Layer Potential Techniques for Sensitivity Analysis.

Habib Zribi

Thèse présentée pour obtenir le grade de

DOCTEUR DE L'ÉCOLE POLYTECHNIQUE

spécialité : Mathématiques Appliquées

par

Habib ZRIBI

La Méthode des Équations Intégrales pour des Analyses de Sensitivité

Soutenue le 12 décembre 2005 devant le jury composé de

MM.	Yves Achdou	Examinateur
	Habib Ammari	Directeur de thèse
	Maury Bertrand	Examinateur
	George Dassios	Rapporteur
	Bruno Després	Examinateur
	Leslie F. Greengard	Rapporteur

Remerciements

Je tiens tout d'abord à exprimer ma profonde gratitude et ma sincère reconnaissnce à M. Habib Ammari pour m'avoir accueilli au CMAP et proposé un sujet de thèse et pour toute l'aide qu'il m'a fournit . Je le remercie beaucoup pour sa gentillesse, ses précieux conseils et sa patience. Sa puissance technique et sa vision claire des mathématiques m'ont incontestablement permis de surmonter les difficultés de ma thèse. Il a développé mon intuition mathématiques et m'a familiarisé aux diverses techniques de problèmes inverses et analyses asymptotiques.

Je tiens à remercier tout particulièrement les professeurs George Dassios et Leslie F. Greengard pour l'attention qu'ils ont portée à mon travail et je les remercie du temps qu'ils ont consacré à sa lecture .

Les professeurs Y. Achdou, M. Bertarnd et B. Després m'ont fait l'honneur d'accepter de faire partie de mon jury. Qu'ils en soient chaleureusement remerciés.

Je tiens également à remercier tous les membres du centre de mathématiques appliquées de l'école polytechnique et son directeur Vincent Giovangigli. J'adresse aussi mes remerciements aux secrétaires du labo Liliane et Jeanne.

Un super grand merci à mes amis et mes collègues pour leur présence et leurs énormes encouragements.

En fin, j'adresse une pensée particulière à toute ma famille qui m'a soutenu et encouragé pendant toutes mes études.

Contents

Présentation Générale

Dans cette thèse, nous menons à l'aide de la méthode des équations intégrales des analyses de sensitivité de solutions ou de spectres de l'équation de conductivité par rapport aux variations géométriques ou de paramètres de l'équation. En particulier, nous considérons le problème de conductivité dans des milieux à forts contrastes, le problème de perturbation du bord d'une inclusion de conductivité, le problème de valeurs propres du Laplacien dans des domaines perturbés et le problème d'ouverture de gap dans le spectre des cristaux photoniques.

Dans la première partie de cette thèse (les trois premiers chapitres), nous obtenons rigoureusement des développements asymptotiques complets des solutions de l'équation de conductivité par rapport à la conductivité de l'inclusion et sa forme. Nos calculs sont basés sur des analyses fines d'opérateurs intégraux singuliers. Les formules démontrées dans cette partie donnent une méthode alternative à celle proposée par Greengard et Lee pour le problème de conductivité à fort contraste, et généralisent celles obtenues par Rundell pour le problème de dérivation de la solution par rapport à la forme de l'inclusion et par Vogelius et ses collaborateurs pour le problème de couche mince de conductivité finie. Notre intérêt pour de telles formules asymptotiques d'ordre élevé est dû au fait qu'elles fournissent des outils très puissants pour la résolution des problèmes d'optimisation.

La seconde partie de cette thèse est consacrée à des problèmes spectraux. Dans le quatrième chapitre, nous étendons les travaux d'Ozawa, de Rauch-Taylor et de Besson sur les comportements asymptotiques des valeurs propres du Laplacien dans un domaine perturbé. Nous proposons une méthode qui permet d'écrire en toute dimension d'espace des développements asymptotiques complets et de traiter le cas des valeurs propres multiples. Nous prouvons que les problèmes spectraux pour le Laplacien dans des domaines perturbés sont équivalents à des systèmes d'équations intégrales posées sur le bord de la perturbation. Ainsi, la recherche des valeurs propres se transforme en la recherche des valeurs caractéristiques d'une fonction méromorphe à valeur d'opérateurs intégraux. En s'appuyant sur les résultats de Gohberg et Sigal, nous calculons explicitement des asymptotiques complètes des valeurs propres. Nous considérons également le cas où l'inclusion est proche du bord ainsi que le cas où elle est de conductivité finie. Lorsque la valeur propre non perturbée est multiple, nous constatons que les valeurs propres perturbées sont les zéros d'un polynôme à coefficients analytiques de degré égal à la multiplicité de la valeur propre non perturbée. Ceci permet ensuite de calculer explicitement leurs asymptotiques complètes.

Le dernier travail effectué est consacré à l'étude des cristaux photoniques. Ces cristaux sont des structures périodiques composées de matériaux diélectriques conçues afin de présenter des propriétés intéressantes, telles que des gaps dans leurs spectres, pour la propagation des ondes électromagnétiques classiques. Le phénomène de bandes interdites de photons peut être réalisé dans des matériaux à forts contrastes structurés périodiquement. Avec un choix adéquat de la structure du cristal photonique, de la dimension de la cellule fondamentale et des matériaux diélectriques composant le cristal, la propagation des ondes électromagnétiques dans certaines bandes de fréquence peuvent être bannies du cristal.

Les résultats des trois premiers chapitres et la technique développée dans le quatrième chapitre nous permettent de fournir une analyse fine de la sensitivité du gap ouvert dans les cristaux photoniques par rapport à la géométrie de l'inclusion et le contraste de conductivité.

Les cinq chapitres de ce document sont autonomes et peuvent être lus indépendemment. Les résultats obtenus dans cette thèse font l'objet de cinq pré-publications [8, 9, 10, 11, 13].

Introduction

This thesis is concerned with the rigorous use of layer potential techniques for sensitivity analysis. Here is an outline of its contents. In Chap. 1 we provide a complete asymptotic expansion of the solution of the conductivity problem in high contrast materials in terms of the conductivity ratio. We prove error estimates for the approximation. Our method can potentially simplify calculations for problems involving highly conducting inclusions.

In Chap. 2 we derive high-order terms in the asymptotic expansions of the boundary perturbations of steady-state voltage potentials resulting from small perturbations of the shape of a conductivity inclusion with $\mathcal{C}^2$-boundary. The asymptotic expansion in this chapter is valid for $\mathcal{C}^1$-perturbations and inclusions with extreme conductivities. It extends those already derived for small volume conductivity inclusions and is expected to lead to very effective algorithms, aimed at determining certain properties of the shape of a conductivity inclusion based on boundary measurements.

The aim of Chap. 3 is to advance the development of asymptotic formulae for steady state voltage potentials associated with thin conductivity inclusions. These formulae recover highly conducting inclusions and those with interfacial resistance.

In Chap. 4 we provide a rigorous derivation of complete asymptotic expansions for eigenvalues of the Laplacian in domains with small inclusions. The inclusions, somewhat apart from or nearly touching the boundary, are of arbitrary conductivity contrast vis-à-vis the background domain, with the limiting perfectly conducting inclusion. By integral equations, we reduce this problem to the study of the characteristic values of integral operators in the complex plane. Powerful techniques from the theory of meromorphic operator-valued functions and careful asymptotic analysis of integral kernels are combined for deriving complete asymptotic expansions for eigenvalues.

In Chap. 5 we investigate the band-gap structure of the frequency spectrum for electromagnetic waves in a high-contrast, two-component periodic medium. We consider two-dimensional photonic crystals consisting of a background medium which is perforated by an array of holes periodic along each of the two orthogonal coordinate axes. We perform a high-order sensitivity analysis with respect to the index ratio and small perturbations in the geometry of the holes. Our method, which is parallel to the one developed in Chap. 4, gives a new tool for the optimal design problem in photonic crystals.

Our general approach in this thesis can be extended to other equations such as the anisotropic conductivity problem, Stokes, the Maxwell and the Lamé systems.

The five chapters of this manuscript are self-contained and can be read independently. Results from this thesis will appear in [8, 9, 10, 11, 13].

Chapter 1

High Contrast Materials

1.1 Introduction

An interesting problem arising in the study of photonic band gap structures concerns the calculation of electrostatic properties of systems made by high contrast materials. See Chap. 5. By high contrast, we mean that the electrical conductivity ratio is high. When the material contrast is high, standard numerical procedures can become ill-conditioned. We refer to Tausch, White, and Wang [89, 90] and Greengard and Lee [45] for effective algorithms for this class of problems. The Tausch-White-Wang approach is based on a perturbation theory while the method of Greengard and Lee is a modification of the classical integral equation.

In this chapter, we derive a complete asymptotic expansion of the solution of the conductivity problem in terms of the conductivity ratio by a boundary integral perturbation method. We provide error estimates for the approximation. Therefore, our method can be viewed as a different computational approach which can potentially simplify calculations for problems involving highly conducting inclusions.

Consider a homogeneous conducting object occupying a bounded domain $\Omega \subset \mathbb{R}^2$, with a connected Lipschitz boundary $\partial\Omega$. We assume, for the sake of simplicity, that its conductivity is equal to 1. Let D with Lipschitz boundary be a conductivity inclusion inside Ω of conductivity equal to some positive constant $k \neq 1$. Let u_k be the solution of

$$\begin{cases} \nabla \cdot (1 + (k-1)\chi_D)\nabla u_k = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u_k}{\partial \nu}\Big|_{\partial\Omega} = g \in L^2_0(\partial\Omega), \\ \displaystyle\int_{\partial\Omega} u_k = 0, \end{cases} \tag{1.1.1}$$

where χ_D is the indicator function of D. We allow k to be 0 or $+\infty$. If $k = 0$, the inclusion D is insulated, and the equation in (1.1.1) is replaced with

$$\begin{cases} \Delta u_0 = 0 \quad \text{in } \Omega \setminus \overline{D}, \\ \dfrac{\partial u_0}{\partial \nu}|_{\partial D} = 0, \\ \dfrac{\partial u_0}{\partial \nu}|_{\partial \Omega} = g, \\ \displaystyle\int_{\partial \Omega} u_0 = 0, \end{cases}$$

and if $k = \infty$, then D is a perfect conductor and the equation in (1.1.1) is replaced with

$$\begin{cases} \Delta u_\infty = 0 \quad \text{in } \Omega \setminus \overline{D}, \\ \nabla u_\infty = 0 \quad \text{in } D, \\ \dfrac{\partial u_\infty}{\partial \nu}|_{\partial \Omega} = g, \\ \displaystyle\int_{\partial \Omega} u_\infty = 0. \end{cases} \tag{1.1.2}$$

It was proved in [41, 54] that u_k converges in $W^{1,2}(\Omega \setminus \overline{D})$ to u_0 or u_∞ as $k \to 0$ or $k \to \infty$. Here the space $W^{1,2}(\Omega \setminus \overline{D})$ is the set of functions $f \in L^2(\Omega \setminus \overline{D})$ such that $\nabla f \in L^2(\Omega \setminus \overline{D})$. The main result of this chapter is a rigorous derivation, based on layer potential techniques, of a complete asymptotic expansion of $u_k|_{\partial \Omega}$ as $k \to +\infty$ or 0. In fact we will derive an asymptotic formula of $u_k|_{\partial \Omega}$ when $k \to k_0$.

This chapter is organized as follows. In the next section we give an explicit asymptotic formula of u_k as $k \to \infty$ or 0 when Ω is a disk and D is a concentric disk. In section 3, we derive a complete asymptotic formula for $u_k - u_{k_0}$ on $\partial \Omega$ when $k \to k_0$. The formula is valid even when $k_0 = 0$ or ∞.

1.2 Explicit formula

In this section, Ω is assumed to be the unit disk centered at the origin, and D to be the concentric disk centered at the origin with radius α. Set

$$g(1, \theta) = \sum_{n \in \mathbb{Z} \setminus \{0\}} g_n e^{in\theta}.$$

Write

$$u_k = \begin{cases} a_0 + b_0 \ln(r) + \displaystyle\sum_{n \in \mathbb{Z} \setminus \{0\}} (a_n r^{|n|} + b_n r^{-|n|}) e^{in\theta} \quad \text{in } \Omega \setminus \overline{D}, \\ \displaystyle\sum_{n \in \mathbb{Z}} \frac{c_n}{\alpha^{|n|}} r^{|n|} e^{in\theta} \quad \text{in } D, \end{cases}$$

where the Fourier coefficients a_n, b_n and c_n are to be found.

Since $g \in L^2_0(\partial\Omega)$ and $\int_{\partial\Omega} u_k = 0$, we have that $a_0 = b_0 = 0$. Using the continuity of u_k across the interface ∂D, we get $c_0 = 0$. Then, for $n \in \mathbb{Z} \setminus \{0\}$, we have

$$\begin{cases} |n|a_n - |n|b_n = g_n, \\ a_n \alpha^{|n|} + b_n \alpha^{-|n|} - c_n = 0, \\ a_n \alpha^{|n|} - b_n \alpha^{-|n|} - k c_n = 0, \end{cases}$$

which yields

$$\begin{aligned} a_n &= \frac{g_n}{|n|} \frac{(k+1)\alpha^{-|n|}}{(k+1)\alpha^{-|n|} + (k-1)\alpha^{|n|}}, \\ b_n &= -\frac{g_n}{|n|} \frac{(k-1)\alpha^{|n|}}{(k+1)\alpha^{-|n|} + (k-1)\alpha^{|n|}}, \\ c_n &= 2\frac{g_n}{|n|} \frac{1}{\alpha^{-|n|}(k+1) + \alpha^{|n|}(k-1)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} u_k(1,\theta) &= \sum_{n \in \mathbb{Z}\setminus\{0\}} (a_n + b_n) e^{in\theta} \\ &= \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{g_n}{|n|} \frac{(k+1)\alpha^{-|n|} - (k-1)\alpha^{|n|}}{(k+1)\alpha^{-|n|} + (k-1)\alpha^{|n|}} e^{in\theta}. \end{aligned}$$

In similar fashion we get

$$u_\infty(1,\theta) = \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{g_n}{|n|} \frac{\alpha^{-|n|} - \alpha^{|n|}}{\alpha^{|n|} + \alpha^{-|n|}} e^{in\theta}.$$

Then the following asymptotic expansion holds as k goes to $+\infty$:

$$u_k(1,\theta) = u_\infty(1,\theta) + \sum_{l=1}^{+\infty} \frac{1}{(k-1)^l} v_\infty^{(l)}(\theta),$$

where

$$v_\infty^{(l)}(\theta) = 2^{l+1}(-1)^{l+1} \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{\alpha^{-(l-1)|n|}}{(\alpha^{|n|} + \alpha^{-|n|})^{l+1}} \frac{g_n}{|n|} e^{in\theta}. \tag{1.2.1}$$

Similarly, we get the following asymptotic formula when $k \to 0$:

$$u_k(1,\theta) = u_0(1,\theta) + \sum_{l=1}^{+\infty} \frac{k^l}{(k-1)^l} v_0^{(l)}(\theta),$$

where

$$v_0^{(l)}(\theta) = 2^{l+1}(-1)^{l+1} \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{\alpha^{-(l-1)|n|}}{(\alpha^{|n|} - \alpha^{-|n|})^{l+1}} \frac{g_n}{|n|} e^{in\theta}. \tag{1.2.2}$$

1.3 The general case

1.3.1 Representation formula

Let $\Gamma(x)$ be the fundamental solution of the Laplacian Δ in $\mathbb{R}^2$: $\Gamma(x) = 1/(2\pi)\ln|x|$. The single and double layer potentials of the density function ϕ on ∂D are defined by

$$\mathcal{S}_D\phi(x) := \int_{\partial D} \Gamma(x-y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^2, \tag{1.3.1}$$

$$\mathcal{D}_D\phi(x) := \int_{\partial D} \frac{\partial}{\partial \nu(y)}\Gamma(x-y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial D. \tag{1.3.2}$$

For a function u defined on $\mathbb{R}^2 \setminus \partial D$, we denote

$$\frac{\partial}{\partial \nu^\pm}u(x) := \lim_{t\to 0^+} \langle \nabla u(x \pm t\nu(x)), \nu(x)\rangle, \quad x \in \partial D,$$

if the limit exists.

The proof of the following trace formula can be found in [42]:

$$\frac{\partial}{\partial \nu^\pm}\mathcal{S}_D\phi(x) = \left(\pm\frac{1}{2}I + \mathcal{K}_D^*\right)\phi(x), \quad x \in \partial D, \tag{1.3.3}$$

$$\mathcal{D}_D\phi|_\pm = \left(\mp\frac{1}{2}I + \mathcal{K}_D\right)\phi(x), \quad x \in \partial D, \tag{1.3.4}$$

where

$$\mathcal{K}_D\phi(x) = \frac{1}{2\pi}\int_{\partial D} \frac{\langle y-x, \nu(y)\rangle}{|x-y|^2}\phi(y)d\sigma(y)$$

and $\mathcal{K}_D^*$ is the L^2-adjoint of $\mathcal{K}_D$. Let $L_0^2(\partial D) := \{f \in L^2(\partial D) : \int_{\partial D} f d\sigma = 0\}$. The following results are of importance to us. For proofs see [42].

Lemma 1.3.1 *The operator $\lambda I - \mathcal{K}_D^*$ is invertible on $L_0^2(\partial D)$ if $|\lambda| \geq \frac{1}{2}$, and for $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, +\infty)$, $\lambda I - \mathcal{K}_D^*$ is invertible on $L^2(\partial D)$.*

Denote by $\mathcal{S}_\Omega, \mathcal{D}_\Omega, \mathcal{K}_\Omega$, and $\mathcal{K}_\Omega^*$ the layer potentials on $\partial\Omega$. Define the functions $H_k(x)$, for $x \in \mathbb{R}^2 \setminus \partial\Omega$, by

$$H_k(x) := \mathcal{D}_\Omega(u_k|_{\partial\Omega})(x) - \mathcal{S}_\Omega g(x), \tag{1.3.5}$$

and introduce $N(\cdot, y)$ to be the Neumann function for Δ in Ω corresponding to a Dirac mass at y, that is, N is the solution to

$$\begin{cases} \Delta_x N(x,y) = -\delta_y \quad \text{in } \Omega, \\ \dfrac{\partial N}{\partial \nu}\Big|_{\partial\Omega} = -\dfrac{1}{|\partial\Omega|}, \\ \displaystyle\int_{\partial\Omega} N(x,y)d\sigma(x) = 0 \quad \text{for } y \in \Omega. \end{cases}$$

Define the background voltage potential, U, to be the unique solution to

$$\begin{cases} \Delta U = 0 \quad \text{in } \Omega, \\ \dfrac{\partial U}{\partial \nu}\Big|_{\partial\Omega} = g, \displaystyle\int_{\partial\Omega} U = 0. \end{cases} \tag{1.3.6}$$

The following representation was proved in [4]:

$$u_k(x) = U(x) - \int_{\partial D} N(x,y)(\lambda I - \mathcal{K}_D^*)^{-1}\left(\frac{\partial H_k}{\partial \nu}\Big|_{\partial D}\right)(y)d\sigma(y), \quad x \in \partial\Omega, \tag{1.3.7}$$

where $\lambda = (k+1)/(2(k-1))$.

Lemma 1.3.2 *Let $k_0 \neq 1$ and $\lambda_0 = (k_0+1)/(2(k_0-1))$. Let $v_k = u_k - u_{k_0}$. Then, for any $x \in \partial\Omega$, we have*

$$\begin{aligned} & v_k(x) + \int_{\partial D} N(x,y)(\lambda I - \mathcal{K}_D^*)^{-1}(\frac{\partial}{\partial \nu}\mathcal{D}_\Omega(v_k)|_{\partial D})(y)d\sigma(y) \\ & = \int_{\partial D} N(x,y)\Big[-(\lambda I - \mathcal{K}_D^*)^{-1} + (\lambda_0 I - \mathcal{K}_D^*)^{-1}\Big](\frac{\partial H_{k_0}}{\partial \nu}\Big|_{\partial D})(y)d\sigma(y). \end{aligned} \tag{1.3.8}$$

Proof. It follows from (1.3.7) that, for $x \in \partial\Omega$,

$$\begin{aligned} & u_k(x) - u_{k_0}(x) + \int_{\partial D} N(x,y)(\lambda I - \mathcal{K}_D^*)^{-1}\left(\frac{\partial (H_k - H_{k_0})}{\partial \nu}\Big|_{\partial D}\right)(y)d\sigma(y) \\ & = \int_{\partial D} N(x,y)\Big[-(\lambda I - \mathcal{K}_D^*)^{-1} + (\lambda_0 I - \mathcal{K}_D^*)^{-1}\Big](\frac{\partial H_{k_0}}{\partial \nu}\Big|_{\partial D})(y)d\sigma(y). \end{aligned}$$

Thanks to (1.3.5), we get

$$H_k(x) - H_{k_0}(x) = \mathcal{D}_\Omega(u_k|_{\partial\Omega} - u_{k_0}|_{\partial\Omega})(x), \quad x \in \Omega,$$

and hence the proof is complete. □

1.3.2 Derivation of the asymptotic expansion

Now, we expand $(\lambda I - \mathcal{K}_D^*)^{-1}$ as k goes to k_0:

$$(\lambda I - \mathcal{K}_D^*)^{-1} = \sum_{n=0}^{\infty}(-1)^n(\lambda - \lambda_0)^n(\lambda_0 I - \mathcal{K}_D^*)^{-n-1}. \tag{1.3.9}$$

Note that the series in the right-hand side of (1.3.9) converges absolutely as an operator on $L_0^2(\partial D)$ as long as $\lambda - \lambda_0$ is small enough. Thus (1.3.8) reads, for any $x \in \partial\Omega$,

$$\begin{aligned} & v_k(x) + \sum_{n=0}^{\infty}(-1)^n(\lambda-\lambda_0)^n \int_{\partial D} N(x,y)(\lambda_0 I - \mathcal{K}_D^*)^{-n-1}(\nabla\mathcal{D}_\Omega(v_k)|_{\partial D}\cdot\nu)(y)d\sigma(y) \\ & = \sum_{n=1}^{\infty}(-1)^{n+1}(\lambda-\lambda_0)^n \int_{\partial D} N(x,y)(\lambda_0 I - \mathcal{K}_D^*)^{-n-1}(\frac{\partial H_{k_0}}{\partial \nu}\Big|_{\partial D})(y)d\sigma(y), \end{aligned}$$

or equivalently,

$$\big(I + \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n T_n\big)(v_k) = \sum_{n=1}^{\infty} (\lambda - \lambda_0)^n F_n, \tag{1.3.10}$$

where

$$T_n(v)(x) = (-1)^n \int_{\partial D} N(x,y)(\lambda_0 I - \mathcal{K}_D^*)^{-n-1}(\nabla \mathcal{D}_\Omega(v)|_{\partial D} \cdot \nu)(y) d\sigma(y), \quad x \in \partial\Omega,$$

and

$$F_n(x) = (-1)^{n+1} \int_{\partial D} N(x,y)(\lambda_0 I - \mathcal{K}_D^*)^{-n-1}\Big(\frac{\partial H_{k_0}}{\partial \nu}\Big|_{\partial D}\Big)(y) d\sigma(y).$$

Note that, since ∂D is away from $\partial\Omega$, we have

$$\begin{aligned} \|T_n v\|_{W^2_{\frac{1}{2}}(\partial\Omega)} &\le C \|(\lambda_0 I - \mathcal{K}_D^*)^{-n-1}(\nabla \mathcal{D}_\Omega(v)|_{\partial D} \cdot \nu)\|_{L^2(\partial D)} \\ &\le C C_0^{n+1} \|\nabla \mathcal{D}_\Omega(v)\|_{L^2(\partial D)} \\ &\le C_1 C_0^{n+1} \|v\|_{L^2(\partial\Omega)}, \end{aligned} \tag{1.3.11}$$

where C_0 is the operator norm of $(\lambda_0 I - \mathcal{K}_D^*)^{-1}$ on $L^2_0(\partial\Omega)$ and C and C_1 are positive constants independent of n. Here $W^2_{\frac{1}{2}}(\partial\Omega)$ is the set of functions $f \in L^2(\partial\Omega)$ such that

$$\int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(x) - f(y)|^2}{|x-y|^2}\, d\sigma(x)\, d\sigma(y) < +\infty.$$

Likewise, we have

$$\|F_n\|_{W^2_{\frac{1}{2}}(\partial\Omega)} \le C C_0^{n+1} \left\| \frac{\partial H_{k_0}}{\partial \nu} \right\|_{L^2(\partial D)}.$$

Note that

$$\left\| \frac{\partial H_{k_0}}{\partial \nu} \right\|_{L^2(\partial D)} \le C(\|u_{k_0}\|_{L^2(\partial\Omega)} + \|g\|_{L^2(\partial\Omega)}) \le C' \|g\|_{L^2(\partial\Omega)},$$

for some C', and hence we get

$$\|F_n\|_{W^2_{\frac{1}{2}}(\partial\Omega)} \le C C_0^{n+1} \|g\|_{L^2(\partial\Omega)}, \tag{1.3.12}$$

for some constant C independent of n. If $\partial\Omega$ is $\mathcal{C}^{1,\beta}$, $\beta > 0$, then we get in the same way

$$\|T_n v\|_{\mathcal{C}^1(\partial\Omega)} \le C C_0^{n+1} \|v\|_{L^2(\partial\Omega)}, \tag{1.3.13}$$

and

$$\|F_n\|_{\mathcal{C}^1(\partial\Omega)} \le C C_0^{n+1} \|g\|_{L^2(\partial\Omega)}, \tag{1.3.14}$$

We need the following lemma. See Chap. 2 for a proof.

Lemma 1.3.3 *If $\partial\Omega$ is Lipchitz, then the operator $I + T_0$ is invertible on $L^2_0(\partial\Omega)$. If $\partial\Omega$ is $\mathcal{C}^{1,\beta}$ for some $\beta > 0$, then it is invertible on $\mathcal{C}^1_0(\partial\Omega)$, where $\mathcal{C}^1_0(\partial\Omega)$ denotes the collection of $f \in \mathcal{C}^1(\partial\Omega)$ with $\int_{\partial\Omega} f = 0$.*

We seek a solution v_k to (1.3.10) in the form

$$v_k(x) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n v_{k_0}^{(n)}(x).$$

Substituting the above expansion of v_k into (1.3.10), we obtain

$$\sum_{n=0}^{\infty} (\lambda-\lambda_0)^n v_{k_0}^{(n)}(x) + \sum_{n=0}^{\infty} (\lambda-\lambda_0)^n \Big(\sum_{p=0}^{n} T_p v_{k_0}^{(n-p)}\Big)(x) = \sum_{n=1}^{\infty} (\lambda-\lambda_0)^n F_n(x), \quad x \in \partial\Omega. \tag{1.3.15}$$

By equating powers of $\lambda - \lambda_0$, we find that $v_{k_0}^{(0)} = 0$ and, for any $n \geq 1$,

$$(I + T_0) v_{k_0}^{(n)} + \sum_{p=1}^{n} T_p v_{k_0}^{(n-p)} = F_n.$$

Using Lemma 1.3.3, it follows that

$$v_{k_0}^{(n)} = (I + T_0)^{-1} \Big(- \sum_{p=1}^{n} T_p v_{k_0}^{(n-p)} + F_n \Big). \tag{1.3.16}$$

Using (1.3.11) and (1.3.12), one can show inductively that

$$\| v_{k_0}^{(n)} \|_{W^2_{\frac{1}{2}}(\partial\Omega)} \leq C_2 C_0^{n+1} n \| g \|_{L^2(\partial\Omega)}, \quad n = 1, 2, \ldots,$$

for some constant C_2 independent of n. The same estimates with the $W^2_{\frac{1}{2}}$-norm replaced with the $\mathcal{C}^1(\partial\Omega)$-norm holds if $\partial\Omega$ is $\mathcal{C}^{1,\beta}$. Since

$$\lambda - \lambda_0 = \frac{k_0 - k}{(k-1)(k_0 - 1)},$$

we finally arrive at the following theorem.

Theorem 1.3.4 *Let C_0 be the operator norm of $(\lambda_0 I - \mathcal{K}_D^*)^{-1}$ on $L^2_0(\partial D)$. Let $0 \leq k_0 \neq 1 \leq \infty$. The following asymptotic expansion holds uniformly and absolutely if $|k - k_0| \leq C < 1/2C_0$ on $\partial\Omega$:*

$$u_k(x) = u_{k_0}(x) + \sum_{n=1}^{\infty} \left[\frac{k_0 - k}{(k-1)(k_0 - 1)} \right]^n v_{k_0}^{(n)}(x), \tag{1.3.17}$$

where the functions $v_{k_0}^{(n)}$ are defined by the recursive formula (1.3.16). The convergence of the series is in $W^2_{\frac{1}{2}}(\partial\Omega)$ if $\partial\Omega$ is Lipschitz, and in $\mathcal{C}^1(\partial\Omega)$ if $\partial\Omega$ is $\mathcal{C}^{1,\beta}$.

In the most significant case, $k_0 = 0$ or ∞, the formula takes the following form:

$$u_k(x) = u_0(x) + \sum_{n=1}^{\infty} \frac{k^n}{(k-1)^n} v_0^{(n)}(x), \tag{1.3.18}$$

and

$$u_k(x) = u_\infty(x) + \sum_{n=1}^{\infty} \frac{1}{(k-1)^n} v_\infty^{(n)}(x). \tag{1.3.19}$$

Moreover, if we interchange the conductivities of $\Omega \setminus D$ and D, the boundary perturbations in the voltage potentials are given by $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(k-1)^n} v_0^{(n)}$ if $k \to +\infty$ and by $\displaystyle\sum_{n=1}^{\infty} \frac{k^n}{(k-1)^n} v_\infty^{(n)}$ if $k \to 0$, where $v_0^{(n)}$ and $v_\infty^{(n)}$ are defined by (1.3.16). This is related to the Keller-Mendelson inversion theorem [57, 58].

Now, if we consider the case when Ω is the unit disk centered at the origin, and D is the concentric disk centered at the origin with radius α then, using

$$\mathcal{K}_D^* \phi(x) = \frac{1}{4\pi\alpha} \int_{\partial D} \phi(y)\, d\sigma(y), \quad \mathcal{K}_\Omega \psi(x) = \frac{1}{4\pi} \int_{\partial \Omega} \psi(y)\, d\sigma(y),$$

and

$$N(x,y) = -2\Gamma(x-y) \text{ modulo constants}, \forall\, x \in \partial\Omega, y \in \partial D,$$

we easily obtain from Theorem 1.3.4 the explicit formulae (1.2.1) and (1.2.2) for $v_\infty^{(n)}$ and $v_0^{(n)}, n \geq 1$.

The formula (1.3.17) holds for all $k_0 \neq 1$. In low contrast case, *i.e.*, $k_0 = 1$, we can get the asymptotic formula trivially. In fact, if $k_0 = 1$, then $H_{k_0} = U$, the background potential. Define

$$\widetilde{T}_n(v)(x) := \int_{\partial D} N(x,y) (\mathcal{K}_D^*)^n \left(\frac{\partial \mathcal{D}_\Omega v}{\partial \nu}\Big|_{\partial D} \right)(y) d\sigma(y), \quad x \in \partial\Omega,$$

and

$$\widetilde{F}_n(x) := -\int_{\partial D} N(x,y) (\mathcal{K}_D^*)^n \left(\frac{\partial U}{\partial \nu}\Big|_{\partial D} \right)(y) d\sigma(y), \quad x \in \partial\Omega,$$

and let the functions $\widetilde{v}^{(n)}$ on $\partial\Omega$, for $n \in \mathbb{N}$, be given by

$$\widetilde{v}^{(0)}(x) = 0, \widetilde{v}^{(n)}(x) = -\sum_{p=0}^{n-1} \widetilde{T}_p \widetilde{v}^{(n-p-1)}(x) + \widetilde{F}_{n-1}(x), \quad n \geq 1.$$

Then we easily get that

$$u_k(x) = U(x) + \sum_{n=1}^{\infty} \frac{2^n (k-1)^n}{(k+1)^n} \widetilde{v}^{(n)}(x), \quad x \in \partial\Omega.$$

The above asymptotic expansion holds uniformly and absolutely on $\partial\Omega$ if $|k - 1| \leq C < 1/2\times$ the operator norm of $\mathcal{K}_D^*$ on $L_0^2(\partial D)$. The convergence of the series is in $W_{\frac{1}{2}}^2(\partial\Omega)$ if $\partial\Omega$ is Lipschitz, and in $\mathcal{C}^1(\partial\Omega)$ if $\partial\Omega$ is $\mathcal{C}^{1,\beta}, \beta > 0$.

Chapter 2

Small Perturbations of an Interface

2.1 Introduction

The main objective is to present a schematic way based on layer potential techniques to derive high-order terms in the asymptotic expansions of the boundary perturbations of steady-state voltage potentials resulting from small perturbations of the shape of a conductivity inclusion with $\mathcal{C}^2$-boundary.

More precisely, consider a homogeneous conducting object occupying a bounded domain $\Omega \subset \mathbb{R}^2$, with a connected $\mathcal{C}^2$-boundary $\partial\Omega$. We assume, for the sake of simplicity, that its conductivity is equal to 1. Let D with $\mathcal{C}^2$-boundary be a conductivity inclusion inside Ω of conductivity equal to some positive constant $k \neq 1$. We assume that $\operatorname{dist}(D, \partial\Omega) \geq C > 0$. The voltage potential in the presence of the inclusion D is denoted by u. It is the solution to

$$\begin{cases} \nabla \cdot \Big(1 + (k-1)\chi_D\Big)\nabla u = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = g, \displaystyle\int_{\partial\Omega} u = 0, \end{cases} \tag{2.1.1}$$

where χ_D is the indicator function of D. Here ν denotes the unit outward normal to the domain Ω and g represents the applied boundary current; it belongs to the set $L^2_0(\partial\Omega) = \{f \in L^2(\partial\Omega), \int_{\partial\Omega} f = 0\}$.

Let D_ϵ be an ϵ-perturbation of D, *i.e.*, let $h \in \mathcal{C}^1(\partial D)$ and ∂D_ϵ be given by

$$\partial D_\epsilon = \{ \ \tilde{x} : \tilde{x} = x + \epsilon h(x)\nu(x), \ x \in \partial D \ \}.$$

Let u_ϵ be the solution to

$$\begin{cases} \nabla \cdot \Big(1 + (k-1)\chi_{D_\epsilon}\Big)\nabla u_\epsilon = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u_\epsilon}{\partial \nu}\Big|_{\partial\Omega} = g, \displaystyle\int_{\partial\Omega} u_\epsilon = 0. \end{cases} \tag{2.1.2}$$

The main achievement of this chapter is a rigorous derivation, based on layer potential techniques, of high-order terms in the asymptotic expansion of $(u_\epsilon - u)|_{\partial\Omega}$ as $\epsilon \to 0$.

The solution u_ϵ to (2.1.2) can be represented using integral operators (see formula (2.3.1)), and hence derivation of asymptotic formula for u_ϵ is reduced to that of the integral operator $\mathcal{K}^*_{D_\epsilon}$ defined by

$$\mathcal{K}^*_{D_\epsilon}\tilde{\varphi}(\tilde{x}) = \frac{1}{2\pi}\text{p.v.}\int_{\partial D_\epsilon} \frac{\langle \tilde{x}-\tilde{y}, \tilde{\nu}(\tilde{x})\rangle}{|\tilde{x}-\tilde{y}|^2}\tilde{\varphi}(\tilde{y})d\sigma_\epsilon(\tilde{y}),$$

where p.v. stands for the Cauchy principal value. The operator $\mathcal{K}^*_{D_\epsilon}$ is a singular integral operator and known to be bounded on $L^2(\partial D_\epsilon)$ [35, 27]. It was proved in [36] that $\mathcal{K}^*_{D_\epsilon}$ converges to the operator $\mathcal{K}^*_D$ on the non-perturbed domain D, defined for a density $\varphi \in L^2(\partial D)$, by

$$\mathcal{K}^*_D\varphi(x) = \frac{1}{2\pi}\int_{\partial D} \frac{\langle x-y, \nu(x)\rangle}{|x-y|^2}\varphi(y)d\sigma(y).$$

In this chapter we will derive a complete asymptotic expansion of the singular integral operator $\mathcal{K}^*_{D_\epsilon}$ on $L^2(\partial D_\epsilon)$ in terms of ϵ. This asymptotic expansion yields an expansion of $u_\epsilon - u$ which extends those already derived for small volume inclusions [4, 5, 21, 22, 41, 60, 97]. Our formula is of significant interest from an "imaging point of view". For instance, if one has a very detailed knowledge of the "boundary signatures" of conductivity inclusions, then it becomes possible to design very effective algorithms to identify certain properties of their shapes. Since it carries very precise information on the shape of the inclusion, it can be efficiently exploited for designing significantly better algorithms. In connection with this, we refer to [23, 50, 92].

2.2 High-order terms in the expansion of $\mathcal{K}^*_{D_\epsilon}$

Let $a, b \in \mathbb{R}$, with $a < b$, and let $X(t) : [a, b] \to \mathbb{R}^2$ be the arclength parametrization of ∂D, namely, X is a $\mathcal{C}^2$-function satisfying $|X'(t)| = 1$ for all $t \in [a, b]$ and

$$\partial D := \{x = X(t), t \in [a, b]\}.$$

Then the outward unit normal to ∂D, $\nu(x)$, is given by $\nu(x) = R_{-\frac{\pi}{2}}X'(t)$, where $R_{-\frac{\pi}{2}}$ is the rotation by $-\pi/2$, the tangential vector at x, $T(x) = X'(t)$, and $X'(t) \perp X''(t)$. Set the curvature $\tau(x)$ to be defined by

$$X''(t) = \tau(x)\nu(x).$$

We will sometimes use $h(t)$ for $h(X(t))$ and $h'(t)$ for the tangential derivative of $h(x)$.

Then, $\tilde{X}(t) = X(t) + \epsilon h(t)\nu(x) = X(t) + \epsilon h(t)R_{-\frac{\pi}{2}}X'(t)$ is a parametrization of ∂D_ϵ. By $\tilde{\nu}(\tilde{x})$, we denote the outward unit normal to ∂D_ϵ at $\tilde{x}$. Then, we have

$$\begin{aligned}\tilde{\nu}(\tilde{x}) &= \frac{R_{-\frac{\pi}{2}}\tilde{X}'(t)}{|\tilde{X}'(t)|} \\ &= \frac{\Big(1-\epsilon h(t)\tau(x)\Big)\nu(x) - \epsilon h'(t)X'(t)}{\sqrt{\epsilon^2 h'(t)^2 + \Big(1-\epsilon h(t)\tau(x)\Big)^2}} \\ &= \frac{\Big(1-\epsilon h(t)\tau(x)\Big)\nu(x) - \epsilon h'(t)T(x)}{\sqrt{\epsilon^2 h'(t)^2 + \Big(1-\epsilon h(t)\tau(x)\Big)^2}},\end{aligned} \tag{2.2.1}$$

and hence $\tilde{\nu}(\tilde{x})$ can be expanded uniformly as

$$\tilde{\nu}(\tilde{x}) = \sum_{n=0}^{\infty} \epsilon^n \nu^{(n)}(x), \quad x \in \partial D, \tag{2.2.2}$$

where the vector-valued functions $\nu^{(n)}$ are bounded. In particular, the first two terms are given by

$$\nu^{(0)}(x) = \nu(x), \quad \nu^{(1)}(x) = -h'(t)T(x).$$

Likewise, we get a uniformly convergent expansion for the length element $d\sigma_\epsilon(\tilde{y})$:

$$d\sigma_\epsilon(\tilde{y}) = |\tilde{X}'(s)|ds = \sqrt{(1-\epsilon\tau(s)h(s))^2 + \epsilon^2 h'^2(s)}ds = \sum_{n=0}^{\infty} \epsilon^n \sigma^{(n)}(y) d\sigma(y), \tag{2.2.3}$$

where $\sigma^{(n)}$ are bounded functions and

$$\sigma^{(0)}(y) = 1, \quad \sigma^{(1)}(y) = -\tau(y)h(y). \tag{2.2.4}$$

Set

$$\begin{aligned} x &= X(t), \qquad \tilde{x} = \tilde{X}(t) = x + \epsilon h(t) R_{-\frac{\pi}{2}} X'(t), \\ y &= X(s), \qquad \tilde{y} = \tilde{X}(s) = y + \epsilon h(s) R_{-\frac{\pi}{2}} X'(s). \end{aligned}$$

Since

$$\tilde{x} - \tilde{y} = x - y + \epsilon\Big(h(t)\nu(x) - h(s)\nu(y)\Big), \tag{2.2.5}$$

we get

$$|\tilde{x}-\tilde{y}|^2 = |x-y|^2 + 2\epsilon\langle x-y, h(t)\nu(x) - h(s)\nu(y)\rangle + \epsilon^2 |h(t)\nu(x) - h(s)\nu(y)|^2,$$

and hence

$$\frac{1}{|\tilde{x}-\tilde{y}|^2} = \frac{1}{|x-y|^2}\frac{1}{1+2\epsilon F(x,y) + \epsilon^2 G(x,y)}, \tag{2.2.6}$$

where

$$F(x,y) = \frac{\langle x-y, h(t)\nu(x) - h(s)\nu(y)\rangle}{|x-y|^2},$$

and

$$G(x,y) = \frac{|h(t)\nu(x) - h(s)\nu(y)|^2}{|x-y|^2}.$$

One can easily see that

$$|F(x,y)| + |G(x,y)|^{\frac{1}{2}} \le C\|X\|_{\mathcal{C}^2}\|h\|_{\mathcal{C}^1}.$$

It follows from (2.2.1), (2.2.3), (2.2.5), and (2.2.6) that

$$\begin{aligned}
\frac{\langle \tilde{x}-\tilde{y}, \tilde{\nu}(\tilde{x})\rangle}{|\tilde{x}-\tilde{y}|^2} d\sigma_\epsilon(\tilde{y}) &= \Big(\frac{\langle x-y, \nu(x)\rangle}{|x-y|^2} + \epsilon\Big[\frac{\langle h(t)\nu(x) - h(s)\nu(y), \nu(x)\rangle}{|x-y|^2} \\
&\quad - \frac{\langle x-y, \tau(x)h(t)\nu(x) + h'(t)T(x)\rangle}{|x-y|^2}\Big] \\
&\quad - \epsilon^2 \frac{\langle h(t)\nu(x) - h(s)\nu(y), \tau(x)h(t)\nu(x) + h'(t)T(x)\rangle}{|x-y|^2}\Big) \\
&\quad \times \frac{1}{1+2\epsilon F(x,y) + \epsilon^2 G(x,y)} \frac{\sqrt{(1-\epsilon\tau(y)h(s))^2 + \epsilon^2 h'^2(s)}}{\sqrt{(1-\epsilon\tau(x)h(t))^2 + \epsilon^2 h'^2(t)}} d\sigma(y) \\
&:= \Big(K_0(x,y) + \epsilon K_1(x,y) + \epsilon^2 K_2(x,y)\Big) \\
&\quad \times \frac{1}{1+2\epsilon F(x,y) + \epsilon^2 G(x,y)} \frac{\sqrt{(1-\epsilon\tau(y)h(s))^2 + \epsilon^2 h'^2(s)}}{\sqrt{(1-\epsilon\tau(x)h(t))^2 + \epsilon^2 h'^2(t)}} d\sigma(y).
\end{aligned}$$

Let

$$\frac{1}{1+2\epsilon F(x,y) + \epsilon^2 G(x,y)} \frac{\sqrt{(1-\epsilon\tau(y)h(s))^2 + \epsilon^2 h'^2(s)}}{\sqrt{(1-\epsilon\tau(x)h(t))^2 + \epsilon^2 h'^2(t)}} = \sum_{n=0}^{\infty} \epsilon^n F_n(x,y),$$

where the series converges absolutely and uniformly. In particular, we can easily see that

$$F_0(x,y) = 1, \quad F_1(x,y) = -2F(x,y) + \tau(x)h(x) - \tau(y)h(y).$$

Then we now have

$$\begin{aligned}
\frac{\langle \tilde{x}-\tilde{y}, \tilde{\nu}(\tilde{x})\rangle}{|\tilde{x}-\tilde{y}|^2} d\sigma_\epsilon(\tilde{y}) &= \frac{\langle x-y, \nu(x)\rangle}{|x-y|^2} d\sigma(y) + \epsilon\Big(K_0(x,y)F_1(x,y) + K_1(x,y)\Big) d\sigma(y) \\
&\quad + \epsilon^2 \sum_{n=0}^{\infty} \epsilon^n \Big(F_{n+2}(x,y)K_0(x,y) + F_{n+1}(x,y)K_1(x,y) + F_n(x,y)K_2(x,y)\Big) d\sigma(y).
\end{aligned}$$

Therefore, we obtain that

$$\frac{\langle \tilde{x}-\tilde{y}, \tilde{\nu}(\tilde{x})\rangle}{|\tilde{x}-\tilde{y}|^2} d\sigma_\epsilon(\tilde{y}) = \sum_{n=0}^{\infty} \epsilon^n \mathbb{k}_n(x,y) d\sigma(y),$$

where

$$\mathbb{k}_0(x,y) = \frac{\langle x-y, \nu(x)\rangle}{|x-y|^2}, \quad \mathbb{k}_1(x,y) = K_0(x,y)F_1(x,y) + K_1(x,y),$$

and for any $n \geq 2$,

$$\mathbb{k}_n(x,y) = F_n(x,y)K_0(x,y) + F_{n-1}(x,y)K_1(x,y) + F_{n-2}(x,y)K_2(x,y).$$

Introduce a sequence of integral operators $(\mathcal{K}_D^{(n)})_{n\in\mathbb{N}}$, defined for any $\phi \in L^2(\partial D)$ by:

$$\mathcal{K}_D^{(n)}\phi(x) = \int_{\partial D} \mathbb{k}_n(x,y)\phi(y)d\sigma(y) \quad \text{for } n \geq 0.$$

Note that $\mathcal{K}_D^{(0)} = \mathcal{K}_D^*$. Observe that the same operator with the kernel $\mathbb{k}_n(x,y)$ replaced with $K_j(x,y)$, $j = 0,1,2$, is bounded on $L^2(\partial D)$. In fact, it is an immediate consequence of the celebrated theorem of Coifman-McIntosh-Meyer [27]. Therefore each $\mathcal{K}_D^{(n)}$ is bounded on $L^2(\partial D)$.

Let Ψ_ϵ be the diffeomorphism from ∂D onto ∂D_ϵ given by $\Psi_\epsilon(x) = x + \epsilon h(t)\nu(x)$, where $x = X(t)$. The following theorem holds.

Theorem 2.2.1 *Let $N \in \mathbb{N}$. There exists C depending only on N, $\|X\|_{\mathcal{C}^2}$, and $\|h\|_{\mathcal{C}^1}$ such that for any $\tilde{\phi} \in L^2(\partial D_\epsilon)$,*

$$||(\mathcal{K}_{D_\epsilon}^*\tilde{\phi}) \circ \Psi_\epsilon - \mathcal{K}_D^*\phi - \sum_{n=1}^{N} \epsilon^n \mathcal{K}_D^{(n)}\phi||_{L^2(\partial D)} \leq C\epsilon^{N+1}||\phi||_{L^2(\partial D)},$$

where $\phi := \tilde{\phi} \circ \Psi_\epsilon$.

2.3 Derivation of the full asymptotic formula for the steady-state voltage potentials

In this section we derive high-order terms in the asymptotic expansion of $(u_\epsilon - u)|_{\partial\Omega}$ as $\epsilon \to 0$.

Suppose that the conductivity of D is k. Let $\lambda := \frac{k+1}{2(k-1)}$. Define the background voltage potential, U, to be the unique solution to

$$\begin{cases} \Delta U = 0 \quad \text{in } \Omega, \\ \dfrac{\partial U}{\partial \nu}\Big|_{\partial\Omega} = g, \displaystyle\int_{\partial\Omega} U = 0. \end{cases}$$

Let $N(x,y)$ be the Neumann function for Δ in Ω corresponding to a Dirac mass at y, that is, N is the solution to

$$\begin{cases} \Delta_x N(x,y) = -\delta_y \quad \text{in } \Omega, \\ \dfrac{\partial N}{\partial \nu}\Big|_{\partial\Omega} = -\dfrac{1}{|\partial\Omega|}, \\ \displaystyle\int_{\partial\Omega} N(x,y)d\sigma(x) = 0 \quad \text{for } y \in \Omega. \end{cases}$$

Let $\mathcal{N}_D$ be defined by

$$\mathcal{N}_D\varphi(x) := \int_{\partial D} N(x,y)\varphi(y)d\sigma(y), \quad x \in \partial\Omega,$$

for $\varphi \in L^2_0(\partial D)$.

Let u_ϵ be the solution to (2.1.2). Then the following representation formula holds [4]:

$$u_\epsilon(x) = U(x) - \mathcal{N}_{D_\epsilon}\tilde{\phi}_\epsilon(x), \quad x \in \partial\Omega \tag{2.3.1}$$

where $\tilde{\phi}_\epsilon \in L^2_0(\partial D_\epsilon)$ is given by

$$H_\epsilon(x) := \mathcal{D}_\Omega(u_\epsilon|_{\partial\Omega})(x) - \mathcal{S}_\Omega g(x), \quad x \in \Omega, \tag{2.3.2}$$

and

$$(\lambda I - \mathcal{K}^*_{D_\epsilon})\tilde{\phi}_\epsilon(x) = \frac{\partial H_\epsilon}{\partial \tilde{\nu}}(x), \quad x \in \partial D_\epsilon.$$

Here and throughout this chapter $\mathcal{S}_\Omega$ and $\mathcal{D}_\Omega$ denote the single and double layer potential on $\partial\Omega$:

$$\mathcal{D}_\Omega\varphi(x) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{\langle y-x, \nu(y)\rangle}{|x-y|^2}\varphi(y)d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial\Omega,$$
$$\mathcal{S}_\Omega\varphi(x) = \frac{1}{2\pi}\int_{\partial\Omega} \log|x-y|\varphi(y)d\sigma(y), \quad x \in \mathbb{R}^2.$$

Likewise the solution u to (2.1.1) has the representation

$$u(x) = U(x) - \mathcal{N}_D\phi(x), \quad x \in \partial\Omega,$$

where $\phi \in L^2_0(\partial D)$ is given by

$$H(x) := \mathcal{D}_\Omega(u|_{\partial\Omega})(x) - \mathcal{S}_\Omega g(x), \quad x \in \Omega, \tag{2.3.3}$$

and

$$(\lambda I - \mathcal{K}^*_D)\phi(x) = \frac{\partial H}{\partial \nu}(x), \quad x \in \partial D. \tag{2.3.4}$$

We then get

$$u_\epsilon(x) - u(x) = -\mathcal{N}_{D_\epsilon}\tilde{\phi}_\epsilon(x) + \mathcal{N}_D\phi(x), \quad x \in \partial\Omega. \tag{2.3.5}$$

We now investigate the asymptotic behavior of $\mathcal{N}_{D_\epsilon}\tilde{\phi}_\epsilon$ as $\epsilon \to 0$. After the change of variables $\tilde{y} = \Psi_\epsilon(y)$, we get from (2.2.3) and the Taylor expansion of $N(x,y)$ that

$$\mathcal{N}_{D_\epsilon}\tilde{\phi}_\epsilon(x) = \int_{\partial D}\Big(\sum_{n=0}^\infty \sum_{|\alpha|=n} \frac{\epsilon^n}{\alpha!}(h(y)\nu(y))^\alpha \partial_y^\alpha N(x,y)\Big)\phi_\epsilon(y)\Big(\sum_{n=0}^\infty \epsilon^n \sigma^{(n)}(y)\Big)d\sigma(y),$$

where $\phi_\epsilon = \tilde{\phi}_\epsilon \circ \Psi_\epsilon$.

One can see from Theorem 2.2.1 that, for each integer N, ϕ_ϵ satisfies

$$\Big(\lambda I - \mathcal{K}_D^* - \sum_{n=1}^{N} \epsilon^n \mathcal{K}_D^{(n)}\Big)\phi_\epsilon + O(\epsilon^{N+1}) = (\nabla H_\epsilon)(\Psi_\epsilon) \cdot \tilde{\nu}(\Psi_\epsilon) \quad \text{on } \partial D.$$

We obtain from (2.2.2) that

$$\begin{aligned}(\nabla H_\epsilon)(\Psi_\epsilon)(y) \cdot \tilde{\nu}(\Psi_\epsilon)(y) &= \Big(\sum_{n=0}^{\infty} \sum_{|\alpha|=n} \frac{\epsilon^n}{\alpha!} (\nabla \partial^\alpha H_\epsilon)(y)(h(y)\nu(y))^\alpha\Big) \cdot \Big(\sum_{n=0}^{\infty} \epsilon^n \nu^{(n)}(y)\Big) \\ &:= \sum_{n=0}^{\infty} \epsilon^n G_n(y). \qquad (2.3.6)\end{aligned}$$

Note that

$$G_0(y) = \frac{\partial H_\epsilon}{\partial \nu}(y), \quad G_1(y) = h(y)\big\langle D^2 H_\epsilon(y)\nu(y), \nu(y)\big\rangle - h'(y)\langle \nabla H_\epsilon, T(y)\rangle, \qquad (2.3.7)$$

where $D^2 H_\epsilon$ is the Hessian of H_ϵ. Therefore, we obtain the following integral equation to solve:

$$\Big(\lambda I - \mathcal{K}_D^* - \sum_{n=1}^{N} \epsilon^n \mathcal{K}_D^{(n)}\Big)\phi_\epsilon + O(\epsilon^{N+1}) = \sum_{n=0}^{+\infty} \epsilon^n G_n \quad \text{on } \partial D. \qquad (2.3.8)$$

The equation (2.3.8) can be solved recursively in the following way: Define

$$\phi^{(0)} = (\lambda I - \mathcal{K}_D^*)^{-1} G_0 = (\lambda I - \mathcal{K}_D^*)^{-1} \Big(\frac{\partial H_\epsilon}{\partial \nu}\big|_{\partial D}\Big), \qquad (2.3.9)$$

and for $1 \leq n \leq N$,

$$\phi^{(n)} = (\lambda I - \mathcal{K}_D^*)^{-1} \Big(G_n + \sum_{p=0}^{n-1} \mathcal{K}_D^{(n-p)} \phi^{(p)}\Big). \qquad (2.3.10)$$

We obtain the following lemma.

Lemma 2.3.1 *Let $N \in \mathbb{N}$. There exists C depending only on N, the $\mathcal{C}^2$-norm of X, and the $\mathcal{C}^1$-norm of h such that*

$$||\phi_\epsilon - \sum_{n=0}^{N} \epsilon^n \phi^{(n)}||_{L^2(\partial D)} \leq C \epsilon^{N+1},$$

where $\phi^{(n)}$ are defined by the recursive relation (2.3.10).

Define, for $n \in \mathbb{N}$ and for $x \in \partial\Omega$,

$$v_n(x) := \sum_{i+j+k=n} \int_{\partial D} \Big(\sum_{|\alpha|=i} \frac{1}{\alpha!} (h(y)\nu(y))^\alpha \partial_y^\alpha N(x,y)\Big) \sigma^{(j)}(y) \phi^{(k)}(y) d\sigma(y). \qquad (2.3.11)$$

It then follows from (2.3.7) and (2.3.11) that

$$\mathcal{N}_{D_\epsilon}\tilde{\phi}_\epsilon(x) = \mathcal{N}_D(\lambda I - \mathcal{K}_D^*)^{-1}\Big(\frac{\partial H_\epsilon}{\partial \nu}\big|_{\partial D}\Big) + \sum_{n=1}^N \epsilon^n v_n(x) + O(\epsilon^{N+1}).$$

Hence we get from (2.3.5) that

$$u_\epsilon(x) - u(x) = -\mathcal{N}_D(\lambda I - \mathcal{K}_D^*)^{-1}\Big(\frac{\partial H_\epsilon}{\partial \nu}\big|_{\partial D} - \frac{\partial H}{\partial \nu}\big|_{\partial D}\Big) - \sum_{n=1}^N \epsilon^n v_n(x) + O(\epsilon^{N+1}), \quad x \in \partial\Omega.$$

Observe from (2.3.2) and (2.3.3) that

$$H_\epsilon(x) - H(x) = \mathcal{D}_\Omega(u_\epsilon|_{\partial\Omega} - u|_{\partial\Omega})(x), \quad x \in \Omega. \tag{2.3.12}$$

If we define the operator $\mathcal{E}$ on $L_0^2(\partial\Omega)$ by

$$\mathcal{E}(v)(x) := \mathcal{N}_D(\lambda I - \mathcal{K}_D^*)^{-1}\Big(\frac{\partial}{\partial \nu}(\mathcal{D}_\Omega v)\big|_{\partial D}\Big)(x), \quad x \in \partial\Omega,$$

then it follows that

$$(I + \mathcal{E})(u_\epsilon - u)(x) = -\sum_{n=1}^N \epsilon^n v_n(x) + O(\epsilon^{N+1}), \quad x \in \partial\Omega. \tag{2.3.13}$$

We need the following lemma.

Lemma 2.3.2 *The operator $I + \mathcal{E}$ is invertible on $L_0^2(\partial\Omega)$.*

Let us continue derivation of asymptotic expansion of $(u_\epsilon - u)|_{\partial\Omega}$ leaving the proof of Lemma 2.3.2 at the end of this section.

We get from (2.3.13) that

$$u_\epsilon(x) - u(x) = -\sum_{n=1}^N \epsilon^n (I + \mathcal{E})^{-1}(v_n)(x) + O(\epsilon^{N+1}), \quad x \in \partial\Omega. \tag{2.3.14}$$

Observe that the function v_n is still depending on ϵ since G_n in (2.3.6) is defined by H_ϵ and hence $\phi^{(n)}$ depends on ϵ. We can remove this dependence on ϵ from the asymptotic formula in an iterative way.

Observe from (2.3.14) that

$$(u_\epsilon - u)|_{\partial\Omega} = O(\epsilon),$$

and hence, by (2.3.12),

$$H_\epsilon(x) - H(x) = O(\epsilon).$$

Thus if we define G_n^1, $n \in \mathbb{N}$, by (2.3.6) with H_ϵ replaced with H, and define $\phi_1^{(n)}$ and v_n^1 by (2.3.9), (2.3.10), and (2.3.11), then $v_n - v_n^1 = O(\epsilon)$. Therefore we get

$$u_\epsilon(x) - u(x) = -\epsilon^1 (I + \mathcal{E})^{-1}(v_1^1)(x) + O(\epsilon^2), \quad x \in \partial\Omega. \tag{2.3.15}$$

Repeat the same procedure with $H - \epsilon \mathcal{D}_\Omega (I+\mathcal{E})^{-1}(v_n^1)$ instead of H to get v_n^2. Then $v_n - v_n^2 = O(\epsilon^2)$ and hence

$$u_\epsilon(x) - u(x) = -\sum_{n=1}^{2} \epsilon^n (I+\mathcal{E})^{-1}(v_n^2)(x) + O(\epsilon^3), \quad x \in \partial\Omega.$$

Repeating the same procedure until we get v_n^N, and we obtain the following theorem.

Theorem 2.3.3 *Let v_n^N, $n = 1, \ldots, N$, be the functions obtained by the above procedure. Then the following formula holds uniformly for $x \in \partial\Omega$:*

$$u_\epsilon(x) - u(x) = -\sum_{n=1}^{N} \epsilon^n (I+\mathcal{E})^{-1}(v_n^N)(x) + O(\epsilon^{N+1}).$$

The remainder $O(\epsilon^{N+1})$ depends only on N, Ω, the $\mathcal{C}^2$-norm of X, the $\mathcal{C}^1$-norm of h, and dist$(D, \partial\Omega)$.

Let us compute the first order approximation of $(u_\epsilon - u)|_{\partial\Omega}$ explicitly. Note that $\phi_1^{(0)} = \phi$ where ϕ is defined by (2.3.4), and

$$\phi_1^{(1)} = (\lambda I - \mathcal{K}_D^*)^{-1}\Big(h\big\langle (D^2 H)\nu, \nu\big\rangle - h'\langle \nabla H, T\rangle + \mathcal{K}_D^{(1)}\phi\Big).$$

Therefore, by (2.2.4) and (2.3.11), v_1^1 takes the form

$$\begin{aligned} v_1^1(x) = &\int_{\partial D} \nabla_y N(x,y)\cdot \nu(y) h(y)\phi(y) d\sigma(y) \\ &- \int_{\partial D} N(x,y)\tau(y)h(y)\phi(y)d\sigma(y) + \int_{\partial D} N(x,y)\phi_1^{(1)}(y)d\sigma(y). \end{aligned}$$

Using this formula and (2.3.15) we find the first-order term in the asymptotic expansion of $u_\epsilon - u$ on $\partial\Omega$.

The first term in the asymptotic expansions is exactly the domain derivative of the solution derived in [48, Theorem 1]. To see that, it suffices to prove that

$$(I+\mathcal{E})w = -\frac{1}{k-1} v_1^1 \quad \text{on } \partial\Omega, \tag{2.3.16}$$

where w is the solution of

$$\begin{cases} \Delta w = 0 \quad \text{in } (\Omega \setminus \overline{D}) \cup D, \\ w|_+ - w|_- = -h \dfrac{\partial u}{\partial \nu}\Big|_- \quad \text{on } \partial D, \\ \dfrac{\partial w}{\partial \nu}\Big|_+ - k\dfrac{\partial w}{\partial \nu}\Big|_- = -\dfrac{\partial}{\partial T}\Big(h\dfrac{\partial u}{\partial T}\Big) \quad \text{on } \partial D, \\ \dfrac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

It is easy to see that

$$w = \mathcal{D}_\Omega(w|_{\partial\Omega}) + \mathcal{D}_D(h\frac{\partial u}{\partial \nu}\Big|_-) + \mathcal{S}_D\theta, \quad x \in \Omega,$$

where the density θ on ∂D is given by

$$\theta = (\lambda I - \mathcal{K}_D^*)^{-1}\Big[- \frac{1}{k-1}(\frac{\partial}{\partial T}h\frac{\partial u}{\partial T}) + \frac{\partial}{\partial \nu}(\mathcal{D}_\Omega w)\big|_{\partial D} + \frac{\partial}{\partial \nu}(\mathcal{D}_D(h\frac{\partial u}{\partial \nu}\Big|_-))\big|_{\partial D}\Big].$$

Thus, for $x \in \partial\Omega$,

$$w(x) + \mathcal{N}_D(\lambda I - \mathcal{K}_D^*)^{-1}\Big(\frac{\partial}{\partial \nu}(\mathcal{D}_\Omega w)\big|_{\partial D}\Big)(x) = - \int_{\partial D} \frac{\partial N}{\partial \nu(y)}(x,y)h(y)\frac{\partial u}{\partial \nu}\Big|_-(y)d\sigma(y)$$
$$+\frac{1}{k-1}\int_{\partial D} N(x,y)(\lambda I - \mathcal{K}_D^*)^{-1}(\frac{\partial}{\partial T}h\frac{\partial u}{\partial T})(y)d\sigma(y)$$
$$-\int_{\partial D} N(x,y)(\lambda I - \mathcal{K}_D^*)^{-1}(\frac{\partial}{\partial \nu}(\mathcal{D}_D(h\frac{\partial u}{\partial \nu}\Big|_-))\big|_{\partial D})d\sigma(y).$$

Since

$$\frac{\partial u}{\partial \nu}\Big|_- = \frac{1}{k-1}\phi,$$

and

$$\frac{\partial}{\partial T}h\frac{\partial u}{\partial T} = h\Big(- \big\langle (D^2H)\nu, \nu\big\rangle - \tau\frac{\partial H}{\partial \nu}\Big) + h'\frac{\partial H}{\partial T} + \frac{\partial}{\partial T}h\frac{\partial}{\partial T}\mathcal{S}_D\phi,$$

where ϕ is defined by (2.3.4), then by using the expression of $\mathcal{K}_D^{(1)}$ it is not difficult to see that (2.3.16) holds.

Proof of Lemma 2.3.2. Since $\mathcal{E}$ is a compact operator, we can apply the Fredholm alternative.

Suppose that $(I + \mathcal{E})v = 0$. Then, first of all, v is smooth on $\partial\Omega$. Since $(-\frac{1}{2}I + \mathcal{K}_\Omega)\mathcal{N}_D = \mathcal{S}_D$ on $L^2_0(\partial D)$ as was proved in [4, 5], we get

$$\Big(- \frac{1}{2}I + \mathcal{K}_\Omega\Big)v + \mathcal{S}_D(\lambda I - \mathcal{K}_D^*)^{-1}\Big(\frac{\partial}{\partial \nu}(\mathcal{D}_\Omega v)\big|_{\partial D}\Big) = 0 \quad \text{on } \partial\Omega.$$

Since

$$(\mathcal{D}_\Omega f)|_-(x) = \Big(\frac{1}{2}I + \mathcal{K}_\Omega\Big)f(x), \quad x \in \partial\Omega,$$

where the subscript $-$ denotes the limit from the inside of Ω [96], we get

$$v(x) = (\mathcal{D}_\Omega v)|_-(x) + \mathcal{S}_D(\lambda I - \mathcal{K}_D^*)^{-1}\Big(\frac{\partial}{\partial \nu}(\mathcal{D}_\Omega v)\big|_{\partial D}\Big)(x), \quad x \in \partial\Omega.$$

Thus v can be extended to whole Ω to satisfy

$$v(x) = (\mathcal{D}_\Omega v)(x) + \mathcal{S}_D(\lambda I - \mathcal{K}_D^*)^{-1}\Big(\frac{\partial}{\partial \nu}(\mathcal{D}_\Omega v)\big|_{\partial D}\Big)(x), \quad x \in \Omega. \qquad (2.3.17)$$

Let the space $W^{1,2}(\Omega)$ be the set of functions $f \in L^2(\Omega)$ such that $\nabla f \in L^2(\Omega)$. We now recall the following facts from [53, 55]: The $W^{1,2}$-solution u to (2.1.1) has the representation

$$u(x) = H(x) + \mathcal{S}_D \phi(x), \quad x \in \Omega, \tag{2.3.18}$$

where the Harmonic function $H \in W^{1,2}(\Omega)$ and $\phi \in L^2_0(\partial D)$ is given by (2.3.3) and (2.3.4). Moreover the harmonic function H and ϕ are unique.

Observe that v given in (2.3.17) takes exactly the form in (2.3.18) with $H = \mathcal{D}_\Omega v$. By (2.3.3) and uniqueness of H, we have $\mathcal{S}_\Omega(\frac{\partial v}{\partial \nu}|_{\partial\Omega}) = 0$ in Ω. It then follows that $\frac{\partial v}{\partial \nu}|_{\partial\Omega} = 0$ on $\partial\Omega$, and hence v is constant in Ω. Since $v \in L^2_0(\partial\Omega)$, we get $v = 0$. So, $I + \mathcal{E}$ is injective, and hence invertible. This completes the proof. $\square$

Chapter 3

Thin Interfaces

3.1 Introduction

Let $a, b \in \mathbb{R}$, with $a < b$, $\beta > 0$, and let $X(t) : [a, b] \to \mathbb{R}^2$ be a $\mathcal{C}^{2,\beta}$-function satisfying $|X'(t)| = 1$ for all $t \in [a, b]$. We consider the bounded $\mathcal{C}^{2,\beta}$-domain D in $\mathbb{R}^2$ parameterized by the function $X(t)$:

$$\partial D := \{x = X(t), t \in [a, b]\}.$$

Then the outward unit normal to D, $\nu(x)$, is given by $\nu(x) = R_{-\frac{\pi}{2}} X'(t)$, where $R_{-\frac{\pi}{2}}$ is the rotation by $-\pi/2$, the tangential vector at x, $T(x) = X'(t)$, and $X'(t) \perp X''(t)$. Set the curvature $\tau(t)$ to be defined by

$$X''(t) = \tau(t)\nu(x).$$

We assume that $\mathrm{dist}(D, \partial\Omega) \geq C > 0$. We will sometimes use $h(t)$ for $h(X(t))$ and $h'(t)$ for the tangential derivative of $h(x)$.

Let D_ϵ be an ϵ-perturbation of D, *i.e.*,

$$\partial D_\epsilon := \{\tilde{x} = \tilde{X}(t) = X(t) + \epsilon h(t)\nu(x) = X(t) + \epsilon h(t) R_{-\frac{\pi}{2}} X'(t)\},$$

where the function $h \in \mathcal{C}^{2,\beta}(\partial D)$. We assume that $h(t) > C > 0$ for all $t \in [a, b]$ and denote by $\tilde{\nu}(\tilde{x})$ the outward unit normal to D_ϵ at $\tilde{x}$.

Consider a homogeneous conducting object occupying a bounded domain $\Omega \subset \mathbb{R}^2$, with a connected $\mathcal{C}^{2,\beta}$-boundary $\partial\Omega$. We assume, for the sake of simplicity, that its conductivity is equal to 1.

Let the conductivity of the thin layer $D_\epsilon \setminus \overline{D}$ inside Ω be equal to some positive constant $k \neq 1$. The voltage potential in the presence $D_\epsilon \setminus \overline{D}$ is denoted u_ϵ. It is the solution to

$$\begin{cases} \nabla \cdot (1 + (k-1)\chi_{D_\epsilon \setminus \overline{D}})\nabla u_\epsilon = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u_\epsilon}{\partial \nu}\Big|_{\partial\Omega} = g, \displaystyle\int_{\partial\Omega} u_\epsilon = 0, \end{cases} \tag{3.1.1}$$

where $\chi_{D_\epsilon \setminus \overline{D}}$ is the indicator function of $D_\epsilon \setminus \overline{D}$, ν denotes the unit outward normal to the domain Ω, and $g \in L^2_0(\partial\Omega)$ represents the applied boundary current. Here $L^2_0(\partial\Omega) = \{f \in L^2(\partial\Omega), \int_{\partial\Omega} f = 0\}$.

Define the background voltage potential, U, to be the unique solution to

$$\begin{cases} \Delta U = 0 \quad \text{in } \Omega, \\ \dfrac{\partial U}{\partial \nu}\Big|_{\partial\Omega} = g, \displaystyle\int_{\partial\Omega} U = 0. \end{cases} \tag{3.1.2}$$

The aim of this chapter is to derive an asymptotic expansion of $(u_\epsilon - U)|_{\partial\Omega}$ as $\epsilon \to 0$. The derivation depends on the values of the conductivity. We consider separately the following three cases:

- case 1: $0 < k \neq 1 < +\infty$ is fixed and $\epsilon \to 0$;
- case 2: $k \to +\infty$ and $\epsilon \to 0$ such that $k\epsilon \to \alpha$, $0 < \alpha < +\infty$;
- case 3: $k \to 0$ and $\epsilon \to 0$ such that $k^{-1}\epsilon \to \delta$, $0 < \delta < +\infty$.

These cases cover highly conducting layer ($k \to \infty$) and those with interfacial resistance ($k \to 0$). The last two cases are degenerating cases as we will see later and it is not so clear what the limit of u_ϵ should be as $\epsilon \to 0$. On the other hand, in the first case it is quite obvious that the leading order term is the background voltage potential, U, which is the unique solution to

$$\begin{cases} \Delta U = 0 \quad \text{in } \Omega, \\ \dfrac{\partial U}{\partial \nu}\Big|_{\partial\Omega} = g, \displaystyle\int_{\partial\Omega} U = 0. \end{cases}$$

So in this case, we look for the first order term in the asymptotic expansion.

In order to have a rough idea about the problems we are interested in, let us consider the following transmission problem in the free space:

$$\begin{cases} \nabla \cdot (1 + (k-1)\chi_{D_\epsilon \setminus \overline{D}})\nabla u_\epsilon = 0 \quad \text{in } \mathbb{R}^2, \\ u_\epsilon(x) - x = O(|x|^{-1}) \quad \text{as } |x| \to \infty, \end{cases} \tag{3.1.3}$$

where D is a disk of radius r centered at 0 and D_ϵ is the concentric disk with radius $r + \epsilon$. By tedious computations we can see that there are constants $a, b, d, C(k, \epsilon)$ such that the solution u_ϵ to (3.1.3) takes the form

$$u_\epsilon(x) = \begin{cases} ax, & x \in D, \\ bx + d\dfrac{x}{|x|^2}, & x \in D_\epsilon \setminus \overline{D}, \\ x + C(k,\epsilon)\dfrac{x}{|x|^2}, & x \in \mathbb{R}^2 \setminus \overline{D_\epsilon}. \end{cases}$$

We will not write down a, b, d explicitly. However, $C(k, \epsilon)$ is given by

$$C(k,\epsilon) = (r+\epsilon)^2 \left[\frac{2(k+1)(r+\epsilon)^2 + 2(k-1)r^2}{(k+1)^2(r+\epsilon)^2 - (k-1)^2 r^2} - 1 \right].$$

If $k \to +\infty$ and $\epsilon \to 0$, and $k\epsilon \to \alpha$, then

$$C(k,\epsilon) \to -\frac{\alpha r^2}{2r+\alpha}. \tag{3.1.4}$$

If $k \to 0$ and $\epsilon \to 0$, and $k^{-1}\epsilon \to \delta$, then

$$C(k,\epsilon) \to \frac{\delta r^2}{2r+\delta}. \tag{3.1.5}$$

Let u be the solution to the following problem:

$$\begin{cases} \nabla \cdot (1 + (\gamma - 1)\chi_D)\nabla u = 0 \quad \text{in } \mathbb{R}^2, \\ u(x) - x = O(|x|^{-1}) \quad \text{as } |x| \to \infty, \end{cases} \tag{3.1.6}$$

Then u in $\mathbb{R}^2 \setminus \overline{D}$ is given by

$$u(x) = x - \frac{(\gamma-1)r^2}{\gamma+1}\frac{x}{|x|^2}, \quad x \in \mathbb{R}^2 \setminus \overline{D}.$$

Thus (3.1.4) shows that in the second case, $u_\epsilon(x)$ converges to the solution to (3.1.6) with $\gamma = (r+\alpha)/r$, for $x \in \mathbb{R}^2 \setminus \overline{D}$. Observe that if $\alpha = 0$, *i.e.*, ϵ goes to 0 faster than k goes to ∞, then then limiting solution is x which is the solution without inclusions. If $\alpha = \infty$, the limiting solution is the one with the perfectly conducting inclusion. In the third case, (3.1.5) shows that u_ϵ converges to the solution to (3.1.6) with the conductivity $\gamma = r/(r+\delta)$.

The leading-order term in the boundary perturbations of the voltage potential due to a thin interface with non extreme conductivity was derived in [19, 18]. The behavior of the voltage potentials in the presence of thin interfaces of either high or low conductivity was investigated in [76, 80]. Various aspects related to the effective conductivity of composite materials with thin interfaces of either low or high conductivity were studied in [93, 25, 30, 62, 63, 67, 84].

To the best of our knowledge, this is the first work to rigorously derive asymptotic expansions of $(u_\epsilon - U)|_{\partial\Omega}$ as $\epsilon \to 0$ using a general unified layer potential technique. Our procedure can in principle be continued and all the terms in the expansion of $(u_\epsilon - U)|_{\partial\Omega}$ can be obtained in the same fashion.

3.2 Derivation of asymptotic formulae for the steady-state voltage potentials

3.2.1 Representation formula

Here we review some basic facts on the layer potentials. Let $\Gamma(x)$ be the fundamental solution of the Laplacian Δ in the two-dimensional case:

$$\Gamma(x) = \frac{1}{2\pi} \ln |x|.$$

The single and double layer potentials of the density function ϕ on ∂D are defined by

$$\mathcal{S}_D \phi(x) := \int_{\partial D} \Gamma(x-y)\phi(y) d\sigma(y), \quad x \in \mathbb{R}^2,$$

$$\mathcal{D}_D \phi(x) := \int_{\partial D} \frac{\partial}{\partial \nu(y)} \Gamma(x-y)\phi(y) d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial D.$$

For a function u defined on $\mathbb{R}^2 \setminus \partial D$, we denote

$$\frac{\partial}{\partial \nu^{\pm}} u(x) := \lim_{t \to 0^+} \langle \nabla u(x \pm t\nu(x)), \nu(x) \rangle, \quad x \in \partial D,$$

if the limit exists.

The proof of the following trace formula can be found in [42]:

$$\frac{\partial}{\partial \nu^{\pm}} \mathcal{S}_D \phi(x) = \left(\pm \frac{1}{2} I + \mathcal{K}_D^* \right) \phi(x), \quad x \in \partial D, \tag{3.2.1}$$

$$\mathcal{D}_D \phi(x)|_{\pm} = \left(\mp \frac{1}{2} I + \mathcal{K}_D \right) \phi(x), \quad x \in \partial D, \tag{3.2.2}$$

where

$$\mathcal{K}_D \phi(x) = \frac{1}{2\pi} \int_{\partial D} \frac{\langle y-x, \nu(y) \rangle}{|x-y|^2} \phi(y) d\sigma(y)$$

and $\mathcal{K}_D^*$ is the L^2-adjoint of $\mathcal{K}_D$. Let $L_0^2(\partial D) := \{ f \in L^2(\partial D) : \int_{\partial D} f d\sigma = 0 \}$. The following results are of importance to us. For proofs see [42].

Lemma 3.2.1 *The operator $\lambda I - \mathcal{K}_D^*$ is invertible on $L_0^2(\partial D)$ if $|\lambda| \geq \frac{1}{2}$, and for $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, +\infty)$, $\lambda I - \mathcal{K}_D^*$ is invertible on $L^2(\partial D)$.*

Let $\lambda = (k+1)/(2(k-1))$. Denote by $\mathcal{S}_{D_\epsilon}, \mathcal{D}_{D_\epsilon}, \mathcal{K}_{D_\epsilon}$, and $\mathcal{K}_{D_\epsilon}^*$ the layer potentials on ∂D_ϵ. The following lemma can be proved in the exactly same manner as the representation formula in [53].

Lemma 3.2.2 *The solution u_ϵ of the problem (3.1.1) can be represented as follows:*

$$u_\epsilon(x) = H_\epsilon(x) + \mathcal{S}_{D_\epsilon} \tilde{\phi}_\epsilon(x) + \mathcal{S}_D \psi_\epsilon(x), \quad x \in \Omega, \tag{3.2.3}$$

where the harmonic part H_ϵ of u_ϵ is given by $H_\epsilon = -\mathcal{S}_\Omega(g)(x) + \mathcal{D}_\Omega(u_\epsilon|_{\partial\Omega})(x)$, $x \in \Omega$, and the pair $(\tilde{\phi}_\epsilon, \psi_\epsilon) \in L^2(\partial D_\epsilon) \times L^2(\partial D)$ is the unique solution of the following system of integral equations:

$$\begin{cases} (\lambda I - \mathcal{K}^*_{D_\epsilon})\tilde{\phi}_\epsilon(\tilde{x}) - \dfrac{\partial \mathcal{S}_D \psi_\epsilon}{\partial \tilde{\nu}}(\tilde{x}) = \dfrac{\partial H_\epsilon}{\partial \tilde{\nu}}(\tilde{x}), \quad \tilde{x} \in \partial D_\epsilon, \\ (\lambda I + \mathcal{K}^*_D)\psi_\epsilon(x) + \dfrac{\partial \mathcal{S}_{D_\epsilon} \tilde{\phi}_\epsilon}{\partial \nu}(x) = -\dfrac{\partial H_\epsilon}{\partial \nu}(x), \quad x \in \partial D. \end{cases} \tag{3.2.4}$$

3.2.2 Asymptotic expansions of layer potentials

It is proved in Chap. 2 that

$$\tilde{\nu}(\tilde{x}) = \nu(x) - \epsilon h'(t) T(x) + O(\epsilon^2), \quad \text{uniformly } x \in \partial D. \tag{3.2.5}$$

Moreover, the following uniformly convergent expansion for the length element $d\sigma_\epsilon(\tilde{y})$ can be obtained easily:

$$d\sigma_\epsilon(\tilde{y}) = (1 - \epsilon\tau(y)h(y) + O(\epsilon^2))d\sigma(y). \tag{3.2.6}$$

Let Ψ_ϵ be the diffeomorphism from ∂D onto ∂D_ϵ given by $\Psi_\epsilon(x) = x + \epsilon h(t)\nu(x)$, where $x = X(t)$. We begin by recalling the following asymptotic formula for $\mathcal{K}^*_{D_\epsilon}$.

Lemma 3.2.3 *Let $N \in \mathbb{N}$. There exists C depending only on N, the $\mathcal{C}^2$-norm of ∂D and the $\mathcal{C}^1$-norm of h such that for any $\tilde{\phi} \in L^2(\partial D_\epsilon)$,*

$$||(\mathcal{K}^*_{D_\epsilon}\tilde{\phi}) \circ \Psi_\epsilon - \mathcal{K}^*_D \phi - \sum_{n=0}^{N-1} \epsilon^{n+1} \mathcal{K}_D^{(n+1)} \phi||_{L^2(\partial D)} \leq C\epsilon^{N+1} ||\phi||_{L^2(\partial D)},$$

where $\phi := \tilde{\phi} \circ \Psi_\epsilon$. The operators $\mathcal{K}_D^{(n+1)}$ are compact on $L^2(\partial D)$ and can be explicitly computed.

In particular, $\mathcal{K}_D^{(1)}$ is given by

$$\begin{aligned} &\mathcal{K}_D^{(1)}\phi(x) \\ &= -\mathcal{K}^*_D(\tau h \phi)(x) - \frac{1}{\pi}\int_{\partial D} \frac{\langle x - y, \nu(x)\rangle \langle x - y, h(t)\nu(x) - h(s)\nu(y)\rangle}{|x - y|^4} \phi(y) d\sigma(y) \\ &+ \frac{1}{2\pi}\int_{\partial D} \frac{\langle h(t)\nu(x) - h(s)\nu(y), \nu(x)\rangle}{|x - y|^2} \phi(y) d\sigma(y) - \frac{h'(t)}{2\pi}\int_{\partial D} \frac{\langle x - y, T(x)\rangle}{|x - y|^2} \phi(y) d\sigma(y). \end{aligned}$$

If $\phi \in \mathcal{C}^{1,\beta}(\partial D)$, then we get

$$
\begin{aligned}
&\mathcal{K}_D^{(1)}\phi(x) = -\mathcal{K}_D^*(\tau h\phi)(x) \\
&+ \frac{h(x)}{2\pi}\Big[\int_{\partial D}\frac{1}{|x-y|^2}\phi(y)d\sigma(y) - 2\int_{\partial D}\frac{\langle x-y,\nu(x)\rangle^2}{|x-y|^4}\phi(y)d\sigma(y)\Big] \\
&- \frac{1}{2\pi}\Big[\int_{\partial D}\frac{\langle \nu(x),\nu(y)\rangle}{|x-y|^2}h(y)\phi(y)d\sigma(y) - 2\int_{\partial D}\frac{\langle x-y,\nu(x)\rangle\langle x-y,\nu(y)\rangle}{|x-y|^4}h(y)\phi(y)d\sigma(y)\Big] \\
&- \frac{h'(t)}{2\pi}\int_{\partial D}\frac{\langle x-y,T(x)\rangle}{|x-y|^2}\phi(y)d\sigma(y) \\
&= -\mathcal{K}_D^*(\tau h\phi)(x) + h(t)\langle D^2(\mathcal{S}_D\phi)(x)\nu(x),\nu(x)\rangle + \frac{\partial(\mathcal{D}_D(h\phi))}{\partial\nu}(x) - h'(t)\frac{\partial(\mathcal{S}_D\phi)}{\partial T}(x).
\end{aligned}
\tag{3.2.7}
$$

Let $\psi \in \mathcal{C}^{1,\beta}(\partial D)$. Let us now derive an asymptotic expansion of $\frac{\partial(\mathcal{S}_D\psi)}{\partial\tilde{\nu}}(\tilde{x})$ for $\tilde{x} \in \partial D_\epsilon$. Since ∂D is $\mathcal{C}^{2,\beta}$, $\mathcal{S}_D\psi$ is $\mathcal{C}^{2,\beta}(\mathbb{R}^2\setminus D)$. Thus, it follows from (3.2.5) and Taylor expansion that

$$
\begin{aligned}
\frac{\partial(\mathcal{S}_D\psi)}{\partial\tilde{\nu}}(\tilde{x}) &= \tilde{\nu}(\tilde{x})\cdot\nabla\mathcal{S}_D\psi(\tilde{x}) \\
&= \Big(\nu(x) - \epsilon h'(t)T(x) + O(\epsilon^2)\Big) \\
&\quad \cdot\Big(\nabla\mathcal{S}_D\psi\big|_+(x) + \epsilon h(t)\sum_{j=1}^2 \partial_j(\nabla\mathcal{S}_D\psi(x))\nu_j(x) + O(\epsilon^{1+\beta})\Big) \\
&= \frac{\partial(\mathcal{S}_D\psi)}{\partial\nu}\bigg|_+(x) + \epsilon\Big[-h'(t)\frac{\partial(\mathcal{S}_D\psi)}{\partial T}(x) + h(t)\langle D^2(\mathcal{S}_D\psi)(x)\nu(x),\nu(x)\rangle\Big] \\
&\quad + O(\epsilon^{1+\beta}), \quad x\in\partial D,
\end{aligned}
$$

where $D^2(\mathcal{S}_D\psi)$ denotes the Hessian of $\mathcal{S}_D\psi$ and $O(\epsilon^{1+\beta})$ term is bounded by $C\epsilon^{1+\beta}\|\psi\|_{\mathcal{C}^{1,\beta}(\partial D)}$. Hence we get

$$
\frac{\partial(\mathcal{S}_D\psi)}{\partial\tilde{\nu}}(\tilde{x}) = \Big(\frac{1}{2}I + \mathcal{K}_D^*\Big)\psi(x) + \epsilon\mathcal{R}_D\psi(x) + O(\epsilon^{1+\beta}),
$$

where

$$
\mathcal{R}_D\psi(x) = -h'(t)\frac{\partial(\mathcal{S}_D\psi)}{\partial T}(x) + h(t)\langle D^2(\mathcal{S}_D\psi)(x)\nu(x),\nu(x)\rangle. \tag{3.2.8}
$$

It is easy to see that

$$
\|\mathcal{R}_D\psi\|_{\mathcal{C}^1(\partial D)} \le C\|\psi\|_{\mathcal{C}^1(\partial D)}.
$$

We now expand $\frac{\partial\mathcal{S}_{D_\epsilon}\tilde{\phi}}{\partial\nu}(x)$ for $x\in\partial D$ when $\tilde{\phi}\in\mathcal{C}^{1,\beta}(\partial D_\epsilon)$. Let f be a $\mathcal{C}^{1,\beta}$ function on ∂D and let u be the solution to $\Delta u = 0$ in D and $u = f$ on ∂D.

Then, we get

$$\begin{aligned}\int_{\partial D} \frac{\partial(\mathcal{S}_{D_\epsilon}\tilde{\phi})}{\partial \nu}(x) f(x) d\sigma(x) &= \int_{\partial D} \mathcal{S}_{D_\epsilon}\tilde{\phi}(x) \frac{\partial u}{\partial \nu}(x) d\sigma(x) \\ &= \int_{\partial D_\epsilon} \tilde{\phi}(\tilde{x}) \mathcal{S}_D\Big(\frac{\partial u}{\partial \nu}\Big)(\tilde{x}) d\sigma_\epsilon(\tilde{x}).\end{aligned}$$

Let $\phi := \tilde{\phi} \circ \Psi_\epsilon$. Then we get from (3.2.6) that

$$\begin{aligned}&\int_{\partial D} \frac{\partial(\mathcal{S}_{D_\epsilon}\tilde{\phi})}{\partial \nu}(x) f(x) d\sigma(x) \\ &= \int_{\partial D} \phi(x) \Big[\mathcal{S}_D\Big(\frac{\partial u}{\partial \nu}\Big)(x) + \epsilon h(x) \frac{\partial}{\partial \nu} \mathcal{S}_D\Big(\frac{\partial u}{\partial \nu}\Big)\Big|_+ (x) + O(\epsilon^{1+\beta})\Big] \\ &\qquad \times (1 - \epsilon\tau(x) h(x) + O(\epsilon^2)) d\sigma(x) \\ &= \int_{\partial D} (\mathcal{S}_D\phi) \frac{\partial u}{\partial \nu} d\sigma + \epsilon \int_{\partial D} \Big[- \mathcal{S}_D(\tau h \phi) + \big(\frac{1}{2} + \mathcal{K}_D\big)(h\phi)\Big] \frac{\partial u}{\partial \nu} d\sigma + O(\epsilon^{1+\beta}) \\ &= \int_{\partial D} \frac{\partial (\mathcal{S}_D\phi)}{\partial \nu}\Big|_- f d\sigma + \epsilon \int_{\partial D} \Big[- \frac{\partial(\mathcal{S}_D(\tau h\phi))}{\partial \nu}\Big|_- + \frac{\partial(\mathcal{D}_D(h\phi))}{\partial \nu}\Big] f d\sigma + O(\epsilon^{1+\beta}).\end{aligned}$$

Thus we get the following expansion:

$$\frac{\partial(\mathcal{S}_{D_\epsilon}\tilde{\phi})}{\partial \nu}(x) = \big(- \frac{1}{2} I + \mathcal{K}_D^*\big)\phi(x) + \epsilon \mathcal{L}_D \phi(x) + O(\epsilon^{1+\beta}),$$

where

$$\mathcal{L}_D \phi = \big(\frac{1}{2} I - \mathcal{K}_D^*\big)(\tau h \phi) + \frac{\partial(\mathcal{D}_D(h\phi))}{\partial \nu} \quad \text{on } \partial D. \tag{3.2.9}$$

3.2.3 Asymptotic expansions of solutions

We now return to the system of integral equations (3.2.4) and derive asymptotic expansions of $(u_\epsilon - U)|_{\partial\Omega}$ as $\epsilon \to 0$, considering separately three cases. To illustrate our method, we restrict for simplicity ourselves to the derivations of the leading-order terms in our asymptotic expansions.

Recall from [4] that

$$||\partial^i H_\epsilon||_{\mathcal{C}^l(\Omega')} \leq C, \quad i \in \mathbb{N}^2, l \in \mathbb{N},$$

for any $\Omega' \subset\subset \Omega$, where C depends only on $\mathrm{dist}(\Omega', \partial\Omega), \Omega, g, i,$ and l. Then the Taylor expansion of $\partial H_\epsilon / \partial \tilde{\nu}(\tilde{x})$ writes:

$$\frac{\partial H_\epsilon}{\partial \tilde{\nu}}(\tilde{x}) = \frac{\partial H_\epsilon}{\partial \nu}(x) + \epsilon\Big(- h'(t) \frac{\partial H_\epsilon}{\partial T}(x) + h(t) \langle D^2 H_\epsilon(x) \nu(x), \nu(x)\rangle\Big) + O(\epsilon^2), \tag{3.2.10}$$

for $\tilde{x} = \Psi_\epsilon(x), x \in \partial D$.

Using Lemma 3.2.3 and (3.2.10), it follows from (3.2.4) that, for $x \in \partial D$,

$$\begin{cases} (\lambda I - \mathcal{K}_D^*)\phi_\epsilon(x) - (\frac{1}{2} I + \mathcal{K}_D^*)\psi_\epsilon(x) - \epsilon\big[\mathcal{K}_D^{(1)}\phi_\epsilon + \mathcal{R}_D\psi_\epsilon(x)\big] + O(\epsilon^{1+\beta}) \\ \qquad = \dfrac{\partial H_\epsilon}{\partial \nu}(x) + \epsilon\Big(- h'(t)\dfrac{\partial H_\epsilon}{\partial T}(x) + h(t)\langle D^2 H_\epsilon(x)\nu(x), \nu(x)\rangle\Big) + O(\epsilon^2), \\ (\lambda I + \mathcal{K}_D^*)\psi_\epsilon(x) + (-\frac{1}{2} I + \mathcal{K}_D^*)\phi_\epsilon(x) + \epsilon\mathcal{L}_D\phi_\epsilon(x) + O(\epsilon^{1+\beta}) = -\dfrac{\partial H_\epsilon}{\partial \nu}(x), \end{cases} \tag{3.2.11}$$

where $\phi_\epsilon = \tilde{\phi}_\epsilon \circ \Psi_\epsilon$. Observe that

$$\det \begin{pmatrix} \lambda I - \mathcal{K}_D^* & -(\frac{1}{2} I + \mathcal{K}_D^*) \\ -\frac{1}{2} I + \mathcal{K}_D^* & \lambda I + \mathcal{K}_D^* \end{pmatrix} = \lambda^2 - \frac{1}{4},$$

which shows that the system of equations (3.2.11) degenerates if $k \to +\infty$ or 0. This causes the most serious difficulty in deriving asymptotic expansions of u_ϵ in extreme conductivity cases.

Since

$$\ln|x - \tilde{y}| = \ln|x - y| - \epsilon h(y)\frac{\langle x - y, \nu(y)\rangle}{|x - y|^2} + O(\epsilon^2)$$

for $\tilde{y} = y + \epsilon h(y)\nu(y) \in \partial D_\epsilon$, we get from (3.2.3) that, for any $x \in \partial\Omega$,

$$\begin{aligned} u_\epsilon(x) &= H_\epsilon(x) + \mathcal{S}_{D_\epsilon}\tilde{\phi}_\epsilon(x) + \mathcal{S}_D\psi_\epsilon(x) \\ &= H_\epsilon(x) + \mathcal{S}_D(\phi_\epsilon + \psi_\epsilon)(x) + \epsilon\big[\mathcal{D}_D(h\phi_\epsilon)(x) - \mathcal{S}_D(\tau h\phi_\epsilon)(x)\big] + O(\epsilon^2). \end{aligned} \tag{3.2.12}$$

Notice that, by adding the two equations in (3.2.11), we get

$$\begin{aligned} &\frac{1}{k-1}(\phi_\epsilon + \psi_\epsilon) + \epsilon(\mathcal{L}_D\phi_\epsilon - \mathcal{K}_D^{(1)}\phi_\epsilon - \mathcal{R}_D\psi_\epsilon) + O(\epsilon^{1+\beta}) \\ &\qquad = \epsilon\Big[- h'(t)\frac{\partial H_\epsilon}{\partial T}(x) + h(t)\langle \partial^2 H_\epsilon(x)\nu(x), \nu(x)\rangle\Big] + O(\epsilon^2), \quad x \in \partial D. \end{aligned} \tag{3.2.13}$$

3.2.4 Case 1: k fixed and $\epsilon \to 0$

Let U be the background solution defined by (3.1.2). Recall that

$$||U - H_\epsilon||_{\mathcal{C}^l(\Omega')} \leq C\epsilon,$$

for any $\Omega' \subset\subset \Omega$, where C depends only on $\mathrm{dist}(\Omega', \partial\Omega), g, \Omega$, and l. See [5].

Define

$$\begin{cases} \phi^{(0)} + \psi^{(0)} = 0 \quad \text{on } \partial D, \\ \phi^{(0)} = (1 - \dfrac{1}{k})\dfrac{\partial U}{\partial \nu} \quad \text{on } \partial D. \end{cases}$$

Then we can easily see that

$$\phi_\epsilon = \phi^{(0)} + O(\epsilon), \quad \text{and} \quad \psi_\epsilon = \psi^{(0)} + O(\epsilon).$$

According to (3.2.13), we then define $\phi^{(1)} + \psi^{(1)}$ by

$$\frac{1}{k-1}(\phi^{(1)} + \psi^{(1)}) + \Big((\mathcal{L}_D - \mathcal{K}_D^{(1)})\phi^{(0)} - \mathcal{R}_D \psi^{(0)} \Big) = -h' \frac{\partial U}{\partial T} + h \langle D^2 U \nu, \nu \rangle$$

on ∂D. Then one can see that

$$\phi_\epsilon + \psi_\epsilon = \epsilon(\phi^{(1)} + \psi^{(1)}) + O(\epsilon^{1+\beta}).$$

By (3.2.7), (3.2.8), and (3.2.9), we have

$$\Big(\mathcal{L}_D - \mathcal{K}_D^{(1)} + \mathcal{R}_D \Big) \phi = \frac{1}{2} \tau h \phi, \tag{3.2.14}$$

and hence

$$(\mathcal{L}_D - \mathcal{K}_D^{(1)})\phi^{(0)} - \mathcal{R}_D \psi^{(0)} = \Big(\mathcal{L}_D - \mathcal{K}_D^{(1)} + \mathcal{R}_D \Big) \phi^{(0)} = \frac{1}{2} \tau h \phi^{(0)}.$$

Since

$$\langle D^2 U \nu, \nu \rangle = \frac{\partial^2 U}{\partial \nu^2} = -\frac{\partial^2 U}{\partial T^2} - \tau \frac{\partial U}{\partial \nu},$$

we get

$$\frac{1}{k-1}(\phi^{(1)} + \psi^{(1)}) = -\frac{\partial}{\partial T} \Big(h \frac{\partial U}{\partial T} \Big) - \frac{3k-1}{2k} \tau h \frac{\partial U}{\partial \nu} \quad \text{on } \partial D.$$

Since $H_\epsilon(x) = -\mathcal{S}_\Omega(g)(x) + \mathcal{D}_\Omega(u_\epsilon|_{\partial\Omega})(x)$ and $U(x) = -\mathcal{S}_\Omega(g)(x) + \mathcal{D}_\Omega(U|_{\partial\Omega})(x)$ for $x \in \Omega$, it follows from (3.2.2) that

$$U(x) - H_\epsilon(x) = (\frac{1}{2} I - \mathcal{K}_\Omega)(u_\epsilon - U)(x), \quad x \in \partial\Omega.$$

It then follows from (3.2.12) that for $x \in \partial\Omega$

$$(\frac{1}{2} I - \mathcal{K}_\Omega)(u_\epsilon - U)(x) = \epsilon \big(\mathcal{S}_D(\phi^{(1)} + \psi^{(1)}) + \mathcal{D}_D(h\phi^{(0)}) - \mathcal{S}_D(\tau h \phi^{(0)}) \big) + O(\epsilon^{1+\beta}).$$

Let $N(\cdot, y)$ be the Neumann function for Δ in Ω corresponding to a Dirac mass at y, that is, N is the solution to

$$\begin{cases} \Delta_x N(x,y) = -\delta_y \quad \text{in } \Omega, \\ \dfrac{\partial N}{\partial \nu}\Big|_{\partial\Omega} = -\dfrac{1}{|\partial\Omega|}, \\ \displaystyle\int_{\partial\Omega} N(x,y) d\sigma(x) = 0 \quad \text{for } y \in \Omega. \end{cases}$$

Define, for $\phi \in L^2(\partial D)$,

$$\mathcal{N}_D \phi(x) := \int_{\partial D} N(x,y)\phi(y)d\sigma(y), \tag{3.2.15}$$

and

$$\mathcal{G}_D \phi(x) := \int_{\partial D} \frac{\partial N(x,y)}{\partial \nu(y)} \phi(y)d\sigma(y). \tag{3.2.16}$$

According to [4],

$$(\frac{1}{2}I - \mathcal{K}_\Omega)(N(\cdot,y))(x) = -\Gamma(x-y) \quad \text{modulo constants}, \quad x \in \partial\Omega, \tag{3.2.17}$$

and hence we get

$$(\frac{1}{2}I - \mathcal{K}_\Omega)(\mathcal{N}_D\phi)(x) = -\mathcal{S}_D\phi(x), \quad (\frac{1}{2}I - \mathcal{K}_\Omega)(\mathcal{G}_D\phi)(x) = -\mathcal{D}_D\phi(x), \quad x \in \partial\Omega.$$

Since $\int_{\partial\Omega}(u_\epsilon - U)d\sigma = 0$, we obtain

$$u_\epsilon(x) = U(x) - \epsilon\Big[\mathcal{N}_D(\phi^{(1)} + \psi^{(1)} - \tau h\phi^{(0)})(x) + \mathcal{G}_D(h\phi^{(0)})(x)\Big] + O(\epsilon^{1+\beta}), \quad x \in \partial\Omega.$$

In conclusion, we obtained the following theorem.

Theorem 3.2.4 *The following asymptotic formula holds as $\epsilon \to 0$:*

$$u_\epsilon(x) = U(x) - \epsilon\Big[-\mathcal{N}_D(\psi)(x) + \mathcal{G}_D(\phi)(x)\Big] + O(\epsilon^{1+\beta}), \quad x \in \partial\Omega, \tag{3.2.18}$$

where

$$\begin{aligned} \phi &= (1 - \frac{1}{k})h\frac{\partial U}{\partial \nu}, \\ \psi &= (k-1)\bigg(\frac{\partial}{\partial T}\Big(h\frac{\partial U}{\partial T}\Big) + \frac{3k+1}{2k}\tau h\frac{\partial U}{\partial \nu}\bigg). \end{aligned}$$

A few words are in order regarding the function $w := -\mathcal{N}_D(\psi) + \mathcal{G}_D(\phi)$. Since $\Delta_x N(x,y) = -\delta_y$, $N(x,y) = -\Gamma(x-y) +$ smooth function and hence $\mathcal{N}_D$ and $\mathcal{G}_D$ obey similar jump formulae as (3.2.1) and (3.2.2). Therefore, w is the solution of

$$\begin{cases} \Delta w = 0 & \text{in } D \cup (\Omega \setminus \overline{D}), \\ w|_+ - w|_- = \phi & \text{on } \partial D, \\ \dfrac{\partial w}{\partial \nu}\Big|_+ - \dfrac{\partial w}{\partial \nu}\Big|_- = \psi & \text{on } \partial D, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial D. \end{cases}$$

3.2.5 Case 2: $\epsilon \to 0$ and $k\epsilon \to \alpha$ with $0 < \alpha < +\infty$

Expanding

$$\begin{cases} \phi_\epsilon = \phi_\infty^{(0)} + \epsilon\phi_\infty^{(1)} + \epsilon^2\phi_\infty^{(2)} + \dots, \\ \psi_\epsilon = \psi_\infty^{(0)} + \epsilon\psi_\infty^{(1)} + \epsilon^2\psi_\infty^{(2)} + \dots, \end{cases}$$

formula (3.2.13) yields

$$\begin{aligned} (\phi_\infty^{(0)} + \psi_\infty^{(0)})(x) \quad & + \alpha((\mathcal{L}_D - \mathcal{K}_D^{(1)})\phi_\infty^{(0)}(x) - \mathcal{R}_D\psi_\infty^{(0)}(x)) \\ & = \alpha(-h'(t)\frac{\partial H_\infty}{\partial T}(x) + h(t)\frac{\partial^2 H_\infty}{\partial \nu^2}(x)), \end{aligned} \tag{3.2.19}$$

where $H_\infty = -\mathcal{S}_\Omega g + \mathcal{D}_\Omega u_\infty$, and u_∞ is the (formal) limit of $u_\epsilon|_{\partial\Omega}$.

From the first equation in (3.2.11), we obtain after sending $\epsilon \to 0$ and $k\epsilon \to \alpha$ that

$$\frac{1}{2}(\phi_\infty^{(0)} - \psi_\infty^{(0)})(x) - \mathcal{K}_D^*(\phi_\infty^{(0)} + \psi_\infty^{(0)})(x) = \frac{\partial H_\infty}{\partial \nu}(x), \quad x \in \partial D. \tag{3.2.20}$$

Using (3.2.14), it follows from (3.2.19) and (3.2.20) that

$$\begin{aligned} & \Big((1 + \frac{\alpha\tau h}{4})I + \frac{\alpha\tau h}{2}\mathcal{K}_D^* - \alpha\mathcal{R}_D\Big)(\phi_\infty^{(0)} + \psi_\infty^{(0)})(x) \\ & = \alpha\Big[-h'(t)\frac{\partial H_\infty}{\partial T}(x) + h(t)(\frac{\partial^2 H_\infty}{\partial \nu^2} - \frac{\tau}{2}\frac{\partial H_\infty}{\partial \nu})(x)\Big], \quad x \in \partial D. \end{aligned} \tag{3.2.21}$$

It is easy to see that for α small enough $(1 + \frac{\alpha\tau h}{4})I + \frac{\alpha\tau h}{2}\mathcal{K}_D^{(1)} - \alpha\mathcal{R}_D$ is invertible and therefore,

$$u_\epsilon(x) \quad = H_\infty(x) + \mathcal{S}_D(\phi_\infty^{(0)} + \psi_\infty^{(0)})(x) + O(\epsilon), \quad x \in \Omega, \tag{3.2.22}$$

where

$$\begin{aligned} \phi_\infty^{(0)} + \psi_\infty^{(0)} = & \Big((1 + \frac{\alpha\tau h}{4})I + \frac{\alpha\tau h}{2}\mathcal{K}_D^* - \alpha\mathcal{R}_D\Big)^{-1}\Big(-h'(t)\frac{\partial H_\infty}{\partial T} \\ & + h(t)(\frac{\partial^2 H_\infty}{\partial \nu^2} - \frac{\tau}{2}\frac{\partial H_\infty}{\partial \nu})\Big) := \mathcal{L}_\infty(u_\infty). \end{aligned}$$

Formula (3.2.22) shows that

$$(\frac{1}{2}I - \mathcal{K}_\Omega)u_\infty = -\mathcal{S}_\Omega(g) + \mathcal{S}_D\mathcal{L}_\infty(u_\infty) \quad \text{on } \partial\Omega.$$

By (3.2.17),

$$(\frac{1}{2} - \mathcal{K}_\Omega)\Big((I + \mathcal{N}_D\mathcal{L}_\infty)(u_\infty) - U|_{\partial\Omega}\Big) = O(\epsilon),$$

where $\mathcal{N}_D$ is defined by (3.2.15), and thus, since $\frac{1}{2}I - \mathcal{K}_\Omega : L^2_0(\partial D) \longrightarrow L^2_0(\partial D)$ is invertible we deduce that

$$(I + \mathcal{N}_D\mathcal{L}_\infty)(u_\infty) = U|_{\partial\Omega} + O(\epsilon).$$

Since $I + \mathcal{N}_D$ is invertible on $L^2_0(\partial\Omega)$, it is easy to prove that for α small enough $I + \mathcal{N}_D \mathcal{L}_\infty$ is invertible and obtain the following formula.

Theorem 3.2.5 *The following asymptotic formula holds as $\epsilon \to 0$, and $k\epsilon \to \alpha$:*

$$u_\epsilon(x) = (I + \mathcal{N}_D \mathcal{L}_\infty)^{-1}(U|_{\partial\Omega})(x) + O(\epsilon), \quad x \in \partial\Omega.$$

3.2.6 Case 3: $\epsilon \to 0$ and $k^{-1}\epsilon \to \delta$ with $0 < \delta < +\infty$

We begin by rewriting the system of equations (3.2.11) as follows:

$$\begin{cases} \frac{k}{k-1}\phi_\epsilon(x) - (\frac{1}{2}I + \mathcal{K}_D^*)(\phi_\epsilon + \psi_\epsilon)(x) + \epsilon\Big[- \mathcal{K}_D^{(1)}\phi_\epsilon - \mathcal{R}_D\psi_\epsilon\Big](x) + O(\epsilon^{1+\beta}) = \frac{\partial H_\epsilon}{\partial \nu}(x) \\ \qquad + \epsilon\Big(- h'(t)\frac{\partial H_\epsilon}{\partial T}(x) + h(t)\langle \partial^2 H_\epsilon(x)\nu(x), \nu(x)\rangle\Big) + O(\epsilon^2), \quad x \in \partial D, \\ \frac{k}{k-1}\psi_\epsilon(x) + (-\frac{1}{2}I + \mathcal{K}_D^*)(\phi_\epsilon + \psi_\epsilon)(x) + \epsilon\mathcal{L}_D\phi_\epsilon(x) + O(\epsilon^{1+\beta}) = -\frac{\partial H_\epsilon}{\partial \nu}(x), \quad x \in \partial D. \end{cases} \tag{3.2.23}$$

If we substitute the expansions

$$\begin{cases} H_\epsilon = H_0^{(0)} + \epsilon H_0^{(1)} + \epsilon^2 H_0^{(2)} + \ldots, \\ \phi_\epsilon = \frac{1}{\epsilon}\phi_0^{(-1)} + \phi_0^{(0)} + \epsilon\phi_0^{(1)} + \epsilon^2\phi_0^{(2)} + \ldots, \\ \psi_\epsilon = \frac{1}{\epsilon}\psi_0^{(-1)} + \psi_0^{(0)} + \epsilon\psi_0^{(1)} + \epsilon^2\psi_0^{(2)} + \ldots, \end{cases}$$

into (3.2.23) we formally get

$$\Big(\frac{1}{\delta} + (\frac{1}{2}I + \mathcal{K}_D^*)(\frac{1}{2}\tau h) + (\mathcal{K}_D^{(1)} - \mathcal{R}_D)\Big)\phi_0^{(-1)} = -\frac{\partial H_0^{(0)}}{\partial \nu} \quad \text{on } \partial D,$$

and

$$\psi_0^{(-1)} = -\phi_0^{(-1)} \quad \text{on } \partial D.$$

Moreover,

$$\phi_0^{(0)} + \psi_0^{(0)} = \frac{1}{2}\tau h \phi^{(-1)} \quad \text{on } \partial D,$$

and

$$H_0^{(0)} = -\mathcal{S}_\Omega g + \mathcal{D}_\Omega(u_0^{(0)}) \quad \text{in } \Omega,$$

where $u_0^{(0)}$ is the limit of $u_\epsilon|_{\partial\Omega}$.

It is easy to see that for δ small enough the above equations for $\phi_0^{(-1)}$ and $\psi_0^{(-1)}$ can be solved.

Following the same arguments as in Theorem 3.2.5, we define

$$\begin{aligned} \mathcal{L}_0(u_0^{(0)})(x) \quad &:= \quad -\Big[\frac{1}{2\pi}\int_{\partial D} h(s)\frac{\langle x-y, \nu(y)\rangle}{|x-y|^2}\phi_0^{(-1)}(y)d\sigma(y) \\ &+ \quad \frac{1}{2\pi}\int_{\partial D} \ln|x-y|\phi_0^{(-1)}(y)\tau(s)h(s)d\sigma(y)\Big], \quad x \in \partial\Omega, \end{aligned}$$

where

$$\phi_0^{(-1)} = -\left(\frac{1}{\delta} + (\frac{1}{2}I + \mathcal{K}_D^*)(\frac{1}{2}\tau h) + (\mathcal{K}_D^{(1)} - \mathcal{R}_D)\right)^{-1} (\frac{\partial H_0^{(0)}}{\partial \nu}),$$

it follows that

$$(\frac{1}{2}I - \mathcal{K}_\Omega)u_0^{(0)} = -\mathcal{S}_\Omega(g) + \mathcal{L}_0(u_0^{(0)}) \quad \text{on } \partial\Omega,$$

and therefore,

$$u_0^{(0)} = \left(I + (\frac{1}{2}I - \mathcal{K}_\Omega)^{-1}\mathcal{L}_0\right)^{-1} (U|_{\partial\Omega}) \quad \text{on } \partial\Omega.$$

Theorem 3.2.6 *The following asymptotic formula holds as* $\epsilon \to 0$, *and* $k^{-1}\epsilon \to \delta$ *and* δ *is small enough:*

$$u_\epsilon(x) = \left(I + (\frac{1}{2}I - \mathcal{K}_\Omega)^{-1}\mathcal{L}_0\right)^{-1} (U|_{\partial\Omega}) + O(\epsilon), \quad x \in \partial\Omega.$$

Chapter 4

Eigenvalues of the Laplacian in Domains with Small Inclusions

4.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with a connected Lipschitz boundary $\partial\Omega$. Let ν denote the unit outward normal to $\partial\Omega$. Suppose that Ω contains a small inclusion D, of the form $D = z + \epsilon B$, where B is a bounded Lipschitz domain in $\mathbb{R}^d$ containing the origin. We also assume that the "background" is homogeneous with conductivity 1. The inclusions, somewhat apart from or nearly touching the boundary, are of arbitrary conductivity contrast vis-à-vis the background domain, with the limiting perfectly conducting case.

Our goal is to find *complete asymptotic expansions* for the eigenvalues of such a domain which had not been established before this work, with the intention of using the expansions as an aid in identifying the inclusions. That is, we would like to find a method for determining the locations and/or shape of small inclusions by taking eigenvalue measurements.

Rauch and Taylor [77] have shown that the spectrum of a bounded domain does not change after imposing Dirichlet conditions on compact subset of capacity zero. After that, many people have studied the asymptotic expansion of the eigenvalues for the case of small holes with the Dirichlet or the Neumann boundary condition. In particular, Ozawa provided in a series of papers [75]-[70] leading-order terms in eigenvalue expansions, see also [98] and [64]. Besson [20] has proved the existence of a complete expansion of the eigenvalue perturbation in the two-dimensional case. Courtois [28] has established a perturbation theory for the Dirichlet spectrum in a compactly perturbed domain in terms of the capacity of the compact perturbation. We shall also mention, in connection with eigenvalue changes under variation of domains, the works by Kato [52], Sanchez Hubert and Sanchez Palencia [81], Ward and Keller [98], Gadyl'shin and Il'in [43], Daners [31], McGillivray [65], and Noll [69].

In this chapter we provide a rigorous derivation of complete asymptotic expansions for eigenvalues of the Laplacian in domains with small inclusions. The inclusions are of arbitrary shape and of arbitrary conductivity contrast. A key difference in our work is the approach we develop: a boundary integral approach with rigorous justification based on the generalized Rouché's theorem. By using layer potential techniques we show that the square roots of the eigenvalues are the characteristic values of meromorphic operator-valued functions that are of Fredholm type with index 0. We then proceed from the generalized Rouché's theorem to construct their complete asymptotic expressions. To the best of our knowledge, the idea of reducing the eigenvalue problem to the study of characteristic values of some integral operators has been introduced by Russian authors; see [83] and the references listed there. See also [16, 3].

In this chapter we confine our attention to the eigenvalues of the Neumann boundary value problem in the bounded domain Ω. The eigenvalue problem with the Dirichlet boundary condition is of equal interest. The asymptotic results for the eigenvalues in such case can be obtained with only minor modifications of the techniques presented here, while the rigorous derivations of similar asymptotic formulae for the full Maxwell's equations or for the equations of linear elasticity require further work.

4.2 Notation and Preliminaries

4.2.1 The generalized Rouché's theorem

In this work the approach we develop is a boundary integral technique with rigorous justification based on the generalized Rouché's theorem. For readers' convenience we recall this theorem due to Ghoberg and Sigal in [44]. We begin by collecting some notations.

Let $\mathcal{H}$ and $\mathcal{H}'$ be two Banach spaces, and let $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ be the algebra of all bounded vector-valued functions acting from $\mathcal{H}$ into $\mathcal{H}'$.

Let ω_0 be a fixed complex value in $\mathbb{C}$. We denote by $\mathcal{A}(\omega)$ an operator-valued function acting from $V_\delta(\omega_0)$ into $\mathcal{L}(\mathcal{H}, \mathcal{H}')$, where $V_\delta(\omega_0)$ is a disc of center ω_0 and radius $\delta > 0$.

ω_0 is called a *characteristic* value of $\mathcal{A}(\omega)$ if

(i) $\mathcal{A}(\omega)$ is holomorphic in some neighborhood of ω_0, except possibly at this point itself;

(ii) there exists a vector-valued function $\phi(\omega)$: $V_\delta(\omega_0) \to \mathcal{H}$ holomorphic at ω_0 and that verifies $\phi(\omega_0) \neq 0$, such that $\mathcal{A}(\omega)\phi(\omega)$ is a holomorphic at ω_0 and vanishes at this point. $\phi(\omega)$ is called a *root* function of $\mathcal{A}(\omega)$ associated with ω_0, and the vector $\phi_0 = \phi(\omega_0)$ is called an eigenvector. The closure of the linear set of eigenvectors corresponding to ω_0 is denoted by $\mathrm{Ker}\mathcal{A}(\omega_0)$.

Suppose that ω_0 is a characteristic value of the function $\mathcal{A}(\omega)$ and $\phi(\omega)$ is a root function satisfying (ii). Then there exists a number $m(\phi) \geq 1$ and a vector-valued function $\psi(\omega) : V_\delta(\omega_0) \to \mathcal{H}$ holomorphic such that

$$\mathcal{A}(\omega)\phi(\omega) = (\omega - \omega_0)^{m(\phi)}\psi(\omega), \quad \psi(\omega_0) \neq 0.$$

The number $m(\phi)$ is called the *multiplicity* of the root function $\phi(\omega)$. Let ϕ_0 be an eigenvector corresponding to ω_0 and let

$$\mathcal{R}(\phi_0) = \{m(\phi); \phi(\omega) \text{ is a root function such } \phi(\omega_0) = \phi_0\}.$$

Then by rank of ϕ_0 we mean $\mathrm{rank}(\phi_0) = \max \mathcal{R}(\phi_0)$.

Suppose that $n = \dim\mathrm{Ker}\mathcal{A}(\omega_0) < +\infty$ and that the ranks of all vectors in $\mathrm{Ker}\mathcal{A}(\omega_0)$ are finite. A system of eigenvectors ϕ_0^j, $j = 1, \ldots, n$, is called a *canonical system of eigenvectors* of $\mathcal{A}(\omega)$ associated to ω_0 if the ranks possess the following property: $\mathrm{rank}(\phi_0^j)$ is the maximum of the ranks of all eigenvectors in some direct complement in $\dim\mathrm{Ker}\mathcal{A}(\omega_0)$ of the linear span of the vectors $\phi_0^1, \ldots, \phi_0^{j-1}$. Let $r_j = \mathrm{rank}(\phi_0^j)$. We call

$$N(\mathcal{A}(\omega_0)) = \sum_{j=1}^{n} r_j$$

the *null multiplicity of the characteristic* value ω_0 of $\mathcal{A}(\omega)$. If ω_0 is not a characteristic value of $\mathcal{A}(\omega)$, we put $N(\mathcal{A}(\omega_0)) = 0$.

Suppose that $\mathcal{A}^{-1}(\omega)$ exists and is holomorphic in some neighborhood of ω_0, except possibly at this point itself. Then the number

$$M(\mathcal{A}(\omega_0)) = N(\mathcal{A}(\omega_0)) - N(\mathcal{A}^{-1}(\omega_0))$$

is called the *multiplicity* of the characteristic value ω_0 of $\mathcal{A}(\omega)$. Suppose that ω_1 is a pole of the operator-valued function. The Laurent expansion of $\mathcal{A}(\omega)$ in ω_1 is given by

$$\mathcal{A}(\omega) = \sum_{j \geq -s} (\omega - \omega_1)^j A_j.$$

If in the last expression the operators A_{-j}, $j = 1, \ldots, s$, are finite-dimensional, then $\mathcal{A}(\omega)$ is called *finitely meromorphic* at ω_1.

The operator-valued function $\mathcal{A}(\omega)$ is said to be of Fredholm type at the point ω_1 if the operator A_0 in the last expansion is a Fredholm operator. If $\mathcal{A}(\omega)$ is holomorphic at the point ω_0 and the operator $\mathcal{A}(\omega_0)$ is invertible, then ω_0 is called a *regular point* of $\mathcal{A}(\omega)$.

The point ω_0 is called a *normal point* of $\mathcal{A}(\omega)$ if there exists a constant $0 < \delta_0$ such that $\mathcal{A}(\omega)$ is finitely meromorphic and of Fredholm type at ω_0 and all the points of a disc of center ω_0 and radius $\delta_0 > 0$ except ω_0 are regular for $\mathcal{A}(\omega)$.

Lemma 4.2.1 *Every normal point ω_0 of $\mathcal{A}(\omega)$ is a normal point of $\mathcal{A}^{-1}(\omega)$. If, in addition, ω_0 is a pole of either $\mathcal{A}(\omega)$ or $\mathcal{A}^{-1}(\omega)$, then it is a characteristic value of finite multiplicity of the other.*

Let ∂V be the contour bounding the domain V. An operator-valued function $\mathcal{A}(\omega)$ which is finitely meromorphic and of Fredholm type in V and continuous at ∂V is called *normal with respect to* ∂V if the operator $\mathcal{A}(\omega)$ is invertible in $\overline{V}$, except for a finite number of points of V which are normal points of $\mathcal{A}(\omega)$. Now, if $\mathcal{A}(\omega)$ is normal with respect to the contour ∂V and ω_i, $i = 1, \ldots, \sigma$, are all its characteristic values and poles lying in V, we put

$$\mathcal{M}(\mathcal{A}(\omega); \partial V) = \sum_{i=1}^{\sigma} M(\mathcal{A}(\omega_i)).$$

Theorem 4.2.2 *Suppose that the operator-valued $\mathcal{A}(\omega)$ is normal with respect to ∂V; then we have*

$$\mathcal{M}(\mathcal{A}(\omega); \partial V) = \frac{1}{2i\pi} \mathrm{tr} \int_{\partial V} \mathcal{A}^{-1}(\omega) \frac{d}{d\omega} \mathcal{A}(\omega) d\omega.$$

By 'tr' we mean the trace of operator which is the sum of all its nonzero characteristic values, see [44, p. 609] for an exact statement. We mention the following property of the trace:

$$\mathrm{tr} \int_{\partial V} \mathcal{A}(\omega)\mathcal{B}(\omega) d\omega = \mathrm{tr} \int_{\partial V} \mathcal{B}(\omega)\mathcal{A}(\omega) d\omega, \tag{4.2.1}$$

where $\mathcal{A}(\omega)$ and $\mathcal{B}(\omega)$ are operator-valued functions which are finitely meromorphic in the neighborhood V of ω_0, which contains no poles of $\mathcal{A}(\omega)$ and $\mathcal{B}(\omega)$ other than ω_0.

The operator generalization of the Rouché theorem is stated below.

Theorem 4.2.3 *Let $\mathcal{A}(\omega)$ be an operator-valued function which is normal with respect to ∂V. If an operator-valued function $S(\omega)$ which is finitely meromorphic in V and continuous at ∂V satisfies the condition*

$$|\mathcal{A}^{-1}(\omega) S(\omega)|_{\mathcal{L}(\mathcal{H},\mathcal{H})} < 1, \quad \omega \in \partial V,$$

then $\mathcal{A}(\omega) + S(\omega)$ is also normal with respect to ∂V, and

$$\mathcal{M}(\mathcal{A}(\omega); \partial V) = \mathcal{M}(\mathcal{A}(\omega) + S(\omega); \partial V).$$

The generalization of the Steinberg theorem is given by the following.

Theorem 4.2.4 *Suppose that $\mathcal{A}(\omega)$ is an operator-valued function which is finitely meromorphic and of Fredholm type in the domain V. If the operator $\mathcal{A}(\omega)$ is invertible at one point of V, then $\mathcal{A}(\omega)$ has a bounded inverse for all $\omega \in V$, except possibly for certain isolated points.*

Finally, the following result due to Ghoberg and Sigal in [44] is central.

Theorem 4.2.5 *Suppose that $\mathcal{A}(\omega)$ is an operator-valued function which is normal with respect to ∂V. Let $f(\omega)$ be a scalar function which is analytic in V and continuous in $\overline{V}$. Then*

$$\frac{1}{2i\pi} \operatorname{tr} \int_{\partial V} f(\omega)\mathcal{A}^{-1}(\omega)\frac{d}{d\omega}\mathcal{A}(\omega)d\omega = \sum_{j=1}^{\sigma} M(\mathcal{A}(\omega_j))f(\omega_j),$$

where ω_j, $j = 1, \ldots, \sigma$, are all the points in V which are either poles or characteristic values of $\mathcal{A}(\omega)$.

4.2.2 Layer potentials for the Helmholtz equation

We will develop a boundary integral formulation for solving the eigenvalue problem. The integral equations applying to this problem will be obtained from a study of the layer potentials for the Helmholtz equation.

For $\omega > 0$, a fundamental solution $\Gamma_\omega(x)$ to the Helmholtz operator $\Delta + \omega^2$ in $\mathbb{R}^d, d = 2, 3$, is given by

$$\Gamma_\omega(x) = \begin{cases} -\dfrac{i}{4}H_0^{(1)}(\omega|x|)\,, & d = 2, \\ -\dfrac{e^{i\omega|x|}}{4\pi|x|}\,, & d = 3, \end{cases}$$

for $x \neq 0$, where $H_0^{(1)}$ is the Hankel function of the first kind of order 0.

For a bounded Lipschitz domain D in $\mathbb{R}^d$ and $\omega > 0$ let $\mathcal{S}_D^\omega$ and $\mathcal{D}_D^\omega$ be the single and double layer potentials defined by Γ_ω, that is,

$$\mathcal{S}_D^\omega\varphi(x) = \int_{\partial D} \Gamma_\omega(x-y)\varphi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d,$$
$$\mathcal{D}_D^\omega\varphi(x) = \int_{\partial D} \frac{\partial \Gamma_\omega(x-y)}{\partial \nu(y)}\varphi(y)\, d\sigma(y)\,, \quad x \in \mathbb{R}^d \setminus \partial D,$$

for $\varphi \in L^2(\partial D)$.

The following formulae give the jump relations obeyed by the double layer potential and by the normal derivative of the single layer potential on general Lipschitz domains:

$$\frac{\partial(\mathcal{S}_D^\omega\varphi)}{\partial\nu}\bigg|_\pm(x) = \Big(\pm\frac{1}{2}I + (\mathcal{K}_D^\omega)^*\Big)\varphi(x) \quad \text{a.e. } x \in \partial D, \tag{4.2.2}$$

$$(\mathcal{D}_D^\omega\varphi)\bigg|_\pm(x) = \Big(\mp\frac{1}{2}I + \mathcal{K}_D^\omega\Big)\varphi(x) \quad \text{a.e. } x \in \partial D, \tag{4.2.3}$$

for $\varphi \in L^2(\partial D)$, where $\mathcal{K}_D^\omega$ is the operator defined by

$$\mathcal{K}_D^\omega\varphi(x) = \text{p.v.}\int_{\partial D} \frac{\partial \Gamma_\omega(x-y)}{\partial \nu(y)}\varphi(y)d\sigma(y),$$

and $(\mathcal{K}_D^\omega)^*$ is the L^2-adjoint of $\mathcal{K}_D^\omega$, that is,

$$(\mathcal{K}_D^\omega)^*\varphi(x) = \text{p.v.} \int_{\partial D} \frac{\partial \Gamma_\omega(x-y)}{\partial \nu(x)} \varphi(y) d\sigma(y).$$

Here p.v. stands for the Cauchy principal value. The singular integral operators $\mathcal{K}_D^\omega$ and $(\mathcal{K}_D^\omega)^*$ are known to be bounded on $L^2(\partial D)$.

Let $0 < \mu_1 \leq \mu_2 \leq \dots$ be the eigenvalues of $-\Delta$ in Ω with the Neumann condition on $\partial\Omega$. Let u_j denote the normalized eigenfunction associated with μ_j, that is, it satisfies $||u_j||_{L^2(\Omega)} = 1$. It is well known that $\{\sqrt{\mu_j}\}_{j\geq 1}$ are exactly the real characteristic values of the operator-valued function $\omega \mapsto (1/2)I - \mathcal{K}_\Omega^\omega$. See [91].

Let $\omega \notin \{\sqrt{\mu_j}\}_{j\geq 1}$. Introduce $N_\Omega^\omega(x,z)$ to be the Neumann function for $\Delta + \omega^2$ in Ω corresponding to a Dirac mass at z. That is, N_Ω^ω is the unique solution to

$$\begin{cases} (\Delta_x + \omega^2) N_\Omega^\omega(x,z) = -\delta_z & \text{in } \Omega, \\ \dfrac{\partial N_\Omega^\omega}{\partial \nu}\Big|_{\partial\Omega} = 0 & \text{on } \partial\Omega. \end{cases}$$

The following identity from [5] relates the fundamental solution Γ_ω to the Neumann function N_Ω^ω:

$$-\left(\frac{1}{2}I - \mathcal{K}_\Omega^\omega\right)^{-1} (\Gamma_\omega(\cdot - z))(x) = N_\Omega^\omega(x,z), \quad x \in \partial\Omega,\ z \in \Omega. \tag{4.2.4}$$

The spectral decomposition,

$$N_\Omega^\omega(x,z) = \sum_{j=1}^{+\infty} \frac{u_j(x)\bar{u_j}(z)}{\omega^2 - \mu_j}, \tag{4.2.5}$$

will be of use to us. We refer the reader to [78, p. 246] for its proof.

Finally, we shall recall the concept of capacity. Suppose $d = 2$ and let $(\varphi_e, a) \in L^2(\partial D) \times \mathbb{R}$ denote the unique solution of the system

$$\begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{\partial D} \ln|x-y| \varphi_e(y) d\sigma(y) + a = 0, \quad \text{on } \partial D, \\ \displaystyle\int_{\partial D} \varphi_e(y) d\sigma(y) = 1. \end{cases} \tag{4.2.6}$$

The logarithmic capacity of ∂D is defined by

$$\text{cap}(\partial D) = e^{2\pi a},$$

where a is given by (4.2.6).

If $d = 3$, there exists a unique $\varphi_e \in L^2(\partial D)$ such that

$$\begin{cases} \displaystyle\int_{\partial D} \frac{\varphi_e(y)}{|x-y|} d\sigma(y) = \text{constant}, \quad \text{on } \partial D, \\ \displaystyle\int_{\partial D} \varphi_e(y) d\sigma(y) = 1. \end{cases} \tag{4.2.7}$$

The capacity of ∂D in three dimensions is defined to be

$$\frac{1}{\mathrm{cap}(\partial D)} = -\frac{1}{4\pi}\int_{\partial D}\frac{1}{|x-y|}\varphi_e(y)d\sigma(y). \tag{4.2.8}$$

4.3 Eigenvalue perturbations caused by small perfectly conducting inclusions

Suppose that the inclusion D is perfectly conducting. Let $0 < \mu_1^\epsilon \leq \mu_2^\epsilon \leq \ldots$ be the eigenvalues of $-\Delta$ in $\Omega_\epsilon := \Omega \setminus \overline{D}$ with the the Neumann condition on $\partial\Omega$ and the Dirichlet condition on ∂D. We arrange them repeatedly according to their multiplicity.

Fix j and suppose that the eigenvalue μ_j is simple. Note that this assumption is not essential in what follows, though its genericity is confirmed in [1, 94]. It is made for ease of exposition. Then there exists a simple eigenvalue μ_j^ϵ near μ_j associated to the normalized eigenfunction u_j^ϵ, that is, u_j^ϵ satisfies the following problem:

$$\begin{cases} \Delta u_j^\epsilon + \mu_j^\epsilon u_j^\epsilon = 0 & \text{in } \Omega_\epsilon, \\ \dfrac{\partial u_j^\epsilon}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u_j^\epsilon = 0 & \text{on } \partial D. \end{cases} \tag{4.3.1}$$

From [6], we know that the solution u_j^ϵ of (4.3.1) can be represented as

$$u_j^\epsilon(x) = \mathcal{D}_\Omega^{\sqrt{\mu_j^\epsilon}}(u_j^\epsilon|_{\partial\Omega})(x) + \mathcal{S}_D^{\sqrt{\mu_j^\epsilon}}(\phi)(x), \quad x \in \Omega_\epsilon,$$

where $\psi := u_j^\epsilon|_{\partial\Omega} \in L^2(\partial\Omega)$ and $\phi \in L^2(\partial D)$ satisfy the following system of integral equations:

$$\begin{cases} \left(\frac{1}{2}I - \mathcal{K}_\Omega^{\sqrt{\mu_j^\epsilon}}\right)(\psi)(x) - \mathcal{S}_D^{\sqrt{\mu_j^\epsilon}}(\phi)(x) = 0, & x \in \partial\Omega, \\ \mathcal{D}_\Omega^{\sqrt{\mu_j^\epsilon}}(\psi)(x) + \mathcal{S}_D^{\sqrt{\mu_j^\epsilon}}(\phi)(x) = 0, & x \in \partial D. \end{cases} \tag{4.3.2}$$

Our strategy for deriving complete asymptotic expansions of the perturbations in the eigenvalues relies on expanding the operator-valued function

$$\omega \mapsto \begin{pmatrix} \left(\frac{1}{2}I - \mathcal{K}_\Omega^\omega\right) & -\mathcal{S}_D^\omega \\ \mathcal{D}_\Omega^\omega & \mathcal{S}_D^\omega \end{pmatrix}$$

in terms of ϵ and then, on calculating the asymptotic expressions of its characteristic values with the help of the generalized Rouché's theorem.

4.3.1 Inclusions far away from the boundary

We assume that the inclusion D is separated from the boundary. More precisely, we assume that there exists a constant $c_0 > 0$ such that $\operatorname{dist}(z, \partial\Omega) \geq 2c_0 > 0$, that ϵ, the order of magnitude of the diameter of the inclusion, is sufficiently small, that the distance of the inclusion to $\mathbb{R}^d \setminus \overline{\Omega}$ is larger than c_0.

We need the following lemma.

Lemma 4.3.1 *Let $\psi \in L^2(\partial\Omega)$ and let $\varphi \in L^2(\partial D)$. Define $\tilde{\varphi}(x) = \epsilon\varphi(\epsilon x + z)$, $x \in \partial B$. Then, for $x \in \partial B$, we have*

$$\mathcal{S}_D^\omega(\varphi)(\epsilon x + z)$$
$$= \frac{1}{2\pi}\sum_{n=0}^{+\infty}(-1)^n \frac{(\omega\epsilon)^{2n}}{2^{2n}(n!)^2}\int_{\partial B}|x-y|^{2n}\left(\ln(\omega\epsilon|x-y|) + \ln\gamma - \sum_{j=1}^{n}\frac{1}{j}\right)\tilde{\varphi}(y)d\sigma(y),$$

for $d = 2$, where $2\gamma = e^{\tilde{\gamma} - i\pi/2}$ and $\tilde{\gamma}$ is Euler's constant, while for $d = 3$,

$$\mathcal{S}_D^\omega(\varphi)(\epsilon x + z) = -\frac{1}{4\pi}\sum_{n=0}^{+\infty}\frac{1}{n!}(i\omega\epsilon)^n\int_{\partial B}|x-y|^{n-1}\tilde{\varphi}(y)d\sigma(y).$$

On the other hand, we have

$$\mathcal{D}_\Omega^\omega(\psi)(\epsilon x + z) = \sum_{n=0}^{+\infty}\epsilon^n\sum_{|\alpha|=n}\frac{1}{\alpha!}\partial^\alpha\mathcal{D}_\Omega^\omega(\psi)(z)x^\alpha, \quad x \in \partial B,\ d = 2, 3,$$

and for $d = 2, 3$,

$$\mathcal{S}_D^\omega(\varphi)(x) = \sum_{n=0}^{+\infty}(-1)^n\epsilon^{n+d-2}\sum_{|\alpha|=n}\frac{1}{\alpha!}\partial^\alpha\Gamma_\omega(x-z)\Big(\int_{\partial B}y^\alpha\tilde{\varphi}(y)d\sigma(y)\Big), \quad x \in \partial\Omega.$$

Proof. For any $\tilde{x}, \tilde{y} \in \partial D$, we have

$$\mathcal{S}_D^\omega(\varphi)(\tilde{x}) = \int_{\partial D}\Gamma_\omega(\tilde{x} - \tilde{y})\varphi(\tilde{y})d\sigma(\tilde{y}).$$

By the change of variables $\tilde{x} = \epsilon x + z$ and $\tilde{y} = \epsilon y + z$, we obtain that

$$\begin{aligned}\mathcal{S}_D^\omega(\varphi)(\tilde{x}) &= \epsilon^{d-1}\int_{\partial B}\Gamma_\omega(\epsilon(x-y))\varphi(\epsilon y + z)d\sigma(y)\\ &= \epsilon^{d-2}\int_{\partial B}\Gamma_\omega(\epsilon(x-y))\tilde{\varphi}(y)d\sigma(y).\end{aligned}$$

The first two formulae immediately follow from the following Taylor expansion of $\Gamma_\omega(\epsilon x)$ as $\epsilon \to 0$:

$$\Gamma_\omega(\epsilon x) = \begin{cases}\displaystyle\frac{1}{2\pi}\sum_{n=0}^{+\infty}(-1)^n\frac{(\omega\epsilon)^{2n}}{2^{2n}(n!)^2}|x|^{2n}\left(\ln(\omega\epsilon|x|) + \ln\gamma - \sum_{j=1}^{n}\frac{1}{j}\right), & d = 2,\\[2ex] \displaystyle-\frac{1}{4\pi\epsilon}\sum_{n=0}^{+\infty}\frac{1}{n!}(i\omega\epsilon)^n|x|^{n-1}, & d = 3.\end{cases}$$

Since $(\Delta + \omega^2)\mathcal{D}_\Omega^\omega(\psi) = 0$ in Ω, $\mathcal{D}_\Omega^\omega(\psi)$ is a smooth function in Ω and its Taylor expansion at z yields

$$\mathcal{D}_\Omega^\omega(\psi)(\epsilon x + z) = \sum_{n=0}^{+\infty} \epsilon^n \sum_{|\alpha|=n} \frac{1}{\alpha!} \partial^\alpha \mathcal{D}_\Omega^\omega(\psi)(z) x^\alpha .$$

Finally, for any $x \in \partial\Omega$, it is easy to see that

$$\begin{aligned}
\mathcal{S}_D^\omega(\varphi)(x) &= \int_{\partial D} \Gamma_\omega(x - \tilde{y})\varphi(\tilde{y}) d\sigma(\tilde{y}) \\
&= \epsilon^{d-2} \int_{\partial B} \Gamma_\omega(x - z - \epsilon y)\tilde{\varphi}(y) d\sigma(y) \\
&= \epsilon^{d-2} \int_{\partial B} \sum_{n=0}^{+\infty} (-1)^n \epsilon^n \sum_{|\alpha|=n} \frac{1}{\alpha!} \partial^\alpha \Gamma_\omega(x - z) y^\alpha \tilde{\varphi}(y) d\sigma(y) \\
&= \sum_{n=0}^{+\infty} (-1)^n \epsilon^{n+d-2} \sum_{|\alpha|=n} \frac{1}{\alpha!} \partial^\alpha \Gamma_\omega(x - z) \int_{\partial B} y^\alpha \tilde{\varphi}(y) d\sigma(y),
\end{aligned}$$

which completes the proof of the lemma. □

Suppose $d = 2$. By Lemma 4.3.1, we have from (4.3.2) that

$$\begin{cases}
\displaystyle \Big(\frac{1}{2}I - \mathcal{K}_\Omega^\omega\Big)(\psi)(x) - \\
\displaystyle \quad \sum_{n=0}^{+\infty} (-1)^n \epsilon^n \times \sum_{|\alpha|=n} \frac{1}{\alpha!} \partial^\alpha \Gamma_\omega(x - z) \Big(\int_{\partial B} y^\alpha \tilde{\phi}(y) d\sigma(y) \Big) = 0, \quad x \in \partial\Omega, \\
\displaystyle \sum_{n=0}^{+\infty} \epsilon^n \sum_{|\alpha|=n} \frac{1}{\alpha!} \partial^\alpha \mathcal{D}_\Omega^\omega(\psi)(z) x^\alpha + \frac{1}{2\pi} \sum_{n=0}^{+\infty} (-1)^n \frac{(\omega\epsilon)^{2n}}{2^{2n}(n!)^2} \\
\displaystyle \quad \times \int_{\partial B} |x - y|^{2n} \Big(\ln(\omega\epsilon|x - y|) + \ln\gamma - \sum_{j=1}^{n} \frac{1}{j} \Big) \tilde{\phi}(y) d\sigma(y) = 0, \quad x \in \partial B,
\end{cases} \tag{4.3.3}$$

where $\omega = \sqrt{\mu_j^\epsilon}, \psi := u_j^\epsilon|_{\partial\Omega}$, and $\tilde{\phi}(x) := \epsilon\phi(\epsilon x + z)$, $x \in \partial B$.

If $d = 3$, then

$$\begin{cases}
\displaystyle \Big(\frac{1}{2}I - \mathcal{K}_\Omega^\omega\Big)(\psi)(x) \\
\displaystyle \quad - \sum_{n=0}^{+\infty} (-1)^n \epsilon^{n+1} \sum_{|\alpha|=n} \frac{1}{\alpha!} \partial^\alpha \Gamma_\omega(x - z) \Big(\int_{\partial B} y^\alpha \tilde{\phi}(y) d\sigma(y) \Big) = 0, \quad x \in \partial\Omega, \\
\displaystyle \sum_{n=0}^{+\infty} \epsilon^n \sum_{|\alpha|=n} \frac{1}{\alpha!} \partial^\alpha \mathcal{D}_\Omega^\omega(\psi)(z) x^\alpha \\
\displaystyle \quad - \frac{1}{4\pi} \sum_{n=0}^{+\infty} \frac{1}{n!} (i\omega\epsilon)^n \int_{\partial B} |x - y|^{n-1} \tilde{\phi}(y) d\sigma(y) = 0, \quad x \in \partial B.
\end{cases} \tag{4.3.4}$$

The systems of integral equations (4.3.3) and (4.3.4) may alternatively be written in the form

$$\mathcal{A}_\epsilon^d(\omega)\begin{pmatrix}\psi\\ \tilde{\phi}\end{pmatrix}=0,$$

where

$$\mathcal{A}_\epsilon^3(\omega)=\sum_{n=0}^{+\infty}(\omega\epsilon)^n\mathcal{A}_n^3(\omega), \tag{4.3.5}$$

with

$$\mathcal{A}_0^3(\omega):=\begin{pmatrix}\frac{1}{2}I-\mathcal{K}_\Omega^\omega & 0\\ \mathcal{D}_\Omega^\omega(\cdot)(z) & -\frac{1}{4\pi}\int_{\partial B}|x-y|^{-1}\cdot\ d\sigma(y)\end{pmatrix},$$

and for $n\geq 1$,

$$\mathcal{A}_n^3(\omega):=\begin{pmatrix}0 & \frac{1}{\omega}A_{n-1}\\ B_n & -\frac{1}{4\pi}\frac{i^n}{n!}\int_{\partial B}|x-y|^{n-1}\cdot\ d\sigma(y)\end{pmatrix},$$

where

$$A_n:=\frac{(-1)^{n+1}}{\omega^n}\sum_{|\alpha|=n}\frac{1}{\alpha!}\partial^\alpha\Gamma_\omega(x-z)\Big(\int_{\partial B}y^\alpha\cdot\ d\sigma(y)\Big),$$

$$B_n:=\frac{1}{\omega^n}\sum_{|\alpha|=n}\frac{1}{\alpha!}\partial^\alpha\mathcal{D}_\Omega^\omega(\cdot)(z)x^\alpha.$$

In the two-dimensional case,

$$\mathcal{A}_\epsilon^2(\omega)=\sum_{n=0}^{+\infty}(\omega\epsilon)^n\Big(\mathcal{A}_n^2(\omega)+\ln(\omega\epsilon)\mathcal{B}_n^2(\omega)\Big),$$

with

$$\mathcal{A}_0^2(\omega):=\begin{pmatrix}\frac{1}{2}I-\mathcal{K}_\Omega^\omega & -\Gamma_\omega(x-z)\Big(\int_{\partial B}\cdot\ d\sigma(y)\Big)\\ \mathcal{D}_\Omega^\omega(\cdot)(z) & \frac{1}{2\pi}\int_{\partial B}\Big(\ln|x-y|+\ln\gamma\Big)\cdot\ d\sigma(y)\end{pmatrix},$$

$$\mathcal{B}_0^2(\omega):=\begin{pmatrix}0 & 0\\ 0 & \frac{1}{2\pi}\int_{\partial B}\cdot\ d\sigma(y)\end{pmatrix},$$

and, for $n\geq 1$,

$$\mathcal{A}_{2n}^2(\omega):=\begin{pmatrix}0 & A_{2n}\\ B_{2n} & C_{2n}\end{pmatrix},\quad \mathcal{A}_{2n-1}^2(\omega):=\begin{pmatrix}0 & A_{2n-1}\\ B_{2n-1} & 0\end{pmatrix},$$

and

$$\mathcal{B}^2_{2n}(\omega) := \begin{pmatrix} 0 & 0 \\ 0 & \displaystyle \frac{1}{2\pi}\frac{(-1)^n}{2^{2n}(n!)^2}\int_{\partial B}|x-y|^{2n}\,\cdot\; d\sigma(y) \end{pmatrix}, \quad \mathcal{B}^2_{2n-1}(\omega) := 0,$$

where A_n and B_n are as above and

$$C_{2n} := \frac{1}{2\pi}\frac{(-1)^n}{2^{2n}(n!)^2}\int_{\partial B}|x-y|^{2n}\left(\ln|x-y| + \ln\gamma - \sum_{j=1}^{n}\frac{1}{j}\right)\cdot\; d\sigma(y).$$

It is therefore obvious that the pair of functions $(\psi, \tilde{\phi})$ is then a characteristic function of the integral operator-valued function $\mathcal{A}^d_\epsilon, d = 2, 3$, associated with the characteristic value $\sqrt{\mu^\epsilon_j}$. We give a rigorous study of the integral operator-valued function $\omega \mapsto \mathcal{A}^d_\epsilon(\omega)$, when ω is in a small complex neighborhood of $\sqrt{\mu_j}$. We proceed from the generalized Rouché's theorem to construct the complete asymptotic expansions for μ^ϵ_j.

At this stage, we emphasize the fact the asymptotic parameter is in fact $\omega\epsilon$ and not ϵ which shows that the asymptotic expansions of $\mu^\epsilon_j - \mu_j$ we are going to derive are not only valid for fixed j when ϵ goes to zero but also uniformly for the set of first j eigenvalues such that $\sqrt{\mu_j}\epsilon$ remain small.

In order to simplify notation we introduce the single layer potential associated with the Laplacian:

$$\mathcal{S}^0_B\phi(x) = \begin{cases} \displaystyle \frac{1}{2\pi}\int_{\partial B}\ln|x-y|\phi(y)d\sigma(y) & \text{if } d = 2, \\ \displaystyle -\frac{1}{4\pi}\int_{\partial B}\frac{1}{|x-y|}\phi(y)d\sigma(y) & \text{if } d = 3. \end{cases}$$

It is easy to see that $\{\sqrt{\mu^\epsilon_j}\}_{j\geq 1}$ are exactly the real characteristic values of the operator-valued function $\mathcal{A}^d_\epsilon$, for $0 \leq \epsilon \leq \epsilon_0, \epsilon_0 > 0$. Conversely, if ω is a real characteristic value of the operator-valued function $\mathcal{A}^d_\epsilon$ then ω^2 is an eigenvalue of (4.3.1).

The next three lemmas can be proved by slightly modifying the arguments given in [17].

Lemma 4.3.2 *The operator-valued function $\mathcal{A}^d_\epsilon(\omega)$ is Fredholm analytic with index 0 in $\mathbb{C} \setminus i\mathbb{R}^-$ and $(\mathcal{A}^d_\epsilon)^{-1}(\omega)$ is a meromorphic function. If ω is a real characteristic value of the operator-valued function $\mathcal{A}^d_\epsilon$ (or equivalently, a real pole of $(\mathcal{A}^d_\epsilon)^{-1}(\omega)$) then there exists j such that $\omega = \sqrt{\mu^\epsilon_j}$.*

Lemma 4.3.3 *Any $\sqrt{\mu_j}$ is a simple pole of the operator-valued function $(\mathcal{A}^d_0)^{-1}(\omega)$.*

Lemma 4.3.4 *Let $\omega_0 = \sqrt{\mu_j}$ and suppose that μ_j is simple. Then there exists a positive constant δ_0 such that for $|\delta| < \delta_0$, the operator-valued function $\omega \mapsto \mathcal{A}^d_\epsilon(\omega)$ has exactly one characteristic value in $\overline{V_{\delta_0}}(\omega_0)$, where $V_{\delta_0}(\omega_0)$ is a disc* *of center ω_0 and radius $\delta_0 > 0$. This characteristic value is analytic with respect to ϵ in $]-\epsilon_0, \epsilon_0[$. Moreover, the following assertions hold:*

$$\begin{cases} \mathcal{M}(\mathcal{A}_\epsilon^d(\omega); \partial V_{\delta_0}) = 1, \\ (\mathcal{A}_\epsilon^d)^{-1}(\omega) = (\omega - \omega_0)^{-1}\mathcal{L}_\epsilon^d + \mathcal{R}_\epsilon^d(\omega), \\ \mathcal{L}_\epsilon^d : Ker((\mathcal{A}_\epsilon^d(\omega_\epsilon))^*) \to Ker(\mathcal{A}_\epsilon^d(\omega_\epsilon)), \end{cases}$$

where $\mathcal{R}_\epsilon^d(\omega)$ is a holomorphic function with respect to $(\epsilon, \omega) \in]-\epsilon_0, \epsilon_0[\times V_{\delta_0}(\omega_0)$ and $\mathcal{L}_\epsilon^d$ is a finite-dimensional operator.

Based on the generalized Rouché's theorem we are now ready to derive complete asymptotic formulae for the characteristic values of $\omega \mapsto \mathcal{A}_\epsilon^d(\omega)$. Applying Theorem 4.2.5, we get the following lemma.

Lemma 4.3.5 *Let $\omega_0 = \sqrt{\mu_j}$ and suppose that μ_j is simple. Then $\omega_\epsilon = \sqrt{\mu_j^\epsilon}$ is given by*

$$\omega_\epsilon - \omega_0 = \frac{1}{2i\pi} \text{ tr } \int_{\partial V_{\delta_0}} (\omega - \omega_0)(\mathcal{A}_\epsilon^d)^{-1}(\omega) \frac{d}{d\omega}\mathcal{A}_\epsilon^d(\omega) d\omega.$$

Following once again [17], we obtain the following complete asymptotic expansion for the eigenvalue perturbations in the three-dimensional case.

Theorem 4.3.6 *Suppose $d = 3$. Then the following asymptotic expansion holds:*

$$\omega_\epsilon - \omega_0 = \frac{1}{2i\pi} \sum_{p=1}^{+\infty} \frac{1}{p} \sum_{n=p}^{+\infty} \epsilon^n \ tr \int_{\partial V_{\delta_0}} (\mathcal{A}_0^3)^{-p}(\omega) B_{n,p}^3(\omega) d\omega, \tag{4.3.6}$$

where

$$B_{n,p}^3(\omega) = (-1)^p \sum_{\substack{n_1+\ldots+n_p=n \\ n_i \geq 1}} \mathcal{A}_{n_1}^3(\omega) \ldots \mathcal{A}_{n_p}^3(\omega) \omega^n.$$

Proof. If ϵ is small enough, then the following Neumann series converges uniformly with respect to ω in ∂V_{δ_0}:

$$(\mathcal{A}_\epsilon^3)^{-1}(\omega) = \sum_{p=0}^{\infty} (\mathcal{A}_0^3)^{-1}(\omega) \left[(\mathcal{A}_0^3(\omega) - \mathcal{A}_\epsilon^3(\omega))(\mathcal{A}_0^3)^{-1}(\omega) \right]^p,$$

and hence we may deduce, using the property (4.2.1) of the trace, that

$$\omega_\epsilon - \omega_0 = \frac{1}{2i\pi} \sum_{p=0}^{\infty} \text{ tr } \int_{\partial V_{\delta_0}} (\omega - \omega_0)(\mathcal{A}_0^3)^{-p-1}(\omega)(\mathcal{A}_0^3(\omega) - \mathcal{A}_\epsilon^3(\omega))^p \frac{d}{d\omega}\mathcal{A}_\epsilon^3(\omega) d\omega.$$

Since

$$(\mathcal{A}_0^3(\omega) - \mathcal{A}_\epsilon^3(\omega))^p \frac{d}{d\omega}\mathcal{A}_\epsilon^3(\omega) =$$
$$(\mathcal{A}_0^3(\omega) - \mathcal{A}_\epsilon^3(\omega))^p \frac{d}{d\omega}\mathcal{A}_0^3(\omega) - \frac{1}{p+1}\frac{d}{d\omega}\Big[(\mathcal{A}_0^3(\omega) - \mathcal{A}_\epsilon^3(\omega))^{p+1}\Big],$$

we have

$$\omega_\epsilon - \omega_0 = \frac{1}{2i\pi}\sum_{p=0}^{\infty} \text{ tr } \int_{\partial V_{\delta_0}} (\omega - \omega_0)(\mathcal{A}_0^3)^{-p-1}(\omega)(\mathcal{A}_0^3(\omega) - \mathcal{A}_\epsilon^3(\omega))^p \frac{d}{d\omega}\mathcal{A}_0^3(\omega) d\omega$$
$$-\frac{1}{2i\pi}\sum_{p=1}^{\infty} \text{ tr } \int_{\partial V_{\delta_0}} (\omega - \omega_0)(\mathcal{A}_0^3)^{-p}(\omega)\frac{1}{p}\frac{d}{d\omega}\Big[(\mathcal{A}_0^3(\omega) - \mathcal{A}_\epsilon^3(\omega))^p\Big] d\omega.$$

Because of Lemma 4.3.3, ω_0 is a simple pole of $(\mathcal{A}_0^3)^{-1}(\omega)$ and $\mathcal{A}_0^3(\omega)$ is analytic, and hence we get

$$\int_{\partial V_{\delta_0}} (\omega - \omega_0)(\mathcal{A}_0^3)^{-1}(\omega)\frac{d}{d\omega}\mathcal{A}_0^3(\omega) d\omega = 0.$$

Therefore, it follows that

$$\omega_\epsilon - \omega_0 = -\frac{1}{2i\pi}\sum_{p=1}^{\infty} \text{ tr } \int_{\partial V_{\delta_0}} (\omega - \omega_0)\frac{d}{d\omega}\Big(\frac{1}{p}(\mathcal{A}_0^3)^{-p}(\omega)(\mathcal{A}_0^3(\omega) - \mathcal{A}_\epsilon^3(\omega))^p\Big) d\omega.$$

Now, a simple integration by parts yields

$$\omega_\epsilon - \omega_0 = \frac{1}{2i\pi}\sum_{p=1}^{\infty}\frac{1}{p} \text{ tr } \int_{\partial V_{\delta_0}} ((\mathcal{A}_0^3)^{-1}(\omega))^p(\mathcal{A}_0^3(\omega) - \mathcal{A}_\epsilon^3(\omega))^p d\omega.$$

Notice from (4.3.5) that

$$(\mathcal{A}_0^3(\omega) - \mathcal{A}_\epsilon^3(\omega))^p = (-1)^p \sum_{n=p}^{\infty}\epsilon^n \sum_{\substack{n_1+\ldots+n_p=n\\ n_i\geq 1}} \mathcal{A}_{n_1}^3(\omega)\ldots\mathcal{A}_{n_p}^3(\omega)\omega^n.$$

Therefore, upon inserting this into the latter formula we arrive at the desired asymptotic expansion. □

The following theorem can be proved in the same way as Theorem 4.3.6.

Theorem 4.3.7 *Suppose $d = 2$. Then the following asymptotic expansion holds:*

$$\omega_\epsilon - \omega_0 = \frac{1}{2i\pi}\sum_{p=1}^{+\infty}\frac{1}{p}\sum_{n=p}^{+\infty}\epsilon^n \ tr \int_{\partial V_{\delta_0}} (\mathcal{A}_0^2(\omega) + \ln(\omega\epsilon)\mathcal{B}_0^2(\omega))^{-p} B_{n,p}^2(\omega) d\omega,$$

where

$$B_{n,p}^2(\omega) = (-1)^p \sum_{\substack{n_1+\ldots+n_p=n\\ n_i\geq 1}} (\mathcal{A}_{n_1}^2(\omega)+\ln(\omega\epsilon)\mathcal{B}_{2n_1}^2(\omega))\ldots(\mathcal{A}_{n_p}^2(\omega)+\ln(\omega\epsilon)\mathcal{B}_{2n_p}^2(\omega))\omega^n.$$

As a simplest case, let us now find the leading-order term in the asymptotic expansion of $\mu_j^\epsilon - \mu_j$ as $\epsilon \to 0$.

Suppose $d = 3$. Recall from (4.3.6) that

$$\omega_\epsilon - \omega_0 = -\frac{\epsilon}{2\pi i} \mathrm{tr} \int_{\partial V_{\delta_0}} (\mathcal{A}_0^3)^{-1}(\omega) \mathcal{A}_1^3(\omega) \omega d\omega.$$

Let the space $W_1^2(\partial B)$ be the set of functions $f \in L^2(\partial B)$ such that $\partial f / \partial T \in L^2(\partial B)$, where $\partial/\partial T$ denotes the tangential derivative on ∂B. It is proved in [96] that $\mathcal{S}_B^0 : L^2(\partial B) \to W_1^2(\partial B)$ is invertible. Therefore, we have

$$(\mathcal{S}_B^0)^{-1}(1) = \mathrm{cap}(\partial B) \varphi_e,$$

where $\mathrm{cap}(\partial B)$ and φ_e defined in (4.2.7) and (4.2.8). It is now easy to see that

$$(\mathcal{A}_0^3)^{-1}(\omega) = \begin{pmatrix} (\frac{1}{2} I - \mathcal{K}_\Omega^\omega)^{-1} & 0 \\ -C \mathcal{D}_\Omega^\omega (\frac{1}{2} I - \mathcal{K}_\Omega^\omega)^{-1}(\cdot)(z) \varphi_e & (\mathcal{S}_B^0)^{-1} \end{pmatrix},$$

where $C := \mathrm{cap}(\partial B)$. It then follows from (4.2.4) that

$$(\mathcal{A}_0^3)^{-1}(\omega) \mathcal{A}_1^3(\omega)$$
$$= \begin{pmatrix} 0 & \frac{1}{\omega} N_\Omega^\omega(x, z) \int_{\partial B} \cdot \, d\sigma(y) \\ (\mathcal{S}_B^0)^{-1}(\nabla \mathcal{D}_\Omega^\omega(\cdot)(z) \cdot x) & -C \mathcal{D}_\Omega^\omega (N_\Omega^\omega(\cdot, z))(z) \varphi_e \int_{\partial B} \cdot \, d\sigma(y) - \frac{i\omega}{4\pi} C \varphi_e \int_{\partial B} \cdot \, d\sigma(y) \end{pmatrix}$$

Since $\int_{\partial B} \varphi_e d\sigma = 1$, we now have

$$\frac{1}{2\pi i} \mathrm{tr} \int_{\partial V_{\delta_0}} (\mathcal{A}_0^3)^{-1}(\omega) \mathcal{A}_1^3(\omega) \omega d\omega = -\frac{C}{2\pi i} \int_{\partial V_{\delta_0}} \mathcal{D}_\Omega^\omega (N_\Omega^\omega(\cdot, z))(z) d\omega.$$

Since $-\Delta u_j = \mu_j u_j$ in Ω and $\frac{\partial u_j}{\partial \nu} = 0$ on $\partial \Omega$, we have

$$\mathcal{D}_\Omega^\omega(u_j)(z) = u_j(z) + (\mu_j - \omega^2) \int_\Omega \Gamma_\omega(z - y) u_j(y) dy.$$

Since Γ_ω is analytic in $\omega \in V_{\delta_0}(\sqrt{\mu_j})$ and $-\sqrt{\mu_l} \notin V_{\delta_0}(\sqrt{\mu_j}), \forall\ l \neq j$, if δ_0 is sufficiently small, the spectral decomposition (4.2.5) and the residue theorem yield that

$$\frac{1}{2\pi i} \int_{\partial V_{\delta_0}} \mathcal{D}_\Omega^\omega (N_\Omega^\omega(\cdot, z))(z) d\omega = \frac{1}{2\pi i} |u_j(z)|^2 \int_{\partial V_{\delta_0}} \frac{1}{\omega^2 - \mu_j} d\omega = \frac{1}{2\sqrt{\mu_j}} |u_j(z)|^2.$$

We thus get

$$\frac{1}{2i\pi} \mathrm{tr} \int_{\partial V_{\delta_0}} (\mathcal{A}_0^3)^{-1}(\omega) \mathcal{A}_1^3(\omega) \omega d\omega = -\frac{1}{2\sqrt{\mu_j}} |u_j(z)|^2 \mathrm{cap}(\partial B),$$

which yields the following corollary.

Corollary 4.3.8 *Suppose $d = 3$. Then the following asymptotic expansion holds:*

$$\mu_j^\epsilon - \mu_j = \epsilon \, cap(\partial B)|u_j(z)|^2 + O(\epsilon^2). \tag{4.3.7}$$

In particular, if B is the unit ball, then $\text{cap}(\partial B) = -4\pi$ and (4.3.7) yields the formula obtained by Ozawa in [71]. It should be mentioned that Ozawa also obtained the ϵ^2 term of the asymptotic expansions of $\mu_j^\epsilon - \mu_j$ when the inclusion is a sphere in [73]. The second order term when the inclusion is of general shape can be explicitly computed using (4.3.6).

Let us now consider the two dimensional case. Recall that

$$\mathcal{A}_0^2(\omega) + \ln(\omega\epsilon)\mathcal{B}_0^2(\omega) = \begin{pmatrix} \dfrac{1}{2}I - \mathcal{K}_\Omega^\omega & -\Gamma_\omega(x-z)\Big(\displaystyle\int_{\partial B} \cdot \, d\sigma(y)\Big) \\ \mathcal{D}_\Omega^\omega(\cdot)(z) & \mathcal{S}_B^0 + \dfrac{1}{2\pi}\displaystyle\int_{\partial B}\Big(\ln(\gamma\omega\epsilon)\Big) \cdot \, d\sigma(y) \end{pmatrix}.$$

Let a and φ_e be as defined by (2.8), and let W_0 be the collection of all functions $\psi \in W_1^2(\partial B)$ such that $\int_{\partial B} \varphi_e \psi d\sigma = 0$. Then it is proved in [96] that $\mathcal{S}_B^0 : L_0^2(\partial B) \to W_0$ is invertible. Let $(\mathcal{S}_B^0)^{-1}$ denote its inverse. Then one can easily see that

$$\Big(\mathcal{A}_0^2(\omega) + \ln(\omega\epsilon)\mathcal{B}_0^2(\omega)\Big)^{-1}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} (\dfrac{1}{2}I - \mathcal{K}_\Omega^\omega)^{-1}(f) - C(f,g)N_\Omega^\omega(x,z) \\ C(f,g)\varphi_e + (\mathcal{S}_B^0)^{-1}(g - \bar{g}) \end{pmatrix},$$

where $\bar{g} = \int_{\partial B} g\varphi_e d\sigma$ and the constant $C(f,g)$ is defined to be

$$C(f,g) = \frac{\bar{g} - \mathcal{D}_\Omega^\omega(\frac{1}{2}I - \mathcal{K}_\Omega^\omega)^{-1}(f)(z)}{\frac{\ln \gamma\omega\epsilon}{2\pi} - \mathcal{D}_\Omega^\omega(N_\Omega^\omega(\cdot,z))(z) - a}.$$

Now, in exactly the same manner as in Corollary 4.3.8, we obtain from Theorem 4.3.7 that the leading-order term in the asymptotic expansion of $\mu_j^\epsilon - \mu_j$ in two dimensions is as follows.

Corollary 4.3.9 *Suppose $d = 2$. Then the following asymptotic expansion holds:*

$$\mu_j^\epsilon - \mu_j = \frac{2\pi}{\ln(\frac{\epsilon\sqrt{\mu_j}}{cap(\partial B)})}|u_j(z)|^2 + o(\frac{1}{\ln(\epsilon)}). \tag{4.3.8}$$

In particular, since the logarithmic capacity of the unit disk $\text{cap}(\partial B) = 1$, (4.3.8) yields the formula derived by Ozawa in [72]. See also Besson [20].

4.3.2 Inclusions nearly touching the boundary

In this section we study the eigenvalue problem in the presence of a diametrically small perfectly conducting inclusion that is nearly touching the boundary. Consider a small perfectly conducting inclusion D inside Ω that is nearly touching the boundary $\partial\Omega$. We assume that $\partial\Omega$ is of class $\mathcal{C}^2$ and $D = \epsilon B + z$, where $z \in \Omega$ is such that $\mathrm{dist}(z, \partial\Omega) = \delta\epsilon$. Here B is a bounded domain in $\mathbb{R}^2$ containing the origin with a connected $\mathcal{C}^2$-boundary and the constant $\delta > \max_{x\in\partial B} |x|$. We show that the leading-order term in the asymptotic expansion of the eigenvalue perturbations is the same as in Corollary 4.3.9.

The following lemma from [2] is of use to us.

Lemma 4.3.10 *Suppose that $\partial\Omega$ is of class $\mathcal{C}^2$ and let $\psi \in \mathcal{C}^0(\partial\Omega)$. Let z_0 be the normal projection of z onto $\partial\Omega$. Then, for any $x \in \partial B$,*

$$\int_{\partial\Omega} \frac{\epsilon}{|y - z_0 - \epsilon(x - \delta\nu(z_0))|^2} \psi(y) d\sigma(y) \to \frac{\pi}{(\delta\nu(z_0) - x)\cdot \nu(z_0)} \psi(z_0).$$

Using Lemma 4.3.10 we prove the following.

Lemma 4.3.11 *Suppose that $\partial\Omega$ is of class $\mathcal{C}^2$ and let $\psi \in \mathcal{C}^0(\partial\Omega)$. Let z_0 be the normal projection of z onto $\partial\Omega$. For any $x \in \partial B$, we have*

$$\mathcal{D}_\Omega^\omega(\psi)(\epsilon x + z) = (\frac{1}{2} I + \mathcal{K}_\Omega^\omega)(\psi)(z_0) + o(1) \quad \text{as } \epsilon \to 0,$$

where the remainder $o(1)$ is uniform in $x \in \partial B$.

Proof. Let $\nu(z_0)$ denote the outward unit normal to $\partial\Omega$ at z_0. Since $z = z_0 + z - z_0 = z_0 - \delta\epsilon\nu(z_0)$, for any $x \in \partial B$, we have

$$\epsilon x + z = \epsilon x + z_0 - \delta\epsilon\nu(z_0) = z_0 + \epsilon(x - \delta\nu(z_0)).$$

Hence, we obtain

$$\begin{aligned}
&\mathcal{D}_\Omega^\omega(\psi)(z_0 + \epsilon(x - \delta\nu(z_0))) \\
&= -\frac{i\omega}{4} \int_{\partial\Omega} \Big({H_0^{(1)}}'(\omega|y - z_0 - \epsilon(x - \delta\nu(z_0))|) \big| y - z_0 - \epsilon(x - \delta\nu(z_0)) \big| \Big) \\
&\quad \times \frac{\langle y - z_0, \nu(y)\rangle - \epsilon\langle x - \delta\nu(z_0), \nu(y)\rangle}{|y - z_0 - \epsilon(x - \delta\nu(z_0))|^2} \psi(y) d\sigma(y).
\end{aligned}$$

Since

$$-\frac{i\omega}{4} {H_0^{(1)}}'(\omega|t - z_0|)|t - z_0| = \frac{1}{2\pi} - \frac{\omega^2}{4\pi}|t - z_0|^2 \ln|t - z_0| - \frac{\omega^2}{4\pi}|t - z_0|^2(\ln(\omega\gamma) - \frac{1}{2}) + \ldots,$$

as $t \to z_0$, we see that

$$\begin{aligned}
&{H_0^{(1)}}'(\omega|z_0 - y + \epsilon(x - \delta\nu(z_0))|) \big| y - z_0 - \epsilon(x - \delta\nu(z_0)) \big| \\
&= {H_0^{(1)}}'(\omega|y - z_0|)|y - z_0| + O(\epsilon),
\end{aligned}$$

as ϵ tends to 0.

Now, since $\partial\Omega$ is of class $\mathcal{C}^2$, we have (see [2])

$$\frac{|\langle y-z,\nu(y)\rangle|}{|y-z|^2}=O(1),$$

and hence

$$\Big|\frac{\langle y-z_0,\nu(y)\rangle-\epsilon\langle x-\delta\nu(z_0),\nu(y)\rangle}{|y-z_0-\epsilon(x-\delta\nu(z_0))|^2}\Big|=O(1),$$

which gives

$$\begin{aligned}
&-\frac{i\omega}{4}\int_{\partial\Omega}\Big(H_0^{(1)'}(\omega|y-z_0-\epsilon(x-\delta\nu(z_0))|)\big|y-z_0-\epsilon(x-\delta\nu(z_0))\big|\Big)\\
&\quad\times\frac{\langle y-z_0,\nu(y)\rangle-\epsilon\langle x-\delta\nu(z_0),\nu(y)\rangle}{|y-z_0-\epsilon(x-\delta\nu(z_0))|^2}\psi(y)d\sigma(y)\\
&=-\frac{i\omega}{4}\int_{\partial\Omega}H_0^{(1)'}(\omega|y-z_0|)|y-z_0|\frac{\langle y-z_0,\nu(y)\rangle-\epsilon\langle x-\delta\nu(z_0),\nu(y)\rangle}{|y-z_0-\epsilon(x-\delta\nu(z_0))|^2}\psi(y)d\sigma(y)+O(\epsilon),\\
&:=I(\epsilon,x)+O(\epsilon)\quad\text{as }\epsilon\to 0.
\end{aligned}$$

Since

$$|y-z_0-\epsilon(x-\delta\nu(z_0))|^2=|y-z_0|^2+\epsilon^2|x-\delta\nu(z_0)|^2-2\epsilon\langle y-z_0,x-\delta\nu(z_0)\rangle,$$

it follows that

$$\begin{aligned}
I(\epsilon,x)-\mathcal{K}_\Omega^\omega\psi(z_0)=&\int_{\partial\Omega}\frac{\epsilon}{|y-z_0-\epsilon(x-\delta\nu(z_0))|^2}\Psi_1(y)d\sigma(y)\\
&+\epsilon|x-\delta\nu(z_0)|^2\int_{\partial\Omega}\frac{\epsilon}{|y-z_0-\epsilon(x-\delta\nu(z_0))|^2}\Psi_2(y)d\sigma(y),
\end{aligned}$$

where

$$\begin{aligned}
\Psi_1(y)=&\frac{-i\omega}{4}H_0^{(1)'}(\omega|y-z_0|)|y-z_0|\Big[2\langle y-z_0,x-\delta\nu(z_0)\rangle\frac{\langle y-z_0,\nu(y)\rangle}{|y-z_0|^2}\\
&-\langle x-\delta\nu(z_0),\nu(y)\rangle\Big]\psi(y)
\end{aligned}$$

and

$$\Psi_2(y)=\frac{i\omega}{4}H_0^{(1)'}(\omega|y-z_0|)|y-z_0|\frac{\langle y-z_0,\nu(y)\rangle}{|y-z_0|^2}\psi(y).$$

By using the fact that $-\frac{i\omega}{4}H_0^{(1)'}(\omega|y-z_0|)|y-z_0|\to\frac{1}{2\pi}$ as $|y-z_0|\to 0$, we deduce the following identities:

$$\Psi_1(z_0)=-\frac{1}{2\pi}\langle x-\delta\nu(z_0),\nu(z_0)\rangle\psi(z_0),\quad \Psi_2(z_0)=-\frac{1}{4\pi}\tau(z_0)\psi(z_0),$$

where $\tau(z_0)$ is the curvature at $z_0\in\partial\Omega$. Applying Lemma 4.3.10 we conclude that

$$I(\epsilon,x)=(\frac{1}{2}I+\mathcal{K}_\Omega^\omega)\psi(z_0)+o(1)\quad\text{as }\epsilon\to 0,$$

which completes the proof of the lemma. $\square$

Let $x, y \in \partial B$. Writing $(z_0 + \epsilon(x - \delta\nu(z_0)) - (z_0 + \epsilon(y - \delta\nu(z_0)) = \epsilon(x - y)$, the following asymptotic formula holds:

$$\mathcal{S}_D^\omega(\varphi)(z_0 + \epsilon(x - \delta\nu(z_0)) = \frac{1}{2\pi}\ln(\gamma\epsilon\omega)\int_{\partial B}\tilde{\varphi}(y)d\sigma(y) + \frac{1}{2\pi}\int_{\partial B}\ln|x-y|\tilde{\varphi}(y)d\sigma(y)$$
$$+ O((\epsilon\omega)^2\ln(\epsilon\omega)),$$

where $\tilde{\varphi}(x) = \epsilon\varphi(\epsilon x + z), x \in \partial B$.

Let $x \in \partial\Omega$. We have

$$\begin{aligned}\mathcal{S}_D^\omega(\varphi)(x) &= -\frac{i}{4}\int_{\partial D} H_0^{(1)}(\omega|x-y|)\varphi(y)d\sigma(y)\\ &= -\frac{i}{4}\int_{\partial B} H_0^{(1)}(\omega|x - z_0 - \epsilon(y - \delta\nu(z_0))|)\tilde{\varphi}(y)d\sigma(y).\end{aligned}$$

We conclude, after lengthy but simple calculation, that in the case of a perfectly conducting inclusion nearly touching the boundary, the leading-order term in the asymptotic expansions of characteristic values of the operator-valued function

$$\omega \mapsto \begin{pmatrix} \left(\frac{1}{2}I - \mathcal{K}_\Omega^\omega\right) & -\mathcal{S}_D^\omega \\ \mathcal{D}_\Omega^\omega & \mathcal{S}_D^\omega \end{pmatrix}$$

is exactly the one given in the previous section. Thus, the following asymptotic expansion holds.

Theorem 4.3.12 *Suppose that ∂B and $\partial\Omega$ are of class $\mathcal{C}^2$ and $D = \epsilon B + z$, where $z \in \Omega$ is such that dist$(z, \partial\Omega) = \delta\epsilon$. Let z_0 be the normal projection of z onto $\partial\Omega$. Then*

$$\mu_j^\epsilon - \mu_j = \frac{2\pi}{\ln(\frac{\epsilon\sqrt{\mu_j}}{cap(\partial B)})}|u_j(z_0)|^2 + o(\frac{1}{\ln(\epsilon)}).$$

4.4 Asymptotic formula of the eigenvalues in the presence of a conductivity inclusion

In this section we provide a rigorous derivation of a full asymptotic formula for perturbations in the eigenvalues caused by the presence of a conductivity inclusion of small diameter with conductivity different from the one of the background.

Suppose that $D \subset\subset \Omega$ is of conductivity equal to some positive constant $k \neq 1$. Let $0 < \mu_1^\epsilon \leq \mu_2^\epsilon \leq \ldots$ be the eigenvalues of $-\nabla \cdot (1 + (k-1)\chi(D))\nabla$ in Ω, where $\chi(D)$ denotes the indicator function of D, with the the Neumann condition on $\partial\Omega$. We arrange them repeatedly according to their multiplicity.

Fix j and suppose that the unperturbed eigenvalue μ_j is simple. Then there exists a simple eigenvalue μ_j^ϵ, near μ_j, associated to the normalized eigenfunction u_j^ϵ, that is, u_j^ϵ satisfies the following problem:

$$\left\{\begin{array}{ll} \Delta u + \omega^2 u = 0 & \text{in } \Omega \setminus \overline{D}, \\ \Delta u + \dfrac{\omega^2}{k} u = 0 & \text{in } D, \\ u|_+ - u|_- = 0 & \text{on } \partial D, \\ \dfrac{\partial u}{\partial \nu}\Big|_+ - k \dfrac{\partial u}{\partial \nu}\Big|_- = 0 & \text{on } \partial D, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega, \end{array}\right. \tag{4.4.1}$$

with $\omega = \sqrt{\mu_j^\epsilon}$.

From once again [6], we know that the solution of (4.4.1) can be represented as

$$u(x) = \left\{\begin{array}{ll} \mathcal{D}_\Omega^\omega(u|_{\partial\Omega})(x) + \mathcal{S}_\Omega^\omega(\phi)(x) & \text{in } \Omega \setminus \overline{D}, \\ \mathcal{S}_D^{\frac{\omega}{\sqrt{k}}}(\theta)(x) & \text{in } D, \end{array}\right.$$

where the triplet of densities $(\psi := u|_{\partial\Omega}, \phi, \theta) \in L^2(\partial\Omega) \times L^2(\partial D) \times L^2(\partial D)$ satisfies the following system of integral equations:

$$\left\{\begin{array}{ll} \Big(\frac{1}{2}I - \mathcal{K}_\Omega^\omega\Big)(\psi)(x) - \mathcal{S}_D^\omega(\phi)(x) = 0, & x \in \partial\Omega, \\ \mathcal{D}_\Omega^\omega(\psi)(x) + \mathcal{S}_D^\omega(\phi)(x) - \mathcal{S}_D^{\frac{\omega}{\sqrt{k}}}(\theta)(x) = 0, & x \in \partial D, \\ \epsilon\left[\dfrac{\partial}{\partial\nu}\Big(\mathcal{D}_\Omega^\omega(\psi)(x) + \mathcal{S}_D^\omega(\phi)(x)\Big)\Big|_+ - k\dfrac{\partial}{\partial\nu}\Big(\mathcal{S}_D^{\frac{\omega}{\sqrt{k}}}(\theta)(x)\Big)\Big|_-\right] = 0, & x \in \partial D, \end{array}\right. \tag{4.4.2}$$

for $\omega = \sqrt{\mu_j^\epsilon}$.

As before, by using the jump formula (4.2.2), we reduce the problem to the calculation of the asymptotic expressions of the characteristic values of the operator-valued function $\mathcal{A}_\epsilon^d(\omega)$ given by

$$\omega \mapsto \mathcal{A}_\epsilon^d(\omega) := \begin{pmatrix} \frac{1}{2}I - \mathcal{K}_\Omega^\omega & -\mathcal{S}_D^\omega & 0 \\ \mathcal{D}_\Omega^\omega & \mathcal{S}_D^\omega & -\mathcal{S}_D^{\frac{\omega}{\sqrt{k}}} \\ \epsilon\frac{\partial}{\partial\nu}\mathcal{D}_\Omega^\omega & \epsilon(\frac{1}{2}I + (\mathcal{K}_D^\omega)^*) & -\epsilon k(-\frac{1}{2}I + (\mathcal{K}_D^{\frac{\omega}{\sqrt{k}}})^*) \end{pmatrix}.$$

We shall expand the operator-valued function $\mathcal{A}_\epsilon^d(\omega)$ in terms of ϵ. With Lemma 4.3.1 on hand, we only need to write the expansion of $\partial\mathcal{D}_\Omega^\omega/\partial\nu$ and $(\mathcal{K}_D^\omega)^*$. On one hand, we have, for $\psi \in L^2(\partial\Omega)$,

$$\frac{\partial}{\partial\nu}\mathcal{D}_\Omega^\omega(\psi)(\epsilon x + z) = \sum_{n=1}^{+\infty} \epsilon^n \sum_{|\alpha|=n} \frac{1}{\alpha!}\partial^\alpha \mathcal{D}_\Omega^\omega(\psi)(z)\frac{\partial x^\alpha}{\partial\nu}, \quad x \in \partial B, \quad d = 2, 3.$$

On the other hand, using the Taylor expansion, we get

$$\frac{\partial}{\partial\nu(x)}\Gamma_\omega(\epsilon(x-y)) = \begin{cases} \dfrac{\langle x-y,\nu(x)\rangle}{2\pi\epsilon|x-y|^2}\Bigg[1+\displaystyle\sum_{n=1}^{+\infty}(-1)^n\frac{(\omega\epsilon)^{2n}}{2^{2n}n!(n-1)!}|x-y|^{2n} \\ \qquad\times\Bigg(\ln(\omega\epsilon|x-y|)+\ln\gamma+\dfrac{1}{2n}-\displaystyle\sum_{j=1}^{n}\frac{1}{j}\Bigg)\Bigg], \quad d=2, \\ -\dfrac{\langle x-y,\nu(x)\rangle}{4\pi\epsilon|x-y|^2}\Bigg[-\dfrac{1}{\epsilon|x-y|} \\ \qquad+\displaystyle\sum_{n=0}^{+\infty}(\frac{1}{n!}-\frac{1}{(n+1)!})(i\omega)^{n+1}\epsilon^n|x-y|^n\Bigg], \quad d=3, \end{cases}$$

we obtain the following expansion.

Lemma 4.4.1 *Let $\varphi\in L^2(\partial D)$. Define $\tilde\varphi(x)=\epsilon\varphi(\epsilon x+z), x\in\partial B$. Then, for $x\in\partial B$, we have*

$$\epsilon(\mathcal{K}_D^\omega)^*(\varphi)(\epsilon x+z)=\mathcal{K}_B^*(\tilde\varphi)(x)+\sum_{n=1}^{+\infty}(-1)^n\frac{(\omega\epsilon)^{2n}}{2^{2n+1}\pi n!(n-1)!}$$
$$\times\int_{\partial B}\langle x-y,\nu(x)\rangle|x-y|^{2(n-1)}\Bigg(\ln(\omega\epsilon|x-y|)+\ln\gamma+\frac{1}{2n}-\sum_{j=1}^{n}\frac{1}{j}\Bigg)\tilde\varphi(y)d\sigma(y),$$

for $d=2$, while for $d=3$,

$$\epsilon(\mathcal{K}_D^\omega)^*(\varphi)(\epsilon x+z)=\mathcal{K}_B^*(\tilde\varphi)(x)$$
$$-\frac{1}{4\pi}\sum_{n=1}^{+\infty}(\frac{1}{n!}-\frac{1}{(n+1)!})(i\omega\epsilon)^{n+1}\int_{\partial B}\langle x-y,\nu(x)\rangle|x-y|^{n-2}\tilde\varphi(y)d\sigma(y),$$

where

$$\mathcal{K}_B^*(\tilde\varphi)(x):=\frac{1}{2(d-1)\pi}p.v.\int_{\partial B}\frac{\langle x-y,\nu(x)\rangle}{|x-y|^d}\tilde\varphi(y)d\sigma(y).$$

As before, define $\tilde\phi(x)=\epsilon\phi(\epsilon x+z)$ and $\tilde\theta(x)=\epsilon\theta(\epsilon x+z)$, $x\in\partial B$. By Lemma 4.3.1 and Lemma 4.4.1, (4.4.2) now takes the form

$$\mathcal{A}_\epsilon^d(\omega)\begin{pmatrix}\psi\\ \tilde\phi\\ \tilde\theta\end{pmatrix}=0,$$

where

$$\mathcal{A}_\epsilon^3(\omega)=\sum_{n=0}^{+\infty}(\omega\epsilon)^n\mathcal{A}_n^3(\omega),$$

with

$$\mathcal{A}_0^3(\omega) := \begin{pmatrix} \frac{1}{2}I - \mathcal{K}_\Omega^\omega & 0 & 0 \\ \mathcal{D}_\Omega^\omega(\cdot)(z) & \mathcal{S}_B^{(0)} & -\mathcal{S}_B^{(0)} \\ 0 & \frac{1}{2}I + \mathcal{K}_B^* & -k(-\frac{1}{2}I + \mathcal{K}_B^*) \end{pmatrix},$$

and, for $n \geq 1$, writing $\mathcal{A}_n^3(\omega) = ((\mathcal{A}_n^3(\omega))_{ll'})_{l,l'=1,2,3}$, we have

$$(\mathcal{A}_n^3(\omega))_{11} = (\mathcal{A}_n^3(\omega))_{13} = 0,$$

$$(\mathcal{A}_n^3(\omega))_{12} = (-1)^n \omega^{-n} \sum_{|\alpha|=n-1} \frac{1}{\alpha!} \partial^\alpha \Gamma_\omega(x-z) \Big(\int_{\partial B} y^\alpha \cdot \, d\sigma(y) \Big),$$

$$(\mathcal{A}_n^3(\omega))_{22} = -\frac{1}{4\pi} \frac{1}{n!} i^n \int_{\partial B} |x-y|^{n-1} \cdot \, d\sigma(y),$$

$$(\mathcal{A}_n^3(\omega))_{23} = \frac{1}{4\pi} \frac{1}{n!} (\frac{i}{\sqrt{k}})^n \int_{\partial B} |x-y|^{n-1} \cdot \, d\sigma(y),$$

$$(\mathcal{A}_n^3(\omega))_{21} = \frac{1}{\omega^n} \sum_{|\alpha|=n} \frac{1}{\alpha!} \partial^\alpha \mathcal{D}_\Omega^\omega(\cdot)(z) x^\alpha,$$

$$(\mathcal{A}_n^3(\omega))_{31} = \frac{1}{\omega^n} \sum_{|\alpha|=n-1} \frac{1}{\alpha!} \partial^\alpha \mathcal{D}_\Omega^\omega(\cdot)(z) \frac{\partial x^\alpha}{\partial \nu},$$

$(\mathcal{A}_1^3(\omega))_{32} = (\mathcal{A}_1^3(\omega))_{33} = 0$, and

$$(\mathcal{A}_n^3(\omega))_{32} = -\frac{1}{4\pi} (\frac{1}{(n-1)!} - \frac{1}{n!}) (i)^n \int_{\partial B} \langle x-y, \nu(x) \rangle |x-y|^{n-3} \cdot d\sigma(y),$$

$$(\mathcal{A}_n^3(\omega))_{33} = \frac{k}{4\pi} (\frac{1}{(n-1)!} - \frac{1}{n!}) (\frac{i}{\sqrt{k}})^n \int_{\partial B} \langle x-y, \nu(x) \rangle |x-y|^{n-3} \cdot d\sigma(y),$$

for $n \geq 2$. Similarly, one can compute $\mathcal{A}_\epsilon^2(\omega)$.

It has been shown in [6] that

$$\begin{pmatrix} \mathcal{S}_B^{(0)} & -\mathcal{S}_B^{(0)} \\ \frac{1}{2}I + \mathcal{K}_B^* & -k(-\frac{1}{2}I + \mathcal{K}_B^*) \end{pmatrix}$$

is invertible. In fact, the inverse is given by

$$\frac{1}{k-1} \begin{pmatrix} k(\lambda I - \mathcal{K}_B^*)^{-1} \big(\frac{1}{2} I - \mathcal{K}_B^* \big) (\mathcal{S}_B^{(0)})^{-1} & (\lambda I - \mathcal{K}_B^*)^{-1} \\ -(\lambda I - \mathcal{K}_B^*)^{-1} \big(\frac{1}{2} I + \mathcal{K}_B^* \big) (\mathcal{S}_B^{(0)})^{-1} & (\lambda I - \mathcal{K}_B^*)^{-1} \end{pmatrix}$$

where $\lambda := \frac{k+1}{2(k-1)}$. Therefore the invertibility of $\mathcal{A}_0^3(\omega)$ holds for any $\omega \notin \{\sqrt{\mu_j}\}_{j \geq 1}$.

The asymptotic expansion can now be constructed in the exactly same manner as in Theorem 4.3.6 and the formula is exactly same as (4.3.6). Analogously, the two-dimensional case can be treated without any new difficulty.

In the same way as in Section 3, we can recover the following result from [15, 14] giving the leading-order term in the asymptotic expansion of the eigenvalue perturbations.

Corollary 4.4.2 *Suppose μ_j is a simple eigenvalue associated with the orthonormal eigenfunction u_j. The following asymptotic expansion holds:*

$$\mu_j^\epsilon - \mu_j = \epsilon^d \nabla u_j(z) \cdot M \nabla u_j(z) + o(\epsilon^d),$$

where

$$M = \int_{\partial B} (\frac{k+1}{2(k-1)} I - \mathcal{K}_B^*)^{-1}(\nu) y d\sigma(y)$$

is the so-called polarization tensor associated with the domain B and the conductivity k. See [5].

We conclude this chapter by making a remark. If we consider the eigenvalue problem in the presence of a diametrically small conductivity inclusion that is nearly touching the boundary then following the arguments given in subsection 4.3.2, we can easily show that the leading-order term in the asymptotic expansion of the eigenvalue perturbations is the same as in Corollary 4.4.2. We leave the details to the reader.

Chapter 5

Sensitivity Analysis of Spectral Properties of High Contrast Band-Gap Materials

5.1 Introduction

Photonic crystals are structures constructed of high-contrast materials arranged in a periodic array. They have attracted enormous interest in the last decade because of their unique optical properties. Such structures have been found to exhibit interesting spectral properties with respect to classical electromagnetic wave propagation, including the appearance of band gaps [99, 51, 79]. Although significant progress has been made, the rigorous analysis of the contrast and geometry dependence of the band-gap of the frequency spectrum for electromagnetic waves in photonic crystals remains problematic. It appears that there are only a few results on the existence of spectral gaps for this type of structures, and these are essentially based on one-dimensional calculations and separation of variables. See [37, 38, 39, 59].

An important example of photonic crystals consists of a background medium which is perforated by an array of arbitrary-shaped holes periodic along each of the two orthogonal coordinate axes in plane. The background medium is of higher index. In this chapter we adopt this specific model to demonstrate our technique and results. We give a full understanding of the relationship between variations in the index ratio or in the geometry of the holes and variations in the band-gap structure of the photonic crystal. We provide such a high-order sensitivity analysis using a boundary integral approach with rigorous justification based on the generalized Rouché's theorem.

Carrying out a band structure calculation for a given photonic crystal involves a family of eigenvalue problems, as the quasi-momentum is varied over the first Brillouin zone. We show that these eigenvalues are the characteristic values of meromorphic operator-valued functions that are of Fredholm type with index zero. We then proceed from the generalized Rouché's theorem to construct their complete asymptotic expressions as the index ratio goes to infinity. We also provide their complete expansions in terms of the infinitesimal changes in the geometry of the holes. We mention, in connection with our approach, the analysis provided by Friedlander in [40] of the Dirichlet-to-Neumann operators in high contrast periodic media.

A range of numerical methods have been developed for band structure calculations in photonic crystals. The most popular ones are based on truncated plane wave decompositions of the electromagnetic fields [49]. These approaches are very natural and commonly used. However, their convergence is slow because of the discontinuity of the underlying medium [87]. Finite element methods, which are suited to handle heterogeneous media, have been successfully introduced for two and three-dimensional photonic crystals [29, 33, 34]. Another approach to the computation of the bandgap structure is based on a combination of boundary element methods and Muller's method [88] for finding complex roots of scalar equations [24, 30]. Our results in this chapter can be used to design a new tool based on a boundary integral perturbation theory for the optimal design problem in photonic crystals.

In this chapter we confine our attention to the two-dimensional case. The asymptotic results for the band-gap structure in three-dimensions can be obtained with only minor modifications of the techniques presented here, while the rigorous derivations of similar asymptotic formulae for the full Maxwell's equations or for the equations of linear elasticity require further work.

5.2 Notation and Preliminaries

5.2.1 The generalized Rouché's theorem

Everything in this section was explained in detail in Chap. 4. Therefore we only recall the following definitions and state the generalized Rouché's theorem.

Let X and Y be two Banach spaces and $\mathcal{L}(X, Y)$ be the space of bounded linear operators acting from X to Y. We denote by 0_X (respectively, 0_Y) the null element of X (respectively, Y). Let $\mathcal{A} : \mathbb{C} \to \mathcal{L}(X, Y)$ be an operator-valued function. The complex number ω_0 is a characteristic value of $\mathcal{A}$ if and only if the function $\mathcal{A}(\omega)$ is holomorphic in a punctured neighborhood of ω_0; there exists a function $x : \mathbb{C} \to X$ such that $x(\omega_0) \neq 0_X, \omega \mapsto x(\omega)$, and $\omega \mapsto \mathcal{A}(\omega)x(\omega)$ are holomorphic in $\omega = \omega_0$, and $\mathcal{A}(\omega_0)x(\omega_0) = 0_Y$. The function $\omega \mapsto x(\omega)$ is a root function associated with ω_0. We call $N(\mathcal{A}(\omega_0))$ the null multiplicity of the characteristic value ω_0 of $\mathcal{A}(\omega)$. If ω_0 is not a characteristic value of $\mathcal{A}(\omega)$, we put $N(\mathcal{A}(\omega_0)) = 0$.

Suppose that $\mathcal{A}^{-1}(\omega)$ exists and is holomorphic in some neighborhood of ω_0, except possibly at this point itself. Then the number

$$M(\mathcal{A}(\omega_0)) = N(\mathcal{A}(\omega_0)) - N(\mathcal{A}^{-1}(\omega_0))$$

is called the multiplicity of the characteristic value ω_0 of $\mathcal{A}(\omega)$.

If $\mathcal{A}(\omega)$ is holomorphic at the point ω_0 and the operator $\mathcal{A}(\omega_0)$ is invertible, then ω_0 is called a regular point of $\mathcal{A}(\omega)$.

The point ω_0 is called a normal point of $\mathcal{A}(\omega)$ if there exists a constant $0 < \delta_0$ such that $\mathcal{A}(\omega)$ is finitely meromorphic and of Fredholm type at ω_0 and all the points of a disc of center ω_0 and radius $\delta_0 > 0$ except ω_0 are regular for $\mathcal{A}(\omega)$.

Let ∂V be the contour bounding the domain V. An operator-valued function $\mathcal{A}(\omega)$ which is finitely meromorphic and of Fredholm type in V and continuous at ∂V is called normal with respect to ∂V if the operator $\mathcal{A}(\omega)$ is invertible in $\overline{V}$, except for a finite number of points of V which are normal points of $\mathcal{A}(\omega)$.

Finally, the generalized Rouché's theorem due to Ghoberg and Sigal [44] reads as follows.

Theorem 5.2.1 *Let ω_0 be a normal point of $\mathcal{A}(\omega)$ and let $V(\omega_0)$ be a neighborhood of ω_0 such that $\mathcal{A}(\omega)$ is normal with respect to $\partial V(\omega_0)$. Let $f(\omega)$ be a scalar function which is analytic in $V(\omega_0)$ and continuous in $\overline{V}(\omega_0)$. Then*

$$\frac{1}{2i\pi} \operatorname{tr} \int_{\partial V(\omega_0)} f(\omega)\mathcal{A}^{-1}(\omega)\frac{d}{d\omega}\mathcal{A}(\omega)d\omega = \sum_{j=1}^{\sigma} M(\mathcal{A}(\omega_j))f(\omega_j),$$

where ω_j, $j = 1, \ldots, \sigma$, are all the points in $V(\omega_0)$ which are either poles or characteristic values of $\mathcal{A}(\omega)$. Here, by tr we mean the trace of operator which is the sum of all its nonzero characteristic values.

5.2.2 Layer potentials for the Helmholtz equation

In this section we collect some notation and well-known results regarding quasi-periodic layer potentials for the Helmholtz equation. We refer to [32, 61, 66, 68, 95] for the details.

In this chapter, the quasi-momentum variable in the Brillouin zone $B = [0, 2\pi[^2$ will be denoted by α. We introduce the two-dimensional quasi-periodic Green's function G_ω^α, which satisfies

$$(\Delta + \omega^2)G_\omega^\alpha(x, y) = \sum_{n \in \mathbb{Z}^2} \delta(x - y - n)e^{in\cdot\alpha}.$$

A function u is said to be quasi-periodic or α-quasi-periodic if $e^{-i\alpha\cdot x}u$ is periodic. If $\omega \neq |2\pi n + \alpha|, \forall\, n \in \mathbb{Z}^2$, then by using the Poisson summation formula:

$$\sum_{n \in \mathbb{Z}^2} e^{i(2\pi n+\alpha)\cdot x} = \sum_{n \in \mathbb{Z}^2} \delta(x - n)e^{i\alpha\cdot x},$$

the quasi-periodic Green's function G_ω^α can be represented as a sum of augmented plane wave over the reciprocal lattice:

$$G_\omega^\alpha(x,y) = \sum_{n\in\mathbb{Z}^2} \frac{e^{i(2\pi n+\alpha)\cdot(x-y)}}{\omega^2 - |2\pi n+\alpha|^2}.$$

Moreover, it can also be shown that

$$G_\omega^\alpha(x,y) = -\frac{i}{4}\sum_{n\in\mathbb{Z}^2} H_0^{(1)}(\omega|x-n-y|)e^{in\cdot\alpha}, \tag{5.2.1}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order 0. The series in (5.2.1) converges uniformly for x,y in compact sets of $\mathbb{R}^2$ and $\omega \neq |2\pi n+\alpha|$ for all $n\in\mathbb{Z}^2$. From (5.2.1) and the well-known fact that $H_0^{(1)}(z) = (2i/\pi)\ln z + O(1)$ as $z\to 0$, it follows that $G_\omega^\alpha(x,y) - (1/2\pi)\ln|x-y|$ is smooth $\forall\, x,y$.

In all the sequel, we assume that $\omega \neq |2\pi n+\alpha|$ for all $n\in\mathbb{Z}^2$. Let D be a bounded domain in $\mathbb{R}^2$, with a connected Lipschitz boundary ∂D. Let ν denote the unit outward normal to ∂D. For $\omega > 0$ let $\mathcal{S}_D^{\alpha,\omega}$ and $\mathcal{D}_D^{\alpha,\omega}$ be the quasi-periodic single and double layer potentials associated with G_ω^α, that is, for a given density $\varphi \in L^2(\partial D)$,

$$\begin{aligned}
\mathcal{S}_D^{\alpha,\omega}\varphi(x) &= \int_{\partial D} G_\omega^\alpha(x,y)\varphi(y)\,d\sigma(y), \quad x\in\mathbb{R}^2,\\
\mathcal{D}_D^{\alpha,\omega}\varphi(x) &= \int_{\partial D} \frac{\partial G_\omega^\alpha(x,y)}{\partial\nu_y}\varphi(y)\,d\sigma(y), \quad x\in\mathbb{R}^2\setminus\partial D.
\end{aligned}$$

Then, $\mathcal{S}_D^{\alpha,\omega}\varphi$ and $\mathcal{D}_D^{\alpha,\omega}\varphi$ satisfy $(\Delta+\omega^2)\mathcal{S}_D^{\alpha,\omega}\varphi = (\Delta+\omega^2)\mathcal{D}_D^{\alpha,\omega}\varphi = 0$ in D and $Y\setminus\overline{D}$ where Y is the periodic cell $[0,1[^2$, and they are α-quasi-periodic.

The next formulae give the jump relations obeyed by the double layer potential and by the normal derivative of the single layer potential on general Lipschitz domains:

$$\frac{\partial(\mathcal{S}_D^{\alpha,\omega}\varphi)}{\partial\nu}\bigg|_\pm(x) = \Big(\pm\frac{1}{2}I + (\mathcal{K}_D^{\alpha,\omega})^*\Big)\varphi(x) \quad \text{a.e. } x\in\partial D, \tag{5.2.2}$$

$$(\mathcal{D}_D^{\alpha,\omega}\varphi)\bigg|_\pm(x) = \Big(\mp\frac{1}{2}I + \mathcal{K}_D^{\alpha,\omega}\Big)\varphi(x) \quad \text{a.e. } x\in\partial D, \tag{5.2.3}$$

for $\varphi\in L^2(\partial D)$, where $\mathcal{K}_D^{\alpha,\omega}$ is the operator on $L^2(\partial D)$ defined by

$$\mathcal{K}_D^{\alpha,\omega}\varphi(x) = \text{p.v.}\int_{\partial D}\frac{\partial G_\omega^\alpha(x,y)}{\partial\nu_y}\varphi(y)d\sigma(y),$$

and $(\mathcal{K}_D^{\alpha,\omega})^*$ is given by

$$(\mathcal{K}_D^{\alpha,\omega})^*\varphi(x) = \text{p.v.}\int_{\partial D}\frac{\partial G_\omega^\alpha(x,y)}{\partial\nu_x}\varphi(y)d\sigma(y).$$

Here p.v. stands for the Cauchy principal value. The singular integral operators $\mathcal{K}_D^{\alpha,\omega}$ and $(\mathcal{K}_D^{\alpha,\omega})^*$ are bounded on $L^2(\partial D)$ as an immediate consequence of the celebrated theorem of Coifman-McIntosh-Meyer [27].

Let the space $W_1^2(\partial D)$ be the set of functions $f \in L^2(\partial D)$ such that $\partial f/\partial T \in L^2(\partial D)$, where $\partial/\partial T$ denotes the tangential derivative on ∂D. The following lemma is of use to us.

Lemma 5.2.2 *Suppose that $\alpha \neq 0$ and ω^2 is neither an eigenvalue of $-\Delta$ in D with the Dirichlet boundary condition on ∂D nor in $Y \setminus \overline{D}$ with the Dirichlet boundary condition on ∂D and the quasi-periodic condition on ∂Y. Then $\mathcal{S}_D^{\alpha,\omega} : L^2(\partial D) \to W_1^2(\partial D)$ is invertible.*

Proof. Suppose that $\phi \in L^2(\partial D)$ satisfies $\mathcal{S}_D^{\alpha,\omega}\phi = 0$ on ∂D. Then $u = \mathcal{S}_D^{\alpha,\omega}\phi$ satisfies $(\Delta + \omega^2)u = 0$ in D and in $Y \setminus \overline{D}$. Therefore, since ω^2 is neither an eigenvalue of $-\Delta$ in D with the Dirichlet boundary condition nor in $Y \setminus \overline{D}$ with the Dirichlet boundary condition on ∂D and the quasi-periodic condition on ∂Y, it follows that $u = 0$ in Y and thus, $\phi = \partial u/\partial\nu|_+ - \partial u/\partial\nu|_- = 0$, as desired. □

Define

$$G_0^\alpha(x,y) = -\sum_{n\in\mathbb{Z}^2} \frac{e^{i(2\pi n+\alpha)\cdot(x-y)}}{|2\pi n + \alpha|^2} \quad \text{for } \alpha \neq 0,$$

and

$$G_0^0(x,y) = -\sum_{n\in\mathbb{Z}^2\setminus\{0\}} \frac{e^{i2\pi n\cdot(x-y)}}{4\pi^2|n|^2}.$$

Note that $G_0^\alpha(x,y), \alpha \neq 0$, is a fundamental solution of the quasi-periodic Laplacian in Y, while $G_0^0(x,y)$ satisfies $\Delta_x G_0^0(x,y) = \delta_y - 1$ in Y with periodic Dirichlet boundary conditions on ∂Y. See [12, 7]. The following lemma is easy to prove.

Lemma 5.2.3 *As $\omega \to 0$,*

$$G_\omega^\alpha(x,y) = G_0^\alpha(x,y) - \sum_{l=1}^{+\infty} \underbrace{\omega^{2l} \sum_{n\in\mathbb{Z}^2} \frac{e^{i(2\pi n+\alpha)\cdot(x-y)}}{|2\pi n + \alpha|^{2(l+1)}}}_{:=-G_l^\alpha(x,y)},$$

for $\alpha \neq 0$, while for $\alpha = 0$,

$$G_\omega^0(x,y) = \frac{1}{\omega^2} + G_0^0(x,y) - \sum_{l=1}^{+\infty} \underbrace{\omega^{2l} \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \frac{e^{i2\pi n\cdot(x-y)}}{(4\pi^2)^{l+1}|n|^{2(l+1)}}}_{:=-G_l^0(x,y)}.$$

Denote by $\mathcal{S}_{D,l}^\alpha, (\mathcal{K}_{D,l}^\alpha)^*$, for $l \geq 0$, and $\alpha \in [0, 2\pi[^2$, the layer potentials associated with the kernel $G_l^\alpha(x,y)$ so that

$$\mathcal{S}_D^{\alpha,\omega} = \sum_{l=0}^{\infty} \mathcal{S}_{D,l}^\alpha \quad \text{and} \quad (\mathcal{K}_D^{\alpha,\omega})^* = \sum_{l=0}^{\infty} (\mathcal{K}_{D,l}^\alpha)^*. \tag{5.2.4}$$

Let $L^2_0(\partial D)$ be the space of L^2-functions on ∂D with zero mean-value.

Lemma 5.2.4 *If $\alpha \neq 0$, then the operator $\frac{1}{2}I + (\mathcal{K}^{\alpha}_{D,0})^* : L^2(\partial D) \to L^2(\partial D)$ is invertible. If $\alpha = 0$, then $\frac{1}{2}I + (\mathcal{K}^{0}_{D,0})^* : L^2_0(\partial D) \to L^2_0(\partial D)$ is invertible.*

Before proving Lemma 5.2.4, let us make a note of the following simple fact: If u and v are α-quasi-periodic, then

$$\int_{\partial Y} \frac{\partial u}{\partial \nu} \overline{v} d\sigma = 0. \tag{5.2.5}$$

To prove this, it is enough to see that

$$\int_{\partial Y} \frac{\partial u}{\partial \nu} \overline{v} = \int_{\partial Y} \Big[\frac{\partial (u e^{-i\alpha\cdot x})}{\partial \nu} - i\alpha \cdot \nu u e^{-i\alpha \cdot x} \Big] \overline{e^{-i\alpha\cdot x} v}.$$

Proof of Lemma 5.2.4. Suppose that $\alpha \neq 0$ and let $\phi \in L^2(\partial D)$ satisfy $(\frac{1}{2}I + (\mathcal{K}^{\alpha,0}_D)^*)\phi = 0$ on ∂D. Then $u = \mathcal{S}^{\alpha,0}_D \phi$ satisfies $\Delta u = 0$ in $Y \setminus \overline{D}$ with $\partial u / \partial \nu |_+ = (\frac{1}{2}I + (\mathcal{K}^{\alpha,0}_D)^*)\phi = 0$ on ∂D, and u is α-quasi-periodic. Therefore, it follows from (5.2.5) that

$$\int_{Y\setminus D} |\nabla u|^2 = \int_{\partial Y} \frac{\partial u}{\partial \nu}\overline{u} - \int_{\partial D} \frac{\partial u}{\partial \nu}\Big|_+ \overline{u} = 0.$$

Thus, u is constant in $Y \setminus \overline{D}$. Since u is α-quasi-periodic and $\alpha \neq 0$, we get $u = 0$ in $Y \setminus \overline{D}$, and hence in D. Thus, we get $\phi = \partial u/\partial \nu|_+ - \partial u/\partial\nu|_- = 0$. By the same argument, one can show that $\frac{1}{2}I + (\mathcal{K}^{0,0}_D)^* : L^2_0(\partial D) \to L^2_0(\partial D)$ is invertible. □

5.2.3 Problem formulation

The photonic crystal we consider in this chapter consists of a homogeneous background medium of constant index k which is perforated by an array of arbitrary-shaped holes periodic along each of the two orthogonal coordinate axes in $\mathbb{R}^2$. These holes are assumed to be of index 1. We assume that the structure has unit periodicity and define the periodic domain $Y = \mathbb{R}^2/\mathbb{Z}^2$, which can be identified with the unit square $[0, 1]^2$.

Suppose $\alpha \neq 0$. We seek eigenfunctions u of

$$\begin{cases} \nabla \cdot (1 + (k-1)\chi_{Y\setminus\overline{D}})\nabla u + \omega^2 u = 0 \quad \text{in } Y, \\ e^{-i\alpha\cdot x} u \text{ is periodic}, \end{cases} \tag{5.2.6}$$

where $\chi_{Y\setminus\overline{D}}$ is the indicator function of $Y \setminus \overline{D}$. The problem (5.2.6) can be rewritten as

$$\begin{cases} k\Delta u + \omega^2 u = 0 & \text{in } Y \setminus \overline{D}, \\ \Delta u + \omega^2 u = 0 & \text{in } D, \\ u|_+ = u|_- & \text{on } \partial D, \\ k\dfrac{\partial u}{\partial \nu}\Big|_+ = \dfrac{\partial u}{\partial \nu}\Big|_- & \text{on } \partial D, \\ e^{-i\alpha\cdot x} u \text{ is periodic}. \end{cases} \tag{5.2.7}$$

For each quasi-momentum variable α and k, let $\sigma_\alpha(D,k)$ be the (discrete) spectrum of (5.2.6). Then the spectral band of the photonic crystal is given by

$$\bigcup_{\alpha\in[0,2\pi]^2} \sigma_\alpha(D,k).$$

We shall investigate the behavior of $\sigma_\alpha(D,k)$ when $k\to\infty$ in Section 5.3 and that under perturbation of D in Section 5.4.

Suppose that ω^2 is not an eigenvalue of $-\Delta$ in $Y\setminus\overline{D}$ with the Dirichlet boundary condition on ∂D and the quasi-periodic condition on ∂Y and ω^2/k is not an eigenvalue of $-\Delta$ in D with the Dirichlet boundary condition. Following the same argument as in [4], one can show that the solution u to (5.2.6) can be represented as

$$u(x)=\begin{cases} \mathcal{S}_D^{\alpha,\omega}\phi(x), & x\in D,\\ H(x)+\mathcal{S}_D^{\alpha,\frac{\omega}{\sqrt{k}}}\psi(x), & x\in Y\setminus\overline{D},\end{cases}$$

for some potentials ϕ and ψ in $L^2(\partial D)$, where the function H is given by

$$H(x)=-\mathcal{S}_Y^{\alpha,\frac{\omega}{\sqrt{k}}}(\frac{\partial u}{\partial\nu}|_{\partial Y})+\mathcal{D}_Y^{\alpha,\frac{\omega}{\sqrt{k}}}(u|_{\partial Y}),\quad x\in Y.$$

But by (5.2.5) we have $H\equiv 0$, and hence

$$u(x)=\begin{cases} \mathcal{S}_D^{\alpha,\omega}\phi(x), & x\in D,\\ \mathcal{S}_D^{\alpha,\frac{\omega}{\sqrt{k}}}\psi(x). & x\in Y\setminus\overline{D}.\end{cases} \tag{5.2.8}$$

See Appendix A for a proof of (5.2.8).

In view of the the transmission conditions in (5.2.7), the pair $(\phi,\psi)\in L^2(\partial D)\times L^2(\partial D)$ satisfies the following system of integral equations:

$$\begin{cases} \mathcal{S}_D^{\alpha,\omega}\phi-\mathcal{S}_D^{\alpha,\frac{\omega}{\sqrt{k}}}\psi=0 & \text{on } \partial D,\\ \Big(-\dfrac{1}{2}I+(\mathcal{K}_D^{\alpha,\omega})^*\Big)\phi-k\Big(\dfrac{1}{2}I+(\mathcal{K}_D^{\alpha,\frac{\omega}{\sqrt{k}}})^*\Big)\psi=0 & \text{on } \partial D.\end{cases} \tag{5.2.9}$$

The converse is also true. If $(\phi,\psi)\in L^2(\partial D)\times L^2(\partial D)$ is a non-zero solution of (5.2.9) then u given by (5.2.8) is an eigenfunction of (5.2.6) associated to the eigenvalue ω^2.

Let $\mathcal{A}_k^\alpha(\omega)$ be the operator-valued function defined by

$$\mathcal{A}_k^\alpha(\omega):=\begin{pmatrix} \mathcal{S}_D^{\alpha,\omega} & -\mathcal{S}_D^{\alpha,\frac{\omega}{\sqrt{k}}}\\ \dfrac{1}{k}\Big(\dfrac{1}{2}I-(\mathcal{K}_D^{\alpha,\omega})^*\Big) & \dfrac{1}{2}I+(\mathcal{K}_D^{\alpha,\frac{\omega}{\sqrt{k}}})^*\end{pmatrix}. \tag{5.2.10}$$

Then, ω^2 is an eigenvalue corresponding to u with given quasi-momentum α if and only if ω is a characteristic value of $\mathcal{A}_k^\alpha$.

For $\alpha = 0$, $\mathcal{A}_k^0$ given by (5.2.10) with $\alpha = 0$ acts on $L_0^2(\partial D) \times L_0^2(\partial D) \to L_0^2(\partial D) \times L_0^2(\partial D)$. Conversely, if ω is a characteristic value of $\mathcal{A}_k^0$, then ω^2 is an eigenvalue of (5.2.6).

Consequently, we have now a new way of looking at the spectrum of (5.2.6) by examining the characteristic values of $\mathcal{A}_k^\alpha$. Expanding this operator-valued function in terms of k and small perturbations of the shape of D, we calculate asymptotic expressions of its characteristic values with the help of the generalized Rouché's theorem.

Lemma 5.2.5 *The operator-valued function $\mathcal{A}_k^\alpha$ is Fredholm analytic with index 0 in $\mathbb{C} \setminus i\mathbb{R}_-$. Moreover, $\omega \mapsto (\mathcal{A}_k^\alpha)^{-1}(\omega)$ is meromorphic function and its poles are on the real axis.*

Proof. To see that the operator-valued function $\mathcal{A}_k^\alpha$ is Fredholm analytic with index 0 in $\mathbb{C} \setminus i\mathbb{R}_-$, it suffices to write

$$\mathcal{A}_k^\alpha(\omega) = \begin{pmatrix} \mathcal{S}_D^{\alpha,0} & -\mathcal{S}_D^{\alpha,0} \\ \dfrac{1}{2k} I & \dfrac{1}{2} I \end{pmatrix} + \begin{pmatrix} \mathcal{S}_D^{\alpha,\omega} - \mathcal{S}_D^{\alpha,0} & -\mathcal{S}_D^{\alpha,\frac{\omega}{\sqrt{k}}} + \mathcal{S}_D^{\alpha,0} \\ \dfrac{1}{k}(\mathcal{K}_D^{\alpha,\omega})^* & (\mathcal{K}_D^{\alpha,\omega})^* \end{pmatrix} := \mathcal{A}^\alpha + \mathcal{B}^\alpha(\omega).$$

Since $\mathcal{A}^\alpha$ is invertible and $\mathcal{B}^\alpha$ is compact and analytic in ω, it follows that $\mathcal{A}_k^\alpha$ is Fredholm analytic with index 0. By the generalization of the Steinberg theorem given in [10, Theorem 2.4], the invertibility of $\mathcal{A}_k^\alpha(\omega)$ at $\omega = 0$ shows that $\omega \mapsto (\mathcal{A}_k^\alpha)^{-1}(\omega)$ is meromorphic function. See [91]. Let ω_0 be a pole of $(\mathcal{A}_k^\alpha)^{-1}(\omega)$. Then ω_0 is a characteristic value of $\mathcal{A}_k^\alpha$. Set (ϕ, ψ) to be a root function associated with ω_0. Define

$$u = \begin{cases} \mathcal{S}_D^{\alpha,\omega_0}\phi(x), & x \in D, \\ \mathcal{S}_D^{\alpha,\frac{\omega_0}{\sqrt{k}}}\psi(x), & x \in Y \setminus \overline{D}. \end{cases}$$

Then, integrating by parts, we obtain that

$$\int_Y (1 + (k-1)\chi_{Y\setminus D})|\nabla u|^2 - \omega_0^2 \int_Y |u|^2 = 0,$$

which shows that ω_0 is real. $\square$

5.3 Sensitivity analysis with respect to the index ratio

The following lemma, which is an immediate consequence of (5.2.4), gives a complete asymptotic expansion of $\mathcal{A}_k^\alpha$ as $k \to +\infty$.

Lemma 5.3.1 *Let*

$$\mathcal{A}_0^\alpha(\omega) = \begin{pmatrix} \mathcal{S}_D^{\alpha,\omega} & -\mathcal{S}_{D,0}^\alpha \\ 0 & \frac{1}{2}I + (\mathcal{K}_{D,0}^\alpha)^* \end{pmatrix}, \quad \mathcal{A}_1^\alpha(\omega) = \begin{pmatrix} 0 & -\mathcal{S}_{D,1}^\alpha \\ \Big(\frac{1}{2}I - (\mathcal{K}_D^{\alpha,\omega})^*\Big) & (\mathcal{K}_{D,1}^\alpha)^* \end{pmatrix},$$

and, for $l \geq 2$,

$$\mathcal{A}_l^\alpha(\omega) = \begin{pmatrix} 0 & -\mathcal{S}_{D,l}^\alpha \\ 0 & (\mathcal{K}_{D,l}^\alpha)^* \end{pmatrix}.$$

If $\alpha \neq 0$, *then we have*

$$\mathcal{A}_k^\alpha(\omega) = \mathcal{A}_0^\alpha(\omega) + \sum_{l=1}^{+\infty} \frac{1}{k^l} \mathcal{A}_l^\alpha(\omega), \tag{5.3.1}$$

If $\alpha = 0$ *then*

$$\mathcal{A}_k^0(\omega) = \frac{k}{\omega^2} \begin{pmatrix} 0 & -\displaystyle\int_{\partial D} \cdot \\ 0 & 0 \end{pmatrix} + \mathcal{A}_0^0(\omega) + \sum_{l=1}^{+\infty} \frac{1}{k^l} \mathcal{A}_l^0(\omega). \tag{5.3.2}$$

We now have the following lemma for the characteristic values of $\mathcal{A}_0^\alpha$.

Lemma 5.3.2 *Suppose* $\alpha \neq 0$. $\omega_0^\alpha \in \mathbb{R}$ *is a characteristic value of* $\mathcal{A}_0^\alpha$ *if and only if* $(\omega_0^\alpha)^2$ *is either an eigenvalue of* $-\Delta$ *in* D *with the Dirichlet boundary condition or an eigenvalue of* $-\Delta$ *in* $Y \setminus \overline{D}$ *with the Dirichlet boundary condition on* ∂D *and the quasi-periodic condition on* ∂Y

Proof. Suppose that $\omega = \omega_0^\alpha \in \mathbb{R}$ is a characteristic value of $\mathcal{A}_0^\alpha$. Then there is $(\phi, \psi) \neq 0$ such that

$$\begin{cases} \mathcal{S}_D^{\alpha,\omega}\phi - \mathcal{S}_{D,0}^\alpha \psi = 0 \\ \Big(\frac{1}{2}I + (\mathcal{K}_{D,0}^\alpha)^*\Big)\psi = 0 \end{cases} \quad \text{on } \partial D. \tag{5.3.3}$$

It then follows from Lemma 5.2.4 that $\psi = 0$ and hence $\mathcal{S}_D^{\alpha,\omega}\phi = 0$ on ∂D. Since $\phi \neq 0$, $\mathcal{S}_D^{\alpha,\omega}\phi \neq 0$ either in D or in $Y \setminus \overline{D}$ and hence $(\omega_0^\alpha)^2$ is either an eigenvalue of $-\Delta$ in D with the Dirichlet boundary condition or an eigenvalue of $-\Delta$ in $Y \setminus \overline{D}$ with the Dirichlet boundary condition on ∂D and the quasi-periodic condition on ∂Y, and its eigenfunction is given by $\mathcal{S}_D^{\alpha,\omega}\phi$.

Conversely if $(\omega_0^\alpha)^2$ is an eigenvalue of $-\Delta$ in D with the Dirichlet boundary condition, then by the Green's representation, we have

$$u(x) = -\mathcal{S}_D^{\alpha,\omega}\Big(\frac{\partial u}{\partial \nu}\Big|_{\partial D}\Big), \quad x \in D.$$

Thus (5.3.3) holds with $(\phi, \psi) = (\frac{\partial u}{\partial \nu}|_{\partial D}, 0)$. The other case can be treated similarly using (5.2.5). This completes the proof. □

At this moment let us invoke the result of Hempel and Lienau [47] (see also [40, 46, 82]). Hempel and Lienau showed, by a completely different argument which involves the convergence of quadratic forms, that the spectrum of (5.2.6) is accumulating near the spectrum of $-\Delta$ in D with the Dirichlet boundary condition on ∂D as $k \to \infty$. According to this result, the eigenvalue of the exterior problem is not realized as a limit of eigenvalues of the problem (5.2.6). In fact, the limit of the corresponding eigenfunctions is given by

$$u(x) = \begin{cases} \mathcal{S}_D^{\alpha,\omega}\phi, & x \in D, \\ \mathcal{S}_{D,0}^{\alpha}\psi = 0, & x \in Y \setminus \overline{D}, \end{cases}$$

where (ϕ, ψ) is defined by (5.3.3). If $(\omega_0^\alpha)^2$ is an eigenvalue for the exterior problem and not for the interior problem, then $\mathcal{S}_D^{\alpha,\omega}\phi = 0$ in D and hence $u = 0$ in Y.

Let ω_0^2 (with $\omega_0 > 0$) be a simple eigenvalue of $-\Delta$ in D with the Dirichlet boundary condition. There exists a unique eigenvalue $(\omega_k^\alpha)^2$ (with $\omega_k^\alpha > 0$) of (5.2.6) lying in a small complex neighborhood V of ω_0 [47, 46]. ω_0 and ω_k^α are simple poles of $(\mathcal{A}_0^\alpha)^{-1}$ and $(\mathcal{A}_k^\alpha)^{-1}$, respectively.

Combining the generalized Rouché's theorem together with Lemma 5.3.1 we are now able to derive complete asymptotic formulae for the characteristic values of $\omega \mapsto \mathcal{A}_k^\alpha(\omega)$. Applying Theorem 4.2.5 yields that

$$\omega_k^\alpha - \omega_0 = \frac{1}{2i\pi} \operatorname{tr} \int_{\partial V} (\omega - \omega_0)(\mathcal{A}_k^\alpha)^{-1}(\omega) \frac{d}{d\omega} \mathcal{A}_k^\alpha(\omega) d\omega.$$

Suppose that the quasi-momentum $\alpha \neq 0$. Analogously to [10, Theorem 3.6], we obtain the following complete asymptotic expansion for the eigenvalue perturbations $\omega_k^\alpha - \omega_0$.

Theorem 5.3.3 *Suppose $\alpha \neq 0$. Then the following asymptotic expansion holds:*

$$\omega_k^\alpha - \omega_0 = \frac{1}{2i\pi} \sum_{p=1}^{+\infty} \frac{1}{p} \sum_{n=p}^{+\infty} \frac{1}{k^n} \, tr \int_{\partial V} (\mathcal{A}_0^\alpha)^{-p}(\omega) B_{n,p}^\alpha(\omega) d\omega,$$

where

$$B_{n,p}^\alpha(\omega) = (-1)^p \sum_{\substack{n_1 + \ldots + n_p = n \\ n_i \geq 1}} \mathcal{A}_{n_1}^\alpha(\omega) \ldots \mathcal{A}_{n_p}^\alpha(\omega), \tag{5.3.4}$$

and

$$(\mathcal{A}_0^\alpha)^{-1}(\omega) = \begin{pmatrix} (\mathcal{S}_D^{\alpha,\omega})^{-1} & (\mathcal{S}_D^{\alpha,\omega})^{-1} \mathcal{S}_{D,0}^\alpha (\frac{1}{2} I + (\mathcal{K}_{D,0}^\alpha)^*)^{-1} \\ 0 & (\frac{1}{2} I + (\mathcal{K}_{D,0}^\alpha)^*)^{-1} \end{pmatrix}. \tag{5.3.5}$$

Let us compute the leading order term in the expansion of $\omega_k^\alpha - \omega_0$. Let u_0 be the (normalized) eigenvector associated to the simple eigenvalue ω_0 and let

$\varphi := \frac{\partial u_0}{\partial \nu}|_-$. Since ω_0 is the only simple zero in V of the mapping $\omega \mapsto \mathcal{S}_D^{\alpha,\omega}$, we can write

$$(\mathcal{S}_D^{\alpha,\omega})^{-1} = \frac{1}{\omega - \omega_0} T + \mathcal{Q}_D^{\alpha,\omega},$$

where T is defined by $Tf = (\varphi, f)\varphi$, the operator-valued function $\mathcal{Q}_D^{\alpha,\omega}$ is holomorphic in ω in V, and $(\cdot,\cdot)$ is the L^2-inner product. Moreover, since $u_0(x) = -\mathcal{S}_D^{\alpha,\omega}(\varphi)(x)$ for $x \in D$, we get

$$(\frac{1}{2}I - (\mathcal{K}_D^{\alpha,\omega_0})^*)(\varphi) = \varphi.$$

It then follows from the Residue theorem that

$$\frac{1}{2i\pi} \operatorname{tr} \int_{\partial V} (\mathcal{A}_0^\alpha)^{-1}(\omega)\mathcal{A}_1^\alpha(\omega)d\omega = \operatorname{tr}\Big[T\mathcal{S}_{D,0}^\alpha\big(\frac{1}{2}I + (\mathcal{K}_{D,0}^\alpha)^*\big)^{-1}\big(\frac{1}{2}I - (\mathcal{K}_D^{\alpha,\omega})^*\big)\Big].$$

Let

$$v_0(x) := \mathcal{S}_{D,0}^\alpha\big(\frac{1}{2}I + (\mathcal{K}_{D,0}^\alpha)^*\big)^{-1}(\varphi)(x), \quad x \in Y \setminus \overline{D}.$$

Then v_0 is the unique α-quasi-periodic solution to

$$\begin{cases} \Delta v_0 = 0 & \text{in } Y \setminus \overline{D}, \\ \dfrac{\partial v_0}{\partial \nu}\Big|_+ = \dfrac{\partial u_0}{\partial \nu}\Big|_- & \text{on } \partial D, \end{cases}$$

and

$$\frac{1}{2i\pi} \operatorname{tr} \int_{\partial V} (\mathcal{A}_0^\alpha)^{-1}(\omega)\mathcal{A}_1^\alpha(\omega)d\omega = (\varphi, v_0) = \int_{Y\setminus\overline{D}} |\nabla v_0|^2.$$

So we have the following corollary.

Corollary 5.3.4

$$\omega_k^\alpha - \omega_0 = -\frac{1}{k}\int_{Y\setminus\overline{D}} |\nabla v_0|^2 + O(k^{-2}) \quad \textit{as } k \to \infty.$$

Turning now to the periodic case ($\alpha = 0$), we first establish a lemma on the characteristic value of $\mathcal{A}_0^0$. For doing so, we need the following notation. Let $W_0^{1,2}(D)$ be the set of functions $f \in L^2(D)$ such that $\nabla f \in L^2(D)$ and $f = 0$ on ∂D and let 1_Y denote the constant function 1 on Y. Let the operator $\widetilde{\Delta}$ be acting on $\operatorname{span}\{1_Y, W_0^{1,2}(D)\}$, with

$$\widetilde{\Delta} u := \begin{cases} -\Delta(u|_D) & \text{in } D, \\ \dfrac{1}{|Y\setminus\overline{D}|}\displaystyle\int_{\partial D} \frac{\partial}{\partial \nu}(u|_D) & \text{in } Y \setminus \overline{D}. \end{cases}$$

See [47].

Lemma 5.3.5 *ω_0^2 (with $\omega_0 > 0$) is an eigenvalue of $\widetilde{\Delta}$ if and only if ω_0 is a characteristic value of the operator-valued function $\mathcal{A}_0^0$ acting on $L_0^2(\partial D) \times L_0^2(\partial D) \to L_0^2(\partial D) \times L_0^2(\partial D)$.*

Analogously to Theorem 5.3.3, the asymptotic formula for $\alpha = 0$ follows from a direct application of Theorem 4.2.5.

Theorem 5.3.6 *Suppose $\alpha = 0$. Let ω_0^2 (with $\omega_0 > 0$) be a simple eigenvalue of $\widetilde{\Delta}$. There exists a unique eigenvalue $(\omega_k^\alpha)^2$ (with $\omega_k^\alpha > 0$) of (5.2.6) lying in a small complex neighborhood of ω_0^2 and the following asymptotic expansion holds:*

$$\omega_k^\alpha - \omega_0 = \frac{1}{2i\pi} \sum_{p=1}^{+\infty} \frac{1}{p} \sum_{n=p}^{+\infty} \frac{1}{k^n} \; tr \int_{\partial V} (\mathcal{A}_0^0)^{-p}(\omega) B_{n,p}^\alpha(\omega) d\omega,$$

where V is a small complex neighborhood of ω_0 and $B_{n,p}^\alpha(\omega)$ and $(\mathcal{A}_0^0)^{-1}(\omega)$ are given by (5.3.4) and (5.3.5) with $\alpha = 0$.

5.4 Sensitivity analysis with respect to small perturbations in the geometry of the holes

Suppose that D is of class $\mathcal{C}^2$. Let D_ϵ be an ϵ-perturbation of D, *i.e.*, let $h \in \mathcal{C}^1(\partial D)$ and ∂D_ϵ be given by

$$\partial D_\epsilon = \{\tilde{x} : \tilde{x} = x + \epsilon h(x)\nu(x), \; x \in \partial D \; \}.$$

Define the operator-valued function $\mathcal{A}_\epsilon^\alpha$ by

$$\mathcal{A}_\epsilon^\alpha : \omega \mapsto \begin{pmatrix} \mathcal{S}_{D_\epsilon}^{\alpha,\omega} & -\mathcal{S}_{D_\epsilon}^{\alpha,\frac{\omega}{\sqrt{k}}} \\ \frac{1}{k}\left(\frac{1}{2}I - (\mathcal{K}_{D_\epsilon}^{\alpha,\omega})^*\right) & \frac{1}{2}I + (\mathcal{K}_{D_\epsilon}^{\alpha,\frac{\omega}{\sqrt{k}}})^* \end{pmatrix}.$$

Write

$$\frac{\partial G_\omega^\alpha}{\partial \nu_x}(x,y) = \frac{1}{2\pi} \frac{\langle x-y, \nu_x \rangle}{|x-y|^2} + R_\omega^\alpha(x,y),$$

where $R_\omega^\alpha(x,y)$ is smooth $\forall \; x, y$. Following Chap. 2, we have a uniformly convergent expansion for the length element $d\sigma_\epsilon(\tilde{y})$ on ∂D_ϵ:

$$d\sigma_\epsilon(\tilde{y}) = \sum_{n=0}^{\infty} \epsilon^n \sigma^{(n)}(y) d\sigma(y),$$

where $\sigma^{(n)}$ are bounded functions, and easily prove that the following lemma holds.

Lemma 5.4.1 *Let Ψ_ϵ be the diffeomorphism from ∂D onto ∂D_ϵ given by $\Psi_\epsilon(x) = x + \epsilon h(x)\nu(x)$. Let $N \in \mathbb{N}$. There exists C depending only on N, $\mathcal{C}^2$-norm of D, and $\|h\|_{\mathcal{C}^1(\partial D)}$ such that for any $\tilde{\varphi} \in L^2(\partial D_\epsilon)$,*

$$||\mathcal{S}_{D_\epsilon}^{\alpha,\omega}\tilde{\varphi} \circ \Psi_\epsilon - \mathcal{S}_D^{\alpha,\omega}\varphi - \sum_{n=1}^{N} \epsilon^n \mathcal{S}_{D,\alpha,\omega}^{(n)}\varphi||_{L^2(\partial D)} \leq C\epsilon^{N+1}||\varphi||_{L^2(\partial D)},$$

and

$$||((\mathcal{K}_{D_\epsilon}^{\alpha,\omega})^*\tilde{\varphi}) \circ \Psi_\epsilon - (\mathcal{K}_D^{\alpha,\omega})^*\varphi - \sum_{n=1}^{N} \epsilon^n \mathcal{K}_{D,\alpha,\omega}^{(n)}\varphi||_{L^2(\partial D)} \leq C\epsilon^{N+1}||\varphi||_{L^2(\partial D)},$$

where $\varphi := \tilde{\varphi} \circ \Psi_\epsilon$. Here

$$\mathcal{S}_{D,\alpha,\omega}^{(n)}\varphi(x) = \sum_{l=0}^{n} \sum_{|\beta|=l} \frac{1}{\beta!} \int_{\partial D} \partial^\beta G_\omega^\alpha(x,y)(h(x)\nu(x) - h(y)\nu(y))^l \sigma^{(n-l)}(y)\varphi(y)d\sigma(y),$$

and

$$\begin{aligned}\mathcal{K}_{D,\alpha,\omega}^{(n)}\varphi(x) &= \mathcal{K}_D^{(n)}\varphi(x)\\ &\quad + \sum_{l=0}^{n} \sum_{|\beta|=l} \frac{1}{\beta!} \int_{\partial D} \partial^\beta R_\omega^\alpha(x,y)(h(x)\nu(x) - h(y)\nu(y))^l \sigma^{(n-l)}(y)\varphi(y)d\sigma(y),\end{aligned}$$

and the bounded operators $\mathcal{K}_D^{(n)}$ are defined in Theorem 2.2.1.

The sensitivity analysis with respect to small perturbations in the geometry of the holes consists of expanding, based on Lemma 5.4.1, $\mathcal{A}_\epsilon^\alpha$ in terms of ϵ to calculate asymptotic expressions of its characteristic values. This can be done in exactly the same manner as in Theorem 5.3.3.

Appendix A

On the representation formula (5.2.8)

In this appendix, we present a proof of the following theorem which played a crucial role in analysis of Chap. 5.

Theorem A.0.2 *Suppose that ω^2 is not an eigenvalue for $-\Delta$ in $Y \setminus \overline{D}$ with Dirichlet boundary condition on ∂D and quasi-periodic boundary condition on ∂Y and $\dfrac{\omega^2}{k}$ is not an eigenvalue for $-\Delta$ in D with Dirichlet boundary condition on ∂D. Then, for any eigenfunction u of (5.2.6), there exists one and only one pair $(\phi, \psi) \in L^2(\partial D) \times L^2(\partial D)$ such that u has the representation (5.2.8). Moreover, (ϕ, ψ) is solution to the integral equation (5.2.9). The mapping $u \longmapsto (\phi, \psi)$ from solutions of (5.2.6) to solutions to the system of integral equations (5.2.9) is one-to-one.*

We first prove the following lemma.

Lemma A.0.3 *Suppose that u is an eigenvalue of (5.2.6), then $u \perp Ker(\mathcal{S}_D^{\alpha,\omega})$.*

Proof. To prove this lemma, we observe that, since $(\Delta + \omega^2)u = 0$ in D,

$$u(x) = \mathcal{D}_D^{\alpha,\omega}\left(u|_{\partial D}\right)(x) - \mathcal{S}_D^{\alpha,\omega}\left(\frac{\partial u}{\partial \nu}\Big|_-\right)(x), \quad x \in D,$$

and consequently,

$$\frac{1}{2}u|_{\partial D} = \mathcal{K}_D^{\alpha,\omega}\left(u|_{\partial D}\right) - \mathcal{S}_D^{\alpha,\omega}\left(\frac{\partial u}{\partial \nu}\Big|_-\right).$$

Let $\phi \in \mathrm{Ker}(\mathcal{S}_D^{\alpha,\omega})$. Because of the assumption on ω^2, we deduce immediately that $\mathcal{S}_D^{\alpha,\omega}\phi = 0$ in $Y \setminus D$, and hence

$$\begin{cases} \mathcal{S}_D^{\alpha,\omega}\phi = 0 \\ \frac{1}{2}\phi + \left(\mathcal{K}_D^{\alpha,\omega}\right)^*\phi = 0 \end{cases} \quad \text{on } D. \tag{A.0.1}$$

Then, we have

$$\begin{aligned}\frac{1}{2}(u|_{\partial D},\phi) &= (\mathcal{K}_D^{\alpha,\omega}(u|_{\partial D}),\phi) - \left(\mathcal{S}_D^{\alpha,\omega}\left(\frac{\partial u}{\partial\nu}\Big|_-\right),\phi\right)\\ &= (u|_{\partial D},(\mathcal{K}_D^{\alpha,\omega})^*\phi) - \left(\frac{\partial u}{\partial\nu}\Big|_-,\mathcal{S}_D^{\alpha,\omega}\phi\right)\\ &= -\frac{1}{2}(u|_{\partial D},\phi) - 0,\end{aligned}$$

which proves the lemma. □

Proof of Theorem A.0.2. We first note that the problem of finding (ϕ,ψ) is equivalent to solving the two equations:

$$\begin{cases}\mathcal{S}_D^{\alpha,\omega}\phi = u|_{\partial D} & \text{on } \partial D,\\ \big(-\frac{1}{2}I + (\mathcal{K}_D^{\alpha,\omega})^*\big)\phi = \frac{\partial u}{\partial\nu}\Big|_- & \text{on } \partial D,\end{cases} \tag{A.0.2}$$

and

$$\begin{cases}\mathcal{S}_D^{\alpha,\frac{\omega}{\sqrt{k}}}\psi = u|_{\partial D} & \text{on } \partial D,\\ \big(\frac{1}{2}I + (\mathcal{K}_D^{\alpha,\frac{\omega}{\sqrt{k}}})^*\big)\psi = \frac{\partial u}{\partial\nu}\Big|_+ & \text{on } \partial D.\end{cases} \tag{A.0.3}$$

Here we only consider the problem of finding ϕ solution to (A.0.2); the problem of finding ψ can be solved in the same way.

Since $\mathcal{S}_D^{\alpha,\omega}$ is self-adjoint and closed as an operator on $L^2(\partial D)$, it follows from Lemma A.0.3 that there exists $\phi_0 \in L^2(\partial D)$ such that

$$\mathcal{S}_D^{\alpha,\omega}(\phi_0+\phi) = u|_{\partial D} \quad \text{on } \partial D, \quad \forall\phi\in \mathrm{Ker}(\mathcal{S}_D^{\alpha,\omega}).$$

Thus to show existence of a solution to (A.0.2), it suffices to prove that there exists $\phi \in \mathrm{Ker}(\mathcal{S}_D^{\alpha,\omega})$ such that the second equation in (A.0.2) is satisfied by $\phi_0+\phi$. Thanks to the second equation in (A.0.1), this equation becomes

$$\phi = \frac{\partial\,(\mathcal{S}_D^{\alpha,\omega}\phi_0 - u)}{\partial\nu}\Big|_-, \tag{A.0.4}$$

and then, we only need to show that

$$\frac{\partial\,(\mathcal{S}_D^{\alpha,\omega}\phi_0 - u)}{\partial\nu}\Big|_- \in \mathrm{Ker}(\mathcal{S}_D^{\alpha,\omega}),$$

which is an immediate consequence of the fact that $\mathcal{S}_D^{\alpha,\omega}\phi_0 - u$ is a solution to $\Delta+\omega^2$ in D with the zero Dirichlet boundary condition. We have then proved the existence of a solution to (A.0.2).

Suppose now that we have two solutions ϕ_1 and ϕ_2 to (A.0.2), then, because of the assumption on ω^2, we have $\mathcal{S}_D^{\alpha,\omega}(\phi_1-\phi_2)=0$ in $Y\setminus\overline{D}$, and hence

$$\big(\frac{1}{2}I + (\mathcal{K}_D^{\alpha,\omega})^*\big)(\phi_1-\phi_2) = 0 \quad \text{on } \partial D.$$

By the second equation in (A.0.2), we have $\phi_1 = \phi_2$.

So far, we have shown that there are unique ϕ and ψ satisfying (A.0.2) and (A.0.3), respectively. The jump conditions satisfied by u immediately show that (ϕ, ψ) satisfy the system of integral equations (5.2.9).

Conversely, suppose that (ϕ, ψ) is a non-zero solution to the system of integral equations (5.2.9), then defining u by (5.2.8), we only need to show that u is not trivial to conclude that u is an eigenfuction of (5.2.6). Suppose that $u = 0$ in Y. Then $\mathcal{S}_D^{\alpha,\omega}\phi = 0$ in D, and by the assumption on ω^2, we deduce that $\mathcal{S}_D^{\alpha,\omega}\phi = 0$ in $Y \setminus \overline{D}$. Finally, from the jump of the normal derivative of $\mathcal{S}_D^{\alpha,\omega}\phi$ on ∂D, we deduce that $\phi = 0$. The assumption on ω^2/k leads to $\psi = 0$. This is in contradiction with the fact that $(\phi, \psi) \neq (0, 0)$. This completes the proof. $\square$

Bibliography

[1] J. Albert, Generic properties of eigenfunctions of elliptic partial differential operators, Trans. Amer. Math. Soc., 238 (1978), 341–354.

[2] H. Ammari, M. Asch, and H. Kang, Boundary voltage perturbations caused by small conductivity inhomogeneities nearly touching the boundary, Adv. Appl. Math., 35 (2005), 368–391.

[3] H. Ammari, N. Béreux, and J.C. Nédélec, Resonances for Maxwell's equations in a periodic structure, Japan J. Indust. Appl. Math., 17 (2000), 149–198.

[4] H. Ammari and H. Kang, High-order terms in the asymptotic expansions of the steady-state voltage potentials in the presence of conductivity inhomogeneities of small diameter, SIAM J. Math. Anal., 34 (2003), 1152–1166.

[5] ———, *Reconstruction of Small Inhomogeneities from Boundary Measurements*, Lecture Notes in Mathematics, Vol. 1846, Springer-Verlag, Berlin, 2004.

[6] ———, Boundary layer techniques for solving the Helmholtz equation in the presence of small inhomogeneities, J. Math. Anal. Appl., 296 (2004), 190–208.

[7] ———, *Polarization and Moment Tensors with Applications to Inverse Problems and Effective Medium Theory*, book, to appear in Applied Mathematical Sciences, Springer-Verlag.

[8] H. Ammari, H. Kang, M. Lim, and H. Zribi, Conductivity interface problems. Part I: small perturbations of an interface, submitted.

[9] ———, Conductivity interface problems. Part II: thin interfaces, in preparation.

[10] ———, Layer potential techniques in spectral analysis. Part I: complete asymptotic expansions for eigenvalues of the Laplacian in domains with small inclusions, submitted.

[11] H. Ammari, H. Kang, S. Soussi, and H. Zribi, Layer potential techniques in spectral analysis. Part II: Sensitivity analysis of spectral properties of high contrast band-gap materials, submitted.

[12] H. Ammari, H. Kang, and K. Touibi, Boundary layer techniques for deriving the effective properties of composite materials, Asymp. Anal., 41 (2005), 119–140.

[13] H. Ammari, H. Kang, and H. Zribi, Electrostatics in high-contrast materials, submitted.

[14] H. Ammari and A. Khelifi, Electromagnetic scattering by small dielectric inhomogeneities, J. Math. Pures Appl., 82 (2003), 749–842.

[15] H. Ammari and S. Moskow, Asymptotic expansions for eigenvalues in the presence of small inhomogeneities, Math. Meth. Appl. Sci. 26 (2003), 67–75.

[16] H. Ammari and F. Triki, Resonances for microstrip transmission lines, SIAM J. Appl. Math., 64 (2004), 601–636.

[17] ———, Splitting of resonant and scattering frequencies under shape deformation, J. Diff. Equat., 202 (2004), 231–255.

[18] E. Beretta, E. Francini, and M.S. Vogelius, Asymptotic formulas for steady state voltage potentials in the presence of thin inhomogeneities. A rigorous error analysis, J. Math. Pures Appl., 82 (2003), 1277–1301.

[19] E. Beretta, A. Mukherjee, and M.S. Vogelius, Asymptotic formuli for steady state voltage potentials in the presence of conductivity imperfections of small area, Z. Angew. Math. Phys., 52 (2001), 543–572.

[20] G. Besson, Comportement asymptotique des valeurs propres du Laplacian dans un domaine avec un trou, Bull. Soc. Math France 113 (1985), 211–230.

[21] Y. Capdeboscq and M.S. Vogelius, A general representation formula for the boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction, Math. Modelling Num. Anal., 37 (2003), 159–173.

[22] D.J. Cedio-Fengya, S. Moskow, and M. Vogelius, Identification of conductivity imperfections of small diameter by boundary measurements. Continuous dependence and computational reconstruction, Inverse Problems 14 (1998), 553–595.

[23] T. Chan and X.-C. Tai, Level set and total variation regularization for elliptic inverse problems with discontinuous coefficients, J. Comput. Phys. 193 (2003), 40–66.

[24] H. Cheng, W.Y. Crutchfield, M. Doery, and L. Greengard, Fast, accurate integral equation methods for the analysis of photonic crystal fibers I: Theory, Optics Express 12 (2004), 3791–3805.

[25] H. Cheng and S. Torquato, Effective conductivity of periodic arrays of spheres with interfacial resistance, Proc. R. Soc. Lond. A, 453 (1997), 145–161.

[26] ———, Effective conductivity of dispersions of spheres with a superconducting interface, Proc. R. Soc. Lond. A, 453 (1997), 1331–1344.

[27] R.R. Coifman, A. McIntosh, and Y. Meyer, L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes, Ann. Math., 116 (1982), 361–387.

[28] G. Courtois, Spectrum of manifolds with holes, J. Funct. Anal. 134 (1995), 194–221.

[29] S.J. Cox and D.C. Dobson, Band structure optimization of two-dimensional photonic crystals in H-polarization, J. Comput. Phys., 158 (2000), 214–224.

[30] W.Y. Crutchfield, H. Cheng, and L. Greengard, Sensitivity analysis of photonic crystal fiber, Optics Express 12 (2004), 4220–4226.

[31] D. Daners, Dirichlet problems on varying domains, J. Diff. Equat. 188 (2003), 591–624.

[32] A. Dienstfrey, F. Hang, and J. Huang, Lattice sums and the two-dimensional, periodic Green's function for the Helmholtz equation, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 457 (2001), 67–85.

[33] D.C. Dobson, An efficient method for band structure calculations in 2D photonic crystals, J. Comput. Phys., 149 (1999), 363–376.

[34] D.C. Dobson, J. Gopalakrishnan, and J.E. Pasciak, *An efficient method for band structure calculations in 3D photonic crystals,* J. Comput. Phys., 161 (2000), 668–679.

[35] E.B. Fabes, M. Jodeit, and N.M. Riviére, Potential techniques for boundary value problems on $\mathcal{C}^1$ domains, Acta Math., 141 (1978), 165–186.

[36] E. Fabes, H. Kang, and J.K. Seo, Inverse conductivity problem: error estimates and approximate identification for perturbed disks, SIAM J. Math. Anal., 30 (4) (1999), 699–720.

[37] A. Figotin and P. Kuchment, Band-gap structure of spectra of periodic dielectric and acoustic media. I: Scalar model, SIAM J. Appl. Math., 56 (1996), 68–88.

[38] ———, Band-gap structure of spectra of periodic dielectric and acoustic media. II: 2D photonic crystals, SIAM J. Appl. Math., 56 (1996), 1561–1620.

[39] ———, Spectral properties of classical waves in high-contrast periodic media, SIAM J. Appl. Math., 58 (1998), 683–702.

[40] L. Friedlander, On the density states of periodic media in the large coupling limit, Commun. Part. Differ. Equat., 27 (2002), 355–380.

[41] A. Friedman and M. Vogelius, Identification of small inhomogeneities of extreme conductivity by boundary measurements: a theorem on continuous dependence, Arch. Rat. Mech. Anal., 105 (1989), 563–579.

[42] G.B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, New Jersey, 1976.

[43] R.R. Gadyl'shin and A.M. Il'in, *Asymptotic behavior of the eigenvalues of the Dirichlet problem in a domain with a narrow slit,* Sbornik Math. 189 (1998), 503–526.

[44] I.T.S. Gohberg and E.I. Sigal, Operator extension of the logarithmic residue theorem and Rouché's theorem, Mat. Sb. (N.S.) 84 (1971), 607–642.

[45] L. Greengard and J.-Y. Lee, Electrostatics and heat conduction in high contrast composite materials, J. Comput. Phys., 211 (2006), 64–76.

[46] R. Hempel and I. Herbst, Strong magnetic fields, Dirichlet boundaries, and spectral gaps, Commun. Math. Phys., 169 (1995), 237–259.

[47] R. Hempel and K. Lienau, Spectral properties of periodic media in the large coupling limit, Commun. Part. Diff. Equat., 25 (2000), 1445–1470.

[48] F. Hettlich and W. Rundell, The determination of a discontinuity in a conductivity from a single boundary measurement, Inverse Problems, 14 (1998), 67–82.

[49] K.M. Ho, C.T. Chan, and C.M. Soukoulis, Existence of a photonic gap in periodic dielectric structures, Phys. Rev. Lett., 65 (1990), 3152.

[50] K. Ito, K. Kunish, and Z. Li, Level-set function approach to an inverse interface problem, Inverse Problems 17 (2001), 1225–1242.

[51] S.G. Johnson and J.D. Joannopoulos,*Photonic Crystals. The Road from Theory to Practice,* Kluwer Acad. Publ., 2002.

[52] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1976.

[53] H. Kang and J.K. Seo, Layer potential technique for the inverse conductivity problem, Inverse Problems 12 (1996), 267–278.

[54] ———, Identification of domains with near-extreme conductivity: global stability and error estimates, Inverse Problems 15 (1999), 851–867.

[55] ———, Recent progress in the inverse conductivity problem with single measurement, in *Inverse Problems and Related Fields*, CRC Press, 2000, 69–80.

[56] H. Kang, J.K. Seo, and D. Sheen, The inverse conductivity problem with one measurement: stability and estimation of size, SIAM J. Math. Anal. 28 (1997), 1389–1405.

[57] J.B. Keller, A theorem on the conductivity of a composite medium, J. Math. Phys., 5 (1964), 548-549.

[58] K.S. Mendelson, Effective conductivity of two-phase material with cylindrical phase boundaries, J. Appl. Phys., 46 (1975), 917-918.

[59] P. Kuchment, The mathematics of photonic crystals, in *Mathematical Modelling in Optical Science*, Bao, Cowsar and Masters, eds., 207–272, Frontiers in Appl. Math. 22, SIAM, Philadelphia, PA, 2001.

[60] O. Kwon, J.K. Seo, and J.R. Yoon, A real-time algorithm for the location search of discontinuous conductivities with one measurement, Commun. Pure Appl. Math. LV (2002), 1–29.

[61] C.M. Linton, The Green's function for the two-dimensional Helmholtz equation in periodic domains, J. Engrg. Math., 33 (1998), 377–402.

[62] R. Lipton, Influence of interfacial surface conduction of the DC electrical conductivity of particle reinforced composites, Proc. R. Soc. Lond. A, 454 (1998), 1371–1382.

[63] ——————, Variational methods, bounds, and size effects for composites with highly conducting interface, J. Mech. Phys. Solids 45 (1997), 361–384.

[64] V.G. Maz'ya, S.A. Nazarov, and B.A. Plamenevskii, Asymptotic expansions of eigenvalues of boundary value problems for the Laplace operator in domains with small holes, Math. USSR 24 (1985), 321–345.

[65] I. McGillivray, Capacitary asymptotic expansion of the groundstate to second order, Comm. Partial Diff. Equat. 23 (1998), 2219–2252.

[66] R.C. McPhedran, N.A. Nicorovici, L.C. Botten, and K.-D. Bao, Green's function, lattice sums and Rayleigh's identity for a dynamic scattering problem, in *Wave Propagation in Complex Media* (Minneapolis, MN, 1994), 155–186, IMA Vol. Math. Appl. 96, Springer-Verlag, New York, 1998.

[67] T. Miloh and Y. Benveniste, On the effective conductivity of composites with ellipsoidal inhomogeneities and highly conducting interfaces, Proc. R. Soc. Lond. A, 455 (1999), 2687–2706.

[68] N.A. Nicorovici, R.C. McPhedran, and L.C. Botten, Photonic band gaps for arrays of perfectly conducting cylinders, Physical Rev. E, 52 (1995), 1135–1145.

[69] A. Noll, Domain perturbations, shift of eigenvalues and capacity, J. Funct. Anal. 170 (2000), 246–263.

[70] S. Ozawa, Singular variation of domains and eigenvalues of the Laplacian, Duke Math. J. 48 (1981), 767–778.

[71] ———, Electrostatic capacity and eigenvalues of the Laplacian, J. Fac. Sci. Univ. Tokyo, Sect IA 30 (1983), 53–62.

[72] ———, An asymptotic formula for the eigenvalues of the Laplacian in a three-dimensional domain with a small hole, J. Fac. Sci. Univ. Tokyo, Sect IA 30 (1983), 243–257.

[73] ———, Spectra of domains with small spherical Neumann boundary, J. Fac. Sci. Univ. Tokyo, Sect IA 30 (1983), 259–277.

[74] ———, Asymptotic property of an eigenfunction of the Laplacian under singular variation of domains-the Neumann condition, Osaka J. Math. 22 (1985), 639–655.

[75] ———, Eigenvalues of the Laplacian under singular variation of domains-the Robin problem with obstacle of general shape, Proc. Japan Acad. Ser. A Math. Sci. 72 (1996), 124–125.

[76] H. Pham Huy and E. Sanchez-Palencia, Phénomènes de transmission à travers des couches minces de conductivité‘élevée, J. Math. Anal. Appl. 47 (1974), 284–309.

[77] J. Rauch and M. Taylor, Potential and scattering theory on wildly perturbed domains, J. Funct. Anal. 18 (1975), 27–59.

[78] G.F. Roach, *Green's Functions: Introductory Theory with Applications*, Van Nostrand Reinhold, London, 1970.

[79] K. Sakoda, *Optical Properties of Photonic Crystals*, Springer Verlag, Berlin, 2001.

[80] E. Sanchez-Palencia, Comportement limite d'un problème de transmission à travers une plaque faiblement conductrice, C.R. Acad. Sci. Paris A 270 (1970), 1026–1028.

[81] J. Sanchez Hubert and E. Sanchez Palencia, *Vibration and Coupling of Continuous Systems*, Springer-Verlag 1989.

[82] J. Selden, Periodic operators in high contrast media and the integrated density of states function, Commun. Part. Diff. Equat., 30 (2005), 1021–1037.

[83] V.P. Shestopalov and Y.V. Shestopalov, *Spectral Theory and Excitation of Open Structures*, IEEE Press, London, 1996.

[84] S. Soussi, Second-harmonic generation in the undepleted-pump approximation, Mult. Scal. Model. Simul., 4 (2005), 115–148.

[85] ———, Convergence of the supercell method for defect modes calculations in photonic crystals, SIAM J. Numer. Anal., 43 (2005), 1175–1201.

[86] ———, Modeling photonic crystal fibers, Adv. Appl. Math., to appear.

[87] H.S. Sözüer, J.W. Haus, and R. Inguva, Photonic bands: Convergence problems with the plane-wave method, Phys. Rev. B; 45 (1992), 13962.

[88] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, Second Edition, Springer-Verlag, New York, 1993.

[89] J. Tausch and J. White, Capacitance extraction of 3-D conductor systems in dielectric media with high permittivity ratios, IEEE Trans. Microwave Theory Tech. 47 (1999), 18–26.

[90] J. Tausch, J. Wang, and J. White, Improved integral formulations for fast 3-D method of moment solvers, IEEE Trans. Comput. Aided Design 20 (2001), 1398–1405.

[91] M.E. Taylor, *Partial Differential Equations II, Qualitative Studies of Linear Equations*, Applied Mathematical Sciences 116, Springer- Verlag, 1996.

[92] C.F. Tolmasky and A. Wiegmann, Recovery of small perturbations of an interface for an elliptic inverse problem via linearization, Inverse Problems 15 (1999), 465–487.

[93] S. Torquato and M.D. Rintoul, Effect of the interface on the properties of composite media, Phys. Rev. Lett. 75 (1995), 4060–4070.

[94] K. Uhlenbeck, Generic properties of eigenfunctions, Amer. J. Math., 98 (1976), 1059–1078.

[95] S. Venakides, M.A. Haider, and V. Papanicolaou, Boundary integral calculations of two-dimensional electromagnetic scattering by photonic crystal Fabry-Perot structures, SIAM J. Appl. Math., 60 (2000), 1686–1706.

[96] G.C. Verchota, Layer potentials and boundary value problems for Laplace's equation in Lipschitz domains, J. Funct. Anal., 59 (1984), 572–611.

[97] M. Vogelius and D. Volkov, Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities, Math. Model. Numer. Anal. 34 (2000), 723–748.

[98] M.J. Ward and J.B. Keller, Strong localized perturbations of eigenvalue problems, SIAM J. Appl. Math. 53 (1993), 770–798.

[99] E. Yablonovitch, Inhibited spontaneous emission in solid-state physic and electronics, Phys. Rev. Lett., 58 (1987), 2059.