Volatility Dynamics

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Abstract

In this thesis we establish some strong asymptotic links between two major classes of stochastic volatility models, when those refer to the same derivative market. On one hand we consider a generic version of stochastic *instantaneous* volatility (SInsV) model, with an SDE system defined formally as an adapted Wiener chaos, and whose state variables are left unspecified. On the other hand we exploit a sliding stochastic *implied* volatility (SImpV) class, which is another *market model* describing explicitly the joint dynamics of an underlying and of the associated European option surface.

Each of these connections is achieved by *layer*, between a group of coefficients defining the SInsV model, and a differential set of the functionals specifying the shape and dynamics of the implied volatility surface. The asymptotic approach leads to these cross-differentials being taken at the zero-expiry, At-The-Money point, and relate the *depth* of the SinsV chaos to the *order* of the SImpV differentiation. We progress from a simple single-underlying and bi-dimensional setup, first to a multi-dimensional configuration, and then to a term-structure framework. In so doing, we expose the structural constraints imposed on each model, as well as the natural asymmetry between the *direct* problem (transferring information from the SInsV to the SImpV class) and the *inverse* one.

We show that this ACE methodology (which stands for *Asymptotic Chaos Expansion*) is a powerful tool for model design and analysis. Focusing on local volatility models and their extensions, we compare its output with the literature and thereby exhibit a systematic bias in some popular heuristics. To illustrate the multi-dimensional context we focus on stochastic-weights baskets, for which ACE easily provides some intuitive results, underlining their embedded inductive features. In the interest rates environment, we derive the first layer of *smile descriptors* for caplets, swaptions and bond options, within both a SV-HJM and a SV-LMM framework.

Also, we prove that ACE can be automated for generic models and at any order, without resorting to symbolic calculus. The interest of such algorithmics is demonstrated by computing manually the second and third layers of smile differentials, in a generic bi-dimensional SInsV model. In that spirit we expose and advocate the considerable applicative potential of ACE for calibration, pricing, hedging or relative value purposes, which we illustrate with some numerical tests on the CEV-SABR model.