Implementational aspects of code-based cryptography

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Introduction
My research domain is cryptology.

**Cryptology** (&omicron;κρυπτός (adj. secret); &omicron;λόγος (n. sayings or body of knowledge)): is the science of making communication incomprehensible to everyone but those who have a right to access it. It has two wings,

→ **Cryptography**: concerns with the system design.

→ **Cryptanalysis**: concerns with system analysis (breaking?).

Two types of systems,

- Symmetric key; same key is used for encryption and decryption.
- Asymmetric key; different keys for encryption and decryption.
Asymmetric systems

The concept of public key (asymmetric) cryptography came to existence to address the following two difficult problems associated with the secret key (symmetric) cryptosystems.

- Key distribution and
- Non-repudiation.

To address both problems, Diffie and Hellman achieved one of the greatest breakthrough in cryptography in 1976. The proposed scheme used a pair of different keys in contrast to one shared key and it was based on mathematical functions instead of substitution and permutation. The use of two keys has profound consequences in facilitating key distribution and providing digital signature.
Public Key Encryption schemes

All the systems have three distinct steps.

1. **Key generation**, generates a set of *public* and *private* keys.

2. **Encryption**, encryption of the *plain text* with the help of *public* key.

3. **Decryption**, decryption of the ciphertext with the help of *private* key, which is hard to deduce from the *public* key.

The security of these systems depend upon some underlying hard problem.
Almost solely used public key encryption scheme is RSA, it is based on the problem of factorization.

Next possible alternative is Elliptic curve systems, because of the better performance for high security levels.

Alternative systems are always of interest. They are attracting more attention as quantum alternatives.

Thus, subjects like lattice based crypto, code based crypto and multivariate crypto are getting more interest.

My research area is code based cryptography.
McEliece encryption scheme

An asymmetric key system, proposed by Robert McEliece in 1978. Based on the well known and well studied problem of “hardness of decoding a general linear code”.

*Private key* is the description of \(t\)-error-correcting code with efficient decoding algorithm (binary Goppa code).

*Public key* is the disguised form of the code selected, as a general linear code (pseudo-randomness of Goppa code).
McEliece encryption scheme cont...

▲ One of the fastest public key system (160 Mbits/s).

▲ Secure; resisted cryptanalysis till date.

▲ Remain secure; will resist the threat posed by quantum computers.

▲ Good scalability; security grows exponentially if we raise parameters.

▼ Rather large key size (100 KB; 88 bit security).

▼ Lesser information rate.

▼ Rarely used in practice, implementation has not been studied before with full state of the art.
Overview of McEliece scheme

Let $\mathcal{F}$ be a family of binary $t$-error-correcting $(n, k)$ codes. We describe McEliece cryptosystem as,

- **public key**: a $k \times n$ binary *generator matrix* $G$ of a code $\in \mathcal{F}$.
- **secret key**: a decoder for the said code.
- **encryption**: $x \mapsto xG + e$ where, hamming weight of $e$ is $w_H(e) \leq t$.
- **decryption**: Essentially the decoding process.

It was introduced by R. McEliece in 1978 with irreducible binary Goppa codes.
Brief time line

- Niederreiter proposed a variant! Knapsack type. 1986.

<table>
<thead>
<tr>
<th>Decoding attacks</th>
<th>Lee and Brickell - ’88.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Information set</td>
<td>Leon - ’88.</td>
</tr>
<tr>
<td>decoding)</td>
<td>J. van Tilburg - ’88.</td>
</tr>
</tbody>
</table>

- Attacks:

<table>
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<th>Structural attacks</th>
<th>Support splitting algorithm</th>
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<td>(Goppa codes)</td>
<td>P. Loidreau and N. Sendrier - ’01.</td>
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</table>

- D. Bernstein, T. Lange and C. Peters - Successful attack, ’08.
- N. Courtois, M. Finiasz and N. Sendrier - Signature Scheme, ’01.
Niederreiter cryptosystem

Proposed by Harald Niederreiter in 1986 a variant of McEliece scheme. Uses parity check matrix instead of generator matrix. Embeds information in the error pattern. The security reduction is identical as the McEliece scheme. Theoretically it is the **dual** system of McEliece.
Our objectives

To study the scope of McEliece system in real life; its practicality, efficiency and security aspects.

- Full state of the art implementation of McEliece public key cryptosystem.
- Reference platform.
- Algorithmic improvement.
- Optimized implementation.
The Hybrid McEliece Encryption Scheme (HyMES)
Our proposal

1. Generator matrix in row echelon form, reduces key size.
2. Encode information in the error, increases the information rate.

In spite of the changes we preserve security.

We improve,

- Key size.
- Encryption speed.
- Information rate.

As we mix the ideas of McEliece and Niederreiter schemes, we call our system Hybrid.
HyMES Description

We define an injective mapping \( \varphi : \{0, 1\}^\ell \to W_{n,t} \), a set of \( n \) length words weight \( t \). Both \( \varphi \) and \( \varphi^{-1} \) should be easy to compute.

- **Key generation:**
  - We construct the necessary tools to design and operate on \( \text{GF} \) along with polynomial representation.
  - generate a support \( L = (\alpha_1, \ldots, \alpha_n) \) of \( n \) distinct elements of \( \mathbb{F}_{2^m} \),
  - generate a monic irreducible generator polynomial \( g(z) \in \mathbb{F}_{2^m}[z] \) of degree \( t \).
  - The secret key is the pair \( (L, g) \) (i.e. the Goppa code \( \Gamma(L, g) \) and its decoder)
  - The public key is a binary \( k \times (n-k) \) matrix \( R \) where \( G = (Id \mid R) \) is a generator matrix of \( \Gamma(L, g) \) in row echelon form.
Description cont...

- Encryption: the plaintext is in $\{0, 1\}^k \times \{0, 1\}^\ell$ and the ciphertext in $\{0, 1\}^n$

\[
\{0, 1\}^k \times \{0, 1\}^\ell \quad \longrightarrow \quad \{0, 1\}^n
\]

\[(x, x') \quad \longmapsto \quad (x, xR) + \varphi(x')\]

- Decryption: the ciphertext has the form $y = xG + e$, with $e = \varphi(x')$ of Hamming weight $\leq t$. Applying the decoder of $\Gamma(L, g)$ on $y$ will provide $x$ and $x' = \varphi^{-1}(e)$.

**Remark:** We try to chose $\ell$, as close as $\log\binom{n}{t}$ for better information rate and security.
Security

- One Way Encryption (OWE) scheme: A public key encryption scheme is a One Way Encryption scheme if the probability of success of any of its adversary running in polynomial time is negligible.

In practice, one needs more than just an OWE scheme.

- McE, though it is OWE, is vulnerable to many attacks.
- Given perfect hash functions exists, there are generic conversions which, starting from an OWE scheme, provide a scheme resistant against adaptative chosen ciphertext attack.

We can prove, given the two following assumptions, the hybrid McEliece encryption scheme is OWE.
Security assumptions

- Hardness of decoding in the average case: The success probability
  \[ \Pr(\mathcal{A}(xG + e, G) = (x, e)) \]
  of any adversary \( \mathcal{A} \) running in polynomial time is negligible.

- Pseudo-randomness of binary Goppa Codes: there exists no efficient distinguisher for Goppa codes. In other words, the generator matrix of a Goppa code looks random.

**Theorem 1** Under the two assumptions stated above, the Hybrid McEliece Encryption Scheme is secure.
Bird’s eye view comparison

- Parameters: \( m, t, n \leq 2^m \), \( W_{n,t} \) set of \( n \) length words of weight \( t \).
- Secret key: The pair \((L, g)\).
  - generator: \( g(z) \in \mathbb{F}_{2^m}[z] \), monic irreducible of degree \( t \).
  - support: \( L \) a sequence of \( n \) distinct elements in \( \mathbb{F}_{2^m}[z] \).
  - code: \( \Gamma(L, g) \), \( t \) error correcting Goppa code.
  - decoder: \( \Psi : \{0, 1\}^n \rightarrow \Gamma(L, g) \), a decoding procedure for \( \Gamma \).
- Public key:
  - A random generator matrix \( G \) of \( \Gamma(L, g) \).
  - A systematic generator matrix \( G = (Id|R) \) of \( \Gamma(L, g) \).
Cont...

- **Encryption:**
  \[
  \{0, 1\}^k \rightarrow \{0, 1\}^n \quad \{0, 1\}^k \times \{0, 1\}^\ell \rightarrow \{0, 1\}^n
  \]

  \[
  x \rightarrow xG + e \quad (x, x') \rightarrow (x, xR) + \varphi(x')
  \]

  Where \(e\) is random error of length \(n\) and weight \(t\).

- **Decryption:**
  \[
  \{0, 1\}^n \rightarrow \{0, 1\}^k \quad \{0, 1\}^n \times \{0, 1\}^\ell \rightarrow \{0, 1\}^n
  \]

  \[
  y \rightarrow \Psi(y)G^* \quad \{0, 1\}^n \times \{0, 1\}^\ell \rightarrow (x, x')
  \]

  Where \(GG^* = Id\) is the identity matrix.

  \(x\) is the first \(k\) bits of \(\Psi(y)\) and \(x' = \varphi^{-1}(y - \Psi(y))\).
Implementation
Running time overview

**Key generation** is one time process, it is fast enough.

**Encryption** is very fast.

For **decryption**, which is the *bottleneck* of the scheme, there are 3 steps and their time consumption are,

- Syndrome computation $O(nt)$; 11.3%.
- Key equation solving $O(t^2)$; 12.0%.
- Error locator polynomial root finding
  - BTA $O(mt^2)$; 72.1%
  - Chien search $O(nt)$; 85.6%
- Other tasks 4.6%
Binary Goppa Code

Let $m > 0$, $n \leq 2^m$ and $a = (a_1, ..., a_n) \in \mathbb{F}_2^n$.

The $n$-length binary Goppa code $\Gamma(L, g)$ is defined by:

- **Support** $L = (\alpha_1, ..., \alpha_n)$ $n$-tuple of distinct elements of $\mathbb{F}_{2^m}$;
- **Goppa polynomial** $g(z) \in \mathbb{F}_{2^m}[z]$, square-free, monic of degree $t > 0$ with no root in $L$.

$\Gamma(L, g)$ is a subfield subcode over $\mathbb{F}_2$ of a particular Goppa code over the binary field $\mathbb{F}_{2^m}$.

We have $a \in \Gamma(L, g)$ if and only if:

$$R_a(z) := \sum_{i=1}^{n} \frac{a_i}{z - \alpha_i} = 0 \text{ over } \mathbb{F}_{2^m}[z]/(g(z)).$$
Decoding Binary Goppa Codes

Let $e$, $x$, $y$ be $n$-length binary vectors. We have to find $x$, the sent codeword knowing $y = x + e$ where $y$ is the received word and $e$ the error word. We can correct up to $t$ errors.

**Algebraic decoding** is carried out in three steps:

1. **Syndrome computation**
   
   $$R_x(z) = R_e(z) = \sum_{i=1}^{n} \frac{e_i}{z - \alpha_i} \text{ over } \mathbb{F}_{2^m}[z]/(g(z)).$$

2. **Solving the Key Equation** to obtain the error locator polynomial
   
   $$R_e(z) \cdot \sigma_e(z) = \sigma'_e(z) \text{ over } \mathbb{F}_{2^m}[z]/(g(z)).$$

3. **Error Locator Polynomial Root Finding**
   
   $$\sigma_e(z) := \prod_{i=1}^{n} (z - \alpha_i)^{e_i}; \quad \sigma_e(\alpha_i) = 0 \Leftrightarrow e_i \neq 0.$$
Step 2 : Patterson algorithm

The Patterson algorithm solves the Goppa code key equation: given \( R(z) \) and \( g(z) \) in \( \mathbb{F}_{2^m}[z] \), with \( g(z) \) of degree \( t \) respectively, find \( \sigma(z) \) of degree \( t \) such that

\[
R(z)\sigma(z) = \frac{d}{dz}\sigma(z) \mod g(z)
\]

We write \( \sigma(z) = \sigma_0(z)^2 + z\sigma_1(z)^2 \). Since \( \frac{d}{dz}\sigma(z) = \sigma_1(z)^2 \), we have

\[
(1 + zR(z))\sigma_1(z)^2 = R(z)\sigma_0(z)^2 \mod g(z).
\]

Because \( g(z) \) is irreducible, \( R(z) \) can be inverted modulo \( g(z) \). We put \( h(z) = z + R(z)^{-1} \mod g(z) \) and we have

\[
h(z)\sigma_1(z)^2 = \sigma_0(z)^2 \mod g(z).
\]
Patterson algorithm cont...

The mapping $f(z) \mapsto f(z)^2 \mod g(z)$ is bijective and linear over $\mathbb{F}_2^{tm}$, there is a unique polynomial $S(z)$ such that $S(z)^2 = h(z) \mod g(z)$. We have

$$S(z)\sigma_1(z) = \sigma_0(z) \mod g(z).$$

The polynomial $\sigma_0(z), \sigma_1(z)$ are the unique solution of the equation

$$\begin{cases} 
S(z)\sigma_1(z) = \sigma_0(z) \mod g(z) \\
\deg \sigma_0 \leq t/2 \\
\deg \sigma_1 \leq (t-1)/2 
\end{cases}$$

(1)
Patterson algorithm cont...

The three steps of the algorithm are the following

1. Compute $h(z) = z + R(z)^{-1} \mod g(z)$ using the extended Euclidian algorithm.

2. Compute $S(z) = \sqrt{h(z)} \mod g(z)$
   If $s(z)$ such that $s(z)^2 = z \mod g(z)$ has been precomputed and $h(z) = h_0 + h_1 z + \ldots + h_{t-1} z^{t-1}$, we have
   \[
   S(z) = \sum_{i=0}^{(t-1)/2} h_{2i}^{2^{m-1}} z^i + \sum_{i=0}^{t/2-1} h_{2i+1}^{2^{m-1}} z^i s(z)
   \]

3. Compute $(\sigma_0(z), \sigma_1(z))$ using the extended Euclidian algorithm.

The polynomial $\sigma(z) = \sigma_0(z)^2 + z \sigma_1(z)^2$ is returned.
Step 3 : Find the roots efficiently

Several approaches are possible, their efficiency depends on the size of the parameters $m$ and $t$.

- **Chien search** computes roots by evaluating artfully the polynomial in all points of $L$. This method is recommended for hardware implementations and coding theory applications in which $m$ is small.

- **BTA** is a **recursive** algorithm using trace function properties. It is a faster method for secure parameters in McEliece-type cryptosystems.
Berlekamp Trace Algorithm (BTA)

Trace function \( \text{Tr}(\cdot) : \mathbb{F}_{2^m} \to \mathbb{F}_2 \)

\[
\text{Tr}(z) := z + z^2 + z^{2^2} + \ldots + z^{2^{m-1}}.
\]

The function \( \text{Tr}(\cdot) \) is \( \mathbb{F}_2 \)-linear and onto.

We know that:

\[
\forall i \in \mathbb{F}_2, \quad \text{Tr}(z) - i = \prod_{\gamma \text{ s.t. } \text{Tr}(\gamma) = i} (z - \gamma).
\]

Moreover, we have:

\[
z^{2^m} - z = \text{Tr}(z) \cdot (\text{Tr}(z) - 1).
\]
Berlekamp Trace Algorithm (BTA) cont...

Let $B = (\beta_1, \ldots, \beta_m)$ a basis of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$.

BTA splits any $f \in \mathbb{F}_{2^m}[z]$ s.t. $f(z)|(z^{2^m} - z)$ into factors of degree 1, by computing iteratively on $\beta \in B$ and recursively on $f$:

$$g(z) := \gcd(f(z), \text{Tr}(\beta \cdot z)) \text{ and } h(z) := \frac{f(z)}{g(z)}.$$ 

**Theorem 2 (Berlekamp)** Berlekamp Trace Algorithm always successfully returns the linear factors of any polynomial.
Reduce time complexity

Though efficient, BTA has a large number of recursive calls when the system parameters grow.

We reduce it by mixing BTA and Zinoviev’s algorithms, ad-hoc methods for finding roots of polynomials of degree \( \leq 10 \) over \( \mathbb{F}_{2^m} \).
We call this process BTZ in the following.

BTZ depends on a parameter \( d_{\text{max}} \) which is the maximum degree up to which we use Zinoviev’s methods.

V.A. Zinoviev, On the solution of equations of degree \( \leq 10 \) over finite fields \( \text{GF}(2^m) \), Research Report INRIA n° 2829, 1996
Pseudocode of a simplified version of BTZ

**Algorithm 1 - BTZ(f, d, i)**

First call: $f \leftarrow \sigma_e; d \leftarrow d_{max} \in \{2, \ldots, 10\}; i \leftarrow 1.$

**if** degree($f$) $\leq d$ **then**

**return** ZINOVIEV($f, d$);

**else**

$g \leftarrow \gcd(f, \text{Tr}(\beta_i \cdot z));$

$h \leftarrow f/g;$

**return** $\text{BTZ}(g, d, i + 1) \cup \text{BTZ}(h, d, i + 1);$  

**end if**
Zinoviev’s algorithms

Zinoviev's methods find an affine multiple of any polynomial of deg $\leq 10$ over $\mathbb{F}_{2^m}$. The methods differ according to this degree.

Affine Polynomial

$$A(z) = L(z) + c$$

where $L$ is a linearized polynomial, $c \in \mathbb{F}_{2^m}$.

Linearized Polynomial

$$L(z) = \sum_{i=0}^{n} l_i \cdot z^{2^i}$$

with $l_i \in \mathbb{F}_{2^m}$ and $l_n = 1$.

After that, finding roots of affine polynomial is easier than in the general case.
Get an affine multiple of a polynomial of degree 2 or 3

Let us have an equation: \( z^2 + \alpha z + \beta = 0, \quad \alpha, \beta \in \mathbb{F}_{2^m} \).
Notice \( z^2 + \alpha z \) is already a linearized polynomial. Nothing to do here.

Now consider the equation: \( z^3 + az^2 + bz + c = 0, \quad a, b, c \in \mathbb{F}_{2^m} \)
We have to decimate the non-linear terms.

For this, we add one particular root by multiplying the left side by \((z + a)\).
We obtain \( z^4 + dz^2 + ez + f = 0 \) with \( d = a^2 + b, \quad e = ab + c, \quad f = ac \).
Got what we wanted, an affine multiple of a polynomial of degree 3.
Results we obtained

We specify a recurrence complexity formula for \( BTZ \).
We then use dynamic programming to estimate its theoretical complexity in the worst case.
We thus determine the best \( d_{\text{max}} \) to use to have the optimal efficiency on the following range of parameters:
\( m = 8, 11, 12, 13, 14, 15, 16, 20, 30, 40 \); \( t = 10\ldots300 \); \( d_{\text{max}} = 2\ldots10 \).
Let \( K \) be the cost function of any operation over \( \mathbb{F}_{2m} \).
We take \( K(+) = 1 \); \( K(\times) = 1 \) or \( K(\times) = m \).
Experimental results

We provide experimental data for finding the roots of a polynomial of degree \( t = 32 \) over \( \mathbb{F}_{2^m} = \mathbb{F}_{2048} \). \( BTZ_d \) means that we use BTZ with \( d_{\text{max}} = d \), for all suitable \( d \).

<table>
<thead>
<tr>
<th>( n = 2048, \ t = 32, \ m = 11 )</th>
<th>Chien</th>
<th>BTA</th>
<th>( BTZ_2 )</th>
<th>( BTZ_3 )</th>
<th>( BTZ_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td># CPU cycles per byte for</td>
<td>root finding</td>
<td>3200</td>
<td>1300</td>
<td>900</td>
<td>800</td>
</tr>
<tr>
<td></td>
<td>decrypting</td>
<td>3700</td>
<td>1800</td>
<td>1400</td>
<td>1300</td>
</tr>
<tr>
<td>percentage of time spent for</td>
<td>syn. computation</td>
<td>5</td>
<td>10</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>solve key equation</td>
<td>7</td>
<td>14</td>
<td>18</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>root finding</td>
<td>87</td>
<td>72</td>
<td>65</td>
<td>61</td>
</tr>
</tbody>
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HyMES simulation results

<table>
<thead>
<tr>
<th>$(m, t)$</th>
<th>cycles/byte</th>
<th>key size</th>
<th>security</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>encrypt</td>
<td>decrypt</td>
<td></td>
</tr>
<tr>
<td>(10, 50)</td>
<td>243</td>
<td>7938</td>
<td>32 kB</td>
</tr>
<tr>
<td>(11, 32)</td>
<td>178</td>
<td>1848</td>
<td>73 kB</td>
</tr>
<tr>
<td>(12, 21)</td>
<td>126</td>
<td>573</td>
<td>118 kB</td>
</tr>
<tr>
<td>(13, 29)</td>
<td>149</td>
<td>535</td>
<td>360 kB</td>
</tr>
<tr>
<td>(14, 15)</td>
<td>132</td>
<td>229</td>
<td>415 kB</td>
</tr>
<tr>
<td>(15, 13)</td>
<td>132</td>
<td>186</td>
<td>775 kB</td>
</tr>
<tr>
<td>(16, 12)</td>
<td>132</td>
<td>166</td>
<td>1532 kB</td>
</tr>
</tbody>
</table>

HyMES: selected parameters at a glance (Pentium Core 2)

<table>
<thead>
<tr>
<th>cycles/byte</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>encrypt</td>
<td>decrypt</td>
</tr>
<tr>
<td>RSA 1024 (1)</td>
<td>800</td>
</tr>
<tr>
<td>RSA 2048 (1)</td>
<td>834</td>
</tr>
<tr>
<td>NTRU (2)</td>
<td>4753</td>
</tr>
</tbody>
</table>

(1) RSA encryption (with malleability defense) using OpenSSL.
(2) ntru-enc1 ees787ep1 NTRU encryption with $N = 787$ and $q = 587$. Software written by Mark Etzel (NTRU Cryptosystem).
Decryption results

![Graph showing decryption cost vs. binary work factor for different extension degrees. The graph plots decryption cost in CPU-cycles per byte on the y-axis and binary work factor (power of 2) on the x-axis. Each line represents a different extension degree: m = 11, m = 12, m = 13, m = 14, m = 15. The graph illustrates the increase in decryption cost as the binary work factor increases. The extension degree m affects the curve's shape, with higher degrees leading to steeper increases in cost.]
Encryption results

Extension degree $m = 11$

Extension degree $m = 12$

Extension degree $m = 13$
Conclusion
Summary

- We made 2 major changes to the original system.
- The security is preserved.
- We implemented the Hybrid system.
- The system is faster than other number theory based PKE’s.
- We have made algorithmic improvement to root finding of polynomials and thus to decode faster.
- Goppa code distinguisher? (Faugère, Otmani, Perret and Tillich)