Interior penalty approximation for optimal control problems. Optimality conditions in stochastic optimal control theory.

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1. Deterministic control
   - Introduction
   - Optimal control of a linear ODE
   - Optimal control of a semilinear PDE

2. Stochastic Control
   - Variational approach
   - The logarithmic penalty in the stochastic case
   - Optimality conditions in stochastic optimal control

Plan

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Introduction

Consider the following optimization problem

$$\mathrm{Min}_{x\in\mathbb{R}^n} \ \ \tfrac{1}{2}x^\top Rx + c^\top x; \quad Ax = b, \ x \geq 0, \qquad (\mathcal{QP})_0$$

with $R \succ 0$. We have that x_0 is a solution of $(\mathcal{QP})_0$ iff there exists (s_0, λ_0) such that $z_0 := (x_0, s_0, \lambda_0)$ solves

$$\begin{cases} x^i s^i = 0, & \text{for all } i \in \{1, ..., m\} \\ Ax = b, & c + Rx + A^\top \lambda = s, \\ x \geq 0, & s \geq 0. \end{cases}$$

Definition

We say that the solution z_0 of $(\mathcal{QP})_0$ is strictly complementary if $x_0^i + s_0^i > 0$ for all $i \in \{1, ..., m\}$.

For $\varepsilon > 0$ the penalized problem $(\mathcal{QP})_\varepsilon$ is defined as

$$\mathrm{Min}_{x\in\mathbb{R}^n} \quad \tfrac{1}{2}x^\top Rx + c^\top x - \varepsilon\sum_{i=1}^{p}\log x_i; \quad Ax = b. \qquad (\mathcal{QP})_\varepsilon.$$

In this case x_ε is a solution of $(\mathcal{QP})_\varepsilon$ iff there exists $(s_\varepsilon, \lambda_\varepsilon)$ such that $z_\varepsilon := (x_\varepsilon, s_\varepsilon, \lambda_\varepsilon)$ solves

$$\begin{cases} x^i s^i = \varepsilon, & \text{for all } i \in \{1, ..., m\} \\ Ax = b, & c + Rx + A^\top\lambda = s, \\ x \geq 0, & s \geq 0. \end{cases}$$

Proposition (Wright-Orban 01)

We have that,

$$|z_\varepsilon - z_0| = \begin{cases} O(\varepsilon) & \textit{if } z_0 \textit{ is strictly complementary,} \\ O(\sqrt{\varepsilon}) & \textit{if not.} \end{cases}$$

Optimal control of a linear ODE

The initial problem:

Consider the following optimal control problem $(\mathcal{CP})_0$

$$\inf_{(u,y)} \frac{1}{2} \int_0^T \left\{ |u|^2 + |y(t) - \bar{y}(t)|^2 \right\} \mathrm{d}t + \frac{1}{2} |y(T) - \bar{y}(T)|^2 ,$$

subject to

$$\begin{aligned} \dot{y}(t) &= y(t) + u(t) \quad \text{for } t \in [0, T]; \\ y(0) &= y_0 \quad u(t) \geq 0, \quad \text{for } t \in [0, T]. \end{aligned}$$

Strong convexity + continuity imply that $(\mathcal{CP})_0$ admits a unique solution $(y_0, u_0) \in W^{1,2} \times L^2$.

Most popular methods: Semismooth methods e.g. M. Ulbrich 00, Hintermüller-Stadler 03, , Hintermüller-Ito-Kunisch 02. Interior point methods e.g. M. Ulbrich-S. Ulbrich 00-09, Weiser 2005.

The penalized problem:

For $\varepsilon > 0$, problem $(\mathcal{CP})_\varepsilon$ is defined as

$$\inf_{(u,y)} \frac{1}{2} \int_0^T \left\{ |u(t)|^2 + |y - \bar{y}(t)|^2 - \varepsilon \log u(t) \right\} \mathrm{d}t + \frac{1}{2} |y - \bar{y}(T)|^2 ,$$

subject to

$$\begin{array}{ll} \dot{y}(t) = y(t) + u(t) & \text{for } t \in [0, T]; \\ y(0) = y_0 \quad u(t) \geq 0, & \text{for } t \in [0, T]. \end{array}$$

Strong convexity + lower semi-continuity imply that $(\mathcal{CP})_\varepsilon$ admits a unique solution $(y_\varepsilon, u_\varepsilon)$. Moreover,

Proposition (Bonnans-Guilbaut 03, Alvarez- Bolte- Bonnans and Silva 08)

There exists a constant $C > 0$, such that for every $\varepsilon > 0$

$$u_\varepsilon(t) \geq c\varepsilon \quad \textit{for a.a. } t \in [0, T].$$

Optimality conditions:

For $\varepsilon \in [0, \infty)$ the integral part of cost of $(\mathcal{CP})_\varepsilon$ is

$$\ell_\varepsilon(t, y, u) := \tfrac{1}{2}|u|^2 + \tfrac{1}{2}\,|y - \bar{y}(t)|^2 - \varepsilon \log u.$$

The Hamiltonian H_ε

$$H_\varepsilon(t, y, p, u) := \ell_\varepsilon(t, y, u) + p \cdot (y + u).$$

The Pontryagin minimum principle yields that there exists p_ε such that

$$\begin{array}{rcl} \dot{y}_\varepsilon(t) & = & y_\varepsilon(t) + u_\varepsilon(t), \quad y_\varepsilon(0) = y_0 \\ -\dot{p}_\varepsilon(t) & = & p_\varepsilon(t) + y_\varepsilon(t) - \bar{y}(t)\ , \quad p_\varepsilon(T) = y_\varepsilon(T) - \bar{y}(T), \\ u_\varepsilon(t) & = & \operatorname{argmin}\{H_\varepsilon(t, y_\varepsilon(t), p_\varepsilon(t), v) : \ v \geq 0\} \end{array}$$

The above conditions yields

$$u_\varepsilon(t) := \varphi_\varepsilon(-p_\varepsilon(t)),$$

where

$$\varphi_\varepsilon(x) := \begin{cases} \frac{1}{2}\left(x + \sqrt{x^2 + 4\varepsilon}\right) & \text{if } \varepsilon > 0, \\ \max\{x, 0\} & \text{if } \varepsilon = 0. \end{cases}$$

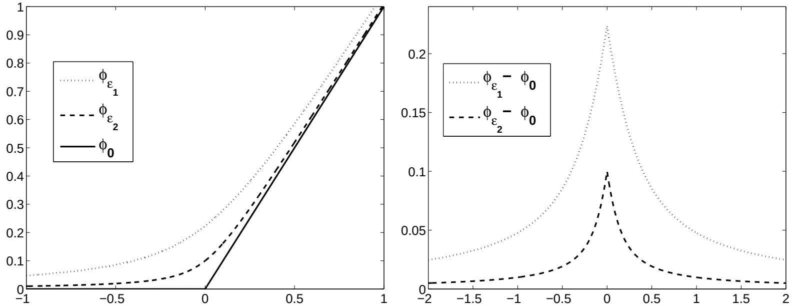


Figure: Left: ϕ_{ε_1}, ϕ_{ε_2} and ϕ_0. Right: $\phi_{\varepsilon_1} - \phi_0$, $\phi_{\varepsilon_2} - \phi_0$, for $\varepsilon_1 = 0.005$, $\varepsilon_2 = 0.001$.

Thus for $\varepsilon \in [0, \infty)$

$$\begin{aligned} \dot{y}_\varepsilon(t) &= y_\varepsilon(t) + \varphi_\varepsilon(-p_\varepsilon(t)), \\ -\dot{p}_\varepsilon(t) &= p_\varepsilon(t) + [y_\varepsilon(t) - \bar{y}(t)], \\ y_\varepsilon(0) &= y_0, \quad p_\varepsilon(T) = y_\varepsilon(T) - \bar{y}(T). \end{aligned}$$

Define $F : W^{1,1} \times W^{1,1} \times \mathbb{R}_+ \to L^1 \times \mathbb{R} \times L^1 \times \mathbb{R}$ by

$$F(y, p, \varepsilon)(\cdot) := \begin{pmatrix} \dot{y}(\cdot) - (\cdot)y(\cdot) - \varphi_\varepsilon(-p(\cdot)) \\ y(0) - y_0 \\ \dot{p}(\cdot) + p(\cdot) + [y(\cdot) - \bar{y}(\cdot)] \\ p(T) - [y(T) - \bar{y}(T)] \end{pmatrix}.$$

It is easy to see that in general F is not differentiable at $(y_0, p_0, 0)$. Therefore, we cannot apply the standard implicit function theorem in order to obtain an expansion of $(y_\varepsilon, p_\varepsilon)$ around (y_0, p_0).

Restoration theorem: [Graves 50]

Data:

- X, Y be Banach spaces and $F : X \times \mathbb{R}_+ \to Y$ continuous.
- $\hat{x} \in X$ such that

$$F(\hat{x}, 0) = 0.$$

Assumptions:

- $D_x F(\hat{x}, 0)$ exists and also its inverse.
- There exists $c : \mathbb{R}_+ \to \mathbb{R}_+$ with $c(\beta) \downarrow 0$ when $\beta \downarrow 0$ such that

$$\|F(x', \varepsilon) - F(x, \varepsilon) - D_x F(\hat{x}, 0)(x' - x)\| \leq c(\beta) \|x' - x\|.$$

for $0 \leq \varepsilon \leq \beta$ and $x, x' \in \overline{B}(\hat{x}, \beta)$.

Theorem

Under the assumptions above, $F(\cdot, \varepsilon)$ has, in a neighborhood of $\hat{x}$, a zero denoted by x_ε and

$$x_\varepsilon = \hat{x} - D_x F(\bar{x}, 0)^{-1} F(\hat{x}, \varepsilon) + r(\varepsilon) \quad \textit{with } ||r(\varepsilon)|| = o\left(\|F(\hat{x}, \varepsilon)\|\right).$$

Strict complementarity assumption

Except for a null Lebesgue set the point $u_0(t)$ satisfies the strict complementarity conditions for the minimization problem

$$\min \left\{ H_0(t, y_0(t), p_0(t), w) : w \in \mathbb{R}_+ \right\}.$$

Alternative formulation

Except for a null Lebesgue set the curve $p_0(t)$ does not intersect the x-axis, i.e. the function $t \in [0, T] \to \frac{\mathrm{d}}{\mathrm{d}t}\varphi_0(-p_0(t))$ is a.s. well defined.

Theorem (Asymptotic expansion)

Under the strict complementarity assumption there exists $r(\varepsilon) = o(||F(y_0, p_0, \varepsilon)||_1)$ *such that*

$$\begin{pmatrix} y_\varepsilon \\ p_\varepsilon \end{pmatrix} = \begin{pmatrix} y_0 \\ p_0 \end{pmatrix} + s_{aux}(\varepsilon) + r(\varepsilon),$$

where $s_{aux}(\varepsilon) = O(||F(y_0, p_0, \varepsilon)||_1)$ *is the state and adjoint state of*

$$\begin{cases} \text{Min } \frac{1}{2}\int_0^T \left(|v(t)|^2 + |\sigma(t)|^2\right) \mathrm{d}t + \frac{1}{2}|\sigma(T)|^2, \\ s.t. \\ \dot{\sigma}(t) = \sigma(t) + v(t) + [\varphi_\varepsilon(-p_0(t)) - \varphi_0(-p_0(t))], \\ \sigma(0) = 0, \quad v(t) = 0 \quad \text{if} \;\; p_0(t) \geq 0. \end{cases}$$

Theorem (Error estimates for interior penalty)

Under the strict complementarity assumption we have that:

(i) *The error estimates for* $u_\varepsilon, y_\varepsilon$ *and* p_ε *are given by*

$$||u_\varepsilon - u_0||_\infty + ||p_\varepsilon - p_0||_{1,\infty} + ||y_\varepsilon - y_0||_{1,\infty} = O(\sqrt{\varepsilon}).$$

(ii) *Assume that* $\{t \in [0, T] \; ; \; p_0(t) = 0\}$ *is finite and that:*

$$p_0(t_0) = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} p_0(t_0) \neq 0 \quad \textit{(Transversality condition)}. \tag{1}$$

Then

$$||u_\varepsilon - u_0||_1 + ||p_\varepsilon - p_0||_{1,1} + ||y_\varepsilon - y_0||_{1,1} = O(\varepsilon |\log \varepsilon|). \tag{2}$$

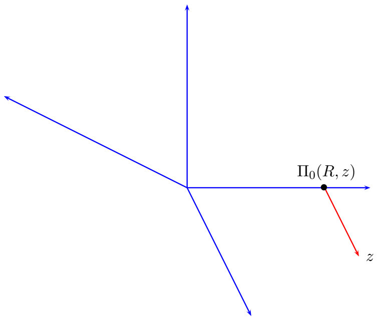
Similar results to (i) have been obtained by e.g. Weiser 05.

A word about the vector control case: Consider two controls and an integral cost including the term

$$u(t)^\top R(t)u(t).$$

The solution $u_0(t)$ is the projection $\Pi(R(t),\cdot)$ on $\mathbb{R}^2_+$ of a "certain" curve "involving" the adjoint state.

The singular zones are illustrated below:



Optimal control of a semilinear PDE

Let $u \in L^s(\Omega)$ ($s \in [2, \infty]$) and $y_u \in W^{2,s}(\Omega)$ the solution

$$\begin{cases} -\Delta y(x) + \phi(y(x)) = f(x) + u(x) & \text{for} \quad x \in \Omega, \\ y(x) = 0 & \text{for} \quad x \in \partial\Omega, \end{cases}$$

where Ω is bounded, open set in $\mathbb{R}^n$ with C^2 boundary, $f \in L^s(\Omega)$ and ϕ is $\mathcal{C}^2$ Lipschitz nondecreasing. For $s > n/2$ ($s = 2$ if $n \leq 3$), define $J_0 : L^s(\Omega) \to \mathbb{R}$ by

$$J_0(u) := \frac{1}{2} \int_\Omega (y_u(x) - \bar{y}(x))^2 dx + \frac{1}{2} \int_\Omega u(x)^2 \mathrm{d}x.$$

Consider the optimization problem

$$\mathrm{Min}\ J_0(u) \quad \text{s.t.} \ \ u \in \mathcal{U}_+^s := \{v \in L^s(\Omega) \ / \ v(x) \geq 0\}. \qquad (\mathcal{CP}_0^s)$$

Problem $(\mathcal{CP}_0^s)$ is non-convex one. Nevertheless, it has at least one solution. For $u \in L^s(\Omega)$ the adjoint state $p_u \in W^{2,s}(\Omega)$, is the solution of

$$\left\{ \begin{array}{rcll} -\Delta p(x) + \phi'(y_u(x))p(x) & = & y_u(x) - \bar{y}(x) & \text{for} \quad x \in \Omega, \\ p(x) & = & 0 & \text{for} \quad x \in \partial\Omega. \end{array} \right.$$

Let $u_0 \in \mathcal{U}_+^s$ be a solution of $(\mathcal{CP}_0^s)$ and y_0 and p_0 the state and adjoint state. Classical techniques yields

$$u_0(x) = \varphi_0(-p_0(x)) \quad \text{for a.a. } x \in \Omega.$$

The penalized local problem

Suppose that u_0 is locally unique in the $L^s(\Omega)$ ball $\bar{B}_s(u_0, b)$. For $\varepsilon > 0$, define

$$J_\varepsilon(u) := J_0(u) - \varepsilon \int_\Omega \log(u(x)),$$

and consider the problem

$$\mathrm{Min}\ \ J_\varepsilon(u) \quad \text{s. t. } u \in \mathcal{U}_+^s \cap \bar{B}_s(u_0, b) \qquad (\mathcal{CP}_\varepsilon^{b,s}),$$

Problem $(\mathcal{CP}_\varepsilon^{b,s})$ has at least one solution u_ε. Note that

$$u \in L^s(\Omega) \to -\int_\Omega \log(u(x))\mathrm{d}x \in \mathbb{R} \cup \{+\infty\}$$

is not continuous. However,

$$c\varepsilon \leq u_\varepsilon(x) \leq K \ \text{ for a.a. } x \in \Omega.$$

Therefore, u_ε solves

$$\mathrm{Min}\ \ J_\varepsilon(u) \quad \text{subject to } u \in \mathcal{U}_+^s \cap \bar{B}_s(u_0, b_0) \cap L^\infty(\Omega)$$

Let y_ε and p_ε be the state and adjoint state associated to u_ε.
Optimality conditions yield

$$u_\varepsilon(x) = \varphi_\varepsilon(-p_\varepsilon(x)) \text{ for a.a. } x \in \Omega.$$

Define $F : W^{1,s} \times W^{1,s} \times \mathbb{R}_+ \to L^s(\Omega) \times L^s(\Omega)$ by

$$F(y, p, \varepsilon)(\cdot) := \begin{pmatrix} \Delta y(\cdot) + \varphi_\varepsilon(-N^{-1} p(\cdot)) + f(\cdot) - \phi(y(\cdot)) \\ \Delta p(\cdot) + y(\cdot) - \bar{y}(\cdot) - \phi'(y(\cdot)) p(\cdot) \end{pmatrix}.$$

We assume

(H1) For the adjoint state p_0, associated to any local solution u_0 of $(\mathcal{CP}_0^s)$, it holds that

$$\text{meas}\left(\{x \in \Omega \ / \ p_0(x) = 0\}\right) = 0.$$

(H2) At any local solution u_0 of $(\mathcal{CP}_0^s)$, the following second-order sufficient condition holds

$$D^2 J_0(u_0)(h,h) > 0 \quad \text{for all } h \in C(u_0) \setminus \{0\}$$

Theorem

Suppose that **(H1)**, **(H2)** *hold. Then, for* $\bar{b}, \varepsilon$ *small enough problem* $(\mathcal{CP}_{\varepsilon}^{\bar{b},s})$ *has a unique solution* u_{ε} *and there exists* $r(\varepsilon) = o(||F(y_0, p_0, \varepsilon)||_s)$ *such*

$$\begin{pmatrix} y_{\varepsilon} \\ p_{\varepsilon} \end{pmatrix} = \begin{pmatrix} y_0 \\ p_0 \end{pmatrix} + s_{aux}(\varepsilon) + r(\varepsilon),$$

where $s_{aux}(\varepsilon)$ *is the state and adjoint state of*

$$\left\{ \begin{array}{l} \mathrm{Min}\ \frac{1}{2} \int_{\Omega} \left[N v^2 + \left(1 - p_0 \phi''(y_0)\right) z^2 \right] \mathrm{d}x, \\ s.t. \\ -\Delta z(x) + \phi'(y_0(x)) z(x) = v + [\varphi_{\varepsilon}(q_0) - \varphi_0(q_0)] \quad \text{for } x \in \Omega, \\ z(x) = 0 \quad \text{for } x \in \partial\Omega, \qquad v(x) = 0 \quad \text{if } p_0(x) \geq 0. \end{array} \right.$$

Theorem

We have

$$||u_\varepsilon - u_0||_\infty + ||p_\varepsilon - p_0||_{2,s} + ||y_\varepsilon - y_0||_{2,s} \quad = \quad O(\sqrt{\varepsilon}).$$

If in addition $n \leq 3$ *(hence* $s = 2$*) and*

- $\{x \in \Omega \ / \ p_0(x) = 0\} = \bigcup_{i=1}^{m} C_i$. *(*$C_i$ *is a closed* $\mathcal{C}^2$ *curve).*
- *There exist positive real numbers* $\alpha > 0$, $0 < \bar{\delta} < 1$ *such that*

$$|p_0(x)| \geq \alpha \, dist(x, C_i) \quad \text{for all } x \in C_i^{\bar{\delta}}.$$

Then

$$||u_\varepsilon - u_0||_2 + ||p_\varepsilon - p_0||_{2,2} + ||y_\varepsilon - y_0||_{2,2} = O(\varepsilon^{\frac{3}{4}}).$$

For the first estimate, similar results found in Weiser, Gänzler and Schiela 08.

Plan

Consider:

- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Consider a d-dimensional Brownian motion $W(t)$ and $\mathbb{F}$ its natural filtration
- $U \subseteq \mathbb{R}^m$ and

$$\mathcal{U} := \{u(t) \text{ is } \mathbb{F} - \text{adapted } u(t,\omega) \in U, a.s., a.e.\}.$$

- Consider the following controlled SDE:

$$\begin{aligned} \mathrm{d}y(t) &= f(t, y(t), u(t))\mathrm{d}t + \sigma(t, y(t), u(t))\mathrm{d}W(t) \quad \text{for } t \in [0, T] \\ y(0) &= y_0 \end{aligned}$$

For the control, we will work we the following spaces

$$\begin{array}{ll} L^2_{\mathcal{F}} & := \{v;\ v \text{ is } \mathbb{F}\text{-adapted}, \ ||v||_2 < +\infty\}, \\ L^\infty_{\mathcal{F}} & := \{v;\ v \text{ is } \mathbb{F}\text{-adapted}, \ ||v||_\infty < +\infty\}, \end{array}$$

where

$$\begin{array}{lll} ||v||_2^2 & := & \mathbb{E}\left(\int_0^T |v(t)|^2 \mathrm{d}t\right) \\ ||v||_\infty & := & \text{ess sup}_{(t,\omega)}|v(t,\omega)| \end{array}.$$

For the state space we work with

$$L^{2,\infty}([0,T];\mathbb{R}^n) \quad := \{y;\ y \text{ is } \mathbb{F}\text{-adapted}, \ ||y||_{2,\infty} < +\infty\},$$

where

$$||y||_{2,\infty}^2 \quad := \quad \mathbb{E}\left(\sup_{t\in[0,T]} |y(t)|^2\right).$$

Lemma

Denoting by y_u the solution associated to u, there exists $C > 0$ such that

$$||y_u||^2_{2,\infty} \leq C\mathbb{E}\left(|y_0|^2 + ||f(0, u(\cdot))||^2_2 + ||\sigma(0, u(\cdot))||^2_2\right).$$

Consider the following stochastic optimal control problem:

$$(\mathcal{SP}) : \begin{cases} \text{Min } J(u) := \mathbb{E}\left(\int_0^T \ell(t, u(t), y_u(t))\mathrm{d}t + \phi(y_u(T))\right) \\ \text{subject to } u \in \mathcal{U}. \end{cases}$$

The existence of a solution for this problem is a very difficult task. Only partial results, specially for the linear quadratic case, have been obtained under our strong formulation.

Variational approach

Let $\bar{u}$ be a solution of $(\mathcal{SP})$ and let $\bar{y}$ its associated state. Define the adjoint state $(\bar{p}, \bar{q})$ as the unique adapted solution of the following *BSDE* (Bismut 73) .

$$
\begin{aligned}
\mathrm{d}p(t) &= -\left[\ell_y(t, y_u, u) + f_y(t, y_u, u)^\top p + \sum_{i=1}^{d} \sigma_y^i(t, y_u, u)^\top q^i\right] \mathrm{d}t \\
&+ q \mathrm{d}W(t) \\
p(T) &= \phi_y(y_u(T))^\top .
\end{aligned}
\tag{3}
$$

Lemma

There exists a constant $C > 0$ such that

$$
||p||_{2,\infty}^2 + ||q||_2^2 \leq C \left(||\phi_y(y_u(T))||_2^2 + ||\ell_y(t, y_u, u)||_2^2 \right) .
$$

Define the Hamiltonian H by

$$H(t, y, u, p, q) := \ell(t, y, u) + p \cdot f(t, y, u) + \sum_{i=1}^{d} q^i \cdot \sigma^i(t, x, u).$$

It holds

Theorem (Minimum principle (Bismut 73, Bensoussan 83))

If $\sigma_u \equiv 0$ then

$$\bar{u}(t) \in argmin \{H(t, \bar{y}, v, \bar{p}, \bar{q}); v \in U\} \quad a.e., \ a.s.$$

When σ depends on u the above result was generalized by Peng in 1990, by introducing a generalized Hamiltonian and a second order adjoint process.

The logarithmic penalty in the stochastic case

We supose here that

$$\begin{array}{rcl} \ell(t,y,u) & = & \frac{1}{2}u^2 + \frac{1}{2}(y-\bar{y}(t))^2 \\ \phi(y) & = & \frac{1}{2}y^2 \\ f(t,y,u) & = & y+u \\ \sigma(t,y,u) & = & y+u \end{array}$$

and

$$\mathcal{U} := \left\{ u \in L^2_{\mathcal{F}} \ / \ u(t,\omega) \geq 0 \ \text{ for a.a } (t,\omega) \in [0,T] \times \Omega \right\}.$$

Since the problem is strongly convex, it has a unique solution u_0. We denote by y_0, (p_0, q_0) for the state and the adjoint state. The SPMP implies yields that

$$u_0(t,\omega) = \phi_0(-p_0(t,\omega) - q_0(t,\omega)).$$

We consider the problem, with the same dynamics than the initial one but the cost ℓ is modified by

$$\ell_\varepsilon(t, y, u) = \ell(t, y, u) - \varepsilon \log u.$$

Strong convexity implies that the new problem admits a unique solution u_ε. Denote by y_ε, $(p_\varepsilon, q_\varepsilon)$ for the state and adjoint state. The SPMP yields that

$$u_\varepsilon(t, \omega) = \phi_\varepsilon(-p_\varepsilon(t, \omega) - q_\varepsilon(t, \omega)).$$

Moreover,

Proposition

There exist $C > 0$ such that

$$u_\varepsilon(t, \omega) \geq \frac{C\varepsilon}{(1 + |p_\varepsilon(t, \omega)| + |q_\varepsilon(t, \omega)|)}. \quad \text{for a.a. } (t, \omega) \in [0, T] \times \Omega.$$

Proposition

It holds that

$$J(u_\varepsilon) - J(u_0) \leq T\varepsilon.$$

Sketch of proof Consider the Lagrangian $\mathcal{L} : L^2_{\mathcal{F}} \times L^2_{\mathcal{F}} \to \mathbb{R}$ defined as

$$\mathcal{L}(u, \lambda) := J_0(u) - \langle \lambda, u \rangle_2,$$

and the dual function $d : \mathcal{U} \to \mathbb{R}$ by

$$d(\lambda) := \inf_{u \in L^2_{\mathcal{F}}} \mathcal{L}(u, \lambda)$$

The SPMP, in its sufficient form, implies that

$$d\left(\varepsilon \frac{1}{u^\varepsilon}\right) = J_0(u_\varepsilon) - \varepsilon T.$$

By weak duality

$$J_0(u_\varepsilon) - \varepsilon T \leq \min_{u \in \mathcal{U}} J_0(u) = J_0(u_0).$$

Strong convexity and regularity result for the BSDE yields

Theorem

For every $\varepsilon > 0$, the following estimates hold

$$\begin{aligned} ||u_\varepsilon - u_0||_2^2 + ||y_\varepsilon - y_0||_{2,\infty}^2 &= O(\varepsilon) \\ ||p_\varepsilon - p_0||_{2,\infty}^2 + ||q_\varepsilon - q_0||_2^2 &= O(\varepsilon) \end{aligned}$$

The above result present a partial extension of the error estimates obtained in the deterministic case.

Optimality conditions in stochastic optimal control

We return to the general problem

$$\begin{aligned} &\mathrm{Min}\ J(u) := \mathbb{E}\left[\int_0^T \ell(t, y_u(t), u(t))\mathrm{d}t + \phi(y_u(T))\right] \\ &\text{subject to} \qquad u \in \mathcal{U}. \end{aligned} \qquad (\mathcal{SP})$$

Now we suppose that $\mathcal{U}$ is a general convex subset of $L^2_{\mathcal{F}}$. It is not necessarily defined by *local constraints*.

Let $\bar{u}$ be a local solution of $(\mathcal{SP})$ and denote by $\bar{y}$ and $(\bar{p}, \bar{q})$ its state and adjoint state.

Set $H_u(t) := H_u(t, \bar{y}(t), \bar{u}(t), \bar{p}(t), \bar{q}(t))$ and define $\Upsilon_1 : L^\infty_{\mathcal{F}} \to \mathbb{R}$ as

$$\Upsilon_1(v) := \mathbb{E}\left(\int_0^T H_u(t)\ v(t)\mathrm{d}t \right).$$

We have

Proposition

Let $v \in L^\infty_{\mathcal{F}}$. Then, the following first order expansion of J around $\bar{u}$ holds

$$J(\bar{u} + v) = J(\bar{u}) + \Upsilon_1(v) + r_1(v)$$

where $\Upsilon_1(v) = O(||v||_2)$ *and* $r_1(v) = O(||v||^2_\infty)$.

The radial cone to $\mathcal{U}$ at $\bar{u}$ is defined as

$$\mathcal{R}_{\mathcal{U}}(\bar{u}) \quad := \quad \{v \in L^2_{\mathcal{F}} \; ; \; \exists \; \sigma > 0 \quad \text{such that} \quad [\bar{u}, \bar{u} + \sigma v] \subseteq \mathcal{U}\}$$

The tangent cone to $\mathcal{U}$ at $\bar{u}$ is defined as

$$T_{\mathcal{U}}(\bar{u}) = \text{closure}(\mathcal{R}_{\mathcal{U}}(\bar{u})).$$

We assume that:

$$\textbf{(H1)} \qquad T_{\mathcal{U}}(\bar{u}) = \text{closure}\left(\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^{\infty}_{\mathcal{F}}\right) .$$

Important case:

$$\mathcal{U} := \left\{u \in L^2_{\mathcal{F}} \; ; \; u(t,\omega) \in U(t,\omega), \quad \text{a.a. } (t,\omega) \in [0,T] \times \Omega\right\}.$$

Using the first order of J expansion + density arguments we obtain:

Proposition

The following first order optimality condition holds

$$\Upsilon_1(v) \geq 0 \quad \textit{for all } v \in T_{\mathcal{U}}(\bar{u}).$$

Let $y_1 = y_1(v)$ be the unique solution of

$$\begin{aligned} \mathrm{d}y_1(t) &= Df(t)(y_1(t), v(t))\mathrm{d}t + D\sigma(t)(y_1(t), v(t))\mathrm{d}W(t), \\ y_1(0) &= 0. \end{aligned}$$

and set

$$\Upsilon_2(v) := \mathbb{E}\left(\int_0^T H_{(y,u)^2}(t)(v(t), y_1(t))^2 \mathrm{d}t + \phi_{yy}(\bar{y}(T))(y_1(T))^2 \right).$$

The following expansion holds:

Proposition

Assume that $\sigma_{uu} \equiv 0$. Then, the following expansion holds:

$$J(\bar{u} + v) = J(\bar{u}) + \Upsilon_1(v) + \tfrac{1}{2}\Upsilon_2(v) + r_2(v) \quad \textit{for all } v \in L^\infty_{\mathcal{F}}.$$

where $\Upsilon_2(v) = O(||v||_2^2)$ and $r_2(v) = O(||v||_\infty ||v||_2^2)$.

The normal and the critical cone to $\mathcal{U}$ at $\bar{u}$ are defined by

$$\begin{array}{rcl} N_{\mathcal{U}}(\bar{u}) & := & \left\{ v^* \in L^2_{\mathcal{F}} \ / \ \langle v^*, v \rangle_2 \leq 0, \ \text{for all } v \in T_{\mathcal{U}}(\bar{u}) \right\}, \\ C(\bar{u}) & := & \left\{ v^* \in T_{\mathcal{U}}(\bar{u}) \ / \ \Upsilon_1(v) \leq 0 \right\}. \end{array}$$

Definition

The set $\mathcal{U}$ is said to be *polyhedric* at $\bar{u} \in \mathcal{U}$ if for all $v^* \in N_{\mathcal{U}}(\bar{u})$, the set $\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap (v^*)^{\perp}$ is dense in $\mathcal{T}_{\mathcal{U}}(\bar{u}) \cap (v^*)^{\perp}$. If $\mathcal{U}$ is polyhedric at each $u \in \mathcal{U}$ we say that $\mathcal{U}$ is *polyhedric*.

(H2) For every $\bar{u} \in \mathcal{U}$ and $v^* \in N_{\mathcal{U}}(\bar{u})$, we have that

$$\text{closure}\left(\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^{\infty}_{\mathcal{F}} \cap (v^*)^{\perp} \right) = \text{closure}\left(\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap (v^*)^{\perp} \right).$$

Theorem

Let $\bar{u}$ be a local solution of $(\mathcal{SP})$ and assume that

(i) *Assumptions* **(H1)-(H2)** *hold.*

(ii) *We have that $\sigma_{uu} = 0$*

(iii) *The constraint set $\mathcal{U}$ is polyhedric.*

Then, the following second order necessary condition hold at $\bar{u}$:

$$\Upsilon_2(v) \geq 0 \quad \text{for all } v \in C(\bar{u}).$$

Important case: Local constrains of the form

$$U(t,\omega) = \{x \in \mathbb{R}^m \ / \ \langle a_i(t,\omega), x\rangle \leq b_i(t,\omega), \ \text{for } i \in \Sigma(t,\omega) \ \}.$$

with $\Sigma(t,\omega)$ finite.

About second order sufficient conditions

We say that J satisfies the quadratic growth condition at u if there exists $\alpha > 0$ and a neighborhood $\mathcal{V}$ of u in $L^\infty_{\mathcal{F}}([0,T];\mathbb{R}^m)$ such that

$$J(u') \geq J(u) + \alpha ||u' - u||_2^2 \quad \text{for all } u' \in \mathcal{V}.$$

Evidently we have

Proposition

Suppose that $\sigma_{uu} = 0$. Then J sastisfies the growth condition at $\bar{u}$ iff there exists $\alpha > 0$ such that $\Upsilon_1(v) = 0$ and $\Upsilon_2(v) \geq \alpha ||v||_2^2$ for all $v \in L^2_{\mathcal{F}}$.