}(x - \lambda_r \nu)\alpha^r(t - \nu, x - \lambda_r \nu) d\nu,$$

with $x \in [0,1]$ and $t \geq \frac{x}{\lambda_r}$. Consequently, this yields

$$\forall x \in [0,1], \quad t \geq \frac{x}{\lambda_p} \Rightarrow \alpha^p(x,t) = 0. \tag{4.46}$$

It achieves the recursion. It is then quite straightforward to prove a similar result for β. Consequently we have

$$\forall t \geq \frac{1}{\lambda_1}, \quad \alpha(x,t) = 0, \quad \forall t \geq \frac{1}{\mu_1}, \quad \beta(x,t) = 0.$$

This concludes the proof. ■

4.2.2 Fredholm transformation

Definition of the transformation

To map the original system (4.31)-(4.33) to the target system (4.38)-(4.40), we use the following Fredholm transformation

$$\alpha(t,x) = u(t,x) - \int_0^1 Q_{11}(x,\xi)u(t,\xi) + Q_{12}(x,\xi)v(t,\xi)d\xi, \tag{4.47}$$

$$\beta(t,x) = v(t,x) - \int_0^1 Q_{21}(x,\xi)u(t,\xi) + Q_{22}(x,\xi)v(t,\xi)d\xi, \tag{4.48}$$

with

$$\begin{aligned}
Q_{11}(x,\xi) &= -K(x,\xi)\mathbb{1}_{[x,-\frac{\bar{\mu}}{\bar{\lambda}}x+1]}(\xi)\mathbb{1}_{[0,\bar{a}]}(x) - M(x,\xi)\mathbb{1}_{[\frac{\bar{\lambda}}{\bar{\mu}}(1-x),x]}(\xi)\mathbb{1}_{]\bar{a},1]}(x), \\
Q_{12}(x,\xi) &= -L(x,\xi)\mathbb{1}_{[x,-\frac{\bar{\mu}}{\bar{\lambda}}x+1]}(\xi)\mathbb{1}_{[0,\bar{a}]}(x) - N(x,\xi)\mathbb{1}_{[\frac{\bar{\lambda}}{\bar{\mu}}(1-x),x]}(\xi)\mathbb{1}_{]\bar{a},1]}(x), \\
Q_{21}(x,\xi) &= -\bar{K}(x,\xi)\mathbb{1}_{[x,-\frac{\bar{\mu}}{\bar{\lambda}}x+1]}(\xi)\mathbb{1}_{[0,\bar{a}]}(x) - \bar{M}(x,\xi)\mathbb{1}_{[\frac{\bar{\lambda}}{\bar{\mu}}(1-x),x]}(\xi)\mathbb{1}_{]\bar{a},1]}(x), \\
Q_{22}(x,\xi) &= -\bar{L}(x,\xi)\mathbb{1}_{[x,-\frac{\bar{\mu}}{\bar{\lambda}}x+1]}(\xi)\mathbb{1}_{[0,\bar{a}]}(x) - \bar{N}(x,\xi)\mathbb{1}_{[\frac{\bar{\lambda}}{\bar{\mu}}(1-x),x]}(\xi)\mathbb{1}_{]\bar{a},1]}(x).
\end{aligned}$$

We define the following triangular domains (which are the analogous to the ones defined in Section 4.1), depicted in Figure 4.3:

$$\mathcal{T}_0 = \{(x,\xi)| \quad x \in [0,\bar{a}], \quad x \leq \xi \leq -\frac{\bar{\mu}}{\bar{\lambda}}x + 1\},$$

$$\tilde{\mathcal{T}}_0 = \{(x,\xi)| \quad x \in [\bar{a},1], \quad \frac{\bar{\lambda}}{\bar{\mu}}(1-x) < \xi \leq x\}.$$

The kernels K, L, $\bar{K}$ and $\bar{L}$ are defined on $\mathcal{T}_0$. The kernels $M, N, \bar{M}$ and $\bar{N}$ are defined on $\tilde{\mathcal{T}}_0$. They all have yet to be defined.

Remark 4.2.5 *One may think that due to the use of the h-functions, the transformation presents a discontinuity in $x = \bar{a}$. Nevertheless, one can check that the right and left limits are equal since the integrals vanish.*

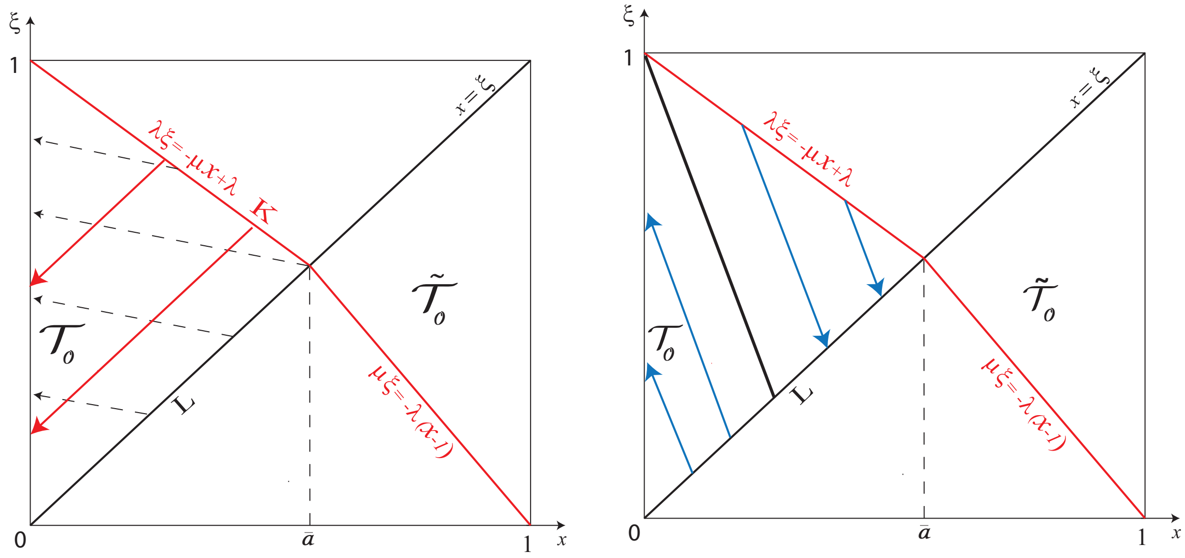

Figure 4.3: Representation of the K-kernels and of the L-kernels ($a_{ij} < \bar{a}$)

Remark 4.2.6 *The fact that one must consider this complex transformation rather than the simple kind of Voltera transformation introduced in the previous chapter to stabilize the system* (4.31)-(4.33) *in time* t_F *is connected to the fact that one* ***must*** *use the two controls at each boundary to reach this minimum stabilization time (see [LR10]). However, as seen in the case of two equations, the transformation* (4.47)-(4.48) *is actually a Volterra transformation. This can be proved using the same techniques as the ones proposed in Remark 4.1.3. However this is extremely technical. For simplicity reasons we choose in what follow to consider the transformation* (4.47)-(4.48) *as a Fredholm transformation rather than a Volterra transformation. Thus, the invertibility of such a transformation still remains to be proved as it is not always granted for a general Fredholm transformation.*

Kernel equations

We now differentiate the Fredholm transformation (4.47)-(4.48) with respect to time and space to determine the equations satisfied by the kernels. We start with the α-transformation (4.47).

<u>If $x \leq \bar{a}$</u>: differentiating (4.47) with respect to space and using the Leibniz rule yields

$$\begin{aligned}\partial_x\alpha(t,x) &= \partial_x u(t,x) - K(x,x)u(t,x) - L(x,x)v(t,x) - \frac{\bar{\mu}}{\bar{\lambda}}K(x,-\frac{\bar{\mu}}{\bar{\lambda}}x+1)u(-\frac{\bar{\mu}}{\bar{\lambda}}x+1)\\ &\quad - \frac{\bar{\mu}}{\bar{\lambda}}L(x,-\frac{\bar{\mu}}{\bar{\lambda}}x+1)v(-\frac{\bar{\mu}}{\bar{\lambda}}x+1) + \int_x^{-\frac{\bar{\mu}}{\bar{\lambda}}x+1}(\partial_x K(x,\xi)u(t,\xi) + \partial_x L(x,\xi)v(t,\xi))d\xi.\end{aligned}$$

Differentiating (4.47) with respect to time, using (4.31), (4.32) and integrating by parts, one obtains

$$\begin{aligned}\partial_t\alpha(t,x) &= -\Lambda^+\partial_x u(t,x) + \Sigma^{++}u(t,x) + \Sigma^{+-}v(t,x) + K(x,x)\Lambda^+u(t,x) - L(x,x)\Lambda^-v(t,x)\\ &\quad - K(x,-\frac{\bar{\mu}}{\bar{\lambda}}x+1)\Lambda^+u(t,-\frac{\bar{\mu}}{\bar{\lambda}}x+1) + L(x,-\frac{\bar{\mu}}{\bar{\lambda}}x+1)\Lambda^-v(t,-\frac{\bar{\mu}}{\bar{\lambda}}x+1)\\ &\quad + \int_x^{-\frac{\bar{\mu}}{\bar{\lambda}}x+1}(\partial_\xi K(x,\xi)\Lambda^+u(t,\xi) - \partial_\xi L(x,\xi)\Lambda^-v(t,\xi) + K(x,\xi)\Sigma^{++}u(t,\xi)\\ &\quad + \int_x^{-\frac{\bar{\mu}}{\bar{\lambda}}x+1}(K(x,\xi)\Sigma^{+-}(\xi)v(t,\xi))d\xi + L(x,\xi)\Sigma^{-+}(\xi)u(t,\xi) + L(x,\xi)\Sigma^{--}v(t,\xi))d\xi.\end{aligned}$$

Plugging these expressions into the target system (4.38)-(4.39) yields the following system of kernel equations

$$\begin{aligned}\Lambda^+\partial_x K(x,\xi)+\partial_\xi K_\xi(x,\xi)\Lambda^+ =& -K(x,\xi)\Sigma^{++}(\xi)-L(x,\xi)\Sigma^{-+}(\xi)\\ &+\Omega(x)K(x,\xi)+\Gamma(x)\bar{K}(x,\xi), \qquad (4.49)\\ \Lambda^+\partial_x L(x,\xi)-\partial_\xi L(x,\xi)\Lambda^- =& -K(x,\xi)\Sigma^{+-}(\xi)-L(x,\xi)\Sigma^{--}(\xi)\\ &+\Omega(x)L(x,\xi)+\Gamma(x)\bar{L}(x,\xi), \qquad (4.50)\end{aligned}$$

along with

$$\Lambda^+K(x,x)-K(x,x)\Lambda^++\Omega(x)=\Sigma^{++}(x), \qquad (4.51)$$
$$\Lambda^+L(x,x)+L(x,x)\Lambda^-+\Gamma(x)=\Sigma^{+-}(x), \qquad (4.52)$$
$$\frac{\bar\mu}{\bar\lambda}\Lambda^+L(x,-\frac{\bar\mu}{\bar\lambda}x+1)-L(x,-\frac{\bar\mu}{\bar\lambda}x+1)\Lambda^-=0, \qquad (4.53)$$
$$\frac{\bar\mu}{\bar\lambda}\Lambda^+K(x,-\frac{\bar\mu}{\bar\lambda}x+1)+K(x,-\frac{\bar\mu}{\bar\lambda}x+1)\Lambda^+=0. \qquad (4.54)$$

<u>If $x>\bar a$</u>: similarly we get

$$\begin{aligned}\Lambda^+\partial_x M(x,\xi)+\partial_\xi M(x,\xi)\Lambda^+ =& -M(x,\xi)\Sigma^{++}(\xi)-N(x,\xi)\Sigma^{-+}(\xi)\\ &+\Omega(x)M(x,\xi)+\Gamma(x)\bar{M}(x,\xi), \qquad (4.55)\\ \Lambda^+\partial_x N(x,\xi)-\partial_\xi N(x,\xi)\Lambda^- =& -M(x,\xi)\Sigma^{+-}(\xi)-N(x,\xi)\Sigma^{--}(\xi)\\ &+\Omega(x)N(x,\xi)+\Gamma(x)\bar{N}(x,\xi), \qquad (4.56)\end{aligned}$$

along with

$$\Lambda^+M(x,x)-M(x,x)\Lambda^++\Omega(x)=-\Sigma^{++}(x), \qquad (4.57)$$
$$\Lambda^+N(x,x)+N(x,x)\Lambda^--\Gamma(x)=-\Sigma^{+-}(x), \qquad (4.58)$$
$$\frac{\bar\lambda}{\bar\mu}\Lambda^+N(x,-\frac{\bar\lambda}{\bar\mu}(x-1))-N(x,-\frac{\bar\lambda}{\bar\mu}(x-1))\Lambda^-=0 \qquad (4.59)$$
$$\frac{\bar\lambda}{\bar\mu}\Lambda^+M(x,-\frac{\bar\lambda}{\bar\mu}(x-1))+M(x,-\frac{\bar\lambda}{\bar\mu}(x-1))\Lambda^+=0. \qquad (4.60)$$

We now compute the kernels for the β-transformation.

<u>If $x\le\bar a$</u>: differentiating (4.48) with respect to space and time and then plugging into the target system (4.38)-(4.39) yields the following system of kernel equations

$$\begin{aligned}\Lambda^-\partial_x\bar K(x,\xi)-\partial_\xi\bar K(x,\xi)\Lambda^+ =& \bar K(x,\xi)\Sigma^{++}(\xi)+\bar L(x,\xi)\Sigma^{-+}(\xi)\\ &-\bar\Omega(x)\bar K(x,\xi)-\bar\Gamma(x)K(x,\xi), \qquad (4.61)\\ -\Lambda^-\partial_x\bar L(x,\xi)+\partial_\xi\bar L(x,\xi)\Lambda^- =& \bar K(x,\xi)\Sigma^{+-}(\xi)+\bar L(x,\xi)\Sigma^{--}(\xi)\\ &-\bar\Omega(x)\bar L(x,\xi)-\bar\Gamma(x)L(x,\xi), \qquad (4.62)\end{aligned}$$

along with

$$\Lambda^-\bar K(x,x)+\bar K(x,x)\Lambda^+-\bar\Gamma(x)=-\Sigma^{-+}(x), \qquad (4.63)$$
$$\Lambda^-\bar L(x,x)-\bar L(x,x)\Lambda^-+\bar\Omega(x)=\Sigma^{--}(x), \qquad (4.64)$$
$$\bar K(x,-\frac{\bar\mu}{\bar\lambda}x+1)\Lambda^+-\frac{\bar\mu}{\bar\lambda}\Lambda^-\bar K(x,-\frac{\bar\mu}{\bar\lambda}x+1)=0, \qquad (4.65)$$
$$\frac{\bar\mu}{\bar\lambda}\Lambda^-\bar L(x,-\frac{\bar\mu}{\bar\lambda}x+1)+\bar L(x,-\frac{\bar\mu}{\bar\lambda}x+1)\Lambda^-=0. \qquad (4.66)$$

If $x > \bar{a}$: similarly, we get

$$\Lambda^-\partial_x\bar{M}(x,\xi) - \partial_\xi\bar{M}(x,\xi)\Lambda^+ = +\bar{M}(x,\xi)\Sigma^{++}(\xi) + \bar{N}(x,\xi)\Sigma^{-+}(\xi) - \bar{\Omega}(x)\bar{M}(x,\xi) - \bar{\Gamma}(x)M(x,\xi), \tag{4.67}$$

$$\Lambda^-\partial_x\bar{N}(x,\xi) + \Lambda^-\partial_\xi\bar{N}(x,\xi) = +\bar{M}(x,\xi)\Sigma^{+-}(\xi) + \bar{N}(x,\xi)\Sigma^{--}(\xi) - \bar{\Omega}(x)\bar{N}(x,\xi) - \bar{\Gamma}(x)N(x,\xi), \tag{4.68}$$

along with

$$\Lambda^-\bar{M}(x,x) + \bar{M}(x,x)\Lambda^+ + \bar{\Gamma}(x) = \Sigma^{-+}(x), \tag{4.69}$$

$$\Lambda^-\bar{N}(x,x) - \bar{N}(x,x)\Lambda^- - \bar{\Omega}(x) = \Sigma^{--}(x), \tag{4.70}$$

$$\frac{\bar{\lambda}}{\bar{\mu}}\Lambda^-\bar{N}(x,-\frac{\bar{\lambda}}{\bar{\mu}}(x-1)) + \bar{N}(x,-\frac{\bar{\lambda}}{\bar{\mu}}(x-1))\Lambda^- = 0, \tag{4.71}$$

$$\frac{\bar{\lambda}}{\bar{\mu}}\Lambda^-\bar{M}(x,-\frac{\bar{\lambda}}{\bar{\mu}}(x-1)) - \bar{M}(x,-\frac{\bar{\lambda}}{\bar{\mu}}(x-1))\Lambda^+ = 0. \tag{4.72}$$

The well-posedness of all these kernel equations is assessed in the following theorems.

Theorem 4.2.1.

Consider system (4.49)-(4.54) and (4.61)-(4.66). There exists a unique solution K, L, $\bar{K}$ and $\bar{L}$ in $L^\infty(\mathcal{T}_0)$.

Theorem 4.2.2.

Consider system (4.55)-(4.60) and (4.67)-(4.72). There exists a unique solution M, N, $\bar{M}$ and $\bar{N}$ in $L^\infty(\mathcal{T}_1)$.

The proofs of these theorems are described in the following section and use the cascade structure of the kernel equations (which is due to the particular shapes of the matrices Ω, $\bar{\Omega}$, Γ and $\bar{\Gamma}$).

4.2.3 An important lemma

In this section, we introduce an important lemma which is central in the proof of Theorem 4.2.1.

Lemma 4.2.2.

Consider the triangle domain $\mathcal{D}$ defined by:

$$\mathcal{D} = \{(x,\xi)|x \le \xi \quad c_1\xi \le c_1 - c_2 x \quad d_1\xi \ge d_1 - d_2 x\},$$

where the coefficients c_1, c_2, d_1, d_2 are all positive. Consider the set of scalar parameters $(\epsilon_1, \dots \epsilon_{n+m}) \in \mathbb{R}^{n+m}$, $(\nu_1, \dots \nu_{n+m}) \in \mathbb{R}^{n+m}$ and the matrix functions $(\Sigma_1, \dots \Sigma_{n+m})$, whose components $\Sigma_{i,j}(\cdot,\cdot) \in L^\infty(\mathcal{D})$ ($1 \le i \le n+m$ and $1 \le j \le n+m$). Consider now the following system of hyperbolic equations

$$\epsilon_j\partial_x F_j(x,\xi) + \nu_j\partial_\xi F_j(x,\xi) = \Sigma_j(x,\xi)F(x,\xi), \tag{4.73}$$

where $F = (F_1 \dots F_{n+m})$ is defined on the triangle $\mathcal{D}$. The corresponding boundary conditions are defined on a closed subset $\mathcal{R}_j$ included on the boundary of the domain $\partial\mathcal{D}$ by

$$F_{j|\mathcal{R}_j} = f_j. \tag{4.74}$$

Assume

- that the homogeneous system, obtained by taking $\Sigma(x,\xi) = 0$ in (4.73) along with boundary conditions (4.74) is well-posed;
- that there exists $\alpha_j > 0$ such that the following inequalities holds for all $j = 1, \dots, n+m$

$$\alpha_j \epsilon_j + \nu_j < -\delta < 0, \tag{4.75}$$

then system (4.73) with boundary conditions (4.74) has a unique solution $F \in L^\infty(\mathcal{D})$.

Proof : Classically (see [Joh60] and [Whi11]), the proof follows three steps:

•First, we compute the characteristic lines.

•In each domain the equations are transformed into integral equations.

•Finally, a method of successive approximations is used to find a solution to the integral equations.

Since the proof is quite similar to the one given in [HDMVK16] or the one of Theorem 3.1.1, we just show here the main differences. The first assumption of Lemma 4.2.2 yields the existence and uniqueness of characteristic curves defined as follows.
For each $1 \le j \le n+m$ and $(x,\xi) \in \mathcal{D}$, we now define the following characteristic lines $(x_j(x,\xi,\cdot), \xi_j(x,\xi,\cdot))$ corresponding to equation (4.73)

$$\begin{cases} \frac{dx_j}{ds}(x,\xi,s) = \epsilon_j \quad s \in [0, s_j^F(x,\xi)] \\ x_j(x,\xi,0) = x_j^0(x,\xi), \quad x_j(x,\xi,s_j^F(x,\xi)) = x, \end{cases} \tag{4.76}$$

$$\begin{cases} \frac{d\xi_j}{ds}(x,\xi,s) = \nu_j \quad s \in [0, s_j^F(x,\xi)] \\ \xi_j(x,\xi,0) = \xi_j^0(x,\xi), \quad \xi_j(x,\xi,s_j^F(x,\xi)) = \xi. \end{cases} \tag{4.77}$$

These lines originate at the point $(x_j^0(x,\xi), \xi_j^0(x,\xi)) \in \mathcal{R}_j$ (i.e located on the boundary of the domain) and terminate at (x,ξ). Such lines are well defined since the homogenous system is assumed to be well-posed. The localization of $(x_j^0(x,\xi), \xi_j^0(x,\xi))$ in $\mathcal{R}_j$ depends on the end of the line (x,ξ) we want to reach. Integrating (4.73) along these characteristics and using boundary conditions (4.74), one can obtain

$$F_j(x,\xi) = f_j(M_j^0(x,\xi)) + \int_0^{s_j^F(x,\xi)} \Sigma_j(M_j(x,\xi,s))F(M_j(x,\xi,s))ds, \tag{4.78}$$

where we denote $M_j(x,\xi,s) = (x_j(x,\xi,s), \xi_j(x,\xi,s))$ and $M_j^0(x,\xi) = (x_j^0(x,\xi), \xi_j^0(x,\xi))$. In order to solve (4.78) we use the method of successive approximations. We define

$$\Phi_j[F](x,\xi) = \int_0^{s_j^F(x,\xi)} \Sigma_j(M_j(x,\xi,s_j))F(M_j(x,\xi,s_j))ds.$$

We now construct the sequence F^p defined by

$$F^0(x,\xi) = 0, \tag{4.79}$$

$$F^{p+1}(x,\xi) = \begin{pmatrix} f_1(M_1^0(x,\xi,s_j)) \\ \vdots \\ f_{n+m}(M_{(n+m)}^0(x,\xi,s_j)) \end{pmatrix} + \begin{pmatrix} \Phi_1[F^p](x,\xi) \\ \vdots \\ \Phi_{(n+m)}[F^p](x,\xi) \end{pmatrix}. \tag{4.80}$$

Consequently, if the sequence F^p has a limit, then this limit is a solution of the integral equation and therefore of the original system.
We define the increment $\Delta F^p = F^p - F^{p-1}$. Provided the limit exists one has

$$F(x,\xi) = \lim_{p\to+\infty} F^p(x,\xi) = \sum_{p=0}^{+\infty} \Delta F^p(x,\xi). \tag{4.81}$$

We now prove the convergence of the series. Similarly to [HDMVK16], the proof of convergence is based on the following properties:

Property 1: Assume that (4.75) holds. then for all $j = 1 \dots n+1$, $(x,\xi) \in \mathcal{D}$, the following function is strictly increasing

$$\phi_{j,x,\xi} : s \in [0, s_j^F(x,\xi)] \longmapsto \alpha_j x_j(x,\xi,s) + \xi_j(x,\xi,s) + (\alpha_j + 1).$$

The proof is this property trivial. Recalling (4.75) yields

$$\phi'_{j,x,\xi}(s) = -\alpha_j \epsilon_j - \nu_j > 0.$$

This concludes the proof of this property.

Property 2: For all $j = 1 \dots n+m$ the following inequalities hold

$$\int_0^{s_j^F(x,\xi)} (-\alpha_j x_j(x,\xi,s) - \xi_j(x,\xi,s) + (\alpha_j+1))^p ds \leq \frac{1}{\delta} \frac{((\alpha_j+1) - \alpha_j x - \xi)^{p+1}}{p+1}.$$

The proof of this property is similar to [HDMVK16, Lemma 6.2], considering the change of variables $\tau = \phi_{x,\xi}(s)$ and using (4.75).

Property 3: Let $M > 0$ be such that

$$M > \bar{\Sigma}\delta, \tag{4.82}$$

where $\bar{\Sigma}$ is defined as

$$\bar{\Sigma} = \max_{(x,\xi)\in\mathcal{D}} \max_{||F||\in\mathbb{R}^{n+m}\neq 0} \frac{||\Sigma(x,\xi)F||}{||F||}.$$

If for some $1 \leq p$ and some $\bar{f} > 0$ one has that for all $(x,\xi) \in \mathcal{T}_0$ and for all $1 \leq j \leq m+n$

$$|\Delta F_j^p(x,\xi)| \leq \bar{f} \frac{M^p(-\alpha_j x - \xi + (\alpha_j+1))^p}{p!}. \tag{4.83}$$

Then one has $\forall j = 1 \dots m+n$

$$|\Delta F_j^{p+1}(x,\xi)| \leq \bar{f} \frac{M^{p+1}(-\alpha_j x - \xi + (\alpha_j+1))^{p+1}}{(p+1)!}. \tag{4.84}$$

Using (4.82), one can easily prove that if (4.83) holds, then (4.84) immediately holds.

We are now able to prove the convergence of the series (4.81). Denoting

$$\bar{f} = \max_{(x,\xi)\in\mathcal{T}_0} \max_{j=1\dots n+m} |f_j(x,\xi)|,$$

one can prove the initialization. Then, using Property 3, one can prove by induction that

$$\sum_{p=0}^{+\infty} |\Delta F^p(x,\xi)| < \bar{f} e^{M(-\alpha_j x - \xi + (\alpha_j+1))}.$$

Defining F as

$$F(x,\xi) = \sum_{p=0}^{+\infty} \Delta F^p(x,\xi) = \lim_{p\to\infty} F^p(x,\xi),$$

taking the limit $p \to \infty$ in (4.80) and using similar arguments as the one presented in [CVKB13] yields the result. ■

Remark 4.2.7 *A necessary and sufficient condition for the first assumption to be satisfied is that, for every $j = 1 \dots n+m$ the characteristic lines defined by the ϵ_j, ν_j uniquely connect each point of $\mathcal{T}_0$ to $\mathcal{D}_j$.*

Remark 4.2.8 *Assumption (4.75) is a simple geometric condition for the well-posedness of the system: the tangent vector (ϵ_j, ν_j) to all the characteristics must lie in the half-space such that the scalar product with $(\alpha_j, 1)^T$ is negative.*

4.2.4 Well-posedness of the kernel equations: proof of Theorem 4.2.1

In this section we prove Theorem 4.2.1. The proof of Theorem 4.2.2 is quite similar and is not detailed here. The general idea is to recursively prove the well-posedness (in the sense of Lemma 4.2.1) of each line of the matrices K, L and $\bar{K}, \bar{L}$. We start by assessing the well-posedness of either the bottom line of the matrices K, L or the last line of the matrices $\bar{K}, \bar{L}$. Then, we prove that at each new iteration of the recursion, we can "go up" one line in either K, L or $\bar{K}, \bar{L}$ and prove the well-posedness of the corresponding line. The order in which we consider the different lines depends on a sequence defined from the matrices Γ and $\bar{\Gamma}$. More precisely, the proof follows four steps:

- First, we develop the kernel equations and the associated boundary conditions.
- Then, we define (equation (4.95)) a particular sequence that depends on the matrices Γ and $\bar{\Gamma}$ in which we solve the equations.
- This sequence determines the order in which we prove the well-posedness of the different lines of the matrices $\bar{K}, \bar{L}$ and $\bar{K}, \bar{L}$. The proof is recursive as the well-posedness of this new line uses the well-posedness of all the previous line.
- At each step of the induction, to assess the well-posedness of the considered line, we divide the spatial domain into multiple sections overwhich we recursively prove the well-posedness.

Development of the kernel equations

We only focus on the kernels $K, L, \bar{K}$ and $\bar{L}$ defined on $\mathcal{T}_0$ since the proof is similar for the remaining kernels. Developing (4.49)-(4.54) and (4.61)-(4.66) we get the following set of kernel PDEs:

<u>For $1 \leq i \leq n$, $1 \leq j \leq n$</u>

$$\begin{aligned}\lambda_i \partial_x K_{ij}(x,\xi) + \lambda_j \partial_\xi K_{ij}(x,\xi) = &-\sum_{k=1}^{n} \sigma_{kj}^{++}(\xi) K_{ik}(x,\xi) - \sum_{p=1}^{m} \sigma_{pj}^{-+}(\xi) L_{ip}(x,\xi) \\ &+ \sum_{i \leq p \leq n} K_{pj}(x,\xi)\omega_{ip}(x) + \sum_{1 \leq p \leq m} \bar{K}_{pj}(x,\xi)\Gamma_{ip}(x). \end{aligned} \tag{4.85}$$

<u>For $1 \leq i \leq n$, $1 \leq j \leq m$</u>

$$\begin{aligned}\lambda_i \partial_x L_{ij}(x,\xi) - \mu_j \partial_\xi L_{ij}(x,\xi) = &-\sum_{k=1}^{m} \sigma_{kj}^{--}(\xi) L_{ik}(x,\xi) - \sum_{p=1}^{n} \sigma_{pj}^{+-}(\xi) K_{ip}(x,\xi) \\ &+ \sum_{i \leq p \leq n} L_{pj}(x,\xi)\omega_{ip}(x) + \sum_{1 \leq p \leq m} \bar{L}_{pj}(x,\xi)\Gamma_{ip}(x). \end{aligned} \tag{4.86}$$

<u>For $1 \leq i \leq m$, $1 \leq j \leq n$</u>

$$\begin{aligned}\mu_i \partial_x \bar{K}_{ij}(x,\xi) - \lambda_j \partial_\xi \bar{K}_{ij}(x,\xi) = &\sum_{k=1}^{n} \sigma_{kj}^{++}(\xi) \bar{K}_{ik}(x,\xi) + \sum_{p=1}^{m} \sigma_{pj}^{-+}(\xi) \bar{L}_{ip}(x,\xi) \\ &- \sum_{i \leq p \leq m} \bar{K}_{pj}(x,\xi)\bar{\omega}_{ip}(x) - \sum_{1 \leq p \leq n} K_{pj}(x,\xi)\bar{\Gamma}_{ip}(x). \end{aligned} \tag{4.87}$$

For $1 \leq i \leq m$, $1 \leq j \leq m$

$$\mu_i \partial_x \bar{L}_{ij}(x,\xi) + \mu_j \partial_\xi \bar{L}_{ij}(x,\xi) = \sum_{k=1}^{m} \sigma_{kj}^{--}(\xi)\bar{L}_{ik}(x,\xi) + \sum_{p=1}^{n} \sigma_{pj}^{+-}\bar{K}_{ip}(x,\xi) - \sum_{i\leq p\leq m} \bar{L}_{pj}(x,\xi)\bar{\omega}_{ip}(x) - \sum_{1\leq p\leq n} L_{pj}(x,\xi)\bar{\Gamma}_{ip}(x). \quad (4.88)$$

We have the following set of boundary conditions (to make the whole content more readable we have removed the domains of definition of the indices).

$$K_{ij}(x, 1 - \frac{\bar{\mu}}{\bar{\lambda}}x) = 0, \quad K_{ij}(x,x) = \frac{\sigma^{++}(x)}{\lambda_i - \lambda_j} \quad i > j,$$
$$\bar{L}_{ij}(x, -\frac{\bar{\mu}}{\bar{\lambda}}x + 1) = 0, \quad \bar{L}_{ij}(x,x) = \frac{\sigma^{--}(x)}{\mu_i - \mu_j} \quad i > j,$$
$$(\frac{\mu_j}{\lambda_i} - \frac{\bar{\mu}}{\bar{\lambda}})L_{ij}(x, 1 - \frac{\bar{\mu}}{\bar{\lambda}}x) = 0, \quad (-\lambda_i + \mu_i\frac{\bar{\mu}}{\bar{\lambda}})\bar{K}_{ij}(x, -\frac{\bar{\mu}}{\lambda_m}x + 1) = 0,$$

$$\text{if } 1 - a_{ji} \geq \bar{a} \quad \bar{K}_{ij}(x,x) = -\frac{\sigma^{-+}(x)}{\lambda_j + \mu_i}. \quad (4.89)$$

We add the following arbitrary boundary conditions (in order to have a well-posed system)

$$\text{if } 1 - a_{ji} < \bar{a} \quad \bar{K}_{ij}(0,\xi) = 0. \quad (4.90)$$

Besides, (4.49) imposes

$$\forall i \leq j \quad \omega_{ij}(x) = (\lambda_j - \lambda_i)K_{ij}(x,x) + \sigma_{ij}^{++}(x), \quad (4.91)$$

and (4.52) imposes

$$\forall a_{ij} < x < 1 \quad \gamma_{ij}(x) = -(\lambda_i + \mu_j)L_{ij}(x,x) + \sigma_{ij}^{+-}(x). \quad (4.92)$$

Similarly (4.64) imposes

$$\forall i \leq j \quad \bar{\omega}_{ij}(x) = (\mu_j - \mu_i)\bar{L}_{ij}(x,x) + \sigma_{ij}^{--}(x), \quad (4.93)$$

and (4.61) imposes

$$\forall 0 \leq x \leq a_{ji} \quad \bar{\gamma}_{ij}(x) = (\mu_i + \lambda_j)\bar{K}_{ij}(x,x) + \sigma_{ij}^{-+}(x). \quad (4.94)$$

This induces a coupling between the kernels through equations (4.85), (4.86), (4.87) and (4.88) that could appear as nonlinear at first sight. However, the coupling has a linear cascade structure due to the particular shapes of the matrices Ω, $\bar{\Omega}$, Γ and $\bar{\Gamma}$. Some of these equations with the corresponding characteristic lines are represented in Figure 4.3. We now define two sequences r_i and $\bar{r}_i$ that will be used in a recursive proof of the well-posedness. The following example gives the intuition on how to define such a sequence.

Example 4.2.2 *(continued from Example 4.2.1) Let us consider the system introduced in Example 4.2.1, for which*

$$n = 3, \ m = 2, \ \Lambda^+ = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \ \Lambda^- = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}.$$

We have shown previously that the matrices Γ *and* $\bar{\Gamma}$ *have the following structure*

$$\Gamma(x) = \begin{pmatrix} * & * \\ * & * \\ 0 & 0 \end{pmatrix} \quad \bar{\Gamma}(x) = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \end{pmatrix},$$

*where the potential non-zero terms are represented by *. As the last line of* Γ *is equal to zero, we choose to prove the well-posedness of the last lines of equations* (4.85)-(4.86)*. They rewrite, for all* $1 \leq j \leq n$ *and all* $1 \leq l \leq m$*, as*

$$\begin{aligned}
\lambda_3 \partial_x K_{3j}(x,\xi) + \lambda_j \partial_\xi K_{3j}(x,\xi) &= -\sum_{k=1}^{3} \sigma_{kj}^{++}(\xi) K_{3k}(x,\xi) - \sum_{p=1}^{2} \sigma_{pj}^{-+}(\xi) L_{3p}(x,\xi) \\
&\quad + K_{3j}(x,\xi)\sigma_{33}^{++}(x). \\
\lambda_3 \partial_x L_{3l}(x,\xi) - \mu_l \partial_\xi L_{3l}(x,\xi) &= -\sum_{k=1}^{2} \sigma_{kl}^{--}(\xi) L_{3k}(x,\xi) - \sum_{p=1}^{3} \sigma_{pl}^{+-}(\xi) K_{3p}(x,\xi) \\
&\quad + L_{3l}(x,\xi)\sigma_{33}^{++}(x).
\end{aligned}$$

These equations are linear and provided that Lemma 4.2.2 is satisfied, one can assess the existence of a unique solution. Let us now consider the last lines of equations (4.87)-(4.88)*. They rewrite, for all* $1 \leq j \leq n$ *and all* $1 \leq l \leq m$*, as*

$$\begin{aligned}
\mu_2 \partial_x \bar{K}_{2j}(x,\xi) - \lambda_j \partial_\xi \bar{K}_{2j}(x,\xi) &= \sum_{k=1}^{3} \sigma_{kj}^{++}(\xi) \bar{K}_{2k}(x,\xi) + \sum_{p=1}^{2} \sigma_{pj}^{-+}(\xi) \bar{L}_{2p}(x,\xi) \\
&\quad - \bar{K}_{2j}(x,\xi)\sigma_{22}^{--}(x) - K_{3j}(x,\xi)[(\mu_2 + \lambda_l)\bar{K}_{23}(x,x) + \sigma_{23}^{-+}(x)], \\
\mu_2 \partial_x \bar{L}_{2l}(x,\xi) - \mu_l \partial_\xi \bar{L}_{2l}(x,\xi) &= -\sum_{k=1}^{2} \sigma_{kl}^{--}(\xi) \bar{L}_{2k}(x,\xi) - \sum_{p=1}^{3} \sigma_{pl}^{+-}(\xi) \bar{K}_{2p}(x,\xi) \\
&\quad + \bar{L}_{2l}(x,\xi)\sigma_{22}^{--}(x) - L_{3l}(x,\xi)[(\mu_2 + \lambda_l)\bar{K}_{23}(x,x) + \sigma_{23}^{-+}(x)].
\end{aligned}$$

As we have already proved the existence of the kernels L_{3l} *and* K_{3j}*, these equations are actually linear. Thus, provided that Lemma 4.2.2 is satisfied, one can assess the existence of a unique solution for this line. We now consider the first lines of equations* (4.87)-(4.88)*. They rewrite, for all* $1 \leq j \leq n$ *and all* $1 \leq l \leq m$*, as*

$$\begin{aligned}
\mu_1 \partial_x \bar{K}_{1j}(x,\xi) - \lambda_j \partial_\xi \bar{K}_{1j}(x,\xi) &= \sum_{k=1}^{3} \sigma_{kj}^{++}(\xi) \bar{K}_{1k}(x,\xi) + \sum_{p=1}^{2} \sigma_{pj}^{-+}(\xi) \bar{L}_{1p}(x,\xi) \\
&\quad - \bar{K}_{1j}(x,\xi)\sigma_{11}^{--}(x) + \bar{K}_{2j}(x,\xi)((\mu_2 - \mu_1)\bar{L}_{12}(x,x) + \sigma_{12}^{++}(x)) \\
&\quad - K_{3j}(x,\xi)[(\mu_1 + \lambda_l)\bar{K}_{13}(x,x) + \sigma_{13}^{-+}(x)], \\
\mu_2 \partial_x \bar{L}_{2l}(x,\xi) - \mu_l \partial_\xi \bar{L}_{2l}(x,\xi) &= -\sum_{k=1}^{2} \sigma_{kl}^{--}(\xi) \bar{L}_{2k}(x,\xi) - \sum_{p=1}^{3} \sigma_{pl}^{+-}(\xi) \bar{K}_{2p}(x,\xi) \\
&\quad + \bar{L}_{1l}(x,\xi)\sigma_{11}^{--}(x) + \bar{L}_{2j}(x,\xi)((\mu_2 - \mu_1)\bar{L}_{12}(x,x) + \sigma_{12}^{++}(x)) \\
&\quad - K_{3l}(x,\xi)[(\mu_1 + \lambda_l)\bar{K}_{1l}(x,x) + \sigma_{1l}^{-+}(x)].
\end{aligned}$$

Once again, since we have proved the existence of the kernels L_{3l}*,* K_{3j}*,* $\bar{K}_{2j}$ *and* $\bar{L}_{2l}$*, these equations are linear and, provided that Lemma 4.2.2 is satisfied, they admit a unique solution. We can then solve in a similar way the second line and finally the first line of equations* (4.85)-(4.86)*. If we define the sequence* r_p *(resp.* $\bar{r}_p$*) as the number of the line of equations* (4.85)-(4.86) *(resp. of equations* (4.87)-(4.88)*) we can solve at the* p^{th} *iteration* $(1 \leq p \leq n+m)$ *we have*

$$\begin{aligned}
&r_0 = 4,\ r_1 = 3,\ r_2 = 3,\ r_3 = 3,\ r_4 = 2,\ r_5 = 1, \\
&\bar{r}_0 = 3,\ \bar{r}_1 = 3,\ \bar{r}_2 = 2,\ \bar{r}_3 = 1,\ \bar{r}_4 = 1,\ \bar{r}_5 = 1.
\end{aligned}$$

By convention we have chosen to start the sequence r at $n+1$ and the sequence $\bar{r}$ at $m+1$. One can notice that these sequences are decreasing and that at each step one (and only one) of these two sequences actually decreases. Once the two sequences have reached one, we have completed the proof of Theorem 4.2.1. Bellow, these sequences are defined in the general case.

Definition of the sequences r_i and $\bar{r}_i$

In this subsection we define two sequences r_i and $\bar{r}_i$ that we are going to use in the recursive proof. They represent the order in which the kernel PDEs have to be solved due to the non-linear couplings. The construction of such sequences is strongly related with the structure of the matrices Γ and $\bar{\Gamma}$. Let us consider the matrices Δ and $\bar{\Delta}$ defined by

$$\forall 1 \le i \le n \ \forall 1 \le j \le m \ \Delta_{ij} = \begin{cases} 0 & \text{if} \quad a_{ij} \ge \bar{a}, \\ 1 & \text{else}, \end{cases}$$

$$\forall 1 \le i \le m \ \forall 1 \le j \le n \ \bar{\Delta}_{ij} = \begin{cases} 0 & \text{if} \quad a_{ji} < 1 - \bar{a}, \\ 1 & \text{else}. \end{cases}$$

These matrices have exactly the same structure as the matrices Γ and $\bar{\Gamma}$, i.e. $\Gamma_{ij} = 0 \Rightarrow \Delta_{ij} = 0$ and $\bar{\Gamma}_{ij} = 0 \Rightarrow \bar{\Delta}_{ij} = 0$. We have the following results (some of the proofs are quite straightforward and are ommited).

Lemma 4.2.3.

The matrix Δ satisfies $\bar{\Delta} = 1 - \Delta^T$.

Proof : The proof relies on the fact that due to the definition of $\bar{a}$ (Equation (4.37)), if $a_{ij} > \bar{a}$ then $a_{ij} > 1 - \bar{a}$. Suppose that $\Delta_{ij} = 0$, then $a_{ij} \ge \bar{a} \Rightarrow a_{ij} \ge 1 - \bar{a}$ (since $\bar{a} > \frac{1}{2}$). This yields $\bar{\Delta}_{ji} = 1$.
Suppose now that $\Delta_{ij} = 1$, then $a_{ij} < \bar{a} \Rightarrow a_{ij} < 1 - \bar{a}$ (due to the definition of $\bar{a}$). This yields $\bar{\Delta}_{ji} = 0$.
■

Lemma 4.2.4.

If $\Delta_{ij} = 0$ then $\forall k > i \quad \Delta_{kj} = 0$.

Lemma 4.2.5.

If $\Delta_{ij} = 0$ then $\forall k < j \quad \Delta_{ik} = 0$.

The two previous lemmas use the fact that $a_{ij} < a_{i+1,j}$ and $a_{ij} > a_{i,j+1}$. Same results hold for $\bar{\Delta}$.

Lemma 4.2.6.

Either the last line of Δ or the last line of $\bar{\Delta}$ is equal to zero.

Proof : Let us assume that the last line of $\bar{\Delta}$ is non-zero. Consequently, $\forall j \in [1, n] \quad a_{jm} \le 1 - \bar{a}$ and particularly $a_{nm} \le 1 - \bar{a}$. This implies $a_{nm} \le \bar{a}$ (due to the definition of $\bar{a}$) and this yields $\Delta_{mn} = 0$. Using the previous lemma, one can conclude the proof. ■

In the following we denote by s_i (resp. $\bar{s}_i$) the number of coefficients which are equal to 1 in the i^{th} line of Δ (resp. $\bar{\Delta}$). For $0 \le i \le n+m$, we define the sequences r_i and $\bar{r}_i$ as

$$\begin{cases} \text{if} \ \ \bar{r}_0 - \bar{r}_i \ge s_{r_i - 1} & \text{then} \quad r_{i+1} = r_i - 1 \quad \bar{r}_{i+1} = \bar{r}_i, \\ \text{if} \ r_0 - r_i \ge \bar{s}_{\bar{r}_i - 1} & \text{then} \quad \bar{r}_{i+1} = \bar{r}_i - 1 \quad r_{i+1} = r_i, \end{cases} \tag{4.95}$$

where $r_0 = n+1$, $\bar{r}_0 = m+1$. We use the convention $s_0 = \bar{s}_0 = \infty$.

Theorem 4.2.3.

The sequences r_i, $\bar{r}_i$ are well defined. Moreover $r_{n+m} = \bar{r}_{n+m} = 1$.

Proof : To prove that the sequences are well defined we need to prove that for any $0 \leq i \leq n+m$ exactly one of the following assumptions is true

$$\bar{r}_0 - \bar{r}_i \geq s_{r_i - 1}, \tag{4.96}$$
$$r_0 - r_i \geq \bar{s}_{\bar{r}_i - 1}. \tag{4.97}$$

We start by proving that at least one of the two assumptions is true. By contradiction let us assume that none of them hold. Consequently we have, for some i

$$\bar{r}_0 - \bar{r}_i < s_{r_i - 1}, \quad r_0 - r_i < \bar{s}_{\bar{r}_i - 1}. \tag{4.98}$$

• By definition of s_{r_i-1}, we have exactly s_{r_i-1} coefficients that are equal to 1 in the $(r_i - 1)^{th}$ line of the matrix Δ. Using Lemma 4.2.3 yields that we have exactly $m - s_{r_i-1}$ coefficients equal to 1 in the $(r_i - 1)^{th}$ column of $\bar{\Delta}$.
• By definition of $\bar{s}_{\bar{r}_i-1}$, we have exactly $\bar{s}_{\bar{r}_i-1}$ coefficients that are equal to 1 in the $(\bar{r}_i - 1)^{th}$ line of $\bar{\Delta}$. Consequently $\bar{\Delta}_{\bar{r}_i-1, \bar{s}_{\bar{r}_i-1}} = 1$. Adjusting Lemma 4.2.4 to $\bar{\Delta}$, we get that in the $(n+1-\bar{s}_{\bar{r}_i-1})^{th}$ column of $\bar{\Delta}$ we have at least $\bar{r}_i - 1$ coefficients equal to 1.
• Since $n+1-\bar{s}_{\bar{r}_i} < r_i$ we get $n+1-\bar{s}_{\bar{r}_i} \leq r_i - 1$. It means that the column $n+1-\bar{s}_{\bar{r}_i}$ is located more on the left than the column $r_i - 1$. Consequently, using Lemma 4.2.4, we must have a larger number of coefficients equal to 1 in the column $r_i - 1$ than in the column $n+1-\bar{s}_{\bar{r}_i}$. This implies

$$m - s_{r_i-1} \geq \bar{r}_i - 1,$$

which is a contradiction to the first inequality of (4.98). To achieve the proof of the well posedness of the two sequences r_i and $\bar{r}_i$, we need to prove that the two assumptions (4.96)-(4.97) cannot both be true. This is quite straightforward using the same ideas. If we assume that (4.97) holds, then $n+1-\bar{s}_{\bar{r}_i} \geq r_i$ implies that the column $n+1-\bar{s}_{\bar{r}_i}$ of $\bar{\Delta}$ is located strictly more on the right than the column $r_i - 1$ and that consequently (Lemma 4.2.5), the number of coefficients equal to 1 in the former column is larger than the number in the later. This implies

$$m - s_{r_i-1} < \bar{r}_i - 1,$$

and consequently (4.96) is false. Using similar ideas, one can easily prove that (4.97) is false when (4.96) holds. ■

The following lemma makes the link between the matrices $\Delta, \bar{\Delta}$, and $\Gamma, \bar{\Gamma}$.

Lemma 4.2.7.

The matrix $\Gamma(x)$ has at least $m - s_i$ zero-coefficients on its i^{th} line. Similarly, the matrix $\bar{\Gamma}(x)$ has at least $n - \bar{s}_i$ zero-coefficients on its i^{th} line.

Corollary 4.2.1.

$\forall i \leq n, \forall j \leq m - s_i$, $\Gamma(x)_{ij} = 0$ and $\forall i \leq m, \forall j \leq n - \bar{s}_i$, $\bar{\Gamma}(x)_{ij} = 0$.

Proof : The proofs of this lemma and of this corollary are quite straightforward noticing that the matrices Γ (resp. $\bar{\Gamma}$) and Δ (resp. $\bar{\Delta}$) have exactly the same structure and that consequently the properties described above can be easily extended to Γ and $\bar{\Gamma}$. ■

Remark 4.2.9 *All these properties are due to the specific structure of the matrices Γ and $\bar{\Gamma}$. This structure is a direct consequence of their definition and of the fact that the velocities λ_i and μ_j are well ordered.*

Using the sequences r_i and $\bar{r}_i$, we now know the order in which we can recursively prove the well-posedness of the different lines of the matrices K, L and $\bar{K}$ $\bar{L}$.

Induction hypothesis

We are now able to prove Theorem 4.2.1, considering the following property $P(q)$ defined for all $1 \leq q \leq m+n$:
$P(q)$: "$\forall r_q \leq i \leq n$, $\forall \bar{r}_q \leq \bar{i} \leq m$, $\forall 1 \leq j \leq n$, $\forall 1 \leq d \leq m$, $\forall 1 \leq \bar{j} \leq n$, $\forall 1 \leq \bar{d} \leq m$, the problem (4.85)-(4.89) where Ω, $\bar{\Omega}$, Γ and $\bar{\Gamma}$ are defined by (4.91)-(4.93) has a unique solution $K_{ij}(\cdot,\cdot), L_{id}(\cdot,\cdot), \bar{K}_{\bar{i}\bar{j}}(\cdot,\cdot), \bar{L}_{\bar{i}\bar{d}}(\cdot,\cdot) \in L^\infty(\mathcal{T}_0)$."
This induction property means that we solve the kernel equations from the bottom line. At each iteration of the recursion, we "go up" one line in either K, L or, $\bar{K}, \bar{L}$ depending on the sequence r_i and $\bar{r}_i$.
<u>Initialization:</u> For $q=1$, we have either ($r_1 = n$ and $\bar{r}_1 = m+1$) or ($r_1 = n+1$ and $\bar{r}_1 = m$). Assuming that $r_1 = n$ and $\bar{r}_1 = m+1$, system (4.85)-(4.89) rewrites as follow.
<u>For $1 \leq j \leq n$</u>

$$\lambda_n \partial_x K_{nj}(x,\xi) + \lambda_j \partial_\xi K_{nj}(x,\xi) = -\sum_{k=1}^{n} \sigma_{kj}^{++}(\xi) K_{nk}(x,\xi) - \sum_{p=1}^{m} \sigma_{pj}^{-+}(\xi) L_{np}(x,\xi) + K_{nj}(x,\xi)\sigma_{nn}^{++}(\xi).$$

<u>For $1 \leq j \leq m$</u>

$$\lambda_n \partial_x L_{nj}(x,\xi) - \mu_j \partial_\xi L_{nj}(x,\xi) = -\sum_{k=1}^{m} \sigma_{kj}^{--}(\xi) L_{nk}(x,\xi) - \sum_{p=1}^{n} \sigma_{pj}^{+-} K_{np}(x,\xi) + L_{nj}(x,\xi)\sigma_{nn}^{++}(\xi),$$

with the corresponding set of boundary conditions. The well-posedness of such a system is a direct consequence of [HDMVK16] or Lemma 4.2.2. The initialization still holds for $r_1 = n+1$ and $\bar{r}_1 = m$.

<u>Induction:</u> Let us assume that the property $P(q-1)$ $(1 < q \leq n+m-1)$ is true. We consequently have that $\forall\ r_{q-1} \leq i \leq n$, $\forall\ \bar{r}_{q-1} \leq \bar{i} \leq n$, $\forall 1 \leq j \leq n$, $\forall 1 \leq d \leq m$, $\forall 1 \leq \bar{j} \leq n$, $\forall 1 \leq \bar{d} \leq m$, $K_{ij}(\cdot,\cdot)$, $L_{id}(\cdot,\cdot)$, $\bar{K}_{\bar{i}\bar{j}}(\cdot,\cdot)$, and $\bar{L}_{\bar{i}\bar{d}}(\cdot,\cdot)$ are bounded.
In the following we assume that $\bar{r}_q = \bar{r}_{q-1}$ (and that consequently $r_q = r_{q-1} - 1$). The result still holds if $r_q = r_{q-1}$ and the proof is similar. We denote $i = r_q$. We now prove that K_{ij} and L_{id} are well-posed
Using the induction hypothesis, one obtains that $\forall 1 \leq \bar{j} \leq n$, $\forall 1 \leq \bar{d} \leq m$, $\forall \bar{r}_q = \bar{r}_{q-1} \leq \bar{i} \leq n$ $\bar{K}_{\bar{i}\bar{j}}(\cdot,\cdot)$, and $\bar{L}_{\bar{i}\bar{d}}(\cdot,\cdot)$ are well-posed. Rewriting equation (4.85) yields

$$\begin{aligned} -\lambda_i \partial_x K_{ij}(x,\xi) - \lambda_j \partial_\xi K_{ij}(x,\xi) &= \sum_{k=1}^{n} \sigma_{kj}^{++}(\xi) K_{ik}(x,\xi) - \sum_{i \leq p \leq m} K_{pj}(x,\xi) \cdot ((\lambda_p - \lambda_i) K_{ip}(x,x) \\ &\quad + \sigma_{ip}^{++}(x))] - \sum_{1 \leq p \leq m} \bar{K}_{pj}(x,\xi) \mathbb{1}_{[a_{ip},1]}(x) \cdot ((\lambda_i + \mu_p) L_{ip}(x,x) + \sigma_{ip}^{+-}(x)) \\ &\quad + \sum_{p=1}^{m} \sigma_{pj}^{-+}(\xi) L_{ip}(x,\xi), \end{aligned} \tag{4.99}$$

with the boundary conditions

$$K_{ij}(x, 1 - \frac{\bar{\mu}}{\bar{\lambda}} x) = 0, \quad K_{ij}(x,x) = \frac{\sigma^{++}(x)}{\lambda_i - \lambda_j} \quad i > j.$$

The one-but-last sum uses the expression of K_{pj} for $i \leq p \leq m$. This term is known and bounded for $p > i$ (induction assumption). For $p = i$, $\lambda_i = \lambda_p$ and the term $(\lambda_p - \lambda_i) K_{ip}(x_{ij}(x,\xi,s), x_{ij}(x,\xi,s))$ cancels.
Using Corollary 4.2.1 and relation (4.92) it is possible to rewrite the last sum as

$$\sum_{m+1-s_i \leq p \leq m} \bar{K}_{pj}(x,\xi)(\sigma_{ip}^{+-} + (\lambda_i + \mu_p) L_{ip}(x,\xi)) \mathbb{1}_{[a_{ij},1]}(x).$$

Using the definition of r_i, we have

$$s_i \leq s_{i-1} = s_{r_q-1} \leq m+1-\bar{r}_q \Rightarrow m - s_i \geq \bar{r}_q - 1.$$

Consequently, the last sum uses the expression of $\bar{K}_{pj}$ for $\bar{r}_q \leq p \leq m$ which is known according to the induction assumption. Therefore, all the non-linearites that could appear at first sight on the kernel equations actually involve terms that have been computed in the previous iterations and that are bounded. We can rewrite (4.99) as

$$-\lambda_i \partial_x K_{ij}(x,\xi) - \lambda_j \partial_\xi K_{ij}(x,\xi) = \sum_{k=1}^{n} C_{kj}^{++}(x,\xi) K_{ik}(x,\xi) + \sum_{k=1}^{m} C_{kj}^{-+}(x,\xi) L_{ik}(x,\xi) ds, \quad (4.100)$$

where the coefficients C_{kj}^{++} and C_{kj}^{-+} are known and bounded (since they are either constants or computed during the previous iteration of the recursion).

Similarly, we can rewrite (4.86) as

$$-\lambda_i \partial_x L_{id}(x,\xi) + \mu_j \partial_\xi L_{id}(x,\xi) = \sum_{k=1}^{n} C_{kj}^{-+}(x,\xi) K_{ik}(x,\xi) + \sum_{k=1}^{m} C_{kj}^{--}(x,\xi) L_{ik}(x,\xi) ds, \quad (4.101)$$

where the coefficients C_{kj}^{-+} and C_{kj}^{--} are known and bounded. Moreover we have the boundary condition

$$(\frac{\mu_j}{\lambda_i} - \frac{\bar{\mu}}{\bar{\lambda}}) L_{ij}\left(x, 1 - \frac{\bar{\mu}}{\bar{\lambda}} x\right) = 0,$$

and

$$\forall x < a_{ij} = \frac{\lambda_i}{\lambda_i + \mu_j}, \quad L_{ij}(x,x) = \frac{\sigma_{ij}^{+-}(x)}{\lambda_i + \mu_j}. \quad (4.102)$$

Each L_{ij} has a discontinuity line defined by $\xi = 1 - \frac{\mu_j}{\lambda_i} x$. The characteristics are integrated in opposite directions on each side of the discontinuity: away from the $\xi = x$ boundary for $\xi \leq 1 - \frac{\mu_j}{\lambda_i} x$ and away from the line $\xi = 1 - \frac{\bar{\mu}}{\bar{\lambda}}$ for $\xi \geq 1 - \frac{\mu_j}{\lambda_i} x$. Therefore the parameters α and δ of Lemma 4.2.2, which have to satisfy (4.75) for all L_{ij} on the domain on which the equations are considered, vary on each side of the discontinuity of all the kernels. In what follows, we define a sequence of triangular domains, depicted in Figure 4.4, on which there exists α and δ satisfying (4.75). More precisely, assuming that $a_{i1} \geq \bar{a}$ (this specific case will be discussed in Remark 4.2.10), for all $k \leq m+1$ such that $a_{ik} < \bar{a}$ (with the convention $a_{i(m+1)} = 0$), consider the domains $\mathcal{T}_k$:

$$\mathcal{T}_k = \{(x,\xi) | x \leq \xi \quad \xi \leq 1 - \frac{\mu_{k-1}}{\lambda_i} x \quad \xi \geq 1 - \frac{\mu_k}{\lambda_i} x\},$$
$$\mathcal{T}_{m+1} = \{(x,\xi) | 0 \leq x \leq \xi \quad \xi \leq 1 - \frac{\mu_m}{\lambda_i} x\}.$$

The equations can be solved successively on these triangles, starting from the rightmost one. These triangles are represented in Figure 4.4.

The trace of the solution on the boundary of a given $\mathcal{T}_k$ provides boundary conditions of the system considered on $\mathcal{T}_{k+1}$.
By induction, let us now consider the property $\mathcal{Q}(k)$ defined for all k such that $a_{ik} < \bar{a}$ by: *The system* (4.100)-(4.102) *is well-posed on* $\mathcal{T}_k \cap \mathcal{T}_0$.

Initialization: Let k_0 such that $a_{ik_0} < \bar{a} \leq a_{i(k_0-1)}$. On $\mathcal{T}_{k_0} \cap \mathcal{T}_0$, equations (4.100) and (4.101) can be simply rewritten, for $l = 1 \ldots n+m$ as

$$\epsilon_l \partial_x F_l(x,\xi) + \nu_l \partial_\xi F_l(x,\xi) = \Sigma_l(x,\xi) F(x,\xi),$$

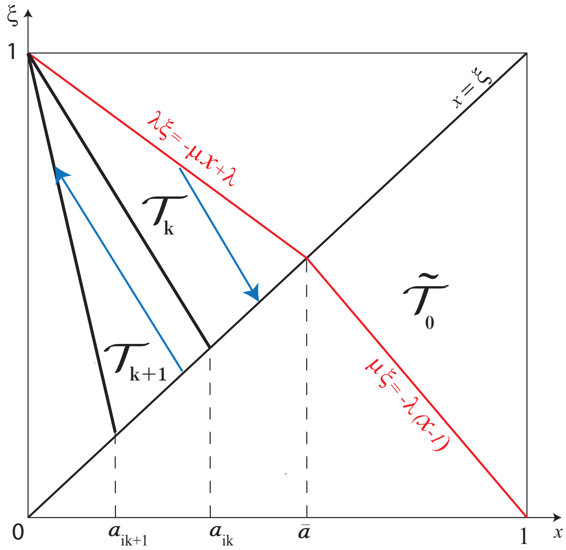

Figure 4.4: Representation of the triangles $\mathcal{T}_k$

where $F = (K_{i,1} \dots K_{i,n}, L_{i,1}, \dots L_{i,m})^T$. The constants ϵ_l and ν_l are defined according to the location of the boundary condition by

$$\epsilon_l = \begin{cases} -\lambda_i & \text{if} \quad l \leq n, \\ -\lambda_i & \text{if} \quad l > n \quad \text{and} \quad l - n < k_0, \\ +\lambda_i & \text{else.} \end{cases}$$

$$\nu_l = \begin{cases} -\lambda_l & \text{if} \quad l \leq n, \\ +\mu_l & \text{if} \quad l > n \quad \text{and} \quad l - n < k_0, \\ -\mu_l & \text{else.} \end{cases}$$

The homogeneous system, obtained by taking $\Sigma_l(x,\xi) = 0$ along with the corresponding boundary conditions is well-posed. If we choose α_{k_0} such that

$$\frac{\mu_{k_0} - 1}{\lambda_i} < \alpha_{k_0} = \frac{\mu_{k_0 - 1} + \mu_{k_0}}{2\lambda_i} < \frac{\mu_{k_0}}{\lambda_i},$$

we easily get

$$\alpha_{k_0}\epsilon_l + \nu_l = \begin{cases} \frac{-\mu_{k_0-1} - \mu_{k_0}}{2} - \lambda_l \text{ if } l \leq n, \\ \frac{-\mu_{k_0-1} - \mu_{k_0}}{2} + \mu_l \text{ if } l > n \text{ and } l - n < k_0, \\ \frac{+\mu_{k_0-1} + \mu_{k_0}}{2} - \mu_l \text{ else.} \end{cases}$$

In the first case, the result is always negative. If $l - n < k_0$, $\mu_l < \mu_{k_0} < \frac{\mu_{k_0} + \mu_{k_0-1}}{2}$. Consequently, for the second case the result is still negative. The same holds for the third case. Consequently, the two hypothesis of Lemma 4.2.2 are verified and we can conclude to the well-posedness of the kernel equations on $\mathcal{T}_{k_0}$. This concludes the initialization.

Recursion If we assume that $\mathcal{Q}(k)$ holds (for $k_0 \geq k < m$) we can easily prove using Lemma 4.2.2 that $\mathcal{Q}(k+1)$ holds. The well-posedness of the homogeneous system is direct using $\mathcal{Q}(k)$ and one can easily check that the second hypothesis of the theorem holds choosing α_{k+1}:

$$\alpha_{k+1} = \frac{\mu_{k+1} + \mu_k}{2},$$

with the convention $\mu_{m+1} = \mu_m + 1$. Moreover this iteration provides us the boundary condition for the next triangle. This concludes the proof.

Remark 4.2.10 *If $a_{i1} < \bar{a}$, the previous result still holds taking $a_{i0} = \bar{a}$.*

This proves the well-posedness of the i^{th} line of the kernels K and L on $\mathcal{T}_0$. Consequently $P(q)$ is true and the well-posedness of the kernels $K, L, \bar{K}, \bar{L}$ on $\mathcal{T}_0$ is proved.

4.2.5 Invertibility of the Fredholm transformation

Unlike the Volterra transformation, the Fredholm transformation is not always invertible [CHO16]. Proving the invertibility of such a transformation is a requirement for the Fredholm backstepping, otherwise the control design cannot be defined. In [CHO16], using the Fredholm alternative, the authors propose sufficient and necessary conditions for the invertibility of the Fredholm transformation for a general class of partial differential equations. In this section, we apply [CHO16, Proposition 2.6] to prove the invertibility of our transformation. Note that, as mentioned in Remark 4.2.6, rewriting the transformation(4.47)-(4.48) as a Volterra transformation would automatically imply its invertibility.

Operator formulation

As the proof developed in this section require an operator formulation of the target system (4.38)-(4.39), we adjust the framework introduced in Section 2.1 to rewrite system (4.38)-(4.39) as

$$\frac{d}{dt}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = A_0\begin{pmatrix}\alpha\\ \beta\end{pmatrix}. \tag{4.103}$$

The operator A_0 is defined by

$$\begin{aligned} A_0 : D(A_0) \subset (L^2(0,1))^2 &\to (L^2(0,1))^2 \\ \begin{pmatrix}\alpha\\ \beta\end{pmatrix} &\longmapsto \begin{pmatrix}-\Lambda^+\alpha_x + \Omega\alpha + \Gamma\beta\\ \Lambda^-\beta_x + \bar{\Omega}\beta + \bar{\Gamma}\alpha\end{pmatrix}, \end{aligned}$$

with

$$D(A_0) = \{(\alpha,\beta) \in (H^1(0,1))^2 |\ \alpha(0) = \beta(1) = 0\}.$$

The operator A_0 is well defined and its adjoint A_0^* is

$$\begin{aligned} A_0^* : D(A_0^*) \subset (L^2(0,1))^2 &\to (L^2(0,1))^2 \\ \begin{pmatrix}\alpha\\ \beta\end{pmatrix} &\longmapsto \begin{pmatrix}\alpha_x^T\Lambda^+ + \alpha^T\Omega + \beta^T\bar{\Gamma}\\ -\beta_x^T\Lambda^- + \beta^T\bar{\Omega} + \alpha^T\Gamma\end{pmatrix}^T, \end{aligned}$$

with

$$D(A_0^*) = \{(\alpha,\beta) \in (H^1(0,1))^2 |\ \alpha(1) = \beta(0) = 0\}.$$

Operator formulation of the Fredholm transformation and properties

We rewrite the previous Fredholm transformation using operators. This will lead to some relations verified by the adjoint operators. The Fredholm transformation (4.47)-(4.48) can be written as an operator P acting on $\begin{pmatrix}u\\ v\end{pmatrix}$. More precisely we have

$$P = Id - Q \quad \text{and} \quad \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = P\begin{pmatrix}u\\ v\end{pmatrix}, \tag{4.104}$$

where $Q : (L^2(0,1))^{n+m} \to (L^2(0,1))^{n+m}$ is the integral operator defined by

$$Q\begin{pmatrix} u \\ v \end{pmatrix} = \int_0^1 \begin{pmatrix} Q_{11}(x,\xi)u(t,\xi) + Q_{12}(x,\xi)v(t,\xi) \\ Q_{21}(x,\xi)u(t,\xi) + Q_{22}(x,\xi)v(t,\xi) \end{pmatrix} d\xi.$$

Its adjoint is:

$$Q^*\begin{pmatrix} u \\ v \end{pmatrix} = \int_0^1 \begin{pmatrix} Q_{11}(\xi,x)u(t,\xi) + Q_{21}(\xi,x)v(t,\xi) \\ Q_{12}(\xi,x)u(t,\xi) + Q_{22}(\xi,x)v(t,\xi) \end{pmatrix} d\xi.$$

One can easily check that $Q^*(D(A^*)) \subset D(A^*)$ (where the operators A and A^* are defined in (2.6)). The control $\begin{pmatrix} U \\ V \end{pmatrix}$ can also be rewritten using operators

$$\begin{pmatrix} U \\ V \end{pmatrix} = \Gamma \begin{pmatrix} u \\ v \end{pmatrix},$$

with

$$\Gamma\begin{pmatrix} u \\ v \end{pmatrix} = \int_0^1 \begin{pmatrix} Q_{11}(0,\xi)u(t,\xi) + Q_{12}(0,\xi)v(t,\xi) \\ Q_{21}(1,\xi)u(t,\xi) + Q_{22}(1,\xi)v(t,\xi) \end{pmatrix} d\xi.$$

Using (4.103) and (4.104) yields

$$\frac{d}{dt}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A_0\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A_0 P\begin{pmatrix} u \\ v \end{pmatrix}.$$

Moreover using (4.103) and (4.104) we get

$$\frac{d}{dt}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{d}{dt}(P\begin{pmatrix} u \\ v \end{pmatrix}) = PA\begin{pmatrix} u \\ v \end{pmatrix} + PB\Gamma\begin{pmatrix} u \\ v \end{pmatrix},$$

since P commutes with the operator $\frac{d}{dt}$. Consequently P and Γ satisfy the following relation:

$$A_0 P = PA + PB\Gamma \Leftrightarrow P^* A_0^* = A^* P^* + \Gamma^* B^* P^*.$$

Using the Fredholm alternative

We give first the following useful lemmas that are required to apply [CHO16, Proposition 2.6]:

Lemma 4.2.8.

Consider the operators P defined by (4.104), A_0 defined by (4.103), A defined by (2.7) and B defined by (2.8). Thus, $\ker P^* \subset D(A_0^*) = D(A^*)$ and $\ker P^* \subset \ker B^*$.

Proof : Let us consider $z \in \ker P^*$. Consequently we have $P^*z = 0$. We can rewrite it

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \int_0^1 \begin{pmatrix} Q_{11}(\xi,x)z_1(t,\xi) + Q_{21}(\xi,x)z_2(t,\xi) \\ Q_{12}(\xi,x)z_1(t,\xi) + Q_{22}(\xi,x)z_2(t,\xi) \end{pmatrix} d\xi.$$

If we evaluate the first line for $x = 1$ and the second one for $x = 0$, using the fact that $Q_{11}(\xi,0) = Q_{21}(\xi,0) = Q_{12}(\xi,1) = Q_{22}(\xi,1) = 0$, we get $z_1(1) = z_2(0) = 0$. Consequently $z \in D(A_0^*)$. The same can be done to prove that $\ker P^* \subset \ker B^*$. ■

Lemma 4.2.9.

Consider the operators A_0 defined by (4.103) and B defined by (2.8). $\forall \lambda \in \mathbb{C}$ $\ker(\lambda Id - A_0^*) \cap \ker B^* = \{0\}$.

Proof : Let us consider $\nu \in \mathbb{C}$ and $z \in \ker(\nu Id - A_0^*) \cap \ker B^*$. Consequently we have

$$0 = \begin{pmatrix} z_1(t,x)_x^T \Lambda^+ + z_1(t,x)^T \Omega + z_2(t,x)^T \bar{\Gamma} - \nu z_1(t,x)^T \\ -z_2(t,x)_x^T \Lambda^- + z_2(t,x)^T \bar{\Omega} + z_1(t,x)^T \Gamma - \nu z_2(t,x)^T \end{pmatrix},$$

with the boundary conditions $z_1(0) = z_2(0) = 0$. Consequently, (using the Cauchy-Lipschitz' theorem) we have $(z = \begin{pmatrix} 0 & 0 \end{pmatrix})^T$. ■

We can now state the following theorem.

Theorem 4.2.4.

Consider the operator P defined by (4.104). The map $P^* = Id - Q^*$ is invertible.

Proof : Since Q^* is a compact operator we can use the Fredholm alternative (e.g [Bre10]): $Id - Q^*$ is either non-injective or surjective. Consequently, it suffices to prove that P^* is injective. This is quite straightforward using Lemma 4.2.8, Lemma 4.2.9 and [CHO16, Proposition 2.6]. ■

4.2.6 Stabilizing control law

In the previous section we have proved that the set of equations (4.49)-(4.54), (4.61)-(4.66), (4.55)-(4.60) and (4.67)-(4.72) admits a unique solution and that consequently, the transformation (4.47)-(4.48) does exist. We choose the following control law.

$$U(t) = -\int_0^1 (K(0,\xi)u(t,\xi) + L(0,\xi)v(t,\xi))d\xi, \tag{4.105}$$

$$V(t) = -\int_0^1 (\bar{M}(1,\xi)u(t,\xi) + \bar{N}(1,\xi)v(t,\xi))d\xi, \tag{4.106}$$

We now prove that this control law fulfills Objective B. We first have the following result,

Lemma 4.2.10.

There exists an invertible bounded linear map $\mathcal{F} : (L^2[0,1])^{n+m} \to (L^2[0,1])^{n+m}$ such that, in presence of the control law (4.105)-(4.106), for every initial condition $(u_0, v_0) \in (L^2([0,1]))^{n+m}$, if $(\alpha_0, \beta_0) \in (L^2([0,1]))^{n+m}$ denotes the solution to (4.38)-(4.40) satisfying the initial data $(\alpha_0(0,\cdot), \beta_0(0,\cdot)) = \mathcal{F}(u_0, v_0)$, then $(u(t), v(t)) = \mathcal{F}^{-1}(\alpha(t), \beta(t))$.

Proof : Let us consider the invertible bounded linear operator,

$$\begin{aligned} \mathcal{F} : (L^2[0,1])^{n+m} &\to (L^2[0,1])^{n+m} \\ \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} &\mapsto \begin{pmatrix} u(t,x) - \int_0^x (Q_{11}(x,\xi)u(\xi) + Q_{12}(x,\xi)v(\xi))d\xi \\ v(t,x) - \int_0^x (Q_{21}(x,\xi)u(\xi) + Q_{22}(x,\xi)v(\xi))d\xi \end{pmatrix}, \end{aligned} \tag{4.107}$$

which is well-defined due to Theorem 4.2.1 and Theorem 4.2.2. Let $(\alpha, \beta) \in \mathcal{C}^0([0,T], (L^2([0,1]))^{n+m})$ be the solution to (4.38)-(4.40) associated with the initial data $(\alpha(0,\cdot), \beta(0,\cdot)) = \mathcal{F}(u_0, v_0)$. Then, $(u(t), v(t)) = \mathcal{F}^{-1}(\alpha(t), \beta(t))$ is the solution to the Cauchy problem (4.31)-(4.33) along with the initial conditions (u_0, v_0). ■

Finally, we have the following theorem.

Theorem 4.2.5.

Consider system (4.31)-(4.32) along with boundary conditions (4.33) and the feedback control law (4.105)-(4.106). Then, for any initial condition $(u_0, v_0) \in (L^2([0,1]))^{n+m}$, it reaches its zero equilibrium in the minimum finite time $t_F = \max\{\frac{1}{\lambda_1}, \frac{1}{\mu_1}\}$. Moreover, if the initial conditions belong to $(\mathcal{C}^1([0,1]))^{n+m}$ (and satisfy the corresponding compatibility conditions), the control law (4.105)-(4.106) ensures the weak exact boundary controllability in the sense of Theorem (2.3.1).

Proof : The proof is a direct consequence of Lemma 4.2.10 , since for any initial condition in $(L^2([0,1]))^{n+m}$, the system (4.38)-(4.40) reaches its zero equilibrium in finite time t_F. Additionally, if the initial conditions (u_0, v_0) belong to $(\mathcal{C}^1([0,1]))^{n+m}$ (and satisfy the corresponding compatibility conditions), then so do the initial data $(\alpha(0,\cdot), \beta(0,\cdot)) = \mathcal{F}(u_0, v_0)$. Thus, due to Lemma 2.1.2, the solution of (4.38)-(4.40) remains in $(\mathcal{C}^1([0,1]))^{n+m}$ and so do (u,v). ■

4.3 Finite-time boundary observability

In this section we design an observer that relies on the measurements of u at the right boundary and v at the left boundary, i.e we measure

$$y_1(t) = u(t,1) \quad \text{and} \quad y_2(t) = v(t,0).$$

Then, using the estimates given by our observer and the control law (4.105)-(4.106), we derive an output feedback controller. The design of the observer is based on the adjoint method introduced in Chapter 3.

4.3.1 Observer design

The observer equations read as follows

$$\begin{aligned}
\partial_t \hat{u}(t,x) + \Lambda^+ \partial_x \hat{u}(t,x) &= \Sigma^{++}(x)\hat{u}(t,x) + \Sigma^{+-}(x)\hat{v}(t,x) - P_{11}(x)(\hat{u}(t,1) - u(t,1)) \\
&\quad - P_{12}(\hat{v}(t,0) - v(t,0)), \qquad (4.108) \\
\partial_t \hat{v}(t,x) - \Lambda^- \partial_x \hat{v}(t,x) &= \Sigma^{-+}(x)\hat{u}(t,x) + \Sigma^{--}(x)\hat{v}(t,x) - P_{21}(x)(\hat{u}(t,1) - u(t,1)) \\
&\quad - P_{22}(\hat{v}(t,0) - v(t,0)), \qquad (4.109)
\end{aligned}$$

with the boundary conditions

$$\hat{u}(t,0) = U(t), \quad \hat{v}(t,1) = V(t), \tag{4.110}$$

where $P_{11}(\cdot)$, $P_{21}(\cdot)$, $P_{12}(\cdot)$ and $P_{22}(\cdot)$ have yet to be designed. Defining the errors estimates $\tilde{u}(t,x) = u(t,x) - \hat{u}(t,x)$ and $\tilde{v}(t,x) = u(v,x) - \hat{v}(t,x)$, we get the following error system

$$\begin{aligned}
\partial_t \tilde{u}(t,x) + \Lambda^+ \partial_x \tilde{u}(t,x) &= \Sigma^{++}(x)\tilde{u}(t,x) + \Sigma^{+-}(x)\tilde{v}(t,x) - P_{11}(x)\tilde{u}(t,1) \\
&\quad - P_{12}(x)\tilde{v}(t,0), \qquad (4.111) \\
\partial_t \tilde{v}(t,x) - \Lambda^- \partial_x \tilde{v}(t,x) &= \Sigma^{-+}(x)\tilde{u}(t,x) + \Sigma^{--}(x)\tilde{v}(t,x) - P_{21}(x)\tilde{u}(t,1) \\
&\quad - P_{22}(x)\tilde{v}(t,0), \qquad (4.112)
\end{aligned}$$

with the boundary conditions

$$\tilde{u}(t,0) = 0, \quad \tilde{v}(t,1) = 0. \tag{4.113}$$

This system evolves in $[0,T] \times [0,x]$ and its initial condition $(\tilde{u}(0,x), \tilde{v}(0,x)) = (\tilde{u}_0(x), \tilde{v}_0(x))$ belongs to $L^2([0,1])^{(n+m)}$. We define the operator $\bar{A}$ by

$$\begin{aligned}
\bar{A} :& D(\bar{A}) \subset (L^2(0,1))^{n+m} \to (L^2(0,1))^{n+m} \\
&\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \longmapsto \begin{pmatrix} -\Lambda^+ \partial_x \bar{u} + \Sigma^{++}(x)\bar{u} + \Sigma^{+-}(x)\bar{v} - P_{11}(x)\bar{u}(t,1) - P_{12}(x)\bar{v}(t,0) \\ \Lambda^- \partial_x \bar{v} + \Sigma^{-+}(x)\bar{u} + \Sigma^{--}(x)\bar{v} - P_{21}(x)\bar{u}(t,1) - P_{22}(x)\bar{v}(t,0) \end{pmatrix}, \qquad (4.114)
\end{aligned}$$

with

$$D(\bar{A}) = \{(u,v) \in (H^1(0,1))^{n+m} | \bar{u}(0) = \bar{v}(1) = 0\}.$$

We now define the following system (which is derived from (4.111)-(4.113) changing t in $T-t$), evolving in $[0,T] \times [0,x]$:

$$-\frac{d}{dt}\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix} = \bar{A}\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix}. \tag{4.115}$$

Its (arbitrary) initial conditions are defined by

$$\begin{pmatrix}\bar{u}(T,x)\\ \bar{v}(T,x)\end{pmatrix} = \begin{pmatrix}\bar{u}_T(x)\\ \bar{v}_T(x)\end{pmatrix} \in L^2([0,1])^{(n+m)}. \tag{4.116}$$

Note that system (4.115) does not include any control operator since they have been canceled (see equation (4.113)). In what follows, we define a control problem that is the dual of the observer problem. The observer gains are then be defined by the gains of the *dual controller*.

A new control problem

Let us consider the following system

$$\partial_t \phi(t,x) - \Lambda^+ \partial_x \phi(t,x) = (\Sigma^{-+}(x))^T \psi(t,x) + (\Sigma^{++}(x))^T \phi(t,x), \tag{4.117}$$

$$\partial_t \psi(t,x) + \Lambda^- \partial_x \psi(t,x) = (\Sigma^{--}(x))^T \psi(t,x) + (\Sigma^{+-}(x))^T \phi(t,x), \tag{4.118}$$

evolving in $\{(t,x)|\ t>0,\ x \in [0,1]\}$, with the following linear boundary conditions

$$\psi(t,0) = U_0(t), \quad \phi(t,1) = V_0(t), \tag{4.119}$$

and the arbitrary initial conditions (belonging to $(L^2([0,1]))^{(n+m)}$)

$$\phi(0,x) = \phi_0(x), \quad \psi(0,x) = \psi_0(x).$$

Using Theorem 4.2.5, we can explicitly compute kernels $K_1, L_1, \bar{M}_1$ and $\bar{N}_1$ such that system (4.117)-(4.118) with the following feedback law

$$U_0(t) = -\int_0^1 (K_1(0,\xi)\phi(t,\xi) + L_1(0,\xi)\psi(t,\xi))d\xi, \tag{4.120}$$

$$V_0(t) = -\int_0^1 (\bar{M}_1(1,\xi)\phi(t,\xi) + \bar{N}_1(1,\xi)\psi(t,\xi))d\xi, \tag{4.121}$$

reaches its zero equilibrium in time t_F. As seen in Section 4.2.5, this system can be rewritten in the abstract form

$$\frac{d}{dt}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = A_1\begin{pmatrix}\phi\\ \psi\end{pmatrix} + B_1\begin{pmatrix}V_0\\ U_0\end{pmatrix}, \tag{4.122}$$

with the initial condition

$$\begin{pmatrix}\phi(0,x)\\ \psi(0,x)\end{pmatrix} = \begin{pmatrix}\phi_0(x)\\ \psi_0(x)\end{pmatrix}, \tag{4.123}$$

where the operators A_1 and B_1 are defined in a similar form as the ones presented in equation (2.6).

Remark 4.3.1 *Using Theorem 4.2.5, we have $\phi(t,x) = \psi(t,x) = 0$ for $t \geq t_F$.*

We now define the gains of the observer as

$$P_{11}(x) = \bar{M}_1(1,x)^T \Lambda^+, \quad P_{21}(x) = \bar{N}_1(1,x)^T \Lambda^+, \tag{4.124}$$

$$P_{12}(x) = K_1(0,x)^T \Lambda^-, \quad P_{22}(x) = L_1(0,x)^T \Lambda^-. \tag{4.125}$$

Theorem 4.3.1.

Consider system (4.31)-(4.32) along with boundary conditions (4.33) and the feedback control law (4.105)-(4.106). Consider the observer system (4.111)-(4.113) (where the observer gains are defined by (4.124)-(4.125)). Then, for any initial condition $(u_0, v_0) \in (L^2([0,1]))^{n+m}$ of the original system, for any observer initial condition $(\hat{u}_0, \hat{v}_0) \in (L^2([0,1]))^{n+m}$ the observer state $(\hat{u}, \hat{v})$ converges (in the sense of the L^2-norm) to the real state in finite time t_F. Moreover, if the initial conditions (u_0, v_0) and $(\hat{u}_0, \hat{v}_0)$ belong to $(\mathcal{C}^1([0,1]))^{n+m}$ (and satisfy the corresponding compatibility conditions), the observer system (4.111)-(4.113) ensures the weak exact boundary observability in the sense of Theorem (2.3.3).

Proof : We recall that we denote $\langle \cdot, \cdot \rangle$ the scalar product associated to the L^2-norm. For every solution $(\bar{u}, \bar{v})$ of (4.115)-(4.116) and every solution (ϕ, ψ) of (4.122)-(4.123) (with any initial conditions) we have

$$\langle \frac{d}{dt}\begin{pmatrix}\phi \\ \psi\end{pmatrix} - A_1 \begin{pmatrix}\phi \\ \psi\end{pmatrix} - B_1 \begin{pmatrix}V_0 \\ U_0\end{pmatrix}, \begin{pmatrix}\bar{u} \\ \bar{v}\end{pmatrix} \rangle = 0, \tag{4.126}$$

since the left part of the scalar product is zero (due to (4.122)). This yields

$$\begin{aligned} 0 = &\int_0^\tau \int_0^1 (-\partial_t \bar{u}^T(t,x) + \partial_x \bar{u}^T(t,x)\Lambda^+ - \bar{u}^T(t,x)(\Sigma^{++})^T - \bar{v}(t,x)^T(\Sigma^{+-})^T)\phi(t,x) + (-\partial_t \bar{v}^T(t,x) - \partial_x \bar{v}^T(t,x)\Lambda^- \\ &- \bar{u}(t,x)(\Sigma^{-+})^T - \bar{v}(t,x)(\Sigma^{--})^T)\psi(t,x)dxdt + \int_0^1 (\bar{v}^T(\tau,x)\psi(\tau,x) - \bar{v}^T(0,x)\psi(0,x) \\ &+ \bar{u}^T(\tau,x)\phi(\tau,x) - \bar{u}^T(0,x)\phi(0,x))dx + \int_0^\tau \int_0^1 (\bar{u}(t,1)^T\Lambda^+(\bar{M}_1(1,x)\phi(t,x) + \bar{N}_1(1,x)\psi(t,x)) \\ &+ \bar{v}(t,0)^T\Lambda^-(K_1(0,x)\phi(t,x) + L_1(0,x)\psi(t,x)))dt. \end{aligned} \tag{4.127}$$

Computing $\langle \begin{pmatrix}\phi \\ \psi\end{pmatrix}, -\frac{d}{dt}\begin{pmatrix}\bar{u} \\ \bar{v}\end{pmatrix} - \bar{A}\begin{pmatrix}\bar{u} \\ \bar{v}\end{pmatrix}\rangle$, one easily obtains

$$\begin{aligned} \langle \begin{pmatrix}\phi \\ \psi\end{pmatrix}, -(\frac{d}{dt} + \bar{A})\begin{pmatrix}\bar{u} \\ \bar{v}\end{pmatrix}\rangle &= \langle (\frac{d}{dt} - A_1)\begin{pmatrix}\phi \\ \psi\end{pmatrix} - B_1\begin{pmatrix}V \\ U\end{pmatrix}, \begin{pmatrix}\bar{u} \\ \bar{v}\end{pmatrix}\rangle \\ &- \int_0^1 (\bar{v}^T(\tau,x)\psi(\tau,x) - \bar{v}^T(0,x)\psi(0,x)dx - \int_0^1 \bar{u}^T(\tau,x)\phi(\tau,x) - \bar{u}^T(0,x)\phi(0,x))dx. \end{aligned}$$

Both scalar products are equal to zero due to (4.115) and (4.122). Choosing $\tau = t_F$, and using Remark 4.3.1, one can cancel some of the remaining integrals and finally obtain

$$0 = \int_0^1 \bar{u}^T(0,x)\phi(0,x) + \bar{v}^T(0,x)\psi(0,x)dx.$$

This has to be true for any initial condition $\phi(0,x)$ and $\psi(0,x)$. It implies that $\bar{u}(0)$ and $\bar{v}(0)$ are equal to zero almost everywhere. Consequently, using the change of variable $r = t_F - t$ we obtain $\tilde{u}(t_F)$ and $\tilde{v}(t_F)$ are equal to zero almost everywhere. This concludes the proof. ■

4.3.2 Output feedback controller

The estimates can be used in an observer-controller to derive an output feedback law yielding finite-time stability of the zero equilibrium.

Theorem 4.3.2.

Consider the system composed of (4.31)-(4.33) and of the observer (4.108)-(4.110) with the following control laws

$$U(t) = -\int_0^1 (K(0,\xi)\hat{u}(t,\xi) + L(0,\xi)\hat{v}(t,\xi))d\xi, \tag{4.128}$$

$$V(t) = -\int_0^1 (\bar{M}(1,\xi)\hat{u}(t,\xi) + \bar{N}(1,\xi)\hat{v}(t,\xi))d\xi, \tag{4.129}$$

where K, L and $\bar{M}, \bar{N}$ are defined by (4.49)-(4.54) and (4.67)-(4.72). Its solution (u, v) converges to zero in finite time $\tau \leq 2t_F$.

Proof : The proof is straightforward and similarly to the proof of Theorem 3.2.2, we start proving the L^2 convergence of the observer before proving the stabilization of the state. The convergence of the observer error states $\tilde{u}, \tilde{v}$ to zero for $t_F \leq t$, in the sense of the L^2-norm, is ensured by Theorem 4.3.1. Thus, once $t_F \leq t$, $v(t,0) = \hat{v}(t,0)$ almost everywhere and one can use Theorem 4.2.5. Therefore for $2t_F \leq t$, one has $(u, v) \equiv 0$. ■

4.4 Application: state estimation during UnderBalanced Drilling

In this section, we illustrate the benefits of our approach by applying it to an industrial problem.

Problem description

Consider the drilling system schematically depicted in Figure 4.5. It consists of a 2530 meter-long drillpipe, rotating around its main axis, through which a drilling fluid is injected, typically water-based mud. At the end of the pipe, the fluid exits through the drillbit (which cuts the rock) and circulates back to the surface inside the annulus, carrying rock cuttings. For numerous reasons [Aar16], it is desirable to produce oil and gas from the reservoir as the drilling process goes on when possible, a technique referred to as UnderBalanced Drilling (UBD). The term *underbalanced* refers to the value of the pressure at the bottom of the annulus, which must be lower than the value of the pressure of hydrocarbons inside the reservoir (called the balance point) for the oil and gas to flow in. To ensure safety and efficiency of operations, it is desirable to monitor at all time the amount of gas inside the well. This is a difficult task since sensors cannot be placed all along the drillpipe, and the dynamics of multiphase flow are known to be complex [ADMG+16]. The presence of gas, in particular, makes the distributed, delay-like nature of the dynamics predominant and may generate instabilities, such as severe slugging.

Modelling of the system

The simulation model is a Drift-Flux Model (DFM) described in [ADMEA14]. It models the flow of liquid (oil, water and drilling fluid being considered as one liquid phase) and gas along the drillstring using two mas conservation laws and one combined momentum conservation law. Along with closure relations, this yields a set of three nonlinear transport PDEs with appropriate boundary conditions [ADMEA14]. For $k = L, G, m$ denoting liquid, gas or mixture, we denote α_k the volume fractions, ρ_k the densities, ν_k the superficial velocities, f the friction factor, D the hydraulic diameter, and P the pressure. The space x corresponds to a curvilinear abscissa with $x = 0$ at the bottom hole and $x = L = 2530$ at the outlet choke position. The equations are as follows:

$$\partial_t(\alpha_L\rho_L) + \partial_x(\alpha_L\rho_L\nu_L) = 0, \tag{4.130}$$

$$\partial_t(\alpha_G\rho_G) + \partial_x(\alpha_G\rho_G\nu_G) = 0, \tag{4.131}$$

$$\partial_t(\alpha_L\rho_L\nu_L + \alpha_G\rho_G\nu_G) + \partial_x(P + \alpha_L\rho_L\nu_L^2 + \alpha_G\rho_G\nu_G^2) = -\rho_m g \sin\phi(x) - \frac{2f\rho_m\nu_m|\nu_m|}{D}. \tag{4.132}$$

In equation (4.132), the term $(\rho_m g \sin\phi(x))$ represents the gravitational source terms, while the term $\frac{2f\rho_m\nu_m|\nu_m|}{D}$ accounts for frictional losses. The mixtures velocity are

$$\rho_m = \alpha_G\rho_G + \alpha_L\rho_L, \quad \nu_m = \alpha_G\nu_G + \alpha_L\nu_L.$$

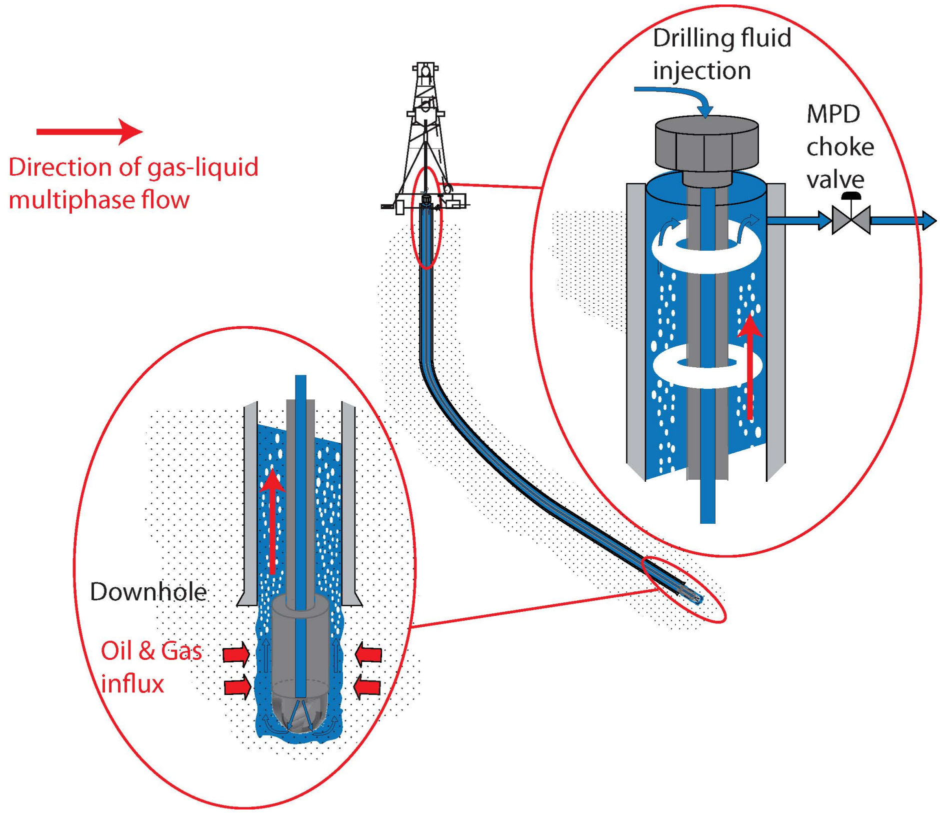

Figure 4.5: Schematic view of drilling facilities. In UnderBalanced Drilling (UBD), oil and gas enter the annulus as the drilling process goes on.

Along with these distributed equations, algebraic relations are needed to define the system.

$$\alpha_L + \alpha_G = 1, \quad V_G = C_0\nu_m + \nu_\infty, \quad \rho_G = Z_G R_G T P, \quad \rho_L = \text{constant},$$

where Z_G, R_G, T are the gas compression factor, specific gas constant and temperature respectively, and C_0, ν_∞ are the slip parameters giving the slip between the velocity of the gas and liquid phase. They are all constant. The boundary conditions on the left (downhole) boundary are given by the mass-rates of gas and liquid injected from the drilling rig and flowing in from the reservoir

$$A\alpha_L(t,0)\rho_L(t,0)\nu_L(t,0) = k_L \max(P(t,0) - P_{res}, 0) + W_{L,inj}(t), \tag{4.133}$$

$$A\alpha_G(t,0)\rho_G(t,0)\nu_G(t,0) = k_G \max(P(t,0) - P_{res}, 0) + W_{G,inj}(t), \tag{4.134}$$

where the injection mass-rates of gas and liquid, $W_{G,inj}$, $W_{L,inj}$ are specified by the driller and can with some constraints be considered as inputs, P_{res} is the reservoir pore pressure and k_G, k_L are the production index of the gas and liquid respectively and A is the area of flow. The topside boundary condition is given by a choke equation relating topside pressure to mass flow rates

$$A(\alpha_L(t,L)\rho_L(t,L)\nu_L(t,L) + \alpha_G(t,L)\rho_G(t,L)\nu_G(t,L)) = C_\nu Z(t)\frac{\sqrt{P(t,L) - P_S}}{\frac{x_L}{\sqrt{\rho_L(t,L)}} + \frac{x_G}{Y^2\sqrt{\rho_G}(t,L)}}, \tag{4.135}$$

where $x_{L,G}$ denotes the mass fraction of liquid and gas, $C_\nu(Z)$ the choke opening given by the manipulated variable, Z and Y is a correction factor for gas flow.

Remark 4.4.1 *This example actually corresponds to the case for which there is no flow of injected gas/drilling mud from the drilling rig. This is a very specific case that corresponds rather* *to an oil production system than a drilling process (for which the circulation of the drilling mud is a normal scenario). The choice of such a specific scenario has been made for sake of simplicity. For the case of constant gas/liquid injection rates, a supplementary state (that would involve a ratio liquid/liquid) would have to be considered. This obviously imply having the possibility to obtain the associated measurement (using for instance a multiphase flowmeter).*

Linearization of the equations

As detailed in, e.g., [ADMG+16], the system of equations (4.130)-(4.132) with the boundary conditions (4.133)-(4.135) can be written in the quasilinear form

$$\partial_t q(t,x) + A[q(t,x)]\partial_x q(t,x) = S[q(t,x)], \tag{4.136}$$

with the boundary conditions

$$h_1[q(t,0),U(t)] = h_2[q(t,0),U(t)] = h_3[q(t,1),V(t)] = 0, \tag{4.137}$$

where $q = [\frac{\alpha_G \rho_G}{\alpha_G \rho_G + \alpha_L \rho_L},\ \nu_G,\ P]$ is the state vector and the actuation acts through the exogenous variable $U(t) = [W_{L,inj}(t)\ W_{G,inj}]^T$ and $V(t) = Z(t)$. The operator A and S are not detailed here. In what follows we consider only the estimation problem and for convenience set $U(t) = V(t) = 0$. Let $\tilde{q}(t,x) = q(t,x) - \bar{q}(t,x)$ denote the distance from some equilibrium profile. Close to this equilibrium profile, the dynamics of the system can be approximated by the linear system [DM11]

$$\partial_t \tilde{q}(t,x) + A(\bar{q}(x))\partial_x \tilde{q}(t,x) = S(x)\tilde{q}(t,x), \tag{4.138}$$

where the matrix A is a diagonal matrix whose components are C^1 functions:

$$A(\bar{q}(x)) = \begin{pmatrix} \lambda_1(\bar{q}(x)) & 0 & 0 \\ 0 & \lambda_2(\bar{q}(x)) & 0 \\ 0 & 0 & -\mu_1(\bar{q}(x)), \end{pmatrix} \tag{4.139}$$

and where $S(x)$ is a continuous matrix whose structure is given by

$$S(x) = \begin{pmatrix} 0 & S_{1,2}(x) & S_{1,3}(x) \\ 0 & 0 & 0 \\ S_{3,1}(x) & S_{3,2}(x) & 0. \end{pmatrix} \tag{4.140}$$

For every $0 < x < L$, $0 < \lambda_1(\bar{q}(x)) < \lambda_2(\bar{q}(x))$ and $\mu_1(\bar{q}(x)) > 0$. Eventually the linearized boundary conditions read

$$q_1(t,0) = k_1 q_3(t,0), \quad q_2(t,0) = k_2 q_3(t,0), \tag{4.141}$$
$$q_3(t,L) = k_3 q_1(t,L) + k_3' q_1(t,L), \tag{4.142}$$

for some constants k_1, k_2, k_3 and k_3' The model (4.138) along with the boundary conditions (4.141)-(4.142) is of the form (2.1)-(2.3) with $n = 2$ and $m = 1$.

Observer design

The model used to design the observer is (4.138) along with the boundary conditions (4.141)-(4.142), starting from a different initial condition. This choice is debatable, since perfect knowledge of the model is unrealistic, however assessing the robustness of the model with respect to model uncertainty is out of the scope of this chapter. These aspects are considered in Chapter 8 in which we show that one may have to renounce to finite-time observability to guarantee the existence of decent robustness margins (and in particular robustness with respect to delays on

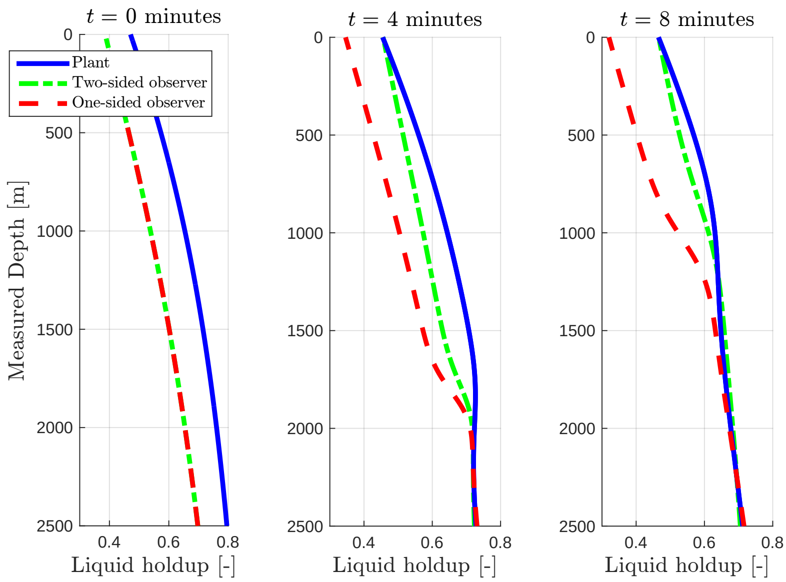

Figure 4.6: Snapshots of the spatial profile of the liquid holdup as a function of depth. The profile of the two-sided observer perfectly converges to the true profile in approximately 25 minutes. Much earlier, it already provides results surpassing alternative techniques using only measurements from one side (red curve).

the measurements). More precisely, the output injection in the observer design do not ensure good filtering properties. This aspect should obviously be considered in details for real implementation. Notice that the nonlinearity of the model, combined with the offset in the initial condition already yields difference in the linearized model parameters.

Similarly to [DMKV13], the observer is designed by copying *the nonlinear* equations and adding the *linear* output error correction terms such that linearizing the observer equations yields (4.108)-(4.110). The observer gains are obtained numerically solving the kernel equations integrating them along their characteristic lines and using a fixed-point algorithm (successive approximations) (the algorithm is similar to the one described in the previous chapter). According to the numerical values of the different velocities that appear in system (4.138), the expected minimum time for the exact convergence would be of 12 minutes.

Simulation results

Figure 4.6 depicts snapshots of the volume fraction of liquid (*holdup*), as a function of well depth at different time instants of a transient simulation. The model used to simulate the real system is the one given by (4.130)-(4.132) while the observer gains are obtained using the linear model (4.138) along with the boundary conditions (4.141)-(4.142). The three curves respectively correspond to the “plant", the “Two-sided observer" (4.108)–(4.110) and the “One-sided observer" described in [DMVK13], that uses only the bottom boundary sensor. The convergence time of the two-sided observer is larger than the theoretical one (which could be expected due to the presence of the non-linearites). However, it presents better performance than the current alternative techniques and in particular than the one-sided observer.

4.5 Conclusion

Using a Fredholm backstepping approach we have presented a stabilizing boundary state feedback law for the general class of linear first-order systems (4.31)-(4.33) controlled at both boundaries. Using the adjoint method introduced in Chapter 3, we have derived the corresponding boundary observer. For L^2 initial conditions, The proposed control law (resp. observer) satisfies Objective B (resp. B') and ensures the convergence of the state (resp. of the estimation error) to zero (in the sense of the L^2-norm) in finite time t_F which is the largest time between the two transport times in each direction. Moreover, in the case of $\mathcal{C}^1$ initial conditions, the proposed observer-controller is an explicit solution to the problems of weak exact boundary controllability and observability stated by Tatsien Li [LR10].

Chapter 5

An explicit mapping from LFOH PDEs to neutral systems

Chapitre 5 Un isomorphisme entre EDPs HLPO et systèmes neutres. *Dans ce chapitre nous utilisons les outils des précédents chapitres (transformations de Volterra et de Fredholm) pour mettre en relation systèmes linéaires hyperboliques du premier ordre et systèmes neutres d'ordre zéro à retards distribués. En considérant le cas de la stabilisation unilatérale (i.e* $U(t) \equiv 0$*), nous montrons que les espaces générés par les solutions des deux systèmes sont isomorphes et que les trajectoires des solutions respectives ont des propriétés de stabilité équivalentes. Cette relation induit un nouvel outil pour l'étude des systèmes hyperboliques et la synthèse de lois de commande.*

Contents

In this chapter we use the tools (Volterra and Fredholm transformations) of the previous chapters to derive a general result on the link between first order linear hyperbolic systems and zero order neutral delay systems with distributed delays. Assuming that the control law $U(t)$ is set to zero (one-sided control situation), we show that the spaces generated by the solutions of both systems are isomorphic and that the trajectories of the solutions have equivalent stability properties. This relation yields a new tool for the study of hyperbolic systems and the design of stabilizing control laws. More precisely, inspired by the notations of [HVL93], we introduce the following definitions. We denote the functional space where the PDE states are defined as

$$\chi \doteq (L^2([0,1];\mathbb{R}))^n \times (L^2([0,1];\mathbb{R}))^m, \tag{5.1}$$

with the associated norm

$$||\phi,\psi||_\chi \doteq \left(\int_0^1 \phi^T(\nu)\phi(\nu)d\nu + \int_0^1 \psi^T(\nu)\psi(\nu)d\nu\right)^{\frac{1}{2}}, \tag{5.2}$$

for any $(\phi, \psi) \in \chi$. Again, we define the **characteristic time** of the system τ as the sum of the two largest transport times in each direction:

$$\tau = \frac{1}{\lambda_1} + \frac{1}{\mu_1}. \tag{5.3}$$

Note that this characteristic time corresponds to the minimal stabilization time given in 3.4. We let $D = L^2([-\tau, 0], \mathbb{R}^m)$ the Banach space of L^2 functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^m$. For a function $\phi : [-\tau, \infty) \to \mathbb{R}$, we define its partial trajectory $\phi_{[t]} \in D$ by $\phi_{[t]}(\theta) \doteq \phi(t+\theta)$, $-\tau \leq \theta \leq 0$. The associated norm is given by

$$||\phi_{[t]}||_D \doteq \left(\int_{-\tau}^{0} \phi^T(t+\theta)\phi(t+\theta)d\theta \right)^{\frac{1}{2}}. \tag{5.4}$$

By convention, we use (without any ambiguity depending on the considered state-space) one of these two norms when referring to the stability of a system. Let us now consider the operator $\mathcal{L}$ defined by

$$\begin{aligned} &\mathcal{L} : D \to \mathbb{R}^m \\ &\phi_{[t]} \mapsto \sum_{k=1}^{p} A_k \phi_{[t]}(-\tau_k) + \int_0^{\tau} f(\nu)\phi_{[t]}(-\nu)d\nu, \end{aligned} \tag{5.5}$$

where $p \in \mathbb{N}$, $A_k \in \mathfrak{M}_{m\times m}(\mathbb{R})$, where the $\tau \geq \tau_k > 0$ represent time-delays and where f is a L^∞ function. The linear operator $\mathcal{L}$ is bounded from D to $\mathbb{R}^m$. We consider the system

$$\phi_{[t]} = \mathcal{L}\phi_{[t]} + V(t), \quad t \geq 0, \tag{5.6}$$

with initial data given by $\phi_0 = g$ where g belongs to D and where we still denote V the input function that has values in $\mathbb{R}^m$. A function $\phi : [-\tau, \infty) \to \mathbb{R}^m$ is called a solution of the initial value problem (5.6) if $\phi_0 = g$ and if (5.6) is satisfied for $t \geq 0$. We prove in this chapter that given a linear feedback law V, there exists an explicit mapping from the space generated by the solutions of (5.6) and the space generated by the solutions of (2.1)-(2.3). The proposed approach is the following: using the Volterra and the Fredholm backstepping transformations introduced [CHO17], we perform a variable change that maps the original system to a neutral system with distributed delays. Its coefficients and kernels are expressed, through the backstepping kernels, as functions of the original system parameters. The content of this chapter has been published in [AAMDM18] for the case of two equations and submitted in [ADMew] for the general case.

5.1 Tutorial case of two equations

In this section we consider the simple case of two coupled equations ($n = m = 1$). The original system (2.1)-(2.3) rewrites

$$\partial_t u(t,x) + \lambda \partial_x u(t,x) = \sigma^{+-}(x)v(t,x), \tag{5.7}$$

$$\partial_t v(t,x) - \mu \partial_x v(t,x) = \sigma^{-+}(x)u(t,x), \tag{5.8}$$

with the following linear boundary conditions

$$u(t,0) = qv(t,0), \quad v(t,1) = \rho u(t,1), \tag{5.9}$$

As above, the velocities λ and μ are assumed to be strictly positive, the distal reflection $q \neq 0$ and the proximal reflection ρ are constant. The in-domain couplings belong to $\mathcal{C}^0([0,1], \mathbb{R})$. The states u and v have values in $\mathbb{R}$ and the corresponding initial condition is denoted (u_0, v_0). It belongs to $(L^2([0,1], \mathbb{R}))^2$. Note that as explained in Remark 2.4.1, the coefficients $\sigma^{++}(x)$ and $\sigma^{--}(x)$ are considered equal to zero, without any loss of generality. As the purpose of this section is to derive the explicit mapping rather than the design of a stabilizing control law, for sake of simplicity we set $V(t) \equiv 0$.

5.1.1 Expression as a neutral system with distributed delays

Considering the backstepping transformation (2.37)-(2.38), it has been proved in section 2.4 that it maps the system (5.7)-(5.9) to the system

$$\partial_t\alpha(t,x)+\lambda\partial_x\alpha(t,x)=0,\quad \partial_t\beta(t,x)-\mu\partial_x\beta(t,x)=0, \tag{5.10}$$

along with the boundary conditions

$$\alpha(t,0)=q\beta(t,0), \tag{5.11}$$

$$\beta(t,1)=\rho\alpha(t,1)-\int_0^1 N^\alpha(\xi)\alpha(t,\xi)+N^\beta(\xi)\beta(t,\xi)d\xi, \tag{5.12}$$

where the functions N^α and N^β are defined by (2.58). We recall that, given an initial condition $(\alpha_0,\beta_0)\in(L^2([0,1]))^2$, the target system (5.10)-(5.12) has a unique weak solution $\alpha(t,x)$ and $\beta(t,x)$ verifying $(\alpha(t,\cdot),\beta(t,\cdot))\in(L^2([0,1]))^2$, $\forall t\geq 0$.

Distributed delay form

We now prove that the state $\beta(t,1)$ can be rewritten as the solution of a neutral equation with distributed delays. Due to the difficulty to deal with the initial condition we start by proving it in the easier case for which for $t\leq\tau$. For all $x\in[0,1]$ and for all $t\geq\tau$, using the method of characteristics on transport equations (5.10) yields

$$\beta(t,x)=\beta(t-\frac{1-x}{\mu},1), \tag{5.13}$$

$$\alpha(t,x)=\alpha(t-\frac{x}{\lambda},0)=q\beta(t-\frac{x}{\lambda},0)=q\beta(t-\frac{x}{\lambda}-\frac{1}{\mu},1). \tag{5.14}$$

Consequently, combining this with the boundary condition (5.12), we get

$$\begin{aligned}\beta(t,1)&=\rho\alpha(t,1)-\int_0^1 N^\alpha(\xi)\alpha(t,\xi)+N^\beta(\xi)\beta(t,\xi)d\xi\\&=\rho q\beta(t-\tau,1)-\int_0^1 qN^\alpha(\xi)\beta(t-\frac{\xi}{\lambda}-\frac{1}{\mu},1)+N^\beta(\xi)\beta(t-\frac{1-\xi}{\mu},1)d\xi\\&=\rho q\beta(t-\tau,1)-\int_{\frac{1}{\mu}}^{\tau} q\lambda N^\alpha(\lambda\nu-\frac{\lambda}{\mu})\beta(t-\nu,1)d\nu-\int_0^{\frac{1}{\mu}}\mu N^\beta(1-\mu s)\beta(t-s,1)ds,\end{aligned}$$

where we have used the change of variable $\nu=\frac{\xi}{\lambda}+\frac{1}{\mu}$ in the first integral and the change of variable $s=\frac{1-\xi}{\mu}$ for the second one. We now define N as

$$N(\nu)=q\lambda N^\alpha(\lambda\nu-\frac{\lambda}{\mu})\mathbb{1}_{[\frac{1}{\mu},\tau]}(\nu)+\mu N^\beta(1-\mu\nu)\mathbb{1}_{[0,\frac{1}{\mu}]}(\nu). \tag{5.15}$$

Thus, we finally get for all $t\geq\tau$,

$$\beta(t,1)=\rho q\beta(t-\tau,1)-\int_0^\tau N(\nu)\beta(t-\nu,1)d\nu. \tag{5.16}$$

Notice that N^α,N^β and N are the trace of H^1 functions along one of the boundary of the triangular domain $\mathcal{T}$ and therefore $N^\alpha,N^\beta\in L^2([0,1])$ and $N\in L^2([0,\tau])\subset L^1([0,\tau])$. It is proved in [BSBAA+18] that N can be expressed in terms of Bessel functions if the parameters σ^{+-} and σ^{-+} are constant. In this case $N\in\mathcal{C}^0([0,\tau])$. The continuity of N at $\frac{1}{\mu}$ is ensured due to the same expression satisfied by $N(\theta)$ for $\theta\in[0,\frac{1}{\mu}]$ and $\theta\in(\frac{1}{\mu},\tau]$. We now prove that $\beta(t,1)$ can be expressed as a function of the initial conditions for $t<\tau$.

Lemma 5.1.1.

For $t < \tau$, the function $\beta(t,1)$ can be expressed as a function of $(\alpha_0(\cdot),\beta_0(\cdot)) = (\alpha(0,\cdot),\beta(0,\cdot))$, the initial conditions of (5.10)-(5.12). Thus, $\beta_{[t]}(\cdot,1) \in L^2((-\tau,0];\mathbb{R})$, $\forall t \geq \tau$.

Proof : We solve $\alpha(t,x)$ and $\beta(t,x)$ for $t \in [0,\tau]$ through the method of characteristics as function of the initial conditions and $\beta(\cdot,1)$. We then solve $\beta(t,1)$ using the solution of $\alpha(t,x)$ and $\beta(t,x)$ and the boundary condition (5.12) and show that it verifies a Volterra integral equation of the second kind, which has a unique square integrable solution.
We consider in the proof that $\lambda \geq \mu$, and the case where $\lambda < \mu$ can be treated similarly. Following the characteristic lines of α and β, we have the following solution for $x \in [0,1]$ and $t \in [0,\tau]$

$$\begin{aligned}\alpha(t,x) &= \alpha_0\left(x-\lambda t\right)\mathbb{1}_{[0,\frac{1}{\lambda}x]}(t) + q\beta_0\left(\mu(t-\frac{x}{\lambda}\right)\mathbb{1}_{(\frac{1}{\lambda}x,\frac{1}{\mu}+\frac{1}{\lambda}x]}(t)\\ &\quad + q\beta(t-\frac{1}{\lambda}x-\frac{1}{\mu},1)\mathbb{1}_{(\frac{1}{\mu}+\frac{1}{\mu}x,\tau]}(t), \qquad (5.17)\\ \beta(t,x) &= \beta_0\left(x+\mu t\right)\mathbb{1}_{[0,\frac{1}{\mu}(1-x)]}(t) + \beta(t-\frac{1}{\mu}(1-x),1)\mathbb{1}_{(\frac{1}{\mu}(1-x),\tau]}(t). \qquad (5.18)\end{aligned}$$

Using (5.17)-(5.18) and boundary condition (5.12), $\beta(t,1)$ verifies

$$\begin{aligned}\beta(t,1) &= \rho\alpha_0\left(1-\lambda t\right)\mathbb{1}_{[0,\frac{1}{\lambda}]}(t) + \rho q\beta_0\left(\mu(t-\frac{1}{\lambda})\right)\mathbb{1}_{(\frac{1}{\lambda},\frac{1}{\mu}]}(t)\\ &+ \int_0^{\lambda t} N^\alpha(y)q\beta_0\left(\mu(t-\frac{y}{\lambda})\right)\mathrm{d}y\mathbb{1}_{[0,\frac{1}{\lambda}]}(t) + \int_{\lambda t}^1 N^\alpha(y)\alpha_0\left(y-\lambda t\right)\mathrm{d}y\mathbb{1}_{[0,\frac{1}{\lambda}]}(t)\\ &+ \int_0^1 N^\alpha(y)q\beta_0\left(\mu(t-\frac{1}{\lambda}y)\right)\mathrm{d}y\mathbb{1}_{(\frac{1}{\lambda},\frac{1}{\mu}]}(t) + \int_0^{\lambda(t-\frac{1}{\mu}} N^\alpha(y)q\beta(t-\frac{1}{\lambda}y-\frac{1}{\mu},1)\mathrm{d}y\mathbb{1}_{(\frac{1}{\mu},\tau]}(t)\\ &+ \int_{\lambda(t-\frac{1}{\mu})}^1 N^\alpha(y)q\beta_0\left(\mu(t-\frac{1}{\lambda}y)\right)\mathrm{d}y\mathbb{1}_{(\frac{1}{\mu},\tau]}(t) + \int_0^{1-\mu t} N^\beta(y)\beta_0\left(y+\mu t\right)\mathrm{d}y\mathbb{1}_{[0,\frac{1}{\mu}]}(t)\\ &+ \int_{1-\mu t}^1 N^\beta(y)\beta\left(t-\frac{1}{\mu}(1-y),1\right)\mathrm{d}y\mathbb{1}_{[0,\frac{1}{\mu}]}(t) + \int_0^1 N^\beta(y)\beta\left(t-\frac{1}{\mu}(1-y),1\right)\mathrm{d}y\mathbb{1}_{(\frac{1}{\mu},\tau]}(t). \qquad (5.19)\end{aligned}$$

Changing the integration variable of the integral terms in (5.19) that depend on $\beta(\cdot,1)$, we have

$$\int_0^{\lambda(t-\frac{1}{\mu})} N^\alpha(y)q\beta(t-\frac{1}{\lambda}y-\frac{1}{\mu},1)\mathrm{d}y = \lambda q\int_0^{t-r_2} N^\alpha(\lambda(t-\theta-\frac{1}{\mu}))\beta(\theta,1)\mathrm{d}\theta, \qquad (5.20)$$

$$\int_{1-\mu t}^1 N^\beta(y)\beta\left(t-\frac{1}{\mu}(1-y),1\right)\mathrm{d}y = \mu\int_0^t N^\beta(1-\mu(t-\theta))\beta(\theta,1)\mathrm{d}\theta, \qquad (5.21)$$

$$\int_0^1 N^\beta(y)\beta\left(t-\frac{1}{\mu}(1-y),1\right)\mathrm{d}y = \mu\int_{t-\frac{1}{\mu}}^t N^\beta(1-\mu(t-\theta))\beta(\theta,1)\mathrm{d}\theta. \qquad (5.22)$$

Plugging (5.20)-(5.22) in (5.19), we get

$$\beta(t,1) = f[\alpha_0(\cdot),\beta_0(\cdot)](t) + \int_0^t \bar{N}(t,\theta)\beta(\theta,1)\mathrm{d}\theta \qquad (5.23)$$

where

$$\begin{aligned}\bar{N}(t,\theta) &\doteq \mu N^\beta(1-\mu(t-\theta))\mathbb{1}_{[0,\frac{1}{\mu}]}(t) \qquad (5.24)\\ &+ \left[\lambda q N^\alpha(\lambda(t-\theta-\frac{1}{\mu}))\mathbb{1}_{[0,t-\frac{1}{\mu}]}(\theta) + \mu N^\beta(1-\mu(t-\theta))\mathbb{1}_{(t-\frac{1}{\mu},t]}(\theta)\right]\mathbb{1}_{(\frac{1}{\mu},\tau]}(t) \qquad (5.25)\end{aligned}$$

and f is a linear operator defined as

$$\begin{aligned}f[\alpha_0(\cdot),\beta_0(\cdot)](t) &\doteq \rho\alpha_0\left(1-\lambda t\right)\mathbb{1}_{[0,\frac{1}{\lambda}]}(t) + \rho q\beta_0\left(\mu(t-\frac{1}{\lambda})\right)\mathbb{1}_{(\frac{1}{\lambda},\frac{1}{\mu}]}(t)\\ &+ \int_0^{\lambda t} N^\alpha(y)q\beta_0\left(\mu(t-\frac{y}{\lambda})\right)\mathrm{d}y\mathbb{1}_{[0,\frac{1}{\lambda}]}(t) + \int_{\lambda t}^1 N^\alpha(y)\alpha_0\left(y-\lambda t\right)\mathrm{d}y\mathbb{1}_{[0,\frac{1}{\lambda}]}(t)\\ &+ \int_0^1 N^\alpha(y)q\beta_0\left(\mu(t-\frac{y}{\lambda})\right)\mathrm{d}y\mathbb{1}_{(\frac{1}{\lambda},\frac{1}{\mu}]}(t) + \int_{\lambda(t-\frac{1}{\mu})}^1 N^\alpha(y)q\beta_0\left(\mu(t-\frac{y}{\lambda})\right)\mathrm{d}y\mathbb{1}_{(\frac{1}{\mu},\tau]}(t)\\ &+ \int_0^{1-\mu t} N^\beta(y)\beta_0\left(y+\mu t\right)\mathrm{d}y\mathbb{1}_{[0,\frac{1}{\mu}]}(t). \qquad (5.26)\end{aligned}$$

The well-posedness of the Volterra equation of the second type (5.23) (see, for instance, [BC16] and [Eva11]) guarantees the existence of a unique solution $\beta(t,1)$ verifying $\beta_{[t]}(\cdot,1) \in L^2((-\tau,0];\mathbb{R})$ for a source term in the right functional space. It follows from the expressions of $\alpha(t,x)$ and $\beta(t,x)$ in (5.17)-(5.18) that there exists a unique weak solution $(\alpha(t,\cdot),\beta(t,\cdot))$. ■

Remark 5.1.1 *If the initial conditions u_0 and v_0 belong to $\mathcal{C}^1([0,1])$ and satisfy the corresponding compatibility conditions, adjusting the proof, [CVKB13, Theorem A.1] to $\mathcal{C}^1([0,1];\mathbb{R})$, it is possible to show that $\beta_{[t]}(\cdot,1) \in \mathcal{C}^1([-\tau,0];\mathbb{R})$, $\forall t \geq \tau$*

Using the backstepping transformation (2.37)-(2.38), we have

$$\alpha(0,x) = u_0(x) - \int_0^x K^{uu}(x,\xi)u_0(\xi) + K^{uv}(x,\xi)v_0(\xi)d\xi,$$
$$\beta(0,x) = v_0(x) - \int_0^x K^{vu}(x,\xi)u_0(\xi) + K^{vv}(x,\xi)v_0(\xi)d\xi.$$

Consequently, this lemma implies the existence a a function $\phi_{u_0,v_0}(\cdot) \in D$ that depends on the initial condition (u_0,v_0) such that $\beta(\cdot,1)$ is the solution of the initial value problem (5.16) with the initial data $\beta(\cdot,1)_0 = \phi_{u_0,v_0}$.

Space isomorphism and stability equivalence

Using the distributed delay equation (5.16), we get the following theorem

Theorem 5.1.1.

Consider the operator $\mathcal{L}_0$ defined by

$$\mathcal{L}_0 : L^2([-\tau,0]) \to \mathbb{R} \tag{5.27}$$
$$\phi_t \mapsto \rho q \phi_t(-\tau) - \int_0^\tau \tilde{N}(\nu)\phi_t(\nu,1)d\nu. \tag{5.28}$$

The space generated by the solutions of

$$\phi_t = \mathcal{L}_0\phi_t. \tag{5.29}$$

with the initial condition $\phi_0 = \phi_{u_0,v_0} \in D$ is isomorphic to the space generated by the solutions of (5.7)-(5.9).

Proof : The proof is differed to the next section where it is done in the general case. ■

A direct corollary is:

Theorem 5.1.2.

Consider the solution (u,v) of (5.7)-(5.9) along with the initial condition (u_0,v_0) and the solution $\phi_{[t]}$ of (5.29) along with the initial condition ϕ_{u_0,v_0}. There exist two constant $C_1 > 0$ and $C_2 > 0$ such that for all $t > \tau$ we have the following inequality

$$C_1||\phi_t||_D \leq ||(u,v)||_\chi \leq C_2||\phi_t||_D. \tag{5.30}$$

Proof : The proof is straightforward and is based on equations (5.13)-(5.14) and on the invertibility of the transformation (2.37)-(2.38). It is done properly in the next section for the general case. ■

This theorem proves that the stability of $\beta(t,1)$ implies the stability of the states (u,v) (and conversely).

Remark 5.1.2 *The equivalence between systems described by a single first-order hyperbolic PDE and systems described by integral delay equations has already been proved in [KK14]. Equation* (5.16) *extends this result for system composed of two coupled hyperbolic PDEs.*

Remark 5.1.3 *One can readily check that the state $v(t,1)$ satisfies exactly the same difference equation with distributed delays as in* (5.16)*.*

5.1.2 A new stability criterion

Beyond stabilization problems, this equivalence between the class of hyperbolic systems (5.7)-(5.9) and the class of delay equations described by (5.29) has important consequences to derive explicit stability criteria in the absence of control. For constant in-domain couplings, it is possible to derive a new explicit stability criterion in the form of a sufficient criterion in terms of the constant parameters of the system. This is done in [BSBAA+18], where the new criterion is validated through academic examples and is compared to other criteria such as the one introduced in [BC16] for instance and recalled in Lemma 2.2.2. In order to simplify the notation, we define the following parameters

$$a \triangleq q\frac{1}{\mu}\sigma^{-+} + \rho\frac{1}{\lambda}\sigma^{+-}, \quad R \triangleq \frac{1}{\lambda\mu}\sigma^{+-}\sigma^{-+}. \tag{5.31}$$

In what follows we denote $I_n, n \in \mathbb{Z}$ the modified Bessel functions of the first kind.

Theorem 5.1.3. [BSBAA+18, Proposition 2]

If the constant parameters of system (5.7)-(5.9) verify one of the following:

(i)

$$\sigma^{+-}\sigma^{-+} \geq 0, \quad \rho q \geq 0 \quad \text{and} \quad |a| + |R|\left(\frac{1}{1+|\rho q|} - \frac{1-|\rho q|}{2}\right) < 1 - |\rho q|, \tag{5.32}$$

(ii)

$$\sigma^{+-}\sigma^{-+} \geq 0, \quad \rho q < 0 \quad \text{and} \quad |a| + |R|\frac{1+|\rho q|}{2} < 1 - |\rho q|, \tag{5.33}$$

(iii)

$$\begin{aligned}&\sigma^{+-}\sigma^{-+} < 0, \quad \rho q \geq 0 \quad \text{and}\\ &|a|I_0\left(\sqrt{|R|}\right) + |R|\left(\frac{1}{1+|\rho q|} - \frac{1-|\rho q|}{2}\right) \times \left[I_0\left(\sqrt{|R|}\right) - I_2\left(\sqrt{|R|}\right)\right] < 1 - |\rho q|,\end{aligned} \tag{5.34}$$

(iv)

$$\begin{aligned}&\sigma^{+-}\sigma^{-+} < 0, \quad \rho q < 0 \quad \text{and}\\ &|a|I_0\left(\sqrt{|R|}\right) + |R|\frac{1+|\rho q|}{2} \times \left[I_0\left(\sqrt{|R|}\right) - I_2\left(\sqrt{|R|}\right)\right] < 1 - |\rho q|,\end{aligned} \tag{5.35}$$

(v)

$$\sigma^{+-}\sigma^{-+} < 0 \quad \text{and} \quad \rho^2 < -\frac{\sigma^{-+}}{\sigma^{+-}}\frac{\lambda}{\mu} < \frac{1}{q^2}, \tag{5.36}$$

then, system (5.7)-(5.9) is exponentially stable in the sense of the L^2-norm.

Following the assumptions of Remark 5.1.1, and according to [BV70, Remark 1], the sufficient condition (5.32)-(5.35) implies the point-wise exponential stability of system (5.16). In other words, there exist μ and $C>0$ such that

$$|\beta(t,1)| \leq C \sup_{-\tau\leq\theta\leq 0} |\beta(\theta,1)| \mathrm{e}^{-\mu(t-\tau)}, \quad t \geq \tau. \tag{5.37}$$

This implies the exponential stability of the original system (5.7)-(5.9) due to Theorem 5.1.2. Let us now compare the presented result (5.32)-(5.35) with the condition required by [BC16, Theorem 5.4], namely the criterion 2.2.2, and apply it to academic examples. This has been done in [BSBAA+18] and the obtained results are given in Table 5.1.

One may check that in the case of $\sigma^{+-} = \sigma^{-+} = 0$, system (5.7)-(5.9) is viewed as a system of two linear conservation laws with two reflection terms at the boundaries. In the case of $\sigma^{+-}\sigma^{-+} < 0$, conditions (5.32)-(5.35) and the one given in Lemma 2.2.2 are both sufficient and can be applied simultaneously. For the sake of clarity, we give some numerical examples in Table 5.2. These results illustrate the fact that the stability of some systems is guaranteed using the conditions required by Theorem 5.1.3 while the condition provided in [BC16, Theorem 5.4] does not hold (and conversely). Consequently, using the neutral approach it has become possible to derive a new explicit criterion different from the ones that can be found in the literature.

5.2 General case of a $n+m$ system

We now consider the general system (2.1)-(2.3) whose equations are rewritten bellow:

$$\partial_t u(t,x) + \Lambda^+ \partial_x u(t,x) = \Sigma^{++}(x)u(t,x) + \Sigma^{+-}(x)v(t,x), \tag{5.38}$$
$$\partial_t v(t,x) - \Lambda^- \partial_x v(t,x) = \Sigma^{-+}(x)u(t,x) + \Sigma^{--}(x)v(t,x), \tag{5.39}$$

along with the linear boundary conditions

$$u(t,0) = Q_0 v(t,0), \quad v(t,1) = R_1 u(t,1) + V(t). \tag{5.40}$$

We assume that the diagonal terms of the matrices Σ^{++} and Σ^{--} are equal to zero. This assumption is non-restrictive as these terms can be removed using a similar transformation as the one introduced in Remark 2.4.1. As for the case of two coupled equations, the objective is to prove that in the absence of control at the left boundary (i.e $U(t) \equiv 0$) the space generated by the solutions of (5.38)-(5.40) is isomorphic to the space generated by the solutions of a neutral system with distributed delays. The proposed method is inspired by [CHO17]. We first combine a Volterra transformation and an invertible Fredholm transformation to move the local coupling terms $\Sigma^{\cdot\cdot}$ to the boundary (in the form of integral terms). Due to these transformations, non-local coupling terms may appear in the system. Using the method of characteristics, the resulting system rewrites in the form given by (5.5).

5.2.1 Volterra transformation: removing the in-domain couplings

Since one of the most important difficulties for the analysis of system (5.38)-(5.40) is due to the presence of the in-domain coupling terms, we remove some of them by way of a Volterra transformation. This transformation, introduced in [HDMVK15], is defined by

$$\alpha(t,x) = u(t,x) - \int_0^x K^{uu}(x,\xi)u(t,\xi)d\xi - \int_0^x K^{uv}(x,\xi)v(t,\xi)d\xi, \tag{5.41}$$
$$\beta(t,x) = v(t,x) - \int_0^x K^{vu}(x,\xi)u(t,\xi)d\xi - \int_0^x K^{vv}(x,\xi)v(t,\xi)d\xi, \tag{5.42}$$

where the kernels $K^{uu}, K^{uv}, K^{vu}, K^{vv}$ belong to $\mathcal{C}(\mathcal{T})$ (where we recall that $\mathcal{T} = \{(x,\xi) \in [0,1]^2 |\ \xi \leq x\}$), and are defined by

$$\Lambda^+ K_x^{uu}(x,\xi) + K_\xi^{uu}(x,\xi)\Lambda^+ = -K^{uu}(x,\xi)\Sigma^{++}(\xi) - K^{uv}(x,\xi)\Sigma^{-+}(\xi), \tag{5.43}$$
$$\Lambda^+ K_x^{uv}(x,\xi) - K_\xi^{uv}(x,\xi)\Lambda^- = -K^{uu}(x,\xi)\Sigma^{+-}(\xi) - K^{uv}(x,\xi)\Sigma^{--}(\xi), \tag{5.44}$$
$$\Lambda^- K_x^{vu}(x,\xi) - K_\xi^{vu}(x,\xi)\Lambda^+ = K^{vu}(x,\xi)\Sigma^{++}(\xi) + K^{vv}(x,\xi)\Sigma^{-+}(\xi), \tag{5.45}$$
$$\Lambda^- K_x^{vv}(x,\xi) + K_\xi^{vv}(x,\xi)\Lambda^- = K^{vu}(x,\xi)\Sigma^{+-}(\xi) + K^{vv}(x,\xi)\Sigma^{--}(\xi), \tag{5.46}$$

along with the following set of boundary conditions

$$\Lambda^+ K^{uu}(x,x) - K^{uu}(x,x)\Lambda^+ = \Sigma^{++}(x), \tag{5.47}$$
$$\Lambda^+ K^{uv}(x,x) + K^{uv}(x,x)\Lambda^- = \Sigma^{+-}(x), \tag{5.48}$$
$$\Lambda^- K^{vu}(x,x) + K^{vu}(x,x)\Lambda^+ = -\Sigma^{-+}(x), \tag{5.49}$$
$$\Lambda^- K^{vv}(x,x) - K^{vv}(x,x)\Lambda^- = -\Sigma^{--}(x). \tag{5.50}$$

To obtain a well-posed system, we add an arbitrary boundary condition for $(K^{uu})_{ij}(x,0)$ when $i \geq j$ and the boundary condition

$$(K^{vv})_{ij}(x,0) = (K^{vu}(x,0)\Lambda^+ Q_0)_{ij} \text{ if } i \geq j. \tag{5.51}$$

Adjusting the proof of Theorem 3.1.1, one can prove that the system (5.43)-(5.51) is well-posed and admits a unique solution in $\mathcal{C}(\mathcal{T})$. Moreover, since the transformation (5.41)-(5.42) is invertible, the inverse transformation exists and can be rewritten as

$$u(t,x) = \alpha(t,x) - \int_0^x L^{\alpha\alpha}(x,\xi)\alpha(t,\xi)d\xi - \int_0^x L^{\alpha\beta}(x,\xi)\beta(t,\xi)d\xi, \tag{5.52}$$
$$v(t,x) = \beta(t,x) - \int_0^x L^{\beta\alpha}(x,\xi)\alpha(t,\xi)d\xi - \int_0^x L^{\beta\beta}(x,\xi)\beta(t,\xi)d\xi, \tag{5.53}$$

where the kernels $L^{\alpha\alpha}, L^{\beta\alpha}, L^{\alpha\beta}, L^{\beta\beta}$ belong to $\mathcal{C}(\mathcal{T})$ and satisfy a similar set of PDEs (which is not given here but can be found in [HDMVK15]).
Defining the continuous matrix $G_1(x) = K^{uu}(x,0)\Lambda^+ Q_0 - K^{uv}(x,0)\Lambda^-$ and the continuous **upper-triangular** matrix $G_2(x) = K^{vu}(x,0)\Lambda^+ Q_0 - K^{vv}(x,0)\Lambda^-$, the invertible transformation (5.41)-(5.42) maps the original system (5.38)-(5.40) to the following target system (see [CHO17] for details)

$$\partial_t \alpha(t,x) + \Lambda^+ \partial_x \alpha(t,x) = G_1(x)\beta(t,0), \tag{5.54}$$
$$\partial_t \beta(t,x) - \Lambda^- \partial_x \beta(t,x) = G_2(x)\beta(t,0), \tag{5.55}$$

along with the boundary conditions

$$\alpha(t,0) = Q_0\beta(t,0), \tag{5.56}$$
$$\beta(t,1) = R_1\alpha(t,1) + \int_0^1 N^\alpha(\xi)\alpha(t,\xi) + N^\beta(\xi)\beta(t,\xi)d\xi + V(t), \tag{5.57}$$

where

$$N^\alpha(\xi) = L^{\beta\alpha}(1,\xi) - R_1 L^{\alpha\alpha}(1,\xi), \quad N^\beta(\xi) = L^{\beta\beta}(1,\xi) - R_1 L^{\alpha\beta}(1,\xi). \tag{5.58}$$

In this target system, a part of the in-domain couplings have been moved at the boundary (in the form of the integral terms N^α and N^β), while the other part has been transformed in non-local coupling terms. These terms will be slightly modified in the next section to be able to rewrite the resulting system as a neutral system.

5.2.2 Fredholm transformation: modifying the non-local coupling terms

Let us consider the following Fredholm transformation

$$\alpha(t,x) = w(t,x), \tag{5.59}$$

$$\beta(t,x) = z(t,x) - \int_0^1 F(x,\xi)z(t,\xi)d\xi, \tag{5.60}$$

where the kernel F is a strict upper triangular matrix (i.e $F_{ij}(x,\xi) = 0$ if $i \geq j$) defined on $\mathcal{T}_F = \{(x,\xi) \in [0,1]^2\}$ by the following set of PDEs

$$\Lambda^- F_x(x,\xi) + F_\xi(x,\xi)\Lambda^- = 0, \tag{5.61}$$

along with the boundary conditions

$$F(x,0) = G_2(x)(\Lambda^-)^{-1}, \quad F(0,\xi) = 0. \tag{5.62}$$

The system (5.61)-(5.62) admits a unique solution $F \in L^2([0,1]\times[0,1])^{m\times m}$ whose components can explicitly be obtained by the method of characteristics [CHO17]: for all $(x,\xi) \in [0,1]^2$, for $i < j$, we have

$$F_{ij}(x,\xi) = -\frac{1}{\mu_j}(G_2)_{ij}(x - \frac{\mu_i}{\mu_j}\xi), \tag{5.63}$$

where $(G_2)_{ij}$ is the component of G_2 located at the intersection of the i^{th} row and the the j^{th} column. The invertibility of this Fredholm transformation is given by the following Lemma.

Lemma 5.2.1. [CHO17, Lemma 2]

Let us consider F defined by (5.61)-(5.62). The transformation defined by (5.59)-(5.60) is invertible. Moreover, its inverse has the same form, i.e., there exists $\tilde{F} \in L^2([0,1]\times[0,1])^{n+m}$, an upper triangular matrix (with 0 diagonal entries) such that

$$w(t,x) = \alpha(t,x), \tag{5.64}$$

$$z(t,x) = \beta(t,x) - \int_0^1 \tilde{F}(x,\xi)\beta(t,\xi)d\xi, \tag{5.65}$$

Proof : Due to the triangular structure of F, we have $z_m(t,x) = \beta_m(t,x)$. Consequently,

$$\beta_{m-1}(t,x) = z_{m-1}(t,x) - \int_0^1 F_{m-1,m}(x,\xi)z_m(t,\xi)d\xi$$

$$\Rightarrow \quad z_{m-1}(t,x) = \beta_{m-1}(t,x) + \int_0^1 F_{m-1,m}(x,\xi)\beta_m(t,\xi)d\xi. \tag{5.66}$$

The rest of the proof is straightforward by induction. ■

We are now able to give the explicit expression of the equations that are satisfied by (w,z). In what follows we define G_3 as the unique L^∞ solution [Yos60] of the Volterra integral equation

$$G_3(x) = F(x,1)\Lambda^- + \int_0^1 F(x,\xi)G_3(\xi)d\xi, \tag{5.67}$$

and $N^z(\xi)$ as

$$N^z(\xi) = F(1,\xi) + N^\beta(\xi) - \int_0^1 N^\beta(\nu)F(\nu,\xi)d\nu. \tag{5.68}$$

We then have the following lemma.

Lemma 5.2.2.

Let us consider the system

$$\partial_t w(t,x) + \Lambda^+ \partial_x w(t,x) = G_1(x) z(t,0), \tag{5.69}$$
$$\partial_t z(t,x) - \Lambda^- \partial_x z(t,x) = G_3(x) z(t,1), \tag{5.70}$$

with the boundary conditions

$$w(t,0) = Q_0 z(t,0), \tag{5.71}$$
$$z(t,1) = R_1 w(t,1) + \int_0^1 N^\alpha(\xi) w(t,\xi) + N^z(\xi) z(t,\xi) d\xi + V(t). \tag{5.72}$$

The invertible transformation (5.59)-(5.60) maps the system (5.69)-(5.72) to the system (5.54)-(5.57).

Proof : We first check that the boundary conditions do correspond. Since $F(0,\xi)=0$, for all $\xi \in [0,1]$, the boundary condition (5.71) is satisfied. We also have

$$\begin{aligned}
\beta(t,1) &= z(t,1) - \int_0^1 F(1,\xi) z(t,\xi) d\xi \\
&= R_1 w(t,1) + \int_0^1 N^\alpha(\xi) w(t,\xi) + N^z(\xi) z(t,\xi) d\xi - \int_0^1 F(1,\xi) z(t,\xi) + V(t) \\
&= R_1 \alpha(t,1) + \int_0^1 N^\alpha(\xi) \alpha(t,\xi) + (N^\beta(\xi) - \int_0^1 N^\beta(\nu) F(\nu,\xi) d\nu) z(t,\xi) d\xi + V(t) \\
&= R_1 \alpha(t,1) + \int_0^1 N^\alpha(\xi) \alpha(t,\xi) d\xi + \int_0^1 (N^\beta(\xi) - \int_0^1 N^\beta(\nu) F(\nu,\xi) d\nu) z(t,\xi) d\xi + V(t) \\
&= R_1 \alpha(t,1) + \int_0^1 N^\alpha(\xi) \alpha(t,\xi) d\xi + \int_0^1 N^\beta(\xi) (z(t,\xi) - \int_0^1 F(\xi,\nu) z(t,\nu) d\nu) d\xi + V(t) \\
&= R_1 \alpha(t,1) + \int_0^1 N^\alpha(\xi) \alpha(t,\xi) d\xi + \int_0^1 N^\beta(\xi) \beta(t,\xi) d\xi + V(t),
\end{aligned}$$

which is exactly the boundary condition (5.57). Differentiating (5.59)-(5.60) with respect to time and space and integrating by parts, we get

$$\begin{aligned}
&\partial_t \beta(t,x) - \Lambda^- \partial_x \beta(t,x) = \partial_t z(t,x) - \int_0^1 F(x,\xi) \partial_t z(t,\xi) d\xi - \Lambda^- \partial_x z(t,x) + \Lambda^- \int_0^1 \partial_x F(x,\xi) z(t,\xi) d\xi \\
&= G_3(x) z(t,1) - \int_0^1 F(x,\xi) G_3(\xi) z(t,1) d\xi - F(x,1) \Lambda^- z(t,1) + F(x,0) \Lambda^- z(t,0) \\
&\quad + \int_0^1 (\Lambda^- \partial_x F(x,\xi) + \partial_\xi F(x,\xi) \Lambda^-) z(t,\xi) d\xi \\
&= (G_3(x) - F(x,1)\Lambda^- \int_0^1 F(x,\xi) G_3(\xi) d\xi) z(t,1) + G_2(x) z(t,0) = G_2 \beta(t,0),
\end{aligned}$$

which is exactly the PDE (5.55). This concludes the proof. ∎

Due to the invertibility of the transformation (5.59)-(5.60), a direct consequence of this lemma is the following theorem.

Theorem 5.2.1.

There exists an invertible bounded linear map $\mathcal{F} : L^2([0,1])^{n+m} \to L^2([0,1])^{n+m}$ such that, for every initial condition $(u_0, v_0) \in L^2([0,1])^{n+m}$, if $(w,z) \in \mathcal{C}^0([0,+\infty), L^2([0,1])^{n+m})$ denotes the solution to (5.69)-(5.72) satisfying the initial data $(w(0,\cdot), z(0,\cdot)) = \mathcal{F}^{-1}(u_0, v_0)$, then $(u(t), v(t)) = \mathcal{F}(w(t), z(t))$ is the solution to (5.38)-(5.40) satisfying $(u(0,\cdot), v(0,\cdot)) = (u_0, v_0)$.

Proof : Let us denote $\mathcal{F}_1 : L^2([0,1])^{n+m} \to L^2([0,1])^{n+m}$ the Fredholm invertible operator defined for all $(w,z) \in L^2([0,1])^{n+m}$, and all $x \in [0,1]$ by

$$\mathcal{F}_1 \begin{pmatrix} w(x) \\ z(x) \end{pmatrix} = \begin{pmatrix} w(x) \\ z(x) - \int_0^1 F(x,\xi)z(\xi)d\xi \end{pmatrix}. \tag{5.73}$$

Similarly we define the operator $\mathcal{F}_1 : L^2([0,1])^{n+m} \to L^2([0,1])^{n+m}$ the Voletrra invertible operator associated to (5.52)-(5.53) and defined for all $(\alpha,\beta) \in L^2([0,1])^{n+m}$, and all $x \in [0,1]$ by

$$\mathcal{F}_2 \begin{pmatrix} \alpha(x) \\ \beta(x) \end{pmatrix} = \begin{pmatrix} \alpha(t,x) - \int_0^x L^{\alpha\alpha}(x,\xi)\alpha(t,\xi)d\xi - \int_0^x L^{\alpha\beta}(x,\xi)\beta(t,\xi)d\xi \\ \beta(t,x) - \int_0^x L^{\beta\alpha}(x,\xi)\alpha(t,\xi)d\xi - \int_0^x L^{\beta\beta}(x,\xi)\beta(t,\xi)d\xi \end{pmatrix}. \tag{5.74}$$

Thus, defining the invertible operator $\mathcal{F} = \mathcal{F}_2 \circ \mathcal{F}_1$, we get the expected result. ■

This theorem proves that the two systems (5.69)-(5.72) and (5.38)-(5.40) are *equivalent* (in the sense that they have equivalent properties), provided the compatibility of the initial conditions holds. We now use the method of characteristics to show that the function $z(t,1)$ satisfies a neutral difference system with distributed delays. Due to the structure of (5.69)-(5.72) the space generated by the solutions of this difference system would be isomorphic to the space generated by the solutions of (5.69)-(5.72) (and consequently isomorphic to the space generated by the solutions of (5.38)-(5.40)).

Remark 5.2.1 *Designing the boundary control law as* $V(t) = -R_1 w(t,1) - \int_0^1 N^\alpha(\xi)w(t,\xi) - N^z(\xi)z(t,\xi)d\xi$ *leads* $z(t,1) = 0$ *in* (5.72). *Thus, due to the underlying cascade structure, the solutions of system* (5.69)-(5.72) *converge to their zero-equilibrium in finite time [CHO17]: for every* $(w_0,z_0) \in (L^2([0,1]))^{n+m}$*, the solution* $(z,w) \in \mathcal{C}^0([0,+\infty);(L^2([0,1]))^{n+m}$ *satisfying* $w(0,\cdot) = w_0$ *and* $z(0,\cdot) = z_0$ *verify* $z(t) = w(t) = 0$ *for every* $t \geq \frac{1}{\lambda} + \frac{1}{\mu}$. *Thus, this control law fulfills Objective A. We prove later in this chapter that this control law and the one defined in Chapter 3 by* (3.38) *are actually the same (and that consequently there exists an invertible transformation that maps the system* (3.5)-(3.7) *to the system* (5.69)-(5.72)*). However, as explained in [CHO17] there may have some advantages to use one approach rather than the other. Even if two successive transformations are required to obtain* (5.69)-(5.72)*, the involved kernels may be simpler to compute as they do not present any recursive structure. Hence the method proposed in [CHO17] might be more reliable for practical implementations since the kernels equations are solved independently and not recursively. Moreover, the structure of the target system* (5.69)-(5.72) *appears simpler than the one of* (3.5)-(3.7) *since it has less coupling terms.*

5.2.3 Neutral system with distributed delays

Inspired by the results obtained in Section 5.1, we use the characteristic method to express, for all $x \in [0,1]$, the states $z(t,x)$ and $w(t,x)$ as functions of (possibly delayed values of) $z(t,1)$. This yields the following theorem.

Theorem 5.2.2.

Consider system (5.69)-(5.72). There exists a matrix function $G \in (L^\infty([0,\tau],\mathbb{R}))^{m\times m}$ such that for all $t > \tau$, for all $1 \leq i \leq m$

$$z_i(t,1) = \sum_{k=1}^{n}\sum_{l=1}^{m}(R_1)_{ik}(Q_0)_{kl}z_l(t - \frac{1}{\lambda_k} - \frac{1}{\mu_l}, 1) + \sum_{l=1}^{m}\int_0^\tau G_{il}(s)z_l(t-s)ds + V_i(t). \tag{5.75}$$

Proof : Using the characteristic method on equation (5.70), we obtain for all $1 \leq j \leq m$

$$z_j(t,x) = z_j(t - \frac{1-x}{\mu_j}, 1) + \sum_{k=1}^{m}\int_0^{\frac{1-x}{\mu_j}} (G_3)_{jk}(x+\mu_j s)z_k(t-s,1)ds. \tag{5.76}$$

Similarly using the characteristic method on equation (5.69), we obtain for all $1 \leq j \leq n$

$$w_i(t,x) = w_i(t - \frac{x}{\lambda_i}, 0) + \sum_{k=1}^{m} \int_0^{\frac{x}{\lambda_i}} (G_1)_{ik}(x - \lambda_i s) z_k(t-s,0) ds.$$

Using the boundary condition (5.71), this rewrites

$$w_i(t,x) = \sum_{k=1}^{m} (Q_0)_{ik} z_k(t - \frac{x}{\lambda_i}, 0) + \sum_{k=1}^{m} \int_0^{\frac{x}{\lambda_i}} (G_1)_{ik}(x - \lambda_i s) z_k(t-s,0) ds. \tag{5.77}$$

Finally, injecting (5.76) inside (5.77), we obtain for all $1 \leq j \leq n$

$$\begin{aligned} w_i(t,x) = &\sum_{k=1}^{m} (Q_0)_{ik} z_k(t - \frac{x}{\lambda_i} - \frac{1}{\mu_k}, 1) + \sum_{k=1}^{m}\sum_{l=1}^{m} \int_0^{\frac{1}{\mu_k}} (Q_0)_{ik}(G_3)_{kl}(\mu_k s) z_l(t - \frac{x}{\lambda_i} - s, 1) ds \\ &+ \sum_{k=1}^{m} \int_0^{\frac{x}{\lambda_i}} (G_1)_{ik}(x - \lambda_i s) z_k(t - s - \frac{1}{\mu_k}, 1) ds \\ &+ \sum_{k=1}^{m}\sum_{l=1}^{m} \int_0^{\frac{x}{\lambda_i}} (G_1)_{ik}(x - \lambda_i s) \int_0^{\frac{1}{\mu_k}} (G_3)_{kl}(\mu_k \nu) z_l(t - s - \nu, 1) d\nu ds. \end{aligned} \tag{5.78}$$

Using the change of variables $\nu = \frac{x}{\lambda_i} + s$ is the first integral, $\nu = \frac{x}{\mu_k} + s$ is the second one and $\eta = s + \nu$ in the third one this rewrites

$$\begin{aligned} w_i(t,x) = &\sum_{k=1}^{m} (Q_0)_{ik} z_k(t - \frac{x}{\lambda_i} - \frac{1}{\mu_k}, 1) + \sum_{k=1}^{m}\sum_{l=1}^{m} \int_{\frac{x}{\lambda_i}}^{\frac{1}{\mu_k} + \frac{x}{\lambda_i}} (Q_0)_{ik}(G_3)_{kl}(\mu_k(\nu - \frac{x}{\lambda_i})) z_l(t - \nu, 1) d\nu \\ &+ \sum_{k=1}^{m} \int_{\frac{1}{\mu_k}}^{\frac{x}{\lambda_i} + \frac{1}{\mu_k}} (G_1)_{ik}(x - \lambda_i(\nu - \frac{1}{\mu_k})) z_k(t - \nu, 1) d\nu \end{aligned} \tag{5.79}$$

$$+ \sum_{k=1}^{m}\sum_{l=1}^{m} \int_0^{\frac{x}{\lambda_i}} (G_1)_{ik}(x - \lambda_i s) \int_s^{s + \frac{1}{\mu_k}} (G_3)_{kl}(\mu_k(\eta - s)) z_l(t - \eta, 1) d\eta ds. \tag{5.80}$$

Using Fubini's theorem, the last integral can be simplified. For all $1 \leq i \leq n$, $1 \leq k \leq m$ and $1 \leq l \leq m$.

$$\begin{aligned} I &= \int_0^{\frac{x}{\lambda_i}} \int_s^{s + \frac{1}{\mu_k}} (G_1)_{ik}(x - \lambda_i s)(G_3)_{kl}(\mu_k(\eta - s)) z_l(t - \eta, 1) d\eta ds \\ &= \int_0^{\frac{x}{\lambda_i}} \int_0^{s + \frac{1}{\mu_k}} (G_1)_{ik}(x - \lambda_i s)(G_3)_{kl}(\mu_k(\eta - s)) z_l(t - \eta, 1) d\eta ds \\ &\quad - \int_0^{\frac{x}{\lambda_i}} \int_0^{s} (G_1)_{ik}(x - \lambda_i s)(G_3)_{kl}(\mu_k(\eta - s)) z_l(t - \eta, 1) d\eta ds \\ &= \int_0^{\frac{x}{\lambda_i}} \int_0^{\frac{1}{\mu_k}} (G_1)_{ik}(x - \lambda_i s)(G_3)_{kl}(\mu_k(\eta - s)) z_l(t - \eta, 1) ds d\eta \\ &\quad + \int_{\frac{1}{\mu_k}}^{\frac{x}{\lambda_i} + \frac{1}{\mu_k}} \int_{\eta - \frac{1}{\mu_k}}^{\frac{x}{\lambda_i}} (G_1)_{ik}(x - \lambda_i s)(G_3)_{kl}(\mu_k(\eta - s)) z_l(t - \eta, 1) ds d\eta \\ &\quad - \int_0^{\frac{x}{\lambda_i}} \int_\eta^{\frac{x}{\lambda_i}} (G_3)_{kl}(\mu_k(\eta - s))(G_1)_{ik}(x - \lambda_i s) z_l(t - \eta, 1) ds d\eta \\ &= \int_0^{\tau} (G_4(x,\eta))_{ikl} z_l(t - \eta, 1) d\eta, \end{aligned} \tag{5.81}$$

where for any $1 \leq i \leq n$, $1 \leq k \leq m$ and $1 \leq l \leq m$, the function $(G_4(x,\eta))_{ikl} \in L^\infty([0,1] \times [0,\tau])$

$$\begin{aligned} (G_4(x,\eta))_{ikl} = &\mathbb{1}_{[0, \frac{x}{\lambda_i}]}(\eta) \int_0^{\frac{1}{\mu_k}} (G_1)_{ik}(x - \lambda_i s)(G_3)_{kl}(\mu_k(\eta - s)) ds \\ &+ \mathbb{1}_{[\frac{1}{\mu_k}, \frac{x}{\lambda_i} + \frac{1}{\mu_k}]}(\eta) \int_{\eta - \frac{1}{\mu_k}}^{\frac{x}{\lambda_i}} (G_1)_{ik}(x - \lambda_i s)(G_3)_{kl}(\mu_k(\eta - s)) ds \\ &- \mathbb{1}_{[0, \frac{x}{\lambda_i}]}(\eta) \int_\eta^{\frac{x}{\lambda_i}} (G_3)_{kl}(\mu_k(\eta - s))(G_1)_{ik}(x - \lambda_i s) ds, \end{aligned} \tag{5.82}$$

We now define for all $1 \leq i \leq n$ and $1 \leq l \leq m$ the matrix function $G_5 \in L^\infty([0,1]\times[0,\frac{1}{\lambda_1}+\frac{1}{\mu_1}]^{n\times m})$ by

$$
\begin{aligned}
(G_5)_{il}(x,\eta) = & \sum_{k=1}^m \mathbb{1}_{[\frac{x}{\lambda_i},\frac{x}{\lambda_i}+\frac{1}{\mu_k}]}(\eta)(Q_0)_{ik}(G_3)_{kl}(\mu_k(\nu-\frac{x}{\lambda_i})) \\
& + \mathbb{1}_{[\frac{1}{\mu_l},\frac{x}{\lambda_i}+\frac{1}{\mu_l}]}(G_1)_{il}(x-\lambda_i(\nu-\frac{1}{\mu_l})) + \sum_{k=1}^m (G_4(x,\eta))_{ikl} \qquad (5.83)
\end{aligned}
$$

Thus, equation (5.80) can be rewritten for every $1 \leq i \leq n$ as

$$
w_i(t,x) = \sum_{k=1}^m (Q_0)_{ik} z_k(t-\frac{x}{\lambda_i}-\frac{1}{\mu_k},1) + \sum_{l=1}^m \int_0^\tau (G_5)_{il}(x,\eta) z_l(t-\eta,1)d\eta. \qquad (5.84)
$$

Let us now rewrite the integral terms that appear in the boundary condition (5.72) as a function of $z(\cdot,1)$. For all $1 \leq i \leq m$, we have

$$
\begin{aligned}
(\int_0^1 N^z(\xi)z(t,\xi)d\xi)_i &= \int_0^1 \sum_{k=1}^m N^z_{ik}(\xi) z_k(t,\xi)d\xi \\
&= \int_0^1 \sum_{k=1}^m N^z_{ik}(\xi) z_k(t-\frac{1-\xi}{\mu_k},1)d\xi \\
&\quad + \int_0^1 \sum_{k=1}^m N^z_{ik}(\xi) \sum_{l=1}^m \int_0^{\frac{1-\xi}{\mu_k}} (G_3)_{kl}(\xi+\mu_k s) z_l(t-s,1)dsd\xi \\
&= \sum_{k=1}^m \int_0^{\frac{1}{\lambda_1}+\frac{1}{\mu_1}} \bar{N}^z_{ik}(s) z_k(t-s,1)ds, \qquad (5.85)
\end{aligned}
$$

where,

$$
\bar{N}^z_{ik}(s) = \mathbb{1}_{[0,\frac{1}{\mu_k}]}(s)\mu_k N^z_{ik}(1-\mu_k s) + \mathbb{1}_{[0,\frac{1}{\mu_k}]}(s) \sum_{l=1}^m \int_0^{1-\mu_k s} N^z_{il}(\xi)(G_3)_{lk}(\xi+\mu_l s)d\xi.
$$

Similarly, for all $1 \leq i \leq m$, we have

$$
\begin{aligned}
\int_0^1 (N^\alpha(\xi)w(t,\xi))_i d\xi &= \int_0^1 \sum_{k=1}^n N^\alpha_{ik}(\xi) w_k(t,\xi)d\xi \\
&= \int_0^1 \sum_{k=1}^n \sum_{l=1}^m (N^\alpha_{ik}(\xi)(Q_0)_{kl} z_l(t-\frac{\xi}{\lambda_k}-\frac{1}{\mu_l},1)d\xi \\
&\quad + \int_0^\tau N^\alpha_{ik}(\xi)(G_5)_{kl}(\xi,\eta) z_l(t-\eta,1)d\eta)d\xi \\
&= \sum_{l=1}^m \int_0^{\frac{1}{\lambda_1}+\frac{1}{\mu_1}} \bar{N}^\alpha_{il}(s) z_l(t-s,1)ds, \qquad (5.86)
\end{aligned}
$$

where,

$$
\bar{N}^\alpha_{il}(s) = \sum_{k=1}^n \mathbb{1}_{[\frac{1}{\mu_l},\frac{1}{\mu_l}+\frac{1}{\lambda_k}]}(s)\lambda_k N^\alpha_{ik}(\lambda_k s - \frac{\lambda_k}{\mu_l})(Q_0)_{kl} + \int_0^1 \sum_{k=1}^n N^\alpha_{ik}(\xi)(G_5)_{kl}(\xi,s)d\xi. \qquad (5.87)
$$

Finally, for all $1 \leq i \leq m$ and $1 \leq l \leq m$, let us denote $G_{il}(\cdot)$ the $L^\infty([0,\tau])$ function defined for all s in $[0,\tau]$ by

$$
G_{il}(s) = \sum_{k=1}^m (R_1)_{ik}(G_5)_{kl}(1,s) + \bar{N}^z_{il}(s) + \bar{N}^\alpha_{il}(s). \qquad (5.88)
$$

Defining the matrix function $G = (G_{il})_{1\leq i,l\leq m}$ yields the expected result. ■

Let us now consider the case $t \leq \tau$. Similar computations can be done and the terms "$z_l(t-\frac{1}{\lambda_k}-\frac{1}{\mu_l},1)$" (and $z_l(t-\nu)$) can be expressed as functions of the initial conditions $z(0,\cdot)$ and $w(0,\cdot)$. Thus, due to Theorem 5.2.1 all these delayed terms can be expressed as functions that linearly depend on the initial conditions of the PDE (2.1)-(2.3), u_0 and v_0. Thus, there exists $\phi_{u_0,v_0} \in D$ that depends on the initial condition (u_0,v_0) (and on the different coupling terms that are present in (2.1)-(2.3)) such that $z(\cdot,1)$ is the solution of the initial value problem (5.75) with the initial date $z(\cdot,1)_0 = \phi_{u_0,v_0}$. This can be stated in the following theorem.

Theorem 5.2.3.

Consider the operator $\mathcal{L}$ defined by

$$\begin{aligned} \mathcal{L} : D &\to \mathbb{R}^m \\ \phi_{[t]} &\mapsto \sum_{k=1}^{n}\sum_{l=1}^{m}(R_1)_{ik}(Q_0)_{kl}\phi_{[t]}(-\frac{1}{\lambda_k}-\frac{1}{\mu_l}) \\ &+\sum_{l=1}^{m}\int_0^{\tau} G_{il}(s)(\phi_{[t]})_l(-s)ds, \end{aligned} \quad (5.89)$$

where G_{il} are $L^{\infty}([0,\tau],\mathbb{R})$-functions defined in (5.88). Consider a given linear feedback control law $U(t)$ and the difference system defined by

$$\phi_{[t]}(0) = \mathcal{L}\phi_{[t]} + V(t), \quad t \geq 0, \quad (5.90)$$

with the initial condition $\phi_0 = \phi_{u_0,v_0} \in D$. For any $(u_0, v_0) \in \chi$, there exists an explicit mapping from the space generated by the solution of (5.7)-(5.9) (along with the initial conditions (u_0, v_0)) and the space generated by the solutions of (5.89) (along with the initial condition $\phi_0 = \phi_{u_0,v_0}$).

Proof : Let us consider an initial condition $(u_0, v_0) \in \chi$ and a linear feedback control law $V(t)$. Consider a solution of (2.1)-(2.3) corresponding to this initial condition. Using Theorem 5.2.3, we can express $z(t,1)$ as a function of u and v. In the other hand, the analysis done above has shown that $z(t,1)$ is a solution of (5.75) with an initial condition $\phi_{u_0,v_0} \in D$. Let us now consider a solution ϕ of (5.89) with a given initial condition $\phi_{u_0,v_0}(t) \in D$. The analysis done above shows that ϕ corresponds to $z(\cdot,1)$, where $z(t,x)$ is the solution of the PDE (5.69)-(5.72). Using equation (5.76) and equation (5.84), one can express for all $t>0$ and for all $x \in [0,1]$ the functions $w(t,x)$ and $z(t,x)$ as functions of ϕ (and of its past values). Using the backstepping transformations (5.60) and (5.64)-(5.65), it then becomes possible to express (u,v) as functions of ϕ (and of its past values) whose initial conditions are in χ. ∎

Theorem 5.2.4.

Consider (u,v) the solution of (5.7)-(5.9) with the initial condition $(u_0, v_0) \in \chi$ and $\phi_{[t]}$, the solution of (5.89) for the initial data ϕ_{u_0,v_0}. Let us define $r = \min\{\frac{1}{\mu_m}, \frac{0.4}{m(||G_3||^2+1)}\}$ and $||\phi_{[t]}||_r$ as

$$||\phi_{[t]}||_r \doteq \left(\int_{-r}^{0} \phi^T(t+\theta)\phi(t+\theta)d\theta\right)^{\frac{1}{2}}. \quad (5.91)$$

Then, there exist two constants $C_1 > 0$ and $C_2 > 0$ such that for all $t > \tau$,

$$C_1||\phi_{[t]}||_r \leq ||(u,v)||_{\chi} \leq C_2||\phi_{[t]}||_D, \quad (5.92)$$

where the norms $||(\cdot,\cdot)||_{\chi}$ and $||\cdot||_D$ are respectively defined in (5.2) and (5.4).

Proof : For all $t>0$, defining $(w(t,\cdot), z(t,\cdot)) = \mathcal{F}^{-1}(u(t,\cdot), v(t,\cdot))$ and using the computations presented above, we have $\phi_{[t]}(0) = z(t,1)$. Using equation (5.76) and Young's inequality, it is straightforward to prove that there exists a constant $M_0 > 0$ that depends on τ and on G_3 such that

$$\int_0^1 z^T(t,x)z(t,x)dx \leq M_0 \int_{-\tau}^{0} z^T(t+\tau,1)z(t+\tau,1)d\tau.$$

Similarly, using (5.84), there exists a constant $M_1 > 0$ such that

$$\int_0^1 w^T(t,x)w(t,x)dx \leq M_1 \int_{-\tau}^{0} z^T(t+\tau,1)z(t+\tau,1)d\tau.$$

This yields

$$||(w,z)||_\chi \leq \sqrt{(M_1+M_0)}||\phi_{[t]}||_D.$$

Using the boundedness of the operator $\mathcal{F}$, we immediately get the second part of the inequality (5.92). Let us now focus on the first part of the inequality (5.92). Using (5.76), for all $x \in [0,1]$, for all $1 \leq j \leq m$, denoting $\nu_j = \frac{1-x}{\mu_j}$ (thus $\nu_j \in [0, \frac{1}{\mu_j}]$), we get

$$z_j(t-\nu_j,1) = z_j(t,1-\mu_j\nu_j) - \sum_{k=1}^{m}\int_0^{\nu_j}(G_3)_{jk}(1-\mu_j\nu_j + \mu_j s)z_k(t-s,1)ds.$$

Thus, since $r \leq \frac{2}{\mu_m}$ ans since the μ_j are increasing, we get for all $1 \leq j \leq m$,

$$\int_0^r (z_j(t-\nu,1))^2 d\nu \leq 2\int_0^r (z_j(t,1-\mu_j\nu))^2 d\nu + 2\sum_{k=1}^{m} \int_0^r \left(\int_0^\nu (G_3)_{jk}^2(1-\mu_j\nu+\mu_j s)(z_k(t-s,1))^2 ds\right) dr.$$

This immediately implies

$$\int_0^r (z_j(t-\nu,1))^2 d\nu \leq 2\mu_j \int_0^1 (z_j(t,x))^2 d\nu + 2||G_3||^2 r\sum_{k=1}^{m}\int_0^r (z_k(t-s,1))^2 ds.$$

Thus,

$$\sum_{k=1}^{m}\int_0^r (z_k(t-\nu,1))^2 d\nu \leq 2\mu_n \sum_{k=1}^{m}\int_0^1 (z_k(t,x))^2 d\nu + 2m||G_3||^2 r\sum_{k=1}^{m}\int_0^r (z_k(t-s,1))^2 ds.$$

Due to the definition or r, we have that $1-2m||G_3||^2 r > 0$. This immediately imply the first part of the inequality (using again the boundedness of the operator $\mathcal{F}$). This concludes the proof. ■

Remark 5.2.2 *The fact that the norms are different on the two sides of the inequality is related to the structure of the neutral equation (see for instance the design of converse Lyapunov-Krasovskii functions [PK13]).*

Remark 5.2.3 *Due to inequality* (5.92)*, any control law that exponentially stabilizes* (5.89)*, exponentially stabilizes* (5.7)-(5.9) *(and reciprocally). Conversely, if one system is unstable, so is the other. Consequently, one can "equivalently" choose one form or the other to design a stabilizing control law. The advantages of* (5.89) *is that we have decoupled the system* (5.7)-(5.9) *regrouping all the contributions of the in-domain terms inside the matrix G.*

We conclude this chapter with the following theorem that proves the uniqueness of the solution of the one-sided weak exact controllability problem stated in Theorem 2.3.2.

Theorem 5.2.5.

For any given initial condition (u_0, v_0) in χ, there exists a unique boundary control $V(t) \in \mathcal{L}((L^2([0,1]))^{n+m}, \Re^m)$ such that the boundary-control problem (5.38)-(5.40) satisfies the final zero-condition (2.25) in time τ.

Proof : We have proved that if the solution of (5.38)-(5.40) is equal to zero for $t \geq \tau$, then it immediately implies that $z(t,1) = 0$ for $t \geq \tau$. Thus, using (5.90), we get that $V(t) = \mathcal{L}y_t$. ■

This theorem proves that the control law defined by (3.38) and the one defined in [CHO17] are identical. This is consistent with [Rus72] where this result has been proved for the specific case of two hyperbolic equations with the same velocities (i.e $\lambda = \mu$).

5.3 Concluding remarks

Using the backstepping transformations introduced in [CHO17], we have successfully removed some of the in-domain couplings at the boundary of the system in the form of integral terms. Then, using the characteristic method, we have proved that the space generated by the solutions of the original PDE (5.7)-(5.9) is isomorphic to the space generated by the solutions of (5.90) which is a neutral system with distributed delays. This latter formulation makes clear the influence of boundary couplings (that correspond to the principal part of the neutral equation) and of in-domain terms (that correspond to the distributed delays) concerning the stability of the system. In that sense, we have decoupled the original system making its analysis and the control design easier. In particular, it has been proved that the trajectories of (5.7)-(5.9) and (5.90) have equivalent stability properties. A direct consequence is the uniqueness of the control law designed in Chapter 3 that solves the problem of weak exact boundary controllability.

Beyond the aspects of system theory, this relation between neutral systems and first order hyperbolic PDEs opens new perspectives in terms of stability analysis, adjusting the methods developed for neutral systems in [DDLM15, HL02, Nic01a, Nic01b] to the case of hyperbolic PDE. In particular, as presented in the next part, this relation is crucial regarding robustness analysis.

Table 5.1: [BSBAA+18] Comparison between the condition (5.32)-(5.35) required by Theorem 5.1.3 and [BC16, Theorem 5.4] for the stability of system (5.7)-(5.9)

Case	Condition required by Theorem (5.1.3) satisfied if	Condition required by Lemma 2.2.2 satisfied if
$\sigma^{+-}\sigma^{-+} > 0$ or only one is equal to zero	$\underline{\rho q \geq 0}$ $\lvert a\rvert + \lvert R\rvert \left(\frac{1}{1+\lvert\rho q\rvert} - \frac{1-\lvert\rho q\rvert}{2}\right) < 1 - \lvert\rho q\rvert$ $\underline{\rho q < 0}$ $\lvert a\rvert + \lvert R\rvert \frac{1+\lvert\rho q\rvert}{2} < 1 - \lvert\rho q\rvert$	(Not satisfied)
$\sigma^{+-} = 0$, $\sigma^{-+} = 0$	$\lvert\rho q\rvert < 1$	$\lvert\rho q\rvert < 1$
$\sigma^{+-}\sigma^{-+} < 0$	$\underline{\rho q \geq 0}$ $\lvert a\rvert I_0\left(\sqrt{\lvert R\rvert}\right) + \lvert R\rvert \left(\frac{1}{1+\lvert\rho q\rvert} - \frac{1-\lvert\rho q\rvert}{2}\right)$ $\times \left[I_0\left(\sqrt{\lvert R\rvert}\right) - I_2\left(\sqrt{\lvert R\rvert}\right)\right] <$ $1 - \lvert\rho q\rvert$ $\underline{\rho q < 0}$ $\lvert a\rvert I_0\left(\sqrt{\lvert R\rvert}\right) + \lvert R\rvert \frac{1+\lvert\rho q\rvert}{2}$ $\times \left[I_0\left(\sqrt{\lvert R\rvert}\right) - I_2\left(\sqrt{\lvert R\rvert}\right)\right] <$ $1 - \lvert\rho q\rvert$	$\rho^2 < -\frac{\lambda\sigma^{-+}}{\mu\sigma^{+-}} < \frac{1}{q^2}$

Table 5.2: Validating examples. All these systems are stable

Example of system (5.7)-(5.9) $(\sigma^{+-}, \sigma^{-+}, r_1, r_2, \rho, q)$	Condition (5.32)-(5.35) of Proposition 5.1.3	Condition required by [BC16, Theorem 5.4]
$(1.1, 0.4, 1, 1.2, 0.4, -0.5)$	Satisfied	Not satisfied ($\sigma^{+-}\sigma^{-+} > 0$)
$(-0.8, 0.7, 1, 1.2, 0.4, 0.25)$	Satisfied	satisfied
$(1.3, -0.95, 1.8, 0.44, 0.45, 0.25)$	Satisfied	Not satisfied
$(1.3, -1.2, 1.8, 1.5, 0.45, 0.25)$	Not Satisfied	satisfied
$(2.3, -3.5, 0.8, 1.1, 0.5, -0.7)$	Not Satisfied	Not satisfied

Part II

Robust stabilization

Introduction

In Part I, we have solved the finite-time boundary stabilization problems (A) and (A') and the finite-time boundary observability problems (B) and (B') for the general class of systems described by equations (2.1)-(2.3). Although the proposed approach is novel and a promising step in terms of control theory, by focusing on finite-time convergence (and therefore maximum performance) it completely neglects the robustness aspects. To envision practical applications, one must be able to perform a trade-off between the two. As the model given by equations (2.1)-(2.3) may be an approximation of a more complex real physical system, some crucial questions have to be taken into accounts while regarding robustness:

1. What if the different parameters of the model are not perfectly known?
2. Do disturbances acting on the system have any impact on the stability properties?
3. What is the impact on stability of potentially neglected dynamics?
4. In particular, what happens in presence of some delays acting on the actuators or on the sensors?

Some of these questions have been the purpose of recent investigations in the exact same context of LFOH PDEs. Among the different challenges, the disturbance rejection problem has been recently considered in [Aam13, AA15, Deu16, Deu17, DSBCdN08, LBL15, TK14]. In [Aam13, AA15], the rejection of a perturbation affecting the uncontrolled boundary side of a linear hyperbolic system composed of two equations is performed using a backstepping approach. In [LBL15], a proportional-integral controller is introduced to ensure the stabilization of a reference trajectory. An integral action is considered in [DSBCdN08] to ensure output rejection and its effectiveness is validated on experimental data. In [TK14], a sliding mode control approach is used to reject a boundary time-varying input disturbance. In presence of uncertainties in the system, the design of adaptive control laws using filter or swapping design is the purpose of [ADAK16a, ADAK16b]. Singularly, a large part of the literature relying on backstepping for PDEs evades the study of the impact of delays on the feedback loop.

Yet, it has been observed [DLP86, LRW96] that for many hyperbolic (among others) systems, the introduction of arbitrarily small time delays in the loop may cause instability for any feedback. In particular, in [LRW96], a systematic frequency domain treatment of this phenomenon for distributed parameter systems is presented. Regarding the computations presented in Chapter 5, linear first order hyperbolic PDEs can be mapped to neutral systems with distributed delays. For these systems the community usually focuses on uncertainties with respect to delays. This has led to the concept of *strong stability* [HVL93, HL02, MN07, MVZ$^+$09]. As the delays that appear in the neutral formulation obtained in Chapter 5 only depends on the velocity matrices Λ^+ and Λ^-, the tools introduced in [HVL93, HL02, MN07, MVZ$^+$09] can be extended to deal with the problem of robustness with respect to uncertainties on the velocities.

The purpose of this part is to analyze the robustness properties of the minimum-time controllers and observers designed in the previous part. More precisely, we prove that by focusing on finite-time stabilization these controllers may have small robustness margins (in particular no delay-margins). Thus, to ensure robust stabilization, we propose some adjustments in the control law by means of additional degrees of freedom enabling a trade-off between convergence rate and robustness. Using these degrees of freedom, we derive sufficient conditions guaranteeing the exponential stability of the controlled system in presence of uncertainties/delays/disturbances and consequently enable practical use of backstepping controllers.

A central theorem of this part is Theorem 6.3.2, as the mechanisms used in its proof are used several times in subsequent proofs. It establishes that the feedback terms that are critical for robustness are terms involving boundary values of the states, contrary to terms depending on the spatial integral of the state. The proof of this theorem consists in analyzing in the RHP of the Laplace domain, the contributions of each component of the characteristic equation associated ot the system.

Chapter 6

Delay-robust stabilization

Chapitre 6: Stabilisation robuste aux retards. *Afin de permettre au lecteur de se familiariser avec la nécessité d'un changement de stratégie pour la stabilisation robuste des systèmes* (2.1)-(2.3)*, nous considérons dans ce chapitre introductif le problème de stabilisation robuste aux retards (au sense de [LRW96]), i.e. nous souhaitons assurer la stabilisation du système* (2.1)-(2.3) *et ce y compris en présence de petits retards dans l'actionneur. En utilisant le formalisme des systèmes neutres introduit au Chapitre 5, nous montrons que les propriétés de robustesse aux retards sont liées à la taille des termes de couplage aux frontières (et à leur annulation éventuelle dans la loi de commande* (3.38)*). En particulier, nous montrons que pour un système composé de deux équations hyperboliques couplées avec des termes anti-diagonaux, si le produit des termes de réflexion distale et proximale est plus grand que 1, alors on ne peut pas stabiliser robustement aux retards. Si ce produit est compris entre* $\frac{1}{2}$ *et* 1*, alors il est possible de stabiliser un tel système de façon robuste aux retards mais pas en temps fini. Enfin, si ce produit est inférieur à* $\frac{1}{2}$*, il est possible de stabiliser ce système en temps fini tout en étant robuste aux retards. Des résultats similaires sont ensuite obtenus dans le cas général de* $n+m$ *équations. Pour garantir la robustesse aux retards de la loi de commande, nous proposons quelques ajustements de la loi de commande* (3.38) *en introduisant un degré de liberté assurant un compromis entre robustesse et performance.*

Contents

To make the reader familiar with the necessity of a change of strategy for the robust control of system (2.1)-(2.3), we consider in this introductory chapter the problem of delay-robust stabilization (in the sense of [LRW96]), i.e. we want to ensure stability even in presence of small delays in the actuation path. These aspects are crucial is so far as we show that in some case the presence of arbitrarily small delays may lead to instability. Using the neutral systems formalism introduced in Chapter 5, we prove that the robustness properties with respect to delays are strongly interwoven with the size of the boundary couplings (and their eventual cancellation in the control law (3.38)). Specifically, we show that, for a system of two heterodirectional linear hyperbolic equations with anti-diagonal source terms, *if the product of the proximal and distal reflection coefficients is*:

- **Greater than one**, the system cannot be stabilized robustly to delays.
- **Smaller than one but greater than one-half**, the system cannot be finite-time stabilized robustly to delays.
- **Smaller than one-half** the system can be finite-time stabilized robustly to delays.

Similar results are then obtained for the general case of $n+m$ equations. To ensure delay-robust stabilization, we finally propose some adjustments in the control law. The content of this chapter has been published in [AAMDM18].

6.1 Definitions and preliminary results

In this section we introduce some definitions and properties that are used throughout this whole part of the thesis. Equations (2.1)-(2.3) rewrite:

$$\partial_t u(t,x) + \Lambda^+ \partial_x u(t,x) = \Sigma^{++}(x)u(t,x) + \Sigma^{+-}(x)v(t,x), \tag{6.1}$$
$$\partial_t v(t,x) - \Lambda^- \partial_x v(t,x) = \Sigma^{-+}(x)u(t,x) + \Sigma^{--}(x)v(t,x), \tag{6.2}$$

with the following linear boundary conditions

$$u(t,0) = Q_0 v(t,0), \quad v(t,1) = R_1 u(t,1) + V(t). \tag{6.3}$$

We recall that $U \equiv 0$, as we consider only one-sided problems in this part. We recall the following definition of delay-robust stabilization.

Definition 6.1.1. Delay-robust stabilization [LRW96]

> The controller $V(t) = \mathcal{K}[\begin{pmatrix} u \\ v \end{pmatrix}]$ where $\mathcal{K} : (L^2([0,1]))^{n+m} \to \mathbb{R}^m$ is a linear operator, delay-robustly stabilizes the system (6.1)-(6.3) in the sense of [LRW96] if the resulting feedback system exponentially stabilizes the system (6.1)-(6.3) in the sense of Definition 2.2.1 and if there exists a delay δ^* (potentially small) such that for any $0 \leq \delta \leq \delta^*$, the system (6.1)-(6.3) along with the delayed control law $V(t-\delta)$ remains exponentially stable. A system is said to be delay-robustly stabilizable if and only if there exists such a $\mathcal{K}$.

Note that there is a shift in term of control objective compared to Objective A, as finite-time stabilization is not required anymore. This notion of delay-robust stabilization has to be compared to the one of strong stabilization [HVL93, HL02, MN07, MVZ+09].To recall definition of the notion of strong stabilization, let uss consider the neutral system with distributed delays

$$z(t) = \sum_{p=1}^{k} A_p z(t-\tau_p) + \int_0^{\tau_0} G(s)z(t-s)ds, \tag{6.4}$$

where G is a L^∞-matrix and A_p are real squared matrices.

Definition 6.1.2. Strong stability [HL02]

> The system (6.4) is said to be strongly stable if it is stable in the sense of the L^2-norm (as defined by (5.4)) and if it remains stable in presence of small variations acting on the

delays τ_0 and τ_p.

Using the expression given in (5.89), one can notice that the problem of strong stability for a closed-loop system is related to the problem of robustness with respect to uncertainties on the transport velocities. Although we focus in this chapter on the problem of delay-robust stabilization, we solve in Chapter 8 in the case of a system composed of two coupled PDEs the combined problems of delay-robust stabilization, strong stabilization and robustness with respect to uncertainties acting on the different coupling terms.

The design of delay-robust control laws and the underlying stability analysis proposed in this part of the thesis are based on the formulation of system (6.1)-(6.3) as a neutral delay differential equation (as detailed in Chapter 5). As explained in Section 2.2, to analyse the stability properties of such systems, one is often lead to write the corresponding complex characteristic equation [HL02, HVL93, Nic01a]. More precisely, the location of the roots of the characteristic equation in the complex plane determines the stability properties of the system (see equation (2.17)). To deal with this root location problem, we give some complex analysis properties below.

In what follows s is a complex number, whose real part is denoted $\Re(s)$ and whose imaginary part is denoted $\Im(s)$. We consider some strictly positive integers p and N, a sequence of constant real matrices $A_k \in \mathcal{M}_{p,p}$ and a sequence of positive scalar constants τ_k. We consider the holomorphic function F defined for every complex number s by

$$F(s) = Id_p - \sum_{k=1}^{N} A_k e^{-s\tau_k}, \tag{6.5}$$

where we recall that Id_p is the identity matrix of dimension p. Theorems 2.2.1 and 2.2.2 provide necessary and sufficient conditions to guarantee that all the roots of the holomorphic function $\det(F(s))$ are located in the left half plane. As proved in section 2.2, in the absence of in-domain couplings, the characteristic equation associated to the uncontrolled (i.e, $V(t) \equiv 0$) neutral equation (5.75) is given by $\det(F(s)) = 0$ (where the structure of F is given by (6.5) and where the matrices A_k depend on the boundary couplings R_1 and Q_0). In presence of in-domain couplings, the characteristic equation has a slightly different structure and Theorem 2.2.1 does not apply. More precisely, as it will appear in the derivations, the characteristic equation rewrites $\det(F(s) + H(s)) = 0$, where H is a holomorphic function with specific properties. To investigate the root location of such an equation, we recall some well-known results.

Lemma 6.1.1. Rouché's theorem [MM07]

If $F(s)$ and $H(s)$ are holomorphic in a simply connected region D containing a closed contour Π and if $|H(s)| < |F(s)|$ on Π, then F and $F + H$ have the same number of zeros inside of Π.

Lemma 6.1.2. Riemann-Lebesgue lemma

Let us consider a function $f \in L^1([0, +\infty[, \mathbb{R})$ supported on $[0, \infty[$. Then

$$\int_0^{\infty} f(t)\mathrm{e}^{-ts}dt \underset{\substack{|s|\to\infty \\ \Re(s)\geq 0}}{\longrightarrow} 0.$$

For any real number $\sigma \in \mathbb{R}$, we denote P_σ the open half-plane $\{s \in \mathbb{C} \mid \Re(s) > \sigma\}$. For any positive η, we denote Z_η the set of complex numbers whose distance to the zeros of $\det(F)$ is at most η:

$$Z_\eta = \{z \in \mathbb{C} \mid \exists s \in \mathbb{C},\ \det(F)(s) = 0 \text{ and } |s - z| < \eta\}. \tag{6.6}$$

The following lemma assesses the existence of a lower bound of the function $|\det(F)|$ on $P_{\sigma-\epsilon}\backslash Z_\eta$.

Lemma 6.1.3. Zero clusters and lower bound

Let us consider $\sigma > 0$ and $\epsilon > 0$. There exists $\eta > 0$ such that any connected component Γ of the set Z_η is bounded and such that $\Gamma \subset P_{\sigma-\epsilon}$ if $\Gamma \cap P_\sigma \neq \emptyset$. Moreover, for any $\eta > 0$ there exists $\kappa > 0$ such that $|\det(F)| \geq \kappa$ on $P_{\sigma-\epsilon}\backslash Z_\eta$.

Proof : This proof is similar to the one given in [Boi13]. Only the main steps are given here. Denoting by $N(\rho)$ the number of zeros of $\det(F)$ whose modulus is smaller than ρ and using [Lev40, Theorem VIII], we have the existence of $K > 0$ such that $\limsup\limits_{\rho \to +\infty} \frac{N(\rho)}{\rho} \leq \frac{K}{\pi}$. Choosing $\eta < \frac{\pi}{2K}$ implies that any connected component of the set Z_η is bounded. The rest of the proof is based on Montel's boundedness theorem and Hurwitz's theorem and is identical to the one given in [Boi13]. ∎

We have the following theorem that extends Lemma 6.1.1.

Lemma 6.1.4.

Let $\sigma > 0$. Let us consider the holomorphic function F defined by (6.5) and an holomorphic function H such that $|H(s)| \underset{|s| \to +\infty}{\longrightarrow} 0$. If the function $\det(F)$ has an infinite number of zeros on P_σ, then the function $\det(F + H)$ has an infinite number of zeros whose real parts are strictly positive.

Proof : Let us consider $0 < \epsilon < \sigma$. The function $\det(F)$ has an infinite number of zeros on P_σ. Let $\eta > 0$ be such that any connected component Γ of Z_η that contains such a zero is bounded and included in P_ϵ. Since the zeros of $\det(F)$ are isolated, every Γ contains a finite number of zeros, and the collection of sets Γ is infinite. Let us consider Γ_k a sequence of connected component of Z_η. Since these components are bounded, we can define Π_k as the boundary contour of Γ_k. Using Lemma 6.1.3 and Rouché's theorem we conclude the proof. ∎

6.2 Delay-robust stabilization: example of two equations

We consider in this section the specific case of two equations. Using the change of variables described in Remark 2.4.1, system (6.1)-(6.3) becomes

$$\partial_t u(t,x) + \lambda \partial_x u(t,x) = \sigma^{+-}(x) v(t,x), \tag{6.7}$$
$$\partial_t v(t,x) - \mu \partial_x v(t,x) = \sigma^{-+}(x) u(t,x), \tag{6.8}$$

with the linear boundary conditions

$$u(t,0) = q v(t,0), \quad v(t,1) = \rho u(t,1) + V(t). \tag{6.9}$$

The objective of this section is to analyze the delay-robustness properties (in the sense of definition 6.1.1) of the closed loop system (6.7)-(6.9) along with the control law $V(t)$ defined in (2.60), that ensures (in the nominal case, i.e without any delay) finite-time stabilization as defined by Objective A. The chosen approach consists in using the isomorphism introduced in Theorem 5.1.1 and the stability equivalence stated in Theorem 5.1.2 to rewrite, in presence of a delay in the actuation, the closed-loop system as a neutral equation whose stability can be analyzed using the characteristic equation. We assume that $q \neq 0$ (although the presented results can be extended to this critical case using similar methods as the one presented in section 2.4). We recall that we denote $\tau = \frac{1}{\lambda} + \frac{1}{\mu}$ the characteristic time of the system. It has been proved in Chapter 2 that with the backstepping transformation (2.37)-(2.38), the system (6.7)-(6.9) is mapped to

$$\partial_t \alpha(t,x) + \lambda \partial_x \alpha(t,x) = 0, \quad \partial_t \beta(t,x) - \mu \partial_x \beta(t,x) = 0, \tag{6.10}$$

and the boundary condition

$$\beta(t,1) = \rho\alpha(t,1) - \int_0^1 N^\alpha(\xi)\alpha(t,\xi) + N^\beta(\xi)\beta(t,\xi)d\xi + V(t). \tag{6.11}$$

where N^α and N^β are defined by (2.58). Using the computations done in Chapter 5, the function $\beta(t,1)$ satisfies for all $t \geq \tau$ the neutral equation

$$\beta(t,1) = \rho q \beta(t-\tau,1) - \int_0^\tau N(\nu)\beta(t-\nu,1)d\nu + V(t), \tag{6.12}$$

where the function N is defined by (5.15).

6.2.1 Open-loop analysis

Let us consider the open-loop system, i.e $V \equiv 0$. The characteristic equation associated to (6.12) is

$$1 - \rho q \mathrm{e}^{-\tau s} + \int_0^\tau N(\nu)\mathrm{e}^{-\nu s} d\nu = 0. \tag{6.13}$$

Lemma 6.2.1.

If $|\rho q| > 1$, the characteristic equation (6.13) has an infinite number of roots with a positive real part.

Proof : This lemma is a direct consequence of Lemma 6.1.4. Let us denote

$$F(s) = 1 - \rho q e^{-s\tau}, \quad H(s) = \int_0^\tau N(\xi)e^{-\xi s}d\xi.$$

Using the Riemann-Lebesgue lemma (see Lemma 6.1.2), we obtain

$$|H(s)| \to 0 \text{ as } |s| \to \infty, \quad \Re(s) > 0.$$

The function F has an infinite number of zeros whose real parts are equal to $\frac{\ln(|\rho q|)}{2\tau}$. The hypothesis of Lemma 6.1.4 are satisfied and we conclude that $F + H$ has an infinite number of zeros whose real parts are strictly positive. ■

A direct consequence of this infinite number of roots in the complex Right Half Plane (RHP) is given by the following theorem.

Theorem 6.2.1.

If $|\rho q| > 1$, system (6.7)-(6.9) cannot be delay-robustly stabilized.

Proof : Due to Lemma 6.2.1, the characteristic equation associated to (6.12) has an infinite number of poles with a positive real part. Thus, this neutral system cannot be delay-robustly stabilized (see [LRW96, Theorem 1.2]). Due to Theorem 5.1.2, neither can be system (6.7)-(6.9). ■

We have proved in this section that if the open-loop gain $|\rho q|$ is greater than one, one cannot find a controller whose delay margin is non-zero. Consequently, there is a whole class of hyperbolic systems (6.7)-(6.9) that cannot be delay-robustly stabilized.

Remark 6.2.1 *The critical case $|\rho q| = 1$ is not considered here. Indeed one cannot simply adjust the previous proof, since the zeros of F are located on the imaginary axis.*

6.2.2 Delay-robustness of the control law

We assume henceforth that $|\rho q| < 1$. The control law (2.60) is defined by

$$\begin{aligned} V(t) &= -\rho u(t,1) + \int_0^1 K^{vu}(1,\xi)u(t,\xi) + K^{vv}(1,\xi)v(t,\xi)d\xi \\ &= -\rho\alpha(t,1) + \int_0^1 N^{\alpha}(1,\xi)\alpha(t,\xi) + N^{\beta}(1,\xi)\beta(t,\xi)d\xi. \end{aligned} \tag{6.14}$$

This control law can be rewritten in terms of $\beta(\cdot,1)$:

$$V(t) = -\rho q\beta(t-\tau,1) + \int_0^{\tau} N(\nu)\beta(t-\nu,1)d\xi. \tag{6.15}$$

Consider now a small delay $\delta > 0$ in the actuation. The neutral equation (6.12) rewrites

$$\beta(t,1) = q\rho\beta(t-\tau,1) - \rho q\beta(t-\tau-\delta,1) - \int_0^{\tau} N(\xi)(\beta(t-\xi,1) - \beta(t-\xi-\delta,1))d\xi. \tag{6.16}$$

The disturbed characteristic equation associated to this neutral delay differential equation is

$$1 - \rho q \mathrm{e}^{-\tau s} + \rho q \mathrm{e}^{-(\tau+\delta)s} + (1-\mathrm{e}^{-\delta s})\int_0^{\tau} N(\xi)\mathrm{e}^{-\xi s}d\xi = 0. \tag{6.17}$$

Theorem 6.2.2.

If $|\rho q| > \frac{1}{2}$, then for any $\delta > 0$, the system (6.1)-(6.3) along with the delayed backstepping control law $V(t-\delta)$ is unstable.

Proof : Let us denote

$$\begin{aligned} F_1(s) &= 1 - \rho q e^{-\tau s} + \rho q e^{-(\tau+\delta)s}, \\ F_2(s) &= 1 - \rho q e^{\tau\epsilon}e^{-\tau s} + \rho q e^{(\tau+d)\epsilon}e^{-(\tau+\delta)s}, \\ H(s) &= (1-\mathrm{e}^{-\delta s})\int_0^{\tau} N(\xi)\mathrm{e}^{-\xi s}d\xi, \end{aligned}$$

where $\epsilon > 0$. Choosing ϵ small enough and δ such that $\tau\epsilon$ and $(\tau+\delta)\epsilon$ are rationally independent, we have that $|\rho q e^{\tau\epsilon}| + |\rho q e^{(\tau+\delta)\epsilon}| > 1$. Consequently, due to Theorem 2.2.2, $F_2(s)$ has an infinite number of roots whose real parts are positive. Moreover, these roots are unbounded (due to the isolated zeros theorem [Boi18]). Thus, $F_1(s)$ has an infinite number of roots whose real parts are larger than ϵ. Using Riemann-Lebesgues' lemma (Lemma 6.1.2), we have that $|H(s)|$ converges to zero for $|s|$ large enough. Lemma 6.1.4, implies that $F_1 + H$ has at least one root whose real part is strictly positive. Thus, the neutral system 6.16 is unstable [HVL93]. Using Theorem 5.1.2 concludes the proof. ■

Theorem 6.2.3.

If $|\rho q| > \frac{1}{2}$, then the system (6.1)-(6.3) cannot be finite-time stabilized robustly to delays.

Proof : This is a direct consequence of Theorem 5.2.5. As the backstepping controller (2.60) is the unique control law that guarantee finite-time stabilization and as the corresponding closed-loop system is not robust to delays if $|\rho q| > \frac{1}{2}$, the system (6.1)-(6.3) cannot be finite-time stabilized robustly to delays. ■

The fact that the backstepping controller (2.60) *proposed in [CVKB13] has zero delay margin when* $|\rho q| > \frac{1}{2}$ *means that it cannot be used for practical applications.* Specifically, $|\rho q| > \frac{1}{2}$ indicates that the resulting feedback system cannot have both finite-time convergence and robustness to delays. This stability limitation is not due to the backstepping method itself but to the cancellation of the proximal reflection term $\rho u(t,1)$. To obtain a tractable implementation of a controller for the system (6.1)-(6.3), one must have robustness to delay and thereby give up finite-time convergence.

Remark 6.2.2 *For systems that do not have a reflection at either boundary, there is no concern with delay-robustness. This is consistent with the delay-robustness results for predictor feedback developed in [BPK10, Krs08].*

It is important to stress that the fundamental limitations of, e.g. Theorem 6.2.1 would not apply to an actual plant in the strict sense. Models of the form (6.1)-(6.3) are obviously simplistic and do not capture, for example, the diffusivity that would stem from Kelvin-Voigt damping, or other phenomena that would be susceptible of making the delay margins non equal to zero. However, these results do indicate

- that the delay-robustness margins may be poor for such systems.
- that controllers of the form (2.60) radically trade off delay-robustness for performance, making them likely to be unusable.

Interstingly, these observations are consistent with reports by industrial practitioners on the limitations of the impedance matching method. For instance, in [KN09], the authors design a controller preventing stick-slip oscillations of a drill-string (a dysfunction of rotary drilling, characterized by large cyclic variations of the drive torque and the rotational bit speed). They observe that completely canceling the proximal reflection coefficient, even in a narrow frequency band, can change the dynamics of the string in a way that makes the system unstable. In this regard, a more quantitative approach to analyzing the performance–delay-robustness trade-offs made available by the use of backstepping is needed, in particular to assess whether the proposed qualitative approach remains valid with more realistic models.

6.2.3 A delay-robust controller

In this section we slightly modify the control law (2.60) to overcome the stability limitation exposed above, while maintaining the same structure for the controller. The control law (2.60) is composed of two parts:

1. the integral part whose objective is to remove the effect of in-domain couplings
2. the term $-\rho u(t,1)$ whose objective is to cancel the proximal reflection and to ensure finite-time convergence (as stated in Theorem 5.2.5).

As seen above, the instability of the feedback system in presence of small delay in the loop is mostly due to the term $-\rho u(t,1)$ in the control law (see Theorem 6.2.2). It appears consequently necessary to *avoid the total cancellation of the proximal reflection* (and thereby to give up finite-time convergence). Let us consider the following control law:

$$\begin{aligned} V_2(t) = -\tilde{\rho}u(t,1) - (\rho - \tilde{\rho}) \int_0^1 K^{uu}(1,\xi)u(t,\xi) + K^{uv}(1,\xi)v(t,\xi)d\xi \\ + \int_0^1 K^{vu}(1,\xi)u(t,\xi) + K^{vv}(1,\xi)v(t,\xi)d\xi, \end{aligned} \tag{6.18}$$

where the coefficient $\tilde{\rho}$ is chosen such that

$$|\tilde{\rho}| < \frac{1-|\rho q|}{|q|}. \tag{6.19}$$

The objective of such a control law is to *preserve a small amount of proximal reflection in the target system to ensure delay-robustness*, while eliminating in-domain couplings. Using the backstepping transformation (2.37)-(2.38) and the inverse transformation (2.49)-(2.50), the control law V_2 can be rewritten, using transformation (2.49)-(2.50), as

$$V_2(t) = -\tilde{\rho}\alpha(t,1) + \int_0^1 N^{\alpha}(1,\xi)\alpha(t,\xi) + N^{\beta}(1,\xi)\beta(t,\xi)d\xi,$$

where N^α and N^β are defined by (2.58). With this transformation, the system (6.1)-(6.3) is mapped to

$$\alpha_t(t,x) + \lambda\alpha_x(t,x) = 0, \quad \beta_t(t,x) - \mu\beta_x(t,x) = 0, \tag{6.20}$$

with the boundary conditions

$$\alpha(t,0) = q\beta(t,0), \quad \beta(t,1) = (\rho - \tilde{\rho})\alpha(t,1). \tag{6.21}$$

Lemma 6.2.2 *The system* (6.20)-(6.21) *is exponentially stable.*

Proof : It is sufficient to prove that $|q(\rho - \tilde{\rho})| < 1$ [BC16]. To do so, let us consider all the cases depending on the signs of q and ρ. If $\rho > 0$ and $q > 0$, we have (using (6.19))

$$-1 + 2\rho q < (\rho - \tilde{\rho})q < 1 \Rightarrow |(\rho - \tilde{\rho})q| < 1,$$

since $|\rho q| < 1$. The other cases can be treated similarly. ■

Consequently, the proposed control law exponentially stabilizes the system (6.1)-(6.3) in the nominal case. We now prove that it is robust with respect to small delays.

Theorem 6.2.4.

Consider the control law V_2 defined by (6.18) with $\tilde{\rho}$ satisfying (6.19). This control law delay-robustly stabilizes the system (6.7)-(6.9) in the sense of Definition 6.1.1.

Proof : Consider a positive delay δ. Consider the two states α and β defined by (6.10)-(6.11). Slightly adjusting the method used in Chapter 5, we get the following equation satisfied by the output $\beta(t,1)$:

$$\beta(t,1) = q\rho\beta(t-\tau,1) - q\tilde{\rho}\beta(t-\tau-\delta,1) + \int_0^\tau N(\xi)(\beta(t-\xi,1) - \beta(t-\delta-\xi,1))d\xi, \tag{6.22}$$

where N is defined by (5.15). The characteristic equation associated to this neutral equation is

$$\boldsymbol{F}(s) \triangleq 1 - q\rho e^{-\tau s} + \tilde{\rho} q e^{-(\tau+\delta)s} - I(s,\delta) = 0,$$

where $I(s,d)$ is defined by

$$I(s,\delta) = \int_0^\tau N(\xi)(\mathrm{e}^{-\xi s} - \mathrm{e}^{-(\xi+\delta)s})d\xi.$$

Let us now consider a complex number s such that $\Re(s) \geq 0$. We then have

$$\begin{aligned} |\boldsymbol{F}(s)| &\geq |1 - q\rho\mathrm{e}^{-\tau s} + \tilde{\rho}q\mathrm{e}^{-(\tau+\delta)s}| - |I(s,\delta)| \geq 1 - |q\rho\mathrm{e}^{-\tau s}| - |\tilde{\rho}q\mathrm{e}^{-(\tau+\delta)s}| - |I(s,\delta)| \\ &\geq 1 - |q\rho| - |\tilde{\rho}q| - |I(s,\delta)|. \end{aligned}$$

Since $\tilde{\rho}$ satisfies (6.19), there exists $\epsilon_0 > 0$ such that

$$1 - |q\rho| - |\tilde{\rho}q| > \epsilon_0.$$

Let us now focus on the term $I(s,\delta)$. Due to Riemann-Lebesgues lemma, we have

$$\forall |s| > M_0, \ |\int_0^\tau N(\xi)\mathrm{e}^{-\xi s}d\xi| < \frac{\epsilon_0}{2}$$

We can now choose δ_0 small enough such that for any $\delta \leq \delta_0$, for all complex s such that $|s| \leq M_0$, $|I(s,\delta)| < \epsilon_0$. With this choice of δ_0, one can easily check that, $\forall \delta \leq \delta_0$, $\forall s \in \mathbb{C}$ such that $\Re(s) \geq 0$

$$|I(s,\delta)| < \epsilon_0.$$

Consequently, for $\delta \leq \delta_0$, we have $|\boldsymbol{F}(s)| > 0$. It means that for $0 < \delta \leq \delta_0$, the function $\boldsymbol{F}(s)$ does not have any root whose real part is positive. Consequently, equation (6.22) is asymptotically stable. Thus, using Theorem 5.1.2, this concludes the proof. ■

Remark 6.2.3 *For a given value of $\tilde{\rho}$, the parameter δ_0 gives a range for admissible delays. However, δ_0 is not necessarily the maximum admissible delay. More precisely, the proposed result is only a qualitative result in the form of a necessary and sufficient condition to guarantee the existence of delay-margins.*

Remark 6.2.4 *Condition* (6.19) *define the maximum admissible cancellation of the boundary reflection term $-\rho u(t,1)$ to ensure the existence of robustness margins. One must be aware that there is a discontinuity line (that corresponds to the critical value given in* (6.19)*) for the existence of delay-robustness margins: if $|\tilde{\rho}| < \frac{1-|\rho q|}{|q|}$ there exists some delay-margin, if not there is no delay-margin. It means that in the case where $|\tilde{\rho}| < \frac{1-|\rho q|}{|q|}$, even if the delays reduce to zero, the norm of the solution of* (6.7)-(6.9) *diverges.*

Remark 6.2.5 *The coefficient $\tilde{\rho}$ can be interpreted as a tuning parameter, enabling a trade-off between performance (convergence rate) and robustness with respect to delays.*

Remark 6.2.6 *Another approach to delay-robustly stabilize* (6.1)-(6.3) *would consist in filtering the control law* (2.60).

6.2.4 Simulation results

We now illustrate our results with simulations on a toy problem. The numerical values of the parameters are as follow.

$$\lambda = \mu = \sigma^{+-} = \sigma^{-+} = q = 1, \quad \rho = 0.85.$$

The (positive) delay in the loop is denoted δ. The parameters values are chosen such that

- the open-loop system is unstable ([BC16])
- the open-loop gain satisfies $\frac{1}{2} < |\rho q| < 1$, so that the control law (2.60) does not guarantee delay-robust stabilization (but the system can be delay-robustly stabilized).

The solver we use to compute the kernels (required for the different control laws) is identical to the one introduced in Section 3.3 and is based on a fixed-point algorithm. The step of the mesh we used is $\frac{1}{60}$ and the chosen precision parameter ϵ is 10^{-4}. The hyperbolic system (6.7)-(6.9) is again simulated using a finite volume method based on a Godunov scheme. The integral part of the control law is computed using a trapezoidal method. The left part of Figure 6.1 pictures the L^2-norm of the state (u, v) using the control law presented in [CVKB13] without any delay ($\delta = 0$ s) and then in presence of a small delay in the loop ($\delta = 0.01$ s). As expected by the theory, with this control law, the system converges in finite-time to its zero-equilibrium when there is no delay in the loop but becomes unstable in presence of a small delay.

The right part of Figure 6.1 pictures the L^2-norm of the state (u, v) using the new control law (6.18) ($\tilde{\rho}$ is chosen equal to 0.1) for the same situations ($\delta = 0$ and $\delta = 0.01$ s) and for a larger delay ($\delta = 0.1$ s). As expected by the theory, the system is now robustly stable to delays in the loop. However, this improvement in terms of delay margin comes at the cost of a diminution of the convergence rate as we have lost finite-time stabilization.

Let us now consider the new following example:

$$\lambda = \mu = 0.5, \ \sigma^{+-} = 0, \ \sigma^{-+} = -2 \ q = 1, \quad \rho = 0.85.$$

The parameters values are chosen such that it is easy to compute the function $N(\cdot)$ defined by (5.15) as simple computations show that for all $x \in [0, 1]$, $N(x) = 1$. Thus, in presence of

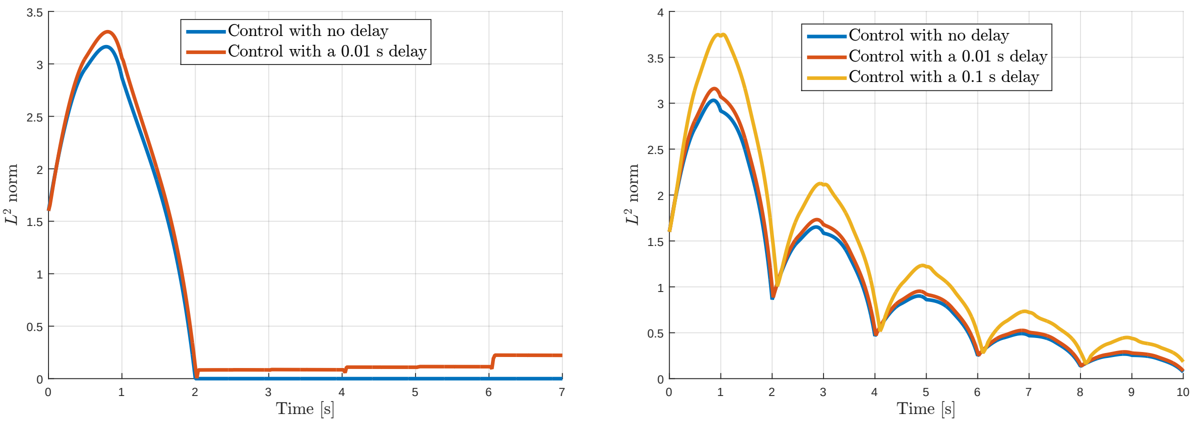

Figure 6.1: Time evolution of the L^2-norm using the control law (2.60) (left) and the control law (6.18) (right) in the nominal case and in presence of delays

the positive delay δ, it has been proved in Theorem 6.2.4 (equation (6.22)) that with the control law V_2, for any $t > 1$, the output $\beta(t, 1)$ satisfy the following equation

$$\beta(t, 1) = 0.85\beta(t-1, 1) - \tilde{\rho}\beta(t-1-\delta) + \int_0^1 (\beta(t-\xi, 1) - \beta(t-\xi-\delta, 1))\, d\xi.$$

The characteristic equation associated to this neutral equation is

$$1 = 0.85\mathrm{e}^{-s} - \tilde{\rho}\mathrm{e}^{-(1+\delta)s} - \frac{1}{s}(1-\mathrm{e}^{-\delta s})(\mathrm{e}^{-s}-1).$$

If we denote $H(s) = 1-(0.85\mathrm{e}^{-s} - \tilde{\rho}\mathrm{e}^{-(1+\delta)s} - \frac{1}{s}(1-\mathrm{e}^{-\delta s})(\mathrm{e}^{-s}-1))$, the delay margin corresponds to the first value of δ such that the holomorphic function H has a zero on the Right Half Plane. It is actually sufficient to consider the apparition of such a zero for $s = j\omega$ ($\omega \in \mathbb{R}$) a point of the imaginary axis. We give in Table 6.1 the maximum admissible delay for different values of $\tilde{\rho}$. Note that, as expected by the analysis done above, there is a discontinuity at $\tilde{\rho} = \frac{1-|\rho q|}{|q|} = 0.15$. Moreover, for this example, the maximum admissible delay depends on the chosen value of $\tilde{\rho}$. The larger $\tilde{\rho}$ is, the smaller the corresponding delay-margin will be. Although we expect this result to be true in the general case, we do not have any proof of it for the moment. We finally picture in Figure 6.2 the function $H(j\omega)$ for $\omega > 0$ for two different values of $\tilde{\rho}$ ($\tilde{\rho} = 0$ and $\tilde{\rho} = 0.14$) in presence of a delay of $0.9s$. One can easily see that in the former case we haven't reached yet the fateful point in terms of stability which is the origin, while in the second case this point have been passed which immediately imply the instability.

6.3 Delay-robust stabilization: case of a $n + m$ system

We now extend these results to the general case of system (6.1)-(6.3). It has been proved in Theorem 5.2.4 that the stability properties of (6.1)-(6.3) are similar to the ones of the neutral

Table 6.1: Maximum admissible delays for various values of $\tilde{\rho}$

Value of $\tilde{\rho}$	0	0.05	0.1	0.14	>0.15
Maximum admissible delay (in second)	0.97	0.91	0.87	0.83	0

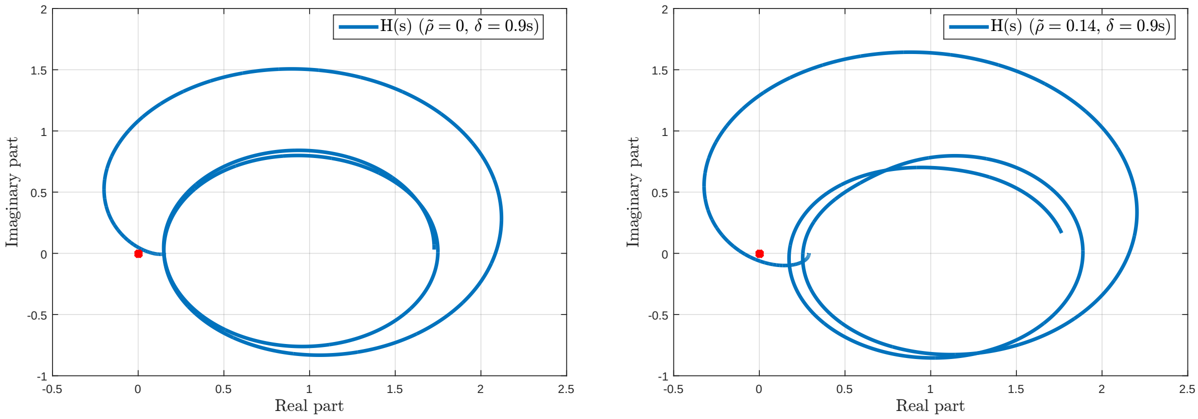

Figure 6.2: Representation on the complex plane of the function $H(j\omega)$ for $\tilde{\rho}=0$ (left) and $\tilde{\rho}=0.14$ (right) in presence of a delay of $0.9s$

system defined for all $1 \leq i \leq m$ by

$$z_i(t,1) = \sum_{k=1}^{n}\sum_{l=1}^{m}(R_1)_{ik}(Q_0)_{kl}z_l(t-\frac{1}{\lambda_k}-\frac{1}{\mu_l},1)+\sum_{l=1}^{m}\int_0^{\tau}G_{il}(s)z_l(t-s)ds+V_i(t), \tag{6.23}$$

where the matrix G is defined by (5.88), along with the initial condition $g_{u_0,v_0}(t)$. Let us consider the system

$$\partial_t u^*(t,x)+\Lambda^+\partial_x u^*(t,x)=0, \tag{6.24}$$
$$\partial_t v^*(t,x)-\Lambda^-\partial_x v^*(t,x)=0, \tag{6.25}$$

evolving in $\{(t,x)| \quad t>0, \quad x\in[0,1]\}$, with the following linear boundary conditions

$$u^*(t,0)=Q_0v^*(t,0), \qquad v^*(t,1)=R_1u^*(t,1). \tag{6.26}$$

This system is analogous to system (6.1)-(6.3), except that all the in-domain couplings have been removed. The next theorem (which is the generalization of Theorem 6.2.1) states that the robustness properties of (6.1)-(6.3) are strongly related to the stability properties of (6.24)-(6.26).

Theorem 6.3.1.

If (6.24)-(6.26) is unstable, then system (6.1)-(6.3) cannot be delay-robustly stabilized.

Proof : Using similar methods as the ones introduced in Chapter 5, the output $v^*(t,1)$ of (6.24)-(6.26) satisfies for all $t\geq\tau$, and for all $1\leq i\leq m$

$$v_i^*(t,1)=\sum_{\substack{1\leq l\leq m\\ 1\leq k\leq n}}(R_1)_{ik}(Q_0)_{kl}v_l^*(t-\frac{1}{\lambda_k}-\frac{1}{\mu_l},1). \tag{6.27}$$

The characteristic matrix associated to Equation (6.27) is denoted $\Delta_0(s)$. Since (6.24)-(6.26) is unstable, $\det(\Delta_0(s))$ has at least one root in the complex Right-Half Plane. Thus using the results of quasi-periodical functions presented in [HL02], the function $\det(\Delta_0)$ has a non-finite number of roots in the complex Right-Half Plane. Let us now denote $H(s)$ the matrix defined for all $1\leq i\leq m$ and $1\leq j\leq m$ by

$$H_{ij}(s)=\int_0^{\tau}G_{ij}(\nu)e^{-s\nu}d\nu, \tag{6.28}$$

where the matrix G is defined by (5.88). The characteristic matrix associated to equation (6.23) is $\Delta_0(s)+H(s)$. Using Riemann-Lebesgue's lemma, we obtain

$$|H(s)| \to 0 \text{ as } |s| \to \infty, \quad \Re(s) > 0.$$

The hypothesis of Lemma 6.1.4 are satisfied and we conclude that $\det(F+H)$ has an infinite number of zeros whose real parts are strictly positive. Thus, system (6.1)-(6.3) cannot be delay-robustly stabilized (see [LRW96, Theorem 1.2]). ■

As detailed in Section 2.2, various methods have been developed to analyze the exponential stability of linear conservation systems of the form (6.24)-(6.26) (or equivalently on the form (6.27)). The necessary and sufficient condition provided by Theorem 2.2.1, although it is not constructive can be used for systems of small dimension. For systems of higher dimensions, the sufficient conditions based on Lyapunov-Krasovskii methods [Car96, DDLMB16, Fri02, Pep05] may be more applicable. Theorem 6.3.1 illustrates the fact that the delay robustness properties of (6.1)-(6.3) are strongly related with the value of the boundary couplings and that some of these systems may not be delay-robustly stabilizable. We assume henceforth that (6.24)-(6.26) is exponentially stable (so that the system (6.1)-(6.3) has a chance to be delay-robustly stabilizable). Similarly, to the case of two equations we slightly modify the control law (3.38) (while maintaining the same structure for the controller) to ensure delay-robustness.

Theorem 6.3.2.

Let us consider the control law:

$$\begin{aligned} V_3(t) = -Pu(t,1) - (R_1 - P)\int_0^1 K^{uu}(1,\xi)u(t,\xi) + K^{uv}(1,\xi)v(t,\xi)d\xi \\ + \int_0^1 K(1,\xi)u(t,\xi) + L(1,\xi)v(t,\xi)d\xi, \end{aligned} \tag{6.29}$$

where the kernels K^{uu}, K^{uv} are defined by (5.43)-(5.51), the kernels K, L are defined by (3.12)-(3.15), and where the matrix P is chosen such that

$$\sup_{\theta_k^l \in [0,2\pi]^{n\times m}, \eta_k^l \in [0,2\pi]^{n\times m\times m}} \mathrm{Sp}\Big(\sum_{k=1}^{n}\sum_{l=1}^{m} A_k^l \exp(i\theta_k^l) + \sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m} B_{j,k}^l \exp(i\eta_k^l)\Big) < 1, \tag{6.30}$$

where for $1 \le k \le n$, and $1 \le l \le m$

$$A_k^l = \begin{pmatrix} 0 & \cdots & \overbrace{(R_1)_{1,k}(Q_0)_{k,l}}^{l^{th}\text{-column}} & \cdots & 0 \\ 0 & \cdots & (R_1)_{2,k}(Q_0)_{k,l} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & (R_1)_{m,k}(Q_0)_{k,l} & \cdots & 0 \end{pmatrix}, \quad B_{j,k}^l = \begin{pmatrix} 0 & \cdots & \overbrace{0}^{l^{th}\text{-column}} & \cdots & 0 \\ 0 & \cdots & (P)_{j,k}(Q_0)_{k,l} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}. \tag{6.31}$$

Then, the control law (6.29) delay-robustly stabilizes (6.1)-(6.3).

Proof : Consider a positive delay vector $\delta = \begin{pmatrix}\delta_1 & \delta_2 & \cdots & \delta_m\end{pmatrix}^T$. It has been proved in Chapter 5 that the control law (3.38) and the control law $\bar{V}(t) = -R_1 w(t,1) - \int_0^1 N^\alpha(\xi)w(t,\xi) - N^z(\xi)z(t,\xi)d\xi$ (where the state (w,z) is defined by the Fredholm transformation (5.59)-(5.60)) are actually the same. Thus, we have the equality

$$-R_1 u(t,1) + \int_0^1 K(1,\xi)u(t,\xi) + L(1,\xi)v(t,\xi)d\xi = -R_1 w(t,1) - \int_0^1 \left(N^\alpha(\xi)w(t,\xi) + N^z(\xi)z(t,\xi)\right)d\xi.$$

Moreover, using the backstepping transformations (5.41) and (5.59), we get

$$w(t,1) = u(t,1) - \int_0^1 \left(K^{uu}(1,\xi)u(t,\xi) + K^{vv}(1,\xi)v(t,\xi)\right)d\xi.$$

Thus, the control law (6.29) rewrites

$$
\begin{aligned}
V_3(t) &= -Pu(t,1) - (R_1 - P)\int_0^1 (K^{uu}(1,\xi)u(t,\xi) + K^{uv}(1,\xi)v(t,\xi))\,d\xi - R_1u(t,1) + R_1u(t,1)\\
&\quad + \int_0^1 (K(1,\xi)u(t,\xi) + L(1,\xi)v(t,\xi))\,d\xi\\
&= (R_1 - P)(u(t,1) - \int_0^1 (K^{uu}(1,\xi)u(t,\xi) + K^{uv}(1,\xi)v(t,\xi))\,d\xi) - R_1w(t,1)\\
&\quad - \int_0^1 (N^\alpha(\xi)w(t,\xi) + N^z(\xi)z(t,\xi))\,d\xi\\
&= -Pw(t,1) - \int_0^1 (N^\alpha(\xi)w(t,\xi) + N^z(\xi)z(t,\xi))\,d\xi,
\end{aligned} \tag{6.32}
$$

where N^α and N^z are defined by (5.58) and (5.68). We now adjust the proof of Theorem 5.2.2. We have that for all $t > \tau$, for all $1 \leq i \leq m$, using (6.32)

$$
z_i(t,1) = \sum_{k=1}^{n}\sum_{l=1}^{m}(R_1)_{ik}(Q_0)_{kl}z_l(t - \frac{1}{\lambda_k} - \frac{1}{\mu_l}, 1) + \sum_{l=1}^{m}\int_0^\tau G_{il}(s)z_l(t-\nu)d\nu + V_i(t-\delta_i),
$$

where the function G is defined by (5.88). This rewrites for all $t > \tau$, for all $1 \leq i \leq m$

$$
\begin{aligned}
z_i(t,1) &= \sum_{k=1}^{n}\sum_{l=1}^{m}(R_1)_{ik}(Q_0)_{kl}z_l(t - \frac{1}{\lambda_k} - \frac{1}{\mu_l}, 1) - \sum_{k=1}^{n}\sum_{l=1}^{m}(P)_{ik}(Q_0)_{kl}z_l(t - \frac{1}{\lambda_k} - \frac{1}{\mu_l} - \delta_i, 1)\\
&\quad + \sum_{l=1}^{m}\int_0^\tau G_{il}(\nu)z_l(t-\nu)d\nu - \sum_{l=1}^{m}\int_0^\tau G_{il}(\nu)z_l(t-\nu-\delta_i)d\nu,
\end{aligned}
$$

This yields

$$
\begin{aligned}
z(t,1) = \sum_{k=1}^{n}\sum_{l=1}^{m}A_k^l z(t - \frac{1}{\lambda_k} - \frac{1}{\mu_l}, 1) &- \sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m}B_{i,k}^l z(t - \frac{1}{\lambda_k} - \frac{1}{\mu_l} - \delta_i, 1)\\
&+ \int_0^\tau DG(\nu)z(t-\nu)d\nu
\end{aligned} \tag{6.33}
$$

where D is the diagonal matrix $\mathcal{M}_{n,n}(\mathbb{R})$ such that $D_{ii} = 1 - \delta_i$. The characteristic equation associated to (6.33) is

$$
\begin{aligned}
\det(Id_m - \sum_{k=1}^{n}\sum_{l=1}^{m}A_k^l \mathrm{e}^{-(\frac{1}{\lambda_k}+\frac{1}{\mu_l})s} &+ \sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m}B_{i,k}^l \mathrm{e}^{-(\frac{1}{\lambda_k}+\frac{1}{\mu_l}+d_i)s}\\
&- \int_0^\tau DG(\nu)\mathrm{e}^{-\nu s}d\nu) = 0.
\end{aligned} \tag{6.34}
$$

Let us denote

$$
F(s) = Id_m - \sum_{k=1}^{n}\sum_{l=1}^{m}A_k^l \mathrm{e}^{-(\frac{1}{\lambda_k}+\frac{1}{\mu_l})s} + \sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m}B_{i,k}^l \mathrm{e}^{-(\frac{1}{\lambda_k}+\frac{1}{\mu_l}+\delta_i)s}, \tag{6.35}
$$

$$
H(s) = -\int_0^\tau G(\nu)\mathrm{e}^{-\nu s}d\nu. \tag{6.36}
$$

The characteristic equation (6.34) rewrites

$$
\det(F(s) + DH(s)) = 0.
$$

The function $\det(F)(s)$ has all its roots in the left-half complex plane due to (6.30) and Theorem 2.2.1. The function $H(s)$ is bounded in the right-half complex plane. By contradiction, assume that there exists $s \in \mathbb{C}$, $s \neq 0$ and $\mathrm{Re}(s) \geq 0$, such that $\det(F(s) + DH(s)) = 0$. There exists $\eta \neq 0$ such that

$$
F(s)\eta = DH(s)\eta.
$$

This yields

$$
\eta^* F^*(s)F(s)\eta = \eta^* H^*(s)DDH(s)\eta,
$$

where * denotes the conjugate transpose. Since $F(s)$ is non singular in $\mathbb{C}^+$, there exists $M_0 > 0$ such that $M_0 < \eta^* F^*(s)F(s)\eta$. Similarly, $H(s)$ is bounded in $\mathbb{C}^+$, so that there exists $M_1 > 0$ such that

$$M_0 \leq \eta^* H^*(s)DDH(s)\eta \leq \sup_i |1 - e^{-\delta_i s}|^2 M_1.$$

Construct $\delta_m(s) = \frac{\bar{\delta}}{|s|}$, for some $\bar{\delta} > 0$ such that $e^{\bar{\delta}} < 1 + \sqrt{\frac{M_0}{M_1}}$. It follows that for any $\delta_i \leq \delta_m(s)$,

$$|1 - e^{-\delta_i s}| \leq e^{\bar{d}} - 1 < \sqrt{\frac{M_0}{M_1}}. \tag{6.37}$$

Since $\det(F(s) + DH(s))$ has only a finite number of zeros in the right-half plane, where the zeros have finite module [HL02], the quantity $\delta^\star = \min_s \delta_m(s)$ is strictly positive. This implies that for any $\delta_i \leq \delta^\star$, (6.37) holds. This leads to a contradiction with the previous inequality. Hence there does not exist any $s \in \mathbb{C}^+$ such that $\det(F(s) + DH(s)) = 0$. Furthermore, since the principal term of $\det(F(s) + DH(s))$ is precisely $F(s)$ which is stable by assumption, the asymptotic vertical chain of zeros of $\det(F(s) + DH(s))$ can not be the imaginary axis. This implies delay-robust stability since all zeros of $\det(F(s) + DH(s))$ are in the open left-half complex plane. ■

Due to the exponential stability of (6.24)-(6.26), condition (6.30) is always satisfied for $P = 0$. Thus the existence of a matrix P such that (6.30) holds is granted. One must be aware that condition (6.30) is only a sufficient condition and becomes a necessary one if the delays $\frac{1}{\lambda_k} + \frac{1}{\mu_l}$ are all rationally independent. However, increasing the dimensions of the matrices A_k^l and $B_{i,k}^l$ one can rewrite it as a necessary and sufficient condition on P (see [HVL93] for details).

Remark 6.3.1 *The method presented in the proof of this theorem is crucial in so far as it is used in multiple proofs presented in the next chapters. The idea will always be the same: if the principal part of the characteristic equation (that corresponds here to the function F) is positively bounded in the right-half complex plane; as it dominates the other terms of this equation, there is only a finite number of candidate roots located in the RHP that can be solutions of the characteristic equation. Using the boundedness of the principal part of the system, we can then choose the delay small enough such that these candidates are not solution of the characteristic equation.*

6.4 Concluding remarks

We have shown in this chapter the necessity of a change of strategy in the design of the backstepping controllers. To deal with delay-robust stabilization we have introduced a tuning parameter enabling a trade-off between convergence rate and delay-robustness. As mentioned in the beginning of the chapter, when dealing with real applications, one must take into account the robustness of the resulting closed-loop system with delays in the loop but also with respect to noise in the measurements, unknown disturbances, uncertainties on the parameters or neglected dynamics. In the next chapters, we introduce additional degrees of freedom in the control design that make possible various trade-off and hopefully an industrial application of the backtepping controllers. The next chapters are organized as follows.

Chapter 7: Delay-robust stabilization of a hyperbolic PDE-ODE system. We develop in this chapter a delay-robust stabilizing state-feedback control law for a linear ordinary differential equation coupled with two linear first order hyperbolic equations in the actuation path. The proposed controller combines the backstepping feedback law (6.18) (to remove the in-domain coupling terms) and a predictor. This control law can be tuned, either by adjusting the reflection coefficient $\tilde{\rho}$ left on the PDE or by choosing the pole placement on the ODE when constructing the predictor. Once again, this enables a trade-off between convergence rate an delay-robustness.

Chapter 8: Disturbance rejection and Input-to-State Stability for a system of two equations. We develop in this chapter a robust stabilizing output-feedback control law for a system of two coupled hyperbolic PDEs. The proposed control law combines the feedback law (6.18) with a state-observer adjusted form [VKC11], but also incorporates an integral action to enable disturbance rejection. The resulting system, under some conditions, is proved to be robust to delays (in the actuation or in measurements) and to uncertainties on the parameters. It is also Input-To-State Stable (ISS) with respect to disturbances and noise. The proposed output-feedback law introduces three degrees of freedom that can be tuned to enable a trade-off between performance and robustness but also between noise sensitivity and disturbance rejection.

Chapter 7

Delay-robust stabilization of a hyperbolic PDE-ODE system

Chapitre 7 Stabilisation robuste d'un système hyperbolique couplé à une équation diffférentielle ordinaire (EDO). *La stabilisation de systèmes d'EDPs hyperboliques couplées à des EDOs est un sujet de recherche très actif (c.f. par exemple [BLK14, BPK14, DMAHK18, SDMKR13]). Il s'agit d'un questionnement naturel lorsque sont considérés les problèmes de retards (que l'on peut considérer comme étant des EDPs hyperboliques du premier ordre) agissant sur l'actionneur ou les capteurs de l'EDOs [BLK14, BP12, FS02, SNA+11, YH05]. Une motivation récurrente pour l'étude de tels systèmes est l'atténuation des vibrations mécaniques dans les tiges de forages. Pour cet exemple, les EDPs hyperboliques représentent les propagations des contraintes axiales et de torsion le long de la tige de forage, tandis que l'EDO modélise la dynamique de fond de puits [BPDM16, DMA15]. L'approche par backstepping a été pour la première fois utilisée pour traiter des problèmes de couplages EDPs-EDOs en [KS08a] dans un cas spécifique pour lequel les retards dans les actionneurs et les capteurs sont explicitement compensés. Si le problème avait été originellement résolu par le prédicteur de Smith [Smi59], la reformulation du retard comme une EDP linéaire a permis de résoudre de nombreux problèmes, en particulier dans le cas de retards non constants ou mal connus [BLK13, BPCP12, BP12, BPCP14]. Récemment, le problème général de stabilisation d'une ODE couplée avec une équation hyperbolique du premier ordre a été résolu dans [DMAHK18] à l'aide d'une transformation intégrale transformant le système initial en une cascade de sous-systèmes exponentiellement stables. Cela a été rendu possible en annulant, entre autres, les termes de reflexion de l'EDP. Comme montré dans le Chapitre 6, cette loi de commande ne garantie pas forcément la robustesse aux retards. Dans ce chapitre, nous proposons la synthèse d'une nouvelle loi de commande robuste aux retards pour un système de deux EDPs hyperboliques linéaires couplées aux frontières avec une EDO. De manière similaire au Chapitre 6, la synthèse de la loi de commande est effectuée à l'aide d'un degré de liberté permettant de réaliser un compromis entre vitesse de convergence et robustesse aux retards. La solution retenue utilise l'approche proposée par [DMAHK18] en la complétant par un prédicteur après une reformulation du problème sous la forme d'un système à retard. L'approche retenue est la suivante: (i) Une transformation de backstepping permet d'enlever les termes de couplages présents dans l'EDPs et d'atténuer la réflexion à la frontière commandée. Le nouveau système se réècrit comme un système neutre. (ii) En utilisant la structure de cette dernière équation, il est possible de construire un opérateur non-inversible qui préserve la détectabilité tout en permettant de se ramener à un problème de stabilisation pour une EDO linéaire présentant un retard sur l'entrée. Un prédicteur est ainsi construit pour cette EDO. (iii) Les propriétés de robustesse aux retards sont étudiées par une analyse algébrique dans le domaine de Laplace.*

Contents

The control of systems of coupled ODEs and hyperbolic PDEs is a very active research topic (see for instance [BLK14, BPK14, DMAHK18, SDMKR13]). It naturally arises when considering delays (that can be seen as first-order hyperbolic PDEs) in the actuating and sensing paths of ODEs [BLK14, BP12, FS02, SNA$^+$11, YH05]. A recurrent practical motivation for the study of such systems is the attenuation of mechanical vibrations in drilling applications, where the hyperbolic PDEs represent axial and torsional stress propagation (waves) along the drill string, while the ODE models the Bottom Hole Assembly (BHA) dynamics [BPDM16, DMA15]. The backstepping approach has first been used in [KS08a] to deal with hyperbolic PDE-ODE couplings where actuator and sensor delays are explicitly compensated. While this problem had already been tackled by the Smith predictor [Smi59], the reformulation of the delay as a linear PDE enabled numerous related problems to be tackled, most notably the presence of non-constant and uncertain delays [BLK13, BPCP12, BP12, BPCP14]. Recently, the general problem of stabilizing an ODE with a system of first-order linear hyperbolic PDEs in the actuator path was solved in [DMAHK18] using a backstepping transformation that maps the fully interconnected system into a cascade of exponentially stable subsystems. This was achieved by canceling, among other terms, the reflection at the controlled boundary. As seen in Chapter 6, this control law may not guarantee delay-robust stabilization. In this chapter we provide a new design for a state-feedback law that achieves delay-robust stabilization of a system of two linear first-order hyperbolic PDEs coupled through the boundary to an ODE. More precisely, we consider systems of the form:

$$\partial_t u(t,x) + \lambda \partial_x u(t,x) = \sigma^{+-}(x) v(t,x) \tag{7.1}$$
$$\partial_t v(t,x) - \mu \partial_x v(t,x) = \sigma^{-+}(x) u(t,x) \tag{7.2}$$
$$\dot{X}(t) = AX(t) + Bv(t,0), \tag{7.3}$$

evolving in $\{(t,x) | \quad t > 0,\ x \in [0,1]\}$, with the boundary conditions

$$u(t,0) = qv(t,0) + CX(t), \quad v(t,1) = \rho u(t,1) + V(t), \tag{7.4}$$

where $X \in \mathbb{R}^p$ is the ODE state, $u(t,x) \in \mathbb{R}$ and $v(t,x) \in \mathbb{R}$ are the PDE states and $U(t)$ is the control input. The other parameters and coefficients are as defined in Chapter 2. The initial condition of the state (u,v) (denoted u_0 and v_0) is still assumed to belong to $(L^2([0,1],\mathbb{R}))^2$ The initial condition of the ODE (7.3) is denoted X_0. The resulting system (7.1)-(7.4) is well-posed [BC16, Theorem A.6, page 254]. This system is schematically depicted in Figure 7.1.

Similarly to Chapter 6, the control design is done by means of an additional degree of freedom enabling a trade-off between convergence rate in the absence of delay and delay-robustness. The proposed design works for all systems within the considered class for which delay-robust stabilization by such a feedback operator can be expected, see [LRW96] and Chapter 6. We achieve this by partially leveraging the backstepping design in [DMAHK18] and complementing it with a predictor-based controller after an adequate reformulation using a time-delay approach.

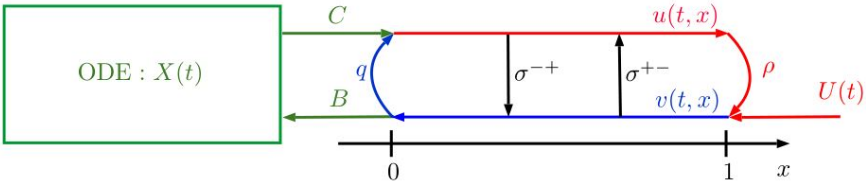

Figure 7.1: Schematic representation of the system (7.1)-(7.3)

The proposed approach is the following: (i) A backstepping transformation (and associated feedback operator) is constructed, removing the in-domain couplings present in the PDEs and possibly attenuating the reflections on the controlled side (depending on the choice of a tuning parameter). Without these in-domain couplings, the new system can be rewritten as a neutral delay differential equation. (ii) Using the structure of the obtained equation, we construct a non-invertible operator that preserves detectability in order to reduce the stabilization problem of the neutral system to that of a linear ODE system with delayed input, for which a state-predictor feedback law is constructed. (iii) Finally, the delay-robustness properties of the system are studied by means of an algebraic analysis in the Laplace domain. Note that the approach introduced in Chapter 6 cannot be directly extended since the system naturally features several feedback loops or couplings that can be sources of instabilities:

- Inside the ODE itself,
- Couplings between hyperbolic states inside the spatial domain,
- Couplings between hyperbolic states at the boundary,
- Couplings between the PDE and the ODE,
- A combination of all the above.

The content of this chapter has been published in [ABABS+18]. We still denote in this chapter $\tau = \frac{1}{\lambda} + \frac{1}{\mu}$.

The goal of this chapter is to design a feedback control law $V(t) = \mathcal{K}[\begin{pmatrix} u & v & X \end{pmatrix}]$ where $\mathcal{K} : (L^2[0,1])^2 \times \mathbb{R}^p \to \mathbb{R}$ is a linear operator, such that:

- the state (u, v, X) of the resulting feedback system (7.1)-(7.4) exponentially converges to its zero equilibrium (**stabilization problem**), *i.e.* there exist $\kappa_0 > 0$ and $\nu > 0$ such that for any initial condition (u_0, v_0, X_0)

$$||(u, v, X)||_2 \leq \kappa_0 \mathrm{e}^{-\nu t} ||(u_0, v_0, X_0)||_2, \ t \geq 0. \tag{7.5}$$

- the resulting feedback system (7.1)-(7.4) is robustly stable with respect to small delays in the loop (**delay-robustness**), *i.e.* there exists $\delta^\star > 0$ such that for any $\delta \in [0, \delta^\star]$, the control law $V(t - \delta)$ still stabilizes (7.1)-(7.4).

We make the two following assumptions:

Assumption 7.0.1 *The pair (A, B) is stabilizable, i.e. there exists a matrix K such that $A + BK$ is Hurwitz.*

Assumption 7.0.2 *The proximal reflection ρ and the distal reflection q satisfy $|\rho q| < 1$.*

The first assumption (stabilizability of the ODE subsystem) is necessary for the stabilizability of the whole system, while the second assumption is required for the existence of a delay-robust linear feedback control. This second assumption is not restrictive since if is not fulfilled, one could prove using arguments similar to those developed in Chapter 6 that the open-loop transfer function has an infinite number of poles in the complex right-half plane. Consequently (see [LRW96, Theorem 1.2]), it would not be possible to find any linear state feedback law $V(\cdot)$ that delay-robustly stabilizes (7.1)-(7.4).

7.1 Design of the control law

In this section we derive a control law that stabilizes of (7.1)-(7.4), following the methodology introduced in Chapter 6. This control law will be shown to be robust to small delays in the next section.

7.1.1 Backstepping transformation

We derive a Volterra transformation to rewrite system (7.1)-(7.4) as a system of transport equations coupled with an ODE. In other words, the purpose of this transformation is to remove the in-domain coupling terms, while conserving (only attenuating) boundary couplings. Let us consider the transformation

$$u(t,x) = \alpha(t,x) + \int_0^x L^{\alpha\alpha}(x,\xi)\alpha(t,\xi)\mathrm{d}\xi + \int_0^x L^{\alpha\beta}(x,\xi)\beta(t,\xi)\mathrm{d}\xi + \gamma_0(x)X(t), \tag{7.6}$$

$$v(t,x) = \beta(t,x) + \int_0^x L^{\beta\alpha}(x,\xi)\alpha(t,\xi)\mathrm{d}\xi + \int_0^x L^{\beta\beta}(x,\xi)\beta(t,\xi)\mathrm{d}\xi + \gamma_1(x)X(t), \tag{7.7}$$

$$X(t) = X(t). \tag{7.8}$$

Note that the equation (7.8) is only added to guarantee the invertibility of (7.6)-(7.7). The kernels $L^{\alpha\alpha}, L^{\alpha\beta}, L^{\beta\alpha}$ and $L^{\beta\beta}$ are defined on $\mathcal{T} = \{(x,\xi) \in [0,1]^2 | \quad \xi \leq x\}$ and γ_0 and γ_1 are row vectors with p components defined on $([0,1])$. They satisfy the following set of PDEs

$$\lambda\partial_x L^{\alpha\alpha}(x,\xi) + \lambda\partial_\xi L^{\alpha\alpha}(x,\xi) = \sigma^{+-}(x)L^{\beta\alpha}(x,\xi), \tag{7.9}$$

$$\lambda\partial_x L^{\alpha\beta}(x,\xi) - \mu\partial_\xi L^{\alpha\beta}(x,\xi) = \sigma^{+-}(x)L^{\beta\beta}(x,\xi), \tag{7.10}$$

$$\mu\partial_x L^{\beta\alpha}(x,\xi) - \lambda\partial_\xi L^{\beta\alpha}(x,\xi) = -\sigma^{-+}(x)L^{\alpha\alpha}(x,\xi), \tag{7.11}$$

$$\mu\partial_x L^{\beta\beta}(x,\xi) + \mu\partial_\xi L^{\beta\beta}(x,\xi) = -\sigma^{-+}(x)L^{\alpha\beta}(x,\xi), \tag{7.12}$$

and ODEs

$$\lambda\gamma_0'(x) = -\gamma_0(x)A + \sigma^{+-}(x)\gamma_1(x) - \lambda L^{\alpha\alpha}(x,0)C, \tag{7.13}$$

$$\mu\gamma_1'(x) = \gamma_1(x)A - \sigma^{-+}(x)\gamma_0(x) + \lambda L^{\beta\alpha}(x,0)C, \tag{7.14}$$

with the boundary conditions

$$L^{\beta\alpha}(x,x) = -\frac{\sigma^{-+}(x)}{\lambda+\mu},\ L^{\alpha\beta}(x,x) = \frac{\sigma^{+-}(x)}{\lambda+\mu},\ \gamma_1(0) = 0,\ \gamma_0(0) = 0, \tag{7.15}$$

$$L^{\alpha\alpha}(x,0) = \frac{\mu}{\lambda q}L^{\alpha\beta}(x,0) - \frac{1}{\lambda q}\gamma_0(x)B, \quad L^{\beta\beta}(x,0) = \frac{\lambda q}{\mu}L^{\beta\alpha}(x,0) + \frac{1}{\mu}\gamma_1(x)B. \tag{7.16}$$

Note that equations (7.9)-(7.12) are identical to equations (2.51)-(2.54) but have different boundary conditions due to the presence of the functions γ_0 and γ_1.

Lemma 7.1.1.

Consider system (7.9)-(7.16). There exists a unique solution $L^{\alpha\alpha}$, $L^{\alpha\beta}$, $L^{\beta\alpha}$ and $L^{\beta\beta}$ in $\mathcal{C}(\mathcal{T})$ and γ_0, γ_1 in $(\mathcal{C}^1([0,1]))^p$.

Proof : This result follows, with some minor adaptations, from [DMAHK18, Theorem 3.2]. The main idea consists on reinterpreting the ODEs in (7.13)-(7.14) as PDEs evolving in the triangular domain $\mathcal{T}$ with horizontal characteristic lines (since there is only an evolution along x) and then solving all the PDEs together. In this case, we extend the ODEs for γ_0 and γ_1, defined for $x \in [0,1]$, to the domain $(x,\xi) \in \mathcal{T}$ as follows:

$$\begin{aligned}\partial_x \tilde{\gamma}^0(x,\xi) &= -\frac{1}{\lambda}\tilde{\gamma}^0(x,\xi)A + \frac{\sigma^{+-}(x)}{\lambda}\tilde{\gamma}^1(x,\xi) - L^{\alpha\alpha}(x,\xi)C,\\ \partial_x \tilde{\gamma}^1(x,\xi) &= \frac{1}{\mu}\tilde{\gamma}^1(x,\xi)A - \frac{\sigma^{-+}(x)}{\mu}\tilde{\gamma}^0(x,\xi) + L^{\beta\alpha}(x,\xi)C,\end{aligned}$$

with boundary conditions

$$\tilde{\gamma}^0(x,x) = \tilde{\gamma}^1(x,x) = 0,$$

and the relations

$$\gamma_0(x) = \tilde{\gamma}^0(x,0), \quad \gamma_1(x) = \tilde{\gamma}^1(x,0).$$

This set of PDEs, together with (7.9)-(7.12) can be solved using the procedure detailed in [DMAHK18, Theorem 3.2]. Furthermore, since all coefficients are continuous, it can be shown that the unique solution obtained is in fact in $\mathcal{C}(\mathcal{T})$ in each component (see [CVKB13]). This regularity of solution to the PDEs implies that the solution to the original ODEs is in $(\mathcal{C}^1([0,1]))^p$. This concludes the proof. ■

Let us now consider the following system

$$\partial_t \alpha_t(t,x) + \lambda \partial_x \alpha(t,x) = 0, \tag{7.17}$$
$$\partial_t \beta_t(t,x) - \mu \partial_x \beta(t,x) = 0, \tag{7.18}$$
$$\dot{X}(t) = AX(t) + B\beta(t,0), \tag{7.19}$$

with the following boundary conditions

$$\alpha(t,0) = q\beta(t,0) + CX(t), \tag{7.20}$$
$$\beta(t,1) = \rho\alpha(t,1) + V(t) + (\rho\gamma_0(1) - \gamma_1(1))X(t) - \int_0^1 (N^\alpha(\xi)\alpha(t,\xi) + N^\beta(\xi)\beta(t,\xi))\mathrm{d}\xi, \tag{7.21}$$

where

$$N^\alpha(\xi) = L^{\beta\alpha}(1,\xi) - \rho L^{\alpha\alpha}(1,\xi), \tag{7.22}$$
$$N^\beta(\xi) = L^{\beta\beta}(1,\xi) - \rho L^{\alpha\beta}(1,\xi). \tag{7.23}$$

The corresponding initial conditions are denoted (α_0, β_0, X_0). Using the inverse transformation of (7.6)-(7.8), they can be expressed as functions of (u_0, v_0, X_0). Differentiating (7.6)-(7.7) with respect to time and space and using the boundary conditions (7.9)-(7.16), one can check that it maps the system (7.17)-(7.21) to the initial system (7.1)-(7.4). Due to the invertibility of the Volterra transformation (7.6)-(7.8), the two systems (7.17)-(7.21) and (7.1)-(7.4) are then equivalent. Thus, the stabilization of (7.17)-(7.21) implies the stabilization of the original system (7.1)-(7.4).

For the control design of the target system (7.17)-(7.19) with the boundary control (7.20)-(7.21), we decompose the input control $V(t)$ as

$$V(t) = V_{ODE}(t) + V_{BS}(t), \tag{7.24}$$

where $V_{ODE}(\cdot)$ has to be designed for the stabilization of the ODE dynamics (7.19), and where

$$\begin{aligned} V_{BS}(t) = &- \tilde{\rho}\,\alpha(t,1) - (\rho\gamma_0(1) - \gamma_1(1))X(t) \\ &+ \int_0^1 (N^\alpha(\xi)\alpha(t,\xi) + N^\beta(\xi)\beta(t,\xi))\mathrm{d}\xi. \end{aligned} \tag{7.25}$$

The coefficient $\tilde{\rho}$ is chosen such that

$$|\tilde{\rho}q| + |\rho q| < 1. \tag{7.26}$$

The existence of such a $\tilde{\rho}$ is a consequence of Assumption 7.0.2. Note that if $\tilde{\rho}$ does not satisfy (7.26), it is straightforward to adjust the proof of Theorem 6.2.1 to prove that the system is not robust to arbitrary small delays. Remark that, due to the invertibility of the Volterra transformation (7.6)-(7.8), $V_{BS}(t)$ can be expressed in terms of u, v and X.

The purpose of such a control law is to dissociate the stabilization of the ODE from the stabilization of the PDE. More precisely, the control law $V_{BS}(t)$ is designed to eliminate in-domain couplings. It preserves some proximal reflection in the target system (with the coefficient $\tilde{\rho}$) to ensure delay-robustness. This control, by itself, would guarantee the delay-robust exponential stabilization of (7.1)-(7.4) if there was not any ODE. In the next section, we use (7.24) and (7.25) to rewrite (7.17)-(7.21) as a neutral delay differential equation with control input $V_{ODE}(t)$. It becomes then possible to derive a control law using classical methods [GKC03, HL02, HVL93] to ensure exponential stabilization.

7.1.2 A neutral delay differential equation

As seen in Chapter 5, the equations (7.17)-(7.18) are transport equations, and consequently, for any $x \in [0,1]$, we get

$$\alpha(t,x) = \alpha\left(t - \frac{x}{\lambda}, 0\right), \; t \geq \frac{x}{\lambda} \tag{7.27}$$

$$\beta(t,x) = \beta\left(t - \frac{1-x}{\mu}, 1\right), \; t \geq \frac{1-x}{\mu}. \tag{7.28}$$

The substitution of (7.24) and (7.25) in the boundary condition (7.21) and the use of (7.27) lead to

$$\beta(t,1) = (\rho - \tilde{\rho})\alpha\left(t - \frac{1}{\lambda}, 0\right) + V_{ODE}(t). \tag{7.29}$$

It follows from (7.20), (7.28) and (7.29) that, for any $t \geq \tau$,

$$\beta(t,1) = q(\rho - \tilde{\rho})\beta(t-\tau,1) + (\rho - \tilde{\rho})CX\left(t - \frac{1}{\lambda}\right) + V_{ODE}(t). \tag{7.30}$$

For $t < \tau$, $\beta(t,1)$ can be expressed as a function of $(\alpha(0,\cdot), \beta(0,\cdot), X(0))$. Consequently (using the inverse of the backstepping transformation (7.6)-(7.7)) it can be expressed as a function of (u_0, v_0, X_0), the initial condition of the PDE (7.1)-(7.3). The ODE dynamics in (7.19) can be written as

$$\dot{X}(t) = AX(t) + B\beta\left(t - \frac{1}{\mu}, 1\right). \tag{7.31}$$

This yields, for any $t \geq \tau + \frac{1}{\mu}$,

$$\begin{aligned} \dot{X}(t) - (\rho - \tilde{\rho})q\dot{X}(t-\tau) = &AX(t) - (\rho - \tilde{\rho})qAX(t-\tau) \\ &+ B\beta(t - \frac{1}{\mu}, 1) - (\rho - \tilde{\rho})qB\beta(t - \frac{1}{\mu} - \tau, 1). \end{aligned}$$

Thus, using equation (7.30), we can substitute the term $\beta(t-\frac{1}{\mu},1)$ by an expression that only depends on X and V_{ODE}, that is

$$\begin{aligned}\dot{X}(t)-(\rho-\tilde{\rho})q\dot{X}(t-\tau) =& AX(t)-(\rho-\tilde{\rho})qAX(t-\tau)+(\rho-\tilde{\rho})BCX(t-\tau)\\ &+B\,V_{ODE}\left(t-\frac{1}{\mu}\right). \end{aligned} \tag{7.32}$$

Note that this expression still holds for $\tau \leq t \leq \tau+\frac{1}{\mu}$. Taking the Laplace transform and denoting

$$\hat{\phi}(s)=1-(\rho-\tilde{\rho})q\,\mathrm{e}^{-\tau s}, \tag{7.33}$$

$$\hat{\hat{V}}_{ODE}(s)=\hat{V}_{ODE}(s)+(\rho-\tilde{\rho})C\mathrm{e}^{-\frac{s}{\lambda}}\hat{X}(s) \tag{7.34}$$

one obtains

$$(sI-A)\hat{\phi}(s)\hat{X}(s)=B\mathrm{e}^{-\frac{s}{\mu}}\hat{\hat{V}}_{ODE}(s). \tag{7.35}$$

Under Assumption 7.0.2 and [HVL93], the function $\hat{\phi}(s)$ is positively bounded in the Right Half complex Plane. Thus, the roots of the characteristic equation associated to (7.32) have right-bounded real parts. Thus, there exists a spectral exponential bound for the existence of the Laplace transform for (7.33)-(7.35).

7.1.3 Spectral stabilization

We are now able to design the control law $\hat{\hat{U}}_{ODE}(s)$ that stabilizes (7.35). Denoting $\hat{Y}(s)=\hat{\phi}(s)\hat{X}(s)$, equation (7.35) can be rewritten as

$$(sI-A)\hat{Y}(s)=B\mathrm{e}^{-\frac{s}{\mu}}\hat{\hat{V}}_{ODE}(s). \tag{7.36}$$

Due to the detectability of X from the new variable Y, we can reduce the stabilization problem of the neutral equation (7.35) into that of a finite-dimensional system with delayed input, that can be rewritten in time domain as

$$\dot{Y}(t)=AY(t)+B\tilde{V}_{ODE}\left(t-\frac{1}{\mu}\right),\ t\geq\frac{1}{\mu}. \tag{7.37}$$

Different methods [Zho06] can be used to design a control law that stabilizes equation (7.37). A classical result from [MR03] states that any control law that stabilizes such an equation is equivalent to a predictor. We then have the following lemma.

Lemma 7.1.2.

Take A, B and K verifying Assumption 1 and any $\tilde{\rho}$ such that (7.26) holds. Then, the control law

$$\tilde{V}_{ODE}(t)=K\left(\mathrm{e}^{\frac{A}{\mu}}Y(t)+\int_{t-\frac{1}{\mu}}^{t}\mathrm{e}^{A(t-\nu)}B\tilde{U}_{ODE}(\nu)\mathrm{d}\nu\right),$$

exponentially stabilizes $Y(t)$ in (7.37). Furthermore, the state feedback

$$V_{ODE}(t)=\tilde{V}_{ODE}(t)-(\rho-\tilde{\rho})CX\left(t-\frac{1}{\lambda}\right)$$

exponentially stabilizes $X(t)$ in (7.32).

Proof : For the state-predictor feedback $\tilde{V}_{ODE}(\cdot)$, the closed-loop system in (7.37) satisfies

$$\dot{Y}(t) = (A + BK)Y(t), \ t \geq \frac{1}{\mu}.$$

Exponential stability is guaranteed by the fact that $(A + BK)$ is Hurwitz. By construction of $Y(t)$ and using (7.33), we have that $X(t)$, solution of (7.32), satisfies for any $t \geq \tau$,

$$X(t) = (\rho - \tilde{\rho})qX(t - \tau) + Y(t).$$

Since $|(\rho - \tilde{\rho})q| < 1$ by (7.26), $X(t)$ is also exponentially stable. ■

We conclude this section with the following theorem.

Theorem 7.1.1.

The control law

$$V(t) = V_{ODE}(t) + V_{BS}(t),$$

where $V_{BS}(t)$ is given in (7.25) and $V_{ODE}(t)$ is defined in Lemma 7.1.2, exponentially stabilizes the system (7.1)-(7.4) to its zero-equilibrium.

Proof : We have proved in Lemma 7.1.2 that the control law $V(t) = V_{ODE}(t) + V_{BS}(t)$ exponentially stabilizes $X(t)$ and $Y(t)$ described by (7.32) and (7.37), respectively. Furthermore, according to the decomposition introduced in (7.35), the state-predictor feedback in Lemma 7.1.2 can be written as

$$\tilde{V}_{ODE}(t) = KY(t + \frac{1}{\mu}),$$

which implies that $\tilde{V}_{ODE}(\cdot)$ exponentially converges to zero. Consequently, using (7.26), the state $\beta(t, 1)$ governed by (7.30) exponentially converges to zero, which in turn implies from (7.28) that $\beta(t, \cdot)$ converges L^2-exponentially to zero in the sense of equation 7.5.
This implies, from (7.27) and the boundary condition (7.20), that $\alpha(t, \cdot)$ converges also L^2-exponentially to zero. This yields the existence of $\kappa_0 > 0$ such that $||(\alpha, \beta, X)||_2 \leq \kappa_0 \mathrm{e}^{-\nu t}||(\alpha_0, \beta_0, X_0)||_2$. Thus the control law $V(t) = V_{ODE}(t) + V_{BS}(t)$ ensures the exponential stabilization of (7.17)-(7.21). Due to the invertibility of the backstepping transformation (7.6)-(7.8), it is straightforward to prove the stabilization of (7.1)-(7.4). ■

Using a backstepping approach combined with a time-delay approach, we have derived a control law ensuring the exponential stabilization of (7.1)-(7.4) to its zero equilibrium. We need now to prove that this control law is delay-robust. This is the purpose of the next section.

7.2 Delay-robust stabilization

In this section we prove the delay-robustness of the control law designed in the previous section. Let us consider a small positive delay $\delta > 0$ on the actuation input $V(\cdot)$. We get from (7.21), (7.20), (7.27) and (7.28)

$$\begin{aligned}
\beta(t,1) =& \rho\alpha(t - \frac{1}{\lambda}, 0) + V(t-\delta) + (\rho\gamma_0(1) - \gamma_1(1))X(t) - \int_0^1 (N^\alpha(\xi)\alpha(t,\xi) + N^\beta(\xi)\beta(t,\xi))\mathrm{d}\xi \\
=& \rho\alpha(t - \frac{1}{\lambda}, 0) + V(t-\delta) + (\rho\gamma_0(1) - \gamma_1(1))X(t) - \int_0^1 (N^\alpha(\xi)\alpha(t - \frac{\xi}{\lambda}, 0)d\xi \\
& - \int_0^1 N^\beta(\xi)\beta(t - \frac{1-\xi}{\mu}, 1))\mathrm{d}\xi \\
=& q\rho\beta(t-\tau, 1) + V(t-\delta) + \rho CX(t - \frac{1}{\lambda}) + (\rho\gamma_0(1) - \gamma_1(1))X(t) - \int_0^\tau N(\xi)\beta(t-\xi, 1)\mathrm{d}\xi \\
& - \int_0^1 N^\alpha(\xi)CX(t - \frac{\xi}{\lambda})\mathrm{d}\xi, \qquad (7.38)
\end{aligned}$$

where N is given by

$$N(\xi) = \begin{cases} \mu N^{\beta}(1-\mu\xi) & \text{for} \quad \xi \in [0, \frac{1}{\mu}) \\ \lambda q N^{\alpha}(\lambda\xi - \frac{\lambda}{\mu}) & \text{for} \quad \xi \in (\frac{1}{\mu}, \tau] \end{cases}.$$

The function $N(\cdot)$ has therefore a unique extension to the whole interval $[0,\tau]$ that is $\mathcal{C}^0$ on $[0, \frac{1}{\mu}]$ and also a unique extension to that interval that is $\mathcal{C}^0$ on $[\frac{1}{\mu}, \tau]$ (depending only on the value assigned at $\frac{1}{\mu}$). These extensions are k_1-Lipschitz on $[0, \frac{1}{\mu}]$ and k_2-Lipschitz on $[\frac{1}{\mu}, \tau]$, respectively. However, there is in general a discontinuity at $\frac{1}{\mu}$ such that

$$N\left(\frac{1}{\mu^-}\right) - N\left(\frac{1}{\mu^+}\right) = (\gamma_1(1) - \rho\gamma_0(1))B.$$

Since for integration purposes these two extensions are equivalent, and to avoid unnecessarily complex notation, depending on the context we may refer to one or the other as $N(\cdot)$.

Substituting the expression of $V(t)$ in (7.24) into (7.38) yields

$$\begin{aligned} \beta(t,1) &= q\rho\beta(t-\tau,1) - \tilde{\rho}q\beta(t-\tau-\delta,1) + V_{ODE}(t-\delta) + \rho C X\left(t - \frac{1}{\lambda}\right) - \tilde{\rho} C X\left(t - \frac{1}{\lambda} - \delta\right) \\ &+ (\rho\gamma_0(1) - \gamma_1(1))(X(t) - X(t-\delta)) - \int_0^{\tau} N(\xi)(\beta(t-\xi,1) - \beta(t-\xi-\delta,1))\mathrm{d}\xi \\ &- \int_0^1 N^{\alpha}(\xi) C\left(X\left(t - \frac{\xi}{\lambda}\right) - X\left(t - \frac{\xi}{\lambda} - \delta\right)\right)\mathrm{d}\xi. \end{aligned} \tag{7.39}$$

Taking the Laplace transform of (7.39) and multiplying by B one can get

$$\begin{aligned} &B\hat{\beta}(s,1) - q\rho B\mathrm{e}^{-\tau s}\hat{\beta}(s,1) + \tilde{\rho}q\mathrm{e}^{-(\tau+\delta)s}B\hat{\beta}(s,1) \int_0^{\tau} N(\xi)(\mathrm{e}^{-\xi s} - \mathrm{e}^{-(\xi+\delta)s})\mathrm{d}\xi B\hat{\beta}(s,1) \\ &= \mathrm{e}^{-\delta s}B\hat{V}_{ODE}(s) + BC(\rho\mathrm{e}^{-\frac{1}{\lambda}s} - \tilde{\rho}\mathrm{e}^{-(\frac{1}{\lambda}+\delta)s})\hat{X}(s) + B(\rho\gamma_0(1) - \gamma_1(1))(1-\mathrm{e}^{-\delta s})\hat{X}(s) \\ &- \int_0^1 N^{\alpha}(\xi)BC(\mathrm{e}^{-\frac{\xi}{\lambda}s} - \mathrm{e}^{-(\frac{\xi}{\lambda}+\delta)s})\mathrm{d}\xi\hat{X}(s). \end{aligned} \tag{7.40}$$

The Laplace transform of equation (7.31) implies that $(sI-A)\hat{X}(s) = B\mathrm{e}^{-\frac{s}{\mu}}\hat{\beta}(s,1)$. Moreover, using the expression of the state feedback in Lemma 7.1.2, we have

$$\begin{aligned} \hat{V}_{ODE}(s) &= \hat{\hat{V}}_{ODE}(s) - (\rho - \tilde{\rho})C\mathrm{e}^{-\frac{s}{\lambda}}\hat{X}(s) \\ &= K_0(s)\hat{\phi}(s)\hat{X}(s) - (\rho-\tilde{\rho})C\mathrm{e}^{-\frac{s}{\lambda}}\hat{X}(s), \end{aligned} \tag{7.41}$$

where $K_0(s)$ stands for the Laplace transform of the predictor state feedback in Lemma 7.1.2, namely

$$K_0(s) = \left[I - K(sI-A)^{-1}(I - \mathrm{e}^{-(sI-A)\frac{1}{\mu}})B\right]^{-1} K\mathrm{e}^{\frac{A}{\mu}}.$$

In what follows, we denote

$$\hat{\phi}_1(s,\delta) = 1 - q\rho\mathrm{e}^{-\tau s} + \tilde{\rho}q\mathrm{e}^{-(\tau+\delta)s} + (1-\mathrm{e}^{-\delta s})\int_0^{\tau} N(\xi)\mathrm{e}^{-\xi s}\mathrm{d}\xi. \tag{7.42}$$

Multiplying equation 7.40 by $\mathrm{e}^{-\frac{s}{\mu}}$ and using (7.41), we obtain

$$\begin{aligned} &(sI-A)(\hat{\phi}_1(s,\delta))\hat{X}(s) = B\mathrm{e}^{-\frac{s}{\mu}}[C\mathrm{e}^{-\frac{s}{\lambda}}(\rho - \tilde{\rho}\mathrm{e}^{-\delta s}) + \mathrm{e}^{-\delta s}K_0(s)\hat{\phi}(s) - (\rho-\tilde{\rho})C\mathrm{e}^{-\frac{s}{\lambda}-s\delta} \\ &+ (\rho\gamma_0(1) - \gamma_1(1))(1-\mathrm{e}^{-\delta s}) - (1-\mathrm{e}^{-\delta s})\int_0^1 N^{\alpha}(\xi)C\mathrm{e}^{-\frac{\xi s}{\lambda}}\mathrm{d}\xi]\hat{X}(s), \end{aligned} \tag{7.43}$$

where $\hat{\phi}$ is defined by (7.33). From [Vid72, Theorem 1], we know that $\phi_1(\cdot,\delta) \in \mathcal{A}$ has a unique inverse in $\mathcal{A}$ if and only if

$$\inf_{\mathrm{Re}(s)\geq 0} |\hat{\phi}_1(s,\delta)| > 0.$$

We have the following lemma on invertibility of $\hat{\phi}_1(s,\delta)$ in $\hat{\mathcal{A}}$ (where the Banach algebra $\hat{\mathcal{A}}$ is defined in the introduction of the chapter).

Lemma 7.2.1.

There exists $\delta^\star \in (0,\tau]$ such that

$$\inf_{\delta\in[0,\delta^\star]} \inf_{\mathrm{Re}(s)\geq 0} |\hat{\phi}_1(s,\delta)| > 0. \tag{7.44}$$

Proof : Consider a fixed $\delta \in [0, \min(\frac{1}{\mu}, \frac{1}{\lambda})]$. The element $\hat{\phi}_1(s,\delta)$ lies in $\hat{\mathcal{A}}$, since $N(\cdot)$ is in $L^1(\mathbb{R}^+,\mathbb{R})$. Furthermore, we have that $\hat{\phi}_1(s,\delta)$ is invertible in the Banach algebra $\hat{\mathcal{A}}$ provided that $\|1-\hat{\phi}_1(s,\delta)\|_{\hat{\mathcal{A}}} < 1$. Since $N(\cdot)$ has support in $[0,\tau]$ belongs to $L^\infty([0,\tau],\mathbb{R})$, a direct calculation using the triangular inequality for the L^1-norm shows that

$$\begin{aligned}\|1-\hat{\phi}_1(s,\delta)\|_{\hat{\mathcal{A}}} &\leq |q\rho| + |\tilde{\rho}q| + \int_0^\delta |N(\xi)|\,\mathrm{d}\xi + \int_\delta^{\frac{1}{\mu}} |N(\xi)-N(\xi-\delta)|\,\mathrm{d}\xi \\ &+ \int_{\frac{1}{\mu}+\delta}^{\tau} |N(\xi)-N(\xi-\delta)|\,\mathrm{d}\xi + \int_{\frac{1}{\mu}}^{\frac{1}{\mu}+\delta} (|N(\xi-\delta)|+|N(\xi)|)\mathrm{d}\xi + \int_\tau^{\tau+\delta} |N(\xi-\delta)|\,\mathrm{d}\xi.\end{aligned}$$

Since $N(\cdot)$ is k_1-Lipschitz in $[0,\frac{1}{\mu}]$ and k_2-Lipschitz in $[\frac{1}{\mu},\tau]$, we get

$$\|1-\hat{\phi}_1(s,\delta)\|_{\hat{\mathcal{A}}} \leq |q\rho| + |\tilde{\rho}q| + \delta\left(4\|N\|_{L^\infty} + \frac{k_1}{\mu} + \frac{k_2}{\lambda}\right). \tag{7.45}$$

Noting that with the condition (7.26) we have $|q\rho|+|\tilde{\rho}q| < 1$, there exists $\delta^\star > 0$ with

$$\delta^\star < \min\left(\frac{1-|q\rho|-|\tilde{\rho}q|}{4\|N\|_{L^\infty}+\frac{k_1}{\mu}+\frac{k_2}{\lambda}}, \min(\frac{1}{\mu},\frac{1}{\lambda})\right),$$

such that for any $\delta \in [0,\delta^\star]$, $\|1-\hat{\phi}_1(s,\delta)\|_{\hat{\mathcal{A}}} < 1$. This implies that $\phi_1(t,\delta)$ is a unit of $\mathcal{A}$, that is (7.44) holds. ■

Regarding equation (7.45), one can now fully understand the importance of the choice of $\tilde{\rho}$ made in (7.26). This choice is possible due to Assumption 7.0.2. Equation (7.43) yields

$$\begin{aligned}(sI-A)(\hat{\phi}_1(s,\delta))\hat{X}(s) &= B\mathrm{e}^{-\frac{s}{\mu}}[C\mathrm{e}^{-\frac{s}{\lambda}}(\rho-\tilde{\rho}\mathrm{e}^{-\delta s}) - (1-\mathrm{e}^{-\delta s})\int_0^1 N^\alpha(\xi)C\mathrm{e}^{-\frac{\xi s}{\lambda}}\mathrm{d}\xi - (\rho-\tilde{\rho})C\mathrm{e}^{-\frac{s}{\lambda}-s\delta} \\ &+ (\rho\gamma_0(1)-\gamma_1(1))(1-\mathrm{e}^{-\delta s}) + \mathrm{e}^{-\delta s}K_0(s)\hat{\phi}(s) - K_0(s)\hat{\phi}_1(s,\delta) + K_0(s)\hat{\phi}_1(s,\delta)]\hat{X}(s),\end{aligned}$$

We consequently get the following characteristic quasipolynomial $p(s)$

$$\begin{aligned}&\det((sI-A-BK_0(s)\mathrm{e}^{-\frac{s}{\mu}})\hat{\phi}_1(s,\delta) - B\mathrm{e}^{-\frac{s}{\mu}}(\rho C\mathrm{e}^{-\frac{1}{\lambda}s} - \tilde{\rho}C\mathrm{e}^{-\frac{1}{\lambda}s-\delta s} - (1-\mathrm{e}^{-\delta s})\int_0^1 N^\alpha(\xi)C\mathrm{e}^{-\frac{\xi}{\lambda}s}\mathrm{d}\xi \\ &+\mathrm{e}^{-\delta s}K_0(s)\hat{\phi}(s) - (\rho-\tilde{\rho})C\mathrm{e}^{-\frac{s}{\lambda}-s\delta} - K_0(s)\hat{\phi}_1(s,\delta) + (\rho\gamma_0(1)-\gamma_1(1))(1-\mathrm{e}^{-\delta s}))=0. \quad (7.46)\end{aligned}$$

Let us now denote

$$F(s) = (sI-(A+BK_0(s)\mathrm{e}^{-\frac{s}{\mu}}))\hat{\phi}_1(s,\delta) \tag{7.47}$$

$$\begin{aligned}H(s) &= B\mathrm{e}^{-\frac{s}{\mu}}(\rho\gamma_0(1)-\gamma_1(1)+\rho C\mathrm{e}^{-\frac{s}{\lambda}} + (\rho q\mathrm{e}^{-\tau s}-1 \\ &-\int_0^\tau N(\xi)\mathrm{e}^{-s\xi}\mathrm{d}\xi)K_0(s) - \int_0^1 N^\alpha(\xi)C\mathrm{e}^{-\frac{\xi}{\lambda}s}\mathrm{d}\xi).\end{aligned} \tag{7.48}$$

Using the definitions of $\hat{\phi}(s)$ and $\hat{\phi}_1(s,\delta)$, equation (7.46) can be rewritten as

$$p(s) = \det(F(s) - (1 - \mathrm{e}^{-\delta s})H(s)) = 0. \tag{7.49}$$

Note that since $K_0(s)$ is bounded in the right-half plane, $H(s)$ is bounded in the right-half plane. We are now finally able to prove that the control law $V(t)$ as defined in (7.24) delay-robustly stabilizes the system (2.1)-(7.4).

Theorem 7.2.1.

The control law $V(t) = V_{ODE}(t) + V_{BS}(t)$ as defined in (7.24) delay-robustly stabilizes the system (7.1)-(7.4). This is, there exists $\delta^\star > 0$ such that, for all $\delta \in [0,\delta^\star]$, $V(t) = V_{ODE}(t-\delta) + V_{BS}(t-\delta)$ exponentially stabilizes the system (7.1)-(7.4).

Proof : The proof follows the same steps as the one of Theorem 6.3.2. The closed-loop characteristic equation can be written as in (7.49), where $F(s)$ has all its roots in the left-half complex plane (see Lemma 7.2.1), and $H(s)$ is bounded in the right-half complex plane. By contradiction, assume that there exists $z \in \mathbb{C}$, $z \neq 0$ and $\mathrm{Re}(z) \geq 0$, such that $p(z) = 0$. There exists $\eta \neq 0$ such that

$$F(z)\eta = (1 - \mathrm{e}^{-\delta z})H(z)\eta.$$

This yields

$$\eta^* F^*(z)F(z)\eta = |1 - \mathrm{e}^{-\delta z}|^2 \eta^* H^*(z)H(z)\eta,$$

where * denotes the conjugate transpose. Since $F(z)$ is non singular in $\mathbb{C}^+$, there exists $M_0 > 0$ such that $M_0 < \eta^* F^*(z)F(z)\eta$. Similarly, $H(z)$ is bounded in $\mathbb{C}^+$, so that there exists $M_1 > 0$ such that

$$M_0 \leq |1 - \mathrm{e}^{-\delta z}|^2 \eta^* H^*(z)H(z)\eta \leq |1 - \mathrm{e}^{-\delta z}|^2 M_1.$$

Construct $\delta_m(z) = \frac{\bar{\delta}}{|z|}$, for some $\bar{\delta} > 0$ such that $\mathrm{e}^{\bar{\delta}} < 1 + \sqrt{\frac{M_0}{M_1}}$. It follows that for any $\delta \leq \delta_m(z)$,

$$|1 - \mathrm{e}^{-\delta z}| \leq \mathrm{e}^{\bar{\delta}} - 1 < \sqrt{\frac{M_0}{M_1}}. \tag{7.50}$$

Since $p(s)$ has only a finite number of zeros in the right-half plane, where the zeros have finite module [HL02], the quantity $\delta^\star = \min_z \delta_m(z)$ is strictly positive. This implies that for any $\delta \leq \delta^\star$, (7.50) holds. This leads to a contradiction with the previous inequality. Hence there does not exist any $z \in \mathbb{C}^+$ such that $p(z) = 0$. Furthermore, since the principal term of $p(s)$ is precisely the principal term of $\hat{\phi}_1(s,\delta)$ which is stable by construction (see Lemma 7.2.1), the asymptotic vertical chain of zeros of $p(s)$ can not be the imaginary axis. This implies delay-robust stability since all zeros of $p(s)$ are in the open left-half complex plane. ■

7.3 Simulation results

In this section we illustrate our results with simulations. Let us consider the unstable system (7.1)-(7.4) for which the coefficients are defined by

$$\lambda = \mu = \sigma^{+-} = \sigma^{-+} = q = 1, \quad \rho = 0.6, \quad A = 0.1,\ B = 0.1,\ C = 0.2. \tag{7.51}$$

The parameters values are chosen such that

- the ODE and the PDE open-loop system are unstable [BC16]
- the reflexion terms satisfies $0 < |\rho q| < 1$, so that Assumption 7.0.2 is satisfied.

We consider the norm $||\cdot||_2$ defined by (1.1). The initial condition is chosen as a C^1 function. The condition (7.26) means that one cannot completely cancel the proximal reflexion term $\rho u(t,1)$ to design a delay-robust control law when $|\rho q| > \frac{1}{2}$. To emphasize this property, we choose $|\rho q| = 0.6 > \frac{1}{2}$ in our simulations. The algorithm we use is adapted from the one proposed in Chapter 3, where additional details are provided. Using the method of characteristics, we write the integral equations associated to the PDE-system (7.9)-(7.16). These integral equations are solved using

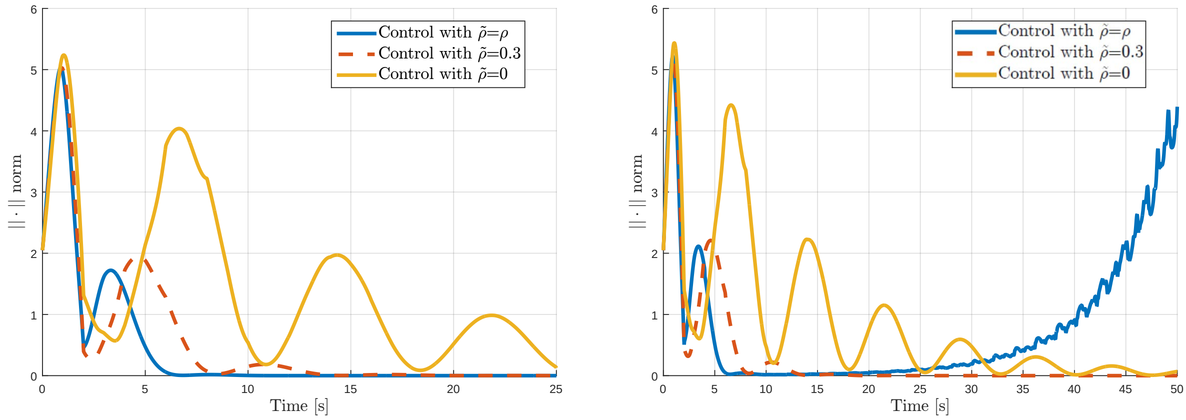

Figure 7.2: Time evolution of the $||\cdot||$-norm of system (7.1)-(7.3) for the parameters (7.51) for different values of $\tilde{\rho}$ without any delay (left) and considering a 0.02s delay.

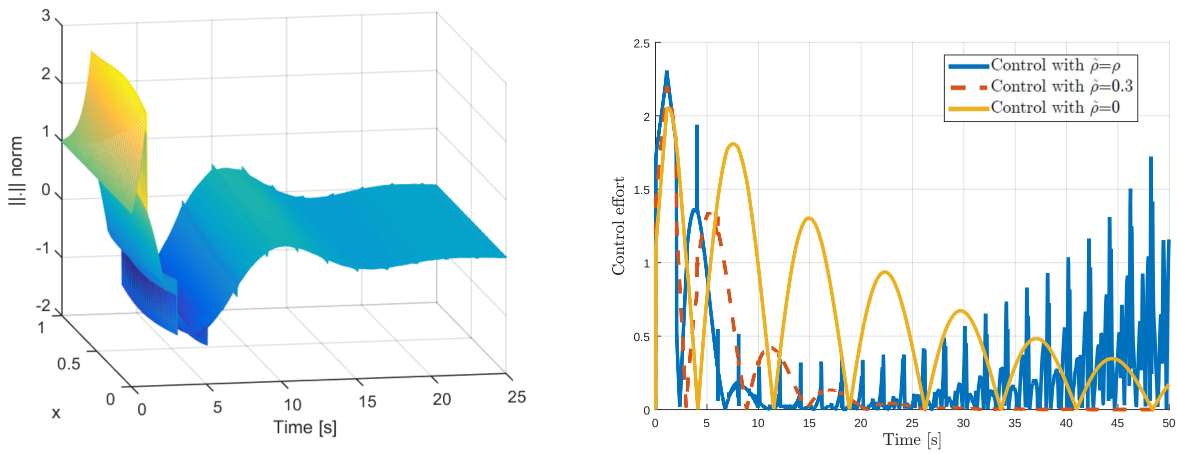

Figure 7.3: Evolution of $u(t, x)$ (left) and of the control effort |V(t)| (right) for various values of $\tilde{\rho}$ in presence of a 0.02s delay.

a fixed-point algorithm. These kernels are then used to compute the control law. Finally, the original system (7.1)-(7.4) is simulated using a Godunov's discretization scheme. The predictor (defined by V_{ODE} in Lemma 7.1.2) is adjusted from the one presented in [MM03]

The left part of Figure 7.2 pictures the $||\cdot||$-norm of the state (u, v, X) using the control law (7.25) for different values of $\tilde{\rho}$ without any delay whereas a small delay in the loop ($\delta = 0.02$ s) is considered in the right part of the figure. Choosing $\tilde{\rho}$ so that equation (7.26) holds, the resulting stabilizing control law is delay-robust. For such a value of $\tilde{\rho}$, due to the definition of the $||\cdot||$, the state X converges to zero. the left part of Figure 7.3 pictures the evolution of $v(t, 0)$ in presence of the delay $\delta = 0.02$ s. The right part of Figure 7.3 pictures the evolution of $u(t, x)$ in presence of the delay $\delta = 0.02$ s for a value of $\kappa = 0.3$. Note that the convergence is only guaranteed in the sense of equation 7.5. Finally, the right part of Figure 7.3 pictures the control effort for various values of $\tilde{\rho}$ in presence of the delay $\delta = 0.02$ s.

7.4 Concluding remarks

In this chapter, a delay-robust stabilizing feedback control law has been developed for a coupled hyperbolic PDE-ODE system. The proposed method combines a feedback constructed using the backstepping approach with a predictor-type feedback. The second feedback control is obtained after a suitable change of variables that reduces the stabilization problem of the PDE-ODE system to that of an ODE with input delay for which classical results for delay equations can be used. Similarly to Chapter 7, the robustness to small delays in the actuation of our combined feedback strategy is ensured by preserving some proximal reflection terms in the PDEs in the backstepping design. The degree to which these reflection terms are canceled introduces a tuning parameter that enables some trade-offs between convergence rates in the nominal system and delay-robustness.

Chapter 8

Disturbance rejection and Input-to-State Stability (ISS) for a system of two equations

Chapitre 8 Rejet de perturbations et stabilité entrée-état pour un système de deux équations *Dans ce chapitre, nous considérons le problème de régulation robuste pour la sortie d'un système hyperbolique de deux équations couplées pour lequel la sortie et l'actionneur sont colloqués et ce, en présence de perturbations et de bruits (ayant toutefois des propriétés particulières). La robustesse du système est analysée par rapport à d'éventuels retards sur l'actionneur ou sur la mesure mais également vis à vis d'éventuelles incertitudes sur les paramètres. Comme établi au Chapitre 6 (équation* (6.18)*), pour assurer la stabilisation de manière robuste aux retards, la loi de commande considérée n'annule qu'une partie de la réflexion à la frontière à l'aide d'un degré de liberté noté* $\tilde{\rho}$*. En outre, pour assurer la stabilité entrée-état et la convergence de la sortie à zéro en présence de perturbations constantes, la loi ce commande présente une action intégrale. La synthèse de l'observateur est adaptée de celle proposée en [VCKB11] (et rappelée au Chapitre 2, équations* (2.63)-(2.65)*). Comme cela a été fait pour la loi de commande, nous introduisons un degré de liberté supplémentaire dans la synthèse de l'observateur qui s'interprète comme une mesure de confiance entre les mesures et le modèle. La loi de commande par retour de sortie ainsi obtenue présente donc trois degrés de liberté: la quantité de réflexion annulée à la frontière, le gain de l'action intégrale et la quantité de réflexion annulée dans l'observateur. Nous donnons des conditions générales suffisantes sur ces trois degrés de liberté pour garantir la robustesse. Sous ces conditions, ces degrés de liberté permettent un compromis entre performance et robustesse, entre rejet de perturbation et atténuation du bruit. L'existence de marge de robustesse et la stabilité entrée-état sont prouvées en adaptant les méthodes introduites dans les chapitres précédents: à l'aide de transformations de backstepping et en utilisant la méthode des caractéristiques, nous prouvons cette fois encore que le système considéré se réécrit comme un système neutre à retards distribués. Ce système est montré comme assurant la stabilité entrée-état et robuste aux retards et incertitudes. En utilisant l'inversibilité de la transformation de backstepping, nous montrons que la sortie reste bornée et converge vers zéro si les perturbations sont constantes. Il est important de noter que la classe de perturbations considérée dans ce chapitre (perturbations bornées) est plus générale que celle proposée en [Deu16, Deu17] (où les perturbations sont générées par un système exogène de dimension finie) ou que les perturbations régulières considérées en [LDM16, LBL15].*

Contents

In this chapter, we consider the problem of robust-output feedback regulation for a system of two hyperbolic equations with collocated input and output in presence of a general class of disturbances and noise. Importantly, the robustness of the controller is considered with respect to delays in the actuation and in the measurements but also with respect to uncertainties on parameters, most importantly transport velocities. It is necessary to consider these three uncertainties simultaneously as it is the only way to ensure the existence of non-zero robustness margins. This approach encompasses the definitions of both delay-robust stabilization and strong stabilization. As established in Chapter 6 (equation (6.18)), to ensure delay-robustness, the proposed controller only cancels part of the boundary reflection by means of a tunable parameter. Moreover, to ensure Input-to-State Stability (ISS) and convergence of the output to zero for constant disturbance, we incorporate an integral action in the control law. The observer design is slightly adjusted from the one given in [VCKB11] (which we have recalled in Chapter 2, see equations (2.63)-(2.65)). As it is done for the controller, we introduce a new tunable parameter that can be interpreted as a measure of trust in our measurements relative to the model. The resulting output feedback controller presents three tuning parameters: the amount of reflection to be canceled at the boundary, the gain of the integral action and the amount of boundary reflection canceled in the observer. We give general conditions on these degrees of freedom that guarantee robustness. Provided that these conditions are satisfied, the parameters enable a trade-off between performance and robustness, between disturbance rejection and sensitivity to noise.

The existence of robustness margins and the ISS of the system are proved by adjusting the methods introduced in the previous chapters: by means of backstepping transformations and using the characteristics method, we prove that the resulting system can be transformed into a Neutral Differential System. This later system is proved to be Input-to-State Stable (ISS) and robust to delays and uncertainties, using classical Laplace analysis techniques. Using the invertibility of the backstepping transformation, we prove the boundedness of the controlled output for the target system and its convergence to zero in presence of constant disturbance. Besides, the class of disturbances considered in this chapter, namely bounded signals, is more general than the one proposed in [Deu16, Deu17] which the disturbance signal is generated by an exosystem of finite dimension, or than the smooth disturbances considered in [LDM16, LBL15].

This Chapter is based on the work published in [LADMA18].

In Section 8.1, we define the problem, and introduce a first "nominal" controller, without any robustness guarantee. Then, we add sufficient conditions on the design parameters to ensure robust stabilization.

8.1 Nominal system and problem under consideration

In this section, we first present the nominal system under consideration (i.e the one without delays, uncertainties or disturbances). For this nominal case, we introduce a stabilizing output feedback law. The proposed control law presents three degrees of freedom ($\tilde{\rho}$, ϵ and k_I) that can be tuned to ensure a trade-off between robustness and performance. We then consider the real system (i.e the system with uncertainties and delays) and define the problems of robust-stabilization and of Input-to-State Stability we solve in this chapter.

8.1.1 Nominal problem

We consider the following linear hyperbolic system ($n = m = 1$)

$$\partial_t u^{nom}(t,x) + \lambda \partial_x u^{nom}(t,x) = \sigma^{+-}(x) v^{nom}(t,x), \tag{8.1}$$
$$\partial_t v^{nom}(t,x) - \mu \partial_x v^{nom}(t,x) = \sigma^{-+}(x) u^{nom}(t,x), \tag{8.2}$$

evolving in $\{(t,x)| \quad t > 0,\ x \in [0,1]\}$, with the following linear boundary conditions

$$\begin{aligned} u^{nom}(t,0) &= q v^{nom}(t,0), \\ v^{nom}(t,1) &= \rho u^{nom}(t,1) + V(t), \end{aligned} \tag{8.3}$$

This system is a copy of system (2.29)-(2.31). The in-domain couplings $\sigma^{-+}(x)$ and $\sigma^{+-}(x)$ belong to $\mathcal{C}([0,1],\mathbb{R})$ and the velocities $\lambda > 0$ and $\mu > 0$ are assumed to be constant. The boundary couplings $q \neq 0$ and ρ satisfy $|\rho q| < 1$ to guarantee the existence of a delay-robust linear feedback control law (see Chapter 6). We still denote the characteristic time $\tau = \frac{1}{\lambda} + \frac{1}{\mu}$. The initial conditions denoted u_0^{nom} and v_0^{nom} are assumed to belong to $L^2([0,1])^2$. The measured output is denoted $y_m(t)$. As we consider the case of collocated measurements, we have $y_m(t) = u^{nom}(t,1)$.

Output feedback stabilization

The observer equations (similar to the ones of [VKC11]) set as follows

$$\partial_t \hat{u} + \lambda \partial_x \hat{u} = \sigma^{+-}(x)\hat{v} - P^{+}(x)\left(\hat{u}(t,1) - y_m(t)\right), \tag{8.4}$$
$$\partial_t \hat{v} - \mu \partial_x \hat{v} = \sigma^{-+}(x)\hat{u} - P^{-}(x)\left(\hat{u}(t,1) - y_m(t)\right), \tag{8.5}$$

with the boundary conditions

$$\hat{u}(t,0) = q\hat{v}(t,0), \tag{8.6}$$
$$\hat{v}(t,1) = \rho(1-\epsilon)\hat{u}(t,1) + \rho\epsilon y_m(t) + V(t). \tag{8.7}$$

The gains $P^{+}(\cdot)$ and $P^{-}(\cdot)$ are defined as

$$P^{+}(x) = -\lambda P^{uu}(x,1) + \mu\rho(1-\epsilon)P^{uv}(x,1), \tag{8.8}$$
$$P^{-}(x) = -\lambda P^{vu}(x,1) + \mu\rho(1-\epsilon)P^{vv}(x,1), \tag{8.9}$$

where the kernels P^{uu}, P^{uv}, P^{vu}, and P^{vv} belong to $\mathcal{C}(\mathcal{T}_1)$ (where $\mathcal{T}_1 = \{(x,\xi) \in [0,1]^2| \quad \xi \leq x\}$) and are defined by (2.71)-(2.76). The initial conditions $\hat{u}_0$ and $\hat{v}_0$ are assumed to belong to $(L^2([0,1]))^2$. The degree of freedom $\epsilon \in [0,1]$ that appears in (8.7) can be seen as a measure of trust in the measurements relative to the model (or unmeasured disturbances), where $\epsilon = 1$ results in relying more on the measurements and $\epsilon = 0$ relying more on the model. Notice that the observer of [VKC11] corresponds to the special case $\epsilon = 1$. Note that for $\epsilon = 1$, the observer gains given by (8.8)-(8.9) correspond to the ones given by (2.77).We now consider the output feedback law

$$V(t) = V_{BS}(t) + k_I V_I(t) + k_I \eta(t), \tag{8.10}$$
$$\dot{\eta}(t) = y_m(t), \tag{8.11}$$

where the initial condition of η is denoted η_0 and where

$$\begin{aligned} V_{BS}(t) = &-\tilde{\rho}(1-\epsilon)\hat{u}(t,1) - \tilde{\rho}\epsilon y_m(t) - (\rho-\tilde{\rho})\int_0^1 (K^{uu}(1,\xi)\hat{u}(t,\xi) + K^{uv}(1,\xi)\hat{v}(t,\xi))d\xi \\ &+ \int_0^1 \left(K^{vu}(1,\xi)\hat{u}(t,\xi) + K^{vv}(1,\xi)\hat{v}(t,\xi)\right) d\xi, \end{aligned} \tag{8.12}$$

$$\begin{aligned} V_I(t) = &-\int_0^1 l_1(\xi)\left(\hat{u}(t,\xi) - \int_0^\xi K^{uu}(\xi,\nu)\hat{u}(t,\nu) + K^{vu}(\xi,\nu)\hat{v}(t,\nu)d\nu\right)d\xi \\ &- \int_0^1 l_2(\xi)\left(\hat{v}(t,\xi) - \int_0^\xi K^{vu}(\xi,\nu)\hat{u}(t,\nu) + K^{vv}(\xi,\nu)\hat{v}(t,\nu)d\nu\right)d\xi. \end{aligned} \tag{8.13}$$

The kernels K^{uu}, K^{uv}, K^{vu}, K^{vv} are defined by (2.43)-(2.48), while the functions l_1 and l_2 are defined on the interval $[0,1]$ as the solutions of the system

$$\lambda l_1'(x) = L^{\alpha\alpha}(1,x), \quad \mu l_2'(x) = -L^{\alpha\beta}(1,x)\,, \tag{8.14}$$

with the boundary conditions

$$l_2(1) = 0\,, \quad l_1(0) = \frac{\mu}{q\lambda} l_2(0), \tag{8.15}$$

where $L^{\alpha\alpha}$, $L^{\alpha\beta}$ are the kernels of the inverse transformation (2.49)-(2.50) defined by equations (2.51)-(2.56). The control law V has three components: $V_{BS}(t)$, $V_I(t)$ and $k_I\eta$. The purpose of the integral term $k_I\eta$ is to enable rejection of constant disturbance, while the control V_{BS} partially cancels potentially destabilizing coupling terms. It corresponds to the extension of the state-feedback control law derived in Chapter 6 (equation (6.18)). It would stabilize the original system in the absence of disturbances and of the integral term $k_I\eta(t)$. The second term of the control law ($k_I V_I(t)$) is related to the presence of the integrator $k_I\eta$. More precisely, the term $k_I\eta$ used to enable disturbance rejection may have an effect on the stability of the system. This has to be compensated by the term ($k_I V_I(t)$). In what follows, we make the following assumption

Assumption 8.1.1

$$1 + \int_0^1 L^{\alpha\alpha}(1,\xi)d\xi + \frac{1}{q}\int_0^1 L^{\alpha\beta}(1,\xi)d\xi \neq 0\,. \tag{8.16}$$

As it will appear in the computations, this assumption is necessary to prove the stability of the closed-loop system. Unfortunately, no physical interpretation has been found for this assumption. The following condition on the tuning parameter k_I is required to guarantee stabilization

Condition 8.1.1 *Let us define $k_1 = (\rho-\tilde{\rho})q$ and $k_2 = k_I q(1+l_1(1)\lambda)$. We assume that $|k_1| < 1$ and $k_2 < 0$. Moreover k_I is chosen such that*

$$\tau < -\frac{\sqrt{1-k_1^2}}{|k_2|}\arctan\left(\frac{\sqrt{1-k_1^2}}{|k_1|}\right) + \frac{\pi\sqrt{1-k_1^2}}{|k_2|}\,, \quad \textit{if } k_1 \in (-1,0)\,, \tag{8.17}$$

$$\tau < \frac{\pi}{2\,|k_2|}\,, \quad \textit{if } k_1 = 0\,, \tag{8.18}$$

$$\tau < \frac{\sqrt{1-k_1^2}}{|k_2|}\arctan\left(\frac{\sqrt{1-k_1^2}}{k_1}\right)\,, \textit{ if } k_1 \in (0,1)\,. \tag{8.19}$$

With this choice of k_I and $\tilde{\rho}$, the complex equation $s-(k_1 s+k_2)\mathrm{e}^{-s\tau}=0$ has all its solutions in the complex left-half plane [CT15]. Finally, we consider the following condition that is required to prove the existence of robustness margins.

Condition 8.1.2 *Let us consider the following matrices*

$$A_1 = \rho q E_{11}^4 - \tilde{\rho}q(1-\epsilon)E_{12}^4 + (q\epsilon(\rho-\tilde{\rho})+q\tilde{\rho})E_{31}^4 - \tilde{\rho}q(1-\epsilon)E_{32}^4, \tag{8.20}$$

$$A_2 = \tilde{\rho}q(1-\epsilon)E_{14}^4 + (\rho-\tilde{\rho})q(1-\epsilon)E_{33}^4 + \tilde{\rho}q(1-\epsilon)E_{34}^4, \tag{8.21}$$

$$A_3 = E_{21}^4 + E_{43}^4, \tag{8.22}$$

$$A_4 = -\tilde{\rho}q\epsilon E_{12}^4 + (\tilde{\rho}-\rho)q\epsilon E_{31}^4 - \tilde{\rho}q\epsilon E_{32}^4. \tag{8.23}$$

The parameters $\tilde{\rho}$ and ϵ are chosen such that the following condition holds:

$$\sup_{\theta_k\in[0,2\pi]^4} Sp(\sum_{k=1}^4 A_k \exp(i\theta_k)) < 1, \tag{8.24}$$

As it will appear in the computations, this condition is strongly related to the conditions stated in Theorem 2.2.1. Note that since $|\rho q|<1$, Assumption 8.1.2 is always satisfied for e.g. $\epsilon=1$ and $\tilde{\rho}=0$. We have the following nominal stabilization theorem, under ideal assumptions.

Theorem 8.1.1. Nominal Stabilization

Consider the nominal system composed of (8.1)-(8.3) and of the observer system (8.4)-(8.7) along with the control law (8.10). If Assumption 8.1.1, and Condition 8.1.1 are satisfied, then, for any initial condition $(u_0^{nom}, v_0^{nom}) \in (L^2([0,1]))^2$, for any observer initial condition $(\hat{u}_0,\hat{v}_0) \in (L^2([0,1]))^2$, the solution $(u^{nom}, v^{nom}, \hat{u}, \hat{v})$ converges (in the sense of the L^2-norm) to zero.

Proof : Denoting $\tilde{u}^{nom}(t,x) = u^{nom}(t,x)-\hat{u}(t,x)$ and $\tilde{v}^{nom}(t,x) = v^{nom}(t,x)-\hat{v}(t,x)$ the error estimates, we get the following error system

$$\partial_t \tilde{u}^{nom}(t,x) + \lambda\partial_x\tilde{u}^{nom}(t,x) = \sigma^{+-}(x)\tilde{v}^{nom}(t,x) - P^+(x)\tilde{u}^{nom}(t,1), \tag{8.25}$$

$$\partial_t \tilde{v}^{nom}(t,x) - \mu\partial_x\tilde{v}^{nom}(t,x) = \sigma^{-+}(x)\tilde{u}^{nom}(t,x) - P^-(x)\tilde{u}^{nom}(t,1), \tag{8.26}$$

along with the boundary conditions

$$\tilde{u}^{nom}(t,0) = q\tilde{v}^{nom}(t,0), \quad \tilde{v}^{nom}(t,1) = \rho(1-\epsilon)\tilde{u}^{nom}(t,1). \tag{8.27}$$

Considering the backstepping transformation (2.69)-(2.70), system (8.25)-(8.27) is mapped to

$$\partial_t\tilde{\alpha}^{nom}(t,x) + \lambda\partial_x\tilde{\alpha}^{nom}(t,x) = 0, \quad \partial_t\tilde{\beta}^{nom}(t,x) - \mu\partial_x\tilde{\beta}^{nom}(t,x) = 0,$$

along with the boundary conditions

$$\tilde{\alpha}^{nom}(t,0) = q\tilde{\beta}^{nom}(t,0), \quad \tilde{\beta}^{nom}(t,1) = \rho(1-\epsilon)\tilde{\alpha}^{nom}(t,1).$$

Using the method of characteristics, we immediately obtain for $t \geq \tau$

$$\tilde{\beta}^{nom}(t,1) = q\rho(1-\epsilon)\tilde{\beta}^{nom}(t-\tau,1). \tag{8.28}$$

Since $\epsilon<1$ and $|\rho q|<1$, $\tilde{\beta}^{nom}(t,1)$ exponentially converges to zero. Considering the backstepping transformation (2.37)-(2.38), system (8.1)-(8.3) is mapped to

$$\partial_t\alpha^{nom}(t,x) + \lambda\partial_x\alpha^{nom}(t,x) = 0, \quad \partial_t\beta^{nom}(t,x) - \mu\partial_x\beta^{nom}(t,x) = 0, \tag{8.29}$$

with the boundary conditions

$$\alpha^{nom}(t,0) = q\beta^{nom}(t,0),$$

$$\beta^{nom}(t,1) = (\rho-\tilde{\rho})\alpha^{nom}(t,1) - k_I\int_0^1 (l_1(\xi)\alpha^{nom}(t,\xi) + l_2(\xi)\beta^{nom}(t,\xi))d\xi + k_I\eta(t) - \tilde{\rho}(1-\epsilon)\tilde{\alpha}^{nom}(t,1)$$

$$-\rho\int_0^1 (L^{\alpha\alpha}(1,\xi)\tilde{\alpha}^{nom}(t,\xi) + L^{\alpha\beta}(1,\xi)\tilde{\beta}^{nom}(t,\xi))d\xi + \int_0^1 (L^{\beta\alpha}(1,\xi)\tilde{\alpha}^{nom}(t,\xi) + L^{\beta\beta}(1,\xi)\tilde{\beta}^{nom}(t,\xi))d\xi,$$

with

$$\dot{\eta} = \alpha^{nom}(t,1) + \int_0^1 \left(L^{\alpha\alpha}(1,\xi)\alpha^{nom}(t,\xi) + L^{\alpha\beta}(1,\xi)\beta^{nom}(t,\xi) \right) d\xi. \tag{8.30}$$

Let us consider the invertible transformation

$$\gamma(t) = \eta(t) - \int_0^1 (l_1(\xi)\alpha^{nom}(t,\xi) + l_2(\xi)\beta^{nom}(t,\xi))d\xi. \tag{8.31}$$

This yields

$$\dot{\gamma}(t) = (1 + \lambda l_1(1))\alpha^{nom}(t,1). \tag{8.32}$$

Using (8.14) and (8.15), we have

$$\begin{aligned} 1 + l_1(1)\lambda &= 1 + l_2(0)\frac{\mu(0)}{q} + \int_0^1 L^{\alpha\alpha}(1,\xi)d\xi \\ &= 1 + \frac{1}{q}\int_0^1 L^{\alpha\beta}(1,\xi)d\xi + \int_0^1 L^{\alpha\alpha}(1,\xi)d\xi. \end{aligned} \tag{8.33}$$

Thus, due to Assumption 8.1.1, $1 + l_1(1)\lambda \neq 0$. Using the characteristic method, we obtain for $t \geq \tau$

$$\begin{aligned} \beta^{nom}(t,1) =& (\rho - \tilde{\rho})q\beta^{nom}(t-\tau,1) + k_I\gamma(t) - \tilde{\rho}q(1-\epsilon)\tilde{\beta}^{nom}(t-\tau,1) \\ &- \rho\int_0^1 (L^{\alpha\alpha}(1,\xi)q\tilde{\alpha}^{nom}(t - \frac{\xi}{\lambda} - \frac{1}{\mu},\xi) + L^{\alpha\beta}(1,\xi)\tilde{\beta}^{nom}(t - \frac{1-x}{\mu},\xi))d\xi \\ &+ \int_0^1 (L^{\beta\alpha}(1,\xi)q\tilde{\alpha}^{nom}(t - \frac{\xi}{\lambda} - \frac{1}{\mu},\xi) + L^{\beta\beta}(1,\xi)\tilde{\beta}^{nom}(t - \frac{1-x}{\mu},\xi))d\xi. \end{aligned} \tag{8.34}$$

By differentiating (8.34) with respect to time, one has

$$\dot{\beta}^{nom}(t,1) = (\rho - \tilde{\rho})q\dot{\beta}^{nom}(t-\tau,1) + k_I(1 + \lambda l_1(1))q\beta^{nom}(t-\tau,1) + \mathcal{R}(\tilde{\beta}^{nom}(t,1)), \tag{8.35}$$

where $\mathcal{R}$ is a linear operator. Due to Condition 8.1.1, the characteristic equation

$$s - (\rho - \tilde{\rho})s\mathrm{e}^{-\tau s} - k_I(1 + \lambda l_1(1))q\mathrm{e}^{-\tau s}$$

has all its roots in the left half plane [CT15]. Using the fact that $\tilde{\beta}^{nom}(t,1)$ exponentially converges to zero, we have that $\beta(t,1)$ exponentially converges to zero. This concludes the proof. ∎

8.1.2 Uncertain system

We now assume that the various parameters are not perfectly known. Moreover, the actuation and the measurements are both delayed and some disturbances and noise act on the system. More precisely, we assume that the plant dynamics are

$$\partial_t u(t,x) + \bar{\lambda}\partial_x u(t,x) = \bar{\sigma}^{+-}(x)v(t,x) + d_1(t)m_1(x), \tag{8.36}$$
$$\partial_t v(t,x) - \bar{\mu}\partial_x v(t,x) = \bar{\sigma}^{-+}(x)u(t,x) + d_2(t)m_2(x), \tag{8.37}$$

evolving in $\{(t,x)| \quad t > 0,\ x \in [0,1]\}$, with the following linear boundary conditions

$$u(t,0) = \bar{q}v(t,0) + d_3(t), \tag{8.38}$$
$$v(t,1) = \bar{\rho}u(t,1) + (1 + \delta_V)V(t - \delta_0) + d_4(t). \tag{8.39}$$

The system parameters are

$$\bar{\lambda} = \lambda + \delta_\lambda, \quad \bar{\mu} = \mu + \delta_\mu, \quad \bar{q} = q + \delta_q, \quad \bar{\rho} = \rho + \delta_\rho, \tag{8.40}$$
$$\bar{\sigma}^{+-}(x) = \sigma^{+-}(x) + \delta_\sigma^{+-}(x), \quad \bar{\sigma}^{-+}(x) = \sigma^{-+}(x) + \delta_\sigma^{-+}(x). \tag{8.41}$$

The terms δ_λ and δ_μ represent constant uncertainties on the velocities. We assume that $\mu - |\delta_\mu| \leq \mu + |\delta_\mu| < \lambda - |\delta_\lambda| \leq \lambda + |\delta_\lambda|$. The terms δ_q and δ_ρ represent constant uncertainties on the distal and proximal reflections. We assume that $\bar{q} \neq 0$ and $|\bar{q}\bar{\rho}| < 1$. The term $\delta_V \neq -1$ is an constant uncertainty on the actuation. The uncertainty functions $\delta_\sigma^{+-}(x)$ and $\delta_\sigma^{-+}(x)$ belong to $\mathcal{C}^0([0,1],\mathbb{R})^2$ and correspond to uncertainties on the coupling terms. We denote

$$\bar{\tau} = \frac{1}{\bar{\lambda}} + \frac{1}{\bar{\mu}}. \tag{8.42}$$

We assume that there is a delay, denoted δ_0, acting on the actuation. The functions d_1 and d_2 correspond to disturbances acting on the right-hand side of (8.36) and (8.37). The locations of these distributed disturbances are given by the unknown functions m_1 and m_2. The functions d_3 and d_4 correspond to disturbances acting on the right-hand side of (8.38) and (8.39), respectively. Moreover, we assume that the delayed measured output is also subject to an unknown noise $n(t)$

$$y_m(t) = u(t-\delta_1, 1) + n(t)\,, \tag{8.43}$$

where we have denoted δ_1 the delay acting on the measurements We denote κ the maximal bound for the uncertainties:

$$\kappa = \max\{\max_{x\in[0,1]}(\delta_\sigma^{+-}(x)), \max_{x\in[0,1]}(\delta_\sigma^{-+}(x)), \delta_\lambda, \delta_\mu, \delta_\rho, \delta_q, \delta_U\}. \tag{8.44}$$

We make the following asusmption on the disturbances and on the noise.

Assumption 8.1.2 *The disturbances d_i, $i = 1,\dots,4$, are in $W^{2,\infty}((0,\infty);\mathbb{R})$, the noise n is assumed to be in $L^\infty((0,\infty);\mathbb{R})$, and the disturbance input locations m_1 and m_2 are in $\mathcal{C}([0,1];\mathbb{R}^+)$.*

With this assumption, using the characteristics method and classical fixed point arguments we have the following result (see e.g. [Bre00]).

Theorem 8.1.2.

The closed-loop system (8.36)-(8.39) along with the control law (8.10) and the observer (8.4)-(8.7) with bounded initial condition $(u_0, v_0, \hat{u}_0, \hat{v}_0)^\top$ admits a unique solution in $\mathcal{C}([0,\infty); L^\infty((0,1);\mathbb{R}^4) \cap L^1((0,1);\mathbb{R}^4))$.

The objective of this chapter is to prove that, provided that the delays and the uncertainties are small, the output of the closed-loop system (8.36)-(8.39) along with the observer (8.4)-(8.7) and the control law (8.10) remains bounded and converges to zero for constant disturbances and in the absence of noise. We give the following definition.

Definition 8.1.1. Input-to-State Stability (ISS) for PDEs

The output of the closed-loop system (8.36)-(8.39) along with the control law (8.10) and the observer (8.4)-(8.7) is ISS with respect to n and d_i, $i = 1,\dots,4$ if there exist a $\mathcal{KL}$ function h_1 and a $\mathcal{K}$ function h_2 such that for any bounded initial condition $(u_0, v_0, \hat{u}_0, \hat{v}_0)^\top$ and any measurable locally essentially bounded input $K(t)$ (that depends on $n(t)$ and $d_i(t)$), the following holds

$$|u(t,1)| \leq h_1\left((u_0, v_0, \hat{u}_0, \hat{v}_0)^\top, t\right) + h_2\left(\|K(t)\|_{L^\infty((0,t);\mathbb{R})}\right). \tag{8.45}$$

We now state the main result of this chapter which is proved in the next sections.

Theorem 8.1.3.

Suppose that Assumption (8.1.1), and Conditions (8.1.1) and (8.1.2) are satisfied. There exist $\delta_{\text{marg}} > 0$ and $\kappa_0 > 0$ such that if $\delta_0 < \delta_{\text{marg}}$, $\delta_1 < \delta_{\text{marg}}$ and $\kappa < \kappa_0$ then, the output of the closed-loop system (8.36)-(8.39) along with the control law (8.10) and the observer (8.4)-(8.7) is ISS. Moreover, for any bounded initial conditions $(u_0, v_0, \hat{u}_0, \hat{v}_0, \eta_0) \in (L^2([0,1]))^4 \times \mathbb{R}$, there exists a positive constant M such that the controlled output $y(t)$ satisfies

$$|y(t)| \leq M\,. \tag{8.46}$$

Furthermore, if $\dot{d}_1(t) = \dot{d}_2(t) = \dot{d}_3(t) = \dot{d}_4(t) = n(t) = 0$, then the controlled output satisfies

$$\lim_{t\to\infty} |y(t)| = 0\,. \tag{8.47}$$

We have the following corollary

Theorem 8.1.4.

Assume that

$$\sup_{\theta_k \in [0,2\pi]^4} \rho(\sum_{k=1}^{4} A_k \exp(i\theta_k)) > 1, \tag{8.48}$$

For any $\delta_0 > 0$, $\delta_1 > 0$ and $\kappa > 0$ the output of the closed-loop system (8.36)-(8.39) along with the control law (8.10) and the observer (8.4)-(8.7) diverges.

8.2 Operator framework and preliminary results

In this section, we introduce some important preliminary results that make the proof of Theorem 8.1.3 simpler.Consider a strictly positive integer p, two collections of strictly positive constants $\mathfrak{R} : (\tau_1, \cdots, \tau_p)$ and $\mathfrak{E} : (\epsilon_1, \cdots, \epsilon_p)$, and a collection of non-negative constants $\mathfrak{U} : (\mathfrak{u}_1, \cdots, \mathfrak{u}_p)$. We define $\tau_{\max}$ as

$$\tau_{\max} = \max_{1\leq i\leq p} (\tau_i). \tag{8.49}$$

The sequence $\mathfrak{E}$ represents a sequence of delays (namely δ_0 and δ_1). The sequence $\mathfrak{R}$ represents a sequence of transport times that appear in the robustness analysis. They are linear combinations of the characteristic transport times of the system (8.36)-(8.39) and (8.4)-(8.7) and of the delays δ_0 and δ_1. Finally, the sequence $\mathfrak{U}$ represents a sequence of small uncertainties. In order to have the same number of elements in every collection $\mathfrak{R}$, $\mathfrak{E}$ and $\mathfrak{U}$, some elements can be repeated. We assume that all the elements of $\mathfrak{E}$ and $\mathfrak{U}$ can be considered as small as wanted if κ (defined by (8.44)) and $\max(\delta_0, \delta_1)$ tend to zero. In the presence of delays and uncertainties, the closed-loop system features operators with specific properties. We classify these in the following three categories.

Definition 8.2.1.

An operator $\mathcal{I}$ belongs to $\mathfrak{I}$ if there exists a real $\tau_{\mathcal{I}} \in \mathfrak{R}$, a compact support function $f_{\mathcal{I}} \in L^1([0, \tau_{\max}])$ whose support is $[0, \tau_{\mathcal{I}}]$ such that

$$\begin{aligned} \mathcal{I} : C([-\tau_{\max}, 0], \mathbb{R}) &\to \mathbb{R} \\ \phi_t &\mapsto \int_{-\tau_{\max}}^{0} f_{\mathcal{I}}(-\nu)\phi_t(\nu)d\nu. \end{aligned} \tag{8.50}$$

For all $n \in \mathbb{N}^*$, we denote $\mathfrak{M}_n(\mathfrak{I})$ the set of square matrix operators such that for any $\mathcal{M} \in \mathfrak{M}_n(\mathfrak{I})$, for all $(i,j) \in [1,n]^2$, $\mathcal{M}_{i,j} \in \mathfrak{I}$.

The class $\mathfrak{I}$ corresponds to integral terms appearing through backstepping transformations and posing no threat to delay-robustness.

Definition 8.2.2.

An operator $\mathcal{D}$ belongs to $\mathfrak{D}$ if there exists a real $\epsilon_{\mathcal{D}} \in \mathfrak{E}$ such that

$$\begin{aligned} \mathcal{D} : \mathcal{C}([-\tau_{\max}, 0], \mathbb{R}) &\to \mathbb{R} \\ \phi_t &\mapsto \phi_t(-\epsilon_{\mathcal{D}}). \end{aligned} \tag{8.51}$$

The class $\mathfrak{D}$ corresponds to delay operators appearing due to the delays in the measurements and in the actuation.

Definition 8.2.3.

An operator $\mathcal{W}$ belongs to $\mathfrak{W}$ if one of the two following conditions is satisfied

1. there exist $\mathcal{I} \in \mathfrak{I}$ and $\mathcal{D} \in \mathfrak{D}$ such that $\mathcal{I} \circ \mathcal{D} \in \mathcal{I}$ and

$$\begin{aligned} \mathcal{W} : \mathcal{C}([-\tau_{\max}, 0], \mathbb{R}) &\to \mathbb{R} \\ \phi_t &\mapsto \mathcal{I}(\phi_t - \mathcal{D}(\phi_t)), \end{aligned} \tag{8.52}$$

2. there exist $\mathcal{D} \in \mathfrak{D}$ and $\mathfrak{u} \in \mathfrak{U}$ such that

$$\begin{aligned} \mathcal{W} : \mathcal{C}([-\tau_{\max}, 0], \mathbb{R}) &\to \mathbb{R} \\ \phi_t &\mapsto \mathfrak{u}\mathcal{D}(\phi_t), \end{aligned} \tag{8.53}$$

3. there exist $\mathcal{I} \in \mathfrak{I}$,and $\mathfrak{u} \in \mathfrak{U}$ such that

$$\begin{aligned} \mathcal{W} : \mathcal{C}([-\tau_{\max}, 0], \mathbb{R}) &\to \mathbb{R} \\ \phi_t &\mapsto \mathfrak{u}\mathcal{I}(\phi_t). \end{aligned} \tag{8.54}$$

For all $n \in \mathbb{N}^*$, we denote $\mathfrak{M}_n(\mathfrak{W})$ the set of square matrix operators such that for any $\mathcal{M} \in \mathfrak{M}_n(\mathfrak{W})$, for all $(i,j) \in [1,n]^2$, $\mathcal{M}_{i,j} \in \mathfrak{W}$.

The class $\mathfrak{W}$ corresponds in the first case to the difference between integral terms and the same delayed terms. In the second case, it corresponds to delayed terms multiplied by arbitrarily small terms; while in the third case are considered integral delayed terms multiplied by an arbitrarily small term. These terms naturally appear in the computations while using the method of characteristics.These terms do not pose any threat for delay-robustness provided the delays and uncertainties are small enough. In what follows we denote $\hat{\mathcal{I}}$ the Laplace transform of the operator $\mathcal{I}$ (provided it is well-defined). We have the following lemma whose proof is straightforward.

Lemma 8.2.1.

Consider a positive constant $\eta > 0$. There exist $\epsilon_0 > 0$ and $\kappa_0 > 0$ such that if for all $i \in [1,p], \epsilon_i < \epsilon_0$ and $\mathfrak{u}_i < \kappa_0$, then for any $\mathcal{W} \in \mathfrak{W}$, its Laplace transform $\hat{\mathcal{W}}$ satisfies $|\hat{\mathcal{W}}(s)| < \eta$ for all $s \in \mathbb{C}^+$.

Proof : If the operator $\mathcal{W}$ satisfies (8.52), the proof is a consequence of Riemann-Lebesgues lemma. Otherwise, it is a consequence of the boundedness of the considered operators. ∎

The two following theorems prove that the operators that belong to $\mathfrak{M}_n(\mathfrak{W})$ do not have any major impact on the stability properties, assuming that the ϵ_i and $\mathfrak{u}_i$ can be chosen as small as we want. The following theorem states that operators in the class $\mathfrak{W}$ cannot destabilize the plant for small uncertainties and delays.

Theorem 8.2.1.

Consider $n \in \mathbb{N}^*$ and an operator $\mathcal{F} : \mathcal{C}([-\tau_{\max}, 0], \mathbb{R}^n) \to \mathfrak{M}_{n,n}(\mathbb{R})$ such that there exists p matrices $A_i \in (\mathfrak{M}_{n,n}(\mathbb{R}))^p$ and $\mathcal{I} \in \mathfrak{M}_n(\mathfrak{I})$, such that for all $\phi \in \mathcal{C}([-\tau_{\max}, 0], \mathbb{R}^n)$

$$\mathcal{F}(\phi_t) = \phi_t - \sum_{i=1}^{p} A_i \phi_t(-\tau_i) - \mathcal{I}(\phi_t). \tag{8.55}$$

Consider an operator $\mathcal{W} \in \mathfrak{M}_n(\mathfrak{W})$. If the semigroup associated to the operator $\mathcal{F}$ is exponentially stable, then there exists $\epsilon_0 > 0$ and $\kappa_0 > 0$ such that the semigroup associated to the operator $\mathcal{F} + \mathcal{W}$ is exponentially stable for all $i \in [1, p], \epsilon_i < \epsilon_0$ and $\mathfrak{u}_i < \kappa_0$.

Proof : Let us denote $\hat{\mathcal{F}}(s)$ (resp. $\hat{\mathcal{W}}(s)$) the Laplace transform of the operator $\mathcal{F}$ (resp $\mathcal{W}$) (see [HVL93]). Adjusting the proof of Theorem 7.2.1, let us assume by contradiction that there exists $s \in \mathbb{C}, s \neq 0$ and $\Re(s) \geq 0$ such that $\det(\hat{\mathcal{F}}(s) + \hat{\mathcal{W}}(s)) = 0$. There exists $\eta \neq 0$ such that

$$\eta^* \hat{\mathcal{F}}^*(s) \hat{\mathcal{F}}(s) \eta = \eta^* \hat{\mathcal{W}}^*(s) \hat{\mathcal{W}}(s) \eta, \tag{8.56}$$

where * denotes the conjugate transpose. Since $\hat{\mathcal{F}}(s)$ is non singular in $\mathbb{C}^+$, there exists $\kappa_1 > 0$ such that $\kappa_1 < \eta^* \hat{\mathcal{F}}^*(s) \hat{\mathcal{F}}(s) \eta$. It is straightforward to prove (using Lemma 8.2.1) that there exists $\epsilon_m(z) > 0$ and $\kappa_m(z) > 0$ such that if for all $i \in [1, p], \epsilon_i < \epsilon_m(z)$ and $\mathfrak{u}_i < \kappa_m(z)$ we obtain a contradiction with (8.56). Since $\det(\hat{\mathcal{F}}(s) + \hat{\mathcal{W}}(s))$ has only a finite number of zeros in the right-half plane, where the zeros have finite module [HL02], the quantities $\epsilon_0 = \min_z \epsilon_m(z)$ and $\kappa_0 = \min_z \kappa_m(z)$ are strictly positive. This leads to a contradiction with (8.56). Thus, the holomorphic function $\hat{\mathcal{G}}(s) = \det(\hat{\mathcal{F}}(s) + \hat{\mathcal{W}}(s))$ does not have any roots on the right-half plane. Furthermore, since the principal term of $\hat{\mathcal{G}}(s)$ is assumed to be stable, the asymptotic vertical chain of zeros of $\hat{\mathcal{G}}(s)$ can not be the imaginary axis. Thus, all the zeros of $\hat{\mathcal{G}}(s)$ are in the open left-half complex plane. This concludes the proof. ■

The following theorem guarantees that systems with a strongly unstable principal part are necessarily unstable.

Theorem 8.2.2.

Consider $n \in \mathbb{N}^*$ and a differential delay matrix operator $\mathcal{F} : \mathcal{C}([-\tau_{\max}, 0], \mathbb{R}^n) \to \mathfrak{M}_{n,n}(\mathbb{R})$ such that there exists p matrices $A_i \in (\mathfrak{M}_{n,n}(\mathbb{R}))^p$ such that for all $\phi \in \mathcal{C}([-\tau_{\max}, 0], \mathbb{R}^n)$

$$\mathcal{F}(\phi_t) = \phi_t - \sum_{i=1}^{p} A_i \phi_t(-\tau_i). \tag{8.57}$$

If the characteristic equation associated to the operator $\mathcal{F}$ has a non finite number of zeros in the open-right half plane, then, for any set of $\epsilon_i > 0$ and $\mathfrak{u}_i > 0$, for any $\mathcal{I} \in \mathfrak{M}_n(\mathfrak{I})$ and $\mathcal{W} \in \mathfrak{M}_n(\mathfrak{W})$, the operator $\mathcal{F} + \mathcal{I} + \mathcal{W}$ generates an unstable semigroup.

Proof : Let us consider an arbitrary set of strictly positive coefficients $\mathfrak{E}$ and positive coefficients $\mathfrak{U}$. Consider $\mathcal{I} \in \mathfrak{M}_q(\mathfrak{I})$ and $\mathcal{W} \in \mathfrak{M}_q(\mathfrak{W})$. The operator $\mathcal{W}$ can be rewritten as $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$ where the components of $\mathcal{W}_1$ are defined either by (8.52) or (8.54) and the components of $\mathcal{W}_2$ are defined by (8.53). Due to 2.2.1, the operator $\mathcal{F} + \mathcal{W}_2$ has an infinite number of zeros in the right half-plane. Using Riemann-Lebesgues' lemma we have that the holomorphic function $|\hat{\mathcal{I}}(s) + \hat{\mathcal{W}}_1(s)|$ converges to 0 for $|s|$ large enough (with $\Re(s) \geq 0$). It implies (see Lemma 6.1.4 for details) that the characteristic equation associated to the operator $\mathcal{F} + \mathcal{I} + \mathcal{W}$ has an infinite number of zeros in the right half-plane and consequently generates an unstable semigroup. ■

Note that due to the vector space structure of $\mathfrak{M}_n(\mathfrak{W})$, similar results hold if the operator $\mathcal{W}$ is replaced by a linear combination of operators that belong to $\mathfrak{M}_n(\mathfrak{W})$. In the next section, to prove Theorem 8.1.3, we express the observer-controller equations as delay-differential equations (with potential integral terms). During the derivations, multiple terms that belong to $\mathfrak{M}_n(\mathfrak{W})$ appear. Since they can be neglected for the stability analysis, for the sake of simplicity and brevity, every time one of this term appears we write it as $\mathcal{O}(X_t)$ (where X_t is related to the state of the system). In other words, all the terms included in $\mathcal{O}(X_t)$ are terms that do not have any influence on the stability if the delays and uncertainties are small enough. This approach is consistent with the one proposed in [CZ12, Chapter 9]. For convenience, the Laplace transform of such terms is denoted $\hat{\mathcal{O}}(X_t)(s)$.

8.3 Output regulation: robustness aspects

In this section we prove a weak version of Theorem 8.1.3 that is the convergence of the output to its equilibrium in presence of delays in the measurement and the actuation and in presence of parameters uncertainties but in the absence of noise and disturbances. In what follows we consequently assume that $d_i \equiv 0$ and $n(t) \equiv 0$. Using the backstepping method, we first rewrite the closed-loop (8.36)-(8.39) along with the observer (8.4)-(8.7) and the control law (8.10) in a simpler set of coordinates in which the in-domain coupling terms have been removed. In this new set of coordinates, it becomes possible (using the method of characteristics) to rewrite the corresponding equations as delay differential equations. We finally analyze the stability properties of this resulting neutral system. In all this section we consider that Assumption (8.1.1), and Conditions (8.1.1) and (8.1.2) are satisfied.

8.3.1 Backstepping transformation of the original system (8.36)-(8.39)

Consider system (8.36)-(8.39) and the invertible Volterra change of coordinates

$$u(t,x) = \alpha(t,x) + \int_0^x (\bar{L}^{\alpha\alpha}(x,\xi)\alpha(t,\xi) + \bar{L}^{\alpha\beta}(x,\xi)\beta(t,\xi))d\xi, \tag{8.58}$$

$$v(t,x) = \beta(t,x) + \int_0^x (\bar{L}^{\beta\alpha}(x,\xi)\alpha(t,\xi) + \bar{L}^{\beta\beta}(x,\xi)\beta(t,\xi))d\xi, \tag{8.59}$$

where the kernels $\bar{L}^{\alpha\alpha}$, $\bar{L}^{\alpha\beta}$, $\bar{L}^{\beta\alpha}$ and $\bar{L}^{\beta\beta}$ belong to $\mathcal{C}(\mathcal{T})$ and are defined by the set of PDEs

$$\begin{aligned} &\bar{\lambda}\partial_x \bar{L}^{\alpha\alpha} + \bar{\lambda}\partial_\xi \bar{L}^{\alpha\alpha} = \bar{\sigma}^{+-}(\xi)\bar{L}^{\beta\alpha},\ \bar{\lambda}\partial_x \bar{L}^{\alpha\beta} - \bar{\mu}\partial_\xi \bar{L}^{\alpha\beta} = \bar{\sigma}^{+-}(\xi)\bar{L}^{\beta\beta}, \\ &\bar{\mu}\partial_x \bar{L}^{\beta\alpha} - \bar{\lambda}\partial_\xi \bar{L}^{\beta\alpha} = -\bar{\sigma}^{-+}(\xi)\bar{L}^{\alpha\alpha},\ \bar{\mu}\partial_x \bar{L}^{\beta\beta} + \bar{\mu}\partial_\xi \bar{L}^{\beta\beta} = -\bar{\sigma}^{-+}(\xi)\bar{L}^{\alpha\beta}, \end{aligned}$$

with the boundary conditions

$$\begin{aligned} \bar{L}^{\alpha\alpha}(x,0) &= \frac{\bar{\lambda}}{\bar{q}\bar{\mu}}\bar{L}^{\alpha\beta}(x,0), \quad \bar{L}^{\beta\beta}(x,0) = \frac{\bar{q}\bar{\lambda}}{\bar{\mu}}\bar{L}^{\beta\alpha}(x,0), \\ \bar{L}^{\alpha\beta}(x,x) &= \frac{\bar{\sigma}^{+-}(x)}{\bar{\lambda}+\bar{\mu}}, \quad \bar{L}^{\beta\alpha}(x,x) = -\frac{\bar{\sigma}^{-+}(x)}{\bar{\lambda}+\bar{\mu}}. \end{aligned}$$

The transformation (2.49)-(2.50) is invertible and the inverse transformation can be expressed as

$$\alpha(t,x) = u(t,x) - \int_0^x (\bar{K}^{uu}(x,\xi)u(\xi) + \bar{K}^{uv}(x,\xi)v(\xi))d\xi, \tag{8.60}$$

$$\beta(t,x) = v(t,x) - \int_0^x (\bar{K}^{vu}(x,\xi)u(\xi) + \bar{K}^{vv}(x,\xi)v(\xi))d\xi, \tag{8.61}$$

where the kernels $\bar{K}^{uu}, \bar{K}^{vu}, \bar{K}^{uv}, \bar{K}^{vv}$ belongs to $\mathcal{C}(\mathcal{T})$ and satisfy similar equations as the ones given in (2.43)-(2.48). We have the following equalities between the two sets of kernels

$$\int_0^x (\bar{K}^{uu}(x,\xi)u(t,\xi) + \bar{K}^{uv}v(t,\xi))d\xi = \int_0^x (\bar{L}^{\alpha\alpha}(x,\xi)\alpha(t,\xi) + \bar{L}^{\alpha\beta}(x,\xi)\beta(t,\xi))d\xi, \quad (8.62)$$
$$\int_0^x (\bar{K}^{vu}(x,\xi)u(t,\xi) + \bar{K}^{vv}v(t,\xi))d\xi = \int_0^x (\bar{L}^{\beta\alpha}(x,\xi)\alpha(t,\xi) + \bar{L}^{\beta\beta}(x,\xi)\beta(t,\xi))d\xi. \quad (8.63)$$

Remark 8.3.1 *Due to the continuity of the kernels with respect to the kernel PDEs parameters, it is straightforward to prove that the kernels $\bar{K}^{uu}, \bar{K}^{vu}, \bar{K}^{uv}$, and $\bar{K}^{vv}$ respectively converge to K^{uu}, K^{vu}, K^{uv}, and K^{vv} and that the kernels $\bar{L}^{\alpha\alpha}, \bar{L}^{\beta\alpha}, \bar{L}^{\alpha\beta}$, and $\bar{L}^{\beta\beta}$ respectively converge to $L^{\alpha\alpha}, L^{\beta\alpha}, L^{\alpha\beta}$, and $L^{\beta\beta}$ when κ goes to zero.*

The dynamics of the system (8.36)-(8.39) in the new coordinates are given by

$$\partial_t\alpha(t,x) + \bar{\lambda}\partial_x\alpha(t,x) = 0, \quad (8.64)$$
$$\partial_t\beta(t,x) - \bar{\mu}\partial_x\beta(t,x) = 0, \quad (8.65)$$

with the following linear boundary conditions

$$\alpha(t,0) = \bar{q}\beta(t,0), \quad (8.66)$$
$$\beta(t,1) = \bar{\rho}\alpha(t,1) + (1+\delta_V)V(t-\delta_0) - \int_0^1 \bar{N}^\alpha(\xi)\alpha(t,\xi) + \bar{N}^\beta(\xi)\beta(t,\xi)d\xi, \quad (8.67)$$

with

$$\bar{N}^\alpha(\xi) = \bar{L}^{\beta\alpha}(1,\xi) - \bar{\rho}\bar{L}^{\alpha\alpha}(1,\xi), \quad \bar{N}^\beta(\xi) = \bar{L}^{\beta\beta}(1,\xi) - \bar{\rho}\bar{L}^{\alpha\beta}(1,\xi). \quad (8.68)$$

Moreover, we have

$$\dot{\eta} = \alpha(t-\delta_1,1) + \int_0^1 \left(\bar{L}^{\alpha\alpha}(1,\xi)\alpha(t-\delta_1,\xi) + \bar{L}^{\alpha\beta}(1,\xi)\beta(t-\delta_1,\xi)\right)d\xi. \quad (8.69)$$

Remark 8.3.2 *Due to the cascade structure of system* (8.64)-(8.67) *and due to the invertibility of the Volterra transformation* (8.58)-(8.59)*, if $\beta(t,1)$ exponentially converges to zero then it implies that the system* (8.36)-(8.39) *is exponentially stable.*

In what follows, we consider the natural extension of Assumption (8.1.1) in presence of uncertainties.

Assumption 8.3.1

$$1 + \int_0^1 \bar{L}^{\alpha\alpha}(1,\xi)d\xi + \frac{1}{\bar{q}}\int_0^1 \bar{L}^{\alpha\beta}(1,\xi)d\xi \neq 0\,. \quad (8.70)$$

Note that due to Remark 8.3.1, provided that Assumption 8.1.1 holds and that the uncertainties are small enough, then Assumption 8.3.1 is guaranteed. As it will appear, Assumption (8.3.1) is required to extend the proof of (8.1.1) in presence of uncertainties and delays. To adjust the proof of Theorem 8.1.1, we define the function $\bar{l}_1$ and $\bar{l}_2$ on the interval $[0,1]$ as the solution of the system

$$\bar{\lambda}\bar{l}_1'(x) = \bar{L}^{\alpha\alpha}(1,x), \quad \bar{\mu}\bar{l}_2'(x) = -\bar{L}^{\alpha\beta}(1,x)\,, \quad (8.71)$$

with the boundary conditions

$$\bar{l}_2(1) = 0\,, \quad \bar{l}_1(0) = \frac{\bar{\mu}}{\bar{q}\bar{\lambda}}\bar{l}_2(0). \quad (8.72)$$

8.3.2 Backstepping transformation of the error system

Combining the observer (8.4)-(8.7) and the real system (8.36)-(8.39) yields the following error system (denoting $\tilde{u}(t,x) = u(t,x) - \hat{u}(t,x)$ and $\tilde{v}(t,x) = v(t,x) - \hat{v}(t,x)$):

$$\begin{aligned}\partial_t \tilde{u}(t,x) + \lambda \partial_x \tilde{u}(t,x) &= \sigma^{+-}(x)\tilde{v}(t,x) - P^+(x)\tilde{u}(t,1) - \delta_\lambda \partial_x u(t,x) \\ &\quad + \delta_{\sigma^{+-}}(x)v(t,x) + P^+(x)(u(t,1) - u(t-\delta_1,1)), \qquad (8.73)\\ \partial_t \tilde{v}(t,x) - \mu \partial_x \tilde{v}(t,x) &= \sigma^{-+}(x)\tilde{u}(t,x) - P^-(x)\tilde{u}(t,1) + \delta_\mu \partial_x v(t,x) \\ &\quad + \delta_{\sigma^{-+}}(x)u(t,x) + P^-(x)(u(t,1) - u(t-\delta_1,1)), \qquad (8.74)\end{aligned}$$

with the boundary conditions

$$\begin{aligned}\tilde{u}(t,0) &= q\tilde{v}(t,0) + \delta_q v(t,0), \qquad (8.75)\\ \tilde{v}(t,1) &= \rho(1-\epsilon)\tilde{u}(t,1) + \rho\epsilon(u(t,1) - u(t-\delta_1,1)) + \delta_\rho u(t,1) \\ &\quad + (1+\delta_V)V(t-\delta_0) - V(t). \qquad (8.76)\end{aligned}$$

Once again, the objective is to find a suitable transformation to remove the in-domain couplings $\sigma^{+-}(x)\tilde{v}(t,x)$ and $\sigma^{-+}(x)\tilde{u}(t,x)$. Let us consider the inverse transformation of (2.69)-(2.70) defined in [VCKB11] by

$$\tilde{\alpha}(t,x) = \tilde{u}(t,x) + \int_x^1 \Big(R^{\alpha\alpha}\tilde{u}(t,\xi) + R^{\beta\alpha}\tilde{v}(t,\xi)\Big)d\xi, \qquad (8.77)$$

$$\tilde{\beta}(t,x) = \tilde{v}(t,x) + \int_x^1 \Big(R^{\beta\alpha}\tilde{u}(t,\xi) + R^{\beta\beta}\tilde{v}(t,\xi)\Big)d\xi, \qquad (8.78)$$

where the kernels $R^{\alpha\alpha}, R^{\alpha\beta}, R^{\beta\alpha}$ and $R^{\beta\beta}$ are continuous functions defined on $\mathcal{T}_1 = \{(x,\xi) \in [0,1]^2,\ \xi \geq x\}$ by the following set of PDEs

$$\begin{aligned}\lambda\partial_x R^{\alpha\alpha}(x,\xi) + \lambda\partial_\xi R^{\alpha\alpha}(x,\xi) &= -\sigma^{-+}(x)R^{\alpha\beta}(x,\xi), \qquad (8.79)\\ \lambda\partial_x R^{\alpha\beta}(x,\xi) - \mu\partial_\xi R^{\alpha\beta}(x,\xi) &= -\sigma^{+-}(x)R^{\alpha\alpha}(x,\xi), \qquad (8.80)\\ \mu\partial_x R^{\beta\alpha}(x,\xi) - \lambda\partial_\xi R^{\beta\alpha}(x,\xi) &= +\sigma^{-+}(x)R^{\beta\beta}(x,\xi), \qquad (8.81)\\ \mu\partial_x R^{\beta\beta}(x,\xi) + \mu\partial_\xi R^{\beta\beta}(x,\xi) &= +\sigma^{+-}(x)R^{\beta\alpha}(x,\xi), \qquad (8.82)\end{aligned}$$

along with the boundary conditions

$$R^{\alpha\alpha}(0,\xi) = qR^{\beta\alpha}(0,\xi),\ R^{\alpha\beta}(x,x) = \frac{\sigma^{+-}(x)}{\lambda+\mu}, \qquad (8.83)$$

$$R^{\beta\beta}(0,\xi) = \frac{1}{q}R^{\alpha\beta}(0,\xi),\ R^{\beta\alpha}(x,x) = -\frac{\sigma^{-+}(x)}{\lambda+\mu}. \qquad (8.84)$$

We have the following lemma

Lemma 8.3.1.

There exist a continuous function f and a continuous function g such that the states α and β defined by (8.77)-(8.78) satisfy the following set of PDEs

$$\begin{aligned}&\partial_t\tilde{\alpha}(t,x) + \lambda\partial_x\tilde{\alpha}(t,x) = \delta_{\sigma^{+-}}(x)v(t,x) - f(x)(u(t,1) - u(t-\delta_1,1)) + \mu R^{\alpha\beta}(x,1)\delta_\rho u(t,1) \\ &- \delta_\lambda\partial_x u(t,x) + \int_x^1 R^{\alpha\beta}(x,\xi)(\delta_{\sigma^{-+}}(\xi)u(t,\xi) + \delta_\mu\partial_x v(t,\xi))d\xi + \mu R^{\alpha\beta}(x,1)((1+\delta_V) \\ &V(t-\delta_0) - V(t)) + \int_x^1 R^{\alpha\alpha}(x,\xi)(\delta_{\sigma^{+-}}(\xi)v(t,\xi) - \delta_\lambda\partial_x u(t,\xi))d\xi, \qquad (8.85)\end{aligned}$$

$$\partial_t\tilde{\beta}(t,x) - \mu\partial_x\tilde{\beta}(t,x) = \delta_{\sigma^{-+}}(x)u(t,x) - g(x)(u(t,1) - u(t-\delta_1,1)) + \mu R^{\alpha\beta}(x,1)\delta_\rho u(t,1)$$
$$+ \delta_\mu\partial_x v(t,x) + \int_x^1 R^{\beta\beta}(x,\xi)(\delta_{\sigma^{-+}}(\xi)u(t,\xi) + \delta_\mu\partial_x v(t,\xi))d\xi + \mu R^{\beta\beta}(x,1)((1+\delta_V)$$
$$V(t-\delta_0) - V(t)) + \int_x^1 R^{\beta\alpha}(x,\xi)(\delta_{\sigma^{+-}}(\xi)v(t,\xi) - \delta_\lambda\partial_x u(t,\xi))d\xi, \tag{8.86}$$

along with the boundary conditions

$$\tilde{\alpha}(t,0) = q\tilde{\beta}(t,0) + \delta_q\beta(t,0), \tag{8.87}$$
$$\tilde{\beta}(t,1) = \rho(1-\epsilon)\tilde{\alpha}(t,1) + \rho\epsilon(u(t,1) - u(t-\delta_1,1)) + \delta_\rho u(t,1) + (1+\delta_V)V(t-\delta_0) - V(t)\,. \tag{8.88}$$

Proof : We recall that since the kernels $R^{\cdot\cdot}$ are the inverse kernels of the kernels $P^{\cdot\cdot}$, we have [KS08b]

$$R^{\alpha\alpha}(x,1) = P^{uu}(x,1) + \int_x^1 R^{\alpha\alpha}(x,\xi)P^{uu}(\xi,1)d\xi + \int_x^1 R^{\alpha\beta}(x,\xi)P^{vu}(\xi,1)d\xi, \tag{8.89}$$
$$R^{\alpha\beta}(x,1) = P^{uv}(x,1) + \int_x^1 R^{\alpha\alpha}(x,\xi)P^{uv}(\xi,1)d\xi + \int_x^1 R^{\alpha\beta}(x,\xi)P^{vv}(\xi,1)d\xi. \tag{8.90}$$

Differentiating (8.77) with respect to space and time and integrating by part yields

$$\partial_t\tilde{\alpha}(t,x) + \lambda\partial_x\tilde{\alpha}(t,x) = \underline{\sigma^{+-}(x)\tilde{v}(t,x)} - \mathbf{P^+(x)\tilde{u}(t,1)} + \delta_{\sigma^{+-}}(x)v(t,x) - \delta_\lambda\partial_x u(t,x)$$
$$\underline{-\lambda R^{\alpha\alpha}(x,x)\tilde{u}(t,x) - \lambda R^{\alpha\beta}(x,x)\tilde{v}(t,x)} + P^+(x)(u(t,1) - u(t-\delta_1,1)) + \underline{\int_x^1 \lambda\partial_x R^{\alpha\alpha}(x,\xi)\tilde{u}(t,\xi)}$$
$$\underline{+\partial_x R^{\alpha\beta}(x,\xi)\tilde{v}(t,\xi)d\xi} + \int_x^1 \underline{\lambda\partial_\xi R^{\alpha\alpha}(x,\xi)\tilde{u}(t,\xi) - \mu\partial_\xi R^{\alpha\beta}(x,\xi)\tilde{v}(t,\xi)d\xi} + R^{\alpha\alpha}(x,\xi)(\underline{\sigma^{+-}(\xi)\tilde{v}(t,\xi)}$$
$$- \mathbf{P^+(\xi)\tilde{u}(t,1)} - \delta_\lambda\partial_x u(t,\xi) + \delta_{\sigma^{+-}}(\xi)v(t,\xi) + P^+(\xi)(u(t,1) - u(t-\delta_1,1)))$$
$$+ R^{\alpha\beta}(x,\xi)(\underline{\sigma^{-+}(x)\tilde{u}(t,x)} - \mathbf{P^-(\xi)\tilde{u}(t,1)} + \delta_\mu\partial_x v(t,\xi) + \delta_{\sigma^{-+}}(\xi)u(t,\xi) + P^-(\xi)(u(t,1) - u(t-\delta_1,1)))d\xi$$
$$+ \underline{\lambda R^{\alpha\alpha}(x,x)\tilde{u}(t,x)} - \underline{\mu R^{\alpha\beta}(x,x)\tilde{v}(t,x)} - \mathbf{\lambda R^{\alpha\alpha}(x,1)\tilde{u}(t,1)}$$
$$+ \mu R^{\alpha\beta}(x,1)\left(\rho(\mathbf{1}-\epsilon)\mathbf{\tilde{u}(t,1)} + \rho\epsilon(u(t,1) - u(t-\delta_1,1)) + \delta_\rho u(t,1) + (1+\delta_V)V(t-\delta_0) - V(t)\right),$$

where the functions P^+ and P^- are defined by (8.8)-(8.9). Due to equations (8.79)-(8.84), all the underlined terms cancel. Due to (8.89)-(8.90) all the bold terms cancel. This leads to equation (8.85) where the function f is defined by

$$f(x) = -P^+(x) - \rho\epsilon\mu R^{\alpha\beta}(x,1) - \int_x^1 R^{\alpha\alpha}(x,\xi)P^+(\xi)d\xi - \int_x^1 R^{\alpha\beta}(x,\xi)P^-(\xi)d\xi$$
$$= -R^{\alpha\alpha}(x,1) - \rho\epsilon\mu R^{\alpha\beta}(x,1).$$

A similar proof can be done to derive equation (8.86). ■

8.3.3 Neutral delay-differential system

We are now able to express the two states $\beta(t,1)$ and $\tilde{\beta}(t,1)$ as the solutions of a neutral delay-differential system that is equivalent to (8.64)-(8.69), (8.85)-(8.76) along with the feedback law (8.10). We define the extended state $X(t)$ as

$$X(t) = \begin{pmatrix}\beta(t,1) & \tilde{\beta}(t,1) & V(t)\end{pmatrix}^T. \tag{8.91}$$

We define the collections $\mathfrak{R}$, $\mathfrak{E}$ and $\mathfrak{U}$ as

$$\mathfrak{R} :(\tau,\bar{\tau},\frac{1}{\lambda}+\frac{1}{\bar{\mu}},\tau+\delta_0,\bar{\tau}+\delta_0,\bar{\tau}+\delta_1,\bar{\tau}+\delta_0+\delta_1,\frac{1}{\lambda}+\frac{1}{\bar{\mu}}+\delta_0),$$
$$\mathfrak{E} :(0,\delta_0,\delta_1),$$
$$\mathfrak{U} :(\delta_\lambda,\delta_\mu,\delta_q,\delta_\rho,\delta_V,\max_{x\in[0,1]}(\delta_\sigma^{+-}(x)),\max_{x\in[0,1]}(\delta_\sigma^{-+}(x))).$$

In what follows, we heavily rely on the definitions of Section 8.2 to ease the notations, grouping terms that correspond to operators in $\mathfrak{I}, \mathfrak{D}$ or $\mathfrak{W}$ and explicitly retaining only the terms that are critical for the robustness analysis. Using equations (8.64)-(8.67), we start expressing $\beta(t,1)$ as the solution of a neutral equation whose terms only depend on the state X_t. It then becomes possible to express $u(t,x)$, $v(t,x)$, $\partial_x u(t,x)$ and $\partial_x v(t,x)$ as functions of X_t. Using these expressions and the observer equations (8.85)-(8.76), we express the state $\tilde{\beta}(t,1)$ as the solution of a neutral equation that depends on the state X_t. It then becomes possible to express $\tilde{u}(t,x)$, $\tilde{v}(t,x)$ as functions of X_t. Finally, we can simplify the expression of the control law (8.10) and express it as a function of X_t. It is then straightforward to obtain the neutral system satisfied by X_t.

Expression of the state $\beta(t,1)$

Considering equations (8.64)-(8.67), using the characteristics method, they simply rewrite for any $t \geq \bar{\tau} + \delta_0 + \delta_1$ as

$$\beta(t,1) = \bar{q}\bar{\rho}\beta(t-\bar{\tau},1) - \int_0^{\bar{\tau}} \tilde{N}(\xi)\beta(t-\xi,1)d\xi + (1+\delta_V)V(t-\delta_0), \tag{8.92}$$

where $\tilde{N}$ is defined by

$$\tilde{N}(\xi) = \begin{cases} \bar{\mu}\bar{N}^{\beta}(1-\bar{\mu}\xi) & \text{for} \quad \xi \in [0, \frac{1}{\bar{\mu}}) \\ \bar{\lambda}\bar{q}\bar{N}^{\alpha}(\bar{\lambda}\xi - \frac{\bar{\lambda}}{\bar{\mu}}) & \text{for} \quad \xi \in (\frac{1}{\bar{\mu}}, \bar{\tau}] \end{cases},$$

with

$$\bar{N}^{\alpha}(\xi) = \bar{L}^{\beta\alpha}(1,\xi) - \bar{\rho}\bar{L}^{\alpha\alpha}(1,\xi), \quad \bar{N}^{\beta}(\xi) = \bar{L}^{\beta\beta}(1,\xi) - \bar{\rho}\bar{L}^{\alpha\beta}(1,\xi).$$

We now give the expression of $u(t,x)$, $\partial_x u(t,x)$, $v(t,x)$ and $\partial_x v(t,x)$ in terms of $\beta(t,1)$, as these terms appear in the observer equations (8.85)-(8.76). This is a necessary step to express $\tilde{\beta}(t,1)$ as the solution of a neutral equation. Using equations (8.64)-(8.67) and the Volterra transformation (8.58), we have that for all $t \leq \bar{\tau}$ and all $x \in [0,1]$,

$$\begin{aligned} u(t,x) &= \alpha(t,x) + \int_0^x (\bar{L}^{\alpha\alpha}(x,\xi)\alpha(t,\xi) + \bar{L}^{\alpha\beta}(x,\xi)\beta(t,\xi))d\xi \\ &= \bar{q}\beta(t - \frac{x}{\bar{\lambda}} - \frac{1}{\bar{\mu}}, 1) + \int_0^1 \left(\bar{q}\bar{L}^{\alpha\alpha}(x,\xi)\beta(t - \frac{\xi}{\bar{\lambda}} - \frac{1}{\bar{\mu}}, 1) + \bar{L}^{\alpha\beta}(x,\xi)\beta(t - \frac{1-\xi}{\bar{\mu}}, 1)d\xi\right). \end{aligned}$$

Using the notations of section 8.2, for every $x \in [0,1]$, there exist $\mathcal{I}_u(x) \in \mathfrak{I}$ and $\mathcal{I}_{u_x}(x) \in \mathfrak{I}$ such that

$$u(t,x) = \bar{q}\beta(t - \frac{1}{\bar{\mu}} - \frac{x}{\bar{\lambda}}, 1) + \mathcal{I}_u(x)(\beta(\cdot,1)_t), \tag{8.93}$$

$$\partial_x u(t,x) = -\frac{1}{\bar{\lambda}}\bar{q}\partial_x\beta(t - \frac{1}{\bar{\mu}} - \frac{x}{\bar{\lambda}}, 1) + \mathcal{I}_{u_x}(x)(\beta(\cdot,1)_t). \tag{8.94}$$

Similarly we obtain the existence of $\mathcal{I}_v(x) \in \mathfrak{I}$ and $\mathcal{I}_{v_x}(x) \in \mathfrak{I}$ such that

$$v(t,x) = \beta(t - \frac{1-x}{\bar{\mu}}, 1) + \mathcal{I}_v(x)(\beta(\cdot,1)_t), \tag{8.95}$$

$$\partial_x v(t,x) = \frac{1}{\bar{\mu}}\partial_x\beta(t - \frac{1-x}{\bar{\mu}}, 1) + \mathcal{I}_{v_x}(x)(\beta(\cdot,1)_t). \tag{8.96}$$

Expression of the observer state $\tilde{\beta}(t,1)$

Let us consider equations (8.85)-(8.88). The objective is to use the method of characteristics and the formalism introduced in Section 8.2 to express $\tilde{\beta}(t,1)$ as the solution of a neutral equation. Regarding the terms in (8.85)-(8.88) that are functions of $u(t,x)$ or $v(t,x)$, one can use equations (8.93)-(8.95) to express them as functions of X_t. The following lemma simplify two other terms that depend on $\partial_x u(\cdot,\cdot)$ and on $\partial_x v(\cdot,\cdot)$ that naturally appear using the method of characteristics.

Lemma 8.3.2.

We have the following relations

$$\begin{aligned}\int_0^{\frac{1}{\bar\lambda}} (-\delta_\lambda \partial_x u(t-s, x-\bar\lambda s) + \int_{x-\bar\lambda s}^1 (R^{\alpha\beta}(x-\bar\lambda s,\xi)\delta_\mu \partial_x v(t-s,\xi) \\ - R^{\alpha\alpha}(x-\bar\lambda s,\xi)\delta_\lambda \partial_x u(t-s,\xi))d\xi)ds = \mathcal{O}(X_t)\end{aligned} \tag{8.97}$$

$$\begin{aligned}\int_0^{\frac{1}{\bar\mu}} (\delta_\lambda \partial_x v(t-s, x+\bar\mu s) + \int_{x+\bar\mu s}^1 (R^{\beta\beta}(x+\bar\mu s,\xi)\delta_\mu \partial_x v(t-s,\xi) \\ - R^{\beta\alpha}(x+\bar\mu s,\xi)\delta_\lambda \partial_x u(t-s,\xi))d\xi)ds = \mathcal{O}(X_t).\end{aligned} \tag{8.98}$$

Proof : We only prove (8.97) as the proof of (8.98) is similar. We have for all $t > \bar\tau$ and all $x \in [0,1]$

$$\beta(t,x) = \beta(t - \frac{1-x}{\bar\mu}, 1) \Rightarrow \partial_x \beta(t,x) = \frac{1}{\bar\mu}\partial_t \beta(t - \frac{1-x}{\bar\mu}, 1). \tag{8.99}$$

Combining (8.99) with (8.94), we obtain

$$\begin{aligned}-\delta_\lambda \partial_x u(t,x) &= -\delta_\lambda \left(-\frac{1}{\bar\lambda}\bar q \partial_x \beta(t - \frac{1}{\bar\mu} - \frac{x}{\bar\lambda}, 1) + \mathcal{I}_{u_x}(x)(\beta(\cdot,1)_t) \right) \\ &= \delta_\lambda \frac{1}{\bar\lambda}\bar q \frac{1}{\bar\mu}\partial_t \beta(t - \frac{1}{\bar\mu} - \frac{x}{\bar\lambda}, 1) + \mathcal{O}(X_t).\end{aligned}$$

Thus,

$$-\delta_\lambda \int_0^{\frac{1}{\bar\lambda}} \partial_x u(t-s, x-\bar\lambda s)ds = \int_0^{\frac{1}{\bar\lambda}} \delta_\lambda \frac{\bar q}{\bar\lambda\bar\mu}\partial_t \beta(t - \frac{1}{\bar\mu} - \frac{x}{\bar\lambda}, 1)ds + \mathcal{O}(X_t) = \mathcal{O}(X_t).$$

Let us now consider the double integral. Using (8.96) and (8.99), we obtain

$$\int_0^{\frac{1}{\bar\lambda}}\int_{x-\bar\lambda s}^1 R^{\alpha\beta}(x-\bar\lambda s,\xi)\delta_\mu \partial_x v(t-s,\xi)d\xi ds = \int_0^{\frac{1}{\bar\lambda}}\int_{x-\bar\lambda s}^1 R^{\alpha\beta}(x-\bar\lambda s,\xi)\delta_\mu \frac{1}{\bar\mu^2}\partial_t\beta(t-s-\frac{1-\xi}{\bar\mu},1)d\xi ds + \mathcal{O}(X_t).$$

Integrating by part, we get

$$\begin{aligned}\int_0^{\frac{1}{\bar\lambda}}\int_{x-\bar\lambda s}^1 R^{\alpha\beta}(x-\bar\lambda s,\xi)\delta_\mu \partial_x v(t-s,\xi)d\xi ds = \int_0^{\frac{1}{\bar\lambda}}\int_{x-\bar\lambda s}^1 \partial_\xi R^{\alpha\beta}(x-\bar\lambda s,\xi)\delta_\mu \frac{1}{\bar\mu}\beta(t-s-\frac{1-\xi}{\bar\mu},1)d\xi ds + \mathcal{O}(X_t) \\ + \int_0^{\frac{1}{\bar\lambda}} \frac{\delta_\mu}{\bar\mu}\left(R^{\alpha\beta}(x-\bar\lambda s,1)\beta(t-s,1) - R^{\alpha\beta}(x-\bar\lambda s, x-\bar\lambda s)\beta(t-s-\frac{1-x+\bar\lambda s}{\bar\mu})\right) ds + \mathcal{O}(X_t) = \mathcal{O}(X_t).\end{aligned}$$

Similar computations can be done to obtain $\int_0^{\frac{1}{\bar\lambda}}\int_{x-\bar\lambda s}^1 R^{\alpha\alpha}(x-\bar\lambda s,\xi)\delta_\lambda \partial_x u(t-s,\xi)d\xi ds = \mathcal{O}(X_t)$. ■

Combining the method of characteristics on equations (8.93)-(8.96) and Lemma 8.3.2, we obtain for any $t \geq \tau + \delta_0 + \delta_1$

$$\begin{aligned}\tilde\beta(t,1) = \rho q(1-\epsilon)\tilde\beta(t-\tau,1) + \rho(1-\epsilon)\delta_q \beta(t - \frac{1}{\lambda}, 0) + \rho\epsilon(u(t,1) - u(t-\delta_1,1)) + \delta_\rho u(t,1) \\ + (1+\delta_V)V(t-\delta_0) - V(t) + \mathcal{O}(X_t).\end{aligned} \tag{8.100}$$

This yields

$$\tilde{\beta}(t,1) = \rho q(1-\epsilon)\tilde{\beta}(t-\tau,1) + \rho q\epsilon(\beta(t-\bar{\tau},1) - \beta(t-\bar{\tau}-\delta_1,1)) + V(t-\delta_0) - V(t) \\ + \mathcal{O}(X_t). \quad (8.101)$$

We can now express the terms $\tilde{u}(t,x)$ and $\tilde{v}(t,x)$ as functions of the state X_t. This is a necessary step to simplify the control law $V(t)$ in the next section. Using the notations of Section 8.2, the relations (8.93)-(8.96), for every $x \in [0,1]$, there exist $\mathcal{I}_{\tilde{u}}(x) \in \mathfrak{I}$ and $\mathcal{I}_{\tilde{v}}(x) \in \mathfrak{I}$ such that

$$\tilde{u}(t,x) = q\tilde{\beta}(t - \frac{1}{\mu} - \frac{x}{\lambda}, 1) + \mathcal{I}_{\tilde{u}}(x)(X_t) + \mathcal{O}(X_t), \quad (8.102)$$

$$\tilde{v}(t,x) = \tilde{\beta}(t - \frac{1-x}{\mu}, 1) + \mathcal{I}_{\tilde{v}}(x)(X_t) + \mathcal{O}(X_t). \quad (8.103)$$

Expression of the control law $V(t)$

We now express the control law $V(t) = V_{BS}(t) + k_I V_I(t) + k_I\eta(t)$ defined in (8.10) as a function of X_t. The part $V_0(t) = V_{BS}(t) + k_I V_I(t)$ given by (8.12) and (8.13) rewrites

$$V_0(t) = -\tilde{\rho}(1-\epsilon)(u(t,1) - \tilde{u}(t,1)) - \tilde{\rho}\epsilon u(t-\delta_1,1) - (\rho-\tilde{\rho})\int_0^1 K^{uu}(1,\xi)(u(t,\xi) - \tilde{u}(t,\xi))d\xi \\ - (\rho-\tilde{\rho})\int_0^1 K^{uv}(1,\xi)(v(t,\xi) - \tilde{v}(t,\xi))d\xi + \int_0^1 (K^{vu}(1,\xi)(u(t,\xi) - \tilde{u}(t,\xi)) \\ + K^{vv}(1,\xi)(v(t,\xi) - \tilde{v}(t,\xi)))d\xi - k_I\int_0^1 l_1(\xi)((u(t,\xi) - \tilde{u}(t,\xi)) \\ - \int_0^\xi K^{uu}(\xi,\nu)(u(t,\nu) - \tilde{u}(t,\nu)) + K^{vu}(\xi,\nu)(v(t,\nu) - \tilde{v}(t,\nu))d\nu))d\xi - k_I\int_0^1 l_2(\xi) \\ \left((v(t,\xi) - \tilde{v}(t,\xi)) - \int_0^\xi K^{vu}(\xi,\nu)(u(t,\nu) - \tilde{u}(t,\nu)) + K^{vv}(\xi,\nu)(v(t,\nu) - \tilde{v}(t,\nu))d\nu\right)d\xi.$$

We simplify this expression using the expression of $u, v, \tilde{u}$ and $\tilde{v}$ as functions of X_t given by relations (8.93)-(8.95) and (8.102)-(8.103). More precisely, we have the following lemma.

Lemma 8.3.3.

There exists $\tilde{F}$ a Lipschitz function such that the control law $V_0(t)$ rewrites

$$V_0(t) = \tilde{\rho}\epsilon q(\beta(t-\bar{\tau},1) - \beta(t-\bar{\tau}-\delta_1,1)) + \tilde{\rho}(1-\epsilon)q\tilde{\beta}(t-\tau,1) \\ - \tilde{\rho}q\beta(t-\bar{\tau},1) - k_I\int_0^1 \left(\bar{l}_1(\xi)\alpha(t,\xi) + \bar{l}_2(\xi)\beta(t,\xi)\right)d\xi + \int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu,1)d\nu \\ + \int_0^{\bar{\tau}} \tilde{N}(\xi)\beta(t-\xi,1)d\xi + \mathcal{O}(X_t). \quad (8.104)$$

Proof : Let us denote

$$F_1(\xi) = (\rho-\tilde{\rho})K^{uu}(1,\xi) - K^{vu}(1,\xi), \quad F_2(\xi) = (\rho-\tilde{\rho})K^{uv}(1,\xi) - K^{vv}(1,\xi),$$

and similarly

$$\bar{F}_1(\xi) = (\bar{\rho}-\tilde{\rho})\bar{K}^{uu}(1,\xi) - \bar{K}^{vu}(1,\xi), \quad \bar{F}_2(\xi) = (\bar{\rho}-\tilde{\rho})\bar{K}^{uv}(1,\xi) - \bar{K}^{vv}(1,\xi).$$

Using Remark 8.3.1, we have that $\bar{F}_1$ and $\bar{F}_2$ uniformly converge to (resp.) F_1 and F_2 if the uncertainties go to zero. It is then straightforward to show (using (8.93)-(8.96)) that

$$\int_0^1 F_1(\xi)u(t,\xi) + F_2(\xi)v(t,\xi)d\xi - \int_0^1 \bar{F}_1(\xi)u(t,\xi) + \bar{F}_2(\xi)v(t,\xi)d\xi = \mathcal{O}(X_t).$$

Using a similar argument and equations (8.58)-(8.59) we obtain:

$$
\begin{aligned}
&- k_I \int_0^1 l_1(\xi)(u(t,\xi) - \int_0^\xi K^{uu}(\xi,\nu)u(t,\nu)K^{vu}(\xi,\nu)v(t,\nu)d\nu)d\xi - k_I \int_0^1 l_2(\xi)(u(t,\xi) - \int_0^\xi K^{vu}(\xi,\nu) \\
&\qquad u(t,\nu)K^{vv}(\xi,\nu)v(t,\nu)d\nu)d\xi = -k_I \int_0^1 \left(\bar{l}_1(\xi)\alpha(t,\xi) + \bar{l}_2(\xi)\beta(t,\xi)\right) d\xi + \mathcal{O}(X_t),
\end{aligned}
$$

Using equations (8.102)-(8.103), there exists a function $\tilde{F}$ such that

$$
\begin{aligned}
&\int_0^1 F_1(\xi)\tilde{u}(t,\xi) + F_2(\xi)\tilde{v}(t,\xi)d\xi + k_I \int_0^1 l_1(\xi)(\tilde{u}(t,\xi) - \int_0^\xi K^{uu}(\xi,\nu)\tilde{u}(t,\nu)K^{vu}(\xi,\nu)\tilde{v}(t,\nu)d\nu)d\xi \\
&+ k_I \int_0^1 l_2(\xi)(\tilde{u}(t,\xi) - \int_0^\xi K^{vu}(\xi,\nu)\tilde{u}(t,\nu)K^{vv}(\xi,\nu)\tilde{v}(t,\nu)d\nu)d\xi = \int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu,1) + \mathcal{O}(X_t).
\end{aligned}
$$

On the other hand, we have

$$
-\tilde{\rho}u(t,1) - \int_0^1 \bar{F}_1(\xi)u(t,\xi) + \bar{F}_2(\xi)v(t,\xi)d\xi = -\tilde{\rho}\bar{q}\beta(t-\bar{\tau},1) + \int_0^1 \bar{N}^\alpha(\xi)\alpha(t,\xi) + \bar{N}^\beta(\xi)\beta(t,\xi)d\xi,
$$

where we have used (8.62)-(8.63). Let us now express the term $u(t,1) - u(t-\delta_1,1)$ as a function of X_t. Using (8.93) we get

$$
u(t,1) - u(t-\delta_1,1) = q(\beta(t-\bar{\tau},1) - \beta(t-\bar{\tau}-\delta_1,1)) + \mathcal{O}(X_t).
$$

We finally obtain

$$
\begin{aligned}
V_0(t) &= \tilde{\rho}(1-\epsilon)\tilde{u}(t,1) + \tilde{\rho}\epsilon(u(t,1) - u(t-\delta_1,1)) - \tilde{\rho}u(t,1) - \int_0^1 \bar{F}_1 u(t,\xi) + \bar{F}_2 v(t,\xi)d\xi \\
&\quad + \int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu,1)d\nu - k_I \int_0^1 \left(l_1(\xi)\alpha(t,\xi) + l_2(\xi)\beta(t,\xi)\right) d\xi + \mathcal{O}(X_t) \\
&= \tilde{\rho}q(1-\epsilon)\tilde{\beta}(t-\tau,1) + \tilde{\rho}q\epsilon(\beta(t-\bar{\tau},1) - \beta(t-\bar{\tau}-\delta_1,1)) \\
&\quad - \tilde{\rho}q\beta(t-\bar{\tau},1) + \int_0^1 \bar{N}^\alpha(\xi)\alpha(t,\xi) + \bar{N}^\beta(\xi)\beta(t,\xi)d\xi + \int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu,1)d\nu \\
&\quad - k_I \int_0^1 \bar{l}_1(\xi)\alpha(t,\xi) + \bar{l}_2(\xi)\beta(t,\xi)d\xi + \mathcal{O}(X_t).
\end{aligned}
$$

This concludes the proof. ■

Similarly to what has been done in the proof of Theorem 8.1.1, we make a change of coordinates to incorporate the term $-k_I \int_0^1 \bar{l}_1(\xi)\alpha(t,\xi) + \bar{l}_2(\xi)\beta(t,\xi)d\xi$ into the integral action. Let us consider the invertible transformation

$$
\gamma(t) = \bar{\eta}(t) - \int_0^1 (\bar{l}_1(\xi)\alpha(t,\xi) + \bar{l}_2(\xi)\beta(t,\xi))d\xi. \tag{8.105}
$$

With this transformation, the control law $V(t)$ rewrites

$$
\begin{aligned}
V(t) &= \tilde{\rho}\epsilon q(\beta(t-\bar{\tau},1) - \beta(t-\bar{\tau}-\delta_1,1)) + \tilde{\rho}(1-\epsilon)q\tilde{\beta}(t-\tau,1) - \tilde{\rho}q\beta(t-\bar{\tau},1) \\
&\quad + \int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu,1)d\nu + \int_0^{\bar{\tau}} \tilde{N}(\xi)\beta(t-\xi,1) + k_I\gamma(t) + \mathcal{O}(X_t).
\end{aligned} \tag{8.106}
$$

Besides, the following lemma gives the ODE that is satisfied by the new variable γ.

Lemma 8.3.4.

The function γ satisfies the following ODE

$$
\dot{\gamma}(t) = q(\beta(t-\bar{\tau}-\delta_1,1) + \lambda l_1(1)\beta(t-\bar{\tau},1)) + \mathcal{O}(\beta(\cdot,1)_t). \tag{8.107}
$$

Proof : Differentiating (8.105) and integrating by part yields

$$\begin{aligned}\dot{\gamma}(t) &= \dot{\eta} - \int_0^1 \bar{l}_1(t,\xi)\partial_t\alpha(t,\xi) + l_2(t,\xi)\partial_t\beta(t,\xi)d\xi \\ &= \alpha(t-\delta_1,1) + \int_0^1 \bar{L}^{\alpha\alpha}(x,\xi)\alpha(t-\delta_1,\xi) + \bar{L}^{\alpha\beta}(x,\xi)\alpha(t-\delta_1,\xi)d\xi - \int_0^1 \bar{\lambda}\bar{l}_1'(\xi)\alpha(t,\xi) - \bar{\mu}\bar{l}_2'(\xi)\beta(t,\xi)d\xi \\ &+ \bar{\lambda}\bar{l}_1(1)\alpha(t,1) - \bar{\lambda}\bar{l}_1(0)\alpha(t,0) - \bar{\mu}\bar{l}_2(1)\beta(t,1) + \bar{\mu}\bar{l}_2(0)\beta(t,0).\end{aligned}$$

Using the definition of $\bar{l}_1$ and $\bar{l}_2$ given in equations (8.71)-(8.72) and the boundary conditions (8.66)-(8.67), we obtain

$$\dot{\gamma}(t) = \alpha(t-\delta_1,1) + \bar{\lambda}\bar{l}_1(1)\alpha(t,1) + \mathcal{O}(\beta(\cdot,1)_t).$$

Finally,

$$\alpha(t-\delta_1,1) + \bar{\lambda}\bar{l}_1(1)\alpha(t,1) = \bar{q}(\beta(t-\bar{\tau}-\delta_1,1) + \bar{\lambda}\bar{l}_1(1))\alpha(t-\bar{\tau},1).$$

Using the continuity of the function $\bar{l}_1$ when the uncertainties go to zero, we have

$$\dot{\gamma}(t) = q(\beta(t-\bar{\tau}-\delta_1,1) + \lambda l_1(1)\beta(t-\bar{\tau},1)) + \mathcal{O}((\beta(\cdot,1)_t)),$$

which is the expected result. ■

Neutral system

Using the previous computations and simplifications we are now able to give the neutral system satisfied by X_t. Injecting (8.104) inside (8.92), (8.101), we obtain the following system

$$\begin{aligned}\beta(t,1) &= \rho q\beta(t-\bar{\tau},1) - \tilde{\rho}q(1-\epsilon)\beta(t-\bar{\tau}-\delta_0,1) - \tilde{\rho}q\epsilon\beta(t-\bar{\tau}-\delta_0-\delta_1,1) + k_I\gamma(t-\delta_0) \\ &\quad + \tilde{\rho}q(1-\epsilon)\tilde{\beta}(t-\tau-\delta_0,1) + \int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu-\delta_0,1)d\nu + \mathcal{O}(X_t), \qquad (8.108)\\ \tilde{\beta}(t,1) &= (\rho-\tilde{\rho})q(1-\epsilon)\tilde{\beta}(t-\tau,1) + \tilde{\rho}q(1-\epsilon)\tilde{\beta}(t-\tau-\delta_0,1) + q((\rho-\tilde{\rho})\epsilon \\ &\quad + \tilde{\rho})\beta(t-\bar{\tau},1) + (\tilde{\rho}-\rho)q\epsilon\beta(t-\bar{\tau}-\delta_1,1) - (1-\epsilon)\tilde{\rho}q\beta(t-\bar{\tau}-\delta_0,1) \\ &\quad - \tilde{\rho}\epsilon q\beta(t-\bar{\tau}-\delta_0-\delta_1,1) + k_I\gamma(t-\delta_0) - k_I\gamma(t) + \mathcal{O}(X_t), \qquad (8.109)\end{aligned}$$

where $\dot{\gamma}$ is given by (8.107):

$$\dot{\gamma}(t) = q(\beta(t-\bar{\tau}-\delta_1,1) + \lambda l_1(1)\beta(t-\bar{\tau},1)) + \mathcal{O}(\beta(\cdot,1)_t). \qquad (8.110)$$

The equation satisfied by $V(t)$ is given in (8.104) and not rewritten here. As equations (8.108)-(8.109) require the expression of $\gamma(t)$ and since only its derivative is available, we choose to differentiate (8.108)-(8.109) with respect to time. This yields

$$\begin{aligned}\dot{\beta}(t,1) &= \rho q\dot{\beta}(t-\bar{\tau},1) - \tilde{\rho}q(1-\epsilon)\dot{\beta}(t-\bar{\tau}-\delta_0,1) - \tilde{\rho}q\epsilon\dot{\beta}(t-\bar{\tau}-\delta_0-\delta_1,1) + k_Iq\beta(t-\bar{\tau}-\delta_1-\delta_0,1) \\ &\quad + \tilde{\rho}q(1-\epsilon)\dot{\tilde{\beta}}(t-\tau-\delta_0) + k_Iq\lambda l_1(1)\beta(t-\bar{\tau}-\delta_1) + \int_0^\tau \tilde{F}(\nu)\dot{\tilde{\beta}}(t-\nu-\delta_0,1)d\nu \\ &\quad + \mathcal{O}(\dot{X}_t) + \mathcal{O}(X_t), \qquad (8.111)\\ \dot{\tilde{\beta}}(t,1) &= (\rho-\tilde{\rho})q(1-\epsilon)\dot{\tilde{\beta}}(t-\tau,1) + \tilde{\rho}q(1-\epsilon)\dot{\tilde{\beta}}(t-\tau-\delta_0,1) + q((\rho-\tilde{\rho})\epsilon \\ &\quad + \tilde{\rho})\dot{\beta}(t-\bar{\tau},1) + (\tilde{\rho}-\rho)q\epsilon\dot{\beta}(t-\bar{\tau}-\delta_1,1) - (1-\epsilon)\tilde{\rho}q\dot{\beta}(t-\bar{\tau}-\delta_0-\delta_1,1) \\ &\quad - \tilde{\rho}\epsilon q\dot{\beta}(t-\bar{\tau}-\delta_0-\delta_1,1) + k_Iq\lambda l_1(1)(\beta(t-\bar{\tau}-\delta_0,1) - \beta(t-\bar{\tau},1)) \\ &\quad + k_Iq(\beta(t-\bar{\tau}-\delta_0-\delta_1,1) - \beta(t-\bar{\tau}-\delta_1)) + \mathcal{O}(\dot{X}_t) + \mathcal{O}(\beta(\cdot,1)_t). \qquad (8.112)\end{aligned}$$

Finally, using equation (8.106) we have

$$\begin{aligned}\dot{V}(t) &= \tilde{\rho}\epsilon q(\dot{\beta}(t-\bar{\tau},1) - \dot{\beta}(t-\bar{\tau}-\delta_1,1)) + \tilde{\rho}(1-\epsilon)q\dot{\tilde{\beta}}(t-\tau,1) - \tilde{\rho}q\dot{\beta}(t-\bar{\tau},1) \\ &\quad + \int_0^\tau \tilde{F}(\nu)\dot{\tilde{\beta}}(t-\nu,1)d\nu + \int_0^{\bar{\tau}} \tilde{N}(\xi)d\xi\dot{\beta}(t-\xi,1) + k_Iq(\beta(t-\bar{\tau}-\delta_1,1) \\ &\quad + \lambda l_1(1)\beta(t-\bar{\tau},1)) + \mathcal{O}(\beta(\cdot,1)_t) + \mathcal{O}(\dot{X}_t). \qquad (8.113)\end{aligned}$$

Consider system (8.111)-(8.113), the objective is now to prove that the first component of the solution X_t, i.e. $\beta(t,1)$, exponentially converges to zero. Note that only the convergence of $\beta(t,1)$ to zero is required and that $\tilde{\beta}(t,1)$ does not necessarily converge to zero (due to the presence of the integral term). To take this into account, and as equations (8.112)-(8.113) only involve the time derivative of the functions $\tilde{\beta}(t,1)$ and $V(t)$, we choose to consider the new state $Y(t)$ defined by

$$Y(t) = \begin{pmatrix}\beta(t,1) & \dot{\beta}(t,1) & \dot{\tilde{\beta}}(t,1), \dot{V}(t)\end{pmatrix}. \tag{8.114}$$

The stability proof is achieved considering the characteristic function associated to the system (8.111)-(8.113).

8.3.4 Complex stability analysis

We start writing the Laplace transform of the equations satisfied by the state Y. We can then easily obtain the characteristic equation associated to this system. The stability is granted provided this characteristic equation does not have any roots in the open complex right plane.

Laplace transform and characteristic equation

The first objective is to express in a simple way the Laplace transform of the system (8.111)-(8.113). Let us introduce the following holomorphic functions

$$\begin{aligned}
F_{11}(s) &= \rho q \mathrm{e}^{-\bar{\tau}s} - \tilde{\rho}q(1-\epsilon)\mathrm{e}^{-(\bar{\tau}+\delta_0)s} - \tilde{\rho}q\epsilon\mathrm{e}^{-(\bar{\tau}+\delta_0+\delta_1)s},\\
F_{12}(s) &= \tilde{\rho}q(1-\epsilon)\mathrm{e}^{-(\tau+\delta_0)s},\\
F_{21}(s) &= q((\rho-\tilde{\rho})\epsilon+\tilde{\rho})\mathrm{e}^{-\bar{\tau}s} + (\tilde{\rho}-\rho)q\epsilon\mathrm{e}^{-(\bar{\tau}+\delta_1)s} - (1-\epsilon)\tilde{\rho}q\mathrm{e}^{-(\bar{\tau}+\delta_0)s} - \tilde{\rho}q\epsilon\mathrm{e}^{-(\bar{\tau}+\delta_0+\delta_1)s},\\
F_{22}(s) &= (\rho-\tilde{\rho})q(1-\epsilon)\mathrm{e}^{-\tau s} + \tilde{\rho}q(1-\epsilon)\mathrm{e}^{-(\tau+\delta_0)s},\\
F_{31}(s) &= \tilde{\rho}\epsilon q\mathrm{e}^{-\bar{\tau}s} - \tilde{\rho}\epsilon q\mathrm{e}^{-(\bar{\tau}+\delta_1)s} - \tilde{\rho}q\mathrm{e}^{-\bar{\tau}s},\\
F_{32}(s) &= \tilde{\rho}(1-\epsilon)q\mathrm{e}^{-\tau s},\\
C_1(s) &= k_I q\mathrm{e}^{-(\bar{\tau}+\delta_1+\delta_0)s} + k_I q\lambda l_1(1)\mathrm{e}^{-(\bar{\tau}+\delta_1)s},\\
C_2(s) &= k_I q\lambda l_1(1)\mathrm{e}^{-\bar{\tau}s}(\mathrm{e}^{-\delta_0 s}-1) + k_I q\mathrm{e}^{-(\bar{\tau}+\delta_1)s}(\mathrm{e}^{-\delta_0 s}-1),\\
C_3(s) &= k_I q\mathrm{e}^{-\bar{\tau}s}(1+\lambda l_1(1)\mathrm{e}^{-\bar{\tau}s}),
\end{aligned}$$

and the matrices

$$F_0(s) = \begin{pmatrix}F_{11}(s) & F_{12}(s)\\ F_{21}(s) & F_{22}(s)\end{pmatrix},\ C_0(s) = \begin{pmatrix}0 & 0 & 0 & 0\\ C_1(s) & 0 & 0 & 0\\ C_2(s) & 0 & 0 & 0\\ C_3(s) & 0 & 0 & 0\end{pmatrix}, G_0(s) = \begin{pmatrix}F_{31}(s) & F_{32}(s)\end{pmatrix},$$

$$F_1(s) = \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & F_0(s) & & 0\\ 0 & G_0(s) & & 0\end{pmatrix}\ E_0(s) = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & \int_0^{\tau}\tilde{F}(\nu)\mathrm{e}^{-(\nu+\delta_0)s}d\nu & 0\\ 0 & 0 & 0 & 0\\ 0 & \int_0^{\bar{\tau}} s\tilde{N}(\nu)\tilde{F}(\nu)\mathrm{e}^{-\nu s}d\nu & \int_0^{\tau}\tilde{F}(\nu)\mathrm{e}^{-\nu s}d\nu & 0\end{pmatrix}.$$

In what follows we denote $I_0(s)$ as the matrix

$$I_0(s) = \begin{pmatrix}s & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

With these notations, the Laplace transform of (8.111)-(8.113) for the state Y defined by (8.114) is given by

$$I_0\hat{Y}(s) = (F_1(s) + C_0(s) + E_0(s))\hat{Y}(s) + \hat{\mathcal{O}}((\hat{Y})_t)(s), \tag{8.115}$$

The characteristic equation associated to (8.111)-(8.113) can be expressed as

$$\det\left(I_0(s) - F_1(s) - C_0(s) - E_0(s) - \hat{\mathcal{O}}(s)\right) = 0, \tag{8.116}$$

where we have (abusively) denoted $\hat{\mathcal{O}}(s)$ the transfer function associated to the operator $\hat{\mathcal{O}}(\hat{Y}_t)(s)$. In what follows, we denote

$$P(s) = |\det\left(I_0(s) - F_1(s) - C_0(s) - E_0(s) - \hat{\mathcal{O}}(s)\right)|. \tag{8.117}$$

The objective is to prove that all the solutions of the characteristic equation (8.116) are located in the complex left-half plane if the delays and uncertainties are small enough. The analysis is different depending on the magnitude of $|s|$. For large values of $|s|$, the strategy is the following. As the principal term $F_1(s)$ may be the main limitation for stability (see Theorem (8.2.2) for details), we start by proving that the function $|\det(I_0(s) - sF_1(s))|$ is positively bounded. Then, we focus on the influence of the integral components of the matrix E_0, which do not belong to $\mathfrak{W}$. We prove that they do not have any incidence in terms of stability if the uncertainties and delays are small enough. Finally, we consider the influence of the term C_0, that corresponds to the integral action. We show that the influence of this term is negligible. For small values of $|s|$, we prove that due to the choice of k_I in Condition 8.1.1, if the delays and uncertainties are small enough then the characteristic function $P(s)$ cannot vanish.

Analysis of the function $\det(I_0(s) - F_1(s))$

We first consider the influence of the principal term F_1 on stability. More precisely we consider the subsystem

$$\begin{pmatrix} sy_1(s) \\ y_2(s) \\ y_3(s) \\ y_4(s) \end{pmatrix} = F_1(s) \begin{pmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \\ y_4(s) \end{pmatrix}. \tag{8.118}$$

Due to the structure of the matrix F_1 (the last column is equal to zero), there is a cascade from the first three lines to the last one. Consequently the last line does not play any role in terms of stability. The characteristic equation associated to (8.118) is given by

$$\det\left(\begin{pmatrix} s & 1 & 0 & 0 \\ 0 & 1 - F_{11}(s) & -F_{12}(s) & 0 \\ 0 & -F_{21}(s) & 1 - F_{22}(s) & 0 \\ 0 & -F_{31}(s) & -F_{32}(s) & 1 \end{pmatrix}\right) = 0.$$

This yields

$$s[(1 - F_{11}(s))(1 - F_{22}(s)) - F_{12}(s)F_{21}(s)] = 0.$$

Consequently to analyze the root location of the characteristic equation associated (8.118), we can consider the simplified system

$$\begin{pmatrix} z_1(s) \\ z_2(s) \end{pmatrix} = \begin{pmatrix} F_{11}(s) & F_{12}(s) \\ F_{21}(s) & F_{22}(s) \end{pmatrix} \begin{pmatrix} z_1(s) \\ z_2(s) \end{pmatrix}, \tag{8.119}$$

which is a neutral system that has the same characteristic equation. Since the different delays involved in the definition of the matrix F_0 are not rationally independent, we cannot directly use Theorem 2.2.1 to conclude to the exponential stability of the temporal system associated to (8.119). Let us define

$$\hat{Z}_p(t) = \begin{pmatrix} z_1(t,1) & z_1(t-\delta_0,1) & z_2(t,1) & z_2(t-\delta_0,1) \end{pmatrix}^T.$$

With these notations, equation (8.119) rewrites

$$\hat{Z}_p(s) = (A_1 \mathrm{e}^{-\bar{\tau}s} + A_2\mathrm{e}^{-\tau s} + A_3 \mathrm{e}^{-\delta_0 s} + A_4 \mathrm{e}^{-(\bar{\tau}+\delta_1)s})\hat{Z}_p(s), \tag{8.120}$$

where the A_i are defined by (8.20)-(8.23). Since the delays are now rationally independent, combining Condition 8.1.2 and Theorem 2.2.1, we can conclude that there exist $\gamma_0 > 0$ such that the function $|(1-F_{11}(s))(1-F_{22}(s)) - F_{12}(s)F_{21}(s)|$ has all its roots on the open complex half plane $\{s \in \mathbb{C} \mid \Re(s) < -\gamma_0\}$. Thus, the function $\det(I_0 - F_1(s))$ has all its roots in the open complex left-half plane except one in $s = 0$. Moreover, adjusting the proof of [HL02, Lemma 2.1], there exists $M_0 > 0$ such that the function $|\det(I_0 - F_1(s))|$ is lower-bounded by a constant $\omega_0 > 0$ on the complex set $\Omega_0 = \{s \in \mathbb{C} \mid \Re(s) \geq 0 \text{ and } |s| > M_0\}$.

Stability analysis of the equation: $I_0(s)\hat{Y}(s) = (F_1(s) + E_0(s))\hat{Y}(s) + \hat{\mathcal{O}}(\hat{Y}_t)(s)$

We now prove that the integral terms of the matrix E_0 and the operator $\hat{\mathcal{O}}((\hat{Y})_t)(s)$ do not affect the stability properties of the previous system if the delays and uncertainties are chosen small enough. As the three first lines of E_0 do not have any component on the last column, there is still a cascade from the three first lines to the last one. Consequently, we only need to focus on the term $\int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu-\delta_0,1)d\nu$. Using successive iterations on this term we prove that, provided the delays and uncertainties are small, its norm is small enough. More precisely, using equation (8.109), direct computations yield

$$\int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu-\delta_0,1)d\nu = \rho q(1-\epsilon)\int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu-\tau-\delta_0,1)d\nu + \mathcal{O}(\hat{Y}_t).$$

Let us consider an integer N_0 that still has to be defined and consider a time $t > (N_0+2)\tau$. Iterating N_0 times the previous computations we obtain

$$\int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu-\delta_0,1)d\nu = \rho q(1-\epsilon)^{N_0}\int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu-N_0\tau-\delta_0,1)d\nu + \mathcal{O}(\hat{X}_t).$$

Consequently, the system $I_0(s)\hat{Y}(s) = (F_1(s)+E_0(s))\hat{Y}(s) + \hat{\mathcal{O}}((\hat{Y})_t)(s)$ can be rewritten

$$I_0(s)\hat{Y}(s) = (F_1(s)+E_1(s))\hat{Y}(s) + \hat{\mathcal{O}}_2((\hat{Y})_t)(s), \tag{8.121}$$

where

$$E_1(s) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (\rho q(1-\epsilon))^{N_0}\int_0^\tau \tilde{F}(\nu)\mathrm{e}^{-(\nu+\tau+\delta_0)s}d\nu & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \int_0^\tau s\tilde{N}(\nu)\tilde{F}(\nu)\mathrm{e}^{-\nu s}d\nu & \int_0^\tau \tilde{F}(\nu)\mathrm{e}^{-\nu s}d\nu & 0 \end{pmatrix},$$

and where we have used the notation $\hat{\mathcal{O}}_2$ to highlight the fact that this $\mathfrak{W}$-term is not the same as before. Since $|(\rho q(1-\epsilon))| < 1$, for N_0 klarge enough, the term The characteristic equation associated to the system (8.121) is $(\rho q(1-\epsilon))_0^N$ can be chosen as small as desired. Adjusting the proof of Theorem 7.2.1, one can easily prove that for N_0 large enough, this function does not have any root on Ω_0 and is positively bounded on Ω_0. Consequently, adjusting the proof Theorem 8.2.1, there exist $\delta_{m_1} > 0$ and $\kappa_{m_1} > 0$ such that if $\delta_0 < \delta_{m_1}$, $\delta_1 < \delta_{m_1}$ and $\kappa < \kappa_{m_1}$ then the function $\det(I_0 - F_1(s) - E_0(s) - \hat{\mathcal{O}}(s))$ is positively bounded by a constant $\omega_1 > 0$ on Ω_0.

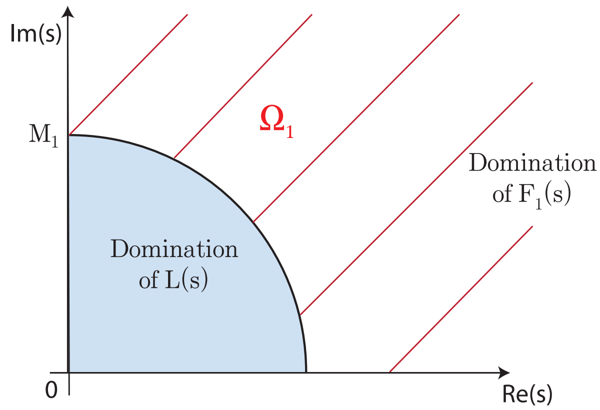

Figure 8.1: Representation of the domain Ω_1.

Stability analysis of equation (8.115)

We can now state the stability properties of equation (8.115). We first prove that for large values of $|s|$, the function $P(s)$ cannot cancel. If $s \in \Omega_0$, the function $|\det(s(I_0 - F_1(s) - E_0(s) - \mathcal{O}(s)))|$ is lower-bounded by $M_0\omega_1$. In the same time, the function $\frac{1}{s}C_0(s)$ converges to zero for $|s|$ large enough. Thus, using the continuity of the determinant, there exists $M_1 > 0$, such that $\forall s \in \Omega_1 = \{s \in \mathbb{C} \mid \Re(s) \geq 0, \text{ and } |s| \geq M_1\}$,

$$|\det\Big((I_0 - F_1(s) - E_0(s) - \hat{\mathcal{O}}(s) - C_0(s)\Big)| > 0.$$

This domain Ω_1 is pictured in Figure 8.1. We now have to prove that $P(s)$ does not cancel on $\mathbb{C}\backslash\Omega_1 \cup \mathbb{C}^+$.

Let us now consider $s \in \mathbb{C}^+$ such that $|s| \leq M_1$. Let us rewrite the function $P(s)$ in a simpler way that highlights the importance of Condition 8.1.1. Defining $L(s)$ and $H(s)$ as

$$\begin{aligned}
L(s) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ k_I q(1+\lambda l_1(1))\mathrm{e}^{-\tau s} & (\rho - \tilde{\rho})q\mathrm{e}^{-\tau s} & \tilde{\rho}q(1-\epsilon)\mathrm{e}^{-\tau s} & 0 \\ 0 & 0 & \rho q(1-\epsilon)\mathrm{e}^{-\tau s} & 0 \\ C_3(s) & F_{31}(s) & F_{32}(s) & 0 \end{pmatrix} \\
H(s) &= F_1(s) - L_1(s) + E_0(s) + C_0(s) + \hat{\mathcal{O}}(\cdot)(s),
\end{aligned}$$

the function $P(s)$ can be expressed as $P(s) = |\det(I_0(s) - L(s) - H(s))|$. As we have seen above, the function $F_1(s)$, as the principal part of the system, imposes the root location for large values of $|s|$, i.e. when $s \in \Omega_1$. However, if s does not belong to Ω_1, then the function $L_1(s)$ is predominant for the root location. Considering the function $I_0(s) - L(s)$, the associated characteristic equation is given by

$$\begin{aligned}
& \det(I_0(s) - L(s)) = 0 \\
\Rightarrow\quad & (1 - \rho q(1-\epsilon))\left(s - s(\rho - \tilde{\rho})\mathrm{e}^{-\tau s} - k_I q(1+\lambda l_1(1)))\mathrm{e}^{-\tau s}\right) = 0.
\end{aligned}$$

As $(1 - \rho q(1-\epsilon)) \neq 0$, this corresponds to characteristic function associated to the system

$$\dot{z}(t) = (\rho - \tilde{\rho})q\dot{z}(t-\tau) + k_I q(1+\lambda l_1(1))z(t-\tau). \tag{8.122}$$

Thus, using Condition 8.1.1, the function $\det(I_0(s) - L(s))$ does not have any zero in the open Right Half Plane. Adjusting the proof of Theorem 7.2.1, we can conclude to the existence of $\delta_m > 0$ and $\kappa_m > 0$ such that if $\delta_0 < \delta_m$, $\delta_1 < \delta_m$ and $\kappa < \kappa_m$ then, $\det(I_0 - L(s) -$

$sH(s)) = \det(I_0 - F_1(s) - C_0(s) - E_0(s) - \hat{\mathcal{O}}(s))$ does not have any zero in $\mathbb{C}^+$. This proves that the function $\beta(t,1)$ converges to zero. Using the transport equations (8.64)-(8.67) and the transformation (8.58)-(8.59), we can conclude to the convergence of the state (u,v) to its zero-equilibrium.

8.4 Output regulation: Input-to-State Stability and proof of Theorem 8.1.3

We have proved that in the absence of disturbances and noise, there exist $\delta_m > 0$ and $\kappa_m > 0$ such that if $\delta_0 < \delta_m$, $\delta_1 < \delta_m$ and $\kappa < \kappa_m$ the state (u,v) exponentially converges to zero (and thus the output regulation is ensured). We consider in this section the influence of the disturbances and of the noise on the regulation. Using the backstepping transformations (8.58)-(8.59) and (8.77)-(8.78) we prove (using similar computations as the ones done in Section 8.3), that the extended states $\beta(t,1)$, $\tilde{\beta}(t,1)$ and $V(t)$ still satisfy (8.111)-(8.113) in which are added some additional terms that vanish if the disturbances are constant. As the computations are extremely similar to what have been done in Section 8.3 we only detail the main differences. We still consider that Assumption 8.3.1 is satisfied.

8.4.1 Backstepping transformation of the original system (8.36)-(8.39) and pseudo-steady state

Considering the Volterra transformation (8.58)-(8.59), we have that the dynamics of the system (8.36)-(8.39) in the new coordinates is given by

$$\partial_t \alpha(t,x) + \bar{\lambda}\partial_x \alpha(t,x) = \mathcal{D}_1(t)M_1(x), \tag{8.123}$$

$$\partial_t \beta(t,x) - \bar{\mu}\partial_x \beta(t,x) = \mathcal{D}_2(t)M_2(x), \tag{8.124}$$

with the following linear boundary conditions

$$\begin{aligned}
&\alpha(t,0) = \bar{q}\beta(t,0) + d_3(t),\\
&\beta(t,1) = \bar{\rho}\alpha(t,1) + (1+\delta_V)V(t-\delta_0) - \int_0^1 \bar{N}^\alpha(\xi)\alpha(t,\xi) + \bar{N}^\beta(\xi)\beta(t,\xi)d\xi + d_4(t),
\end{aligned} \tag{8.125}$$

where $\bar{N}^\alpha$ and $\bar{N}^\beta$ are defined by (8.68) and where

$$\mathcal{D}_1(t)M_1(x) = d_1(t)m_1(x) - \bar{K}^{uu}(x,0)\bar{\lambda}d_3(t) - \int_0^x \Big(\bar{K}^{uu}(x,\xi)d_1(t)m_1(\xi) + \bar{K}^{uv}(x,\xi)d_2(t)m_2(\xi)\Big)\,d\xi,$$

$$\mathcal{D}_2(t)M_2(x) = d_2(t)m_2(x) - \bar{K}^{vu}(x,0)\bar{\lambda}d_3(t) - \int_0^x \Big(\bar{K}^{vu}(x,\xi)d_1(t)m_1(\xi) + \bar{K}^{vv}(x,\xi)d_2(t)m_2(\xi)\Big)\,d\xi.$$

Moreover, we have

$$\dot{\eta} = \alpha(t-\delta_1,1) + \int_0^1 \Big(\bar{L}^{\alpha\alpha}(1,\xi)\alpha(t-\delta_1,\xi) + \bar{L}^{\alpha\beta}(1,\xi)\beta(t-\delta_1,\xi)\Big)\,d\xi + n(t). \tag{8.126}$$

For convenience, in what follows we consider the analysis of the pseudo-steady state associated to the target system (8.123)-(8.125). This pseudo-steady state corresponds to what the system would converge if the disturbances were constant. Namely, it corresponds to $u^{ss}(t,1) = \alpha^{ss}(t,1) + \int_0^1 \bar{L}^{\alpha\alpha}(1,\xi)\alpha^{ss}(t,\xi)d\xi + \int_0^1 \bar{L}^{\alpha\beta}(1,\xi)\beta^{ss}(t,\xi)d\xi = 0$. The pseudo steady-state is defined by

$$\frac{d}{dx}\begin{pmatrix}\alpha^{ss}(t,x)\\ \beta^{ss}(t,x)\end{pmatrix} = \begin{pmatrix}\frac{\mathcal{D}_1(t)M_1(x)}{\bar{\lambda}}\\ -\frac{\mathcal{D}_2(t)M_2(x)}{\bar{\mu}}\end{pmatrix}, \tag{8.127}$$

along with the initial conditions

$$\beta^{ss}(t,0) = \frac{1}{\bar{q}}(\alpha^{ss}(t,0) - d_3(t)), \tag{8.128}$$

$$\alpha^{ss}(t,1) = -\int_0^1 \bar{L}^{\alpha\alpha}(1,\xi)\alpha^{ss}(t,\xi)d\xi - \int_0^1 \bar{L}^{\alpha\beta}(1,\xi)\beta^{ss}(t,\xi)d\xi\,. \tag{8.129}$$

We have the following lemma regarding the existence of a solution to the ODE (8.127), with the boundary condition (8.128)-(8.129).

Lemma 8.4.1.

If Assumption 8.3.1 holds, the ordinary differential equation (8.127) with boundary conditions (8.128) and (8.129) has a unique solution. Moreover, for every $x \in [0,1]$, one has $\alpha^{ss}(\cdot,x)$ and $\beta^{ss}(\cdot,x)$ in $W^{2,\infty}((0,\infty);\mathbb{R})$.

Proof : Let us define the matrix A_1 by

$$A_1 = \begin{pmatrix} 1 + \int_0^1 \bar{L}^{\alpha\alpha}(1,\xi)d\xi & \int_0^1 \bar{L}^{\alpha\beta}(1,\xi)d\xi \\ -\frac{1}{\bar{q}} & 1 \end{pmatrix}. \tag{8.130}$$

Due to (8.70), this matrix is invertible. We then define $a = \begin{pmatrix} a_1 & a_2 \end{pmatrix}^\top$ by $a = A_1^{-1}b$ with $b = \begin{pmatrix} b_1 & b_2 \end{pmatrix}^\top$ where

$$b_1 = \int_0^1 \bar{L}^{\alpha\alpha}(1,\xi)\int_\xi^1 \frac{\mathcal{D}_1(t)M_1(\nu)}{\bar{\lambda}}d\nu d\xi + \int_0^1 \bar{L}^{\beta\alpha}(1,\xi)\int_0^\xi \frac{\mathcal{D}_2(t)M_2(\nu)}{\bar{\mu}}d\nu d\xi, \tag{8.131}$$

$$b_2 = -\frac{d_3(t)}{\bar{q}} - \int_0^1 \frac{\mathcal{D}_1(t)M_1(\xi)}{q\bar{\lambda}}d\xi\,. \tag{8.132}$$

One can then check that the function

$$\begin{pmatrix} \alpha^{ss}(t,x) \\ \beta^{ss}(t,x) \end{pmatrix} = \begin{pmatrix} a_1 - \int_x^1 \frac{\mathcal{D}_1(t)M_1(\xi)}{\bar{\lambda}}d\xi \\ a_2 - \int_0^x \frac{\mathcal{D}_2(t)M_2(\xi)}{\bar{\mu}}d\xi \end{pmatrix}, \tag{8.133}$$

is a solution of (8.127) with the boundary conditions (8.128) and (8.129). This concludes the proof. ■

Let us state

$$(1+\delta_V)\eta^{ss}(t) = \frac{\beta^{ss}(t+\delta_0,1) - \bar{\rho}\alpha^{ss}(t+\delta_0,1) - d_4(t+\delta_0)}{k_I}\,. \tag{8.134}$$

By defining the error variables $\bar{\alpha} = \alpha - \alpha^{ss}$, $\bar{\beta} = \beta - \beta^{ss}$, and $\bar{\eta} = \eta - \eta^{ss}$, one gets the following system

$$\partial_t\bar{\alpha}(t,x) + \bar{\lambda}\partial_x\bar{\alpha}(t,x) = -\partial_t\alpha^{ss}(t,x), \tag{8.135}$$

$$\partial_t\bar{\beta}(t,x) - \bar{\mu}\partial_x\bar{\beta}(t,x) = -\partial_t\beta^{ss}(t,x)\,, \tag{8.136}$$

with the boundary conditions

$$\bar{\alpha}(t,0) = \bar{q}\bar{\beta}(t,0) \tag{8.137}$$

$$\begin{aligned}\bar{\beta}(t,1) = {}& \bar{\rho}\bar{\alpha}(t,1) + (1+\delta_V)V(t-\delta_0) - \int_0^1 \bar{N}^\alpha(\xi)\bar{\alpha}(t,\xi) + \bar{N}^\beta(\xi)\bar{\beta}(t,\xi)d\xi \\ & - (1+\delta_V)k_I\eta^{ss}(t-\delta_0) - \int_0^1 \bar{N}^\alpha(\xi)\alpha^{ss}(t,\xi) + \bar{N}^\beta(\xi)\beta^{ss}(t,\xi)d\xi\,. \end{aligned} \tag{8.138}$$

Noticing that $\alpha^{ss}(t,1) = -\int_0^1 \bar{L}^{\alpha\alpha}(1,\xi)\alpha^{ss}(t,\xi)d\xi - \int_0^x \bar{L}^{\alpha\beta}(1,\xi)\beta^{ss}(t,\xi)d\xi$, we also have that

$$\begin{aligned}\dot{\bar{\eta}}(t) = {}& \bar{\alpha}(t-\delta_1,1) + \int_0^1 \left(\bar{L}^{\alpha\alpha}(1,\xi)\bar{\alpha}(t-\delta_1,\xi) + \bar{L}^{\alpha\beta}(1,\xi)\bar{\beta}(t-\delta_1,\xi)\right)d\xi \\ & + n(t) - \dot{\eta}^{ss}(t)\,. \end{aligned} \tag{8.139}$$

The initial condition of equations (8.135)-(8.136) is denoted $(\overline{\alpha}_0, \overline{\beta}_0)$. It belongs to $(L^2([0,1]))^2$ and can be expressed as a function of (u_0, v_0).

8.4.2 Backstepping transformation of the error system

As it has been done in Section 8.3.2, combining the observer (8.4)-(8.7) to the real system (8.36)-(8.39) yields the following error system (denoting $\tilde{u}(t,x) = u(t,x) - \hat{u}(t,x)$ and $\tilde{v}(t,x) = v(t,x) - \hat{v}(t,x)$):

$$\begin{aligned}\partial_t \tilde{u}(t,x) + \lambda \partial_x \tilde{u}(t,x) &= \sigma^{+-}(x)\tilde{v}(t,x) - P^+(x)\tilde{u}(t,1) + d_1(t)m_1(x) - P^+(x)n(t) \\ &\quad - \delta_\lambda \partial_x u(t,x) + \delta_{\sigma^{+-}}(x)v(t,x) + P^+(x)(u(t,1) - u(t-\delta_1,1)), \quad (8.140)\\ \partial_t \tilde{v}(t,x) - \mu \partial_x \tilde{v}(t,x) &= \sigma^{-+}(x)\tilde{u}(t,x) - P^-(x)\tilde{u}(t,1) + d_2(t)m_2(x) - P^-(x)n(t) \\ &\quad + \delta_\mu \partial_x v(t,x) + \delta_{\sigma^{-+}}(x)u(t,x) + P^-(x)(u(t,1) - u(t-\delta_1,1)), \quad (8.141)\end{aligned}$$

with the boundary conditions

$$\begin{aligned}\tilde{u}(t,0) &= q\tilde{v}(t,0) + \delta_q v(t,0) + d_3(t), \quad (8.142)\\ \tilde{v}(t,1) &= \rho(1-\epsilon)\tilde{u}(t,1) + \rho\epsilon(u(t,1) - u(t-\delta_1,1)) + \delta_\rho u(t,1) \\ &\quad + (1+\delta_V)V(t-\delta_0) - V(t) + d_4(t) - \rho\epsilon n(t)\,. \quad (8.143)\end{aligned}$$

Using the Volterra transformation (8.77)-(8.78), this system can be rewritten in a simpler way. More precisely, we have the following lemma whose proof is not given here as it is identical to the one of Lemma 8.3.1.

Lemma 8.4.2.

There exist four continuous functions (f_1, f_2, f_3, f) and four continuous functions (g_1, g_2, g_3, g) such that the states α and β defined by (8.77)-(8.78) satisfy the following set of PDEs

$$\begin{aligned}&\partial_t \tilde{\alpha}(t,x) + \lambda \partial_x \tilde{\alpha}(t,x) = f_1(x)d_1(t) + f_2(x)d_2(t) + f_3(x)d_4(t) + f(x)n(t)\\ &+ \delta_{\sigma^{+-}}(x)v(t,x) - f(x)(u(t,1) - u(t-\delta_1,1)) + \mu R^{\alpha\beta}(x,1)\delta_\rho u(t,1) - \delta_\lambda \partial_x u(t,x)\\ &+ \mu R^{\alpha\beta}(x,1)((1+\delta_V)V(t-\delta_0) - V(t)) + \int_x^1 R^{\alpha\alpha}(x,\xi)(\delta_{\sigma^{+-}}(\xi)v(t,\xi) - \delta_\lambda \partial_x u(t,\xi))d\xi\\ &+ \int_x^1 R^{\alpha\beta}(x,\xi)(\delta_{\sigma^{-+}}(\xi)u(t,\xi) + \delta_\mu \partial_x v(t,\xi))d\xi, \quad (8.144)\end{aligned}$$

$$\begin{aligned}&\partial_t \tilde{\beta}(t,x) - \mu \partial_x \tilde{\beta}(t,x) = g_1(x)d_1(t) + g_2(x)d_2(t) + g_3(x)d_4(t) + g(x)n(t)\\ &+ \delta_{\sigma^{-+}}(x)u(t,x) - g(x)(u(t,1) - u(t-\delta_1,1)) + \mu R^{\alpha\beta}(x,1)\delta_\rho u(t,1) + \delta_\mu \partial_x v(t,x)\\ &+ \mu R^{\beta\beta}(x,1)((1+\delta_V)V(t-\delta_0) - V(t)) + \int_x^1 R^{\beta\alpha}(x,\xi)(\delta_{\sigma^{+-}}(\xi)v(t,\xi) - \delta_\lambda \partial_x u(t,\xi))d\xi\\ &+ \int_x^1 R^{\beta\beta}(x,\xi)(\delta_{\sigma^{-+}}(\xi)u(t,\xi) + \delta_\mu \partial_x v(t,\xi))d\xi, \quad (8.145)\end{aligned}$$

along with the boundary conditions

$$\begin{aligned}\tilde{\alpha}(t,0) &= q\tilde{\beta}(t,0) + \delta_q \beta(t,0) + d_3(t), \quad (8.146)\\ \tilde{\beta}(t,1) &= \rho(1-\epsilon)\tilde{\alpha}(t,1) + \rho\epsilon(u(t,1) - u(t-\delta_1,1)) + \delta_\rho u(t,1) \\ &\quad + (1+\delta_V)V(t-\delta_0) - V(t) + d_4(t) - \rho\epsilon n(t)\,. \quad (8.147)\end{aligned}$$

8.4.3 Neutral delay-differential system

As it has been done in Section 8.3.3, we are now able to express the two states $\bar{\beta}(t,1)$ and $\tilde{\beta}(t,1)$ as the solutions of a neutral delay-differential system that is equivalent to (8.135)-(8.138), (8.144)-

(8.143) along with the feedback law (8.10). In what follows we define the state $X(t)$ as

$$X(t) = \begin{pmatrix}\bar{\beta}(t,1) & \tilde{\beta}(t,1) & V(t)\end{pmatrix}^T. \tag{8.148}$$

The definition of the collections $\mathfrak{R}$, $\mathfrak{E}$ and $\mathfrak{U}$ is given in Section 8.3.3. Once again, in what follows, we heavily rely on the definitions of Section 8.2 to ease the notations, grouping terms that correspond to operators in $\mathfrak{I}, \mathfrak{D}$ or $\mathfrak{W}$ and explicitly retaining only the terms that are critical to the robustness analysis. We also group the terms depending on the disturbances. Using equations (8.123)-(8.125), we start expressing $\bar{\beta}(t,1)$ as the solution of a neutral equation whose terms only depend on the state X_t and on the disturbances. It then becomes possible to express $u(t,x)$, $v(t,x)$, $\partial_x u(t,x)$ and $\partial_x v(t,x)$ as functions of X_t and of the disturbances. Using these expressions and the observer equations (8.144)-(8.143), we express the state $\tilde{\beta}(t,1)$ as the solution of a neutral equation that depends on the state X_t and on the disturbances. It then becomes possible to express $\tilde{u}(t,x)$, $\tilde{v}(t,x)$ as functions of X_t and of the disturbances. Finally, we can simplify the expression of the control law (8.10) and express it as a function of X_t of the disturbances. It is then straightforward to obtain the neutral system satisfied by X_t. Differentiating it, provided the disturbances are constant, we obtain system (8.111)-(8.113). To simplify the notations, we regroup all the disturbances, their derivatives and the noise in a single vector:

$$\zeta(t) = \begin{pmatrix}d_1(t) & d_2(t) & d_3(t) & d_4(t) & \dot{d}_1(t) & \dot{d}_2(t) & \dot{d}_3(t) & \dot{d}_4(t) & n(t)\end{pmatrix}^T. \tag{8.149}$$

Using this notation, most of the terms that appear considering the influence of the disturbances can be expressed as linear operators of ζ. For instance, there exist two row matrices $H_1(x)$ and $H_2(x)$ such that $\mathcal{D}_1(t)M_1(x) = H_1(x)\zeta(t)$ and $\mathcal{D}_2(t)M_2(x) = H_2(x)\zeta(t)$. Similarly, using equation (8.133), one can easily prove the existence of row matrices functions $H_3(x)$, $H_4(x)$, $H_5(x)$, $H_6(x)$ and of a row matrix H_7 such that

$$\begin{aligned}&\alpha^{ss}(t,x) = H_3(x)\zeta(t), \quad \beta^{ss}(t,x) = H_4(x)\zeta(t), \quad \partial_t\alpha^{ss}(t,x) = H_5(x)\zeta(t),\\ &\partial_t\alpha^{ss}(t,x) = H_6(x)\zeta(t), \quad \eta^{ss}(t) = H_7\zeta(t+\delta_0).\end{aligned} \tag{8.150}$$

Expression of the state $\bar{\beta}(t,1)$

Considering equations (8.135)-(8.138), using the characteristics method, they simply rewrite for any $t \geq \bar{\tau} + \delta_0 + \delta_1$ as

$$\bar{\beta}(t,1) = \bar{q}\bar{\rho}\bar{\beta}(t-\bar{\tau},1) - \int_0^{\bar{\tau}} \tilde{N}(\xi)\bar{\beta}(t-\xi,1)d\xi + (1+\delta_V)V(t-\delta_0) + K_0(\zeta(t)), \tag{8.151}$$

where the operator $K_0(\zeta)$ is defined by

$$\begin{aligned}K_0(\zeta) = &-(1+\delta_V)k_i\eta^{ss}(t-\delta_0) - \int_0^1 \left(\bar{N}^{\alpha}(\xi)\alpha^{ss}(t,\xi) + \bar{N}^{\beta}(\xi)\beta^{ss}(t,\xi)\right)d\xi\\ &- \int_0^{\frac{1}{\bar{\lambda}}} \partial_t\alpha^{ss}(t-s, 1-\bar{\lambda}s)ds - \int_0^{\frac{1}{\bar{\mu}}} \bar{q}\partial_t\beta^{ss}(t-\frac{1}{\bar{\lambda}}-s, \bar{\mu}s)ds.\end{aligned}$$

Using (8.150), we have that the operator $K_0(\zeta)$ linearly depends on ζ. Using the Volterra transformation (8.58), we can get the analogous of equations (8.93)-(8.96) in presence of disturbances. Namely, for all $t \leq \bar{\tau}$ and all $x \in [0,1]$ there exist four operators $K_1(x,\cdot)$, $K_2(x,\cdot)$, $K_3(x,\cdot)$

and $K_4(x,\cdot)$ that are linear in their second argument such that,

$$u(t,x) = \bar{q}\bar{\beta}(t-\frac{1}{\bar{\mu}}-\frac{x}{\bar{\lambda}},1) + \mathcal{I}_u(x)(\bar{\beta}(\cdot,1)_t) + K_1(x,\zeta(t)) \tag{8.152}$$

$$\partial_x u(t,x) = -\frac{1}{\bar{\lambda}}\bar{q}\partial_x\bar{\beta}(t-\frac{1}{\bar{\mu}}-\frac{x}{\bar{\lambda}},1) + \mathcal{I}_{u_x}(x)(\bar{\beta}(\cdot,1)_t) + K_2(x,\zeta(t)) \tag{8.153}$$

$$v(t,x) = \bar{\beta}(t-\frac{1-x}{\bar{\mu}},1) + \mathcal{I}_v(x)(\bar{\beta}(\cdot,1)_t) + K_3(x,\zeta(t)) \tag{8.154}$$

$$\partial_x v(t,x) = \frac{1}{\bar{\mu}}\partial_x\bar{\beta}(t-\frac{1-x}{\bar{\mu}},1) + \mathcal{I}_{v_x}(x)(\bar{\beta}(\cdot,1)_t) + K_4(x,\zeta(t)). \tag{8.155}$$

The operators $\mathcal{I}_u$, $\mathcal{I}_{u_x}$, $\mathcal{I}_v$ and $\mathcal{I}_{v_x}$ have been defined in (8.93)-(8.96).

Expression of the observer state $\tilde{\beta}(t,1)$

Let us consider equations (8.144)-(8.147). Using the method of characteristics and the formalism introduced in Section 8.2 we can express $\tilde{\beta}(t,1)$ as the solution of a neutral equation.

$$\begin{aligned}\tilde{\beta}(t,1) &= \rho q(1-\epsilon)\tilde{\beta}(t-\tau,1) + \rho(1-\epsilon)\delta_q\beta(t-\frac{1}{\bar{\lambda}},0) + \rho\epsilon(u(t,1)-u(t-\delta_1,1)) + \delta_\rho u(t,1) \\ &\quad + (1+\delta_V)V(t-\delta_0) - V(t) + \mathcal{O}(X_t) + K_5(\zeta(t)),\end{aligned} \tag{8.156}$$

where K_5 is an operator that linearly depends on ζ. This yields

$$\begin{aligned}\tilde{\beta}(t,1) &= \rho q(1-\epsilon)\tilde{\beta}(t-\tau,1) + \rho q\epsilon(\beta(t-\bar{\tau},1)-\beta(t-\bar{\tau}-\delta_1,1)) + V(t-\delta_0) - V(t) \\ &\quad + K_6(\zeta(t)) + \mathcal{O}(X_t),\end{aligned} \tag{8.157}$$

where $K_6(\zeta(t)) = K_5(\zeta(t)) + K_1(1,\zeta(t)) - K_1(1,\zeta(t-\delta_1))$. We can now express the terms $\tilde{u}(t,x)$ and $\tilde{v}(t,x)$ as functions of the state X_t and of ζ. This is a necessary step to simplify the control law $V(t)$. Using the notations of Section 8.2, the relations (8.152)-(8.155), for every $x\in[0,1]$, there exist two operators $K_7(x,\cdot)$ and $K_8(x,\cdot)$ linear in their second variable such that

$$\tilde{u}(t,x) = q\tilde{\beta}(t-\frac{1}{\mu}-\frac{x}{\lambda},1) + \mathcal{I}_{\tilde{u}}(x)(X_t) + K_7(x,\zeta(t)) + \mathcal{O}(X_t), \tag{8.158}$$

$$\tilde{v}(t,x) = \tilde{\beta}(t-\frac{1-x}{\mu},1) + \mathcal{I}_{\tilde{v}}(x)(X_t) + K_8(x,\zeta(t)) + \mathcal{O}(X_t), \tag{8.159}$$

where the operators $\mathcal{I}_{\tilde{u}}(x)$ and $\mathcal{I}_{\tilde{v}}(x)$ are defined in (8.93)-(8.96)

Expression of the control law $V(t)$

We now express the control law $V(t) = V_{BS}(t) + k_I V_I(t) + k_I\eta(t)$ defined in (8.10) in terms of $\beta(t,1)$ and $\tilde{\beta}(t,1)$ and of the disturbance vector ζ. More precisely, we have the following lemma, whose proof is identical to the ones of Lemma 8.3.3 and Lemma 8.3.4 and are not given here.

Lemma 8.4.3.

There exist $\tilde{F}$ a Lipschitz function, K_9 and K_{10} two linear operators such that the control law $V(t)$ rewrites

$$\begin{aligned}V(t) &= \tilde{\rho}\epsilon q(\bar{\beta}(t-\bar{\tau},1)-\bar{\beta}(t-\bar{\tau}-\delta_1,1)) + \tilde{\rho}(1-\epsilon)q\tilde{\beta}(t-\tau,1) - \tilde{\rho}q\bar{\beta}(t-\bar{\tau},1) \\ &\quad + \int_0^{\tau}\tilde{F}(\nu)\tilde{\beta}(t-\nu,1)d\nu + \int_0^{\bar{\tau}}\tilde{N}(\xi)\bar{\beta}(t-\xi,1) + k_I\gamma(t) + K_9(\zeta(t)) + \mathcal{O}(X_t).\end{aligned} \tag{8.160}$$

where the function γ satisfies the ODE

$$\dot{\gamma}(t) = q(\bar{\beta}(t-\bar{\tau}-\delta_1,1) + \lambda l_1(1)\bar{\beta}(t-\bar{\tau},1)) + K_{10}(\dot{\zeta}(t)) + \mathcal{O}(X_t). \tag{8.161}$$

Neutral system

We can now give the neutral system satisfied by X_t. Combining equations (8.151), (8.157) and (8.160), we obtain the following system

$$\begin{aligned}
\bar{\beta}(t,1) = {}& \tilde{\rho}q\bar{\beta}(t-\bar{\tau},1) + \tilde{\rho}q(1-\epsilon)\bar{\beta}(t-\bar{\tau}-\delta_0) - \tilde{\rho}q\epsilon\bar{\beta}(t-\bar{\tau}-\delta_0-\delta_1,1) + k_I\gamma(t-\delta_0) \\
& + \tilde{\rho}q(1-\epsilon)\tilde{\beta}(t-\tau-\delta_0) + \int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu-\delta_0,1)d\nu + K_0(\zeta(t)) \\
& + (1+\delta_V)K_9(\zeta(t-\delta_0)) + \mathcal{O}(X_t), \qquad (8.162)\\
\tilde{\beta}(t,1) = {}& (\rho-\tilde{\rho})q(1-\epsilon)\tilde{\beta}(t-\tau,1) + \tilde{\rho}q(1-\epsilon)\tilde{\beta}(t-\tau-\delta_0,1) + q((\rho-\tilde{\rho})\epsilon \\
& + \tilde{\rho})\bar{\beta}(t-\bar{\tau},1) + (\tilde{\rho}-\rho)q\epsilon\bar{\beta}(t-\bar{\tau}-\delta_1) - (1-\epsilon)\tilde{\rho}q\beta(t-\bar{\tau}-\delta_0) \\
& - \tilde{\rho}\epsilon q\beta(t-\bar{\tau}-\delta_0-\delta_1) + k_I\gamma(t-\delta_0) - k_I\gamma(t) + K_9(\gamma(t-\delta_0)) \\
& - K_9(\zeta(t)) + \mathcal{O}(X_t), \qquad (8.163)
\end{aligned}$$

The equation satisfied by $V(t)$ is given in (8.160) and not rewritten here. As equations (8.162)-(8.163) require the expression of $\gamma(t)$ and since only its derivative is available, we choose to differentiate (8.162)-(8.163) with respect to time. This yields

$$\begin{aligned}
\dot{\bar{\beta}}(t,1) = {}& \tilde{\rho}q\dot{\bar{\beta}}(t-\bar{\tau},1) + \tilde{\rho}q(1-\epsilon)\dot{\bar{\beta}}(t-\bar{\tau}-\delta_0) - \tilde{\rho}q\epsilon\dot{\bar{\beta}}(t-\bar{\tau}-\delta_0-\delta_1,1) + k_Iq\bar{\beta}(t-\bar{\tau}-\delta_1-\delta_0) \\
& + \tilde{\rho}q(1-\epsilon)\dot{\tilde{\beta}}(t-\tau-\delta_0) + k_Iq\lambda l_1(1)\bar{\beta}(t-\bar{\tau}-\delta_1) + \int_0^\tau \tilde{F}(\nu)\dot{\tilde{\beta}}(t-\nu-\delta_0,1)d\nu \\
& + K_0(\dot{\zeta}(t)) + (1+\delta_V)K_9(\dot{\zeta}(t-\delta_0)) + k_IK_{10}(\dot{\zeta}(t)) + \mathcal{O}(\dot{X}_t) + \mathcal{O}(X_t), \qquad (8.164)\\
\dot{\tilde{\beta}}(t,1) = {}& (\rho-\tilde{\rho})q(1-\epsilon)\dot{\tilde{\beta}}(t-\tau,1) + \tilde{\rho}q(1-\epsilon)\dot{\tilde{\beta}}(t-\tau-\delta_0,1) + q((\rho-\tilde{\rho})\epsilon \\
& + \tilde{\rho})\dot{\bar{\beta}}(t-\bar{\tau},1) + (\tilde{\rho}-\rho)q\epsilon\dot{\bar{\beta}}(t-\bar{\tau}-\delta_1) - (1-\epsilon)\tilde{\rho}q\dot{\bar{\beta}}(t-\bar{\tau}-\delta_0-\delta_1) \\
& - \tilde{\rho}\epsilon q\beta(t-\bar{\tau}-\delta_0-\delta_1) + k_Iq\lambda l_1(1)(\bar{\beta}(t-\bar{\tau}-\delta_0) - \bar{\beta}(t-\bar{\tau})) + k_Iq(\bar{\beta}(t-\bar{\tau}-\delta_0-\delta_1) \\
& - \bar{\beta}(t-\bar{\tau}-\delta_1)) + k_IK_{10}(\dot{\zeta}(t-\delta_0)) - k_IK_{10}(\dot{\zeta}(t)) + K_9(\dot{\zeta}(t-\delta_0)) \\
& - K_9(\dot{\zeta}(t)) + \mathcal{O}(\dot{X}_t) + \mathcal{O}(X_t). \qquad (8.165)
\end{aligned}$$

Finally, using equation (8.106) we have

$$\begin{aligned}
\dot{V}(t) = {}& \tilde{\rho}\epsilon q(\dot{\bar{\beta}}(t-\bar{\tau},1) - \dot{\bar{\beta}}(t-\bar{\tau}-\delta_1,1)) + \tilde{\rho}(1-\epsilon)q\dot{\tilde{\beta}}(t-\tau,1) - \tilde{\rho}q\dot{\bar{\beta}}(t-\bar{\tau},1) \\
& + \int_0^\tau \tilde{F}(\nu)\dot{\tilde{\beta}}(t-\nu,1)d\nu + \int_0^{\bar{\tau}} \tilde{N}(\xi)d\xi\dot{\bar{\beta}}(t-\xi,1) + k_Iq(\bar{\beta}(t-\bar{\tau}-\delta_1,1) \\
& + \lambda l_1(1)\bar{\beta}(t-\bar{\tau},1)) + k_IK_{10}(\dot{\zeta}(t)) + K_9(\dot{\zeta}) + \mathcal{O}(X_t) + \mathcal{O}(\dot{X}_t). \qquad (8.166)
\end{aligned}$$

Input-to-State Stability

As the system (8.164)-(8.166) corresponds to the system (8.111)-(8.113) to which is added a disturbance term, we choose to rewrite it with an operator formulation so that we can apply the variation-of-constants formula. Let us denote $\mathcal{K}$ the operator defined by

$$\mathcal{K}(\zeta(t)) = \begin{pmatrix} K_0(\dot{\zeta}(t)) + (1+\delta_V)K_9(\dot{\zeta}(t-\delta_0)) + k_IK_{10}(\dot{\zeta}(t)) \\ k_IK_{10}(\dot{\zeta}(t-\delta_0)) - k_IK_{10}(\dot{\zeta}(t)) + K_9(\dot{\zeta}(t-\delta_0)) - K_9(\dot{\zeta}(t)) \\ k_IK_{10}(\dot{\zeta}(t)) + K_9(\dot{\zeta}(t)) \end{pmatrix}, \tag{8.167}$$

and $\mathcal{P}$ the operator associated to (8.111)-(8.113) (i.e (8.111)-(8.113) can be rewritten $X_t = \mathcal{P}(X_t)$). With these notations, the system (8.164)-(8.166) rewrites

$$X_t = \mathcal{P}(X_t) + K(\zeta(t)).$$

The variation-of-constants formula for this system reads (see [HVL93] page 173)

$$X\left(\left(\overline{\alpha}_0, \overline{\beta}_0\right), \mathcal{K}\right)(t) = X\left(\left(\overline{\alpha_0}, \overline{\beta}_0\right), 0\right)(t) + \int_0^t X_0(t-s)\mathcal{K}(\zeta(s))ds\,, \tag{8.168}$$

where $X\left(\left(\overline{\alpha}_0, \overline{\beta}_0\right), 0\right)(t)$ denotes the solution of the homogeneous NDE system $X_t = \mathcal{P}(X_t)$ in term of the fundamental solution X_0 (see [HVL93] for a definition of the fundamental solution). As $\delta_0 < \delta_m$, $\delta_1 < \delta_m$ and $\kappa < \kappa_m$, the first component of X converges to zero. This implies the ISS of the system.

To conclude the proof, we now have to show that equation (8.47) holds when $\dot{d}_1(t) = \dot{d}_2(t) = \dot{d}_3(t) = \dot{d}_4(t) = n(t) = 0$. In this case the operator $\mathcal{K}$ satisfies $\mathcal{K} \equiv 0$. Thus, $\bar{\beta}(t,1)$ exponentially converges to zero. Using the cascade structure of system (8.135)-(8.138), this implies the convergence to zero of $\bar{\alpha}(t,x)$ and $\bar{\beta}(t,x)$ for every $x \in [0,1]$. Finally, using the transformation (8.58) we obtain

$$\begin{aligned}\lim_{t\to\infty} |u(t,1)| &= \lim_{t\to\infty}\left|\alpha(t,1) + \int_0^1 \bar{L}^{\alpha\alpha}(1,\xi)\alpha(t,\xi)d\xi + \int_0^1 \bar{L}^{\alpha\beta}(1,\xi)\beta(t,\xi)d\xi\right| \\ &= \lim_{t\to\infty}\left|\bar{\alpha}(t,1) + \int_0^1 L^{\alpha\alpha}(1,\xi)\bar{\alpha}(t,\xi)d\xi + \int_0^1 L^{\alpha\beta}(1,\xi)\bar{\beta}(t,\xi)d\xi\right| = 0. \end{aligned}\tag{8.169}$$

This concludes the proof of Theorem 8.1.3.

Remark 8.4.1 *Using the same computations, the proof of Theorem 8.1.4 is straightforward. More precisely, if* (8.48) *holds, then, the function* $\det(Id_2 - F_0(s))$ *has an infinite number of zeros in the right half plane (see Theorem 2.2.1). Thus, using Lemma 6.1.4 and Theorem 8.2.2 yields the result stated in Theorem 8.1.4.*

Remark 8.4.2 *To be completely rigorous, we should consider the presence of uncertainties acting on the functions σ^{++} and σ^{--} as the transformation done in Remark 2.4.1 is now subject to uncertainties. However, as the computations would be similar, for sake of simplicity, we have not considered these two uncertainties in this chapter.*

8.5 Simulation results and concluding remarks

In this section, we numerically illustrate the results of this chapter by explicitly computing the admissible $\tilde{\rho}$, ϵ and k_I that guarantee robustness in the case of a simple example. Let us consider the following set of parameters

$$\lambda = \mu = q = 1,\ \sigma^{-+} = -1,\ \sigma^{+-} = 0,\ \rho = 0.6.$$

Equations (2.51)-(2.56) can be solved explicitly. This yields for any $(x,\xi) \in \mathcal{T}$

$$L^{\alpha\alpha}(x,\xi) = L^{\alpha\beta}(x,\xi) = 0.$$

Consequently, for any $x \in [0,1]$, we obtain

$$l_1(x) = l_2(x) = 0.$$

With this set of parameters, Assumption 8.16 is obviously satisfied. It has been proved in Chapter 6 that the maximal amount of reflection that can be canceled is given by

$$|\tilde{\rho}_{\max}| = \frac{1-|\rho q|}{|q|} = 0.4.$$

In what follows we consider $\tilde{\rho} \in [0, \tilde{\rho}_{\max}[$. Condition 8.1.1 implies the following condition for k_I:

$$0 > k_I > -\frac{\sqrt{1-(0.6-\tilde{\rho})^2}}{2} \arctan\left(\frac{\sqrt{1-(0.6-\tilde{\rho})^2}}{(0.6-\tilde{\rho})}\right). \tag{8.170}$$

Figure 8.2 pictures the condition (8.24) that the parameters $\tilde{\rho}$ and ϵ (left) have to satisfy and the condition (8.170) that the coefficient k_I (right) has to satisfy. These conditions are required to guarantee the existence of robustness margins. The $(\tilde{\rho}, \epsilon)$ domain is obtained using an iterative algorithm that computes condition (8.24). This algorithm uses the convexity properties of the stability domain described in Appendix B. It is also based on the fact that it is easier to check that (8.24) is false rather than the converse. This domain is compared with the stability domain we would have obtained considering only a delay in the actuation δ_0 and neglecting the influence of the terms δ_1, δ_λ, and δ_μ. This illustrates the property stated in [BC16] that uncertainties on the velocities have a non-negligible impact on delay-robustness and emphasizes the necessity to study these two problems simultaneously. The $(k_I, \tilde{\rho})$ domain is obtained by computing the inequalities given in (8.170).

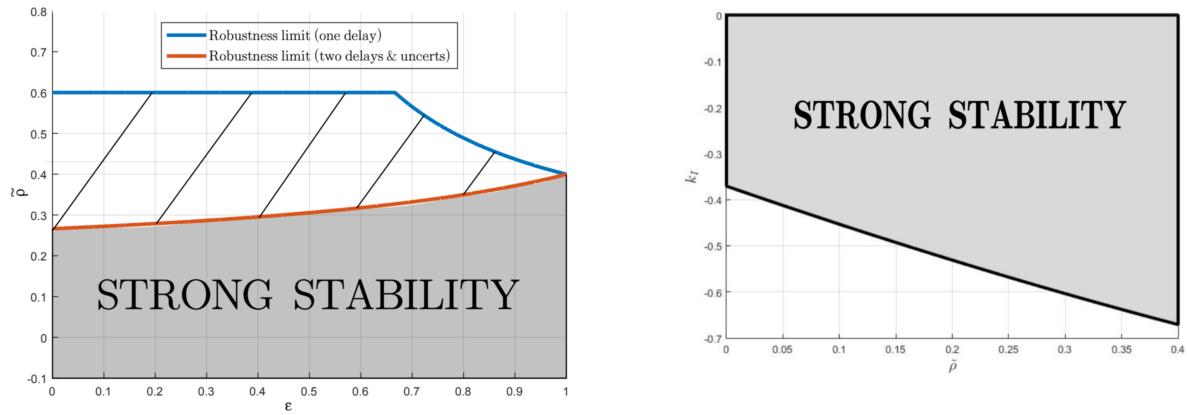

Figure 8.2: Representation of the robustness domain in the plane $(\epsilon, \tilde{\rho})$ (left) and in the plane $(\tilde{\rho}, k_I)$ (right) for $q = 1$ and $\rho = 0.6$.

To analyze the effect of the tuning parameters k_I and ϵ, we picture in Figure 8.3-8.6 the temporal response of the output for different situations:

- In Figure 8.3, we picture the evolution of the output in the absence of disturbance or noise and without any integral compensation for different values of ϵ (left) and $\tilde{\rho}$ (right).
- In Figure 8.4, we picture the evolution of the output in the absence of noise and with a disturbance $d_3(t) = 1$ for different values of ϵ (left) and $\tilde{\rho}$ (right). The integral gain k_I is set to zero.
- In Figure 8.5, we picture the evolution of the output in the absence of disturbance with a high frequency noise for different values of ϵ (left) and $\tilde{\rho}$ (right). The integral gain k_I is set to zero.

- In Figure 8.6, we fix the tuning parameters $\tilde{\rho} = 0.3$ and $\epsilon = 1$. We consider the constant disturbances $d_1 = d_2 = n = 0$, $d_3 = d_4 = 1$. We picture the evolution of the output for different admissible values of k_I.

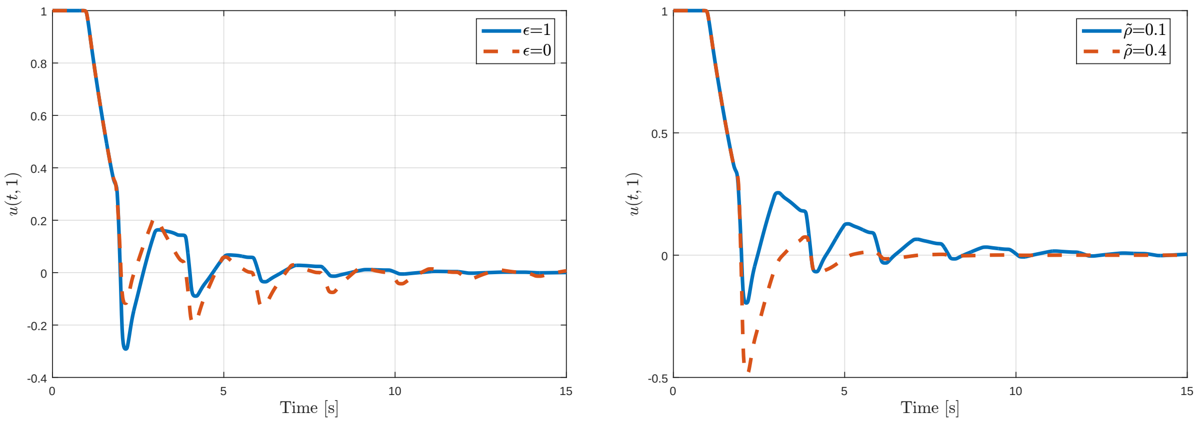

Figure 8.3: Evolution of the output $u(t, 1)$ in the absence of noise and disturbance for different values of ϵ with $k_I = 0$ and $\tilde{\rho} = 0.2$ (left) and for different values of $\tilde{\rho}$ with $k_I = 0$ and $\epsilon = 1$ (right).

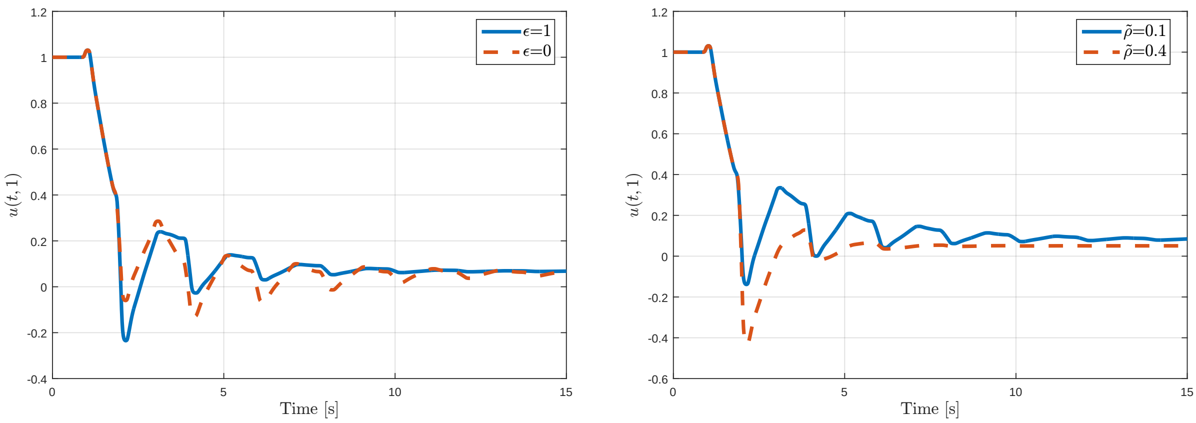

Figure 8.4: Evolution of the output $u(t, 1)$ in the absence of noise with a disturbance $d_3 = 0.1$ for different values of ϵ with $k_I = 0$ and $\tilde{\rho} = 0.2$ (left) and for different values of $\tilde{\rho}$ with $k_I = 0$ and $\epsilon = 1$ (right).

From Figure 8.2 and Figure 8.6, we can make the following remarks

1. The coefficient k_I has a direct impact on disturbance rejection but also on the raise-time and on the settle time. Classically, increasing k_I improves the disturbance rejection but generates oscillations.

2. Choosing a high absolute value for k_I implies to have $\tilde{\rho}$ large enough (right part of Figure 8.2). Thus, as $\tilde{\rho}$ enables a trade-off between performance and robustness, choosing an arbitrary value for k_I may have some negative impact on robustness. Consequently, there is a trade-off between disturbance rejection and (delay-)robustness.

3. Choosing a small value for ϵ seems to improve the noise rejection (even if the convergence is slower). However, a reduction of ϵ may cause a loss of phase margin which must be amended

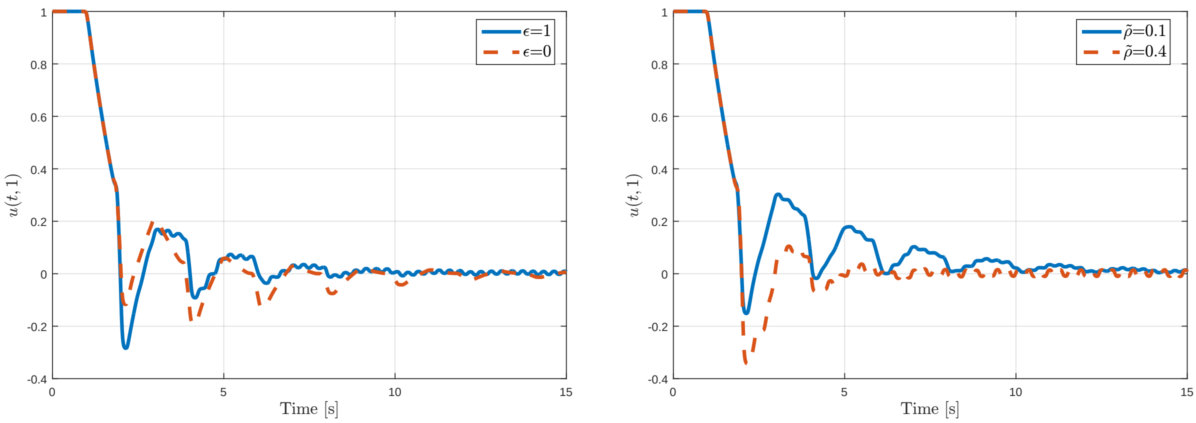

Figure 8.5: Evolution of the output $u(t, 1)$ in the absence disturbance along with a high frequency noise for different values of ϵ with $k_I = 0$ and $\tilde{\rho} = 0.2$ (left) and for different values of $\tilde{\rho}$ with $k_I = 0$ and $\epsilon = 1$ (right).

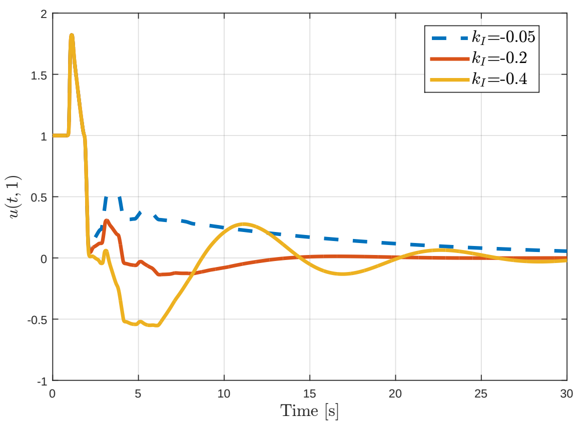

Figure 8.6: Evolution of the output $u(t, 1)$ for different values of k_I with $\tilde{\rho} = 0.3$ and $\epsilon = 1$ in presence of a disturbance $d_3 = 0.1$.

by also reducing the integral gain k_I to avoid a potential unacceptably high controller induced resonance. There is consequently a complex trade-off between performance and robustness, noise sensitivity and disturbance rejection.

These remarks illustrate the fact that the degrees of freedom introduced in this chapter enable various trade-offs and have to be specifically tuned depending on the problem considered. A deeper analysis can only be done for a case by case basis. More precisely, deriving the transfer function of the controller and of the observer, using classical controller analysis techniques (including the analysis of the rise time, of the response time, computing Nyquist charts, or the Gang of Six [ÅM10] for instance), it is possible to derive some tuning heuristics giving a trade-off between noise sensitivity versus disturbance rejection performance or between delay-robustness (especially for high frequencies) and nominal performance.

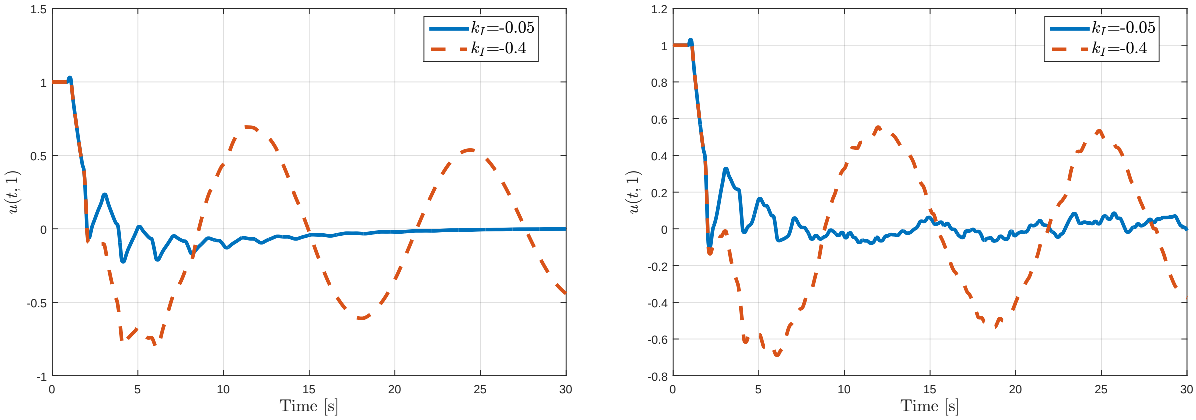

Figure 8.7: Evolution of the output $u(t,1)$ for different values of k_I with $\tilde{\rho} = 0.1$ and $\epsilon = 0$ (left) and $\epsilon = 1$ (right) in presence of a disturbance $d_3 = 0.1$ and of a high frequency noise.

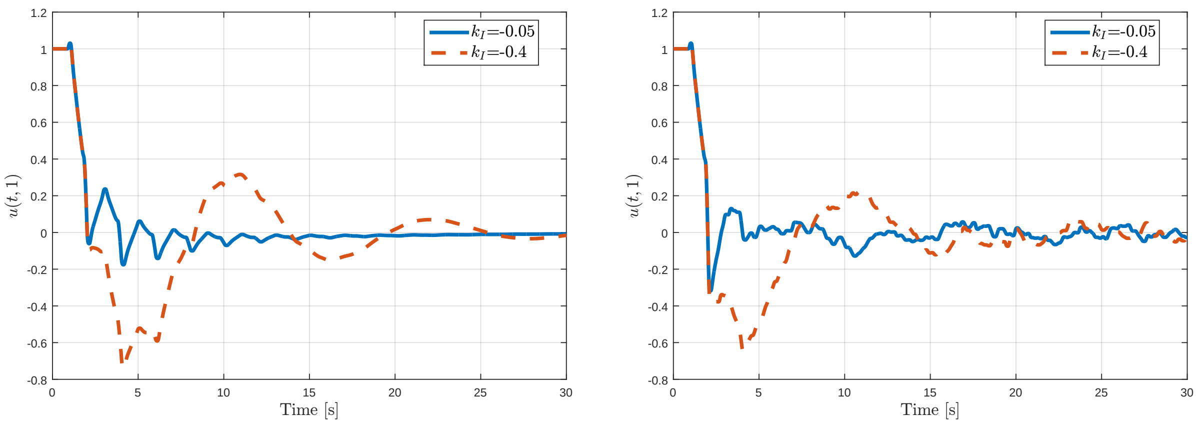

Figure 8.8: Evolution of the output $u(t,1)$ for different values of k_I with $\tilde{\rho} = 0.4$ and $\epsilon = 0$ (left) and $\epsilon = 1$ (right) in presence of a disturbance $d_3 = 0.1$ and of a high frequency noise.

Conclusions and perspectives

Conlusions et perspectives. Dans ce chapitre, nous résumons les contributions de la thèse et proposons des axes pour d'éventuels prolongements.

In this thesis, we have addressed several problems related to the control and observability of LFOH PDEs. The first objective in terms of control theory has been to provide explicit control laws and observers that solve the problem of minimum finite-time boundary stabilization and observability defined in Chapter 2. In Chapter 3, we have extended the existing literature results to solve the problem of finite-time one-sided boundary stabilization in the general case of a system composed of $n + m$ equations; while the two-sided problem has been addressed in Chapter 4. In both cases, the idea consists in decoupling the original system by means of a backstepping transformation that moves most of the destabilizing terms at the boundary. The corresponding boundary observers have been designed through a dual approach. An important by-product of this analysis (detailed in Chapter 5) is an explicit mapping from the solutions of the considered PDEs to the solutions of a neutral system with distributed delays.

In Part II of the thesis, we have developed a range of tools and methods that enable tuning of the backstepping controllers. A limited set of parameters can be adjusted to perform various classical trade-offs (performance versus robustness, convergence rate versus noise sensitivity,...) and we have provided bounds for these parameters that guarantee robust stability in closed-loop. However, if the *qualitative* effect of these parameters is understood, we believe that it is now time to *quatitatively* assess the benefits of the proposed control laws. A backstepping observer-controller algorithm requires know-how and some computing power to be implemented. The incentive to do so, rather than relying on finite-dimensional controllers is a performance criterion: explicitly taking into account the delays and high-frequency content in the model should lead to increased performance. We have provided tools to do so while retaining some robustness. One must now assess whether the overall trade-off can be favorable compared to industry standards, by applying these methods on an industrial problem. A good candidate is the control of stick-slip vibrations in drilling, where torsional vibrations propagate, along a several kilometer-long drill-string. For this example, other questions naturally arise. As modern technologies (as wired drill pipe) also allow additional measurements along the drill pipe, how is it possible to use these excess information to improve the performance or the robustness of the observer? This raise the general question of sensor placement: having several sensors available, what is the best way (according to some criteria that have to be defined depending on the considered problem) to place them? Moreover, for such industrial examples, disturbances and model uncertainties can sometimes be very significant and the clean formulation of ISS property as given in Chapter 8 may not always be the most appropriate. In this extent, alternative methods such as adaptive control [AA15, ADAK16a] could be considered (although such methods increase the computational complexity). Regarding the computational burden inherent to the implementation of the different control laws and observers presented in this thesis, one must be aware that most of the computations can be done offline and that, as described in Appendix A, the corresponding output feedback law can be simplified for practical use.

Finally, more general target systems (and thus additional degrees of freedom) could potentially be constructed by keeping dissipative in-domain couplings, although this would require a precise knowledge of the influence of each coupling in term of stability. In this sense the port-Hamiltonian approach [LGZM04, LGZM05] could be a path to follow, as this method has been proved to represent a powerful framework for modeling, simulation and control of physical systems described by PDEs [DMSB09, MLGRZ17], using the natural energy dissipation of the system.

Bibliography

[AA15] H. Anfinsen and O. M. Aamo. Disturbance rejection in the interior domain of linear 2×2 hyperbolic systems. *IEEE Transactions on Automatic Control*, 60(1):186–191, 2015.

[AAHP13] U.J.F Aarsnes, O.M Aamo, E. Hauge, and A. Pavlov. Limits of controller performance in the heave disturbance attenuation problem. In *European Control Conference (ECC), 2013*, pages 1071–1076. IEEE, 2013.

[Aam13] O.M Aamo. Disturbance rejection in 2x2 linear hyperbolic systems. *IEEE Transactions on Automatic Control*, 58(5):1095–1106, May 2013.

[Aam16] O. M. Aamo. Leak detection, size estimation and localization in pipe flows. *IEEE Transactions on Automatic Control*, 61(1):246–251, 2016.

[AAMDM18] J. Auriol, U. J. F. Aarsnes, P. Martin, and F. Di Meglio. Delay-robust control design for heterodirectional linear coupled hyperbolic pdes. *IEEE Transactions on Automatic Control*, 2018.

[AAP17] Pedro Ascencio, Alessandro Astolfi, and Thomas Parisini. Backstepping pde design: A convex optimization approach. *IEEE Transactions on Automatic Control*, 2017.

[Aar16] U.J.F. Aarsnes. *Modeling of Two-Phase Flow for Estimation and Control of Drilling Operations*. PhD thesis, Norwegian University of Science and Technology, 2016.

[ABABS+18] J. Auriol, F. Bribiesca Argomedo, D. Bou Saba, M. Di Loreto, and F. Di Meglio. Delay-robust stabilization of a hyperbolic pde-ode system. *Automatica, provisionally accepted*, 2018.

[ADAK16a] H. Anfinsen, M. Diagne, O. M. Aamo, and M. Krstic. An adaptive observer design for $n+1$ coupled linear hyperbolic pdes based on swapping. *IEEE Transactions on Automatic Control*, 61(12):3979–3990, 2016.

[ADAK16b] H. Anfinsen, M. Diagne, O. M. Aamo, and M. Krstic. Boundary parameter and state estimation in general linear hyperbolic pdes. *IFAC-PapersOnLine*, 49(8):104–110, 2016.

[ADM16a] J. Auriol and F. Di Meglio. Minimum time control of heterodirectional linear coupled hyperbolic PDEs. *Automatica*, 71:300–307, 2016.

[ADM16b] J. Auriol and F. Di Meglio. Two-sided boundary stabilization of two linear hyperbolic pdes in minimum time. In *Decision and Control (CDC), 2016 IEEE 55th Conference on*, pages 3118–3124. IEEE, 2016.

[ADM17] J. Auriol and F. Di Meglio. Two sided boundary stabilization of heterodirectional linear coupled hyperbolic pdes. *IEEE Transactions on Automatic Control*, 2017.

[ADMEA14] U.J.F Aarsnes, F. Di Meglio, S. Evje, and O.-M. Aamo. Control-oriented drift-flux modeling of single and two-phase flow for drilling. In *Proceedings of the ASME 2014 Dynamic Systems and Control Conference, San Antonio, USA, October 22-24*, 2014.

[ADMG+16] U.J.F Aarsnes, F. Di Meglio, R. Graham, O.-M. Aamo, et al. A methodology for classifying operating regimes in underbalanced-drilling operations. *SPE Journal*, 21(02):423–433, 2016.

[ADMew] J. Auriol and F. Di Meglio. An explicit mapping from linear first order hyperbolic pdes to difference systems. *Systems & Control Letter*, 2018 (under review).

[AGA+14] U. J. F. Aarsnes, M. S. Gleditsch, O. M. Aamo, A. Pavlov, et al. Modeling and avoidance of heave-induced resonances in offshore drilling. *SPE Drilling & Completion*, 29(04):454–464, 2014.

[AHB08] S. Amin, F. M. Hante, and A. M. Bayen. On stability of switched linear hyperbolic conservation laws with reflecting boundaries. In *Hybrid Systems: Computation and Control*, pages 602–605. Springer, 2008.

[ÅM10] K.J. Åström and R. M. Murray. *Feedback systems: an introduction for scientists and engineers*. Princeton university press, 2nd edition, 2010.

[AMDM18] J. Auriol, K. Morris, and F. Di Meglio. Late-lumping backstepping control of partial differential equations. *Automatica, under review*, 2018.

[BAK15] Federico Bribiesca-Argomedo and Miroslav Krstic. Backstepping-forwarding control and observation for hyperbolic pdes with fredholm integrals. *Automatic Control, IEEE Transactions on*, 60(8):2145–2160, 2015.

[BC63] R. E. Bellman and K. L. Cooke. Differential-difference equations. 1963.

[BC11] G. Bastin and J-M Coron. On boundary feedback stabilization of non-uniform linear 2×2 hyperbolic systems over a bounded interval. *Systems & Control Letters*, 60(11):900–906, 2011.

[BC16] G. Bastin and J.-M. Coron. *Stability and boundary stabilization of 1-d hyperbolic systems*. Springer, 2016.

[BDPDM07] A. Bensoussan, G. Da Prato, M. Delfour, and S. Mitter. Unbounded control operators: Hyperbolic equations with control on the boundary. *Representation and Control of Infinite Dimensional Systems*, pages 459–475, 2007.

[BI97] H.T. Banks and K. Ito. Approximation in LQR problems for infinite dimensional systems with unbounded input operators. *Journal of Mathematical Systems Estimation and Control*, 7:119–119, 1997.

[BIG88] J.A. Burns, K. Ito, and G.Propst. On non-convergence of adjoint semigroups for control systems with delays. *SIAM Jour. Control and Optim.*, 26(6):1442–1454, 1988.

[BK84] H.T. Banks and K. Kunisch. The linear regulator problem for parabolic systems. *SIAM Journal on Control and Optimization*, 22(5):684–698, 1984.

[BK14] P. Bernard and M. Krstic. Adaptive output-feedback stabilization of non-local hyperbolic pdes. *Automatica*, 50(10):2692–2699, 2014.

[BLK13] N. Bekiaris-Liberis and M. Krstic. *Nonlinear control under nonconstant delays.* SIAM, 2013.

[BLK14] N. Bekiaris-Liberis and M. Krstic. Compensation of wave actuator dynamics for nonlinear systems. *IEEE Transactions on Automatic Control*, 59(6):1555–1570, June 2014.

[Boi13] S. Boisgérault. Growth bound of delay-differential algebraic equations. *Comptes Rendus Mathematique*, 351(15):645–648, 2013.

[Boi18] S. Boisgérault. *Analyse complexe et applications.* Mines ParisTech, PSL Research University, 2018.

[BP12] D. Bresch-Pietri. *Commande robuste de systèmes à retard variable: Contributions théoriques et applications au contrôle moteur.* PhD thesis, Ecole Nationale Supérieure des Mines de Paris, 2012.

[BPCP12] D. Bresch-Pietri, J. Chauvin, and N. Petit. Adaptive control scheme for uncertain time-delay systems. *Automatica*, 48(8):1536–1552, 2012.

[BPCP14] D. Bresch-Pietri, J. Chauvin, and N. Petit. Prediction-based stabilization of linear systems subject to input-dependent input delay of integral-type. *IEEE Transactions on Automatic Control*, 59(9):2385–2399, 2014.

[BPDM16] D. Bresch-Pietri and F. Di Meglio. Prediction-based control of linear input-delay system subject to state-dependent state delay-application to suppression of mechanical vibrations in drilling. *IFAC-PapersOnLine*, 49(8):111–117, 2016.

[BPK10] D. Bresch-Pietri and M. Krstic. Delay-adaptive predictor feedback for systems with unknown long actuator delay . *IEEE Transactions on Automatic Control*, 55(9):2106–2112, 2010.

[BPK14] D. Bresch-Pietri and M. Krstic. Adaptive output feedback for oil drilling stick-slip instability modeled by wave pde with anti-damped dynamic boundary. In *American Control Conference (ACC), 2014*, pages 386–391. IEEE, 2014.

[Bre00] A. Bressan. *Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem*, volume 20. Oxford University Press on Demand, 2000.

[Bre10] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Springer Science & Business Media, 2010.

[BSBAA+18] D. Bou Saba, F. Bribiesca Argomedo, J. Auriol, M. Di Loreto, and F. Di Meglio. Stability analysis of linear 2×2 hyperbolic pdes. *submitted to International Journal of Robust and Nonlinear Control*, 2018.

[BSZ08] J.A. Burns, E. W. Sachs, and L. Zietsman. Mesh independence of Kleinman-Newton iterations for Riccati equations in Hilbert space. *SIAM Jour. Control and Optim.*, 47(5):2663–2692, 2008.

[BV70] R. Baker and D. Vakharia. Input-output stability of linear time-invariant systems. *IEEE Transactions on Automatic Control*, 15(3):316–319, 1970.

[CANB99] J.-M. Coron, B. Andréa-Novel, and G. Bastin. A lyapunov approach to control irrigation canals modeled by saint-venant equations. In *Proc. European Control Conference, Karlsruhe*, 1999.

[Car96] L. Carvalho. On quadratic liapunov functionals for linear difference equations. *Linear Algebra and its applications*, 240:41–64, 1996.

[CBdN08] J-M. Coron, G. Bastin, and B. d'Andréa Novel. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. *SIAM Journal on Control and Optimization*, 47(3):1460–1498, 2008.

[CHO16] J.-M. Coron, L. Hu, and G. Olive. Stabilization and controllability of first-order integro-differential hyperbolic equations. *Journal of Functional Analysis*, 271(12):3554–3587, 2016.

[CHO17] J.-M. Coron, L. Hu, and G. Olive. Finite-time boundary stabilization of general linear hyperbolic balance laws via fredholm backstepping transformation. *Automatica*, 84:95–100, 2017.

[Cir69] M. Cirina. Boundary controllability of nonlinear hyperbolic systems. *SIAM Journal on Control*, 7(2):198–212, 1969.

[CM03] A. Cheng and K. A. Morris. Well-posedness of boundary control systems. *SIAM Jour. Control & Optim.*, 42(4):1244–1265, 2003.

[Cor09] J.-M. Coron. *Control and nonlinearity.* Number 136. American Mathematical Soc., 2009.

[CP78] R.F. Curtain and AJ. Pritchard. Infinite dimensional linear systems theory, vol. 8 of. *Lecture Notes in Control and Information Sciences*, 1978.

[CS86] R. F. Curtain and D. Salamon. Finite-dimensional compensators for infinite-dimensional systems with unbounded input operators. *SIAM journal on control and optimization*, 24(4):797–816, 1986.

[CT15] J.-M. Coron and S. O. Tamasoiu. Feedback stabilization for a scalar conservation law with PID boundary control. *Chinese Annals of Mathematics, Series B*, 36(5):763–776, 2015.

[CVKB13] J-M Coron, R. Vazquez, M. Krstic, and G. Bastin. Local exponential h^2 stabilization of a $2\times$ 2 quasilinear hyperbolic system using backstepping. *SIAM Journal on Control and Optimization*, 51(3):2005–2035, 2013.

[CZ12] R.F Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*, volume 21. Springer Science & Business Media, 2012.

[D'A49] J. D'Alembert. Suite des recherches sur la courbe que forme une corde tendue, mise en vibration. *Histoire de l'Académie Royale des Sciences et des Belles Lettres de Berlin*, pages 220–249, 1749.

[DBAR12] S. Dudret, K. Beauchard, F. Ammouri, and P. Rouchon. Stability and asymptotic observers of binary distillation processes described by nonlinear convection/diffusion models. In *American Control Conference (ACC), 2012*, pages 3352–3358. IEEE, 2012.

[DBC12] A. Diagne, G. Bastin, and J.-M. Coron. Lyapunov exponential stability of 1-d linear hyperbolic systems of balance laws. *Automatica*, 48(1):109–114, 2012.

[DBVdHJ10] S. Djordjevic, O.H. Bosgra, P.M.J. Van den Hof, and D. Jeltsema. Boundary actuation structure of linearized two-phase flow. In *American Control Conference (ACC), 2010*, pages 3759–3764. IEEE, 2010.

[DDLM15] S. Damak, M. Di Loreto, and S. Mondié. Stability of linear continuous-time difference equations with distributed delay: Constructive exponential estimates. *International Journal of Robust and Nonlinear Control*, 25(17):3195–3209, 2015.

[DDLMB16] S. Damak, M. Di Loreto, S. Mondié, and X. Brun. Exponential stability with decay rate estimation for linear difference equations. *IEEE Transactions on Automatic Control*, 61(1):252–257, 2016.

[Deu16] J. Deutscher. Backstepping design of robust state feedback regulators for linear 2×2 hyperbolic systems. *IEEE Transactions on Automatic Control*, 2016.

[Deu17] J. Deutscher. Finite-time output regulation for linear 2×2 hyperbolic systems using backstepping. *Automatica*, 75:54–62, 2017.

[dHPC+03] J. de Halleux, C. Prieur, J.-M. Coron, B. d'Andréa Novel, and G. Bastin. Boundary feedback control in networks of open channels. *Automatica*, 39(8):1365–1376, 2003.

[DLP86] R. Datko, J. Lagnese, and M.P. Polis. An example on the effect of time delays in boundary feedback stabilization of wave equations. *SIAM Journal on Control and Optimization*, 24(1):152–156, 1986.

[DLS85] W. Desch, I. Lasiecka, and W. Schappacher. Feedback boundary control problems for linear semigroups. *Israel Journal of Mathematics*, 51(3):177–207, 1985.

[DM11] F. Di Meglio. *Dynamics and control of slugging in oil production*. PhD thesis, École Nationale Supérieure des Mines de Paris, Centre Automatique et Systèmes (CAS), 2011.

[DMA15] F. Di Meglio and U.J.F. Aarsnes. A distributed parameter systems view of control problems in drilling. In *to appear in 2nd IFAC Workshop on Automatic Control in Offshore Oil and Gas Production, Florianópolis, Brazil*, 2015.

[DMAHK18] F. Di Meglio, F.B. Argomedo, L. Hu, and M. Krstic. Stabilization of coupled linear heterodirectional hyperbolic pde–ode systems. *Automatica*, 87:281–289, 2018.

[DMBPA14] F. Di Meglio, D. Bresch-Pietri, and U. J. F. Aarsnes. An adaptive observer for hyperbolic systems with application to underbalanced drilling. *IFAC Proceedings Volumes*, 47:11391–11397, 2014.

[DMKV13] F. Di Meglio, M. Krstic, and R. Vazquez. A backstepping boundary observer for a class of linear first-order hyperbolic systems. In *Control Conference (ECC), 2013 European*, pages 1597–1602. IEEE, 2013.

[DMSB09] V. Duindam, A. Macchelli, S. Stramigioli, and H. Bruyninckx. *Modeling and control of complex physical systems: the port-Hamiltonian approach*. Springer Science & Business Media, 2009.

[DMVK13] F. Di Meglio, R. Vazquez, and M. Krstic. Stabilization of a system of $n+1$ coupled first-order hyperbolic linear pdes with a single boundary input. *IEEE Transactions on Automatic Control*, 58(12):3097–3111, 2013.

[DMVKP12] F. Di Meglio, R. Vazquez, M. Krstic, and N. Petit. Backstepping stabilization of an underactuated $3\times$ 3 linear hyperbolic system of fluid flow equations. In *American Control Conference (ACC), 2012*, pages 3365–3370. IEEE, 2012.

[DSBCdN08] V. Dos Santos, G. Bastin, J.-M. Coron, and B. d'Andréa Novel. Boundary control with integral action for hyperbolic systems of conservation laws: Stability and experiments. *Automatica*, 44(5):1310–1318, 2008.

[EG02] O. Egeland and J.T. Gravdahl. *Modeling and simulation for automatic control*, volume 76. Marine Cybernetics Trondheim, Norway, 2002.

[Eva11] G.C. Evans. VolterraŠs integral equation of the second kind, with discontinuous kernel. ii. *Transactions of the American Mathematical Society*, 12(4):429–472, 1911.

[FI96] A.V. Fursikov and O.Y. Imanuvilov. *Controllability of evolution equations.* Number 34. Seoul National University, 1996.

[Fri02] E. Fridman. Stability of linear descriptor systems with delay: a lyapunov-based approach. *Journal of Mathematical Analysis and Applications*, 273(1):24–44, 2002.

[FS02] E. Fridman and U. Shaked. H_∞-control of linear state-delay descriptor systems: an LMI approach. *Linear Algebra and its Applications*, 351:271–302, 2002.

[GD11] M. Gugat and M. Dick. Time-delayed boundary feedback stabilization of the isothermal euler equations with friction. *Math. Control Relat. Fields*, 1(4):469–491, 2011.

[GFF17] V. Glizer, E. Fridman, and Y. Feigin. A novel approach to exact slow-fast decomposition of linear singularly perturbed systems with small delays. *SIAM Journal on Control and Optimization*, 55(1):236–274, 2017.

[GKC03] K. Gu, V.L. Kharitonov, and J. Chen. *Stability of time-delay systems.* Birkhäuser, 2003.

[GT84] J.M. Greenberg and L.T. Tsien. The effect of boundary damping for the quasilinear wave equation. *Journal of Differential Equations*, 52(1):66–75, 1984.

[HDMVK15] L. Hu, F. Di Meglio, R. Vazquez, and M. Krstic. Boundary exponential stabilization of 1-d inhomogeneous quasilinear hyperbolic systems. *arXiv preprint arXiv:1512.03539*, 2015.

[HDMVK16] L. Hu, F. Di Meglio, R. Vazquez, and M. Krstic. Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs. *IEEE Transactions on Automatic Control*, 61(11):3301–3314, 2016.

[HL02] J. Hale and S.M. Verduyn Lunel. Strong stabilization of neutral functional differential equations. *IMA Journal of Mathematical Control and Information*, 19(1 and 2):5–23, 2002.

[HR83] L.F. Ho and D.L. Russell. Admissible input elements for systems in Hilbert space and a carleson measure criterion. *SIAM Journal on Control and Optimization*, 21(4):614–640, 1983.

[HVL93] J.K. Hale and S.M. Verduyn Lunel. *Introduction to functional differential equations.* Springer-Verlag, 1993.

[IK98] K. Ito and F. Kappel. The Trotter-Kato theorem and approximation of PDEs. *Mathematics of Computation of the American Mathematical Society*, 67(221):21–44, 1998.

[Jac30] D. Jackson. *The theory of approximation*, volume 11. American Mathematical Soc., 1930.

[JMK12] L. Jadachowski, T. Meurer, and A. Kugi. An efficient implementation of backstepping observers for time-varying parabolic pdes. *IFAC Proceedings Volumes*, 45(2):798–803, 2012.

[Joh60] F. John. Continuous dependence on data for solutions of partial differential equations with a prescribed bound. *Communications on pure and applied mathematics*, 13(4):551–585, 1960.

[Kai80] T. Kailath. *Linear systems*, volume 156. Prentice-Hall Englewood Cliffs, NJ, 1980.

[KGBS08] M. Krstic, B-Z. Guo, A. Balogh, and A. Smyshlyaev. Output-feedback stabilization of an unstable wave equation. *Automatica*, 44(1):63–74, 2008.

[KK14] I. Karafyllis and M. Krstic. On the relation of delay equations to first-order hyperbolic partial differential equations. *ESAIM: Control, Optimisation and Calculus of Variations*, 20(3):894–923, 2014.

[KMV08] M. Krstic, L. Magnis, and R. Vazquez. Nonlinear stabilization of shock-like unstable equilibria in the viscous burgers pde. *IEEE Transactions on Automatic Control*, 53(7):1678–1683, 2008.

[KMV09] M. Krstic, L. Magnis, and R. Vazquez. Nonlinear control of the viscous burgers equation: Trajectory generation, tracking, and observer design. *Journal of Dynamic Systems, Measurement, and Control*, 131(2):021012, 2009.

[KN09] A. Kyllingstad and P.J. Nessjøen. A new stick-slip prevention system. In *SPE/IADC Drilling Conference and Exhibition*. Society of Petroleum Engineers, 2009.

[Krs08] M. Krstic. Lyapunov tools for predictor feedbacks for delay systems: Inverse optimality and robustness to delay mismatch. *Automatica*, 44(11):2930–2935, 2008.

[KS08a] M. Krstic and A. Smyshlyaev. Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays. *Systems & Control Letters*, 57(9):750–758, 2008.

[KS08b] M. Krstic and A. Smyshlyaev. *Boundary control of PDEs: A course on backstepping designs*, volume 16. Siam, 2008.

[LADMA18] P.-O Lamare, J. Auriol, F. Di Meglio, and U.J.F Aarsnes. Robust output regulation of 2 x 2 hyperbolic systems: Control law and input-to-state stability. In *American and Control Conference*, 2018.

[Las86] I. Lasiecka. Approximations of riccati equation for abstract boundary control problems applications to hyperbolic systems. *Numerical functional analysis and optimization*, 8(3-4):207–243, 1986.

[LBL15] P.-O. Lamare and N. Bekiaris-Liberis. Control of 2×2 linear hyperbolic systems: Backstepping-based trajectory generation and PI-based tracking. *Systems & Control Letters*, 86:24–33, 2015.

[LDM16] P.-O. Lamare and F. Di Meglio. Adding an integrator to backstepping: Output disturbances rejection for linear hyperbolic systems. In *American Control Conference (ACC), 2016*, pages 3422–3428. IEEE, 2016.

[Lev40] N. Levinson. *Gap and density theorems*, volume 26. American mathematical society New York, 1940.

[LF09] X. Litrico and V. Fromion. Boundary control of hyperbolic conservation laws using a frequency domain approach. *Automatica*, 45(3):647–656, 2009.

[LGZM04] Y. Le Gorrec, H. Zwart, and B. Maschke. A semigroup approach to port hamiltonian systems associated with linear skew symmetric operator. In *16th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2004)*, 2004.

[LGZM05] Y. Le Gorrec, H. Zwart, and B. Maschke. Dirac structures and boundary control systems associated with skew-symmetric differential operators. *SIAM journal on control and optimization*, 44(5):1864–1892, 2005.

[Li94] D. Li. *Global classical solutions for quasilinear hyperbolic systems*, volume 32. John Wiley & Sons, 1994.

[Li08] T. Li. Exact boundary observability for quasilinear hyperbolic systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 14(4):759–766, 2008.

[Li10] D. Li. *Controllability and observability for quasilinear hyperbolic systems.* American Institute of Mathematical Sciences Springfield, Ill, USA, 2010.

[Lio71] J.L. Lions. *Optimal control of systems governed by partial differential equations problèmes aux limites.* Springer, 1971.

[Lio88a] J.L. Lions. Contrôlabilité exacte, stabilisation et perturbations de systemes distribués. tome 1. contrôlabilité exacte. *Rech. Math. Appl*, 8, 1988.

[Lio88b] J.L Lions. Exact controllability, stabilization and perturbations for distributed systems. *SIAM review*, 30(1):1–68, 1988.

[LJ01] T. Li and Y. Jin. Semi-global c1 solution to the mixed initial-boundary value problem for quasilinear hyperbolic systems. *Chinese Annals of Mathematics*, 22(03):325–336, 2001.

[LP61] G. Lumer and R.S. Phillips. Dissipative operators in a banach space. *Pacific Journal of Mathematics*, 11(2):679–698, 1961.

[LR02] T. Li and B. Rao. Local exact boundary controllability for a class of quasilinear hyperbolic systems. *Chinese Annals of Mathematics*, 23(02):209–218, 2002.

[LR03] T. Li and B. Rao. Exact boundary controllability for quasi-linear hyperbolic systems. *SIAM Journal on Control and Optimization*, 41(6):1748–1755, 2003.

[LR10] T. Li and B. Rao. Strong (weak) exact controllability and strong (weak) exact observability for quasilinear hyperbolic systems. *Chinese Annals of Mathematics, Series B*, 31(5):723–742, 2010.

[LRW96] H. Logemann, R. Rebarber, and G. Weiss. Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop. *SIAM Journal on Control and Optimization*, 34(2):572–600, 1996.

[LT83a] I. Lasiecka and R. Triggiani. Feedback semigroups and cosine operators for boundary feedback parabolic and hyperbolic equations. *Journal of Differential Equations*, 47(2):246–272, 1983.

[LT83b] I. Lasiecka and R. Triggiani. Regularity of hyperbolic equations under l^2 (0, t; $l^2(\gamma)$)-dirichlet boundary terms. *Applied mathematics & optimization*, 10(1):275–286, 1983.

[LT91] I. Lasiecka and R. Triggiani. Differential and algebraic riccati equations with application to boundary/point control problems: continuous theory and approximation theory. *Lecture notes in control and Information Sciences*, 164:1–160, 1991.

[LT00a] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, volume I. Cambridge University Press, 2000.

[LT00b] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, volume II. Cambridge University Press, 2000.

[LY03] T. Li and L. Yu. Contrôlabilité exacte frontière pour les équations des ondes quasi linéaires unidimensionnelles. *Comptes Rendus Mathematique*, 337(4):271–276, 2003.

[MA13] D. Melchor-Aguilar. Exponential stability of linear continuous time difference systems with multiple delays. *Systems & Control Letters*, 62(10):811–818, 2013.

[MLGRZ17] A. Macchelli, Y. Le Gorrec, H. Ramirez, and H. Zwart. On the synthesis of boundary control laws for distributed port-hamiltonian systems. *IEEE transactions on automatic control*, 62(4):1700–1713, 2017.

[MM03] S. Mondié and W. Michiels. Finite spectrum assignment of unstable time-delay systems with a safe implementation. *IEEE Transactions on Automatic Control*, 48(12):2207–2212, 2003.

[MM07] R.K. Miller and A.N. Michel. *Ordinary differential equations.* Dover, 2007.

[MN07] W. Michiels and S.-I. Niculescu. *Stability and stabilization of time-delay systems: an eigenvalue-based approach.* SIAM, 2007.

[Mor94a] K.A. Morris. Convergence of controllers designed using state-space methods. *IEEE Trans. Auto. Cont.*, 39(10):2100–2104, 1994.

[Mor94b] K.A. Morris. Design of finite-dimensional controllers for infinite-dimensional systems by approximation. *Jour. of Mathematical Systems, Estimation and Control*, 4(2):1–30, 1994.

[Mor01] K.A. Morris. H_∞ output feedback control of infinite-dimensional systems via approximation. *Systems and Control Letters*, 44(3):211–217, 2001.

[Mor10] K.A Morris. Control of systems governed by partial differential equations. *The Control Theory Handbook*, 2010.

[MR03] L. Mirkin and N. Raskin. Every stabilizing dead-time controller has an observer-predictor-based structure. *Automatica*, 39:1747–1754, 2003.

[MVZ+09] W. Michiels, T. Vyhlídal, P. Zítek, H. Nijmeijer, and D. Henrion. Strong stability of neutral equations with an arbitrary delay dependency structure. *SIAM Journal on Control and Optimization*, 48(2):763–786, 2009.

[MWTA00] P. C. Magnusson, A. Weisshaar, V. K. Tripathi, and G. C. Alexander. *Transmission lines and wave propagation.* CRC Press, 2000.

[MZ04] T. Meurer and M. Zeitz. Flatness-based feedback control of diffusion-convection-reaction systems via k-summable power series. *IFAC Proceedings Volumes*, 37(13):177–182, 2004.

[Nic01a] S.-I. Niculescu. *Delay effects on stability: a robust control approach*, volume 269. Springer Science & Business Media, 2001.

[Nic01b] S.-I. Niculescu. On delay-dependent stability under model transformations of some neutral linear systems. *International Journal of Control*, 74(6):609–617, 2001.

[Paz12] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44. Springer Science & Business Media, 2012.

[Pep05] P. Pepe. On the asymptotic stability of coupled delay differential and continuous time difference equations. *Automatica*, 41(1):107–112, 2005.

[PGW12] C. Prieur, A. Girard, and E. Witrant. Lyapunov functions for switched linear hyperbolic systems. *IFAC Proceedings Volumes*, 45(9):382–387, 2012.

[Pin02] M.A Pinsky. *Introduction to Fourier analysis and wavelets*, volume 102. American Mathematical Soc., 2002.

[PK13] P. Pepe and I. Karafyllis. Converse lyapunov–krasovskii theorems for systems described by neutral functional differential equations in hale's form. *International Journal of Control*, 86(2):232–243, 2013.

[PW78] AJ. Pritchard and A. Wirth. Unbounded control and observation systems and their duality. *SIAM Journal on Control and Optimization*, 16(4):535–545, 1978.

[Qin85] T.H Qin. Global smooth solutions of dissipative boundary-value problems for 1st order quasilinear hyperbolic systems. *Chinese annals of mathematics series B*, 6(3):289–298, 1985.

[RNM15] D. Rager, R. Neumann, and H. Murrenhoff. Simplified fluid transmission line model for pneumatic control applications. In *Proc. 14th Scandinavian International Conference on Fluid Power (SICFP15). Tampere, Finland*, 2015.

[RR06] M. Renardy and R.C. Rogers. *An introduction to partial differential equations*, volume 13. Springer Science & Business Media, 2006.

[Rus67a] D.L. Russell. Nonharmonic fourier series in the control theory of distributed parameter systems. *Journal of Mathematical Analysis and Applications*, 18(3):542–560, 1967.

[Rus67b] D.L. Russell. On boundary-value controllability of linear symmetric hyperbolic systems. *Mathematical Theory of Control*, pages 312–321, 1967.

[Rus72] D.L. Russell. Control theory of hyperbolic equations related to certain questions in harmonic analysis and spectral theory. *Journal of Mathematical Analysis and Applications*, 40(2):336–368, 1972.

[Rus78a] D.L. Russell. Canonical forms and spectral determination for a class of hyperbolic distributed parameter control systems. *Journal of Mathematical Analysis and Applications*, 62(1):186–225, 1978.

[Rus78b] D.L. Russell. Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. *Siam Review*, 20(4):639–739, 1978.

[Rus91] D.L. Russell. Neutral pde canonical representations of hyperbolic systems. *The Journal of Integral Equations and Applications*, pages 129–166, 1991.

[Sal87] D. Salamon. Infinite-dimensional linear systems with unbounded control and observation: a functional analytic approach. *Transactions of the American Mathematical Society*, 300(2):383–431, 1987.

[SCK10] A. Smyshlyaev, E. Cerpa, and M. Krstic. Boundary stabilization of a 1-d wave equation with in-domain antidamping. *SIAM Journal on Control and Optimization*, 48(6):4014–4031, 2010.

[SDMKR13] C. Sagert, F. Di Meglio, M. Krstic, and P. Rouchon. Backstepping and flatness approaches for stabilization of the stick-slip phenomenon for drilling. *IFAC Proceedings Volumes*, 46(2):779–784, 2013.

[SK04] A. Smyshlyaev and M. Krstic. Closed-form boundary state feedbacks for a class of 1-d partial integro-differential equations. *IEEE Transactions on Automatic Control*, 49(12):2185–2202, 2004.

[SK05] A. Smyshlyaev and M. Krstic. Backstepping observers for a class of parabolic PDEs. *Systems & Control Letters*, 54(7):613–625, 2005.

[SK09] A. Smyshlyaev and M. Krstic. Boundary control of an anti-stable wave equation with anti-damping on the uncontrolled boundary. *Systems & Control Letter*, 58(8):617–623, 2009.

[Sle83] M. Slemrod. Boundary feedback stabilization for a quasi-linear wave equation. In *Control Theory for Distributed Parameter Systems and Applications*, pages 221–237. Springer, 1983.

[Smi59] O. Smith. A controller to overcome dead time. *ISA J.*, 6:28–33, 1959.

[SNA$^+$11] R. Sipahi, S.-I. Niculescu, C.T. Abdallah, W. Michiels, and K. Gu. Stability and stabilization of systems with time delay. *IEEE Control Systems*, 31(1):38–65, 2011.

[Tie85] Q. Tiehu. Global smooth solutions of dissipative boundary value problems for first order quasilinear hyperbolic systems. *Chinese annals of mathematic, series B*, 6(3):289–298, 1985.

[TK14] S. Tang and M. Krstic. Sliding mode control to the stabilization of a linear 2×2 hyperbolic system with boundary input disturbance. In *American Control Conference*, pages 1027–1032, Portland, OR, USA, 2014. IEEE.

[VCKB11] R. Vazquez, J.-M. Coron, M. Krstic, and G. Bastin. Local exponential h-2 stabilization of a $2\times$ 2 quasilinear hyperbolic system using backstepping. In *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, pages 1329–1334. IEEE, 2011.

[Vid72] M. Vidyasagar. Input-output stability of a broad class of linear time-invariant multivariable systems. *SIAM J. Control*, 10(1):203–209, 1972.

[VK14] R. Vazquez and M. Krstic. Marcum q-functions and explicit kernels for stabilization of $2\times$ 2 linear hyperbolic systems with constant coefficients. *Systems & Control Letters*, 68:33–42, 2014.

[VK16] R. Vazquez and M. Krstic. Bilateral boundary control of one-dimensional first- and second-order PDEs using infinite-dimensional backstepping. In *Decision and Control (CDC), 2016 IEEE 55th Conference on*, pages 537–542. IEEE, 2016.

[VK17] R. Vazquez and M. Krstic. Taking a step back: A brief history of PDE backstepping. *IFAC Proceedings Volumes*, 2017.

[VKC11] R. Vazquez, M. Krstic, and J.-M. Coron. Backstepping boundary stabilization and state estimation of a 2×2 linear hyperbolic system. In *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, pages 4937–4942. IEEE, 2011.

[Was79] D. Washburn. A bound on the boundary input map for parabolic equations with application to time optimal control. *SIAM Journal on Control and Optimization*, 17(5):652–671, 1979.

[Wei89] G. Weiss. Admissibility of unbounded control operators. *SIAM Journal on Control and Optimization*, 27(3):527–545, 1989.

[Whi11] G.B. Whitham. *Linear and nonlinear waves*, volume 42. John Wiley & Sons, 2011.

[WRE17] F. Woittennek, M. Riesmeier, and S. Ecklebe. On approximation and implementation of transformation based feedback laws for distributed parameter systems. *IFAC Proceedings Volumes*, 2017.

[XS02] C.-Z. Xu and G. Sallet. Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 7:421–442, 2002.

[YH05] D. Yue and Q.-L. Han. Delayed feedback control of uncertain systems with time-varying input delay. *Automatica*, 41(2):233–240, 2005.

[Yos60] K. Yoshida. *Lectures on differential and integral equations*, volume 10. Interscience Publishers, 1960.

[Zha86] Y.C Zhao. The boundary value problem for systems of first-order quasilinear hyperbolic equations. *Chinese Annals of Mathematics Serie A*, 7:629–643, 1986.

[Zho06] Q.-C. Zhong. *Robust control of time-delay systems*. Springer-Verlag, 2006.

Appendix A

Late-lumping approximation

In this thesis, all the controllers developed are based on the backstepping approach. Considering applications of such controllers to industrial problems, in most cases, only an approximation of the state is available for controller design. This direct design approach is sometimes referred to as late-lumping since the last step in the design is to approximate the controller by a finite-dimensional, or lumped parameter system. The other possible approach is early-lumping or reduced order design, where the controller design is based on a finite-dimensional approximation of the PDE. Numerous results ensuring the convergence of early-lumping controllers can be found in the literature; see for example [BI97, BK84, Las86, LT00a, LT00b, Mor01, Mor94a] and the tutorial paper [Mor10]. However, the question of the convergence of late-lumping backstepping controllers has not been well-investigated. In [WRE17], a method for computing the bounded part of the control operator is proposed. It relies on a finite-dimensional approximation of the state and enables efficient computing of the feedback law. However, the unbounded part of the operator is not approximated and no guarantees of convergence are provided. In this appendix, we give sufficient conditions guaranteeing the convergence of backstepping-based late-lumping controller (2.60) for system (2.29)-(2.31). For this example, we consider that the approximation of the state satisfies some specific assumptions. The resulting feedback system is mapped to a simpler target system using backstepping-like transformations. An explicit Lyapunov function is used to prove exponential stability. Note that in this appendix, the design is based on the boundary control formulation; the system is not converted to state space form. Moreover, we only consider a state-feedback control law and the results presented in this appendix still have to be adjusted for output feedback control law. The performance of the resulting late-lumping controller is compared to an early-lumping controller in simulations using a high order approximation of the PDE as the system.

This content of this chapter is extracted from a submission in [AMDM18].

A.1 Presentation of the method

As the method we present in this appendix can be generalized to other systems that (2.29)-(2.31) we introduce a general framework that could be easily adjusted for similar problems.

Let us consider the following boundary control system [Sal87]

$$\begin{aligned} &\frac{dz}{dt} = \mathfrak{A}z(t), \quad z(0) = z_0, \quad t \in [0,T] \\ &\mathfrak{B}z(t) = u(t), \end{aligned} \tag{A.1}$$

where $\mathfrak{A} \in \mathcal{L}(\mathcal{Z},\mathcal{H})$, $\mathfrak{B} \in \mathcal{L}(\mathcal{Z},\mathcal{U})$ with $\mathcal{Z}$ and $\mathcal{U}$ separable Hilbert spaces. The space $\mathcal{Z}$ is a dense subspace of $\mathcal{H}$ with continuous, injective embedding $i_{\mathcal{Z}}$. We assume that the boundary control system (A.1) satisfies the following assumptions [Sal87]:

- The operator $\mathfrak{B}$ is onto, its kernel is dense in $\mathcal{H}$ and there exists $\mu \in \mathbb{R}$ such that $\ker(\mu\mathfrak{I} - \mathfrak{A}) \cap \ker \mathfrak{B} = 0$, and $\mu\mathfrak{I} - \mathfrak{A}$ is onto $\mathcal{H}$ (where $\mathfrak{I}$ is the identity operator).

- For any $z_0 \in \mathcal{Z}$ with $\mathfrak{B}z_0 = 0$, there exists a unique solution to (A.1) in $C^1([0, T; \mathcal{H}]) \cap C([0, T; \mathcal{Z}])$ depending continuously on z_0 (where T is a positive time).

The initial condition z_0 is assumed to belong to $\mathcal{Z}$. Note that this formalism is slightly different from the one introduced in Chapter 2 since we do not considered the abstract space form 2.6 anymore. The space $\mathcal{Z}$ has to satisfy the following additional assumption.

Assumption A.1.1 *The domain of definition $D(\mathfrak{A})$ satisfies $D(\mathfrak{A}) \subset (H^1([0, 1]))^p$ where p is a positive integer.*

The value of p obviously depends on the PDE considered. Since the space $H^1([0, 1])$ is embedded in the Holder space $\mathcal{C}^{0,\frac{1}{2}}([0, 1])$, using Morrey's inequality (see e.g [Bre10, Theorem 9.12]), a direct consequence of Assumption A.1.1 is the existence of an constant $\alpha > 0$ such that for all $z \in D(\mathfrak{A})$, for all $1 \leq i \leq p$,

$$\sup_{x\in[0,1]} |z_i(x)| \leq \alpha(||z_i||_{H^1([0,1])})^p. \tag{A.2}$$

Let us adjust the definition of exponential stabilization given in 2.2.1 for the considered abstract problem.

Definition A.1.1.

The system (A.1) is **exponentially stabilizable** if there exists $K \in L(\mathcal{Z}, \mathcal{V})$ such that if $V(t) = Kz(t)$ the semigroup $\mathcal{S}$ associated to (A.1) is exponentially stable semigroup, i.e there exist $M \geq 1$ and $\omega > 0$ such that

$$||\mathcal{S}(t)|| \leq Me^{-\omega t}. \tag{A.3}$$

The early-lumping approach consists in approximating the original PDE (A.1) using standard methods (such as finite elements for instance). This yields a system of ordinary differential equations. Controller design is based on this finite-dimensional approximation. Consider finite-dimensional subspace $\mathcal{Z}_n$ of the state-space $\mathcal{Z}$ and P_n the orthogonal projection $P_n : \mathcal{Z} \to \mathcal{Z}_n$ such that

$$\forall\ z\ \in \mathcal{Z},\ \lim_{n\to\infty} ||P_n z - z|| = 0. \tag{A.4}$$

Although the orthogonality of the operators P_n is not necessary, it is convenient as it makes the computation easier; that is, the reconstruction of the full state from its projections. The subspaces $\mathcal{Z}_n$ are equipped with the norm inherited from $\mathcal{Z}$. Considering this approximation scheme and defining the operator $\mathfrak{A}_n \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Z})$ by some method method while $\mathfrak{B}_n = \mathfrak{B}P_n$, this leads to the following finite-dimensional approximation:

$$\begin{aligned}&\frac{d\tilde{z}}{dt} = \mathfrak{A}_n\tilde{z}(t), \quad \tilde{z}(0) = P_n z_0, \quad t \in [0, T].\\&\mathfrak{B}_n\tilde{z}(t) = V(t).\end{aligned} \tag{A.5}$$

Define the operator on $\mathcal{H}$

$$Az = \mathfrak{A}z, \quad D(A) = \{z \in \mathcal{Z}; z \in \ker \mathfrak{B}\},$$

and let $S(t)$ be the C_0-semigroup generated on $\mathcal{H}$ by A. Denote similarly by $\mathcal{S}_n(t)$ the semigroups generated by $\mathfrak{A}_n : D(\mathfrak{A}_n) \mapsto \mathcal{Z}_n$ with $D(\mathfrak{A}_n) = \mathcal{Z}_n \cap \ker \mathfrak{B}_n$ and $\mathfrak{A}_n z = \mathfrak{A}_n z$ for $z \in \mathcal{Z}_n$. We make the following classical assumption that ensure the uniform convergence on bounded intervals of the open-loop approximating state $\tilde{z}(t)$ to the exact state: for each $z \in \mathcal{Z}$, and all intervals of time $[t_1, t_2]$

$$\lim_{n\to\infty} \sup_{t\in[t_1,t_2]} ||\mathcal{S}_n(t) P_n z - \mathcal{S}(t) z|| = 0. \tag{A.6}$$

Equation (A.6), which is often satisfied by ensuring that the conditions of the Trotter-Kato Theorem hold (see [IK98, Paz12]), along with equation (A.4) imply open loop convergence of the approximatng systems. However it is not sufficient to guarantee that a control sequence u_n that stabilizes the approximations (A.5) will stabilize the original system and provide good performance (see [BIG88, Mor94b, Mor10]). For bounded control operators, a large number of tools and techniques are available for controller design using this approach (see for example[BK84, BSZ08, Las86, LT00a, LT00b, Mor94b, Mor01] and the tutorial paper [Mor10]). However, boundary control typically leads to an unbounded control operator when put in state space form and only a few results can be found in the literature [BI97, Las86].

Late-lumping control

For numerous systems, it is possible to directly derive from the PDE infinite-dimensional state feedback insuring stabilization, that is, to find an operator $K \in \mathcal{L}(\mathcal{Z}, \mathcal{V})$ such that the semigroup associated to (A.1) along with the control law $V(t) = Kz(t)$ is stable. The backstteping controllers previously derived in this thesis are not the only type of late lumping controllers. Other examples include the flatness-based controllers derived in [MZ04, SDMKR13], the optimization controllers in [Lio71], the controller in [LF09] based on a frequency-domain approach

In this appendix, we only focus on the control aspects, neglecting the design of the observer. However, to reflect the fact that we do not have fully-distributed measurements, we assume that only an approximation of the state is available to synthesize the control law. More precisely, considering a stabilizing control law $u(t) = Kz$, the late-lumping assumption implies that the real control law that will be used is

$$u(t) = KP_n z = Kz^n, \tag{A.7}$$

denoting $z^n = P_n z$ where P_n is the orthogonal projection (A.4) onto some subspace. The proof we propose for the uniform convergence of the late-lumping controller relies on the following assumption on the approximation sequence.

Assumption A.1.2 *Let p be the integer in Assumption A.1.1. There exists a sequence C_n such that $\forall z \in \mathcal{Z} \subset (H^1([0,1]))^p$,*

1. $\lim_{n\to\infty} C_n = 0$,
2. $\forall n \in \mathbb{N},\ ||KP_n z - Kz|| \leq C_n ||z||_{(H^1([0,1]))^p}$.

This assumption means that the approximation scheme has to be chosen accurate enough to ensure the uniform convergence of the approximated control operator to the real one. The fact that C_n does not depend on z is crucial to ensure this uniform convergence. It will be shown throughout the next sections, that common approximation methods, such as finite elements, satisfy this assumption.

A.2 Application to the system of two linear PDEs (2.29)-(2.31)

Let us consider the system (2.29)-(2.31) defined by

$$\partial_t u(t,x) + \lambda \partial_x v(t,x) = \sigma^{+-}(x) v(t,x) \tag{A.8}$$
$$\partial_t v(t,x) - \mu \partial_x v(t,x) = \sigma^{-+}(x) u(t,x), \tag{A.9}$$

evolving in $\{(t,x)| \quad t>0,\ x \in [0,1]\}$, with the following linear boundary conditions

$$u(t,0) = qv(t,0), \quad v(t,1) = \rho u(t,1) + V(t). \tag{A.10}$$

The boundary coupling term q is assumed to be non-zero. Moreover, we assume $|\rho q| < 1$ to guarantee delay-robustness (see Chapter 6). We recall that depending on σ^{+-}, σ^{-+} and q, the system may be unstable [BC16] (the eigenvalues can curve over). The initial conditions denoted u_0 and v_0 are assumed to belong to $(H^1([0,1]))^2$ and satisfy the compatibility conditions. As proved in in Chapter 6, the system (A.8)-(A.10) is delay-robustly stabilizable and has a finite number of poles in the right half-plane.

Remark A.2.1 *It is important to highlight the fact that the initial conditions now belong to* $(H^1([0,1]))^2$ *instead of* $(L^2([0,1]))^2$.

Late-lumping controller

Let us consider the control law

$$V_{BS}(t) = K_{BS}\begin{pmatrix} u & v \end{pmatrix}^T, \tag{A.11}$$

where

$$K_{BS}\begin{pmatrix} u & v \end{pmatrix}^T = -\tilde{\rho} u(t,1) - (\rho - \tilde{\rho}) \int_0^1 K^{uu}(1,\xi) u(t,\xi) + K^{uv}(1,\xi) v(t,\xi) d\xi$$
$$+ \int_0^1 K^{vu}(1,\xi) u(t,\xi) + K^{vv}(1,\xi) v(t,\xi) d\xi, \tag{A.12}$$

where the kernels $K^{uu}, K^{uv}, K^{vu}, K^{vv}$ are defined on $\mathcal{T} = \{(x,\xi) \in [0,1]^2| \quad \xi \leq x\}$ by the set of hyperbolic PDEs (2.43)-(2.48) and where $\tilde{\rho}$ satisfies

$$|\tilde{\rho}| < \frac{1 - |\rho q|}{|q|}. \tag{A.13}$$

Considering transformation (2.37)-(2.38), it has been proved in Chapter 6 that this transformation maps the original system (A.8)-(A.10) to the target system

$$\partial_t \alpha(t,x) + \lambda \partial_x \alpha(t,x) = 0, \quad \partial_t \beta(t,x) - \mu \partial_x \beta(t,x) = 0, \tag{A.14}$$

with the boundary conditions

$$\alpha(t,0) = q\beta(t,0), \quad \beta(t,1) = (\rho - \tilde{\rho}) \alpha(t,1). \tag{A.15}$$

Since the initial condition belong to $(H^1([0,1]))^2$, we can extend the result given by Lemma 6.2.2.

Lemma A.2.1.

For any initial condition $(u(0,\cdot), v(0,\cdot)) \in H^1(0,1) \times H^1(0,1)$ that satisfies the compatibility conditions, the system (A.8)-(A.10) along with the control law V_{BS} defined by (A.11), has a unique solution $(u,v) \in \mathcal{C}([0,\infty), H^1(0,1) \times H^1(0,1))$ which is exponentially stable in the sense of the L^2-norm.

We recall that there exist two constants C_1 and C_2 such that

$$C_1(||\alpha||_{H^1([0,1])} + ||\beta||_{H^1([0,1])}) \leq ||u||_{H^1([0,1])} + ||v||_{H^1([0,1])}, \tag{A.16}$$
$$||u||_{H^1([0,1])} + ||v||_{H^1([0,1])} \leq C_2(||\alpha||_{H^1([0,1])} + ||\beta||_{H^1([0,1])}). \tag{A.17}$$

Let us consider an approximation scheme satisfying Assumption A.1.2 (an example of such an approximating scheme is given by (A.29)). Denoting by P_n the projection on this approximating space, and considering the system (A.8)-(A.10) along with the control law

$$V^n_{BS}(t) = K_{BS}P_n \begin{pmatrix} u & v \end{pmatrix}^T = K_{BS}P^n \begin{pmatrix} u & v \end{pmatrix}^T, \tag{A.18}$$

we then have the following theorem.

Theorem A.2.1.

There exists $N \in \mathbb{N}$ such that for any $n \geq N$, for any initial condition $(u_0, v_0) \in (H^1([0,1]))^2$, the system (A.8)-(A.10) along with the approximated control law (A.18) is exponentially stable at the origin.

Proof : Using the backstepping transformation (2.37)-(2.37), the system (A.8)-(A.10) can be mapped to

$$\partial_t \alpha(t,x) = -\lambda \partial_x \alpha(t,x) \quad \partial_t \beta(t,x) = \mu \partial_x \beta(t,x) \tag{A.19}$$
$$\alpha(t,0) = q\beta(t,0), \quad \beta(t,1) = (\rho - \tilde{\rho})\alpha(t,1) + (K_{BS}P^n - K_{BS}) \begin{pmatrix} u & v \end{pmatrix}^T. \tag{A.20}$$

Since the approximation scheme satisfies Assumption A.1.2, we obtain

$$|(K_{BS}P^n - K_{BS}) \begin{pmatrix} u \\ v \end{pmatrix} | \leq C_n C_2(||(\alpha,\beta)||_{H^1}). \tag{A.21}$$

We now prove the stability of the system (A.19)-(A.20) with a Lyapunov analysis. Let us consider the Lyapunov function candidate

$$R(t) = \int_0^1 \frac{1}{\lambda} \mathrm{e}^{-\nu x} \alpha^2(t,x) + \frac{q^2}{\mu} \mathrm{e}^{\nu x} \beta(t,x) dx \tag{A.22}$$

where ν is a strictly positive parameter. Using the Cauchy Schwartz and Young's inequalities, one can show that there exist $m_1 > 0$ and $m_2 > 0$ such that

$$m_1(||\alpha(t,\cdot)||^2_{H^1} + ||\beta(t,\cdot)||^2_{H^1}) \leq R(t) \leq m_2(||\alpha(t,\cdot)||^2_{H^1} + ||\beta(t,\cdot)||^2_{H^1}). \tag{A.23}$$

Differentiating R with respect to time and integrating by part yields

$$\begin{aligned}
\dot{R}(t) &= -\int_0^1 \nu \mathrm{e}^{-\nu x} \alpha^2(t,x) + \nu q^2 \mathrm{e}^{\nu x} \beta^2(t,x) dx + [-\mathrm{e}^{-\nu x}\alpha^2(t,x) + q^2 \mathrm{e}^{\nu x}\beta^2(t,x)]_0^1 \\
&\leq -\int_0^1 \nu e^{-\nu x} \alpha^2(t,x) + \nu q^2 e^{\nu x} \beta^2(t,x) dx - (1 - (\rho - \tilde{\rho})^2 q^2 \mathrm{e}^{\nu}) \alpha(t,1)^2 + q^2 \mathrm{e}^{\nu} ((K_{BS}P^n - K_{BS}) \begin{pmatrix} u \\ v \end{pmatrix})^2 \\
&\leq -\int_0^1 \nu \mathrm{e}^{-\nu x} \alpha^2(t,x) + \nu q^2 e^{\nu x} \beta^2(t,x) dx + C_n^2 C_2^2 q^2 e^{\nu} (||\alpha||^2 + ||\beta||^2),
\end{aligned} \tag{A.24}$$

since $(\rho - \tilde{\rho})^2 q^2 \mathrm{e}^{\nu} < 1$. Since C_n converges to zero, we easily obtain using (A.21) that there exist $M > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\dot{R}(t) \leq -MR(t) \tag{A.25}$$

This implies the exponential stability of the system (A.14)-(A.20). Due to (A.15), the original state (u, v) has the same properties. This concludes the proof. ■

Convergence of the late-lumping controller and design of an early-lumping controller

We now give the abstract formulation of (A.8)-(A.10) in terms of operators. This abstract formulation, although it was not required for the design of the backstepping controller is useful while designing an early-lumping controller. We can rewrite the system in the abstract form as

$$\begin{pmatrix} \dot{u}(t) \\ \dot{v}(t) \end{pmatrix} = \mathfrak{A} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \quad \mathfrak{B} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = V(t). \tag{A.26}$$

The operator $\mathfrak{A}$ is defined by

$$\begin{aligned} \mathfrak{A} : \mathcal{Z} &\to (\mathcal{L}^2([0,1]))^2 \\ \begin{pmatrix} u \\ v \end{pmatrix} &\longmapsto \begin{pmatrix} -\lambda \partial_x u(t) + \sigma^{+-}(x) v(t) \\ \mu \partial_x v(t) + \sigma^{-+}(x) u(t) \end{pmatrix}, \end{aligned} \tag{A.27}$$

with $\mathcal{Z} = \{\begin{pmatrix} u & v \end{pmatrix}^T \in (H^1([0,1]))^2 |\ u(0) = qv(0)\}$. This operator and densely defined. We equipe its domain of definition with the scalar product associated with the norm of $(H^1([0,1]))^2$. The control operator $\mathfrak{B}$ is defined on $\mathcal{Z}$ by

$$\mathfrak{B} \begin{pmatrix} u \\ v \end{pmatrix} = v(1) - \rho u(1) \tag{A.28}$$

: To prove the convergence of the late-lumping controller and to design early lumping controllers, a Galerkin approximation based on eigenfunctions is used. The approximation scheme is based on a Riesz basis. Consider the family ϕ_k defined for all $k \in \mathbb{N}^*$ by

$$\phi_k(x) = \begin{pmatrix} \phi_k^1(x) \\ \phi_k^2(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{k\pi} \cos(k\pi x) \\ \cos(k\pi x) \end{pmatrix} \tag{A.29}$$

Define ϕ_0 and $\phi_{0,1}$ as

$$\phi_0(x) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T, \quad \phi_{0,1}(x) = \begin{pmatrix} 0 & 1 \end{pmatrix}^T \tag{A.30}$$

The family $\{\phi_{0,1}, \phi_k,\ k \in \mathbb{N}\}$ forms a Riesz basis on the state space $H^1(0,1) \times L^2(0,1)$ [CZ12]. Let us consider $n \in \mathbb{N}$, we define $\chi_n = \text{span}\{\text{span}_{i=-n,\cdots,n}\{\phi_i\}, \phi_{0,1}\}$ and denote P_n, the orthogonal projection onto χ_n. The space χ_n is equipped with the H^1-norm. Define $\mathfrak{A}_n$ by the Galerkin approximation

$$\langle \mathfrak{A}_n \phi_j, \phi_k \rangle = \langle \mathfrak{A} \phi_j, \phi_k \rangle, \quad (j,k) \in [0,n]^2 \tag{A.31}$$

and $\mathfrak{B}_n = P_n \mathfrak{B}$. We have the following lemma

Lemma A.2.2.

The considered approximation scheme combined with the control law (A.31) satisfies Assumption A.1.2.

Proof : Due to Jackson's inequality [Jac30],[Pin02, Exercise 1.5.14] there exists a constant $C_1 > 0$ such that for all $u \in H^1([0,1])$, if we denote u^n its projection on the basis defined by (A.31), we have for all $x \in [0,1]$

$$|u(x) - u^n(x)| \leq \frac{C \ln(n)}{\sqrt{n}} \omega(\frac{1}{n}, u),$$

where $\omega(\frac{1}{n}, u)$ denotes the modulus of continuity of u with the step $\frac{1}{n}$. As the function u is in $H^1([0,1])$ which is embedded in the Holder space $\mathcal{C}^{0,\frac{1}{2}}([0,1])$, using Morrey's inequality (A.2) we have

$$\omega(\frac{1}{n}, z) \leq 2 \sup |u(x)| \leq 2||u||_{H^1}.$$

This yields the expected result, using the linearity of the control law (A.31). ■

This implies the convergence of the late-lumping backstepping controller introduced above. Using this approximation scheme, it is straightforward to design early-lumping controllers based on a pole placement or early-lumping LQR controllers that can be numerically compared with the late-lumping one. For instance, inspired by the backstepping controller, a natural way to design an early-lumping controller is to approximate the (exponentially stable) target system (A.19)-(A.20), find the eigenvalues of the resulting ODE and place the eigenvalues of (A.31) at the same location. This sequence of control law will be denoted $V^n_{BS_{early}}$. A second method to design an early lumping controller is linear quadratic control. Consider the quadratic functional

$$J(u^n, z_0) = \int_0^\infty \langle \mathbf{z}^n(t), \mathbf{z}^n(t) \rangle + \alpha((u^n)(t))^2 dt, \tag{A.32}$$

where $\alpha > 0$ is a tuning coefficient. One can design an early-lumping designing a LQ controller associated with minimizing the cost (A.32) for the Galerkin approximation. However, since the underlying semigroup is not analytic, the convergence or performance of the controllers on the PDE is not guaranteed.

A.3 Simulation results

The real system is simulated using the Galerkin's approximation with a number of modes $N = 40$. The basis we use for the approximating spaces is the same as the one introduced in the previous section (i.e the family ϕ_k defined in equation (A.29)-(A.30)). We finally compare the controller (A.18) with the two early-lumping controllers proposed in the previous section (early-lumping backstepping controller and early-lumping LQR controller). These control laws are designed using only $M < 40$ modes (different values of M are used). The system parameters are chosen as follow: $\lambda = 0.25$, $\mu = 0.5$, $\sigma^{+-} = 0$, $\sigma^{-+} = 0.5$, $q = 1$, $\rho = \tilde{\rho} = 0.3$. The initial conditions are defined by $u_0(x) = v_0(x) = 1$. The LQR early-lumping controller did not stabilize the system when using more than 10 modes. Therefore, in Figure A.1-A.2, we compare the time evolution of the L^2 norm (performance) and the control effort for only the early-lumping pole placement controller and the late-lumping backstepping controller. The late-lumping backstepping controller still stabilizes the system in **finite-time** even with a few number of modes. Moreover, the control effort is better for this controller when enough modes are used. The early-lumping backstepping controller also stabilizes the system (even with one mode) but the performance are not as good as the convergence is slower. However, when the number of modes increases, we obtain similar results in term of performance and control efforts for the two controllers. For this class of hyperbolic systems, it seems that the late-lumping approach allows better performance with low order controllers. Moreover, for these systems, early lumping is problematic as the convergence may require an important number of modes or may not exist (the LQR control law does not convergence for low values of M). Of course, a more complete analysis (computing the transfer functions, analyzing the robustness margins,...) would be necessary.

Note that this Galerkin approximation may not be the best method to approximate the control law. More precisely, it has been proved in [AAMDM18, BC16] that the considered class of hyperbolic equations can be rewritten as neutral equations with distributed delays. As multiple accurate solvers exist for such equations, it may be interesting to use them. The Galerkin approximation has been chosen here to fairly compare the early-lumping controller with the late-lumping one.

Note that the simulations(see Figures A.1-A.2) have comparable computation times (the late-lumping approach requires the computations of the kernels but this can be done once offline).

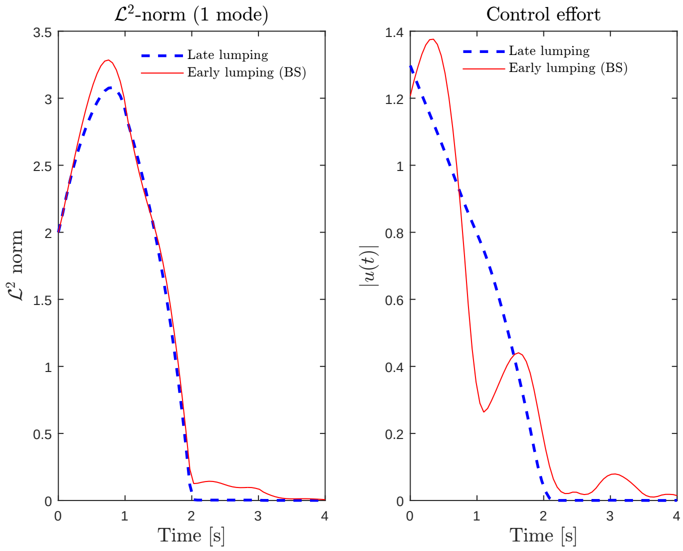

Figure A.1: Time evolution of the $\mathcal{L}^2$-norm and of the control efforts for different controllers (system (2.1)-(2.3), N=40, M=1). The late-lumping controller has a better convergence rate and a better control effort. The early-lumping LQR controller did not converge.

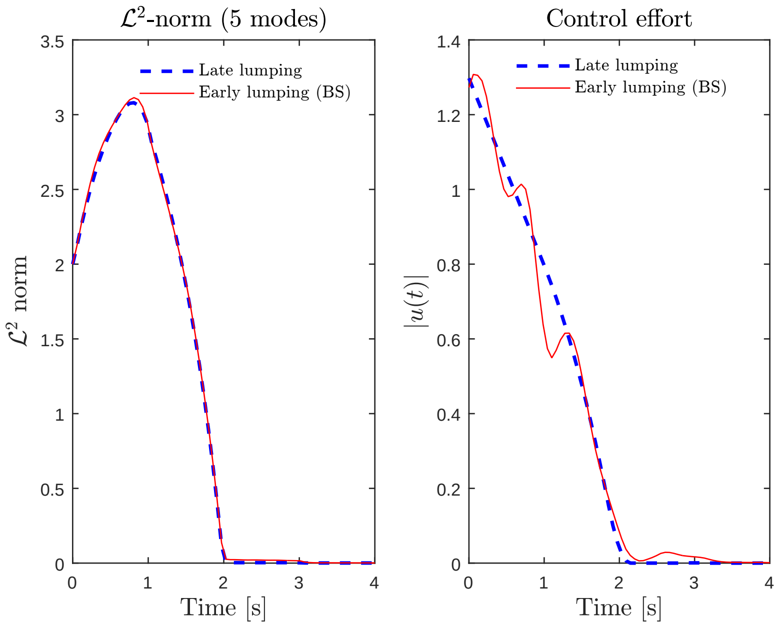

Figure A.2: Time evolution of the $\mathcal{L}^2$-norm and of the control efforts for different controllers (system (2.1)-(2.3), N=40, M=5). The late-lumping controller has a slightly better control effort. The early-lumping LQR controller did not converge.

A.4 Remarks on observer design

All the backstepping control laws presented above are state-feedback control laws and require the value of the (approximated) state for all $x \in [0,1]$. However, the measurements, in distributed parameter systems, are rarely available across the domain. It is more common for the sensors to be placed only at the boundaries. Consequently, to envision industrial applications, for each problem presented above a state-observer has to be designed. The corresponding state estimation can then be used to derive an output-feedback control law. As these observers are usually designed as the duals to the backstepping controllers presented above, they are defined through PDEs that are similar to the ones describing the original systems. Regarding the late-lumping approximation, the solutions of these observer systems have to be approximated. Thus, the second line of Assumption 2 has to be changed by

$$\forall n \in \mathbb{N}, ||K\hat{z}_n - Kz|| \leq C_n ||z||_{(\mathcal{H}^1([0,1]))^p}, \tag{A.33}$$

where $\hat{z}_n$ is the approximation of order n of the observer state $\hat{z}$. Proving that backstepping observers satisfy such an assumption may not be an easy task and is out of the scope of this paper.

In this section, we just give some remarks for reflection in perspective of future work.

Let us consider the system (A.8)-(A.10). We assume that only boundary measurements at the right boundary of the spatial domain are available (i.e. we measure $u(t,1)$). For such a system, the following backstepping state observer has been designed in [VKC11] (see (2.63)-(2.65)):

$$\begin{aligned} \hat{u}_t(t,x) + \lambda \hat{u}_x(t,x) &= \sigma^{+-}\hat{v}(t,x) \\ &\quad + P_1(x)(u(t,1) - \hat{u}(t,1)), \end{aligned} \tag{A.34}$$

$$\begin{aligned} \hat{v}_t(t,x) - \mu \hat{v}_x(t,x) &= \sigma^{+-}\hat{u}(t,x) \\ &\quad + P_2(x)(u(t,1) - \hat{u}(t,1)), \end{aligned} \tag{A.35}$$

with the boundary conditions

$$\hat{u}(t,0) = q\hat{v}(t,0), \quad \hat{v}(t,1) = \rho u(t,1) + V(t), \tag{A.36}$$

where the output injection gains P_1 and P_2 are continuous functions defined by (2.77). As seen int he previous chapters, for any control law $V(t)$, the solutions of (A.34)-(A.36) exponentially converge to the solutions of (A.8)-(A.10) (they actually converges in finite time). Moreover, combining this observer with the control law (A.11) leads to the design of a stabilizing output feedback law [VKC11, Theorem 2]. The observer system (A.34)-(A.36) can be approximated using the Galerkin approximation based on the Riesz basis (A.29)-(A.30). The corresponding state is denoted $(\hat{w}_n, \hat{z}_n)$. Once projected the projection done, the observer system (A.34)-(A.36) admits the following abstract representation

$$\frac{d}{dt}\begin{pmatrix} \hat{u}_n \\ \hat{v}_n \end{pmatrix} = \mathfrak{F}_n \mathfrak{C} \begin{pmatrix} u \\ v \end{pmatrix} + (\mathfrak{A}_n - \mathfrak{F}_n \mathfrak{C}) \begin{pmatrix} \hat{u}_n \\ \hat{v}_n \end{pmatrix} \tag{A.37}$$

$$\mathfrak{B}_n \begin{pmatrix} \hat{u}_n \\ \hat{v}_n \end{pmatrix} = V(t), \tag{A.38}$$

where $\mathfrak{A}_n \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Z}_n)$ is the Galerkin approximation of the operator $\mathfrak{A}$, where $\mathfrak{C}$ is the output operator (i.e. $\mathfrak{C}\begin{pmatrix} u \\ v \end{pmatrix} = u(1)$), and where $\mathfrak{F}$ corresponds to the projection of the output injection operator $\mathfrak{F}$ that appears in (A.34)-(A.36) and which is defined by $\mathfrak{F} : y \in \mathbb{R} \to \begin{pmatrix} p_1(x)y(t) \\ p_2(x)y(t) \end{pmatrix}$. It is straightforward to prove that $\mathfrak{F}_n$ uniformly converges to $\mathfrak{F}$. This is not however sufficient to conclude to the convergence of the late-lumping observer.

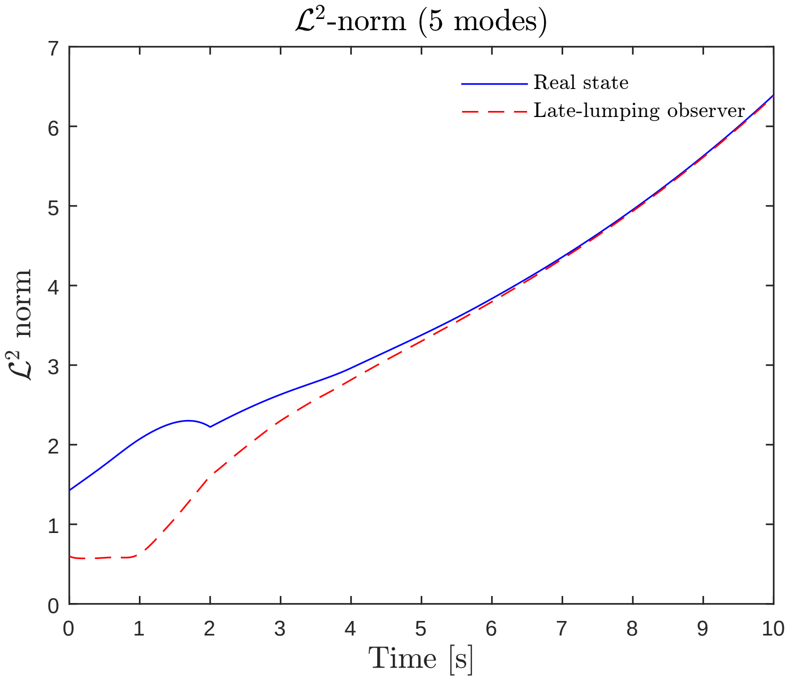

Figure A.3: Time evolution of the $\mathcal{L}^2$-norm of the state of the real system (2.1)-(2.3) and of the state of the approximated observer system (A.37) without any actuation (M=5).

As an illustration we have pictured in Figure A.3 the evolution of the state of the real system (A.8)-(A.10) and of the state of the approximated observer system (A.37) without any actuation for $M = 5$. The system parameters are chosen as follow: $\sigma^{+-} = 0.4$, $\sigma^{-+} = 1$, $q = 1$. The initial conditions of the real system are defined by $w_0(x) = z_0(x) = 1$, while the ones for the observer are arbitrarily chosen. It appears that, for this example, the observer state converges to the real state. This motivates further investigations. Showing convergence of the late-lumping observers, as well as the stability properties of the output feedback control law, will be the purpose of future work.

Appendix B

Convexity properties of the robustness domain

In this appendix, we give an interesting extension to the results proved Chapter 8 that is, knowing that the uncertain system (8.36)-(8.39),with the observer (8.4)-(8.7) and the control law (8.10) is exponentially stable for a specific finite number of sets of uncertainties is sufficient to conclude to exponential stability for any set of smaller uncertainties. The proof of this convexity property is based on an operator framework and on Lumer-Phillips' theorem. In what follows we denote $\langle \cdot, \cdot \rangle$ the usual scalar product in $(L^2([0,1]))^6 \times \mathbb{R}$.

B.1 Operator formulation of the problem

Let us consider the uncertain system (8.36)-(8.39), with the observer (8.4)-(8.7) and the control law (8.10). Let us consider $\delta_0 > 0$ and $\delta_1 > 0$ the two delays acting respectively on the actuation and on the measurements. Representing them as transport equations [KS08a], this system rewrites as

$$\begin{cases} \partial_t u(t,x) + \bar{\lambda}\partial_x u(t,x) = \bar{\sigma}^{+-}(x)v(t,x), \\ \partial_t v(t,x) - \bar{\mu}\partial_x v(t,x) = \bar{\sigma}^{-+}(x)u(t,x), \\ \partial_t \hat{u}(t,x) + \lambda\partial_x \hat{u}(t,x) = \sigma^{+-}(x)\hat{v} - P^{+}(x)\left(\hat{u}(t,1) - z_1(t,1)\right), \\ \partial_t \hat{v}(t,x) - \mu\partial_x \hat{v}(t,x) = \sigma^{-+}(x)\hat{u} - P^{-}(x)\left(\hat{u}(t,1) - z_1(t,1)\right), \\ \partial_t z_1(t,x) + \frac{1}{\delta_1}\partial_x z_1(t,x) = 0, \quad \partial_t z_2(t,x) + \frac{1}{\delta_0}\partial_x z_2(t,x) = 0, \\ \partial_t \eta(t) = z_1(t,1), \end{cases} \tag{B.1}$$

evolving in $\{(t,x) | \quad t>0,\ x \in [0,1]\}$, along with the following linear boundary conditions

$$\begin{aligned} &u(t,0) = \bar{q}v(t,0), \quad v(t,1) = \bar{\rho}u(t,1) + (1+\delta_V)z_2(t,1), \\ &\hat{u}(t,0) = q\hat{v}(t,0), \quad \hat{v}(t,1) = \rho(1-\epsilon)\hat{u}(t,1) + \rho\epsilon z_1(t,1) + z_2(t,0), \\ &z_1(t,0) = u(t,1), \\ &z_2(t,0) = k_I\eta(t) - \tilde{\rho}(1-\epsilon)\hat{u}(t,1) - \tilde{\rho}\epsilon z_1(t,1) + \int_0^1 K_u(\xi)\hat{u}(t,\xi) + K_v(\xi)\hat{v}(t,\xi)d\xi, \end{aligned} \tag{B.2}$$

where we have denoted

$$K_u(\xi) = K^{vu}(1,\xi) - (\rho - \tilde{\rho})K^{uu}(1,\xi) - k_I l_1(\xi) + k_I(\int_\nu^1 \left(K^{uu}(\nu,\xi)l_1(\nu) + K^{uv}(\nu,\xi)l_2(\nu)\right)d\nu),$$

$$K_v(\xi) = K^{vv}(1,\xi) - (\rho - \tilde{\rho})K^{uv}(1,\xi) - k_I l_2(\xi) + k_I(\int_\nu^1 \left(K^{vu}(\nu,\xi)l_1(\nu) + K^{vv}(\nu,\xi)l_2(\nu)\right)d\nu).$$

Multiplying formally (B.1) by smooth test functions, we obtain the weak formulation of the equation (see Definition 2.1.1). We consider the boundary couplings as source terms combined with a Dirac's distribution. More precisely, let us define $D(A)$ as

$$D(A) = \{w \in (H^1(0,1))^6 \times \mathbb{R} \mid w_1(0) = w_2(1) = w_3(0) = w_4(1) = 0,\\ w_5(0) = w_1(1), w_6(0) = 0\}.$$

Consider the operator $A : D(A) \subset (L^2(0,1))^6 \times \mathbb{R} \to (L^2(0,1))^6 \times \mathbb{R}$ such that for all $w \in D(A)$ we have

$$\begin{aligned}
A_1(w) &= -\bar{\lambda}\partial_x(w_1) + \bar{\sigma}^{+-}(x)w_2 + \bar{\lambda}\bar{q}w_2(0)\boldsymbol{\delta}_0(x),\\
A_2(w) &= \bar{\mu}\partial_x(w_2) + \bar{\sigma}^{-+}(x)w_1 + \bar{\mu}\bar{\rho}w_1(1)\boldsymbol{\delta}_1(x) + \bar{\mu}(1+\delta_V)\boldsymbol{\delta}_1(x)w_6(1),\\
A_3(w) &= -\lambda\partial_x(w_3) + \sigma^{+-}(x)w_4 - P^+(x)(w_3(1) - w_5(1)) + \lambda q w_4(0)\boldsymbol{\delta}_0(x),\\
A_4(w) &= \lambda\partial_x(w_4) + \sigma^{-+}(x)w_3 - P^-(x)(w_3(1) - w_5(1)) + (\mu\rho(1-\epsilon) + \rho\epsilon w_5(1) + w_6(0))\boldsymbol{\delta}_1(x),\\
A_5(w) &= -\frac{1}{\delta_1}\partial_x(w_5) + \boldsymbol{\delta}_0(x)w_1(1),\\
A_6(w) &= -\frac{1}{\delta_0}\partial_x(w_6) + \boldsymbol{\delta}_0(x)(-\tilde{\rho}(1-\epsilon)w_3(1) - \tilde{\rho}\epsilon w_5(1) - \int_0^1 (K_u(\nu)w_3(\nu) + K_v(\nu)w_4(\nu))\,d\nu),\\
A_7(w) &= w_5(1),
\end{aligned} \tag{B.3}$$

where we have denoted $\boldsymbol{\delta}_s(\cdot)$ the Dirac-distribution in s. The system (B.1) can be rewritten as

$$\frac{d}{dt}w = Aw. \tag{B.4}$$

If the delay δ_0 or the delay δ_1 is equal to zero, we still use these (abuse of) notations (even if there is a division by δ_0 or δ_1 in the equations of (B.3)). Actually, one must aware that if $\delta_1 = 0$, then we choose $w_5(1) = w_1(1)$ and if $\delta_0 = 0$, then we choose $w_6(1) = -\tilde{\rho}(1-\epsilon)w_3(\delta_1) - \tilde{\rho}\epsilon w_5(1) - \int_0^1 K_u(\nu)w_3(\nu) + K_v(\nu)w_4(\nu)d\nu$.

B.2 Elementary convexity property

In this section, we prove the following result: knowing that the semigroup associated to equation (B.4) is exponentially stable for a given set of uncertainties and delays, we prove that it remains exponentially stable if one (and one only) uncertainty or delay is smaller. We first state the following preliminary result.

Lemma B.2.1.

> Let us consider two operators A_1 and A_2 that generate exponentially stable semigroups. Then, for every $s \in [0,1]$, the operator $(1-s)A_1 + sA_2$ generates an exponentially stable semigroup.

Proof : Since both A_1 and A_2 generate exponentially stable semigroups T_1 and T_2, there exist $\omega_1 < 0, \omega_2 < 0$, $M_1 > 0$ and $M_2 > 0$ such that

$$||exp(A_1 t)|| \leq M_1 \exp(\omega_1 t), \quad ||exp(A_2 t)|| \leq M_2 \exp(\omega_2 t).$$

Let us denote $\omega = \max(\omega_1, \omega_2) < 0$, $M = 2\max(M_1, M_2) > 0$ and T the semigroup generated by $(1-s)A_1 + sA_2$. By linearity, we immediately have

$$||T(t)|| \leq (1-s)||T_1(t)|| + s||T_2(t)|| \leq M\exp(\omega t).$$

This concludes the proof. ■

A direct consequence of this lemma is given bellow

Lemma B.2.2.

Let us assume that the semigroup associated to system (B.4) is exponentially stable in presence of a given set of uncertainties and delays. If it remains exponentially stable replacing δ_λ by $-\delta_\lambda$, then it remains exponentially stable if δ_λ is replaced by $s\delta_\lambda$ (with $s \in [-1, 1]$). Similar results hold substituting δ_λ by δ_μ, δ_q, δ_ρ, δ_V, δ_0 or δ_1.

Proof : Let us consider a set of uncertainties and delays. The operator A associated to (B.3) generates an exponentially stable semigroup by assumption. Similarly, the operator A^- associated to (B.3) but for which the uncertainty δ_λ acting on λ has been replaced by $-\delta_\lambda$ generates an exponentially stable semigroup. Let us now consider $s \in [-1, 1]$ and denote A_s the operator obtained from (B.3) but for which the uncertainty δ_λ acting on λ has been replaced by $s\delta_\lambda$. Since $A_s = (\frac{1+s}{2})A + \frac{1-s}{2}A^-$, we obtain the expected result using Lemma B.2.1. The other cases can be proved in a similar way. ■

Let us now consider the cases of the uncertainties acting on σ^{+-} and σ^{-+}. Since these uncertainties depend on space, the convexity property is slightly different. More precisely, we have the following lemma.

Lemma B.2.3.

Let us assume that the semigroup associated to the system (B.4) is exponentially stable in presence of a given set of uncertainties and delays. If this semigroup remains exponentially stable when the continuous uncertainty $\delta_{\sigma^{+-}}(x)$ acting on $\sigma^{+-}(x)$ is replaced by the continuous uncertainty $-\delta_{\sigma^{+-}}(x)$, then it remains exponentially stable replacing $\delta_{\sigma^{+-}}(x)$ by $s(x)\delta_{\sigma^{+-}}(x)$ where $s(x)$ is a continuous function whose image is embedded in $[-1, 1]$. The same result holds substituting σ^{-+} to σ^{+-}.

Proof : Let us consider a given set of uncertainties. We denote A the operator associated to (B.3). We also denote A^- the operator obtained replacing $\delta_{\sigma^{+-}}(x)$ by the continuous uncertainty $-\delta_{\sigma^{+-}}(x)$. Since A^- and A are exponentially stable, their decay bound is strictly negative. Thus, there exists $\omega < 0, M > 0$ such that $||\exp(At)|| \leq M\exp(\omega t)$ and $||\exp(A^-t)|| \leq M\exp(\omega t)$. Thus [RR06, Theorem 12.21], there exists an equivalent norm on $D(A)$ such that in the operator norm corresponding to this new norm, A generates a contraction semigroup. For the sake of simplicity and without any loss of generality, we can now assume that $M = 1$. Due to Lumer-Phillip's (see Theorem 2.1.1), $D(A)$ is dense, A and A^- are closed and dissipative (and so are their adjoints). Let us now consider a continuous function $s : [0, 1] \to [-1, 1]$ and denote A_s the operator associated to (B.3) in presence of the uncertainty $s(x)\delta_{\sigma^{+-}}(x)$ acting on $\sigma^{+-}(x)$. In what follows, we prove that A_s satisfy all the hypothesis of Lumer-Phillips' theorem. Adjusting the proof of Lemma B.2.2, this result holds for $A_0 = \frac{1}{2}(A + A^-)$. Using the expression of A_0 given in (B.3), we have for all $w \in (H^1([0, 1]))^6 \times \mathbb{R}$

$$\langle A_s w, w\rangle = \langle A_0 w, w\rangle + \int_0^1 s(x)\delta_{\sigma^{+-}}(x)w_1(x)w_2(x)dx$$

Density of $D(A_s)$: since A_s and A have the same domain of definition, $D(A_s)$ is dense.

Dissipativity of A_s: let us consider w an arbitrary element of $(H^1([0, 1]))^6 \times \mathbb{R}$. We start proving the dissipativity property of A_s assuming that w_1, w_2 and δ_σ are polynomial functions. In this case, their product $P(x)$ is a polynomial function on $[0, 1]$. Let us assume that P is not equal to zero (otherwise the result is straightforward). Thus, its sign changes a finite number of times on $]0, 1[$. Let us denote N the number of times P cancels on $]0, 1[$ and a_i $(1 \leq i \leq N)$ the points of $]0, 1[$ for which $P(x) = 0$. We can assume that $a_1 < a_2 < \cdots < a_N$. We denote $a_0 = 0$ and $a_{N+1} = 1$. Let us define

$$I_1 = \bigcup_i [a_{2i}, a_{2i+1}], \quad \text{with } 2i \leq N, \tag{B.5}$$

$$I_2 = \bigcup_i [a_{2i+1}, a_{2i+2}], \quad \text{with } 2i + 1 \leq N. \tag{B.6}$$

By construction P is either positive on I_1 and negative on I_2 or positive on I_2 and negative on I_1. We have $I_1 \cup I_2 = [0, 1]$. We assume that P is positive on I_2 and negative on I_1 (the proof can easily be adjusted for the other case). We define $\epsilon^* = \min |a_{i+1} - a_i|$ $(0 \leq i \leq N)$. For all $1 \leq i \leq 7$ and for all k such that $2 \leq k+2 \leq N$, let us now define the function $z \in (H^1([0,1]))^6 \times \mathbb{R}$ f by

$$\begin{aligned} z_i(x) = & w_i(x)\mathbb{1}_{[a_0,a_1-\epsilon]}(x) + w_i(a_{2k+1}-\epsilon)(1 - \frac{1}{2\epsilon}(x - a_{2k+1}+\epsilon))\mathbb{1}_{[a_{2k+1}-\epsilon,a_{2k+1}+\epsilon]}(x) \\ & + w_i(x)\mathbb{1}_{[a_{2k+2}+\epsilon,a_{2k+3}-\epsilon]}(x) + w_i(a_{2k+2}-\epsilon)(1 + \frac{1}{2\epsilon}(x - a_{2k+2}-\epsilon))\mathbb{1}_{[a_{2k+2}-\epsilon,a_{2k+2}+\epsilon]}(x) \\ & + w_i(x)\mathbb{1}_{[a_{2N}+\epsilon,a_{2N+1}]\cap I_1}(x), \end{aligned}$$

where $0 < \epsilon < \epsilon^*$. The function z corresponds roughly to P on I_1 and is equal to zero on I_2. Linear junctions have been added to avoid brutal jumps. One can easily check that with this definition $z \in (H^1([0,1]))^6 \times \mathbb{R}$. The function z can be rewritten in the simpler way

$$z(x) = w(x)\mathbb{1}_{I_1}(x) + f_{I_\epsilon}(x)\mathbb{1}_{I_\epsilon}(x),$$

where I_ϵ is a measurable set that does not contain 0 and 1, whose measure tends to zero if ϵ goes to zero and with f_{I_ϵ} a $H^1([0,1])^6 \times \mathbb{R}$ function. Similarly, denoting $\bar{z} = w - z$, we have

$$\bar{z}(x) = w(x)\mathbb{1}_{I_2}(x) + \bar{f}_{\bar{I}_\epsilon}(x)\mathbb{1}_{\bar{I}_\epsilon}(x), \tag{B.7}$$

where $\bar{I}_\epsilon$ is a measurable set that does not contain 0 and 1, whose measure tends to zero if ϵ goes to zero and with $\bar{f}_{\bar{I}_\epsilon}$ a $H^1([0,1])^6 \times \mathbb{R}$-function. Thus, there exist a measurable set $\tilde{I}_\epsilon$ that does no contain 0 and 1, whose measure tends to zero if ϵ goes to zero and a $H^1([0,1])^6$-function $\tilde{f}_{\tilde{I}_\epsilon}$ such that

$$w(x) = z(x) + \bar{z}(x) + \tilde{f}_{\tilde{I}_\epsilon}(x)\mathbb{1}_{\tilde{I}_\epsilon}(x). \tag{B.8}$$

One can notice that due to the expression of A_0 given in (B.3), we have $\langle A_0 z, \bar{z}\rangle = 0$ and $\langle A_0 \bar{z}, z\rangle = 0$. Using the dissipativity property of the operator A^-, we have the existence of a constant $k_0 < 0$ such that

$$\begin{aligned} & \langle A_0 z, z\rangle - \int_0^1 \delta_{\sigma^{+-}}(x) z_1(x) z_2(x) dx < k_0\langle z, z\rangle \\ \Leftrightarrow & \langle A_0 z, z\rangle - \int_{I_1} \delta_{\sigma^{+-}}(x) z_1(x) z_2(x) dx - \int_{I_\epsilon} \delta_{\sigma^{+-}}(x) z_1(x) z_2(x) dx < k_0\langle z, z\rangle. \end{aligned}$$

The range of the function $\frac{1-s(x)}{2}$ belongs to $[0,1]$. Thus, since $\delta_{\sigma^{+-}}(x) z_1(x) + z_2(x)$ is negative on I_1, we have

$$\langle A_0 z, z\rangle + \int_{I_1} \frac{1-s(x)}{2}\delta_{\sigma^{+-}}(x) z_1(x) z_2(x) dx - \int_{I_\epsilon} \delta_{\sigma^{+-}}(x) z_1(x) z_2(x) dx < k_0\langle z, z\rangle.$$

Since the range of the function $\frac{1+s(x)}{2}$ belongs to $[0,1]$, the function $\frac{1+s(x)}{2}\delta_{\sigma^{+-}}(x)z_1(x)+z_2(x)$ is negative on I_1. Thus,

$$\begin{aligned} & \langle A_0 z, z\rangle + \int_{I_1} s(x)\delta_{\sigma^{+-}}(x) z_1(x) z_2(x) dx - \int_{I_\epsilon} \delta_{\sigma^{+-}}(x) z_1(x) z_2(x) dx \\ = & \langle A_0 z, z\rangle + \int_{I_1} \frac{1+s(x)}{2}\delta_{\sigma^{+-}}(x) z_1(x) z_2(x) dx + \int_{I_1} \frac{1-s(x)}{2}\delta_{\sigma^{+-}}(x) z_1(x) z_2(x) dx \\ & - \int_{I_\epsilon} \delta_{\sigma^{+-}}(x) z_1(x) z_2(x) dx < k_0\langle z, z\rangle. \end{aligned} \tag{B.9}$$

Similarly, using the dissipativity of the operator A and similar computations, we obtain the existence of a constant k_1.

$$\langle A_0 \bar{z}, \bar{z}\rangle + \int_{I_2} s(x)\delta_{\sigma^{+-}}(x)\bar{z}_1(t,x)\bar{z}_2(t,x)dx + \int_{\bar{I}_\epsilon} \delta_{\sigma^{+-}}(x)\bar{z}_1(t,x)\bar{z}_2(t,x)dx < k_1\langle \bar{z}, \bar{z}\rangle. \tag{B.10}$$

Using (B.8) and the expression of the operator A_{nom}, we obtain

$$\langle A_0 w, w\rangle = \langle A_0(z + \bar{z} + \tilde{f}_{\tilde{I}_\epsilon}(\cdot)\mathbb{1}_{\tilde{I}_\epsilon}(\cdot)), z + \bar{z} + \tilde{f}_{\tilde{I}_\epsilon}(\cdot)\mathbb{1}_{\tilde{I}_\epsilon}(\cdot)\rangle = \langle A_0 z, z\rangle + \langle A_0 \bar{z}, \bar{z}\rangle + h_\epsilon,$$

where h_ϵ tends to zero when ϵ goes to zero. Combining (B.9) and (B.10), we obtain

$$\begin{aligned} \langle A_0 w, w\rangle + \int_0^1 s(x)\delta_{\sigma^{+-}}(x) w_1(x) w_2(x) dx + h_\epsilon + \int_{\bar{I}_\epsilon} \delta_{\sigma^{+-}}(x)\bar{z}_1(x)\bar{z}_2(x)dx \\ - \int_{I_\epsilon} \delta_{\sigma^{+-}}(x) z_1(x) z_2(x) dx < \max(k_1, k_2)\langle w, w\rangle. \end{aligned}$$

Choosing ϵ small enough, one can easily prove that there exists a constant $k_3 < 0$ such that

$$\langle A_0 w, w\rangle + \int_0^1 s(x)\delta_{\sigma^{+-}}(x) w_1(t,x) w_2(t,x)dx < k_3\langle w, w\rangle, \tag{B.11}$$

which is the expected result. Let us now consider the case where w_1 is not a polynomial function. Since $w_1 \in H^1([0,1])$, it can be approximated by a sequence of polynomial functions $\tilde{w}_n$, i.e for all $\epsilon_1 > 0$ there exists $N > 0$ such that for any $n \geq N$, $||w_1 - \tilde{w}_1|| < \epsilon_1$. Using inequality (B.11), it is straightforward to prove that for ϵ_1 small enough, there exists $k_4 < 0$ such that

$$\langle A_0 w, w\rangle + \int_0^1 s(x)\delta_{\sigma^{+-}}(x) w_1(t,x) w_2(t,x)dx < k_4\langle w, w\rangle. \tag{B.12}$$

The same proof can be repeated for the case where w_2 or δ_σ are not polynomial function. This proves the dissipativity of the operator A_s. The dissipativity of its adjoint can be proved in a similar way.

A_s is closed Since $A_0 = \frac{1}{2}(A + A^-)$, the property obviously holds for A_0. Let us consider a sequence $w_p \in D(A_s)$ such that w_p converges to w and $A_s w_p$ converges to y. We have (using the expression of the operator A given by (B.3))

$$A_s w_p = A_0 w_p + \begin{pmatrix} s(x)\delta_{\sigma^{+-}}(x)(w_p)_2(x) \\ 0_{6\times 1} \end{pmatrix}.$$

Since w_p converges to w, we immediately get that $A_0 w_p$ converges to $y_1 = y - \begin{pmatrix} s(x)\delta_{\sigma^{+-}}(x) w_2(x) \\ 0_{6\times 1} \end{pmatrix}$. Thus, since A_0 is closed, $w \in D(A_0)$ and $A_0 w = y_1$. It implies that $w \in D(A_s)$ and $A_s w = y$. This proves that A_s is closed. Using Lumer-Phillips (see Theorem 2.1.1) concludes the proof. ■

B.3 Global convexity property

We conclude this section giving the following general convexity theorem.

Theorem B.3.1.

Consider a set of uncertainties and delays $(\delta_\lambda, \delta_\mu, \delta_{\sigma^{+-}}(x), \delta_{\sigma^{-+}}(x), \delta_q, \delta_\rho, \delta_U, \delta_0, \delta_1)$. Assume that for all set $r = (r_1, \cdots, r_9) \in \{-1,1\}^7 \times \{0,1\}^2$, the system (8.36)-(8.39) along with the observer (8.4)-(8.7) and the control law (8.10) with the uncertain parameters

$$\begin{aligned}
&\bar{\lambda} = \lambda + r_1\delta_\lambda,\ \bar{\mu} = \mu + r_2\delta_\mu,\ \bar{q} = \lambda + r_3\delta_q,\ \bar{\rho} = \rho + r_4\delta_\rho,\\
&\delta_U = r_5\delta_U,\ \delta_0 = r_8\delta_0,\ \delta_1 = r_9\delta_1,\\
&\bar{\sigma}^{+-}(x) = \sigma^{+-}(x) + r_6\delta_{\sigma^{+-}}(x),\\
&\bar{\sigma}^{-+}(x) = \sigma^{-+}(x) + r_7\delta_{\sigma^{-+}}(x),
\end{aligned}$$

is exponentially stable. Then for all $(s_1, \cdots, s_5) \in [-1,1]^5 \times [0,1]^2$, for all continuous functions $s_6 : [0,1] \to [-1,1]$ and $s_7 : [0,1] \to [-1,1]$, for all $(s_8, s_9) \in [0,1]^2$ the system (8.36)-(8.39) along with the observer (8.4)-(8.7) and the control law (8.10), for which the real parameters and delays are given by

$$\begin{aligned}
&\bar{\lambda} = \lambda + s_1\delta_\lambda,\ \bar{\mu} = \mu + s_2\delta_\mu\ \bar{q} = \lambda + s_3\delta_q,\ \bar{\rho} = \rho + s_4\delta_\rho,\\
&\delta_U = s_5\delta_U,\ \delta_0 = s_8\delta_0,\ \delta_1 = s_9\delta_1,\\
&\bar{\sigma}^{+-}(x) = \sigma^{+-}(x) + s_6(x)\delta_{\sigma^{+-}}(x),\\
&\bar{\sigma}^{-+}(x) = \sigma^{-+}(x) + s_7(x)\delta_{\sigma^{-+}}(x),
\end{aligned}$$

is exponentially stable.

Proof : The proof is a direct consequence of the results stated in the previous section. ■

Remark B.3.1 *This theorem means that knowing that the system remains exponentially stable for a set of uncertainties located on the vertices of an k-orthotope (i.e a hyper rectangle) is sufficient to conclude to exponential stability for any set of uncertainties and delays located inside the k-orthotope.*

Résumé

Les systèmes d'Equations aux Dérivées Partielles Hyperboliques Linéaires du Premier Ordre (EDPs HLPO) permettent de modéliser des dynamiques impliquant des phénomènes de transport ou des lois de conservation. Ils apparaissent, par exemple, lors de la modélisation de problèmes de trafic routier, d'échangeurs de chaleurs, ou de problèmes multiphasiques. Différentes approches ont été proposées pour stabiliser ou observer de tels systèmes. Parmi elles, la méthode de backstepping consiste à transformer le système originel en un système découplé pour lequel la synthèse de la loi de commande est plus simple. Les contrôleur obtenus par cette méthode sont explicites. Dans la première partie de cette thèse, nous présentons des résultats généraux de théorie des systèmes. Plus précisément, nous résolvons les problèmes de stabilisation en temps fini pour une classe générale d'EDPs HLPO. Le temps de convergence minimal atteignable dépend du nombre d'actionneurs disponibles. Les observateurs associés à ces contrôleurs (nécessaires pour envisager une utilisation industrielle de tels contrôleurs) sont obtenus via une approche duale. Un des avantages importants de l'approche considérée dans cette thèse est de montrer que l'espace généré par les solutions de l'EDPs HLPO considérée est isomorphe à l'espace généré par les solutions d'une système neutre à retards distribués. Dans la seconde partie de cette thèse, nous montrons la nécessité d'un changement de stratégie pour résoudre les problèmes de contrôle robuste. Ces questions surviennent nécessairement lorsque sont considérées des applications industrielles pour lesquelles les différents paramètres du système peuvent être mal connus, pour lesquelles des dynamiques peuvent avoir été négligées, de même que des retards agissant sur la commande ou sur la mesure, ou encore pour lesquelles les mesures sont bruitées. Nous proposons ainsi quelques modifications sur les lois de commandes précédemment développées en y incorporant plusieurs degrés de liberté permettant d'effectuer un compromis entre performance et robustesse. L'analyse de stabilité et de robustesse sous-jacente est rendue possible en utilisant l'isomorphisme précédemment introduit.

Mots Clés

EDPs hyperboliques, contrôlabilité, observabilité, backstepping, systèmes neutres, retard distribués, robustesse

Abstract

Linear First-Order Hyperbolic Partial Differential Equations (LFOH PDEs) represent systems of conservation and balance law and are predominant in modeling of traffic flow, heat exchanger, open channel flow or multiphase flow. Different control approaches have been tackled for the stabilization or observation of such systems. Among them, the backstepping method consists to map the original system to a simpler system for which the control design is easier. The resulting controllers are explicit.
In the first part of this thesis, we develop some general results in control theory. More precisely, we solve the problem of finite-time stabilization of a general class of LFOH PDEs using the backstepping methodology. The minimum stabilization time reachable may change depending on the number of available actuators. The corresponding boundary observers (crucial to envision industrial applications) are obtained through a dual approach. An important by-product of the proposed approach is to derive an explicit mapping from the space generated by the solutions of the considered LFOH PDEs to the space generated by the solutions of a general class of neutral systems with distributed delays. This mapping opens new prospects in terms of stability analysis for LFOH PDEs, extending the stability analysis methods developed for neutral systems.
In the second part of the thesis, we prove the necessity of a change of strategy for robust control while considering industrial applications, for which the major limitation is known to be the robustness of the resulting control law to uncertainties in the parameters, delays in the loop, neglected dynamics or disturbances and noise acting on the system. In some situations, one may have to renounce to finite-time stabilization to ensure the existence of robustness margins. We propose some adjustments in the previously designed control laws by means of several degrees of freedom enabling trade-offs between performance and robustness. The robustness analysis is fulfilled using the explicit mapping between LFOH PDEs and neutral systems previously introduced.

Keywords

hyperbolic PDEs, controllability, observability, backstepping, neutral systems, distributed delays, robustness