Méthodes de Contrôle Stochastique pour la Gestion Optimale de Portefeuille

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MÉTHODES DE CONTRÔLE STOCHASTIQUE POUR LA GESTION OPTIMALE DE PORTEFEUILLE

STOCHASTIC CONTROL METHODS FOR OPTIMAL PORTFOLIO INVESTMENT

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Résumé

Cette thèse présente trois sujets de recherche indépendants, le dernier étant décliné sous la forme de deux problèmes disctincts. Ces différents sujets ont en commun d'appliquer des méthodes de contrôle stochastique à des problèmes de gestion optimale de portefeuille.

Dans une première partie, nous nous intéressons à un modèle de gestion d'actifs prenant en compte des taxes sur les plus-values. Pour une fonction d'utilité générale, nous obtenons un développement au premier ordre de la fonction valeur du problème d'optimisation associé, lorsque le taux de taxation et le taux d'intérêt tendent vers zéro.

Dans une seconde partie, nous étudions un problème de détection du maximum d'un processus de retour à la moyenne. Nous cherchons le temps d'arrêt qui minimise, en espérance, l'écart quadratique entre la valeur maximale que le processus atteindra entre l'instant initial et le premier instant de retour à sa valeur moyenne et la valeur du processus au temps d'arrêt. Nous parvenons à résoudre explicitement ce problème sous forme d'un problème à frontière libre. De façon inattendue, la frontière que nous obtenons est, en général, constituée de deux morceaux, l'un croissant et l'autre décroissant.

Dans les troisième et quatrième parties, nous regardons un problème d'investissement optimal lorsque les agents se regardent les uns les autres. Un investisseur cherche à maximiser l'espérance de l'utilité non pas de sa richesse comme dans le cas classique, mais de l'utilité d'une combinaison convexe entre sa richesse (absolue) et l'écart entre sa richesse et la richesse moyenne de ses pairs. Ajoutant de possibles contraintes sur les portefeuilles des investisseurs, nous montrons, pour une dynamique de prix de type avec drift et volatilité déterministes et des utilités exponentielles, l'existence d'un équilibre de Nash ainsi qu'une caractérisation des équilibres de Nash possibles. Une conséquence économique notable est que plus les agents s'observent, plus le risque du marché augmente.

Enfin dans une cinquième partie, nous étudions une variante de cette problématique, en remplaçant la contrainte sur les portefeuilles par un terme de pénalisation dans le critère d'optimisation. Pour une certaine classe de fonctions de pénalisation englobant de nombreux exemples intéressants, nous obtenons l'existence et l'unicité d'un équilibre de Nash, encore une fois pour des fonctions d'utilité exponentielles, mais pour des dynamiques de prix générales.

Abstract

This PhD dissertation presents three independant research topics, the third one being divided into two distinct problems. Those topics all use stochastic control methods in order to solve optimal investment problems.

In a first part, we consider a financial model which includes capital gains taxes. For general utility functions, we obtain a first order expansion of the value function of the optimization problem, when the tax rate and the interest rate go to zero.

In a second part, we study a problem in which we want to detect the maximum of a mean-reverting process. We look for the stopping time that minimizes the expectation of the quadratic error between the maximum that will be reached by the process between the initial time and the first time when the process hits zero and the value of the process at the stopping time. We manage to solve explicitly this problem as a free boundary problem. Surprisingly, the boundary is, in general, made of two pieces, one increasing and one decreasing.

In the third and fourth parts, we consider an optimal investment problem in which the agents take a look at their peers. In the classical case, an investor tries to maximize the expected utility of his wealth, but here he tries to maximize a convex combination between his (absolute) wealth and the difference between his wealth and the average wealth of his peers. Adding some possible constraints of the portfolios of the investors, we show, for a dynamics of the price process with deterministic drift and volatility and for exponential utilities, the existence of a Nash equilibrium as well as a characterization of Nash equilibria. An important economical consequence is that the more agents look at each others, the riskier the market is.

Finally in a fifth part, we study a variation of the previous problem, where we replace the constraints on the portfolios by a penalization component in the optimization criterion. For a certain class of penalization functions which includes many interesting examples, we show the existence and uniqueness of a Nash equilibrium, again for exponential utility functions, but for general dynamics of the price process.

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Chapter 1

Introduction

Un problème classique en mathématiques financières est celui d'un agent économique pouvant investir son argent dans un marché financier, choisissant entre un actif non risqué, communément appelé compte en banque, et un actif risqué, tel qu'une action. Etant donné un horizon d'investissement T fini, l'agent cherche à maximiser l'espérance de l'utilité de sa richesse à la date T. Partant d'une richesse initiale x, on note π le portefeuille de l'investisseur, parfois appelé stratégie d'investissement, et qui représente la richesse investie dans l'actif risqué. Quitte à considérer les grandeurs actualisées, supposons le taux d'intérêt égal à zéro et notons S l'actif risqué. Alors la richesse (ou valeur du portefeuille) de l'agent à l'instant t s'écrit:

$$X_t^{x,\pi} = x + \int_0^t \pi_u \frac{dS_u}{S_u}.$$

Selon les cas, il peut être plus pratique de considérer un portefeuille défini en nombre d'actifs et non en richesse comme nous l'avons fait.

Suivant l'approche classique introduite par Von Neumann et Morgenstern, on suppose que les préférences de l'investisseur peuvent être représentées par une fonction d'utilité U et une mesure de probabilité $\mathbb{P}$. U est supposée strictement croissante et strictement concave. Ainsi, l'investisseur préfère toujours avoir plus d'argent, mais plus il en a et plus la satisfaction d'avoir un euro supplémentaire est faible. Le problème que l'agent cherche à résoudre est alors le problème de contrôle stochastique suivant:

$$V(x) := \sup_{\pi \in \mathcal{A}} \mathbb{E} U(X_T^{x,\pi}),$$

où $\mathcal{A}$ est l'ensemble des portefeuilles admissibles.

On appelle ce problème d'optimisation le problème d'**investissement optimal**. Une variante, appelée problème d'**investissement et consommation optimaux**, ajoute la possibilité pour l'investisseur de consommer une part de sa richesse avant l'horizon T. D'un point de vue financier, on peut également voir cette consommation comme une rente versée par un manager de fond à ses clients. Notons C_t le taux de consommation à l'instant t, c'est-à-dire que sur la période $[t, t+dt]$, l'agent consomme $C_t dt$. La richesse de l'agent est alors donnée par:

$$X_t^{x,C,\pi} = x + \int_0^t \pi_u \frac{dS_u}{S_u} - \int_0^t C_u du.$$

On se donne encore une fonction d'utilité U ainsi qu'un éventuel taux d'escompte psychologique $\beta \geq 0$ qui traduit la préférence pour le présent. Le problème d'investissement-consommation optimaux est alors:

$$V_C(x) := \sup_{(C,\pi)\in\mathcal{B}} \mathbb{E} \int_0^T e^{-\beta t} U(C_t) dt,$$

où $\mathcal{B}$ est l'ensemble des stratégies admissibles. En réalité, $e^{-\beta t}U(c)$ peut être remplacée par une fonction d'utilité dépendant du temps plus générale.

Alors que cela n'avait pas de sens pour le problème d'investissement optimal, ici l'horizon de temps T peut aussi être pris infini: $T = +\infty$, à condition de faire quelques hypothèses d'intégrabilité supplémentaires. Enfin pour T fini, le problème mixte peut aussi être considéré (encore appelé problème d'investissement-consommation optimaux):

$$\tilde{V}_C(x) := \sup_{(C,\pi)\in\mathcal{B}} \mathbb{E} \left[\int_0^T e^{-\beta t} U_1(C_t) dt + U_2(X_T) \right],$$

U_1 et U_2 étant deux fonctions d'utilité. Signalons enfin qu'il est possible de prendre en compte plusieurs actifs risqués, ce qui, dans la plupart des cas, ne change pas grand chose au problème.

Merton [58, 59] fut le premier à introduire ces problématiques dans les années 60, et parvint à les résoudre dans un cas particulier en appliquant des techniques classiques de contôle. Ainsi, supposant que l'actif risqué suivait une dynamique de Black-Scholes, et donc en particulier que le marché était complet, et en supposant de plus que la fonction d'utilité était une fonction puissance:

$$U(x) = \frac{x^p}{p}, \text{ pour } x \geq 0 \text{ avec } p \in (0,1),$$

il a résolu l'équation de Bellman ou Hamilton-Jacobi-Bellman associée au problème. Mais il n'y est pas parvenu dans un cadre plus général. Ce n'est qu'au milieu des années 80 que le problème a été résolu pour des fonctions d'utilité générales mais toujours en marché complet grâce à des techniques probabilistes, par Pliska [67] pour le problème d'investissement optimal et par Cox et Huang [10] et Karatzas, Lehoczky et Shreve [46] pour le problème d'investissement et consommation. Dans ces travaux, l'introduction des (fonctions) duales convexes à la fois pour la fonction d'utilité et la fonction valeur au moyen de la transormée de Fenchel-Legendre conjointement avec l'existence d'une unique probabilité martingale pour S (puisque le marché est complet) jouent un rôle crucial.

Par la suite, de très nombreuses études ont tenté de se débarasser des limites de la formulation de Merton, et en particulier de l'hypothèse d'un marché complet et parfait, qui n'est pas réaliste et trop restrictive pour de nombreuses applications. Ces généralisations ont été faites dans plusieurs directions et à l'aide de différentes techniques. L'une de ces directions est l'introduction de contraintes sur les portefeuilles admissibles, comme l'on fait Cvitanic et Karatzas [12], ou Zariphopoulou [79], qui imposent que π reste dans un certain ensemble. Pour ce faire, les premiers cités utilisent des techniques probabilistes généralisant le cas du marché complet et ont recours là encore à la théorie de la dualité (par l'intermédiaire du dual convexe). Le second papier en revanche utilise des techniques déterministes provenant de la théorie du contrôle, se plaçant dans un cadre markovien. On peut aussi mentionner les travaux d'El Karoui, Jeanblanc et Lacoste [24], d'Elie et Touzi [22] et d'Elie [21] dans lesquels le portefeuille, pour être admissible, doit rester à tout instant supérieur à un certain processus. Une autre direction importante est la question des coûts de transaction, supposés proportionnels pour simplifier leur étude. Cette problématique a été introduite par Constantinides et Magill [9], qui ont mis en évidence un changement radical par rapport au problème de Merton qui supposait un trading continu: en présence de coûts de transaction il est optimal de trader "par à-coups", ce qui mathématiquement parlant se traduit par un processus à variations bornées. Par la suite, la question des coûts de transaction a été reprise et approfondie entre autres par Davis et Norman [15], Shreve et Soner [75], Duffie et Sun [17] ou Akian, Menaldi et Sulem [2], utilisant essentiellement des techniques d'EDP. Finalement, l'étude des marchés incomplets d'un point de vue général a été introduite par He et Pearson [?] en temps discret, puis par Karatzas, Lehoczky, Shreve et Xu [47] dans un modèle en temps continu et plus récemment dans un cadre très général où les prix des actifs sont simplement supposés être des semi-martingales, par Kramkov et Schachermayer [53, 54] et Kramkov et Sirbu [55]. Mentionnons également un cas particulier d'incomplétude traité par Zariphopoulou [80], pour une fonction d'utilité exponentielle ou puissance et un actif risqué (tradable) dont la dynamique dépend d'un actif non-tradable (comme par exemple dans le cas d'une volatilité stochastique). Grâce à un changement de variable astucieux, l'auteur parvient à transformer l'équation de HJB du problème en une EDP linéaire et obtient ainsi la solution sous forme explicite. Nous présentons la question des taxes dans la section suivante.

Cette thèse est constituée de cinq parties indépendantes; exceptées les parties trois et quatre qui se suivent; présentant différentes extensions de cette problématique de gestion optimale de portefeuille. Dans la première partie, nous étudions un modèle d'optimisation de portefeuille incluant des taxes sur les plus-values. Dans la seconde partie, nous considérons un problème de détection du maximum d'un processus de retour à la moyenne. Dans la troisième partie, nous considérons un problème d'optimisation de portefeuille avec un critère prenant en compte la richesse moyenne des autres agents du marché. Dans la quatrième partie, nous développons un certain nombre d'exemples illustrant les résultats de la partie précédente. Enfin dans la cinquième partie, nous étudions une variante de la troisième partie en incluant une pénalisation dans le critère d'optimisation.

1.1 Investissement et consommation optimaux avec taxes: développement du premier ordre pour des fonctions d'utilité générales

Bien que de nombreuses généralisations du problème d'optimisation de portefeuille aient été étudiées depuis les travaux de Merton, la question des taxes a reçu une attention assez limitée jusqu'à ces dernières années, alors qu'en pratique, leur impact est très important. Nous donnons ci-dessous un aperçu rapide des principaux travaux sur le sujet.

1.1.1 Principaux travaux sur les taxes et article source

La première étude pertinente sur les taxes a été faite par Constantinides [8]. Ce dernier montre que les décisions d'investir et de consommer peuvent être prises indépendamment et que la stratégie optimale consiste à réaliser immédiatement les pertes et différer les gains. Ce résultat peu satisfaisant car il ne correspond pas à ce qui est constaté en pratique, est très lié à l'hypothèse qu'il est possible de vendre à découvert l'actif risqué, sans aucune limite. Par conséquent, les travaux suivants ont tous rejeté cette hypothèse. Ainsi dans ce qui suit, il sera toujours supposé qu'il est impossible de vendre à découvert. Etant donné qu'il est question de taxes sur les plus-values, le montant des taxes est calculé en comparant le prix de vente aujourd'hui et une "base de taxe" reliée au(x) prix d'achat antérieur(s). Cette base de taxe est fixée par le code de taxation du pays de l'agent. Bien entendu, la définition de cette base de taxe influe très fortement sur l'étude du problème. En pratique, elle est définie par l'une des deux règles suivantes:
(i) elle est égale au prix d'achat spécifique de l'actif qui est vendu;
(ii) elle est égale à une moyenne pondérée des prix aux différentes dates d'achat.

Sans hypothèse supplémentaire, Dybvig et Koo [20] ont étudié la règle (i) pour un modèle binomial, et ont fourni des résultats numériques limités. Par la suite, en ajoutant la règle du "premier arrivé, premier sorti", Jouini, Koehl et Touzi [44, 45] ont prouvé un résultat d'existence. Mais là encore, les résultats numériques sont assez limités en raison de la grande complexité héritée de la dépendance trajectorielle.

De leur côté, Damon, Spatt et Zhang [13] ont considéré la règle (ii), dans le contexte d'un modèle binomial. Cette étude a ensuite été généralisée au cas de plusieurs actifs par Gallmeyer, Kaniel et Tompaidis [35]. Dans ce cadre, la dépendance trajectorielle est grandement simplifiée. Dans [3, 4], Ben Tahar, Soner et Touzi ont formulé l'analogue en temps continu du modèle précédent et ont réussi à démontrer des résultats à la fois théoriques et numériques. Notre travail étant une généralisation d'une partie de leurs résultats, nous fournissons ci-dessous une revue plus détaillée de leurs papiers.

Le marché est constitué d'un actif sans risque ayant pour taux d'intérêt $r \geq 0$ et d'un actif risqué P qui suit une dynamique de Black-Scholes:

$$dP_t = P_t[(r + \theta\sigma)dt + \sigma dW_t],$$

où W est un mouvement brownien, θ et $\sigma > 0$ sont des constantes, appelées respectivement la prime de risque et la volatilité.

Les ventes de l'action (l'actif risqué) sont soumises à des taxes sur les plus-values. En d'autres termes, le montant de taxes à payer pour la vente d'une action à l'instant t est calculé en comparant le prix courant P_t à l'indice B_t que l'on a appelé plus haut la base de taxes. En quelques mots, l'index est défini comme moyenne des prix d'achat de l'action aux dates antérieures à t, pondérée par la quantité achetée. Si le prix P_t est supérieur à l'index B_t, alors l'investisseur réalise un gain en vendant ses actions, alors que dans la situation inverse il réalise une perte. La règle de taxation est supposée linéaire, c'est-à-dire qu'il existe une constante $\alpha \in (0, 1)$ telle que pour chaque unité d'action vendue, le montant (algébrique) de taxes à payer est:

$$\alpha(P_t - B_t).$$

Un montant négatif correspond dans la réalité à un crédit de taxes, comme c'est le cas dans certains pays, le Canada notamment.

L'horizon d'investissement est supposé infini et le problem de consommation-investissement est formulé comme suit:

$$V(s) = \sup_{\nu \in \mathcal{A}(s)} \mathbb{E} \int_0^\infty e^{-\beta t} U(C_t) dt,$$

où la donnée initiale s prend en compte la richesse initiale détenue dans le compte en banque, la richesse initiale détenue dans l'actif risqué et la valeur initiale de l'index B. C est la consommation de l'agent et $\mathcal{A}(s)$ l'ensemble des stratégies admissibles qui partent de s. Enfin U est une fonction d'utilité puissance, autrement dit il existe $p \in (0, 1)$ tel que:

$$U(c) = \frac{c^p}{p}.$$

Après avoir montré des propriétés élémentaires de la fonction valeur V, les auteurs fournissent des bornes de V à l'aide de problèmes de Merton pour lesquels la solution est explicite. Grâce à ces bornes explicites, il est alors possible d'obtenir un développement au premier ordre de V lorsque le taux d'intérêt r et le taux de taxe α tendent vers 0 simultanément. Nous généralisons ce résultat dans notre étude. Un autre résultat important de [3, 4] est la caractérisation de la solution V à l'aide des solutions de viscosité. Nous renvoyons à l'article de Crandall, Ishii et Lions [11] pour plus de détails sur les solutions de viscosité. Dans les travaux de Ben Tahar, Soner et Touzi, un résultat de comparaison fort qui permettrait de montrer l'unicité et la continuité de la fonction valeur n'est pas démontré car une irrégularité de la condition au bord sur une partie du bord complique la situation. Afin de surmonter cette difficulté, ils définissent des problèmes approchés plus réguliers. Chacun de ces problèmes est alors l'unique solution de viscosité de l'équation de HJB approchée, et ils montrent ensuite un résultat de convergence uniforme sur les compacts de ces solutions approchées.

1.1.2 Nouveaux résultats

Dans notre étude, nous considérons un marché financier complet constitué d'un actif sans risque évoluant avec un taux d'intérêt $r \geq 0$ et n actifs risqués dont la dynamique est donnée par:

$$dP_t = diag(P_t)[b(P_t)dt + \sigma(P_t)dW_t],$$

étendant ainsi le cadre précédent à un processus markovien multi-dimensionnel assez général.

Ensuite, nous prenons la même règle de taxation que dans [3, 4], mais en revanche, et c'est là la principale différence, nous considérons une fonction d'utilité générale U qui est supposée vérifier des conditions classiques, essentiellement qu'elle est strictement croissante, strictement concave et suffisamment régulière. La plupart des propriétés élémentaires sont encore vérifiées dans ce cadre plus général et les preuves peuvent être adaptées presque immédiatement. La plus importante de ces propriétés est l'optimalité des "wash sales". Sous certaines hypothèses supplémentaires, cela permet là encore de borner la fonction valeur à l'aide de deux problèmes de Merton bien choisis.

La principale difficulté vient de l'étude de la régularité de ces bornes et de leurs développements au premier ordre. Tandis que pour une dynamique de Black-Scholes et une fonction d'utilité puissance cette question est immédiate car la fonction valeur du problème de Merton est explicite, dans notre cadre une étude de ces fonctions valeurs simplifiées est nécessaire. Si l'on note $I(y) := (U')^{-1}(y)$, en utilisant des résultats classiques, nous savons que V est donnée par:

$$V = G \circ \mathcal{Y},$$

où G admet la représentation probabiliste suivante:

$$G(y) := \mathbb{E}\left[\int_0^{+\infty} e^{-\beta t} U \circ I(ye^{\beta t} M_t) dt\right],$$

tandis que $\mathcal{Y}$ est définie implicitement par: $\mathcal{X} \circ \mathcal{Y}(x) = x$, où $\mathcal{X}$ admet la représentation:

$$\mathcal{X}(y) := \mathbb{E}\left[\int_0^{+\infty} M_t I(ye^{\beta t} M_t) dt\right],$$

sachant que M est défini par:

$$M_t := e^{-rt - \int_0^t \theta_u^* dW_u - \frac{1}{2}\int_0^t \|\theta_u\|^2 du}.$$

Sous des hypothèses relativement faibles qui garantissent de l'intégrabilité uniforme, nous démontrons que G et $\mathcal{X}$ sont C^1 par rapport au taux de taxe α et au taux d'intérêt r, et qu'il est possible de dériver les expressions précédentes sous les signes espérance/intégrale. Pour finir, à l'aide du théorème des fonctions implicites, nous concluons que V est C^1 et que les deux bornes admettent le même développement au premier ordre, fournissant par la même occasion le développement de la fonction valeur pour le problème avec taxes.

1.1.3 Perspectives

Bien que le développement au premier ordre soit l'un des principaux résultats de [3, 4], les auteurs prouvent plusieurs autres résultats importants dans leurs papiers. Pour l'essentiel, ils montrent d'abord que la fonction valeur V est solution de viscosité de l'équation de HJB associée au probléme. Mais cette formulation ne leur permet pas de prouver un résultat de comparaison, et ne fournit pas en particulier la continuité de V. Cependant, utilisant la propriété d'homogénéité que V hérite de la fonction d'utilité puissance, ils sont capables de montrer directement que la fonction valeur est Lipschitzienne. En outre, ils prouvent que V est limite uniforme des fonctions valeurs de problèmes approchés, et que chacune de ces fonctions valeurs est l'unique solution de viscosité de l'équation de HJB associé au problème approché. Ce résultat implique la convergence des schémas numériques.

Toutes les preuves permettant de démontrer ces résultats peuvent être étendues presque immédiatement à notre cadre, mis à part le caractère Lipschitz de V et la convergence uniforme (sur les compacts) des fonctions valeurs approchées. La preuve de la première propriété utilisant de manière cruciale l'homogénéité de V dans le cas d'une fonction d'utilité puissance, il y a peu d'espoir de pouvoir utiliser les mêmes arguments pour une fonction d'utilité générale. En ce qui concerne la seconde propriété, l'adaptation des arguments de [3, 4] permet d'obtenir la convergence simple. La convergence uniforme sur les compacts est alors une conséquence du théorème de Dini grâce à la continuité de V. Il est donc essentiel de pouvoir montrer la continuité de V dans un cadre général pour pouvoir étendre leur preuve.

1.2 Détection du maximum d'un processus scalaire de retour à la moyenne

Cette partie ne s'inscrit pas exactement dans le cadre de la gestion optimale de portefeuille telle que nous l'avons définie précédemment. Néanmoins elle considère un problème d'arrêt optimal dont la solution peut être appliquée pour mettre en place une stratégie de gestion de portefeuille par un trader ou un manager de fond d'investissement. En ce sens elle constitue bien un problème de contrôle stochastique appliqué à de la gestion optimale de portefeuille.

1.2.1 Motivation

Dans les années 60, Shiryaev [71] a considéré ce que l'on appelle le problème d'écart. Supposons qu'un processus soit la somme d'un mouvement brownien et d'une constante a, et qu'il soit possible qu'à un moment donné, la valeur de cette constante passe de a à b. Shiryaev a cherché à trouver comment détecter ce changement de régime. Bien que les problèmes d'arrêt optimal et les problèmes à frontière libre, également appelés problèmes de Stefan, ont été étudiés en détail, voir notamment Shiryaev [72], Peskir et Shiryaev [66] ou Karatzas et Shreve, appendice D [49], et que leur utilité pour des problèmes financiers tels que les options américaines, voir par exemple Myneni [61], ou les options russes, voir [69, 70], a été mise en évidence, très peu de travaux se sont intéressés à des questions de détection jusqu'au début des années 2000.

Pourtant, au-delà de l'intérêt théorique, cette famille de problèmes peut s'avérer intéressante pour un manager de fond ou pour un trader, et peut en particulier servir de justification théorique a ce que les praticiens appellent "l'analyse technique". Dans la théorie de la réplication, le gestionnaire annule le risque de son portefeuille en le répliquant parfaitement à l'aide des actifs échangeables présents sur le marché, et est de fait, au moins sur le plan théorique, indifférent à l'évolution du marché. Bien sûr en pratique cela présuppose que le marché se comporte comme le modèle et de plus que le marché soit complet. Cependant dans le cas d'un marché incomplet, la surréplication ou encore la gestion optimale de portefeuille dans le cadre introduit par Merton prolongent cette idée. Mais ce point de vue ne représente qu'une partie de l'activité de gestion et laisse de côté la prise de position. Dans cette optique, le gestionnaire tente de prédire l'évolution du marché pour en profiter. C'est exactement le but de l'analyse technique. Ainsi les problèmes de détection permettent de développer des outils mathématiques donnant aux gestionnaires l'opportunité de définir la "meilleure" réponse pour s'adapter aux évolutions du marché et prendre de façon rationnelle une position.

En particulier, nous considérons dans ce chapitre un processus qui oscille autour de zéro (un processus de retour à la moyenne) et partant d'une valeur initiale strictement positive. Supposons que l'on soit capable de détecter l'instant où ce processus atteint son maximum avant de devenir négatif, et symétriquement de détecter le moment où il atteint son minimum avant de redevenir positif, et ainsi de suite. Du point de vue d'un trader cela peut être considéré comme une stratégie optimale puisqu'il peut ainsi réaliser un gain maximal.

Si l'on écrit ce problème en termes de problème d'optimisation, notre formulation ressemble à celle introduite par Graversen, Peskir et Shiryaev [37], que nous présentons brièvement dans la section suivante, mais en réalité ces deux points de vue diffèrent profondément. Contrairement à ce qui est fait ici, ces derniers considèrent en effet le maximum sur une période de temps fixe, ce qui présente plusieurs inconvénients pour un gestionnaire:
- ils dérivent leur résultat uniquement pour un mouvement brownien et la solution explicite de ce problème est très difficile à étendre, tandis que la stratégie qui nous intéresse est intéressante essentiellement pour un processus de retour à la moyenne;
- un investisseur qui prend une position n'a intérêt à sortir de sa position que s'il a réalisé un gain minimum, ne serait-ce que pour couvrir ses coûts de transaction. Or, pour un mouvement brownien partant de 0, pour tout $\varepsilon > 0$, l'espérance du premier temps de passage en ε est infinie, ce qui signifie que pour réaliser un gain minimum de ε, après avoir acheté ou vendu un actif qui est supposé être un mouvement brownien, le trader devra en moyenne garder sa position pendant un temps infini;
- il n'y a pas de raison théorique de choisir une maturité particulière;
- même si l'on admet que la maturité puisse être choisie de manière satisfaisante, elle doit l'être a priori, c'est à dire au début de la période. Sur des réalisations, le processus peut se comporter de façon très différente. Parfois il oscillera très vite, tandis que d'autres il oscillera très lentement. En fixant a priori la fin de la période, l'investisseur ne tient pas compte de ce phénomène et investira régulièrement soit trop souvent, soit trop peu souvent.

Ces problèmes sont résolus par notre formulation qui considère une maturité aléatoire définie comme le premier temps de passage en zéro et s'intéresse à des processus de retour à la moyenne tels que le processus de Ornstein-Uhlenbeck.

Nous donnons dans la section suivante un aperçu des principaux travaux liés au nôtre.

1.2.2 Aperçu de certains travaux connexes

Comme nous l'avons brièvement évoqué, Graversen, Peskir and Shiryaev [37] considèrent un problème qui a inspiré le nôtre: ils cherchent à détecter le maximum d'un mouvement brownien sur une période de temps fixe, disons $[0,1]$. Notons X un mouvement brownien standard, et, pour un maximum hérité z, Z le processus de maximum courant associé:

$$Z_t = \sup_{s \leq t} X_s \vee z.$$

Alors, pour une fonction de perte ℓ, ils considèrent le problème d'optimisation suivant:

$$V(x) = \inf_{0 \leq \tau \leq 1} \mathbb{E}[\ell(Z_1 - X_\tau) | X_0 = Z_0 = x] = V(0),$$

où la minimisation est sur l'ensemble des temps d'arrêt τ.

Essayer d'utiliser des techniques standard d'arrêt optimal afin de tenter de résoudre l'équation d'Hamilton-Jacobi-Bellman associée au problème aboutirait à un problème à frontière libre de dimension trois (le temps t et deux variables d'espace x et z) qu'il semble impossible de résoudre explicitement. Cependant, en utilisant des propriétés du mouvement brownien et de son maximum, les auteurs parviennent astucieusement à simplifier considérablement la situation et dérivent en fin de compte un problème à frontière libre en dimension un. En d'autres termes, ils obtiennent une équation différentielle ordinaire dont la frontière est inconnue, qu'ils parviennent à résoudre explicitement.

Plus précisément, pour des fonctins de perte puissance $\ell(x) = x^p$, avec $p > 0$ et $p \neq 1$, ils montrent qu'il existe un réel $z_p > 0$, défini comme unique solution d'une certaine équation, tel que le temps d'arrêt:

$$\tau^p = \inf\{t \leq 1,\ Z_t - X_t \geq z_p \sqrt{1-t}\}$$

est optimal. De plus, la valeur du problème d'optimisation est également explicite, et est fonction de z_p. Bien que cela soit sans grande importance, notons cependant que la fonction valeur n'est pas connue pour $z \neq x$.

La preuve peut être décomposée en deux étapes principales. Dans un premier temps, grâce à l'indépendance des accroissements de X, ils réécrivent le problème en fonction de $Z_\tau - X_\tau$ uniquement. Ensuite, utilisant le fait que pour un mouvement brownien, si $x = z$, alors $Z_t - X_t$ et $|X|$ ont même loi, ils réduisent le nombre de variables spatiales de deux à une. Puis, à l'aide du changement de temps:

$$s = -\frac{1}{2} \ln(1-t),$$

ils transforment le problème en un problème stationnaire et du même coup se débarassent de la variable t.

Ensuite, la seconde étape est classique: ils dérivent l'équation de HJB associée au problème simplifié, la reformule en terme de problème à frontière libre qu'il résolve explicitement et finalement utilise un argument de vérification.

Ces travaux ont été quelque peu généralisés par Pedersen [64] puis étendus au cas d'un mouvement brownien avec drift, pour une fonction de perte quadratique $\ell(x) = x^2$, par Du Toit et Peskir [18]. Là encore la preuve utilise des propriétés spécifiques du mouvement brownien drifté, mais elle est plus complexe. De plus, la forme de la solution n'est pas classique, puisqu'ils montrent que deux frontières sont mises en jeu, le temps d'arrêt optimal qu'ils explicitent étant de la forme:

$$\tau^* = \inf\{t \leq 1,\ b_1(t) \leq Z_t - X_t \leq b_2(t)\}.$$

Ainsi, pour un drift négatif, partant de $x = z > 0$, comme dans [37], si X redescend trop après avoir atteint un niveau Z, il est optimal de s'arrêter, car l'écart devenant trop grand, il est improbable de remonter avant le temps $t = 1$ jusqu'au niveau Z, mais désormais à cause du drift négatif, si $Z - X$ est petit et t suffisamment proche de 1, il est également improbable d'atteindre un niveau supérieur à Z ou même supérieur à la valeur présente.

Une variation sur le sujet est le problème d'optimisation suivant, introduit par Shiryaev [73]:

$$\inf_{0\leq\tau\leq 1} \mathbb{E}|\theta - \tau|,$$

sur l'ensemble des temps d'arrêt τ et où θ est le premier ou dernier instant où le maximum d'un mouvement brownien est atteint avant $t = 1$. Il est intéressant de noter qu'Urusov [77] a montré que pour un mouvement brownien ce problème est équivalent au précédent. En effet l'auteur a démontré l'identité suivante pour tout temps d'arrêt τ:

$$\mathbb{E}(X_\theta - X_\tau)^2 = \mathbb{E}|\theta - \tau| + \frac{1}{2}. \tag{1.1}$$

Il est à noter que θ **n'est pas un temps d'arrêt** car il dépend en réalité de tout le processus jusqu'à l'instant 1, par conséquent le théorème d'arrêt de Doob ne s'applique pas. C'est d'ailleurs pour cette raison que la constante $\frac{1}{2}$ apparaît. Cette fois encore, cette étude a été généralisée pour des mouvements browniens avec drift par Du Toit et Peskir [19], qui sont parvenus à caractériser un temps d'arrêt optimal comme premier temps d'atteinte d'une certaine barrière. Notons cependant que (1.1) n'est pas vrai dans ce cas plus général, tandis que leurs arguments s'étendent en grande partie aux processus de Lévy.

Mentionnons également le papier de Shiryaev, Xu et Zhou [74] dans lequel est étudié, pour

X un mouvement brownien géométrique, la variante suivante:

$$\inf_{0\leq\tau\leq 1} \mathbb{E}\left(\frac{Z_1 - X_\tau}{Z_1}\right).$$

Un autre problème intéressant; qui n'est certes pas un problème de détection mais malgré tout un problème d'arrêt optimal relié au maximum d'un processus; présente certaines similitudes dans sa résolution avec le nôtre. Peskir [65] a introduit le problème d'optimisation suivant:

$$\sup_\tau \mathbb{E}\left(Z_\tau - \int_0^\tau c(X_t)dt\right),$$

où X est une diffusion stationnaire, Z son maximum, et c peut être vue comme une fonction de coût (et en particulier est positive). Le principal résultat est une condition nécessaire et suffisante pour que la fonction valeur soit finie et qu'un temps d'arrêt optimal existe. Cette condition est appelée principe de maximalité par l'auteur car elle est satisfaite si et seulement si le flot d'une certaine équation différentielle ordinaire admet une solution maximale en un certain sens lorsque l'on fait varier sa condition initiale. Il est amusant de constater que dans notre étude également nous devons considérer la solution maximale d'une certaine équation différentielle ordinaire dans un sens très proche de celui introduit par Peskir.

Ces travaux ont par la suite été étendus à des processus X et des fonctions c plus généraux par Obloj [62], tandis qu'Hobson [41] considère le problème opposé:

$$\inf_\tau \mathbb{E}\left(Z_\tau - \int_0^\tau c(X_t)dt\right).$$

Bien qu'utilisant des techniques différentes, la solution de ce dernier problème est plus ou moins de même nature que celle de [65]. Nous aimerions signaler que pour ces différents problèmes, un changement d'échelle grâce à la fonction d'échelle, voir Karlin et Taylor [50], permet de réduire le cas d'une diffusion stationnaire au cas du mouvement brownien, ce qui simplifie l'analyse. Dans notre étude, un tel changement est possible, mais il n'apporte pas de simplification significative.

De nombreux autres problèmes sont plus ou moins liés à ce sujet. Mentionnons pour finir le problème de "switching" optimal (commutation optimale) introduit pour des processus d'Ornstein-Uhlenbeck par Zhang et Zhang [81] qui traduit une problématique naturelle pour un trader. Bien que naturel, leur critère ne prend en compte que l'espérance du rendement mais aucune forme de risque.

1.2.3 Nouveaux résultats

Considérons un processus dont la dynamique est donnée par l'EDS stationnaire suivante:

$$dX_t \;=\; \mu(X_t)dt + \sigma(X_t)dW_t,\; t \geq 0,$$

ainsi qu'une valeur initiale $X_0 > 0$. W est un mouvement brownien tandis que μ et σ satisfont des hypothèses assez générales (elles sont lipschitziennes en particulier). On dira que X est un processus de retour à la moyenne si:

$$\mu(x) \;\leq\; 0 \quad \text{pour tout} \quad x \geq 0,$$

Cela signifie en effet que le processus X est ramené vers l'origine lorsque $X > 0$.
Introduisons ensuite le premier temps d'atteinte de la barrière 0:

$$T_0 \;:=\; \inf\{t > 0 :\; X_t = 0\}\,,$$

et le maximum courant: $X_t^* := \max_{0\leq s\leq t} X_s$. $\mathcal{T}_0$ désignant l'ensemble des temps d'arrêt inférieurs à T_0, on considère le problème d'optimisation suivant:

$$V_0 \;:=\; \inf_{\theta\in\mathcal{T}_0} \mathbb{E}\left[\ell\left(X_{T_0}^* - X_\theta\right)\right],$$

où ℓ est appelée fonction de perte et satisfait des conditions assez faibles. En particulier, elle est supposée croissante, convexe et suffisamment régulière.
Afin de se placer dans le cadre de la programmation dynamique, définissons (Z a été introduit plus haut):

$$V(x,z) \;:=\; \inf_{\theta\in\mathcal{T}_0} \mathbb{E}_{x,z}\left[\ell\left(Z_{T_0} - X_\theta\right)\right],$$

où $\mathbb{E}_{x,z}$ est l'espérance conditionnelle sachant $X_0 = x$ and $Z_0 = z$.
Définissons également la récompense en cas d'arrêt:

$$g(x,z) := \mathbb{E}_{x,z}\left[\ell\left(Z_{T_0} - x\right)\right], \qquad 0 \leq x \leq z,$$

qui correspondrait à la fonction valeur s'il était optimal d'arrêter immédiatement.
A l'aide de la fonction d'échelle, voir par exemple Karlin et Taylor [50], on peut déterminer explicitement la loi de Z_{T_0} en réinterprétant Z_{T_0} en termes de temps d'atteinte de barrière. Cela permet ensuite d'obtenir une formule explicite pour la fonction g et de déduire des conditions suffisantes pour que V soit finie. Nous dérivons ensuite l'équation de HJB associée au problème. Si on introduit la fonction $\alpha := -\frac{2\mu}{\sigma^2}$ qui se révèle être le seul paramètre lié à

la diffusion X qui soit pertinent pour notre problème et l'operateur infinitésimal L associé à X:

$$Lv(x) = v''(x) - \alpha(x)v'(x),$$

l'équation de HJB est l'inégalité variationnelle suivante:

$$\begin{aligned}
&\min\{Lv, g - v\} = 0\\
&v(0, z) = \ell(z)\\
&v_z(z, z) = 0.
\end{aligned}$$

Malheureusement, une telle inégalité variationnelle est difficile à résoudre en général, mais de même que pour les options américaines ou de nombreux autres problèmes d'arrêt optimal, nous pensons que la solution doit résoudre un problème à frontière libre associé. Après avoir résolu ce problème plus simple, nous devrons vérifier que la solution trouvée est égale à la fonction valeur du problème initial. Ainsi nous cherchons une frontière libre $\gamma(x)$ telle qu'il soit optimal d'arrêter dans la région $\{z \geq \gamma(x)\}$ alors qu'il est optimal de continuer dans la région $\{z < \gamma(x)\}$. On cherche donc à résoudre:

$$\begin{aligned}
&Lv(x, z) = 0 \text{ for } 0 < z < \gamma(x)\\
&v(x, z) = g(x, z) \text{ and } Lg(x, z) \geq 0 \text{ for } z \geq \gamma(x)\\
&v(0, z) = \ell(z)\\
&v_z(z, z) = 0.
\end{aligned}$$

Etant donné que dans l'expression de L seules les dérivées par rapport à x interviennent, l'espoir de résoudre cette équation explicitement est permis. Afin de pouvoir appliquer la formule d'Itô, imposons également des conditions de continuité et de "smooth-fit" sur la frontière:

$$\begin{aligned}
&v(x, \gamma(x)) = g(x, \gamma(x))\\
&v_x(x, \gamma(x)) = g_x(x, \gamma(x)).
\end{aligned}$$

Nous restreignons dans un premier temps notre étude à une fonction de perte quadratique:

$$\ell(x) = \frac{x^2}{2}.$$

Une des difficutés pour déterminer γ vient du fait que celle-ci est faite en général de deux parties régies par des équations distinctes, et n'est pas monotone. Ainsi il existe $\zeta \geq 0$ tel que γ est décroissante sur $[0, \zeta]$ et croissante sur $[\zeta, +\infty)$. L'équation de la partie croissante est donnée par l'équation différentielle ordinaire suivante:

$$\gamma' = \frac{Lg(x,\gamma)}{1 - \frac{S(x)}{S(\gamma)}},$$

sans condition initiale a priori. Nous obtenons de fait une famille de candidats potentiels et devons choisir la "bonne" équation, puisqu'une seule nous permettra de trouver la solution du problème. La "bonne" équation se révèle être celle fournissant la "plus grande" fonction γ qui croise la diagonale $\{(x, z);\ x = z\}$.

Nous devons ensuite déterminer la partie décroissante. Pour ce faire nous devons résoudre à z fixé l'équation ayant x pour inconnue:

$$f(x,z) = g(x,z) - g_x(x,z)\frac{S(x)}{S'(x)} - \frac{z^2}{2},$$

qui correspond à l'équation satisfaite par l'inverse de γ sur sa partie décroissante. Cette fois il n'y a pas de problème d'indétermination et nous montrons que les deux parties s'intersectent.

Puis nous obtenons une expression explicite pour notre candidat et montrons un théorème de vérification. Nous montrons par la même occasion l'existence d'un temps d'arrêt optimal θ^* donné par:

$$\theta^* := \inf\{t \geq 0;\ Z_t \geq \gamma(X_t)\}.$$

Nous donnons par la suite quelques exemples de processus qui entrent dans le cadre de notre étude, tels que les processus d'Ornstein-Ulhenbeck, de Cox-Ingersoll-Ross ou le mouvement brownien avec un drift strictement négatif. En revanche, contrairement au cadre de [37], pour le mouvement brownien standard la fonction valeur est infinie. Nous expliquons également comment généraliser ces résultats à des fonctions de perte plus générales.

Bien que l'idée de réduire l'équation de HJB à un problème à frontière libre soit classique, la résolution de ce dernier problème dans notre cadre est inhabituelle. D'une part, nous sommes confrontés à une famille d'équations possibles parmi lesquelles nous devons choisir la "bonne". Comme nous l'avons évoqué plus haut, c'est également le cas dans [65]. D'autre part, la frontière est divisée en deux parties disctinctes, l'une décroissante, l'autre croissante, et en particulier, si l'on cherche à définir la frontière γ comme fonction de z, alors elle n'est pas bien d'éfinie. Remarquons cependant qu'en pratique si l'on part de $x = z$, alors la partie décroissante de γ ne sera jamais atteinte.

Finalement, nous avons appliqué ce résultat à une stratégie de trading, pour laquelle nous avons calibré un processus d'Ornstein-Uhlenbeck sur des données de marché. La stratégie consiste à vendre au maximum détecté par notre étude, fermer la position quand $X = 0$, et symétriquement, acheter au minimum détecté puis fermer la position à $X = 0$.

1.2.4 Perspectives

Tout d'abord, pour une fonction de perte générale ℓ, nous n'avons pas pu définir des conditions suffisantes totalement satisfaisantes sous lesquelles le résultat est vrai. Nous supposons ainsi qu'une fonction que nous notons Γ et dont le graphe représente plus ou moins l'ensemble $\{(x,z);\ Lg(x,z) = 0\}$, doit avoir une certaine forme. Bien que cela soit relativement facile à vérifier sur des cas concrets, il serait plus satisfaisant d'avoir une condition portant directement sur α et ℓ.

Une autre question intéressante est celle de la dépendance de la fonction valeur V et de la frontière γ vis-à-vis des paramètres du modèle. Implicitement, V et γ dépendent du processus X au travers de la fonction α. Dans des cas simples tels que le processus d'Ornstein-Uhlenbeck ou le mouvement brownien avec un drift négatif, cette dépendance se résume à un seul paramètre $\alpha \in \mathbb{R}$. Si on considère une suite (α_n) convergeant vers α, on peut s'interroger sur l'existence d'une limite à (γ_n) et (V_n). Ensuite dans le cas général, si on se donne une suite de fonctions (α_n) qui converge pour une certaine topologie, qu'en est-il?

Enfin, une généralisation intéressante est le cas inhomogène, c'est-à-dire avec des coefficients μ et σ qui dépendent également du temps. Sur le plan pratique, cela permettrait de considérer un drift et une volatilité qui dépendent du temps. Mais le plus important n'est pas là. En réalité, l'hypothèse la plus contraignante dans le cas stationnaire est le fait que la "moyenne" autour de laquelle oscille le processus est supposée constante, et cela peut remettre en cause la viabilité de la stratégie de trading que nous avons considérée.

En effet, supposons que X oscille autour d'une moyenne $b(t)$. Pour pouvoir appliquer les résultats précédents, il faudrait considérer le processus $Y_t := X_t - b(t)$ qui n'est pas forcément tradable. Le problème c'est que, partant de $Y_0 > 0$, si l'on note T_0 le premier instant où Y revient en zéro, on aura $Y_{\theta^*} \geq Y_{T_0}$, mais il n'y a aucune raison pour que $X_{\theta^*} \geq X_{T_0}$. En revanche, l'étude du problème avec coefficients inhomogènes résoudrait ce problème. Notons cependant que dans ce cadre, il semble difficile d'obtenir plus qu'une caractérisation de la fonction valeur comme unique solution de viscosité de l'équation de HJB associée au problème (voir [11] pour plus de détails sur les solutions de viscosité).

1.3 Investissement optimal avec des considérations de performance relative

1.3.1 Motivation

Comme nous l'avons déjà vu, depuis les articles de Merton, le problème d'investissement optimal a été généralisé dans bien des directions, mais dans aucun de ces travaux l'investisseur ne tient compte de ses semblables dans sa prise de décision. Autrement dit, l'agent n'est pas influencé par le comportement des autres. Or cette hypothèse n'est pas réaliste, et ce pour différentes raisons. En premier lieu bien sûr, posséder cent euros en France n'est pas équivalent à posséder la même somme en Afrique. Mais ceci ne constitue pas un réel problème puisqu'il est toujours possible par un changement de variable d'en tenir compte (en considérant $U(\frac{x}{K})$ pour un facteur d'échelle K). Mais ce qui en revanche est beaucoup plus problématique est qu'une performance n'a pas de sens dans l'absolu, mais selon le contexte ou encore ce que font les concurrents, en particulier dans le monde des banques et des hedgefunds. Ainsi, en période de crise, un trader qui se contente de ne pas perdre d'argent est perçu comme brillant, alors que pendant la bulle internet, il aurait été considéré comme médiocre. Et puis il semble indéniable qu'il est dans la nature humaine de se comparer à ses voisins, ses amis ou encore ses collègues.

Pour ce qui est du monde financier, on peut même aller plus loin. Si l'on s'intéresse au marketing des produits dérivés ou de manière équivalente au discours des hedgefunds à leurs clients, certes la performance absolue est mentionnée, mais elle est presque toujours comparée à un benchmark. Ce phénomène intervient également dans les rémunérations des traders, qui n'est pas uniquement indexée sur leur performance absolue. Les traders qui performent nettement mieux que les autres recevront toujours un bonus conséquent, même les mauvaises années, tandis que ceux qui performent nettement moins bien risquent même en plein essor de recevoir un bonus qui n'est que symbolique ou d'être renvoyés au profit de quelqu'un de plus performant.

Il est intéressant de mentionner que dans la littérature économique mais aussi sociologique, des considérations de performance ou richesse relative ont déjà été étudiées. Ainsi l'impact sur les comportements a été mis en évidence depuis le début du vingtième siècle, notamment dans les travaux de l'économiste et sociologue Veblen [78], qui parle de consommation "ostentatoire", qui ne provient pas d'un besoin de posséder un bien, mais de la volonté de montrer que l'on peut se permettre de l'acheter. Le premier à avoir regardé les implications financières de considérations de performance relative est Abel [1] qui étudie leur influence sur le prix d'un actif. Plusieurs articles ont par la suite utilisé plus ou moins le même cadre, notamment ceux de Gali [34] et Gomez, Priestley et Zapatero [36]. Dans tous ces papiers, les considérations relatives sont de nature exogène, au sens où les préférences des agents prennent directement en compte la richesse moyenne des autres investisseurs. Nous adoptons le même point de vue. Par contre DeMarzo, Kaniel et Kremer [16] ont montré que la source de considérations relatives peut aussi être endogène, c'est-à-dire qu'elles peuvent provenir des interactions entre investisseurs sans pour autant être modélisées directement.

Bien que très intéressants, tous ces travaux considèrent des modèles très simples, statiques ou en temps discret. L'idée est donc d'essayer de développer une théorie d'optimisation de portefeuille qui prenne en compte ces considérations de performance relative.

1.3.2 Aperçu de sujets connexes

D'un point de vue mathématique, les considérations relatives impliquent des interactions entre investisseurs et sont de fait reliées à la théorie des jeux. Cependant dans notre cadre, le seul concept de théorie des jeux que nous utilisons est celui d'équilibre de Nash, par conséquent nous ne nous étendrons pas sur le sujet. Rappelons simplement qu'un équilibre de Nash dans un jeu à n joueurs est un n-uplet constitué d'une stratégie pour chaque joueur tel que, pour chaque individu, cette stratégie correspond au meilleur choix possible sachant les stratégies des autres joueurs.

Dans notre cadre, l'étude des équilibres de Nash n'est pas triviale. Ainsi, suivant les idées de Lasry et Lions [56] sur les jeux à champ moyen, il est naturel de se demander si les choses se simplifient lorsque le nombre de joueurs tend vers l'infini. En particulier, on peut tenter d'expliciter le système d'équations progressive-rétrograde (forward-backward), appelé équations de Kolmogorov-HJB. Malheureusement notre modèle n'entre pas dans le cadre développé dans [56], et nous ne savons pas écrire les équations de Kolmogorov-HJB. En effet deux hypothèses essentielles énoncées dans [56] ne sont pas vérifiées ici. D'abord, les auteurs requièrent de la symétrie entre les agents (voir section 2.2 de cet article), ce qui reviendrait dans notre problème à avoir pour tout i, $U^i = U$, $A_i = A$ et $\lambda_i = \lambda$, ce qui reste tout de même un cas particulier intéressant. Mais surtout, les mouvements browniens intervenant dans la dynamique de la richesse de deux agents différents doivent être indépendants, ce qui correspond au cas particulier d'agents investissant chacun dans un actif indépendant des autres. Le cas d'un brownien commun peut aussi être traité, même s'il ne figure pas dans [56], mais en revanche le cas général reste un problème ouvert. Néanmoins, nous serons en mesure de calculer explicitement la limite dans un certain sens, et nous pourrons constater que le passage à la limite simplifie les expressions.

Dans la prochaine section, nous rappelons les principaux résultats concernant les équations différentielles stochastiques rétrogrades (EDSR) puisque nous aurons besoin de certains de ces résultats, mais auparavant, passons en revue quelques travaux de la littérature économique sur les considérations relatives. En simplifiant quelque peu, Abel [1] considère un modèle en temps discret, avec un horizon infini et une fonction d'utilité puissance dépendant de la consommation propre de l'investisseur à l'instant t (c_t) et de la consommation moyenne à l'instant $t-1$ (C_{t-1}):

$$U(c_t, C_{t-1}) := \frac{1}{\alpha}\left(\frac{c_t}{C_{t-1}}\right)^{\alpha}.$$

Les équilibres de Nash ne sont pas étudiés, mais à la place l'auteur explique que si tous les agents sont identiques, à l'équilibre chaque investisseur doit avoir la même consommation, ce qui permet de trouver le prix d'équilibre.

Gali [34], quant à lui, considère un cadre voisin du précédent, mais au lieu de comparer sa consommation à la date t avec la consommation moyenne de la date $t-1$, ici l'investisseur compare ces deux quantités à l'instant t, ce qui aboutit à un certain nombre de différences qualitatives.

Enfin, dans l'article de DeMarzo, Kaniel et Kremer [16], les auteurs considèrent un modèle dans lequel les considérations sont d'origine endogène, alors que dans les articles cités ci-dessus l'origine est exogène. Plus précisément, ils modélisent les préférences de chaque investisseur de manière classique, c'est-à-dire uniquement à travers sa richesse absolue. Puis ils considèrent un bien en quantité limitée, dont le prix d'équilibre augmente avec la richesse moyenne. Dès lors, l'investisseur qui ne peut suivre l'augmentation de la richesse moyenne risque de se retrouver exclus, ce qu'il cherche à éviter. C'est ainsi que se fait l'interaction. Les auteurs montrent, dans le cadre d'un modèle à deux périodes, comment ce phénomène peut favoriser l'apparition de bulles financières et accroit le risque global du marché.

1.3.3 Principaux résultats sur les EDSR

Le dernier champ important dont nous avons besoin dans cette partie est la théorie des équations différentielles stochastiques rétrogrades (EDSR). Contrairement au monde déterministe, le fait d'avoir une condition finale n'est pas équivalent à avoir une condition initiale. En effet, il n'est pas possible d'inverser le temps car cela changerait la filtration vis-à-vis de laquelle la solution doit être adaptée. Utilisons la notation suivante: ξ est la condition finale, f_t est le "driver", et (Y, Z) la solution de l'EDSR:

$$\begin{aligned} dY_t &= -f_t(t, Y_t, Z_t)dt + Z_t dW_t \\ Y_T &= \xi, \end{aligned}$$

avec $Y \in \mathbb{R}^n$, $Z \in \mathbb{R}^{n\times d}$.

Les EDSR furent d'abord introduites et résolues pour un driver f linéaire dans les années 70 par Bismut [6], mais c'est seulement au début des années 90 que Pardoux et Peng [63] ont établi le premier résultat général d'existence et d'unicité pour un driver lispchitzien et une condition finale dans L^2. Par la suite les EDSR ont fait l'objet de nombreux travaux parce que leurs applications, notamment en finance, sont à la fois naturelles et multiples. Nous renvoyons en particulier à l'article d'El Karoui, Peng et Quenez [26] pour une revue détaillée des EDSR dans le cas lipschitzien et un aperçu de nombreuses applications à la finance. L'idée principale pour montrer l'existence est de démontrer tout d'abord des estimations a priori sur Y et Z, après avoir appliqué la formule d'Itô à $|Y|^2$, avant d'utiliser le théorème du point fixe de Picard. En outre, pour $n = 1$, un résultat de comparaison peut être démontré, établissant que si $f^1 \leq f^2$ et $\xi^1 \leq \xi^2$, alors $Y^1 \leq Y^2$
Dans le papier d'Hamadene [38], les hypothèses de Lipschitz continuité du driver sont affaiblies à localement Lipschitz avec croissance strictement sous-quadratique par rapport à Z. Néanmoins ce résultat n'est valable que pour la dimension 1 ($n = 1$).

Puis Kobylanski [52] a démontré l'existence d'une solution pour un driver continu à croissance quadratique en Z, à condition que la condition finale ξ soit bornée. Elle a également prouvé un résultat de comparaison sous des hypothèses un peu plus restrictives. Malheureusement, ces résultats ne sont valables qu'en dimension un ($n = 1$ avec les notations précédentes). Les principales étapes de la preuve sont les suivantes. Dans un premier temps, à l'aide d'un changement de variable exponentiel, l'auteur parvient à établir des estimations a priori de Y et Z, c'est-à-dire que ces variables sont "contrôlées". Puis elle approche le problème quadratique à l'aide d'une suite de problèmes lipschitziens, et grâce au théorème de comparaison valable dans le cadre lipschitzien, montre que la suite (Y_n) est monotone. A l'aide des estimations établies précédemment, il est alors possible d'obtenir la convergence des suites (Y_n) et (Z_n) en un sens suffisamment fort pour pouvoir passer à la limite dans les EDSR. Ceci garantit l'existence. La preuve de l'unicité est moins intéressante et sera par la suite grandement simplifiée par l'utilisation du caractère BMO de Z, voir [51] sur la propriété BMO et par example [60] pour des résultats généraux pour les EDSR quadratiques.

Nous insistons sur le fait que dans ce qui précède, à la fois les estimations a priori et l'utilisation d'un théorème de comparaison ne sont valables que dans le cas unidimensionnel.

Une autre preuve intéressante et très différente de celle de Kobylanski est due à Tevzadze [76], qui utilise un argument de point fixe pour une condition terminale suffisamment petite, coupe son EDSR en morceaux rentrant dans ce cadre et réussit ensuite à recoller les différents morceaux pour résoudre l'EDSR initiale. Malheureusement là encore la démonstration ne s'applique qu'à la dimension un.

Malgré cette restriction, ce résultat sur les EDSR quadratiques a de nombreuses applications. En particulier, l'une d'entre elles a inspiré la méthode que nous appliquons pour rechercher les équilibres de Nash dans notre modèle: il s'agit du problème de maximisation d'utilité quand le portefeuille de l'investisseur est soumis à des contraintes, introduit par El Karoui et Rouge [27] puis complété par la suite par Hu, Imkeller et Müller [42]. Bien qu'utilisant en grande partie les mêmes idées que dans [27], Hu, Imkeller et Müller sont les premiers à avoir remarqué que la partie martingale de la solution d'une EDSR quadratique (Z avec les notations précédentes) est en fait une martingale BMO, propriété plus forte que celle énoncée initialement dans [52]. Cette propriété de BMO garantit de l'uniforme intégrabilité et par conséquent le théorème de Girsanov peut être appliqué à l'exponentielle de Doléans-Dade de Z.

Enfin, assez récemment, Briand et Hu [7] ont étendu ces résultats au cas de conditions finales non bornées. Plus exactement, ils demandent uniquement à ξ d'avoir des moments exponentiels d'un certain ordre (dépendant des paramètres de l'EDSR et de l'horizon temporel T). Une fois encore, l'idée est de montrer des estimations a priori grâce au même changement de variable exponentiel, d'approcher l'EDSR initiale par des EDSR avec condition finale bornée, puis de montrer qu'il est possible de passer à la limite grâce aux estimations précédentes. Ils ont également montré un théorème de comparaison, mais sous des hypothèses beaucoup plus fortes, notamment une hypothèse de convexité du driver f par rapport à Z.

1.3.4 Nouveaux résultats

Considérons un marché financier constitué d'un actif sans risque, de taux d'intérêt égal à zéro et d actifs risqués dont la dynamique est donnée par:

$$dS_t = \operatorname{diag}(S_t)\sigma_t(\theta_t dt + dW_t),$$

où W est un mouvement brownien sous une probabilité $\mathbb{P}$.

Considérons également N agents particuliers qui se regardent les uns les autres. Plus précisément, de même que dans la théorie classique d'optimisation de portefeuille, les préférences de l'agent i sont caractérisées par une fonction d'utilité U^i, qui est supposée satisfaire des hypothèses classiques, à savoir essentiellement qu'elle est strictement croissante, strictement concave et C^1. Mais au lieu de ne prendre en compte que sa richesse absolue, chaque investisseur utilise comme critère une combinaison convexe entre sa richesse absolue et l'écart entre sa richesse et la richesse moyenne de ses sembables. Si π^i est le portefeuille de l'agent i, introduisons pour chaque i la richesse des autres investisseurs:

$$\bar{X}_t^i = \frac{1}{N-1} \sum_{j \neq i} X_t^{\pi^j},$$

alors l'agent i cherche à résoudre le problème d'optimisation suivant:

$$\begin{aligned} V_i &:= \sup_{\pi^i \in \mathcal{A}_i} \mathbb{E} U_i \left((1-\lambda_i) X_T^{\pi^i} + \lambda_i (X_T^{\pi^i} - \bar{X}_T^i) \right) \\ &= \sup_{\pi^i \in \mathcal{A}_i} \mathbb{E} U_i \left(X_T^{\pi^i} - \lambda_i \bar{X}_T^i \right), \end{aligned}$$

en considérant que les stratégies π^j des autres agents sont connues et fixées. Autrement dit, on suppose l'information parfaite. $\mathcal{A}_i$ désigne l'ensemble des portefeuilles admissibles pour l'investisseur i. En plus d'hypothèses techniques relativement faibles, nous supposons que les portefeuilles sont soumis à des contraintes. Etant donné un ensemble A_i, on impose:

$$\pi_t^i \in A_i, \ \forall t, \ \mathbb{P}\text{-p.s.}$$

Le paramètre λ_i est la sensibilité à l'égard de la performance relative de l'agent, et est supposé être dans $[0,1]$.
Le but principal de ce chapitre est de montrer l'existence d'un équilibre de Nash ainsi qu'une caractérisation des équilibres de Nash possibles pour le problème à N investisseurs, dès lors que les λ_i et les A_i vérifient une condition assez générale.

En l'absence de contraintes sur les portefeuilles, ce qui correspond à $A_i = \mathbb{R}^d$ pour tout i, on peut montrer en utilisant des techniques classiques pour l'optimisation de portefeuille en marché complet que si:

$$\prod_{i=1}^N \lambda_i < 1,$$

alors l'existence et l'unicité de l'équilibre de Nash est garantie.
En revanche, en présence de contraintes, il semble difficile de dire quoi que ce soit pour des fonctions d'utilité générales. C'est pourquoi nous supposons que les fonctions d'utilités des agents sont de type exponentiel:

$$U^i(x) = -e^{-\frac{1}{\eta_i} x},$$

où $\dfrac{1}{\eta_i}$ est appelé aversion au risque ou sensibilité au risque de l'agent i.

Suivant les idées développées dans [27] ou [42], nous cherchons dans une première étape à caractériser au moyen d'EDSR les portefeuilles optimaux et à donner une expression de la fonction valeur du problème d'optimisation simple de l'agent i (les autres portefeuilles étant fixes). Si cela était possible, dans une seconde étape, le but serait de combiner ces équations pour obtenir une caractérisation des équilibres de Nash. Malheureusement, les travaux cités précédemment ne prennent en compte que des conditions finales bornées, ce qui n'est pas le cas dans notre modèle. En outre, pour mener à bien la seconde étape, il faudrait pouvoir résoudre une EDSR quadratique multidimensionnelle, ce qui est à l'heure actuelle une question ouverte. Nous n'avons pu prouver aucun de ces résultats, par conséquent, afin de surmonter ces difficultés nous supposons que θ et σ sont des fonctions déterministes de t, ce qui permet de traiter notamment le modèle de Black-Scholes, mais pas des diffusions markoviennes générales.

Notons néanmoins que même dans ce cadre, la démonstration est loin d'être triviale, en particulier pour la caractérisation d'un équilibre de Nash. En pratique, nous montrons que pour un équilibre de Nash, une certaine paire de processus (Y, Z) doit être solution de la même EDSR que celle obtenue avec le raisonnement précédent, sauf que dans ce cadre, il est possible d'exhiber une solution de cette N-EDSR. Au final, nous obtenons le résultat suivant, dans le cas où pour chaque i A_i est un espace vectoriel:

Théorème:
Supposons que θ et σ sont déterministes et que:

$$\prod_{i=1}^{N} \lambda_i < 1 \text{ ou } \bigcap_{i=1}^{N} A_i = \{0\}.$$

Alors il existe un équilibre de Nash et le portefeuille d'équilibre pour l'agent i est:

$$\hat{\pi}_t^i = \hat{\pi}_t^{i,N} := \sigma(t)^{-1} P_t^i M_t^i \theta(t)$$

avec:

$$M_t^i := \left[I - \sum_{j \neq i} \frac{\frac{\lambda_j}{N-1}}{1 + \frac{\lambda_j}{N-1}} P_t^j \left(I + \frac{\lambda_i}{N-1} P_t^i \right) \right]^{-1} \left[\frac{1}{\eta_i} I + \frac{1}{N-1} \sum_{j \neq i} \frac{\frac{\lambda_i}{\eta_j} - \frac{\lambda_j}{\eta_i}}{1 + \frac{\lambda_j}{N-1}} P_t^j \right].$$

De plus, la fonction valeur pour l'agent i à l'équilibre est donnée par:

$$V_i = V_i^N = -e^{-\frac{1}{\eta_i}(x^i - \lambda_i \bar{x}^i - Y_0^i)}$$

$$\text{avec } Y_0^i = -\frac{\eta_i}{2} \int_0^T |\theta(t)|^2 dt + \frac{1}{2\eta_i} \int_0^T |Q_t^i M_t^i \theta(t)|^2 dt,$$

et P^i, Q^i sont des projecteurs orthogonaux.

De plus pour tout équilibre de Nash $(\tilde{\pi}^1, ..., \tilde{\pi}^N)$ on a:

$$\tilde{\pi}^i_t = \sigma(t)^{-1} P^i_t \left[\psi_t(\tilde{Z}_t)\right]^i \qquad \text{et} \qquad V_i = -e^{-\frac{1}{\eta_i}(x^i - \lambda_i \bar{x}^i - \tilde{Y}^i_0)}$$

où $(\tilde{Y}, \tilde{Z})$ est solution d'une certaine EDSR quadratique N-dimensionnelle et ψ_t un certain opérateur.

Nous donnons également une généralisation de ce résultat au cas où les A_i sont simplement des convexes fermés.

Suivant les idées provenant de l'étude des jeux à champ moyen, pour lesquels quand le nombre de joueurs tend vers l'infini les expressions se simplifient considérablement, nous étudions la limite du portefeuille d'équilibre quand N tend vers l'infini. Une question naturelle qui surgit alors est de savoir si le nombre d'actifs reste borné ou s'il tend lui aussi vers l'infini. Le second cas soulève la question de la définition des EDS en dimension infinie ainsi que celui de l'existence de la projection sur un sous-espace qui n'est pas forcément fermé en dimension infinie. De fait, nous ne considérons que le premier cas qui revient à prendre d fixe. On montre alors l'existence d'un portefeuille d'équilibre "limite":

$$\hat{\pi}^{i,\infty}_t := \sigma(t)^{-1} P^i_t \left[\eta_i I + \lambda_i \bar{\eta}(I - \bar{\lambda} U_t)^{-1} U_t\right] \theta(t),$$

avec $U := \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} P^i$ dans $\mathcal{L}(\mathbb{R}^d)$, tandis que $\bar{\lambda}$ et $\bar{\eta}$ sont les moyennes des λ_i et des η_i respectivement.
Enfin, parce que les opérateurs manipulés sont linéaires quand les A_i sont des espaces vectoriels, nous sommes en mesure de montrer que, dans certains cas, la volatilité de la richesse de l'agent i est croissante vis-à-vis de n'importe quel λ_j. En outre, si l'on définit le portefeuille d'indice de marché comme étant la moyenne des portefeuilles:

$$\bar{\pi} := \frac{1}{N} \sum_{i=1}^{N} \pi^i,$$

alors, en toute généralité, si l'on prend la limite quand N tend vers l'infini, on peut montrer que la volatilité et le drift de la richesse limite sont tous deux croissants en $\bar{\lambda}$, tandis que la quantité

$$\frac{\text{espérance des rendements}}{\text{variance des rendements}}$$

est décroissante en $\bar{\lambda}$. En d'autres termes, d'un point de vue de ce critère de risque, quand le nombre d'agent tend vers l'infini, il est globalement inefficace pour le marché que les agents se regardent les uns les autres (même si cela peut être efficace pour certains agents particuliers).

1.3.5 Perspectives

De nombreuses questions intéressantes restent en suspens. Nous en mentionnons un certain nombre ci-après. Tout d'abord, comme nous l'avons évoqué, la résolution de la question d'existence et d'unicité d'EDSR multidimensionnelles, pas forcément dans un cadre général mais englobant tout de même celles qui interviennent dans notre problème, permettrait d'envisager des coefficients θ et σ non déterministes. L'autre ingrédient nécessaire serait alors de prouver une généralisation du problème de maximisation d'utilité en marché incomplet, comme énoncé dans [27], pour une condition finale non bornée. Pour des coefficients déterministes, nous avons pu nous en passer, mais la méthode ne semble pas pouvoir s'appliquer au cas général.

Ensuite, dans cette partie nous montrons que même si l'on impose que les ensembles de contraintes A_i soient des espaces vectoriels, l'influence des λ_i pour N fixé n'est pas claire en toute généralité. Par contre, à la limite N tend vers l'infini, les choses se simplifient. Si les A_i sont supposés être des convexes fermés, peut-on au moins dans ce second cas mettre en évidence une influence des λ_i?
On peut aussi étudier de plus près le cas d'ensembles non convexes. Nous donnons des contre-exemples pour lesquels le résultat énoncé précédemment est faux. Cela ne signifie pas pour autant qu'il n'existe pas d'équilibre de Nash. Et de plus, peut-être existe-t-il des hypothèses plus faibles que la convexité sous lesquelles le théorème est encore vérifié.

Un autre domaine d'investigation est la question de la limite $N \to \infty$. En premier lieu, quand le nombre d'actifs d tend vers l'infini avec le nombre d'agents N, est-il possible de définir un portefeuille limite? La question des EDS de dimension infinie a été étudiée par Da Prato et Zabczyk [14], mais nous l'avons laissée de côté. De plus, que d soit fixe ou non, si l'on définit un jeu limite dans un certain sens, la limite du portefeuille d'équilibre serait-elle un équilibre de Nash, et si oui, cet équilibre sera-t-il unique? De surcroît, si l'on voit ce problème limite comme un jeu à champ moyen, peut-on parvenir à écrire les équations de Kolmogorov-HJB associées comme c'est le cas dans [56]?

Enfin, on peut s'intéresser à deux extensions du modèle de départ. La première concerne la fonction d'utilité. Nous avons considéré des fonctions exponentielles car cela permet souvent des simplifications. Mais il est probable qu'il en soit de même avec des fonctions puissances, en prenant le critère suivant:

$$\left(\frac{X_T^i}{\left(\bar{X}_T^i\right)^{\lambda_i}}\right)^{\beta_i}.$$

Une seconde extension du modèle, bien plus délicate, serait d'inclure de la consommation.

1.4 Investissement optimal avec des considérations de performance relative et une pénalisation locale du risque

1.4.1 Motivation

Partant de la même idée que pour la partie précédente, ce chapitre, bien qu'indépendant, peut être vu comme le prolongement naturel du précédent. En effet, une manière naturelle de généraliser le cadre de la partie précédente est de remplacer les contraintes strictes de la forme $\pi \in A$ par des contraintes relâchées en soustrayant dans le problème d'optimisation de l'investisseur un terme de pénalisation $g(\pi)$ tandis que l'on autorise π à décrire $\mathbb{R}^d$. Plus précisément, nous remplaçons l'argument $X_T^{\pi^i} - \lambda_i \bar{X}_T^i$ de la fonction d'utilité par:

$$X_T^{\pi^i} - \lambda_i \bar{X}_T^i - \int_0^T g^i(\sigma_u \pi_u^i) du.$$

Puisque nous cherchons un supremum, il est clair qu'en prenant g égale à $+\infty$ sur le complémentaire de l'ensemble A, nous forçons le contrôle optimal ou une suite maximisante de contrôles à éviter A.

De plus, sur le plan financier, il semble naturel de considérer ce type de contraintes "faibles". Ainsi la fonction d'utilité peut être considérée comme une transcription mathématique des préférences de l'agent, tandis que les contraintes peuvent être vues comme des facteurs exogènes de différentes natures qui affectent l'agent. On peut penser par exemple à des restrictions légales, comme par exemple l'interdiction de vendre à découvert, ou l'interdiction d'investir dans un actif particulier pour des problèmes de confidentialité. Dans ces cas, une contrainte stricte semble appropriée en général, mais pas nécessairement tout le temps, car les services de régulation n'ont pas systématiquement des règles strictes. Mais on peut aussi penser à des raisons commerciales, par exemple pour un hedgefund qui garantit un certain type de profil de risque à ses clients, ou pour des raisons managériales, quand le chef d'une équipe de trading ou d'un fond ne veut pas que ses traders prennent trop de risques ou leveragent trop leurs portefeuilles. Pour tous ces types de contraintes, cette nouvelle formulation plus faible semble plus réaliste.

Toutefois, puisque nous n'avons pas pu résoudre le problème précédent autrement que lorsque les coefficients intervenant dans la dynamique des actifs étaient déterministes, nous serons confrontés au même problème ici. Rappelons que le principal problème était lié au fait que l'existence et l'unicité d'une solution pour des EDSR quadratiques multidimensionnelles sont des questions ouvertes. Mais en choisissant de façon appropriée la fonction de pénalisation

g, nous pouvons avoir affaire à des EDSR lipschitziennes, et ainsi nous pouvons énoncer des résultats pour des actifs ayant une dynamique générale, pouvant même ne pas être markovienne.

1.4.2 Nouveaux résultats

Comme dans la partie précédente, nous considérons un marché financier constitué d'un actif sans risque ayant un taux d'intérêt égal à zéro et d actifs risqués de dynamique:

$$dS_t = \mathrm{diag}(S_t)\sigma_t(\theta_t dt + dW_t),$$

où W est un mouvement brownien sous une probabilité $\mathbb{P}$, et l'on considère N agents particuliers qui se regardent les uns les autres. Mais le critère d'optimisation est différent. Là encore, au lieu de ne prendre en compte que sa richesse propre, chaque agent utilise une combinaison convexe de sa richesse et de la différence entre sa richesse et la richesse moyenne des autres, mais de plus il soustrait une pénalisation sur le portefeuille. Cette pénalisation remplace les contraintes de la partie précédente. Plus précisément, chaque agent est caractérisé par sa fonction d'utilité U^i, qui résume ses préférences, et par sa fonction de pénalisation $g^i : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, qui peut être vue comme représentative des facteurs extérieurs qui incitent l'investisseur à préférer certains types de portefeuilles, comme nous l'avons expliqué auparavant. Dans ce cadre, mis à part pour $g^i = 0$, les arguments usuels du marché complet ne s'appliquent pas, c'est pourquoi nous considérons encore des fonctions d'utilité exponentielles:

$$U^i(x) = -e^{-\frac{1}{\eta_i}x}.$$

Le problème d'optimisation de l'agent i est alors formulé de la manière suivante:

$$\begin{aligned} V_i &:= \sup_{\pi^i \in \mathcal{A}_i} \mathbb{E} e^{-\frac{1}{\eta_i}\left((1-\lambda_i)X_T^{\pi^i} + \lambda_i(X_T^{\pi^i} - \bar{X}_T^i) - \int_0^T g^i(\sigma_u \pi_u^i)du\right)} \\ &= \sup_{\pi^i \in \mathcal{A}_i} \mathbb{E} e^{-\frac{1}{\eta_i}\left(X_T^{\pi^i} - \lambda_i \bar{X}_T^i - \int_0^T g^i(\sigma_u \pi_u^i)du\right)}, \end{aligned}$$

en considérant que les stratégies π^j des autres agents sont connues et fixées. Ainsi nous supposons que l'information est parfaite. $\mathcal{A}_i$ est l'ensemble des portefeuilles admissibles pour l'agent i, qui doivent satisfaire des conditions techniques raisonnables. Le paramètre λ_i est là encore la sensibilité à l'égard de la performance relative, et appartient à $[0, 1]$.

Comme nous l'avons dit, il s'agit a priori d'une généralisation du travail de la partie précédente, mais pour éviter les difficultés techniques que nous avions rencontrées alors (à savoir des EDSR quadratiques multidimensionnelles), et être en mesure d'établir des résultats pour des dynamiques d'actifs assez générales, nous ne considérons ici qu'une classe particulière de fonctions de pénalisation. En particulier, nous supposerons g^i lipschitzienne. Le but principal de ce chapitre est de montrer l'existence et l'unicité d'un équilibre de Nash pour le problème à N investisseurs avec pénalisation, dès lors que les λ_i vérifient des conditions assez faibles.

Si g^i est à valeurs dans $\mathbb{R}$, l'agent peut investir dans le marché tout entier, mais la présence du terme de pénalisation ne permet pas de considérer ce modèle comme du marché complet. En réalité, la pénalisation joue le même rôle qu'une imperfection de type coût de transaction. Par conséquent, nous commençons par transposer à notre cadre les résultats sur la maximisation d'utilité en marché incomplet présents dans [27] et [42]. Nous considérons ainsi le problème d'optimisation suivant:

$$V = \sup_{\pi \in \mathcal{A}} \mathbb{E} - e^{-\frac{1}{\eta}\left(X_T^\pi - \int_0^T g(\sigma_u \pi_u) du - F\right)},$$

où F est un actif contingent que nous souhaitons imiter ("tracker"). Cet actif est simplement supposé être $\mathcal{F}_T$-mesurable et avoir des moments exponentiels de tous ordres, ce qui signifie:

$$\forall \delta > 0, \; \mathbb{E} e^{\delta |F|} < \infty.$$

En particulier, nous ne supposons pas que F est borné. Suivant les idées introduites dans [27], nous relions cette optimisation à un problème d'EDSR. Sous certaines hypothèses techniques sur g, garantissant essentiellement que le driver de l'EDSR est lipschitzien, nous caractérisons la solution du problème d'optimisation et l'unique portefeuille optimal par:

$$V(x) = -e^{-\frac{1}{\eta}(x - Y_0)},$$

où (Y, Z) est la solution unique de l'EDSR lipschitzienne suivante:

$$\begin{aligned} dY_t &= -f_t(Z_t)dt + Z_t.dW_t \\ Y_T &= F \end{aligned}$$

et l'unique portefeuille optimal est donné par:

$$\hat{\pi}_t = \sigma_t^{-1} \circ \Gamma_t(Z_t),$$

où f_t et Γ_t ne dépendent que de g et θ_t.

Ce résultat nous permet alors de prouver, avec quelques hypothèses techniques supplémentaires, qu'il existe un unique équilibre de Nash au problème initial à N agents, dès lors que les λ_i

ne sont pas trop proches de 1. Plus précisément, il existe $\lambda_m^N \in \left(1-\frac{2}{N},1\right)$ tel que pour tout i:

$$\lambda_i < \frac{\lambda_m^N}{B},$$

où B est la constante de Lipschitz commune à des fonctions Γ_i, définies comme Γ ci-dessus. De même que dans la partie précédente, nous obtenons une caractérisation de la fonction valeur de l'agent i sous la forme:

$$V_i = -e^{-\frac{1}{\eta_i}(x^i-\lambda_i\bar{x}^i-Y_0^i)},$$

mais cette fois (Y,Z) est l'unique solution de l'EDSR N-dimensionnelle **lipschitzienne** suivante:

$$\begin{aligned} dY_t^i &= -[f_t^i \circ \psi_t^i(Z_t) + \psi_t^i(Z_t).\theta_t]dt + Z_t^i.dB_t \\ Y_T^i &= 0. \end{aligned}$$

De plus, le portefeuille d'équilibre est donné par:

$$\hat{\pi}_t^{i,N} = \sigma_t^{-1} \circ \Gamma_t^i \circ \psi_t^i(Z_t),$$

avec:

$$\begin{aligned} \psi_t^i(Z) = &\left[I - \frac{1}{N-1}\sum_{j\neq i}\Gamma_t^j \circ \left(I+\lambda_j^N\Gamma_t^j\right)^{-1} \circ \left[\lambda_j\left(I+\frac{\lambda_i}{N-1}\Gamma_t^i\right)\right]\right]^{-1} \\ &\circ \left[Z^i + \frac{1}{N-1}\sum_{j\neq i}\Gamma_t^j \circ \left(I+\lambda_j^N\Gamma_t^j\right)^{-1}(\lambda_i Z^j - \lambda_j Z^i)\right]. \end{aligned}$$

Nous nous intéressons de nouveau à la question d'une éventuelle limite au portefeuille d'équilibre quand N tend vers l'infini. Malheureusement, pour un θ général, cela implique de résoudre une EDSR de dimension infinie, même dans le cas où le nombre d'actifs d est fixe. Certes de telles EDSR ont déjà été étudiées, notamment par Fuhrman et Hu [33], mais cela dépasse le champ de notre étude. De fait, nous nous restreignons au cas d'une prime de risque θ supposée être une fonction déterministe du temps t. Dans ce cas, on peut résoudre explicitement la N-EDSR, ce qui fournit une formule explicite pour le portefeuille d'équilibre $\hat{\pi}^i = \hat{\pi}^{i,N}$. Dès lors, sous des hypothèses assez faibles, nous prouvons un résultat de convergence de $\sigma_t\hat{\pi}^{i,N}$ vers:

$$\sigma_t\hat{\pi}^{i,\infty} := \circ\Gamma_t^i \circ (I-\chi_t)^{-1} \circ \chi_t(0),$$

si l'on suppose que $\frac{1}{N}\sum_{j=1}^N \Gamma_t^j \circ \lambda_j I$ converge uniformément sur les compacts vers χ_t.

1.4.3 Perspectives

Si on oublie les restrictions sur les fonctions de pénalisation permettant d'obtenir des EDSR lipschitziennes, alors toutes les extensions que nous avons évoquées pour la partie précédente s'appliquent à cette partie. Mais il y en a au moins deux qui sont spécifiques à cette étude.

D'abord, comme nous l'avons expliqué ci-dessus, lorsque la prime de risque θ est détermniniste, puisque le portefeuille d'équilibre est explicite, il n'y a pas de problème d'ordre théorique lorsque l'on regarde la limite N tend vers l'infini (au moins pour d fixe). Par contre, dans le cas général, le portefeuille d'équilibre est défini comme fonction de la solution d'une EDSR de dimension N, et par conséquent il faut pouvoir être capable de définir proprement des EDSR de dimension infinie. Dans le cas Lipschitz, de telles EDSR ont été étudiées par Fuhrman et Hu [33] par exemple, et peut-être leurs travaux s'appliquent-ils à notre cadre de travail.

Une seconde question intéressante concerne le dernier exemple que nous considérons dans cette partie. Plus précisément, nous prenons une suite d'approximations g_n^i des contraintes strictes $\pi \in A_i$ utilisées dans le chapitre précédent, pour des ensembles convexes fermés A_i et telles que (g_n^i) converge simplement vers $\infty \times 1_{A_i}$. Quand n tend vers l'infini, peut-on trouver une limite à la suite des portefeuilles d'équilibre? Peut-on trouver une limite à la suite des solutions des EDSR approchées (Y_n, Z_n), et si oui cette limite est-elle solution de l'EDSR quadratique?

Chapter 2

Optimal investment-consumption with taxes: first order expansion for general utility functions

2.1 Introduction

Since the seminal papers of Merton [58, 59], the problem of optimal investment has been extensively studied in order to generalize the original framework. Those generalizations have been made in different directions and using different techniques. Pliska [67], Cox and Huang [10] and Karatzas, Lehoczky and Shreve [46] have extended the study to general utility functions in a complete market. The next step was to deal with incompleteness. Cvitanic and Karatzas [12] or Zariphopoulou [79] have studied incompleteness due to constraints on the portfolio, as they impose that the portfolio must remain in a certain set. Constantinides and Magill [9] introduced the problem with proportional transaction costs. See also Davis and Norman [15], Shreve and Soner [75], Duffie and Sun [17] or Akian, Menaldi and Sulem [2] on the subject. Another important direction is the incomplete market case from a general point of view, see He and Pearson [39, 40] in discrete time, and Karatzas, Lehoczky, Shreve and Xu [47], Kramkov and Schachermayer [53, 54] or Kramkov and Sirbu [55] for the continuous-time case.

However, the question of taxes has received limited attention whereas it has an important economical impact. The first relevant work on the subject was done by Constantinides [8], but the assumption of no limit on short sales gave a trivial and non realistic solution. Therefore, later studies have always rejected the possibility of short sales. Dybvig and Koo [20] considered, in a binomial model, a taxation rule with a very complex path dependancy:

when selling the asset, the tax basis of a unit of the asset was equal to the purchase price of this specific unit. They only provided limited numerical results. Jouini, Koehl and Touzi [44, 45] then proved an existence result in the continuous-time framework, but again the complexity of this rule allowed them to provide only limited numerical and theoretical results.

Damon, Spatt and Zhang [13] introduced, in a binomial model, another taxation rule with a simplified path dependancy. Indeed, when selling the asset, the tax basis is computed as the average of past purchase prices, weighted by the proportion of asset purchased at that price. Gallmeyer, Kaniel and Tompaidis [35] generalized this study to multi-assets. Then Ben Tahar, Soner and Touzi [3, 4] introduced the continuous-time version of the previous model, with a power utility function and a single risky asset following a Black-Scholes dynamics. They provided both theoretical and numerical results.

In this chapter, we generalize some of the results stated in [3, 4], for a general utility function satisfying mild conditions and for a multi-dimensional risky asset following a more general Markov diffusion. After checking that some basic properties of the value function still hold in our framework, we show how we can here again derive upper and lower bounds to the value function using tax-free problems with different parameters. Although not explicit in general, the value functions of those tax-free (or Merton's) problems can be expressed by means of probabilistic representations. This allows us to show that, as the interest rate r and the tax rate α simultaneously go to zero, the value function of the problem with taxes admits a first order expansion around the value of the tax-free problem.

2.2 Model and problem formulation

2.2.1 Assets

Let $n \geq 1$ be an integer, $W = \{W_t, t \geq 0\}$ be an n-dimensional Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the corresponding completed canonical filtration. We also consider an infinite time horizon $T = +\infty$.

We then consider a (complete) financial market consisting of:
- one non-risky asset (a bank account), evolving with the interest rate $r \geq 0$, which will be assumed to be constant, in order to simplify notations; an easy generalization could be done for a deterministic interest rate;

- n risky assets with dynamics that are given by:

$$dP_t = diag(P_t)[b(P_t)dt + \sigma(P_t)dW_t] \tag{2.1}$$

where $P_t : \Omega \to \mathbb{R}^n$, $b : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to M_n(\mathbb{R})$. For a vector x, $diag(x)$ stands for the diagonal matrix with i-th term on the diagonal equal to x^i.
We assume that b and σ are Lipschitz continuous and bounded, which guarantees that the previous SDE has a strong solution with continuous paths. We also assume that σ is regular, σ^{-1} is bounded and we define the risk premium by:

$$\theta := \sigma^{-1}[b - r1_n], \text{ with } 1_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n.$$

We focus here on the infinite horizon case, but the results can be extended almost without any change to the finite horizon case. In fact notations are a bit more complex for the finite case (functions depend also on time t), but things work simpler as we will not have to assume integratibility hypotheses unlike in the infinite horizon framework. We will make a few remarks about that along this work. Another easy extension would be to consider the interest rate r to be a deterministic continuous and bounded function.

2.2.2 Taxation rule

As in [3], the aim of this work is to study a problem of optimal consumption and investment, when the capital gains are subject to taxes. We use the same taxation model as theirs, which is the continuous version of the one introduced first by Damon, Spatt and Zhang [13]. We assume that the sales of stocks are subject to taxes on capital gains. The amount of tax for a sale at time t of the i-th risky asset is computed by comparing the current price P_t^i to the current i-th index value B_t^i, defined as the weighted average price of shares purchased by the investor up to time t. If $P_t^i < B_t^i$, the investor realizes a *capital gain*, while if $P_t^i > B_t^i$ he realizes a *capital loss*. When the investor sells a certain number of shares, the proportions of shares are kept constant in order to compute the tax basis, which means that it will not change the tax basis (but it will change the number of shares used to compute it).

For example, if the agent buys x_1 shares of the i-th risky asset at time t_1 at the price P_1 and x_2 shares of the same asset at time t_2 at the price P_2, the tax basis at time t will be $B_t^i = \frac{x_1 P_1 + x_2 P_2}{x_1 + x_2}$ (if nothing else occurs). If he then sells at time t_3 z shares, we consider that he still owns $x_1(1 - \frac{z}{x_1+x_2})$ of the t_1-shares and $x_2(1 - \frac{z}{x_1+x_2})$ of the t_2-shares. Thus $B_{t_3^+} = B_{t_3^-}$.

Then the sale of one unit share of the i-th asset at time t will breed the (algebraic) payment of an amount denoted by $\ell(P_t^i - B_t^i)$ which we assume to be linear:

$$\ell(P_t^i - B_t^i) = \alpha(P_t^i - B_t^i) \tag{2.2}$$

where $\alpha \in [0,1)$ is constant and is called the tax rate coefficient.

Remark 2.1 As claimed before, taxes can be negative. It is almost true in certain markets. Indeed, in some countries, if P is lower than B, the investor will have a tax deduction for the following year. A linear model is not of course perfectly realistic, but not absolutely inconsistent.

Remark 2.2 As it is pointed out in [3], such a taxation rule might a priori allow better portfolios than the optimal one of the tax-free model, which would not be economically acceptable. Hopefully, in our framework too the upper bound result will show that this will never happen.

2.2.3 Strategies

We assume that the agent allocates his wealth between the bank account (non-risky asset) and the risky assets. The amount of money he owns in the bank account is denoted by (X_t), the amount in the risky assets by (Y_t) which takes values in $\mathbb{R}^n$. We also denote by (C_t) the consumption (rate) of the agent. In other words, $C_t dt$ represents the amount consumed within $[t, t+dt]$. Recall that essentially for notational purposes, we have taken an infinite time horizon $T = +\infty$.

We then introduce the position in the risky assets evaluated at the basis prices:

$$K_t^i = B_t^i \frac{Y_t^i}{P_t^i}. \tag{2.3}$$

Comparing P to B is equivalent to comparing Y to K. Notice that B_t^i is not defined if $Y_t^i = 0$. We assume that in that case $B_t^i = Y_t^i$, but such a choice has no influence on the value of K_t^i.

Moreover we make the following assumptions:

- (X_t), (Y_t) and (C_t) are progressively measurable with respect to $\mathbb{F}$;
- $C_t \geq 0,\ \mathbb{P}-\text{a.s},\ \forall t \geq 0$;
- $\displaystyle\int_0^S C_t dt < +\infty,\ \mathbb{P}-\text{a.s},\ \text{for all } S \in [0, +\infty)$.

Remark 2.3 As we will see, X and Y are both càdlàg by definition, therefore assuming that they are $\mathbb{F}$-adapted is equivalent to assuming that they are progressively measurable.

We will denote (Z_t) the total wealth of the investor after liquidation of the risky asset positions:

$$Z_t = X_t + \sum_{i=1}^{n} [(1-\alpha)Y_t^i + \alpha K_t^i]. \tag{2.4}$$

Transfers on the financial market are described by means of the transfers between the bank account towards the risky assets and in the opposite direction. Because of taxes, considering only the sum of algebraic transactions does not make sense. Therefore, we denote by $(L_t) \in \mathbb{R}^n$ the **absolute** transfer from X to Y, and by $(M_t) \in \mathbb{R}^n$ the **relative** transfer from Y to X (M_t^i is the proportion of Y_t^i transfered, $Y_{t-}^i dM_t^i$ is the amount).
We assume that M and L are $\mathbb{F}$-adapted, right-continuous and non-decreasing processes. We also assume that $M_{0-} = L_{0-} = 0$. Finally, we assume that short-sales are not allowed, so that we restrict the jumps of M by:

$$\Delta M_t^i \leq 1, \forall i, \forall t \geq 0, \mathbb{P}\text{-a.s.} \tag{2.5}$$

We assume that the dynamics of Y is given for each component i by:

$$dY_t^i = Y_t^i \frac{dP_t^i}{P_t^i} + dL_t^i - Y_{t^-}^i dM_t^i \tag{2.6}$$

as the variation of the wealth in the i-th risky asset is due to three factors: the relative variation of the stock $\dfrac{dP_t^i}{P_t^i}$ multiplied by the amount held Y_t^i, the absolute transfers from the bank account to the risky asset dL_t^i and the relative transfers from the risky asset to the bank account dM_t^i multiplied by the amount held just before the transfer (as jumps are allowed) Y_{t-}^i.
The dynamics of K is assumed to be given, for each i by:

$$dK_t^i \;\; = dL_t^i - K_{t^-}^i dM_t^i \tag{2.7}$$

as the variation of K^i is only due to the transfers in one direction or the other.
Then the dynamics of X under the self-financing condition is:

$$dX_t = (rX_t - C_t)dt + \sum_{i=1}^{n} [-dL_t^i + \{(1-\alpha)Y_{t^-}^i + \alpha K_{t^-}^i\} dM_t^i] \tag{2.8}$$

as the variation of the wealth in the bank account is due to five factors. The non-risky rate brings the rX_tdt term. The consumption brings the $-C_tdt$ term. The absolute transfers from the bank account to the risky assets bring the terms $-dL_t^i$, while the tranfers from the risky assets to the bank account bring the term $Y_{t-}^i dM_t^i$ to which we must substract the taxes, explaining the expression $-\alpha(Y_{t-}^i - K_{t-}^i)dM_t^i$.
And this implies:

$$dZ_t = (rZ_t - C_t)dt + \sum_{i=1}^{n} [(1-\alpha)Y_t^i(\frac{dP_t^i}{P_t^i} - rdt) - r\alpha K_t^i dt]. \tag{2.9}$$

Notice that L and M do not appear in the dynamics of Z.

Denoting $x.y$ the scalar product of x and y, we will use the following notations:
(i) For any $m \in \mathbb{N}^*$, if $x, x' \in \mathbb{R}^m$, we will denote $x \geq x'$ (respectively $x > x'$) if and only if $x - x' \in (\mathbb{R}_+)^m$ (respectively $x - x' \in (\mathbb{R}_+^*)^m$).
(ii) Let

$$S = \left\{(x, y, k) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n;\ x + [(1-\alpha)y + \alpha k]\,.1_n > 0,\ \forall i,\ y^i > 0,\ k^i > 0\right\}. \tag{2.10}$$

Then $\overline{S}$ is the closure of S, and $\partial^z S = \left\{(x, y, k) \in \overline{S},\ z = x + [(1-\alpha)y + \alpha k]\,.1_n = 0\right\}$ the boundary corresponding to a zero initial wealth after liquidation of the risky positions.

Definition 2.4 *(i) A (consumption-investment) strategy is a triple (C, L, M) satisfying the previous hypotheses.*

(ii) Given an initial condition $s \in \overline{S}$, and a strategy ν, we note $S^{s,\nu} = (X^{s,\nu}, Y^{s,\nu}, K^{s,\nu})$ the unique strong solution of the dynamics above such that $S_{0-} = s$.

(iii) Let $s \in \overline{S}$, a strategy ν is s-admissible if $S^{s,\nu}$ satisfies the no-bankruptcy condition:

$$Z_t \geq 0,\ \mathbb{P}\text{-}a.s,\ \forall t \geq 0. \tag{2.11}$$

We denote by $\mathcal{A}(s)$ the set of all s-admissible strategies. A strategy ν is said to be admissible if there exists $s \in \overline{S}$ such that ν is s-admissible.

2.2.4 The consumption-investment problem

Now we are able to define the agent's optimization problem. We assume that the investor's preferences are described by a utility function $U : \mathbb{R}_+ \to \mathbb{R}$, C^1, (strictly) increasing, strictly concave, and satisfying the Inada conditions:

$$\lim_{x \to 0} U'(x) = +\infty \text{ and } \lim_{x \to +\infty} U'(x) = 0.$$

Moreover, we assume that $U(0) = 0$.

Then, given the discount parameter $\beta > 0$, for any initial data $s \in \overline{S}$ and strategy $\nu := (C, L, M) \in \mathcal{A}(s)$, the associated utility for the agent is defined by:

$$J_\infty(s,\nu) = \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(C_t) dt\right]. \tag{2.12}$$

Finally, the value function associated to the consumption-investment problem is:

$$V_\infty(s) = \sup_{\nu \in \mathcal{A}(s)} J_\infty(s,\nu). \tag{2.13}$$

Remark 2.5 The power utility function $U(x) = \dfrac{x^p}{p}$, $p \in (0,1)$ used in [3] is a particular case of our framework, but the logarithmic or exponential utility functions do not fit in our framework.

2.3 First properties of the value function

Although the proofs of [3] were written assuming a one-dimensional Black-Scholes dynamics and a power utility function, the proofs of this section are very similar, and are therefore omitted here. For completeness, we rewrite them in our framework in the appendix.

First, as one could expect, we have the following monotonicity result.

Proposition 2.6 *V is nondecreasing with respect to each of the variables x, y_i and k_i (for any $1 \le i \le n$).*

Then it is quite intuitive that starting from a zero wealth breeds a value function equal to 0, as it is impossible in this case to guarantee the admissible condition $Z \ge 0$ after borrowing some money.

Proposition 2.7 *Let $s = (x, y, k) \in \partial^z S$ (which means $z = 0$), then we have $V(s) = 0$.*

Remark 2.8 Recall that we have assumed $U(0) = 0$. In general, we would have $V(s) = U(0)$.

The next result is quite striking. Indeed it shows that the zero interest rate case ($r = 0$) can be seen as a tax-free problem, or in other words can be reduced to the Merton's problem. As we will use it again:

$\bar{V}$ denotes the value function of Merton's problem.

Proposition 2.9 *Assume $r = 0$. Let $s = (x, y, k) \in \overline{S}(\subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$, then we have:*

$$V(s) = \bar{V}\left(x + \left[(1-\alpha)y + \alpha k\right].1_n\right).$$

Finally we end this section with a phenomenon that can be seen in the real world: wash sales can be optimal. A wash sale (of the i-th risky asset) is the action of selling every asset i and buying (at the same time) the same amount. In other words, it is a way to reinitialize the tax basis. We will see that in our framework, if $K_t^i > Y_t^i$ (or in other words $B_t^i > P_t^i$), then it is better to perform a wash sale.

Definition 2.10 *We will call* non-infinite stopping time *a stopping time τ such that:*

$$\mathbb{P}\{\tau < +\infty\} > 0.$$

Proposition 2.11 *(Optimality of "wash sales") Let $s \in \overline{S}$, $\nu = (C, L, M) \in \mathcal{A}(s)$. Assume that there exists a non-infinite stopping time τ and an index i_0 such that:*

$$K_\tau^{(s,\nu)i_0} > Y_\tau^{(s,\nu)i_0}, \; P\text{-}a.s.$$

Then there exists an admissible strategy $\tilde{\nu} = (\tilde{C}, \tilde{L}, \tilde{M}) \in \mathcal{A}(s)$ satisfying:

$$Y^{s,\tilde{\nu}} = Y^{s,\nu}, \; \Delta\tilde{M}_t^{i_0} - \Delta M_t^{i_0} = (1 - \Delta M_t^{i_0})1_{\{t\}}(\tau) \textit{ and } J_\infty(s, \tilde{\nu}) > J_\infty(s, \nu).$$

Remark 2.12 This phenomenon is highly dependent on the tax model used. In a non-negative tax model (for example of the form $\alpha(P_t - B_t)^+$), it would probably not occur. Moreover, it is also due to the absence of transaction costs.

2.4 Bounds using Merton's problem

In contrast with a frictionless model (Merton's problem), even in simple cases as in [3] (Black-Scholes dynamics for P, power utility function), we have very few knowledge of the value function as soon as $\alpha > 0$. And in our more general framework, we know even less. But using bounds thanks to the Merton's problem, we will be able to derive a first order expansion of the value function as the tax rate α and the interest rate r go to zero. Therefore we first make a quick review of Merton's problem before showing that considering two Merton's problems with slightly different parameters, we obtain upper and lower bounds.

2.4.1 Review of Merton's problem

We first recall classical results about Merton's problem without proofs. We refer the reader to Karatzas and Shreve [49], chapter 3, for the proofs or any further detail.

In this section, we denote by $\bar{V}^\alpha$ the value function for Merton's problem with modified parameters $(\theta^\alpha, \sigma^\alpha)$. Moreover we call the two problems associated with $\bar{V}$ and $\bar{V}^\alpha$ respectively *standard Merton's problem* and *modified Merton's problem*.

Merton's problem is the same as the previous problem, but without taxes. Consequently, we only consider the dynamics of Z (X and Y bring no useful information for the optimization problem), and the strategies take the form $(C_t, \Pi_t) \in \mathbb{R}_+ \times \mathbb{R}^n$, where Π_t^i is the position in the i-th risky asset, so that it corresponds to the Y_t^i of the model with taxes. We write $\bar{Z}$ the portfolio value for this problem, and its dynamics is given by:

$$d\bar{Z}_t = (r\bar{Z}_t - C_t)dt + \Pi_t.\sigma_t(\theta_t dt + dW_t). \tag{2.14}$$

The set of admissible strategies is bigger than the one defined before. In fact, we do not limit ourselves to transfers with bounded variations, whereas it was the case for L and M previously.

The classical approach consists in considering the optimization dual problem. First, let us define the dual problem using Legendre-Fenchel transform:

$$\tilde{V}(\zeta) := \sup_{z\in\mathbb{R}} \{\bar{V}(z) - z\zeta\}, \ \forall \zeta \in \mathbb{R}$$

As $\bar{V}$ is concave, we have:

$$\bar{V}(z) = \inf_{\zeta\in\mathbb{R}} \{\tilde{V}(\zeta) + z\zeta\}, \ \forall z \in \mathbb{R}$$

We define in the same way the dual utility function, denoted by $\tilde{U}$.

We denote by $\mathbb{P}^0$ the martingale measure, defined by its Radon-Nikodym density:

$$\left.\frac{d\mathbb{P}^0}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{-\int_0^t \theta_u^* dW_u - \frac{1}{2}\int_0^t \|\theta_u\|^2 du}, \tag{2.15}$$

where we have written θ_u instead of $\theta(P_u)$ for notational purposes. We denote by $\mathbb{E}^0$ the expectation under $\mathbb{P}^0$, while $\mathbb{E}$ stands for the expectation under $\mathbb{P}$. We also define W^0 by:

$W_t^0 = W_t + \displaystyle\int_0^t \theta_u du$, and using Girsanov's Theorem W^0 is a Brownian motion under $\mathbb{P}^0$. Finally, we introduce the following process:

$$\begin{aligned} H_t &:= e^{-rt-\int_0^t \theta_u^* dW_u - \frac{1}{2}\int_0^t \|\theta_u\|^2 du} \\ &= e^{-rt} \left.\frac{d\mathbb{P}^0}{d\mathbb{P}}\right|_{\mathcal{F}_t}. \end{aligned} \tag{2.16}$$

The assumptions on U ensure that U' has an inverse which we denote by I and we define for $y > 0$:

$$\begin{aligned} \mathcal{X}(\zeta) &:= \mathbb{E}\left[\int_0^{+\infty} H_t I(\zeta e^{\beta t} H_t) dt\right] \\ &= \mathbb{E}^0\left[\int_0^{+\infty} e^{-rt} I(\zeta e^{\beta t} H_t) dt\right]. \end{aligned} \tag{2.17}$$

We assume that:

$$\mathcal{X}(\zeta) < \infty, \ \forall \zeta > 0. \tag{2.18}$$

Then, $\mathcal{X}$ inherits the decrease of I, $\lim\limits_{0+} \mathcal{X} = +\infty$ and $\lim\limits_{+\infty} \mathcal{X} = 0$. Moreover we know that $\mathcal{X}$ is $C^2(0, +\infty)$ and admits an inverse function $\mathcal{Y}$ which is decreasing and C^2:

$$\mathcal{X} \circ \mathcal{Y}(z) = z.$$

Then we introduce for $\zeta \in (0, \infty)$:

$$G(\zeta) := \mathbb{E}\left[\int_0^{+\infty} e^{-\beta t} U \circ I(\zeta e^{\beta t} H_t) dt\right]. \tag{2.19}$$

We also know that G is $C^2(0, +\infty)$.

Remark 2.13 If $T < +\infty$, we define similar quantities. See [49] for more details.

We summarize here the important properties for us:

Theorem 2.14 *For $z > 0$, $V(z) = G \circ \mathcal{Y}(z)$, and in particular V is C^2 on $(0, T)$. Moreover there exists an optimal control $(\hat{C}, \hat{\Pi})$, given by:*

$$\begin{aligned} \hat{C}_t &= I(e^{\beta t} \mathcal{Y}(z) H_t) \\ \hat{\Pi}_t &= (\sigma_t^*)^{-1} \left[\hat{Z}_t \theta_t + \frac{\psi_t}{H_t}\right], \end{aligned}$$

where $\hat{Z}$ is the associated optimal wealth satisfying:

$$\hat{Z}_t = \frac{1}{H_t}\mathbb{E}\left[\int_t^\infty H_u \hat{C}_u du \Big| \mathcal{F}_t\right],$$

and ψ in the (unique) adapted and L^2 integrand in the martingale representation of:

$$\mathbb{E}\left[\int_0^\infty H_u \hat{C}_u du | \mathcal{F}_t\right].$$

And we have:

$$V'(\zeta) = \mathcal{Y}(z),\ \tilde{V}(\zeta) = G(\zeta) - \zeta\mathcal{X}(\zeta),\ \tilde{V}'(\zeta) = -\mathcal{X}(\zeta).$$

Remark 2.15 Notice that assumption (2.18) implies that $\mathbb{E}\left[\int_0^\infty H_u \hat{C}_u du\right] < \infty$.

Remark 2.16 We also know that, in this Markov framework, V satisfies an HJB equation, but we will not use it.

Remark 2.17 Again, this theorem can be stated for $T < \infty$.

2.4.2 Upper and lower bounds

Now we derive upper and lower bounds from two Merton's problems, with different parameters. Again the proofs of the results of this section are very close to the ones given in [3], except for the end of the proof of Proposition 2.20 below. The proofs are given in the appendix, stated in our framework and with the additional arguments for Proposition 2.20. First we give the upper bound.

Proposition 2.18 *Let $s = (x, y, k) \in \overline{S}$, we have the following upper bound:*

$$V(s) \leq \bar{V}\left(x + \left[(1-\alpha)y + \alpha k\right].1_n\right)$$

Then we introduce a "modified" Merton's problem. More precisely, we consider Merton's problem but with parameters $(\theta^\alpha, \sigma^\alpha)$ defined by:

$$\theta^\alpha = \theta - \frac{r\alpha}{1-\alpha}\sigma^{-1}1_n \quad (2.20)$$

$$\sigma^\alpha = (1-\alpha)\sigma. \quad (2.21)$$

We write $\bar{V}$ and $\bar{V}^\alpha$ respectively the value functions of Merton's problem with parameters (θ, σ) and $(\theta^\alpha, \sigma^\alpha)$ respectively.

Recall that we note $J(s,\nu) = \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(C_t)dt\right]$ where (C_t) is associated to the (admissible) strategy ν. For Merton's problem, we will note as well $\bar{J}(z,\gamma)$.

For Merton's problem, we define the relative consumption and portofolio:

$$c_t := \frac{C_t}{\bar{Z}_t}$$
$$\pi_t := \frac{\Pi_t}{\bar{Z}_t}, \text{ if } \bar{Z}_t \neq 0.$$

We denote $\hat{c}$ and $\hat{\pi}$ (resp. $\hat{c}^\alpha$ and $\hat{\pi}^\alpha$) the optimal relative consumption and portfolio associated to Merton's problem (resp. Merton's modified problem).
Together with assumption (2.18), we make the following hypotheses for the optimal control of any Merton's modified problem (any $\alpha \geq 0$):

- for any $\alpha \geq 0$ and any $T > 0$, $\hat{c}^\alpha$ and $\hat{\pi}^\alpha$ are bounded on $[0,T]$; (2.22)
- for any $\alpha \geq 0$, there exists a version of $\hat{\pi}^\alpha$ that has continuous paths; (2.23)
- for any $\alpha \geq 0$, $\hat{\pi}^\alpha$ is a diffusion with uniformly bounded drift and volatility. (2.24)

As H has continuous paths, and I is continuous, it is immediate that for any α, $\hat{c}^\alpha$ has continuous paths.

Remark 2.19 When θ and σ are constant (resp. deterministic) and $T = +\infty$ (resp. $T < +\infty$), the feedback form given by Corollary 3.9.15 in [49] (resp. 3.8.8) implies that $\hat{\pi}^\alpha$ has continuous paths, as it is a continuous function of the optimal wealth. Moreover, the optimal terminal wealth is a continuous function of H_t, so that the dynamics of $\hat{\pi}^\alpha$ is explicit.

Finally we give a lower bound using this modified problem.

Proposition 2.20 *Let $s = (x,y,k)$ such that $z = x + [(1-\alpha)y + \alpha k].1_n \geq 0$. Then:*

$$\bar{V}^\alpha(z) \leq V(s).$$

Here, the proof is a bit different from the one given in [3]. More precisely, the main idea is the same, but in the end, as we are dealing with a general utility function, it is a little harder to conclude. This additional part corresponds to the part from Lemma 2.36 until the end of the appendix.

Remark 2.21 We could have taken a specific tax rate for each risky asset: $\alpha = (\alpha_i) \in \mathbb{R}^n$. Everything would work exactly the same way, we would just need to define σ^α and θ^α the following way: $\sigma^\alpha = (I - diag(\alpha))\sigma$, which means that the i-th line is multiplied by $(1-\alpha_i)$, and we will have $(\sigma^\alpha)^{-1} = diag\left(\frac{1}{1-\alpha_i}\right)_i \sigma^{-1}$. Then we would also have $\theta^\alpha = \theta - r(\sigma^\alpha)^{-1}\alpha$.

2.5 First order expansion

As a consequence of the previous results, we have:

$$\overline{V}^{\alpha}\left(x+[(1-\alpha)y+\alpha k].1_n\right) \leq V(x,y,k) \leq \overline{V}\left(x+[(1-\alpha)y+\alpha k].1_n\right),$$

where the left and the right terms are associated to Merton's problems, which are "well-known".
Implicitly, all the terms depend on r and α. If we are able, as in [3] to show that the left and right terms have the same first order expansion when r and α converge to 0, then we will have an expansion for the problem with taxes as well. We now study possible first order expansions as r and α go to 0 for the standard and modified Merton's problems, and try to compare them.

2.5.1 Regularity of G and $\mathcal{X}$ with respect to r and α

Since σ^{-1} and b are bounded, by constants that we write respectively Σ and Υ, θ is bounded as well, by:

$$\Theta = \Theta(r) := \Sigma(\Upsilon + nr) > 0.$$

Recall that: $\theta_t^{\alpha} = \theta_t - \dfrac{r\alpha}{1-\alpha}\sigma_t^{-1}1_n$, so that θ^{α} is also bounded, by:

$$\Theta_{\alpha} = \Theta_{\alpha}(r,\alpha) := \Sigma\left(\Upsilon + n\frac{r}{1-\alpha}\right) > 0.$$

Notice that $\Theta_{\alpha} \geq \Theta = \Theta_0$. We make the following additional assumptions:

- U is C^2; (2.25)
- $\exists K>0,\ \delta>1,\ |I(\zeta)|+|I'(\zeta)| \leq K\zeta^{-\delta}$; (2.26)
- $\exists R_0>0,\ \Lambda_0\in(0,1),\ \beta\delta > R_0(\delta-1)+(\delta-1)(\delta-\frac{1}{2})\Theta_{\Lambda_0}^2$; (2.27)
- $\forall r\in\{0,R_0\},\ \alpha\in\{0,\Lambda_0\},\ \beta(\delta-1) > r(\delta-2)+(\delta-\frac{3}{2})(\delta-2)\Theta_{\alpha}^2.$ (2.28)

We will abusively use the notation θ_t instead of $\theta(P_t)$. We do the same for σ and b.
For any $\alpha\in[0,\Lambda_0]$, $\Theta_{\Lambda_0}\geq\Theta_{\alpha}\geq\Theta$. As $\delta>1$, (2.27) and (2.28) imply that for any $(r,\alpha)\in[0,R_0]\times[0,\Lambda_0]$,

$$\beta\delta > r(\delta-1)+(\delta-1)(\delta-\frac{1}{2})\Theta_{\alpha}^2$$
$$\beta\delta > r(\delta-2)+(\delta-\frac{3}{2})(\delta-2)\Theta_{\alpha}^2.$$

Remark 2.22 Let $\delta > 1$, Θ, $R_0 > 0$ and $\Lambda_0 > 0$ be given. Then for any $\alpha \in [0, \Lambda_0]$, $\Theta \leq \Theta_\alpha \leq \Theta_{\Lambda_0} < +\infty$, and it is possible to choose $\beta > 0$ such that for any $(r, \alpha) \in [0, R] \times [0, \Lambda]$, (2.27) and (2.28) are satisfied. In other words, they only mean that β cannot be too small. We will see that those assumptions are needed for integrability reasons. In the case $T < +\infty$, they can be removed.

Remark 2.23 Notice that for power functions $U(x) = \dfrac{x^p}{p}$, with $p \in (0,1)$, those assumptions are satisfied, as well as for the logarithm $U(x) = \ln x$, but not for for exponential functions $U(x) = -e^{-\eta x}$ because in that case we would have $\delta = 1$, but this function does not satisfy $U'(0) = +\infty$ either.

Recall the definition of H_t: $H_t = e^{-rt - \int_0^t \theta_u^* dW_u - \frac{1}{2}\int_0^t \|\theta_u\|^2 du}$.
We will denote H^α, $\mathcal{G}^\alpha$, $\mathcal{X}^\alpha$ and $\mathcal{Y}^\alpha$ the processes defined as H, $\mathcal{G}$, $\mathcal{X}$ and $\mathcal{Y}$ respectively when replacing θ and σ by θ^α and σ^α.

Lemma 2.24 *H_t and H_t^α are differentiable (in fact C^∞) with respect to r and α, for $\alpha \in [0,1)$ and we have:*

$$\frac{\partial H_t}{\partial r} = -tH_t;\ \frac{\partial H_t}{\partial \alpha} = 0;$$

$$\frac{\partial H_t^\alpha}{\partial r} = H_t^\alpha \left[-t + \frac{\alpha}{1-\alpha} 1_n^* \int_0^t (\sigma_u^{-1})^* (\theta_u du + dW_u) - r\left(\frac{\alpha}{1-\alpha}\right)^2 \int_0^t \|\sigma_u^{-1} 1_n\|^2 du \right];$$

$$\frac{\partial H_t^\alpha}{\partial \alpha} = H_t^\alpha \left[r\left(\frac{1}{1-\alpha}\right)^2 1_n^* \int_0^t (\sigma_u^{-1})^* (\theta_u du + dW_u) - r^2 \frac{\alpha}{(1-\alpha)^3} \int_0^t \|\sigma_u^{-1} 1_n\|^2 du \right].$$

Proof. This follows from direct computation, as:

$$H_t^\alpha = H_t e^{r\frac{\alpha}{1-\alpha} 1_n^* \int_0^t (\sigma_u^{-1})^* (\theta_u du + dW_u) - \frac{1}{2} r^2 (\frac{\alpha}{1-\alpha})^2 \int_0^t \|\sigma_u^{-1} 1_n\|^2 du}.$$

□

Recall the expressions of G and $\mathcal{X}$:

$$G(\zeta) = \mathbb{E}\left[\int_0^{+\infty} e^{-\beta t} U \circ I(\zeta e^{\beta t} H_t) dt\right] \text{ and } \mathcal{X}(\zeta) = \mathbb{E}\left[\int_0^{+\infty} H_t I(\zeta e^{\beta t} H_t) dt\right].$$

Proposition 2.25 *Under assumptions* (2.26)-(2.28)*, G and $\mathcal{X}$ are C^1 with respect to r and α, and we have the following expressions:*

$$\frac{\partial G}{\partial r} = -\mathbb{E}\left[\int_0^\infty t I'(\zeta e^{\beta t} H_t) \zeta H_t dt\right];$$

$$\frac{\partial \mathcal{X}}{\partial r} = -\mathbb{E}\left[\int_0^\infty tH_t\big(I(\zeta e^{\beta t}H_t) + I'(\zeta e^{\beta t}H_t)\zeta e^{\beta t}H_t\big)dt\right];$$

$$\frac{\partial G}{\partial \alpha} = \frac{\partial \mathcal{X}}{\partial \alpha} = 0;$$

$$\frac{\partial G}{\partial \zeta} = \mathbb{E}\left[\int_0^\infty I'(\zeta e^{\beta t}H_t)H_t dt\right] \quad \textit{and} \quad \frac{\partial G}{\partial \zeta} = \mathbb{E}\left[\int_0^\infty e^{\beta t}H_t^2 I'(\zeta e^{\beta t}H_t)dt\right].$$

Proof. Let us begin with the partial derivative of G w.r.t r. We use Fubini's theorem to write:

$$\mathbb{E}\int_0^\infty |tI'(\zeta e^{\beta t}H_t)\zeta H_t|dt = \int_0^\infty t\mathbb{E}|I'(\zeta e^{\beta t}H_t)H_t|dt.$$

We will show that, for any fixed t and $R \in (0, R_0]$, the family

$$\{I'(\zeta e^{\beta t}H_t)H_t;\ (r,\alpha) \in [0,R] \times \mathbb{R}_+\}$$

is uniformly integrable (with respect to $(r,\alpha) \in [0,R] \times \mathbb{R}_+$), by proving that it is bounded in L^2. From assumption (2.26), we get:

$$|I'(\zeta e^{\beta t}H_t)H_t|^2 \le K^2\zeta^{-2\delta}e^{-2\delta\beta t}H_t^{2(1-\delta)}.$$

Let $a \ge 0$ and let us compute: $\mathbb{E}[(H_t)^a] = \mathbb{E}[e^{-rat-\int_0^t a\theta_u^* dW_u - \frac{1}{2}\int_0^t a\|\theta_u\|^2 du}]$. We define:

$$Z^a = e^{-\int_0^t a\theta_u^* dW_u - \frac{1}{2}\int_0^t a^2\|\theta_u\|^2 du} (> 0).$$

We have $\mathbb{E}Z^a = 1$, so that we can define the probability $\mathbb{Q}^a$, equivalent to $\mathbb{P}$, by its Radon-Nikodym derivative: $\dfrac{d\mathbb{Q}^a}{d\mathbb{P}} = Z^a$. We then have:

$$\mathbb{E}[H_t^a] = \mathbb{E}^{\mathbb{Q}^a}\left[\frac{(H_t)^a}{Z^a}\right] = e^{-rat}\mathbb{E}^{\mathbb{Q}^a}\left[e^{\frac{1}{2}\int_0^t (a^2-a)\|\theta_u\|^2 du}\right].$$

As $\|\theta\| \le \Theta$, we deduce that:

$$\mathbb{E}[(H_t)^a] \le \begin{cases} e^{-rat} \text{ if } a \in [0,1] \\ e^{-rat+\frac{1}{2}(a^2-a)\Theta^2 t} \text{ otherwise.} \end{cases}$$

Applying the previous computation for $a = 2(1-\delta)$, we see that $\mathbb{E}[(I'(\zeta e^{\beta t}H_t)H_t)^2]$ is bounded independently from r and α, on any set of the form $[0,R] \times \mathbb{R}_+$, which guarantees the claimed uniform integrability. As a consequence, we can apply Lebesgue's theorem to get that $-t\mathbb{E}[I'(\zeta e^{\beta t}H_t)\zeta H_t]$ is the derivative of $\mathbb{E}[e^{-\beta t}U \circ I(\zeta e^{\beta t}H_t)]$ with respect to r (recall that $U' \circ I(x) = x$) and moreover we get the following bound, as $\delta > 1$:

$$\begin{aligned}[t\mathbb{E}I'(\zeta e^{\beta t}H_t)H_t]^2 &\le t^2\mathbb{E}(I'(\zeta e^{\beta t}H_t)H_t)^2 \\ &\le t^2 e^{-2\left(\beta\delta - R(\delta-1)-(\delta-1)(\delta-\frac{1}{2})\Theta^2\right)t},\end{aligned}$$

which is integrable on $\mathbb{R}_+$ because of assumption (2.27).

Therefore, we can apply Lebesgue's theorem to the integral with respect to t in order to conclude.

We proceed similarly for the partial derivative of $\mathcal{X}$ w.r.t r. First we have:

$$\begin{aligned}\left|H_t\big(I(\zeta e^{\beta t}H_t)+I'(\zeta e^{\beta t}H_t)\zeta e^{\beta t}H_t\big)\right|^2 &\leq 2K^2\zeta^{-2\delta}e^{-2\beta\delta t}H_t^{2(1-\delta)}\left[1+\zeta^2 e^{2\beta t}(H_t)^2\right]\\ &\leq A\left[e^{-2\beta\delta t}H_t^{2(1-\delta)}+e^{-2\beta(\delta-1)t}H_t^{2(2-\delta)}\right],\end{aligned}$$

where $A>0$ is a constant independent from t.

The first term is the same as before. Using the computation of $\mathbb{E}[(H_t)^a]$ made before, with $a=2(2-\delta)$, we deal with the second term and get exactly as previously that the derivative of $\mathbb{E}H_tI(\zeta e^{\beta t}H_t)$ is $-t\mathbb{E}H_t\big(I(\zeta e^{\beta t}H_t)+I'(\zeta e^{\beta t}H_t)\zeta e^{\beta t}H_t\big)$, with the following bounds: if $\delta\in[\frac{3}{2},2]$ then:

$$t^2\mathbb{E}e^{-2\beta(\delta-1)t}H_t^{2(2-\delta)}\leq t^2e^{-2(\beta(\delta-1)+R(2-\delta))t},$$

if $\delta<\dfrac{3}{2}$ then:

$$t^2\mathbb{E}e^{-2\beta(\delta-1)t}H_t^{2(2-\delta)}\leq t^2e^{-2(\beta(\delta-1)-R(\delta-2)-(\delta-\frac{3}{2})(\delta-2)\Theta^2)t},$$

while if $\delta>2$ then:

$$t^2\mathbb{E}e^{-2\beta(\delta-1)t}H_t^{2(2-\delta)}\leq t^2e^{-2(\beta(\delta-1)-(\delta-\frac{3}{2})(\delta-2)\Theta^2)t},$$

In any case, thanks to assumption (2.28), the bound is integrable on $\mathbb{R}_+$, so that we can apply Lebesgue's theorem and conclude.

The derivatives with respect to ζ are dealt with exactly in the same way, and finally, as $\dfrac{\partial H_t}{\partial\alpha}=0$, the results for the derivatives with respect to α are immediate. $\square$

Proposition 2.26 *Under assumptions* (2.26)-(2.28), *G^α and $\mathcal{X}^\alpha$ are C^1 with respect to r, α and ζ, and the expressions of their derivatives are obtained by differentiating under the expectation and integral signs.*

Proof. The proof is close to the previous one, but a little more technical. We first prove that, for any t, $R \in (0, R_0)$ and $\Lambda \in (0, \Lambda_0)$, the family:

$$\left\{ I'(\zeta e^{\beta t} H_t^\alpha) H_t^\alpha \left(-t + \frac{\alpha}{1-\alpha} \; 1_n^* \int_0^t (\sigma_u^{-1})^* (\theta_u du + dW_u) \right.\right.$$
$$\left.\left. -r \left(\frac{\alpha}{1-\alpha} \right)^2 \int_0^t \|\sigma_u^{-1} 1_n\|^2 du \right) ; \; (r, \alpha) \in [0, R] \times [0, \Lambda] \right\},$$

is bounded in L^p for a certain $p > 1$.

Let us denote:

$$B_t := -t + \frac{\alpha}{1-\alpha} 1_n^* \int_0^t (\sigma_u^{-1})^* (\theta_u du + dW_u) - r \left(\frac{\alpha}{1-\alpha} \right)^2 \int_0^t \|\sigma_u^{-1} 1_n\|^2 du,$$

and let $p \in (1, 2)$. We compute:

$$A_t := \left(I'(\zeta e^{\beta t} H_t^\alpha) H_t^\alpha B_t \right)^p \leq K^p \zeta^p e^{-p\beta\delta t} (H_t^\alpha)^{p(1-\delta)} B_t^p.$$

We define q by $\dfrac{1}{q} + \dfrac{p}{2} = 1$. As $q > 1$, $pq > p > 1$. We can apply Holder's inequality, so that there exists a constant C (independent from t) such that:

$$\mathbb{E} A_t \leq C e^{-p\beta\delta t} \left[\mathbb{E} (H_t^\alpha)^{2(1-\delta)} \right]^{\frac{p}{2}} \left[\mathbb{E} |B_t|^{pq} \right]^{\frac{1}{q}}.$$

We show that $\mathbb{E}\left(|B_t|^{pq}\right)$ is bounded by a sum of power functions in t, r and $\dfrac{\alpha}{1-\alpha}$, with (strictly) positive powers. Indeed by convexity of $x \mapsto |x|^{pq}$:

$$\mathbb{E}|B_t|^{pq} \leq 4^{pq-1} \mathbb{E} \left[t^{pq} + \left| \frac{\alpha}{1-\alpha} 1_n^* \int_0^t (\sigma_u^{-1})^* \theta_u du \right|^{pq} + \left| \frac{\alpha}{1-\alpha} 1_n^* \int_0^t (\sigma_u^{-1})^* dW_u \right|^{pq} \right.$$
$$\left. + \left| r \left(\frac{\alpha}{1-\alpha} \right)^2 \int_0^t \|\sigma_u^{-1} 1_n\|^2 du \right|^{pq} \right].$$

As θ and σ^{-1} are bounded, the only term that needs some more details is the stochastic integral. But using Burkholder-Davis-Gundy's inequality, we get for a certain constant $D > 0$ (depending only on pq):

$$\mathbb{E} \left| \int_0^t (\sigma^{-1} 1_n)^* dW_u \right|^{pq} \leq D \mathbb{E} \left(\int_0^t |(\sigma^{-1}) 1_n|^2 du \right)^{\frac{pq}{2}},$$

so there is a constant $F > 0$ such that:

$$\mathbb{E}|B_t|^{pq} \leq F \left(t^{pq} + \left(\frac{\alpha}{1-\alpha} \right)^{pq} + \left(\frac{\alpha}{1-\alpha} \right)^{\frac{pq}{2}} + r^{pq} \left(\frac{\alpha}{1-\alpha} \right)^{2pq} \right). \tag{2.29}$$

Now exactly as in the previous proof, we get the expression:

$$\mathbb{E}\left(H_t^\alpha\right)^a \leq \begin{cases} e^{-rat} \text{ if } a \in [0,1] \\ e^{-rat+\frac{1}{2}(a^2-a)\Theta_\alpha^2 t} \text{ otherwise,} \end{cases}$$

Therefore, we have uniform integrability and we can differentiate inside the expectation. Then the previous computations also give us the following bound, for any $(r, \alpha) \in [0, R] \times [0, \Lambda]$:

$$\mathbb{E}A_t \leq Ce^{-p(\beta\delta - R(\delta-1)-(\delta-1)(\delta-\frac{1}{2})\Theta_\Lambda^2)t}\left(t^{pq} + D\right),$$

for some constants $C > 0$ and $D > 0$ (independant from r, α and t). Condition (2.27) allows us to apply Lebesgue's theorem to the integral with respect to t.

The derivative of $\mathcal{X}$ with respect to r can be treated in the same way. Then the derivatives with respect to α and ζ can be treated exactly in the same way. □

Corollary 2.27 *Writing $G = G(\zeta, r, \alpha)$ and $\mathcal{X} = \mathcal{X}(\zeta, r, \alpha)$, we have, for all $\zeta > 0$, at the point $(\zeta, 0, 0)$:*

$$G_r = G_r^\alpha \; ; \; G_\alpha = G_\alpha^\alpha = 0 \text{ and } G_\zeta = G_\zeta^\alpha.$$

The same holds for $\mathcal{X}$.

Proof. If $\alpha = 0$, then $H_t = H_t^\alpha$ and $\dfrac{\partial H_t}{\partial r} = \dfrac{\partial H_t^\alpha}{\partial r}$, $dt \otimes d\mathbb{P}$-a.e, while if $r = 0$, $\dfrac{\partial H_t}{\partial \alpha} = \dfrac{\partial H_t^\alpha}{\partial \alpha} = 0$. Therefore, using Propositions 2.25 and 2.26, we get the result. □

2.5.2 Regularity of $\overline{V}$

Recall now that $\overline{V}(z) = G \circ \mathcal{Y}(z)$ and that $\mathcal{Y}$ is implicitly defined by the equation: $\mathcal{X} \circ \mathcal{Y}(z) = z$. We can rewrite this last expression as follows $\mathcal{X}\left(\mathcal{Y}(z, r, \alpha), r, \alpha\right) = z$.

Proposition 2.28 *$\mathcal{Y}$ and $\mathcal{Y}^\alpha$ are C^1 with respect to r, α and z and we have for any $z > 0$, at the point $(z, 0, 0)$:*

$$\mathcal{Y}_r = \mathcal{Y}_r^\alpha \; ; \; \mathcal{Y}_\alpha = \mathcal{Y}_\alpha^\alpha = 0 \text{ and } \mathcal{Y}_z = \mathcal{Y}_z^\alpha.$$

Proof. Let z be fixed, and consider the function $(\zeta, r, \alpha) \mapsto \mathcal{X}(\zeta, r, \alpha) - z$, which, as we have seen before, is C^1. The same claim is true for $\mathcal{X}^\alpha$. As $\mathcal{X}$ and $\mathcal{X}^\alpha$ are decreasing in ζ (because of the definition of $\mathcal{X}$ and the hypotheses on I), we can apply the implicit functions theorem and deduce that, for any z, $\mathcal{Y}(z,.,.)$ is C^1 with respect to (r,α). Therefore, in a neighborhood of $(r,\alpha)=(0,0)$:

$$\mathcal{Y}_r = -\frac{\mathcal{X}_r \circ \mathcal{Y}}{\mathcal{X}_\zeta \circ \mathcal{Y}} \text{ and } \mathcal{Y}_r^\alpha = -\frac{\mathcal{X}_r^\alpha \circ \mathcal{Y}^\alpha}{\mathcal{X}_\zeta^\alpha \circ \mathcal{Y}^\alpha}.$$

As $\mathcal{X} = \mathcal{X}^\alpha$ for $\alpha = 0$, we also have $\mathcal{Y} = \mathcal{Y}^\alpha$. Therefore thanks to the previous results we get that $\mathcal{Y}_r = \mathcal{Y}_r^\alpha$ at any point $(\zeta, 0, 0)$.

The same holds for the derivative with respect to α, and as $\mathcal{X}_\alpha = 0$, $\mathcal{Y}_\alpha = 0$ as well. Finally, the derivative with respect to z is immediate:

$$\mathcal{Y}_z = \frac{1}{\mathcal{X}_\zeta \circ \mathcal{Y}} \text{ and } \mathcal{Y}_z^\alpha = \frac{1}{\mathcal{X}_\zeta^\alpha \circ \mathcal{Y}^\alpha},$$

and the expressions are equal for any $(z,0,0)$. □

Proposition 2.29 *$\overline{V}$ and $\overline{V}^\alpha$ are C^1 with respect to (z,r,α) and we have:*

$$\overline{V}_r = \overline{V}_r^\alpha \;;\; \overline{V}_\alpha = \overline{V}_\alpha^\alpha = 0 \textit{ and } \overline{V}_z = \overline{V}_z^\alpha.$$

Proof. By definition, $\overline{V}(z) = G \circ \mathcal{Y}(z)$ and $\overline{V}^\alpha(z) = G^\alpha \circ \mathcal{Y}^\alpha(z)$, so that they are both C^1 with respect to (z,r,α) and we have:

$$\begin{aligned}\overline{V}_r(z) &= G_r \circ \mathcal{Y}(z) + G_\zeta \circ \mathcal{Y}(z)\mathcal{Y}_r(z)\\ \overline{V}_r^\alpha(z) &= G_r^\alpha \circ \mathcal{Y}^\alpha(z) + G_\zeta^\alpha \circ \mathcal{Y}^\alpha(z)\mathcal{Y}_r^\alpha(z).\end{aligned}$$

So the previous propositions tells us that for $r=\alpha=0$, $\overline{V}_r(z) = \overline{V}_r^\alpha(z)$. The same holds for the other derivatives. Moreover, $\mathcal{G}_\alpha = \mathcal{Y}_\alpha = 0$ implies $\overline{V}_\alpha = 0$. □

Theorem 2.30 *We have the following bounds:*

$$\overline{V}^\alpha\left(x + [(1-\alpha)y + \alpha k].1_n\right) \le V(x,y,k) \le \overline{V}\left(x + [(1-\alpha)y + \alpha k].1_n\right).$$

The right and left terms are C^1 w.r.t (ζ,r,α) and their derivatives are equal if $(r,\alpha)=(0,0)$. We note $z = x + y.1_n$. V is therefore differentiable with respect to r and α in $(r,\alpha)=(0,0)$ and admits the following expansion when (r,α) converge to $(0,0)$:

$$V(x,y,k) = \left\{\overline{V}(z) + \alpha(k-y).1_n\overline{V}_z(z) + r\overline{V}_r(z)\right\}\Big|_{(r,\alpha)=(0,0)} + o(r+\alpha).$$

Proof. We denote $\overline{V}(z', r, \alpha)$ instead of $\overline{V}(z')$ to emphasize the dependence w.r.t r and α. $\overline{V}(z', 0, 0) = \overline{V}^{\alpha}(z', 0, 0)$ for any $z' \geq 0$ is immediate from the preceeding computations. We fix x and y and make an expansion in r and α, which affects $z' = x + [(1-\alpha)y + \alpha k].1_n$. Using Taylor-Young's formula, we get:

$$\overline{V}(z', r, \alpha) = \overline{V}(z, 0, 0) + (z' - z)\overline{V}_z(z, 0, 0) + r\overline{V}_r(z, 0, 0) + \alpha\overline{V}_\alpha(z, 0, 0) + o(r + \alpha)$$
$$\overline{V}^{\alpha}(z', r, \alpha) = \overline{V}^{\alpha}(z, 0, 0) + (z' - z)\overline{V}^{\alpha}_z(z, 0, 0) + r\overline{V}^{\alpha}_r(z, 0, 0) + \alpha\overline{V}^{\alpha}_\alpha(z, 0, 0) + o(r + \alpha).$$

Moreover we have $z' - z = \alpha(k - y).1_n$, so using Proposition 2.29, we get:

$$\begin{aligned}\overline{V}(z', r, \alpha) &= \overline{V}(z, 0, 0) + \alpha(k - y).1_n\overline{V}_z(z, 0, 0) + r\overline{V}_r(z, 0, 0) + o(r + \alpha)\\ &= \overline{V}^{\alpha}(z', r, \alpha) + o(r + \alpha).\end{aligned}$$

Therefore, for any (x, y, k), V is differentiable w.r.t r and α at the point $(r, \alpha) = (0, 0)$, and we have the expansion claimed. □

Appendix

For the sake of completeness, we give here the missing proofs of part 2 and 3. They are direct generalizations of the ones given in [3], except for Proposition 2.20.

Proof of Proposition 2.6

Let $s = (x, y, k) \in \overline{S}$ and $s' = (x', y', k')$ such that $s' \geq s$. We have $s' \in \overline{S}$ and we will prove that $\mathcal{A}(s) \subset \mathcal{A}(s')$. Let $\nu = (C, L, M) \in \mathcal{A}(s)$, we have:

$$Z_t^{s',\nu} - Z_t^{s,\nu} = X_t^{s',\nu} - X_t^{s,\nu} + \sum_i \left[(1-\alpha)(Y_t^{(s',\nu)i} - Y_t^{(s,\nu)i}) + \alpha(K_t^{(s',\nu)i} - K_t^{(s,\nu)i})\right].$$

Therefore, if we show that $X_t^{s',\nu} - X_t^{s,\nu} \geq 0$, $Y_t^{(s',\nu)i} - Y_t^{(s,\nu)i} \geq 0$ and $K_t^{(s',\nu)i} - K_t^{(s,\nu)i} \geq 0$ for all i, we will have what we want. But using Itô's formula we have $\mathbb{P}$-a.s:

$$Y_t^{(s',\nu)i} - Y_t^{(s,\nu)i} = (y'^i - y^i)e^{\int_0^t (b_i(P_u) - \frac{\sigma_i(P_u)^2}{2})du + \int_0^t \sigma_i(P_u)dW_u - M_t^{i,c}} \prod_{0 \leq u \leq t}(1 - \Delta M_u^i) \geq 0,$$

where σ_i stands for the i-th line of σ and $M^{i,c}$ stands for the continuous part of M^i,

$$K_t^{(s',\nu)i} - K_t^{(s,\nu)i} = (k'^i - k^i)e^{-M_t^{i,c}} \prod_{0 \leq u \leq t}(1 - \Delta M_u^i) \geq 0 \ \mathbb{P}\text{-a.s},$$

$$\text{and } e^{-rt}(X_t^{s',\nu} - X_t^{s,\nu}) = x' - x + \sum_i \int_0^t e^{-ru} \left[(1-\alpha)(Y_u^{(s',\nu)i} - Y_u^{(s,\nu)i}) + \alpha(K_u^{(s',\nu)i} - K_u^{(s,\nu)i})\right] dM_u^i \geq 0 \ \mathbb{P}\text{-as.}$$

And we have the result.

Proof of Proposition 2.7

Let $s = (x, y, k) \in \partial^z S$ and $\nu \in \mathcal{A}(s)$. As $z = 0$, using Itô's formula, we have:

$$e^{-rt} Z_t^{s,\nu} = -\int_0^t e^{-ru} \left[C_u + r\alpha \sum_i K_u^{(s,\nu)i} du\right] + \int_0^t e^{-ru}(1-\alpha) \sum_i Y_u^{(s,\nu)i} \sigma_i(\theta du + dW_u).$$

We know that $C \geq 0$ and the L^i's are nondecreasing, so as ν is admissible, the K^i's and Z are nonnegative, and we have:

$$0 \leq e^{-rt} Z_t^{s,\nu} \leq \int_0^t e^{-ru}(1-\alpha) \sum_i Y_u^{(s,\nu)i} \sigma_i(\theta du + dW_u)$$

Recall that $\mathbb{P}^0$ is the probability equivalent to $\mathbb{P}$ under which $\int_0^t \theta du + W_t$ is a (n-dimensional) brownian motion (given by Girsanov's theorem). The right term in the previous expression is a $\mathbb{P}^0$-local martingale which is nonnegative, so (using Fatou's lemma) it is a $\mathbb{P}^0$-supermartingale. Therefore we have: $\mathbb{E}[Z_t^{s,\nu}] = 0$ for all t, and then as $Z^{s,\nu} \geq 0$, $Z_t^{s,\nu} = 0$, $\forall t \geq 0$ $\mathbb{P}$-as. By taking the expectation of the above expression of Z, we see that $C_t = 0$, $K_t^{(s,\nu)i} = 0$ and $L_t^i = L_0^i = 0$ (for all t, $\mathbb{P}$-as). Thus we have $V(s) = 0$ because $U(0) = 0$.

Proof of Proposition 2.9

Let $s \in \overline{S}$, and $\nu \in \mathcal{A}(s)$. We define $\tilde{X}^\nu = X^\nu + \sum_i \alpha K^{\nu,i}$, $\tilde{Y}^\nu = (1-\alpha)Y^\nu$ and $\tilde{L} = (1-\alpha)L$.

We have:

$$d\tilde{X}_t^\nu = -C_t dt - \sum_i d\tilde{L}_t^i + \tilde{Y}_{t^-}^{\nu,i} dM_t^i$$
$$d\tilde{Y}_t^{\nu,i} = \tilde{Y}_t^{\nu,i} \frac{dP_t^i}{P_t^i} + d\tilde{L}_t^i - \tilde{Y}_{t^-}^{\nu,i} dM_t^i.$$

But we also have $Z_t^\nu = \tilde{X}_t^\nu + \sum_i \tilde{Y}_t^{\nu,i}$. As K is not involved in the dynamics of X, Y or Z, it leads us to a tax-free problem, with the strategy $(C, (1-\alpha)L + M)$. In Merton's problem, the allocations in risky assets are not assumed to have bounded variations like it is the case here, but it is well-known that, for this problem, restricting the set of admissible strategies in this way leads to the same value function.

Proof of Proposition 2.11

We will use the following lemma:

Lemma 2.31 *Using the notations of the previous proposition, we define* (L', M') *by:*

$$(L'^{i_0}, M'^{i_0}) = (L^{i_0}, M^{i_0}) + (1 - \Delta M_\tau^{i_0}) 1_{t \geq \tau} (Y_\tau^{(s,\nu)i_0}, 1).$$

For other indices $i \neq i_0$, $L'^i = L^i$ *et* $M'^i = M^i$. *Then* $\nu' = (C, L', M') \in \mathcal{A}(s)$ *we have:*

$$\begin{aligned} &Y^{s,\nu'} = Y^{s,\nu},\ Z^{s,\nu'} \geq Z^{s,\nu},\ K^{s,\nu'} \leq K^{s,\nu} \ \text{a.s} \\ &\text{and } Z^{s,\nu'} > Z^{s,\nu} \ \text{a.s on } \{t > \tau\}. \end{aligned}$$

For the indices $i \neq i_0$, $K^{(s,\nu')i} = K^{(s,\nu)i}$.

Proof. (lemma) Let us note i instead of i_0. For the other indices, our transformation obviously changes nothing. Let s and ν be given as in the previous proposition and ν' defined in the lemma. ν and ν' are different only because of the jump at τ and $\Delta L_\tau'^i = Y_\tau^i \Delta M_\tau'^i$, so it gives: $Y_t^{(s,\nu')i} = Y_t^{(s,\nu)i}$. We also have $Z_t^{s,\nu'} = Z_t^{s,\nu}$ and $K_t^{(s,\nu')i} = K_t^{(s,\nu)i}$ for $t < \tau$ ($\mathbb{P}$-as). Because of the dynamics of Z, we can claim that $Z_t^{s,\nu'}$ and $Z_t^{s,\nu}$ have continuous paths. So $Z_\tau^{s,\nu'} = Z_\tau^{s,\nu}$.
Then if $t \geq \tau$ (using Itô):

$$K_t^{(s,\nu')i} - K_t^{(s,\nu)i} = (K_\tau^{(s,\nu')i} - K_\tau^{(s,\nu)i}) e^{-M_t^{i,c} + M_\tau^{i,c}} \prod_{\tau < u \leq t} (1 - \Delta M_u^i)$$

But as $\Delta M_\tau' = 1 \in \mathbb{R}^n$, we have $K_\tau^{s,\nu'} = Y_\tau^{s,\nu'} = Y_\tau^{s,\nu}$, which leads to:

$$K_t^{(s,\nu')i} - K_t^{(s,\nu)i} = (Y_\tau^{(s,\nu)i} - K_\tau^{(s,\nu)i}) e^{-M_t^{i,c} + M_\tau^{i,c}} \prod_{\tau < u \leq t} (1 - \Delta M_u^i) \leq 0 \text{ pour } t \geq \tau.$$

As M is right continuous, there exists (a stopping time) $\eta : \Omega \to \mathbb{R}_+^*$ such as $\forall u \in [\tau, \tau + \eta], \forall i$, $\Delta M_u^i < 1$. Therefore, if $\tau \leq t \leq \tau + \eta$, then: $K_t^{s,\nu'} - K_t^{s,\nu} < 0$ because $Y_\tau^{(s,\nu)i} - K_\tau^{(s,\nu)i} < 0$ (hypothesis).

Finally, taking again a look at the dynamic of Z, for $t > \tau$ we have (using Itô'sformula and $Z^{s,\nu'}_\tau = Z^{s,\nu}_\tau$):

$$e^{-rt}(Z^{s,\nu'}_t - Z^{s,\nu}_t) = -r\alpha\left[\int_\tau^{t\wedge(\tau+\eta)} e^{-ru}(K^{(s,\nu')i}_t - K^{(s,\nu)i}_t)du + \int_{\tau+\eta}^{t\vee(\tau+\eta)} e^{-ru}(K^{(s,\nu')i}_t - K^{(s,\nu)i}_t)du\right].$$

The first term is positive and the second nonnegative. So $\nu' \in \mathcal{A}(s)$ and $Z^{s,\nu'}_t > Z^{s,\nu}_t$ $\mathbb{P}$-as on $\{t > \tau\}$. □

Proof. (proposition) Let $\nu' = (C, L', M')$ be given by the previous lemma. Let us define $Z^{s,\tilde{\nu}}$ by the same dynamics as $Z^{s,\nu'}$, but with C_t being replaced by $C_t + \xi(Z^{s,\tilde{\nu}}_t - Z^{s,\nu}_t)1_{t\geq\tau}$ for a certain $\xi > 0$. This guarantees that $Z^{s,\tilde{\nu}}$ is well defined.
We then define $\tilde{C}_t = C_t + \xi(Z^{s,\tilde{\nu}}_t - Z^{s,\nu}_t)1_{t\geq\tau}$ and $\tilde{\nu} = (\tilde{C}, L', M')$. We have $(Y^{s,\tilde{\nu}}, K^{s,\tilde{\nu}}) = (Y^{s,\nu'}, K^{s,\nu'})$ and $Z^{s,\tilde{\nu}}_t = Z^{s,\nu'}_t = Z^{s,\nu}_t$ pour $t \leq \tau$. So $K^{s,\tilde{\nu}} - K^{s,\nu} \leq 0$. Using Itô's formula:

$$\begin{aligned} e^{-r(t-\tau)} &= -r\alpha\int_\tau^t e^{-r(u-\tau)}(K^{(s,\tilde{\nu})i_0}_u - K^{(s,\nu)i_0}_u)du + \xi\int_\tau^t e^{-r(u-\tau)}(Z^{s,\tilde{\nu}}_u - Z^{s,\nu}_u)du \\ &\geq \xi\int_\tau^t e^{-r(u-\tau)}(Z^{s,\tilde{\nu}}_u - Z^{s,\nu}_u)du. \end{aligned}$$

Finally, thanks to Gronwall's lemma we get $Z^{s,\tilde{\nu}}_t > Z^{s,\nu}_t$ on $\{t > \tau\}$ and then $\tilde{C} > C$ on $\{t > \tau\}$. As a consequence, $\tilde{\nu}$ is admissible (because $\tilde{C}_t > 0$ and $Z^{s,\tilde{\nu}}_t > 0$) and a.s $\{t > \tau\}$ has a positive $dx\otimes d\mathbb{P}$ measure (because τ is non-infinite). Therefore, we get $J_\infty(s,\tilde{\nu}) > J_\infty(s,\nu)$. □

Remark 2.32 If we have $K^i_\tau > Y^i_\tau$ for several indices i, a possible strategy for wash sales is the one that modifies as described above simultaneously for all the concerned indices (because it affects only the index associated to i_0).

Proof of Proposition 2.18

Let $s = (x, y, k) \in \overline{S}$ and $\nu = (C, L, M) \in \mathcal{A}(s)$. We define $\tilde{\nu} = (C, (1-\alpha)L + M)$ and denote by $(\tilde{X}, \tilde{Y})$ the processes associated to the tax-free problem, with initial conditions $x + \sum_i (1-\alpha)y_i + \alpha k_i$. Taking a look at the dynamics of Y and $\tilde{Y}$, we see that $\tilde{Y}_t = (1-\alpha)Y^{s,\nu}_t \geq 0\ \forall t$, $\mathbb{P}$-as. But we have $\tilde{Z}_{0-} = Z^{s,\nu}_{0-}$ therefore using Itô's formula:

$$e^{-rt}(\tilde{Z}_t - Z^{s,\nu}_t) = \int_0^t e^{-ru} r\alpha K^{s,\nu}_u du \geq 0 \ \mathbb{P}\text{-a.s.}$$

So $J_\infty(s,\nu) \leq \bar{V}\left(x + \sum_{i=1}^{n}(1-\alpha)y_i + \alpha k_i\right)$, and we get the requested result by taking the upper bound.

Proof of Proposition 2.20

As in [3], we fix $T < \infty$, and we will in the end take the limit $T \to \infty$. $\hat{c}^\alpha = \hat{c}^{\alpha,T}$ and $\hat{\pi}^\alpha = \hat{\pi}^{\alpha,T}$ denote respectively the optimal consumption and portfolio for Merton's modified problem. Recall that we have assumed that they were both continuous and bounded.
For n fixed, we define a sequence of consumption-investment strategies $\left(\nu^{T,n,m}\right)_{m\in\mathbb{N}}$ the following way. In order to alleviate notations, we do not write the dependance towards T and n below, until N is not fixed anymore.

1. $C_t = \hat{c}^\alpha_t Z_t$ for $t \in [0,T]$. The investment strategies L, M for the problem with taxes are piecewise constant and defined by:

$$dL_t = dM_t = 0 \text{ for all } t \notin \{\tau_j,\ j \in \mathbb{N}\},$$

where the stopping times τ_j are defined in step 3 below.
2. $\Delta L_0 = \hat{\pi}^\alpha_0 z$ and $\Delta M_0 = 1$, so that:

$$K_0 = Y_0;\ \pi_0 = \hat{\pi}^\alpha_0;\ Z_0 = z.$$

3. The sequence of stopping times $(\tau_j)_j$ is defined the following way:

$$\tau_0 = 0 \text{ and } \tau_j = T \wedge \tau^\pi_j \wedge \tau^B_j, \text{ for } j \geq 1,$$

where:

$$\tau^\pi_j = \inf\{t \geq \tau_{j-1};\ |\pi_t - \hat{\pi}^\alpha_t| > \frac{1}{n}\},$$
$$\tau^B_j = \inf\{t \geq \tau_{j-1};\ \left|1 - \frac{K_t}{Y_t}\right| > \frac{1}{n}\}.$$

4. At each $t = \tau_j$, for $j \geq 1$ we set:

$$\Delta L_t = \hat{\pi}^\alpha_t Z_t \text{ and } \Delta M_t = 1,$$

so that if $t \in \{\tau_j;\ j \geq 1\}$:

$$\pi_t = \hat{\pi}^\alpha_t \text{ and } B_t = P_t.$$

The problem is that we do not know if such a strategy (C, L, M) is well defined as we could have an accumulation point preventing from having $\tau_j \to T$ a.s. Therefore we define the sequence $(\nu^m)_m = (\nu^{T,n,m})_m$ as follows: $\nu^m = (C^m, L^m, M^m)$ such that $C_t^m = C_t$ for any $t \in [0, T]$ and (L^m, M^m) is equal to (L, M) for $t < \tau_m$. Then, for $t = \tau_m$, we set:

$$\Delta L_t^m = 0 \text{ and } \Delta M_t^m = 1,$$

and for $t > \tau_m$, $dL_t^m = dM_t^m = 0$, so that for $\tau_m \leq t \leq T$:

$$Y_t^m = K_t^m = 0 \text{ and } X_t^m = Z_{\tau_m}^m e^{r(T-\tau_m)}.$$

As the number of jumps here is finite, these strategies are by construction well-defined.

Lemma 2.33 *Let $n \geq 1$ be fixed. Then $\lim_{m\to\infty} \tau_m = T$, $\mathbb{P}$-a.s.*

Proof. Let $n \geq 1$. For n sufficiently large, $(\tau_m^\pi)_m$ is equivalently defined by:

$$\tau_{m+1}^\pi = \inf\{t \geq \tau_m;\ |\pi_t^m - \hat{\pi}_t^\alpha| > \frac{1}{n}\},$$

and π^m has continuous paths after τ_m and is a diffusion with bounded drift and volatility. By assumption, the same is true for $\hat{\pi}^\alpha$, and $\pi_{\tau_m}^m = \hat{\pi}_{\tau_m}^\alpha$. Therefore we can compute, using Bienaymé-Tchebytchef's inequality:

$$\begin{aligned}\mathbb{P}\left[\tau_{m+1}^\pi \leq \tau_m + \frac{1}{m}\right] &= \mathbb{P}\left[\sup_{\tau_m \leq t \leq \tau_m + \frac{1}{m}} |\pi_t^m - \hat{\pi}_t^\alpha| > \frac{1}{n}\right] \\ &\leq n^3 \mathbb{E}\left[\sup_{\tau_m \leq t \leq \tau_m + \frac{1}{m}} |\pi_t^m - \hat{\pi}_t^\alpha|^3\right].\end{aligned}$$

Now:

$$\mathbb{E}\left[\sup_{\tau_m \leq t \leq \tau_m + \frac{1}{m}} |\pi_t^m - \hat{\pi}_t^\alpha|^3\right] \leq 4\left(\frac{1}{m^3}||f||_\infty^3 + \frac{1}{m^{\frac{3}{2}}}||g||_\infty^3\right),$$

where f is the drift in the dynamics of $\pi^m - \hat{\pi}^\alpha$, while g is its volatility and we used Burkholder-Davis-Gundy's inequality.

The same reasoning brings also $\mathbb{P}\left[\tau_{m+1}^\pi \leq \tau_m + \frac{1}{m}\right] \leq A\left(\frac{1}{m^3} + \frac{1}{m^{\frac{3}{2}}}\right)$, so that:

$$\sum_{m\in\mathbb{N}} \mathbb{P}\left[\tau_{m+1} \leq \tau_m + \frac{1}{m}\right] < \infty.$$

Then Borel-Cantelli's lemma implies that $\lim_{m\to\infty} \tau_m = T$. $\square$

Lemma 2.34 *For each $n \in \mathbb{N}$, $\nu^{T,n,m} \in \mathcal{A}(s)$.*

Proof. We have:

$$dZ_t^m = Z_t^m \left[(r - \hat{c}_t^\alpha)dt + \pi_t^m \sigma_t^\alpha (\theta_t^\alpha dt + dW_t) + r\alpha\pi_t^m (1 - B_t^m) dt\right].$$

We also have $0 < \left(1 - \frac{1}{n}\right)\hat{\pi}_t^\alpha \leq \pi_t^m \leq \left(1 + \frac{1}{n}\right)\hat{\pi}_t^\alpha$, so that π^m is bounded, as well as $\hat{c}^\alpha$, σ^α and θ^α. Therefore the above dynamics implies that Z^m is positive and $Y_t^m = \pi_t^m Z_t^m > 0$, $\mathbb{P}$-a.s, so that $\nu^{T,n,m} \in \mathcal{A}(s)$. □

Lemma 2.35 *There exists a constant A depending only on T such that:*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|Z_t^m - \bar{Z}_t^\alpha\right|^2\right] \leq \left(\frac{1}{n^2} + \mathbb{E}[T - \tau_m^m]\right) Ae^{AT}.$$

Proof. We decompose the difference $D_t := Z_t^m - \bar{Z}_t^\alpha$ into:

$$D_t = F_t + G_t + H_t,$$

where:

$$\begin{aligned}
F_t &:= \int_0^t D_t \left[(r - \hat{c}_u^\alpha)du + \pi_u^m \left(\sigma_u^\alpha\theta_u^\alpha du + \alpha r\left(1 - \frac{K_u^m}{Y_u^m}\right)du + \sigma_u^\alpha dW_u\right)\right]\\
G_t &:= \int_0^t \bar{Z}_u^\alpha \sigma_u^\alpha \left(\pi_u^m - \hat{\pi}_u^\alpha\right)\left(\theta_u^\alpha + dW_u\right)\\
H_t &:= \alpha r \int_0^t \pi_u^m \bar{Z}_u^\alpha \left(1 - \frac{K_u^m}{Y_u^m}\right) du.
\end{aligned}$$

For any process V, we shall denote by $V_t^* := \sup_{u\in[0,t]} |V_u|$. As π^m is bounded by $2\hat{\pi}^\alpha$, we have, by convexity of $x \mapsto |x|^2$ and using Cauchy-Schwarz inequality:

$$|F_t|^2 \leq 6\left(\int_0^t |D_u^*|^2 du\right)\left((r - \hat{c}_u^\alpha)^2 + ||\pi_u^m||^2\left(||\sigma_u^\alpha\theta_u^\alpha||^2 + \left(\frac{\alpha r}{n}\right)^2\right)\right) + 2\left(\int_0^t D_u\pi_u^m\sigma_u^\alpha dW_u\right)^2.$$

Using Burkholder-Davis-Gundy's inequality, we get for a certain constant A depending on T (and the bounds for $\hat{c}^\alpha$, $\hat{\pi}^\alpha$, σ_u^α and θ_u^α):

$$\mathbb{E}|F_t^*|^2 \leq A\int_0^t \mathbb{E}|D_u^*|^2 du.$$

Similarly, there exist two constants B and C such that:

$$\mathbb{E}|G_t^*|^2 \leq B\left(\frac{1}{n^2} + \mathbb{E}[T - \tau_m^m]\right) \text{ and } \mathbb{E}|H_t^*|^2 \leq C\left(\frac{1}{n^2} + \mathbb{E}[T - \tau_m^m]\right).$$

Finally, we get for a constant that we again write A, depending on T:

$$\mathbb{E}|D_t^*|^2 \leq A\left(\int_0^t \mathbb{E}|D_u^*|^2 du + \frac{1}{n^2} + \mathbb{E}[T - \tau_m^m]\right),$$

and the result follows by use of Gronwall's lemma. □

Lemma 2.36 *For any $z \geq 0$, we have: $\lim_{T\to\infty} \bar{V}_T(z) = \bar{V}_\infty(z)$.*

Proof. If $z = 0$, it is immediate. Let $z > 0$ be given.
First, if γ^T is admissible for the problem with horizon T, we can define γ equal to γ^T on $[0, T]$, then for $t > T$:

$$C_t = \Pi_t = 0.$$

Then we immediately have γ is admissible for the infinite horizon and $\bar{J}_T(z, \gamma^T) = \bar{J}_\infty(z, \gamma)$, so that $\bar{V}_T(z) \leq \bar{V}_\infty(z)$ for any T, which gives: $\limsup_{T\to\infty} \bar{V}_T \leq \bar{V}_\infty$.

Then recall that for any $y > 0$ and $T \in \mathbb{R}_+ \cup \{+\infty\}$:

$$\mathcal{X}_T(y) = \mathbb{E}\left[\int_0^T M_t I(ye^{\beta t} M_t) dt\right],$$

so that $\mathcal{X}_T$ is nondecreasing w.r.t T and, using the monotone convergence theorem, converges pointwise to $\mathcal{X}_\infty$. Then the same holds for $\mathcal{Y}_T$. Using the definition of $\hat{C}^T$ given in Theorem 2.14, as I is nonincreasing, $\hat{C}^T$ is nonincreasing w.r.t T (in the sense that, for any t, $\mathbb{P}$-a.s, $\hat{C}_t^{T_1} \leq \hat{C}_t^{T_2}$ if $T_1 \geq T_2$) and converges, for any t almost surely, to $\hat{C}^\infty$.

Therefore, as U is increasing, we have for any $T < \infty$:

$$\mathbb{E}\int_0^T e^{-\beta t} U(\hat{C}_t^\infty) dt \leq \mathbb{E}\int_0^T e^{-\beta t} U(\hat{C}_t^T) dt.$$

As $\mathbb{E}\displaystyle\int_0^\infty e^{-\beta t} U(\hat{C}_t^\infty) dt < \infty$, for any $\varepsilon > 0$, for T large enough we have:

$$\bar{V}_\infty(z) - \varepsilon \leq \mathbb{E}\int_0^T e^{-\beta t} U(\hat{C}_t^\infty) dt \leq \bar{V}_T(z),$$

so that $\liminf_{T\to\infty} \bar{V}_T \geq \bar{V}_\infty$. □

Back to the proof of Proposition 2.20.

Let s be given. As U is C^1 on $[0,+\infty)$, and concave, it is Lipschitz continuous on $[\eta,+\infty)$, for any $\eta > 0$. We denote by K_η the associated Lipschitz constant. Then, as U is increasing, we have for any x and y:

$$|U(x) - U(y)| \leq K_\eta |x-y| + |U(\eta) - U(0)|.$$

Indeed, if $x, y \geq \eta$, the Lipschitz property holds, whereas if $x, y < \eta$, the increase of U gives the result. Finally, if for example $y < \eta \leq x$ we have:

$$\begin{aligned}|U(x) - U(y)| &\leq |U(x) - U(\eta)| + |U(\eta) - U(0)| \\ &\leq K_\eta |x-\eta| + |U(\eta) - U(0)| \\ &\leq K_\eta |x-y| + |U(\eta) - U(0)|.\end{aligned}$$

Therefore we compute:

$$\begin{aligned}\mathbb{E}\left|J_T(s,\nu^{T,n,m}) - \int_0^T e^{-\beta t} U(\hat{c}_t^{\alpha,T}\bar{Z}_t^{\alpha,T})dt\right| &= \mathbb{E}\left|\int_0^T e^{-\beta t}\left(U(\hat{c}_t^{\alpha,T}Z_t^{T,n,m}) - U(\hat{c}_t^{\alpha,T}\bar{Z}_t^{\alpha,T})\right)dt\right| \\ &\leq K_\eta T ||\hat{c}^{\alpha,T}||_\infty \mathbb{E}\sup_{t\in[0,T]}|Z_t^{T,n,m} - \bar{Z}_t^{\alpha,T}| + T|U(\eta) - U(0)|.\end{aligned}$$

Now if $\varepsilon > 0$, we can choose $\eta > 0$ so that $T|U(\eta) - U(0)| < \dfrac{\varepsilon}{2}$ and then, using Lemmas 2.33 and 2.35, we can choose n and m such that $K_\eta T ||\hat{c}^{\alpha,T}||_\infty \mathbb{E}\sup_{t\in[0,T]}|Z_t^{T,n,m} - \bar{Z}_t^{\alpha,T}| < \dfrac{\varepsilon}{2}$. As a consequence:

$$\lim_{n,m\to\infty} J_T(s,\nu^{T,n,m}) = \int_0^T e^{-\beta t} U(\hat{c}_t^{\alpha,T}\bar{Z}_t^{\alpha,T})dt = \bar{V}_T^\alpha(z).$$

As $V(s) \geq J_T(s,\nu^{T,n,m})$ for any T, taking the limit $T\to\infty$ and using Lemma 2.36, we get the result.

Remark 2.37 We see here why we rejected the possibility for a logarithm utility function, as we would not control $U(x) - U(y)$ by controlling $x - y$.

Chapter 3

Detecting the maximum of a mean-reverting scalar process

3.1 Introduction

Shiryaev [71] introduced in the 60's a problem of earliest detection called the discrepancy problem. Considering a Brownian motion X and two constants a and b, the issue was to detect a switch from the state $a + X$ to the state $b + X$. Although of many interest from a financial point of view, very few works have been interested in earliest detection until recently. Graversen, Peskir and Shiryaev [37] considered the problem of detecting the maximum of a Brownian motion between time zero and a fixed maturity. The problem was formulated as an optimal stopping problem where the aim is to minimize the error at the power $p > 0$ between the value of the maximum over the period and the value of the Brownian motion at a stopping time. This work was generalized by Pedersen [64] and especially extended to a Brownian motion with drift by Du Toit and Peskir [18].

A different criterion was also introduced by Shiryaev [73] and the solution of this problem was very similar to the solution of the previous one. Urusov [77] proved an identity explaining the link between those two problems. Then Du Toit and Peskir [19] studied the same question for a Brownian motion with drift, and in that case, this formulation is not equivalent to the previous one. Let us also mention the work of Shiryaev, Xu and Zhou [74] where they consider yet another criterion.

In this chapter, we consider another point of view. For a process X which "mean reverts" towards zero, and starting from a positive initial data, we define a random maturity T_0 as the first hitting time of zero. Then we study the detection of the maximum of X in the sense of [37] but on the random period $[0, T_0]$ instead of the fixed period $[0, T]$. We solve explicitly this problem as a free boundary problem. However, the form of the solution is not classical. Indeed, the boundary is made of two different parts and is not monotonic in general.

The interpretation of this property is quite unusual. Indeed, if we consider for example an Ornstein-Uhlenbeck process that we write X and its maximum that we write Z, if Z becomes big enough, and then $Z - X$ becomes big enough, it is optimal to stop as there is no hope to reach Z again because of the mean reversion. This is quite intuitive. But if we start from $x < z$, for certain values of z not too big and not too small, and with x close to zero, as the mean-reversion is negligible in comparison with the martingale part for small values of an OU process, it is not optimal to stop immediately. We did not expect this second behavior. However, if we start from $x = z$, this second phenomenon will never be encountered, and the boundary will appear as an increasing boundary.

It is also interesting to notice that the resolution of this problem has some similarities with a problem introduced by Peskir [65], see also Obloj [62] and Hobson [41].

3.2 Problem formulation

Let W be a scalar Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the corresponding completed canonical filtration. Given two Lipschitz functions $\mu, \sigma : \mathbb{R} \longrightarrow \mathbb{R}$, we consider the scalar diffusion defined by the stochastic differential equation

$$dX_t \quad = \quad \mu(X_t)dt + \sigma(X_t)dW_t, t \geq 0$$

together with some initial data $X_0 > 0$. We assume throughout that

$$\mu(x) \ \leq \ 0 \quad \text{for every} \quad x \geq 0, \tag{3.1}$$

meaning that the process X is reverted towards the origin. For the purpose of this paper, the following stronger restrictions on the coefficients μ and σ are needed:

$$\text{the function} \quad \alpha := \frac{-2\mu}{\sigma^2} : \ (0, \infty) \longrightarrow \mathbb{R} \quad \text{is } C^2, \text{ positive and concave.} \tag{3.2}$$

We introduce the so-called scale function S:

$$S(x) := \int_0^x e^{\int_0^u \alpha(r)dr} du \tag{3.3}$$

Since α is non-negative and non-decreasing it follows that $\int_0^u \alpha(r)dr < \infty$. Notice that the mean reversion condition (3.1) is equivalent to the convexity of S, and implies that

$$\lim_{x\to\infty} S(x) \quad = \quad \infty \tag{3.4}$$

Remark 3.1 For later use, we observe that the restriction (3.2) has the following useful consequences:
(i) α is a nondecreasing function and $\left(\frac{1}{\alpha}\right)'(x) \to 0$ as $x \to \infty$,
(ii) $2S' - \alpha S - 2$ is a non-negative increasing function.

We denote by

$$T_y \quad := \quad \inf\{t > 0 : \; X_t = y\}$$

the first hitting time of the barrier y. We recall that, for a homogeneous scalar diffusion, we have

$$\mathbb{P}_x\left[T_y < T_0\right] = \frac{S(x)}{S(y)} \quad \text{for} \quad 0 \le x < y, \tag{3.5}$$

Our main objective is to solve the optimization problem

$$V_0 \quad := \quad \inf_{\theta\in\mathcal{T}_0} \mathbb{E}\left[\ell\left(X^*_{T_0} - X_\theta\right)\right], \tag{3.6}$$

where $X^*_t := \max_{s\le t} X_s,\; t \ge 0$ is the running maximum process of X, $\ell : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a non-decreasing convex function, and $\mathcal{T}_0$ is the collection of all $\mathbb{F}-$stopping times θ with $\theta \le T_0$ a.s.
We shall approach this problem by the dynamic programming technique. We then introduce the dynamic version:

$$V(x,z) \quad := \quad \inf_{\theta\in\mathcal{T}_0} \mathbb{E}_{x,z}\left[\ell\left(Z_{T_0} - X_\theta\right)\right], \tag{3.7}$$

where $\mathbb{E}_{x,z}$ denotes the expectation operator conditional on $X_0 = x$ and an inherited maximum z, and

$$Z_t := z \vee X^*_t, \qquad t \ge 0.$$

Defining the reward from stopping

$$g(x,z) := \mathbb{E}_{x,z}\left[\ell\left(Z_{T_0} - x\right)\right], \qquad 0 \le x \le z, \tag{3.8}$$

we may re-write this problem in the standard form of an optimal stopping problem:

$$V(x,z) \quad := \quad \inf_{\theta\in\mathcal{T}_0} \mathbb{E}_{x,z}\left[g\left(X_\theta, Z_\theta\right)\right]. \tag{3.9}$$

Using (3.5), we immediately calculate that

$$\mathbb{P}_{x,z}[Z_{T_0} \le u] \;=\; \mathbb{P}_x[T_u \ge T_0]\mathbf{1}_{u\ge z} \;=\; \left(1-\frac{S(x)}{S(u)}\right)\mathbf{1}_{u\ge z},$$

so that

$$\begin{aligned} g(x,z) &= \ell(z-x)\left(1-\frac{S(x)}{S(z)}\right)+S(x)\int_z^\infty \ell(u-x)\frac{S'(u)}{S(u)^2}du & (3.10)\\ &= \ell(z-x)+S(x)\int_z^\infty \frac{\ell'(u-x)}{S(u)}du,\ 0<x\le z, & (3.11)\end{aligned}$$

where ℓ' is the generalized derivative of ℓ, and the latter expression is obtained by integration by parts together with the observation that

$$\int^\infty \ell(u)\frac{S'(u)}{S(u)^2}du<\infty \quad \text{iff} \quad \int^\infty \frac{\ell'(u)}{S(u)}du<\infty. \tag{3.12}$$

Indeed, since

$$\int_z^A \ell(u)\frac{S'(u)}{S(u)^2}du = \frac{\ell(z)}{S(z)}-\frac{\ell(A)}{S(A)}+\int_z^A\frac{\ell'(u)}{S(u)}du \tag{3.13}$$

we clearly have $\int^\infty \ell(u)\frac{S'(u)}{S(u)^2}du=\infty$ implies $\int^\infty\frac{\ell'(u)}{S(u)}du=\infty$. Conversely, if $\int^\infty \ell(u)\frac{S'(u)}{S(u)^2}du<\infty$, then, since $\lim_{A\to\infty}S(A)=\infty$, it follows from (3.13) that $\int_z^\infty \ell(u)\frac{S'(u)}{S(u)^2}du\ge \ell(z)S(z)^{-1}$, and therefore $\ell(z)S(z)^{-1}\le\int_1^\infty \ell(u)\frac{S'(u)}{S(u)^2}du$ for $z\ge 1$ by (3.4). Combined with (3.13), this shows that (3.12) holds true.

We now provide necessary and sufficient conditions on the loss function ℓ which ensure that V is finite on $\mathbb{R}_+$. Recall that $V(0,z)=g(0,z)=\ell(z)$ is always finite.

Proposition 3.2 *Assume that $\alpha\ge 0$ and*

$$\sup_{u\ge z}\frac{\ell(u)}{\ell(u-x)}<\infty \quad \textit{for every} \quad z\ge x\ge 0. \tag{3.14}$$

Then, the following statements are equivalent:
(i) $V(x,z)<\infty$ *for every* $0\le x\le z$,

(i') $V(x_0, z_0) < \infty$ *for some* $0 < x_0 \le z_0$,
(ii) $g(x, z) < \infty$ *for every* $0 \le x \le z$,
(ii') $g(x_0, z_0) < \infty$ *for some* $0 < x_0 \le z_0$,
(iii) *either one of the equivalent conditions of* (3.12) *holds true.*

Proof. For $\theta \in \mathcal{T}_0$, set $J(\theta, x, z) := \mathbb{E}_{x,z}\ell(Z_{T_0} - X_\theta)$. The implications (ii) $\Longleftrightarrow$ (ii') $\Longleftrightarrow$ (iii) follow immediately from the definition of g in (3.10) together with Condition (3.14). Also the implications $(i) \Longrightarrow (i')$ and $(ii) \Longrightarrow (i)$ are immediate as $V \le g$.

We conclude the proof by showing that $(i') \Longrightarrow (iii)$. Let (i') hold true and assume to the contrary that $\int^\infty \frac{\ell'(u)}{S(u)}du = \infty$. For abritrary $0 < x \le z$ and $\theta \in \mathcal{T}_0$, we have:

$$\mathbb{E}[\ell(Z_{T_0} - X_\theta)|X_\theta] = g(X_\theta, Z_\theta) = \begin{cases} +\infty \text{ if } X_\theta > 0 \\ \ell(Z_\theta) \text{ if } X_\theta = 0 \end{cases}$$

Let $A := \{\theta \ne T_0\}$. Then,
• either $\mathbb{P}(A) > 0$, and:

$$\begin{aligned} J(\theta, x, z) &= \mathbb{E}_{x,z}\ell(Z_{T_0} - X_\theta) = \mathbb{E}_{x,z}\mathbb{E}[\ell(Z_{T_0} - X_\theta)|X_\theta] \\ &\ge \mathbb{E}_{x,z}\mathbf{1}_A\mathbb{E}[\ell(Z_{T_0} - X_\theta)|X_\theta] = +\infty, \end{aligned}$$

• or $\mathbb{P}(A) = 0$, i.e. $\theta = T_0$ a.s. and $J(\theta, x, z) = J(T_0, x, z) = \ell(z) + S(x)\int_z^\infty \frac{\ell'(u)}{S(u)}du = +\infty$
By arbitrariness of $0 < x \le z$ and $\theta \in \mathcal{T}_0$, this shows that $V = +\infty$ everywhere. □

Notice that if (3.12) holds, then the expression (3.10) is also true for $x = 0$ $(z > 0)$ or $x = z = 0$.

Remark 3.3 Without assuming (3.14), we see from the previous proof that (3.12) is still a sufficient condition for (i) or (ii) to hold true. But in general, it is not a necessary condition. Indeed consider for example a process with scale function $S(x) = e^{x^2}$, and the loss function $\ell(x) = \displaystyle\int_0^x e^{u^2}du$. Then $\displaystyle\int_z^\infty \frac{\ell'(u)}{S(u)}du = +\infty$ while for $x > 0$, $\displaystyle\int_z^\infty \frac{\ell'(u-x)}{S(u)}du = \frac{e^{x^2+2xz}}{2x}$, so that (i) and (ii) are satisfied.

Remark 3.4 Condition (3.14) is satisfied by power and exponential loss functions $\ell(x) = x^p$ for some $p \ge 1$, or $e^{\eta x}$ for some $\eta > 0$. Without Condition (3.14), one can not hope to prove that (i') $\Longrightarrow$ (i) or (ii') $\Longrightarrow$ (ii). Consider for instance the process with scale function $S(x) = e^{x^2}$ and, for $\varepsilon > 0$, the loss function $\ell(x) = \displaystyle\int_0^x e^{(u+\varepsilon)^2}du$. Then if $x \le \varepsilon$, $\displaystyle\int_z^\infty \frac{\ell'(u-x)}{S(u)}du = \infty$, while if $x > \varepsilon$, $\displaystyle\int_z^\infty \frac{\ell'(u-x)}{S(u)}du = \frac{e^{(x-\varepsilon)^2+2(x-\varepsilon)z}}{2(x-\varepsilon)}$. So $g(x, z) < \infty$ if and only if $x > \varepsilon$ or $x = 0$. In other words (ii') is true while (ii) is false. Adapting the proof of (i')$\Longrightarrow$(iii) by replacing the set A by $\{X_\theta \in (0, \varepsilon)\}$, which has a nonzero probability if $x \in (0, \varepsilon)$ and θ is not almost surely equal to T_0, we see that we also have (i') but not (i) (so that $V(x, z) < \infty$ if and only if $x \geq \varepsilon$ or $x = 0$).

Remark 3.5 From the previous proof, we also observe that we have $g = +\infty$ everywhere except for $x = 0$ implies $V = +\infty$ everywhere except for $x = 0$. This statement does not require Condition (3.14).

We conclude this section by considering the linear case, which turns out to be degenerate.

Proposition 3.6 *Assume that $\alpha \geq 0$ and let $\ell(x) = x$. Then $V = g$.*

Proof. If $\ell(x) = x$, the problem can be rewritten as:

$$V(x, z) = \inf_\theta \mathbb{E}_{x,z}(Z_{T_0} - X_\theta) = \mathbb{E}_{x,z} Z_{T_0} - W(x)$$

where:

$$W(x) = \sup_{\theta \leq T_0} \mathbb{E}_x X_\theta$$

Now as $\alpha \geq 0$, $X_{t \wedge T_0}$ is a local supermartingale, bounded from below. By Fatou's lemma, this implies that $\mathbb{E}_x X_\theta \leq x$ for $\theta \leq T_0$. $\square$

3.3 Preliminary properties

Our general approach to solve the optimal detection problem is to exhibit a candidate solution for the dynamic programming equation corresponding to the optimal stopping problem (3.9) which is:

$$\min\{Lv, g - v\} = 0 \tag{3.15}$$
$$v(0, z) = \ell(z) \tag{3.16}$$
$$v_z(z, z) = 0, \tag{3.17}$$

where L is the second order differential operator

$$Lv(x) = v''(x) - \alpha(x)v'(x), \tag{3.18}$$

and α is defined in (3.3). Notice that $LS = 0$. We do not intend to prove directly that V satisfies this differential equation. Instead, we shall guess a candidate solution v of (3.15), and show that v indeed coincides with the value function V by a verification argument.

From now on, we will assume that one of the equivalent relations of (3.12) is satisfied, so that g and V are finite everywhere.

In order to exhibit a solution of (3.15), we guess that there should exist a free boundary $\gamma(x)$ so that stopping is optimal in the region $\{z \geq \gamma(x)\}$, while continuation is optimal in the remaining region $\{z < \gamma(x)\}$. If such a stopping boundary exists, then the above dynamic programming equation reduces to:

$$Lv(x,z) = 0 \text{ for } 0 < z < \gamma(x) \tag{3.19}$$

$$v(x,z) = g(x,z) \text{ and } Lg(x,z) \geq 0 \text{ for } z \geq \gamma(x) \tag{3.20}$$

$$v(0,z) = \ell(z) \tag{3.21}$$

$$v_z(z,z) = 0. \tag{3.22}$$

The verification step requires that the value function be C^1 and piecewise C^2 in order to allow for the application of Itô's formula. We then complement the above system by the continuity and the smoothfit conditions

$$v(x,\gamma(x)) = g(x,\gamma(x)) \tag{3.23}$$

$$v_x(x,\gamma(x)) = g_x(x,\gamma(x)) \tag{3.24}$$

Our objective is to find a candidate v which satisfies (3.19) to (3.24) and an optimal stopping boundary γ so as to apply the following verification result:

Theorem 3.7 *Let γ be continuous and let v be a classical solution of* (3.19) *to* (3.24), *bounded from below, satisfying $\lim_{z\to\infty} v(z,z) - g(z,z) = 0$, such that $v \leq g$ on $\{(x,z);\ 0 \leq x \leq z\}$ and $v < g$ on the continuation region $\{(x,z);\ 0 < x \leq z \text{ and } z < \gamma(x)\}$.*
Then $v = V$ and $\theta^* = \inf\{t \geq 0; Z_t \geq \gamma(X_t)\}$ is an optimal stopping time.
Moreover if τ is another optimal stopping time, then $\theta^* \leq \tau$ a.s.

Proof.

$V \geq v$:

Let $\theta \in \mathcal{T}_0$ and for $n \in \mathbb{N}$, define $\theta_n = n \wedge \theta \wedge \inf\{t \geq 0;\ |Z_t| \geq n\}$. Then as v is sufficiently regular we have (Itô):

$$v(x,z) = v(X_{\theta_n}, Z_{\theta_n}) - \int_0^{\theta_n} Lv(X_t,Z_t)dt - \int_0^{\theta_n} v_x(X_t,Z_t)\sigma(X_t)dW_t - \int_0^{\theta_n} v_z(X_t,Z_t)dZ_t$$

Taking expectations and using the fact that $v_z(X_t,Z_t)dZ_t = v_z(Z_t,Z_t)dZ_t = 0$, $Lv \geq 0$ and $v \leq g$:

$$\begin{aligned} v(x,z) &\leq \mathbb{E}_{x,z} v(X_{\theta_n}, Z_{\theta_n}) \\ &\leq \mathbb{E}_{x,z} g(X_{\theta_n}, Z_{\theta_n}) = \mathbb{E}_{x,z}[\mathbb{E}_{X_{\theta_n},Z_{\theta_n}} \ell(Z_{T_0} - X_{\theta_n})] = \mathbb{E}_{x,z}\ell(Z_{T_0} - X_{\theta_n}) \end{aligned}$$

Clearly $\theta_n \underset{n\to\infty}{\to} \theta$ a.s. Now $(0 \leq)\ell(Z_{T_0} - X_{\theta_n}) \leq \ell(Z_{T_0})$, which is in L^1 in our framework. Now $(\mathbb{E}[\ell(Z_{T_0})|X_{\theta_n}, Z_{\theta_n}])_n$ is uniformly integrable so $(\mathbb{E}_{X_{\theta_n},Z_{\theta_n}}\ell(Z_{T_0} - X_{\theta_n}))_n$ is also UI and so it gives:

$$\mathbb{E}_{x,z}\ell(Z_{T_0} - X_{\theta_n}) \underset{n\to\infty}{\to} \mathbb{E}_{x,z}\ell(Z_{T_0} - X_\theta)$$

So that:

$$v(x, z) \leq V(x, z).$$

$V \leq v$:

If $z \geq \gamma(x)$, then $v = g \geq V$.

Assume now that $z < \gamma(x)$. Let $\theta^* = \inf\{t \geq 0;\ Z_t \geq \gamma(X_t)\}$. Thanks to the regularity of v, we have $Lv(X_t, Z_t) = 0$ if $t \in [0, \theta^*)$. As before define $\theta_n = n \wedge \theta^* \wedge \inf\{t \geq 0;\ |Z_t| \geq n\}$, then as previously we get for any n:

$$v(x, z) = \mathbb{E}_{x,z}v(X_{\theta_n}, Z_{\theta_n})$$

As v is bounded from below and $v \leq g$, we have $|v| \leq c+g$ for a constant c, so $(\mathbb{E}_{X_{\theta_n},Z_{\theta_n}}v(X_{\theta_n}, Z_{\theta_n}))_n$ is UI too.

So we can claim the following:

$$\begin{aligned} v(x, z) = \mathbb{E}_{x,z}v(X_{\theta^*}, Z_{\theta^*}) = \mathbb{E}_{x,z}v(X_{\theta^*}, \gamma(X_{\theta^*})) &= \mathbb{E}_{x,z}g(X_{\theta^*}, \gamma(X_{\theta^*})) \\ &= \mathbb{E}_{x,z}\ell(Z_{T_0} - X_{\theta^*}) \geq V(x, z). \end{aligned}$$

And therefore $v = V$ and θ^* is optimal.

Finally we show the minimality of θ^*. Assume to the contrary that there exists τ satisfying $\mathbb{P}(\tau < \theta^*) > 0$ and $\mathbb{E}_{x,z}\ell(Z_{T_0} - X_\tau) = \inf_\theta \mathbb{E}_{x,z}\ell(Z_{T_0} - X_\theta) = V(x, z)$.

But on $\{\tau < \theta^*\}$, we have by assumption $V(X_\tau, Z_\tau) < g(X_\tau, Z_\tau)$, while we always have $V(X_\tau, Z_\tau) \leq g(X_\tau, Z_\tau)$. Therefore:

$$\begin{aligned} V(x, z) = \mathbb{E}_{x,z}\ell(Z_{T_0} - X_\tau) &= \mathbb{E}_{x,z}g(X_\tau, Z_\tau) \\ &> \mathbb{E}_{x,z}V(X_\tau, Z_\tau) \geq V(x, z), \end{aligned}$$

where the last inequality comes immediately from the definition of V, and so we have a contradiction which guarantees the minimality of θ^*. □

In the rest of this paper, our objective is to exhibit functions γ and v satisfying the assumptions of the previous theorem.

In view of (3.20), the stopping region satisfies

$$\{(x, z) :\ z \geq \gamma(x)\} \quad \subset \quad \Gamma^+ := \{(x, z) :\ Lg(x, z) \geq 0\}\,. \tag{3.25}$$

We therefore need to study the structure of the set Γ^+.

Now we give some asymptotic results that will be useful later on. As we will focus first on the case of a quadratic loss function, we first give the useful results for this case.

Proposition 3.8 *We have the following asymptotic behaviors, as $z \to \infty$:*
(i) $S(z) \sim \dfrac{S'(z)}{\alpha(z)}$;
(ii) $\displaystyle\int_z^\infty \frac{du}{S(u)} \sim \frac{1}{S'(z)}$; $\displaystyle\int_z^\infty \frac{u}{S(u)}du \sim \frac{z}{S'(z)}$ *and* $\displaystyle\int_z^\infty \frac{u-z}{S(u)} \sim \frac{1}{\alpha(z)S'(z)}$.

Proof. Recall that $\left(\dfrac{1}{\alpha}\right)' \to 0$ at infinity. All the limits and equivalents are when $z \to +\infty$.

(i): As $S(z) \to +\infty$, $S(z) = \displaystyle\int_0^z e^{\int_0^u \alpha(v)dv} \sim \int_1^z e^{\int_0^u \alpha(v)dv}$. Integrating by parts, we get:

$$\int_1^z e^{\int_0^u \alpha(v)dv} = \left[\frac{e^{\int_0^u \alpha(v)dv}}{\alpha(u)}\right]_1^z - \int_1^z \left(\frac{1}{\alpha}\right)'(u)e^{\int_0^u \alpha(v)dv}du.$$

And as $\left(\dfrac{1}{\alpha}\right)' \to 0$, $\displaystyle\int_1^z \left(\frac{1}{\alpha}\right)'(u)e^{\int_0^u \alpha(v)dv}du = o\left(\int_1^z e^{\int_0^u \alpha(v)dv}\right)$, so that $S(z) \sim \frac{S'(z)}{\alpha(z)}$.

(ii): Using (i) and integrating by parts, we get:

$$\int_z^\infty \frac{du}{S(u)} \sim \int_z^\infty \frac{\alpha(u)}{S'(u)}du = \int_z^\infty \alpha(u)e^{-\int_0^u \alpha(v)dv}du \quad = \frac{1}{S'(z)};$$

$$\int_z^\infty \frac{udu}{S(u)} \sim \int_z^\infty \frac{u\alpha(u)}{S'(u)}du = \frac{z}{S'(z)} + \int_z^\infty \frac{1}{S'(u)}du.$$

But $u\alpha(u) \to \infty$ as $u \to \infty$, so that

$$\int_z^\infty \frac{1}{S'(u)}du = o\left(\int_z^\infty \frac{u\alpha(u)}{S'(u)}du\right),$$

and therefore:

$$\int_z^\infty \frac{udu}{S(u)} \sim \frac{z}{S'(z)}.$$

Finally integrating by parts twice, we get:

$$\begin{aligned}\int_z^\infty \frac{u-z}{S(u)}du &\sim \int_z^\infty \frac{(u-z)\alpha(u)}{S'(u)}du = \int_z^\infty \frac{1}{S'(u)}du \\ &= \int_z^\infty \frac{\alpha(u)}{\alpha(u)S'(u)}du = \frac{1}{\alpha(z)S'(z)} + \int_z^\infty \left(\frac{1}{\alpha}\right)'(u)\frac{1}{S'(u)}du.\end{aligned}$$

As $\left(\dfrac{1}{\alpha}\right)'(u) \to 0$ as $u \to \infty$, we get the result. $\square$

If ℓ is not the quadratic loss function, we will make the following assumptions on the loss function:

$$- \text{ or } \ell \text{ is } C^3,\ \ell' > 0,\ \ell'' > 0,\ \ell^{(3)} \geq 0 \text{ and } \ell,\ \ell',\ \ell'' \text{ satisfy (3.12)} \tag{3.26}$$

$$- \ K_1 := \sup_{y\geq 0} \frac{\ell^{(3)}(y)}{\ell''(y)} < \infty \text{ and } \lim_{x\to\infty} \alpha(x) > K_1 \tag{3.27}$$

$$- \ K_2 := \sup_{y\geq 0} \frac{\ell''(y)}{\ell'(y)} < \infty \text{ and } \lim_{x\to\infty} \alpha(x) > K_2. \tag{3.28}$$

Notice that those assumptions are satisfied for exponential loss functions $\ell(x) = \lambda e^x$ with $\lambda > 0$ or for power loss functions of the form $\lambda(x+\varepsilon)^p$ with $\varepsilon > 0$ and $p \geq 2$.

Proposition 3.9 *Assume* (3.26)-(3.28). *Let φ be a measurable function such that $0 \leq \varphi(z) \leq z$ for all z (big enough). We then have the following asymptotic behaviors, as $z \to \infty$:* (i) *there exists a bounded function δ (depending on φ) satisfying $\delta(z) \geq 1$, for z big enough, and such that:*

$$\int_z^\infty \frac{\ell''(u-\varphi(z))}{S(u)}du \sim \delta(z)\frac{\ell''(z-\varphi(z))}{S'(z)};$$

(ii) *there exists a bounded function ν satisfying $\nu(z) \geq 1$, for z big enough, and such that:*

$$\int_z^\infty \frac{\ell'(u-\varphi(z))}{S(u)}du \sim \nu(z)\frac{\ell'(z-\varphi(z))}{S'(z)}.$$

Moreover if $\lim_{x\to\infty} \alpha(x) = \infty$, then for any φ, δ and ν are constant and equal to 1.

Proof. (i): The proof is close to the proof of Proposition 3.8-(ii). First as φ is measurable and satisfies $0 \leq \varphi(z) \leq z$, the expressions make sense and the integrals exist. Then, thanks to (i) and integrating by parts, we have:

$$\begin{aligned}\int_z^\infty \frac{\ell''(u-\varphi(z))}{S(u)}du &\sim \int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)}du = \int_z^\infty \alpha(u)e^{-\int_0^u \alpha(v)dv}\ell''(u-\varphi(z))du \\ &= \frac{\ell''(z-\varphi(z))}{S'(z)} + \int_z^\infty \frac{\ell^{(3)}(u-\varphi(z))}{S'(u)}du.\end{aligned}$$

According to assumption (3.26), all the terms above are non-negative. Moreover, using (3.27) we get:

$$\int_z^\infty \frac{\ell^{(3)}(u-\varphi(z))}{S'(u)}du \leq K_1 \int_z^\infty \frac{\ell''(u-\varphi(z))}{S'(u)}du$$

$$\text{while } \int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)}du \geq \alpha(z)\int_z^\infty \frac{\ell''(u-\varphi(z))}{S'(u)}du\ (>0),$$

so that $A := \limsup_{z\to\infty} \dfrac{\int_z^\infty \frac{\ell^{(3)}(u-\varphi(z))}{S'(u)}du}{\int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)}du} < 1$, which means that, for z large enough, there exists a certain $k(z) \in \left[0, \dfrac{1+A}{2}\right)$ such that

$$\int_z^\infty \frac{\ell^{(3)}(u-\varphi(z))}{S'(u)}du = k(z)\int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)}du + o\left(\int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)}du\right).$$

As $\varphi(z) < z$ if $z > 0$, $\ell''(z-\varphi(z)) > 0$, and this implies that

$$(1-k(z))\int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)}du \sim \frac{\ell''(z-\varphi(z))}{S'(z)}.$$

Setting $\delta(z) = \dfrac{1}{1-k(z)} \in \left[1, \dfrac{2}{1-A}\right]$, we have the result. We also see that if $\alpha(x) \to \infty$ as $x \to \infty$, then $k(z) = 0$, so $\delta(z) = 1$. If $\ell(x) = \lambda x^2$, then $\ell^{(3)}(x) = 0$, so we also have $\delta(z) = 1$.

(ii): Follows the lines of Proposition 3.8-(ii) above, replacing ℓ'' by ℓ', we conclude similarly thanks to assumption (3.28). $\square$

In the subsequent paragraphs we shall first focus on quadratic loss functions. For general loss functions, we shall provide some conditions which guarantee that the structure of the solution agrees with that of the quadratic case.

3.4 The set Γ^+ for a quadratic loss function

Recall that the definition of Γ^+ was given by (3.25). Throughout this section, we consider the quadratic loss function

$$\ell(x) := \frac{1}{2}\,x^2 \quad \text{for} \quad x \geq 0.$$

In order to study the set Γ^+, we compute that:

$$Lg(x,z) = 1 + \alpha(x)(z-x) - (2S'(x) - \alpha(x)S(x))\int_z^\infty \frac{du}{S(u)}, \qquad 0 \leq x \leq z, \quad (3.29)$$

which takes values in $\mathbb{R} \cup \{-\infty\}$. Since $\alpha \geq 0$ and $2S' - \alpha S \geq 2$ by Remark 3.1, it follows that for every fixed $x \geq 0$, the function $z \longmapsto Lg(x,z)$ is strictly increasing on $[x,\infty)$. Now as $\displaystyle\int_z^\infty \frac{du}{S(u)} \to 0$ when $z \to \infty$, we see that $\lim_{z\to\infty} Lg(x,z) > 0$ for any $x \geq 0$. This shows that $\Gamma^+ \neq \emptyset$ and that $\Gamma^+ = Epi(\Gamma) := \{(x,z);\ z \geq \Gamma(x)\}$ where

$$\Gamma(x) \ := \ \inf\{z \geq x :\ Lg(x,z) \geq 0\}, \quad (3.30)$$

Moreover, $\Gamma^+ \setminus graph(\Gamma) = Int(\Gamma^+) \subset \{(x,z);\ Lg(x,z) > 0\}$ and Γ is continuous.

We also directly compute that for $x > 0$:

$$\frac{\partial^2}{\partial x^2} Lg(x,z) \;=\; -2\alpha'(x) + \alpha''(x)(z-x) - (\alpha^2(x)S'(x) - \alpha''(x)S(x)) \int_z^\infty \frac{du}{S(u)} \;<\; 0$$

by the concavity, the non-decrease, and the positivity of α on $(0,\infty)$. This implies that the function Γ is U-shaped in the sense of (i) of the Proposition 3.10 below. We also prove that the diagonal $\{x = z\}$ is an asymptote for Γ when x goes towards infinity.

Consider the following additional assumption on α:

$$\text{either } \exists K \geq 0, \text{ for } x \geq K,\ \alpha'(x) = 0, \text{ or, as } x \to \infty,\ \alpha''(x) = o\left([\alpha^2]'(x)\right) \tag{3.31}$$

We already know that $\alpha'(x) = o(\alpha^2(x))$, so this assumption is not that strong.

Proposition 3.10 (i) $\Gamma(0) > 0$ *and there is a constant* $\zeta \geq 0$ *such that* Γ *is decreasing on* $[0,\zeta]$ *and increasing on* $[\zeta, +\infty)$.
(ii) $\lim_{x \to +\infty} \Gamma(x) - x = 0$
(iii) *Assume* (3.31). *Then there exists* $\Gamma^{\max} > 0$ *such that either for any* $x \geq \Gamma^{\max}$, $\Gamma(x) > x$ *or for any* $x \geq \Gamma^{\max}$, $\Gamma(x) = x$. *Moreover, if* $\lim_\infty \alpha = \infty$, *then* $\Gamma^\infty < \infty$.

Proof. (i): We first show that for $x_1 < x_3$, $\lambda \in (0,1)$ and $x_2 = \lambda x_1 + (1-\lambda)x_3$, we have $\Gamma(x_2) < \max(\Gamma(x_1), \Gamma(x_3))$.

Indeed, assuming to the contrary that $\Gamma(x_2) \geq \max(\Gamma(x_1), \Gamma(x_3))$, it follows from the strict concavity of Lg w.r.t x and its non-decrease w.r.t z that:

$$\begin{aligned} Lg(x_2, \Gamma(x_2)) &> \lambda Lg(x_1, \Gamma(x_2)) + (1-\lambda) Lg(x_3, \Gamma(x_2)) \\ &\geq \lambda Lg(x_1, \Gamma(x_1)) + (1-\lambda) Lg(x_3, \Gamma(x_3)) \geq 0 \end{aligned}$$

By continuity of Lg, $Lg(x_2, \Gamma(x_2)) > 0$ implies that $\Gamma(x_2) = x_2$, which is in contradiction with $\Gamma(x_2) \geq \Gamma(x_3) \geq x_3 > x_2$.

Therefore Γ is either increasing, either decreasing, or first decreasing and then increasing. But as $\Gamma(x) \geq x$, the second one is impossible, so we have the result (where $\zeta = 0$ corresponds to the increasing case).

Finally, we show that $\Gamma(0) > 0$.

We have $S(x) \sim x$ as $x \to 0$, which implies that $\int_0^\infty \frac{du}{S(u)} = \infty$, and so for z small, $Lg(x,z) < 0$ for any $x \leq z$. In particular for $z > 0$ and small enough $Lg(0,z) < 0$, so that

$\Gamma(0) > 0$ and by continuity of Lg, $Lg(0, \Gamma(0)) = 0$.

(ii): Let $a > 0$, then it follows from Proposition 3.8 that:

$$\begin{aligned} Lg(z-a, z) &= 1 + a\alpha(z-a) - \frac{S'(z-a)}{S'(z)} + o(1) \\ &= 1 + a\alpha(z-a) - e^{-\int_{z-a}^{z} \alpha(u)du} + o(1) \end{aligned}$$

where $o(1) \to 0$ as $z \to \infty$.

If $\lim\limits_{x\to\infty} \alpha(x) = +\infty$, then $\dfrac{S'(z-a)}{S'(z)} \to 0$ (as $z \to \infty$).

If $\lim\limits_{x\to\infty} \alpha(x) = M > 0$ then $\dfrac{S'(z-a)}{S'(z)} \sim e^{-aM} < 1$.

In both cases, $Lg(z-a, z) > 0$ for z large enough, and so $(0 \leq)\Gamma(z) - z < a$.

(iii): We compute $Lg(x, x) = 1 - [2S'(x) - \alpha(x)S(x)] \displaystyle\int_x^\infty \frac{du}{S(u)}$. Using Proposition 3.8, we can see that $\lim\limits_{x\to+\infty} Lg(x, x) = 0$, but it is not enough.

Using assumption (3.31) and the fact that $\ell(x) = \dfrac{x^2}{2}$, we will refine the asymptotic expansions of Proposition 3.8. Recall that we have for $x \geq 1$:

$$S(x) = S(1) + \frac{S'(x)}{\alpha(x)} - \frac{S'(1)}{\alpha(1)} - \int_1^x \left(\frac{1}{\alpha}\right)'(u)S'(u)du.$$

Case 1: $\left|\displaystyle\int_1^\infty \left(\frac{1}{\alpha}\right)'(u)S'(u)du\right| < \infty$. Then there is $A \in \mathbb{R}$ such that $S(x) = A + \dfrac{S'(x)}{\alpha(x)}$.

Subcase 1.1: $A = 0$. Then we compute $\displaystyle\int_x^\infty \frac{du}{S(u)} = \int_x^\infty \frac{\alpha(u)du}{S'(u)} = \frac{1}{S'(x)}$. So that for $x \geq 1$, $Lg(x, x) = 0$, which means that $\Gamma(x) = x$ for $x \geq 1$.

Subcase 1.2: $A \neq 0$. We compute:

$$\begin{aligned} \int_x^\infty \frac{du}{S(u)} &= \int_x^\infty \frac{du}{A + \frac{S'(u)}{\alpha(u)}} \\ &= \frac{1}{S'(x)} - A\int_x^\infty \frac{\alpha^2(u)}{[S'(u)]^2}du \\ &= \frac{1}{S'(x)} - A\int_x^\infty \alpha^2(u)e^{-2\int_0^u \alpha(v)dv}du \end{aligned}$$

Now integrating by part:

$$2\int_x^\infty \frac{\alpha^2(u)}{[S'(u)]^2}du = \frac{\alpha(x)}{[S'(x)]^2} + \int_x^\infty \frac{\alpha'(u)}{[S'(u)]^2}du$$

As $\alpha'(u) = o(\alpha^2(u))$, we get:

$$\int_x^\infty \frac{du}{S(u)} = \frac{1}{S'(x)}\left[1 - \frac{A\alpha(x)}{2S'(x)} + o\left(\frac{\alpha(x)}{S'(x)}\right)\right]$$

So that:

$$\begin{aligned} Lg(x,x) &= 1 - [2S'(x) - \alpha(x)S(x)]\int_x^\infty \frac{du}{S(u)} \\ &= 1 - (S'(x) - A\alpha(x))\frac{1 - \frac{A\alpha(x)}{2S'(x)} + o\left(\frac{\alpha(x)}{S'(x)}\right)}{S'(x)} \\ &= \frac{3A\alpha(x)}{2S'(x)} + o\left(\frac{\alpha(x)}{S'(x)}\right) \end{aligned}$$

Which means that for x large enough $ALg(x,x) > 0$. If $A > 0$, $\Gamma(x) = x$ for x large enough, while if $A < 0$, $\Gamma(x) > x$ for x large enough.

Case 2: $\left|\int_1^\infty \left(\frac{1}{\alpha}\right)'(u)S'(u)du\right| = \infty$. Then we have necessarily: for all $x \geq 0$, $\alpha'(x) > 0$. We compute:

$$\int_1^x \left(\frac{1}{\alpha}\right)'(u)S'(u)du = \left[\left(\frac{1}{\alpha}\right)'(u)\frac{S'(u)}{\alpha(u)}\right]_1^x - \int_1^x \left[\frac{1}{\alpha}\left(\frac{1}{\alpha}\right)'\right]'(u)\frac{S'(u)}{\alpha(u)}du.$$

But thanks to assumption (3.31):

$$\frac{1}{\alpha}\left[\frac{1}{\alpha}\left(\frac{1}{\alpha}\right)'\right]' = \frac{\alpha\alpha'' - (\alpha')^2}{\alpha^5} = o\left(\frac{\alpha'}{\alpha^2}\right).$$

Therefore we have:

$$\int_1^x \left(\frac{1}{\alpha}\right)'(u)S'(u)du = \left(\frac{1}{\alpha}\right)'(x)\frac{S'(x)}{\alpha(x)} + o\left(\left(\frac{1}{\alpha}\right)'(x)\frac{S'(x)}{\alpha(x)}\right),$$

which implies:

$$S(x) = \frac{S'(x)}{\alpha(x)}\left[1 - \left(\frac{1}{\alpha}\right)'(x) + o\left(\left(\frac{1}{\alpha}\right)'(x)\right)\right],$$

and:

$$\int_x^\infty \frac{du}{S(u)} = \frac{1}{S'(x)} + \int_x^\infty \frac{\alpha(u)}{S'(u)}\left(\frac{1}{\alpha}\right)'(u)du + o\left(\int_x^\infty \frac{\alpha(u)}{S'(u)}\left(\frac{1}{\alpha}\right)'(u)du\right).$$

Again integrating by part and using assumption (3.31):

$$\begin{aligned}\int_x^\infty \frac{\alpha(u)}{S'(u)}\left(\frac{1}{\alpha}\right)'(u)du &= \left(\frac{1}{\alpha}\right)'(x)\frac{1}{S'(x)} + \int_x^\infty \left(\frac{1}{\alpha}\right)''(u)\frac{du}{S'(u)} \\ &= \left(\frac{1}{\alpha}\right)'(x)\frac{1}{S'(x)} + o\left(\left(\frac{1}{\alpha}\right)'(x)\frac{1}{S'(x)}\right).\end{aligned}$$

So finally:

$$\begin{aligned}Lg(x,x) &= 1 - [2S'(x) - \alpha(x)S(x)]\int_x^\infty \frac{du}{S(u)} \\ &= 1 - \left[1 + \left(\frac{1}{\alpha}\right)'(x) + o\left(\left(\frac{1}{\alpha}\right)'(x)\right)\right]\left[1 + \left(\frac{1}{\alpha}\right)'(x) + o\left(\left(\frac{1}{\alpha}\right)'(x)\right)\right] \\ &= -2\left(\frac{1}{\alpha}\right)'(x) + \left(\left(\frac{1}{\alpha}\right)'(x)\right).\end{aligned}$$

Since $\left(\frac{1}{\alpha}\right)' < 0$, this implies that for x large enough, $Lg(x,x) > 0$ and therefore $\Gamma(x) = x$.

We now show that if $\Gamma^\infty = \infty$, then α is bounded. Indeed, the previous computations show that, if $\Gamma^\infty = \infty$ then:

$$\left|\int_1^\infty \left(\frac{1}{\alpha}\right)'(u)S'(u)du\right| < \infty.$$

Assume to the contrary that α is not bounded, which means that $\lim_{x\to\infty}\alpha(x) = \infty$. Let $x \geq 1$. As α is non-decreasing

$$S'(x) = e^{\int_0^x \alpha(u)du} \geq e^{\alpha(x-1)}.$$

Since α' is non-increasing and non-negative, α' is bounded, therefore there exists $K > 0$, such that $0 \leq \alpha(x) - \alpha(x-1) \leq K$, and therefore $S'(x) \geq e^{\alpha(x)-K}$ (for $x \geq 1$).

On the other hand, $\lim_{x\to\infty}\alpha(x) = \infty$ implies that $\alpha(x)^2 = o(e^{\alpha(x)-K})$, which means that $\frac{S'(x)}{\alpha(x)^2} \to \infty$, and finally we get, as $x \to \infty$:

$$\alpha'(x) = o\left(-\left(\frac{1}{\alpha}\right)'(x)S'(x)\right).$$

As the right-hand side is integrable at infinity, the left-hand side is integrable as well, which means that α is bounded, which brings the required contradiction. $\square$

Remark 3.11 The two possibilities stated in point (iii) of the previous proposition can be encountered. Indeed, for a brownian motion with negative drift, we have $Lg(z, z) < 0$ for all z, while for an Ornstein-Uhlenbeck process, there exists $K > 0$ such that for any $z \geq K$, $Lg(z, z) > 0$. See the examples section below for more details.

Let us introduce a few notations. We write:

1. $\Gamma^0 = \Gamma(0)$
2. $\Gamma^\infty = \sup\{x > 0,\ Lg(x, x) < 0\} \in (0, +\infty]$
3. $\Gamma_\downarrow = \Gamma_{|_{[0,\zeta]}}$, the restriction of Γ to the interval $[0, \zeta]$ and $\Gamma_\uparrow = \Gamma_{|_{[\zeta,\infty)}}$, the restriction of Γ to the interval $[\zeta, \infty)$, where ζ is given by Proposition 3.10.

We have the following result:

Lemma 3.12 *$0 < \Gamma^0 < \Gamma^\infty$.*

Proof. The first property follows from Proposition 3.10.

For the second one, we show that $Lg(\Gamma^0, \Gamma^0) < 0$. Using Remark 3.1 (ii) and the fact that $Lg(0, \Gamma^0) = 0$, we compute:

$$\begin{aligned}
Lg(\Gamma^0, \Gamma^0) &= 1 - (2S' - \alpha S)(\Gamma^0) \int_{\Gamma^0}^{\infty} \frac{du}{S(u)} \\
&< 1 - 2 \int_{\Gamma^0}^{\infty} \frac{du}{S(u)} \\
&= Lg(0, \Gamma^0) - \alpha(0)\Gamma^0 \\
&= -\alpha(0)\Gamma^0 \leq 0.
\end{aligned}$$

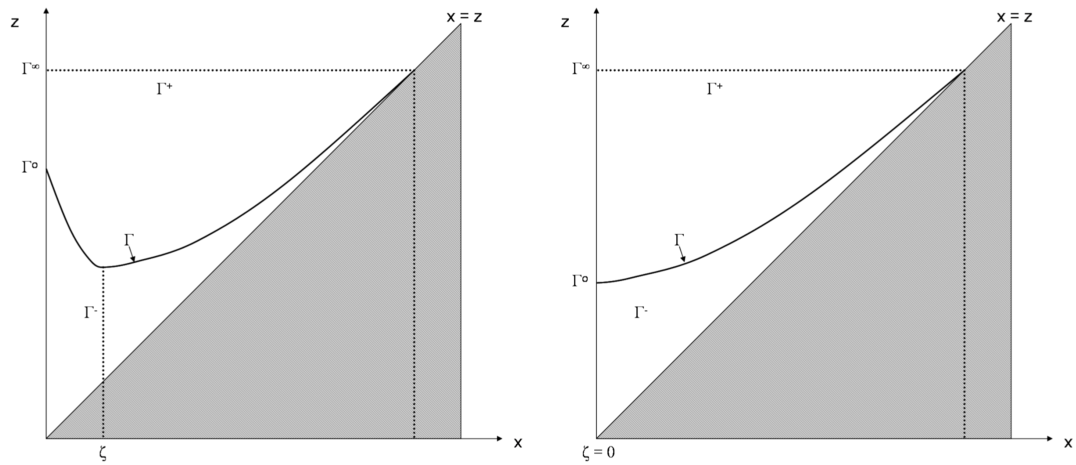
Now by continuity of Lg, this implies that $\Gamma^\infty > \Gamma^0$. □

Remark 3.13 This shows that, in the quadratic case, $\Gamma_\uparrow$ will never be reduced to a subset of the diagonal, or in other words that $\Gamma(\zeta) > \zeta$.

The figures below exhibit the two possible shapes of the function Γ and the location of Γ^+. Notice that in both cases Γ^∞ can be finite or not. We refer the reader to the examples section for examples of both cases.

3.5 The stopping boundary in the quadratic case

We now turn to the characterization of the stopping boundary γ.



(a) $\zeta > 0$ (for example Ornstein-Uhlenbeck process)

(b) $\zeta = 0$ (for example Brownian motion with negative drift)

Figure 3.1: The two possible shapes of Γ

3.5.1 The increasing part

Let $\Gamma^- := \{(x,z) : \ Lg(x,z) \leq 0\}$. From the previous section, we know that Γ^- is below the graph of Γ (and above the diagonal $\{x = z\}$) and $\Gamma^+ \cap \Gamma^-$ is the graph of Γ. Recall that, as stated in Remark 3.13, the graph of $\Gamma_\uparrow$ is not reduced to the diagonal and therefore $D^+ := \{x > 0; \ \Gamma(x) > x\} \neq \emptyset$. Moreover, we know that $\Gamma^0 > 0$.

Therefore we can write $D^+ = [0,a) \cup D'$ with $a \in \mathbb{R}_+ \cup \{+\infty\}$, $[0,a) \cap D' = \emptyset$ and $a > \zeta$.

We are looking here to find v that will satisfy (3.19) to (3.24) in the area that lies between the graph of γ and the diagonal $\{x = z\}$, so we must take into account the Neumann condition (3.22). From a heuristic point of view, $Lv = 0$ brings a function v of the form:

$$v(x,z) = A(z) + B(z)S(x).$$

Then, assuming that γ is bijective, the continuity and smoothfit conditions (3.23) and (3.24) imply that $v(x,z) = g(\gamma^{-1}(z),z) + \dfrac{g_x(\gamma^{-1}(z),z)}{S'(\gamma^{-1}(z))}[S(x) - S(\gamma^{-1}(z))]$. Putting it together with (3.22), we finally get that γ should solve the following ODE:

$$\gamma' = \frac{Lg(x,\gamma)}{1 - \frac{S(x)}{S(\gamma)}} \tag{3.32}$$

In fact we take this ODE as a starting point and we will construct γ. Notice that there is no a priori initial condition here. We will have to find the "good" one. If $0 < x_i < z_i$, then the Cauchy problem with initial condition $\gamma(x_i) = z_i$ is well defined, and Cauchy-Lipschitz theorem can be applied. We give here the main result of this section:

Proposition 3.14 *There exists γ continuous and defined on $\mathbb{R}_+$ such that:*
(i) *γ satisfies ODE* (3.32) *on* $(0, \Gamma^\infty)$ *(and therefore $\gamma(x) > x$ on that set)*
(ii) *if $x \geq \zeta$ then $(x, \gamma(x)) \in \Gamma^+$ and γ is increasing on* $(\zeta, +\infty)$
(iii) $\lim_{x\to\infty} \gamma(x) - x = 0$

Before we can prove this result, we must show some additional properties. Let $x_i \in (\zeta, a)$ be given. By definition, $Lg(x_i, x_i) < 0$.
If $z_i > x_i$, we write γ^{z_i} the maximal solution of the Cauchy problem with initial condition $\gamma(x_i) = z_i$ and we denote $(\xi_0^{z_i}, \xi_1^{z_i})$ the associated (open) interval. Notice that as the right-hand side of the ODE is locally lipschitz on the set $\{(x, \gamma),\ 0 < x < \gamma\}$, the maximal solution will be defined as long as $0 < x < \gamma$. Moreover, the flow of γ is a bijection onto the half-space $E = \{(x, z),\ 0 < x < z\}$.

Remark 3.15 We do not prove that the there exists a unique γ satisfying the properties of Proposition 3.14. However, we explain in Remark (3.18) why the function that we construct does not depend on the choice of x_i.

We first state a few properties concerning γ^{z_i} and $(\xi_0^{z_i}, \xi_1^{z_i})$ in the following lemma, that will allow us to construct the requested boundary:

Lemma 3.16 *We have:* (i) $\xi_0^{z_i} = 0$;
(ii) $\exists \delta > 0,\ \forall z_i \in (x_i, x_i + \delta),\ \xi_1^{z_i} < a$;
(iii) $\exists D > 0, \forall z_i \geq D, \xi_1^{z_i} = +\infty$.

Proof. (i): The right-hand side of (3.32) is locally Lipschitz as long as $0 < x < \gamma(x)$. Now γ is non-increasing if $(x, \gamma(x)) \in \Gamma^-$. So as $x_i < a$ while $\Gamma(x) > x$ for any $x < a$, the minimality of $\xi_0^{z_i}$ implies that $\xi_0^{z_i} = 0$.

(ii): We know that $(x_i, x_i) \in Int(\Gamma^-)$, and $x_i > \zeta$, so that Γ is non-decreasing on $[x_i, +\infty)$. Therefore, as γ is non-increasing as long as $(x, \gamma(x)) \in \Gamma^-$, if $z_i \in (x_i, \Gamma(x_i))$, γ^{z_i} is non-increasing on $[x_i, \xi_1^{z_i})$. Therefore $\xi_1^{z_i} \leq z_i$. Now if we take $\delta \leq \min(\Gamma(x_i) - x_i, a - x_i)$, we have the result.

(iii): Let $\varepsilon > 0$ be given. From Proposition 3.8-(ii), we get as $x \to \infty$:

$$Lg(x, (1+\varepsilon)x) = 1 + \varepsilon x \alpha(x) - \frac{S'(x)}{S'((1+\varepsilon)x)} + o(1)$$

Now $\dfrac{S'(x)}{S'((1+\varepsilon)x)} = e^{-\int_x^{x+\varepsilon x}\alpha(v)dv} \to 0$ as $x \to \infty$, and $\varepsilon x\alpha(x) \to +\infty$, so that:

$$\exists A \geq 0, \forall x \geq A, Lg(x,(1+\varepsilon)x) \geq 1+2\varepsilon.$$

In particular, $(A,(1+\varepsilon)A) \in Int\Gamma^+$. Then, as γ is non-decreasing as long as $(x,\gamma(x)) \in \Gamma^+$, if $z \geq \max((1+\varepsilon)A, \Gamma^0) =: D$, the shape of Γ implies that γ^z will be defined and non-decreasing on an open interval containing $[x_i, A]$ (and therefore $\xi_1^z > A$) and that $\gamma^z(A) > (1+\varepsilon)A$.

Now we show that for $z \geq D$, for any $x \geq A$, $\gamma^z(x) \geq (1+\varepsilon)x$.

Assume to the contrary that $\exists x_0 > A$, such that $\gamma^z(x_0) \leq (1+\varepsilon)x_0$. Let us define:

$$x_1 := \inf\{x > A;\ \gamma^z(x) = (1+\varepsilon)x)\}.$$

As γ^z is continuous, we have $A < x_1 \leq x_0$, and in particular, $Lg(x_1,(1+\varepsilon)x_1) \geq 1+2\varepsilon$. As Lg is also continuous, there is a neighborhood W of $(x_1,(1+\varepsilon)x_1)$ such that for $(x,z) \in W$, $Lg(x,z) \geq 1+\frac{3}{2}\varepsilon$, and so there exists $\eta > 0$ such that for any $x \in [x_1-\eta, x_1+\eta]$:

$$\gamma'(x) \geq Lg(x,\gamma(x)) \geq 1+\frac{3}{2}\varepsilon.$$

Now so if $x \in (x_1-\eta, x_1)$ and $x > A$, we have:

$$\gamma(x_1)-\gamma(x) = \int_x^{x_1}\gamma'(u)du \geq (1+\frac{3}{2}\varepsilon)(x_1-x),$$

and so:

$$\begin{aligned}\gamma(x) &\leq (1+\varepsilon)x_1 - (1+\frac{3}{2}\varepsilon)x_1 + (1+\frac{3}{2}\varepsilon)x \\ &\leq (1+\varepsilon)x + \frac{\varepsilon}{2}(x-x_1) \\ &< (1+\varepsilon)x.\end{aligned}$$

But as $\gamma(A) > (1+\varepsilon)A$, it contradicts the minimality of x_0.

So finally we get that if $z_i \geq D$, $\xi_1^{z_i} = +\infty$. □

We then provide results on the location of γ^{z_i} with respect to Γ. f and g being two continuous functions, we say that f crosses g at x if there exists $\varepsilon > 0$ such that $f-g$ has a unique zero at x on $[x-\varepsilon, x+\varepsilon]$.

Lemma 3.17 *If $(x,\gamma^{z_i}(x)) \in Int(\Gamma^+)$, then γ^{z_i} cannot cross $\Gamma_\uparrow$ at $y \leq x$ and it cannot cross $\Gamma_\downarrow$ at $y \geq x$.*

If $(x,\gamma^{z_i}(x)) \in Int(\Gamma^-)$, then γ^{z_i} cannot cross $\Gamma_\uparrow$ at $y \geq x$ and it cannot cross $\Gamma_\downarrow$ at $y \leq x$. As a consequence, γ^{z_i} can cross at most once $\Gamma_\uparrow$ and at most once $\Gamma_\downarrow$, $\lim\limits_{x\to\xi_1^{z_i}} \gamma^{z_i}(x) = \xi_1^{z_i}$ and $\lim\limits_{x\to 0} \gamma^{z_i}(x) > 0$ (and in particular it exists).

Proof. If γ crosses $\Gamma_\uparrow$ at x, we necessarily have for some $\varepsilon > 0$, $[x-\varepsilon, x] \subset \Gamma^+$ and $[x, x+\varepsilon] \subset \Gamma^-$, because $\Gamma_\uparrow$ is increasing, and γ nondecreasing (resp. non-increasing) if $(x,\gamma(x)) \in \Gamma^+$ (resp. $(x,\gamma(x)) \in \Gamma^-$). Therefore it cannot happen more than once. Similarly, if γ crosses $\Gamma_\downarrow$ at x, we necessarily have for some $\varepsilon > 0$, $[x-\varepsilon, x] \subset \Gamma^-$ and $[x, x+\varepsilon] \subset \Gamma^+$.

Then γ^{z_i} is monotonic after the possible crossing. So the limit exists. Now $\xi_1^{z_i} < +\infty$ means exactly that $\gamma(x) - x$ cannot be bounded from below by a positive constant when $x \to \xi_1^{z_i}$, and if $\xi_1^{z_i} = +\infty$, as $\gamma(x) > x$, the conclusion is immediate. The same kind of argument stands for the limit at $\xi_0^{z_i}$. $\square$

Now we can prove Proposition 3.14:

Proof. (Proposition 3.14)

Recall that $x_i \in (\zeta, a)$. We define $A := \{z > x_i;\ \gamma^z \text{ crosses } \Gamma_\uparrow\}$. We claim that $A \neq \emptyset$.

Indeed, consider the Cauchy problem for $z_i = \Gamma(x_i)$. Assume that $z_i \notin A$. We cannot have $\gamma^{z_i} = \Gamma$ locally, because it would imply that $\Gamma'(x) = 0$ locally which is impossible ($\Gamma'(x) > 0$ for $x > \zeta$). Therefore if γ^{z_i} does not cross $\Gamma_\uparrow$, we have either $\gamma^{z_i}(x) > \Gamma(x)$ for $x \neq x_i$ close enough or $\gamma^{z_i}(x) < \Gamma(x)$ for $x \neq x_i$ close enough.

Assume for example that $\gamma^{z_i}(x) > \Gamma(x)$ for $x \neq x_i$ close enough. As the flow describes the entire half-space E and is one-to-one, it means that there exists $z' < z_i$ and $x' < x_i$ such that $\gamma^{z'}(x') > \Gamma(x')$ while $\gamma^{z'}(x_i) = z' < z_i = \Gamma(x_i)$. Which means that $z' \in A$. The case $\gamma^{z_i}(x) < \Gamma(x)$ is done using similar arguments.

Now according to Lemma 3.16-(iii), A is bounded, so we can define $z_0 := \sup A$. We write γ (respectively ξ_1) instead of γ^{z_0} (resp. $\xi_1^{z_0}$). We know that $\xi_0 = 0$, thanks to Lemma 3.16-(i). Thanks to Lemma 3.17, $\lim\limits_{x\to\xi_1} \gamma(x) = \xi_1$ (possibly infinite) and $\lim\limits_{x\to 0} \gamma(x)$ exists. We write it $\gamma(0) (> 0)$.

Now we show that $\xi_1 = \Gamma^\infty$. If $\Gamma^\infty = +\infty$, the properties of the flow bring immediately that $\xi_1 = +\infty$ as well. So assume that $\xi_1 < \infty$. By continuity of the flow and of Γ, there exists $\varepsilon > 0$, such that $(z_0 - \varepsilon, z_0) \subset A$. Therefore, according to Lemma 3.17, if $z \in (z_0 - \varepsilon, z_0)$, γ^z stays in Γ^- for $x \geq x_i$, so that γ^z is non-increasing on $[x_i, \xi_1^z)$, so that for any $z \in (z_0 - \varepsilon, z_0)$, and any $x \in (0, \xi_1^z)$, if we write x_c^z the point where γ crosses $\Gamma_\uparrow$ we have:

$$\gamma^z(x) \leq \sup_{[\xi_0^z, \xi_1^z]} \gamma = \gamma(x_c^z) = \Gamma(x_c^z) < \Gamma^\infty$$

In particular, it means that $\xi_1^z < \Gamma^\infty$.

Assume that $\xi_1 (= \xi_1^{z_0}) > \Gamma^\infty$, then we also have $\gamma(\Gamma^\infty) > \Gamma^\infty$, so that if $z \in (\Gamma^\infty, \gamma(\Gamma^\infty))$, (Γ^∞, z) will not be attained by the flow, which is impossible. Assume then that $\xi_1 < \Gamma^\infty$, there exists $x \in (\xi_1, \Gamma^\infty)$ and $z > x$ such that $(x, z) \in Int(\Gamma^-)$, so that it cannot be attained by the flow, which again is impossible. So we have $\xi_1 = \Gamma^\infty$.

Then we show that on $[x_i, \Gamma^\infty)$, γ does not cross $\Gamma_\uparrow$. Indeed if it was the case, as said before, γ would be non-increasing after the crossing, so that we would not have $\lim_{x \to \Gamma^\infty} \gamma(x) = \Gamma^\infty$. Therefore $(x, \gamma(x)) \in \Gamma^+$ on $[x_i, \Gamma^\infty)$.

If $\Gamma^\infty < \infty$, then we define for $x \geq \Gamma^\infty$, $\gamma(x) = x$. By construction and with what we have just shown we have:
- γ is defined on $\mathbb{R}_+$ and is continuous
- if $x \geq \zeta$, $(x, \gamma(x)) \in \Gamma^+$ and therefore γ is nondecreasing on (ζ, Γ^∞)
- on $(0, \Gamma^\infty)$, γ satisfies ODE (3.32)
- on $(\Gamma^\infty, +\infty)$, $\gamma(x) = x$.

So we need to prove that γ is increasing on (ζ, Γ^∞) and study the limit at infinity of $\gamma(x) - x$. We cannot have locally $\gamma = \Gamma$, because then γ would cross $\Gamma_\uparrow$. Therefore the set $\{x < \Gamma^\infty;\ \gamma(x) = \Gamma(x))\}$ has an empty interior and as a consequence, γ is increasing on (ζ, ∞).

Finally we study $\lim_{x \to \infty} \gamma(x) - x$. If $\Gamma^\infty < \infty$, then $\gamma(x) = x$ for $x > \Gamma^\infty$ so it is immediate. Assume that $\Gamma^\infty = \infty$. Let $a > 0$, we prove that for x large enough, $\gamma(x) \leq x + a$.

Using Proposition 3.8, we compute:

$$Lg(x, x+a) = 1 + a\alpha(x) - e^{-\int_x^{x+a} \alpha(u)du} + o(1)$$

- If $\lim_{x \to \infty} \alpha(x) = \infty$, then, for $\varepsilon > 0$, we get that $Lg(x, x+a) > 1 + \varepsilon$ for x large enough.
- If $\lim_{x \to \infty} \alpha(x) = M > 0$, then $\dfrac{Lg(x, x+a)}{1 - \frac{S(x)}{S(x+a)}} = \dfrac{1 - e^{-aM} + aM}{1 - e^{-aM}} + o(1)$, so that for $\varepsilon > 0$ small enough, we get that $\dfrac{Lg(x, x+a)}{1 - \frac{S(x)}{S(x+a)}} > 1 + \varepsilon$ for x large enough.

Now assume to the contrary that there exists x_0 as large as needed and such that $\gamma(x_0) > x_0 + a$. Then as $\gamma(x) > x$ on $[x_i, +\infty)$ (because $\xi_1 = \infty$), by continuity of the flow with respect to the initial data, we can find $z < z_0$ such that $\gamma^z(x) > x$ on $[x_i, x_0]$ and $\gamma^z(x_0) > x_0 + a$.

But because of the previous calculation, we have for $x \in [x_0, +\infty)$:

$$\gamma^z(x) - \gamma^z(x_0) = \int_{x_0}^{x} (\gamma^z)'(u)du \geq (1+\varepsilon)(x - x_0)$$
$$\text{and so } \gamma^z(x) > (1+\varepsilon)(x - x_0) + x_0 + a \geq x + a,$$

so that $\xi_1^z = +\infty$ and therefore γ^z does not cross $\Gamma_\uparrow$, and neither does γ^y for $y \in [z, z_0]$, which contradicts the definition of z_0 as $\sup A$.

In conclusion, $\lim_{x\to\infty} \gamma(x) - x = 0$ and we have the result. □

Remark 3.18 It is quite interesting to wonder if for another starting point x_i' such that $\Gamma(x_i') > x_i'$, the function γ defined in the previous proof would have been the same or not. We can show that it is the case.
If $x_i' \in (\zeta, a)$ this is quite obvious. But it is also true if $x_i' > a$ as long as $\Gamma(x_i') > x_i'$. Let us write γ and γ' the associated boundaries constructed as in the previous proof. As $\xi_1 = \Gamma^\infty$, then the definition of z_0' implies that γ is above γ'. Similarly if $\xi_0' = 0$, then γ' is above γ because of the definition of z_0. Assume that γ is strictly above γ'. Then by continuity of the flow, there exists $z < z_0$, such that (we write γ^z the solution of ODE (3.32) with initial condition $\gamma^z(x_i) = z$) γ^z is strictly between γ and γ' and is defined on $[x_i, x_i']$. But then $\xi_0^z = 0$ for the same reason as $\xi_0 = 0$, while γ^z does not cross $\Gamma_\uparrow$ after x_i' as it is above γ'. Therefore γ^z is defined on $(0, \Gamma^\infty + \varepsilon)$ for some $\varepsilon > 0$, so that it cannot cross $\Gamma_\uparrow$, which contradicts the definition of z_0. Finally we see that $\gamma = \gamma'$.

3.5.2 The decreasing part

The problem now is that there is no reason for the function γ constructed in the previous paragraph to be entirely in Γ^+ as it can cross graph($\Gamma_\downarrow$). Indeed for an OU process it will not be the case. See the examples section for more details. In fact, the boundary is in general made of two parts as shown on the following plot. Therefore we need to consider the area that lies between the axis $\{x = 0\}$ and graph(γ). While the right part of γ is characterized by the ODE above because of the Neumann condition, here we must take into account the Dirichlet condition (3.21).
Therefore, we consider the following problem, for a fixed $z > 0$:

$$g(x(z), z) - g_x(x(z), z)\frac{S(x(z))}{S'(x(z))} - \frac{z^2}{2} = 0 \tag{3.33}$$

We write $f(x, z) = g(x, z) - g_x(x, z)\dfrac{S(x)}{S'(x)} - \dfrac{z^2}{2}$.

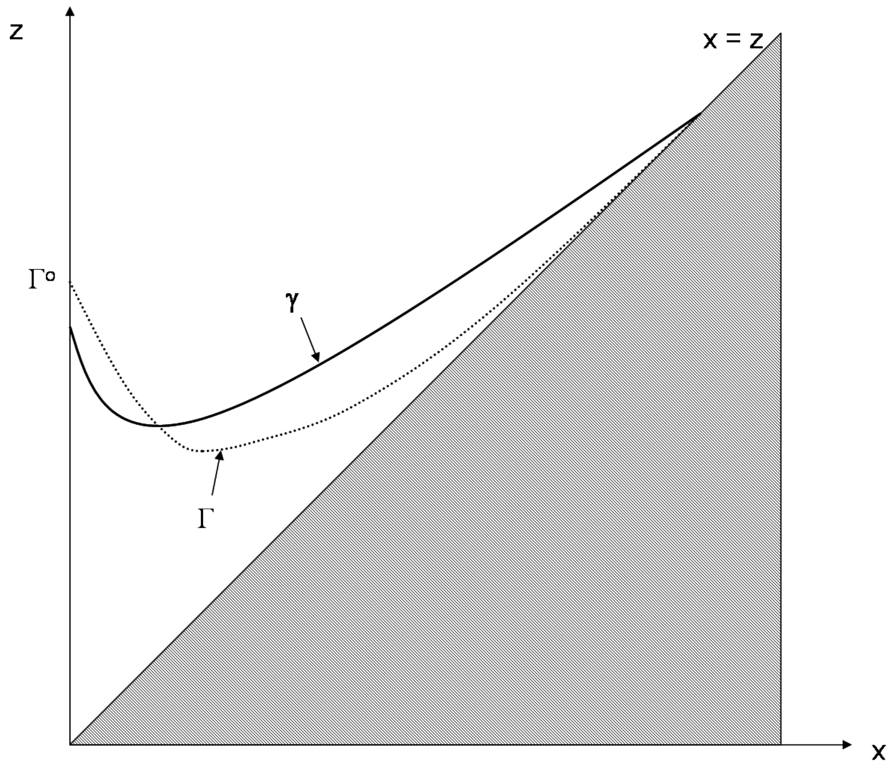


Figure 3.2: On the left part, the graph of γ is inside $Int(\Gamma^-)$ and γ is decreasing.

We will see that this problem has a unique positive solution for z in a certain interval, and it will allow us to define the second part of our boundary. First notice that $x(z) = 0$ is always solution of (3.33).

Proposition 3.19 *Assume that $\Gamma_\downarrow$ is not degenerate (ie $\zeta > 0$). Then there exists $x^* > 0$ and a function $\gamma_\downarrow$ defined on $[0, x^*]$, which is decreasing, C^0 on $[0, x^*]$, C^1 on $(0, x^*)$ and such that:*
(i) $\forall x \in [0, x^*],\ f(x, \gamma_\downarrow(x)) = 0$
(ii) $\forall x \in [0, x^*],\ (x, \gamma_\downarrow(x)) \in \Gamma^+$
(iii) $\gamma_\downarrow(0) = \Gamma^0$
(iv) $(x^*, \gamma_\downarrow(x^*)) \in graph(\Gamma_\uparrow)$

We first prove the following lemma, which will be also used later.

Lemma 3.20

$$\frac{\partial}{\partial x}\left(\frac{g_x(x,z)}{S'(x)}\right) = \frac{Lg(x,z)}{S'(x)}$$

Proof. (lemma) Direct calculation as $LS = 0$. □

Proof. (proposition) By definition of g and S, for any z, $f(0, z) = 0$. Then:

$$f_x(x, z) = g_x(x, z) - S(x)\frac{Lg(x, z)}{S'(x)} - g_x(x, z) = -S(x)\frac{Lg(x, z)}{S'(x)}$$

Therefore, $f_x(x, \Gamma^0) < 0$ for any $x \in]0, \Gamma_\uparrow^{-1}(\Gamma^0)]$, thus $f(x, \Gamma^0) < 0$ if $x \in]0, \Gamma_\uparrow^{-1}(\Gamma^0)]$. On the contrary, if $z < \Gamma^0$, then $f(x, z) > 0$ for x in a certain $]0, \varepsilon(z)]$ (with $\varepsilon(z) > 0$).

Now by continuity of f, there exists $\varepsilon > 0$ and $x > 0$ such that for any $z \in]\Gamma^0 - \varepsilon, \Gamma^0]$, $f(x, z) < 0$. Therefore there exists $x \in]\Gamma_\downarrow^{-1}(z), \Gamma_\uparrow^{-1}(z)]$ satisfying $f(x, z) = 0$. Let z_i be in such a neighborhood and write $x_i \in]\Gamma_\downarrow^{-1}(z_i), \Gamma_\uparrow^{-1}(z_i)]$ satisfying $f(x_i, z_i) = 0$. By definition, $(x_i, z_i) \in Int(\Gamma^+)$.

We consider now the following Cauchy problem:

$$\gamma_\downarrow'(x) = \frac{Lg(x, \gamma_\downarrow(x))S(x)}{S(x) - xS'(x) - \frac{(S(x))^2}{S(\gamma_\downarrow)}}$$

With initial condition $\gamma_\downarrow(x_i) = z_i$. This ODE is obtained by a formal derivation of the equation $f(x, \gamma(x)) = 0$. Indeed, assuming γ is C^1, it brings:

$$f_x(x, \gamma(x)) + \gamma'(x)f_z(x, \gamma(x)) = 0$$

We compute:

$$\begin{aligned}
f_z(x, z) &= g_z(x, z) - g_{xz}(x, z)\frac{S(x)}{S'(x)} - z \\
&= z - x - \frac{S(x)}{S(z)}(z - x) + \frac{S(x)}{S'(x)}(1 + \frac{S'(x)(z - x)}{S(z)} - \frac{S(x)}{S(z)}) - z \\
&= -x + \frac{S(x)}{S'(x)} - \frac{(S(x))^2}{S'(x)S(z)}
\end{aligned}$$

So we get:

$$\gamma'[-xS'(x) + S(x) - \frac{(S(x))^2}{S(\gamma)}] = S(x)Lg(x, \gamma)$$

As long as $x > 0$, $S(x) - xS'(x) - \frac{(S(x))^2}{S(\gamma_\downarrow)} \leq S(x) - xS'(x) < 0$, so the Cauchy problem is well defined ($0 < x_i \leq z_i$). The maximal solution will be defined on an interval (x_-, x_+), with $x_i \in (x_-, x_+)$. We also have $\gamma_\downarrow' < 0$ as long as $(x, \gamma_\downarrow(x)) \in Int(\Gamma^+)$ and $(x_i, z_i) \in Int(\Gamma^+)$, so we have $graph(\gamma_\downarrow) \cap \Gamma \neq \emptyset$.

Now if $(x_\Gamma, z_\Gamma) \in graph(\gamma_\downarrow) \cap \Gamma$, then $\gamma_\downarrow'(x_\Gamma) = 0$, so (x_Γ, z_Γ) can only be on $graph(\Gamma_\uparrow)$. This implies that $x_- = 0$ and we can define $x^* = \inf\{x \geq x_i, (x, \gamma_\downarrow(x)) \in \Gamma\}$. $\gamma_\downarrow$ is defined on $(0, x^* + \varepsilon)$ for a certain $\varepsilon > 0$.

Now by construction $f(x, \gamma_\downarrow(x)) = \text{constant} = f(x_i, z_i) = 0$, $(x, \gamma_\downarrow(x)) \in \Gamma^+$ and $(x^*, \gamma_\downarrow(x^*)) \in graph(\Gamma_\uparrow)$.

Finally, as $\gamma_\downarrow$ is decreasing it has a limit at 0. The fact that $(x, \gamma_\downarrow(x)) \in \Gamma^+$ implies that $\gamma_\downarrow(0) \geq \Gamma^0$, and if we had $\gamma_\downarrow(0) > \Gamma^0$, then by continuity of $\gamma_\downarrow$, there will be $x \in (0, \Gamma_\uparrow^{-1}(\Gamma^0)]$, such that $f(x, \Gamma^0) = 0$, which is impossible. So we have the result. □

$\gamma_\downarrow$ defined in the previous proposition will be the second part of our boundary. Now we have to wonder if those two parts will have an intersection. This is provided in the following proposition. We write $\gamma_\uparrow$ the boundary constructed in the previous paragraph.

Proposition 3.21 *We have either $\gamma_\uparrow$ is increasing on $[0, +\infty)$, or $|graph(\gamma_\downarrow) \cap graph(\gamma_\uparrow)| = 1$. In the first case we write $\bar{x} = 0$ and $\bar{z} = \gamma_\uparrow(0)$. In the second case, we write $(\bar{x}, \bar{z}) = graph(\gamma_\downarrow) \cap graph(\gamma_\uparrow)$. In both cases we have $(\bar{x}, \bar{z}) \in \Gamma^+$.*

Proof. $\gamma_\uparrow$ is increasing as long as $Lg(x, \gamma_\uparrow(x)) > 0$. If we do not have $\gamma_\uparrow$ increasing on $[0, +\infty)$, it means that there exists x_0 such as $Lg(x_0, \gamma_\uparrow(x_0)) = 0$ while $\gamma_\uparrow$ is increasing on $(x_0, +\infty)$. This implies that $\gamma_\uparrow'(x_0) = 0$ so that necessarily $(x_0, \gamma_\uparrow(x_0)) \in \Gamma_\downarrow$ and $(x, \gamma_\uparrow(x))$ is in $Int(\Gamma^-)$ in a left neighborhood of x_0. But if $\gamma_\uparrow$ crosses $\Gamma_\downarrow$ at x, then $(x, \gamma_\uparrow(x))$ is in $Int(\Gamma^+)$ in a right neighborhood of x, so $\gamma_\uparrow$ does not cross $\Gamma_\downarrow$ on $(0, x_0)$.

Finally we have $\gamma_\downarrow$ is decreasing, continuous, starts from $(0, \Gamma^0)$ and ends on $\Gamma_\uparrow$ while $\gamma_\uparrow$ is increasing, continuous, crosses once $\Gamma_\downarrow$ and is defined until $+\infty$. Therefore we have $|graph(\gamma_\downarrow) \cap graph(\gamma_\uparrow)| = 1$ and this intersection is in Γ^+.

Now if $\gamma_\uparrow$ is increasing on $[0, +\infty)$, then $(x, \gamma_\uparrow(x)) \in \Gamma^+$ for all $x > 0$, so by continuity of $\gamma_\uparrow$ and as Γ^+ is a closed set, it is still true for $x = 0$. □

From now on, we write γ the concatenation of $\gamma_\downarrow$ and $\gamma_\uparrow$, which is continuous, and even piecewise C^1:

$$\gamma(x) = \begin{cases} \gamma_\downarrow(x) \text{ if } x < \bar{x} \\ \gamma_\uparrow(x) \text{ if } x \geq \bar{x} \end{cases}$$

We also introduce $\phi_\downarrow = \gamma_\downarrow^{-1}$ and $\phi_\uparrow = \gamma_\uparrow^{-1}$.
Notice that if $\gamma_\downarrow$ is degenerate, then $\gamma = \gamma_\uparrow$.

3.6 Definition of v and verification result

Now we are able to define our candidate function v and we will then prove that it is indeed the wanted value function V.

Let us first define four different parts of the half-space E. We define:

$$\begin{aligned} A_1 &= \{(x,z),\ x \in [0, \bar{x}[\text{ and } \bar{z} \le z < \gamma(x)\} \\ A_2 &= \{(x,z),\ x \ge \bar{x} \text{ and } \bar{z} \le z < \gamma(x)\} \\ A_3 &= \{(x,z),\ 0 \le x \le z < \bar{z}\} \\ A_4 &= \{(x,z),\ x \ge 0 \text{ and } z \ge \gamma(x)\}. \end{aligned}$$

(A_1, A_2, A_3, A_4) is a partition of E. Notice that if $(x,z) \in A_2$, then $x \le \Gamma^\infty$, and recall that $\bar{x} < \bar{z}$.

The first term corresponds to the area 1 in the plot below, which is on the left of the boundary, the second term (area 2) is on the right of the boundary, the third term (area 3) is below areas 1 and 2 and finally the fourth term (area 4) is above the boundary.

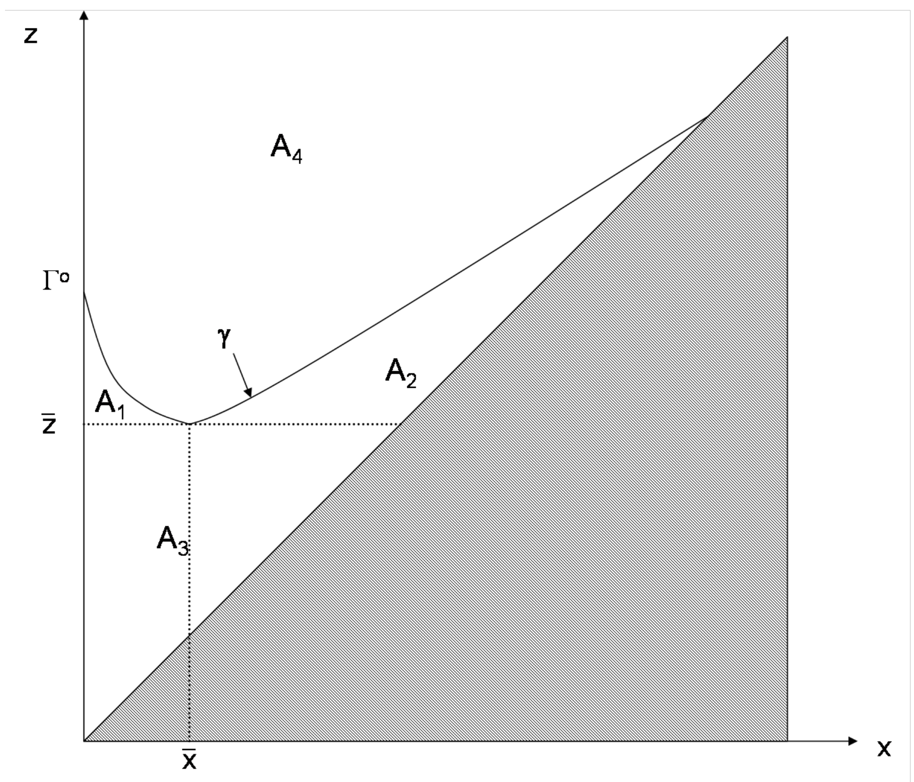


Figure 3.3: The different areas

Recall that $\bar{x}$ and $\bar{z}$ have been defined in Proposition 3.21 and $\phi_\downarrow$, $\phi_\uparrow$ have been defined just after it. Let $K = \int_{\bar{z}}^{\infty} \frac{u}{S(u)} du - \frac{g_x(\bar{x},\bar{z})}{S'(\bar{x})}$, we define v in the following way:

$$v(x,z) = \frac{z^2}{2} + g_x(\phi_1(z),z)\frac{S(x)}{S'(\phi_1(z))} \text{ if } (x,z) \in A_1 \tag{3.34}$$

$$v(x,z) = g(\phi_2(z),z) + g_x(\phi_2(z),z)\frac{S(x)-S(\phi_2(z))}{S'(\phi_2(z))} \text{ if } (x,z) \in A_2 \tag{3.35}$$

$$v(x,z) = \frac{z^2}{2} + S(x)[\int_{z}^{\infty} \frac{u}{S(u)} du - K] \text{ if } (x,z) \in A_3 \tag{3.36}$$

$$v(x,z) = g(x,z) \text{ if } (x,z) \in A_4. \tag{3.37}$$

First we need to prove that v is sufficiently regular, in order to be able to apply Itô's formula. We denote by $\bar{A}$ the closure of a set A.

Lemma 3.22 *v is C^0 w.r.t (x,z), C^1 w.r.t x and piecewise $C^{2,1}$ w.r.t. (x,z). More precisely, except on $\cup_{i\neq j}(\bar{A}_i \cap \bar{A}_j)$, it is $C^{2,1}$.*

Proof. From the definition of v, it is immediate that v can be extended as a $C^{2,1}$ function on any $\bar{A}_i$.

Let us write v_i the expression of v on $\bar{A}_i$. As $\phi_\downarrow$ satisfies (3.33), it is immediate to see that v is C^0 in (x,z) and C^1 in x on the boundary (v_1 with v_4 and v_2 with v_4). On $z=\bar{z}$, it is easy to check that the expressions of v_2 and v_3 coincide. It is also true for v_1 and v_3 as $\phi_\downarrow$ satisfies (3.33) and $\bar{x} = \phi_\downarrow(\bar{z})$. It is straightforward that it is also C^1 and even C^2 w.r.t x. □

We now show that v satisfies the limit conditions.

Lemma 3.23 $\forall z \geq 0,\ v(0,z) = \frac{z^2}{2}$ *and* $v_z(z,z) = 0$.

Proof. $v(0,z) = \frac{z^2}{2}$ is immediate. Then for $(z,z) \in Int(A_4)$, as $g_z(z,z) = 0$, we have $v_z(z,z) = 0$ and for $(z,z) \in Int(A_3)$ it is immediate that $v_z(z,z) = 0$. For $(z,z) \in Int(A_2)$, as γ_2 satisfies the ODE (3.32), then $\phi'_\uparrow(z)Lg(\phi_\uparrow(z),z) = 1 - \frac{S(\phi_\uparrow(z))}{S(z)}$. And we compute

$$\begin{aligned} v_z(z,z) &= g_z(\phi_\uparrow(z),z) + g_{xz}\frac{S(z)-S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} + \phi'_\uparrow(z)Lg(\phi_\uparrow(z),z)\frac{S(z)-S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} \\ &= -(1-\frac{S(\phi_\uparrow(z))}{S(z)})\frac{S(z)-S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} + (1-\frac{S(\phi_\uparrow(z))}{S(z)})\frac{S(z)-S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} \\ &= 0 \end{aligned}$$

The only remaining points are $(\bar{z}, \bar{z})$ and, if $\Gamma^\infty < \infty$, $(\Gamma^\infty, \Gamma^\infty)$. But the previous computation and the definitions of v on A_3 and A_4 show that at those points, v_z has right and left limits that are equal to 0, so we have the result. □

Then we prove a result that is non trivial in the case $\Gamma^\infty = \infty$.

Proposition 3.24 *Assume that either α is bounded or* (3.31) *is satisfied. Then v is bounded from below and* $\lim_{z\to\infty} v(z,z) - g(z,z) = 0$

Proof. If $\Gamma^\infty < \infty$, it is obvious since in this case $v = g$ outside a compact set, v is continuous and g is non-negative. So let's assume $\Gamma^\infty = \infty$. If (3.31) is satisfied, Proposition 3.10-(iii), we know that α is bounded. We write $\alpha \leq M$.

A_1 is bounded because of the definition of $\gamma_\downarrow$, and A_3 is bounded too. As $v = g$ on A_4 and $g \geq 0$, we only need to check that v is bounded from below on A_2.

Now $v = g$ on the set $\{(x, \gamma_\uparrow(x));\ x \in [\bar{x}, \infty)\}$ and for $(x,z) \in A_2$ we have:

$$v(x,z) = g(\phi_\uparrow(z), z) + g_x(\phi_\uparrow(z), z)\frac{S(x) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))}.$$

In particular, we see that v is monotonic w.r.t x on A_2. Therefore it is sufficient to check that v is bounded from below on the diagonal $\{(z,z);\ z \in [\bar{x}, \infty)\}$.

From Proposition 3.14 we know that $\lim_{x\to\infty}(\gamma_\uparrow(x) - x) = 0$, so that $\lim_{z\to\infty}(z - \phi_\uparrow(z)) = 0$, therefore we get as $z \to \infty$:

$$g_x(\phi_\uparrow(z), z) = -(z - \phi_\uparrow(z)) + S'(\phi_\uparrow(z))\int_z^\infty \frac{u - \phi_\uparrow(z)}{S(u)}du - S(\phi_\uparrow(z))\int_z^\infty \frac{du}{S(u)}.$$

And using Proposition 3.8 and the fact that $\phi_\uparrow(z) < z$ as $\Gamma^\infty = \infty$:

$$S'(\phi_\uparrow(z))\int_z^\infty \frac{u - \phi_\uparrow(z)}{S(u)}du \sim S'(\phi_\uparrow(z))\frac{z - \phi_\uparrow(z)}{S'(z)} = O(1)$$
$$S(\phi_\uparrow(z))\int_z^\infty \frac{du}{S(u)} \sim \frac{S'(\phi_\uparrow(z))}{\alpha(\phi_\uparrow(z))S'(z)} = O(1)$$

So that $g_x(\phi_\uparrow(z), z) = O(1)$.

We also have, as $\alpha \leq M$:

$$\begin{aligned} S(z) - S(\phi_\uparrow(z)) &= \int_{\phi_\uparrow(z)}^z S'(u)du \\ &\leq (z - \phi_\uparrow(z))S'(z)) \\ &\leq (z - \phi_\uparrow(z))S'(\phi_\uparrow(z))e^{M(z - \phi_\uparrow(z))} \end{aligned}$$

So that: $0 \leq \dfrac{S(z) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} \leq (z - \phi_\uparrow(z))e^{M(z-\phi_\uparrow(z))} = o(1)$.

And so, as v is continuous and $g \geq 0$, v it is bounded from below and $v(z,z) - g(\phi_\uparrow(z), z) \to 0$.

Finally, we show that $g(z,z) - g(\phi_\uparrow(z), z) \to 0$. Indeed:

$$g(z,z) - g(\phi_\uparrow(z), z) = -\frac{(z-\phi_\uparrow(z))^2}{2} + (S(z) - S(\phi_\uparrow(z))) \int_z^\infty \frac{u-z}{S(u)} du - S(\phi_\uparrow(z)) \int_z^\infty \frac{z-\phi_\uparrow(z)}{S(u)} du.$$

Using Proposition 3.8, we get: $\displaystyle\int_z^\infty \frac{u-z}{S(u)} du \sim \frac{1}{\alpha(z)S'(z)}$, so that

$$\begin{aligned} (S(z) - S(\phi_\uparrow(z))) \int_z^\infty \frac{u-z}{S(u)} du &\sim \frac{S(z) - S(\phi_\uparrow(z))}{\alpha(z)S'(z)} \\ &\frac{S(z) - S(\phi_\uparrow(z))}{\alpha(z)S'(\phi_\uparrow(z))} = o(1) \end{aligned}$$

as we have seen before. Using again Proposition 3.8, we finally get:

$$S(\phi_\uparrow(z)) \int_z^\infty \frac{z-\phi_\uparrow(z)}{S(u)} du \sim (z - \phi_\uparrow(z)) \frac{S'(\phi_\uparrow(z))}{\alpha(\phi_\uparrow(z))S'(z)} = o(1),$$

and in consequence:

$$g(z,z) - g(\phi_\uparrow(z), z) = o(1).$$

Therefore $\lim\limits_{z\to\infty} v(z,z) - g(z,z) = 0$. □

Then the fourth important property of v that is required is the following one.

Proposition 3.25 *$v \leq g$ on $\{(x,z);\ 0 \leq x \leq z\}$ and $v < g$ on the contiuation region $\{(x,z);\ 0 < x \leq z$ and $z < \gamma(x)\}$.*

Proof.

On A_1:

Here for $\bar{z} \leq z < \Gamma^0$ and $0 \leq x \leq \phi_\downarrow(z)$:

$$\begin{aligned} v(x,z) - g(x,z) &= \frac{z^2}{2} + g_x(\phi_\downarrow(z), z) \frac{S(x)}{S'(\phi_\downarrow(z))} - g(x,z) \\ v_x(x,z) - g_x(x,z) &= g_x(\phi_\downarrow(z), z) \frac{S'(x)}{S'(\phi_\downarrow(z))} - g_x(x,z) \\ &= S'(x) \int_x^{\phi_\downarrow(z)} \frac{Lg(u,z)}{S'(u)} du \end{aligned}$$

For $\bar{z} \leq z < \Gamma^0$, $(0, z) \in \Gamma^-$ while $(\phi_\downarrow(z), z) \in \Gamma^+$, so we can a priori have three behaviors for $v(., z) - g(., z)$:
- it is increasing on $[0, \phi_\downarrow(z)]$
- or it is decreasing on $[0, \phi_\downarrow(z)]$
- or it is decreasing on $[0, \delta)$ and increasing on $(\delta, \phi_\downarrow(z)]$, for a certain $\delta \in (0, \phi_\downarrow(z))$.

But we know that $v(0, z) = g(0, z)$ and $v(\phi_\downarrow(z), z) = g(\phi_\downarrow(z), z)$, so $v \leq g$ in the area and in fact only the last behavior can occur and $v < g$ except if $x = \phi_\downarrow(z)$ or $x = 0$.

On A_2:

Here for $x \geq \phi_\uparrow(z)$:

$$v(x, z) - g(x, z) = g(\phi_\uparrow(z), z) + g_x(\phi_\uparrow(z), z)\frac{S(x) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} - g(x, z)$$

So similarly:

$$v_x(x, z) - g_x(x, z) = -S'(x)\int_{\phi_\uparrow(z)}^{x} \frac{Lg(u, z)}{S'(u)}du$$

Here again only three behaviors are a priori possible, for $(v-g)(., z)$, increasing on $[\phi_\uparrow(z), z]$, decreasing on $[\phi_\uparrow(z), z]$ or decreasing on $[\phi_\uparrow(z), \delta)$ and increasing on $(\zeta, z]$. As $v(\phi_\uparrow(z), z) = g(\phi_\uparrow(z), z)$, we need only to prove that $v(z, z) \leq g(z, z)$.

We write $n(z) = v(z, z) - g(z, z)$. Now as $v_z(z, z) = g_z(z, z) = 0$:

$$\begin{aligned}\frac{\partial}{\partial z}(v(z, z) - g(z, z)) = n'(z) &= v_x(z, z) - g_x(z, z)\\ &= -S'(z)\int_{\phi_\uparrow(z)}^{z} \frac{Lg(u, z)}{S'(u)}du\end{aligned}$$

We find the same expression as before, with $x = z$. So if $n'(z) < 0$, we have $\displaystyle\int_{\phi_\uparrow(z)}^{z} \frac{Lg(u, z)}{S'(u)}du >$ 0 which implies that for any $x \in (\phi_\uparrow(z), z]$, $\displaystyle\int_{\phi_\uparrow(z)}^{x} \frac{Lg(u, z)}{S'(u)}du > 0$. Therefore $(v - g)(., z)$ is decreasing on $[\phi_\uparrow(z), z]$, and as $(v - g)(\phi_\uparrow(z), z) = 0$, we get $n(z) \leq 0$.

Now assume that there exists z such that $n(z) > 0$. In regard of what we have just seen, it means that $n'(z) \geq 0$. Therefore as n is continuous it implies that n is nondecreasing on $[z, \Gamma^\infty)$. If $\Gamma^\infty < \infty$ this is impossible as $v(\Gamma^\infty, \Gamma^\infty) = g(\Gamma^\infty, \Gamma^\infty)$, and if $\Gamma^\infty = \infty$, Proposition 3.24 gives $\lim_{z\to\infty} n(z) = 0$, so again this is impossible. Finally, $n(z) \leq 0$ and therefore $v \leq g$ on A_2 and $v < g$ except if $x = \phi_\uparrow(z)$.

On A_3:

There we have for $x \le z \le \bar{z}$:

$$v(x,z) - g(x,z) = \frac{z^2}{2} - KS(x) + \frac{(z-x)^2}{2} + xS(x)\int_z^{+\infty}\frac{du}{S(u)}$$
$$\text{so } v_z(x,z) - g_z(x,z) = x(1 - \frac{S(x)}{S(z)}) > 0$$

Now as v is C^0, the result for areas 1 and 2 tells us $v(.,\bar{z}) \le g(.,\bar{z})$ so $v \le g$ and it is strict if $x \ne 0$. $\square$

Finally, v and γ satisfy the assumptions of Theorem 3.7, therefore $v = V$ and

$$\theta^* := \inf\{t \ge 0; Z_t \ge \gamma(X_t)\}$$

is an optimal stopping time. Moreover, if τ is another optimal stopping time, then $\theta^* \le \tau$ a.s.

In the next section, we will provide a few examples.

3.7 Examples

3.7.1 Brownian motion

In this case, $\alpha(x) = 0$ and $S(x) = x$. As (3.12) will never be satisfied for a nondecreasing and convex function ℓ, proposition (3.2) tells us that V and g will be infinite if ℓ satisfies (3.14). But moreover we have the following result.

Proposition 3.26 *For any $0 < x \le z$ and any convex and nondecreasing ℓ, we have:*
(i) $\mathbb{E}_{x,z}T_0 = +\infty$
(ii) $\mathbb{E}_{x,z}Z_{T_0} = \mathbb{E}_{x,z}(Z_{T_0})^2 = +\infty$
(iii) *V and g are infinite everywhere except for $x = 0$.*

Proof. (i) We can compute it directly using the law of T_0:

$$\mathbb{P}_x[\theta \ge t] = \mathbb{P}_x[\inf_{[0,t]} X_u > 0] = \mathbb{P}_0[X_t^* > x] = 2\mathbb{P}_0[X_t > x].$$

So that $\mathbb{E}_{x,z}\theta = \displaystyle\int_0^\infty \frac{xte^{-\frac{x^2}{2t\sigma^2}}dt}{\sqrt{2\pi}t^{\frac{3}{2}}\sigma} = +\infty$ as $\displaystyle\frac{2e^{-\frac{x^2}{2t\sigma^2}}}{\sqrt{2\pi t}\sigma} \sim \sqrt{\frac{2}{\pi t\sigma^2}}$ when $t \to +\infty$.

(ii) We have of course $\mathbb{E}_{x,z}Z_{T_0} = +\infty \Rightarrow \mathbb{E}_{x,z}(Z_{T_0})^2 = +\infty$. As we saw in the general case, we have the law of Z_{T_0}, with $S(x) = x$ here. So:

$$\begin{aligned}\mathbb{E}_{x,z}Z_{T_0} &= (1 - \frac{S(x)}{S(z)})z + \int_z^\infty y\frac{S(x)S'(y)}{S^2(y)}dy \\ &= (z-x) + \int_z^\infty \frac{x}{y}dy = +\infty.\end{aligned}$$

(iii) as ℓ is nondecreasing and convex, $\exists C > 0, D \in \mathbb{R}, \ell(x) \geq Cx + D$. Therefore (ii) implies that g is infinite everywhere except for $x = 0$. Now as in the proof of Proposition (3.2), it implies that V is infinite everywhere (except if $x = 0$). $\square$

Here we see a huge difference between our framework and the framework used in both [Shiryaev] and [Peskir and du Toit] where everything was done for a standard Brownian motion and a Brownian motion with drift respectively.

3.7.2 Brownian motion with negative drift

Now we consider the following diffusion, for $\mu < 0$ and $\sigma > 0$ constant:

$$dX_t = \mu dt + \sigma dW_t.$$

Therefore $\alpha(x) = -\dfrac{2\mu}{\sigma^2} = \alpha > 0$, $S(x) = \dfrac{e^{\alpha x} - 1}{\alpha}$, $S'(x) = e^{\alpha x}$.
We have an interesting homogeneity result for this process, as long as ℓ is a power function, which allows us to assume that $\alpha(x) = 1$.

Proposition 3.27 *For a Brownian motion with negative drift, and $\ell(x) = x^p$ $(p \geq 1)$, if $\alpha > 0$, then:*

$$\gamma_\alpha\left(\frac{z}{\alpha}\right) = \frac{\gamma_1(z)}{\alpha}.$$

Proof. Let X be a brownian motion with parameter $\alpha_X = \alpha$, and define $\bar{X} = \alpha X$. The dynamics of $\bar{X}$ is

$$d\bar{X}_t = \alpha dX_t = \alpha\mu dt + \alpha\sigma dW_t.$$

So that $\alpha_{\bar{X}} = \frac{-2\mu\alpha}{\sigma^2\alpha^2} = 1$. We write $\bar{Z}$ the "maximum" process associated with $\bar{X}$, starting from αz. Then $\bar{Z} = \alpha Z$, $T_0(X) = T_0(\bar{X}) = T_0$ and for any θ:

$$\mathbb{E}_{kx,kz}(\bar{Z}_{T_0} - \bar{X}_\theta)^p = \alpha^p \mathbb{E}_{x,z}(Z_{T_0} - X_\theta)^p.$$

This equality implies that if τ is optimal for one problem, it is also optimal for the other one. Together with the minimality of θ^*, it means that:

$$\begin{aligned} Z_t = \gamma_\alpha(X_t) &\Leftrightarrow \bar{Z}_t = \gamma_1(\bar{X}_t) \\ &\Leftrightarrow \alpha Z_t = \gamma_1(kX_t) \end{aligned}$$

And so we get the result. $\square$

In the quadratic case $\ell(x) = \dfrac{x^2}{2}$, we have $Lg(x,z) = 1 + \alpha(z-x) + (1+e^{\alpha x})\ln(1-e^{-\alpha z})$. We can see that $\frac{\partial}{\partial x}Lg < 0$, so that Γ is increasing (ie $\zeta = 0$). Moreover, for any $x > 0$, $\ln(1-x) < -x$, so that for $z > 0$, $Lg(z,z) < -e^{-\alpha z} < 0$, so that $\Gamma^\infty = +\infty$.

The plot below is a numerical computation of γ for $\ell(x) = \frac{x^2}{2}$. As Γ is increasing, γ is necessarily increasing too ($\gamma_\downarrow$ is degenerate).

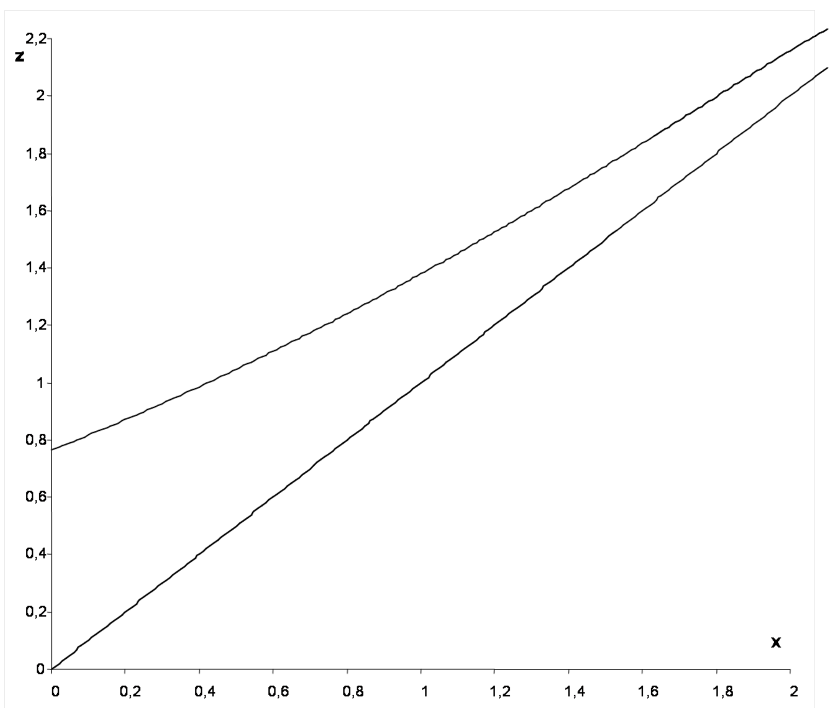


Figure 3.4: γ for a Brownian motion with negative drift and $\ell(x) = \frac{x^2}{2}$

3.7.3 CIR

Let $b \geq 0$, $\mu < 0$ and $\sigma > 0$, then the dynamics of X is:

$$dX_t = \mu X_t dt + \sigma \sqrt{b + X_t} dW_t.$$

Here, $\alpha(x) = \alpha \frac{x}{x+b}$ with $\alpha > 0$. If $b = 0$, then it is exactly the same as for the Brownian with negative drift. But if $b > 0$, then it is slightly different. In particular, for $\ell(x) = \frac{x^2}{2}$, proceeding as in the proof of Proposition 3.10-(iii), we can see that $\Gamma^\infty < \infty$, unlike in the case $b = 0$.
Moreover, as $x \to 0$, $\alpha(x) \sim \frac{\alpha x}{b}$, $\alpha'(x) \sim \frac{\alpha}{b}$, so that we can see that for any $z > 0$, $\frac{\partial}{\partial x} Lg > 0$ for x small enough, which means that $\Gamma_\downarrow$ is not degenerate, or equivalently that $\zeta > 0$.

3.7.4 Ornstein-Uhlenbeck process

Finally we end this examples section with the most classical mean-reverting process: the Ornstein-Uhlenbeck process. The dynamics of X is given by:

$$dX_t = \mu X_t dt + \sigma dW_t.$$

This case and the Brownian motion with negative drift case can be seen as the extreme cases of our framework. Indeed here $\alpha(x) = \alpha x$ is the "most increasing" concave function, while $\alpha(x) = \alpha$ is the "least nondecreasing" function.
As for the Brownian motion with negative drift, we have an homogeneity result for this process, as long as ℓ is a power function, which allows us to assume that $\alpha(x) = x$.

Proposition 3.28 *For OU processes, and $\ell(x) = x^p$, let $\alpha > 0$, then*

$$\gamma_\alpha \left(\frac{z}{\sqrt{\alpha}} \right) = \frac{\gamma_1(z)}{\sqrt{\alpha}}$$

Proof. We follow the proof in the case of a brownian motion with negative drift. Let X be an OU of parameter $\alpha_X = \alpha$, and define $\bar{X} = kX$ where $k = \sqrt{\alpha}$. Then $\alpha_{\bar{X}} = \frac{\alpha}{k^2} = 1$. We write $\bar{Z}$ the "maximum" process associated with $\bar{X}$. Then $\bar{Z} = kZ$, $T_0(X) = T_0(\bar{X}) = T_0$ and for any θ:

$$\mathbb{E}_{kx,kz}(\bar{Z}_{T_0} - \bar{X}_\theta)^p = k^p \mathbb{E}_{x,z}(Z_{T_0} - X_\theta)^p$$

So that using the minimality of θ^* we have:

$$\begin{aligned} X_t = \gamma_\alpha(Z_t) &\Leftrightarrow \bar{X}_t = \gamma_{\frac{\alpha}{k^2}}(\bar{Z}_t) \\ &\Leftrightarrow kX_t = \gamma_{\frac{\alpha}{k^2}}(kZ_t) \end{aligned}$$

And so we get the result. $\square$

Then, focusing on the case $\ell(x) = \frac{x^2}{2}$, we show that Γ is decreasing in a neighborhood of 0 (so that $\zeta > 0$), and that $\Gamma^\infty < +\infty$.

Proposition 3.29 *For an OU process, $Lg(x, \Gamma^0) > 0$ for $x > 0$ in a neighborhood of 0 (so that $\Gamma_\downarrow$ is not degenerate) and $Lg(z, z) > 0$ in a neighborhood of $+\infty$ (so that $\Gamma^\infty < +\infty$).*

Proof. If x is small, we have the $S(x) \sim x$, $S'(x) = 1 + S''(0)x + o(x) = 1 + o(x)$ and by definition of Γ^0, $\displaystyle\int_{\Gamma^0}^\infty \frac{du}{S(u)} = \frac{1}{2}$. So as $x \to 0$, we can write:

$$Lg(x, \Gamma^0) = 1 + \alpha x \Gamma^0 - 1 + o(x)$$

So as $\alpha > 0$, $Lg(x, \Gamma^0) > 0$ for $x > 0$ and small enough.

Then as $\alpha(x) \to \infty$ as $x \to \infty$, Proposition 3.10-(iii) implies that $\Gamma^\infty < \infty$. $\square$

Finally, the plot below is a numerical computation of the boundary γ for $\ell(x) = \dfrac{x^2}{2}$. While we do not prove it, we can see that γ is in this case decreasing first and then increasing.

3.8 Extension to general loss functions

The previous analysis only considered the case of the quadratic loss function $\ell(x) = \dfrac{x^2}{2}$. Of course all these results can be generalized for any quadratic function $\ell(x) = \lambda x^2$. In fact, as the reader has probably noticed, the quadratic loss function plays a special role here, as we then have $\ell^{(3)}(x) = 0$, which simplifies a lot the study of the set Γ^+, as well as the asymptotic behavior of Lg. Some crucial properties that we estabished in the quadratic case seem very hard to derive in the general case stated in the first 2 sections.
Nevertheless, if we make a few additional assumptions, our results still hold true in a more general framework. We explain here how to do so. Recall the assumptions (3.26)-(3.28).

Let us compute:

$$\begin{aligned} Lg(x,z) =& \ell''(z-x) + \alpha(x)\ell'(z-x) - (2S'(x) - \alpha(x)S(x)) \int_z^\infty \frac{\ell''(u-x)}{S(u)} du \\ &+ S(x) \int_z^\infty \frac{\ell^{(3)}(u-x)}{S(u)} du \end{aligned}$$

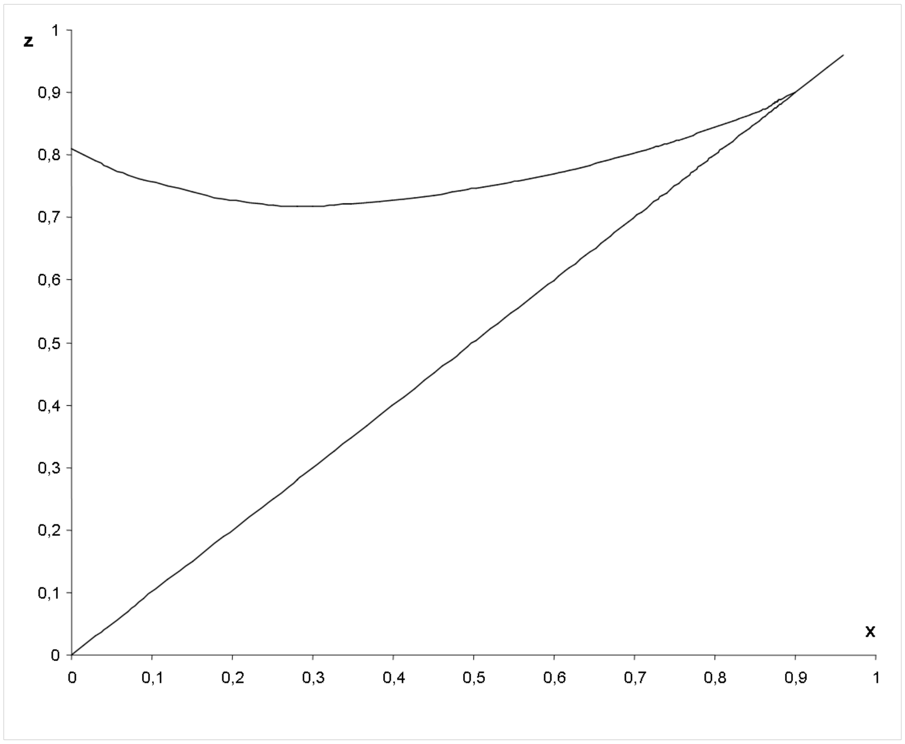


Figure 3.5: γ for an OU process with $\ell(x) = \frac{x^2}{2}$

As $\ell''(x) > 0$ for $x > 0$ and $\ell^{(3)} \geq 0$, for any $x \geq 0$, $z \mapsto \dfrac{\partial}{\partial z} Lg(x,z)$ is increasing. Moreover, we have $\ell'(x) \to \infty$ as $x \to \infty$, so that for any $x \geq 0$, $\lim\limits_{z\to\infty} Lg(x,z) > 0$. As a consequence, $\Gamma^+ \neq \emptyset$, the definition of Γ (3.30) can be extended, as well as the following remarks.

But Lg is no longer concave with respect to x, and it seems hard to show that Γ is U-shaped. In fact Proposition 3.10 (i) and (iii) are crucial but are very hard to prove in general. Therefore we make the following additional assumptions:

$$- \ \exists \zeta \geq 0, \text{ such that } \Gamma \text{ is decreasing on } [0,\zeta] \text{ and increasing on } [\zeta,+\infty) \quad (3.38)$$

$$- \text{ if } \lim_{x\to\infty} \alpha(x) = \infty, \text{ then } \Gamma^\infty < \infty. \quad (3.39)$$

The other statements in Proposition 3.10 are not needed.

Lemma 3.12 is not true in general, but this is not important. It just means that we have a new possibility for the shape of γ: $\gamma_\uparrow(x) = x$ for every $x \geq \bar{x}$, as we will explain later.

In order to determine the increasing part of γ, ODE (3.32) is replaced by:

$$\gamma' = \frac{Lg(x,\gamma)}{\ell''(\gamma - x)\left(1 - \frac{S(x)}{S(\gamma)}\right)} \tag{3.40}$$

As $\ell'' > 0$, the Cauchy problem is well defined for any $x_i > 0$ and $\gamma(x_i) > x_i$, and the maximal solution is defined as long as $\gamma(x) > x$.
Lemma 3.16 still holds true. The only point to be checked is (iv).
If $\varepsilon > 0$, from Proposition 3.9 (i), there is a bounded function δ such that, as $x \to \infty$, we get:

$$Lg(x,(1+\varepsilon)x) \geq \ell''(\varepsilon x)\left[1 - \delta((1+\varepsilon)x)\frac{S'(x)}{S'((1+\varepsilon)x)}\right] + \alpha(x)\ell'(\varepsilon x) + o(1)$$

Now $\frac{S'(x)}{S'((1+\varepsilon)x)} \to 0$ as $x \to \infty$, while $\alpha(x)\ell'(\varepsilon x) \to \infty$, so that $Lg(x,(1+\varepsilon)x) \geq 1 + 2\varepsilon$ for x large enough, and the proof can be achieved as before.

Then Lemma 3.17 still hold. Finally we make a last assumption:

$$\text{either } \alpha(x) \to \infty \text{ as } x \to \infty, \text{ or in Proposition 3.9 (ii), for any } a > 0, \text{ and } \varphi(z) = z - a, \ \delta \equiv 1 \tag{3.41}$$

Thanks to this, the proof of Proposition 3.14 is still valid. Indeed for $a > 0$, we compute:

$$\begin{aligned} Lg(x, x+a) =& \ell''(a) + \alpha(x)\ell'(a) - S'(x)\int_{x+a}^{\infty}\frac{\ell''(u-x)}{S(u)}du \\ &+ S(x)\int_{x+a}^{\infty}\frac{\ell^{(3)}(u-x)}{S(u)}du + o(1) \\ &\geq \ell''(a) + \alpha(x)\ell'(a) - \eta\ell''(a)\frac{S'(x)}{S'(x-a)} + o(1) \end{aligned}$$

If $\alpha(x) \to \infty$ as $x \to \infty$, then the previous expression goes towards infinity as well. If $\alpha \to M$ with $M > 0$, then our assumption guarantees that $\eta = 1$, so that:

$$\begin{aligned} \frac{Lg(x,x+a)}{\ell''(a)\left(1 - \frac{S(x)}{S(x+a)}\right)} &= \frac{\ell''(a)(1-e^{-aM}) + M\ell'(a)}{\ell''(a)(1-e^{-aM})} + o(1) \\ &= 1 + \frac{M\ell'(a)}{\ell''(a)(1-e^{-aM})} + o(1) \end{aligned}$$

As $\frac{M\ell'(a)}{\ell''(a)(1-e^{-aM})} > 0$, as in the proof of Proposition 3.14, we have in both cases α bounded or not, for $\varepsilon > 0$ small enough, we have as $x \to \infty$:

$$\frac{Lg(x,x+a)}{\ell''(a)\left(1 - \frac{S(x)}{S(x+a)}\right)} > 1 + \varepsilon.$$

And then we can conclude as in the proof of the proposition.

Although we do not need it, this also implies Proposition 3.10 (ii).

Then we look at the decreasing part of γ. There equation (3.33) is replaced by:

$$g(x(z), z) - g_x(x(z), z)\frac{S(x(z))}{S'(x(z))} - \ell(z) = 0 \tag{3.42}$$

Now in the proof of Proposition 3.19, the new ODE for the Cauchy problem is:

$$\gamma'(x) = \frac{Lg(x,\gamma)S(x)}{(\ell'(\gamma - x) - \ell'(\gamma))S'(x) + \ell''(\gamma - x)S(x)\left(1 - \frac{S(x)}{S(\gamma)}\right)}$$

Now for any x and γ, there exists $y \in (\gamma - x, \gamma)$ such that:

$$\begin{aligned}(\ell'(\gamma - x) - \ell'(\gamma))S'(x) + \ell''(\gamma - x)S(x)\left(1 - \frac{S(x)}{S(\gamma)}\right) &= -x\ell''(y)S'(x) + \ell''(\gamma - x)S(x)\left(1 - \frac{S(x)}{S(\gamma)}\right) \\ &\leq -x\ell''(y) + \ell''(\gamma - x)S(x) \\ &\leq \ell''(\gamma - x)(S(x) - xS'(x))\end{aligned}$$

The last inequality coming from the fact that $\ell^{(3)} \geq 0$. So as $\ell''(x) > 0$ for $x > 0$, we can proceed as in the proof of Proposition 3.19. And finally Proposition 3.21 is replaced by the following:

Proposition 3.30 *We have of the following cases:*
- $\gamma_\uparrow$ *is increasing on* $[0, +\infty)$, *and this implies* $\Gamma^0 < \Gamma^\infty$*;*
- $\gamma_\downarrow(x^*) = x^*$ *and* $x^* \geq \Gamma^\infty$, *which implies* $\Gamma^0 > \Gamma^\infty$*;*
- $|graph(\gamma_\downarrow) \cap graph(\gamma_\uparrow)| = 1$.

In the first case we write $\bar{x} = 0$ *and* $\bar{z} = \gamma_\uparrow(0)$. *In the second case we write* $\bar{x} = \bar{z} = x^*$. *In the third case, we write* $(\bar{x}, \bar{z}) = graph(\gamma_\downarrow) \cap graph(\gamma_\uparrow)$. *In all three cases we have* $(\bar{x}, \bar{z}) \in \Gamma^+$.

Remark 3.31 In the second case of the previous proposition, the condition $x^* \geq \Gamma^\infty$ is not a priori a consequence of $\gamma_\downarrow(x^*) = x^*$, as there is no reason in general for the set $Int\Gamma^-$ to be connected.

Finally, all the results of section 5 can be proved in the same way in general, using the asymptotic expansions of Proposition 3.9.

We define v by:

$$v(x,z) = \ell(z) + g_x(\phi_1(z),z)\frac{S(x)}{S'(\phi_1(z))} \text{ if } (x,z) \in A_1 \tag{3.43}$$

$$v(x,z) = g(\phi_2(z),z) + g_x(\phi_2(z),z)\frac{S(x) - S(\phi_2(z))}{S'(\phi_2(z))} \text{ if } (x,z) \in A_2 \tag{3.44}$$

$$v(x,z) = \ell(z) + S(x)[\int_z^\infty \frac{\ell'(u)}{S(u)}du - K] \text{ if } (x,z) \in A_3 \tag{3.45}$$

$$v(x,z) = g(x,z) \text{ if } (x,z) \in A_4, \tag{3.46}$$

where $K = \displaystyle\int_{\bar{z}}^\infty \frac{\ell'(u)}{S(u)}du - \frac{g_x(\bar{x},\bar{z})}{S'(\bar{x})}$.

The proof of Lemmas 3.22 and 3.23 still work in this case, with the new definition of v, and the new equations for γ. In the proof of Proposition 3.24, the assumptions made allow us to use the asymptotic expansions of Proposition 3.9 in order to get $g_x(\phi_\uparrow(z),z) = O(1)$, $v(z,z) - g(\phi_\uparrow(z),z) = o(1)$. and $v(z,z) - g(\phi_\uparrow(z),z) = o(1)$.

But in both cases we get as in th quadratic case that $v(z,z) - g(\phi_\uparrow(z),z) = o(1)$, and the result. Finally, the proof of Proposition 3.25 still holds, so that we can apply Theorem 3.7.

Appendix: application to a hedging strategy

We have applied this result with the following strategy. Assume that X is an OU process with parameter α. We compute γ for $\ell(x) = \frac{x^2}{2}$. Assume at $t = 0$, $X_0 > 0$, then the first time $t \in [0, T_0]$ such that $X_t \geq \gamma(Z_t)$, we sell 1 stock of X. At $t = T_0$, we close the position. Then we reinitialize everything and do the same with the minimum (of course we buy instead of sell in this case).
We compare it to a family of strategies that we call "fixed barriers". We fix an a priori barrier level $b > 0$, and if $X > 0$, we sell 1 stock the first time $X_t \geq b$, then close the position at $t = T_0$, and do the symmetric if $X_t \leq -b$. We have tested those strategies in two cases. First a theoretical example, where we simulate the OU process X and use the "right" parameter α, then a market data example, where we took a process X computed from market data, assumed it behaved as an OU process and tried to estimate the parameter. More precisely in this second case, we took two stocks A and B, and computed:

$$X = \frac{\frac{A}{B}}{MA(\frac{A}{B})} - 1$$

where $MA(Y)$ is the moving average of Y (on a 3-month period).

We present hereafter the annualized Sharpe ratios obtained. What we call "a posteriori best barrier" is the best result we obtained with a fixed barrier b while b described $\mathbb{R}_+$, so there is no way to know how to fix it. In fact, in every simulation that we made, this "a posteriori best barrier" b_0 was close to Γ^0, which is not very surprising as the plot of γ for an OU process is quite flat. We emphasize on the fact that for a random barrier b, the Sharpe ratio is most of the time very small and can even be 0.

Data	Detection method	Sharpe ratio
Theoretical data	optimal stopping	2,1
	a posteriori best barrier	2
Market data	optimal stopping	1,8
	a posteriori best barrier	1,6

Appendix: some comments on the computation of γ

The numerical computations of the boundaries γ were made using Visual C++. We made them only for a quadratic loss function $\ell(x) = \dfrac{x^2}{2}$, but the method used could be generalized to other functions. In order to compute γ, it is necessary to solve ODE (3.32) and equation (3.33), and therefore to compute S, S', and the following integral:

$$\int_z^\infty \frac{du}{S'(u)}.$$

For a Brownian motion with drift both the expressions of S and S' are "classical" functions, for an OU process, S is only defined as the integral of "classical". For any integral of the form:

$$I(z) = \int_0^z f(u)du,$$

where f is non-decreasing, we derived an easily computable equivalent $J(z)$ as $z \to \infty$ together with an estimation of the error in order to determine a constant $M > 0$ such that the approximation:

$$I(z) \approx J(z)$$

was "satisfying" for $z \geq M$. Then for $z \leq M$, we bounded from below and above the integral using series by means of the classical relation:

$$\sum_{i=0}^{n-1} f(i\delta) + (z - n\delta)f(n\delta) \leq I(z) \leq \sum_{i=1}^{n} f(i\delta) + (z - n\delta)f(z),$$

with $\delta > 0$ was chosen small enough so that the difference between the previous right and left hand sides was small enough, and $n := E(\frac{z}{\delta})$.

For an integral of the form:

$$I(z) = \int_z^{\infty} f(u)du,$$

where f is non-increasing, we derive again an equivalent J and a constant $M > 0$ such that for z large enough we can make the approximation $I(z) \approx J(z)$, then we approximate:

$$\int_z^M f(u)du$$

using series as before (but inverting the left and right hand sides as f is non-increasing this time).

Then, for a given initial condition, ODE (3.32) was solved using a finite differences scheme, and then the value of the "good" initial condition approximately found by dychotomy.

Chapter 4

Optimal investment under relative performance concerns

4.1 Introduction

Since the seminal papers of Merton [58, 59], the problem of optimal investment has been extensively studied in order to generalize the original framework. Those generalizations have been made in different directions and using different techniques. We refer to [67], [10] or [46] for the complete market situation, to [12] or [79] for constrained portfolios, to [9], [15], [75], [17] or [2] for transactions costs, to [8], [44, 45], [13], [3, 4] or the first chapter for taxes, and to [43], [53, 54] or [55] for general incomplete markets.

But in all those works, no interaction between agents is taken into account, whereas economical and sociological studies have emphasized the importance of relative concerns in human behaviors, see Veblen [78], Abel [1], Gali [34], Gomez, Priestley and Zapatero [36] or DeMarzo, Kaniel and Kremer [16]. Indeed, a performance makes sense in a specific context and in comparison to competitors or a benchmark. A return of 5% during a crisis is not equivalent to the same return during a financial bubble. Moreover, human beings tend to compare themselves to their peers.

In this chapter, we study the optimal investment problem when including relative performance concerns. More precisely, there are N specific investors that compare themselves to each other. Instead of considering only his absolute wealth, each agent takes into account a convex combination of his wealth (with weight $1-\lambda$, $\lambda \in [0,1]$) and the difference between his wealth and the average wealth of the other investors (with weight λ). This creates interactions between agents and therefore can be seen as a differential game with N players.

We also consider that each agent's portfolio must stay in a set of constraints.

In an exponential utility framework, assuming mainly that those sets of constraints are vector spaces and that the drift and volatility of the risky assets are deterministic, we show the existence of a Nash equilibrium as well as a characterization of any Nash equilibrium, using BSDE techniques inspired by El Karoui and Rouge [27] or Hu, Imkeller and Müller [42]. Then we study the limit when the number of players N goes to infinity. In the spirit of mean field games, see Lasry and Lions [56], the situation becomes simpler at the limit.

Finally, we study the influence of the parameter λ, which represents the weight of relative concerns. We show that under some additional assumptions which are satisfied in many examples, the local volatility of the wealth of each agent is nondecreasing with respect to λ. In other words, the more investors look at each other (λ big), the more risky is the portfolio of each investor. However in some cases, this can fail to hold. But in the limit N goes to infinity, the same phenomenon holds for the average portfolio of the market. Roughly speaking, this means that, in general, the global risk of the market increases with λ, although it can fail for the portfolio of some specific agents.

4.2 Problem formulation

Let W be a d-dimensional Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the corresponding completed canonical filtration. Let $T > 0$ be the horizon of investment, so that $t \in [0, T]$. Given a continuous function θ defined on $[0, T]$ and taking values in $\mathbb{R}^d$, together with an $\mathbb{F}$-predictable process σ taking values in $M_d(\mathbb{R})$ and satisfying:

$$\int_0^T |\sigma_t|^2 dt < +\infty \text{ a.s,}$$

we consider a market with a non risky asset with interest rate $r = 0$ and a d-dimensional risky asset $S = (S^1, ..., S^d)$ given by the following dynamics:

$$dS_t = \operatorname{diag}(S_t)\sigma_t(\theta(t)dt + dW_t), \tag{4.1}$$

where for a vector $X \in \mathbb{R}^d$, $\operatorname{diag}(X)$ is the diagonal matrix with i-th diagonal term equal to X^i.

We assume that σ is invertible and, as we can always reduce to this case, we also assume that σ is symmetric definite positive.

A portfolio is an $\mathbb{F}$-predictable process $\{\pi_t,\ t \in [0,T]\}$ taking values in $\mathbb{R}^d$. Here π_t^i is the amount invested in the i-th risky asset at time t. The associated wealth X_t^π follows the following dynamics:

$$dX_t^\pi = \sum_{j=1}^d \pi_t^j \frac{dS_t^j}{S_t^j}.$$

Let $N \in \mathbb{N}$ be given, we consider a set of N agents that will interact with each other. We will always assume that $N \geq 2$. We assume that each agent is "small" in the sense that his actions do not impact the market prices S. For each $1 \leq i \leq N$, agent i wants to maximize his expected utility taking into account two criteria: his absolute wealth on the one hand, and his wealth compared to the average wealth of other agents. More precisely, we assume that each agent has a utility function $U_i : \mathbb{R} \to \mathbb{R}$, which we will assume to be C^1, increasing, strictly concave and satisfying Inada conditions:

$$U_i'(-\infty) = +\infty,\ U_i'(+\infty) = 0. \tag{4.2}$$

We also assume that agent i has a relative wealth sensitivity $\lambda_i \in [0,1]$. We write:

$$\bar{X}_t^{i,\pi} = \frac{1}{N-1} \sum_{j \neq i} X_t^{\pi^j}$$

the average wealth of agents other than i, where the π^j are given. We will assume that the portfolio of agent i is subject to constraints. More precisely, there is A_i, subset of $\mathbb{R}^d$ such that for all $t \in [0,T]$, $\pi_t^i \in A_i$.
Given π^j for $j \neq i$, then his optimization problem is:

$$\begin{aligned} V_i = V_i(\pi^j, j \neq i) &= \sup_{\pi^i \in \mathcal{A}_i} \mathbb{E} U_i \left((1-\lambda_i) X_T^{\pi^i} + \lambda_i (X_T^{\pi^i} - \bar{X}_T^{i,\pi}) \right) && (4.3) \\ &= \sup_{\pi^i \in \mathcal{A}_i} \mathbb{E} U_i \left(X_T^{\pi^i} - \lambda_i \bar{X}_T^{i,\pi} \right) && (4.4) \end{aligned}$$

where the set of admissible portfolios $\mathcal{A}_i$ is the set of predictable processes π and will be defined later. Roughly speaking, we impose integrability conditions as well as the constraints $\pi^i \in A_i$.
Our main objective is to find if the agents can simultaneously solve their optimization problems, and then see what happens when the number of agents N goes to infinity. We therefore introduce the definition of Nash equilibrium in our framework:

Definition 4.1 *(Nash equilibrium) A Nash equilibrium for our problem is an N-uple $(\hat{\pi}^1, ..., \hat{\pi}^N)$ of elements of $\mathcal{A}_1 \times ... \times \mathcal{A}_N$ such that, for each $1 \leq i \leq N$:*

$$V_i(\hat{\pi}^j, j \neq i) = \mathbb{E} U_i (X_T^{\hat{\pi}^i} - \lambda_i \bar{X}_T^{i, \hat{\pi}^j, j \neq i}). \tag{4.5}$$

In order to simplify notations, from now on, we will write X_t^i and $\bar{X}_t^i$.
In the first section of this paper, we shall consider the complete market situation in which the portfolios will be free of constraints (in other words, $A_i = \mathbb{R}^d$ for each i). This will be solved for general utility functions. In the next sections, we will derive results for more general types of constraints, but we will focus on the case of exponential utility functions: $U_i(x) = -e^{-\frac{x}{\eta_i}}$. We will first consider the A_i to be (vector) subspaces of $\mathbb{R}^d$, and then we shall extend our results to general closed convex sets.

4.3 The complete market situation

In this section, we consider the case:

$$A_i = \mathbb{R}^d, \text{ for all } i = 1, ..., N.$$

In contrast with the general results in the subsequent sections, the complete market situation can be solved for general utility functions. In this case, the set of admissible strategies $\mathcal{A} = \mathcal{A}_i$ is the set of predictable processes π such that:

$$\mathbb{E}\int_0^T |\sigma_t \pi_t|^2 dt < \infty. \tag{4.6}$$

Remark 4.2 The admissible set defined by (4.6) is classical, and this assumption guarantees the absence of arbitrage. Indeed it implies that, for any admissible π, X^π is a martingale under the martingale measure $\mathbb{Q}$ defined by:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\int_0^T \theta(u).dW_u - \frac{1}{2}\int_0^T |\theta(u)|^2 du}.$$

Then, starting from an initial wealth x, if an admissible strategy π guarantees $X_T^\pi \geq x$ $\mathbb{P}$-a.s, as $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, this is true $\mathbb{Q}$-a.s as well. But we have $\mathbb{E}^{\mathbb{Q}} X_T^\pi = x$, so that $X_T^\pi - x$ is a non-negative random variable with null expectation, so $X_T^\pi = x$ $\mathbb{Q}$-a.s or $\mathbb{P}$-a.s. As a consequence, $\mathbb{P}(X_t^\pi > x) = 0$, which means that there is no arbitrage.

Remark 4.3 As it is often done, we could have taken the following admissibility conditions:

$$\bullet \quad \int_0^T |\sigma_t \pi_t|^2 dt < \infty \tag{4.7}$$

$$\bullet \quad \exists\, C \in \mathbb{R},\ X_t^\pi \geq C,\ \forall t,\ \mathbb{P}\text{-a.s.} \tag{4.8}$$

This would still guarantee the absence of arbitrage. Indeed the first one implies that, for any admissible π, X^π is a local martingale under $\mathbb{Q}$, while the second one, thanks to Fatou's lemma, implies that X^π is in fact a supermartingale. And the arguments of the previous remark still apply for supermartingales. However, the admissibility conditions in the general case will be an extension of assumption (4.6) and not the other conditions.

To simplify the notations and presentation in this introductory example, we also assume that:

$$\lambda_i = \lambda \in [0, 1), \text{ for all } i = 1, ..., N. \tag{4.9}$$

In other words all agents have the same relative wealth sensitivity λ, and in order to guarantee the uniqueness of an equilibrium we need to assume that $\lambda < 1$. The case of specific relative sensitivities in the exponential utility framework will be a consequence of the general case described in the subsequent sections. We also observe that the assumption $\lambda < 1$ can be weakened in our general analysis (depending on the A_i's).
However, we allow the investors to have different utility functions U_i and different initial endowment $x^i \in \mathbb{R}$. We denote:

$$\bar{x}^i = \frac{1}{N-1} \sum_{j \neq i} x^j.$$

4.3.1 Single agent optimization

The first step is to find the optimal portfolio and wealth (if they exist) of each agent, the other agents strategies being given. In other words, we try to find the best response of agent i to the strategies of others. We will do this using classical techniques of optimal investment in complete markets.
We recall the definition of the convex dual of $-U(-x)$:

$$\tilde{U}(y) = \sup_{x \in \mathbb{R}} \{U(x) - xy\},$$

which is convex. Since U is strictly concave and C^1, we can define:

$$I = (U')^{-1}$$

which is a bijection from $\mathbb{R}_+^*$ onto $\mathbb{R}$. Then:

$$\tilde{U}(y) = U \circ I(y) - yI(y).$$

Finally, as the market is complete, we denote by $\mathbb{Q}$ the (unique) martingale measure.

Lemma 4.4 *Given the other strategies $\pi^j \in \mathcal{A}_j$ for $j \neq i$, there exists an optimal portfolio for the optimization problem* (6.4) *of agent i and the optimal wealth is given by:*

$$X_T^{i*} = I_i\left(y^i \frac{d\mathbb{Q}}{d\mathbb{P}}\right) + \lambda \bar{X}_T^i \qquad \textit{where } \bar{X}_T^i = \frac{1}{N-1} \sum_{j \neq i} X_T^j \tag{4.10}$$

and y^i is the unique solution of $\mathbb{E}^{\mathbb{Q}} I_i\left(y^i \frac{d\mathbb{Q}}{d\mathbb{P}}\right) = x^i - \lambda \bar{x}^i$

Proof. We report the proof of this standard result for completeness.

For ease of presentation, we omit the indices i for X^i, $\bar{X}^i$, x^i and $\bar{x}^i$. For $y>0$, $\pi\in\mathcal{A}$ and $X:=X^\pi$, we have a.s:

$$U_i(X_T-\lambda\bar{X}_T)-y\frac{d\mathbb{Q}}{d\mathbb{P}}(X_T-\lambda\bar{X}_T)\le\tilde{U}_i\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)=U_i\circ I_i\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)-y\frac{d\mathbb{Q}}{d\mathbb{P}}I_i\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right).$$

As $\frac{d\mathbb{Q}}{d\mathbb{P}}>0$ a.s, the previous expressions are well defined. So by taking expectations under $\mathbb{P}$ we get, for any $y>0$ and any $\pi\in\mathcal{A}$:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}U_i(X_T-\lambda\bar{X}_T)&\le\mathbb{E}^{\mathbb{P}}U_i\circ I_i\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)+y\mathbb{E}^{\mathbb{Q}}(X_T-\lambda\bar{X}_T)-y\mathbb{E}^{\mathbb{Q}}I_i\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\\&\le\mathbb{E}^{\mathbb{P}}U_i\circ I_i\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)+y\left[(x-\lambda\bar{x})-\mathbb{E}^{\mathbb{Q}}I_i\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right].\end{aligned}$$

Now as I_i is a bijection from $\mathbb{R}_+^*$ onto $\mathbb{R}$, there exists a unique $y^i>0$ such that:

$$\mathbb{E}^{\mathbb{Q}}I_i\left(y^i\frac{d\mathbb{Q}}{d\mathbb{P}}\right)=x-\lambda\bar{x}. \tag{4.11}$$

Indeed, $f:y\mapsto\mathbb{E}^{\mathbb{Q}}I_i(y\frac{d\mathbb{Q}}{d\mathbb{P}})$ is decreasing, and $\lim_{y\to0}f(y)=+\infty$ while $\lim_{y\to+\infty}f(y)=-\infty$.

And so we have for any portfolio π:

$$U_i(X_T-\lambda\bar{X}_T)\le\mathbb{E}^{\mathbb{P}}U_i\circ I_i\left(y^i\frac{d\mathbb{Q}}{d\mathbb{P}}\right).$$

So that $V(x^i)\le\mathbb{E}^{\mathbb{P}}U_i\circ I_i(y^i\frac{d\mathbb{Q}}{d\mathbb{P}})$. But now as the market is complete, it follows from (4.11) that there exists π^* such that a.s $X_T^{\pi^*}=I_i(y^i\frac{d\mathbb{Q}}{d\mathbb{P}})+\lambda\bar{X}_T$. Thus π^* is an optimal portfolio and the associated wealth satisfies $X_T^*=I_i(y^i\frac{d\mathbb{Q}}{d\mathbb{P}})+\lambda\bar{X}_T^i$. □

4.3.2 Partial Nash equilibrium

The second step is now to find if there is a Nash equilibrium between the N agents. Let $X_N=(X_T^i)_{1\le i\le N}$ be the vector of optimal final wealth of the investors, from the previous lemma we have:

$$A_NX_N=J_N,\quad\text{where } A_N=\begin{pmatrix}1 & & -\frac{\lambda}{N-1}\\ & \ddots & \\ -\frac{\lambda}{N-1} & & 1\end{pmatrix}\in M_N(\mathbb{R});\ J_N\ =\left(I_i\left(y^i\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right)_{1\le i\le N}.$$

Since $\lambda\neq1$ by (4.9), it follows that A_N is invertible and we can compute explicitly that:

$$A_N^{-1}=\begin{pmatrix}1+\frac{\lambda^2}{(1-\lambda)(N+\lambda-1)} & & \frac{\lambda}{(1-\lambda)(N+\lambda-1)}\\ & \ddots & \\ \frac{\lambda}{(1-\lambda)(N+\lambda-1)} & & 1+\frac{\lambda^2}{(1-\lambda)(N+\lambda-1)}\end{pmatrix},$$

thus providing a unique Nash equilibrium:

Theorem 4.5 *There exists a unique Nash equilibrium, with optimal final wealth for each* $i = 1, ..., N$ *given by:*

$$X_T^i = \left(1 + \frac{\lambda^2}{(1-\lambda)(N+\lambda-1)}\right) I_i\left(y^i \frac{d\mathbb{Q}}{d\mathbb{P}}\right) + \frac{\lambda}{(1-\lambda)(N+\lambda-1)} \sum_{j\neq i} I_j\left(y^j \frac{d\mathbb{Q}}{d\mathbb{P}}\right). \quad (4.12)$$

4.3.3 The exponential utility case: impact of λ

In order to push further the analysis of the complete market situation, we now consider the special exponential utility case:

$$U_i(x) = -e^{-\frac{x}{\eta_i}}, \ \eta_i > 0.$$

Recall that the risk premium θ is a deterministic function of t. Then $I_i(y) = -\eta_i \ln(\eta_i y)$. So that the optimal wealth process is:

$$X_T^i = a^i - \frac{\eta_i}{1-\lambda} \ln \frac{d\mathbb{Q}}{d\mathbb{P}}$$

where $a^i = x^i + \frac{\eta_i}{1-\lambda}\mathbb{E}^{\mathbb{Q}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}}$ is a constant. Thus the exposure to the market of this portfolio is entirely described by $\frac{\eta_i}{1-\lambda} \ln \frac{d\mathbb{Q}}{d\mathbb{P}}$.

In particular for $\lambda = 0$, we find the same wealth as in the classical case, which is $\eta_i \ln \frac{d\mathbb{Q}}{d\mathbb{P}}$. We can compute explicitly the optimal portfolio. We write $\hat{\pi}^{N,\lambda}$ the optimal portfolio in order to emphasize the dependence with respect to both N and λ, and we introduce the average of the η_i's:

$$\eta^N = \frac{1}{N} \sum_{j=1}^{N} \eta_j. \quad (4.13)$$

Proposition 4.6 *The optimal portfolio for agent* i *is given by:*

$$\hat{\pi}_t^{i,N,\lambda} = \frac{1}{1-\lambda}\left[\left(1 - \frac{\lambda N}{N+\lambda-1}\right) + \frac{\lambda N}{N+\lambda-1}\frac{\eta^N}{\eta_i}\right]\hat{\pi}_t^{0,i},$$

where $\hat{\pi}_t^{0,i} := \eta_i \sigma_t^{-1}\theta(t)$ *is the optimal portfolio for the classical problem* ($\lambda = 0$)*.*
Moreover as $N \to \infty$*, if* $\eta^N \to \eta > 0$ *then:*

$$\left|\left| \sup_{0\le t\le T} \hat{\pi}_t^{i,N,\lambda} - \hat{\pi}_t^{i,\infty,\lambda}\right|\right|_\infty \to 0, \quad \textit{where } \hat{\pi}_t^{i,\infty,\lambda} := \frac{1}{1-\lambda}\left[(1-\lambda) + \lambda\frac{\eta}{\eta_i}\right]\hat{\pi}_t^{0,i}.$$

Proof. We can compute: $\ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) = -\int_0^T \theta(s).dB_s + \frac{1}{2}\int_0^T |\theta(s)|^2 ds$ where B is a Brownian motion under $\mathbb{Q}$. Therefore we have (recall that θ is deterministic):

$$\begin{aligned}
I_i(y^i \frac{d\mathbb{Q}}{d\mathbb{P}}) &= x^i - \lambda \bar{x}^i + \eta_i \mathbb{E}^{\mathbb{Q}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}} - \eta_i \ln \frac{d\mathbb{Q}}{d\mathbb{P}} \\
&= x^i - \lambda \bar{x}^i + \eta_i \int_0^T \theta(s).dB_s.
\end{aligned}$$

Now using Theorem 4.5, we compute the optimal portfolio for N and λ:

$$\begin{aligned}
X_T^{N,i} &= \left(1 + \frac{\lambda^2}{(1-\lambda)(N+\lambda-1)}\right)(x^i - \lambda\bar{x}^i) + \frac{\lambda}{(1-\lambda)(N+\lambda-1)} \sum_{j\neq i}(x^j - \lambda \bar{x}^j) \\
&\quad + \left[\left(1 + \frac{\lambda^2}{(1-\lambda)(N+\lambda-1)}\right)\eta_i + \frac{\lambda}{(1-\lambda)(N+\lambda-1)}\sum_{j\neq i}\eta_j\right]\int_0^T \theta(s).dB_s \\
&= x^i + \left[\frac{N-1}{N+\lambda-1}\eta_i + \frac{\lambda N}{(1-\lambda)(N+\lambda-1)}\eta^N\right]\int_0^T \theta(s).dB_s \\
&= x^i + \left[\frac{N-1}{N+\lambda-1}\eta_i + \frac{\lambda N}{(1-\lambda)(N+\lambda-1)}\eta^N\right]\int_0^T \sigma_s^{-1}\theta(s).\mathrm{diag}(S_s)^{-1}dS_s \\
&= x^i + \int_0^T \hat{\pi}_t^{N,i}.\mathrm{diag}(S_s)^{-1}dS_s.
\end{aligned}$$

Finally the convergence is immediate. □

Corollary 4.7 *Suppose that all agents have the same risk aversion coefficient $\eta_i = \eta > 0$. Then for all i:*

$$\hat{\pi}^{i,N,\lambda} = \hat{\pi}^{\lambda} := \frac{1}{1-\lambda}\hat{\pi}_t^0.$$

Remark 4.8 If $\lambda = 1$, we can see from the previous computations, that there exist either an infinity of Nash equilibria or no Nash equilibrium, depending on the initial wealth of the investors x^i's. Indeed, using the notations of the explanations before Theorem 4.5, A_N is of rank equal to $N-2$. Therefore if J_N belongs to the image of A_N, then there is an affine space of dimension one of Nash equilibria, while if J_N is not in the image of A_N, then there is no Nash equilibrium.

In the general case, developped in the following sections, we cannot conclude anything on the behavior of every agent, therefore we introduce the following definition:

Definition 4.9 *(Market index and portfolio) The market index is the average wealth of the market:*

$$\bar{X}_t = \frac{1}{N}\sum_{i=1}^{N} X_t^i.$$

The market portfolio is the portfolio associated to the market index, or equivalently the average portfolio of the market:

$$\bar{\pi}_t = \frac{1}{N}\sum_{i=1}^{N} \pi_t^i.$$

For Europe, one can think of the market index as the Dow Jones Euro Stoxx 50. Recall that the Sharpe ratio SR is defined by:

$$SR = \frac{\text{expected excess return}}{\text{volatility}}, \tag{4.14}$$

but we think that a better risk ratio is a mean-variance one. Therefore we introduce the VRR (variance risk ratio) by:

$$VRR := \frac{\text{expected excess return}}{\text{variance}}. \tag{4.15}$$

When considering a time period of size kL instead of L, SR is multiplied by k, whereas VRR remains the same. On the other hand, when considering a portfolio kX instead of X, SR remains the same, whereas VRR is multiplied by $\frac{1}{k}$. Although this last property could seem to be an advantage of the Sharpe ratio, this is not true, as in reality a bigger portfolio should be considered as more risky than a smaller one.
The following results are straightforward:

Proposition 4.10 *The dynamics of the market index under $\mathbb{P}$ is given by:*

$$d\bar{X}_t = \frac{\eta^N}{1-\lambda}\theta(t).[\theta(t)dt + dW_t].$$

Thus the drift and volatility of the market index are both increasing w.r.t λ. Its Sharpe ratio is $|\theta(t)|$, independent from λ while its VRR is $\frac{1-\lambda}{\eta^N}$, which is decreasing w.r.t λ.
The market portfolio is:

$$\bar{\pi}_t = \frac{\eta^N}{1-\lambda}\sigma_t^{-1}\theta(t).$$

Therefore for any linear form φ, $|\varphi(\bar{\pi}_t)|$ is increasing w.r.t λ.

Proof. We compute:

$$
\begin{aligned}
d\bar{X}_t &= \frac{1}{N}\sum_{i=1}^{N} dX_t^i \\
&= \frac{1}{N}\sum \left[\frac{N-1}{N+\lambda-1}\sum_{i=1}^{N}\eta_i + \frac{\lambda N^2}{(1-\lambda)(N+\lambda-1)}\eta^N\right]\sigma_t^{-1}\theta(t).\sigma_t[\theta(t)dt + dW_t] \\
&= \frac{\eta^N}{1-\lambda}\theta(t).[\theta(t)dt + dW_t].
\end{aligned}
$$

□

Remark 4.11 If all the η_i's are equal, then it is obvious that the same conclusions hold true for each $\hat{\pi}^i$ and $\hat{X}^i$.

Proposition 4.10 means that the more investors look at each other, the more risk they will take globally. Moreover, on each direction of investment, the global position of agents, described by $|\varphi(\bar{\pi}_t)|$, will increase with λ and in the limit $\lambda \to 1$, we even have a limit of infinite positions $|\varphi(\bar{\pi}_t)| \to \infty$ a.s. The perverse aspect is that together with the volatility, the drift of the market index will also increase with λ, encouraging the emergence of financial bubbles. Notice that the Sharpe ratio is independent of λ, which means that it does not capture this phenomenon. On the contrary, the VRR is decreasing w.r.t λ, therefore we see that, considering this risk ratio, this kind of behavior is in fact inefficient.

4.3.4 General equilibrium

Another interesting question is the existence of a general equilibrium on the financial market. Indeed one could think that the previous influence of λ is due to the fact that we have taken the prices S to be given, and that it would be different in the case where the interactions of the N agents define the price processes.

The idea is the following. Assume that the whole market consists of our N agents, and that N is large enough for each agent to be considered as an "atom". Now assume that each agent optimizes its wealth given S and the other strategies, as done before, and that it leads to a Nash equilibrium. Can we find a dynamics for the assets S coherent with the previous Nash equilibrium? If so we call it an equilibrium market or general equilibrium. We refer for example to Karatzas and Shreve [49] for further details on the subject. As noted by the authors, such an equilibrium is not unique in general.

We want to find S such that:

$$\text{for each } 1 \leq j \leq d \text{ and all t, } \sum_{i=1}^{N} \pi_t^{i,j} = K^j S_t^j \tag{4.16}$$

$$\sum_{i=1}^{N} X_t^i - \pi_t^i . \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 0 \tag{4.17}$$

where K^j is a constant such that $K^j S_t^j$ is the market capitalization of the firm j. In order to simplify notations, we set $K^j = N$. Indeed, setting $K^j = 1$ would not be consistent if N is varying (and especially if $N \to \infty$).
The first equation means that the total amount invested in the stocks of the firm j is equal to the market capitalization of this firm. The second one means that the total amount of money in the non-risky asset is equal to 0.
In fact (4.17) is implied by (4.16) together with:

$$\sum_{i=1}^{N} x^i = \sum_{j=1}^{d} K^j S_0^j, \tag{4.18}$$

which means that at time $t = 0$, the global market capitalizations of the different firms are equal to the global wealth.
Indeed, assuming (4.16) and (4.18), we have:

$$\begin{aligned} \sum_{i=1}^{N} X_t^i &= \sum_{i=1}^{N} \left(x^i + \int_0^t \sum_{j=1}^{d} \pi_t^{i,j} \frac{dS_t^j}{S_t^j} \right) = \sum_{i=1}^{N} x^i + \sum_{j=1}^{d} \int_0^t K^j dS_t^j \\ &= \sum_{i=1}^{N} x^i + \sum_{j=1}^{d} K^j (S_t^j - S_0^j) = \sum_{j=1}^{d} K^j S_t^j = \sum_{j=1}^{d} \sum_{i=1}^{N} \pi_t^{i,j} \\ &= \sum_{i=1}^{N} \pi_t^i . \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} . \end{aligned}$$

We have the following result:

Proposition 4.12 *Assume* (4.16) *and* (4.18). *Then for any deterministic (continuous) function θ, there exists an equilibrium market whose risk premium is θ. In such a market, the dynamics of the market index is given by:*

$$d\bar{X}_t = \frac{\eta^N}{1-\lambda} \theta(t).[\theta(t)dt + dW_t]$$

Proof. Necessary conditions:
Using (4.16) we must have:

$$\begin{aligned}
S_t &= \frac{1}{N}\left[\frac{N-1}{N+\lambda-1}\sum_{i=1}^{N}\eta_i + \frac{\lambda N^2}{(1-\lambda)(N+\lambda-1)}\eta^N\right]\sigma_t^{-1}\theta(t) \\
&= \left(\frac{N-1}{N+\lambda-1} + \frac{\lambda N}{(1-\lambda)(N+\lambda-1)}\right)\eta^N\sigma_t^{-1}\theta(t) \\
&= \frac{\eta^N}{1-\lambda}\sigma_t^{-1}\theta(t)
\end{aligned}$$

Notice that as soon as $d > 1$, the previous equation does not define σ uniquely.

Sufficient conditions:
Now let a continuous function θ be given, then we can choose $\sigma_t = \sigma(t, S_t)$ diagonal such that (a.s):

$$\begin{aligned}
\sigma^{ii}(t, S_t) &= \frac{\eta^N}{(1-\lambda)S_t^i}\theta^i(t); \\
\sigma^{ij}(t, S_t) &= 0, \text{ if } i \neq j.
\end{aligned}$$

Then we compute:

$$\mathrm{diag}(S_t)\sigma(t, S_t) = \begin{pmatrix} \frac{\eta^N}{1-\lambda}\theta^1(t) & & 0 \\ & \ddots & \\ 0 & & \frac{\eta^N}{1-\lambda}\theta^d(t) \end{pmatrix}.$$

Therefore σ satisfies the conditions for S to be a strong solution of (4.1). Then, thanks to Proposition 4.6 we have:

$$\begin{aligned}
d\bar{X}_t &= \frac{1}{N}\sum_{i=1}^{N} dX_t^i = \frac{1}{N}\sum_{i=1}^{N}\hat{\pi}_t^i.\mathrm{diag}(S_t)^{-1}dS_t \\
&= \frac{1}{N}\sum\left[\frac{N-1}{N+\lambda-1}\sum_{i=1}^{N}\eta_i + \frac{\lambda N^2}{(1-\lambda)(N+\lambda-1)}\eta^N\right]\sigma_t^{-1}\theta(t).\sigma_t[\theta(t)dt + dW_t] \\
&= \frac{\eta^N}{1-\lambda}\theta(t).[\theta(t)dt + dW_t]
\end{aligned}$$

□

We can wonder what is the influence of λ on the drift and the volatility of the market index. As there are several equilibria, there are several ways to compare it but they all lead to similar conclusions. Let us for example assume that the risk premium is independent of λ.

Then the drift of the market index is $\dfrac{\eta|\theta(t)|^2}{1-\lambda}$ and the volatility is $\dfrac{\eta|\theta(t)|}{1-\lambda}$, thus both are increasing w.r.t λ, and with the same order. This can be seen as a financial bubble, where the return is higher than in the classical case, but so is the volatility. Another point of view is the one of a fund manager. For the same return, looking at the market brings a higher volatility than in the classical case.

Notice that the VRR defined by (4.15) is $\dfrac{(1-\lambda)}{\eta}$, thus decreasing in λ. At the limit $\lambda \to 1$, it tends to 0. Therefore such behavior can be seen as inefficient.

Remark 4.13 We could relax the hypothesis $r = 0$ and try to find S and the non-risky asset S_0. By replacing $\theta(t)$ by $\theta(t) - \sigma^{-1}r(t)\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$, this would bring the same results as before.

Remark 4.14 In the particular case where all agents are similar, ie $\forall i$, $\eta_i = \eta$ and $\lambda_i = \lambda$, we can see the result very easily. Indeed in that case, by symmetry, at the equilibrium, all the X^i will be equal, so that $X^i = \bar{X}^i$ and so the optimization problem becomes:

$$\sup_{\pi} -\mathbb{E}e^{-\frac{1-\lambda}{\eta}X_T^i}$$

This is the classical case with η replaced by $\frac{\eta}{1-\lambda}$, so that the optimal portfolio is:

$$\hat{\pi}_t = \frac{\eta}{1-\lambda}\sigma_t^{-1}\theta(t)$$

as claimed before.

4.4 The general case with exponential utility

Now we consider a more general case, with the assumptions stated in the first paragraph, but with exponential utility functions:

$$U_i(x) = -e^{-\frac{x}{\eta_i}}.$$

We assume that for each i, the set of constraints A_i is a vector subspace of $\mathbb{R}^d$, and we will assume that θ and σ are deterministic functions of the time.

We denote by P_t^i the orthogonal projection on $\sigma(t)A_i$, and by $Q_t^i = I - P_t^i$ the orthogonal projection on $(\sigma(t)A_i)^\perp$. We define the set of admissible strategies $\mathcal{A}_i$ to be the set of predictable processes π satisfying:

- $\mathbb{E}\int_0^T |\sigma_t\pi_t|^2 dt < \infty$ (4.19)
- $\pi_t(\omega) \in A_i \; dt \otimes d\mathbb{P}$-a.e (4.20)
- for any $p > 1$, $\{e^{-\frac{1}{\eta_i}(X_\tau^\pi - X_\nu^\pi)};\ \nu,\ \tau$ stopping times on $[0,T]$ with $\nu \leq \tau$ a.s$\}$ is uniformly bounded in $\mathbb{L}^p(\mathbb{P})$. (4.21)

Remark 4.15 As we will need to derive a dynamic principle, as in Lim and Quenez [57], in (4.21) we need to have uniform boundedness in any L^p for $p > 1$. Although we have not formulated this condition in the same way as in [57], the two conditions are essentially the same.

Remark 4.16 Assumption (4.19) guarantees the absence of arbitrage. Indeed under this assumption, for any admissible π, X^π is a martingale under the martingale measure $\mathbb{Q}$ so we can make the same reasonning as in Remark 4.2. Moreover, because of the concavity of U_i, it also guarantees that $\mathbb{E}U_i(X_T^{\pi^i} - \lambda_i \bar{X}_T^{i,\pi}) < +\infty$ for any $\pi_j \in \mathcal{A}_j$, $1 \leq j \leq N$

Remark 4.17 In comparison with the admissibility conditions of section 4.3, (4.19) is the same, (4.20) is always satisfied for $A_i = \mathbb{R}^d$ but we have added assumption (4.21).

Finally we make the following assumption:

$$\prod_{i=1}^N \lambda_i < 1 \text{ or } \bigcap_{i=1}^N A_i = \{0\}. \tag{4.22}$$

4.4.1 Equilibrium

As this quantity will be used many times, we introduce the following notation:

$$\lambda_i^N := \frac{\lambda_i}{N-1}. \tag{4.23}$$

We introduce the following linear operator on $\mathbb{R}^d$ (which is well-defined thanks to Lemma 4.27 below):

$$M_t^i := \left[I - \sum_{j\neq i} \frac{\lambda_j^N}{1+\lambda_j^N} P_t^j \left(I + \lambda_i^N P_t^i\right)\right]^{-1} \left[\eta_i I + \sum_{j\neq i} \frac{\lambda_i^N \eta_j - \lambda_j^N \eta_i}{1+\lambda_j^N} P_t^j \right]. \tag{4.24}$$

We also introduce a linear operator on $M_{N,d}(\mathbb{R})$, φ_t, such that for any $m \in M_{N,d}(\mathbb{R})$:

$$\text{for each } 1 \le k \le N,\ \varphi_t(m)^k := m^k - \lambda_k^N \sum_{j \neq k} P_t^j(m^j).$$

Again because of Lemma 4.27 below, φ is invertible and we denote its inverse:

$$\psi_t := \varphi_t^{-1}. \tag{4.25}$$

The main result of this section is the existence of a Nash equilibrium together with a characterization of Nash equilibria.

Theorem 4.18 *Assume that σ and θ are deterministic. Then there exists a Nash equilibrium and the equilibrium portfolio for agent i is:*

$$\hat{\pi}_t^i = \hat{\pi}_t^{i,N} = \sigma(t)^{-1} P_t^i M_t^i \theta(t)$$

where M^i is defined by (4.24).
The value function for agent i at equilibrium is given by:

$$V_i = V_i^N = -e^{-\frac{1}{\eta_i}(x^i - \lambda_i \bar{x}^i - Y_0^i)}$$

$$\text{where } Y_0^i = -\frac{\eta_i}{2}\int_0^T |\theta(t)|^2 dt + \frac{1}{2\eta_i}\int_0^T |Q_t^i M_t^i \theta(t)|^2 dt.$$

Moreover, for any Nash equilibrium $(\tilde{\pi}^1, ..., \tilde{\pi}^N)$ we have:

$$\tilde{\pi}_t^i = \sigma(t)^{-1} P_t^i \left[\psi_t(\tilde{Z}_t)\right]^i \qquad \text{and} \qquad V_i = -e^{-\frac{1}{\eta_i}(x^i - \lambda_i \bar{x}^i - \tilde{Y}_0^i)},$$

where ψ_t is given by (4.25) *and $(\tilde{Y}, \tilde{Z})$ is a solution of the following N-dimensional BSDE:*

$$d\tilde{Y}_t^i = -\frac{1}{2\eta_i}\left|Q_t^i \left[\psi_t(\tilde{Z}_t)\right]^i\right|^2 dt + \tilde{Z}_t^i.dB_t, \qquad \tilde{Y}_T^i = -\eta_i \ln \frac{d\mathbb{Q}}{d\mathbb{P}},$$

which has a unique solution such that $\tilde{Z}$ is deterministic.

Remark 4.19 As expected, if $\lambda_i = 0$, we find the classical optimal portfolio (with constraints): $\eta_i \sigma(t)^{-1} P_t^i \theta(t)$.

Remark 4.20 We do not have uniqueness of the Nash equilibrium in general, because we do not know have a uniqueness result for the N-dimensional BSDE. However in the complete market case, the drift of this BSDE is equal to zero and therefore we have uniqueness thanks to the uniqueness in the martingale representation theorem.

Corollary 4.21 *(Similar agents with different investment constraints)*
Assume that σ and θ are deterministic. Assume also that there exist λ and η such that, $\forall j$, $\lambda_j = \lambda$ and $\eta_j = \eta$. Then there exists a Nash equilibrium and the optimal portfolio for agent i is:

$$\hat{\pi}_t^{i,N} = \eta\sigma(t)^{-1}P_t^i \left[I - \frac{\lambda^N}{1+\lambda^N}\left(\sum_{j\neq i} P_t^j\right)\left(I + \lambda^N P_t^i\right)\right]^{-1} \theta(t).$$

Proof. (Theorem) The main idea of this proof is to derive the necessary conditions of individual optimality using the BSDE approach as in [27] or [42]. We then show the existence of Nash equilibrium. Notice that we have one major difference with those papers in that the final data for each agent's optimization problem is not bounded. In order to adapt their argument to our context, we need to define accordingly the set of admissible strategies.

We denote by $B_t = W_t + \displaystyle\int_0^t \theta(u)du$, the Brownian motion under $\mathbb{Q}$.

Necessary conditions:

First we will characterize Nash equilibria. It will give us a candidate equilibrium and at the same time give the characterization.

Assume that $(\tilde{\pi}^1, ..., \tilde{\pi}^N)$ is a Nash equilibrium for our problem. Let $\mathcal{T}$ be the set of all stopping times with values in $[0, T]$, we define the following family of random variables:

$$W^i(\tau) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_i} \mathbb{E}\left(-e^{-\frac{1}{\eta_i}\left(\int_\tau^T \sigma(u)\pi_u.dB_u - \lambda_i(\bar{X}_T^i - \bar{x}^i)\right)}|\mathcal{F}_\tau\right) \text{ for any } \tau \in \mathcal{T},$$

where we wrote $\bar{X}_T^i$ instead of $\bar{X}_T^{i,\tilde{\pi}}$.

Step 1: we exhibit a family of processes indexed by the π's such that they all are supermartingales and for $\tilde{\pi}^i$ it is a martingale.

Notice that:

$$W^i(0) = e^{\frac{1}{\eta_i}(x^i - \lambda_i \bar{x}^i)} V_i.$$

Using Lemma 4.25 below, the family $\{W^i(\tau);\ \tau \in \mathcal{T}\}$ satisfies a supermartingale property. Indeed, writing $\beta_t^{i,\pi} = e^{-\frac{1}{\eta_i}\int_0^t \sigma(u)\pi_u.dB_u}$, for any $\pi \in \mathcal{A}_i$, we have for $\tau \leq \theta$:

$$\beta_\tau^{i,\pi} W_\tau^i \geq \mathbb{E}(\beta_\theta^{i,\pi} W_\theta^i | \mathcal{F}_\tau).$$

Therefore, we can extract a process (W_t^i) which is càdlàg and consistent with the family defined previously in the sense that $W_\tau^i = W^i(\tau)$ a.s (see Karatzas and Shreve [48], Proposition I.3.14 p.16, for more details). Moreover, this process also satisfies the dynamic programming principle stated in Lemma 4.25, so that for any $\pi \in \mathcal{A}_i$, $(\beta_t^{i,\pi} W_t^i)$ is a supermartingale (under $\mathbb{P}$). We will show that for $\pi = \tilde{\pi}^i$, it is in fact a martingale.

Indeed, the definition of a Nash equilibrium implies that $\tilde{\pi}^i$ is optimal for agent i, in other words:

$$\begin{aligned} W_0^i &= \sup_{\pi \in \mathcal{A}_i} \mathbb{E} - e^{-\frac{1}{\eta_i}[X_T^\pi - x^i - \lambda_i(\bar{X}_T^i - \bar{x}^i)]} \\ &= \mathbb{E} - e^{-\frac{1}{\eta_i}[X_T^{\tilde{\pi}^i} - x^i - \lambda_i(\bar{X}_T^i - \bar{x}^i)]}. \end{aligned}$$

By definition of W_t^i:

$$\beta_t^{i,\tilde{\pi}^i} W_t^i \geq \mathbb{E}\left(-e^{-\frac{1}{\eta_i}[X_T^{\tilde{\pi}^i} - x^i - \lambda_i(\bar{X}_T^i - \bar{x}^i)]} | \mathcal{F}_t \right) \text{ a.s.}$$

Now using the supermartingale property of $\beta^{i,\tilde{\pi}^i} W^i$, taking expectations of the previous inequality and using the optimality of $\tilde{\pi}^i$, we get:

$$W_0^i \geq \mathbb{E}\beta_t^{i,\tilde{\pi}^i} W_t^i \geq \mathbb{E} - e^{-\frac{1}{\eta_i}[X_T^{\tilde{\pi}^i} - x^i - \lambda_i(\bar{X}_T^i - \bar{x}^i)]} = W_0^i.$$

So that there must be equality almost surely:

$$\beta_t^{i,\tilde{\pi}^i} W_t^i = \mathbb{E}\left(-e^{-\frac{1}{\eta_i}[X_T^{\tilde{\pi}^i} - x^i - \lambda_i(\bar{X}_T^i - \bar{x}^i)]} | \mathcal{F}_t \right) \text{ a.s}$$

and therefore $(\beta_t^{i,\tilde{\pi}^i} W_t^i)$ is a martingale (as the conditional expectation of a L^1 random variable).

Step 2: we define the candidate Y^i and show that it is a diffusion process.

Let us now define an adapted and continuous process (Y_t^i) by:

$$Y_t^i := X_t^{\tilde{\pi}^i} - x^i + \eta_i \ln(-\beta_t^{\tilde{\pi}^i} W_t^i).$$

This is indeed possible as $W_t^i < 0$ almost surely, and by construction, Y^i is adapted and continuous. Notice that we have:

$$\beta_t^{\tilde{\pi}^i} W_t^i = -e^{-\frac{1}{\eta_i}(X_t^{\tilde{\pi}^i} - x^i - Y_t^i)} \text{ and } Y_T^i = \lambda_i(\bar{X}_T^i - \bar{x}^i).$$

Now we define for each $\pi \in \mathcal{A}_i$, the process $M_t^\pi := -e^{-\frac{1}{\eta_i}(X_t^\pi - x^i - Y_t^i)}$. In particular, $M^{\tilde{\pi}^i} = \beta^{\tilde{\pi}^i} W^i$ is a martingale, and is in L^2 because of the admissibility condition (4.19). We show hereafter that for any π, M^π is a supermartingale.

Assume to the contrary that there exists $\pi \in \mathcal{A}_i$, $t \geq s$ and $A \in \mathbb{F}_s$, with $\mathbb{P}(A) > 0$ and such that:

$$\mathbb{E}\left(-e^{-\frac{1}{\eta_i}(X_t^\pi - x^i - Y_t^i)}|\mathcal{F}_s\right) > -e^{-\frac{1}{\eta_i}(X_s^\pi - x^i - Y_s^i)} \text{ on } A,$$

we will proceed towards a contradiction. We define:

$$\hat{\pi}_u(\omega) = \begin{cases} \pi_u(\omega) \text{ if } u \in [s,t] \text{ and } \omega \in A \\ \tilde{\pi}_u(\omega) \text{ otherwise} \end{cases}$$

As $A \in \mathcal{F}_s$, we can apply Lemma 4.26, so that $\hat{\pi} \in \mathcal{A}_i$ and we have:

$$\begin{aligned} W_0^i &\geq \mathbb{E} - e^{-\frac{1}{\eta_i}(X_T^{\hat{\pi}} - x^i - Y_T^i)} \\ &= \mathbb{E}\left[\mathbb{E}\left(-e^{-\frac{1}{\eta_i}(X_T^{\hat{\pi}} - x^i - Y_T^i)}|\mathcal{F}_t\right)\right] \\ &= \mathbb{E} - e^{-\frac{1}{\eta_i}(X_t^{\hat{\pi}} - x^i - Y_t^i)} \text{ as } \hat{\pi} = \tilde{\pi} \text{ on } [t,T] \\ &= \mathbb{E}\left[\mathbb{E}\left(-e^{-\frac{1}{\eta_i}(X_t^{\hat{\pi}} - x^i - Y_t^i)}|\mathcal{F}_s\right)\right] \\ &> \mathbb{E} - e^{-\frac{1}{\eta_i}(X_s^{\hat{\pi}} - x^i - Y_s^i)} \text{ as } \mathbb{P}(A) > 0 \\ &= -e^{\frac{1}{\eta_i}Y_0^i} = W_0^i \end{aligned}$$

which provides the required contradiction.

Now we can compute the dynamics of Y^i. As:

$$Y_t^i = X_t^{i,\tilde{\pi}^i} - x^i + \eta_i \ln\left(-M_t^{\tilde{\pi}^i}\right),$$

with $M_t^{\tilde{\pi}^i} < 0$, for all t, a.s, and $M^{\tilde{\pi}^i}$ is a L^2 martingale, the martingale representation theorem tells us that $M^{\tilde{\pi}^i}$ is a diffusion process, so that Y^i is also a diffusion process. Therefore, we write its dynamics as follows:

$$dY_t^i = -b_t^i dt + Z_t^i . dW_t.$$

Moreover, Y^i is adapted and continuous, while Z^i is predictable.

Step 3: we derive the BSDE satisfied by (Y^i, Z^i).

After using Itô's formula in order to compute the drift of M^π, the previous supermartingale and martingale properties can be rewritten as:

$$\begin{aligned} \text{for any } \pi \in \mathcal{A}_i,\ b_t^i &\leq \frac{1}{2\eta_i}\left|\sigma(t)\pi_t - (Z_t^i + \eta_i\theta(t))\right|^2 - \frac{\eta_i}{2}|\theta(t)|^2 - Z_t^i.\theta(t) \\ b_t^i &= \frac{1}{2\eta_i}\left|\sigma(t)\tilde{\pi}_t^i - (Z_t^i + \eta_i\theta(t))\right|^2 - \frac{\eta_i}{2}|\theta(t)|^2 - Z_t^i.\theta(t). \end{aligned}$$

Writing $d(x,E)$ the distance of a vector x to a set E:

$$d(x,E) := \inf_{z\in E} |x-z|,$$

this implies that:

$$\begin{aligned}
\tilde{\pi}^i_t &= \sigma(t)^{-1} P^i_t \left(Z^i_t + \eta_i \theta(t) \right);\\
b^i_t &= f_i(t, Z^i_t) = \frac{1}{2\eta_i} d(Z^i_t + \eta_i\theta(t), \sigma(t)A_i)^2 - \frac{\eta_i}{2}|\theta(t)|^2 - Z^i_t.\theta(t).
\end{aligned}$$

Finally, (Y^i, Z^i) is a solution of the following BSDE:

$$\begin{aligned}
dY^i_t &= \left[Z^i_t.\theta(t) + \frac{\eta_i|\theta(t)|^2}{2} - \frac{1}{2\eta_i} \left| Q^i_t(Z^i_t + \eta_i\theta(t)) \right|^2 \right] dt + Z^i_t.dW_t\\
Y^i_T &= \lambda_i(\bar{X}^i_T - \bar{x}_i) = \lambda^N_i \sum_{j\neq i} \int_0^T \pi^j_u.\sigma(u)(dW_u + \theta(u)du).
\end{aligned}$$

Recalling that $dB_t = dW_t + \theta(t)dt$, we can write it:

$$\begin{aligned}
dY^i_t &= \left[\frac{\eta_i|\theta_t|^2}{2} - \frac{1}{2\eta_i} \left| Q^i_t(Z^i_t + \eta_i\theta_t) \right|^2 \right] dt + Z^i_t.dB_t\\
Y^i_T &= \lambda_i(\bar{X}^i_T - \bar{x}_i) = \lambda^N_i \sum_{j\neq i} \int_0^T \pi^j_u.\sigma(u)dB_u.
\end{aligned}$$

And the portfolio $\tilde{\pi}^i$ is necessarily given by:

$$\sigma(t)\tilde{\pi}^i_t = P^i_t \left(Z^i_t + \eta_i\theta(t) \right).$$

Step 4: we put the N BSDEs together.

As $(\tilde{\pi}^1, ..., \tilde{\pi}^N)$ is a Nash equilibrium, this is simultaneously the case for all agents, and so replacing the value of $\tilde{\pi}^j$ in the expression of Y^i and writing $\Gamma^i := Z^i + \eta_i\theta$, we see that (Y^i, Γ^i) must satisfy for each $t\in[0,T]$:

$$\begin{aligned}
Y^i_t = \lambda^N_i \sum_{j\neq i} \int_0^T P^j_u(\Gamma^j_u).dB_u - \frac{\eta_i}{2}\int_t^T |\theta(u)|^2 du + \frac{1}{2\eta_i}\int_t^T |Q^i_u(\Gamma^i_u)|^2 du - \int_t^T \Gamma^i_u.dB_u\\
+ \eta_i \int_t^T \theta(u).dB_u,
\end{aligned}$$

so that if we write $\gamma_t^i := Y_t^i - \frac{\eta_i}{2}\int_0^t |\theta(u)|^2 du + \frac{\eta_i}{2}\int_0^t \theta(u).dB_u - \lambda_i^N \sum_{j\neq i}\int_0^t P_u^j(\Gamma_u^j).dB_u$, (γ_t^i) is $(\mathcal{F}_t)$-adapted, continuous and (γ^i, Γ^i) must satisfy:

$$\gamma_t^i = -\eta_i \ln \frac{d\mathbb{Q}}{d\mathbb{P}} + \frac{1}{2\eta_i}\int_t^T |Q_u^i(\Gamma_u^i)|^2 du - \int_t^T \left[\Gamma_u^i - \lambda_i^N \sum_{j\neq i} P_u^j(\Gamma_u^j)\right].dB_u.$$

Moreover, it is immediate that Γ^i is predictable.

Now writing φ_t such that:

$$\zeta^i = \varphi_t(Z)^i = Z^i - \lambda_i^N \sum_{j\neq i} P_t^j(Z^j),$$

under hypothesis (4.22), using Lemma 4.27 below, we know that φ_t is invertible, with inverse ψ_t which is continuous (in t). So that we get that (γ, ζ) must be a solution of the following system of BSDEs:

$$\gamma_0^i = -\eta_i \ln \frac{d\mathbb{Q}}{d\mathbb{P}} + \frac{1}{2\eta_i}\int_0^T |Q_t^i([\psi_t(\zeta_t)]^i)|^2 dt - \int_0^T \zeta_t^i.dB_t.$$

Moreover, for each i, the optimal investment at equilibrium is given by:

$$\sigma(t)\tilde{\pi}_t^i = P_t^i\left[\psi_t(\zeta)^i\right].$$

Finally, we get the characterization claimed. Moreover from the uniqueness part of Lemma 4.28 below, we see that there exists at most one Nash equilibrium such that $\tilde{Z}$ is deterministic, which is the one given in the statement of the Theorem. As $Y_0^i = \gamma_0^i$, we also get the expression of V_i.

Sufficient conditions:

Now we need to show that the candidate is indeed a Nash equilibrium. The idea is to show that we can make the previous computations in the reverse sense.

Recall that M_t^i is defined by:

$$M_t^i := \left[I - \sum_{j\neq i} \frac{\lambda_j^N}{1+\lambda_j^N} P_t^j\left(I + \lambda_i^N P_t^i\right)\right]^{-1} \left[\eta_i I + \sum_{j\neq i} \frac{\lambda_i^N \eta_j - \lambda_j^N \eta_i}{1+\lambda_j^N} P_t^j\right]$$

and let us consider the adapted and continuous process γ and the predictable process ζ

defined by:

$$\begin{aligned}
\zeta_t^i &:= \eta_i \theta(t) \\
\gamma_t^i &:= \eta_i \int_0^t \theta(u).dB_u - \frac{\eta_i}{2}\int_0^T |\theta(u)|^2 du + \frac{1}{2\eta_i}\int_t^T |Q_u^i[\psi_u(\zeta_u)^i]|^2 du \\
&= \eta_i \int_0^t \theta(u).dB_u - \frac{\eta_i}{2}\int_0^T |\theta(u)|^2 du + \frac{1}{2\eta_i}\int_t^T |Q_u^i M_u^i \theta(u)|^2 du.
\end{aligned}$$

And let define the following portfolios for the N agents by:

$$\hat{\pi}_t^i = \sigma(t)^{-1} P_t^i \left[I - \sum_{j\neq i} \frac{\lambda_j^N}{1+\lambda_j^N} P_t^j \left(I + \lambda_i^N P_t^i\right)\right]^{-1} \left[\eta_i I + \sum_{j \neq i} \frac{\lambda_i^N \eta_j - \lambda_j^N \eta_i}{1+\lambda_j^N} P_t^j \right] \theta(t).$$

We introduce the equivalent measure $\mathbb{Q}^i$, defined by its Radon-Nikodym density:

$$R_t^i = \frac{d\mathbb{Q}^i}{d\mathbb{P}}_{|\mathcal{F}_t} = e^{\int_0^t \left(\frac{1}{\eta_i} M_u^i - I\right)\theta(u).dW_u - \frac{1}{2}\int_0^t \left|\left(\frac{1}{\eta_i}M_u^i - I\right)\theta(u)\right|^2 du}. \tag{4.26}$$

By construction, $\hat{\pi}^i \in \mathcal{A}_i$ for each i. Indeed $\hat{\pi}_t^i \in A_i$ for all t is immediate. The fact that $\int_0^T |\sigma(t)\hat{\pi}_t^i|^2 dt < \infty$ a.s is also immediate as σ and $\hat{\pi}^i$ are deterministic and continuous on $[0,T]$. Finally, as $\hat{\pi}^i$ is deterministic, it is immediate that the family $\{e^{-\frac{1}{\eta_i}X_\tau^{\hat{\pi}^i}};\ \tau \in \mathcal{T}\}$ is uniformly bounded in $L^p(\mathbb{P})$ for any $p > 1$. Because of Lemma 4.29, this implies the uniform integrability of the family $\{e^{-\frac{1}{\eta_i}X_\tau^{\hat{\pi}^i}};\ \tau \in \mathcal{T}\}$ under $\mathbb{Q}^i$.

From Lemma 4.28 below, we know that (γ, ζ) satifies the following N-dimensional BSDE:

$$\begin{aligned}
d\gamma_t^i &:= -\frac{1}{2\eta_i}|Q_t^i[\psi_t(\zeta_t)^i]|^2 dt + \zeta_t^i dB_t \\
\gamma_T^i &= -\eta_i \ln \frac{d\mathbb{Q}}{d\mathbb{P}}.
\end{aligned}$$

If we set:

$$\begin{aligned}
Y_t^i &= \gamma_t^i + \frac{\eta_i}{2}\int_0^t |\theta(u)|^2 du - \eta_i \int_0^t \theta(u).dB_u + \lambda_i^N \sum_{j\neq i}\int_0^t P_u^j(\psi_u(\zeta_u)^j).dB_u \\
Z_t^i &= \psi_t(\zeta_t)^i - \eta_i\theta(t) = \left(M_t^i - \eta_i I\right)\theta(t),
\end{aligned}$$

the previous computations in the reverse sense, show that for each i, (Y^i, Z^i) is a solution of the 1-dimensional BSDE:

$$\begin{aligned}
dY_t^i &= \left[Z_t^i.\theta(t) + \frac{\eta_i|\theta(t)|^2}{2} - \frac{1}{2\eta_i}\left|Q_t^i(Z_t^i + \eta_i\theta(t))\right|^2\right] dt + Z_t^i.dW_t \\
Y_T^i &= \lambda_i^N \sum_{j\neq i}\int_0^T \hat{\pi}_u^j.\sigma(u)(dW_u + \theta(u)du).
\end{aligned}$$

We define for each $\pi \in \mathcal{A}_i$:

$$M_t^\pi = -e^{-\frac{1}{\eta_i}\left(X_t^{i,\pi} - x^i - Y_t^i\right)}.$$

Again as in the previous computations, using Itô's formula, we see that M^π is a local supermartingale for each π and a local martingale for $\hat{\pi}^i$, under $\mathbb{P}$, so that there exists a sequence of stopping times (τ_n) in $\mathcal{T}$, such that $\tau_n \to T$ a.s and for each n and any $s \le t$:

$$\begin{aligned} \mathbb{E}[M^\pi_{t\wedge\tau_n} | \mathcal{F}_s] &\le M^\pi_{s\wedge\tau_n} \\ \mathbb{E}[M^{\hat{\pi}^i}_{t\wedge\tau_n} | \mathcal{F}_s] &= M^{\hat{\pi}^i}_{s\wedge\tau_n}. \end{aligned}$$

Then using the definition of φ and ψ we can rewrite Y^i as:

$$\begin{aligned} Y_t^i &= -\frac{\eta_i}{2}\int_0^t |\theta(u)|^2 du + \frac{1}{2\eta_i}\int_t^T |Q_u^i M_u^i \theta(u)|^2 du + \lambda_i^N \sum_{j\neq i}\int_0^t P_u^j [\psi_u(\zeta_u)]^j . dB_u \\ &= -\frac{\eta_i}{2}\int_0^t |\theta(u)|^2 du + \frac{1}{2\eta_i}\int_t^T |Q_u^i M_u^i \theta(u)|^2 du + \int_0^t Z_u^i . dB_u \\ &= -\frac{\eta_i}{2}\int_0^t |\theta(u)|^2 du + \frac{1}{2\eta_i}\int_t^T |Q_u^i M_u^i \theta(u)|^2 du + \int_0^t \left(M_u^i - \eta_i I\right)\theta(u) . dB_u. \end{aligned}$$

From Lemma 4.29, if $\pi \in \mathcal{A}_i$, then the family $\{e^{-\frac{1}{\eta_i}X_\tau^{\hat{\pi}^i}}\ ;\ \tau \in \mathcal{T}\}$ is uniformly integrable under $\mathbb{Q}^i$.

We write $\mathbb{E}$ the expectation under $\mathbb{P}$ and $\mathbb{E}^i$ the expectation under $\mathbb{Q}^i$. As θ is a deterministic and continuous function on $[0,T]$, the definition of M^i guarantees that $-\frac{\eta_i}{2}\int_0^t |\theta(u)|^2 du + \frac{1}{2\eta_i}\int_t^T |Q_u^i M_u^i \theta(u)|^2 du$ is bounded (under either one of the equivalent measures $\mathbb{P}$ or $\mathbb{Q}^i$).

Thus, for any stopping time τ_n and $t \geq 0$, we have:

$$\begin{aligned}
\mathbb{E}M^{\pi}_{t\wedge\tau_n} &= \mathbb{E} - e^{-\frac{1}{\eta_i}\left(X^{\pi}_{t\wedge\tau_n} - x^i - Y^i_{t\wedge\tau_n}\right)} \\
&= \mathbb{E}\left[-e^{-\frac{1}{\eta_i}(X^{\pi}_{t\wedge\tau_n} - x^i) - \frac{1}{2}\int_0^{t\wedge\tau_n}|\theta(u)|^2 du + \frac{1}{2\eta_i^2}\int_{t\wedge\tau_n}^T |Q^i_u M^i_u \theta(u)|^2 du}\right. \\
&\qquad \left. e^{\int_0^{t\wedge\tau_n}\left(\frac{1}{\eta_i}M^i_u - I\right)\theta(u).(dW_u + \theta(u)du)}\right] \\
&= \mathbb{E}\left[-R^i_{t\wedge\tau_n} e^{-\frac{1}{\eta_i}(X^{\pi}_{t\wedge\tau_n} - x^i) - \frac{1}{2}\int_0^{t\wedge\tau_n}|\theta(u)|^2 du + \frac{1}{2\eta_i^2}\int_{t\wedge\tau_n}^T |Q^i_u M^i_u \theta(u)|^2 du + \int_0^{t\wedge\tau_n}\left(\frac{1}{\eta_i}M^i_u - I\right)\theta(u).\theta(u)du}\right. \\
&\qquad \left. e^{\frac{1}{2}\int_0^{t\wedge\tau_n}\left|\left(\frac{1}{\eta_i}M^i_u - I\right)\theta(u)\right|^2 du}\right] \\
&= \frac{1}{R^i_s}\mathbb{E}^i\left[-e^{-\frac{1}{\eta_i}(X^{\pi}_{t\wedge\tau_n} - x^i) - \frac{1}{2}\int_0^{t\wedge\tau_n}|\theta(u)|^2 du + \frac{1}{2\eta_i^2}\int_{t\wedge\tau_n}^T |Q^i_u M^i_u \theta(u)|^2 du + \int_0^{t\wedge\tau_n}\left(\frac{1}{\eta_i}M^i_u - I\right)\theta(u).\theta(u)du}\right. \\
&\qquad \left. e^{\frac{1}{2}\int_0^{t\wedge\tau_n}\left|\left(\frac{1}{\eta_i}M^i_u - I\right)\theta(u)\right|^2 du}\right].
\end{aligned}$$

Now all the terms are bounded, so that thanks to the admissibility condition 4.21, we have uniform integrability under $\mathbb{Q}^i$, so that:

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{E}M^{\pi}_{t\wedge\tau_n} &= \frac{1}{R^i_s}\mathbb{E}^i\left[-e^{-\frac{1}{\eta_i}(X^{\pi}_t - x^i) - \frac{1}{2}\int_0^t|\theta(u)|^2 du + \frac{1}{2\eta_i^2}\int_t^T |Q^i_u M^i_u \theta(u)|^2 du + \int_0^t\left(\frac{1}{\eta_i}M^i_u - I\right)\theta(u).\theta(u)du}\right. \\
&\qquad \left. e^{\frac{1}{2}\int_0^t\left|\left(\frac{1}{\eta_i}M^i_u - I\right)\theta(u)\right|^2 du}\right] \\
&= \mathbb{E}\left[-R^i_t e^{-\frac{1}{\eta_i}(X^{\pi}_t - x^i) - \frac{1}{2}\int_0^t|\theta(u)|^2 du + \frac{1}{2\eta_i^2}\int_t^T |Q^i_u M^i_u \theta(u)|^2 du + \int_0^t\left(\frac{1}{\eta_i}M^i_u - I\right)\theta(u).\theta(u)du}\right. \\
&\qquad \left. e^{\frac{1}{2}\int_0^t\left|\left(\frac{1}{\eta_i}M^i_u - I\right)\theta(u)\right|^2 du}\right] \\
&= \mathbb{E}M^{\pi}_t.
\end{aligned}$$

Therefore we conclude that:

$$\begin{aligned}
\text{for each } \pi \in \mathcal{A}_i,\ \mathbb{E} - e^{-\frac{1}{\eta_i}\left(X^{\pi}_t - x^i - Y^i_t\right)} &\leq -e^{\frac{1}{\eta_i}Y^i_0} \\
\mathbb{E} - e^{-\frac{1}{\eta_i}\left(X^{\hat{\pi}}_t - x^i - Y^i_t\right)} &= -e^{\frac{1}{\eta_i}Y^i_0}
\end{aligned}$$

Multiplying by $e^{-\frac{1}{\eta_i}(x^i - \lambda_i \bar{x}^i)}$, we finally get for each i:

$$V_i = -e^{-\frac{1}{\eta_i}(x^i - \lambda_i \bar{x}^i - Y^i_0)}$$

and $\hat{\pi}^i$ is optimal for agent i. Hence $(\hat{\pi}^1, ..., \hat{\pi}^N)$ is a Nash equilibrium. □

Remark 4.22 As noticed by Ramon van Handel, the existence part of Theorem 4.18 is still true if one assumes that θ and σ are deterministic but such that the filtration generated by θ and σ is independent from $(\mathcal{F}_t)$, and the problem remains Markovian, which would for example be the case of stochastic volatilities driven by independent Brownian motions. Indeed, writing $(\mathcal{F}_t^S)$ the filtration generated by W and $(\mathcal{F}_t^{\theta,\sigma})$, if one enlarges the set of admissible processes to be $(\mathcal{F}_t^S \vee \mathcal{F}_T^{\theta,\sigma})$-predictable, then, as we are in a Markov framework, the problem conditioned by $\mathcal{F}_T^{\theta,\sigma}$ is equivalent to the deterministic one. So the previous formula would hold. Now as the previous formula gives in fact an $(\mathcal{F}_t^S \vee \mathcal{F}_t^{\theta,\sigma})$-predictable process $\hat{\pi}^i$ and $(\mathcal{F}_t^S \vee \mathcal{F}_t^{\theta,\sigma}) \subset (\mathcal{F}_t^S \vee \mathcal{F}_T^{\theta,\sigma})$, the optimality in the set of $(\mathcal{F}_t^S \vee \mathcal{F}_t^{\theta,\sigma})$-predictable processes is also guaranteed. Unfortunately, we do not know about the uniqueness of this Nash equilibrium.

Remark 4.23 In the previous proof, we see that the driver of the N-dimensional BSDE is quadratic (in Z), and unfortunately, there is no results of existence nor uniqueness of such multi-dimensional quadratic BSDEs, even for bounded terminal conditions (which is not even the case here). We refer to Frei and dos Reis [32] for further discussions on the subject. If we had such results, the previous proof would almost work for general non-deterministic θ and σ, but the justification of the martingale/supermartingale property of the M^π would still be an issue. Indeed, in the proof of Hu-Imkeller-Müller, they strongly use the fact that for bounded terminal conditions, the martingale part of the BSDE Z is BMO. In our proof above, the fact that Z is deterministic is crucial. But with non-deterministic θ and σ, there is no reason for Z to be BMO, even if the existence was guaranteed, see Briand and Hu [7]. Nevertheless, we can provide the following necessary condition for existence of a Nash equilibrium.

In the following proposition, we allow θ and σ to be general $\mathbb{F}$-predictable processes satisfying

$$\int_0^T |\sigma_t|^2 dt < +\infty \text{ a.s, and } \int_0^T |\theta_t|^2 dt < +\infty \text{ a.s,}$$

in order to guarantee that equation (4.1) admits a unique strong solution.

Proposition 4.24 *If a Nash equilibrium $(\hat{\pi}^1, ..., \hat{\pi}^N)$ exists for our problem, then there exist at least one solution (Y, Z) to the following BSDE:*

$$\begin{aligned} dY_t^i &:= -\frac{1}{2\eta_i} |Q_t^i[\psi_t(Z_t)^i]|^2 dt + Z_t^i dB_t \\ Y_T^i &= -\eta_i \ln \frac{d\mathbb{Q}}{d\mathbb{P}}. \end{aligned}$$

and moreover we have $\hat{\pi}^i = \sigma_t^{-1} P_t^i(\psi_t(Z_t)^i)$ for a certain solution (Y, Z).

Proof. The necessary conditions in the previous proof does not use the fact that θ and σ were deterministic, so we have the result. $\square$

We provide hereafter the different lemmas used in the proof of Theorem 4.18. First we give a dynamic programming principle, see El Karoui [23], or El Karoui, Jeanblanc and Nguyen [25] for general results.

Lemma 4.25 *(Dynamic Programming) Define for any stopping time τ on $[0,T]$:*

$$W^i_\tau = ess \sup_{\pi \in \mathcal{A}_i} \mathbb{E}\left(-e^{-\frac{1}{\eta_i}\left(\int_\tau^T \sigma(u)\pi_u.dB_u - \lambda_i(\bar{X}^i_T - \bar{x}^i)\right)}|\mathcal{F}_\tau\right).$$

Then for any stopping times $\tau \leq \theta$, we have:

$$W^i_\tau = ess \sup_{\pi \in \mathcal{A}_i} \mathbb{E}(e^{-\frac{1}{\eta_i}\int_\tau^\theta \sigma(u)\pi_u.dB_u} W^i_\theta|\mathcal{F}_\tau)$$

Proof. Let $\tau \leq \theta \leq T$ a.s.

$\leq$:

$$\begin{aligned}
W^i_\tau &= \text{ess}\sup_{\pi \in \mathcal{A}_i} \mathbb{E}\left(-e^{-\frac{1}{\eta_i}\left(\int_\tau^T \sigma(u)\pi_u.dB_u - \lambda_i(\bar{X}^i_T - \bar{x}^i)\right)}|\mathcal{F}_\tau\right)\\
&= \text{ess}\sup_{\pi \in \mathcal{A}_i} \mathbb{E}\left[\mathbb{E}\left(-e^{-\frac{1}{\eta_i}\left(\int_\theta^T \sigma(u)\pi_u.dB_u - \lambda_i(\bar{X}^i_T - \bar{x}^i)\right)} e^{-\frac{1}{\eta_i}\int_\tau^\theta \sigma(u)\pi_u.dB_u}|\mathcal{F}_\theta\right)|\mathcal{F}_\tau\right]\\
&= \text{ess}\sup_{\pi \in \mathcal{A}_i} \mathbb{E}\left[\mathbb{E}\left(-e^{-\frac{1}{\eta_i}\left(\int_\theta^T \sigma(u)\pi_u.dB_u - \lambda_i(\bar{X}^i_T - \bar{x}^i)\right)}|\mathcal{F}_\theta\right) e^{-\frac{1}{\eta_i}\int_\tau^\theta \sigma(u)\pi_u.dB_u}|\mathcal{F}_\tau\right]\\
&\leq \text{ess}\sup_{\pi \in \mathcal{A}_i} \mathbb{E}(e^{-\frac{1}{\eta_i}\int_\tau^\theta \sigma(u)\pi_u.dB_u} W^i_\theta|\mathcal{F}_\tau)
\end{aligned}$$

$\geq$: Let $\pi_0 \in \mathcal{A}_i$. Define:

$$J^\pi_\theta := \mathbb{E}\left[-e^{-\frac{1}{\eta_i}\left(\int_\theta^T \sigma(u)\pi^n_u.dB_u - \lambda_i(\bar{X}^i_T - \bar{x}^i)\right)}|\mathcal{F}_\theta\right].$$

Recall the definition of the stochastic interval $[\![0,\theta]\!]$:

$$[\![0,\theta]\!] = \{(t,\omega);\ 0 \leq t \leq \theta(\omega)\}.$$

We first prove that there exists a sequence $(\hat{\pi}_n)$ satisfying:

- $\forall n,\ \hat{\pi}_n = \pi_0$ on $[\![0,\theta]\!]$ (4.27)
- $(J^{\hat{\pi}_n}_\theta)$ is nondecreasing and converges to W^i_θ. (4.28)

We write $\mathcal{A}_i(\theta) := \{\pi \in \mathcal{A}_i;\ \pi = \pi_0 \text{ on } [\![0,\theta]\!],\ dt \otimes d\mathbb{P}\text{-a.e}\}$. For any π, J^π_θ depends on π only through its values on $[\![\theta, T]\!]$, therefore we have the identity:

$$W^i_\theta = \operatorname{ess}\sup_{\pi \in \mathcal{A}_i(\theta)} J^\pi_\theta.$$

We show that the family $\{J^\pi_\theta,\ \pi \in \mathcal{A}_i(\theta)\}$ is closed under pairwise maximization. Let π_1, π_2 in $\mathcal{A}_i(\theta)$, then define $A := \{\omega \in \Omega;\ J^{\pi_1}_\theta(\omega) \geq J^{\pi_2}_\theta(\omega)\}$, and $\pi := 1_A\pi_1 + 1_{\Omega\setminus A}\pi_2$. As $A \in \mathcal{F}_\theta$, using Lemma 4.26 below, $\pi \in \mathcal{A}_i(\theta)$ and $J^\pi_\theta = \max(J^{\pi_1}_\theta, J^{\pi_2}_\theta)$. We can therefore apply Theorem A.3, p.324 in Karatzas and Shreve [49], providing a sequence $(\hat{\pi}^n)$ satisfying (4.27) and (4.28).

Then we have:

$$\begin{aligned}
W^i_\tau &\geq \mathbb{E}\left(-e^{-\frac{1}{\eta_i}\left(\int_\tau^T \sigma(u)\hat{\pi}^n_u.dB_u - \lambda_i(\bar{X}^i_T - \bar{x}^i)\right)}|\mathcal{F}_\tau\right)\\
&\geq \mathbb{E}\left[\mathbb{E}\left(-e^{-\frac{1}{\eta_i}\left(\int_\theta^T \sigma(u)\pi^n_u.dB_u - \lambda_i(\bar{X}^i_T - \bar{x}^i)\right)}|\mathcal{F}_\theta\right) e^{-\frac{1}{\eta_i}\int_\tau^\theta \sigma(u)\pi_u.dB_u}|\mathcal{F}_\tau\right]\\
&\geq \mathbb{E}\left[J^{\hat{\pi}_n}_\theta e^{-\frac{1}{\eta_i}\int_\tau^\theta \sigma(u)\pi_u.dB_u}|\mathcal{F}_\tau\right].
\end{aligned}$$

Now $J^{\hat{\pi}_n}_\theta e^{-\frac{1}{\eta_i}\int_\tau^\theta \sigma(u)\pi_u.dB_u}$ converges almost surely to $e^{-\frac{1}{\eta_i}\int_\tau^\theta \sigma(u)\pi_u.dB_u}W^i_\theta$ and is nondecreasing, so that we can apply the monotone convergence theorem to conclude that:

$$W^i_\tau \geq \mathbb{E}(e^{-\frac{1}{\eta_i}\int_\tau^\theta \sigma(u)\pi_u.dB_u}W^i_\theta|\mathcal{F}_\tau).$$

As π was chosen arbitrarily, we have the result. □

Lemma 4.26 *Let π^1 and π^2 be two strategies in $\mathcal{A}_i$, and $\theta \in \mathcal{T}$. We define:*

$$\pi^3_t := \begin{cases} \pi^1_t & \textit{if } t \leq \theta \\ \pi^2_t & \textit{otherwise.}\end{cases}$$

Then $\pi^3 \in \mathcal{A}_i$.

Proof. We need only to check that the family $\{e^{-\frac{1}{\eta_i}X^{\pi^3}_\tau};\ \tau \in \mathcal{T}\}$ is uniformly bounded in any L^p, $p > 1$. Let $p > 1$ and $\tau \in \mathcal{T}$, we write $\nu := \tau \wedge \theta$, and we have using Cauchy-Schwarz inequality:

$$\begin{aligned}
\mathbb{E}e^{-\frac{p}{\eta_i}X^{\pi^3}_\tau} &= \mathbb{E}e^{-\frac{p}{\eta_i}\left(\int_0^\nu \sigma(t)\pi^1_t.dB_t + \int_\nu^\tau \sigma(t)\pi^2_t.dB_t\right)}\\
&\leq \left(\mathbb{E}e^{-\frac{2p}{\eta_i}\int_0^\nu \sigma(t)\pi^1_t.dB_t}\right)^{\frac{1}{2}}\left(\mathbb{E}e^{-\frac{2p}{\eta_i}\int_\nu^\tau \sigma(t)\pi^2_t.dB_t}\right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

□

Lemma 4.27 *Let $\Sigma \in \mathcal{L}(\mathbb{R}^d)$ be invertible, and for each i, P^i be the orthogonal projection on ΣA_i. Define $\varphi : M_{N,d}(\mathbb{R}) \to M_{N,d}(\mathbb{R})$ by:*

$$\forall i \in [1, N],\ \varphi^i(z) = z^i - \lambda_i^N \sum_{j \neq i} P^j z^j.$$

Then φ is invertible if and only if assumption (4.22) is satisfied and then its inverse ψ is given by:

$$\psi^i(\zeta) = \left[I - \left(\sum_{j \neq i} \frac{\lambda_j^N}{1 + \lambda_j^N} P^j \right) \left(I + \lambda_i^N P^i \right) \right]^{-1} \left(\zeta^i + \frac{1}{N-1} \sum_{j \neq i} \frac{1}{1 + \lambda_j^N} P^j (\lambda_i \zeta^j - \lambda_j \zeta^i) \right).$$

Proof. We write, for any $1 \leq i \leq N$, $\zeta^i = z^i - \lambda_i^N \sum_{j \neq i} P^j z^j$. We want to inverte this system.

If we define $\xi = \sum_{j=1}^{N} P^j z^j$, we get:

$$\zeta^i = z^i + \lambda_i^N P^i z^i - \lambda_i^N \xi. \tag{4.29}$$

So for any i, j:

$$\begin{aligned} \lambda_j \zeta^i - \lambda_i \zeta^j &= \lambda_j z^i - \lambda_i z^j + \frac{\lambda_i \lambda_j}{N-1} (P^i z^i - P^j z^j) \\ \text{so } \lambda_i \left(I + \lambda_j^N P^j \right) z^j &= \lambda_j \left(I + \lambda_i^N P^i \right) z^i + \lambda_i \zeta^j - \lambda_j \zeta^i. \end{aligned}$$

Now for any j, as $\lambda_j \geq 0$, $I + \lambda_j^N P^j$ is (symmetric) definite positive so it is invertible. Therefore we get:

$$\begin{aligned} \xi &= \sum_{j=1}^{N} P^j z^j \\ &= P^i z^i + \frac{1}{\lambda_i} \sum_{j \neq i} P^j \left(I + \lambda_j^N P^j \right)^{-1} \left(\lambda_j \left(I + \lambda_i^N P^i \right) z^i + \lambda_i \zeta^j - \lambda_j \zeta^i \right). \end{aligned}$$

We next observe that: $P^j \left(I + \lambda_j^N P^j \right)^{-1} = \frac{1}{1 + \lambda_j^N} P^j$. Indeed, if $(e_1, ..., e_k)$ is a basis of $Ker P^j$, while $(e_{k+1}, ..., e_d)$ is a basis of $Im P^j$, it is easy to check that:

$$P^j \left(I + \lambda_j^N P^j \right)^{-1} e_i = \begin{cases} 0 \text{ if } 1 \leq i \leq k \\ \frac{1}{1+\lambda_j^N} e_i \text{ if } k+1 \leq i \leq d. \end{cases}$$

Another way to see this, for $N \geq 3$ or $\lambda_j < 1$, is to compute $\left(I + \lambda_j^N P^j\right)^{-1} = I + \sum_{k\geq 1} \left(-\lambda_j^N\right)^k P^j$, as $(P^j)^2 = P^j$.

Thus:

$$\xi = P^i z^i + \frac{1}{\lambda_i} \sum_{j\neq i} \frac{1}{1+\lambda_j^N} P^j \left[\lambda_j \left(I + \lambda_i^N P^i\right) z^i + \lambda_i \zeta^j - \lambda_j \zeta^i\right].$$

And:

$$\begin{aligned}
\zeta^i &= z^i + \lambda_i^N (P^i z^i - \xi) \\
&= z^i - \frac{1}{N-1} \sum_{j\neq i} \frac{1}{1+\lambda_j^N} P^j \left[\lambda_j \left(I + \lambda_i^N P^i\right) z^i + \lambda_i \zeta^j - \lambda_j \zeta^i\right].
\end{aligned}$$

So that we can write (4.29) equivalently into:

$$\left(I - \sum_{j\neq i} \frac{\lambda_j^N}{1+\lambda_j^N} P^j \left(I + \lambda_i^N P^i\right)\right) z^i = \zeta^i + \frac{1}{N-1} \sum_{j\neq i} \frac{1}{1+\lambda_j^N} P^j \left(\lambda_i \zeta^j - \lambda_j \zeta^i\right).$$

Now as the ζ^j's describe $\mathbb{R}^d$, the right-hand side describes $\mathbb{R}^d$. Indeed, let $\zeta^i \in \mathbb{R}^d$, if we take for any $j \neq i$, $\zeta^j = \frac{\lambda_j}{\lambda_i} \zeta^i$, the right-hand side is equal to ζ^i. So it is necessary and sufficient for φ to be invertible that $\left(I - \sum_{j\neq i} \frac{\lambda_j^N}{1+\lambda_j^N} P^j \left(I + \lambda_i^N P^i\right)\right)$ be invertible for any i.

Now we will show that this is true if and only if assumption 4.22 is satisfied, ie if and only if: $\prod_{j=1}^N \lambda_j < 1$ or $\bigcap_{j=1}^N A_j = \{0\}$. Let us write $U_i = \sum_{j\neq i} \frac{\lambda_j^N}{1+\lambda_j^N} P^j \left(I + \lambda_i^N P^i\right)$.

<u>Necessary condition:</u>

Assume that $\forall j$, $\lambda_j = 1$ and $\bigcap_{j=1}^N A_j \neq \{0\}$. Then let $x \neq 0$ such as $x \in \bigcap_{j=1}^N A_j$. We have for any j, $P^j x = x$ and so:

$$U_i x = \frac{1}{N-1} \sum_{j\neq i} \frac{1}{1+\frac{1}{N-1}} P^j \left(I + \frac{1}{N-1} P^i\right) x = x.$$

Then $x \in Ker(I - U_i)$ and $I - U_i$ is not invertible.

<u>Sufficient condition:</u>

Case 1: assume that $\exists j,\ \lambda_j < 1$, then we have: $\dfrac{\lambda_j^N}{1+\lambda_j^N} < \dfrac{\frac{1}{N-1}}{1+\frac{1}{N-1}}$, while for any i,k:

$$\frac{\lambda_k^N}{1+\lambda_k^N} \le \frac{\frac{1}{N-1}}{1+\frac{1}{N-1}}$$
$$\left|P^k(I+\lambda_i^N P^i)x\right| \le \left(1+\frac{1}{N-1}\right)|x|,$$

so that if $i \neq j$, $|U_i x| < |x|$ if $x \neq 0$.

Case 2: on the contrary, assume that $\forall j,\ \lambda_j = 1$, but $\displaystyle\bigcap_{j=1}^N A_j = \{0\}$. Let $x \in Ker(I - U_i)$, we have:

$$\begin{aligned}|x| = |U_i x| &= \left|\frac{1}{N-1}\sum_{j\neq i}\frac{1}{1+\frac{1}{N-1}}P^j\left(I+\frac{1}{N-1}P^i\right)x\right| \\ &\le \frac{1}{N-1}\sum_{j\neq i}|x| = |x|\end{aligned}$$

But if the inequality is an equality, it implies that: $P^i x = x$ and $|P^j x| = |x|$, which implies that $P^j x = x$. So $x \in \displaystyle\bigcap_{j=1}^N A_j$, which means $x = 0$. □

We write $L^2(\mathbb{R}^m)$ the space of adapted processes (X_t) such that for each t, X_t is in $L^2(\mathbb{R}^m)$, and we write $Det^2(\mathbb{R}^m)$ the space of deterministic processes (X_t) with values in $\mathbb{R}^m$, such that $\displaystyle\int_0^T |X_t|^2 dt < \infty$.

Lemma 4.28 *Let B a d-dimensional Brownian motion and $A : [0,T] \to \mathcal{L}(M_{N,d}(\mathbb{R}))$ a continuous function. Then the following system of N BSDEs:*

$$\forall i \in 1,...,N,\ Y_t^i = \eta_i\left(\int_0^T \theta(u).dB_u - \frac{1}{2}\int_0^T |\theta(u)|^2 du\right) + \int_t^T |[A(u)(Z_u)]^i|^2 du - \int_t^T Z_u^i.dB_u$$

has a unique solution $(Y,Z) \in L^2(\mathbb{R}^N) \times Det^2(M_{N,d}(\mathbb{R}))$ given by:

$$\begin{aligned}Z_t^i &= \eta_i\theta(t) \\ Y_t^i &= \eta_i\left(\int_0^t \theta(u).dB_u - \frac{1}{2}\int_0^T |\theta(u)|^2 du\right) + \int_t^T |[A(u)(Z_u)]^i|^2 du\end{aligned}$$

Proof. Let $(Y, Z) \in L^2(\mathbb{R}^N) \times Det^2(M_{N,d}(\mathbb{R}))$ be a solution of the previous BSDE. Then for each i:

$$Y_t^i = \eta_i \left(\int_0^T \theta(u).dB_u - \frac{1}{2} \int_0^T |\theta(u)|^2 du \right) + \int_t^T |[A(u)(Z_u)]^i|^2 du - \int_t^T Z_u^i.dB_u.$$

As θ is a continuous deterministic function, and as $Z^i \in Det^2(\mathbb{R}^d)$, $\int_0^t \theta(u).dB_u$ and $\int_0^t Z_u^i.dB_u$ are $\mathbb{Q}$-martingales, while $\int_t^T |[A(u)(Z_u)]^i|^2 du$ is deterministic, so that taking conditional expectations under $\mathbb{Q}$, we get:

$$Y_t^i = \eta_i \left(\int_0^t \theta(u).dB_u - \frac{1}{2} \int_0^T |\theta(u)|^2 du \right) + \int_t^T |[A(u)(Z_u)]^i|^2 du.$$

Now using Itô's formula, we also compute:

$$Y_t^i = Y_0^i - \int_0^t |[A(u)(Z_u)]^i|^2 du + \int_0^t Z_u^i.dB_u,$$

So that by uniqueness in the semi-martingale decomposition it implies:

$$\int_0^t Z_u^i.dB_u = \int_0^t \theta(u).dB_u.$$

Now because θ and Z^i both belong to $Det^2(\mathbb{R}^d)$, the uniqueness in the martingale representation theorem implies that:

$$Z_t^i = \eta_i \theta(t) \ a.s.$$

And therefore Y^i is necessarily the one in the statement of the lemma, which gives the uniqueness part.

Now as θ is deterministic, it is obvious that this is a solution satisfying $(Y, Z) \in L^2(\mathbb{R}^N) \times Det^2(M_{N,d}(\mathbb{R}))$. □

Finally, we show here that condition (4.21) is stronger than the needed condition. Recall the definition of the equivalent probability $\mathbb{Q}^i$ given by its Radon-nikodym density:

$$R_t^i = \left. \frac{d\mathbb{Q}^i}{d\mathbb{P}} \right|_{\mathcal{F}_t} = e^{\int_0^t \left(\frac{1}{\eta_i} M_u^i - I\right)\theta(u).dW_u - \frac{1}{2}\int_0^t \left|\left(\frac{1}{\eta_i} M_u^i - I\right)\theta(u)\right|^2 du},$$

where M^i was defined by (4.24). It is immediate that $R^i \in \mathbb{L}^p(\mathbb{P})$ for any $p > 1$.

Lemma 4.29 *Assume that $\pi \in \mathcal{A}_i$. Then $\{e^{-\frac{1}{\eta_i} X_\tau^{i,\pi}} ;\ \tau \in \mathcal{T}\}$ is uniformly integrable under $\mathbb{Q}^i$.*

Proof. Let $p > 1$ such that $\{e^{-\frac{1}{\eta_i}X_\tau^{i,\pi}};\ \tau \in \mathcal{T}\}$ is uniformly bounded in $L^p(\mathbb{P})$. Let $\tau \in \mathcal{T}$, we write:

$$Y_\tau := e^{-\frac{1}{\eta_i}X_\tau^{i,\pi}}.$$

Then, for $r = \dfrac{1+p}{2} \in (1,p)$, q such that $\dfrac{1}{q} + \dfrac{1}{r} = 1$ and $c > 0$, we compute using Holder's inequality:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^i}\left[Y_\tau 1_{Y_\tau \geq c}\right] &= \mathbb{E}\left[R_T^i Y_\tau 1_{Y_\tau \geq c}\right] \\ &\leq \left(\mathbb{E}\left(R_T^i\right)^q\right)^{\frac{1}{q}} \left(\mathbb{E}\left[Y_\tau^r 1_{Y_\tau^r \geq c^r}\right]\right)^{\frac{1}{r}}\end{aligned}$$

But as $p > r$, the last term uniformly goes to 0 as $c \to \infty$, so that we have the uniform integrability of $\{Y_\tau;\ \tau \in \mathcal{T}\}$ under $\mathbb{Q}^i$. □

4.4.2 Limit as N goes to infinity

We see that we have a general result of existence and uniqueness of a Nash equilibrium together with an explicit formula for the equilibrium portfolios. Unfortunately, except in some simple cases, those expressions are quite complicated. Following the ideas of the theory of Mean-Field Games, an interesting question is to see if we can find a "limit" portfolio when the number of agents N goes to infinity, and if the expression gets simpler and helps us to derive some behavioral implications. As the number of agents goes to infinity, a natural question is: what happens to the assets? In what follows we will only consider the case of a fixed number of assets d. However in the examples developped in the next chapter, we will see that we can also get some convergence results when d goes to infinity.
Recall that we denote by $\mathcal{L}(\mathbb{R}^d)$ the space of linear mappings on $\mathbb{R}^d$. Let $|.|$ be the canonical euclidean norm on $\mathbb{R}^d$, we then introduce the classical norm $||.||$ on $\mathcal{L}(\mathbb{R}^d)$ defined for any $U \in \mathcal{L}(\mathbb{R}^d)$ by:

$$||U|| = \sup_{|x|=1} |U(x)|.$$

Remark 4.30 Considering that d goes to infinity with N raises several problems, in particular the definition of an infinite dimensional "limit" market and the existence of orthogonal projections in infinite dimensional spaces.

Proposition 4.31 *($N \to \infty$, d fixed)*
Let d be fixed and assume that $(\eta_i)_i$ is bounded in $\mathbb{R}$.
(i) Assume that for each $t \in [0,T]$ as $N \to \infty$:

$$\frac{1}{N}\sum_{i=1}^N \lambda_i P_t^i \to U_t^\lambda \text{ in } \mathcal{L}(\mathbb{R}^d) \text{ with } ||U_t^\lambda|| < 1 \quad \text{and} \quad \frac{1}{N}\sum_{i=1}^N \eta_i P_t^i \to U_t^\eta \text{ in } \mathcal{L}(\mathbb{R}^d).$$

Then for each $t \in [0,T]$:

$$\hat{\pi}_t^{i,N} \to \hat{\pi}_t^{i,\infty} := \sigma(t)^{-1} P_t^i (I - U_t^\lambda)^{-1} \left[\eta_i (I - U_t^\lambda) + \lambda_i U_t^\eta\right] \theta(t).$$

(ii) Assume that, as $N \to \infty$:

$$\sup_{t\in[0,T]} \left|\left| \frac{1}{N}\sum_{i=1}^N \lambda_i P_t^i - U_t^\lambda \right|\right| \to 0 \text{ with } \forall t,\ ||U_t^\lambda|| < 1 \quad \text{and} \quad \sup_{t\in[0,T]} \left|\left| \frac{1}{N}\sum_{i=1}^N \eta_i P_t^i - U_t^\eta \right|\right| \to 0.$$

Then:

$$\sup_{t\in[0,T]} |\hat{\pi}_t^{i,N} - \hat{\pi}_t^{i,\infty}| \to 0.$$

Proof. Let $t \in [0,T]$. Using the expression of $\hat{\pi}^{i,N}$ in Theorem 4.18, we have:

$$\hat{\pi}_t^{i,N} = \sigma(t)^{-1} P_t^i A_t^i B_t^i \theta(t)$$

where:

$$\begin{aligned} A_t^i &:= \left[I - \sum_{j\neq i} \frac{\lambda_j^N}{1+\lambda_j^N} P_t^j \left(I + \lambda_i^N P_t^i\right) \right]^{-1}; \\ B_t^i &:= \left[\eta_i I + \sum_{j\neq i} \frac{\lambda_i^N \eta_j - \lambda_j^N \eta_i}{1+\lambda_j^N} P_t^j \right]. \end{aligned}$$

As $||P_t^j|| \le 1$, we also have:

$$\begin{aligned} \left|\left| \frac{1}{N-1}\sum_{j\neq i} \frac{\lambda_j}{1+\lambda_j^N} P_t^j - \frac{1}{N}\sum_{j=1}^N \lambda_j P_t^j \right|\right| &\le \frac{1}{N-1} \left|\left| \sum_{j\neq i} \frac{\lambda_j}{1+\lambda_j^N} P_t^j - \lambda_j P_t^j \right|\right| \\ &\quad + \left|\left| \frac{1}{N-1}\sum_{j\neq i} \lambda_j P_t^j - \frac{1}{N}\sum_{j=1}^N \lambda_j P_t^j \right|\right| \\ &\le \left|\left| \frac{1}{(N-1)^2}\sum_{j\neq i} \frac{\lambda_j^2}{1+\lambda_j^N} P_t^j \right|\right| + \left|\left| \frac{1}{N}\lambda_i P_t^i + \frac{1}{N(N-1)}\sum_{j\neq i} \lambda_j P_t^j \right|\right| \\ &\le \frac{3}{N}, \end{aligned}$$

and similarly, writing C such that $\forall j$, $\eta_j \le C$:

$$\left|\left| \frac{1}{N-1}\sum_{j\neq i} \frac{\eta_j}{1+\lambda_j^N} P_t^j - \frac{1}{N}\sum_{j=1}^N \eta_j P_t^j \right|\right| \le \frac{3C}{N}.$$

Therefore as $N \to \infty$, we have the following convergences in $\mathcal{L}(\mathbb{R}^d)$, which are uniform in t in (ii):

$$
\begin{aligned}
& I + \lambda_i^N P_t^i \to I; \\
& \frac{1}{N-1} \sum_{j \neq i} \frac{\lambda_j}{1+\lambda_j^N} P_t^j \to U_t^\lambda; \\
& \frac{1}{N-1} \sum_{j \neq i} \frac{\eta_j}{1+\lambda_j^N} P_t^j \to U_t^\eta.
\end{aligned}
$$

So that $A_t^i \to (I - U_t^\lambda)^{-1}$ and $B_t^i \to \eta_i I + \lambda_i U_t^\eta - \eta_i U_t^\lambda$, uniformly in t in (ii), and we get the result. □

Another way to see this is to adopt a probabilistic point of view. Indeed, we can assume that there is a continuum of players, and that each player is a realization of the random agent, independent from the others. Therefore we assume here that we have a probability space $(\Delta, \mathcal{D}, \mu)$ which describes the players, and which is independent from the space used to describe the market. Then we have two random variables λ, η and a process $P = (P_t)$ taking values respectively in $[0,1]$, $(0, +\infty)$ and $\mathcal{L}(\mathbb{R}^d)$. If we assume independence between them, we can reformulate the previous proposition in the following way:

Corollary 4.32 *Let d be fixed, and assume that λ, η and for any $t \in [0,T]$ P_t are independent random variables on the probability space $(\Delta, \mathcal{D}, \mu)$ in $L^1(\mu)$. We write:*

$$\bar{\lambda} = \mathbb{E}^\mu \lambda; \quad \bar{\eta} = \mathbb{E}^\mu \eta; \quad U_t = \mathbb{E}^\mu P_t.$$

Also assume that for all t, $||\bar{\lambda} U_t|| < 1$, then for each t we have:

$$\hat{\pi}_t^{i,N} \to \hat{\pi}_t^{i,\infty}$$

$$\textit{where } \hat{\pi}_t^{i,\infty} = \sigma(t)^{-1} P_t^i \left[\eta_i I + \lambda_i \bar{\eta} (I - \bar{\lambda} U_t)^{-1} U_t\right] \theta(t).$$

Remark 4.33 The condition $||\bar{\lambda} U|| < 1$ means that either $\bar{\lambda} < 1$ or $||U|| < 1$, which can be seen as the extension of assumption 4.22. Notice though that it cannot be forgotten as it is not a consequence of assumption 4.22. First we can have $\prod_{i=1}^N \lambda_i < 1$ for any N while $\bar{\lambda} = 1$, and then $\prod_{i=1}^N A_i = \{0\}$ for any N but $||U|| = 1$, even in simple cases. To see that, consider for example $d = 2$ and the P^j's given by $P^j = P_{\mathbb{R}e_1}$ if $j = n^2$, for any $n \in \mathbb{N}$, and $P^j = P_{\mathbb{R}e_2}$ on the contrary (where (e_1, e_2) is the canonical basis of $\mathbb{R}^2$). Then if $n^2 \leq N < (n+1)^2$, $\left| \frac{1}{N} \sum_{j=1}^N P^j e_2 \right| \geq \frac{n^2 - n}{(n+1)^2} \to 1$, so that $|U e_2| = 1$.

If the agents share the same sensitivities, we obtain the following simpler formulation:

Corollary 4.34 *(Similar agents) Let d be fixed. Assume that there exist λ and η such that for all j, $\lambda_j = \lambda$ and $\eta_j = \eta$. Assume moreover that:*

$$\text{for each } t \text{ (resp. uniformly in } t\text{)}, \ \frac{1}{N}\sum_{i=1}^{N} P_t^i \to U_t \text{ in } \mathcal{L}(\mathbb{R}^d) \text{ with } ||\lambda U_t|| < 1.$$

Then $\hat{\pi}_t^{i,N} \to \hat{\pi}_t^{i,\infty}$ for all t (resp. uniformly in t) where:

$$\hat{\pi}_t^{i,\infty} = \eta\sigma(t)^{-1} P_t^i (I - \lambda U_t)^{-1} \theta(t).$$

We give a final particular case where things get simpler. It corresponds to a finite number of different agents.

Corollary 4.35 *Let d be fixed, and assume that there is only a finite number of possible A_i's. We reindex the A_i's, $1 \leq i \leq p$, and we write k_i^N the number of agents that can invest along A_i. Assume that:*

$$\text{for all } 1 \leq i \leq p, \ \frac{k_i^N}{N} \to \kappa_i \in [0,1]$$
$$\text{either } \lambda < 1 \text{ or } \bigcap_{i,\kappa_i \neq 0} A_i = \{0\}.$$

Then $\sup_{t\in[0,T]} |\hat{\pi}_t^{i,N} - \hat{\pi}_t^{i,\infty}| \to 0$, where:

$$\hat{\pi}_t^{i,\infty} = \eta\sigma(t)^{-1} P_t^i \left(I - \lambda \sum_{j=1}^{p} \kappa_j P_t^j \right)^{-1} \theta(t).$$

Proof. It this case, $\frac{\lambda}{N}\sum_{j=1}^{N} P_t^j \to \lambda \sum_{j=1}^{p} \kappa_j P_t^j$, uniformly in $t \in [0,T]$, and if $\lambda < 1$ or $\bigcap_{j,\kappa_j \neq 0} A_j = \{0\}$, $||\lambda \sum_{j=1}^{p} \kappa_j P^j|| < 1$, so we have the result thanks to Corollary 4.35. □

Remark 4.36 We obviously have $\sum_i \kappa_i = 1$.

4.4.3 Risk of the market index and influence of λ

We now study the influence of the sensitivities on the optimal portfolios and in particular the influence of the λ_i's. Unfortunately, we cannot say a lot in general for the portfolio of any agent, this is why we introduced the concepts of market index and market portfolio. Recall their definitions:

$$\bar{X}_t^N = \frac{1}{N}\sum_{i=1}^N X_t^i;$$
$$\bar{\pi}^N = \frac{1}{N}\sum_{i=1}^N \pi_N^i.$$

When N is fixed, we will write $\bar{X}$ (resp. $\bar{\pi}$) instead of $\bar{X}^N$ (resp. $\bar{\pi}^N$). We have the following relation between them:

$$d\bar{X}_t^N = \sigma(t)\bar{\pi}_t^N.[dW_t + \theta(t)dt],$$

so that $|\sigma(t)\bar{\pi}_t^N|$ is the volatility of the market index.

On the contrary, we will see in the examples in the next chapter that in some specific cases we can say a lot more than the general results below.

As in the proof of Proposition 4.31, we introduce the following linear mappings:

$$A_t^i := \left[I - \sum_{j\neq i}\frac{\lambda_j^N}{1+\lambda_j^N}P_t^j\left(I+\lambda_i^N P_t^i\right)\right]^{-1};$$
$$B_t^i := \left[\eta_i I + \sum_{j\neq i}\frac{\lambda_i^N\eta_j - \lambda_j^N\eta_i}{1+\lambda_j^N}P_t^j\right].$$

Recall that we also denoted: $M^i = A^iB^i$. From Theorem 4.18, we have: $\hat{\pi}_t^{i,N} = \sigma(t)^{-1}P_t^i\theta(t)$, and $d\hat{X}_t^i = \sigma(t)\bar{\pi}_t^{i,N}.[dW_t + \theta(t)dt]$. Implicitly, A^i depends on all the λ_j, $1\le j\le N$, and B^i depends on all the λ_j and η_j, $1\le j\le N$.

We first give a result for each agent but under a quite restrictive hypothesis, but which still covers many examples. Moreover, in this case the influence of the λ_i's is very strong.

Proposition 4.37 *Let N be given and assume that for each $t\in[0,T]$, the P^i's pairwise commute.*
Then, for any i and any t, $|\sigma(t)\hat{\pi}_t^i|$ is nondecreasing w.r.t λ_j and η_j for any $1\le j\le N$.

An immediate consequence of Proposition 4.37 is that if there exist λ and η such that for all j, $\lambda_j = \lambda$ and $\eta_j = \eta$, then $|\sigma(t)\hat{\pi}^i_t|$ is nondecreasing w.r.t λ and η.

Remark 4.38 The assumption that the P^i's pairwise commute is equivalent to the assumption that there are all diagonalizable in the same orthonormal basis.

Proof. (proposition) Let i be fixed. As the P^j's pairwise commute, it means that the P^j's are jointly diagonalizable in the same orthonormal basis that we write (u_j). And A^i, B^i, M^i are also diagonalizable in the same basis. Every eigenvalue of P^iM^i is in $[0,1]$, but we show moreover that there are nondecreasing w.r.t any λ_j and any η_j. Now let j be also fixed (possibly equal to i).

First the dependence w.r.t η_j is easy to establish. Indeed there exist $(\varepsilon_1, ..., \varepsilon_N)$ taking values in $\{0,1\}$ such that:

$$\begin{aligned} B^i u_k &= \left(\eta_i + \sum_{m \neq i} \frac{\lambda_i^N \eta_m - \lambda_m^N \eta_i}{1 + \lambda_m^N} \varepsilon_m \right) u_k \\ &= b_k u_k. \end{aligned}$$

We see that b_k is nondecreasing w.r.t η_j for any j and $b_k \geq 0$ (notice that $1 \geq \sum_{m \neq i} \frac{\lambda_m^N}{1+\lambda_m^N}$). Then $P^iA^iB^iu_k = a_kb_ku_k$ where a_k is non-negative and independent of η_j, so that we have the result. If $j = i$, then the dependence w.r.t λ_i is also obvious because in the previous expression both a_k and b_k are nondecreasing w.r.t λ_i and non-negative. For $j \neq i$, it is not that immediate as b_k is nonincreasing while a_k is nondecreasing.

Let $j \neq i$ and k be given. If $P^ju_k = 0$, then both a_k and b_k are independent of λ_j. If $P^ju_k = u_k$, then using the previous notations we have $P^iM^iu_k = a_kb_ku_k$ with:

$$\begin{aligned} a_k &= \varepsilon_i \left[1 - \sum_{m \neq i} \frac{\lambda_m^N}{1 + \lambda_m^N} \varepsilon_m \left(1 + \lambda_i^N \varepsilon_i \right) \right]^{-1}; \\ b_k &= \eta_i + \sum_{m \neq i} \frac{\lambda_i^N \eta_m - \lambda_m^N \eta_i}{1 + \lambda_m^N} \varepsilon_m. \end{aligned}$$

If $\varepsilon_i \neq 0$, then $\varepsilon_i = 1$ and we compute:

$$\begin{aligned}\frac{1}{a_k^2}\frac{\partial a_k b_k}{\partial \lambda_j} &= \frac{1}{(N-1)\left(1+\lambda_j^N\right)^2}\left[\left(\eta_i + \sum_{m\neq i}\frac{\lambda_i^N\eta_m - \lambda_m^N\eta_i}{1+\lambda_m^N}\varepsilon_m\right)\left(1+\lambda_i^N\right)\right.\\ &\qquad\left. -\eta_i\left(1-\sum_{m\neq i}\frac{\lambda_m^N}{1+\lambda_m^N}\varepsilon_m\left(1+\lambda_i^N\right)\right)\right]\\ &= \frac{1}{(N-1)\left(1+\lambda_j^N\right)^2}\left[\frac{\lambda_i\eta_i}{N-1} + \lambda_i^N\sum_{m\neq i}\frac{\eta_m}{1+\lambda_m^N}\varepsilon_m\left(1+\lambda_i^N\right)\right]\\ &> 0.\end{aligned}$$

Therefore we have the nondecrease with respect to λ_j. Then as (u_k) is orthonormal and the eigenvalues are all nonnegative, we have the result for $|\sigma(t)\hat{\pi}_t^i|$. □

Remark 4.39 We cannot say more in general, when the P^i's do not commute. Indeed we provide here a very simple example for which most of the previous properties fail. Take $N = d = 2$, $A_1 = \mathbb{R}e_1$, $A_2 = \mathbb{R}(e_1+e_2)$ and $\sigma = I$. Then if we write the matrices in (e_1, e_2), we have:

$$P^1 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \quad ; \quad P^2 = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix};$$

$$\left[I - \frac{\lambda_2}{1+\lambda_2}P^2(I+\lambda_1P^1)\right]^{-1} = \frac{1}{2-\lambda_1\lambda_2}\begin{pmatrix}2+\lambda_2 & \lambda_2\\ (1+\lambda_1)\lambda_2 & 1-\lambda_1\lambda_2\end{pmatrix};$$

$$P^1\left[I - \frac{\lambda_2}{1+\lambda_2}P^2(I+\lambda_1P^1)\right]^{-1}\left(\eta_1 I + \frac{\lambda_1\eta_2-\lambda_2\eta_1}{1+\lambda_2}P_2\right) = \frac{1}{2-\lambda_1\lambda_2}\begin{pmatrix}2\eta_1+\lambda_1\eta_2 & \lambda_1\eta_2\\ 0 & 0\end{pmatrix}.$$

So that $|\hat{\pi}^1| = \dfrac{1}{2-\lambda_1\lambda_2}\left|(2\eta_1+\lambda_1\eta_2)\,\theta_1 + \lambda_1\eta_2\theta_2\right|$.
If $\theta_1 > 0$ but $(2\eta_1+\lambda_1\eta_2)\,\theta_1 + \lambda_1\eta_2\theta_2 < 0$, then $|\hat{\pi}^1|$ is locally decreasing w.r.t η_1.
If $\lambda_1 > 0$, $\theta_1+\theta_2 > 0$ but $(2\eta_1+\lambda_1\eta_2)\,\theta_1 + \lambda_1\eta_2\theta_2 < 0$, then $|\hat{\pi}^1|$ is locally decreasing w.r.t η_2.
Finally, for any $\theta_1 > 0$, λ_1 and λ_2, we can choose θ_2 such that:

$$0 < \left(\frac{\eta_1}{\eta_2\lambda_1}+1\right)\theta_1 < -\theta_2 < \left(\frac{2\eta_1}{\eta_2\lambda_1}+1\right)\theta_1,$$

so that locally:

$$\begin{aligned}\frac{\partial|\hat{\pi}^1|}{\partial\lambda_1} &= 2\lambda_2\eta_1\theta_1 + \lambda_1\lambda_2\eta_2(\theta_1+\theta_2) + (2-\lambda_1\lambda_2)(\theta_1+\theta_2)\eta_2\\ &\leq 2\lambda_2\left[\theta_1\eta_1 + \lambda_1\eta_2(\theta_1+\theta_2)\right] < 0.\end{aligned}$$

Therefore $|\hat{\pi}^1|$ is locally decreasing w.r.t λ_1 (but it is nondecreasing w.r.t λ_2).

On the other hand, when we consider the limit $N \to \infty$, things get simpler. We use the notations of Corollary 4.32.

Proposition 4.40 *Under the assumptions of Corollary 4.32, we can define a limit market portfolio $\bar{\pi}^\infty$ and a limit market index $\bar{X}^\infty$. $\bar{\pi}^\infty$ is given by:*

$$\bar{\pi}_t^\infty = \bar{\eta}\sigma(t)^{-1}U_t(I - \bar{\lambda}U_t)^{-1}\theta(t),$$

and for any t, $|\sigma(t)\bar{\pi}_t^\infty|$ is nondecreasing w.r.t $\bar{\lambda}$ and $\bar{\eta}$.
The dynamics of $\bar{X}^\infty$ is given by:

$$d\bar{X}_t^\infty = \bar{\eta}U_t(I - \bar{\lambda}U_t)^{-1}\theta(t).[dB_t + \theta(t)dt].$$

In particular its drift is always nonnegative. Moreover, for any t, U_t is symmetric and independent of λ and η. We write $\frac{1}{\bar{\lambda}} > \mu_1(t) \geq ... \geq \mu_d(t) \geq 0$ its eigenvalues (with the convention $\frac{1}{0} = +\infty$), $(u_{1,t}, ..., u_{d,t})$ an orthonormal basis of associated eigenvectors, and we write $\theta(t) = \sum_{i=1}^{d} \theta_i(t)u_{i,t}$. Then the volatility and the drift of the market index are given for any t by:

$$\bar{\sigma}(t) = |\sigma(t)\bar{\pi}_t^\infty| = \bar{\eta}\sqrt{\sum_{i=1}^{N}(\frac{\mu_i(t)}{1 - \bar{\lambda}\mu_i(t)})^2\theta_i(t)^2}$$

$$\bar{b}(t) = \bar{\eta}\sum_{i=1}^{N}\frac{\mu_i(t)}{1 - \bar{\lambda}\mu_i(t)}\theta_i(t)^2$$

In particular, they are both nondecreasing w.r.t $\bar{\lambda}$ and $\bar{\eta}$, while the $VRR = \frac{\bar{b}}{\bar{\sigma}^2}$ is nonincreasing w.r.t $\bar{\lambda}$ and $\bar{\eta}$.

Proof. By definition, we have in $\mathcal{L}(\mathbb{R}^d)$:

$$\bar{\lambda}U = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\lambda_i P^i$$

$$\bar{\eta}U = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\eta_i P^i$$

Noticing that $\bar{\lambda}U(I - \bar{\lambda}U)^{-1} = (I - \bar{\lambda}U)^{-1} - I$ and that U and $(I - \bar{\lambda}U)^{-1}$ commutate, we get (in $\mathcal{L}(\mathbb{R}^d)$):

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\sigma^{-1}P^i[\eta_i I + \lambda_i\bar{\eta}(I - \bar{\lambda}U)^{-1}U] = \bar{\eta}U + \bar{\lambda}\bar{\eta}U(I - \bar{\lambda}U)^{-1}U$$

$$= \bar{\eta}(I - \bar{\lambda}U)^{-1}U = \bar{\eta}U(I - \bar{\lambda}U)^{-1}$$

So we have the expressions of $\bar{\pi}^{\infty}$ and $\bar{X}^{\infty}$.

For all t, as the P^i's are all symmetric and positive, U is also symmetric and positive. So it is diagonalizable in an orthonormal basis that we write $(u_1, ..., u_d)$, with associated eigenvalues $\mu_1 \geq ... \geq \mu_d$. As U is positive, $\mu_d \geq 0$, and as $||\bar{\lambda} U|| < 1$, we also have $\mu_1 < \frac{1}{\bar{\lambda}}$ with the convention $\frac{1}{0} = +\infty$. Moreover U only depends on the A_i, not on the λ_i or the η_i. As $||P^i|| \leq 1$, we also have $\mu_1 \leq 1$.

But $(I - \bar{\lambda} U)^{-1}$ is also diagonalizable in the same basis $(u_1, ..., u_d)$, with associated eigenvalues $1 \geq \frac{1}{1-\bar{\lambda}\mu_1} \geq ... \geq \frac{1}{1-\bar{\lambda}\mu_d} > 0$ so that $U(I - \bar{\lambda} U)^{-1}$ is diagonalizable in $(u_1, ..., u_d)$ with eigenvalues $1 \geq \frac{\mu_1}{1-\bar{\lambda}\mu_1} \geq ... \geq \frac{\mu_d}{1-\bar{\lambda}\mu_d} \geq 0$. And we get the claimed formulas for $\bar{\sigma}(t) = \sigma(t)\bar{\pi}_t^{\infty}$ and $\bar{b}(t)$.

Finally if we define:

$$S(\bar{\lambda}, \bar{\eta}) = \frac{\bar{b}}{\bar{\sigma}^2} = \frac{\sum \frac{\mu_i}{1-\bar{\lambda}\mu_i}\theta_i^2}{\bar{\eta} \sum \left(\frac{\mu_i}{1-\bar{\lambda}\mu_i}\right)^2 \theta_i^2},$$

it is immediate that S is nonincreasing w.r.t $\bar{\eta}$ and

$$\bar{\eta}\frac{\partial S}{\partial \bar{\lambda}} = \frac{\sum \left(\frac{\mu_i}{1-\bar{\lambda}\mu_i}\right)^2 \theta_i^2}{\sum \left(\frac{\mu_i}{1-\bar{\lambda}\mu_i}\right)^2 \theta_i^2} - \frac{\sum \frac{\mu_i}{1-\bar{\lambda}\mu_i}\theta_i^2 \sum \left(\frac{\mu_i}{1-\bar{\lambda}\mu_i}\right)^3 \theta_i^2}{\left(\sum \left(\frac{\mu_i}{1-\bar{\lambda}\mu_i}\right)^2 \theta_i^2\right)^2}.$$

But using Cauchy-Scharwz inequality, we get:

$$\left(\sum \left(\frac{\mu_i}{1-\bar{\lambda}\mu_i}\right)^2 \theta_i^2\right)^2 \leq \sum \frac{\mu_i}{1-\bar{\lambda}\mu_i}\theta_i^2 \sum \left(\frac{\mu_i}{1-\bar{\lambda}\mu_i}\right)^3 \theta_i^2,$$

so that

$$\frac{\partial S}{\partial \bar{\lambda}} \leq 0.$$

□

From a financial point of view, the last sentence of the previous proposition means that the bigger $\bar{\lambda}$ is, the bigger the drift will be, but at the same time the bigger the volatility will be too. So for example if an agent uses a criterion such as $\mathbb{E} - mV$, and wants to choose between a market where investors do not look at each other and a market where investors look at each other, the second market could artificially seem more attractive than the first one just because of this phenomenon, while this second market is a lot more risky. With a criterion such as $\mathbb{E} - m\sqrt{V}$, or as the Sharpe ratio defined by (4.14), the second market will seem even more attractive.

On the contrary, the VRR is nonincreasing with respect to $\bar{\lambda}$, therefore a manager who uses it to evaluate its strategies will consider that a bigger $\bar{\lambda}$ is inefficient.

In fact, Proposition 4.40 gives a lot of information on the way $\bar{\lambda}$ influences $\bar{\sigma}$ and $\bar{b}$. We see that the more μ_i is close to 1, the bigger influence it has for the i-th component of θ, written in the basis (u_j), while when $\mu_i = 0$, there is no influence at all on the i-th component.
μ_i represents the accessibility of (the associated part of) the market. $\mu_i = 1$ means that, except maybe for a negligible part of the population, everyone can invest on this part of the market, while on the contrary, $\mu_i = 0$ means that, except for a negligible part, no-one can invest on this part. More generally, the dispersion of the μ_i is a measure of the heterogeneity of the market.

4.5 General sets of constraints

In this section we generalize the results established before for more general A_i. We will be able to show some existence results, but unfortunately it is very hard to compute the optimal portfolio.

4.5.1 Closed convex sets

If the sets of constraints fail to be vector spaces, but are still closed convex sets, then the existence of a unique projection that minimizes the distance allows us to derive some general results. We still denote by P_t^i the projection on $\sigma(t)A_i$, but it is not linear anymore.
We introduce the following assumption that plays the role of assumption (4.22):

$$\prod_{1 \le i \le N} \lambda_i < 1 \tag{4.30}$$

Let us define for each $t \in [0,T]$, $f_t^i : \mathbb{R}^d \to \mathbb{R}^d$ by:

$$\begin{aligned} f_t^i(X) := &\left[I - \frac{1}{N-1} \sum_{j \neq i} P_t^j \circ \left(I + \lambda_j^N P_t^j\right)^{-1} \circ \left[\lambda_j \left(I + \lambda_i^N P_t^i\right)\right] \right]^{-1} \\ &\circ \left[\eta_i X + \frac{1}{N-1} \sum_{j \neq i} P_t^j \circ \left(I + \lambda_j^N P_t^j\right)^{-1} \left[(\lambda_i \eta_j - \lambda_j \eta_i) X\right] \right]. \end{aligned} \tag{4.31}$$

Theorem 4.41 *Assume that σ and θ are deterministic. Assume also that assumption* (4.30) *holds. Then there exists a Nash equilibrium and the equilibrium portfolio for agent i is given by:*

$$\hat{\pi}_t^{i,N} = \sigma(t)^{-1} \circ P_t^i \circ f_t^i(\theta(t)),$$

where f^i : is defined by (4.31)*, and the value function for agent i at equilibrium is given by:*

$$V_i = V_i^N = -e^{-\frac{1}{\eta_i}(x^i - \lambda_i \bar{x}^i - Y_0^i)}$$

$$\text{where } Y_0^i = -\frac{\eta_i}{2}\int_0^T |\theta(t)|^2 dt + \frac{1}{2\eta_i}\int_0^T |(I - P_t^i) \circ f_t^i(\theta(t))|^2 dt.$$

Moreover, for any Nash equilibrium $(\tilde{\pi}^1, ..., \tilde{\pi}^N)$ we have:

$$\tilde{\pi}_t^i = \sigma(t)^{-1} P_t^i \left[\psi_t(\tilde{Z}_t)\right]^i \qquad \text{and} \qquad V_i = -e^{-\frac{1}{\eta_i}(x^i - \lambda_i \bar{x}^i - \tilde{Y}_0^i)},$$

where ψ_t is given in Lemma 4.42 below and $(\tilde{Y}, \tilde{Z})$ is a solution of the following N-dimensional BSDE:

$$d\tilde{Y}_t^i = -\frac{1}{2\eta_i}\left|(I - P_t^i)\left[\psi_t(\tilde{Z}_t)\right]^i\right|^2 dt + \tilde{Z}_t^i . dB_t, \qquad \tilde{Y}_T^i = -\eta_i \ln \frac{d\mathbb{Q}}{d\mathbb{P}},$$

which has a unique solution such that $\tilde{Z}$ is deterministic.

Proof. If A_i is a closed convex set, so is $\sigma(t)A_i$ and then the existence and uniqueness of the projection on $\sigma(t)A_i$ is guaranteed, and P^i has the same properties as for a vector space except that it is no longer a linear mapping. In particular we can no longer write $I - P^i = Q^i$, the projection on $(\sigma(t)A_i)^\perp$, but it is only a notational issue for our problem. The important point is that P^i is still a contraction. Therefore the proof of Theorem 4.18 still holds here, except for Lemma 4.27 that we replace below by Lemma 4.42, and we get the expressions for $\hat{\pi}^i$ and V_i. □

Lemma 4.42 *For each i, let B_i be a convex closed set and P^i be the orthogonal projection on B_i. Define $\varphi : M_{N,d}(\mathbb{R}) \to M_{N,d}(\mathbb{R})$ by:*

$$\forall i \in [1, N],\ \varphi^i(z) = z^i - \lambda_i^N \sum_{j \neq i} P^j z^j.$$

If assumption (4.30) *is satisfied, then φ is a bijection from $M_{N,d}(\mathbb{R})$ onto itself and its inverse ψ is given by:*

$$\psi^i(\zeta) = \left[I - \frac{1}{N-1}\sum_{j \neq i} P_t^j \circ \left(I + \lambda_j^N P_t^j\right)^{-1} \circ \left[\lambda_j \left(I + \lambda_i^N P_t^i\right)\right]\right]^{-1}$$
$$\circ \left[\zeta^i + \frac{1}{N-1}\sum_{j \neq i} P_t^j \circ \left(I + \lambda_j^N P_t^j\right)^{-1}(\lambda_i \zeta_j - \lambda_j \zeta^i)\right].$$

Moreover, ψ is Lipschitz continuous with a constant depending only on N and the λ_i's.

Proof. Following the steps of Lemma 4.27, we compute:

$$\lambda_i \left[I + \lambda_j^N P^j\right](z^j) = \lambda_j \left(I + \lambda_i^N P^i\right)(z^i) + \lambda_i \zeta^j - \lambda_j \zeta^i.$$

1) Let us prove that $I + \lambda_j^N P^j$ is a bijection of $\mathbb{R}^d$ and its inverse is a contraction.

As B_j is a closed convex set, P^j is a contraction of $\mathbb{R}^d$ and we have for any x, y, $(x-y).(P^j(x) - P^j(y)) \geq |P^j(x) - P^j(y)|^2 \geq 0$.

Let $y \in \mathbb{R}^d$, we set $f_y(x) = y - \lambda_j^N P^j(x)$.

As P^j is a contraction, we compute:

$$\begin{aligned}
|f_y(x) - f_y(x')| &= \lambda_j^N |P^j(x) - P^j(x')| \\
&\leq \lambda_j^N |x - x'|.
\end{aligned}$$

If $N \geq 3$, then thanks to assumption (4.30), $\lambda_j^N < 1$, which means that f_y is a strict contraction of $\mathbb{R}^d$, and so it has a unique fixed point x_0. This means that $f(x) = y - \lambda_j^N P^j(x) = x$ has a unique solution x_0, for any y, so that $I + \lambda_j^N P^j$ is a bijection of $\mathbb{R}^d$.

Now we prove that its inverse is a contraction. Indeed if $x \neq y$, we have:

$$\begin{aligned}
\left|x - y + \lambda_j^N (P^j(x) - P^j(y))\right|^2 &= |x-y|^2 + \left(\lambda_j^N\right)^2 |P^j(x) - P^j(y)|^2 \\
&\quad + 2\lambda_j^N (x-y).(P^j(x) - P^j(y)) \\
&\geq |x-y|^2 > 0.
\end{aligned}$$

So we get the contraction property of the inverse function.

If $N = 2$, f_y is no longer a strict contraction if $\lambda_j = 1$. But the previous computation still holds, which implies that $I + P^j$ is one-to-one. Using Lemma 4.43 below, we get the bijection property of $I + P^j$ and the contraction property of the inverse function is obtained as before.

2) Therefore we can compute:

$$\begin{aligned}
\xi &:= \sum_{j=1}^N P^j z^j \\
&= P^i z^i + \frac{1}{N-1} \sum_{j \neq i} P^j \left(I + \lambda_j^N P^j\right)^{-1} \left(\lambda_j \left(I + \lambda_i^N P^i\right)(z^i) + \lambda_i \zeta^j - \lambda_j \zeta^i\right).
\end{aligned}$$

And:

$$\begin{aligned}\zeta^i :=& \varphi^i(z) = z^i + \lambda_i^N(P^i z^i - \xi)\\ =& z^i - \frac{1}{N-1}\sum_{j\neq i} P^j\left(I+\lambda_j^N P^j\right)^{-1}\left[\lambda_j\left(I+\lambda_i^N P^i\right)(z^i) + \lambda_i\zeta^j - \lambda_j\zeta^i\right].\end{aligned}$$

So that finally:

$$\begin{aligned}&\left[I - \frac{1}{N-1}\sum_{j\neq i} P^j\left(I+\lambda_j^N P^j\right)^{-1}\ \left[\lambda_j\left(I+\lambda_i^N P^i\right)\right]\right](z^i)\\ &\qquad = \zeta^i + \frac{1}{N-1}\sum_{j\neq i} P^j\left(I+\lambda_j^N P^j\right)^{-1}(\lambda_i\zeta^j - \lambda_j\zeta^i).\end{aligned}$$

3) Now set: $g(x) = \left[I - \frac{1}{N-1}\sum_{j\neq i} P^j\left(I+\lambda_j^N P^j\right)^{-1}\left[\lambda_j\left(I+\lambda_i^N P^i\right)\right]\right](x)$. We show that under assumption (4.30), g is a bijection of $\mathbb{R}^d$.

First we compute:

$$\begin{aligned}\left|\left(I+\lambda_i^N P^i\right)(x) - \left(I+\lambda_i^N P^i\right)(y)\right| &\leq |x-y| + \lambda_i^N|P^i(x) - P^i(y)|\\ &\leq \left(1+\lambda_i^N\right)|x-y|,\end{aligned}$$

which gives the Lipschitz constant for $I + \lambda_i^N P^i$

Then we show that $P^j(I+\lambda_j^N P^j)^{-1}$ is $\frac{1}{1+\lambda_j^N}$-Lipschitz.

$$\begin{aligned}\left|\left(I+\lambda_j^N P^j\right)(x) - \left(I+\lambda_j^N P^j\right)(y)\right|^2 &= |x-y|^2 + \frac{2\lambda_j}{N-1}(x-y).(P^j(x) - P^j(y))\\ &\quad + \left(\lambda_j^N\right)^2|P^j(x) - P^j(y)|^2\\ &\geq \left(1 + \frac{2\lambda_j}{N-1} + \left(\lambda_j^N\right)^2\right)|P^j(x) - P^j(y)|^2\\ &\geq \left(1+\lambda_j^N\right)^2|P^j(x) - P^j(y)|^2.\end{aligned}$$

Therefore we get that $P^j\left(I+\lambda_j^N P^j\right)^{-1}\left[\lambda_j\left(I+\lambda_i^N P^i\right)\right]$ is $\frac{\lambda_j}{1+\lambda_j^N}(1+\lambda_i^N)$-Lipschitz.

Finally, let $y\in\mathbb{R}^d$, we define:

$$h_y(x) = \frac{1}{N-1}\sum_{j\neq i} P^j\left(I+\lambda_j^N P^j\right)^{-1}\left[\lambda_j\left(I+\lambda_i^N P^i\right)\right](x) + y.$$

$$\begin{aligned}|h_y(x)-h_y(x')| &\le \frac{1}{N-1}\sum_{j\neq i}\Big|P^j\left(I+\lambda_j^N P^j\right)^{-1}\left[\lambda_j\left(I+\lambda_i^N P^i\right)\right](x) \\ &\qquad -P^j\left(I+\lambda_j^N P^j\right)^{-1}\left[\lambda_j\left(I+\lambda_i^N P^i\right)\right](x')\Big| \\ &\le \frac{1}{N-1}\sum_{j\neq i}\frac{\lambda_j}{1+\lambda_j^N}\left(1+\lambda_i^N\right)|x-x'|.\end{aligned}$$

Now $\frac{\lambda_j}{1+\lambda_j^N}\left(1+\lambda_i^N\right) \le \max(\lambda_i,\lambda_j)$, with equality if and only if $\lambda_i=\lambda_j$, so that thanks to assumption (4.30), $K := \frac{1}{N-1}\sum_{j\neq i}\frac{\lambda_j}{1+\lambda_j^N}\left(1+\lambda_i^N\right) < 1$. As a consequence, h_y is a strict contraction on $\mathbb{R}^d$. Therefore it admits a unique fixed point, x_0, which means that x_0 is the unique solution of $g(x)=y$. Moreover, the previous computation shows that g^{-1} is $\frac{1}{1-K}$-Lipschitz, which ends the proof. $\square$

We used the following lemma:

Lemma 4.43 *Let A be a closed convex set of $\mathbb{R}^d$. We write*

$$B = 2A := \{y\in\mathbb{R}^d;\ \exists x\in A,\ y=2x\},$$

and P_A (resp. P_B) the projection on A (resp. B). Let $y\in\mathbb{R}^d$, we define $x=\frac{1}{2}P_B(y)$ and $z=y-2x$, then we have:

$$P_A(x+z)=x.$$

In particular, the image of $\mathbb{R}^d$ by $I+P$ is $\mathbb{R}^d$.

Proof. Let $y\in\mathbb{R}^d$, we define x and z as in the statement of the lemma. Then $P_B(y)$ is the only point satisfying:

$$\forall b\in B,\ (y-P_B(y)).(b-P_B(y))\le 0.$$

In other words, we have for all $b\in B$, $z.(b-2x)\le 0$, or by definition of B, for any $a\in A$, $z.(2a-2x)\le 0$, so that:

$$\forall a\in A,\ (x+z-x).(a-x)\le 0,$$

which means that $x=P_A(x+z)$. Now $(I+P_A)(x+z)=x+z+x=y$, which completes the proof. $\square$

Remark 4.44 Notice that, in contrast with the case of vector subspaces, even if for all j's, $\lambda_j = \lambda$ and $\eta_j = \eta$, we cannot provide a simpler expression in general. We just have:

$$\hat{\pi}^{i,N} = \sigma^{-1} P^i \circ \left[I - \frac{1}{N-1} \sum_{j \neq i} P^j \circ (I + \lambda^N P^j)^{-1} \circ [\lambda(I + \lambda^N P^i)] \right]^{-1}$$
$$\circ \left(\eta\theta + \frac{1}{N-1} \sum_{j \neq i} P^j \circ (I + \lambda^N P^j)^{-1}(0) \right] .$$

Once again, we are interested in the limit as $N \to \infty$. We formulate it only in the case of similar agents, and we do not give the proof as it is a particular case of Proposition 6.20 in the last chapter.

Proposition 4.45 *Assume that for all j, $\lambda_j = \lambda$ and $\eta_j = \eta$ and that $\lambda < 1$.*

(i) *If for all t, there exists a function f_t such that, as $N \to \infty$, $\frac{1}{N} \sum_{i=1}^N P_t^i$ converges uniformly on any compact set towards f_t, then:*

$$\hat{\pi}_t^{i,N} \to \hat{\pi}_t^{i,\infty} := \sigma(t)^{-1} \circ P_t^i \circ (I - f_t \circ (\lambda I))^{-1} (\eta_i \theta(t) + f_t(0)) .$$

(ii) *If there exists a function f_t such that for any compact set K, as $N \to \infty$:*

$$\sup_{(t,x) \in [0,T] \times K} \left| \frac{1}{N} \sum_{i=1}^N P_t^i(x) - f_t(x) \right| \to 0, \text{ then:}$$

$$\sup_{t \in [0,T]} |\hat{\pi}_t^{i,N} - \hat{\pi}_t^{i,\infty}| \to 0.$$

Unfortunately, in this case of non-linear operators, we are not able to analyze the influence of λ.

4.5.2 Non convex closed sets

If the assumption of convexity fails, we do not know if the results can be extended. There is no definition of a projection in general. We still can define things using the distance to the set A_i, and, assuming the axiom of choice, choose simultaneously a particular vector in all the sets of minimizers. Therefore we can define a representation that we write P^i. Proceeding as in the proof of Theorem 4.18, an optimal portfolio should be represented using this P^j's. But even with this, there is no reason that φ is one-to-one nor surjective onto

$M_{N,d}(\mathbb{R})$. Moreover, P^j is not continuous, which could be a problem.

The lack of one-to-one property is not the biggest problem. It would just mean that there could exist more than one Nash equilibrium. But the fact that it could not map onto $M_{N,d}(\mathbb{R})$ is a real problem, as it would lead to a constrained (N-dimensional) BSDE. But with no additional nondecreasing term, there is no solution to such BSDEs in general. And so there is no Nash equilibrium in general.

We derive here a very simple example in which the one-to-one property is satisfied, but not the surjectivity onto $\mathbb{R}^d$. Consider $N = 2$, $\sigma = I_d$ and for each i, $\lambda_i = \lambda$ and $A_i = A := \{x \in \mathbb{R}^d;\ |x_1| \geq 1\}$, where x_1 is the first component of x. The projection is uniquely determined for $x_1 \neq 0$, and we can take for example the following:

$$P(x) = \begin{cases} x, \text{ if } x \in A \\ (1, x_2, ..., x_d)^t, \text{ if } x_1 \in [0,1) \\ (-1, x_2, ..., x_d)^t, \text{ if } x_1 \in (-1,0). \end{cases}$$

If φ was surjective onto $M_d(\mathbb{R})$, then substracting the expressions of φ^1 and φ^2 we see that $I + \lambda P$ would be surjective onto $\mathbb{R}^d$. Let $y \in \mathbb{R}^d$, we want to find x such that $x + \lambda P(x) = y$.

- If $x_1 \geq 1$, then $(1 + \lambda)x_1 = y_1$, so that $y_1 \geq 1 + \lambda$;
- if $x_1 \in [0, 1)$, then $x_1 + \lambda = y_1$, so that $y_1 \in [\lambda, 1 + \lambda)$;
- if $x_1 \in (-1, 0)$, then $x_1 - \lambda = y_1$, so that $y_1 \in (-1 - \lambda, -\lambda)$;
- if $x_1 \leq -1$, then $(1 + \lambda)x_1 = y_1$, so that $y_1 \leq -1 - \lambda$.

Therefore $\{x \in \mathbb{R}^d;\ x_1 \in [-\lambda, \lambda)\}$ is not attained by $I + \lambda P$, so that as soon as $\lambda > 0$, φ is not surjective. Moreover, the interior of the complementary of its image is non empty.

We can give another simple example, where the projection is well defined on $\mathbb{R}^d \backslash \{0\}$, and yet the image of φ is not $\mathbb{R}^d$. Consider this time that for each i, $A_i = B := \{x \in \mathbb{R}^d;\ |x| \geq 1\}$, the complementary of the unit (open) ball. The projection is uniquely determined for $x \neq 0$, on we can for example take:

$$P(x) = \begin{cases} x, \text{ if } x \in B \\ \frac{1}{|x|}x, \text{ if } |x| \in (0,1) \\ 1_d, \text{ if } x = 0. \end{cases}$$

As before, in order to have φ surjective, we need $I + \lambda P$ surjective onto $\mathbb{R}^d$. If $y \in \mathbb{R}^d$, and $x + \lambda P(x) = y$, we compute:

- If $|x| \geq 1$, then $(1 + \lambda)x = y$, so that $|y| \geq 1 + \lambda$;

- if $|x| \in (0,1)$, then $\left(1 + \dfrac{\lambda}{|x|}\right) x = y$, so that $|y| \in (\lambda, 1+\lambda)$;
- if $x = 0$, then $y = \lambda 1_d$.

Therefore $\{x \in \mathbb{R}^d;\ |x| < \lambda\}$ is not attained by $I + \lambda P$, so again as soon as $\lambda > 0$, φ is not surjective. Moreover the interior of the complementary of its image is non empty.

Chapter 5

Examples for the optimal investment under relative performance concerns

5.1 Introduction

In this chapter, we consider several examples that illustrate the theoretical study of optimal investment under relative performance concerns made in the previous chapter.

In the previous study, for exponential utility functions and deterministic coefficients in the dynamics of the risky assets, we proved the existence and uniqueness of a Nash equilibrium. We also proved a convergence result of the equilibrium portfolio of an agent when the number of assets d is fixed while the number of agents N goes to infinity. Moreover if the sets of constraints are vector spaces, we showed two results on the influence of the parameter λ. First, under some additional assumptions, we showed that for N fixed the risk of the wealth of any agent was nondecreasing with respect to λ. Then, in general, we proved that in the limit N goes to infinity, the global risk of the market was nondecreasing with respect to λ.

We hereafter develop several examples for which we can apply all or a part of the previous results. We will also see that in specific cases we can say more than the general results formulated before. These examples allow us to derive some interesting economical consequences.

5.2 A simple example

First let us consider a very simple example with $N = d = 3$. Let $\mathbf{e} = (e_1, e_2, e_3)$ be the canonical basis of $\mathbb{R}^3$. Assume that $A_1 = \mathbb{R}e_1 + \mathbb{R}e_2$, $A_2 = \mathbb{R}e_2 + \mathbb{R}e_3$ and $A_3 = \mathbb{R}e_3$. Assume

also that $\sigma = \sigma I_3$. We see that agent 1 and 3 invest in separate assets that are independent. Without agent 2, they would invest as in the classical case (see "independent investments" below), for any λ_i, in other words, they will have no influence on each other.
In order to simplify, we take $\forall i$, $\lambda_i = \lambda$ and $\eta_i = \eta$.

Notice that $A_1 \cap A_2 \cap A_3 = \{0\}$. Using Corollary 4.21, for any $\lambda \in [0,1]$ we have the following equilibrium portfolio:

$$\hat{\pi}^i_t = \eta P^i \left[I - \frac{\frac{\lambda}{2}}{1+\frac{\lambda}{2}} \sum_{j\neq i} P^j \left(I + \frac{\lambda}{2} P^i \right) \right]^{-1} \theta(t).$$

We denote by $Mat_{\mathbf{e}}(U)$ the matrix of the operator U, expressed in the basis $\mathbf{e}$. We get:

$$Mat_{\mathbf{e}}\left(I - \frac{\frac{\lambda}{2}}{1+\frac{\lambda}{2}} \sum_{j\neq 1} P^j \left(I + \frac{\lambda}{2} P^1 \right) \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\lambda}{2} & 0 \\ 0 & 0 & 1-\frac{\lambda}{1+\frac{\lambda}{2}} \end{pmatrix}$$

$$Mat_{\mathbf{e}}\left(P^1 \left[I - \frac{\frac{\lambda}{2}}{1+\frac{\lambda}{2}} \sum_{j\neq 1} P^j \left(I + \frac{\lambda}{2} P^1 \right) \right]^{-1} \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{1-\frac{\lambda}{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Mat_{\mathbf{e}}\left(P^2 \left[I - \frac{\frac{\lambda}{2}}{1+\frac{\lambda}{2}} \sum_{j\neq 2} P^j \left(I + \frac{\lambda}{2} P^2 \right) \right]^{-1} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{\lambda}{2}} & 0 \\ 0 & 0 & \frac{1}{1-\frac{\lambda}{2}} \end{pmatrix}$$

$$Mat_{\mathbf{e}}\left(P^3 \left[I - \frac{\frac{\lambda}{2}}{1+\frac{\lambda}{2}} \sum_{j\neq 3} P^j \left(I + \frac{\lambda}{2} P^3 \right) \right]^{-1} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\frac{\lambda}{2}} \end{pmatrix},$$

leading to the equilibrium portfolios:

$$\hat{\pi}^1_t = \eta\theta^1(t)e_1 + \frac{\eta\theta^2(t)}{1-\frac{\lambda}{2}}e_2; \quad \hat{\pi}^2_t = \frac{\eta\theta^2(t)}{1-\frac{\lambda}{2}}e_2 + \frac{\eta\theta^3(t)}{1-\frac{\lambda}{2}}e_3; \quad \hat{\pi}^3_t = \frac{\eta\theta^3(t)}{1-\frac{\lambda}{2}}e_3.$$

In order to emphasize the influence of other agents, recall that the classical case would correspond to the following equilibrium portfolios:

$$\hat{\pi}^1_t = \eta\left(\theta^1(t)e_1 + \theta^2(t)e_2\right); \quad \hat{\pi}^2_t = \eta\left(\theta^2(t)e_2 + \theta^3(t)e_3\right); \quad \hat{\pi}^3_t = \eta\theta^3(t)e_3.$$

We see that the component with respect to the e_1 is not affected as there is only one investor that has access to it. However the two other directions e_2 and e_3 are both multiplied by a factor $\dfrac{1}{1-\frac{\lambda}{2}} \in [1,2]$ as two investors have access to each of them.

We can also compare it to the situation where there are only agents 1 and 2 ($N = 2$, $d = 3$ in that case). This time $A_1 \cap A_2 \neq \{0\}$, so we need $\lambda < 1$. Then, we have:

$$Mat_{\mathbf{e}}\left(P^1\left[I - \frac{\lambda}{1+\lambda}P^2(I+\lambda P^1)\right]^{-1}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{1-\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Mat_{\mathbf{e}}\left(P^2\left[I - \frac{\lambda}{1+\lambda}P^1(I+\lambda P^2)\right]^{-1}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{1-\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that:

$$\hat{\pi}_t^1 = \eta\theta^1(t)e_1 + \frac{\eta\theta^2(t)}{1-\lambda}e_2; \qquad \hat{\pi}_t^2 = \frac{\eta\theta^2(t)}{1-\lambda}e_2 + \eta\theta^3(t)e_3.$$

The component with respect to e_1 or e_3 is the same as in the classical case, and in particular for agent 2, the component with respect to e_3 is smaller in absolute value than in the three agents world. The component with respect to e_2 is the classical one multiplied by a factor $\frac{1}{1-\lambda} \in [1, +\infty)$, which is greater than the one of the three agents world whereas in both cases, two investors have access to it.

Remark 5.1 Taking specific λ_i's (but the same η), we would get the following equilibrium portfolios:

$$\hat{\pi}_t^1 = \eta\theta^1(t)e_1 + \frac{\eta(2+\lambda_1)}{2-\frac{\lambda_1\lambda_2}{2}}\theta^2(t)e_2; \qquad \hat{\pi}_t^2 = \frac{\eta(2+\lambda_2)}{2-\frac{\lambda_1\lambda_2}{2}}\theta^2(t)e_2 + \frac{\eta(2+\lambda_2)}{2-\frac{\lambda_2\lambda_3}{2}}\theta^3(t)e_3;$$
$$\hat{\pi}_t^3 = \frac{\eta(2+\lambda_3)}{2-\frac{\lambda_2\lambda_3}{2}}\theta^3(t)e_3,$$

which is more or less the same as before, but we see that two agents investing on the same area, will not have exactly the same component on that area because the λ_i's are not all the same.

5.3 The complete market case

We derive here again the complete market case in order to check the results of section 4.3. We also consider new situations, as we can take specific λ_i's as well as specific η_i's. From a financial point of view, this example corresponds to a unique complete market, with no restrictions at all.

In this case, $A_i = \mathbb{R}^d$, and we need to have $\prod_{i=1}^{N} \lambda_i < 1$, so that using Theorem 4.18, we get the following equilibrium portfolio:

$$\hat{\pi}_t^{i,N} = \left[1 - \sum_{j\neq i} \frac{\lambda_j^N}{1+\lambda_j^N}\left(1+\lambda_i^N\right)\right]^{-1} \left(\eta_i + \sum_{j\neq i} \frac{\lambda_i^N \eta_j - \lambda_j^N \eta_i}{1+\lambda_j^N}\right) \sigma(t)^{-1}\theta(t).$$

One can check that for the same λ_i's it is the same as in section 4.3. Recall that the classical optimal portfolio is $\eta_i \sigma^{-1}\theta$ (ie all λ_j's equal to 0).

We can apply Proposition 4.37, but in this case, we can say more about the influence of the λ_j's, as the equilibrium portfolio is a simple dilation (homothety) of the classical portfolio. Not only $|\sigma(t)\hat{\pi}_t^{i,N}|$ is increasing w.r.t any λ_j and nondecreasing w.r.t any η_j but we can see (exactly as in the proof of Proposition 4.37) that in absolute value, any component of this equilibrium portfolio has the same behavior.

Then, using Proposition 4.31, we get the limit portfolio as $N \to \infty$, while d remains constant:

$$\hat{\pi}_t^{i,\infty} = \left(\eta_i + \bar{\eta}\frac{\lambda_i}{1-\bar{\lambda}}\right)\sigma(t)^{-1}\theta(t)$$

where $\bar{\lambda} < 1$ and $\bar{\eta}$ are respectively the average λ and η of the market.

Again the absolute value of each component is increasing w.r.t λ_i, increasing w.r.t η_i, nondecreasing w.r.t $\bar{\eta}$, and it is also nondecreasing w.r.t $\bar{\lambda}$. We also see that if $\lambda_i > 0$, when $\bar{\lambda}$ goes to 1, even if $\lambda_i < 1$, any component of π^i_∞ goes a.s towards ∞ (in absolute value). In other words, it means that even if agent i is "reasonable" in the sense that he does not only take care about relative performance, if the rest of the market mainly takes only care about relative performance, then agent i's behavior will be contaminated by the stupidity of the market and will have to take "infinite" risk.

For N fixed, several particular cases are interesting to study. There was of course the case with a common λ, already discussed in section 4.3. Then, in the spirit of the previous remark when $N \to \infty$, there is the case where agent i has a $\lambda_i \in (0,1)$ while other agents have $\lambda_j = 1$. There is also the case where $\lambda_i > 0$ while other agents have $\lambda_j = 0$.

If $\lambda_i \in (0,1)$ while $\lambda_j = 1$, $j \neq i$ and $1 \leq j \leq N$. Then, writing $\eta^{N,i} = \frac{1}{N-1}\sum_{j\neq i}\eta_j$, we compute for agent i:

$$\begin{aligned}\hat{\pi}^{i,N} &= \left(1 - \frac{N-1}{N}\left(1+\lambda_i^N\right)\right)^{-1}\left(\eta_i - \eta_i\frac{N-1}{N} + \frac{\lambda_i}{N}\sum_{j\neq i}\eta_j\right)\sigma^{-1}\theta \\ &= \frac{1}{1-\lambda_i}\left[\eta_i + \lambda_i\eta^{N,i}(N-1)\right]\sigma^{-1}\theta.\end{aligned}$$

So that, as $N \to \infty$, if $\eta^{N,i} \to \bar{\eta}$, then each component of the portfolio goes a.s towards ∞ (in absolute value), which gives the same result as before when we took first the limit over N and then the limit $\bar{\lambda} \to 1$.

Finally, if we take $\lambda_i > 0$, and $\lambda_j = 0$, $j \neq i$ and $1 \leq j \leq N$, we get:

$$\hat{\pi}^{i,N} = \left(\eta_i + \lambda_i\eta^{N,i}\right)\sigma^{-1}\theta.$$

So, as $N \to \infty$, if $\eta^{N,i} \to \bar{\eta}$, then we get $\hat{\pi}^{i,\infty} = (\eta_i + \lambda_i\bar{\eta})\,\sigma^{-1}\theta$ So we see that for agent i, the number of agents in the market does not affect his behavior. But as for other agents, for any N, $\pi^{j,N} = \eta_j\sigma^{-1}\theta$, the limit market index will be the same as in the classical case (no additional risk).
Those different cases also bring the following intuitive conclusion: if investors mostly behave "cautiously", then the limit $N \to \infty$ will hide the few black sheep, so that the market will behave as in the classical case, while if investors mostly behave "riskily", then the limit will hide the few good investors, so that the market will go crazy (infinite positions).

5.4 Specific and independent investments

Now let us take a look at the case where agent i invests in the stock S^i only, where S^i and S^j are assumed to be independent. In other words, $d = N$ and we have $\sigma = \sigma I_N$ ($\sigma > 0$), $A_i = \mathbb{R}e_i$, where (e_i) is the canonical (orthonormal) basis of $\mathbb{R}^N$. More generally, we could assume that (e_i) is a basis of $\mathbb{R}^N$ and $(\sigma_t e_i)$ is an orthogonal basis of $\mathbb{R}^N$ for any t. As $\bigcap_{i=1}^{N} A_i = \{0\}$, we do not need an additional assumption on the λ_i's. From a financial point of view, we can see it as each agent investing in his own market (for example its country) or in a specific sector of activities, and that those markets or sectors are independent. Of course it is not realistic to consider that only one agent invests in a given sector or country, but we will see later a generalization of this hypothesis. The problem of no correlation is also dealt with in the next section.

As $P^iP^j = 0$ for $i \neq j$, and $\sum_{j=1}^{N} P^j = I$ we get:

$$\left[I - \sum_{j\neq i} \frac{\lambda_j^N}{1+\lambda_j^N} P^j \left(I + \lambda_i^N P^i\right)\right]^{-1} = P^i + \sum_{j\neq i} \left(1 - \frac{\lambda_j^N}{1+\lambda_j^N}\right)^{-1} P^j$$

$$\text{so } P^i \left[I - \sum_{j\neq i} \frac{\lambda_j^N}{1+\lambda_j^N} P^j \left(I + \lambda_i^N P^i\right)\right]^{-1} = P^i.$$

And finally:

$$\hat{\pi}^{i,N} = \eta_i \sigma^{-1} P^i \theta = \eta_i \sigma^{-1} \theta_i e_i$$

which is the optimal portfolio in the classical case, and in particular is independent from λ_i and of N.

In other words, if every person invests in assets that are independent from the assets available for others, then λ has no influence on their behavior.

5.5 A case of specific and correlated investments

Let us generalize a bit the previous case, as it is more realistic to consider that different markets or sectors are correlated. Indeed, we want to see what happens if we put some correlation between the assets, but in order to be able to make computations, we consider a very symmetric situation. So here again $d = N$, $A_i = \mathbb{R}e_i$, but now

$$\sigma^2 = \sigma_N^2 \begin{pmatrix} 1 & & \rho^2 \\ & \ddots & \\ \rho^2 & & 1 \end{pmatrix}$$

for a certain $\rho \in (-1, 1)$ and $\sigma_N > 0$. σ_N indeed depends on N for normalization reasons, because on the contrary, for $\rho \neq 0$, the overall volatility would explode as $N \to \infty$. We want to have the same volatility as in the case $\rho = 0$: $\bar{\sigma}^2 = |\sigma 1|^2 = N\sigma_0^2$, we will take $\sigma_N = \dfrac{\sigma_0}{\sqrt{1+(N-1)\rho^2}}$.

Here the computation of the equilibrium portfolio is not straightforward. We provide it in the particular case where $\lambda_j = \lambda$, $\eta_j = \eta$ and $\theta = \theta_N 1_N$, 1_N being the vector of ones of $\mathbb{R}^N$, and θ_N is a normalization factor. In fact, we want $|P^i\theta|$ to be equal to its value in the case $\rho = 0$, so we take $\theta_N = \dfrac{1}{1+(N-1)\rho^2}$. In fact the method applied hereafter also works for the general case, but the expressions are really complicated, so we thought it would be more convenient to make this assumption in order to derive interesting conclusions. We also consider the limit as $N \to \infty$, which is not a consequence of Proposition 4.31 as $d = N$ is not fixed.

Proposition 5.2 *For N fixed, we get the following equilibrium portfolio:*

$$\hat{\pi}^{i,N} = \frac{\eta\theta_0}{\sigma_0\left(1+\rho^2(N-1)\right)}\left[1+\frac{\rho^2(N-1)(1+\gamma\rho^2)}{(1-\delta[1+\rho^2(N-2)+\gamma\rho^4(N-1)])}\right]e_i,$$

where $\gamma = \gamma_N = \lambda^N$ and $\delta = \delta_N = \dfrac{\gamma}{1+\gamma}$.
As $N \to \infty$, $\hat{\pi}^{i,N}$ converges to:

$$\hat{\pi}^{i,\infty} = \frac{\eta\theta_0}{\sigma_0}\frac{1}{1-\lambda\rho^2}e_i.$$

Proof. As σ is assumed to be symmetric positive definite, we have: $\sigma = \sigma_N \begin{pmatrix} a_N & & b_N \\ & \ddots & \\ b_N & & a_N \end{pmatrix}$ with $a = \sqrt{1-(N-1)b^2}$ and $b = \sqrt{\dfrac{2+\rho^2(N-2)-2\sqrt{1+\rho^2(N-2)-\rho^4(N-1)}}{4(N-1)+(N-2)^2}}$. In particular for $\rho = 0$, we have $a = 1$ and $b = 0$, as in the previous section.

And we also have: $\sigma^{-1} = \frac{1}{\sigma_N}\begin{pmatrix} c_N & & d_N \\ & \ddots & \\ d_N & & c_N \end{pmatrix}$, for some c_N and d_N. We do not need their values.

But noticing that $|\sigma 1_N|^2 = 1_N.\sigma^2 1_N$ and similarly $|\sigma^{-1}1_N|^2 = 1_N.\left(\sigma^2\right)^{-1}1_N$, brings the following identities:

$$\begin{aligned} a_N + (N-1)b_N &= \sqrt{1+(N-1)\rho^2} \\ c_N + (N-1)d_N &= \frac{1}{\sqrt{1+(N-1)\rho^2}}. \end{aligned}$$

We write $u_i = \sigma e_i = \sigma_N \left(a_N e_i + b_N \sum_{j\neq i} e_j\right)$. We symmetrically have $e_i = \sigma^{-1} e_i = \dfrac{1}{\sigma_N}\left(c_N e_i + d_N \sum_{j\neq i} e_j\right)$. Thus if $\theta = \theta_N \sum_{j=1}^{N} e_i = \sum_{j=1}^{N} \bar{\theta}_i u_i$, we get for all i:

$$\bar{\theta}_i = \frac{\theta_N}{\sigma_N}\left(c_N + (N-1)d_N\right) = \frac{\theta_0}{\sigma_0(1+(N-1)\rho^2)}.$$

We compute $u_i.u_j = \sigma_N^2(2a_Nb_N+(N-2)b_N) = \sigma_N^2\rho^2$ and $|u_i^2| = \sigma_N^2(a_N^2+(N-1)b_N^2) = \sigma_N^2$. Therefore, if $x = \sum_{j=1}^N x_j u_j$, we compute:

$$P^i(x) = \Big(x_i + \rho^2 \sum_{j\neq i} x_j\Big) u_i,$$

and if $i \neq j$, $P^jP^i(x) = \rho^2\left(x_i + \rho^2 \sum_{j\neq i} x_j\right) u_j$. We see here why we had to take $\theta_N = \dfrac{\theta_0}{\sigma_0(1+(N-1)\rho^2)}$ in order to guarantee that $|P^i(1_N)|$ is independent of ρ.

In order to simplify notations, we write $\gamma := \lambda^N$ and $\delta := \dfrac{\gamma}{1+\gamma}$. Let i be given. We want to solve:

$$\left[I - \sum_{j\neq i} \frac{\lambda_j^N}{1+\lambda_j^N} P^j\left(I + \lambda_i^N P^i\right)\right]^{-1}\left(\sum_{j=1}^N u_j\right) = Y.$$

Lemma 4.27 tells us that Y exists and is unique. We look for Y of the form: $Y = y_iu_i + y_0\sum_{j\neq i} u_j$. And we get:

$$\begin{aligned} y_i &= 1 \\ \text{for } j \neq i,\ y_0 - \delta(N-1)&\left[(1+\rho^2(N-2))y_0 + \rho^2 y_i + \gamma\rho^2(y_i + (N-1)\rho^2 y_0)\right] = 1. \end{aligned}$$

Therefore, as $\delta(1+\gamma) = \gamma$:

$$y_0 = \frac{1+\gamma\rho^2}{1-\delta[1+\rho^2(N-2)+\gamma\rho^4(N-1)]},$$

and we compute:

$$\begin{aligned} \sigma\hat{\pi} &= \frac{\eta\theta_0}{\sigma_0}\frac{1+\rho^2(N-1)y_0}{1+\rho^2(N-1)} u_i \\ &= \frac{\eta\theta_0}{\sigma_0\left(1+\rho^2(N-1)\right)}\left[1 + \frac{\rho^2(N-1)(1+\gamma\rho^2)}{(1-\delta[1+\rho^2(N-2)+\gamma\rho^4(N-1)])}\right] u_i. \end{aligned}$$

And as $N \to \infty$, we get the convergence to $\hat{\pi}^{i,\infty}$. □

Notice that here the P^i's do not commute except for $\rho = 0$, so Proposition 4.37 cannot be applied, but from the previous proposition we see that $|\hat{\pi}^{i,N}|$ and $|\hat{\pi}^{i,\infty}|$ are nondecreasing w.r.t λ, the relevant variable being more or less $\lambda\rho^2$. In the case $N \to \infty$, it is clearer: the optimal portfolio is the classical one, dilatated by the factor $\frac{1}{1-\lambda\rho^2}$. So we see that there are two factors that increase the risk:

- the more you look at other agents (impact of λ)
- the more correlated the assets are (impact of ρ^2)

In particular for $\lambda = 0$ we find the results of the previous example. It is striking to notice that as $N \to \infty$, if one takes the limit $\rho \to \pm 1$, we find the same Nash equilibrium as in the complete market case (which is not true for fixed N), whereas there is no reason for it to be true, as we have assumed everywhere that (u_i) was a basis of $\mathbb{R}^N$.

5.6 Groups of managers investing in independent sectors

Another interesting example would be to consider a mix between the previous cases. More precisely, let us assume that there are different groups of investors, such that in the same group, everyone can invest on the same asset (or the same assets), while the assets accessible for investors from different groups are independent. This is a more realistic extension of the case where each agent can only invest in his own market. It means that we have different markets and/or sectors of activities, and the agents can invest in one of those. Notice also that the common investment and the specific investment are particular (but interesting) cases of this example.

We will assume that there are d groups, with k_i $(1 \leq i \leq m)$ agents in each group and d assets as well, satisfying $N = \sum_{i=1}^{d} k_i$. For an agent j in the group i, we have $A_j = \mathbb{R}e_i$. In order to simplify, we take $\sigma = \sigma_0 I_d$ and assume that inside the same group, agents share the same λ and η. We use again the notations $\gamma_j = \lambda_j^N$ and $\delta_j = \frac{\gamma_j}{1+\gamma_j}$.

So we index by the number of the group. For an agent in group i we have:

$$\begin{aligned}
\left[I - \sum_{j\neq i} k_j\delta_j P^j(I+\lambda_i P^i) - (k_i-1)\delta_i P^i(I+\gamma_i P^i)\right]^{-1} &= \left[I - \sum_{j\neq i} k_j\delta_j P^j - (k_i-1)\gamma_i P^i\right]^{-1} \\
&= \frac{1}{1-(k_i-1)\gamma_i}P^i + \sum_{j\neq i}\frac{1}{1-k_j\delta_j}P^j;
\end{aligned}$$

$$P^i\left[I - \sum_{j\neq i} k_j\delta_j P^j(I+\lambda_i P^i) - (k_i-1)\delta_i P^i(I+\gamma_i P^i)\right]^{-1} = \frac{1}{1-(k_i-1)\gamma_i}P^i.$$

So that finally for group i:

$$\hat{\pi}^{i,N} = \frac{\eta_i}{\sigma_0} \frac{1}{1 - \frac{\lambda_i(k_i-1)}{N-1}} \theta_i e_i.$$

In particular if $d = 1$, we find the same result as in the complete market case, and for $k_i = 1$, the same as in the specific independent investment case. In general, we see that an agent is affected by the people of his own group, but with a factor $\lambda \dfrac{k_i - 1}{N - 1}$ instead of λ in the complete market case. In other words, the bigger is the group you belong to, the more risk you will take. Once again we can take the limit as $N \to \infty$, assuming $\dfrac{k_i^N}{N} \to \alpha_i (\in [0, 1])$, which gives:

$$\hat{\pi}^{i,\infty} = \frac{\eta_i}{\sigma_0} \frac{1}{1 - \lambda_i \alpha_i} \theta_i e_i.$$

Notice that if $\alpha_i = 0$, which means that at the limit $N \to \infty$, the group i is not representative (or the proportion of this group in the population tends to 0), then $\hat{\pi}^i_\infty$ is the classical optimal portfolio: there is no influence of λ_i for agents inside this group.

Remark 5.3 Even if we did not do it in order to simplify notations, this could be done exactly in the same way for λ and η not constant inside a group, and it would mainly lead to the same conclusions. In particular, the last remark for $\alpha_i = 0$ still holds.

5.7 Investment with respect to hyperplanes

Another interesting example is the one where each A_i is an hyperplane of $\mathbb{R}^d$, and such that the $(\sigma A_i)^\perp$'s generate an orthogonal basis. In order to simplify, we consider $\sigma = \sigma_0 I_N$ ($\sigma_0 > 0$), and $A_i = (\mathbb{R} e_i)^\perp$, even though what follows can be done as long as (σA_i) is an orthogonal basis of $\mathbb{R}^d$. From a financial point of view, it can be seen as the following: a bank or a hedgefund can invest in the whole market except for its own stock or the stocks of firms on which it has access to non public information (for example if you manage the money of a firm it will be the case).
As before, we define $\gamma_i = \lambda_i^N$. We compute easily that:

$$\hat{\pi}^{i,N} = \frac{\eta_i}{\sigma_0} \sum_{j \neq i} \left(1 - \sum_{k \neq i,j} \frac{\gamma_k}{1 + \gamma_k} (1 + \gamma_i) \right)^{-1} \left(\eta_i + \sum_{k \neq i,j} \frac{\gamma_i \eta_j - \gamma_j \eta_i}{1 + \gamma_j} \right) \theta_j e_j.$$

In particular if $\lambda_j = \lambda$ and $\eta_j = \eta$ for all j, we get:

$$\hat{\pi}^{i,N} = \frac{\eta}{\sigma_0} \frac{1}{1 - \lambda + \lambda^N} \sum_{j \neq i} \theta_j e_j.$$

As in the complete market situation, the equilibrium portfolio is a dilatation of what would be obtained in the classical case ($\lambda = 0$), but with a factor $\dfrac{1}{1-\lambda+\lambda^N}$, instead of $\dfrac{1}{1-\lambda}$, which means that the more investors there are, the bigger this factor is.

As $N \to \infty$, as long as $\sum_{j \neq i} \theta_j e_j$ is well-defined, we get:

$$\hat{\pi}^{i,\infty} = \frac{\eta}{\sigma_0} \frac{1}{1-\lambda} \sum_{j \neq i} \theta_j e_j,$$

which is the same as in the complete maket case. This is not surprising, because when $d = N \to \infty$, investing in the whole market except one asset is almost the same as investing in the whole market.

5.8 Simple examples with more general convex sets

Although it is harder to compute explicitly the optimal portfolio, we can still do it in simple cases when the sets of constraints are convex sets instead of vector spaces. We briefly develop very simple examples. In particular, we always assume that $\sigma = I$, $\lambda_j = \lambda$ and $\eta_j = \eta$ for all j's.

5.8.1 Common investment

First assume that all agents share the same set $A_j = \bar{B}(x,r)$ for a certain $x \in \mathbb{R}^d$, where $\bar{B}(x,r)$ is the closed ball of center x and radius $r > 0$, for the canonical euclidean norm of $\mathbb{R}^d$. Then we find the following optimal portfolio:

$$\hat{\pi}_t^i = P\left(\frac{\eta}{1-\lambda}\theta(t)\right) = \begin{cases} \frac{\eta}{1-\lambda}\theta(t) \text{ if } \frac{\eta}{1-\lambda}\theta(t) \in \bar{B}(x,r) \\ x + \frac{r}{\left|\frac{\eta}{1-\lambda}\theta(t) - x\right|}\left[\frac{\eta}{1-\lambda}\theta(t) - x\right] \text{ otherwise.} \end{cases}$$

Notice in particular that, as one could expect, $\hat{\pi}_t^i - x$ is colinear to $\dfrac{\eta}{1-\lambda}\theta(t) - x$ and that as soon as $\dfrac{\eta}{1-\lambda}\theta(t) \notin \bar{B}(x,r)$, then $\hat{\pi}_t^i$ is in the boundary of $\bar{B}(x,r)$. One can prove that $|\hat{\pi}|$ is nondecreasing w.r.t λ and η.

5.8.2 Specific independent investments

Now assume that $A_i = [a_i, b_i]e_i$ with $a_i \leq b_i$ for each i, (e_j) being the canonical basis of $\mathbb{R}^d$. Then we have the following optimal portfolio for agent i:

$$\hat{\pi}_t^i = P^i\left(\eta\theta(t)\right) = \begin{cases} \eta\theta^i(t) \text{ if } \eta\theta^i(t) \in [a_i, b_i] \\ a_i \text{ if } \eta\theta^i(t) < a_i \\ b_i \text{ otherwise.} \end{cases}$$

Chapter 6

Optimal investment under relative performance concerns with local risk penalization

6.1 Introduction

Since the seminal papers of Merton [58, 59], the problem of optimal investment has been extensively studied in order to generalize the original framework. Those generalizations have been made in different directions and using different techniques. We refer to [67], [10] or [46] for the complete market situation, to [12] or [79] for constrained portfolios, to [9], [15], [75], [17] or [2] for transactions costs, to [8], [44, 45], [13], [3, 4] or the first chapter for taxes, and to [43], [53, 54] or [55] for general incomplete markets.

But in all those works, no interaction between agents is taken into account, whereas economical and sociological studies have emphasized the importance of relative concerns in human behaviors, see Veblen [78], Abel [1], Gali [34], Gomez, Priestley and Zapatero [36] or DeMarzo, Kaniel and Kremer [16]. Indeed, a performance makes sense in a specific context and in comparison to competitors or a benchmark. A return of 5% during a crisis is not equivalent to the same return during a financial bubble. Moreover, human beings tend to compare themselves to their peers.

In the two previous chapters, we studied the optimal investment problem including relative performance concerns and possible constraints on the portfolios. In this chapter, we consider a variant of this previous problem by including a local risk penalization in the investor's optimization criterion. More precisely, there are N specific investors that compare themselves to each other. As before, each agent takes into account a convex combination of his wealth (with weight $1-\lambda$, $\lambda \in [0,1]$) and the difference between his wealth and the average wealth of the other investors (with weight λ), but he also substracts a penalization term which is equal to the integral of a function of the volatility of the wealth. On the other hand, there are no constraints anymore on the portfolio of an agent. Again, this criterion generates interactions between agents and leads to a differential game between N players.

This could be seen as a generalization of the previous chapter, but we restrict our work to a certain class of penalization functions that guarantee that we will have to deal with Lipschitz BSDEs. We first solve the problem of utility maximization with a risk penalization for an exponential utility function, in the spirit of El Karoui and Rouge [27] or Hu, Imkeller and Müller [42]. Then, for exponential utility functions and under mild conditions, we use it to show that there exists a unique Nash equilibrium for a general dynamics of the assets. Finally, we also show that for a deterministic risk premium, we can take the limit when the number of investors N goes to infinity.

6.2 Problem formulation

Let W be a d-dimensional Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the corresponding completed canonical filtration. Let $T > 0$ be the horizon of investment. Given two $\mathbb{F}$-predictable processes θ and σ taking values respectively in $\mathbb{R}^d$ and $M_d(\mathbb{R})$, and satisfying:

$$\bullet\ \theta \text{ is bounded a.s,} \tag{6.1}$$

$$\bullet\ \int_0^T |\sigma_t|^2 dt < +\infty \text{ a.s,} \tag{6.2}$$

we consider a market with a non risky asset with interest rate $r = 0$ and a d-dimensional risky asset $S = (S_1, ..., S_d)$ given by the following dynamics:

$$dS_t = \operatorname{diag}(S_t)\sigma_t(\theta_t dt + dW_t). \tag{6.3}$$

We assume that σ is symmetric definite positive, so in particular invertible.

A portfolio is an $\mathbb{F}$-predictable process $\{\pi_t,\ t \in [0,T]\}$ taking values in $\mathbb{R}^d$. Here π_t^i is the amount invested in the i-th risky asset at time t. The associated wealth X_t^π follows the following dynamics:

$$dX_t^\pi = \sum_{j=1}^d \pi_t^j \frac{dS_t^j}{S_t^j}.$$

Then, let $N \in \mathbb{N}$ be given, we consider a set of N agents that will interact with each other. We will always assume that $N \geq 2$. We still assume that each agent is "small" in the sense that his actions do not impact the market prices S. For each $1 \leq i \leq N$, agent i wants to maximize his expected utility taking into account two criteria: his absolute wealth on the one hand, and his wealth compared to the average wealth of other agents. More precisely, we assume that each agent has an exponential utility function with risk aversion $\eta_i > 0$:

$$U_i(x) = -e^{-\frac{1}{\eta_i}x}.$$

We also assume that agent i has a relative wealth sensitivity $\lambda_i \in [0,1]$. We write:

$$\bar{X}_t^{i,\pi} = \frac{1}{N-1}\sum_{j \neq i} X_t^{\pi^j}$$

the average wealth of agents other than i, where the π^j are given. We also consider a penalization function $g^i : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, which is assumed to be Lipschitz continuous. Given π^j for $j \neq i$, then his optimization problem is:

$$V_i = V_i(\pi^j, j \neq i) = \sup_{\pi^i \in \mathcal{A}_i} \mathbb{E}e^{-\frac{1}{\eta_i}\left((1-\lambda_i)X_T^{\pi^i} + \lambda_i(X_T^{\pi^i} - \bar{X}_T^i) - \int_0^T g^i(\sigma_u \pi_u^i)du\right)} \tag{6.4}$$

$$= \sup_{\pi^i \in \mathcal{A}_i} \mathbb{E}e^{-\frac{1}{\eta_i}\left(X_T^{\pi^i} - \lambda_i \bar{X}_T^i - \int_0^T g^i(\sigma_u \pi_u^i)du\right)} \tag{6.5}$$

where the set of admissible portfolio $\mathcal{A}_i$ is the set of predictable processes π such that:

- $\int_0^T |\sigma_t \pi_t|^2 dt < \infty$ a.s (6.6)
- $\{e^{-\frac{1}{\eta_i}\left(X_\tau^{i,\pi} - \int_0^\tau g^i(\sigma_u \pi_u^i)du\right)};\ \tau$ stopping time on $[0,T]\}$ is uniformly bounded in $L^p(\mathbb{P})$, for a certain $p > 1$. (6.7)

In (6.7), p may depend on i and π, so that the assumption is quite weak. Notice though that it is slightly stronger than assuming that the family is uniformly integrable.

Remark 6.1 A financial interpretation of the penalization function could be the following one: while the agent is allowed to invest in the whole market, he is rewarded by his management not only with regard to his absolute and relative performances, but also to the local risk generated by his investment strategy.

Our main objective is to find if the agents can simultaneously solve their optimization problem, and then see what happens when the number of agents N goes to infinity. We therefore introduce the definition of Nash equilibrium in our framework:

Definition 6.2 *(Nash equilibrium) A Nash equilibrium for our problem is an N-uple $(\hat{\pi}^1, ..., \hat{\pi}^N)$ of elements of $\mathcal{A}_1 \times ... \times \mathcal{A}_N$ such that, for each $1 \leq i \leq N$:*

$$V_i(\hat{\pi}^j, j \neq i) = \mathbb{E}e^{-\frac{1}{\eta_i}\left(X_T^{\hat{\pi}^i} - \lambda_i \bar{X}_T^{i,\hat{\pi}} - \int_0^T g^i(\sigma_u \hat{\pi}_u^i)du\right)}. \tag{6.8}$$

In order to simplify notations, from now on we will write X_t^i and $\bar{X}_t^i$. We say that a random variable has exponential moments of any order if:

$$\forall \delta > 0, \ \mathbb{E}e^{\delta |X| < \infty}, \tag{6.9}$$

and we introduce the set:

$$\mathcal{E} := \{(Y_t); \ (\mathcal{F}_t)\text{-predictable and } \forall \delta > 0, \ \mathbb{E}e^{\delta \sup_{t\in[0,T]} |Y_t| < \infty}\}. \tag{6.10}$$

Remark 6.3 Notice that, without further assumptions, the framework of the previous chapter is a particular case of this one. Indeed it corresponds to the functions:

$$g^i(x) = \begin{cases} 0 \text{ if } x \in A_i \\ +\infty \text{ if } x \notin A_i. \end{cases}$$

But in fact we will make stronger assumptions later which will exclude this setting.

In the first section of this paper, we shall characterize the value function of a problem of optimal investment with a penalization function. Then we will apply this result to our problem of optimal investment under relative performance concerns and in particular show the existence and uniqueness of a Nash equilibrium for general dynamics of the assets. Then we will take a closer look at the case of deterministic θ and σ. Finally we will derive a few examples.

6.3 Optimal investment with a risk penalization

In this section, we consider a single agent who wants to track a contingent claim F, which is assumed to be $\mathcal{F}_T$-measurable and to have exponential moments of any order (recall the definition given by (6.9)). Given $\eta > 0$ and a Lipschitz penalization function $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, we consider the following problem:

$$V = \sup_{\pi \in \mathcal{A}} \mathbb{E} - e^{-\frac{1}{\eta}\left(X_T^\pi - \int_0^T g(\sigma_u \pi_u)du - F\right)} \tag{6.11}$$

where $\mathcal{A}$ in the set of admissible processes, which are processes satisfying assumptions (6.6)-(6.7) with η_i replaced by η.

Let us introduce the following function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$:

$$f(\theta, z) := \inf_{p \in \mathbb{R}^d} m(\theta, z, p) := \inf_{p \in \mathbb{R}^d} \{g(p) + \frac{1}{2\eta}|p - z|^2 - p.\theta\} \tag{6.12}$$

Introducing the convex dual of $h(p) := g(p) + \frac{1}{2\eta}|p|^2$ by means of the Fenchel-Legendre transform (which is convex):

$$\forall x \in \mathbb{R}^d,\ \tilde{h}(x) = \sup_{p \in \mathbb{R}^d} \{p.x - h(p)\}, \tag{6.13}$$

we notice that f can be rewritten as:

$$f(\theta, z) = \frac{1}{2\eta}|z|^2 - \tilde{h}(\frac{1}{\eta}z + \theta), \tag{6.14}$$

and f is continuous w.r.t (θ, z). In fact, as $\tilde{h}$ is convex, for any $\theta \in \mathbb{R}^d$, the gradient of $f(\theta, .)$ is defined for almost very $z \in \mathbb{R}^d$ and f is locally Lipschitz w.r.t z, uniformly on compact sets w.r.t θ, which means that for all compact set $K \subset \mathbb{R}^d$:

$$\exists L_K > 0,\ \forall (\theta, z, z') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d,\ |f(\theta, z) - f(\theta, z')| \leq L_K |z - z'|.$$

Moreover, as g is Lipschitz, for any fixed $(\theta, z) \in \mathbb{R}^d \times \mathbb{R}^d$, $m(\theta, z, p) \to \infty$ as $|p| \to \infty$ so that the infimum in the definition of f is attained, which implies that for any $\theta \in \mathbb{R}^d$ and $z \in \mathbb{R}^d$, there exists $p^* \in \mathbb{R}^d$ satisfying $f(\theta, z) = m(\theta, z, p^*)$. We introduce:

$$I(\theta, z) := \arg\min_{p \in \mathbb{R}^d} m(\theta, z, p).$$

Assuming the axiom of choice, we can simultaneously choose a representant in each $I(\theta, z)$ and therefore define a function $p^* : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ such that $f(\theta, z) = m(\theta, z, p^*(\theta, z))$. We hereafter call such a function a representation of I.

In order to simplify notations, we write $f_t(z) := f(\theta_t, z)$, $m_t(z) := m(\theta_t, z)$, $I_t(z) := I(\theta_t, z)$.

Consider the following assumptions:

- f_t is Lipschitz w.r.t z, uniformly in t, ω; (H_1)
- f_t is locally Lipschitz w.r.t z, uniformly in t, ω and has linear growth w.r.t z, uniformly in t, ω; (H_1')
- for any representation p^* of I, there exists $C > 0$, $\forall z, t, \omega$, $|p^*(\theta_t, z) - z| \leq C$ a.s; (H_2)
- for any $(t, \omega, z) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d$, $I_t(z)$ is a singleton. (H_3)

In (H_1'), by uniform linear growth we mean that:

$$\exists L_1, L_2 > 0,\ \forall t, \omega, z,\ |f_t(z)| \leq L_1 + L_2|z|.$$

We derive here a few sufficient conditions for assumptions (H_1) to (H_3) to hold.

Proposition 6.4 (i) *If there exists a compact K such that g is C^1 on $\mathbb{R}^d \setminus K$ and ∇g is bounded on $\mathbb{R}^d \setminus K$, then (H_1') and (H_2) are satisfied;*

(ii) *If moreover $p \mapsto \nabla g(p) + \frac{1}{\eta} p$ is one-to-one on $\mathbb{R}^d \setminus K$, and its inverse is Lipschitz on its domaine, then (H_1) and (H_2) are satisfied;*

(iii) *If $h(p) = g(p) + \frac{1}{2\eta}|p|^2$ is strictly convex, then (H_3) is satisfied.*

Remark 6.5 In particular, if $g(x) = |x|$ is the canonical euclidean norm, assumptions (H_1) to (H_3) are satisfied.
If $g(x) = 0$, then we are in the same setting as in complete market situation of the previous chapter, see section 4.3.

Proof. (i): As g is Lipschitz, for any $(t, \omega, z) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d$, the minimum is attained. Therefore it is either attained for a certain $p^* \in K$ or for a p^* satisfying the first order condition:

$$\nabla g(p^*) = \theta_t + \frac{1}{\eta}(z - p^*).$$

Let us show that for z large enough p^* cannot be in K. Indeed by convexity we have $2|p - z|^2 \geq |z|^2 - 2|p|^2$ so that:

$$\begin{aligned}\inf_{p\in K}\{g(p) + \frac{1}{2\eta}|p - z|^2 - p.\theta_t\} &\geq \inf_{p\in K}\{g(p) - \frac{1}{2\eta}|p|^2 - ||\theta||_\infty |p|\} + \frac{1}{4\eta}|z|^2 \\ &\geq A + \frac{1}{4\eta}|z|^2,\end{aligned}$$

for a certain constant A, independant from z, while for $p = z$, B being the Lipschitz constant for g:

$$g(z) - z.\theta_t \leq |g(0)| + (B + ||\theta||_\infty)|z|.$$

Therefore:

$$\begin{aligned}\inf_{p\in K}\{g(p) + \frac{1}{2\eta}|p - z|^2 - p.\theta_t\} - \inf_{p\in\mathbb{R}^d}\{g(p) + \frac{1}{2\eta}|p - z|^2 - p.\theta_t\}& \\ \geq A - |g(0)| + \frac{1}{4\eta}|z|^2 - (B + ||\theta||_\infty)|z|& \\ \to \infty \text{ as } |z| \to \infty,&\end{aligned}$$

which implies that there exists a compact K' of $\mathbb{R}^d$ such that for $z \notin K'$, any minimizer of f_t is not in K. And so for $z \notin K'$, let $t \in [0,T]$ and $p^* = p_t^*(z)$ such that $f_t(z) = m_t(z, p^*)$, then we must have:

$$\nabla g(p^*) = \theta_t + \frac{1}{\eta}(z - p^*).$$

so that as ∇g is bounded by a certain $C > 0$, for all t, $z \in K'$ and $p^*(z)$, we get that:

$$|p_t^*(z) - z| \leq \eta \left(C + ||\theta||_\infty\right).$$

Finally, if $z \in K'$, we show as before that p^* cannot be too large. Indeed $g(z) - z.\theta_t$ is bounded for $z \in K'$, while:

$$g(p) + \frac{1}{2\eta}|p-z|^2 - p.\theta_t \geq -(C + ||\theta||_\infty)|p| - \frac{1}{2\eta}|z|^2 + \frac{1}{4\eta}|p|^2 \to \infty,$$

as $|p| \to \infty$, uniformly in $z \in K'$.

Then we already know that f_t is uniformly locally Lipschitz, so we only need to check that it has uniform linear growth. We know that $p_t^*(z) - z$ is uniformly bounded by C and g is B-Lipschitz, so we have:

$$\begin{aligned} |f_t(z)| &= \left| g\left(p_t^*(z)\right) + \frac{1}{2\eta}|p_t^*(z) - z|^2 - p_t^*(z).\theta_t \right| \\ &\leq |g(0)| + \left[B + ||\theta||_\infty\right] |p_t^*(z)| + \frac{1}{2\eta}C^2 \\ &\leq |g(0)| + \frac{1}{2\eta}C^2 + \left[B + ||\theta||_\infty\right](C + |z|), \end{aligned}$$

so we have the result.

(ii): Now if $p \mapsto \nabla g(p) + \frac{1}{\eta}p$ is one-to-one outside K, it is a bijection from $\mathbb{R}^d \setminus K$ into a certain D and we write its inverse $\varphi : D \to \mathbb{R}^d \setminus K$. As we have seen before, there exists a compact C such that $(\mathbb{R}^d \setminus C) \subset D$. Moreover, $I_t(z)$ is a singleton for $|z|$ large enough, uniformly in (t, ω). Now if φ is Lipschitz, then $z \mapsto p^*(z)$ is Lipschitz too for $|z|$ large enough, uniformly in (t, ω), and as (H_2) is satisfied, $z \mapsto |p^*(z) - z|^2$ is also uniformly Lipschitz, so that f_t is uniformly Lipschitz for z large enough. Now because of (i), we know that f_t is also uniformly locally Lipschitz so it is uniformly globally Lipschitz on $\mathbb{R}^d$.

(iii): Using the expression (6.14) for f_t, it is immediate that the minimum is attained only at one point if h is strictly convex. □

For any $m \in \mathbb{N}^*$, we denote by $\mathbb{H}^2(\mathbb{R}^m)$ and $\mathbb{S}^2(\mathbb{R}^m)$ the following spaces of processes:

$$\mathbb{H}^2(\mathbb{R}^m) := \{(Y_t);\ \mathbb{R}^m\text{-valued, predictable process with } \mathbb{E}\int_0^T |Y_t|^2 dt < \infty\}, \tag{6.15}$$

$$\mathbb{S}^2(\mathbb{R}^m) := \{(Y_t);\ \mathbb{R}^m\text{-valued, continuous and adapted process with } \mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t|^2\Big] < \infty\}, \tag{6.16}$$

and by $\mathcal{P}$ the σ-field of predictable sets of $[0,T]\times\Omega$. Finally let $\mathcal{T}$ be the family of stopping times less or equal to T.
Then we have the following result:

Theorem 6.6 *Assume that g satisfies (H_1) and (H_2) and F has exponential moments of any order. Then the value of the optimization problem* (6.11) *is given by:*

$$V(x) = -e^{-\frac{1}{\eta}(x-Y_0)},$$

where (Y,Z) is the unique solution in $\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R}^d)$ of the following BSDE:

$$\begin{aligned} dY_t &= -f_t(Z_t)dt + Z_t.dW_t \\ Y_T &= F \end{aligned}$$

with f_t defined by (6.12)*.*
Moreover $Y \in \mathcal{E}$ and there exists an optimal portfolio $\hat{\pi} \in \mathcal{A}$ such that for each $t \in [0,T]$:

$$\hat{\pi}_t \in \sigma_t^{-1} I_t(Z_t),\ \mathbb{P}\text{-}a.s.$$

Assume also that (H_3) is satisfied, then the optimal portfolio is unique.

Proof. From the definition of f_t, we immediately see that f is $\mathcal{P}\times\mathcal{B}(\mathbb{R}^d)$-measurable, that $f_\cdot(0) \in \mathbb{S}^2(\mathbb{R})$ and thanks to assumption (H_1), f_t is uniformly Lipschitz. $F \in L^2$ is clear too. Therefore the existence and uniqueness of $(Y,Z) \in \mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R}^d)$ are a well-known result (see Pardoux and Peng [63] or El Karoui, Peng and Quenez [26]). Moreover, as F has exponential moments of any order, we can apply Corollary 4 of Briand and Hu [7], which guarantees that $Y \in \mathcal{E}$ (again it is straightforward that their assumptions are satisfied in our case).

Then we define for $\pi \in \mathcal{A}$:

$$M_t^\pi := -e^{-\frac{1}{\eta}(X_t^\pi - \int_0^t g(\sigma_u\pi_u)du - Y_t)},$$

and we will show that M^π is a $\mathbb{P}$-supermartingale for any $\pi \in \mathcal{A}$, and a $\mathbb{P}$-martingale for a certain $\hat{\pi}$, which will then be an optimal control.

We write $p_t = \sigma_t \pi_t$ and we compute:

$$\frac{dM_t^\pi}{M_t^\pi} = -\frac{1}{\eta}\left\{(p_t.\theta_t - g(p_t) + f_t(Z_t) - \frac{1}{2\eta}|p_t - Z_t|^2)dt + (p_t - Z_t).dW_t\right\}.$$

By definition of f_t, $-\frac{1}{\eta}\left(p_t.\theta_t - g(p_t) + f_t(Z_t) - \frac{1}{2\eta}|p_t - Z_t|^2\right) > 0$, while $M_t^\pi < 0$, so that M^π is a local supermartingale for any $\pi \in \mathcal{A}$. If $\hat{\pi} = \sigma^{-1}p^*$, where p^* is the process constructed in Lemma 6.9, then $M^{\hat{\pi}}$ is a local martingale. Moreover, if (H_3) is satisfied, then M^π is a strict local supermartingale for any $\pi \neq \hat{\pi}$.

In other words there exists a sequence $(\tau_n = \tau_n^\pi)$ of stopping times such that $\tau_n \to \infty$ a.s and for any n and $0 \leq s \leq t \leq T$:

$$\begin{aligned}
&\mathbb{E}[M_{t\wedge\tau_n}^\pi | F_s] \leq M_{s\wedge\tau_n}^\pi \text{ for any } \pi \in \mathcal{A}\\
&\mathbb{E}[M_{t\wedge\tau_n}^{\hat{\pi}} | F_s] = M_{s\wedge\tau_n}^{\hat{\pi}}.
\end{aligned}$$

Now if $\pi \in \mathcal{A}$, because of condition (6.7), there exists $p > 1$ such that $\{e^{-\frac{1}{\eta}X_\tau^\pi};\ \tau \in \mathcal{T}\}$ is uniformly bounded in L^p by a constant C while $Y \in \mathcal{E}$. Let $r \in (1, p)$, then $\frac{p}{r} > 1$, so we define the conjugate of $\frac{p}{r} > 1$ by q such that $\frac{r}{p} + \frac{1}{q} = 1$. Then using Holder's inequality we get for any $\tau \in \mathcal{T}$:

$$\begin{aligned}
\mathbb{E}e^{-\frac{r}{\eta}\left(X_\tau^\pi - \int_0^\tau g(\sigma_u\pi_u)du - Y_\tau\right)} &\leq \left(\mathbb{E}e^{-\frac{p}{\eta}\left(X_\tau^\pi - \int_0^\tau g(\sigma_u\pi_u)du\right)}\right)^{\frac{r}{p}} \left(\mathbb{E}e^{\frac{rq}{\eta}Y_\tau}\right)^{\frac{1}{q}}\\
&\leq C^{\frac{r}{p}} \left(\mathbb{E}e^{\frac{rq}{\eta}\sup_{t\in[0,T]}|Y_t|}\right)^{\frac{1}{q}}.
\end{aligned}$$

As $rq\eta > 0$, $\{e^{-\frac{r}{\eta}\left(X_\tau^\pi - \int_0^\tau g(\sigma_u\pi_u)du - Y_\tau\right)};\ \tau \in \mathcal{T}\}$ is uniformly bounded in L^r $(r > 1)$ and therefore is uniformly integrable. As a consequence we can apply Lebesgue's theorem while we send n to infinity and we have:

$$\mathbb{E}[M_t^\pi | F_s] = \lim_{n\to\infty} \mathbb{E}[M_{t\wedge\tau_n}^\pi | F_s] \leq \lim_{n\to\infty} M_{s\wedge\tau_n}^\pi = M_s^\pi.$$

Finally let us show that $\hat{\pi} \in \mathcal{A}$. Then as previously we could apply Lebesgue's theorem which would guarantee the martingale property of $M^{\hat{\pi}}$, and so the optimality of $\hat{\pi}$.

By definition, $f_t(Z_t) = m_t(Z_t, \hat{\pi}_t)$ so that we have for any $\tau \in \mathcal{T}$ and $r > 1$:

$$\begin{aligned}
e^{-\frac{r}{\eta}\left(X_\tau^{\hat{\pi}} - \int_0^\tau g(\sigma_u\hat{\pi}_u)du - Y_\tau\right)} &= e^{-\frac{r}{\eta}(x - Y_0)} e^{-\frac{r}{\eta}\left(\int_0^\tau [\sigma_u\hat{\pi}_u.\theta_u - g(\sigma_u\hat{\pi}_u) + f_t(\sigma_u\hat{\pi}_u)]du + \int_0^\tau (\sigma_u\hat{\pi}_u - Z_u).dW_u\right)}\\
&= e^{-\frac{r}{\eta}(x - Y_0)} e^{-\frac{r}{\eta}\left(\int_0^\tau \frac{\eta}{2}|\sigma_u\hat{\pi}_u - Z_u|^2du + \int_0^\tau (\sigma_u\hat{\pi}_u - Z_u).dW_u\right)}.
\end{aligned}$$

Thanks to assumption (H_2), we know that $|\sigma_t \hat{\pi}_t - Z_t| \leq C$. As θ is bounded, it implies that $(e^{-\frac{r}{\eta}\int_0^t(\sigma_u\hat{\pi}_u - Z_u).dW_u - \frac{r^2}{2\eta^2}\int_0^t \frac{1}{2\eta}|\sigma_u\hat{\pi}_u - Z_u|^2 du})$ is a $\mathbb{P}$-martingale and that we can define the equivalent measure $\mathbb{Q}^\eta$ by its Radon-Nikodym density:

$$\frac{d\mathbb{Q}^\eta}{d\mathbb{P}} = e^{-\frac{r}{\eta}\int_0^T(\sigma_u\hat{\pi}_u - Z_u).dW_u - \frac{r^2}{2\eta^2}\int_0^T \frac{1}{2\eta}|\sigma_u\hat{\pi}_u - Z_u|^2 du}.$$

Then we have:

$$\begin{aligned}\mathbb{E}e^{-\frac{r}{\eta}\left(X_\tau^{\hat{\pi}} - \int_0^\tau g(\sigma_u\hat{\pi}_u)du - Y_\tau\right)} &= e^{-\frac{r}{\eta}(x-Y_0)}\mathbb{E}^{\mathbb{Q}^\eta} e^{\frac{r(r-1)}{2\eta^2}\int_0^\tau |\sigma_u\hat{\pi}_u - Z_u|^2 du} \\ &\leq e^{-\frac{r}{\eta}(x-Y_0)} e^{\frac{r(r-1)}{2\eta^2}TC^2}.\end{aligned}$$

Let $p \in (1, r)$, and q defined by $\frac{p}{r} + \frac{1}{q} = 1$, applying Holder's inequality and using the fact that $Y \in \mathcal{E}$, we get:

$$\mathbb{E}e^{-\frac{p}{\eta}\left(X_\tau^{\hat{\pi}} - \int_0^\tau g(\sigma_u\hat{\pi}_u)du\right)} \leq \left(\mathbb{E}e^{-\frac{r}{\eta}\left(X_\tau^{\hat{\pi}} - \int_0^\tau g(\sigma_u\hat{\pi}_u)du - Y_\tau\right)}\right)^{\frac{p}{r}} \left(\mathbb{E}e^{-\frac{pq}{\eta}Y_\tau}\right)^{\frac{1}{q}} \leq A,$$

where A is a constant independent of τ, so that $\hat{\pi} \in \mathcal{A}$.

Finally, if (H_3) is satisfied, we see that $\hat{\pi}$ is the unique optimal portfolio as for any other $\pi \in \mathcal{A}$, M^π is a strict supermartingale. □

Remark 6.7 From the previous proof, we can see that in fact for any $p > 1$, the family:

$$\{e^{-\frac{1}{\eta_i}\left(X_\tau^{i,\hat{\pi}} - \int_0^\tau g^i(\sigma_u\hat{\pi}_u^i)du\right)};\ \tau \in \mathcal{T}\}$$

is uniformly bounded in $L^p(\mathbb{P})$.

Remark 6.8 In fact this result is also true if assumption (H_1) is replaced by (H_1'), as Hamadene [38] showed that under this weaker assumption, the existence and uniqueness of a BSDE still hold true. But as this is true only in dimension 1, we will not be able to use this result in the next section, and therefore we stated our result in the Lipschitz case.

We show here that we can indeed select in a predictable way the process p^* in the previous proof.

Lemma 6.9 *Let (Z_t) be a predictable process. Then there exists a predictable process (p_t^*) satisfying for all $t \in [0, T]$, $p_t^* \in I_t(Z_t)$, $\mathbb{P}$-a.s.*

Proof. We will show that there exists a $\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R}^d)$-$\mathcal{B}(\mathbb{R}^d)$ measurable mapping ψ such that $\psi(\theta,z)\in I(\theta,z)$.

Recall that: $m(\theta,z,p)=g(p)+\frac{1}{2\eta}|p-z|^2-p.\theta$. Moreover, m is continuous w.r.t (θ,z,p) and, as g is Lipschitz, for any $n\in\mathbb{N}$, there exists Y_n compact of $\mathbb{R}^d$ such that, for any $p\notin Y_n$ and $(\theta,z)\in\bar{B}(0,n)$, we have:

$$m(\theta,z,p)>m(\theta,z,0),$$

where $\bar{B}(0,n)$ is the closed ball of radius equal to n in $\mathbb{R}^{2d}$. Therefore, for any $(\theta,z)\in\bar{B}(0,n)$, we have:

$$f(\theta,z)=\inf_{p\in\mathbb{R}^d}m(\theta,z,p)=\inf_{p\in Y_n}m(\theta,z,p).$$

Let us define the following sequence of functions (f_n):

$$f_n(\theta,z):=\inf_{p\in Y_n}m(\theta,z,p).$$

We therefore have $f_n=f$ on $\bar{B}(0,n)$, and in particular (f_n) converges pointwise to f. As Y_n is compact, we can use Proposition 7.33, p.153 of Bertsekas and Shreve [5], which tells us that, for each n, there exists a $\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R}^d)$-$\mathcal{B}(\mathbb{R}^d)$ measurable function

$$\varphi_n: X\to Y_n,$$

such that $m\left(\theta,z,\varphi_n(\theta,z)\right)=f_n(\theta,z)$ for any $(\theta,z)\in\mathbb{R}^{2d}$. Moreover, as $f_n=f$ on $\bar{B}(0,n)\subset\bar{B}(0,n+1)$, $f_n=f_{n+1}$ on $\bar{B}(0,n)$. Therefore if we define $\psi_0=\varphi_0$ and for $n\geq 0$:

$$\psi_{n+1}=\begin{cases}\psi_n \text{ on } \bar{B}(0,n)\\ \varphi_{n+1} \text{ otherwise,}\end{cases}$$

then ψ_n is $\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R}^d)$-$\mathcal{B}(\mathbb{R}^d)$ measurable as well, and $m\left(\theta,z,\psi_n(\theta,z)\right)=f_n(\theta,z)$. Finally, as $\bigcup_{n\in\mathbb{N}}\bar{B}(0,n)=\mathbb{R}^{2d}$ while $\psi_{n+1}=\psi_n$ on $\bar{B}(0,n)$, (ψ_n) convergences pointwise to a function ψ which is $\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R}^d)$-$\mathcal{B}(\mathbb{R}^d)$ measurable and such that $\psi(\theta,z)\in I(\theta,z)$ for any θ, and z.

As (θ_t) and (Z_t) are predictable processes, $p_t^*=\psi(\theta_t,Z_t)$ satisfies the required conditions. □

We next introduce the dynamic value function:

$$V(\tau,X_\tau)=\operatorname*{ess\,sup}_{\pi\in\mathcal{A}}\mathbb{E}\left[-e^{-\frac{1}{\eta}\left(X_\tau+\int_\tau^T\sigma_u\pi_u.(dW_u+\theta_u du)-\int_\tau^T g(\sigma_u\pi_u)du-F\right)}|\mathcal{F}_\tau\right] \tag{6.17}$$

The following result shows that $\hat{\pi}$ is optimal is a stronger way.

Proposition 6.10 *(Dynamic programming) For any stopping time $\tau \in \mathcal{T}$, we have:*

$$V(\tau, X_\tau) = -e^{-\frac{1}{\eta}(X_\tau - Y_\tau)},$$

where Y is the solution of the BSDE given in Theorem 6.6. Moreover an optimal portfolio for the problem starting at τ is the one given in Theorem 6.6.

Proof. Using Doob's optional sampling theorem, we can do exactly the same as in the proof of Theorem 6.6, but starting from τ instead of 0. $\square$

6.4 Relative performance concerns

Now we turn back to the problem of portfolio optimization under relative performance concerns in a financial market with N interacting agents.

6.4.1 Nash equilibrium

We give here the main result of this work, namely the existence and uniqueness, under quite general conditions, of a Nash equilibrium for problem (6.4).

If (H_3) holds true, then we define $\Gamma_t^i(z)$ by:

$$I_t^i(z) = \{\Gamma_t^i(z)\}, \tag{6.18}$$

and we consider three new assumptions:

– (H_3) holds true, and Γ_t^i is Lipschitz, uniformly in (t, ω) (H_4)

– Γ_t^i is uniformly elliptic in the sense that there exist $\varepsilon_i \geq 0$, for any z, z', $(\Gamma_t^i(z) - \Gamma_t^i(z')).(z - z') \geq \varepsilon_i |z - z'|^2$. (H_5)

If in (H_5) we have $\varepsilon^i = 0$, then we can rewrite it in the simpler following way:

$$\text{for any } z, z', (\Gamma_t^i(z) - \Gamma_t^i(z')).(z - z') \geq 0,$$

and we say that Γ^i is nondecreasing.

Remark 6.11 Notice that (H_2), (H_3) and (H_4) imply (H_1). Indeed, we then have:

$$f_t^i(z) = g^i \circ \Gamma_t^i(z) + \frac{1}{2\eta_i}|\Gamma_t^i(z) - z|^2 - \Gamma_t^i(z).\theta_t.$$

Assumption (H_2) implies that $|\Gamma_t^i(z) - z| \leq C$, so that for any z and z':

$$\begin{aligned}\left||\Gamma_t^i(z) - z|^2 - |\Gamma_t^i(z') - z'|^2\right| &= \left||\Gamma_t^i(z) - z| - |\Gamma_t^i(z') - z'|\right| \times \left(|\Gamma_t^i(z) - z| + |\Gamma_t^i(z') - z'|\right) \\ &\leq 2C \left||\Gamma_t^i(z) - z| - |\Gamma_t^i(z') - z'|\right| \\ &\leq 2C \left|(\Gamma_t^i(z) - z) - (\Gamma_t^i(z') - z')\right|.\end{aligned}$$

Because of (H_4), g^i and Γ^i are Lipschitz, uniformly in (t, ω), and so the same holds for f^i.

As we will see, assumption (H_5) is not required, it just allows us to relax a little bit some assumptions. But (H_4) is crucial. However, even in very simple cases, this last property can fail to hold. We give here an example of function g where (H_3) is satisfied, but not (H_4), although g is C^1, with a bounded gradient. Indeed let:

$$g(p) = \begin{cases} \frac{|p|^3}{3} - \frac{1}{2\eta}|p|^2, \text{ if } |p| \leq \frac{1}{\eta} \\ -\frac{1}{6\eta^3}, \text{ otherwise.} \end{cases}$$

Then g is Lipschitz, and $h(p) = g(p) + \dfrac{1}{2\eta}|p|^2$ is given by:

$$h(p) = \begin{cases} \frac{|p|^3}{3}, \text{ if } |p| \leq \frac{1}{\eta} \\ \frac{1}{2\eta}|p|^2 - \frac{1}{6\eta^3}, \text{ otherwise.} \end{cases}$$

which is strictly convex, so that (H_3) is satisfied and the minimizer is unique. As g is C^1 with gradient:

$$\nabla g(p) = \begin{cases} (|p| - \frac{1}{\eta})p, \text{ if } |p| \leq \frac{1}{\eta} \\ 0, \text{ otherwise.} \end{cases}$$

We see that ∇g is bounded, so that (H_1') and (H_2) are also satisfied and for z and θ_t, the minimizer $\Gamma_t(z)$ is:

$$\Gamma_t(z) = \begin{cases} 0, \text{ if } z + \eta\theta_t = 0 \\ \frac{1}{\sqrt{\eta(z+\eta\theta_t)}}\left(z + \eta\theta_t\right), \text{ if } 0 < |z + \eta\theta_t| \leq \frac{1}{\eta} \\ z + \eta\theta_t, \text{ if } |z + \eta\theta_t| > \frac{1}{\eta}. \end{cases}$$

In particular, $|\Gamma_t(z)| = \sqrt{\dfrac{|z + \eta\theta_t|}{\eta}}$ if $|z + \eta\theta_t| \leq \dfrac{1}{\eta}$, which is not Lipschitz in any neighborhood of $z = -\eta\theta_t$, so that (H_4) cannot be true.

However, we can provide sufficient conditions under which it is satisfied.

Lemma 6.12 *Assume that* (H_3) *holds,* g^i *is* C^1*,* ∇g^i *is bounded,* $p \mapsto \nabla g^i(p) + \frac{1}{\eta_i} p$ *is one-to-one and its inverse is Lipschitz. Then* (H_4) *holds true.*

Proof. very close to the proof of Proposition 6.4-(ii). $\square$

We define:

$$\lambda_m = \lambda_m^N := \frac{\sqrt{N^2 + 4(N-1)} - N}{2}. \tag{6.19}$$

Notice that $\lambda_m \in (0,1)$, is increasing in N and as $N \to \infty$, $\lambda_m^N = 1 - \frac{1}{N} + o\left(\frac{1}{N}\right)$. Notice also that $\lambda_m^N \geq 1 - \frac{2}{N}$.

As this quantity will be used many times, we introduce the following notation:

$$\lambda_i^N := \frac{\lambda_i}{N-1}. \tag{6.20}$$

We also introduce the following mapping $\psi : [0,T] \times M_{N,d}(\mathbb{R}) \to M_{N,d}(\mathbb{R})$ such that for each $t \in [0,T]$, its i-th line is defined by:

$$\begin{aligned} \psi_t^i(Z) = &\left[I - \frac{1}{N-1} \sum_{j \neq i} \Gamma_t^j \circ \left(I + \lambda_j^N \Gamma_t^j\right)^{-1} \circ \left[\lambda_j \left(I + \lambda_i^N \Gamma_t^i\right)\right] \right]^{-1} \\ &\circ \left[Z^i + \frac{1}{N-1} \sum_{j \neq i} \Gamma_t^j \circ \left(I + \lambda_j^N \Gamma_t^j\right)^{-1} \left(\lambda_i Z^j - \lambda_j Z^i\right) \right]. \end{aligned} \tag{6.21}$$

Now we are able to state the following result:

Theorem 6.13 *Assume that for each* i*,* g^i *satisfies assumptions* (H_1) *to* (H_4)*. We write* B_i *the Lipschitz constant for* Γ^i *and* $B := \max_{1 \leq i \leq N} B_i$*. Assume also that:*

$$\sup_{1 \leq i \leq N} \lambda_i < \frac{\lambda_m^N}{B}. \tag{6.22}$$

Then problem (6.4) *admits a unique Nash equilibrium* $(\hat{\pi}^1, ..., \hat{\pi}^N)$ *defined by:*

$$V_i = V_i^N = -e^{-\frac{1}{\eta_i}(x^i - \lambda_i \bar{x}^i - Y_0^i)} \quad \text{and} \quad \hat{\pi}_t^{i,N} = \sigma_t^{-1} \circ \Gamma_t^i \circ \psi_t^i(Z_t),$$

where (Y,Z) is the unique solution in $\mathbb{S}^2(\mathbb{R}^d)\times\mathbb{H}^2(M_{N,d}(\mathbb{R}))$ of the N-dimensional BSDE:

$$Y_t^i = \int_t^T [f_u^i \circ \psi_u^i(Z_u) + \psi_u^i(Z_u).\theta_u]du - \int_t^T Z_u^i.dB_u. \tag{6.23}$$

In the case where (H_5) is also satisfied for each i, then the same is true when replacing (6.22) *by the weaker assumption that:*

$$\sup_{1\le i\le N}\lambda_i < \max\left(\frac{\lambda_{m,\varepsilon}^N}{B}, \frac{N-1}{B}\right), \tag{6.24}$$

where $\varepsilon := \min_{1\le i\le N}\varepsilon_i \in [0,B]$, and $\lambda_{m,\varepsilon} = \lambda_{m,\varepsilon}^N$ is the unique positive solution of:

$$xB\left(1+\frac{xB}{N-1}\right) = \sqrt{1+2\frac{\varepsilon}{B}\frac{xB}{N-1}+\left(\frac{xB}{N-1}\right)^2}. \tag{6.25}$$

Finally, $\lambda_{m,\varepsilon}$ is increasing w.r.t ε and $\lambda_m < \lambda_{m,0} < \lambda_{m,B} = \frac{1}{B}$.

Remark 6.14 If $B<1$, then we see that for ε in a (left) neighborhood of B, $\lambda_{m,\varepsilon}>1$, so that we really need the maximum of the two terms as stated in assumption (6.24).

Remark 6.15 If (H_3) fails for one or several i's, but (H_4) for the same i's are replaced by the uniform Lipschitz property of the representation of the I^i's coming from Theorem 6.13, then we still have the existence part of the previous Theorem. But the uniqueness fails.

Proof. (theorem) We write:

$$-\mathbb{E}e^{-\frac{1}{\eta_i}\left(X_T^{i,\pi^i}-\lambda_i\bar{X}_T^i-\int_0^T g^i(\sigma_u\pi_u^i)du\right)} = -e^{-\frac{1}{\eta_i}(x^i-\lambda_i\bar{x}^i)}$$
$$\times\ \mathbb{E}e^{-\frac{1}{\eta_i}\left(\int_0^T\sigma_u\pi_u^i.(dW_u+\theta_u du)-\lambda_i^N\int_0^T\sum_{j\neq i}\sigma_u\pi_u^j.(dW_u+\theta_u du)-\int_0^T g^i(\sigma_u\pi_u^i)du\right)},$$

which allows us to forget about the initial conditions and recall that $B_t=\int_0^t\theta_u du+W_t$ is a Brownian motion under $\mathbb{Q}$.

Necessary conditions:

Let $(\tilde{\pi}^1,\ldots,\tilde{\pi}^N)$ be a Nash equilibrium. Then for each i it means that $\tilde{\pi}^i$ is optimal for the problem of agent i, given the other agents strategies. As (H_1) to (H_3) are satisfied, the uniqueness in Theorem 6.6 implies that:

$$\sigma_t\tilde{\pi}_t^i = \Gamma_t^i(Z_t^i),$$

where (Y_t^i, Z_t^i) is the unique solution of the following BSDE:

$$\begin{aligned} dY_t^i &= -f_t^i(Z_t^i)dt + Z_t^i dW_t \\ Y_T^i &= \frac{\lambda_i}{N-1}\int_0^T \sum_{j\neq i} \sigma_u \tilde{\pi}_u^j.(dW_u + \theta_u du). \end{aligned}$$

As this is true for each i, putting everything together it implies that, writing Y the vector of $\mathbb{R}^N$ which i-th component is Y^i and Z the matrix of $M_{N,d}(\mathbb{R})$ which i-th line is Z^i, we get that (Y, Z) must satisfy:

$$Y_t^i = \int_0^T \lambda_i^N \sum_{j\neq i} \Gamma_u^j(Z_u^j)du.dB_u + \int_t^T f_u^i(Z_u^i)du - \int_t^T Z_u^i.dB_u + \int_t^T Z_u^i.\theta_u du.$$

Let us define:

$$\gamma_t^i := Y_t^i - \int_0^t \lambda_i^N \sum_{j\neq i} \Gamma_u^j(Z_u^j)du.dB_u.$$

We see that γ is adapted, continuous and as for all j, Γ^j is Lipschitz and θ_t bounded, $\gamma \in \mathbb{S}^2(\mathbb{R}^d)$, and (γ, Z) satisfies the following BSDE, which is not in a classical form:

$$\begin{aligned} d\gamma_t^i &= -[f_t^i(Z_t^i) + Z_t^i.\theta_t]dt + \left[Z_t^i - \lambda_i^N \sum_{j\neq i} \Gamma_t^j(Z_t^j) \right].dB_t \\ \gamma_T^i &= 0. \end{aligned}$$

Let us define $\varphi : M_{N,d}(\mathbb{R}) \to M_{N,d}(\mathbb{R})$ by:

$$\text{for all } 1 \leq i \leq N,\ \varphi^i(Z_t) := Z_t^i - \lambda_i^N \sum_{j\neq i} \Gamma_t^j(Z_t^j),$$

and we set $\zeta_t := \varphi_t(Z_t)$. By construction and because of assumption (H_4), φ_t is Lipschitz, uniformly in t, so that $\zeta \in \mathbb{H}^2(M_{N,d}(\mathbb{R}))$. Using Lemma 6.16 below, $\psi_t = \varphi_t^{-1}$ exists, so that (γ, ζ) satisfies the BSDE in canonical form (6.23):

$$\begin{aligned} d\gamma_t^i &= -[f_t^i \circ \psi_t^i(\zeta_t) + \psi_t^i(\zeta_t).\theta_t]dt + \zeta_t^i.dB_t \\ \gamma_T^i &= 0. \end{aligned}$$

Moreover, by Lemma 6.16, ψ_t is uniformly Lipschitz as the constant given by the lemma only depends on B, ε and N. Therefore f_t^i and ψ_t are Lipschitz uniformly in (t, ω) and using expression (6.14), $f_t^i(0) = \tilde{h}(\theta_t)$ is a continuous function w.r.t θ_t which is bounded, so that $f_t^i(0) \in \mathbb{S}^2(\mathbb{R})$. Therefore, the previous N-dimensional BSDE satisfies classical assumptions for Lipschitz BSDEs (see [63] or [26]), so that (γ, ζ) exists and is unique. The relations $\tilde{\pi}_t^i = \sigma_t^{-1} \circ \Gamma_t^i \circ \psi_t^i(\zeta_t)$ for each i show that the unique possible candidate for a Nash equilibrium is the one claimed, and uniqueness is proved.

Sufficient conditions:

Now let us make the previous computations in the reverse sense. Let (γ, ζ) be the unique solution in $\mathbb{S}^2(\mathbb{R}^d) \times \mathbb{H}^2(M_{N,d}(\mathbb{R}))$ of BSDE (6.23) and for each i, $\hat{\pi}_t^i := \sigma_t^{-1} \circ \Gamma_t^i \circ \psi_t^i(\zeta_t)$. We then define $Z_t^i := \psi_t^i(\zeta)$ and:

$$Y_t^i := \gamma_t^i + \int_0^t \lambda_i^N \sum_{j \neq i} \Gamma_u^j(Z_u^j) du.dB_u.$$

We easily check that Y is continuous and adapted, Z is predictable, (Y, Z) belongs to $\mathbb{S}^2(\mathbb{R}^d) \times \mathbb{H}^2(M_{N,d}(\mathbb{R}))$, and for any i, (Y^i, Z^i) solves the BSDE:

$$\begin{aligned} dY_t^i &= -f_t^i(Z_t^i)dt + Z_t^i dW_t \\ Y_t^i &= \frac{\lambda^i}{N-1} \int_0^T \sum_{j \neq i} \sigma_u \hat{\pi}_u^j.(dW_u + \theta_u du). \end{aligned}$$

Using Theorem 6.6, we see that for any i, $\hat{\pi}^i$ is optimal for agent i, given the other $\hat{\pi}^j$'s, which means that $(\hat{\pi}^1, ..., \hat{\pi}^N)$ is a Nash equilibrium.

Finally as $Y_0^i = \gamma_0^i$, we get the expression of V_i.

The end of the statement, concerning $\lambda_{m,\varepsilon}$ is proved in Lemma 6.17 below. □

We show here that the bijection property of φ as well as the Lipschitz property of its inverse function.

Lemma 6.16 *For each i, let Γ^i be a Lipschitz function with Lipschitz constant $B > 0$, independent from i. Define $\varphi : M_{N,d}(\mathbb{R}) \to M_{N,d}(\mathbb{R})$ by:*

$$\forall i \in [0, N], \ \varphi^i(z) = z^i - \lambda_i^N \sum_{j \neq i} \Gamma^j(z^j).$$

Under the assumptions of Theorem 6.13, φ is a bijection from $M_{N,d}(\mathbb{R})$ onto itself and its inverse ψ is given by:

$$\begin{aligned} \psi^i(\zeta) = &\left[I - \frac{1}{N-1} \sum_{j \neq i} \Gamma^j \circ \left(I + \lambda_j^N \Gamma^j \right)^{-1} \circ \left[\lambda_j \left(I + \lambda_i^N \Gamma^i \right) \right] \right]^{-1} \\ &\circ \left[\zeta^i + \frac{1}{N-1} \sum_{j \neq i} \Gamma^j \circ \left(I + \lambda_j^N \Gamma^j \right)^{-1} \left(\lambda_i \zeta^j - \lambda_j \zeta^i \right) \right]. \end{aligned}$$

Moreover, writing $\Lambda := \max_{1\le i\le N} \lambda_i$ *and* $\Lambda^N := \dfrac{\Lambda}{N-1}$, ψ *is Lipschitz continuous with Lipschitz constant:*

$$\left(1+\frac{2\Lambda B}{1-\Lambda^N B}\right)\frac{1}{1-K}$$

with K *equal to:*

- $\dfrac{\Lambda B}{1-\Lambda^N B}\left(1+\Lambda^N B\right)$, *in general;*
- $\dfrac{\Lambda B\left(1+\Lambda^N B\right)}{\sqrt{1+2\Lambda^N\varepsilon+\left(\Lambda^N B\right)^2}}$ *if* (H_5) *holds for every* i.

Proof. Let

$$\zeta^i := \varphi^i(z) = z^i - \lambda_i^N \sum_{j\ne i} \Gamma^j(z^j).$$

We compute:

$$\lambda_i\left[I+\lambda_j^N\Gamma^j\right](z^j) = \left[\lambda_j\left(I+\lambda_i^N\Gamma^i\right)(z^i)+\lambda_i\zeta^j-\lambda_j\zeta^i\right].$$

1) Let us prove that $I+\lambda_j^N\Gamma^j$ is a bijection of $\mathbb{R}^d$.

Indeed let $y\in\mathbb{R}^d$, we define $f_y(x) = y-\lambda_j^N\Gamma^j(x)$.
As Γ^j is B-Lipschitz, we compute:

$$\begin{aligned}|f_y(x)-f_y(x')| &= \lambda_j^N|\Gamma^j(x)-\Gamma^j(x')|\\ &\le \lambda_j^N B|x-x'|.\end{aligned}$$

As we either have (6.22) or (6.24), $\lambda_j^N B<1$, which means that f_y is a strict contraction of $\mathbb{R}^d$, and so it has a fixed point that we write x_y. In other words, x_y is the only solution of $f(x_y) = y-\lambda_j^N\Gamma^j(x_y) = x_y$, so that as this is the case for any $y\in\mathbb{R}^d$, $I+\lambda_j^N\Gamma^j$ is a bijection of $\mathbb{R}^d$.

2) Therefore we compute:

$$\begin{aligned}\xi :=& \sum_{j=1}^N \Gamma^j(z^j)\\ =&\Gamma^i(z^i)+\frac{1}{N-1}\sum_{j\ne i}\Gamma^j\circ\left(I+\lambda_j^N\Gamma^j\right)^{-1}\circ\left[\lambda_j\left(I+\lambda_i^N\Gamma^i\right)(z^i)+\lambda_i\zeta^j-\lambda_j\zeta^i\right].\end{aligned}$$

And:

$$\begin{aligned}\zeta^i &= z^i + \lambda_i^N(\Gamma^i(z^i) - \xi)\\ &= z^i - \frac{1}{N-1}\sum_{j\neq i}\Gamma^j \circ \left(I + \lambda_j^N\Gamma^j\right)^{-1} \circ \left[\lambda_j\left(I + \lambda_i^N\Gamma^i\right)(z^i) + \lambda_i\zeta^j - \lambda_j\zeta^i\right].\end{aligned}$$

So that finally:

$$\begin{aligned}&\left[I - \frac{1}{N-1}\sum_{j\neq i}\Gamma^j \circ \left(I + \lambda_j^N\ \Gamma^j\right)^{-1} \circ \left[\lambda_j\left(I + \lambda_i^N\Gamma^i\right)\right]\right](z^i)\\ &\qquad\qquad = \zeta^i + \frac{1}{N-1}\sum_{j\neq i}\Gamma^j \circ \left(I + \lambda_j^N\Gamma^j\right)^{-1}(\lambda_i\zeta^j - \lambda_j\zeta^i).\end{aligned}$$

3) Let us now define:

$$F(x) = \left[I - \frac{1}{N-1}\sum_{j\neq i}\Gamma^j \circ (I + \lambda_j^N\Gamma^j)^{-1} \circ [\lambda_j(I + \lambda_i^N\Gamma^i)]\right](x). \tag{6.26}$$

We show that F is a bijection of $\mathbb{R}^d$, that its inverse is Lipschitz and compute the associated Lipschitz constant. Again the idea is to use Picard's fixed point theorem. We will see that if assumption (H_5) is satisfied, the Lipschitz constant is smaller.

We compute:

$$\begin{aligned}\left|\left(I + \lambda_i^N\Gamma^i\right)(x) - \left(I + \lambda_i^N\Gamma^i\right)(y)\right| &\leq |x-y| + \lambda_i^N|\Gamma^i(x) - \Gamma^i(y)|\\ &\leq \left(1 + \lambda_i^N B\right)|x-y|,\end{aligned}$$

so that $I + \lambda_i^N\Gamma^i$ is $\left(1 + \lambda_i^N B\right)$-Lipschitz.

Then we take a look at $\Gamma^j \circ (I + \lambda_j^N\Gamma^j)^{-1}$. In the general case, we have:

$$\begin{aligned}\left|\left(I + \lambda_j^N\Gamma^j\right)(x) - \left(I + \lambda_j^N\Gamma^j\right)(y)\right| &\geq |x-y| - \lambda_j^N|\Gamma^j(x) - \Gamma^j(y)|\\ &\geq \left(\frac{1}{B} - \lambda_j^N\right)|\Gamma^j(x) - \Gamma^j(y)|.\end{aligned}$$

Again we know that $\dfrac{1}{B} - \lambda_j^N > 0$, so that $\Gamma^j \circ (I + \lambda_j^N\Gamma^j)^{-1}$ is $\dfrac{1}{\frac{1}{B} - \lambda_j^N}$-Lipschitz.

If assumption (H_5) holds true, then we can say more:

$$\begin{aligned}\left|\left(I + \lambda_j^N\Gamma^j\right)(x) - \left(I+\ \lambda_j^N\Gamma^j\right)(y)\right|^2 &= |x-y|^2 + 2\lambda_j^N(x-y).(\Gamma^j(x) - \Gamma^j(y))\\ &\quad + \left(\lambda_j^N\right)^2|\Gamma^j(x) - \Gamma^j(y)|^2\\ &\geq \left(\frac{1}{B^2} + \frac{2\varepsilon\lambda_j^N}{B^2} + \left(\lambda_j^N\right)^2\right)|\Gamma^j(x) - \Gamma^j(y)|^2.\end{aligned}$$

Therefore under (H_5), $\Gamma^j \circ (I+\lambda_j^N \Gamma^j)^{-1}$ is $\dfrac{1}{\sqrt{\frac{1}{B^2}+\frac{2\varepsilon\lambda_j^N}{B^2}+\left(\lambda_j^N\right)^2}}$-Lipschitz (the constant is strictly smaller than the previous one as $\varepsilon \geq 0$).

So we get that $\Gamma^j \circ (I+\lambda_j^N \Gamma^j)^{-1} \circ [\lambda_j(I+\lambda_i^N\Gamma^i)]$ is Lipschitz with constant:

- $\dfrac{\lambda_j B}{1-\lambda_j^N B}\left(1+\lambda_i^N B\right)$ in general;
- $\dfrac{\lambda_j B\left(1+\lambda_i^N B\right)}{\sqrt{1+2\lambda_j^N\varepsilon+\left(\lambda_j^N B\right)^2}}$ if (H_5) holds.

Notice that both expressions are increasing w.r.t λ_i and λ_j. For the first one, this is immediate. The second one is also clearly increasing w.r.t λ_i and its derivative w.r.t λ_j has the same sign as:

$$1+2\varepsilon\lambda_j+(\lambda_j B)^2-\varepsilon\lambda_j-(\lambda_j B)^2 = 1+\varepsilon\lambda_j > 0,$$

so we have the result. As a consequence, if we write $\Lambda := \max\limits_{1\leq j\leq N}\lambda_j$, for any i, j, $\Gamma^j \circ (I+\lambda_j^N\Gamma^j)^{-1} \circ [\lambda_j(I+\lambda_i^N\Gamma^i)]$ is Lipschitz with the same constant given by the previous expressions, replacing both λ_i and λ_j by Λ.

4) For any $y \in \mathbb{R}^d$, let us now define:

$$\begin{aligned} G_y(x) &= \frac{1}{N-1}\sum_{j\neq i}\Gamma^j \circ \left(I+\lambda_j^N\Gamma^j\right)^{-1} \circ \left[\lambda_j\left(I+\lambda_i^N\Gamma^i\right)\right](x)+y \\ &= (I-F)(x)+y, \end{aligned}$$

where the definition of F is given by (6.26).

From the previous computations, we see that G_y is Lipschitz with constant K independent of y, with K equal to:

- $\dfrac{\Lambda B}{1-\Lambda^N B}\left(1+\Lambda^N B\right)$, in general; (6.27)
- $\dfrac{\Lambda B\left(1+\Lambda^N B\right)}{\sqrt{1+2\Lambda^N\varepsilon+\left(\Lambda^N B\right)^2}}$ if (H_5) holds. (6.28)

Using now Lemma 6.17 below, under assumption (6.22) in the general case or (6.24) if (H_5) holds, then G_y is a strict contraction of $\mathbb{R}^d$, thus admits a unique fixed point, that we write x_y. Therefore for any $y \in \mathbb{R}^d$, x_y is the unique solution of $x_y - G_y = 0 = F(x_y) - y$, which means that F is a bijection, and therefore φ is a bijection as well with inverse function ψ given in the statement of the Lemma.

Moreover, as $K < 1$ (the Lipschitz constant for G_y) and noticing that:

$$|F(x) - F(x')| = |(I - G_y)(x) - (I - G_y)(x')| \geq (1 - K)|x - x'|,$$

we see that F^{-1} is $\dfrac{1}{1-K}$-Lipschitz, and the expression of K is given by (6.27).

Finally using again the previous computations, we easily derive that

$$\zeta \mapsto \zeta^i + \frac{1}{N-1} \sum_{j \neq i} \Gamma^j \circ \left(I + \lambda_j^N \Gamma^j\right)^{-1} (\lambda_i \zeta^j - \lambda_j \zeta^i)$$

is Lipschitz with Lipschitz constant equal to $1 + \dfrac{2\Lambda B}{1 - \Lambda^N B}$. So we have the result. □

The following lemma was used in the previous proof.

Lemma 6.17 (i) *If assumption* (6.22) *is satisfied, then*

$$\frac{\Lambda B}{1 - \Lambda^N B}\left(1 + \Lambda^N B\right) < 1.$$

(ii) *For any $\varepsilon \in [0, B]$, the following equation (in x):*

$$xB\left(1 + \frac{xB}{N-1}\right) = \sqrt{1 + 2\frac{\varepsilon}{B}\frac{xB}{N-1} + \left(\frac{xB}{N-1}\right)^2}$$

has a unique positive solution that we write $\lambda_{m,\varepsilon}$, which is increasing w.r.t ε and satisfies:

$$\lambda_m < \lambda_{m,0} < \lambda_{m,B} = \frac{1}{B}.$$

If assumption (6.24) *is satisfied, then*

$$\frac{\Lambda B\left(1 + \Lambda^N B\right)}{\sqrt{1 + 2\Lambda^N \varepsilon + \left(\Lambda^N B\right)^2}} < 1.$$

Proof. (i): If we write $x = \Lambda^N B$, this is equivalent to showing that $x(1+x) < \dfrac{1}{N-1}(1-x)$, ie:

$$x^2 + \left(1 + \frac{1}{N-1}\right)x - \frac{1}{N-1} < 0,$$

which is equivalent to $x \in \left(-\dfrac{\sqrt{N^2 + 4(N-1)} + N}{2(N-1)}, \dfrac{\sqrt{N^2 + 4(N-1)} - N}{2(N-1)}\right)$, so that if $\Lambda \in \left[0, \frac{\lambda_m}{B}\right)$, then $\dfrac{\Lambda B}{1 - \Lambda^N B}\left(1 + \Lambda^N B\right) < 1$.

(ii): If we write $y := \dfrac{xB}{N-1}$, then x is a positive solution of:

$$xB\left(1+\frac{xB}{N-1}\right) = \sqrt{1+2\frac{\varepsilon}{B}\frac{xB}{N-1}+\left(\frac{xB}{N-1}\right)^2}$$

if and only if y is a positive solution of:

$$y(1+y) = \frac{1}{N-1}\sqrt{1+2\frac{\varepsilon}{B}y+y^2},$$

and this is equivalent to y is a positive solution of:

$$y^4+2y^3+y^2 = \left(\frac{1}{N-1}\right)^2\left(1+2\frac{\varepsilon}{B}y+y^2\right).$$

Writing:

$$f(y) := y^4+2y^3+y^2 - \left(\frac{1}{N-1}\right)^2\left(1+2\frac{\varepsilon}{B}y+y^2\right),$$

we compute $f''(y) = 12y^2+12y+2\left(1-\dfrac{1}{(N-1)^2}\right)$, which has two non-positive solution, is therefore positive on $(0,+\infty)$. Thus f' is increasing on $\mathbb{R}_+$. As $f'(0) = -2\dfrac{\varepsilon}{B(N-1)^2} \leq 0$ while $\lim_{+\infty} f' = +\infty$, f' has a unique positive zero, say α, so that f is decreasing on $[0,\alpha]$ and increasing on $[\alpha,+\infty]$. Finally as $f(0) = -\dfrac{1}{(N-1)^2} < 0$ while $\lim_{+\infty} f = +\infty$, we get the existence and uniqueness of the solution of the required equation, so that $\lambda_{m,\varepsilon}$ is well-defined.

Then, as $f = f_\varepsilon$ is decreasing w.r.t ε, so that if $\varepsilon < \varepsilon'$, we have $f_\varepsilon(\lambda_{m,\varepsilon'}) > f'_\varepsilon(\lambda_{m,\varepsilon'}) = 0$. Then, the previous analysis implies that $\lambda_{m,\varepsilon'} > \lambda_{m,\varepsilon}$. The fact that $\lambda_m < \lambda_{m,0}$ is then immediate as $\lambda_m = \lambda_{m,-B}$, and if $\varepsilon = B$, then $\dfrac{\lambda_{m,B}B}{N-1}$ is the positive solution of:

$$x(1+x) = \frac{1}{N-1}(1+x),$$

so that $\lambda_{m,B} = \dfrac{1}{B}$.

Finally, if $\Lambda \in \left[0, \dfrac{\lambda_{m,\varepsilon}}{B}\right)$, as $f(x) < 0$ on $[0,\lambda_{m,\varepsilon})$, we have the result. $\square$

6.4.2 Deterministic θ and limit as N goes to infinity

In this section, we analyze the asymptotics as $N \to \infty$, when the risk premium process θ is deterministic. In other words, we assume that $\theta : [0, T] \to \mathbb{R}^d$ is a continuous function. We first restate Theorem 6.13 in this setting.

Corollary 6.18 *In addition to the assumptions of Theorem 6.13, assume that θ is a deterministic function of t. Then Y^i is given by:*

$$Y_t^i = \int_t^T (f_u^i \circ \psi_u^i(0) + \psi_u^i(0).\theta(u))du,$$

and the equilibrium portfolio for agent i is:

$$\hat{\pi}_t^i = \sigma_t^{-1} \circ \Gamma_t^i \circ \psi_t^i(0).$$

Proof. The definition of f^i implies that if θ is deterministic, so is f^i for each i. Then Γ^i is deterministic too, and therefore ψ as well. As a consequence, if Y^i is given as in the statement of the corollary, then $(Y, 0)$ is the solution of BSDE (6.23). So we have the expression of Y^i and the expression of $\hat{\pi}^i$. □

Remark 6.19 Notice that under the assumption that θ is a deterministic function of t, for any i, Γ^i is also a deterministic function of t. As a consequence, $\sigma_t \hat{\pi}_t^i$ is deterministic too.

Then we have the following result:

Proposition 6.20 *Assume that θ is there exists $B > 0$ such that for all $j \in \mathbb{N}$, Γ^j is B-Lipschitz, uniformly in t, and that:*

$$\Lambda = \max_{1 \le j \le N} \lambda_j < \frac{1}{B}.$$

(i) *If for $t \in [0, T]$, there exists a function χ_t such that, as $N \to \infty$:*

$$\frac{1}{N} \sum_{j=1}^N \Gamma_t^j \circ \lambda_j I \to \chi_t \text{ uniformly on any compact of } \mathbb{R}^d \text{ and if } \sup_{j \in \mathbb{N}} |\Gamma_t^j(0)| < \infty, \text{ then:}$$

$$\sigma_t \hat{\pi}_t^{i,N} \to \sigma_t \hat{\pi}_t^{i,\infty} := \Gamma_t^i \circ (I - \chi_t)^{-1} \circ \chi_t(0).$$

(ii) *If for all* $t \in [0,T]$*, there exists a function* χ_t*, such that for any compact* K *of* $\mathbb{R}^d$*, as* $N \to \infty$*:*

$$\sup_{(t,x)\in[0,T]\times K} \left| \frac{1}{N}\sum_{j=1}^{N} \Gamma_t^j(\lambda_j x) - \chi_t(x) \right| \to 0, \ \ \textit{and} \ \ \sup_{(t,j)\in[0,T]\times\mathbb{N}} |\Gamma_t^j(0)| < \infty,$$

then:

$$\sup_{t\in[0,T]} |\sigma_t \hat{\pi}_t^{i,N} - \sigma_t \hat{\pi}_t^{i,\infty}| \to 0.$$

Proof. First $\Gamma_t^j \circ \lambda_j I$ is ΛB-Lipschitz, so that χ_t also, and as $\Lambda B < 1$, it implies that $I - \chi_t$ is a bijection and moreover its inverse is Lipschitz. We write $\Gamma_{0,t} := \sup_{j\in\mathbb{N}} |\Gamma_t^j(0)|$.

From Corollary 6.18, we have: $\sigma_t \hat{\pi}_t^{i,N} = \Gamma_t^i \circ \psi_t^{N,i}(0)$, which is a deterministic function of t as θ is deterministic. Recall the expression of $\psi_t^N(0)$:

$$\begin{aligned}\psi_t^{N,i}(0) = &\left[I - \frac{1}{N-1}\sum_{j\neq i} \Gamma_t^j \circ \left(I + \lambda_j^N \Gamma_t^j\right)^{-1} \circ \left[\lambda_j\left(I + \lambda_i^N \Gamma_t^i\right)\right] \right]^{-1} \\ &\circ \left[\frac{1}{N-1}\sum_{j\neq i} \Gamma_t^j \circ \left(I + \lambda_j^N \Gamma_t^j\right)^{-1}(0) \right].\end{aligned}$$

Let $1 \leq j \leq N$. We have:

$$\begin{aligned} \left|\left(I + \lambda_j^N \Gamma_t^j\right)(x) - x\right| &\leq \Lambda^N B(|x| + \Gamma_{0,t}), \\ \text{so that: } \left|\left(I + \lambda_j^N \Gamma_t^j\right)^{-1}(x) - x\right| &\leq \Lambda^N B\left(\left|\left(I + \lambda_j^N \Gamma_t^j\right)^{-1}(x)\right| + \Gamma_{0,t}\right) \\ &\leq \frac{\Lambda B}{N-1-\Lambda B}(|x| + \Gamma_{0,t}), \end{aligned}$$

as we compute that:

$$\left|\left(I + \lambda_j^N \Gamma_t^j\right)(x)\right| \geq \left(1 - \Lambda^N B\right)|x| - \Lambda^N B |\Gamma_t^j(0)|.$$

This also implies that:

$$\begin{aligned} &\left|\left(I + \lambda_j^N \Gamma_t^j\right)^{-1} \circ \left[\lambda_j\left(I + \lambda_i^N \Gamma_t^i\right)\right](x) - x\right| \\ &\qquad \leq \frac{\Lambda B}{N-1-\Lambda B}\left(\left|\left(I + \lambda_i^N \Gamma_t^i\right)(x)\right| + \Gamma_{0,t}\right) + \left|\left(I + \lambda_i^N \Gamma_t^i\right)(x) - x\right| \\ &\qquad \leq \left[\frac{\Lambda B}{N-1-\Lambda B}\left(1 + \Lambda^N B\right) + \Lambda^N B \right](|x| + \Gamma_{0,t}) \end{aligned}$$

Therefore we have:

$$\left|\Gamma_t^j \circ \left(I + \lambda_j^N \Gamma_t^j\right)^{-1} (\lambda_j x) - \Gamma_t^j(\lambda_j x)\right| \le \frac{\Lambda B^2}{N-1-\Lambda B}(\Lambda|x| + \Gamma_{0,t}),$$

so that:

$$\left|\frac{1}{N-1}\sum_{j\neq i} \Gamma_t^j \circ \left(I + \lambda_j^N \Gamma_t^j\right)^{-1} (\lambda_j x) - \frac{1}{N-1}\sum_{j\neq i}\Gamma_t^j(\lambda_j x)\right| \le \frac{\Lambda B^2}{N-1-\Lambda B}(\Lambda|x| + \Gamma_{0,t}).$$

In other words, $\frac{1}{N-1}\sum_{j\neq i}\Gamma_t^j \circ \left(I + \lambda_j^N \Gamma_t^j\right)^{-1} \circ \lambda_j I$ converges uniformly on compact sets to χ_t, and similarly $f_N := \frac{1}{N-1}\sum_{j\neq i}\Gamma_t^j \circ \left(I + \lambda_j^N \Gamma_t^j\right)^{-1} \circ \left[\lambda_j \left(I + \lambda_i^N \Gamma_t^i\right)\right]$ converges uniformly on compact sets to χ_t as well. In case (ii), this convergence is uniform in $t \in [0,T]$. Moreover, we know from Theorem 6.13 that the f_N's are uniformly Lipschitz with a constant $D < 1$. Therefore, using Lemma 6.21 below, for any compact K, in case (i) (resp. (ii)), we get the uniform convergence on K (resp. on $[0,T]\times K$) of:

$$\left[I - \frac{1}{N-1}\sum_{j\neq i}\Gamma_t^j \circ \left(I + \lambda_j^N \Gamma_t^j\right)^{-1} \circ \left[\lambda_j \left(I + \lambda_i^N \Gamma_t^i\right)\right]\right]^{-1}$$
$$\circ \left[\frac{1}{N-1}\sum_{j\neq i}\Gamma_t^j \circ \left(I + \lambda_j^N \Gamma_t^j\right)^{-1} \circ \lambda_j I\right],$$

which brings the required result. □

Lemma 6.21 (i) *Let (f_n) be a sequence of uniformly Lipschitz functions, with constant $D \in (0,1)$, and assume that (f_n) converges uniformly on compact sets (resp. pointwise converges) to a function f. Then $((I - f_n)^{-1})$ converges uniformly on compact sets (resp. pointwise converges) to $(I-f)^{-1}$.*
(ii) *Let (f_n) and (g_n) be sequences which converge uniformly on compact sets respectively towards f and g. Assume that (g_n) is uniformly Lipschitz with constant D, and that f is continuous. Then $(g_n \circ f_n)$ converges uniformly on compact sets towards $g \circ f$.*

Proof. (i): As f_n is a strict contraction, $I - f_n$ is a bijection and its inverse is $\frac{1}{1-D}$-Lipschitz. We write $g_n = I - f_n$. As the sequence shares the same Lipschitz constant, the same holds for f. We also write $g = I - f$. Let $x \in \mathbb{R}^d$, we write $y = g^{-1}(x)$, then we have:

$$\begin{aligned}
|g_n^{-1}(x) - g^{-1}(x)| &= |g_n^{-1} \circ g(y) - y| \\
&= |g_n^{-1} \circ g(y) - g_n^{-1} \circ g_n(y)| \\
&\le \frac{1}{1-D}|f(y) - f_n(y)|.
\end{aligned}$$

So we have the pointwise convergence, and as g^{-1} is Lipschitz, the uniform convergence on compact sets of f_n implies the uniform convergence on compact sets of $((I-f_n)^{-1})$ towards $((I-f)^{-1})$.

(ii): We compute:

$$\begin{aligned}|g_n \circ f_n(x) - g \circ f(x)| &\le |g_n \circ f_n(x) - g_n \circ f(x)| + |g_n \circ f(x) - g \circ f(x)| \\ &\le D|f_n(x) - f(x)| + |g_n \circ f(x) - g \circ f(x)|.\end{aligned}$$

Now as f is continuous, for any compact set K, $f(K)$ is also compact, so that we have the result. □

6.5 Examples

In this section, we provide some examples of penalization functions g, so that assumptions (H_1) to (H_4) are satisfied. We also show that some of these assumptions fail in some cases.

6.5.1 A volatility penalization

Let us first consider the following penalization function:

$$g^i(p) = \alpha_i |p|, \tag{6.29}$$

where $\alpha_i \in \mathbb{R}$ and $|.|$ is the canonical euclidean norm of $\mathbb{R}^d$. Recall that the (global) volatility of the wealth associated to a portfolio π is equal to $\displaystyle\int_0^T |\sigma_t \pi_t| dt = \frac{1}{\alpha_i}\int_0^T g(\sigma_t \pi_t) dt$. So that this penalization function is a way to penalize portfolios that have a large volatility.
g^i is Lipschitz. We show hereafter that g^i satisfies (H_1) to (H_3) if $\alpha_i \neq 0$ and we compute explicitly f_t^i and the minimizing function Γ^i. g^i is C^1 on $\mathbb{R}^*$, and its gradient is given by:

$$\nabla g^i(p) = \frac{\alpha_i}{|p|} p, \; p \neq 0,$$

which is bounded on $\mathbb{R}^*$, so that (H_2) is satisfied. Then $h^i(p) = \alpha_i |p| + \dfrac{1}{2\eta_i}|p|^2$ is strictly convex, so that (H_3) is also satisfied. Finally, $\nabla g + \dfrac{1}{\eta} I$ is one-to-one on $\mathbb{R}^*$ and the inverse of its restriction to $\mathbb{R} \setminus [-\varepsilon, \varepsilon]$ is Lipschitz, for every $\varepsilon > 0$, so that (H_1) is satisfied as well. Moreover, we can compute f_t^i and I_t^i explicitly, and we will see that if $\alpha_i < 0$, then (H_4) does not hold true.

Lemma 6.22 *Let g^i be given by* (6.29)*. Then:*

$$f_t^i(z) = \begin{cases} \frac{1}{2\eta_i}|z|^2 & \text{if } |z+\eta_i\theta_t| < \alpha_i\eta_i \\ \alpha_i\,|z+\eta_i\theta_t| - z.\theta_t - \frac{\eta_i}{2}(|\theta_t|^2+\alpha_i^2) & \text{otherwise;} \end{cases}$$

$$\Gamma_t^i(z) = \begin{cases} 0 & \text{if } |z+\eta_i\theta_t| < \alpha_i\eta_i \\ \left(1 - \frac{\alpha_i\eta_i}{|z+\eta_i\theta_t|}\right)(z+\eta_i\theta_t) & \text{otherwise.} \end{cases}$$

Proof. If we write p^* a point where the minimum is attained, then if $p^* \neq 0$, the first order condition must be satisfied, or in other words:

$$\left(1 + \frac{\alpha_i\eta_i}{|p^*|}\right) p^* = z + \eta_i\theta_t.$$

But this equation has a solution if and only if $|z+\eta_i\theta_t| \geq \alpha_i\eta_i$, which is always the case if $\alpha_i \leq 0$. And if this condition holds, then we have a unique solution for this equation which is:

$$p^* = \left(1 - \frac{\alpha_i\eta_i}{|z+\eta_i\theta_t|}\right)(z+\eta_i\theta_t)\,.$$

We must also consider the possibility $p^* = 0$. Then $m_t(z,0) = \dfrac{1}{2\eta_i}|z|^2$. While if $|z+\eta_i\theta_t| \geq \alpha_i\eta_i$ for the previous expression:

$$m_t(z,p^*) = \alpha_i\,|z+\eta_i\theta_t| - z.\theta_t - \frac{\eta_i}{2}(|\theta_t|^2+\alpha_i^2).$$

Noticing that $\dfrac{1}{2\eta_i}\,|z+\eta_i\theta_t|^2 = \dfrac{1}{2\eta_i}|z|^2 + \dfrac{\eta_i}{2}|\theta_t|^2 + z.\theta_t$, we have:

$$\frac{1}{2\eta_i}|z|^2 \geq \alpha_i\,|z+\eta_i\theta_t| - z.\theta_t - \frac{\eta_i}{2}(|\theta_t|^2+\alpha_i^2),$$

with equality if and only if $|z+\eta_i\theta_t| = \alpha_i\eta_i$, in which case $p^* = 0$, so we have the result. $\square$

Then we want to apply Theorem 6.13 in this case. In fact if $\alpha_i \geq 0$, it is easy to check that Γ_t^i is Lipschitz, uniformly in (t,ω), but if $\alpha_i < 0$, then Γ_t^i is not even locally Lipschitz. Indeed, if $\alpha_i < 0$, consider (z_n) a sequence in $\mathbb{R}^d \setminus \{0\}$ such that $|z_n| \to 0$, then we have:

$$\left|\Gamma_t^i(z_n - \eta_i\theta_t) - \Gamma_t^i(-z_n - \eta_i\theta_t)\right| = 2\left(1 - \frac{\alpha_i\eta_i}{|z_n|}\right)|z_n|,$$

and $\left(1 - \dfrac{\alpha_i\eta_i}{|z_n|}\right) \to \infty$ as $n \to \infty$, so it cannot be Lipschitz in any neighborhood of $-\eta_i\theta_t$.

6.5.2 Penalization in one direction

If one chooses $\alpha_i \in \mathbb{R}^d$, and $g^i(p) = \alpha_i.p$, then it means that we only penalize the component in the direction of α_i. Then g^i is C^1, its gradient is constant $\nabla g^i(p) = \alpha_i$, and $h^i(p) = g^i(p) + \frac{1}{2\eta_i}|p|^2$ is strictly convex, so that Proposition 6.4 tells us that (H_1) to (H_3) hold. Moreover we have the following explicit expressions for f^i and I^i:

$$
\begin{aligned}
f^i_t(z) &= -\frac{\eta_i}{2}|\theta_t - \alpha_i|^2 - (\theta_t - \alpha_i).z \\
\Gamma^i_t(z) &= z + \eta_i(\theta_t - \alpha_i).
\end{aligned}
$$

And we can apply Theorem 6.13.

6.5.3 Penalization on the leverage

Another interesting example is the 1-norm on $\mathbb{R}^d$:

$$
g^i(p) = \alpha_i \sum_{j=1}^{d} |p^j|,
$$

with $\alpha_i \in \mathbb{R}$, and p^j the j-th component in the canonical euclidean base of $\mathbb{R}^d$. From a financial point of view, the global amount invested in risky assets is called the leverage. It would correspond to $g^i(\pi_t)$, but as we use $g^i(\sigma_t\pi_t)$, this can be seen as the leverage rescaled by the volatility.

To compute f^i, let us notice that:

$$
\min_{p\in\mathbb{R}^d}\left\{\sum_{j=1}^{N} \alpha_i|p^j| + \frac{1}{2\eta_i}(p^j - z^j)^2 - p^j\theta^j_t\right\} = \sum_{j=1}^{N} \min_{p^j\in\mathbb{R}}\{\alpha_i|p^j| + \frac{1}{2\eta_i}(p^j - z^j)^2 - p^j\theta^j_t\},
$$

so that the solution can be deduced from the volatility penalization with $d = 1$ (section 6.5.1). In particular, assumptions (H_1) to (H_3) are satisfied.

6.5.4 Nearly constrained investors

In this example, we consider an approximation of the case dealt with in the previous chapter. More precisely, if A_i is a closed convex subset of $\mathbb{R}^d$, we approximate the case of restrictive constraints $\pi^i \in A_i$ which would correspond to functions:

$$
g^i(p) = \begin{cases} 0 \text{ if } p \in A_i \\ +\infty \text{ if } p \notin A_i \end{cases}
$$

by the following functions g_n^i, $n \in \mathbb{N}$:

$$g_n^i(p) = nd(p, A_i).$$

For any n, g_n^i is Lipschitz, which guarantees the existence of a minimizer for f_t^i. Moreover $h(p) = g(p) + \dfrac{1}{2\eta_i}|p|^2$ is strictly convex, so using Proposition 6.4, (H_3) holds. For the other assumptions required, it is not easy to see it immediately, but we can compute explicitly the expressions for f_t^i and Γ_t^i. As A_i is a closed convex set, the orthogonal projection on A_i exists.

Proposition 6.23 *We write P^i the orthogonal projection on A_i and $\zeta_t^i(z) := z+\eta_i\theta_t$. Then:*

$$f_t^{i,n}(z) = \begin{cases} \frac{1}{2\eta_i} d^2(\zeta_t^i(z), A_i) - z.\theta_t - \frac{\eta_i}{2}|\theta_t|^2, & \textit{if } d(\zeta_t^i(z), A_i) \leq n\eta_i \\ nd(\zeta_t^i(z), A_i) - \frac{n^2\eta_i}{2} - z.\theta_t - \frac{\eta_i}{2}|\theta_t|^2, & \textit{if } d(\zeta_t^i(z), A_i) > n\eta_i; \end{cases}$$

$$\Gamma_t^{i,n}(z) = \begin{cases} P^i(\zeta_t^i(z)), & \textit{if } d(\zeta_t^i(z), A_i) \leq n\eta_i \\ \left(1 - \frac{n\eta_i}{d(\zeta_t^i(z), A_i)}\right)\zeta_t^i(z) + \frac{n\eta_i}{d(\zeta_t^i(z), A_i)} P^i(\zeta_t^i(z)), & \textit{if } d(\zeta_t^i(z), A_i) > n\eta_i. \end{cases}$$

As a consequence, assumptions (H_1) to (H_4) are satisfied.

Remark 6.24 Notice that for any $z \in \mathbb{R}^d$ and $t \in [0, T]$, as $n \to \infty$, we have $\Gamma_t^i(z) \to P^i(z + \eta_i\theta_t)$ a.s.

Proof. Let t and z be given. As g_n^i is Lipschitz, the minimum of $f_t^i(z)$ is attained. So we are looking for:

$$\min_{p\in\mathbb{R}^d} nd(p, A_i) + \frac{1}{2\eta_i}|p - z|^2 - p.\theta_t.$$

Noticing that:

$$\frac{1}{2\eta_i}|p^* - z|^2 - p^*.\theta_t = \frac{1}{2\eta_i}\left|p^* - (z + \eta_i\theta_t)\right|^2 - \frac{\eta_i}{2}|\theta_t|^2 - z.\theta_t,$$

we are in fact looking for:

$$\min_{p\in\mathbb{R}^d} nd(p, A_i) + \frac{1}{2\eta_i}\left|p - (z + \eta_i\theta_t)\right|^2 - z.\theta_t - \frac{\eta_i}{2}|\theta_t|^2,$$

and we see that if $d(z + \eta_i\theta_t, A_i) = 0$, then the previous expression will always be minimum for $p^* = z + \eta_i\theta_t$ and only for this point.

Let us from now on write $\zeta := z + \eta_i\theta_t$. Assume now that $d(\zeta, A_i) > 0$. Then $P^i(\zeta) \neq \zeta$. Let $d(p, A_i) = \rho \geq 0$ be fixed, let us prove that

$$\min_{p,\ d(p,A_i)=\rho} nd(p, A_i) + \frac{1}{2\eta_i}|p-\zeta|^2$$

is attained at a unique point which is the unique intersection between $C_i(\rho) := \{x \in \mathbb{R}^d,\ d(x, A_i) = \rho\}$ and the half-line $L_i(\zeta) := \{x \in \mathbb{R}^d;\ \exists\alpha \geq 0,\ x = \alpha\zeta + (1-\alpha)P^i(\zeta)\}$.

Indeed, if we write $x_\alpha := \alpha\zeta + (1-\alpha)P^i(\zeta)$ let us prove that $P^i(x_\alpha) = P^i(\zeta)$. We know that $P^i(x_\alpha)$ is the only point that satisfies:

$$\text{for all } y \in A_i,\ (x_\alpha - P^i(x_\alpha)).(y - P^i(x_\alpha)) \leq 0,$$

but as $\alpha \geq 0$, we compute for $y \in A_i$:

$$(x_\alpha - P^i(\zeta)).(y - P^i(\zeta)) = \alpha(\zeta - P^i(\zeta)).(y - P^i(\zeta)) \leq 0,$$

so we have the equality needed.

Therefore we can compute $d(x_\alpha, A_i) = |x(\alpha) - P^i(\zeta)| = \alpha d(\zeta, A_i)$, and so the existence and uniqueness of the intersection is proved.

Now let us write x_ρ this unique itersection and let $p \in C_i(\rho)$. As $P^i(\zeta) = P^i(x_\rho)$, we compute, if $\rho \leq d(\zeta, A_i)$:

$$\begin{aligned}
&|\zeta - P^i(p)| \leq |\zeta - p| + |p - P^i(p)| = |\zeta - p| + \rho \\
&|\zeta - P^i(\zeta)| = |x_\rho - P^i(\zeta)| + |\zeta - x_\rho| = \rho + |\zeta - x_\rho|.
\end{aligned}$$

If $P^i(p) \neq P^i(\zeta)$, then $|\zeta - P^i(\zeta)| < |\zeta - P^i(p)|$, so that $|\zeta - p| > |\zeta - x_\rho|$. If $P^i(p) = P^i(\zeta)$, but $p \neq x_\rho$, then $|\zeta - P^i(p)| < |\zeta - p| + |p - P^i(p)| = |\zeta - p| + \rho$, and again $|\zeta - P^i(\zeta)| < |\zeta - P^i(p)|$.

If $\rho \geq d(\zeta, A_i)$:

$$\begin{aligned}
&|p - P^i(\zeta)| \leq |p - \zeta| + |\zeta - P^i(\zeta)| \\
&|\zeta - P^i(\zeta)| = |x_\rho - P^i(\zeta)| - |\zeta - x_\rho| = \rho - |\zeta - x_\rho|.
\end{aligned}$$

And as before if $p \neq x_\rho$, we either have $|p - P^i(\zeta)| > \rho$ or $|p - P^i(\zeta)| < |p - \zeta| + |\zeta - P^i(\zeta)|$, so that $|\zeta - p| > |\zeta - x_\rho|$.

As a conclusion, recalling that $d(x_\alpha, A_i) = |x_\alpha - P^i(\zeta)| = \alpha d(\zeta, A_i)$, we have reduced the minimization problem to:

$$\inf_{\alpha \geq 0} n\alpha d(\zeta, A_i) + \frac{1}{2\eta_i}(\alpha - 1)^2 d^2(\zeta, A_i).$$

If $d(\zeta, A_i) > n\eta_i$, then the minimum is attained at $\alpha = 1 - \frac{n\eta_i}{d(\zeta,A_i)}$, which brings $\Gamma_t^i(z) = \left(1 - \frac{n\eta_i}{d(\zeta,A_i)}\right)\zeta + \frac{n\eta_i}{d(\zeta,A_i)}P^i(\zeta)$ and the claimed expression of $f_t^i(z)$, while if $d(\zeta, A_i) \leq n\eta_i$, then $\Gamma_t^i(z) = P^i(\zeta)$ and again the claimed expression of $f_t^i(z)$.

Finally, it is easy to check that assumptions (H_1) to (H_4) are satisfied. $\square$

Then we can apply Theorem 6.13 guaranteeing the existence and uniqueness of a Nash equilibrium.

An open question is what happens when n goes to infinity. For any i, the sequence of functions $(g^{i,n})_n$ is nondecreasing, therefore the same holds for $(f^{i,n})_n$. Unfortunately, as there is no general comparison result for coupled multidimensional BSDEs (see Hu and Peng [43]), we do not know how to do it.

Bibliography

[1] A.B. Abel (1990). Asset prices under habit formation and catching up with the Joneses. *American Economic Review* 80, 38-42.

[2] M. Akian, J.-L. Menaldi, A. Sulem (1995). Multi-asset portfolio selection problem with transaction costs. *Probabilités numériques (Paris, 1992), Mathematics and Computers in Simulation* 38, 163-172.

[3] I. Ben Tahar, H.M. Soner and N. Touzi (2008). Modelling continuous-time financial markets with capital gains taxes. Preprint.

[4] I. Ben Tahar, H.M. Soner and N. Touzi (2008). The dynamic programming equation for the problem of optimal investment under capital gains taxes. *SIAM J. on Control and Optimization* 46, 1779-1801.

[5] D.P. Bertsekas and S.E Shreve (1978). *Stochastic Optimal Control: the Discrete Time Case*. Academic Press.

[6] J.-M. Bismut (1973). Conjugate convex functions in optimal stochastic control. *J. Math. Ann. Appl.* 44, 384-404.

[7] P. Briand and Y. Hu (2008). Quadratic BSDEs with convex generators and unbounded terminal value. *Probab. Theory Relat. Fields* 141, 543-567.

[8] G.M. Constantinides (1983). Capital market equilibrium with personal taxes. *Econometrica* 51, 611-636.

[9] G.M. Constantinides and M.J.P. Magill (1976). Portfolio selection with transaction costs. *J. Econ. Theory* 13, 245-263.

[10] J. Cox and C.F. Huang (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *J. Econ. Theory* 49, 33-83.

[11] M.G. Crandall, H. Ishii and P.-L. Lions (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.* 27 (1), 1-67.

[12] J. Cvitanic and I. Karatzas (1992). Convex duality in constrained portfolio optimization. *Ann. Appl. Porbab.* 2, 767-818.

[13] R.M. Damon, C.S. Spatt and H.H. Zhang (2001). Optimal consumption and investment with capital gains taxes. *The Review of Financial Studies* 14, 583-616.

[14] G. Da Prato and J. Zabczyk (1992). *Stochastic Equations in Infinite Dimensions.* Cambridge University Press.

[15] M.H.A. Davis and A.R. Norman (1990). Portfolio selection with transaction costs. *Math. of Operations Research* 15 (4), 676-713.

[16] P. DeMarzo, R. Kaniel and I. Kremer (2007). Relative wealth concerns and financial bubbles. *Rev. Financ. Stud.*

[17] D. Duffie and T. Sun (1990). Transaction costs and portfolio choice in a discrete-continuous-time setting. *Journal of Economic Dynamics and Control* 14, 35-51.

[18] J. Du Toit and G. Peskir (2007). The trap of complacency in predicting the maximum. *Ann. Probab.* 35, 340-365.

[19] J. Du Toit and G. Peskir (2008). Predicting the time of the ultimate maximum for Brownian motion with drift. *Proc. Math. Control Theory Finance* (Lisbon 2007), 95-112, Springer Berlin.

[20] P. Dybvig and H. Koo (1996). Investment with taxes. Working paper, Washington University, Saint Louis, MO.

[21] R. Elie (2008). Finite time strategy under drawdown constraint: a viscosity approach. *Appl. Mathematics and Optimization*, 58 (3), 411-431.

[22] R. Elie and N. Touzi (2008). Optimal lifetime consumption-investment strategy under drawdown constraint, *Finance and Stochastics*, 12 (3), 299-330.

[23] N. El Karoui (1981). Les aspects probabilistes du contrôle stochastique. *Ecole d'Eté de Probabilités de Saint-Flour 1979, Lectures notes in Math.* 876, 73-238, Springer.

[24] N. El Karoui, M. Jeanblanc and V. Lacoste (2005). Optimal portfolio management with American capital guarantee. *J. Econ. Dyn. Control* 29 (3), 409-440.

[25] N. El Karoui, D. Hu Nguyen and M. Jeanblanc-Piqué (1987). Compactification methods in the control of degenerate diffusions: existence of an optimal control. *Stochastics* 20, 169-219.

[26] N. El Karoui, S. Peng and M.-C. Quenez (1997). Backward stochastic differential equations in finance. *Math. Finance* 7 (1), 1-71.

[27] N. El Karoui and R. Rouge (2000). Pricing via utility maximization and entropy. *Math. Finance* 10, 259-276.

[28] G.-E. Espinosa (2010). Optimal investment-consumption with taxes: first order expansion for general utility functions. Preprint.

[29] G.-E. Espinosa (2010). Optimal investment under relative performance concerns with local risk penalization. Preprint.

[30] G.-E. Espinosa and N. Touzi (2010). Detecting the maximum of a mean-reverting scalar process. Preprint.

[31] G.-E. Espinosa and N. Touzi (2010). Optimal investment under relative performance concerns. Preprint.

[32] C. Frei and G. dos Reis (2010). Existence issues for multidimensional BSDEs and their relation to finance. In progress.

[33] M. Fuhrman and Y. Hu (2007). Backward stochastic differential equations in infinite dimensions with continuous driver and applications. *Appl. Math. Optim.* 56 (2), 265-302.

[34] J. Gali (1994). Keeping up with the Joneses: consumption externalities, portfolio choice and asset prices. *Journal of Money Credit and Banking* 26, 1-8.

[35] M. Gallmeyer, R. Kaniel and S. Tompaidis (2006). Tax management strategies with multiple risky assets. *Journal of Financial Economics* 80, 243-291.

[36] J.P. Gomez, R. Priestley and F. Zapatero (2007). The impact of keeping up with the Joneses behavior on asset prices and portfolio choice. *Fin. Res. Lett.* 4 (2), 95-103.

[37] S.E. Graversen, G. Peskir and A.N. Shiryaev (2001). Stopping Brownian motion without anticipation as close as possible to its ultimate maximum. *Theory Probab. Appl.* 45 (125-136).

[38] S. Hamadene (1996). Equations stochastiques rétrogrades: le cas localement lipschitzien. *Annales de l'IHP, Probabilités et Statistiques* 32 (5), 645-659.

[39] H. He and N.D. Pearson (1991). Consumption and portfolio policies with incomplete markets and short-sale constraints: the finite-dimensional case. *Math. Finance* 1, 1-10.

[40] H. He and N.D. Pearson (1991). Consumption and portfolio policies with incomplete markets and short-sale constraints: the infinite-dimensional case. *J. Econom. Theory* 54, 259-304.

[41] D. Hobson (2007). Optimal stopping of the maximum process: a converse to the results of Peskir. *Stochastics* 79 (1), 85-102.

[42] Y. Hu, P. Imkeller and M. Müller (2005). Utility maximization in incomplete markets. *Ann. Appl. Probab.* 15 (3), 1691-1712.

[43] Y. Hu and S. Peng (2006). On the comparison theorem for multidimensional BSDEs. *Comptes-Rendus Mathématiques* 343 (2), 135-140.

[44] E. Jouini, P.-F. Koehl and N. Touzi (1997). Optimal investment with taxes: an optimal control problem with endogeneous delay. *Nonlinear Analysis: Theory, Methods and Applications* 37, 31-56.

[45] E. Jouini, P.-F. Koehl and N. Touzi (1999). Optimal investment with taxes: an existence result. *Journal of Mathematical Economics* 33, 373-388.

[46] I. Karatzas, J.P. Lehoczky and S.E. Shreve (1987). Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. *SIAM J. Control and Optimization* 25, 1557-1586.

[47] I. Karatzas, J.P. Lehoczky, S.E. Shreve and G. Xu (1991). Martingale and duality methods for utility maximization in an incomplete market. *SIAM J. Control and Optimization* 29, 702-730.

[48] I. Karatzas and S.E. Shreve (1991). *Brownian Motion and Stochastic Calculus.* Springer.

[49] I. Karatzas and S.E. Shreve (1998). *Methods of Mathematical Finance*. Springer.

[50] S. Karlin and H.M. Taylor (1981). *A Second Course in Stochastic Processes*. Academic Press.

[51] N. Kazamaki (1994). *Continuous Exponential Martingales and BMO*. Lecture notes in Math., Springer.

[52] M. Kobylanski (2000). Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Probab.* 28 (2), 558-602.

[53] D. Kramkov and W. Schachermayer (1999). The condition on the asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab.* 9, 904-950.

[54] D. Kramkov and W. Schachermayer (2003). Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Ann. Appl. Probab.* 13 (4), 1504-1516.

[55] D. Kramkov and M. Sirbu (2006). On the two-times differentiability of the value functions in the problem of optimal investment in incomplete markets. *Ann. Appl. Probab.* 16 (3), 1352-1384.

[56] J.-M. Lasry and P.-L. Lions (2007). Mean field games. *Japanese Journal of Mathematics* 2, 229-260.

[57] T. Lim and M.-C. Quenez (2009). Utility maximization in incomplete markets with default. Preprint.

[58] R.C. Merton (1969). Lifetime portfolio selection under uncertainty: the continuous-time model. *Rev. Econ. Statist.* 51, 247-257.

[59] R.C. Merton (1971). Optimum consumption and portfolio rules in a continuous-time model. *J. Econ. Theory* 3, 373-413.

[60] M.-A. Morlais (2009). Quadratic BSDEs driven by a continuous martingale and application to utility maximization. *Finance and Stochastics* 13 (1), 121-150.

[61] R. Myneni (1992). The pricing of American option. *Ann. Appl. Probab.* Vol. 2, No. 1 (1-23).

[62] J. Obloj (2007). The maximality principle revisited: on certain optimal stopping problems. *Séminaire de Probabilités XL, Lect. Notes in Math.* 1899, 309-328.

[63] E. Pardoux and S. Peng (1990). Adapted solution of a backward stochastic differential equation. *Syst. Control Lett.* 14(1), 55-61.

[64] J.L. Pedersen (2003). Optimal prediction of the ultimate maximum of Brownian motion. *Stoch. Stoch. Rep.* 75 (205-219).

[65] G. Peskir (1998). Optimal stopping of the maximum process: the maximality principle. *Ann. Probab.* Vol. 26, No. 4 (1614-1640).

[66] G. Peskir and A.N. Shiryaev (2006). *Optimal Stopping and Free-Boundary Problems.* Lectures in Mathematics, ETH Zurich, Birkhauser.

[67] S.R. Pliska (1986). A stochastic calculus model of continuous trading: optimal portfolios. *Math. Operations Research* 11, 371-382.

[68] R.T. Rockafellar (1996). *Convex Analysis.* Princeton University Press.

[69] L.A. Shepp and A.N. Shiryaev (1993). The Russian option: reduced regret. *Ann. Appl. Probab.* 3 (631-640).

[70] L.A. Shepp and A.N. Shiryaev (1994). A new look at the Russian option. *Theory Probab. Appl.* 39 (103-119).

[71] A.N. Shiryaev (1967). Two problems of sequential analysis. *Cybernetics* 3, No.2 (79-86).

[72] A.N. Shiryaev (1978). *Optimal Stopping Rules.* Springer-Verlag, Berlin-New York.

[73] A.N. Shiryaev (2002). Quickest detection problems in the technical analysis of the financial data. *Proc. Math. Finance Bachelier Congress* (Paris, 2000), Springer (487-521).

[74] A.N. Shiryaev, Z. Xu and X. Zhou (2008). Thou shalt buy and hold. *Quantitative Finance* 8, 765-776.

[75] S.E. Shreve and H.M. Soner (1994). Optimal investment and consumption with transaction costs. *Ann. Appl. Probab.* 4, 609-692.

[76] R. Tevzadze (2008). Solvability of backward stochastic differential equations with quadratic growth. *Stoch. Proc. Appl.* 118 (3), 503-515.

[77] M.A. Urusov (2005). On a property of the moment at which Brownian motion attains its maximum and some optimal stopping problems. *Theory Probab. Appl.* 49 (169-176).

[78] T.B. Veblen (1899). *The Theory of the Leisure Class: an Economic Study of Institutions.* New York, Penguin.

[79] T. Zariphopoulou (1994). Consumption-investment models with constraints. *SIAM J. Control and Optimization* 32 (1), 59-85.

[80] T. Zariphopoulou (2001). A solution approach to valuation with unhedgeable risks. *Finance and Stochastics* 5, 61-82.

[81] H. Zhang and Q. Zhang (2008). Trading a mean-reverting asset: buy low and sell high. *Automatica* 44 (1511-1518).