

Dynamic copulas: applications to finance & economics

Daniel Totouom Tangho

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T H E S E

pour obtenir le grade de
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Spécialité "Economie et Finance"

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Daniel TOTOUOM TANGHO

le 6 novembre 2007

COPULES DYNAMIQUES:
APPLICATIONS EN FINANCE & EN ECONOMIE

DYNAMIC COPULAS:
APPLICATIONS TO FINANCE & ECONOMICS

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Jury

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Table of Contents

Acknowledgements **i**

Résumé en français **vii**

Chapter 1 : Introduction **1**

1.1 Recent subprime crisis 2

1.2 Structure of the thesis 9

1.3 Brief history of credit derivatives up to 2000 12

1.4 Three generations of credit derivatives 16

1.5 Pricing first generation products 26

1.6 Archimedean copulas within a credit derivatives framework 45

Chapter 2 : Dynamic Copula Model **49**

2.1 Levy processes including gamma processes 49

2.2 Dynamic copulas from a gamma process perspective 52

2.3 Dynamic copula processes seen from a Levy perspective 56

2.4 Modelling default times 62

Chapter 3 Combining Credit Migration and Copulas **75**

3.1 Construction of single name credit migration & default processes 75

3.2 Construction of risk neutral single name credit migration proceseses 79

3.3 Copula approach for dependent credit migration & default processes 81

Chapter 4 Multi-factor & time-changed approach **85**

4.1 Multi-factor approach for dependent default times 85

4.2 Time-changed Levy process for dependent spread dynamics 92

Applications

Chapter 5 A New Way of Modelling CDO Tranches **99**

5.1 Dynamic Archimedean copula processes 100

5.2 Specific dynamic Archimedean copula process 102

5.3 Pricing a correlation product: CDO 104

5.4 Conclusions 109

Chapter 6 Comparison with five 1-factor models **111**

6.1 Dynamic copula process 113

6.2 Pricing a correlation product 115

6.3 Conclusions 117

Chapter 7 Conclusions **123**

Chapter 8 References **127**

Résumé

Les dérivés de crédit ont connu en quelques années un développement fulgurant en finance : les volumes de transactions ont augmenté exponentiellement, de nouveaux produits ont été créés. La récente crise du sub-prime a mis en évidence l'insuffisance des modèles actuels. Le but de cette thèse est de créer de nouveaux modèles mathématiques qui prennent en compte la dynamique de dépendance (« *tail dependence* ») des marchés.

Après une revue de la littérature et des modèles existants, nous nous focalisons sur la modélisation de la « corrélation » (ou plus exactement la dynamique de la structure de dépendance) entre différentes entités dans un portefeuille de crédit (CDO). Dans une première phase, une formulation simple des copules dynamiques est proposée. Ensuite, nous présentons une seconde formulation en utilisant des processus de Lévy à spectre positif (i.e. gamma Ornstein-Uhlenbeck). L'écriture de cette nouvelle famille de copules archimédiennes nous permet d'obtenir une formule asymptotique simple pour la distribution des pertes d'un portefeuille de crédit granulaire. L'une des particularités du modèle proposé est sa capacité de reproduire des dépendances extrêmes comparables aux phénomènes récents de contagion sur les marchés comme la crise du « subprime » aux Etats-Unis. Une application sur l'estimation des prix des tranches de CDOs est aussi présentée.

Dans cette thèse, nous proposons également d'utiliser des copules dynamiques pour modéliser des migrations jointes des qualités de crédit afin de prendre en compte les co-migrations extrêmes. En effet, les copules nous permettent d'étendre notre connaissance des processus de migration mono-variable à un cadre multi-variables. Afin de tenir compte de multiples sources de risques systémiques, nous développons des copules dynamiques à plusieurs facteurs. Enfin, nous montrons que la brique élémentaire de structure de dépendance induite par une mesure du temps aléatoire « *Time Changed Process* » rentre dans le cadre des copules dynamiques.

Mots-clés : crédit dérivés, CDO, copules archimédiennes, processus de Lévy, processus Ornstein-Uhlenbeck non gaussiens, chaînes de Markov « *credit migration* ».

Structure de la Thèse

Dans la première partie de la thèse, nous analysons :

- L'évolution récente du marché des dérivés de crédit et de la titrisation
- Les événements récents du subprime sur le marché
- La revue de la littérature

La deuxième partie est composée de trois chapitres (deux a quatre)

- Construction des copules dynamiques
- Modélisation des migrations jointes de qualité de crédit avec des copules
- Développement des copules a plusieurs facteurs

Dans la troisième partie, nous présentons deux applications pour la tarification des tranches de CDOs (chapitres cinq et six)

- *Dynamic Copula Processes: A new way of modelling CDO tranches*
- *Dynamic copulas processes: Comparison with five 1-factor models for pricing CDOs*

Enfin le chapitre sept du document de thèse présente les principales conclusions et perspectives.

Evolution récente des dérivés de crédit

Les dérivés de crédit ont connus un des développements les plus rapides en finance du marché jusqu'à présent. Les montants nominaux des transactions ont eus une croissance exponentielle, ce qui a entraîné naturellement la création continuelle de nouveaux produits pour répondre aux besoins du marché. Nous présenterons ces produits en commençant par les CDS introduit sur les marchés au début des années 90. Nous montrons que ce développement a été à l'initiative du marché, avec des modèles mathématiques utilisés pas toujours à jour de ces innovations financières. Un exemple typique est l'utilisation des modèles structurels de la firme (Merton, 1974) pour estimer les prix des CDS. La standardisation des CDS est devenue une réalité grâce aux nouvelles normes et définitions mises en place par l'ISDA[1], le prix du CDS devenant un juste équilibre de l'offre et de la demande.

[1] International Swap & Derivatives Association.

Ce travail de thèse a commencé en octobre 2003. A ce moment-là, les sujets d'actualité étaient l'estimation de la *Value at Risk* (VaR), ainsi que la contribution a la VAR pour la gestion de risque et les dérivées de crédit. Les limitations de modèle de Merton avaient déjà conduits a l'utilisation des modèles à intensité avec prise en compte des corrélations implicites, la *base correlation* venait à peine de faire son chemin sur le marché. A ce moment, Jon Gregory, alors Responsable de l'Equipe de Recherche Crédit chez BNP Paribas suggère a mon directeur de thèse que le vrai problème est de trouver un modèle dynamique pour les dépendances extrêmes, qui permettrai de prendre en compte les fondamentaux du risque de crédit. Des tests effectués par Burtschell, Gregory & Laurent (2005a) montrent que la copule de Clayton peut donner des résultats nettement plus intéressants que ceux obtenus avec les copules de Gauss et de Student, tout en les surpassant pour l'explication des dépendances extrêmes.

Les copules gaussiennes suggèrent que l'événement de contagion extrême sur les marchés a une probabilité quasi nulle d'occurrence. Dans des périodes de marasme économique, un défaut a tendance à déclencher d'autres, ce qui n'est pas le cas dans des conditions économiques standard avec présence de liquidité. La forme classique en « cône» de la copule de Clayton avec sa dépendance de queue inférieure (figure 0.1, gauche) capture cette asymétrie ; la copule de Gauss (la distribution normale) (figure 0.1, la droite) ne nous permet pas de reproduire cette caractéristique.

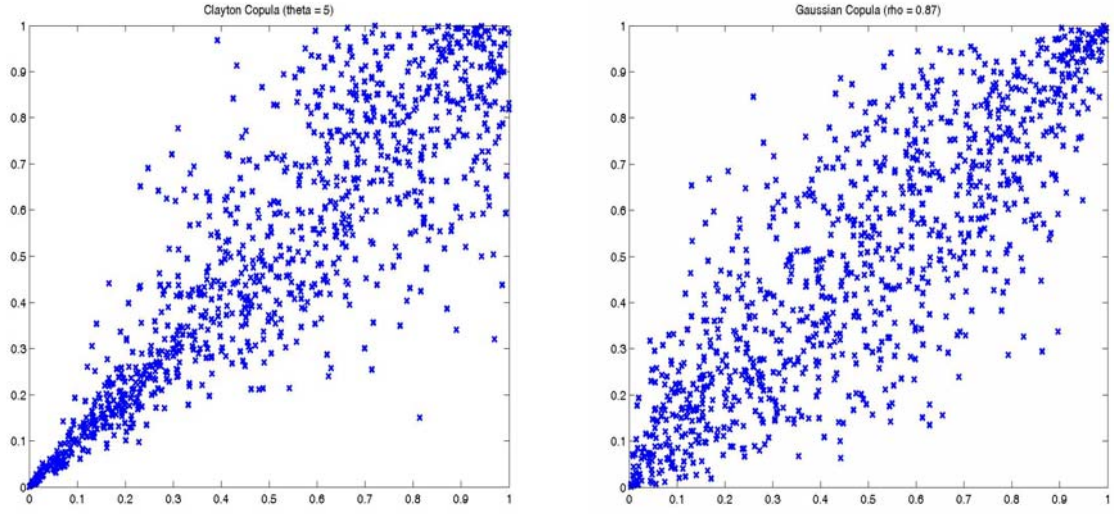


Figure 0.1: Copule de Clayton avec pour parametre $\theta = 5$ (gauche) et une copule Gaussienne avec $\rho = 0.87$ (droit)

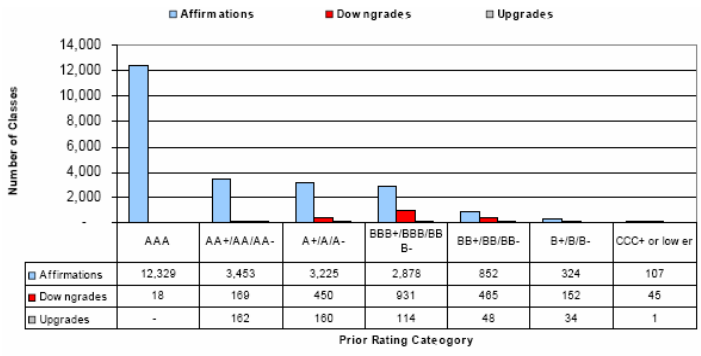
Notre premier objectif était donc de construire un modèle dynamique de copule présentant dépendance de queue inférieure afin d'effectuer la tarification de différents de dérivées de crédit.

Les évenements recents du Subprime

Depuis la fin du premier trimestre 2007, des rumeurs de plus en plus persistantes sur la qualité de titres adossés sur le subprime aux Etats-Unis ont gagnés les marchés. Un « subprime » *(subprime loan* ou *subprime mortgage* en anglais) est un crédit à risque, offert à un emprunteur qui n'offre pas les garanties suffisantes pour bénéficier du taux d'intérêt le plus avantageux (*prime rate*). Le terme est employé plus particulièrement pour désigner une forme de crédit hypothécaire (*mortgage*), apparu aux États-Unis et destiné aux emprunteurs à risque. Ce crédit immobilier est gagé sur le logement de l'emprunteur).

Une hausse des taux de défaut prévus sur les actifs sous-jacents, combinée, le cas échéant, à des changements de méthodologie, a conduit les principales agences de notation à dégrader courant juillet et août 2007 la notation de nombreuses tranches de subprime RMBS[2], quelle que soit d'ailleurs leur position initiale sur l'échelle de notation ce qui a entraînée inéluctablement un mouvement extrême de co-migration des qualités de crédit, combinée à une baisse de prix ou plus exactement une augmentation du taux de défaut implicite.

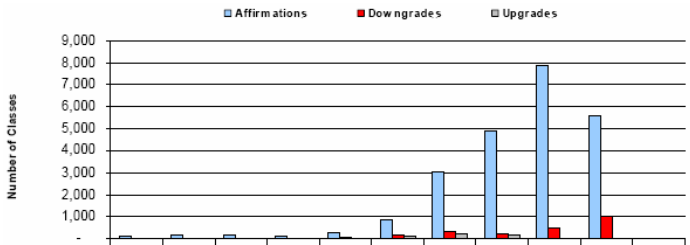
La figure 0.2 montre l'évolution des notations des tranches. On observe qu'en moyenne, il y a plus de « downgrades » que d' « upgrades », d'où une anticipation de détérioration globale de la qualité de crédit de cette classe d'actifs. La figure 0.3 nous montre d'une part que les volumes de transactions notées par l'agence Fitch sont plus importantes de 2003 à 2006 qu'avant 2002, et d'autre part que les transactions émises sur le marche en 2005 et 2006 sont de moins bonne qualité en moyenne, d'où un taux d'abaissement de la note de crédit plus élevé.



	AAA	AA+/AA/AA-	A+/A/A-	BBB+/BBB/BBB-	BB+/BB/BB-	B+/B/B-	CCC+ or lower
Affirmations	12,329	3,453	3,225	2,878	852	324	107
Downgrades	18	169	450	931	465	152	45
Upgrades	-	162	160	114	48	34	1

Figure 0.2 Source Fitch: « US RMBS Rating Actions » par catégorie de Rating au 07 septembre 2007

[2] RMBS (*Residential mortgage-backed securities*) : Titrisation des prêts adossés aux créances hypothécaires



	Pre-1998	1998	1999	2000	2001	2002	2003	2004	2005	2006	YTD2007
Affirmations	118	148	132	116	289	887	3,069	4,909	7,877	5,631	6
Downgrades	11	3	8	10	40	132	315	220	492	996	3
Upgrades	2	9	18	5	21	88	220	121	15	20	-

Figure 0.3 Source Fitch: « US RMBS Rating Actions » par « vintage » au 07 septembre 2007

L'indice ABX HE est un indice de CDS sur les titrisations subprime RMBS, composé de 20 des 25 plus importantes titrisations subprime RMBS mises en place dans les 6 mois précédant son lancement. La figure 0.4 montre son évolution, ou plus exactement celle de différentes tranches de AAA (rouge) au BBB- (marron), du 19 janvier au 06 juillet 2007. La baisse des prix indique une augmentation de la probabilité de défaut.

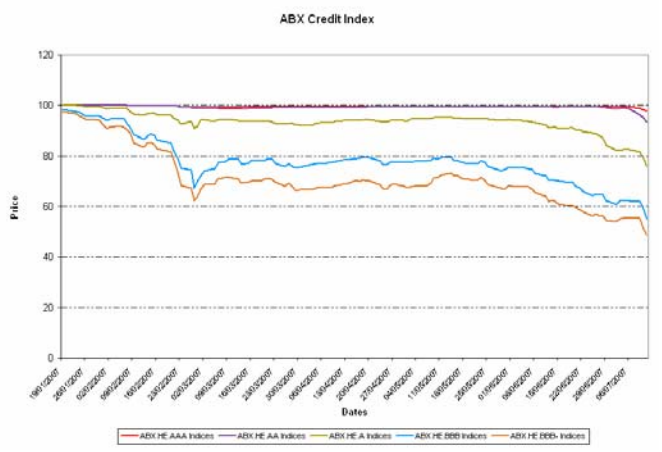


Figure 0.4: Evolution des prix des tranches ABS AAA (en rouge) jusqu'à BBB- (en marron) pendant la periode du 19 janvier au 06 juillet 2007. La baisse des prix indique une augmentation de la probabilite de defaut implicite.

La figure 0.5 nous montre l'évolution journalière des prix de trois tranches de l'index ABX (AAA, A et BBB) dans les six premiers mois de 2007. On note que d'une part que les variations importantes de prix sont des événements rares, et d'autre part que ces types d'événements ont tendance à avoir une très forte corrélation.

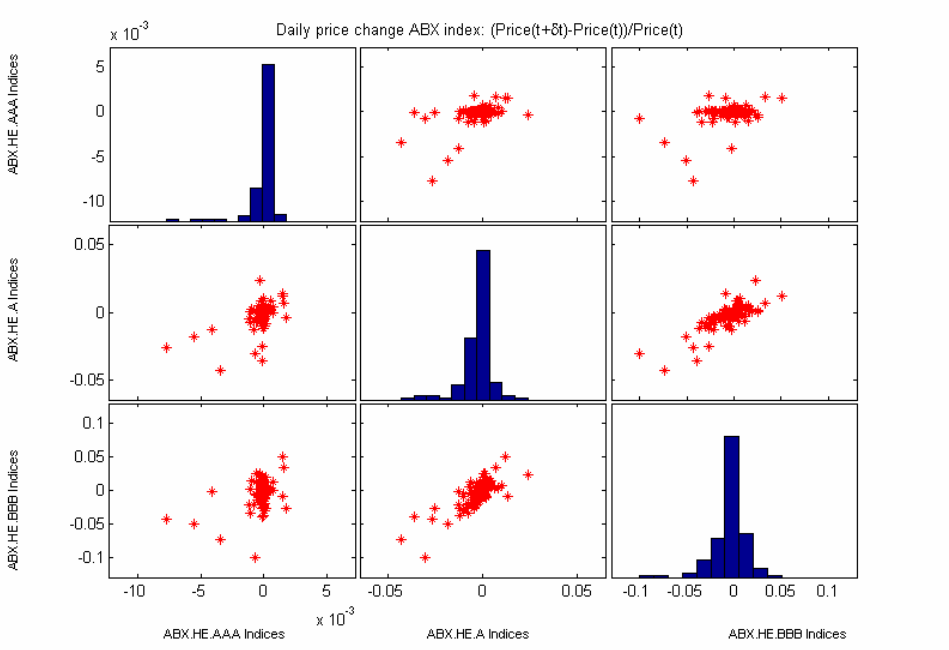


Figure 0.5: Analyse de l'histogramme des marginales et de la dependance empirique des variations des prix des tranches ABS AAA à BBB pendant la periode du 19 janvier au 06 juillet 2007. Les baisses extremes sont celles qui apparaissent en juillet.

Alors qu'elles n'avaient pas été atteintes par les mouvements de prix de février 2007, avec la crise de liquidité en juillet, les tranches de meilleure notation (AAA et AA) ont vu pour la première fois leur prix baisser, dans le sillage des dégradations et mises sous surveillance survenues les 10 et 11 juillet, entraînant ainsi un vent de panique et une très forte incertitude sur l'ensemble des marchés financiers. La figure 0.6 se focalise sur la période de mars a septembre 2007, sur les indices les plus liquides du crédit. On peut voir clairement qu'en juillet et août, la dynamique de ces indices est identique, d'où une forte dépendance, que celle due à la crise de crédit et de liquidité de l'été 2007.

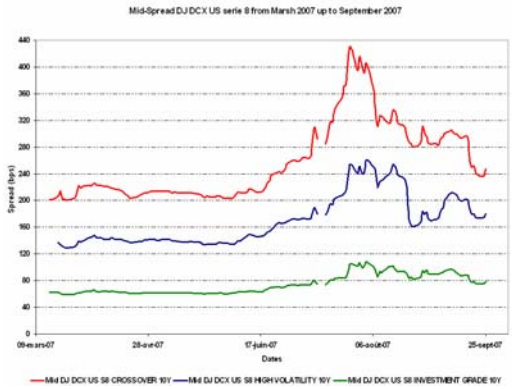


Figure 0.6: Source Bloomberg : Spread DJ CDX cross-Over, High volatility & Investment grade de mars 2007 a septembre 2007

Deux crises ont secoué le monde du crédit dans les trois dernières années : celle du subprime en 2007, et celle de Ford et General Motors en 2005. Pendant ces crises, le mouvement des marges, « *spreads* », de crédit sont tout à fait asymétriques et présente un cône de dépendance comparable a celle que l'on obtient avec une copule de type Clayton (voir figure 0.7). Sur la copule empirique de dépendance entre l'indice iTraxx série S3 Auto & Senior Financial (CDS 10 ans), on observe qu'en période de crise, on a des dépendances extrêmes : en été 2005 avec le downgrade de Ford et General Motor (points verts), et en été 2007, crise du subprime et crise de liquidité (points rouges).

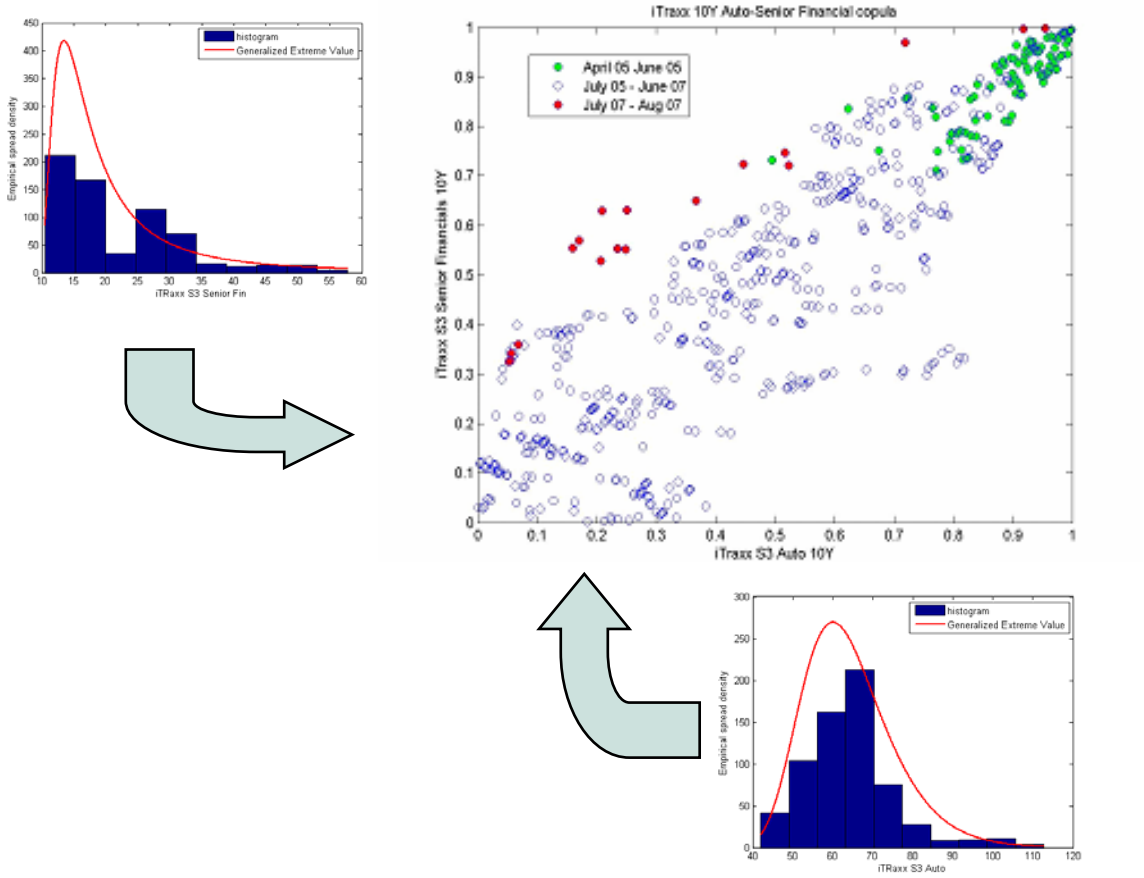


Figure 0.7: Analyse empirique de la distribution de valeurs des CDS et de la dynamique de la dependance entre mars 2005 a septembre 2007. Copule des spreads de l'indice iTraxx S3 series, Auto10Y versus Senior Financials 10Y, ainsi que leur histogrammess (Senior Financials à gauche & Autos en bas). Les points verts correspondent à la crise GM-Ford en 2005, tandis les rouges sont de la crise subprime

Trois générations de produits dérivés de crédit

Dans le document de thèse, nous donnons dans la première partie un bref aperçu de l'histoire récente des dérivés de crédit et de la titrisation, une revue de la littérature sur les modèles existants, et nous présentons également les principales générations de produits :

1. Les obligations risques et les Credit Default Swap
2. Les Basket Default Swap
3. Les Produits *forward-starting* (exposition future au risque de risque de crédit)

Le sujet central de cette thèse, la modélisation de la structure de dépendance dans les produits de seconde et de troisième génération, est développé dans les chapitres deux, trois et quatre.

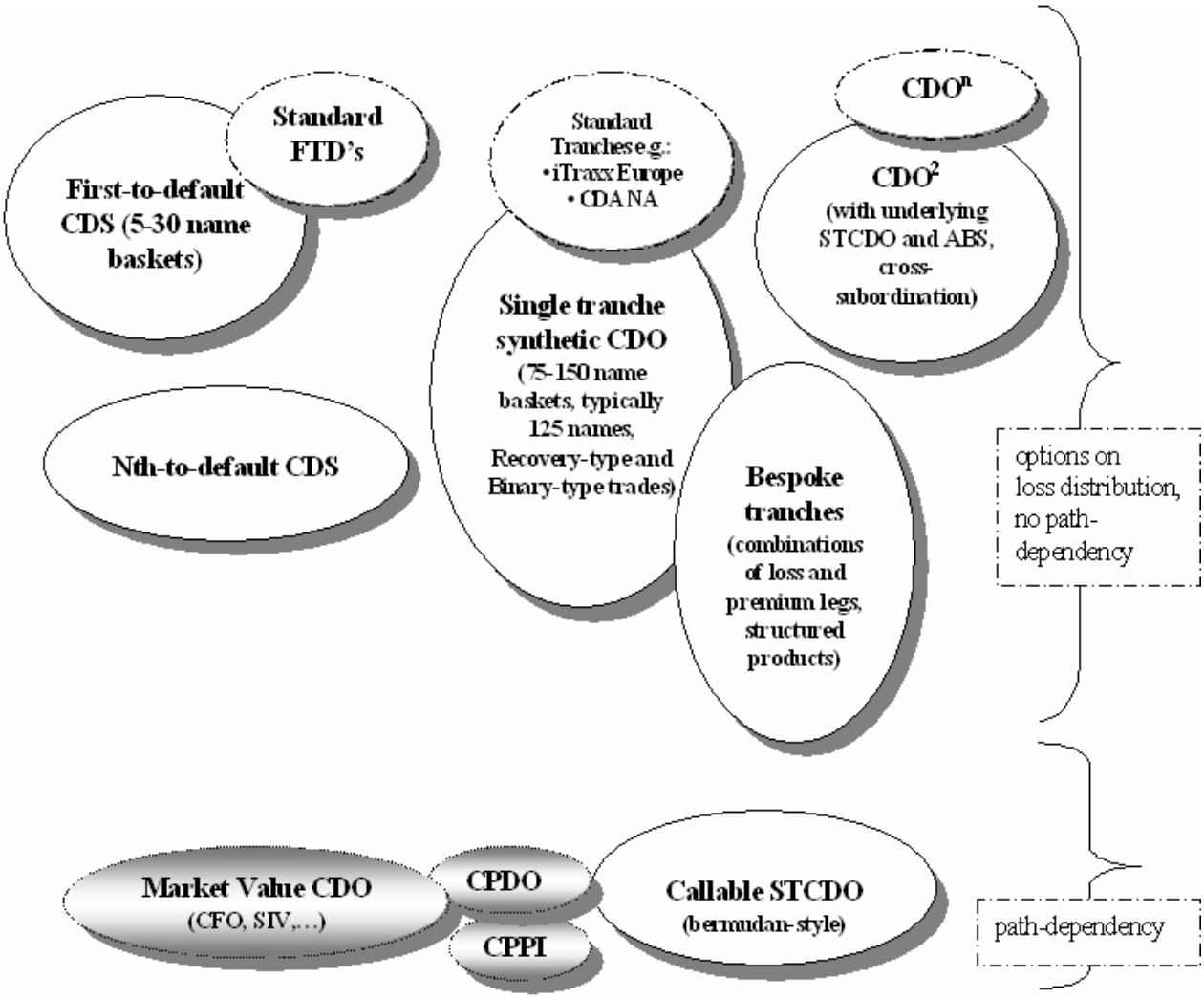


Figure 0.8 Evolution des produits de seconde et de troisième generation des derives de credit

Chapitre 2 : Modélisation des copules archimediennes dynamiques

Les fonctions copules font partie des nouvelles méthodologies de plus en plus significatives, qui offrent une flexibilité dans l'analyse des co-mouvements entre les facteurs de risque et d'autres variables importantes étudiées en finance. Les premières applications des copules se sont effectuées en statistiques.

Avec le développement des marches financiers et l'apparition des événements extrêmes, l'hypothèse de non normalité des variables utilisées en finance devenant de plus en plus prédominante, des développements sont effectués pour capturer la non normalités des distributions marginales d'une part, et d'autre part, l'asymétrie et la dynamique de la dépendance, ce qui l'objet de ce chapitre qui commence par une construction simple des copules archimédiennes dynamique.

Une littérature vaste existe sur les copules, surtout les copules archimédiennes, avec des applications dans une grande variété de champs. Mais quand l'on regarde de plus près, la plupart des publications scientifiques sur le sujet traitent seulement des copules à deux variables. Très peu de documents sont disponibles sur les copules de multi variables. Dans cette thèse, nous expliquons les difficultés théoriques inhérentes à l'extension des copules à deux variables aux copules à trois variables, et par conséquent à plusieurs variables.

Il est clair que c'est encore plus difficile de construire des processus dynamiques sur des copules à plusieurs variables que sur les copules statiques.

Notre objectif est donc de construire les procédés de copule dynamiques, basé sur des processus stochastiques.

Les copules multi variables

Le manuel canonique sur les copules, Nelson (1999) énumère plusieurs types de copules archimédiennes a deux variables (avec un ou plus de paramètres) mais comme la plupart de ces copules a deux variables ne sont pas strictes, très peux d'équivalents multi variables existent.

Les générateurs de copules strictes peuvent être reliées à une transformée de Laplace spécifique. Par exemple, le générateur de la copule de Clayton, correspond à la transformée de Laplace d'une distribution gamma et la copule de Gumbel est reliée à une distribution alpha-stable.

Nelsen (1999) donne plusieurs contre-exemples afin de montrer la difficulté de trouver des copules a plusieurs variables. Un autre livre de référence majeure, Joe (1997), fournit

quelques résultats pour le cas a trois variables et a quatre variables. Lindskog (2000) fournit également quelques extensions intéressantes et montre les contraintes sur les valeurs des paramètres dans le cas non-échangeable. Mais toutes ces copules étudiées sont statiques.

Copules archimediennes

On appelle copule archimédienne, une fonction copule telle qu'il existe une fonction inversible $\varphi^{-1}:[0,1]\to\Re_+\cup\{\infty\}$ vérifiant :

$$C(u_1,...,u_N)=\varphi\left(\sum_{i=1}^{N}\varphi^{-1}(x_i)\right).$$

Dans ce cas, $\forall x\in\Re_+\cup\{\infty\}$, on a $\varphi'(x)\prec 0,\varphi''(x)\succ 0$.

Une façons assez simple de générer des copules archimédiennes, c'est de considérer les transformées de Laplace des distributions de variables aléatoires. Soit Y une variable aléatoire a spectre positif ayant pour transformée de Laplace :

$$\varphi(s)=E\left[\exp(-s\times Y)\right]=\int_0^{+\infty}\exp(-s\times y)\times f_Y(y)\,dy,$$

Alors nous avons :

$$\forall s\succ 0,\varphi^{(n)}(s)=(-1)^n\int_0^{+\infty}y^n\times\exp(-s\times y)\times f_Y(y)\,dy$$
$$\Rightarrow(-1)^n\varphi^{(n)}(s)\succ 0$$

On obtient ainsi les conditions nécessaires et suffisantes de construction des copules.

Quelques proprietes des copules archimediennes

Dépendance inférieure

La dépendance extrême dans les queues s'écrit facilement dans le cas des copules archimédiennes :

$$\lambda_{Inf}=\lim_{t\to 0^+}P(x_1\le t/x_2\le t)=\lim_{t\to 0^+}\frac{C(t,t)}{t}$$
$$=\lim_{t\to 0^+}\frac{\varphi\left(2\varphi^{-1}(t)\right)}{t}=\lim_{z\to+\infty}\frac{\varphi(2z)}{\varphi(z)}$$

Dans le cas d'une copule de Clayton, on a $\lambda_{Inf}=\lim_{z\to+\infty}\frac{\varphi(2z)}{\varphi(z)}=2^{-1/\theta}$. Il peut être facilement prouvé que pour une copule gaussienne, $\lambda_{Inf}=0$ voir Embrechts, Lindskog & Mc Neil (2001).

Par construction, λ_{Inf} c'est la probabilité d'avoir une réalisation extrême dans une direction sachant que nous avons déjà un extrême dans la même direction.

Si $\lambda_{Inf}=0$, alors les extrêmes sont dits indépendants, et si $\lambda_{Inf}=1$, les extrêmes sont parfaitement "corrélés".

λ_{Inf} Est donc une "mesure de corrélation" des extrêmes. C'est le coefficient de dépendance de queue. Pour la distribution normale multidimensionnelle, vaut λ_{Inf} systématiquement 0 sauf lorsque la corrélation est égale `a 1. La distribution normale ne présente pas de dépendance de queue, ce qui n'est pas le cas de la distribution de Student multidimensionnelle. Mais cette dernière a une dépendance en queue symétrique.

Dépendance supérieure

$$\lambda_{Sup}=\lim_{t\to 1^-}P\left(x_1\succ t/x_2\succ t\right)=\lim_{t\to 1^-}\left\{\frac{P\left(x_1\succ t\right)+P\left(x_2\succ t\right)-1+P\left(x_1\le t,x_2\le t\right)}{P\left(x_2\succ t\right)}\right\}$$

$$=2-\lim_{t\to 1^-}\frac{1-C\left(t,t\right)}{1-t}=2-\lim_{t\to 1^-}\frac{1-\varphi\left(2\varphi^{-1}\left(t\right)\right)}{1-t}=2-\lim_{z\to 0^+}\frac{1-\varphi\left(2z\right)}{1-\varphi\left(z\right)}$$

La mesure de dépendance Tau de Kendall

$$\tau_{X_1,X_2}=\tau_C=\begin{cases}4\int_0^1\int_0^1 C\left(x_1,x_2\right)dC\left(x_1,x_2\right)\\ 1-4\int_0^1\int_0^1\frac{\partial C\left(x_1,x_2\right)}{\partial x_1}\frac{\partial C\left(x_1,x_2\right)}{\partial x_2}dx_1dx_2\end{cases}$$

Dans le cadre des copules archimédiennes, cette relation s'écrie simplement :

$$\tau_{X_1,X_2}=\tau_C=1+4\int_0^1\frac{\varphi^{-1}\left(u\right)}{\left(\varphi^{-1}\right)'\left(u\right)}du=1-4\int_0^\infty s\left[\frac{d}{ds}\varphi\left(s\right)\right]^2ds$$

Dans le cas d'une copule de Clayton, on a $\tau_C=\frac{\theta}{\theta+2}$

Les copules du point de vue des series temporelles

Les économétriciens ont développé les modèles de copule dynamiques d'un point de vue des séries temporelles. Patton (2001 & 2003) a développé une approche basé des processus type ARMA et l'a appliqué aux taux de change des devises. Fermanian & Wegkamp (2004) ont étendu l'approche de Patton. Duan (1995) et plus récemment Goorbergh et al (2005) ont utilisés les processus GARCH. Un défaut majeur de ces papiers est qu'ils considèrent seulement les cas à deux variables. Chen & Evente (2005) considèrent une classe de copule semi-paramétriques à plusieurs variables. Le modèle dynamique est calibré sur des données

intra-day. Aucun de ces derniers ne semble être convenable pour évaluer les tranches de CDOs sur les grands paniers d'actifs. Berd, Engle & Voronov (2005) ont développés un modèle hydride dans lequel les variables latentes fondamentales suivent soit un processus GARCH, soit un processus TARCH. Ceci a l'avantage de produire les distributions de retour de total qui sont asymétriques et clairement non gaussiens. Les auteurs ont utilisé des données historiques sur le SP500 a partir de 1962 jusqu'à la période pré-1990. Ces données historiques n'incluent pas forcement la prime de risque du marche d'une part et d'autre part, le marche actions n'est pas entièrement correle avec le marche du crédit, d'où la difficulté a pouvoir calibrer les probabilités implicites de défaut de chaque nom d'une part, et d'autre estimer les prix de tranches de CDO.

Les copules dynamiques

Très peu de publications portent sur l'analyse des copules en temps continu. L'un des premiers papiers à traiter du sujet semble être Darsow & al (1992) qui a étudié les copules archimédiennes et les processus de Markov d'un point de vue théorique. Plusieurs auteurs ont modélisé la dynamique de risque de crédit en utilisant des modèles a intensités implicites de défaut. Rogge & Schonbucher (2003) ont développé une méthode de modélisation du risque de portefeuille de crédit de façon dynamique avec les copules archimédiennes. Ils fournissent quelques résultats très utiles qui relient les copules archimédiennes aux transformées de Laplace spécifiques, et présentent également une méthode rapide et efficace de simulation Monte Carlo, cela n'est pas mentionné par Nelsen (1999).

Madan analyse la dynamique du marché actions (Madan & Seneta, 1990, et Madan & Milne, 1991). Cariboni & Schouten (2004) sur le marché du crédit. Joshi & Stacey (2004) utilise un modèle a intensité pour la tarification des tranches de CDOs, ils ont trouvé qu'avec deux processus gamma combinés, il était possible de reproduire les niveaux de « base correlation » observées dans le marché correctement. Une des difficultés des modèles a intensité c'est l'exigence du calibrage nom par nom de la courbe implicite des probabilités de défaut.

Les copules archimediennes dynamiques

En partant des développements de Rogge & Schonbucher (2003), nous considérons une variable aléatoire Y positive avec pour transformée de Laplace $\varphi(s)$. Soit U_i N variable aléatoires uniformes sur [0,1] mutuellement indépendantes, et indépendantes de Y, Alors les N variables aléatoires définies par V_i telle que :

$$V_i = \varphi\left(\frac{-\mathrm{Ln}(U_i)}{Y}\right) \qquad \text{for } i = 1,\ldots N$$

Sont uniformes sur [0,1], et leur distribution jointe est données par :

$$\mathrm{Prob}(V_1 \le v_1,\cdots,V_N \le v_N) = \varphi\left(\sum_{i=1}^{N}\varphi^{-1}(v_i)\right)$$

La distribution jointe de ces variables est une copule archimédiennes statique. Nous définition la copula archimédienne dynamique en considérant que Y n'est plus une variable aléatoire, mais un processus stochastique.

$$V_i(t) = \varphi_t\left(\frac{-\mathrm{Ln}(U_i(t))}{Y(t)}\right) \qquad \text{for } i = 1,\ldots N$$

Les propriétés classiques des copules archimédiennes sont conservées.

Exemple de construction des copules dynamiques

Dans cette exemple (figure 0.9), nous utilisons un processus non gaussien avec retour a la moyenne type gamma Ornstein-Uhlenbeck. Le choix de ce processus est motive d'une part par l'existence d'une solution analytique connue pour sa transformée de Laplace, et d'autre part, sa caractéristique d'être a spectre positif, ce qui lui permet d'être aussi bien utilisable pour la modélisation des taux d'intérêt. Barndorff-Nielsen & Shephard (2001) ont utilisé un model similaire pour modéliser la volatilité stochastique sur les marchés actions.

Plus récemment ce processus a été étudié par Schoutens, Simonsy & Tistaertz (2003). Leurs résultats démontrent que ce processus permet de bien capturer la dynamique de la volatilité des prix d'actions. Ce processus est aussi connu en géostatistique comme « processus d'Ambartzumian » d'après le mathématicien Soviétique du même nom depuis plusieurs années (Chilès & Delfiner 1999 p489, et Matheron, 1969).

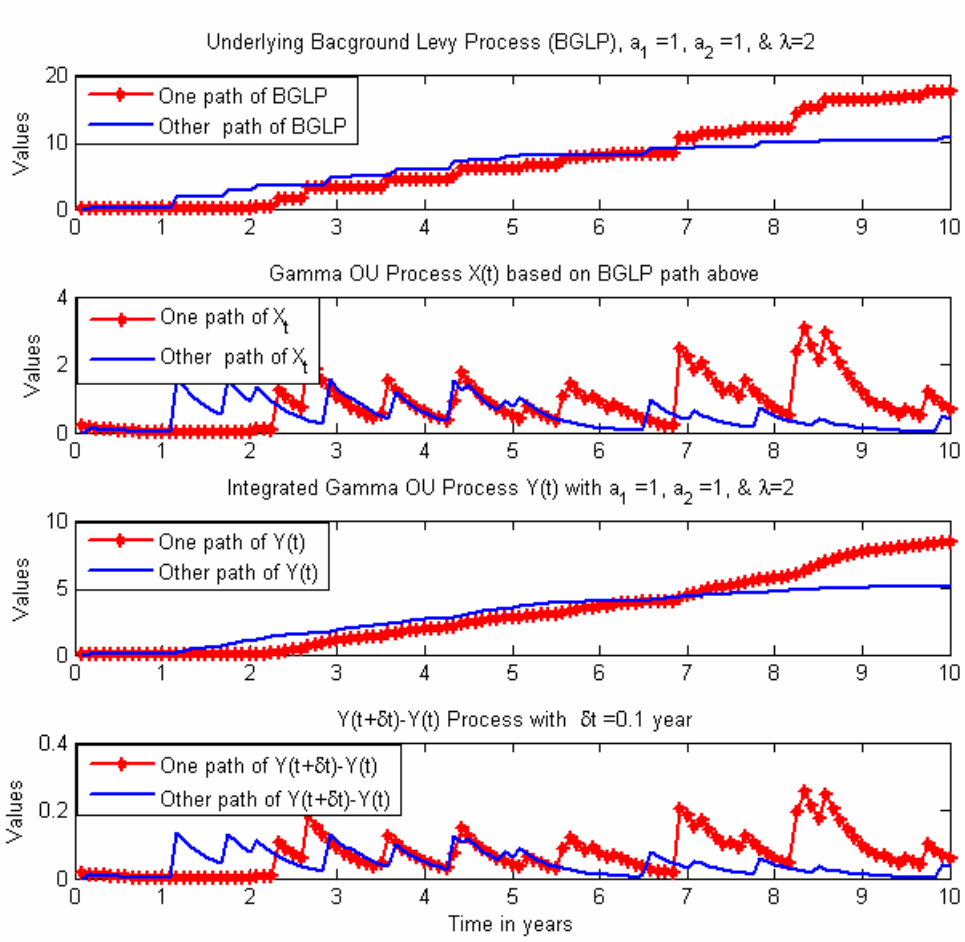


Figure 0.9: Construction d'un processus gamma O-U et de son integrale temporelle

Pour les copules spot

La transformée de Laplace du processus est donnée par :

$$\varphi_t(s) = \exp\left(-\frac{sx_0}{\lambda}\left(1-\exp(-\lambda t)\right) - \frac{s\lambda a_2 t}{\lambda a_1 + s} + \frac{s\lambda a_2}{\lambda a_1 + s} Ln\left(1+\frac{s}{\lambda a_1}\left(1-\exp(-\lambda t)\right)\right)\right)$$

Nous en déduisons l'écriture de la variable latente de construction de la copule archimédienne dynamique correspondante.

$$V_t = \varphi_t\left(-\frac{Ln(U_t)}{Y(t)}\right)$$

Pour les copules forward

La transformée de Laplace du processus sachant que nous connaissons la filtration jusqu'en t_0 est donnée par :

$$\varphi_{t_0,t}(s)=\exp\left(-\frac{sx_{t_0}}{\lambda}\left(1-\exp\left(-\lambda\times\left(t-t_0\right)\right)\right)-\frac{s\lambda a_2(t-t_0)}{\lambda a_1+s}+\right)$$
$$\times\exp\left(\frac{s\lambda a_2}{\lambda a_1+s}Ln\left(1+\frac{s}{\lambda a_1}\left(1-\exp\left(-\lambda\times\left(t-t_0\right)\right)\right)\right)\right)$$

Nous en déduisons l'écriture de la variable latente de construction de la copule archimédienne dynamique forward correspondante.

$$V_{t_0,t}=\varphi_{t_0,t}\left(-\frac{\mathrm{Ln}\left(U_t\right)}{\left(Y(t)-Y\left(t_0\right)\right)}\right)$$

Distribution asymptotique d'un portefeuille granulaire dans le cadre des copules dynamiques

Considérons un portefeuille de crédit ayant N noms avec des notionnels P_i =P/N, et des taux de récupérations fixes $R_i=R,\ \ (i=1,...,N)$. Le taux agrégé de perte au titre du risque de crédit dans le portefeuille est donne par la formule suivante :

$$\mathrm{Loss}_N(t)=\sum_{i=1}^{N}(1-R_i)P_i\,1_{\{\tau_i\prec t\}}=\frac{(1-R)P}{N}\sum_{i=1}^{N}1_{\{\tau_i\prec t\}}$$

où $1_{\{\tau_i\prec t\}}$ est la fonction indicatrice du défaut du $i^{ième}$ nom dans le portefeuille.

La transformée de Laplace de la distribution des pertes est donnée par :

$$E\left\{\exp\left(-s\,\mathrm{Loss}_N(t)\right)\right\}=E\left\{\prod_{i=1}^{N}\left\{\left(1-1_{\{\tau_i\prec t\}}\right)+1_{\{\tau_i\prec t\}}\,\eta^{\frac{1}{N}}\right\}\right\}$$
$$=E\left\{\left[\left(1-\eta^{\frac{1}{N}}\right)\exp\left(-Y_t\,\varphi_t^{-1}\left(PD(t)\right)\right)+1\right]^N\right\}$$

avec $\eta=\exp\left(-sN(1-R)\right)>0$.

En utilisant les propriétés classiques des calculs aux limites,

$$\lim_{N\to+\infty}\left\{N\left(\eta^{\frac{1}{N}}-1\right)\right\}=\left|\frac{\partial\eta^x}{\partial x}\right|_{x=0}=\mathrm{Ln}(\eta)=-sN(1-R)$$

On en déduit :

$$\mathrm{Loss}_N(t)\underset{N\to+\infty}{\approx}P(1-R)\exp\left(-Y_t\varphi_t^{-1}\left(PD(t)\right)\right)$$

Chapitre 3 : Utilisation des copules archimediennes pour les migrations jointes de qualité de credit

Nous proposons un modèle pour la dynamique des migrations jointes de la qualité de crédit avec des copules. Les migrations individuelles des qualités de crédit sont modélisées par une chaîne de Markov en temps continue, pendant que leur dynamique commune repose sur

l'utilisation des copules. L'usage de copules nous permet d'étendre notre connaissance des « upgrades » et « downgrades » individuels à un cadre multi varie.

Nous revisitons alors les lois communes des temps implicites de défaut de toutes les entreprises dans un portefeuille de crédit. Le développement de nouveaux produits (CPDO) exige une parfaite prise en compte des migrations de crédit, et la dépendance.

La probabilité de migration d'un état donné d'une chaîne de Markov vers un autre état à une écriture simple dans le cadre des copules archimédiennes.

Soit $\left(X^{(1)},\cdots,X^{(N)}\right)=\left(\left(X_t^{(1)}\right)_{t\geq 0},\cdots,\left(X_t^{(N)}\right)_{t\geq 0}\right)$ une chaîne de Markov en temps continu sur un espace $S^{(N)}$, avec pour matrice génératrice de transition $\left(\Lambda^{(1)},\cdots,\Lambda^{(N)}\right)$:

$$S^{(N)}=\begin{Bmatrix} S_1 \\ \cdots \\ S_N \end{Bmatrix}=\begin{Bmatrix} 1 & \cdots & m \\ \cdots & \cdots & \cdots \\ 1 & \cdots & m \end{Bmatrix}.$$

Nous définissons ci-dessous les seuils correspondant a chaque état ou Notation (Rating) dans la chaîne de Markov.

Soit Q la matrice définie par :

$$Q_{t,t+\delta t}^{(k)}=\exp\left(\delta t\times\Lambda_t^{(k)}\right)$$

Alors les seuils de transition a chaque date sont données par :

$$K_{t,t+\delta t}=cumQ_{t,t+\delta t(RN)}$$

$$=\begin{Bmatrix} \left(cumQ_{t,t+\delta t_{(RN)}}^{(1)}\right)_{i,1} & \cdots & 1 \\ \vdots & \ddots & \iddots \quad \vdots \\ \left(cumQ_{t,t+\delta t_{(RN)}}^{(k)}\right)_{i,1} & \left(cumQ_{t,t+\delta t_{(RN)}}^{(k)}\right)_{i,j} & 1 \\ \vdots & \iddots & \ddots \quad \vdots \\ \left(cumQ_{t,t+\delta t(RN)}^{(N)}\right)_{i,1} & \cdots & 1 \end{Bmatrix}=\begin{Bmatrix} K_{t,t+\delta t}^{(1)} \\ \vdots \\ K_{t,t+\delta t}^{(k)} \\ \vdots \\ K_{t,t+\delta t}^{(N)} \end{Bmatrix}$$

Avec $\left(cumQ_{t,t+\delta t_{(RN)}}^{(k)}\right)_{i,j}=\sum_{j=1}^{i}\left(Q_{t,t+\delta t_{(RN)}}^{(k)}\right)_{i,j}$

Pour chaque couple $\eta,\upsilon\in S^{(N)}$, $\vec{V}=\left(V^{(1)},\cdots,V^{(N)}\right)$ est un ensemble de variable aléatoires uniformes où la distribution jointe est donnée par la fonction copule C, dont la transformée de Laplace de la variable latente systémique Y est φ.

Pour des raisons de simplicité de calcul, nous considérons que:

$$\left(K_{t,t+\delta t}{}^{(k)}\right)_{i,0} = 0$$

La fonction copula de la migration jointe de migration ou de défaut de toute transition entre deux états quelconques de la chaîne de Markov

$$\{\eta,\upsilon\} = \begin{Bmatrix} \eta_1 & \upsilon_1 \\ \vdots & \vdots \\ \eta_m & \upsilon_m \end{Bmatrix}, where \quad \eta_i,\upsilon_i \in (1,\cdots,m)$$

est donnée par :

$$\begin{aligned} &\mathrm{P}_{RN}\left(\mathrm{X}_{t+h} = \upsilon / \mathrm{X}_t = \eta\right) \\ &= \mathrm{P}_{RN}\left(\left(K_{t,t+\delta t}{}^{(1)}\right)_{\eta_1,\upsilon_1-1} \le \mathrm{X}_{t+h}{}^{(1)} \prec \left(K_{t,t+\delta t}{}^{(1)}\right)_{\eta_1,\upsilon_1}, \cdots, \left(K_{t,t+\delta t}{}^{(N)}\right)_{\eta_N,\upsilon_N-1} \le \mathrm{X}_{t+h}{}^{(N)} \prec \left(K_{t,t+\delta t}{}^{(N)}\right)_{\eta_N,\upsilon_N-1}\right) \\ &= C\left(\left(K_{t,t+\delta t}{}^{(1)}\right)_{\eta_1,\upsilon_1-1} \le \mathrm{X}_{t+h}{}^{(1)} \prec \left(K_{t,t+\delta t}{}^{(1)}\right)_{\eta_1,\upsilon_1}, \cdots, \left(K_{t,t+\delta t}{}^{(N)}\right)_{\eta_N,\upsilon_N-1} \le \mathrm{X}_{t+h}{}^{(N)} \prec \left(K_{t,t+\delta t}{}^{(N)}\right)_{\eta_N,\upsilon_N-1}\right) \end{aligned}$$

Ce qui peut s'écrire dans le cadre des copules archimédiennes:

$$\mathrm{P}_{RN}\left(\mathrm{X}_{t+h} = \upsilon / \mathrm{X}_t = \eta\right) = \begin{vmatrix} \varphi\left(\varphi^{-1}\left(\left(K_{t,t+\delta t}{}^{(1)}\right)_{\eta_1,\upsilon_1}\right) + \cdots + \varphi^{-1}\left(\left(K_{t,t+\delta t}{}^{(N)}\right)_{\eta_N,\upsilon_N}\right)\right) \\ -\varphi\left(\varphi^{-1}\left(\left(K_{t,t+\delta t}{}^{(1)}\right)_{\eta_1,\upsilon_1-1}\right) + \cdots + \varphi^{-1}\left(\left(K_{t,t+\delta t}{}^{(N)}\right)_{\eta_N,\upsilon_N-1}\right)\right) \end{vmatrix}$$

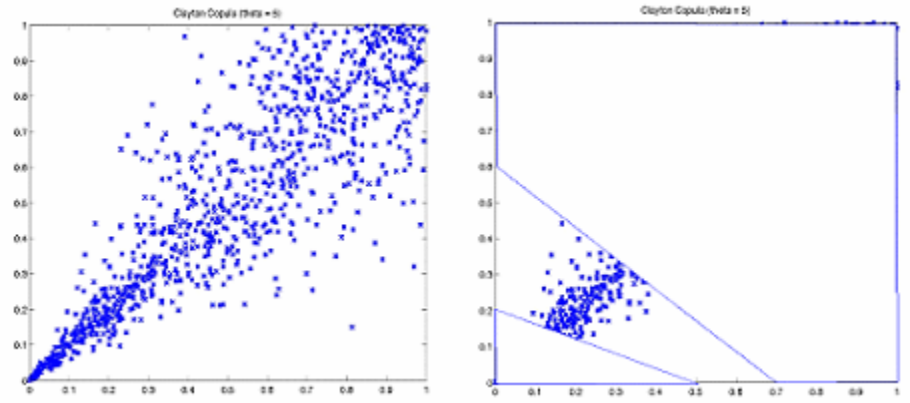


Figure 0.10 : à gauche 'une copule de Clayton ; à droite la zone correspondante a la probabilite jointe de migration dans le cas de cette copule

Chapitre 4 : Copules archimediennes generalisées ou multifactorielles.

Dans ce chapitre, nous développons des copules archimédiennes généralisées.

Modelisation multifactorielle avec les copules dynamique

Les modèles a facteurs sont des constructions mathématiques qui tentent d'expliquer la corrélation entre une grande série de variables a partir d'un nombre restreint de facteurs fondamentaux. Une hypothèse majeure d'analyse par facteurs avec les copules archimédiennes est que ce n'est pas possible d'observer ces facteurs directement. Dans le cadre de travail des copules dynamiques, nous supposons également que les facteurs ne sont pas observables. Soit $Y^k(t), k \in [1..m]$ un processus à spectre positif, processus de Levy indépendants ou aussi un « *compound Levy process* » avec un spectre positif.

Nous supposons que la transformée de Laplace de chaque $Y^k(t), k \in [1..m]$ existe et est définie par

$$\varphi^k(s) = E\left[\exp\left(-sY^k(t)\right)\right]$$

Supposons que la variable, Y(t), puisse s'écrire comme une combinaison linéaire de facteurs :

$$Y(t) = \sum_k^m \eta^k Y^k(t)$$

Nous pouvons démontrer facilement que:

$$\tilde{\varphi}^{\eta}(s) = E\left[\exp\left(-sY(t)\right)\right] = \prod_{k=1}^{m} \varphi^k\left(s\eta^k\right)$$

Definition d'une copule multi-facteurs

A partir de maintenant, nous analyserons la structure de dépendance dans un portefeuille de N variables (noms) qui dépendent a leurs tours des facteurs. Le $i^{ième}$ nom dans le portefeuille aura une variable de dépendance systémique, $Y_i(t)$, définie par :

$$Y_i(t) = \sum_k \eta_i^k Y^k(t), \qquad k \in [1..m], i \in [1..N]$$

Si $U_i(t)$ est une variable uniforme ou un processus alors

$$V_i(t) = \tilde{\varphi}^{\eta_i}\left(-\frac{Ln\left(U^i(t)\right)}{Y_i(t)}\right), i \in [1..N]$$

est la variable latente uniforme d'une copule archimédienne à plusieurs facteurs.

Preuve:

$$\Pr\left(V_1(t) < x_1, ..., V_N(t) < x_N\right)$$
$$= \Pr\left(U_1(t) < \exp\left\{-Y_1(t)\left(\tilde{\varphi}^{\eta_1}\right)^{-1}(x_1)\right\}, ..., U_N(t) < \exp\left\{-Y_N(t)\left(\tilde{\varphi}^{\eta_N}\right)^{-1}(x_N)\right\}\right)$$

Conditionnellement à $Y_i(t)$, les $U_i(t)$ sont mutuellement indépendants, d'où

$$\Pr\left(V_1(t) < x_1, \ldots, V_N(t) < x_N\right) = E\left[\prod_i \exp\left\{-Y_i(t)\left(\tilde{\varphi}^{\eta_i}\right)^{-1}(x_i)\right\}\right]$$

$$= \prod_k \varphi^k\left(\sum_i \left(\tilde{\varphi}^{\eta_i}\right)^{-1}(x_i) \times \eta_i^k\right), k \in [1..m],, i \in [1..N]$$

Exemple avec un "compound gamma process"

$$\varphi^k(s) = E\left[\exp\left(-sY^k(t)\right)\right] = \left(1 + a_2^k \mathrm{Ln}\left(1 + s\beta(t)\right)\right)^{-a_1^k t}$$

$$= \exp\left[-a_1^k t \,\mathrm{Ln}\left(1 + a_2^k \mathrm{Ln}\left(1 + s\beta(t)\right)\right)\right]$$

La figure 0.11 montre le cas où 6 processus dépendent de deux facteurs fondamentaux comme donné ci-dessous :

$$\begin{bmatrix} Y_1(t) \\ Y_2(t) \\ Y_3(t) \\ Y_4(t) \\ Y_5(t) \\ Y_6(t) \end{bmatrix} = \begin{bmatrix} \eta_1^1 & \eta_1^2 \\ \eta_2^1 & \eta_2^2 \\ \eta_3^1 & \eta_3^2 \\ \eta_4^1 & \eta_4^2 \\ \eta_5^1 & \eta_5^2 \\ \eta_6^1 & \eta_6^2 \end{bmatrix} \times \begin{bmatrix} Y^1(t) \\ Y^2(t) \end{bmatrix}, \qquad \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{bmatrix} = \begin{bmatrix} \eta_1^1 & \eta_1^2 \\ \eta_2^1 & \eta_2^2 \\ \eta_3^1 & \eta_3^2 \\ \eta_4^1 & \eta_4^2 \\ \eta_5^1 & \eta_5^2 \\ \eta_6^1 & \eta_6^2 \end{bmatrix} = \begin{bmatrix} 0.00 & 1.00 \\ 0.00 & 1.00 \\ 0.50 & 0.50 \\ 0.50 & 0.50 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix}$$

$$V_i(t) = \tilde{\varphi}^{\eta_i}\left(-\frac{\mathrm{Ln}\left(U^i(t)\right)}{Y_i(t)}\right), a_1 = 0.15, a_2 = 0.20 \;\; \& T = 5\,\text{ans} \quad i \in [1..6]$$

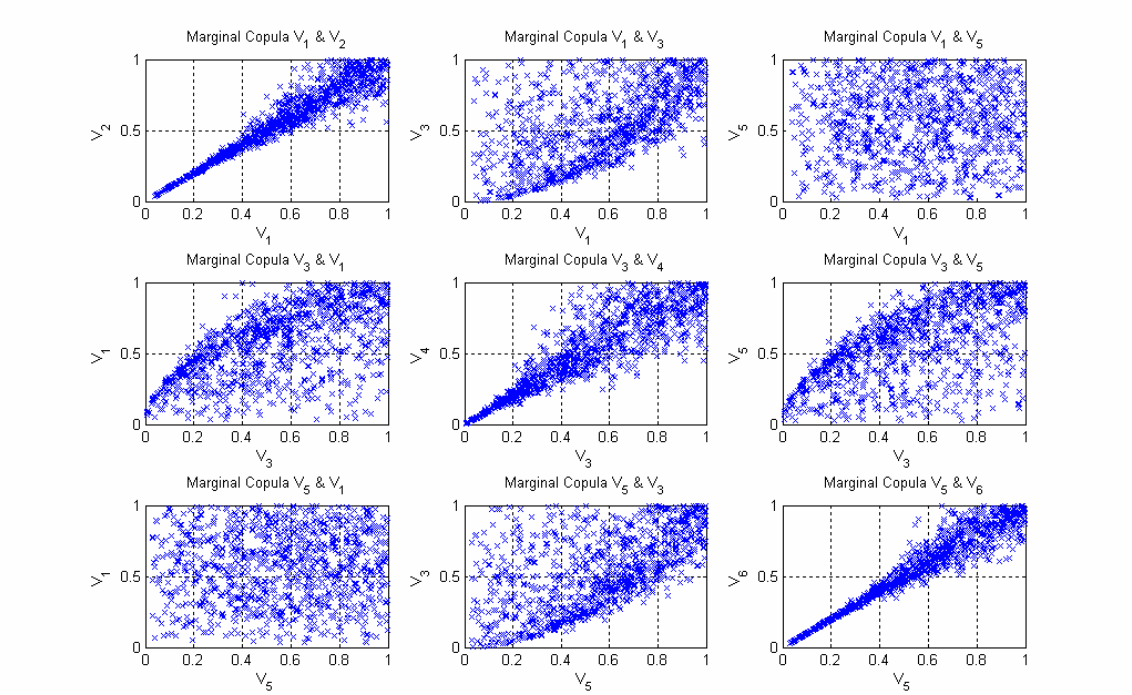


Figure 0.11: La copula dynamique a plusieurs facteurs nous permet de representer plusieurs formes de dependances entre les differentes distributions marginales. Les variables ne sont plus forcement interchangeables

Partie II : Applications

La deuxième moitié de la thèse est subdivisée en deux chapitres, chacun présente une application différente.

1. Le premier (**Chapitre 5**) sur les copules dynamiques et leur application à la tarification des tranches de CDOs a été présentés au congrès Advances in Econometrics, 5th Annual Advances in Econometrics Conference in Baton Rouge, Louisiane, November 3-5 2006 » et a été accepté pour publication dans « Advances in Econometrics: Econometrics of Risk Management", Volume 22, 2007 ».
2. Le second (**Chapitre 6**) présente une analyse comparée des résultats obtenus avec les copules dynamiques et d'autres modèles a un facteur. « Dynamic copulas process: Comparison with five 1-factor models for pricing CDOs ». Il a été soumis au « Journal of Derivatives ».
3. Une troisième application « Pricing Forward Starting CDOs using Dynamic Copula Processes » non présente dans cette thèse est acceptée pour publication dans « Credit derivatives Handbook » MacGraw Hill Edition 2007 ».

Conclusions et perspectives

Nous avons dans cette thèse créé de nouveaux modèles mathématiques qui prennent en compte entre autre la dynamique de dépendance (« tail dependence ») des marchés. Avec cette nouvelle famille de copules archimédiennes, nous avons effectué une première extension au cadre multi facteurs, et en suite nous avons montré que ce modèle peut être utilisé pour la prise en compte des migrations jointes qui peuvent être extrêmes.

Perspectives pour le travail futur :

La croissance fulgurante des dérivés de crédit dans l'industrie de la finance continuera à induire de nouvelles questions, tout en posant de nouveaux défis pour la modélisation mathématique. Un champ d'application intéressant pour les copules dynamiques est la modélisation de la structure de dépendance des fonds d'investissements alternatifs. Les dépendances extrêmes sur les marches de taux et de changes pourraient être également modélisées avec les copules dynamiques.

Chapter 1 : Introduction

Credit derivatives are one of the fastest developing parts of market finance at present. Not only have the nominal amounts increased phenomenally, but new products are continually being created. In the second part of this introductory chapter, we present these products starting out from the credit default swap, CDS, and show how they have evolved since their inception in the early 90s. Our aim is demonstrate this development has been market driven, with the mathematical models used for pricing lagging behind. At the outset models, such as Merton's model of the firm (1974), were required for pricing credit default swaps. As the market developed the weak points of the model became apparent and improved models were developed to overcome these points. Then as the credit derivative sector matured and the CDS market became more & more liquid, there was no point in using a model to price them; their default spreads were available directly in the market.

The next generation of products was based on a basket or a portfolio of underlying assets rather a single entity as had been the case in the first generation. Examples of such products are the first-to-default (or more generally the n^{th} to default) and then the collateralised debt obligation (or CDO). The first CDOs, called cash CDOs, required the product structurers to purchase the underlying bonds, which started to distort the bond markets. Synthetic CDOs were then created. Since it was no longer necessary to physically acquire the underlying assets, it became possible to sell individual tranches in the structure. Nowadays the standard CDOs (e.g. iTraxx and CDX) are based on portfolios of 125 entities (or names). There are two aspects to pricing tranches out of these structures

- Modelling the default probabilities of the individual entities (which can be done via the CDS spreads if the CDS for that name is sufficiently liquid)
- Modelling the "correlation" (or more accurately the dependency structure) between the entities in the portfolio

In this thesis we focus on the second question. Many different models have been developed for this. They include the Gaussian copula and the interpolation of base correlations, parametric factor copulas (local correlation), non-parametric factor approaches (implied copula), loss dynamics (top down, bottom up), with different strengths and weaknesses. Base correlation, first developed by JP Morgan McGinty & Ahluwalia (2004) became the industry standard for pricing CDO tranches. The base correlation model is based on Homogeneous

Large Pool Gaussian Copula Model, which is a simplified version of the Gaussian copula widely used in the market. This model is not new, the methodology is almost identical to the original Credit Metrics model (Gupton et al, 1997). It is a simplified form of earlier one-factor models (Vasicek, 1987) and is described in numerous places, for example Rogge & Schonbucher (2003), Schonbucher (2003).

By about 2003-2004 CDOs were becoming standard products. A new generation of products which we will refer to as third generation credit derivatives were starting to come on line: these include forward-starting CDS, forward-starting CDOs, options on CDOs and so forth. In contrast to early products, these derivatives require a dynamic model of the evolution of the "correlation" between the names over time, something which base correlation was not designed to do. The aim of this doctorate has been to develop a mathematically consistent framework for pricing these types of products.

Now let's step back in time to when this work started in October 2003. At that time, the "hot" topics were linked to computing the VaR for credit derivatives. The limitations of Merton's model of the firm were well known. It had been superseded by intensity models and by implied correlation, but base correlation was still in the future. At that point Jon Gregory, then the global head of credit derivatives research analytics at BNP Paribas, told my supervisor that the real problem was to find a dynamic model with lower tail dependence, which would be capable of modelling large baskets of underlying assets. Tests by Burtschell, Gregory & Laurent (2005a) had shown that the Clayton copula gave better results than most of other copulas, notably the Gaussian and Student's t which is computationally intensive. Copulas based on elliptic distributions (such as the Gaussian or Student's t) have symmetric upper and lower tails. They are effectively saying that defaults occur in the same way in bull and bear markets. In tough times, one default tends to trigger others, which is not the case in normal times. The classic "icecream cone" shape of the Clayton copula with its lower tail dependence (Figure 1.1, left) captures this insight; the symmetric Gaussian (normal distribution) copula (Figure 1.1, right) does not. So our objective right from the start was to build a dynamic copula model with lower tail dependency like this, and to test it on different types of credit derivatives.

1.1 Recent subprime crisis in the US

Since March 2007 there have been persistent rumours about the quality of subprime rated portfolios in the US. (The rising interest rates have meant that some Americans were unable to make mortgage repayments and were declared bankrupt). In the mid July, it was announced

that two Bear Stearns asset-backed (AB) funds based on these subprime mortgages were worthless. Our analysis of this crisis shows that as expected, downward movements in credit ratings (and upward rises in spreads) are quite asymmetric, with tighter correlation in the downward spiral.

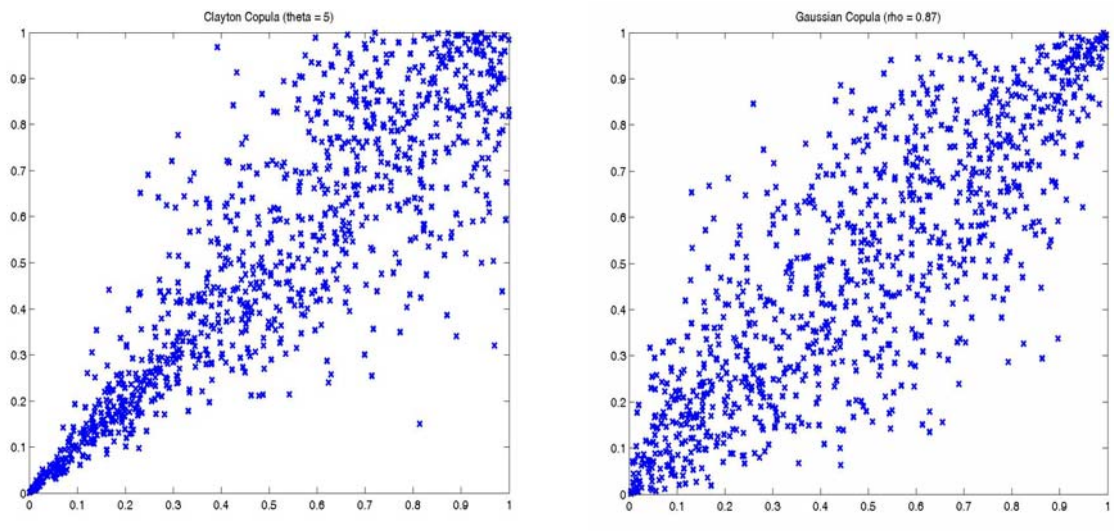


Figure 1.1: Clayton copula with parameter θ = 5 (left) and Gaussian copula with ρ = 0.87 (right)

Background to ABX

The ISDA ABCDS documentation which was completed in 2005, paved the way for the creation of a standardized synthetic index (ABX) in January 2006. Just one year later, the ABS market took another step forward via trading standardized ABX tranches which are based on a specific portfolio of 40 ABS tranches. Two products are traded: one based on BBB tranches and another based on BBB- tranches. The underlying ABS portfolio (40 tranches) for each product is the combination of tranches from ABX 06-2 and ABX 07-1. The BBB and BBB- portfolios use different attachment/detachment points. Table 1.1 compares these ABX BBB and ABX BBB- tranches with CDX and with a mezzanine tranche from a CDO of ABS. In normal market conditions, there is an almost no correlation between the ABX AAA price and the ABX BBB tranches, but when market conditions deteriorate, there is a significant increase in the dependency during the price decline. Figure 1.2 shows the price of five ABX tranches rated from AAA down to BBB over the period from 19 January to 6 July 2007. As expected the AAA tranche is rated at par (100%) almost the whole time except for a slight drop near the end of June. In contrast the other four tranches have consistently lost value. This decline indicates an increasing the likelihood of default. Note the correlation in price drops especially between the tranches with lower ratings.

Table: 1.1: Comparison of ABX, CDX and a CDO of ABS. (Source: JP Morgan based on 06-02 spread only)

	ABX BBB / ABX BBB-	CDX	Mezzanine CDO of ABS
Number of Names	40	125(IG), 100(HY)	70-120
Underlying Assets	100% Home Equity ABS	Diversified Corporate CDS	Home Equity ABS CDOs, CMBS Alt A RMBS
Index Spread	289/471 bps	33 bps (IG 5y), 241 bps (HY 5y)	180-250 bps target
Management	Unmanaged	Unmanaged	Managed
Cash Flow	Linear write-down	Linear write-down	Payment prioritiy varies: Interest/Principal can be diverted to senior tranches

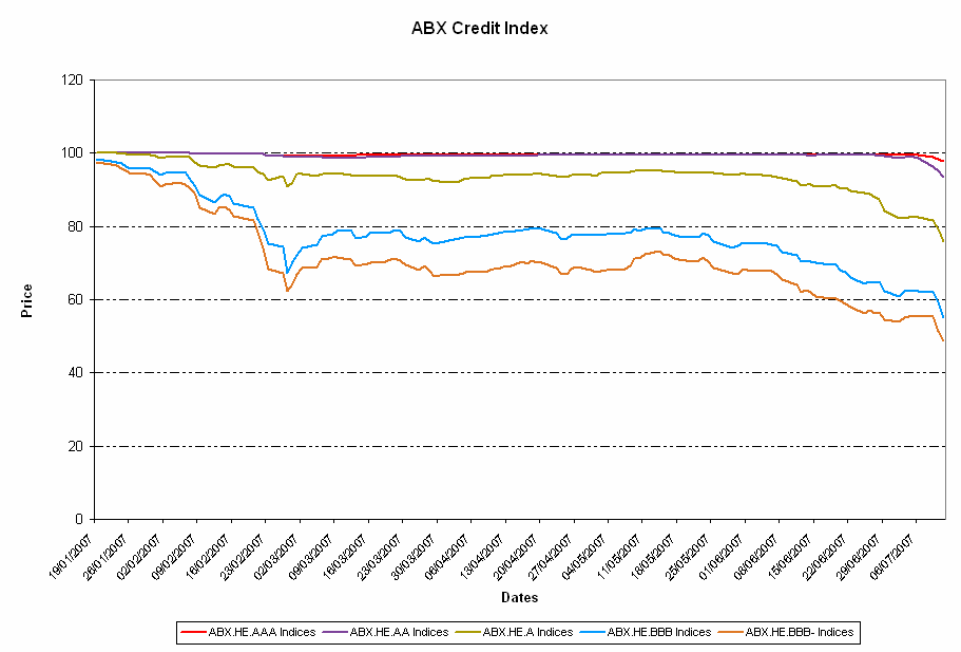


Figure 1.2: Evolution of the price of five ABX tranches rated from AAA (in red) through to BBB- (in brown) over the period from 19 January to 6 July 2007. The price decline indicates an increasing probability of default.

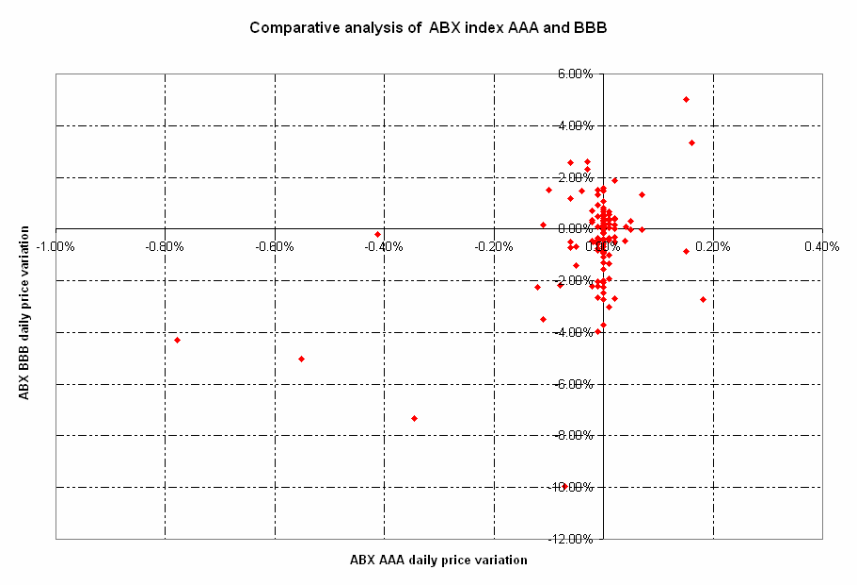


Figure 1.3: The daily variation of ABX index AAA and BBB. A negative variation of the price means a deterioration of the credit quality. (Source Bloomberg)

The asymmetric nature of downward changes compared to upward changes can be seen in Figure 1.3 which shows the daily variation of the ABX index AAA (X-axis) compared to that of the ABX index BBB (Y axis). ABX are quoted in terms of price instead of spread as are credit indexes such as iTraxx and CDX. So a negative variation of the spread means a deterioration of the credit quality. Here the extreme daily variations of -4% or more for ABX BBB were mainly caused by the sub-prime effect in the US in the second quarter 2007.

The effect of the deterioration in the sub-prime class can also be seen in Figure 1.4 which shows the spreads for four indexes over the second quarter 2007:

- DJ DCX America S8 Investment Grade 5Y (blue)
- DJ DCX America S8 High Yield 5Y (brown)
- DJ DCX America S8 High Volatility 5Y (purple)
- DJ DCX America S8 Crossover 5Y (red)

Note how the investment grade (in blue) has remained virtually flat compared to the other three. In comparison, the high yield index (in brown) jumped sharply from 200bps to 300 bps in mid-March and rose again to nearly 400bps at the end of the quarter. Figure 1.5 shows the histograms of the changes in daily spread change of these four US indexes over the same period. The impact of the subprime is much more marked for the high yield index compared to the investment grade. In fact we can even observe a daily jump of 40% for the high yield.

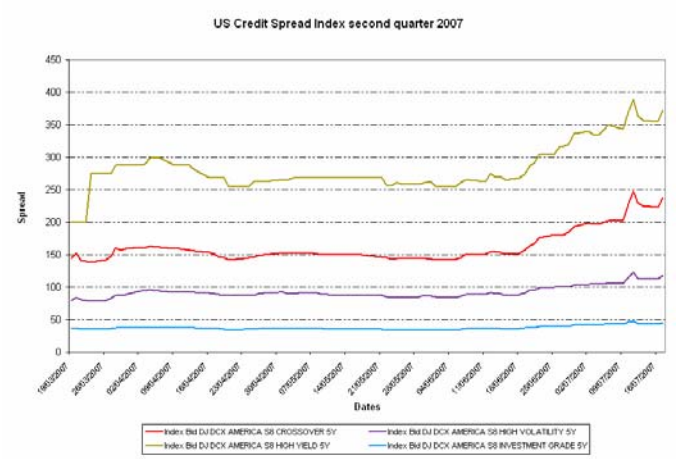


Figure 1.4 : The evolution of four US indexes during the second quarter 2007. Note how much the high yield index (brown) has deteriorated compared to the investment grade (blue).

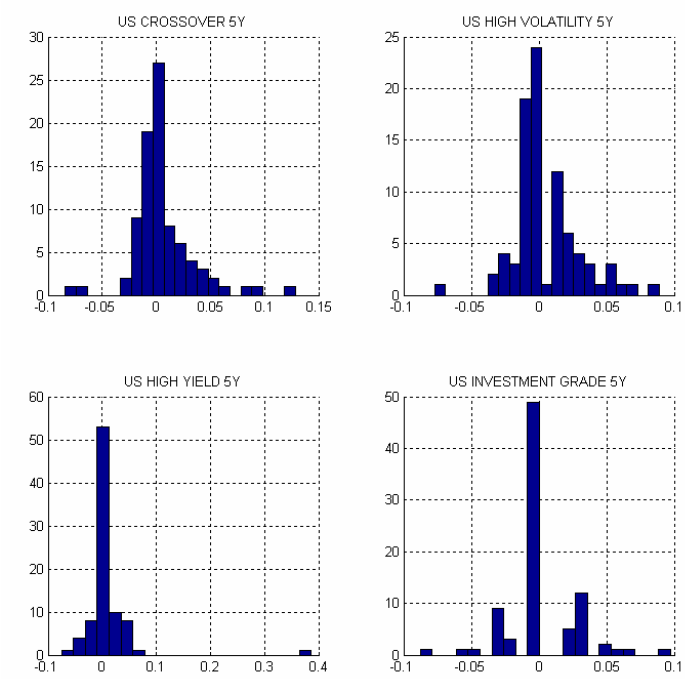


Figure 1.5: The histogram of daily spread (iTraxx and CDX are quoted in spread) change of four US indexes during the second quarter 2007. Note that the impact of the subprime is much more higher for the high yield index compared to the investment grade. In fact we can even observe a daily jump of 40%, the Invest grade index become slightly asymmetric conditional to the subprime event.

The daily spread variations of the CrossOver Index (X-Over) for the US versus the same index for Europe during the second quarter 2007 are presented in Figure 1.6(a) & (b) firstly as a cross-plot then as a copula. As before, the extreme variations of 7% or more are mainly due to the sub-prime effect in the US. As these data give the daily variation in spread, positive values correspond to deterioration in credit quality, which explains the upper tail dependence instead on lower tail dependence in standard CDOs.

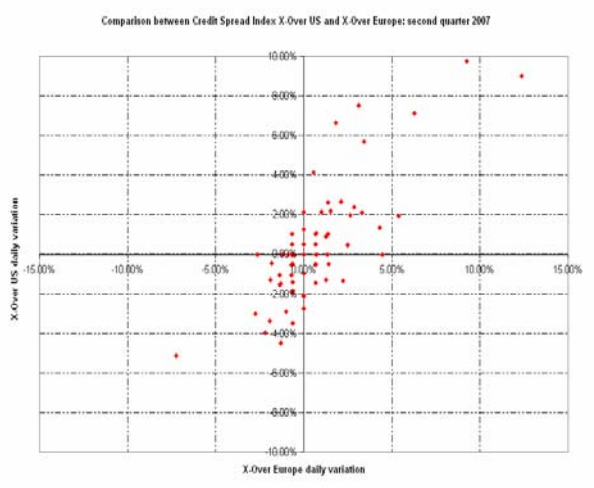


Figure 1.6(a): The daily variation of the CrossOver Index (X-Over) US and Europe seems to have an asymmetric dependence. A positive variation of the spread means a deterioration of the credit quality. The extreme daily variations of 7% or more are mainly due by the Sub Prime crisis in the US. (Source Bloomberg)

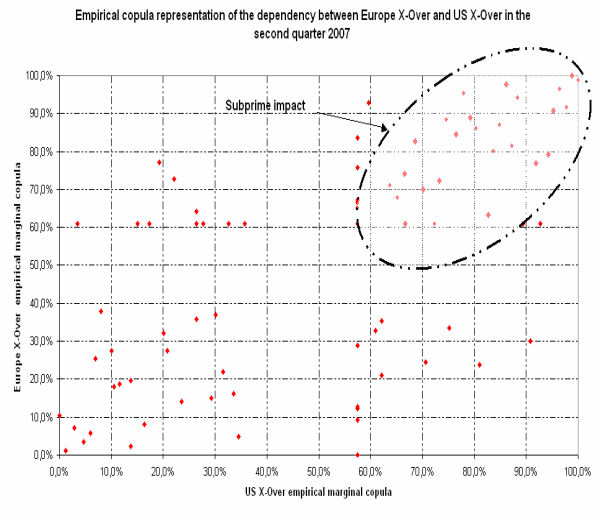


Figure 1.6(b): Copula representation based on empirical cumulative distribution of the daily variation of the CrossOver Index (X-Over) US and US High Yield index shows upper tail dependence for spread variation, which is equivalent to a lower tail depdendence of default. As spread is a decreasing function of default probability, an increase in the spread means a drop in credit quality.

A clearer picture of credit crises can be obtained by taking the iTraxx S3 spreads over a longer period, for example by starting in April 2005 during the GM-Ford crisis through to the end of August 2007. Figure 1.8 compares the spreads for the Auto 10Y S3 series with those for Senior Financials 10Y. As expected, the histograms are markedly skew. The experimental copula shows the characteristic "ice-cream cone" shape. The points shown in green correspond to the GM-Ford crisis while the red ones go from July to August 2007. These show that during the subprime crisis the spreads for the Senior Financials 10Y rose much faster than those for Autos 10Y.

The events that occurred during the recent subprime crisis have confirmed our impression about the importance of being able to incorporate upper/lower tail dependence when modeling correlation products.

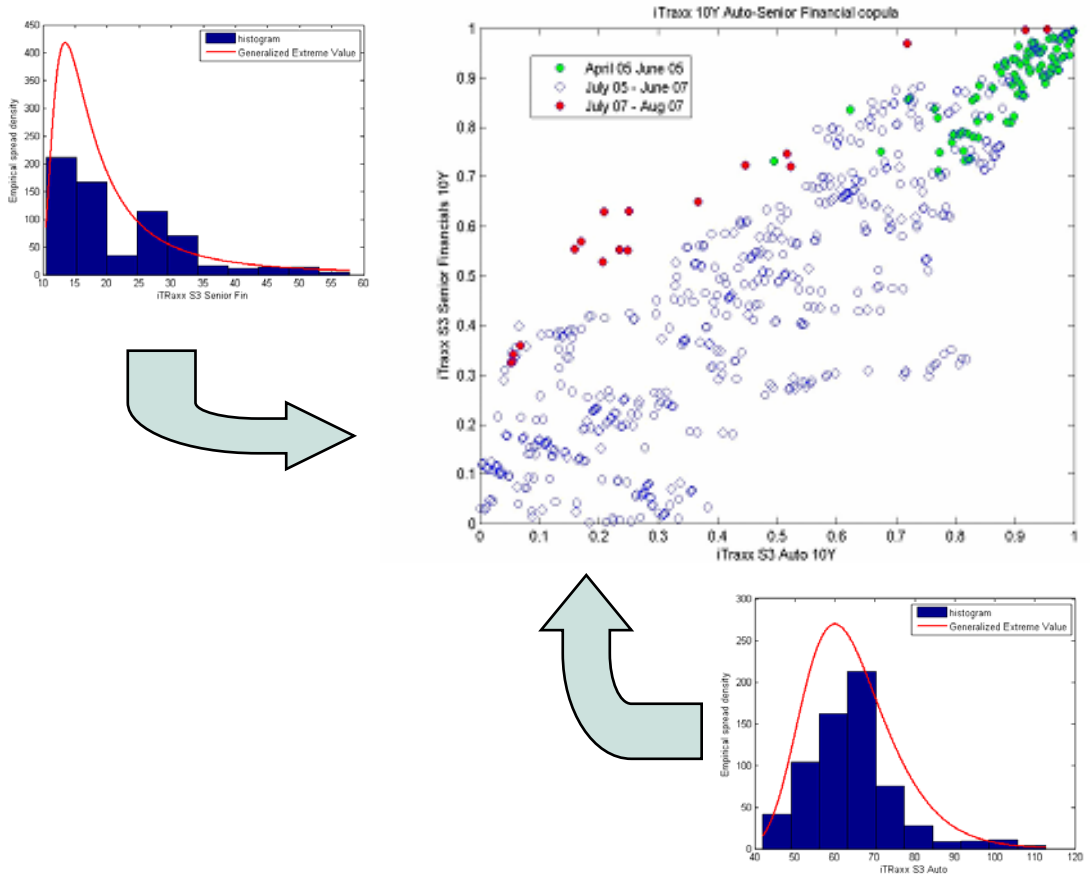


Figure 1.8: Empirical copula computed from the spreads for the iTraxx S3 series, Auto10Y versus Senior Financials 10Y, together with their histograms (senior Financials on left & Autos below). The data cover the period April 2005 to August 2007. The green points correspond to the GM-Ford crisis in 2005, the red one to the subprime crisis

1.2 Structure of the thesis

In the rest of this chapter, we give a brief history of credit derivatives and review the market trends in the credit derivatives & securitization. Then we present the characteristics of the main products which can be split into three generations:

1. Single name protection
2. Basket products (where the correlation between the different names becomes important)
3. Forward-starting products

Although the thrust of this thesis is modelling the correlation found second and third generation products, we describe the main steps involved in pricing first generation products because we will use these techniques when fitting the marginal distributions for individual names within portfolios. It is clear that as the market evolved, more advanced models were required in order to capture idiosyncratic risk and dynamic dependence or correlation risk. The last part of the chapter reviews the literature on modelling correlation and on Archimedean copulas.

Ch 2: Dynamic copula model

We present the first formulation of the dynamic copula processes and then a new model for pricing CDO tranches based on a dynamic copula process with low tail dependence as in the Clayton copula

A formula for the asymptotic loss distribution, similar to the Vasicek formula, is derived. This simplifies the computations for the case of a fine grained portfolio.

We finally in this section, explore the use as a building block of a dynamic copula a gamma Ornstein–Uhlenbeck process, where a closed form solution is known for the Laplace transform. The mathematical constructions in this part of the thesis no longer use Brownian motion.

Ch 3: Combining credit migration and copula

We propose a model for the joint dynamics of credit ratings of several firms based on copulas and dynamic copulas. Namely, individual credit ratings are modelled by a continuous time Markov chain, while their joint dynamics is modelled using copulas. The use of copulas allows us to incorporate our knowledge of the modelling of single name credit migration processes, into a multivariate framework. We then revisit the joint laws of the default times of all the firms in the portfolio. The development of new products such as Constant Proportion

Debt Obligation (CPDO) requires the modelling of credit migration, and correlation in order to handle substitution rule on index during the roll.

Ch 4: Multi-factor & Time changed approach

A multifactor approach is developed within the new formulated dynamic copula processes, and a time changed levy process is used to introduce dependency on spreads dynamics. We show in this chapter that the building block of time changed approach fail within the frameworks of dynamic copulas.

Part II: Two practical applications

The second half of the thesis is split into two chapters, each presenting work on a different application. The first one on pricing synthetic CDOs at different maturities was presented at the 5th Annual Advances in Econometrics Conference in Baton Rouge, Louisiane, November 3-5 2006 and has been submitted for publication in "Advances in Econometrics: Econometrics of Risk Management", Volume 22, 2007. The second one presents a comparison of the pricing given by these dynamic copulas with five well-known copula models. It has been submitted to the Journal of Derivatives. These two chapters present developments of work on dynamic copulas with applications to credit risk derivatives.

Ch 5: Dynamic Copula Processes: A new way of modelling CDO tranches

In this chapter, the dynamic copula framework is used for pricing tranches of synthetic CDOs at different maturities, which requires a model of the dynamic dependence between default times. While the base correlation approach gives good results at any given maturity, it does not link prices and spreads at different times. A stochastic process with time-dependence between the different names is needed. Existing factor models using Gaussian or Student's t copulas do not price CDO tranches correctly, because their upper and lower tails are symmetric. An "icecream cone" shaped copula with lower tail dependence (such as in the Clayton copula, Fig 1) would be more appropriate. After presenting a new family of dynamic copula processes, we focus on a specific one with lower tail dependence. Using CDS data as at July 2005, we show that the base correlations given by this model at the standard detachment points are very similar to those quoted in the market for a maturity of 5 years.

Ch 6: Dynamic copulas processes: Comparison with five 1-factor models for pricing CDOs

The tests carried out in the previous chapter showed that the model could reproduce the base correlations. But how does it compare to other models? In this chapter using market data as at 30 January 2006 we demonstrate that it outperforms five well-known one factor copula models. Following the approach used by Van der Voort (2006), we fitted the parameters to match the base correlation for a maturity of 5Y for iTraxx and for CDX, then computed the model-implied correlation skew for a maturity of 10Y. In both cases the results obtained from the dynamic copula model were much closer to the market correlation skews, than those given by existing copula factor models.

Ch 7: Conclusions

The final chapter reviews the work that has been carried out and suggests several perspectives for further work.

Ch 8: References

1.3 Brief history of credit derivatives up to 2000

Credit derivatives are among the fastest growing products in the capital markets arena, rapidly assuming more complex structures and forms. Banks, insurance companies, hedge funds, pension funds, asset managers and structured finance vehicles are increasingly using credit derivatives for arbitrage, speculation and hedging.

Credit derivatives[1] had emerged by early 1993. In March 1993, Global Finance carried an article saying that three Wall Street firms - J. P. Morgan, Merrill Lynch, and Bankers Trust - were marketing some form of credit derivatives. Prophetically, this article also said that credit derivatives could, within a few years, rival the $4-trillion market for interest rate swaps. In retrospect, we know that this was right. According to a new report to be published by the British Bankers' Association (BBA) at its Credit Derivatives conference in September 2006, the global market in Credit Derivatives is expected to rise to $33 trillion by the end of 2008 (Fig 1.8).

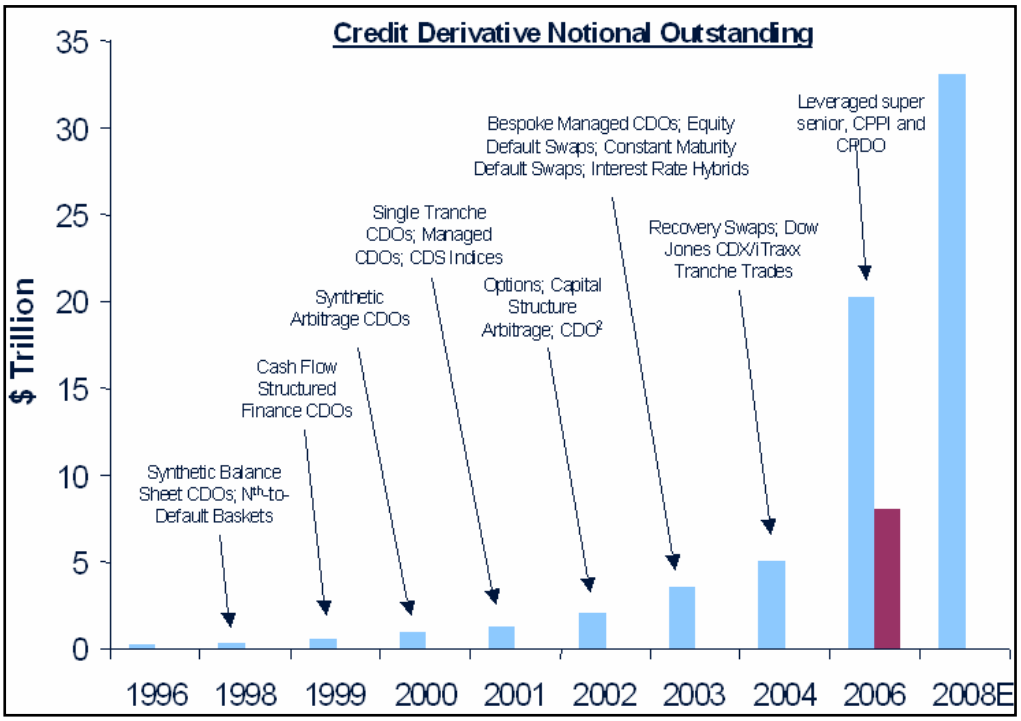


Figure 1.8: The evolution of the credit derivatives market since 1996. Note: Notional excludes asset swaps. Source: British Bankers Association Credit Derivatives Report 2006; Barclays Capital Credit Research

[1] Readers interested in the history of how credit derivatives were developed, can consult *Credit Risk Modelling* edited by M. Gordy (2003) which is a compendium of 34 technical papers published in Risk from 1999 to 2003, or *Credit Derivatives: The Definitive Guide* edited by Jon Gregory (2003) which is a collection of 21 articles on credit risk derivatives.

Although credit derivatives were already a frequent topic in the financial press by 1993, they initially faced resistance. In November 1993, the Investment Dealers Digest carried an article entitled *Derivatives pros snubbed on latest exotic product* which claimed that a number of private credit derivative deals had been seen in the market but it was doubted if they were ever completed. The article also said that Standard and Poor's had refused to rate credit derivative products and this refusal may put a permanent damper on the fledgling market. S&P seems to have issued some kind of a document which said that in essence, these securities represent a bet by the investor that none of the corporate issuers in the reference group will default or go bankrupt. One commentator quoted in the said article said:

> *It (credit derivatives) is like Russian roulette. It doesn't make a difference if there's only one bullet: If you get it you die.*

Almost 3 years later, Euromoney[2] reported that a lot of credit derivatives deals were already happening. From a product that was branded as a "touted" product in 1993, the market perception had changed into one of unbridled optimism. The article said:

> *The potential of credit derivatives is immense. There are hundreds of possible applications: for commercial banks which want to change the risk profile of their loan books; for investment banks managing huge bond and derivatives portfolios; for manufacturing companies over-exposed to a single customer; for equity investors in project finance deals with unacceptable sovereign risk; for institutional investors that have unusual risk appetites (or just want to speculate); even for employees worried about the safety of their deferred remuneration. The potential uses are so widespread that some market participants argue that credit derivatives could eventually outstrip all other derivative products in size and importance.*

Key Milestones

Here are some significant milestones in the development of credit derivatives:

- **1992 - Credit derivatives emerge. ISDA[3] first uses the term "credit derivatives" to describe a new, exotic type of over-the-counter contract.**
- **1993 -KMV introduces the first version of its Portfolio Manager model, the first credit portfolio model.**

The KMV model is an extension of the Merton model that allows the estimation of the loss

[2] March 1996: *Credit derivatives get cracking*
[3] ISDA: International Swap Dealers Association

distribution of a portfolio of loans based on the calibrated default probability. The best known representative of this type of default probabilities is the Expected Default Frequency (EDF) from Moody's KMV.

- **1994 - Credit derivatives market begins to evolve.**
- **September 1996 - The first CLO by UK's National Westminster Bank.**

A balance-sheet collateralized loan obligation (CLO) is a form of securitization in which assets (bank loans) are removed from a bank's balance sheet and packaged into marketable securities that are sold on to investors[4]. Different tranches of the CLO have different risk-return characteristics and can be targeted at specific investor classes. One appeal of certain CLO tranches has been that they can offer more attractive yields than similarly rated securities.

The first significant step in the development of the CLO market was the $5bn ROSE Funding #1 issue by the UK's National Westminster Bank in September 1996[5]. This CLO was backed by an international portfolio of more than 200 commercial loans. One year later, Nations Bank launched a $4bn CLO, the first significant deal in the US. Japanese and Continental European banks soon followed. Deutsche Bank's first Core CLO was largely backed by loans to medium-sized German companies. In the absence of a CLO-type structure, selling loans made to Mittelstand companies would have been difficult because of the strong lending relationships built up by German banks with their corporate clients.

- **April 1997 - J P Morgan launches CreditMetrics**

CreditMetrics is a tool for assessing portfolio risk due to changes in debt value caused by changes in obligor credit quality. It includes changes in value caused not only by possible default events, but also by upgrades and downgrades in credit quality.

- **October 1997 - Credit Suisse launches CreditRisk+**

Developed by Credit Suisse, CreditRisk+ is based on a portfolio approach to modelling credit default risk that takes into account information relating to size and maturity of an exposure and the credit quality and systematic risk of an obligor. The CreditRisk+ Model is a statistical model of credit default risk that makes no assumptions about the causes of default. Actuarial techniques applied widely in the

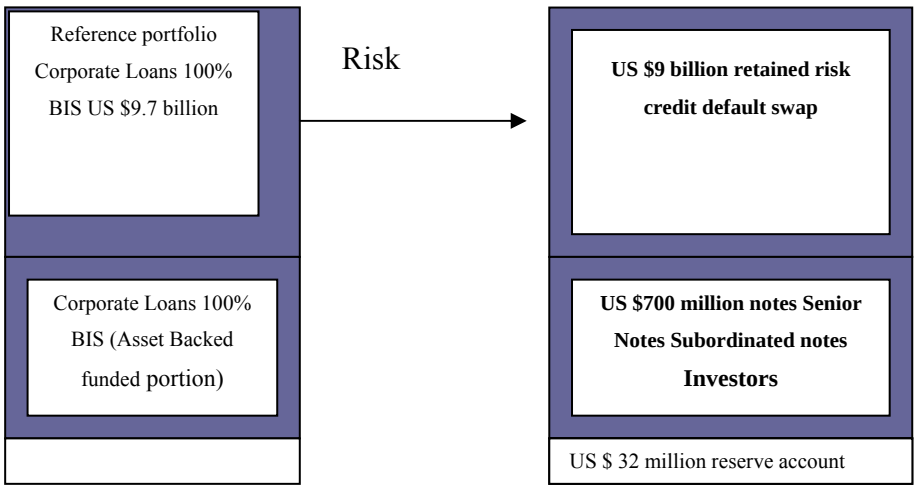
[4] The main alternative to a balance sheet CLO is an arbitrage CLO. In these, an asset management firm will buy credit risk in the market before selling claims on the repackaged risk. The originator of the deal profits from the yield differential between the assets in the portfolio and the cost of funding the assets through the sale of securities.

[5] Continental Bank's FRENDS issue in 1988 is often cited as a precursor to the CLO market, but this was a relatively isolated deal..

insurance industry are used to model the sudden event of an obligor default. This approach contrasts with the mathematical techniques typically used in finance where one is usually concerned with modelling continuous price changes rather than sudden events. Applying insurance modelling techniques, the analytic CreditRisk+ Model captures the essential characteristics of credit default events and allows explicit calculation of a full loss distribution for a portfolio of credit exposures.

- **December 1997 - The first synthetic securitisation, JP Morgan's Bistro deal.**

Generally viewed as the first synthetic securitisation, BISTRO was a JP Morgan vehicle brought to market in December 1997. The aim of the transaction was to remove the credit risk on a portfolio of corporate credits held on JP Morgan's books, with no funding or balance sheet impact.



Source: JP Morgan

Figure 1.9: The structure of the first synthetic securitisation brought to market in Dec 1997 by JP Morgan

- **1998 -McKinsey introduced CreditPortfolioView , credit risk portfolio model**
- **July 1999 - Credit derivative definitions issued by ISDA.**

The development of credit derivatives had been hampered by the lack of standardised contracts and the amount of legal work required to set up each deal. In July 1999 ISDA, the International Association of Swap Dealers, provided a standardised legal framework with agreed-upon definitions of default events. This considerably simplified the back office procedures for booking these deals.

1.4 Three generations of credit derivatives

Credit derivatives have gone from strength to strength since then. The products that have been developed can be split into three generations:

1. Single name protection
2. Basket products (where the correlation between the names first became important)
3. Forward-starting products

First-generation credit derivatives

Credit derivatives can be defined as arrangements that allows one party (protection buyer or originator) to transfer the credit risk of a reference asset (or assets), which it may or may not own, to one or more other parties (the protection sellers).
The main first generation credit derivatives currently being used in the market are:

Total return swap:

Total return swap, or total rate of return swap, or TRORS, is a contract in which one party receives interest payments on a reference asset plus any capital gains and losses over the payment period, while the other receives a specified fixed or floating cash flow unrelated to the credit worthiness of the reference asset, especially where the payments are based on the same notional amount. The reference asset may be any asset, index, or basket of assets. TRORS are particularly popular on bank loans, which do not have a liquid repo market.
So TRORS allows one party to derive the economic benefit of owning an asset without putting that asset on its balance sheet, and allows the other (which does retain that asset on its balance sheet) to buy protection against loss in its value.
The essential difference between a TRORS and a credit default swap is that the latter provides protection not against loss in asset value but against specific credit events. In a sense, a TRORS is not a credit derivative at all, in the same sense that a CDS is. ***A TRORS is funding-cost arbitrage.***

Credit default swap:

A credit default swap (CDS) is a swap designed to transfer the credit exposure of fixed income products between parties. It is an agreement between a protection buyer and a protection seller in which the buyer pays a periodic fee in return for a contingent payment by the seller if a credit event (such as a certain default) happens to the reference entity. Most CDS contracts are physically settled: if a credit event occurs, the protection seller must pay

the par amount of the contract to the protection buyer who is obliged to deliver a bond or loan of the name against which protection is being sold. Figure 1.10 summarises the situation.

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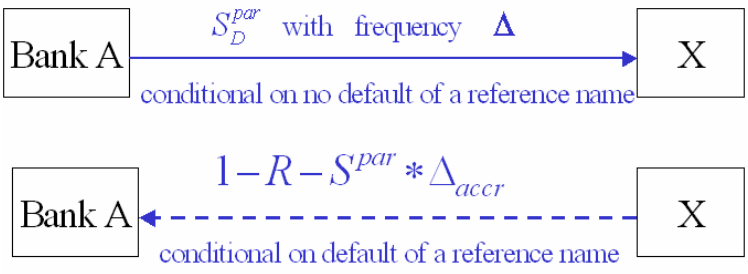


Figure 1.10: Basic principles in a credit default swap (CDS) in which a protection buyer, Bank A, pays a fixed amount to the protection provided, X, provided that no credit event occurs. However conditional on default Bank A received a reference bond (generally the cheapest-to deliver) from a specified set. Here R is a recovery value of a reference bond. It is assumed to be the same for all CDS on a given name.

A CDS is often used like an insurance policy, or hedge for the holder of debt, except that because there is no requirement to actually hold any asset or suffer a loss. So a CDS is not actually insurance. Over recent years, the CDS market has become extremely liquid. The typical maturities available for CDS contract are 3, 5, 7 and 10 years with 5Y being the most liquid. Almost any maturity is possible for an over-the-counter CDS. Box N° 1 describes the standard method for valuing a CDS, by computing the value of the two legs: fee and contingent. For a par spread, the net present value of both legs must equal to zero.

Credit default swaption or credit default option:

A default option, credit default swaption or credit default option is an option to buy protection (payer option) or sell protection (receiver option) as a credit default swap on a specific reference credit with a specific maturity. The option is usually European, (i.e. exercisable at only one date in the future). The strike price is defined as a coupon on the credit default swap. Credit default options on single credits are extinguished upon default without any cashflows. Therefore buying a payer option is not a good protection against an actual default; it merely provides protection against a rise in the credit spread. Having said that, options on credit indices such as iTraxx and iBoxx, include any defaulted entities in the intrinsic value of the option when exercised.

Box N°1: Determining the Par Spread of a Credit Default Swap

The value of the fee leg is approximated by:

$$\text{PV No Default} = S_n \text{Annuity}_n = S_n \sum_{i=1}^{n} DF_i \times PND_i \times \Delta_i$$

where S_n is the Par Spread for maturity n

DF_i is the Riskless Discount Factor from T_o to T_i

PND_i is the No Default Probability from T_o to T_i

Δ_i is the Accrual Period from T_{i-1} to T_i

If accrual fee is paid upon default, then the value of the fee leg is approximated by:

$$\text{PV No Default} + \text{PV Default Accruals} = S_n \text{ Annuity} + S_n \left(\text{Default Accruals}_n\right)$$

$$= S_n \sum_{i=1}^{n} DF_i \times PND_i \times \Delta_i + S_n \sum_{i=1}^{n} DF_i \times \left(PND_{i-1} - PND_i\right) \times \frac{\Delta_i}{2}$$

where $(PND_{i-1} - PND_i)$ is the probability of a credit event occurring during period T_{i-1} to T_i

$\frac{\Delta_i}{2}$ is the Average Accrual from T_{i-1} to T_i

The value of the contingent leg is approximated by:

$$\text{PV Of Contingent} = \text{Contingent}_n = \left(1 - R\right) \sum_{i=1}^{n} DF_i \times \left(PND_{i-1} - PND_i\right)$$

where R is the Recovery Rate of the reference obligation

Therefore, for a par credit default swap is the solution of the equation,

$$\text{PV No Default} + \text{PV Default Accruals} = \text{PV Of Contingent}$$

$$S_n = \frac{\left(1 - R\right) \sum_{i=1}^{n} DF_i \times \left(PND_{i-1} - PND_i\right)}{\sum_{i=1}^{n} DF_i \times PND_i \times \Delta_i + \sum_{i=1}^{n} DF_i \times \left(PND_{i-1} - PND_i\right) \times \frac{\Delta_i}{2}}$$

Credit linked notes:

A credit-linked note (CLN) is a security issued by a special purpose company or trust, designed to offer investors par value at maturity unless a referenced credit defaults. In the case of default, the investors receive a recovery rate. The trust will also have entered into a default swap with a dealer. In case of default, the trust will pay the dealer par minus the recovery rate, in exchange for an annual fee which is passed on to the investors in the form of a higher yield on their note.

The purpose of the arrangement is to pass the risk of specific default onto investors willing to bear that risk in return for the higher yield it makes available. The CLNs themselves are typically backed by very highly-rated collateral, such as U.S. Treasury securities. CLN is a security with an embedded credit default swap allowing the issuer to transfer a specific credit risk to credit investors.

CLNs are created through a Special Purpose Company (SPC), or trust, which is collateralised with AAA-rated securities. Investors buy securities from a trust that pays a fixed or floating coupon during the life of the note. At maturity, the investors receive par unless the referenced credit defaults or declares bankruptcy, in which case they receive an amount equal to the recovery rate.

Second generation credit derivatives

The development of credit derivatives parallels that of fixed income and interest rate products. The increased use of credit default swaps and other basic credit derivatives helped to build the critical mass and liquidity levels required for constructing the next generation credit derivative products. Portfolio products such as synthetic CDOs and nth-to-default baskets, with enhanced returns and customized risks, were among the first extensions of CDS. These products borrowed techniques from the securitization business. In a conventional cash flow CDO, a portfolio of loans or other debt obligations are transferred to a vehicle that issues liabilities (rated notes and equity). Repayment of the liabilities is collateralised by the portfolio of loans or debts. However, in the case of synthetic CDOs, the tranches are collateralized by a portfolio of investment-grade assets and the additional credit yield is acquired by selling credit protection.

By purchasing a combination of AAA-rated instruments and selling CDS on individual single-name instruments, the vehicle can create the desired maturities, together with the concentration and industry focus required to issue liabilities with strong ratings.

First-to-default basket:

In a first-to-default basket, the risk buyer typically takes a credit position in each credit equal to the notional at stake. After the first credit event, the first-to-default note (swap) stops and the investor no longer bears the credit risk to the basket. First-to-default Credit Linked Note will either be unwound immediately after the Credit Event – this is usually the case when the notes are issued by an SPV - or remain outstanding – this is often the case with issuers - in which case losses on default will be carried forward and settled at maturity. Losses on default are calculated as the difference between par and the final price of a reference obligation, as determined by a bid-side dealer poll for reference obligations, plus or minus, in some cases, the mark-to-market on any embedded currency/interest rate swaps transforming the cash flows of the collateral.

Collateralised debt obligation

Collateralised debt obligations (CDOs) are similar to asset-backed securities and structured finance products. In a CDO a portfolio of bonds, loans, or other fixed income securities is gathered together, and used to create a new set of fixed income securities. This allows for a technique called "credit tranching" by which losses from the portfolio are repackaged.

CDOs typically issue four classes of securities designated as senior debt, mezzanine debt, subordinate debt and equity. Any losses from the portfolio of investments are applied to equity first before being applied to earlier ones. As a result, products ranging from the risky equity debt to the relatively low risk senior debt can be created from one basket of bonds or loans. See Fig 1.11. CDOs can be split into two broad classes: balance sheet CDOs and arbitrage CDOs. Balance sheet CDOs are those which result into transfer of loans from the balance sheet and hence, which impact the balance sheet of the originator. Arbitrage CDOs are those where the originator is merely a repackager: buying loans or bonds or ABS from the market, pooling them together and securitising the same. The prime objective in balance sheet CDOs is the reduction of regulatory capital, while the purpose in arbitrage CDOs is making arbitraging profits.

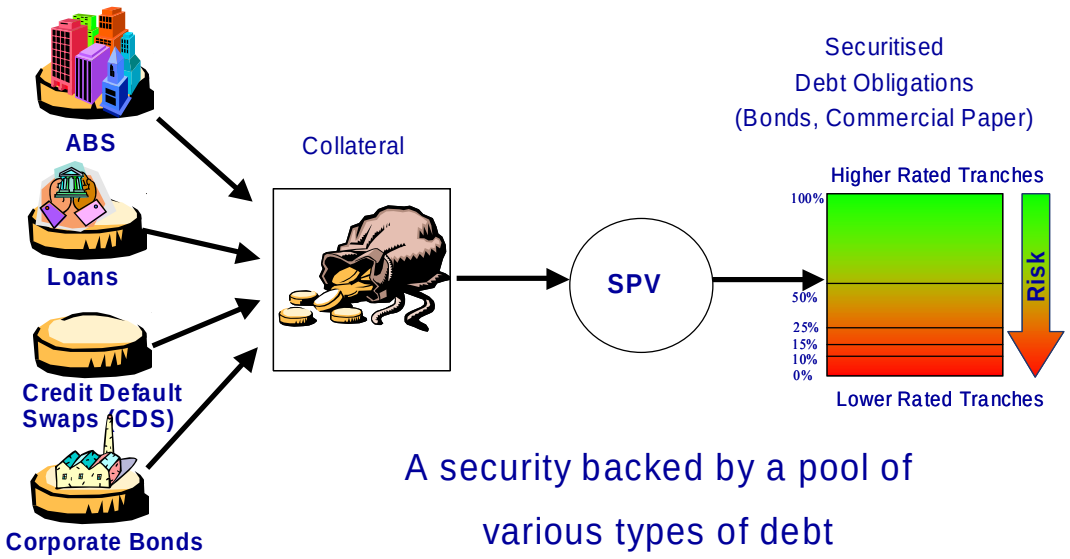


Figure 1.11: Typical CDO showing the asset on the left and the tranching structure on the right

(Source BNPParibas)

From a legal point of view, most CDOs are constructed via a special purpose vehicle (SPV), a bankruptcy remote company. See Figure 1.12.

The term CDO is often used as a generic term that includes:

- Collateralised bond obligations (CBOs) -- CDOs backed primarily by bonds
- Collateralised loan obligations (CLOs) -- CDOs backed primarily by leveraged loans
- Structured finance CDOs (SFCDOs) -- CDOs backed primarily by asset-backed securities
- Collateralised mortgage obligation (CMOs) -- CDOs backed primarily by residential or commercial mortgages.
- Commercial Real Estate CDOs (CRE CDOs) -- backed primarily by real estate assets
- CDO-Squared -- CDOs backed primarily by securities issued by other CDO vehicles.
- CDO^n -- Generic term for CDO^3 (CDO cubed) and higher. These are particularly difficult vehicles to model due to the possible repetition of exposures in the underlying.

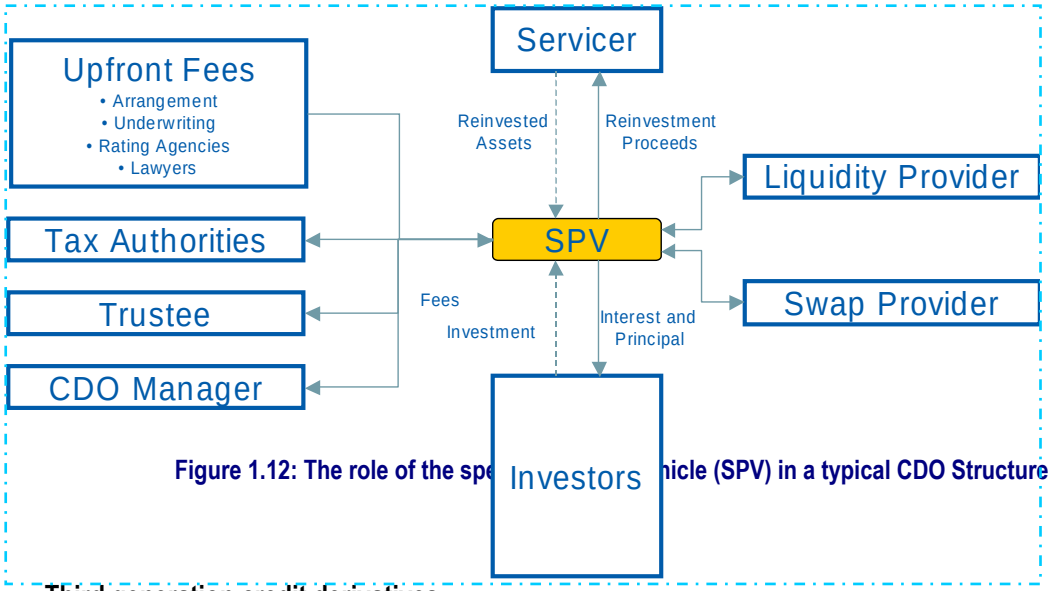


Figure 1.12: The role of the spe[illegible]nicle (SPV) in a typical CDO Structure

Third generation credit derivatives

Figure 1.13 shows the evolution of the second and third generation credit derivatives from basket derivatives through CDO^n and latest products. Market value CDOs transactions such as CFO (Colletarised Fund Obligations) are also mentioned as an innovative product, alongside CPPI and CPDO

- **Forward starting CDS**

The only difference between a forward starting contract and a regular contract is that regular CDS starts immediately, while a forward CDS starts on a future date. Box 2 shows how the present value of a CDS contract starting at time t_k with maturity t_n, can be expressed from a protection seller's point of view. Forward spread is defined as the par spread of a forward starting CDS contract.

- **Forward starting CDO and possibly options on CDO tranches**

The availability of CDO data for multiple time horizons presents researchers with an interesting and important challenge. This is to develop a dynamic model that fits market data and tracks the evolution of the credit risk of a portfolio. Dynamic models are important for the valuation of some structures. For example, options on tranches of CDOs cannot be valued in a satisfactory way without a dynamic model.

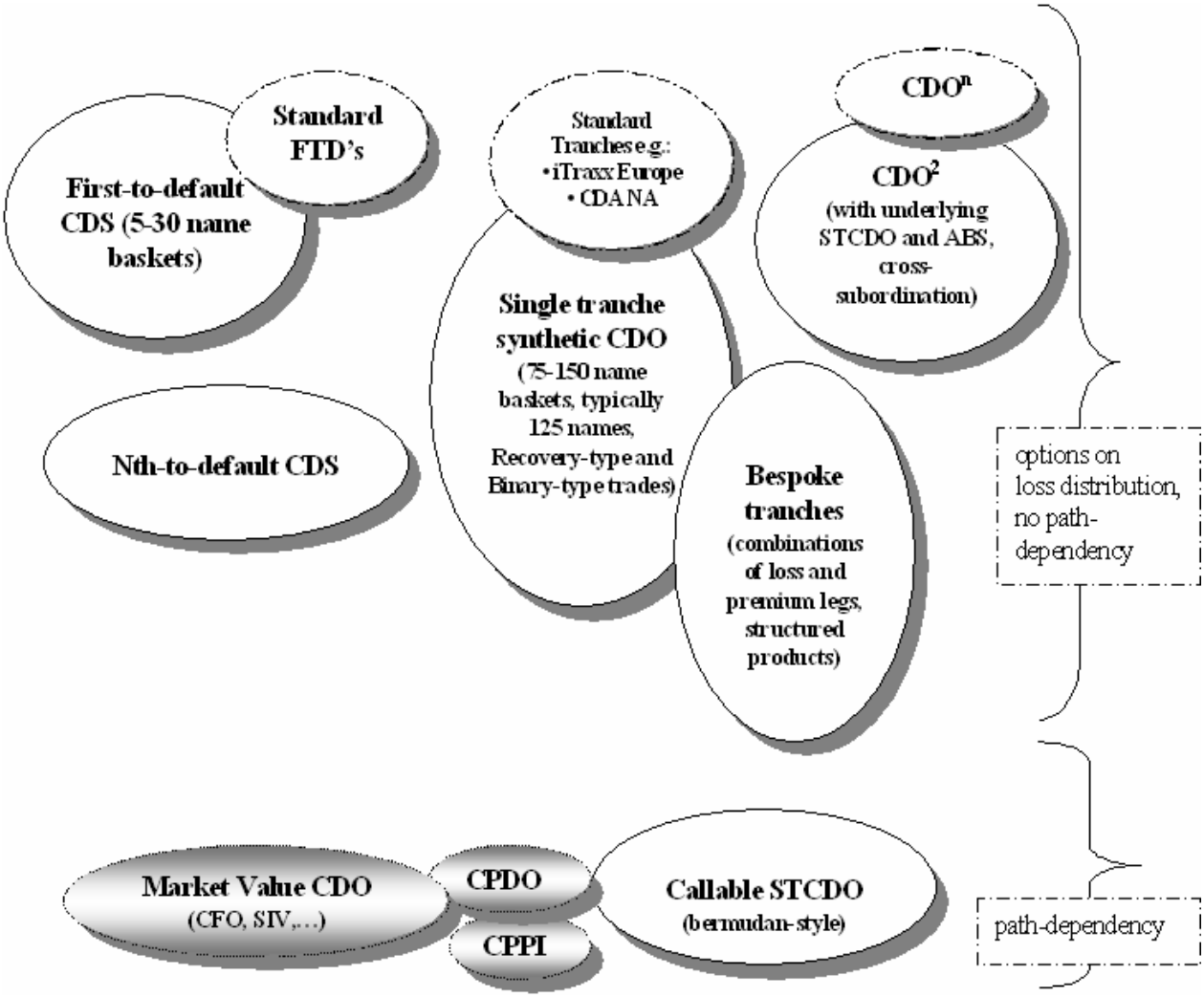


Figure 1.13 shows the evolution of second and third generation credit derivatives from basket derivatives through CDO^n and latest products.

Box N°2: Determining the Forward Spread

Using the same methodology as per standard credit default swap, the Forward Par credit default swap is the solution of the equation,

PV No Default+ PV Default Accruals = PV Of Contingent

$$FWS_n = \frac{(1-R)\sum_{i=k+1}^{n} DF_i \times (PND_{i-1} - PND_i)}{\sum_{i=k+1}^{n} DF_i \times PND_i \times \Delta_i + \sum_{i=k+1}^{n} DF_i \times (PND_{i-1} - PND_i) \times \frac{\Delta_i}{2}}$$

- **Market Values CDO**

Although closely resembling a hedge fund, market value CDOs are considered to be ABS (Asset Backed Securities) due to their fundamental structure. As with other ABS, a market value CDO is a debt obligation issued by a bankruptcy remote SPV secured by some form of receivable. CDOs also often use overcollateralization and subordinated notes to achieve the desired credit quality.

Where a market value CDO really differs from a traditional ABS structure is with respect to:

(i) the servicer's level of involvement,

(ii) the diversity of assets, and

(iii) the number of tranches issued. In a market value CDO, the portfolio manager does more than simply collect and service the portfolio, but also actively trades the asset pool.

Common market value CDO portfolios include cash, treasuries, bonds, loans, CP (Commercial Papers), mezzanine debt, distressed debt, equity, emerging market debt, and even other CDOs.

The primary impetus for issuing a CDO is to take advantage of either an arbitrage opportunity or to improve a financial institution's capital ratios. The issuance of a market value CDO provides an insurance company or investment manager with the means to rapidly expand assets under management, leading in turn to increased management fees. Demand for market value CDOs typically originates from investors who seek some exposure to the high yield market, but are constrained by minimum rating requirements. Market value CDOs also provide an investment offering not commonly available in the traditional ABS, such as notes whose credit ratings cover a broader credit spectrum (AAA to B) and long terms to maturity (4 to 15 years).

- **Collaterised Fund Obligations**

Unlike cash flow deals, the underlying asset pools for Collaterised Fund Obligations are Hedge Funds assets and their value at each valuation date are marked-to-market. The market value CFO meets principal (if applicable) and interest liabilities by generating cash from trading assets and from interest or dividends on invested assets.

Hedge fund assets do not generate predictable cash flow streams, but have significant market value upside potential as well as significant downside potential. Market value deal managers trade actively and aggressively and can employ leverage. Of course, not every trade results in a gain. The ratio of the

market value of assets to the face value of liabilities is a key risk metric of a market value CFO. The amount of debt or note tranches that a CFO can issue as a percentage of market value is limited by a "haircut" to maintain a set level of theoretical over- collateralisation.

The CFO methodology enhances the framework of market value transaction as opposed to classical cash transaction where one disregards the market value of the asset. The Net Asset Value, and liquidity profile monitoring are key factors in CFO methodology, and also in the following market new types of transactions.

- *SIV (Structured Investment Vehicle, can be seen as long or long-short credit only hedge fund strategy)*
- *Market Value CDO*
- *Equity Default Swap, Equity CDO*
- *FX CDO*
- *Commodities CDO*

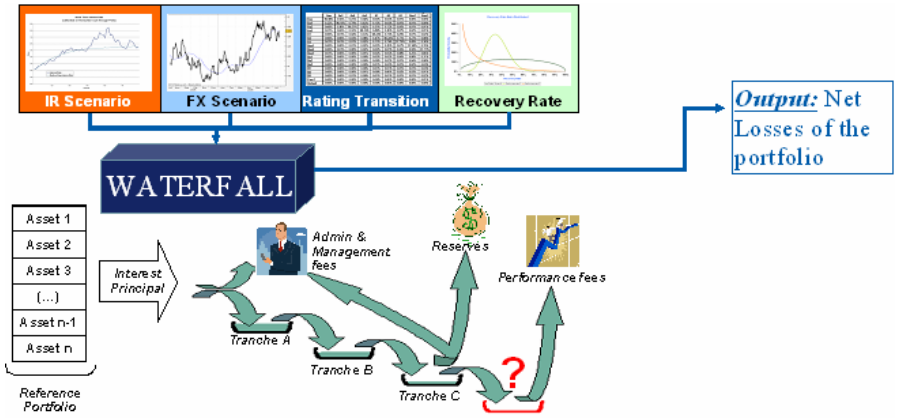


Figure 1.14: Generic market value CDO structure (Source: BNP Paribas)

- **Constant Proportion Portfolio Insurance (CPPI)**

The constant proportion portfolio insurance (CPPI), first established by Black and Jones (1986), is a technique for leveraging up investments while providing full or partial protection. This method has been used extensively in equities and hedge funds and is now applied to the credit market. The investment is comprised of two parts:-

Risky asset + Risk Free Asset (Zero coupon bonds)

The higher the amount of risky assets in the portfolio, the higher the potential returns over the principal amount. The fraction of risky asset in the dynamic basket is referred to as

"Exposure". It is adjusted from time to time to maximize potential return and ensure principal protection. CPPI rebalances between an investment in the risky asset and a zero coupon bond to provide principal protection

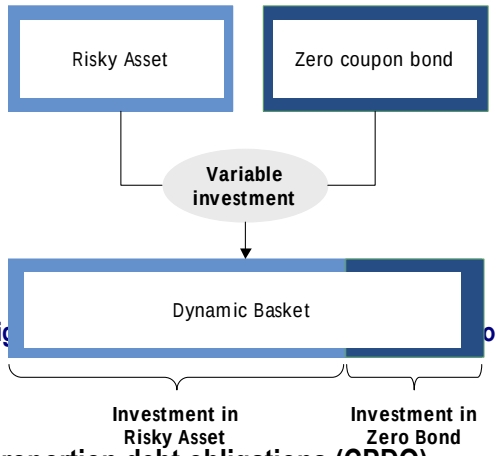


Fig o insurance CPPI structure

- **Constant proportion debt obligations (CPDO)**

Constant proportion debt obligations (CPDOs) are a recent innovation in the credit market answering the growing need for a rated coupon. CPDOs are essentially a variant of CPPI. The main differences are a fixed coupon with no upside and different leverage rules. Like credit CPPIs, CPDOs give leveraged exposure to credit portfolios, although they do not offer principal protection to investors. Constant proportion debt obligations (CPDOs) are engineered to combine attractive yield and high ratings for investors who typically take exposure to a large diversified portfolio of credits without engendering the traditional correlation risk present on single-tranche CDO transactions (STCDO).

A Constant Proportion Debt Obligation could be backed by an index of debt securities (such as CDX or Itraxx) or could be deal specific). This is periodically rolled, thus introducing market risk through the rollover. The leverage of the CPDO is periodically re-adjusted to match asset and liability spread.

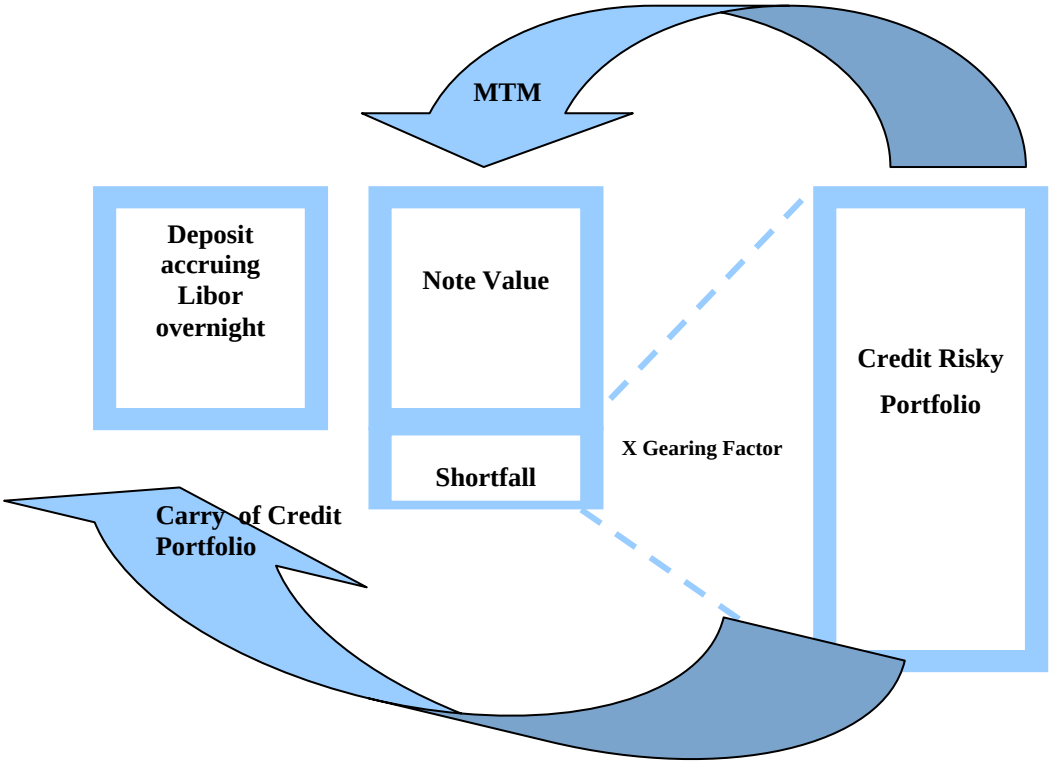


Figure 1.16: Typical structure of a constant proportion debt obligations (CPDO). (Source: BNP Paribas)

Terminology used in CPDO

Note Value (NV): This is the current market value of the CPDO and is calculated as the present value of all outstanding positions which includes the cash deposit and any other unrealised gains/losses.

Cash Deposit: Like the CPPI, the cash deposit account holds the investment proceeds, interest, premiums and any mark-to-market gains achieved. Any losses which arise are also settled using cash from this account.

Target Redemption Value (TRV): This is the present value of all promised future liabilities (coupons and principal payments) payable under the CPDO strategy.

Shortfall: The shortfall is calculated as the difference between the TRV and the NV, and represents the value that still needs to be extracted from the strategy to enable it to 'cash-in'. The size of the shortfall therefore drives the allocation to the credit risky portfolio; as the shortfall increases a more leveraged strategy is adopted. The overall aim of the CPDO strategy is to reduce this shortfall to zero prior to maturity.

Gearing Factor (GF): The (static) gearing factor determines the credit exposure, and hence the risk, of the CPDO strategy. A high factor increases the exposure and hence the probability of triggering either a cash-in or a cash-out event.

Investment Portfolio Premium Value (IPPV): This is the present value of all future premiums that will be paid by the credit assets held in the investment portfolio. The IPPV is used to determine the target credit exposure, and hence the leverage, of the strategy. A higher IPPV essentially means that less exposure needs to be taken.

1.5 Pricing first generation products

At the outset of credit derivatives, the key problem was in being able to price products based on single assets. The increased liquidity of the CDS market for the standard maturities (3Y, 5Y, 7Y and even 10Y) means that this is no longer the main problem. Market values are used, at least for the well-known names.

However, the problem of data availability and quality is still an issue for more illiquid reference assets or names. Under ideal conditions, firms would price a credit derivative from a name-specific credit curve. In the absence of reliable market data, generic credit curves (e.g., AA Utilities) are used. Whatever the approach, curves are generally constructed using available market information, whether it is a term structure of spreads for a generic curve or market quoted CDS rates for a specific name. The next section shows how this is done.

Constructing the swap curves

A bootstrap method is selected in constructing swap curves. This is done mainly because there are only a handful of benchmark swap inputs covering the maturity horizons. Since these swaps are highly liquid, they should be priced exactly. The bootstrap procedure processes consecutive swap instruments individually to ensure exact fits. Furthermore, the method, along with low-order polynomial parameterization, ensures that the constructed curve does not have excessive wiggles due to large spacing between swap maturities. In contrast, the yield curve for treasuries is usually built with least-square optimization of cubic splines. This is because in the U.S. Treasury market there are over a hundred bonds densely distributed over the maturity horizon, and the supply-demand for different bonds causes differences in their relative prices. The least squares method searches for the best fit, not for exact pricing of input bonds, but rather for overall accuracy and stability. The stability is achievable because of the large number of bond inputs. In comparison, the method would not be effective for constructing swap curves due to the limited number of swap inputs. Using splines with relatively few inputs usually leads to wavy and unstable curves.

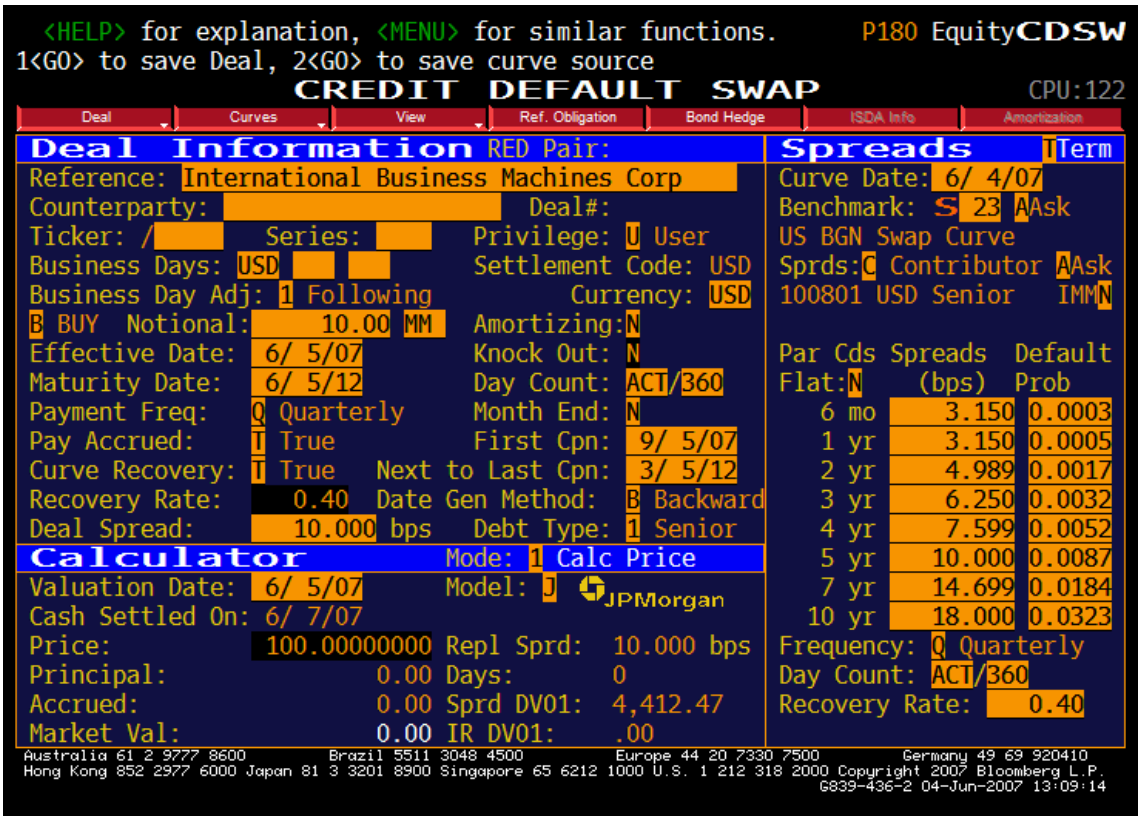


Figure 1.17: Standard CDS maturities quoted in the market for liquid company. Source Bloomberg

After the curve of benchmark securities is constructed, the credit derivative pricing model must then be calibrated to the market which –assuming deterministic recovery rates – generates the hazard rates.

Figure 1.17 shows the quotes on Bloomberg for the standard CDS maturities for a liquid company. The implied hazard rates for the company above given different recovery rates assumptions are given in Table 1.2.

Table 1.2 : Implied default hazard rate for different recovery rates assumptions

Year	Hazard Rate (30% RR)	Hazard Rate (40% RR)	Hazard Rate (50% RR)	Hazard Rate (60% RR)
0.5	0.00045	0.00053	0.00063	0.00079
1	0.00045	0.00053	0.00063	0.00079
2	0.00071	0.00083	0.00100	0.00125
3	0.00089	0.00104	0.00125	0.00156
4	0.00109	0.00127	0.00152	0.00190
5	0.00143	0.00167	0.00200	0.00250
7	0.00210	0.00245	0.00294	0.00367
10	0.00257	0.00300	0.00360	0.00450

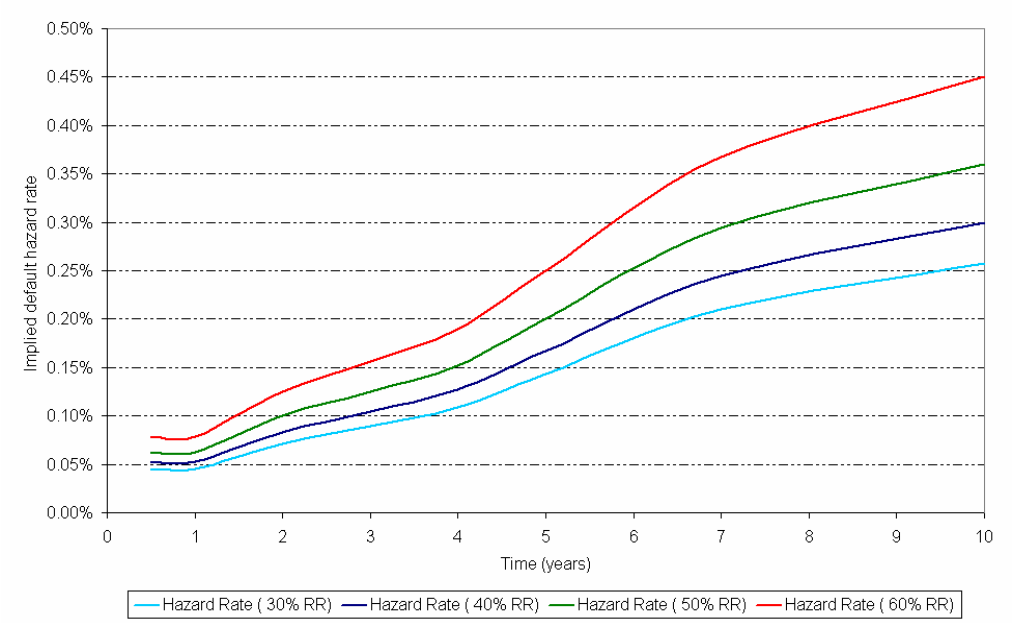


Figure 1.18 : Implied default hazard rate for different recovery rates assumptions

Calibration flexibility is a critical piece of the risk management equation, especially with the variability and unreliable quality of input data. For example, recovery rate assumptions are generally based around historically observed recoveries (by sector and seniority) provided by the rating agencies. Because these recovery rates are a very difficult to estimate and since there is no liquid market on recovery rate such as Recovery Default Swap, one has to use the recovery rates with care in the calibration framework. The ability to stress test these assumptions and see their effect on model risk is crucial.

The recovery rate is usually modelled as a stochastic variable derived from a beta distribution with two parameters: mean and standard deviation, which depend on the country and the seniority.

Table 1.3 : Corporate: Information on 500 non-financial public and private US Companies that have defaulted since 1998, Structured Finance Instruments: Information on 2,000 defaulted Banks, loans, high yield bonds and other debt instruments (Source S&P)

		U.K.	*U.S.*	*France*
Senior Secured	Mean	60%	50%	40%
	Standard deviation	20%	20%	20%
Senior Unsecured	Mean	33%	38%	29%
	Standard deviation	15%	20%	15%
Subordinate	Mean	17%	20%	14%
	Standard deviation	13%	15%	11%
Sovereigns	Mean	25%	25%	25%
	Standard deviation	12%	12%	12%

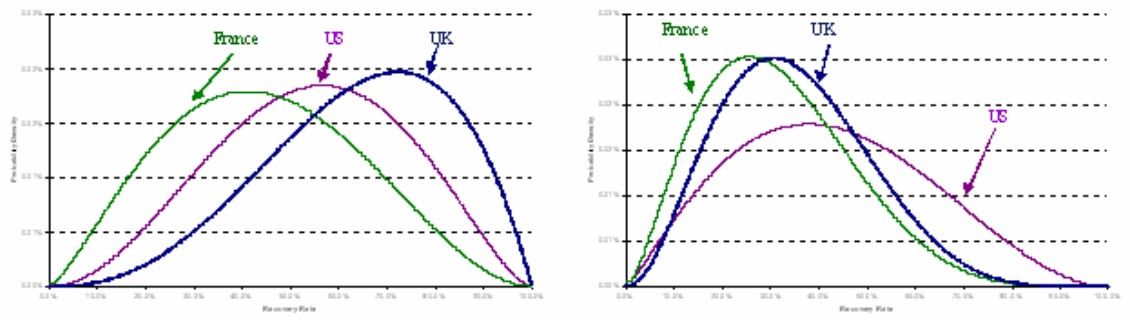


Figure 1.19 : Source S&P: on the left figure, Beta Distributed Recovery Rates for Senior Secured Debt and on the right: Beta Distributed Recovery Rates for Senior Unsecured Debt

The lack of quality market data and hence the potential inconsistencies in valuation and risk management are important issues that must be addressed in order to accurately manage interest rate risk, foreign exchange risk and credit risk at a portfolio level across different assets, liabilities and derivatives.

The gap in reliable market data is being addressed by a number of new information providers that have entered the market. Many of these providers stem from broker-dealers, who have years of compiled data directly from their businesses.

However, even as the market develops more consistent methodologies for valuing single-name products, the proliferation of basket trades and other portfolio products is making new demands for more consistency in default probability correlation models and challenging previous assumptions about credit correlation.

1.6 Literature review

This thesis focuses on the question of modelling the "correlation" (or more accurately the dependency structure) between the entities in the portfolio. Over the years many different models have been developed for this. They include the Gaussian copula and interpolation of base correlations, parametric factor copulas (local correlation), non-parametric factor approaches (implied copula), loss dynamics (top down, bottom up, etc). In this section we review the strengths and weaknesses of these models and explain why a new model is required.

As the solutions proposed in this thesis are stochastic processes based on Archimedean copulas, we review the literature on copulas, focussing on Archimedean copula in the second part of this section.

Early models for risk management

Merton (1974) was the first to model the default of a company. Default probabilities are calculated based on the firm's capital structure and asset volatility. The model says that a firm defaults when the value of its liabilities exceeds its assets at the debt's maturity date. It uses an option framework to calculate this risk neutral default probability. The equity holders of the firm have a call option on the underlying value of the firm with a strike price equal to the face value of the firm's debt. Figure 1.20 illustrates the concept.

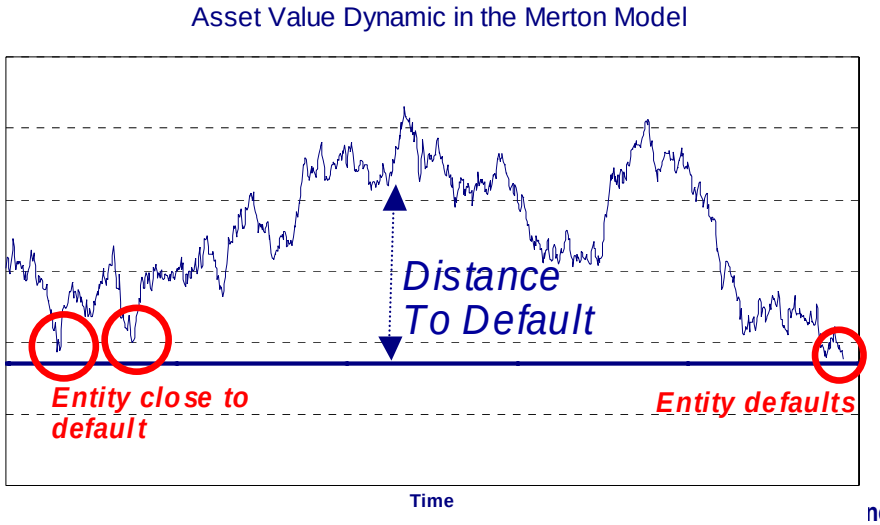


Figure 1. near misses (on left)
the firm finally defaults (on right)

The model recognizes that neither the underlying value of the firm nor its volatility are directly observable. Under the model's assumptions both can be inferred from the value of equity, the volatility of equity and several other observable variables by solving two nonlinear simultaneous equations. After inferring these values, the model specifies that the probability of default is the cumulative normal distribution function of a z-score depending on the firm's underlying value, the firm's volatility and the face value of the firm's debt.

As the model assumes that asset returns follow a geometric Brownian motion process, the joint distribution of asset returns is multivariate normal (and hence is uniquely determined by its correlation matrix). The default correlation is taken into account through the correlation asset returns of the companies.

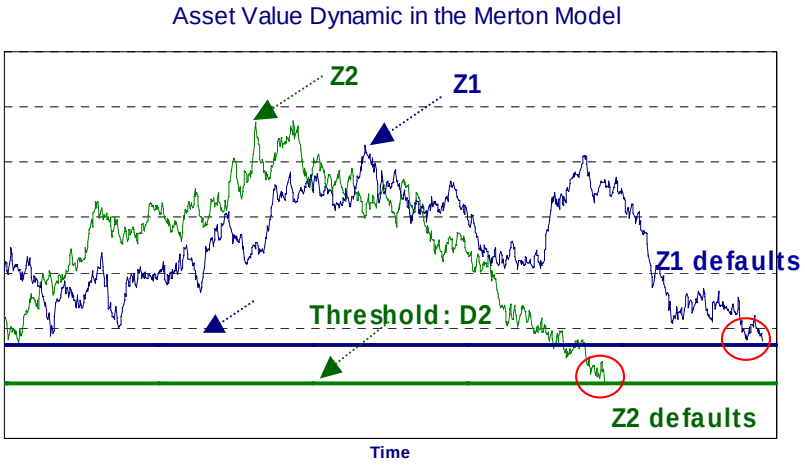


Figure 1.21 : Asset value dynamics of two correlated firms in Merton's model. The illustration shows the z-scores of the entities. The correlation between Z1 & Z2 means that Z2's defaults makes Z1's default much more likely

As KMV and CreditMetrics are extensions of Merton's model, the joint distribution of asset returns (after transformation) is the standard multivariate normal with a common correlation between the asset returns. If Z_i denotes the (transformed) asset returns, and if ρ denotes the correlation between assets returns, then Z_i can be written in terms of a common term Y plus an uncorrelated terms U_i, which are i.i.d. standard normal:

$$Z_i = \sqrt{\rho}Y + \sqrt{1-\rho}U_i$$

Then conditional on Y, the default rate for an asset with a mean default rate p is

$$R(Y) = \mathbf{P}(Z_i < \Phi^{-1}(p) \mid Y) = \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right),$$

The default rate volatility can be expressed in terms of the bivariate normal integral, using:

$$\mathbf{E}[R^2] = \Phi_2(\Phi^{-1}(p), \Phi^{-1}(p); \rho)\,.$$

Moody's KMV approach

KMV applies the framework developed by Merton (1974). The structural model used by Moody's KMV in modelling default risk provides a framework to help in understanding the factors affecting credit quality and is based on an in-depth analysis of market information in the form of valuations (prices) and volatility of valuations (business risk) as observed amongst public firms.

Moody's KMV approach, also called Vasicek's approach, has been used widely in the market, and its one factor asymptotic approach for homogeneous loan portfolio loss distribution is a

benchmark. One of its strong points is that it provides an asymptotic expression for the percentage of portfolio loss for a large homogeneous portfolio.

Asymptotic Vasicek's formula in a one-factor model for a homogeneous loan portfolio

Following the Merton model, the value of the i^{th} borrower's assets A_i follows a geometric Brownian process :

$$\frac{\mathrm{d}\,A_i(t)}{A_i(t)} = \mu_i\,\mathrm{dt} + \sigma_i A_i\,\mathrm{d}\,w_i(\mathrm{t})$$

The asset value at maturity T can be represented as:

$$\mathrm{Ln}\left[A_i(\mathrm{T})\right] = \mathrm{Ln}\left[A_i(0)\right] + \mu_i\,\mathrm{T} - \frac{1}{2}\sigma_i^2\,\mathrm{T} + \sigma_i\sqrt{\mathrm{T}}\varepsilon_i$$

where ε_i is a standard normal variable.

A default occurs if the value of the borrower's assets at the loan maturity T falls below the contractual value $B_i(T) = B_i$ of its obligations payable. Thus, a default situation occurs when $\mathrm{A_i(T)} < \mathrm{B_i(T)}$; that is,

$$\varepsilon_i < c_i(\mathrm{T}) \quad \text{where} \quad c_i(\mathrm{T}) = \frac{\mathrm{Ln}\left[\mathrm{B_i(T)}\right] - \mathrm{Ln}\left[\mathrm{A_i(0)}\right] - \mu_i\,\mathrm{T} + \frac{1}{2}\sigma_i^2\,\mathrm{T}}{\sigma_i\sqrt{T}}$$

The probability of default of the i^{th} loan is then:

$$\mathrm{p_i(T)} = \mathrm{P}\left[\mathrm{A_i(T)} < \mathrm{B_i(T)}\right] = \mathrm{N}\left(\mathrm{c_i(T)}\right)$$

where N is the cumulative normal distribution.

Let $E_i(T)$ be an indicator variable, such that $E_i(T) = 1$ if the i^{th} firm has defaulted between time 0 and T, and 0 otherwise. We can write: $\mathrm{p_i(T)} = \mathrm{P}\left[\mathrm{E_i(T)} = 1\right]$. With a loss-given-default LGD_i for the firm i, the percentage loss is:

$$L_{\%i}(T) = E_i(T) \cdot LGD_i$$

Consider a portfolio consisting of N loans with the same term T. If the ε_i are jointly standard normal variables with the same pair wise correlation, ρ, they can therefore be expressed in terms of a common factor Y and mutually independent idiosyncratic errors $\varepsilon_{F_1}, \varepsilon_{F_2}, \ldots, \varepsilon_{F_n}$.

$$\varepsilon_i = Y\sqrt{\rho} + \varepsilon_{F_i}\sqrt{1-\rho}$$

The variable Y can be interpreted as a portfolio common factor, such as an economic index, over the interval (0, T). Then the term $Y\sqrt{\rho}$ is the company's exposure to the common

factor and the term $\varepsilon_{F_i}\sqrt{1-\rho}$ represents the company specific risk. When the common factor is fixed, it is not difficult to show that the conditional probability of an event of default on any one loan is given by:

$$\Pr_i(Y) = N\left(\frac{N^{-1}(p_i) - Y\sqrt{\rho}}{\sqrt{1-\rho}}\right)$$

The probability $\Pr_i(Y)$ is the loan default probability under scenario Y; the unconditional default probability p_i is the average of the conditional probabilities over the scenarios. The percentage loss $L_{\%i}(Y)$ conditional on the factor, Y : $L_{\%i}(Y) = p_i(Y)\cdot LGD_i$.

If $L_{\%}$ denotes the percentage loss for the portfolio and w_i are the weights:

$$L_{\%} = \sum_{i=1}^{N} w_i \cdot L_{\%i}\ , \qquad \text{where } \sum_{i=1}^{N} w_i = 1\,.$$

Vasicek worked out an asymptotic expression for $L_{\%}$ if the portfolio is **homogeneous** (the same probability of default p for all loans). The portfolio percentage loss $L_{\%}(Y)$ conditional on the factor Y converges towards its expectation P, as N tends to infinity (the infinite granular portfolio hypothesis) if and only if the sum of the squares of the weights tends to zero.

$$L_{\%}(Y) \xrightarrow[\sum_{i=1}^{N} w_i^2 \to 0]{N\to\infty} \sum_{i=1}^{N} w_i \cdot L_{\%i}(Y)$$

Provided that the condition $\sum_{i=1}^{N} w_i^2 \to 0$ holds, the portfolio conditional on the factor Y is:

$$L_{\%}(Y) = N\left(\frac{N^{-1}(p) - Y\sqrt{\rho}}{\sqrt{1-\rho}}\right)\cdot LGD$$

CreditMetrics approach

CreditMetrics is based on the risk measurement methodology developed by JP Morgan for the measurement, management and control of the credit risk in its trading, arbitrage and investment account activities. It can be seen as a tool for professional risk managers in the financial markets. It is widely used now as a benchmark for the credit risk measurement.

The principal aim of CreditMetrics was to capture counterparty default risk as per the available data on credit quality in the market. In this approach it is much more complicated to assess credit risk than market risk. In fact, the market risk is easy to compute because we easily have the daily price observations available to calculate the VaR. The fundamental

technique in CreditMetrics is the migration analysis from one credit state to another through time. JP Morgan developed these transition matrices in 1987 and this was the starting point of all the other methods based on transition matrices to build their own portfolio credit risk calculation.

CreditMetrics method is an analytic method based on the Monte Carlo simulations which can be described in 3 steps:

- Generation of scenarios: in a time horizon the aim is to generate future credit ratings for the obligors in the overall portfolio. This is done through a basic procedure which consists on the calculating the asset return limit for the obligors, generating the scenarios on the asset returns according to a normal distribution and finally the mapping of the asset return scenarios to credit rating scenarios (transition probabilities).
- Portfolio valuation: in each scenario, the portfolio is revalued in order to reflect the changes in credit ratings. This step will return a large number of possible future portfolio values.
- Aggregation of the results: having the distribution generated into different scenarios, the model allows us to estimate risk parameters at the portfolio level.

CreditPortfolioView: an econometric approach

In 1998 McKinsey introduced the CreditPortfolioView approach. Tom Wilson, formerly of McKinsey, developed a credit portfolio model which takes account of the current macro-economic environment. Rather than using historical default rate averages calculated from decades of data, CreditPortfolioView uses default probabilities conditional on the current state of the economy. The portfolio loss distribution is conditioned by the current state of the economy for each country and industry segment.

The macro-economic variable, Y, is assumed to be normally distributed. The default rate is given as a function of Y by the logit function:

$$R(Y) = 1/(1+e^{a+bY})$$

Had the cumulative normal function been used instead of the logit function for R(Y), the results would have been similar to those of the Merton model.

CreditRisk+ : an actuarial approach

Unlike the Merton-based approach used by Portfolio Manager and CreditMetrics, the CreditRisk+ methodology developed by Tom Wilde (1997) at Credit Suisse is based on mathematical models used in the insurance industry. CreditRisk+ models default rates as continuous random variables. Observed default rates for credit ratings vary over time, and the uncertainty in these rates is captured by the default rate volatility estimates (standard deviations).

Default correlation is generally caused by external factors such as regional economic strength or industry weakness. Credit Suisse argues that default correlations are difficult to observe and are unstable over time. Instead of trying to model these correlations directly, CreditRisk+ uses the default rate volatilities to capture the effect of default correlations and produce a long tail in the portfolio loss distribution. CreditRisk+ can handle thousands of exposures and uses a portfolio approach which reduces risk for diversification.

CreditRisk+ modelling framework.

Consider a portfolio with N loans or credits. The default or non-default of the i^{th} loan by a fixed time horizon is indicated by a random variable D_i which can only take the values 1 (for default) or 0 otherwise. Its exposure, E_i, is measured as a positive integer multiple of a basic currency unit. Consequently $p_i = P(D_i = 1) \in [0,1]$ is the default probability of the i^{th} loan. The realized total loss L0 of the portfolio is then given as:

$$L_0 = \sum_{i=1}^{N} E_i D_i$$

If the default probabilities p_i are small, the total loss distribution does not change too much when the Bernoulli variables D_i are replaced by random variables $N_1, \cdots, N_N$ with non-negative integer values. Conditional on some factors these are assumed to be independent Poisson-distributed with measurable intensities $R_0, \cdots, R_N \succ 0$ such that

$$E[R_i] = p_i, \quad i = 1, \cdots, N$$

CreditRisk+ uses the following approximate distribution:

$$L_0 = \sum_{i=1}^{N} E_i N_i$$

The stochastic intensities R_i of the N_i are given as:

$$R_i = \sum_{j=0}^{K} r_{j,i} S_j, \quad i = 1, \cdots, N, r_{j,i} \succ 0, \sum_{j=1}^{K} r_{j,i} \succ 0, \sum_{i=1}^{N} r_{j,i} = 1, S_0 \succ 0 \ \& \ \text{constant}$$

The S_j are K independent gamma-distributed with parameters (α_j, β_j). They are usually interpreted as default intensities which are characteristic for sectors, countries, or branches.

$$E[R_i] = p_i = r_{0,i}S_0 + \sum_{j=1}^{K} r_{j,i}\alpha_j\beta_j, \quad i = 1, \cdots, N$$

Table 1.4 compares these risk management model used by banks.

	Credit Migration Approach		***Contingent claim approach***	***Actuarial Approach***
Model	**CreditMetrics**	**CreditPortfolioView**	**Moody's KMV**	**CreditRisk+**
Risks definition	Change in market value	Change in market value	Default losses	Default losses
Credit events	Downgrade/ Default	Downgrade/ Default	Continuos default probabilities	Default
Risk drivers	Asset Value	Macro-factors	Asset Values	Expected default rate
Transition probabilities	Constant	Driven by macro-factors	Driven by: Individual term structure of EDF Asset value processes	N/A
Correlation of credit events	Standard multivariate normal distribution (Equity factor model)	Conditional default probabilities as a function of macro-factors	Standard multivariate normal distribution (Asset factor model)	Conditional default probabilities as a function of common risk factors
Recovery rates	Random (Beta distribution)	Random (empirical distribution)	Random (Beta distribution)	Deterministic Loss Given default
Numerical approach	Simulation/Analytic	Simulation	Simulation/Analytic	Analytic

Gordy (2001) and Tasche & al (2002) have computed the characteristic function of the loss:

$$g(z) = \mathrm{E}\left(z^{L_0}\right) = \exp\left(S_0 \left(\left(\sum_{i=1}^{N} r_{0,i} z^{E_0} \right) - 1 \right) \right) \prod_{j=1}^{K} \left(1 + \beta_j - \beta_j \left(\sum_{i=1}^{N} r_{j,i} z^{E_0} \right) \right)^{-\alpha_j}$$

As we have shown above, exposures can be allocated to industrial or geographical sectors and different time horizons of exposure can be incorporated. The minimal data requirements make the model easy to implement, and the analytical calculation of the portfolio loss distribution is very fast.

Single versus multi-name products ; single versus multiperiod

When analysing credit derivatives it is interesting to split them into four broad categories depend on whether they are single or multi-name products and whether they refer to single time periods or several. Table 1.5 presents the different credit derivatives in these four categories. We now present the pricing models in the same categories.

Table 1.5: Schematic comparison of credit derivatives

	Single Name ***Payoff depends only on one risky asset***	***Multi Names*** ***Payoff depends only on the pool of underlying credits***
Static Credit model ***Single Period***	• Corporate Bonds • CDS (Credit default Swap) • (Index Swap) • …	**Depends only on actual loss distribution** • CDO Tranche • CDO^n • NtD (N^{th}-To-Default basket) • …
Dynamic Credit model ***Multi Period***	• CDS option • Forward Start products • …	**Depends on forward loss distribution** • Forward Tranche • Tranche Options • Leverage Super Senior • …

Single -period static credit models

Gaussian copula – the current industry standard

Base correlation, first developed by JP Morgan: McGinty & Ahulwalia, (2004a & b) has become the industry standard for pricing CDO tranches. The base correlation model is based on Homogeneous Large Pool Gaussian Copula Model, which is a simplified version of the Gaussian copula widely used in the market. This model is not new, it is a simple methodology that is almost identical to the original CreditMetrics model (Gupton et al, 1997). It is a simplified form of earlier one-factor models (Vasicek, 1987).

Hyperbolic copulas induced by a one factor Levy model

In the same way that Vasicek's one-factor model results in a Gaussian copula, a more general approach using a Lévy factor model has been introduced and provides an endless variety of different copulas. To understand the extent of the class of copulas that we can choose from, we start by presenting some characteristics of the Lévy process.

The generalized hyperbolic (GH) distribution was introduced by Barndorff-Nielsen (1977b) for describing dune movements and was first applied to financial time series by Eberlein and Keller (1995). Today, the GH distribution and its subclasses are very popular within finance since they give almost exact fits to different log returns. See for example Prause (1999). Barndorff-Nielsen (1977a) proved that the GH distribution is infinitely divisible and thus induces a Lévy process. This is why hyperbolic distributions are used for correlated default modelling for CDO pricing. Levy copulas do not necessarily introduce asymmetric dependence and are only used in practice as static copula for vanilla CDO.

Multi Period Dynamic Credit model

Classical reduced form models

Reduced-form models provide an alternative to structural models (Merton type model). They were introduced by Duffie and Gârleanu (2001). They assume that the hazard rate of a company is the sum of an idiosyncratic component, a component common to all companies, and a component common to all companies in the same sector. Each component follows a process with both a diffusion and a jump component.

Dynamic generalized-Poisson loss model

Brigo & al (2007 & 2006) considered a dynamic model for the number of defaults in a pool of names. In contrast to most other approaches, their model which is based on generalized Poisson processes, allows for more than one default in small time intervals.

They model the pool loss directly and introduce extensions based on piecewise-gamma, scenario-based or CIR random intensities, leading to richer spread dynamics. According to the authors, their model was related to insurance shock models leading to a "bottom-up" approach where single name default dependence is represented through a Marshal-Olkin copula, but they have not yet provided a valid proof.

Affine point processes

Giesecke & al (2006 & 2007) proposed an affine point process as a dynamic model of portfolio loss. The recovery at each default is random and events are governed by an intensity that is driven by affine jump diffusion risk factors. The portfolio loss itself is a risk factor so past defaults and their recoveries influence future loss dynamics. This specification incorporates feedback from events and a dependence structure among default and recovery rates. Hawkes model (Hawkes & al, 1971 & 1974) and Moller, Jesper & Rasmussen (2004), are examples of a computationally tractable affine point process used by Giesecke & al.

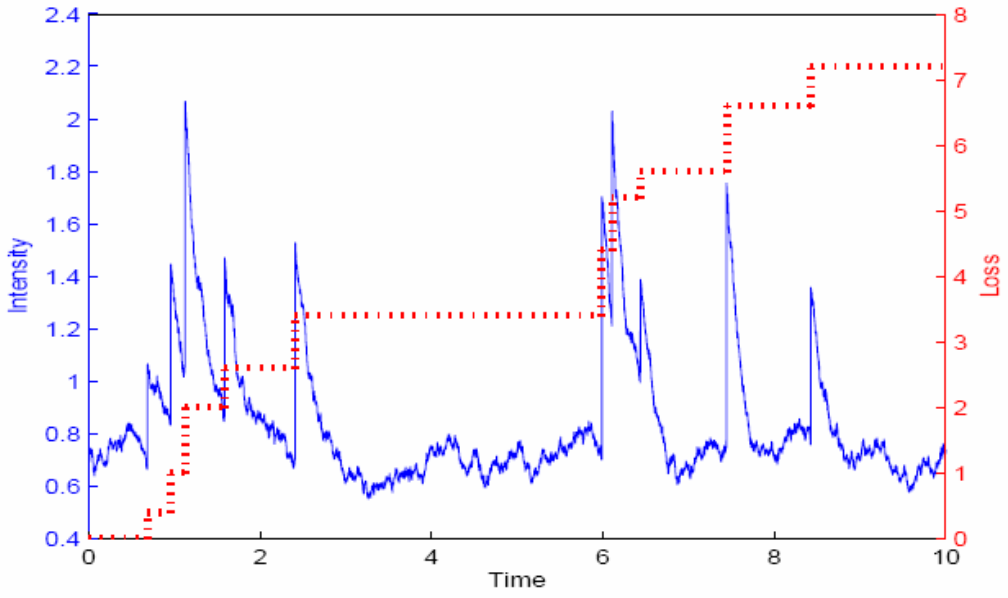


Figure 1.22: Sample paths of the intensity and the associated loss process L. A jumpin the intensity represents the impact of a default. The jump size is the product of the loss at default and the sensitivity parameter δ = 1. The loss at default is drawn from an independent uniform distribution on {0:4; 0:6; 0:8; 1}. The reversion rate κ= 5 and the reversion level $\lambda_{\text{long term}}$ = 0:7. The volatility σ= 0:2 controls the diffusiveusive uctuation of the intensity between events. Source: Giesecke & al (2006 & 2007)

The intensity based model for portfolio default and loss is defined by:

Simple form

$$d\lambda_t = \underbrace{\kappa}_{\text{mean reversion coeff}} \times (\lambda_\infty - \lambda_t)\,dt + \underbrace{\delta dL_t}_{\text{jump term}}$$

General form

$$d\lambda_t = \underbrace{\kappa}_{\text{mean reversion coeff}} \times (\lambda_\infty - \lambda_t)\,dt + \underbrace{\sigma\sqrt{\lambda_t}\,dW_t}_{\text{diffusion term}} + \underbrace{\delta dL_t}_{\text{jump term}}$$

The authors show that the model leads to analytically tractable transform based pricing, hedging and calibration of credit derivatives. But it is difficult to analyse idiosyncratic risk within this framework.

This model does not specify the real granularity in the portfolio (example different names with different spreads and recovery rates). It is also unclear how one could analyses idiosyncratic risk, or how we can do a mapping between the correlation or dependence and the volatility and jump of the underlying process.

Markovian Bivariate Spread-Loss Model for Portfolio Credit Derivatives

The BSLP model developed by Arnsdorf and Halperin (2007) is a two-dimensional dynamic model of interacting portfolio-level loss and loss intensity processes. It is constructed as a Markovian, short-rate intensity model, which facilitates fast lattice methods for pricing various portfolio credit derivatives such as tranche options, forward-starting tranches, leveraged super-senior tranches, but only on a very liquid index, without taking into account the dynamics of single names. As for the previous model, this one does not specify the real granularity in the portfolio (e.g. different names with different spreads and recovery rates). It is also unclear how one could analyse idiosyncratic risk, or how we can do a mapping between the correlation or dependence and the volatility and jump of the underlying process.

Local Intensity Model

This model can be used only with local intensity incorporating contagion and assuming that only one default can happen per time period, dt. The link between the marginal default distribution over time dt and the local intensity is given by:

$$\pi(k,t) = \underbrace{\lambda(k-1,t)\,dt\,\pi(k-1,t)}_{\text{default}} + \underbrace{\left(1-\lambda(k-1,t)\,dt\right)\pi(k,t)}_{\text{no default}}$$

where k is the number of defaults in the portfolio and the intensity is given by:

$$\lambda(k,t) = -\frac{1}{\pi(k,t)}\sum_{i=1}^{k}\frac{d}{dt}\pi(i,t)$$

Hence, starting at k = 0, the local intensity can be solve iteratively. So it is of the form:

$$\lambda(K_t,t) = \underbrace{f(K_t,t)}_{Contagion\ factors} \quad \underbrace{(N-K_t)}_{Number\ names\ in\ portfolio}$$

Stochastic Intensity Bivariate Spread Loss Process; (BSLP) Model

A stochastic driven factor Y_t is introduced.

$$\lambda(K_t,t) = \underbrace{f(K_t,t)}_{Contagion\ factors} \times \underbrace{(N-K_t)}_{Number\ names\ in\ portfolio} \times Y_t$$

Its dynamics are given by the following equation:

$$\frac{dY_t}{Y_t} = a\left(Ln(\theta_t) - Ln(Y_t)\right) + \sigma dW_t + \gamma dK_t$$

The one-dimensional local intensity model obtained in the zero volatility limit of the stochastic intensity is useful in its own right for pricing non-standard index tranches by arbitrage-free interpolation. But it is difficult to analyse idiosyncratic risk within this framework.

Dynamic Credit Correlation Model (derived from credit barrier model)

The dynamic credit correlation model proposed by Albanese et al (2006) is based on continuous time lattice models correlated by conditioning to a non-recombining tree. The model describes rating transitions and spread dynamics as well as default events, while preserving single name marginals. Its strengths are that it can be made consistent with many data sources such as rating transition probabilities, historical default probabilities, single name credit spread curves and equilibrium recovery swap rates. The authors found that tranche spreads for the CDX and iTraxx index portfolios in the period subsequent to the summer 2005 could be calibrated simultaneously.

The problem with this model is that it requires a full calibration of credit spreads and the correlation. It lacks the flexibility of a copula approach where the dynamics of the margins are segregated from that of the joint distribution. It is very difficult to translate pricing on very liquid names or index with different set of products for calibration into a pricing of bespoke portfolio. This model is very computer intensive and it lacks a simple asymptotic properties.

First Passage Model under Stochastic Volatility

Fouque & al (2006) extended the first passage model to model default dependency by extending it to multi-dimensions and by incorporating stochastic volatility. The authors consider a pool of N defaultable bonds whose underlying firm process $\left\{X_t^{(i)}\right\}_{i=1}^{N}$ exhibits the

following multi-factor stochastic volatility dynamics under the real world probability measure:

$$\begin{aligned} dX_t^{(1)} &= \mu_1 X_t^{(1)} dt + f_1(Y_t, Z_t) X_t^{(1)} dW_t^{(1)} \\ dX_t^{(2)} &= \mu_2 X_t^{(i)} dt + f_2(Y_t, Z_t) X_t^{(2)} dW_t^{(2)} \\ \ldots &\quad \ldots \quad \ldots\ldots\ldots \\ dX_t^{(N)} &= \mu_N X_t^{(N)} dt + f_N(Y_t, Z_t) X_t^{(N)} dW_t^{(N)} \end{aligned}$$

Where

$$\begin{aligned} dY_t &= \frac{1}{\varepsilon}\left(m_Y - Y_t\right) dt + \frac{\upsilon_Y \sqrt{2}}{\sqrt{\varepsilon}} Y_t \, dW_t^{(Y)} \\ dZ_t &= \delta\left(m_Z - Z_t\right) dt + \upsilon_Z \sqrt{2\delta} Y_t \, dW_t^{(Z)} \end{aligned}$$

$$E\left[dW_t^{(i)} dW_t^{(Y)}\right] = \rho_{iY}, \quad E\left[dW_t^{(i)} dW_t^{(Z)}\right] = \rho_{iZ}$$

$$E\left[dW_t^{(Z)} dW_{tZ}^{(Y)}\right] = \rho_{iY}, \quad E\left[dW_t^{(i)} dW_t^{(j)}\right] = 1_{\{i \neq j\}} dt$$

The default time of the ith firm is defined as:

$$\tau_t^{(i)} = \inf\left\{s \geq t, X_t^{(i)} \leq B^{(i)}(s)\right\}, \quad B^{(i)}(s) = K^{(i)} e^{\eta_i t},$$

where $B^{(i)}(t)$ is the pre-specified default threshold at time t, as in Black and Cox (1976).

This modelling approach falls into the conditional independence framework which simplifies the computation. Using regular and singular perturbation techniques as in an earlier paper on single names (Fouque & al, 2003), they obtained an approximation for the joint survival probabilities. The asymptotic formulation of this model could help for a fast computation of standard vanilla CDO, but is very computer intensive and lacks simple asymptotic properties when modelling more exotic products.

1.6 Archimedean copulas within the credit derivatives framework

Our overall objective is to develop a family of multivariate copula processes with different types of upper and lower tail dependence so as to be able to reproduce the correlation smiles/skews observed in practice. We have chosen to work with Archimedean copulas because they encompass a wide range of types of upper and lower tail dependence.

Archimedean copulas are defined via a generator, f[6] :

$$C(u_1,\ldots u_n) = f^{-1}\left[f(u_1)+\ldots+f(u_n)\right]$$

An enormous amount of literature exists on copulas especially Archimedean copulas, with applications in a wide range of fields. But when one looks closely at it, the most papers concern bivariate copulas. Much less has been published on multivariate copulas. In the next section we explain the theoretical difficulties involving in extending from bivariate to trivariate and hence to multivariate copulas. Clearly it is even more difficult to construct dynamic multivariate copula processes than static copulas. Our aim is to construct dynamic copula processes based on continuous stochastic processes.

Static multivariate copulas

The canonical textbook on copulas, Nelson (1999) lists many bivariate Archimedean copulas (with one or more parameters) but as most are not strict 2-copulas no corresponding n-copula exists. The generators of strict copulas are related to specific Laplace transforms. For example, the generator for the Clayton copula, corresponds to a gamma distribution with parameter $1/\theta$ and the Gumbel copula is related to an α-stable distribution. Table 1.6 lists a selection of n-copulas together with their Laplace transform. Nelsen (1999) also contains several counter-examples which demonstrate how wrong "intuition" can be and how difficult it is to find multivariate Archimedean copulas. The other major reference book, Joe (1997), provides some results for the trivariate & quadrivariate cases. Lindskog (2000) provides some interesting extensions and shows the level of constraints on the parameter values in the non-exchangeable case. But these are all static copulas.

[6] To avoid confusion, in this paper we use φ for a Laplace transform, and f for the generator of an Archimedean copula, whereas Nelsen (1994) uses φ for the generator of an Archimedean copula.

Table 1.6: The Laplace Transforms corresponding to selected strict bivariate copulas (which can be extend to n-copulas). The names LTE, LTF, LTG, LTH and LTI are derived from the naming system used by Joe (1997).

Copula Name	Generator (Laplace Transform)
Clayton	$\varphi(s)=(1+s)^{-1/\theta}, \theta > 0$
Gumbel	$\varphi(s) = \exp\left(-s^{1/\theta}\right), \theta > 1$
Frank	$\varphi(s)=-\frac{1}{\theta}Ln[1-\exp(-s)\{1-\exp(-\theta)\}], \theta \neq 0$
LTE	$\varphi(s) = \left(1 + s^{1/\delta}\right)^{-1/\theta}$
LTF	$\varphi(s)=(1+\delta^{-1}Ln(1+s))^{-1/\theta}, \ \delta > 0, \theta \geq 1$
LTG	$\varphi(s)=\exp\left\{1-\left[\delta^{-1}Ln(1+s)\right]^{1/\theta}\right\}, \ \delta > 0, \theta \geq 1$
LTH	$\varphi(s)=1-\left[1-\exp\left\{-s^{1/\delta}\right\}\right]^{1/\theta}, \ \delta > 0, \theta \geq 1$
LTI	$\varphi(s)=1-\left[1-(1+s)^{-1/\delta}\right]^{1/\theta}, \ \delta > 0, \theta \geq 1$

Dynamic copulas seen from a time series point of view

Econometricians have developed dynamic copula models from a time series point of view. Patton (2001 & 2003) developed an approach based on an ARMA-type process and applied it to foreign exchange data. Fermanian & Wegkamp (2004) extended Patton's approach which was based on pseudo-copulas. Duan (1995) and more recently Van den Goorbergh et al (2005) have used GARCH processes. A major shortcoming of these papers have is that they only consider bivariate cases. Chen & Fan (2005) consider a class of semi-parametric copula

based multivariate dynamic models which they apply to daily exchange rates. None of the latter seems to be suitable for pricing CDOs on large portfolios.
Berd, Engle & Voronov (2005) have developed a more promising hydrid model in which the dynamics of the underlying latent variables are governed by a GARCH or a TARCH process.

This has the advantage of producing aggregate return distributions that are asymmetric and clearly non-Gaussian. The authors used historical data on the SP500 going back to 1962 as a proxy for market returns (pre and post 1990). While they mention using some market data (e.g. the level of hazard rates and the expected default probabilities) to calibrate parameters, they do not seem to use the available CDS data.

Dynamic copula processes

Fewer papers have tackled the question from a continuous point of view. The earliest paper on copula processes seems to be Darsow et al (1992) who studied Archimedean copulas and Markov processes from a theoretical point of view. Several authors have modelled credit risk dynamics using default intensities rather than default times. Rogge & Schonbucher (2003) developed a method for modelling dynamic portfolio credit risk based on Archimedean copulas. They provide some very useful results that link Archimedean copulas with Laplace transforms, and also a fast and efficient method for simulating realisations, one that is not mentioned by Nelsen (1999). Following Madan's work on stock prices (Madan & Seneta, 1990, and Madan & Milne, 1991) and that of Cariboni & Schouten (2004) on the value of the firm, Joshi & Stacey (2004) used a gamma process when modelling default intensities and pricing CDOs. They found that a double gamma process was required to match the base correlations observed in the market correctly. One disadvantage of working with intensities is that it requires calibrating the default functions for each of the names.
To summarize, very little work is available on multivariate copula. We will take information from Rogge & Schonbucher (2003) as the starting point for our models.

Chapter 2 : Dynamic copula model

Our objective is to develop a family of multivariate copula processes with different types of upper and lower tail dependence so as to be able to reproduce the correlation smiles/skews observed in practice. We chose to work with Archimedean copulas because they encompass a wide range of types of tail dependence. In this chapter we present two different approaches for developing these processes. The first model developed is a non-additive jump process based on a background gamma process; the second approach is based on time changed spectrally positive Levy process. The first approach is very convenient for simulations; the second approach is based on additive building blocks and hence is a more general. To start with we give a brief overview of Levy processes with the gamma process being a special case.

2.1 Levy processes including gamma processes

The emergence of Levy processes in finance literature is due to empirical observations that the distribution of equity returns is in general skewed and leptokurtotic. Empirical data show evidence for non-normally distributed credit spreads together with the presence of stochastic volatility and/or jumps in spread. Nowadays, a battery of models is available which capture non-normality and integrate stochastic volatility. After defining a Levy process, we focus on a model that has a gamma process as its building block. Barndorff-Nielsen and Shephard (2001) were first to introduce gamma processes for modelling stock market data.

Definition of a Levy process

A cadlag[1] stochastic process $(X_t)_{t\geq 0}$ on (Ω, F, P) with values in $\Re^n$ such that $X_0 = 0$ is said to a Levy process if it possesses the following properties:

- Independent increments: for every increasing sequence of times $t_0,...,t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0},..., X_{t_n} - X_{t_{n-1}}$ are independent.
- Stationary increments $X_{t+h} - X_t$ do not depend on t
- Stochastic continuity: $\forall \varepsilon \succ 0, \quad \lim_{h\to 0} P(|X_{t+h} - X_t| \geq \varepsilon) = 0$.

[1] Some authors do not impose the cadlag (right-continuity and left limit) property in the definition of the Levy process but it can be shown that every Levy Process (defined without a cadlag property) has a unique modification that is cadlag. Therefore the cadlag property can be assumed without loss of generality.

For more information on Levy processes and their applications in finance, the reader can consult Sato (1999) on Levy processes and infinitely divisible distribution. Carr, Geman, Madan and Yor (2001) developed Levy models with stochastic time. Cont and Tankov (2004) generalized this framework for exponential Levy Processes. This class of models is built out of a Levy process which is time-changed by a stochastic clock. This induces the desired stochastic volatility effect.

Gamma processes

Gamma processes belong to a special class of Levy process called subordinators. These consist of a purely non-Gaussian Levy process $\{X_t\}$ in $(-\infty,+\infty)$, whose Levy measure υ satisfies the following two conditions:

$$\upsilon((-\infty,0]) = 0$$

$$\int_0^1 x\,\nu(dx) \prec +\infty .$$

In particular, the trajectories $t \mapsto X_t(\omega)$ are increasing functions a.s. and the transition kernels $\mathrm{K}_{t-s}(\mathrm{x},\mathrm{dy}) \equiv \mathrm{Prob}(X_t \in dy / X_s = x)$ are supported in $[x,+\infty)$.

Definition of a gamma process

Let $\alpha(\mathrm{t})$ be a process with independent increments following a gamma distribution such that

$$\begin{aligned} &\alpha(t+\delta t) - \alpha(t) \equiv \Gamma(a_1 \delta t, a_2) \\ &\alpha(0) = 0 \end{aligned} \qquad [2.01]$$

Its value at time t has the following gamma distribution:

$$\alpha(t) \equiv \Gamma(a_1 t, a_2) \qquad [2.02]$$

and its density is given by:

$$f(x, a_1 t, a_2) = \frac{x^{a_1 t - 1}}{(a_2)^{a_1 t}\,\Gamma(a_2)} exp\left(-\frac{x}{a_2}\right) \qquad [2.03]$$

where $\Gamma(x)$ is the gamma function. See Joshi & Stacey (2005) for details.

This process has an interesting scaling property. The gamma process $\alpha(\mathrm{t})$ with parameters a_1 and a_2, can be rewritten as

$$\frac{1}{\mathrm{a}_2}\alpha(\mathrm{t}) \cong \Gamma(\mathrm{a}_1\mathrm{t}, 1) \qquad [2.04]$$

where $\Gamma(\alpha_1 t,1)$ is a gamma process with parameters a_1 and 1. So in order to simulate this process we merely need to be able to simulate standard gamma distributions. Two algorithms for doing this are given in Box 2.1

Box N° 2.1: Two algorithms for generating a standard gamma variable

There are two well-known algorithms for simulating standard gamma variables, depending on the value of the parameter.

Johnk's generator of gamma variables, for $a \leq 1$

Repeat

Generate i.i.d. uniform [0,1] random variables U,V

Set $X = U^{\frac{1}{a}}$ and $Y = V^{\frac{1}{(1-a)}}$

Until $X + Y \leq 1$

Generate an exponential random variable E

$$\Gamma(a,1) \equiv \frac{XE}{X+Y}$$

Best's generator of gamma variables, for $a \succ 1$

Set $b = a - 1$ and $c = 3a - \frac{3}{4}$

Repeat

Generate i.i.d. uniform [0,1] random variables U,V

Set $W = U(1-U), Y = \sqrt{\frac{c}{W}}\left(U - \frac{1}{2}\right), X = b + Y$

If $X < 0$,go to Repeat

Set $Z = 64\,W^3 V^3$

Until $\mathrm{Ln}(Z) \leq 2\left(b\,\mathrm{Ln}\left(\frac{x}{b}\right) - Y\right)$

Generate an exponential random variable E

$\Gamma(a,1) \equiv X$

2.2 Dynamic copulas from a gamma process perspective

The first approached consisted of developing a family of dynamic Archimedean copula processes to model the default times. For simplicity we assume that the portfolio is large but not necessarily homogeneous. We require a stochastic process in which the copula defining the defaults amongst the N names is a valid N-dimensional copula at any point in time. As was explained in the literature review in Chapter 1, while there is a vast literature on bivariate or 2-dimensional copulas, much less has been published on multivariate copulas. Our approach is based on one of Rogge & Schonbucher's observations, from which we developed a family of dynamic copula processes based on a generalised compound gamma process. In Chapter 5 we show how this can be used for pricing CDOs.

Following Rogge & Schonbucher (2003), let Y be a positive random variable whose Laplace transform is $\varphi(s)$ and let U_i be n uniform random variables on [0,1] that are mutually independent and also independent of Y. Then the n random variables V_i defined as

$$V_i = \varphi\left(\frac{-\mathrm{Ln}(U_i)}{Y}\right) \qquad \text{for } i = 1,\ldots n \qquad [2.05]$$

are uniform on [0,1], and their cumulative distribution function is given

$$\mathrm{Prob}\left(V_1 \le v_1,\cdots,V_N \le v_N\right) = \varphi\left(\sum_{i=1}^{N}\varphi^{-1}(v_i)\right) \qquad [2.06]$$

Consequently their multivariate copula is the Archimedean copula having φ^{-1} as its generator.

The extension from the static case to dynamic processes is straightforward. Let Y(t) be a stochastic process that takes positive values. It represents the state of the economy. Let $\varphi_t(s)$ be its Laplace transform. Note that as Y(t) need not be stationary, its distribution (and hence φ) depends on t. Let $U_i(t)$ be n mutually independent stochastic processes taking values that are uniform on [0,1]. As before, these must also be independent of Y(t).

The procedure for simulating the dynamic copula process $V_i(t)$ is simple. Simulate the processes Y(t) and the $U_i(t)$. The copula process $V_i(t)$ is given by:

$$V_i(t) = \varphi_t\left(\frac{-\mathrm{Ln}(U_i(t))}{Y(t)}\right) \qquad \text{where } Y(t) > 0 \qquad [2.07]$$

At any given time t, the multivariate structure between $V_i(t)$ and $V_j(t)$ (for $i \neq j$) is just the Archimedean copula having φ_t^{-1} as its generator. In the next sections we study the dynamic properties of these processes.

Evolution of $V_i(t)$ over time

In this section we compute the bivariate distribution function of $V_i(t)$ and $V_i(t+\delta t)$ for two different cases. To simplify the notation, let

$$K_{V_i(t),V_i(t+\delta t)}(a,b) = \text{Prob}\left(V_i(t) < a, V_i(t+\delta t) < b\right)$$
$$H_{U_i(t),U_i(t+\delta t)}(a,b) = \text{Prob}\left(U_i(t) < a, U_i(t+\delta t) < b\right) \quad [2.08]$$

Note that these can also be viewed as integral transforms. In the multi-period case, we extend this notation in the obvious way:

$$K_{V_i(t),\ldots V_i(t+k\delta t),\ldots,V_i(t+n\delta t)}\left(v_i^t,\ldots,v_i^{t+k\delta t},\ldots,v_i^{t+n\delta t}\right)$$
$$= \text{Prob}\left(V_i(t) < v_i^t,\ldots,V_i(t+k\delta t) < v_i^{t+k\delta t},\ldots,V_i(t+n\delta t) < v_i^{t+n\delta t}\right)$$

By conditioning on the values of Y(t) and Y(t+δt) and noting that

$$\varphi\left(-\ln(U_i)/Y\right) \le V_i \Leftrightarrow U_i \le \exp\left(-Y\varphi^{-1}(v_i)\right)$$

it is easy to show that

$$\text{Prob}\left(V_i(t) < w, V_i(t+\delta t) < z \mid Y(t), Y(t+\delta t)\right) = H_{U_i(t),U_i(t+\delta t)}\left(e^{-\varphi_t^{-1}(w)Y(t)}, e^{-\varphi_{t+\delta t}^{-1}(z)Y(t+\delta t)}\right) \quad [2.09]$$

Hence

$$K_{V_i(t),V_i(t+\delta t)}(w,z)) = E\left[H_{U_i(t),U_i(t+\delta t)}\left(e^{-\varphi_t^{-1}(w)Y(t)}, e^{-\varphi_{t+\delta t}^{-1}(z)Y(t+\delta t)}\right)\right] \quad [2.10]$$

Case N° 1 : One time step analysis

We assume $U_i(t)$ and $U_i(t+\delta t)$ are independent and that Y(t) is a stochastic process with independent identically distributed increments. Because of the independence and because the process U(t) is uniform on [0,1]

$$H_{U_i(t),U_i(t+\delta t)}(a,b) = \text{Prob}\left(U_i(t) < a\right) \times \text{Prob}(U_i(t+\delta t) < b) = a \times b$$

If we let $1_{\delta t>0}$ be an indicator function that takes the value 1 if $\delta t > 0$, and 0 otherwise, then

$$H_{U_i(t),\,U_i(t+\delta t)}\left(u_i^t, u_i^{t+\delta t}\right) = 1_{\delta t>0} \times u_i^t \times u_i^{t+\delta t} + \left(1-1_{\delta t>0}\right) \times \mathrm{Min}\left(u_i^t, u_i^{t+\delta t}\right)$$

Consequently

$$K_{V_i(t),V_i(t+\delta t)}(w,z) = E\left[1_{\delta t>0} \times e^{-\left[\varphi_t^{-1}(w)Y(t)+\varphi_{t+\delta t}^{-1}(z)Y(t+\delta t)\right]} + \left(1-1_{\delta t>0}\right) \times e^{-Y(t)\mathrm{Max}\left[\varphi_t^{-1}(w),\varphi_{t+\delta t}^{-1}(z)\right]}\right] \quad [2.11]$$

Since Y(t) is a stochastic process with independent identically distributed increments

$$K_{V_i(t),V_i(t+\delta t)}(w,z) = \begin{cases} 1_{\delta t>0} \times E\left[e^{-\left[\varphi_t^{-1}(w)+\varphi_{t+\delta t}^{-1}(z)\right]Y(t)}\right] \times E\left[e^{-\varphi_{t+\delta t}^{-1}(z)\left[Y(t+\delta t)-Y(t)\right]}\right] \\ + \\ \left(1-1_{\delta t>0}\right) \times E\left[e^{-Y(t)\mathrm{Max}\left[\varphi_t^{-1}(w),\varphi_{t+\delta t}^{-1}(z)\right]}\right] \end{cases} \quad [2.12]$$

Since φ_t is a decreasing function, this simplifies to

$$K_{V_i(t),V_i(t+\delta t)}(w,z) = \begin{cases} 1_{\delta t>0} \times \varphi_t\left(\varphi_t^{-1}(w)+\varphi_{t+\delta t}^{-1}(z)\right) \times \varphi_{\delta t}\left(\varphi_{t+\delta t}^{-1}(z)\right) \\ + \\ \left(1-1_{\delta t>0}\right) \times \mathrm{Min}(w,z) \end{cases} \quad [2.13]$$

If $\delta t = 0$

$$K_{V_i(t),V_i(t+0)}(w,z) = \mathrm{Min}[w,z]$$

As $\delta t \to 0$

$$\begin{aligned} \text{As } \delta t \to 0^+ \quad & K_{V_i(t),V_i(t+\delta t)}(w,z) \to \varphi_t\left(\varphi_t^{-1}(w)+\varphi_t^{-1}(z)\right) \\ & \le K_{V_i(t),V_i(t+0)}(w,z) = \mathrm{Min}[w,z] \end{aligned} \quad [2.14]$$

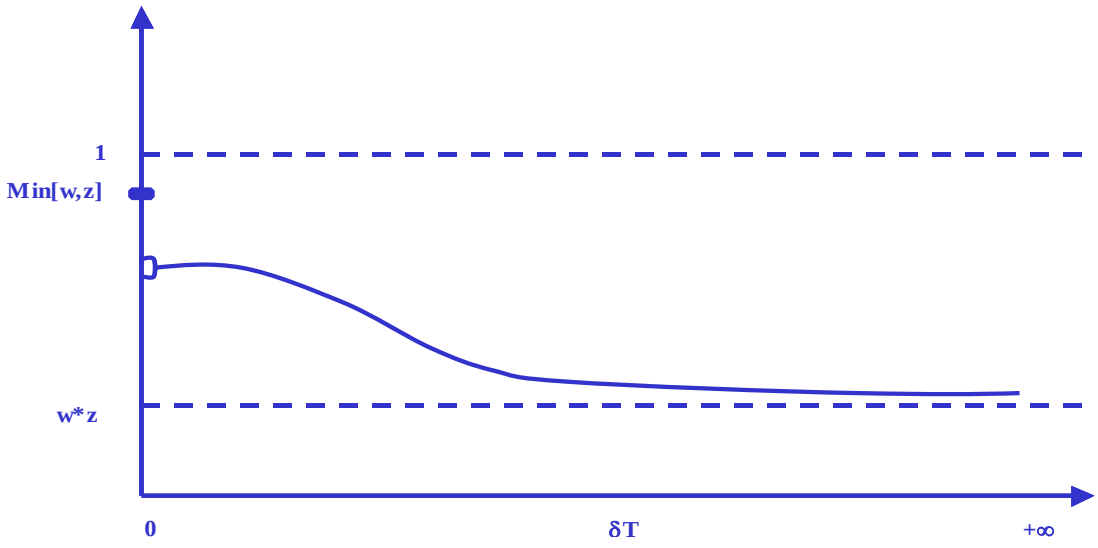


Figure 2.1: the relation between the probability when $\delta t = 0$ and when $\delta t \to 0$. Note the discontinuity at zero.

Case N° 2 : Multi-time step analysis

As before, we assume $U_i(t)$ and $U_i(t+\delta t)$ are independent and that Y(t) is a stochastic process with independent identically distributed increments. Because of the independence,

$$H_{U_i(t),\ldots,U_i(t+k\delta t),\ldots,U_i(t+n\delta t)}\left(u_i^t,\ldots,u_i^{t+k\delta t},\ldots,u_i^{t+n\delta t}\right) = 1_{\delta t>0}\times\prod_{k=0}^{n}u_i^{t+k\delta t}+\left(1-1_{\delta t>0}\right)\times \mathrm{Min}\left[u_i^t,\ldots,u_i^{t+k\delta t},\ldots,u_i^{t+n\delta t}\right]$$

$$K_{V_i(t),\ldots,V_i(t+k\delta t),\ldots,V_i(t+n\delta t)}\left(v_i^t,\ldots,v_i^{t+k\delta t},\ldots,v_i^{t+n\delta t}\right) = E\begin{bmatrix} 1_{\delta t>0}\times\exp\left(-\sum_{k=0}^{n}\varphi_{t+k\delta t}^{-1}\left(v_i^{t+k\delta t}\right)Y\left(t+k\delta t\right)\right) \\ + \\ \left(1-1_{\delta t>0}\right)\times\exp\left(-Y(t)\,\mathrm{Max}\left[\varphi_t^{-1}\left(v_i^t\right),\ldots,\varphi_{t+n\delta t}^{-1}\left(v_i^{t+n\delta t}\right)\right]\right)\end{bmatrix} \quad [2.15]$$

Remark: since

$$\sum_{k=0}^{n}\varphi_{t+k\delta t}^{-1}\left(v_i^{t+k\delta t}\right)Y\left(t+k\delta t\right) = Y(t)\times\sum_{k=0}^{n}\varphi_{t+k\delta t}^{-1}\left(v_i^{t+k\delta t}\right)+\sum_{m\geq 1}^{n}\left\{\left(Y\left(t+k\delta t\right)-Y(t)\right)\times\varphi_{k\delta t}^{-1}\left(v_i^{t+k\delta t}\right)\right\}$$

we obtain

$$K_{V_i(t),\ldots,V_i(t+k\delta t),\ldots,V_i(t+n\delta t)}\left(v_i^t,\ldots,v_i^{t+k\delta t},\ldots,v_i^{t+n\delta t}\right) = E\begin{bmatrix} 1_{\delta t>0}\times\varphi_t\left(\sum_{k=0}^{n}\varphi_{t+k\delta t}^{-1}\left(v_i^{t+k\delta t}\right)\right)\prod_{m=1}^{n}\varphi_{k\delta t}\left(\varphi_{t+k\delta t}^{-1}\left(v_i^{t+k\delta t}\right)\right) \\ + \\ \left(1-1_{\delta t>0}\right)\times\mathrm{Min}\left[v_i^t,\ldots,v_i^{t+k\delta t},\ldots,v_i^{t+n\delta t}\right]\end{bmatrix} \quad [2.16]$$

Properties of dynamic Archimedean copula when i≠j

In this section we focus on the case of two different names, i & j. Using the same notation

$$\mathrm{Prob}\left(V_i(t)<w,V_j(t+\delta t)<z\right)=K_{V_i(t),V_j(t+\delta t)}(w,z)$$

$$\mathrm{Prob}\left(U_i(t)<w,U_j(t+\delta t)<z\right)=H_{u_i(t),U_j(t+\delta t)}(w,z) \quad [2.17]$$

The simplest case is when $U_i(t)$ and $U_j(t+\delta t)$ are independent, and when Y(t) has independent identically distributed increments:

$$K_{V_i(t),V_j(t+\delta t)}(w,z)=H_{U_i(t),U_j(t+\delta t)}\left(\exp\left[-\varphi_t^{-1}(w)Y(t)\right],\exp\left[-\varphi_{t+\delta t}^{-1}(z)Y(t+\delta t)\right]\right) \quad [2.18]$$

As Y(t) & Y(t+δt) need not be independent these formulas cannot be simplified. However the conditional probability can be factorized:

$$\begin{aligned}&\text{Prob}\left(V_i\ (t) < w, V_j\ (t+\delta t) < u \middle/ Y(t), Y(t+\delta t)\right)\\&= \exp\left(-\varphi_t^{-1}(w)\,Y(t)\right) \times \exp\left(-\varphi_{t+\delta t}^{-1}(z)\,Y(t+\delta t)\right)\end{aligned} \qquad [2.19]$$

We compute the conditional probability given Y(t) & Y(t+δt) and then decondition over all possible Y(t) & Y(t+δt). As $\delta t \to 0$, this gives

$$K_{V_i\ (t),V_j\ (t+\delta t)}(w,z) = E\left[e^{-\varphi_t^{-1}(w)Y(t)} \times e^{-\varphi_{t+\delta t}^{-1}(z)Y(t+\delta t)}\right] \qquad [2.20]$$

In this case the results are different from those obtained earlier:

$$K_{V_i(t),V_j(t+\delta t)}(w,z) \underset{\delta t \to 0^+}{=} \varphi_t\left(\varphi_t^{-1}(w) + \varphi_t^{-1}(z)\right)$$

Consequently

$$\text{Prob}\left(V_i\ (t) < w, V_i\ (t+\delta t) < z\right) \underset{\delta t \to 0^+}{=} \text{Prob}\left(V_i\ (t) < w, V_i\ (t+\delta t) < z\right)_{\delta t = 0} \qquad [2.21]$$

This approach was used in Chapter 6 for pricing CDOs.

2.3 Dynamic copulas seen from a Levy process perspective: compound gamma process

In the next two sections we define two compound gamma processes; firstly the standard version, then a more general version which is rescaled by a deterministic factor to have the same mean. In both cases the standard building blocks are additive.

Definition of a standard ompound gamma process

First we define a standard compound gamma process, X(t) or X_t for short. This is stochastic process with independent increments having a gamma distribution whose first parameter takes a value equal to the corresponding increment of the gamma defined earlier:

$$X(t+\delta t) - X(t) = \Gamma\left(\delta\alpha(t), 1\right) \qquad [2.22]$$

At time t, conditional on the realization of the gamma process, $\alpha(t)$, X(t) has a gamma distribution $\Gamma(\alpha(t),1)$. Figure 2.2 shows a typical realization of $\alpha(t)$ in red, together with two realizations of the standard compound gamma X(t) based on this. Table 2.1 gives its mean & variance. We will now show that X(t) is a Levy process.

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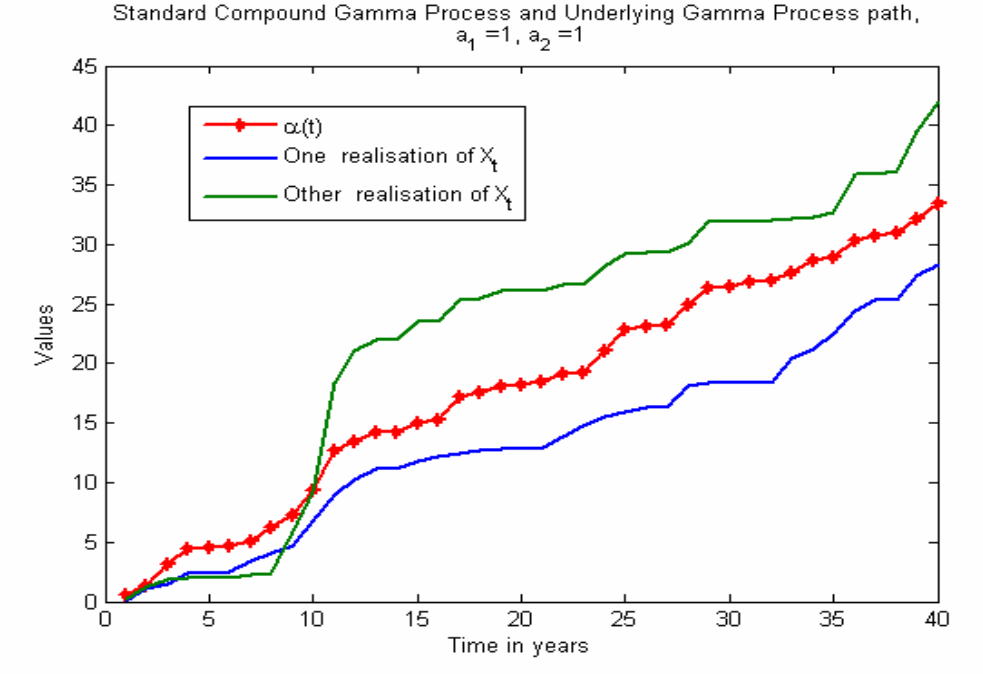


Figure 2.2: Simulation of a gamma process (red) together with two realisations of the standard compound gamma process constructed using this realisation of α(t)

Table 2.1: Mean & Variance of a standard compound gamma process

Mean	$a_1\, a_2\, t$
Variance	$a_1\, a_2\, t\,(1+a_2)$

Proof

Following the line of reasoning in Cont and Tankov [2004], we prove that the standard compound process is a Levy process. The first two properties are obvious from the definition given earlier. By construction the process $(X_t)_{t\geq 0}$ is defined on (Ω, F, P):

$$\delta X_t \equiv X_{t+\delta t} - X_t = \Gamma(\delta\alpha(t), 1) \qquad [2.23]$$

Since $\alpha(t)$ is a gamma process with independent increments, for $i \neq j$ the increments

$$\alpha(t_i) - \alpha(t_{i-1}) \text{ and } \alpha(t_j) - \alpha(t_{j-1})$$

are independent and so are $(X_{t_i} - X_{t_{i-1}})$ and $(X_{t_j} - X_{t_{j-1}})$.

Now we show that the increments are stationary. The increment $X_{t+h} - X_t$ has the gamma distribution $\Gamma(\alpha(t+h)-\alpha(t),1)$ while the corresponding increment of the α process has the following gamma distribution:

$$\alpha(t+h)-\alpha(t) \equiv \Gamma(a_1 h, a_2). \quad [2.24]$$

Consequently the increments have a compound gamma distribution:

$$X_{t+h}-X_t \equiv \Gamma(\Gamma(a_1 h, a_2),1) \quad [2.25]$$

whose parameters depend on the time difference h but not on t itself. So the stationary increment $X_{t+h} - X_t$ does not depend on t.

Stochastic continuity:

In order to prove that $\forall \varepsilon > 0, \quad \lim_{h\to 0} P(|X_{t+h}-X_t| \geq \varepsilon) = 0$, we compute the Laplace transform of $|X_{t+h}-X_t| = X_{t+h}-X_t$:

$$\begin{aligned} E\left[\exp\left(-s|X_{t+h}-X_t|\right)\right] &= E\left[\exp\left(-s(X_{t+h}-X_t)\right)\right] \\ &= E\left[\exp\left(-s\Gamma(\Gamma(a_1 h, a_2),1)\right)\right], \quad a_1 \geq 0, a_2 \geq 0 \\ &= E_{\Gamma(a_1 h, a_2)}\left[\exp\left(-s\left(\Gamma(\Gamma(u,1),1)\right)\right) / \Gamma(a_1 h, a_2) = u\right] \\ &= \exp\left(-a_1 h \, Ln\left(1+Ln(1+s)a_2\right)\right) \end{aligned}$$

$\forall s \geq 0, \forall h > 0,$

$$\lim_{h\to 0}\left\{E\left[\exp\left(-s|X_{t+h}-X_t|\right)\right]\right\} = \lim_{h\to 0}\left\{\exp\left(-a_1 h \, Ln\left(1+Ln(1+s)a_2\right)\right)\right\} = 1$$

Therefore:

$$\forall \varepsilon > 0, \quad \lim_{h\to 0} P(|X_{t+h}-X_t| \geq \varepsilon) = 0 \, [\quad [2.26]$$

Definition of a more general compound gamma process

We now use this standard compound gamma process to define a more general compound gamma process, Y(t), by rescaling X(t). Let $\beta(t)$ be a positive deterministic function, i.e. $\beta(t) > 0$ for $t > 0$. Two obvious candidates are $\beta(t) = 1/t$ and $\beta(t) = \exp(-t)$.

On an incremental basis, Y(t) starts from 0 and follows the dynamic equation below.

$$\delta Y(t) = Y(t+\delta t) - Y(t) = \beta\left(t+\frac{\delta t}{2}\right)\delta X_t \quad [2.27]$$

Using the scaling property of the gamma process, the value Y(t) has the following distribution

$$Y(t_n) = Y(t_0) + \sum_{i=1}^{n} \beta\left(t - \frac{\delta t_i}{2}\right) \times \Gamma\left(\alpha(\delta t_i), 1\right) \qquad [2.28]$$

In continuous time, the process Y(t) can be written as the integral of an increment of a Levy process (the standard compound gamma X(t) times the continuous function α(t)):

$$Y(t) = \sum_{0<s<t} \Delta Y_s \qquad [2.29]$$

Figure 2.3 shows the steps of construction of a general compound gamma process. Figures 2.4 & 2.5 show several realisations of this process for the cases where β(t) = 1/t and β(t) = exp(-t) respectively.

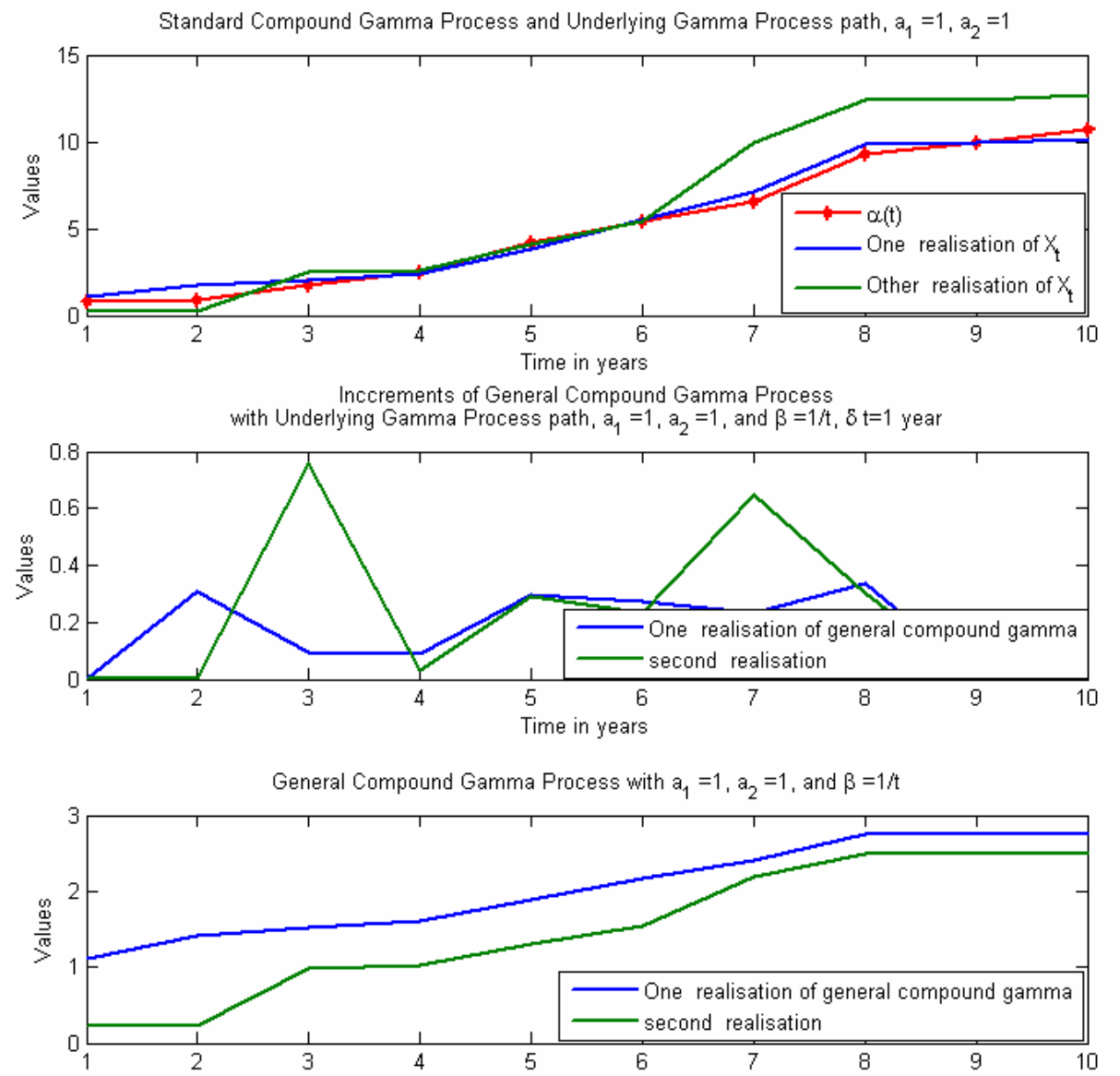


Figure 2.3: Simulation of general compound gamma process. The standard compound gamma process has more

volatility than the underlying gamma process; the general coumpound gamma is a smooth process through time.

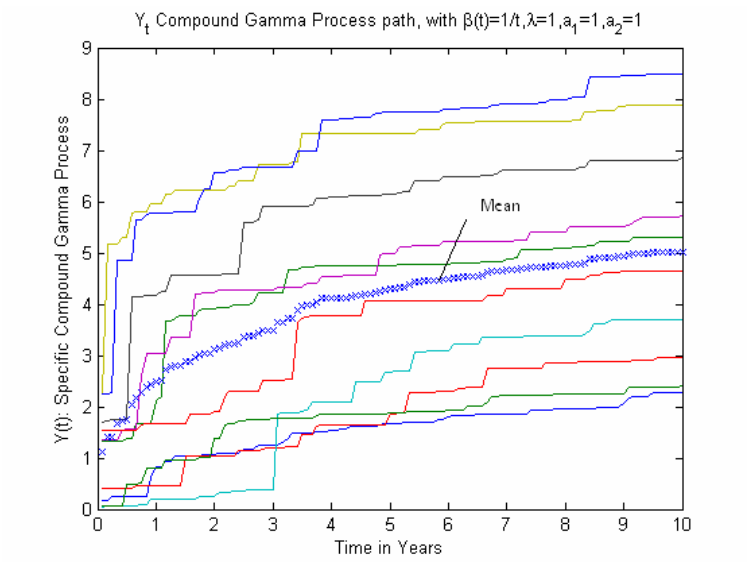


Figure 2.4: Simulation of a specific compound gamma process, β(t) = 1/t and Y(0) = Γ(a1,a2)

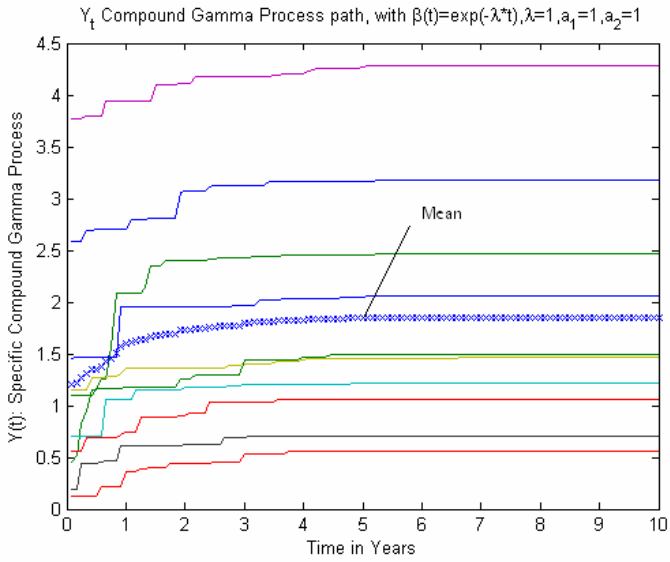


Figure 2.5: Simulation of a specific compound gamma process, β(t) = exp(-t), and Y(0) = Γ(a1,a2)

Some properties of the process Y(t)

As the increments are independent, the first two moments of Y(t) are just

$$E\left[Y\left(t_n\right)\right] = a_1\, a_2 \sum_{i=1}^{n} \beta\left(\frac{t_i + t_{i-1}}{2}\right)\delta t_i \qquad [2.30]$$

$$\mathrm{Var}\left[Y(t_n)\right] = \mathrm{Var}\left[Y(t_0)\right] + a_1 a_2 (1+a_2) \sum_{i=1}^{n} \beta\left(\frac{t_i + t_{i-1}}{2}\right)^2 \delta t_i \qquad [2.31]$$

The next step is to compute its Laplace transform $\varphi(s)$.

$$\begin{aligned}\varphi(s) &= E\left[\exp\left(-s\,Y(t_n)\right)\right] \\ &= E\left[\exp\left(-s\,Y(t_0)\right)\right] \prod_{i=1}^{n} \left[1 + a_2\,\mathrm{Ln}\left(1 + s\beta\left(\frac{t_i + t_{i-1}}{2}\right)\right)\right]^{-a_1 \delta t_i}\end{aligned} \qquad [2.32]$$

This is used in the construction and simulation of the $V_i(t)$. It is also important because it guarantees that the resulting copula is "strict" (see Nelsen 1997) and hence the construction gives rise to a valid multivariate Archimedean copula.

Its continuous time limit which will be used in the next section is :

$$\begin{aligned}&\lim_{\max\{\delta t_i\} \to 0^+} \varphi(s) \\ &= E\left[\exp\left(-s\,Y(t_0)\right)\right] \exp\left(\exp\left(-\int_0^{t_n} a_1\,\mathrm{Ln}\left[1 + \mathrm{Ln}\left(1 + s\beta(u)\right) a_2\right] du\right)\right)\end{aligned} \qquad [2.33]$$

Asymptotic continuous time properties of the process Y (t)

Since we only need to know the behaviour of Y(t) between t_0 and t_n,we analyse the asymptotic behaviour of the Laplace transform of $Y(t_n) - Y(t_0)$ instead of $Y(t_n)$,

$$Y(t_n) - Y(t_0) = \sum_{i=1}^{n} \beta\left(\frac{t_i + t_{i-1}}{2}\right) \Gamma\left(\alpha(\delta t_i), 1\right) :$$

$$\tilde{\varphi}(s) = \prod_{i=1}^{n} \left[1 + a_2\,\mathrm{Ln}\left(1 + s\beta\left(\frac{t_i + t_{i-1}}{2}\right)\right)\right]^{-a_1 \delta t_i} \qquad [2.34]$$

$$\lim_{\max\{\delta t_i\} \to 0^+} \left\{-\mathrm{Ln}\left(\tilde{\varphi}(s)\right)\right\} = \lim_{\max\{\delta t_i\} \to 0^+} \left\{-\sum_{i=1}^{n} a_1\,\delta t_i \mathrm{Ln}\left[1 + a_2\,\mathrm{Ln}\left(1 + s\,\beta\left(\frac{t_i + t_{i-1}}{2}\right)\right)\right]\right\}$$

Based on classical results on Riemann integral theory, the Riemann sums above converge to the integral below.

$$\lim_{\max\{\delta t_i\} \to 0^+} \left\{-\mathrm{Ln}\left(\tilde{\varphi}(s)\right)\right\} = \int_{t_0}^{t_n} a_1\,\mathrm{Ln}\left[1 + a_2\,\mathrm{Ln}\left(1 + s\,\beta(u)\right)\right] du \qquad [2.35]$$

Hence we have computed the Laplace transform of $Y(t_n) - Y(t_0)$ on a continuous time:

$$\lim_{\max\{\delta t_i\}\to 0^+} \tilde{\varphi}(s) = \exp\left(-\int_{t_0}^{t_n} a_1 \, Ln\left[1 + a_2 Ln\left(1 + s\beta(u)\right)\right] du \right) \quad [2.36]$$

Its moments can be obtained by differentiating and setting s = 0

$$\begin{aligned} &Mean\left(\tilde{Y}(t_n) - \tilde{Y}(t_0)\right) = a_1 a_2 \int_{t_0}^{t_n} \beta(u)\, du \\ &Var\left(\tilde{Y}(t_n) - \tilde{Y}(t_0)\right) = a_1 \, a_2 (1 + a_2) \int_{t_0}^{t_n} \beta(u)^2 \, du \end{aligned} \quad [2.37]$$

2.4 Modelling default times

A single name will default during a time period $[t_{i-1}, t_i[$ if and only if the variable

$$V_{t_{i-1},t_i} = \varphi_{t_{i-1},t_i}\left(-\frac{Ln(U_{t_i})}{(Y(t_i) - Y(t_{i-1}))} \right) \quad [2.38]$$

falls below a threshold equal to its forward default probability. V_{t_{i-1},t_i} is a latent variable; it cannot be observed directly. Let us assume, as is usual in credit derivatives that the survival default probability of the company is given by:

$$S(t) = \exp\left(-\int_0^t \lambda(u) du \right) \quad [2.39]$$

The forward default probability between during the time period $[t_{i-1}, t_i[$ is given by:

$$FwdDP(t_{i-1}, t_i) = 1 - \exp\left(-\int_{t_{i-1}}^{t_i} \lambda(u) du \right) \quad [2.40]$$

The default event of a company is then defined by:

$$V_{t_{i-1},t_i} < FwdDP(t_{i-1}, t_i),$$

That is, if

$$\varphi_{t_{i-1},t_i}\left(-\frac{Ln(U_{t_i})}{(Y(t_i) - Y(t_{i-1}))} \right) < 1 - \exp\left(-\int_{t_{i-1}}^{t_i} \lambda(u) du \right) \quad [2.41]$$

Default only occurs at discrete payment dates

Although default events can occur at any time in practice, the legal contracts defining CDOs specify a finite set of n payment dates at which default may occur. Consequently, the default

time is the smallest time at which the value of V_{t_{i-1},t_i} falls below the threshold for that time interval:

$$\tau(t_n) = \inf\left\{ t_i < 0, i \le n \quad \varphi_{t_{i-1},t_i}\left(-\frac{\mathrm{Ln}(U_{t_i})}{(Y(t_i) - Y(t_{i-1}))} \right) < 1 - \exp\left(-\int_{t_{i-1}}^{t_i} \lambda(u)\,du \right) \right\} \qquad [2.42]$$

Lemma: $\Pr\{\tau(t_n) = +\infty\} = 0$

Proof: Since V_{t_{i-1},t_i} is a uniform variable on [0,1] at any given time, we only need to show that in at least one interval $[t_{i-1}, t_i[$, the probability of default is different from zero.

$$t_n < +\infty, \text{ and hence } \forall i \in \{1,...,n\}, 0 < 1 - \exp\left(-\int_{t_{i-1}}^{t_i} \lambda(u)\,du \right) < 1$$

Let k_q be the maximum of the forward default probabilities on the intervals $[t_{i-1}, t_i[$

$$k_q = \max_{i \in \{1,...,n\}} \left\{ 1 - \exp\left(-\int_{t_{i-1}}^{t_i} \lambda(u)\,du \right) \right\} = \left\{ 1 - \exp\left(-\int_{t_{q-1}}^{t_q} \lambda(u)\,du \right) \right\} \Rightarrow 0 \prec k_q \prec 1 \qquad [2.43]$$

Since V_{t_{i-1},t_i} is a uniform on [0,1],

$$\tau(t_n) = +\infty$$
$$\Rightarrow \left\{ \forall i \in \{1,...,n\}, \mathrm{prob}\left(\varphi_{t_{i-1},t_i}\left(-\frac{\mathrm{Ln}(U_{t_i})}{(Y(t_i) - Y(t_{i-1}))} \right) < 1 - \exp\left(-\int_{t_{i-1}}^{t_i} \lambda(u)\,du \right) \right) = 0 \right\}$$

But

$$\mathrm{prob}\left\{ V_{t_{q-1},t_q} = \varphi_{t_{q-1},t_q}\left(-\frac{\mathrm{Ln}(U_{t_q})}{(Y(t_q) - Y(t_{q-1}))} \right) \prec 1 - \exp\left(-\int_{t_{q-1}}^{t_q} \lambda(u)\,du \right) \right\} \qquad [2.44]$$
$$= 1 - \exp\left(-\int_{t_{q-1}}^{t_q} \lambda(u)\,du \right) \neq 0$$

Hence $prob\{\tau(t_n) = +\infty\} = 0$.

By construction our model is automatically calibrated to the forward default probability and hence to the cumulative probability. This is a desirable feature since the CDS spread or the forward CDS spread (which give the implied term structure) can have a wide range of shapes. To illustrate the procedure we have taken the historical term structure of the probability of default for names in Moody's Baa2 ratings class because this is the average rating of CDO portfolios traded in the market. Any other class (Aaa, A2 … Ba2 Caa) could have been used; our model would automatically be calibrated to the term structure.

Box 2.2: Marginal Default Probability reconstitution with Forward Dynamic Copula

Algorithm

Extraction of the Survival probability curve associated with a Company:

$$S(t) = \exp\left(-\int_0^t \lambda(u)du \right)$$

For piecewise constant hazard rate or clean spread:

$$S(t) = \exp\left(-\int_0^t \lambda(u)du \right) = \exp(-\lambda t) = \exp(-\text{CleanSpread} \times t) = 1 - PD(t)$$

where $\text{CleanSpread} = \dfrac{\text{Ln}(1 - PD(t))}{t}$

Construction of the default probability curve associated with a Company:

$PD(t) = 1 - S(t)$

Construction of the forward default probability curve given a time step δt :

$$\text{FwdDP}(t, t+\delta t) = 1 - \frac{S(t+\delta t)}{S(t)} = 1 - \exp\left(-\int_t^{t+\delta t} \lambda(u)du \right)$$

For j =1 to maximum number of simulations

For i=1 to n (find if there is any default from today to maturity)

$$\text{If } V_{t_{i-1},t_i} = \varphi_{t_{i-1},t_i}\left(-\frac{\text{Ln}(U_{t_i})}{(Y(t_i) - Y(t_{i-1}))} \right) \prec 1 - \exp\left(-\int_{t_{i-1}}^{t_i} \lambda(u)du \right)$$

Then the default happens before the maturity

Next j simulation

Aggregate results.

Figure 2.6 compares the forward default probability produced by the forward dynamic copula (green) and the one extracted from idealized default probability term structure provided by Moody's for a Baa2 rating based on historical data (blue). Ten thousand MC simulations were used to compute the model. Here, we can see that with this many simulations that we have convergence.

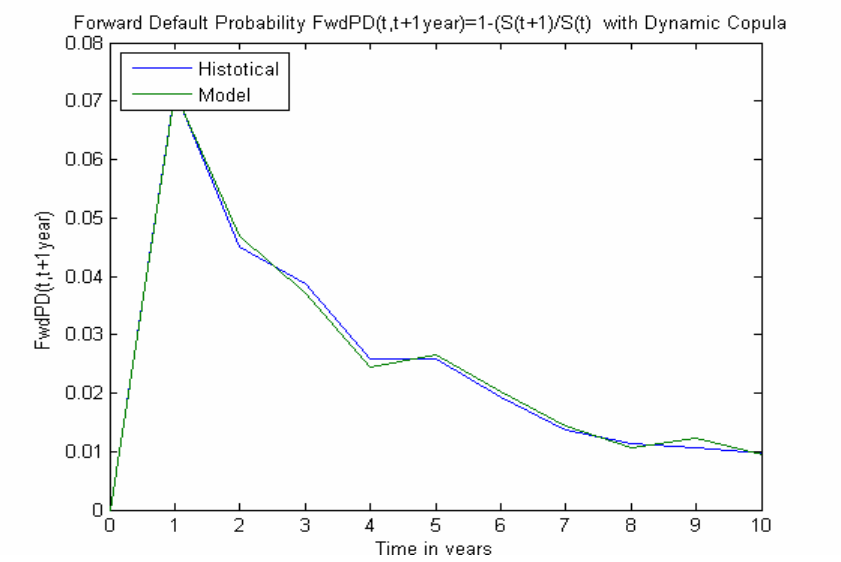


Figure 2.6 presents the forward default probability produced by the forward dynamic copula (green) and the one extract from idealized default probability term structure provided by Moody's for a Baa2 rating based on historical data (blue). Here, we can see that with 10 000 simulations, we have a convergence.

Figure 2.7 compares the cumulative default probability produced by the forward dynamic copula (green) and the one extracted from idealized default probability term structure provided by Moody's for a Baa2 rating based on historical data (blue). As before, with 10 000 simulations, the convergence is excellent.

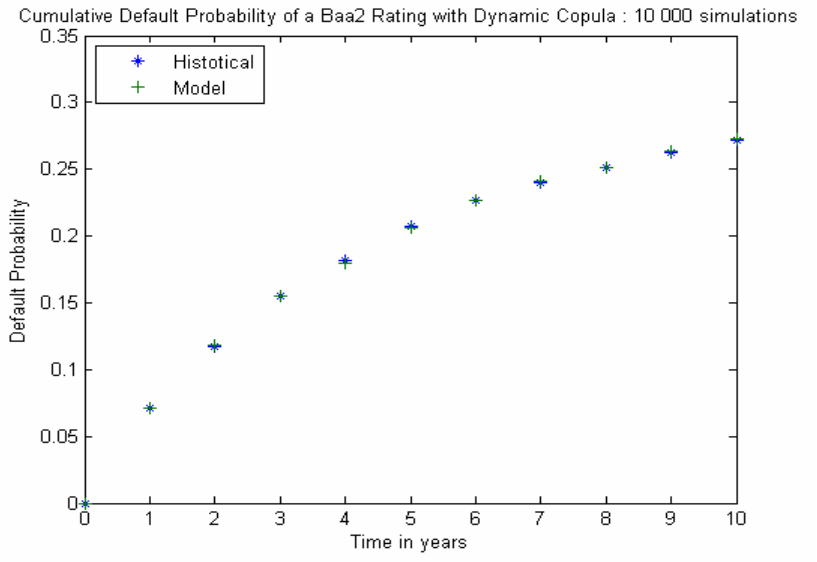


Figure 2.7 presents the cumulative default probability produced by the forward dynamic copula (green) and the one extract from idealized default probability term structure provided by Moody's for a Baa2 rating based on historical data (blue) . Here, we can see that with 10 000 simulations, we have a very good convergence.

Asymptotic spot loss distribution with bullet exposure

Assume that the credit portfolio consists of N underlying credits whose notionals are $P_i = P/N$, with fixed recovery rates are $R_i = R, \quad (i = 1,...,N)$. The aggregate loss from today to time t is a fixed sum of random variables:

$$\text{Loss}_N(t) = \sum_{i=1}^{N} (1 - R_i) P_i \, 1_{\{\tau_i \prec t\}} = \frac{(1-R)P}{N} \sum_{i=1}^{N} 1_{\{\tau_i \prec t\}} \qquad [2.45]$$

where $1_{\{\tau_i \prec t\}}$ is the indicator function for the default of the i^{th} name. Its Laplace transform is

$$E\left\{\exp\left(-s\,\text{Loss}_N(t)\right)\right\} = E\left\{\exp\left(-\frac{sP(1-R)}{N}\sum_{i=1}^{N} 1_{\{\tau_i \prec t\}}\right)\right\}$$

$$= E\left\{\prod_{i=1}^{N}\left\{\left(1 - 1_{\{\tau_i \prec t\}}\right) + 1_{\{\tau_i \prec t\}}\,\eta^{\frac{1}{N}}\right\}\right\}$$

[2.46]

Letting $\eta = \exp\left(-s\,N(1-R)\right) > 0$

$$E\left\{\exp\left(-s\text{Loss}_N(t)\right)\right\} = E\left\{\left[\left(1 - \eta^{\frac{1}{N}}\right)\exp\left(-Y_t\,\varphi_t^{-1}\left(PD(t)\right)\right) + 1\right]^N\right\}$$

We now compute its limit as N the number of names tends to infinity. Since

$$\lim_{N\to+\infty}\left\{N\left(\eta^{\frac{1}{N}} - 1\right)\right\} = \lim_{n\to+\infty}\left\{\frac{\left(\eta^{\frac{1}{N}} - 1\right)}{\frac{1}{N}}\right\} = \left|\frac{\partial \eta^x}{\partial x}\right|_{x=0} = \text{Ln}(\eta) = -sN(1-R)$$

we obtain

$$\text{Loss}_N(t) \underset{N\to+\infty}{\approx} P(1-R)\exp\left(-Y_t\varphi_t^{-1}\left(PD(t)\right)\right) \qquad [2.47]$$

Asymptotic forward loss distribution

Here we want to compute the aggregate loss from t to time T, **assuming no default prior to time t**. If k defaults had occurred in the portfolio, the analysis below would still hold, we need only replace N by (N-k).

As before the credit portfolio consists of N underlying credits with equal notional amounts and with a fixed recovery rate R. The individual notional is independent of the number, k, of

prior defaults. For simplicity, we only consider the case where we have zero defaults prior to time t. As before,

$$\text{Loss}_N(t,T) = \sum_{i=1}^{N} L_i \quad = \frac{(1-R)P}{N} \sum_{i=1}^{N} 1_{\{t<t_i\le T\}}$$

With $\eta = \exp\left(-s\,N(1-R)\right) > 0$, we compute its Laplace transform

$$E\left\{\exp\left(-s\,\text{Loss}_N(t,T)\right)\right\}$$
$$= E\left\{\left[\left(1-\exp\left(-\left(Y_T-Y_t\right)\varphi_{t,T}^{-1}\left(\text{FwdPD}(t,T)\right)\right)\right)+\eta^{\frac{1}{N}}\exp\left(-\left(Y_T-Y_t\right)\varphi_{t,T}^{-1}\left(\text{FwdPD}(t,T)\right)\right)\right]^N\right\}$$

where

$$\text{FwdPD}(t,T) = \frac{\text{PD}(T)-\text{PD}(t)}{1-\text{PD}(t)}$$

Hence,

$$E\left\{\exp\left(-s\,\text{Loss}_N(t,T)\right)\right\} = E\left\{\left[\left(1-\eta^{\frac{1}{N}}\right)\exp\left(-\left(Y_T-Y_t\right)\varphi_{t,T}^{-1}\left(\text{FwdPD}(t,T)\right)\right)+1\right]^N\right\}$$

So its limit as N tends to infinity is:

$$\text{Loss}_{\mathbf{N}}(t,T) \underset{\mathbf{N}\to+\infty}{\approx} \mathbf{P}(1-\mathbf{R})\exp\left(-\left(Y_T-Y_t\right)\boldsymbol{\varphi}_{t,T}^{-1}\left(\text{FwdPD}(t,T)\right)\right) \qquad [2.48]$$

The generator function of the forward copula $\varphi_{t-\delta t,t}(s)$ is the Laplace transform of the probability distribution of the forward compound gamma process $\left(Y_t - Y_{t-\delta t}\right)$ with the filtration $F_{t-\delta t}$.

The equation for the forward loss is quite similar to that for the spot loss. The difference is that the total notional of the portfolio from which it should be deducted, decreases whenever there is a loss.

2.5 Dynamic copulas from a Levy process perspective: gamma Ornstein–Uhlenbeck process

In this section we construct another building block for dynamic copulas, but this time based on a gamma Ornstein–Uhlenbeck process. This process was chosen because a closed form solution is known for its Laplace transform. Secondly it is a positive process and has been used for modelling stochastic spreads and interest rates. For example Barndorff-Nielsen and Shephard (2001) applied it when modelling the stock market. In that case the volatility was

modelled by an Ornstein Uhlenbeck (OU) process driven by a subordinator. More recently this process was studied by Schoutens, Simonsy & Tistaertz (2003). Their results were impressive in term of capturing the dynamics of the volatility of the stock prices. The process has also been known to geostatisticians working in the earth sciences for nearly 40 years. According to Chilès & Delfiner (1999, p489), Matheron (1969) called this an Ambartzumian after the Soviet mathematician of the same name.

Definition of gamma Ornstein–Uhlenbeck process

We use the classical and tractable example of the gamma-OU process. The marginal law of the volatility is gamma-distributed. Volatility can only jump upwards and then it decays exponentially. A co-movement effect between up-jumps in volatility and (down)-jumps in the stock price is also incorporated to make the price of the asset jump downwards when the volatility jumps upward. In the absence of a jump, the asset price process moves continuously and the volatility decays also continuously. Other choices for OU-processes can be made; we mention especially the Inverse Gaussian OU process; this is also tractable.

The squared volatility now follows a SDE of the form:

$$d\sigma^2(t) = -\lambda\sigma^2(t)dt + dz(\lambda t), \qquad [2.49]$$

where $\lambda \succ 0$ and $z = \{z_t, t \geq 0\}$ is a subordinator.

The risk-neutral dynamics of the log-price $Z_t = Ln(S_t)$ are given by:

$$dZ_t = \left(r - q - \lambda(\rho) - \frac{\sigma^2(t)}{2}\right)dt + \sigma(t)dW_t + \rho dz(\lambda t), \qquad [2.50]$$

where $Z_0 = Ln(S_0)$, $W = \{W_t, t \geq 0\}$ is a Brownian motion and is independent of the subordinator process $z = \{z_t, t \geq 0\}$. The parameter ρ controls the correlation between the volatility and the stock price dynamics.

In our case, the factor Y(t) will be a gamma-OU process; that is,

$$dY(t) = -\lambda Y(t)dt + dz(\lambda t) \qquad [2.51]$$

With $\lim_{t\to+\infty} Y(t) \equiv \Gamma(a_1, a_2)$ where

$$f_{Y(t),t\to+\infty}(x)=\frac{(a_1)^{a_2}}{\Gamma(a_2)}x^{a_2-1}\exp(-a_1x)1_{\{x\geq 0\}} \qquad [2.52]$$

Box 2.3: Reminders on the Poisson process

A Poisson process N(t) is an increasing process at the integers 0,1,2,… Let η be the intensity of the Poisson process. We assume that

- the probability of a jump in the next small time interval δt is proportional to δt
- jumps by more than one do not occur,
- jumps in disjoint time intervals happen independently of each other. Conversely, this means that the probability of the process remaining constant is:

$$prob\left(N(t+\delta t)-N(t)=0\right)=1-\eta\delta t$$

Figure 2.8 presents a typical simulation of a Poisson process.

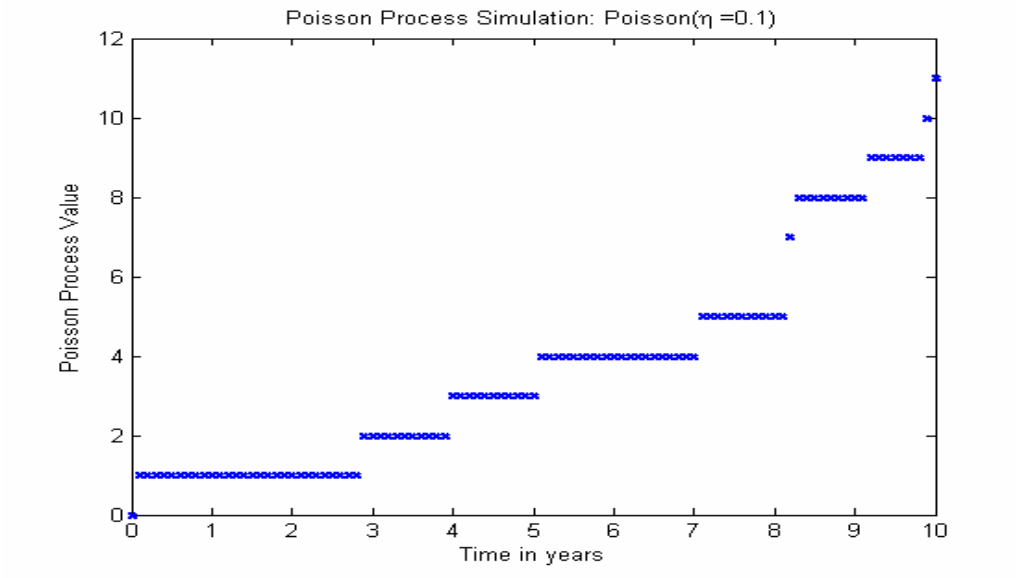


Figure 2.8 presents a typical simulation of a Poisson process.

This means that the Levy measure has zero drift and has the density:

$$\upsilon^{y}(x)=\frac{a_2}{x}\exp(-a_1x)1_{\{x\geq 0\}} \qquad [2.53]$$

We can deduce that the Levy density of the Levy process Z(t) has zero drift; its density

$$\upsilon(x)=a_2\exp(-a_1x)1_{\{x\geq 0\}}. \qquad [2.54]$$

This Levy process is a compound Poisson subordinator with exponential jump size. See Box 2.3 for information on Poisson processes.

Next we will analyse the integrated process

$$\widetilde{Y}(t)=\int_0^t Y(s)ds \quad . \qquad [2.55]$$

Based on the properties of non-Gaussian Ornstein-Uhlenbeck processes, we have the following:

$$\widetilde{Y}(t)\equiv\int_0^t Y(s)ds=\frac{y_0}{\lambda}(1-\exp(-\lambda t))+\frac{1}{\lambda}\int_0^t(1-\exp(-\lambda(t-s)))dz(s) \qquad [2.56]$$

$$\widetilde{Y}(t)\equiv\int_0^t Y(s)ds=\frac{1}{\lambda}(z(\lambda t)-Y(t)+Y(0)) \qquad [2.57]$$

Characteristic function of the integrated process

Cont and Tankov (2004) gave an explicit formula for the Laplace transform of a positive Ornstein-Uhlenbeck process, but we derive a different formulation of the Laplace transform here.

$$E\left[\exp\left(-u\tilde{Y}(t)\right)\right] = \exp\left(-\frac{uy_0}{\lambda}\left(1-\exp\left(-\lambda t\right)\right)+\int_0^t l\left(\frac{u}{\lambda}\left(1-\exp\left(-\lambda\left(t-s\right)\right)\right)\right)ds\right) \qquad [2.58]$$

Where:

$$l(u)=E[\exp(-uz(1))]=\frac{\lambda a_2 u}{(a_1+u)}$$

$$E\left[\exp\left(-u\tilde{Y}(t)\right)\right] = \exp\left(-\frac{uy_0}{\lambda}\left(1-\exp\left(-\lambda t\right)\right)-\frac{u\lambda a_2 t}{\lambda a_1+u}+\frac{u\lambda a_2}{\lambda a_1+u}Ln\left(1+\frac{u}{\lambda a_1}\left(1-\exp\left(-\lambda t\right)\right)\right)\right)$$

The Laplace transform of the joint distribution is.

$$E\left[\exp\left(-u\tilde{Y}(t)-vz(t)\right)\right] = \exp\left(-\frac{uy_0}{\lambda}\left(1-\exp\left(-\lambda t\right)\right)+\int_0^t l\left(v+\frac{u}{\lambda}\left(1-\exp\left(-\lambda\left(t-s\right)\right)\right)\right)ds\right) \qquad [2.59]$$

Box 2.3: Fast simulation for the gamma-OU process

Algorithm

The gamma OU process is as follows:

$$dY(t) = -\lambda Y(t)\,dt + dz(\lambda t), \quad with \lim_{t\to+\infty} Y(t) \equiv \Gamma(a_1, a_2)$$

To simulate the gamma OU process in discrete time $t_n = n\delta t, \quad n = 1,2,3,...,$ we first simulate a Poisson process $N = \{N_t, t \geq 0\}$ with intensity parameter λa_1 at the same time points. Then, with the convention that an empty sum equals zero),

$$y_{t_n} = (1-\lambda\delta t)\, y_{t_{n-1}} + \sum_{k=N_{t_{n-1}}+1}^{N_{t_n}} x_k \exp(-\lambda \times \delta t \times \tilde{u}_k),$$

$$with\ marginal\ \Gamma(a_1, a_2), \quad u_k, \tilde{u}_k\ uniform[0,1], \qquad [2.60]$$

$$x_k = -\frac{Ln(u_k)}{a_2}, \quad N(t) \quad Poisson \quad \lambda a_1 t$$

The integrated process is given by the following formula:

$$\tilde{Y}(t) \equiv \int_0^t Y(s)ds = \frac{y_0}{\lambda}(1-\exp(-\lambda t)) + \frac{1}{\lambda}\int_0^t (1-\exp(-\lambda(t-s)))dz(s) \qquad [2.61]$$

Mean reverting gamma O-U process simulation

A mean reverting gamma O-U process can be seen as a classical mean reverting process where the stochastic part is driven by a compound Poisson process. The number of jumps is a Poisson counting process during the time interval, and the size of the jumps follow an exponential distribution.

Our simulation is a direct implementation of the algorithm described in Box 2.3. The background Levy process (BGLP) is as follows:

$$BGLP_{t_n} = BGLP_{t_{n-1}} + \sum_{k=N_{t_{n-1}}+1}^{N_{t_n}} x_k \exp(-\lambda\,\delta t\,\tilde{u}_k) \qquad [2.62]$$

Figure 2.9 shows the background Levy process that is a compound Poisson process with an exponentially distributed jump size. The second graph is the corresponding gamma O-U process, and we can see clearly that the resulting process is mean reverting.

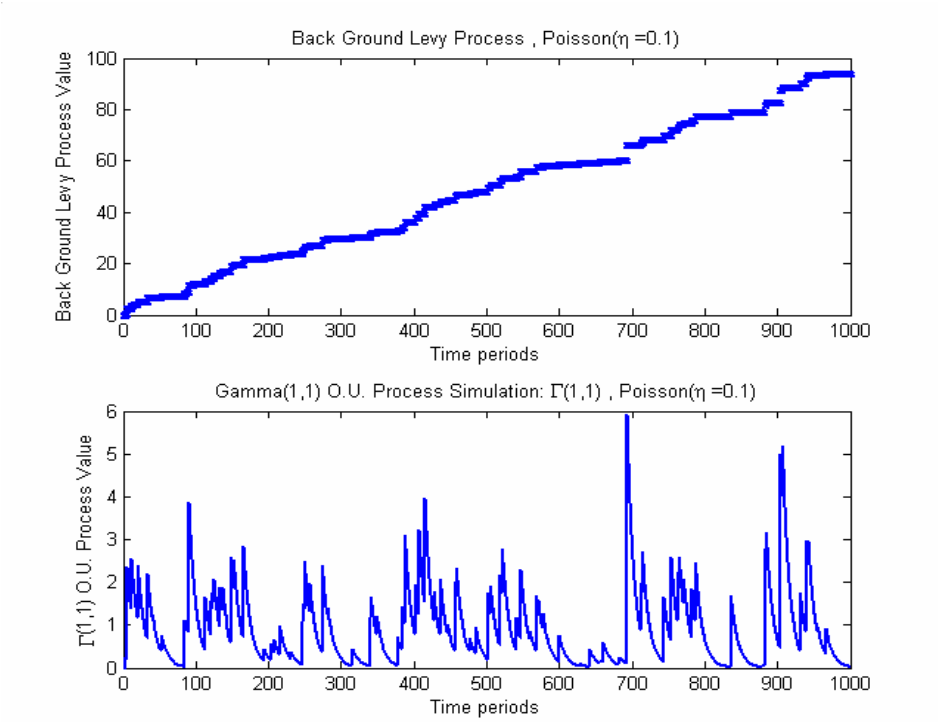


Figure 2.9 As expected the process is stationary and mean reverting distribution.

Building block based on integrated gamma Ornstein–Uhlenbeck process

For spot copula

$$\varphi_t(s)=\exp\left(-\frac{sy_0}{\lambda}\left(1-\exp(-\lambda t)\right)-\frac{s\lambda a_2 t}{\lambda a_1+s}+\frac{s\lambda a_2}{\lambda a_1+s}Ln\left(1+\frac{s}{\lambda a_1}\left(1-\exp(-\lambda t)\right)\right)\right) \quad [2.63]$$

As usual the marginal spot copula variable is given by:

$$V_t=\varphi_t\left(-\frac{\mathrm{Ln}(U_t)}{\tilde{Y}(t)}\right) \quad [2.64]$$

For the forward dynamic copula

$$\varphi_{t_0,t}(s)=\exp\left(-\frac{sy_0}{\lambda}\left(1-\exp\left(-\lambda\times(t-t_0)\right)\right)-\frac{s\lambda a_2 t}{\lambda a_1+s}+\right)\times\exp\left(\frac{s\lambda a_2}{\lambda a_1+s}Ln\left(1+\frac{s}{\lambda a_1}\left(1-\exp\left(-\lambda\times(t-t_0)\right)\right)\right)\right) \quad [2.65]$$

And its copula variable is given by:

$$V_{t_0,t}=\varphi_{t_0,t}\left(-\frac{\mathrm{Ln}(U_t)}{\left(\tilde{Y}(t)-\tilde{Y}(t_0)\right)}\right) \quad [2.66]$$

Box 2.4: Alternative fast simulation for the gamma-OU process

Algorithm

The integrated process is given by the following formula:

$$\tilde{Y}(t) \equiv \int_0^t Y(s)ds = \frac{y_0}{\lambda}(1-\exp(-\lambda t)) + \frac{1}{\lambda}\int_0^t (1-\exp(-\lambda(t-s)))dz(s)$$

Based on the result below, we will simulate the integrated gamma OU process

$$\exp(-\lambda t)\int_0^t \exp(\lambda s)\, d\alpha(\lambda s) = \frac{1}{a_2}\exp(-\lambda t)\sum_{i=1}^{N(1)} Ln\left(\frac{1}{c_i}\right)\exp(\lambda t r_i) \qquad [2.67]$$

With marginal distribution $\Gamma(a_1, a_2)$, r_i *uniform* $[0,1]$

$N(t)$ *Poisson* $\lambda a_1 t$, $c_i \prec c_{i+1} \leq 1$

c_i *are arrival times of the Poisson process*

$$\tilde{Y}(t) \equiv \frac{y_0}{\lambda}(1-\exp(-\lambda t)) + \frac{1}{a_2}\frac{1}{\lambda}\sum_{i=1}^{N(1)} Ln\left(\frac{1}{c_i}\right)\exp(\lambda t(1-r_i)) \qquad [2.68]$$

The back ground Levy process (BGLP) is given by the following formula:

$$BGLP_{t_n} = BGLP_{t_{n-1}} + \frac{1}{a_2}\frac{1}{\lambda}\sum_{i=1}^{N(1)} Ln\left(\frac{1}{c_i}\right)\exp(\lambda t(1-r_i)) \qquad [2.69]$$

Chapter 3: Combining credit migration and copulas

The development of new products such as Constant Proportion Debt Obligation (CPDO) requires being able to model credit migration and correlation in order to handle substitutions on the index during the roll, so we propose a model for the joint dynamics of credit ratings of several firms. Individual credit ratings are modelled by a continuous time Markov chain, and their joint dynamics are modelled using a copula process. Copulas allow us to incorporate our knowledge of single name credit migration processes, into a multivariate framework. We now revisit the joint distributions of the default times of all the firms in the portfolio.

3.1 Construction of the single name credit migration and default process

Consider a portfolio that is composed of N risky bonds with respective ratings $X_t^{(1)},\cdots,X_t^{(N)}$. The ratings take values in the ordered set $S=(1,\cdots,m)$, where 1 is the best rating, and m is the default state. The goal is to model the joint behaviour of the stochastic processes $X^{(1)},\cdots,X^{(N)}$. The assumptions made below are natural in a credit rating migration context. The first assumption concerns the dynamics of the individual ratings while the second relates to their joint distribution.

For any $1\le k\le N$, $X^{(k)}=\left(X_t^{(k)}\right)_{t\ge 0}$ is a continuous time Markov chain on S. Its infinitesimal generator is $\Lambda^{(k)}$, that is

$$Q^{(k)}=P\left(X_{t+h}^{(k)}=j/X_t^{(k)}=i\right)=\begin{cases} h\,\Lambda_{ij}^{(k)}+o(h),\ j\neq i \\ 1+h\,\Lambda_{ii}^{(k)}+o(h),\ j=i \end{cases},\ i,j\in S \qquad [3.01]$$

Furthermore, for any k, the only absorbing state is state m (default). This is standard continuous-time modelling of the individual credit ratings $X^{(k)}$, where ratings can change at any time according to a Markov chain. As the only absorbing state is default, setting $\lambda_i^{(k)}=-\Lambda_{ii}^{(k)}$, implies that $\lambda_i^{(k)}\succ 0$ if $i\prec m$ and $\lambda_m^{(k)}=0\ \ 1\le k\le N$.

We then assume this distribution evolves according to the Kolmogorov equation

$$\frac{\partial}{\partial t}Q_t^{(k)}=\Lambda^{(k)}Q_t^{(k)} \qquad [3.02]$$

This equation leads to:

$$Q_t^{(k)}=\exp\left(t\times\Lambda^{(k)}\right) \qquad [3.03]$$

Estimation of the transition intensity

By using a one-year average historical rating transition for example, one can obtain an estimate, $\hat{\Lambda}$, of the generator matrix Λ. If the rating transitions of the obligors or instruments were observed over a period of T years, then a classical estimator of the $\lambda_{i,j}$'s of the generator matrix (see Andersen and Borgan, 1995) is given by:

$$\lambda_{i,j} = \frac{N_{i,j}(T)}{\int_0^T Nbr_i(s)\,ds} \qquad [3.04]$$

where $N_{i,j}(T)$ is the total number of transitions migrating from rating category i to rating category j during the T-year period and $Nbr_i(s)$ is the number of firms in rating category i at time s.

Usually, rating agencies do not publish the generating matrix Λ of the rating migration process, but only the transition probability

$$Q = \left\{ P\left(X_{t+1\,year}^{(k)} = j / X_t^{(k)} = i \right) \right\}_{1 \le i,j \le m}. \qquad [3.05]$$

Public historical transition matrices do not include the exact timing of transitions. They merely give estimates of, say, one-year $(t = 1)$ transition probabilities that are obtained by observing the credit ratings of cohorts of firms or structured finance securities at the beginning and at the end of the year. Table 3.1 gives a typical example of an average 1-year migration transition matrix (as at 1997). We use these estimates to obtain an approximation $\hat{Q}(1)$ of $Q(1)$, which in turn provides an estimate $\hat{\Lambda}$ of the generator matrix Λ. The latter is used to obtain transition probabilities $Q(t)$ for every t.

Table 3.1 Estimates of average 1-year migration matrix (conditional upon no rating withdrawal): Moody's Investors Service, July 1997. "Moody's Rating Migration and Credit Quality Correlation, 1920-1996"

One year(Average)	Rating To							
Rating From	AAA	AA	A	BBB	BB	B	CCC-C	Default
AAA	92,18%	6,51%	1,04%	0,25%	0,02%	0,00%	0,00%	0,00%
AA	1,29%	91,62%	6,11%	0,70%	0,18%	0,03%	0,00%	0,07%
A	0,08%	2,50%	91,36%	5,11%	0,69%	0,11%	0,02%	0,14%
BBB	0,04%	0,27%	4,22%	89,16%	5,25%	0,68%	0,07%	0,31%
BB	0,02%	0,09%	0,44%	5,11%	87,08%	5,57%	0,46%	1,25%
B	0,00%	0,04%	0,14%	0,69%	6,52%	85,20%	3,54%	3,87%
CCC-C	0,00%	0,02%	0,04%	0,37%	1,45%	6,00%	78,30%	13,82%
Default	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	100,00%

Following Jarrow, Lando and Turnbull (1997), we have the following approximation for $Q_t^{(k)} = \exp\left(t \times \Lambda^{(k)}\right)$: [3.06]

$$e^{\Lambda t} \approx$$

$$\widehat{Q}(t) = \begin{pmatrix} e^{\lambda_{1,1}t} & \lambda_{1,2}\varphi_{1,1}(t) & \lambda_{1,3}\varphi_{1,1}(t) & \cdots & \lambda_{1,16}\varphi_{1,1}(t) & \lambda_{1,m-1}\varphi_{1,1}(t) & \lambda_{1,m}\varphi_{1,1}(t) \\ \lambda_{2,1}\varphi_{2,2}(t) & e^{\lambda_{2,2}t} & \lambda_{2,3}\varphi_{2,2}(t) & \cdots & \lambda_{2,16}\varphi_{2,2}(t) & \lambda_{2,m-1}\varphi_{2,2}(t) & \lambda_{2,m}\varphi_{2,2}(t) \\ \lambda_{3,1}\varphi_{3,3}(t) & \lambda_{2,3}\varphi_{3,3}(t) & e^{\lambda_{3,3}t} & \cdots & \lambda_{3,16}\varphi_{3,3}(t) & \lambda_{3,m-1}\varphi_{3,3}(t) & \lambda_{3,m}\varphi_{3,3}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda_{m-2,1}\varphi_{m-2,m-2}(t) & \lambda_{m-2,2}\varphi_{m-2,m-2}(t) & \lambda_{m-2,3}\varphi_{m-2,m-2}(t) & \cdots & e^{\lambda_{m-2,m-2}t} & \lambda_{m-2,m-1}\varphi_{m-2,m-2}(t) & \lambda_{m-2,m}\varphi_{m-2,m-2}(t) \\ \lambda_{m-1,1}\varphi_{m-1,m-1}(t) & \lambda_{m-1,2}\varphi_{m-1,m-1}(t) & \lambda_{m-1,3}\varphi_{m-1,m-1}(t) & \cdots & \lambda_{m-1,m-2}\varphi_{m-1,m-1}(t) & e^{\lambda_{m-1,m-1}t} & \lambda_{m-1,m}\varphi_{m-1,m-1}(t) \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

where $\varphi_{i,i}(t) = \dfrac{e^{\lambda_{i,i}t} - 1}{\lambda_{i,i}}$ for $i = 1, ..., m-1$

So the 1-year transition probability matrix $\widehat{Q}(t=1)$ can be computed from the 1-year rating transition matrix $P = \left(p_{i,j}\right)$ published by a rating agency. The values of $\widehat{Q}(1) = \left(\widehat{q}_{i,j}\right)$ can be read directly from $P = (p_{i,j})$, where $p_{i,j}$ has been adjusted for the "Withdrawn" ratings and normalised so that the sum is 1.0.

From $\widehat{Q}(1)$, we can estimate the generator matrix $\widehat{\Lambda}$. Then solving $\widehat{Q}(1) \approx e^{\widehat{\Lambda}}$, gives the following $e^{\widehat{\lambda}_{i,i}} \approx \widehat{q}_{i,i}$. Hence

$$\lambda_{i,i} \approx \widehat{\lambda}_{i,i} = \ln(\widehat{q}_{i,i}) \text{ for } i = 1, ..., m \qquad [3.07]$$

For the off-diagonal elements, we have:

$$\widehat{q}_{i,j} \approx \underset{i \neq j}{\widehat{\lambda}_{i,j}} \widehat{\varphi}_{i,i} = \underset{i \neq j}{\widehat{\lambda}_{i,j}} \frac{e^{\widehat{\lambda}_{i,i}} - 1}{\widehat{\lambda}_{i,i}} \approx \underset{i \neq j}{\widehat{\lambda}_{i,j}} \frac{\widehat{q}_{i,i} - 1}{\ln(\widehat{q}_{i,i})},$$

hence

$$\underset{i \neq j}{\lambda_{i,j}} \approx \underset{i \neq j}{\widehat{\lambda}_{i,j}} = \widehat{q}_{i,j} \frac{\ln(\widehat{q}_{i,i})}{\widehat{q}_{i,i} - 1} \text{ for } i = 1, ..., m \qquad [3.08]$$

Table 3.2 shows the real world transition intensities computed from the 1-year migration matrix (conditional on no rating withdrawal) that were computed in this way from the data in Table 3.1. Table 2.4 shows the intensities of the transition from one state of rating to another.

Table 3.2 Estimates of the real world transition intensitiess matrix from the average 1-year transition matrix (conditional upon no rating withdrawal) using Jarrow, Lando and Turnbull (1997) method. Data source: Moody's Investors Service, July 1997. "Moody's Rating Migration and Credit Quality Correlation, 1920-1996"

Real world Intensity	Rating To							
Rating From	AAA	AA	A	BBB	BB	B	CCC-C	Default
AAA	-8,14%	6,78%	1,08%	0,26%	0,02%	0,00%	0,00%	0,00%
AA	1,35%	-8,75%	6,38%	0,73%	0,19%	0,03%	0,00%	0,07%
A	0,08%	2,61%	-9,04%	5,34%	0,72%	0,12%	0,02%	0,15%
BBB	0,04%	0,29%	4,47%	-11,47%	5,56%	0,72%	0,07%	0,33%
BB	0,02%	0,10%	0,47%	5,47%	-13,83%	5,96%	0,49%	1,34%
B	0,00%	0,04%	0,15%	0,75%	7,06%	-16,02%	3,83%	4,19%
CCC-C	0,00%	0,02%	0,05%	0,42%	1,63%	6,76%	-24,46%	15,57%
Default	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%

Table 3.3 Estimation of the real world transition intensitiess matrix from the average one year transition matrix (conditional upon no rating withdrawal) using Jarrow, Lando and Turnbull (1997) method. Data source: Moody's Investors Service, July 1997. "Moody's Rating Migration and Credit Quality Correlation, 1920-1996"

Cumulative Quaterly Real world Migration	Rating To							
Rating From	AAA	AA	A	BBB	BB	B	CCC-C	Default
AAA	97,98%	99,66%	99,93%	99,99%	100,00%	100,00%	100,00%	100,00%
AA	1,66%	98,19%	99,75%	99,93%	99,97%	99,98%	99,98%	100,00%
A	0,26%	1,82%	98,45%	99,75%	99,93%	99,96%	99,96%	100,00%
BBB	0,06%	0,24%	1,56%	98,36%	99,72%	99,90%	99,92%	100,00%
BB	0,01%	0,05%	0,23%	1,60%	98,09%	99,55%	99,67%	100,00%
B	0,00%	0,01%	0,04%	0,21%	1,68%	98,03%	98,97%	100,00%
CCC-C	0,00%	0,00%	0,01%	0,02%	0,15%	1,10%	96,18%	100,00%
Default	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	100,00%

Table 3.4 Estimates of the real world cumulative quaterly migration matrix threshold using Jarrow, Lando and Turnbull (1997) method. Data source: Moody's Investors Service, July 1997. "Moody's Rating Migration and Credit Quality Correlation, 1920-1996"

Quaterly Real world Migration	Rating To							
Rating From	AAA	AA	A	BBB	BB	B	CCC-C	Default
AAA	97,98%	1,68%	0,27%	0,06%	0,01%	0,00%	0,00%	0,00%
AA	1,66%	96,54%	1,56%	0,18%	0,05%	0,01%	0,00%	0,02%
A	0,26%	1,56%	96,62%	1,31%	0,18%	0,03%	0,01%	0,04%
BBB	0,06%	0,18%	1,32%	96,80%	1,36%	0,18%	0,02%	0,08%
BB	0,01%	0,05%	0,18%	1,37%	96,49%	1,46%	0,12%	0,33%
B	0,00%	0,01%	0,03%	0,18%	1,47%	96,35%	0,94%	1,03%
CCC-C	0,00%	0,00%	0,01%	0,02%	0,12%	0,95%	95,09%	3,82%
Default	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	100,00%

This table should not be compared directly to Table 2.3 since the later gives directly the probability of migration on a one year horizon. However, the Table 2.4 allows us to estimate the migration probability on any time horizon assuming that the migration process is an homogeneous Markov process.

3.2 Construction of the risk neutral single name credit migration and default process

A good credit risk pricing model is one that can be calibrated to benchmark market prices, or alternatively can be specified under the martingale measure and not under historical probabilities. By construction, we have extracted the risk neutral default and the forward risk neutral default probability from CDS market prices. (See Box 3.1).

It is important to adjust the historical probabilities to martingale probabilities because of the presence of large risk premiums in credit market. By construction, we have extracted the risk neutral default and the forward risk neutral default probability from CDS market prices. See below.

Risk neutral probability migration.

Each of the N companies has its own ratings migration matrix with its infinitesimal generator $\Lambda^{(k)}$ in the real world measure. Following Lando (2000), let $X^{(k)} = \left(X_t^{(k)}\right)_{t\geq 0}$ be a continuous time Markov chain on the set of ratings, S. The risk neutral transition matrix is given by:

$$Q_{RN}{}^{(k)} = P_{RN}\left(X_{t+h}^{(k)} = j / X_t^{(k)} = i\right) = \begin{cases} h\,\Lambda_{ij}{}^{(k)} + o(h),\ j \neq i, j \prec m \\ h\,\mu_i(t)\,\Lambda_{im}{}^{(k)} + o(h),\ j \neq i, j \leq m \\ 1 - h\left(\mu_i(t)\,\Lambda_{im}{}^{(k)} + \sum_{j\neq i, j<m} \Lambda_{ij}{}^{(k)}\right) + o(h),\ j = i \end{cases}, i, j \in S \qquad [3.09]$$

The middle line in eqn [3.09] corresponds to a default event; the last one corresponds to staying in the same state. The term $\mu(t)$ is a correction term from the historic probability to the risk neutral one.

Box 3.1: Algorithm for risk neutral marginal default probability extraction from CDS prices

Algorithm

Extraction of the survival probability curve associated with a company:

$$S(t) = \exp\left(-\int_0^t \lambda(u)du\right) \qquad [3.10]$$

For piecewise constant hazard rate or clean spread:

$$S(t) = \exp\left(-\int_0^t \lambda(u)du\right) = \exp(-\lambda t) = \exp(-\text{CleanSpread} \times t) = 1 - PD(t)$$

Where

$$\text{CleanSpread} = \frac{\text{Spread}}{LGD} = \frac{\text{Ln}(1 - PD(t))}{t} \qquad [3.11]$$

Construction of the risk neutral default probability curve associated with a Company

$$PD(t) = 1 - S(t)$$

Construction of the forward default probability curve given a time step δt :

$$\text{FwdDP}(t, t+\delta t) = 1 - \frac{S(t+\delta t)}{S(t)} = 1 - \exp\left(-\int_t^{t+\delta t} \lambda(u)du\right) \qquad [3.12]$$

Alternatively we could have linked the risk neutral probability migration equation proposed by Lando with the real world probability migration in the following way.

$$\begin{cases} \Lambda_{(RN)ij}^{(k)} = \Lambda_{ij}^{(k)},\ j \neq i, j \prec m \\ \Lambda_{(RN)im}^{(k)} = \mu_i(t)\Lambda_{im}^{(k)}, \textit{and}\ \Lambda_{(RN)ii}^{(k)} = \sum_{j \neq i, j \prec m} \Lambda_{(RN)ij}^{(k)}, j = i \end{cases}, i, j \in S \qquad [3.13]$$

The estimate of the adjustment parameter that leads the real world measure and the risk neutral measure is:

$$\Lambda_{(RN)im}^{(k)} = \mu_i(t)\Lambda_{im}^{(k)} \Rightarrow \mu_i(t) = \frac{\lim_{\delta t \to 0}\left(\frac{1}{\delta t}\text{FwdDP}(t, t+\delta t)\right)}{\Lambda_{im}^{(k)}} \qquad [3.14]$$

Illustration of marginal rating migration by simulation in the real world

We need first of all to construct a discrete time matrix migration.

$$Q_{t,t+\delta t}^{(k)} = \exp\left(\delta t \times \Lambda_t^{(k)}\right) \qquad [3.15]$$

Every quarter, we consider a uniform random variable between 0 and 1, that could be correlated within a copula framework and we compare that value with the appropriate thresholds for the appropriate rating (its rating at the beginning of quarter). This allows us to determine the rating at the end of the period. This can lead to upgrades, no-action, downgrades, or defaults. The thresholds used in Table 3.5 for the simulation are the ones obtained in Table 3.4 that are the quarterly cumulative migration probability.

3.3 Copula approach for dependent credit migrations and default processes

Once we have built the model for individual credit rating migrations, we need to model their joint evolution. This multivariate problem has been one of the most technically challenging in the credit risk literature. For example Greg et al. (1997) use a single Poisson-based transition matrix for all the individual securities, and a Gaussian copula to model their joint behaviour.

Table 3.5 Illustration of rating migration

Quarter	Rating at the beginning of the period	*Uniform Random*	Rating at the end of the period	Simulated rating action
1	**AAA**	**98,00%**	**AA**	Downgrade
2	**AA**	**10,00%**	**AA**	No action
3	**AA**	**99,80%**	**A**	Downgrade
4	**A**	**30,00%**	**A**	No action
5	**A**	**99,94%**	**BB**	Downgrade
6	**BB**	**80,00%**	**BB**	No action
7	**BB**	**50,00%**	**BB**	No action
8	**BB**	**99,56%**	**B**	Downgrade
9	**B**	**20,00%**	**BB**	Upgrade
10	**BB**	**99,90%**	**Default**	Default
10	**Default**	**10,00%**	**Default**	Non reversible
12	**Default**	**2,00%**	**Default**	Non reversible

Since the individual Markov chains all evolve according to the same transition matrix, the value of the portfolio is determined strictly by the number of bonds initially in each state, without allowing these bonds to have different characteristics. This elimination of idiosyncratic default risk and risk premiums in the credit derivatives market is quite a severe

simplification. We now show how to estimate the actual rating at each time step from the initial state until maturity.

Construction of a multivariate migration threshold for any given time period and interval δ t

Let $\left(X^{(1)},\cdots,X^{(N)}\right)=\left(\left(X_t^{(1)}\right)_{t\geq 0},\cdots,\left(X_t^{(N)}\right)_{t\geq 0}\right)$ be a continuous time Markov chain on the state space $S^{(N)}$, with infinitesimal generator $\left(\Lambda^{(1)},\cdots,\Lambda^{(N)}\right)$:

$$S^{(N)}=\begin{Bmatrix}S_1\\ \cdots\\ S_N\end{Bmatrix}=\begin{Bmatrix}1 & \cdots & m\\ \cdots & \cdots & \cdots\\ 1 & \cdots & m\end{Bmatrix}. \qquad [3.16]$$

We define the threshold that corresponds to each rating in the state space. Let Q be the matrix defined in eqn [3.15]

$$\left(cumQ_{t,t+\delta t_{(RN)}}^{(k)}\right)_{i,j}=\sum_{j=1}^{i}\left(Q_{t,t+\delta t_{(RN)}}^{(k)}\right)_{i,j}$$

$$K_{t,t+\delta t}=cumQ_{t,t+\delta t(RN)}$$

$$=\begin{Bmatrix}\left(cumQ_{t,t+\delta t_{(RN)}}^{(1)}\right)_{i,1} & \cdots & 1\\ \vdots & \ddots & \iddots\ \vdots\\ \left(cumQ_{t,t+\delta t_{(RN)}}^{(k)}\right)_{i,1} & \left(cumQ_{t,t+\delta t_{(RN)}}^{(k)}\right)_{i,j} & 1\\ \vdots & \iddots & \ddots\ \vdots\\ \left(cumQ_{t,t+\delta t(RN)}^{(N)}\right)_{i,1} & \cdots & 1\end{Bmatrix}=\begin{Bmatrix}K_{t,t+\delta t}^{(1)}\\ \vdots\\ K_{t,t+\delta t}^{(k)}\\ \vdots\\ K_{t,t+\delta t}^{(N)}\end{Bmatrix} \qquad [3.17]$$

For any $\eta,\upsilon\in S^{(N)}$, we define $\vec{V}=\left(V^{(1)},\cdots,V^{(N)}\right)$ as a set of uniform random variables where the joint distribution is a copula function C with the associated generating function φ. For simplicity, we define:

$$\left(K_{t,t+\delta t}^{(k)}\right)_{i,0}=0 \qquad [3.18]$$

Hence, the copula function of the joint migration and default for any transition between two states is:

$$\{\eta,\upsilon\}=\begin{Bmatrix}\eta_1 & \upsilon_1\\ \vdots & \vdots\\ \eta_m & \upsilon_m\end{Bmatrix}, where \quad \eta_i,\upsilon_i\in(1,\cdots,m) \qquad [3.19]$$

is given by:

$$\begin{aligned}&\mathrm{P}_{RN}\left(\mathrm{X}_{\mathrm{t+h}}=\upsilon/\mathrm{X}_{\mathrm{t}}=\eta\right)\\&=\mathrm{P}_{RN}\left(\left(K_{t,t+\delta t}^{(1)}\right)_{\eta_1,\upsilon_1-1}\le \mathrm{X}_{\mathrm{t+h}}^{(1)}\prec\left(K_{t,t+\delta t}^{(1)}\right)_{\eta_1,\upsilon_1},\cdots,\left(K_{t,t+\delta t}^{(N)}\right)_{\eta_N,\upsilon_N-1}\le \mathrm{X}_{\mathrm{t+h}}^{(N)}\prec\left(K_{t,t+\delta t}^{(N)}\right)_{\eta_N,\upsilon_N-1}\right)\\&=C\left(\left(K_{t,t+\delta t}^{(1)}\right)_{\eta_1,\upsilon_1-1}\le \mathrm{X}_{\mathrm{t+h}}^{(1)}\prec\left(K_{t,t+\delta t}^{(1)}\right)_{\eta_1,\upsilon_1},\cdots,\left(K_{t,t+\delta t}^{(N)}\right)_{\eta_N,\upsilon_N-1}\le \mathrm{X}_{\mathrm{t+h}}^{(N)}\prec\left(K_{t,t+\delta t}^{(N)}\right)_{\eta_N,\upsilon_N-1}\right)\end{aligned}$$

To illustrate the concept, we assume that the joint distribution between two names is given by a Clayton copula (Figure 3.6 left). The thresholds for a given rating class for the first name are shown as 0.5 and 0.7; similarly for the other variable they are 0.2 and 0.6. The figure on the right shows the points that correspond to a pair of ratings.

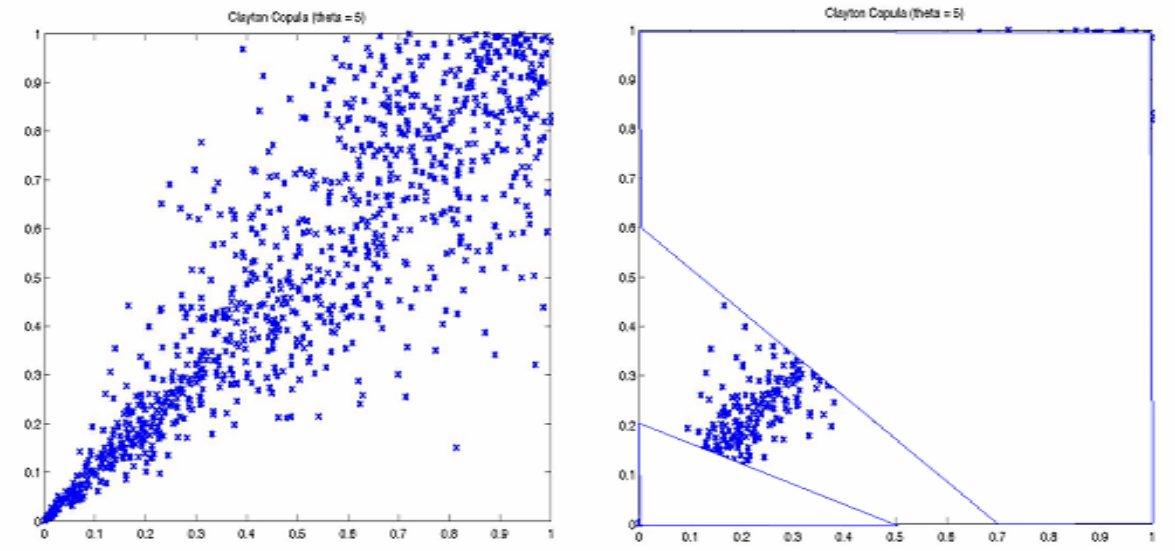


Figure 3.6: Joint transition with a migration dependence driven by Clayton copula with parameter θ = 5 (left) and probability estimation with two variables (right)

The joint transition probability has an Archimedean form copula as expected:

$$\mathrm{P}_{RN}\left(\mathrm{X}_{\mathrm{t+h}}=\upsilon/\mathrm{X}_{\mathrm{t}}=\eta\right)=\begin{vmatrix}\varphi\left(\varphi^{-1}\left(\left(K_{t,t+\delta t}^{(1)}\right)_{\eta_1,\upsilon_1}\right)+\cdots+\varphi^{-1}\left(\left(K_{t,t+\delta t}^{(N)}\right)_{\eta_N,\upsilon_N}\right)\right)\\-\varphi\left(\varphi^{-1}\left(\left(K_{t,t+\delta t}^{(1)}\right)_{\eta_1,\upsilon_1-1}\right)+\cdots+\varphi^{-1}\left(\left(K_{t,t+\delta t}^{(N)}\right)_{\eta_N,\upsilon_N-1}\right)\right)\end{vmatrix} \qquad [3.20]$$

Box 3.2: Modelling dependent credit migration and default with forward dynamic copula

Algorithm

Extraction of the survival probability curve associated with a company:

$$S(t) = \exp\left(-\int_0^t \lambda(u)du \right)$$

Construction of the forward default probability curve given a time step δt with the standard algorithm:

$$\text{FwdDP}(t, t+\delta t) = 1 - \frac{S(t+\delta t)}{S(t)} = 1 - \exp\left(-\int_t^{t+\delta t} \lambda(u)du \right)$$

For each time period, adjust the risk neutral transition matrix.

$$\Lambda_{(RN)im}^{(k)} = \mu_i(t)\Lambda_{im}^{(k)} \Rightarrow \mu_i(t) = \frac{\lim_{\delta t \to 0}\left(\frac{1}{\delta t}\text{FwdDP}(t, t+\delta t) \right)}{\Lambda_{im}^{(k)}}$$

For j =1 to maximum number of simulations

For i=1 to n (find migration or default from today to maturity)

For each company

Compute the risk neutral migration matrix for the given time step

Compute the risk neutral multivariate migration threshold $K_{t,t+\delta t}$

compute $V_{t,t+\delta t}$, the latent marginal copula

Estimate the new rating of the company including default

Handle if need be the action of the new rating on the transaction

If the new rating is the default state, draw the recovery rate

Next j simulation

Aggregate results.

Chapter 4: Multi-factor & time changed approach

In this chapter, a multifactor approach is developed within the new formulated dynamic copula processes, and a time changed levy process is used to introduce dependency on spreads dynamics. The spread dynamics are very important to transactions such as CPDO (Mark-to-Market risk on Index roll), Leverage Super Senior (LSS) and also cash CDO transaction when one to model more accurately reinvestment risk. The first section focuses on multi-factor approach while the second one focuses on the time changed.

4.1 Multi-factor approach for dependent defaults

We now extend the dynamic copula process to handle cases where several market factors explain the returns on an asset over time. These factors can be thought of as capturing the effects of potentially unobservable, economic forces that affect certain groups of assets (Industry, Region, Country …). In most areas of applied finance, factor models have established themselves as the most predictive approach to estimating correlations. The advantage is that they reduce the dimensionality of the problem to be solved.

For example, Moody's KMV uses a factor model to measure correlations between the asset returns of firms because this produces better predictive estimates than simple historical measures of correlations. Historical correlations are subject to a large amount of sampling or random error. The predictive power of the factor model results largely from its control over these errors. As is shown in Figure 3.1, there are three levels of factors in the structure: (1) a composite company specific factor, (2) country and industry factors, (3) global, regional and industrial sector factors.

In a similar way, the rating agency, Fitch, highlighted the fact that the overwhelming majority of global structured finance defaults over the 1991–2005 period (more than 97% of the total) were from the U.S., and what is more, they were concentrated within the ABS sector. The remaining 27 international defaults were primarily from the CDO sector. This confirms how important it is to take account of the impact of geography and industrial sector on credit portfolios. See Figure 4.2.

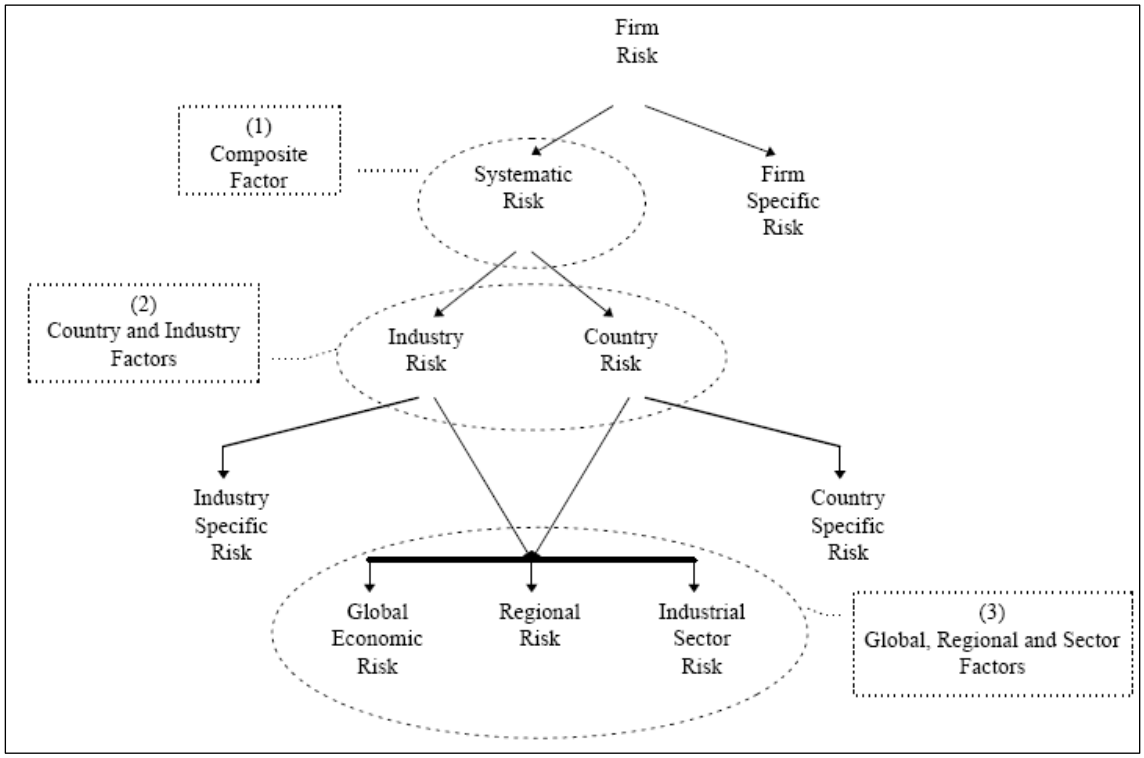


Figure 4.1: Factors used within the Moody's KMV framework (Source KMV)

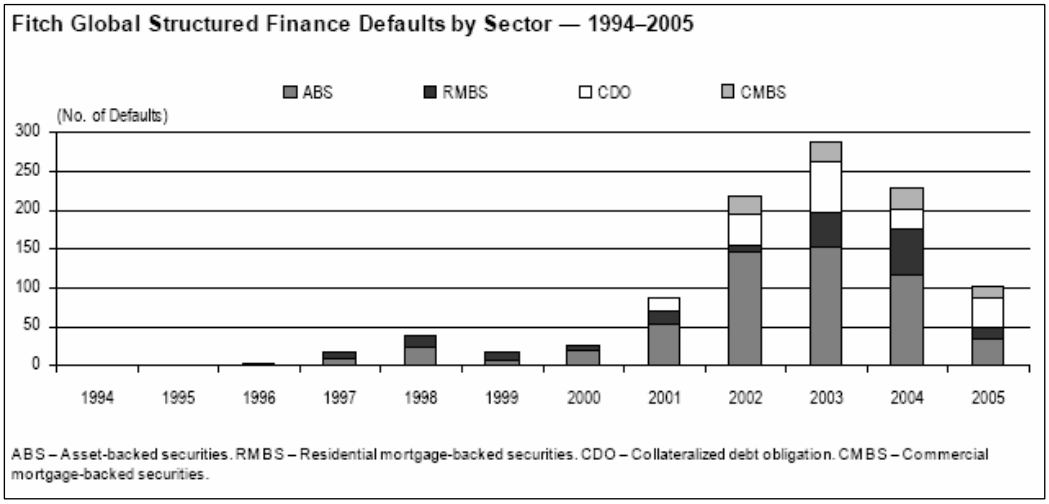


Figure 4.2: Default drivers depends on sectors (Source Fitch Ratings)

Factor analysis can be based either on known economic variables or on statistically determined factors which are implicit in the data set. Within the dynamic copula framework, the parameters or the distribution of the factors are implicit, since they are not directly observable.

Factors in a Gaussian framework

KMV uses a multivariate Gaussian copula for portfolio loss modeling. This means that a unit normal variable is assigned to each obligor. When its value, X_i, hits a critical threshold, a default event ($D_i = 1$ if default occurs, 0 if not) is generated.

$$D_i = 1[X_i < \overline{N}^{-1}(p_i)]$$

The underlying factor model is a multivariate linear regression:

$$X_i = \alpha_{i0}\varepsilon_i + \alpha_{i1}Y_1 + \cdots + \alpha_{im}Y_m \qquad [4.01]$$

Here the Y are systemic factors while the ε are obligor-specific. As usual these are mutually independent unit normal variables. To ensure that the variance of X_i is 1,

$$\sum_{k=0}^{m} \alpha_{ik}{}^2 = 1$$

The variable X_i is interpreted as the asset return of the i^{th} obligor (after a logarithmic transformation and rescaling); the weights, α_{ij}, can be inferred (estimated) from stock market correlations.

In credit derivatives and hedge funds, systemic risk is commonly used to describe the possibility of a series of correlated defaults among financial institutions - typically bank runs that occur over a short period of time - are often provoked by a single major event.

Multi-factor dynamic spot copula

As has been said, factor models are mathematical constructions which attempt to explain the correlation between a large set of variables in terms of a small number of underlying factors. A major assumption of factor analysis is that it is not possible to observe these factors directly; the variables depend upon the factors but are also subject to random errors.

In the dynamic copula framework, we also assume that the factors are not observable. Let $Y^k(t), k \in [1..m]$ be a set of positive-valued, independent Levy processes or compounded Levy processes with a positive spectrum. Secondly we assume that the Laplace transform of each $Y^k(t), k \in [1..m]$ exists and is defined by:

$$\varphi^k(s) = E\left[\exp\left(-sY^k(t)\right)\right] \qquad [4.02]$$

Suppose that the variable of interest, Y(t), can be written in terms of these factors:

$$\mathrm{Y}(\mathrm{t}) = \sum_{k}^{m} \eta^{k}\, \mathrm{Y}^{\mathrm{k}}(\mathrm{t})$$

It is easy to show that its Laplace transformation is

$$\tilde{\varphi}^{\eta}(\mathrm{s}) = \mathrm{E}\left[\exp\left(-\mathrm{sY}(\mathrm{t})\right)\right] = \prod_{k=1}^{m} \varphi^{\mathrm{k}}\left(\mathrm{s}\eta^{k}\right) \qquad [4.03]$$

Definition of the multi-factor spot copula

From now on, we want to analyze the dependence structure within N variables (names) which depend in turn on the m factors. The i^{th} name, $Y_i(t)$, can be expressed as

$$\mathrm{Y}_{i}(\mathrm{t}) = \sum_{k} \eta_{i}^{k}\, \mathrm{Y}^{\mathrm{k}}(\mathrm{t}), \qquad \mathrm{k} \in [1..\mathrm{m}], i \in [1..N]$$

If $U_i(t)$ are uniform random variables or processes, then the variable

$$\mathrm{V}_{i}(\mathrm{t}) = \tilde{\varphi}^{\eta_{\mathrm{i}}}\left(-\frac{\mathrm{Ln}\left(\mathrm{U}^{\mathrm{i}}(\mathrm{t})\right)}{\mathrm{Y}_{i}(\mathrm{t})}\right), \mathrm{i} \in [1..\mathrm{N}] \qquad [4.04]$$

is the conditional marginal of the multi-factor Archimedean copula.

Proof:

From equation [2.50] it is clear that

$$\begin{aligned}
&\Pr\left(\mathrm{V}_{1}(\mathrm{t}) < \mathrm{x}_{1}, ..., \mathrm{V}_{\mathrm{N}}(\mathrm{t}) < \mathrm{x}_{\mathrm{N}}\right) \\
&= \Pr\left(\mathrm{U}_{1}(\mathrm{t}) < \exp\left\{-\mathrm{Y}_{1}(\mathrm{t})\left(\tilde{\varphi}^{\eta_{\mathrm{i}}}\right)^{-1}(\mathrm{x}_{1})\right\}, ..., \mathrm{U}_{\mathrm{N}}(\mathrm{t}) < \exp\left\{-\mathrm{Y}_{\mathrm{N}}(\mathrm{t})\left(\tilde{\varphi}^{\eta_{\mathrm{N}}}\right)^{-1}(\mathrm{x}_{\mathrm{N}})\right\}\right)
\end{aligned}$$

Conditional on the factors $Y_i(t)$, the $U_i(t)$ are mutually independent and hence

$$\begin{aligned}
\Pr\left(\mathrm{V}_{1}(\mathrm{t}) < \mathrm{x}_{1}, ..., \mathrm{V}_{\mathrm{N}}(\mathrm{t}) < \mathrm{x}_{\mathrm{N}}\right) &= \mathrm{E}\left[\prod_{\mathrm{i}} \exp\left\{-\mathrm{Y}_{i}(\mathrm{t})\left(\tilde{\varphi}^{\eta_{\mathrm{i}}}\right)^{-1}(\mathrm{x}_{\mathrm{i}})\right\}\right] \\
&= \mathrm{E}\left[\prod_{\mathrm{i}} \exp\left\{-\left(\tilde{\varphi}^{\eta_{\mathrm{i}}}\right)^{-1}(\mathrm{x}_{\mathrm{i}}) \sum_{\mathrm{k}} \eta_{\mathrm{i}}^{\mathrm{k}} \mathrm{Y}^{\mathrm{k}}(\mathrm{t})\right\}\right] \qquad [4.05] \\
&= \mathrm{E}\left[\exp\left\{-\sum_{\mathrm{k}} \mathrm{Y}^{\mathrm{k}}(\mathrm{t}) \sum_{\mathrm{i}} \left(\tilde{\varphi}^{\eta_{\mathrm{i}}}\right)^{-1}(\mathrm{x}_{\mathrm{i}}) \times \eta_{\mathrm{i}}^{\mathrm{k}}\right\}\right] \\
&= \prod_{\mathrm{k}} \varphi^{\mathrm{k}}\left(\sum_{\mathrm{i}} \left(\tilde{\varphi}^{\eta_{\mathrm{i}}}\right)^{-1}(\mathrm{x}_{\mathrm{i}}) \times \eta_{\mathrm{i}}^{\mathrm{k}}\right), \mathrm{k} \in [1..\mathrm{m}],, \mathrm{i} \in [1..\mathrm{N}]
\end{aligned}$$

Now we are going to consider two cases: firstly a compound gamma process, then a CIR process.

(1) Based on a compound gamma process

Let $U_i(t)$ be a uniform random process that is independent of $Y^k(t)$ $k\in [1, \ldots m]$.In its simplest form it could be a series of independent uniform random numbers taking values between 0 and 1. But other possibilities exist. Each $Y^k(t)$ has a compound gamma distribution

$$Y^k(t) = \beta(t)\Gamma\left(\alpha^k(t),1\right) \qquad [4.06]$$

where $\alpha^k(t)$ is a gamma process with independent increments.

$$\alpha^k(t+\delta t)-\alpha^k(t) \equiv \Gamma\left(a_1^k\,\delta t, a_2^k\right) \qquad [4.07]$$

Its Laplace transform is

$$\begin{aligned}\varphi^k(s) &= E\left[\exp\left(-sY^k(t)\right)\right] = \left(1+a_2^k \mathrm{Ln}\left(1+s\beta(t)\right)\right)^{-a_1^k t} \\ &= \exp\left[-a_1^k t\,\mathrm{Ln}\left(1+a_2^k \mathrm{Ln}\left(1+s\beta(t)\right)\right)\right]\end{aligned} \qquad [4.08]$$

Example 1:

Consider the case where 6 processes depend on two underlying factors as given below:

$$\begin{bmatrix}Y_1(t)\\Y_2(t)\\Y_3(t)\\Y_4(t)\\Y_5(t)\\Y_6(t)\end{bmatrix} = \begin{bmatrix}\eta_1^1 & \eta_1^2\\ \eta_2^1 & \eta_2^2\\ \eta_3^1 & \eta_3^2\\ \eta_4^1 & \eta_4^2\\ \eta_5^1 & \eta_5^2\\ \eta_6^1 & \eta_6^2\end{bmatrix} \times \begin{bmatrix}Y^1(t)\\Y^2(t)\end{bmatrix}, \qquad \begin{bmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\end{bmatrix} = \begin{bmatrix}\eta_1^1 & \eta_1^2\\ \eta_2^1 & \eta_2^2\\ \eta_3^1 & \eta_3^2\\ \eta_4^1 & \eta_4^2\\ \eta_5^1 & \eta_5^2\\ \eta_6^1 & \eta_6^2\end{bmatrix} = \begin{bmatrix}0.00 & 1.00\\ 0.00 & 1.00\\ 0.50 & 0.50\\ 0.50 & 0.50\\ 1.00 & 0.00\\ 1.00 & 0.00\end{bmatrix} \qquad [4.09]$$

Both factors are composed of independent identically distributed increments having a gamma distribution:

$$\alpha^k(t+\delta t)-\alpha^k(t) = \Gamma\left(a_1^k t, a_2^k\right),$$

So

$$V_i(t) = \tilde{\varphi}^{\eta_i}\left(-\frac{\mathrm{Ln}\left(U^i(t)\right)}{Y_i(t)}\right), \qquad i\in[1..6] \qquad [4.10]$$

Figure 4.3 shows simulations of the $V_i(t)$ for the case where $a_1 = a_2 = 0.15$. Note how different the dependence is between the different copulas. Considering factors introduces more flexibility into the model.

(2) Based on a CIR Process

In this case each of the m factors $Y^k(t)$ has a non-central chi-squared distribution

$$dY^k(t) = a^k\left(\theta^k - Y^k(t)\right)dt + \sigma^k\sqrt{Y^k(t)}\,dW_t \qquad [4.11]$$

The relation $a^k\theta^k \succ \frac{1}{2}\left(\sigma^k\right)^2$ has to be satisfied, to ensure that the factors are positive

$$\forall t, \quad Y^k(0) \succ 0 \Rightarrow Y^k(t)$$

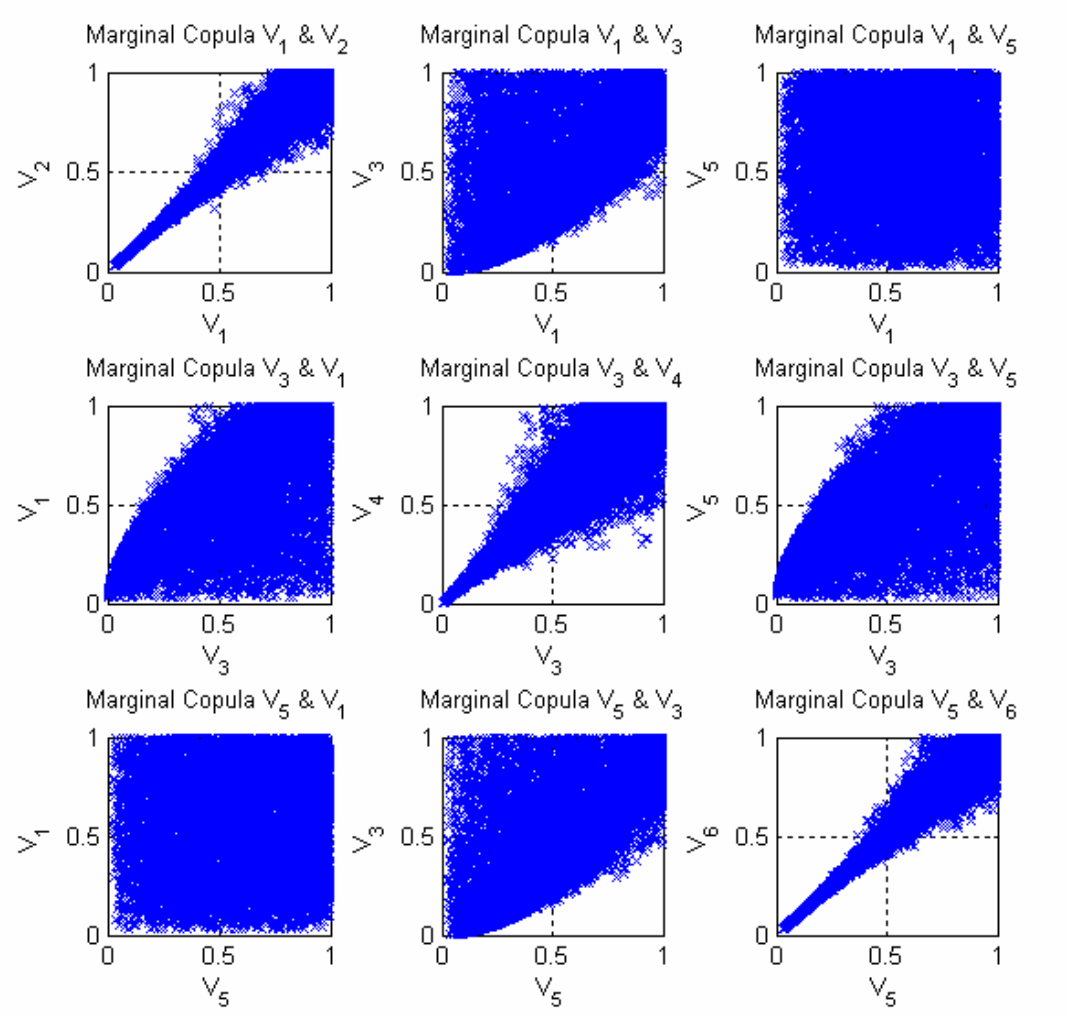


Figure 4.3: The various copula related to each factor can have quite different types of dependence between them

The parameters of the non-central chi-squared distribution are

Variance: $$\eta^k = \frac{(\sigma^k)^2}{4\,\mathrm{a}^{\mathrm{k}}}\left(1-\exp\left(-\mathrm{a}^{\mathrm{k}}\times t\right)\right), \qquad [4.12]$$

Non-centrality parameter $$\tilde{\lambda}^k = \frac{4\,\mathrm{a}^{\mathrm{k}}\exp\left(-\mathrm{a}^{\mathrm{k}}\times t\right)}{(\sigma^k)^2\left(1-\exp\left(-\mathrm{a}^{\mathrm{k}}\times t\right)\right)}, \qquad [4.13]$$

Degrees of freedom: $$\upsilon^k = \frac{4\,\mathrm{a}^{\mathrm{k}}\theta^{\mathrm{k}}}{(\sigma^k)^2} \qquad [4.14]$$

Figure 4.4 illustrates the construction of the CIR process; Figure 4.5 shows the resulting copula.

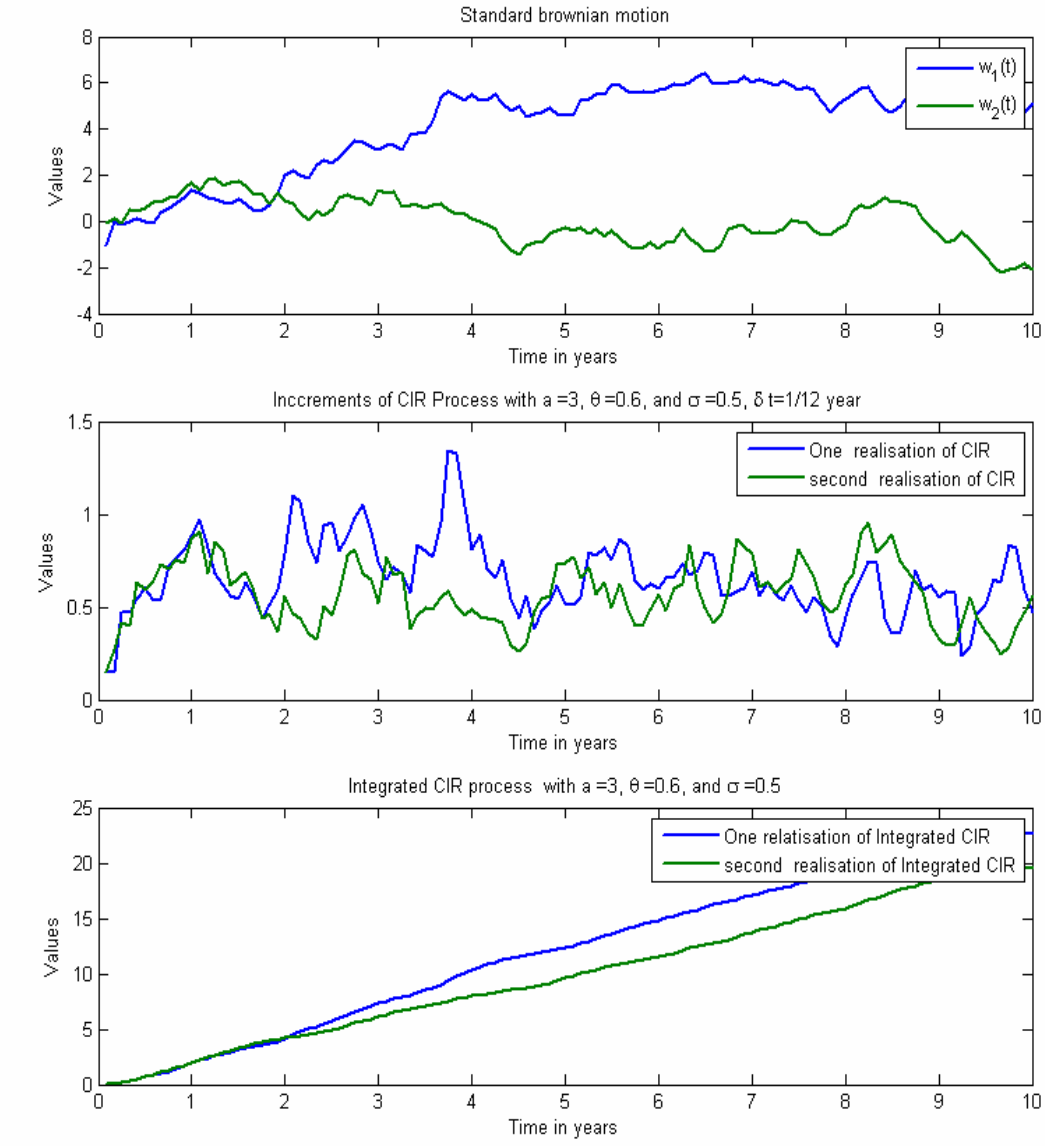


Figure 4.4: Construction of a CIR process by simulation. The integrated CIR process is an increasing function of time. We could also observe that the shape of the CIR process that is a mean reverting process with positive spectrum is very similar to the one of the mean reverting gamma O-U process.

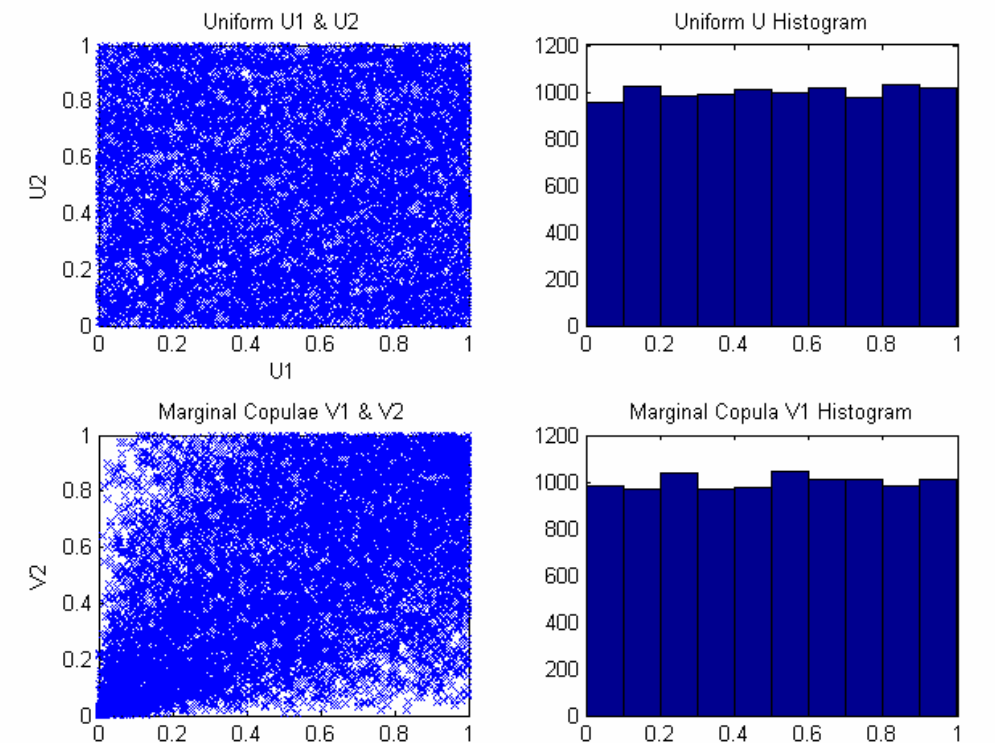


Figure 4.5: Example of CIR copula

4.2 Time changed Levy process for dependent spreads dynamics

Some products such as CPDO need a multivariate dynamic spread model. This work is based on the results obtained by Cariboni and Schoutens (2006). They presented an intensity-based credit risk model where the default intensity of the point process was modelled by an Ornstein-Uhlenbeck type process completely driven by jumps. Two particular cases were considered: firstly, one where the stationary distribution is a gamma (hence the name gamma-OU) and secondly an inverse Gaussian one (denoted by IG-OU). Following Barndorff-Nielsen & Sheppard they also considered an integrated version of the process. They computed the default probability over time by linking it to the characteristic function of the integrated intensity process. In case of the gamma-OU and IG-OU processes this leads to closed form expressions for the default probability and to a straightforward estimate of credit default swap prices.

In this thesis, we will only focus on the result obtained with gamma O-U process, knowing that it could be extended to a large class of Levy process with positive spectrum.

In the intensity models framework, the marginal survival functions are given by specifying the default intensity so that it matches the market. As the gamma-OU process takes positive values it is particularly interesting for modelling stochastic spreads and interest rates.

The default probability here is defined as:

$$S(t) = E\left[\exp\left(-\int_0^t \lambda(s)\,ds\right)\right] = E\left[\exp\left(-\tilde{\lambda}(\mathrm{t})\right)\right] \qquad [4.15]$$

Here $\lambda(t)$ is the stochastic intensity of the Poisson process that defines the default intensity.

$$\tilde{\lambda}(t) = \int_0^t \lambda(s)\,ds \text{ and}$$

$$d\lambda(t) = -\eta\lambda(t)\,dt + dz(\eta t), \quad with \lim_{t\to+\infty} \lambda(t) \equiv \Gamma(a_1, a_2) \qquad [4.16]$$

Substituting the result obtained in the equation [2.59] in chapter 2 on the integrated OU process, into eqn [4.15] gives a closed form formula for the survival probability. The most widely used reduced-form approach is based on the work of Jarrow and Turnbull (1995), who characterise a credit event as the first event of a Poisson counting process which occurs at some time t with a probability defined as $prob(\tau \le \mathrm{t} + \delta t / \tau \succ t) = \lambda(t)\,dt$ .ie, the probability of a default occurring within the time interval [t, t+dt] conditional on surviving to time t, is proportional to some time dependent function λ(t), known as the hazard rate, and the length of the time interval dt. Over a finite time period T, it is possible to show that the probability of surviving is given by

$$\mathrm{S(t)} = prob(\tau \succ \mathrm{t}) = E_0\left\{\exp\left(-\int_0^{\mathrm{t}} \lambda(\mathrm{u})\,\mathrm{du}\right)\right\}$$
$$\tau = \inf\left\{\mathrm{t} \succ 0\int_0^{\mathrm{t}} \lambda(\mathrm{u})\,\mathrm{du} \succ \mathrm{U}\right\},\ \mathrm{U}\ uniform \qquad [4.17]$$

$$\mathrm{S(t)} = \exp\left(-\frac{\lambda_0}{\eta}\left(1-\exp(-\eta t)\right) - \frac{\lambda a_2 t}{\eta a_1 + 1} + \frac{\eta a_2}{\eta a_1 + 1} Ln\left(1 + \frac{1}{\lambda a_1}\left(1-\exp(-\eta t)\right)\right)\right) \qquad [4.18]$$

Box 4.1: CDS price computation

Consider a CDS with maturity T and a continuous spread c. Let P(t) be the risk-neutral probability of no-default between 0 and t. The price of this CDS is then given by.

$$CDS = (1-RR)\left(-\int_0^T \exp(-rs)\,dP(s)\,ds\right) - Spread \times \int_0^T \exp(-rs)\,dP(s)\,ds \quad [4.20]$$

where RR is the asset specific recovery rate and r is the default-free discount rate. From this, we find the par spread, denoted by Spread* that makes the CDS price equal to zero:

$$Spread^* = \frac{(1-RR)\left(-\int_0^T \exp(-rs)\,dP(s)\,ds\right)}{\int_0^T \exp(-rs)\,dP(s)\,ds} = \frac{(1-RR)\left(1-\exp(-rT)P(T)-r\int_0^T \exp(-rs)P(s)\,ds\right)}{\int_0^T \exp(-rs)\,dP(s)\,ds} \quad [4.21]$$

Calibration on CDS term structure curve

The calibration of the model to the market data is straight forward and is carried out on a series of CDS term structures taken from the market. See Box 3.3. The Levenberg-Macquart algorithm available in MATLAB Package could be used to minimize the difference between market CDS prices and calibrate the model in the least-squares sense: minimizing the root mean square error (rmse):

$$\text{rmse} = \sqrt{\sum_{\text{Options}} \frac{(\text{Market Price} - \text{Model Price})^2}{\text{Number of Options}}} \quad [4.19]$$

Cariboni and Schoutens (2006) carried out a study in which they calibrated the 125 credit default swaps constituting the Itraxx Europe Index, using both the gamma-OU and the IG-OU models. The capabilities of the OU models were compared with the homogeneous and inhomogeneous Poisson models and with the CIR model, using data from January 2005 to February 2006. They showed that while homogeneous and inhomogeneous Poisson models failed to replicate real market structures, the CIR, gamma-OU and IG-OU models could be

calibrated to market data quite satisfactorily. In addition the calibration time for both OU models was quite short. At the end of their paper Carboni & Schoutens comment that it would be interesting to extend this to a multivariate setting. The next section will be devoted to this. As was noted earlier, this is important in practice for modelling CPDO.

Dynamics copula representation of time changed intensity model

Following in the footsteps of Joshi and Stacey (2005), Luciano and Schoutens (2005) and Cariboni and Schoutens (2006), we introduce dependency by time-changing. Consider N assets described by N independent individual intensity models:

$$\lambda^{(i)} = \left\{ \lambda^{(i)}{}_t, t \geq 0 \right\}, i \in [1,...,N] \qquad [4.22]$$

The default of each asset is defined by the first jump-time of a Cox process

$$M^{(i)} = \left\{ M^{(i)}{}_t, t \geq 0 \right\}, i \in [1,...,N] \qquad [4.23]$$

We assume that the corresponding default intensities are described by the OU model:

$$d\lambda^{(i)}(t) = -\eta^{(i)}\lambda^{(i)}(t)dt + dz\left(\eta^{(i)}t\right) \qquad [4.24]$$

Where $\lim_{t\to+\infty} \lambda(t) \equiv \Gamma\left(a^{(i)}{}_1, a^{(i)}{}_2\right), i \in [1,...,N]$

In the next two sections we propose two different possibilities for the subordinator: firstly a gamma process then a compound gamma process.

Time-changed gamma process

Here, we introduce dependency by time-changing the individual Cox processes by a common subordinator. A tractable choice for this subordinator is the gamma process:

$$M_{\bar{\alpha}}{}^{(i)} = \left\{ M^{(i)}{}_{\bar{\alpha}(t)}, t \geq 0 \right\}, i \in [1,...,N]$$

Where $\bar{\alpha}(t+\delta t) - \bar{\alpha}(t) \equiv \Gamma\left(\bar{a}_1 \delta t, \bar{a}_2\right)$ [4.25]

The time to default $\tau^{(i)}$ of the ith firm is again defined as:

$$\tau^{(i)} = inf\left\{ t \geq 0, M^{(i)}{}_{\bar{\alpha}(t)} \succ 0 \right\} \qquad [4.26]$$

The implied survival probability up to time t of the ith firm is given by

$$
\begin{aligned}
S^{(i)}(t) &= E\left[\exp\left(-\int_0^{\bar{\alpha}(t)} \lambda^{(i)}(s)\,ds\right)\right] \\
&= E\left[\exp\left(\begin{array}{l} -\dfrac{\lambda^{(i)}{}_0}{\eta^{(i)}}\left(1-\exp\left(-\eta^{(i)}\times\bar{\alpha}(t)\right)\right)-\dfrac{\lambda^{(i)}a^{(i)}{}_2\times\bar{\alpha}(t)}{\eta^{(i)}a^{(i)}{}_1+1} \\ +\dfrac{\eta^{(i)}a^{(i)}{}_2}{\eta^{(i)}a^{(i)}{}_1+1}Ln\left(1+\dfrac{1}{\lambda^{(i)}a^{(i)}{}_1}\left(1-\exp\left(-\eta^{(i)}\times\bar{\alpha}(t)\right)\right)\right) \end{array}\right)\right] \\
&= E\left[\varphi^{(i)}{}_{\bar{\alpha}(t)}(1)\right]
\end{aligned}
\qquad [4.27]
$$

That is, it is the expectation of the Laplace transform evaluated at s = 1.

Now we compute the joint survival probability in a conditional independence framework. We denote the joint survival probability by

$$S^{(1,\cdots,N)}(t) = P\left(\tau^{(1)} \succ t,\cdots,\tau^{(N)} \succ t\right)$$

Using the same line of reasoning as above, this is equal to

$$
\begin{aligned}
&S^{(1,\cdots,N)}(t) \\
&= E\left[\prod_{i=1}^{N}\left\{E_{\bar{\alpha}(t)}\left[\exp\left(-\int_0^{\bar{\alpha}(t)}\lambda^{(i)}(s)\,ds\right)\right]\right\}\right] \\
&= E\left[\prod_{i=1}^{N}\varphi^{(i)}{}_{\bar{\alpha}(t)}(1)\right] \\
&= \int_0^{+\infty}\left(\left\{\prod_{i=1}^{N}\varphi^{(i)}{}_x(1)\right\}\frac{x^{\bar{a}_1 t-1}}{(\bar{a}_2)^{a_1 t}\,\Gamma(\bar{a}_2)}\exp\left(-\frac{x}{\bar{a}_2}\right)\right)dx
\end{aligned}
\qquad [4.28]
$$

The join survival probability has a quasi-closed form solution.

Time-changed standard compound gamma process

This time, the standard compound gamma process that was introduced earlier in this thesis is used as the subordinator. Its increments have a compound gamma distribution:

$$X_{t+h} - X_t \equiv \beta \times \Gamma\left(\Gamma(\bar{a}_1 h, \bar{a}_2), 1\right) \equiv \Gamma\left(\bar{\alpha}(t), \beta\right) \qquad [4.29]$$

The implied survival probability up to time t of the i^{th} firm becomes

$$S^{(i)}(t) = E\left[\varphi^{(i)}{}_{\bar{X}(t)}(1)\right] \qquad [4.30]$$

Now we compute the joint survival probability in a conditional independence framework.

$$
\begin{aligned}
&S^{(1,\cdots,N)}(t) = E\left[\prod_{i=1}^{N} \varphi^{(i)}{}_{\bar{X}(t)}(1)\right] \\
&= E\left[E_{\bar{\alpha}(t)}\left[\prod_{i=1}^{N} \varphi^{(i)}{}_{\bar{X}(t)/\bar{\alpha}(t)}(1)\right]\right] \\
&= \int_0^{+\infty}\left(\left\{\int_0^{+\infty}\left(\left\{\prod_{i=1}^{N} \varphi^{(i)}{}_{x}(1)\right\}\frac{x^{u-1}}{(\beta)^{u\times t}\,\Gamma(\beta)}\exp\left(-\frac{x}{\beta}\right)\right)dx\right\}\frac{u^{\bar{a}_1 t-1}}{(\bar{a}_2)^{a_1 t}\,\Gamma(\bar{a}_2)}\exp\left(-\frac{u}{\bar{a}_2}\right)\right)du
\end{aligned}
\qquad [4.31]
$$

The join survival probability has here also a quasi-closed form solution, in fact we only need to compute a two dimensional integration.

Time-changed mean reverting gamma O-U process

This time, the standard compound gamma process that was introduced earlier in this thesis is used as the subordinator. Its increments have a compound gamma distribution:

$$
\begin{aligned}
&X_t = \int_0^t \delta x_u, \quad \delta x_t = -\eta x_t dt + \delta z(\eta t), \\
&with \lim_{t\to+\infty} x_t \equiv \Gamma(\bar{a}_1, \bar{a}_2), \; where \quad \bar{a}_1 \times \bar{a}_2 = 1
\end{aligned}
\qquad [4.32]
$$

The implied survival probability up to time t of the i^{th} firm becomes

$$
S^{(i)}(t) = E\left[\varphi^{(i)}{}_{\bar{X}(t)}(1)\right] \qquad [4.33]
$$

Now we compute the joint survival probability in a conditional independence framework.

$$
\begin{aligned}
&S^{(1,\cdots,N)}(t) \\
&= E\left[\prod_{i=1}^{N}\left\{E_{X(t)}\left[\exp\left(-\int_0^{X(t)} \lambda^{(i)}(s)\,ds\right)\right]\right\}\right] \\
&= E\left[\prod_{i=1}^{N} \varphi^{(i)}{}_{X(t)}(1)\right] \\
&= \int_0^{+\infty}\left(\left\{\prod_{i=1}^{N} \varphi^{(i)}{}_{x}(1)\right\}\times f(x)\right)dx
\end{aligned}
\qquad [4.34]
$$

where f(x) is the distribution function of the mean reverting gamma O-U process. This function could be evaluated numerically by using the inverse Laplace transform or inverse Fourier transform techniques. The joint survival probability here also has a quasi-closed form solution. In fact we only need to compute a 1-dimensional integral.

We have proposed two possible extensions here on the framework developed by Cariboni and Schoutens (2006).Further improvement could be made by using a multi factor approach where the background driven positive process is a linear combination of a small number of factors, with coefficients specifics to each obligor. This further extension could then allow to taking into account regional and sector systemic risk.

Chapter 5: A new way of modelling CDO tranches

Since its creation in 2004, base correlation (McGinty & Ahulwalia, 2004 a & b) has become popular with market practioners for pricing CDOs because it reproduces prices and spreads. Unfortunately it does not link prices/spreads at different times, which is needed for pricing different maturities and more importantly for forward starting CDOs. Ideally we would like a mathematically consistent model of the dependence structure between default times (as in factor copulas) that reproduces market prices and spreads (as base correlations do).

Over the past five years the factor copulas first proposed by (Li, 2000) have been widely used for pricing CDOs. See Andersen, Sidenius & Basu (2003 & 2005), Gregory & Laurent (2003), Hull & White (2003) and Burtschell Gregory & Laurent (2005a). Their strong points are that the pricing is semi-analytic and that the dependence structure between default times can be specified independent of the marginal credit curves. But as the CDS market became more liquid, it became clear that a flat correlation model did not price CDO tranches correctly. See Burtschell et al (2005b) for an example. Tests by Burtschell, Gregory & Laurent (2005a) showed that the Clayton copula gave better results than other copulas, notably the Gaussian and student's t. Why is this?

Factor copulas based on the normal distribution (or student's t) have symmetric upper and lower tails. They are effectively saying that defaults occur in the same way in bull and bear markets. In tough times, one default tends to trigger others, which is not the case in normal times. The classic "icecream cone" shape of the Clayton copula with its lower tail dependence (Figure 5.1, left) captures this insight; the symmetric gaussian (normal distribution) copula (Figure 5.1, right) does not. The Clayton copula belongs to a special family of copulas called Archimedean copulas. While books have been written about their statistical properties (Nelsen (1999) and Joe (1997)), very little work has been done on stochastic processes based on them. In this paper we present a family of dynamic Archimedean copula processes suitable for pricing synthetic CDO tranches.

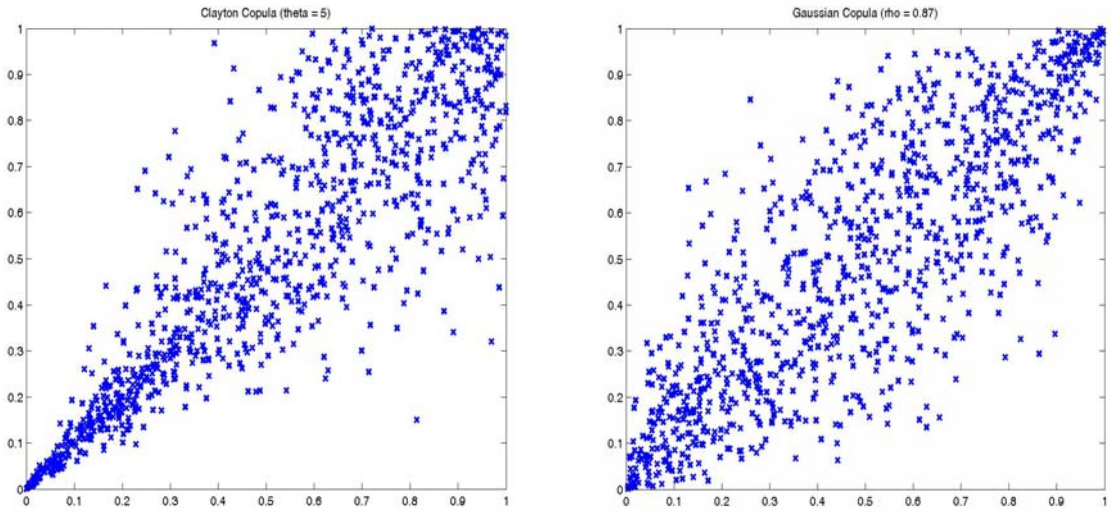


Figure 5.1: Clayton copula with parameter $\theta = 5$ (left) and Gaussian copula with $\rho = 0.87$ (right)

This section of the thesis is organized as follows. In the next section, after giving an overview of Archimedean copulas we introduce the new family of dynamic copula processes. In Section 3, we present a specific copula process related to the Clayton, which is lower tail dependent but not upper tail dependent. In Section 4 this model is used to price standard CDO tranches assuming a bullet exposure at maturity (5 years) and a large but not necessarily homogeneous portfolio. Using market data (Anon, 2005) we show that a correlation skew similar to that observed in the market in July 2005, can be obtained with a suitable set of parameter values. In fact a wide range of correlation skews (both convex & concave) can obtained, depending on the parameter values. The conclusions follow in the last section.

5.1 Dynamic Archimedean copula processes

Copulas express the dependence structure between two or more variables $X_1, \ldots X_n$ separately from their marginal distributions. The original variables $X_1, \ldots X_n$ are replaced their cumulative distribution functions $V_1, \ldots V_n$ which are uniform on [0,1]. In our case they will represent the default probabilities of n names in a credit portfolio. Archimedean copulas are a special type of copula that are defined via a generator, f [1]:

[1] To avoid confusion, note that in this paper, we use φ to denote a Laplace transform, and f for the generator of an Archimedean copula, whereas Nelsen (1999) uses φ for the generator of an Archimedean copula. The function f must be a continuous and strictly decreasing.

$$C(v_1,\dots v_n) = f^{-1}\left[f(v_1)+\dots+f(v_n)\right] \qquad [5.01]$$

While many bivariate Archimedean copulas are known, few multivariate ones exist because their generators have to be Laplace transforms. Table 5.1 lists selected multivariate Archimedean copulas with their Laplace transforms. For example, the Clayton copula corresponds to a gamma distribution.

Table 5.1: The Laplace Transforms corresponding to selected strict bivariate copulas (which can be extend to n-copulas). The names LTE, LTF, LTG, LTH and LTI are derived from the naming system used by Joe (1997).

Copula Name	Generator (Laplace Transform)
Clayton	$\varphi(s)=(1+s)^{-1/\theta} \quad \theta>0$
Gumbel	$\varphi(s)=\exp\left(-s^{1/\theta}\right) \quad \theta>1$
Frank	$\varphi(s)=-\frac{1}{\theta}\mathrm{Ln}\left[1-\exp(-s)\{1-\exp(-\theta)\}\right], \quad \theta\neq 0$
LTE	$\varphi(s)=\left(1+s^{1/\delta}\right)^{-1/\theta}$
LTF	$\varphi(s)=(1+\delta^{-1}Ln(1+s))^{-1/\theta}, \ \delta>0,\theta\geq 1$
LTG	$\varphi(s)=\exp\left\{1-\left[\delta^{-1}Ln(1+s)\right]^{1/\theta}\right\}, \ \delta>0,\theta\geq 1$
LTH	$\varphi(s)=1-\left[1-\exp\left\{-s^{1/\delta}\right\}\right]^{1/\theta}, \ \delta>0,\theta\geq 1$
LTI	$\varphi(s)=1-\left[1-(1+s)^{-1/\delta}\right]^{1/\theta}, \ \delta>0,\theta\geq 1$

Burtschell, Gregory & Laurent (2005a) showed that the Clayton copula was useful for modeling the correlation smile at a fixed point in time. The question was how to develop a dynamic continuous time stochastic process whose values at any given time have a given Archimedean copula (in our case, one with lower tail dependence). Our approach is based on an observation found in Rogge & Schonbucher (2003): let Y be a positive random variable whose Laplace transform is φ(s) and let U_i be n uniform random variables on [0,1] that are mutually independent and also independent of Y. Then the n random variables V_i defined by

$$V_i = \varphi\left(\frac{-\mathrm{Ln}(U_i)}{Y}\right) \qquad \text{for } i=1,\dots n \qquad [5.02]$$

are uniform on [0,1], and their cumulative distribution function is given

$$\text{Prob}\left(V_1 \le v_1, \cdots, V_n \le v_n\right) = \varphi\left(\sum_{i=1}^{n} \varphi^{-1}\left(v_i\right)\right) \qquad [5.03]$$

Consequently their multivariate copula is the Archimedean copula having φ-1 as its generator (See Rogge & Schonbucher for details). This provides a fast and efficient method for simulating realisations, one that is not mentioned by Nelsen (1999)2. At this point we diverge from their approach. We let Y(t) be a stochastic process that represents the state of the economy, and so the Vi become stochastic processes, Vi(t). Provided the Ui(t) are mutually independent and independent of Y(t), then the static copula of the Vi(t) is the Archimedean copula given in [3]. Ui(t) can be interpreted as the prior probability of default, which is then updated by the current state of the economy Y(t). So Vi(t) is the expected posterior probability of default, since the Laplace transform computes the expectation depending on the distribution of Y(t).

The specific properties of $V_i(t)$ and $V_i(t+\delta t)$, and of $V_i(t)$ and $V_j(t+\delta t)$ for $i \neq j$ depend on the way the $U_i(t)$ are constructed. See Totouom & Armstrong (2005) for details.

5.2 Specific dynamic Archimedean copula process

First we construct a new type of compound gamma process Y(t) conditional on an underlying gamma process α(t). As usual α(0) = 0. For t > 0, its increments are independent gammas[3]:

$$\alpha\left(t+\delta t\right) - \alpha\left(t\right) \equiv \Gamma\left(a_1 \times \delta t, a_2\right) \qquad [5.04]$$

The parameters a_1 and a_2 are constant over time. For t > 0, α(t) has the gamma distribution:

$\Gamma(a_1 t, a_2)$.

The values of Y(t) are drawn from the gamma distribution: $\Gamma(\alpha(t), \beta(t))$ where β(t) is a strictly positive, deterministic function of time. There are two obvious choices: β(t) = 1 and β(t) = 1/t. While the first one leads to a Levy process, the second does not. To the best of our knowledge this process has not been studied before. In the next section we compute the Laplace transform of Y(t), and hence its moments.

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[2] Nelsen (1999) gives several general methods for simulating bivariate Archimedean copulas and some ad-hoc methods for specific copulas. These are presented in Exercises N° 4.13, 4.14 & 4.15 p108

[3] To make it simpler to use standard software to do the simulations, we have changed to the standard notation:
if the random variable X has the gamma distribution $\Gamma\left(a,b\right)$, its density is

$$f(x) = \frac{x^{a-1}}{\Gamma(a) b^a} e^{-x/b} \qquad x \ge 0$$

The mean and variance of the two processes given in Table 5.2 will be used later when calibrating the model to market data.

Table 5.2 : Moments of the processes $\alpha(t)$ and Y(t)

	Mean	Variance
$\alpha(t)$	$a_1 a_2 t$	$a_1 a_2^2 t$
Y(t)	$a_1 a_2 t \beta(t)$	$a_1 a_2\, t\beta(t)\left[1+a_2\,\beta(t)\right]$

Laplace Transform of Y(t)

As the process Y(t) is always positive, its Laplace transform is given by:

$$\varphi_t(s) = E\left\{\exp\left[-s \times Y(t)\right]\right\} = E\left[E_{\alpha(t)}\left\{\exp\left[-s \times Y(t)\right]\right\}\right] \qquad [5.05]$$

We first compute the conditional Laplace transform of Y(t) given $\alpha(t)$.

$$\varphi_t(s \mid \alpha(t)) = \left(1+s\beta(t)\right)^{-\alpha(t)} = \exp\left\{-\lambda\alpha(t)\right\} \quad \text{where } \lambda = \mathrm{Ln}\left(1+s\beta(t)\right) \qquad [5.06]$$

Deconditioning over all values of $\alpha(t)$ gives the Laplace transform of Y(t):

$$\begin{aligned}\varphi_t(s) &= \left(1+a_2\mathrm{Ln}\left(1+s\beta(t)\right)\right)^{-a_1 t} \\ &= \exp\left[-a_1 t\ \mathrm{Ln}\left(1+a_2\mathrm{Ln}\left(1+s\beta(t)\right)\right)\right]\end{aligned} \qquad [5.07]$$

For any given time t, the associated static copula is not a standard Clayton copula but it has the same type of lower tail dependence (Figure 1). For want of a better name we call it an extended Clayton copula. The shape can be interpreted as follows. When Y(t) takes low values, the values of the $V_i(t)$ will be low and hence correlated. If one of the names defaults, others are likely to follow suit. Conversely when Y(t) takes high values, the $V_i(t)$ will be poorly correlated. So if one name defaults the others are unlikely to do so. So this dynamic copula process effectively reproduces what one would intuitively expect.

Simulating $V_i(t)$

A simple three-step procedure is used for simulating $V_i(t)$

(a) Simulate the process $\alpha(t)$

- Initialize $\alpha(0)$ to 0
- For any $t > 0$ and $\delta t > 0$, simulate an increment

$$\alpha(t+\delta t)-\alpha(t)\equiv\Gamma(a_1\times\delta t,a_2)$$

- Compute $\alpha(t+\delta t)$

(b) Simulate the compound gamma process Y(t)

- At time t > 0, draw a value of Y(t) with the conditional gamma distribution

$$\Gamma(\alpha(t),\beta(t))$$

- The values at different times, $Y(t_1)$ and $Y(t_2)$, are drawn conditional on the values of the underlying process, $\alpha(t_1)$ and $\alpha(t_2)$, but otherwise independent of each other. This adds random noise around $\alpha(t)$.

(c) Simulate the $U_i(t)$ then deduce the $V_i(t)$

- For each of the N realizations of Y(t) simulate n $U_i(t)$ where n is the number of names in the portfolio.

5.3 Pricing a correlation product: CDO

Pricing synthetic CDOs involves computing aggregate loss distributions over different time horizons. So CDO tranche premiums depend upon the individual credit risk of names in the underlying portfolio and the dependence structure between default times.

Notation & Definitions

i =1, …n:	Single name credits in the base portfolio for CDO pricing
$\tau_1,\tau_2\ldots\tau_n$:	Default times
LGD_i :	Loss given default on the Name *i*
$PD_i(t)$:	Cumulative default probability of on the Name *i* at time t
N_i :	Nominal of the Name *i*

The aggregated loss in the portfolio at time t is given by:

$$Loss(t)=\sum_{i=1}^{n}N_i\,LGD_i\,1_{\{\tau_i<t\}} \qquad [5.08]$$

If K_u and K_d are the upper and lower detachment points, the loss in the tranche $[K_d, K_u]$ at time t is:

$$\text{Loss}_t\left(K_d, K_u\right) = \text{Min}\left[K_u, \text{Loss}(t)\right] - \text{Min}\left[K_d, \text{Loss}(t)\right] \quad [5.09]$$

The Expected Loss (EL) in the base tranche [0, K] at time t is just:

$$EL = E\left\{\text{Min}\left[K, \text{Loss}(t)\right]\right\} \quad [5.10]$$

Having analytic expressions for the expected loss makes it easy to compute the Greeks for the portfolio.

Data source

We used market data (Anon, 2005a & b) as of July 22 2005. (See Table 5.3). Figure 5.2 shows the base correlation as a function of the detachment point. The next step was to compute the cleanspreads and the default probabilities for the 125 names, for different horizons: 1 year, 3 years, 5 years, 7 years or 10 years, from the second spreadsheet. A constant loss given default of 60% was assumed on all names. The cumulative default probability at any time horizon was computed as follows:

$$\text{cleanSpread}_i\left(\text{Horizon}\right) = \frac{\text{Spread}_i\left(\text{Horizon}\right)}{\text{LGD}_i} \quad [5.11]$$

$$\text{PD}_i\left(\text{Horizon}\right) = 1 - \exp\left[-\frac{\text{cleanSpread}_i\left(\text{Horizon}\right)}{10000} \times \text{Horizon}\right]$$

Table 5.3 : Attachment & detachment points for the market data for 22 July 2005 extracted from excel file on Wilmott website, together with the correlation expressed as a percentage.

22 July 2005 Extracted			Dealer Source (a)		
Attach %	Detach %	Correl %	26 July	25 July	18 July
0	3%	12,08	12,3	11,4	11,6
0	7%	33,8	32,9	32,3	33.6
0	10%	44,14	43.2	42.5	44.1
0	15%	58,14	56.2	55.4	57.2
0	30%	79,78	80.2	78.8	80.5
$Index Not	$Index EL	Index (bps)	Index (bps)	Index (bps)	Index (bps)
$125,000	$2,933.69	53	53	53	56

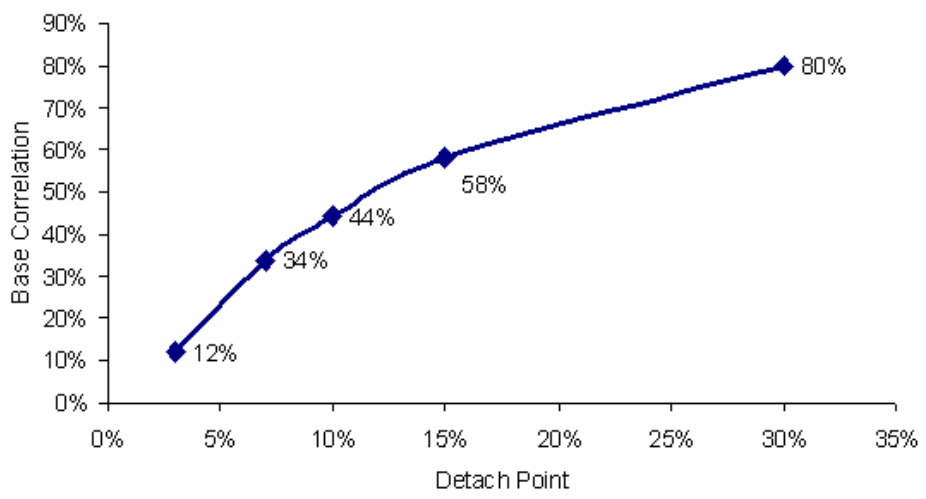


Figure 5.2: Base correlation as a function of the detachment point for the market data as at 22 July 2005

The average 5 year default probability in the portfolio is 4.30%. Table 5.4 gives the summary statistics of default probabilities at a 5 year horizon.

Table 5.4 : Summary statistics of 125 5-year default probabilities

Minimum	0.09%	Median	3.05%
Maximum	36.80%	Mode	2.26%
Mean	4.30%	Skewness	4.30
Standard Deviation	4.78%	Kurtosis	22.13

Calibrating the parameters for Monte Carlo pricing of the CDO

A simple iterative procedure was used to calibrate the parameters of the gamma distribution. The base correlation was computed by running Monte Carlo simulations of the portfolio and comparing this with the market base correlation. Typically 10,000 simulations were carried out. For simplicity all the exposures are bullet. Further work will be needed to improve the calibration procedure.

The riskfree rate shown in Figure 5.4 was used within the model and within the gaussian copula model to obtain the base correlation but as the two cancel out, this choice has no impact on the base correlation. So the result is the same as if we assumed that it was zero as does JP Morgan. See McGinty & Ahulwalia, (2004a & b).

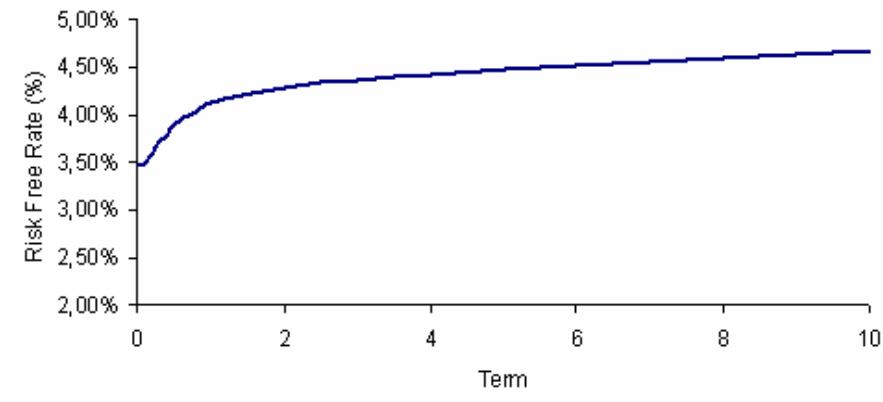


Figure 5.4: Term structure of the riskfree rate as at 22 July 2005

The parameters were calibrated for a maturity of 5 years because this is the most liquid. There is no unique optimum. Three possible sets of values are shown in Table 5.5. The resulting base correlations (blue, black and red) are compared to those for the market (pink), at the standard detachment points. As different parameter values give comparable base correlations for this maturity, other maturities should be used to choose the most appropriate set overall. Table 5.6 shows the term structure for the equity tranche for the standard maturities (5yr, 7yr & 10yr) for the same sets of parameters.

Figure 5.4 presents the base correlation as a function of the detachment point, for the four maturities (3yr, 5yr, 7 yr & 10 yr) for different values of the second parameter a_2. Note how the convexity of the curve changes with the maturity. The model can produce convex curves as well as concave ones.

Table 5.5 : The base correlations computed from the model using 3 sets of parameter estimates for the process $\alpha(t)$, together with the market values. The parameters are shown below

Detach Pt	Market	Set 1	Set 2	Set 3
3%	12%	16.6%	15.5%	17.9%
7%	34%	28.7%	28.3%	30.6%
10%	44%	38.9%	38.8%	41.0%
15%	58%	52.9%	52.8%	55.2%
30%	80%	80.7%	80.7%	82.7%
	a_1	5	5.55	4.55
	a_2	60	90	110

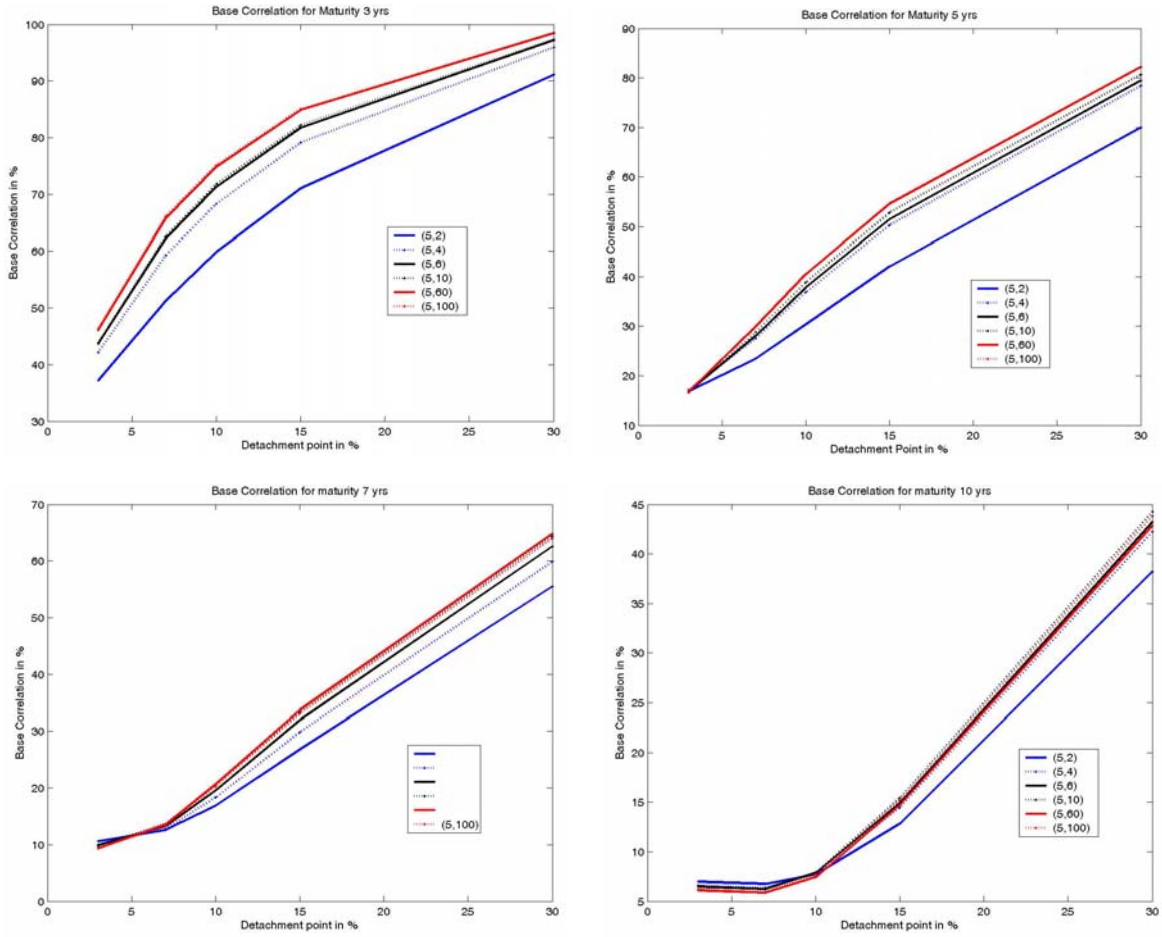


Figure 5.4: The base correlation as a function of the detachment point for the four standard maturities: 3yr top left, 5yr top right, 7yr lower left & 10 yr lower right. Note that the change in the convexity with maturity

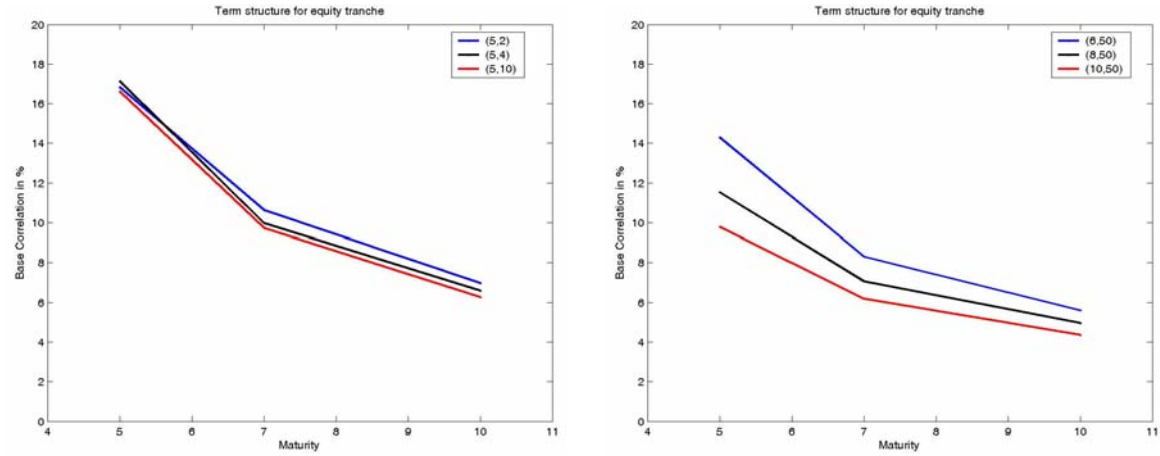


Figure 5.5: The term structure of the equity tranche for different maturities; on the left, for a fixed value of the first parameter a_1, on the right, for a fixed value of the second parameter a_2

Figure 5.6 illustrates the impact of these two ratios on the terms structure. In both cases, increasing one parameter for a fixed value of the other one, leads to a decrease in the base correlation of the equity tranche.

Table 5.6 : The term structure (i.e. the base correlation for the equity tranche for the same parameters)

Maturity Market	Set 1	Set 2	Set 3
3 yr	46.1%	44.1%	49.2%
5 yr	16.6%	15.5%	17.9%
7 yr	9.3%	8.6%	9.9%
10 yr	6.1%	5.9%	6.6%
a_1	5	5.55	4.55
a_2	60	90	110

5.4 Conclusions

In this paper we have chosen to model default times rather than intensities, and have developed a new class of dynamic copula processes, based on the well-known relation between Archimedean copulas and Laplace transforms:

$$V_i = \varphi\left(\frac{-\mathrm{Ln}(U_i)}{Y}\right) \qquad \text{for } i = 1,\ldots n$$

Replacing the random variables Y and U_i, by suitably chosen processes Y(t) and $U_i(t)$, provides a simple way of constructing and simulating a wide range of dynamic copula processes. This effectively overcomes the difficulties of constructing multivariate copulas that have been well documented in the literature on copulas (Nelsen, 1999 & Joe, 1997).

After presenting the procedure for simulating this class of copula processes (Section 5.2), we focus on a particular case: where Y(t) is a new type of compound gamma process, because this gives rise to a dynamic process in which the copulas have lower tail dependence but not upper tail dependence. As we use Y(t) to represent the current economic climate, this means that defaults are correlated in unfavourable times but not during normal times, as one would intuitively expect. The $U_i(t)$ can be interpreted as the prior probability of default which is updated given the state of the economy to obtain the posterior probability of default $V_i(t)$.

In Section 5.4 we use market data to calibrate the model. We show that the model reproduces the base correlations observed at that time. We have also studied the types of term structure given by the model. One advantage of this approach compared to those based on default intensities is that it provides a simple way of computing base correlations without having to specify or calibrate the marginal densities, but its primary strong point is that it provides a mathematically consistent framework for modelling the structure of defaults over different time horizons.

Chapter 6: Comparison with five 1-factor models

Over recent years, major changes have occurred in the credit derivatives industry. The liquidity of single name CDS has increased dramatically, in parallel with the volume of transactions. Quotes are now available for standard tranches for reference baskets, iTraxx in Europe and CDX in North America, for the standard maturities of 3Y, 5Y, 7Y and 10Y. Base correlation (McGinty et al, 2004) has become the industry standard for pricing CDOs. Many new types of derivatives including forward starting CDOs, options on CDOs and leveraged super-senior tranches have been developed. It is now recognized that existing models for pricing credit derivatives are static and as such not suitable for pricing these new products (Hull & White, 2006). A dynamic model of portfolio losses over time is required.

This model should be a properly defined stochastic process and should correctly reproduce the correlation structure observed in the market. Andersen and Sidenius (2004) noted that the correlation between the various names was not the same during bearish and bullish periods. In difficult economic times, defaults tended to occur in cascades, whereas in better times they were more or less independent of each other. With this in mind we developed a dynamic copula process model where the correlation between defaults depends on an underlying factor, which is a proxy for the state of the economy (Totouom and Armstrong, 2007). Conditional on this factor, the default copula has lower tail dependence similar to a Clayton copula. In preliminary tests using market data from July 2005, this model gave acceptable results for the base correlation as a function of the detachment point and for the term structure of the base correlation. The question is: How does it perform compared to other models, notably one factor copula models?

Van der Voort (2006) ran performance tests on five well-known one factor models chosen from those considered by Burtschell, Gregory & Laurent (2005):

- External defaults model (Li, 2000; Laurent and Gregory, 2005; Hull and White, 2004)
- Random factor loadings model (Andersen and Sidenius, 2004)
- Mixture model
- Mixture model with random recovery (Andersen and Sidenius, 2004)
- Double t copula model (Hull and White, 2004)

Using the term structure of the CDS for individual names and the risk-free rate deduced from swap market quotes on 30 January 2006, he estimated the parameters of the models so as to match the 5Y market skews and then used these values to compute the model implied base correlation skews for iTraxx and CDX for a maturity of 10 years. Parameters chosen to match the 5Y market skews resulted in a poor fit for the 10Y skew. All models had problems matching the steepness of the base correlation.

To test our model we used the same procedure and the same data as van der Voort (2006); that is, we fitted the parameters to the 5Y market skews for both iTraxx and CDX, and then used these to predict the 10Y base correlation skew. We demonstrate that our model outperforms the five models listed above, correctly predicting the base correlation skews for iTraxx 10Y and CDX 10Y, as well as reproducing those of iTraxx 5Yand CDX 5Y.

The paper is structured as follows. The new dynamic copula model is presented in the next chapter; the results of the comparisons are given in Chapter 3. The conclusions follow in Chapter 4.

6.1 Dynamic copula process

Our approach is based on an observation found in Rogge & Schonbucher (2003): let Y be a positive random variable whose Laplace transform is φ(s) and let U_i be n uniform random variables on [0,1] that are mutually independent and also independent of Y. Then the n random variables V_i defined by

$$V_i = \varphi\left(\frac{-\mathrm{Ln}(U_i)}{Y}\right) \qquad \text{for } i = 1,\ldots n \qquad [6.01]$$

are uniform on [0,1], and their cumulative distribution function is given

$$\mathrm{Prob}\left(V_1 \le v_1,\cdots,V_n \le v_n\right) = \varphi\left(\sum_{i=1}^{n}\varphi^{-1}(v_i)\right) \qquad [6.02]$$

Consequently their multivariate copula is the Archimedean copula having φ^{-1} as its generator (See Rogge & Schonbucher for details). This provides a fast and efficient method for simulating realisations. At this point we diverge from their approach, by letting Y(t) be a stochastic process that represents the state of the economy. So the V_i become stochastic processes, $V_i(t)$. Provided the $U_i(t)$ are mutually independent and independent of Y(t), then the static copula of the $V_i(t)$ is as in [2].

The process Y(t) will be used as a proxy for the state of the economy at time t; the $U_i(t)$ represent idiosyncratic variations and the resulting $V_i(t)$ indicate each firm's creditworthiness at time t on a uniform scale from 0 to 1. If $V_i(t)$ falls below a critical threshold then the firm is considered to have defaulted. In contrast to Merton's model of the firm (1974) where default occurred when its debt exceeded its equity, the threshold in our model is determined indirectly from the firm's CDS spread at that maturity (i.e. by the market's appreciation of its creditworthiness).

A proxy for the state of the economy, Y(t)

We use a two-step procedure for generating Y(t), based on an underlying gamma process α(t) with independent increments:

$$\alpha(t+\delta t) - \alpha(t) \equiv \Gamma(a_1\delta t, a_2) \qquad [6.03]$$

As usual α(0) = 0. The parameters a_1 and a_2 are constant over time. For t > 0, α(t) has the gamma distribution: $\Gamma(a_1 t, a_2)$. The values of Y(t) are drawn from the gamma distribution:

$\Gamma(\alpha(t), 1/t)$; that is, conditional on the realisation of $\alpha(t)$. This results in a new type of compound gamma process for Y(t). Its properties (including its moments) can be found by computing the Laplace transform of Y(t). See Totouom & Armstrong (2007) for details. Table 6.1 gives the mean and variance of the two processes which are used when calibrating the model to market data.

Table 6.1: Moments of the processes $\alpha(t)$ and Y(t)

	Mean	Variance
$\alpha(t)$	$a_1 a_2 t$	$a_1 a_2^2 t$
Y(t)	$a_1 a_2$	$a_1 a_2 \left(1 + a_2 / t\right)$

Copula with lower tail dependence

Once the Y(t) have been simulated at all times of interest, it is easy to generate the $V_i(t)$. The procedure is described in the box. For any given time t, the associated static copula is lower tail dependent and is similar to a Clayton copula (Figure 6.1). For want of a better name we call it an extended Clayton copula. The shape can be interpreted as follows. When Y(t) takes low values, the values of the $V_i(t)$ will all be low and hence correlated. If one of the names defaults, others are likely to follow suit. Conversely when Y(t) takes high values, the $V_i(t)$ will be poorly correlated. So if one name defaults the others are unlikely to do so. So this dynamic copula process effectively reproduces what one would intuitively expect.

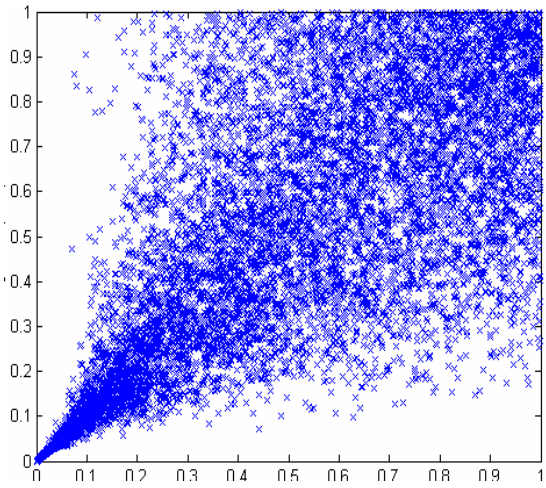


Figure 6.1: Simulations of the extended Clayton copula showing the lower tail dependence

Three-step procedure for simulating $V_i(t)$

1. Simulate the process $\alpha(t)$
   - Initialize $\alpha(0)$ to 0
   - For any $t > 0$ and $\delta t > 0$, simulate an increment
     $$\alpha(t+\delta t) - \alpha(t) \equiv \Gamma(a_1\,\delta t, a_2)$$
   - Compute $\alpha(t+\delta t)$.
2. Simulate the compound gamma process Y(t)
   - At time $t > 0$, draw a value of Y(t) with the conditional gamma distribution
     $$\Gamma(\alpha(t), 1/t)$$
   - The values at different times, $Y(t_1)$ and $Y(t_2)$, are drawn conditional on the values of the underlying process, $\alpha(t_1)$ and $\alpha(t_2)$, but otherwise independent of each other. This adds random noise around $\alpha(t)$.
3. Simulate the $U_i(t)$ then deduce the $V_i(t)$
   - For each of the N realizations of Y(t) simulate n $U_i(t)$ where n is the number of names in the portfolio.

6.2 Pricing a correlation product: CDO

Pricing synthetic CDOs involves computing aggregate loss distributions over different time maturities. So CDO tranche premiums depend upon the credit risk of the individual names in the underlying portfolio and the dependence structure between default times. In the previous section we saw how to simulate the $V_i(t)$; in this section we focus on determining whether the i^{th} name has defaulted at time t.

The cumulative risk neutral default probability $PD_i(t)$ is derived from the term structure of the CDS spread at maturity, assuming 40% gross recovery rate which means a loss given default LDG_i of 60%. If the simulated value of $V_i(t)$ is less than $PD_i(t)$, the name is considered to be in default at time t.

Notation & Definitions

The aggregate loss in the portfolio at time t is just the sum of the nominals of those names which have defaulted up to time t. As usual the loss in the tranche $[K_d, K_u]$ at time t is:

$$\text{Loss}_t\left(K_d, K_u\right) = \text{Min}\left[K_u, \text{Loss}(t)\right] - \text{Min}\left[K_d, \text{Loss}(t)\right] \quad [6.04]$$

and the Expected Loss (EL) in the base tranche [0, K] at time t is just:

$$EL = E\left\{\text{Min}\left[K, \text{Loss}(t)\right]\right\} \quad [6.05]$$

Data Source

To facilitate comparisons with the models already tested by Van der Voort (2006), we used the same data as he did; that is, quotes for the standard tranches on the iTraXX and CDX baskets for 5Y, 7Y and 10Y, for 30 January 2006. Figure 6.2 shows the base correlations as a function of the attachment point for iTraxx (left) and CDX (right) for these three maturities.

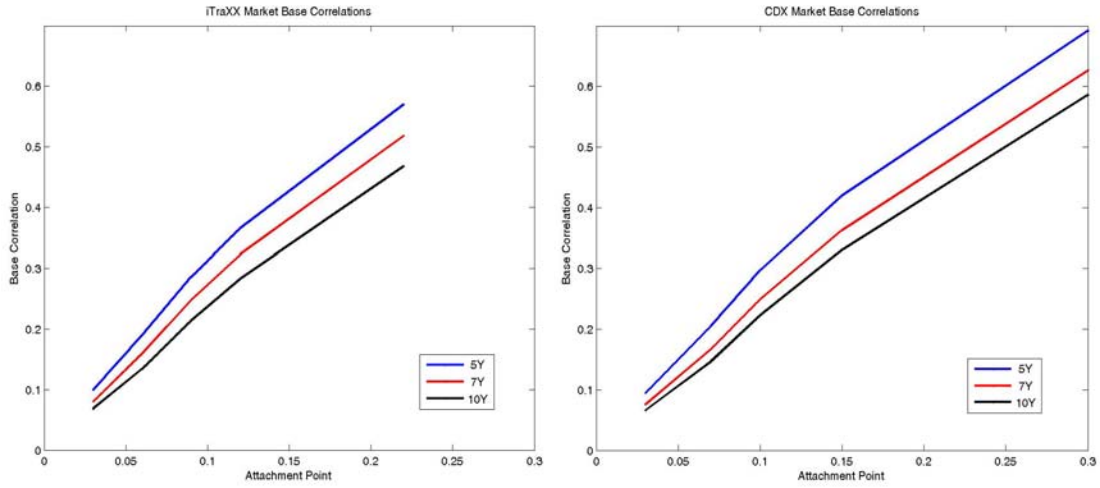


Figure 6.2: Base correlation as a function of the detachment point for the market data as at 30 January 2006, for the standard maturities of 5Y (blue), 7Y (red) and 10Y (black) for iTraxx (left) and for CDX (right)

We computed the cleanspreads and the cumulative default probabilities for the 125 names, for the three maturities: 5 years, 7 years or 10 years, using a constant loss given default of 60%. As the spread is quoted in basis points, the clean spread is divided by 10000 to get the absolute value of the spread.

$$\text{cleanSpread}_i\left(\text{Maturity}\right) = \frac{\text{Spread}_i\left(\text{Maturity}\right)}{\text{LGD}_i} \quad [6.06]$$

$$\text{PD}_i\left(\text{Maturity}\right) = 1 - \exp\left[-\frac{\text{cleanSpread}_i\left(\text{Maturity}\right)}{10,000} \times \text{Maturity}\right] \quad [6.07]$$

Calibrating the parameters for pricing of the CDO

An iterative least squares procedure was used to calibrate the parameters (a_1,a_2) of the gamma distribution, using data with a 5 year maturity because it is the most liquid. The base correlation was computed by running Monte Carlo simulations of the portfolio and comparing this with the market base correlation. Typically 10,000 simulations were carried out. For simplicity all the exposures are bullet. The resulting base correlations are compared to those for the market at the standard detachment points. Table 6.2 gives the parameter values for iTraxx (left) and CDX (right). These were then used to compute the base correlations for a maturity of 10Y.

Table 6.2: Values of parameters fitted to iTraxx and CDX quotes on 30 January 2006

	iTraxx	CDX
a_1	14	13
a_2	44	63

Figure 6.3 presents the market base correlation (solid red line) and the base correlation given by the dynamic copula model (solid black line) as a function of the standard attachment points for a maturity of 10Y, for iTraxx (above) and for CDX (below). In addition the base correlations computed by van der Voort (2006) for five common models are also presented in grey. It is clear that overall the dynamic copula reproduces the market results better than any of those models.

6.3 Conclusions

One of the challenges currently facing the credit risk industry is to develop dynamic models for pricing credit derivatives such as CDOs, and then to test them against market data. In a previous paper (Totouom & Armstrong, 2007) we proposed a new family of dynamic copula processes for modelling the evolution of the credit risk of a large portfolio over time. Preliminary tests using market data indicated that it gave acceptable results for the base correlation as a function of the detachment point and for the term structure of the base correlation. The next step was to test its performance compared to other one-factor copula models.

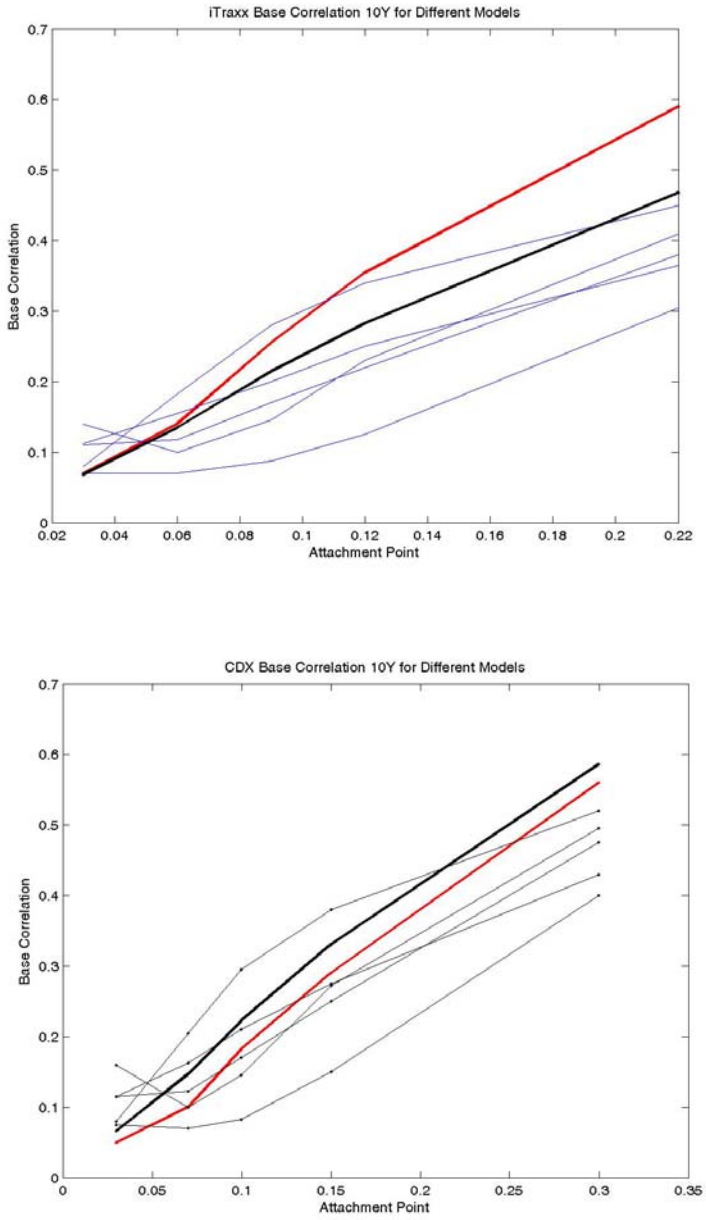


Figure 6.3: Comparing the market base correlation (red) at 10Y iTraxx (above) and 10Y CDX (below) with predictions given different models, those tested by van der Voort (2006) in grey and dynamic copula (thick black line), for 30 January 2006. In all cases parameter values were fitted using 5Y maturity. The dynamic copula tracks the market values more closely than the other five 1-factor copula

Market data as at 30 January 2006 for the most liquid maturity 5Y were used to calibrate the model. We then ran 10,000 simulations of the $V_i(t)$ for a maturity of 10Y using the parameters fitted to the 5Y data, together with the individual CDS spreads for 10Y. Comparing the cumulative probability of default computed from each firm's CDS spread with the simulated $V_i(t)$ tells us which firms have defaulted in each simulation, and hence the loss in any given tranche. The model-implied base correlation can be computed from this. The model reproduced the market base correlations for iTraxx and for CDX quite well. Having used the same market data (30 January 2006) as van der Voort (2006) this demonstrates that it outperforms the five well-known one factor copula models that he considered.

This new class of dynamic copula processes was developed by extending on a well-known relation between Archimedean copulas and the Laplace transforms of random variables to stochastic processes:

$$V_i(t) = \varphi\left(\frac{-\mathrm{Ln}(U_i(t))}{Y(t)}\right) \qquad \text{for } i = 1,\ldots n$$

where $U_i(t)$ is a uniform process on [0,1], Y(t) is a type of gamma process and φ is the Laplace transform of the distribution of Y. The processes Y(t) and $U_i(t)$ must be mutually independent. This provides a simple way of constructing and simulating dynamic copula processes which effectively overcomes the difficulties of constructing multivariate copulas that have been well documented in the literature (Nelsen 1999 & Joe 1997).

Having chosen Y(t) to be a compound gamma process the copula between any two names $V_i(t)$ and $V_j(t)$ at time t, is lower tail dependent but not upper tail dependent (rather like the Clayton copula). In our model Y(t) acts as a proxy for the current economic climate. So the lower tail dependence means that defaults are correlated in unfavourable times but not during normal times, as one would intuitively expect. The $U_i(t)$ represent idiosyncratic variations in each firm's credit status. Combining these two gives each firm's creditworthiness $V_i(t)$ at time t on a uniform scale from 0 to 1. If this value falls below a critical threshold determined from its CDS spread at that maturity, then that name is considered to have defaulted.

To summarize, this new family of dynamic copula processes has the type of lower tail dependence needed to model credit risk, as well as being a properly defined stochastic

process. In addition the two processes Y(t) and U(t) that are used in its construction are economically meaningful. Lastly we have shown that outperforms a range of well-known one-factor copula models.

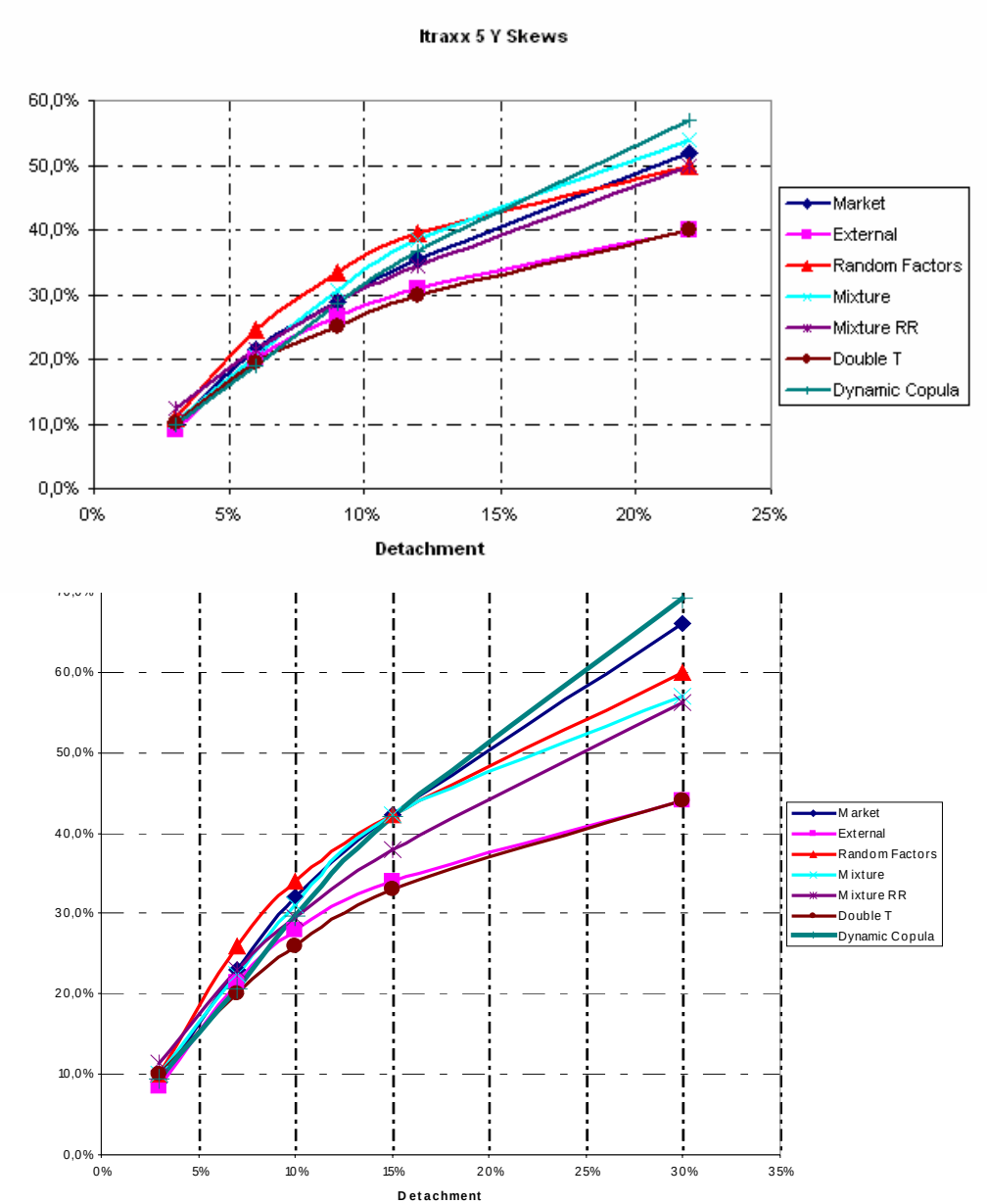


Figure 6.4: Comparing the market base correlation (blue) at 5Y iTraxx (above) and 5Y CDX (below) with calibrated results given by different models, those tested by van der Voort (2006) (external, random factors, mixture, double T) and dynamic copula (thick green line), for 30 January 2006. In all cases parameter values were fitted using 5Y maturity. The dynamic copula tracks the market values more closely than the other five 1-factor copula

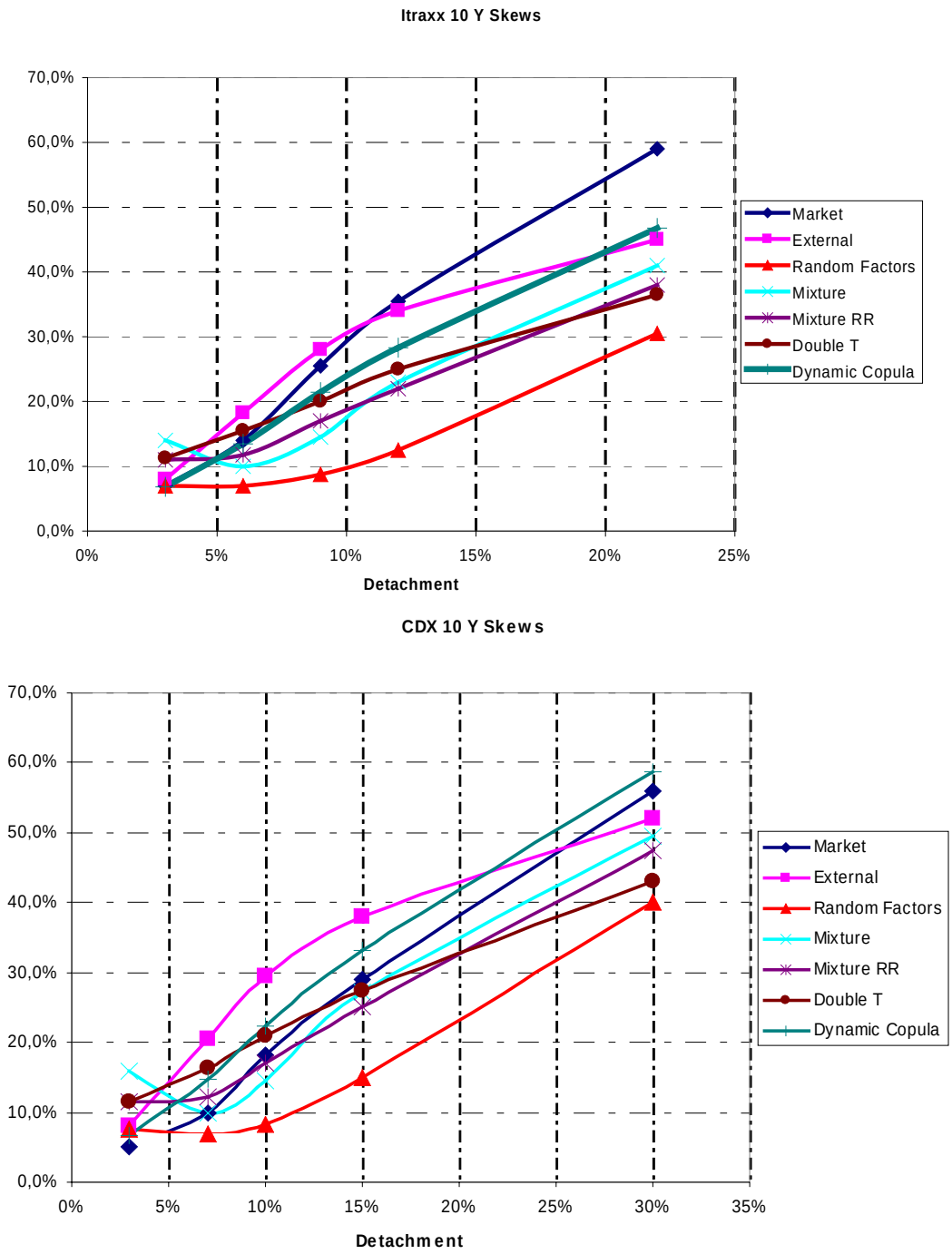


Figure 6.5: Comparing the market base correlation (blue) at 10Y iTraxx (above) and 10Y CDX (below) with parameters obtained by calibration on 5Y Index. predictions given different models are those tested by van der Voort (2006) (external, random factors, mixture, double T) and dynamic copula (thick green line), for 30 January 2006. In all cases parameter values were fitted using 5Y maturity, and the calibrated results are in the figure 6.4. The dynamic copula tracks the market values more closely than the other five 1-factor copula

Chapter 7: Conclusions

In this thesis, we show that with the growth of credit derivatives markets, new products are continually being created and market liquidity is increasing. After reviewing these products starting out from the credit default swap, CDS, and describing their evolution since their inception in the early 90s, we demonstrate that this development has been market driven, with the mathematical models used for pricing lagging behind. As the market developed, the weak points of the models became apparent and improved models had to be developed. In October 2003 when the work on this thesis started, CDOs (Collateralised Debt Obligations) were becoming standard products. A new generation of products which we will refer to as third generation credit derivatives were starting to come on line: these include forward-starting CDS, forward-starting CDOs, options on CDOs, CPDO (in full) and so forth. In contrast to early products, these derivatives require a dynamic model of the evolution of the "correlation" between the names over time, something which base correlation was not designed to do. The aim of this doctorate has been to develop a mathematical consistent framework for pricing these types of products.

After reviewing the literature it became clear that

- A dynamic (rather static) model would be required to handle multiple maturities simultaneously and third generation credit derivatives (especially forward starting products)
- Models based on copulas have the advantage of separating the modelling of the marginal distribution from that of the correlation structure between companies (names) in the CDO basket. So we decided to develop a dynamic copula model
- Work by Burtschell, Gregory & Laurent (2005a) had shown that for a given maturity, a (static) copula with lower tail dependence such as the Clayton copula gave better results than the other copulas (notably the Gaussian copula and Student's t)
- The Clayton copula belongs to a broad family of copulas called Archimedean copulas which encompass a wide range of types of tail dependence.
- Although many models exist for bivariate copulas, very few have multivariate equivalents. Archimedean copulas are amongst the few "strict" copulas with this property.

Consequently our objective was to develop a family of multivariate copula processes with different types of upper and lower tail dependence so as to be able to reproduce the correlation smiles/skews observed in credit derivatives in practice. We chose to work with a dynamic version of Archimedean copulas because unlike many other copulas found in the literature, they are mathematically consistent multivariate models. Chapter 2 presents two different approaches for developing these processes. The first model developed is a non-additive jump process based on a background gamma process; the second approach is based on time changed spectrally positive Levy process. The first approach is very convenient for simulations; the second approach is based on additive building blocks and hence is a more general. Two applications of these models to credit risk derivatives were carried out. The first one on pricing synthetic CDOs at different maturities (Chapter 5) was presented at the 5th Annual Advances in Econometrics Conference in Baton Rouge, Louisiane, November 3-5 2006 and has been submitted for publication. The second one which presents a comparison of the pricing given by these dynamic copulas with five well-known copula models, has been submitted to the Journal of Derivatives (see Chapter 6).

Having tested the basic dynamic copula models in a credit derivative context, we went on to combine this framework with matrix migration approach (Chapter 3). In order to market structured credit derivatives, banks have to get them rated by rating agencies such as S&P, Moody's and Fitch. A key question is the evolution of the rating over time (i.e. its migration). As the latest innovations in the credit derivatives markets such as Constant Proportion Debt Obligation (CPDO) require being able to model credit migration and correlation in order to handle substitutions on the index during the roll, we propose a model for the joint dynamics of credit ratings of several firms.

We then proposed a mathematical framework were individual credit ratings are modelled by a continuous time Markov chain, and their joint dynamics are modelled using a copula process. Copulas allow us to incorporate our knowledge of single name credit migration processes, into a multivariate framework.

This is further extended with the multi-factor and time changed approach. A multifactor approach is developed within the new formulated dynamic copula processes, and a time changed Levy process is used to introduce dependency on spread dynamics. The latter are very important to transactions such as CPDO (Mark-to-Market risk on Index roll), Leverage

Super Senior (LSS) and also cash CDO transaction in order to model reinvestment risk more accurately. We showed that the building block of time changed approach falls within the framework of dynamic copulas.

Perspectives for future work

Credit risk derivatives are one of the most rapidly developing parts of the finance industry. These rapid changes mean that new and interesting questions arise constantly, posing new challenges for mathematical modellers. Below we cite one potential application that it would be interesting to tackle from the point of view of dynamic copulas.

Potential application to Hedge Funds and Collaterized Fund Obligation (CFO) modelling.

A Collateralized Fund Obligation can be regarded as a financial structure with equity investors and lenders where all the assets, equity and bonds, are invested in a portfolio of hedge funds. The lenders earn a spread over interest rates and the equity holders, usually the managers of the CFO, earn the total return of the fund minus the financing fees. The dynamic copula framework might allow us to incorporate two well known characteristics from the return series of hedge funds: first, the skewed and leptokurtic nature of the marginal distribution functions, and second, the asymmetric correlation or correlation breakdown phenomenon (Longin and Solnik, 2001). The correlation between the different hedge funds depends on the direction of the market. For instance, correlations tend to be larger in a bear market than in a bull market.

As hedge funds report returns only on a monthly basis, this leads to a lack of data for estimation purposes. Therefore it becomes necessary to consider multivariate models with as few parameters as possible. One way to achieve this (that is, to capture both the one-dimensional leptokurtic and asymmetric nature of hedge fund returns and the correlation breakdown phenomenon), would be by using dynamic copula for the dependence process and a time changed Levy or self similar process for modelling the returns of a single or pool of hedge funds.

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