Some contributions to stochastic control and backward stochastic differential equations in finance.

Emilie Fabre

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présentée par

Émilie Fabre

Quelques contributions au contrôle et aux équations rétrogrades en finance

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Bruno Bouchard	Examinateur
Stéphane Crépey	Rapporteur
Emmanuel Gobet	Examinateur
Anis Matoussi	Rapporteur
Erik Taflin	Examinateur
Nizar Touzi	Directeur

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Résumé

Cette thèse est divisée en trois parties indépendantes. Le premier chapitre est consacré à un cas particulier de problèmes mixtes de gestion de portefeuille et de liquidation optimale d'un actif indivisible. Le deuxième chapitre est dédié à l'étude des équations rétrogrades du second ordre sous contraintes convexes pour le processus de gain tandis que le troisième chapitre est voué à l'étude d'un modèle à volatilité stochastique.

Le problème mixte de gestion de portefeuille consiste à étudier le comportement d'un particulier possédant un portefeuille d'actifs financiers ainsi qu'un actif indivisible dont il veut réaliser la vente. On suppose que le risque associé à l'actif indivisible ne peut être couvert par une gestion dynamique du portefeuille. L' objectif de cet agent est de maximiser l'espérance de l'utilité de son gain lors de la vente. Nous donnons une formulation explicite de la fonction valeur ainsi que des stratégies optimales associées pour une fonction d'utilité générale.

Par ailleurs, nous nous sommes penchés sur le problème des équations rétrogrades du second ordre avec contraintes convexes sur le processus de gain. Comme pour le cas du premier ordre, on montre via une approche trajectorielle que les EDSRs du second ordre avec contraintes convexes admettent une solution minimale. Nous donnons, de plus, une formule de représentation stochastique pour cette solution.

Enfin nous avons étudié un modèle de prix d'actif où la volatilité instantanée dépend de la courbe de volatilité forward. Notre proposons un développement asymptotique du prix d'une option pour de petites variations de la volatilité. Notre approche a consisté à utiliser l'analogie avec les modèles de taux et utiliser la stabilité des solutions de viscosité dans les espaces de Hilbert.

Mots clés : maximisation d'utilité, enveloppe concave, solution de viscosité dans des espaces de Hilbert, EDSR du second ordre, volatilité stochastique.

Abstract

We consider three independent topics. First, we study a mixed investment sell problem. It consists in studying the behavior of a risk-averse agent who has to manage a portfolio and an indivisible asset to sell. The indivisible asset is assumed to be independent of the portfolio of assets. The aim of the agent is to maximize the expected utility of its total wealth at the sell time. We give an explicit calculation of the value function and its associated optimal rules.

Secondly, we consider second order backward SDEs with convex constraints on the gain process. We show via a pathwise approach that this problem admits a minimal solution. We provide a stochastic representation formula for this solution.

Finally, we study a pricing model where the instantaneous volatility depends on the forward variance curve. We propose an asymptotic expansion of the option price for small variations of the volatility. Our approach consists in using the analogy between the bond price model and the stability of viscosity solutions in Hilbert spaces.

Key words: mixed investment/sell problem, concave hull, viscosity solutions in Hilbert spaces, constrained 2BSDEs, stochastic volatility model.

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Chapitre 1

Introduction Générale

1.1 Le problème mixte d'investissement et d'arrêt optimal

Cette thèse se divise en trois parties indépendantes. Le premier chapitre est consacré à un cas particulier de problèmes mixtes de gestion de portefeuille et de liquidation optimale d'un actif indivisible. Le deuxième chapitre analyse le cas d'équations rétrogrades du second ordre sous contraintes convexes pour le processus de gain tandis que le troisième chapitre est dédié à l'étude d'un modèle à volatilité stochastique où la volatilité de la volatilité dépend d'une courbe de volatilité forward.

1.1.1 Le problème de gestion de portefeuille

Le problème de gestion de portefeuille est un problème classique en mathématiques financières. Un investisseur cherche à gérer dynamiquement sa richesse sur une durée limitée ou illimitée. C'est à dire qu'il cherche à réaliser la meilleure stratégie d'investissement tout en optimisant la consommation de son portefeuille. Sa richesse est constituée d'une partie risquée, typiquement un portefeuille d'actions mais aussi d'une partie non risquée placée sur un compte bancaire. Le cour des valeurs du portefeuille peut être décrit par un processus stochastique S de dimension d où d représente le nombre d'actifs du portefeuille. On modélise la stratégie d'investissement par un processus d dimensionnel π et la consommation par un processus positif c. Ainsi sous hypothèse que le taux d'actualisation est nul, la richesse du portefeuille sous stratégie d'investissement π et pour un capital initial x, $X^{x,\pi,c}$ s'exprime par :

$$X_t^{x,\pi,c} = x + \int_0^t \pi_u \frac{dS_u}{S_u} - \int_0^t c_u du \tag{1.1}$$

La modélisation sous espérance d'utilité permet de prendre en compte le comportement de l'investisseur. En effet, à espérance égale, l'investisseur préfèrera obtenir de petits gains avec une plus grande chance de réussite plutôt que de toucher une somme plus importante mais avec une probabilité plus faible de succès. Il est donc nécessaire de prendre en compte cette attitude via l'introduction d'une fonction, appelée fonction d'utilité, strictement croissante et concave de la richesse $X^{x,\pi,c}$. Ainsi au début, l'agent a une forte appétence au gain mais ses exigences deviennent moindres au fur et à mesure que sa richesse s'accroit. Le problème générique de gestion de portefeuille avec consommation est donc défini par :

$$V(t,x) = \sup_{(c,\pi)\in\mathcal{X}} \mathbb{E}_t\left[\int_0^T U_1(c_u)du + U_2(X_T^{x,\pi,c})\right] \tag{1.2}$$

où $\mathcal{X}$ représente l'ensemble des stratégies d'investissement et de consommation possibles, U_1, U_2 sont deux fonctions d'utilité et $T \in \mathbb{R}^+$ est une maturité donnée.

Sous hypothèse de marché complet, en considérant une fonction d'utilité puissance :

$$U(x) = \frac{x^p}{p},\ p \in [0,1].$$

et en modélisant la richesse du portefeuille par une diffusion Black-Scholes, Merton [63] donne le premier une réponse explicite à ce problème. Par des méthodes de contrôle stochastique, il parvient à calculer explicitement la fonction valeur V et calcule la stratégie optimale. D'autres études suivront et étendront le précédent résultat à des fonctions d'utilité plus générales tout en conservant la complétude du marché. La résolution de ce genre de problèmes utilise la concavité de la transformée de Fenchel et exploite la concavité de la fonction d'utilité. C'est le cas notamment de Karatzas, Lehoczky, Sethi, et Shreve[47] ainsi que Cox et Huang [14] ou encore de Pliska [73].
L'introduction d'incomplétude dans le marché est aussi source de nombreuses pistes de recherche. He and Pearson dans [39], Karatzas, Lehoczky, Shreve et Xu dans [48] ainsi que Kramkov et Schachermayer dans [54] proposent un problème variationnel dual et retrouvent la solution du problème originel via la dualité. Par ailleurs, Cvitanic et Karatzas dans [19] introduisent des contraintes sur les stratégies d'investissement.

Étudions maintenant une première famille de problèmes mixtes d'investissement optimal et d'arrêt optimal. Il apparaît clairement qu'une possible extension de (1.2) consiste à considérer un horizon T aléatoire, à déterminer. Dans ce contexte, la fonction V ne dépend plus du temps. On peut alors définir le problème suivant :

$$V(x) = \sup_{(\pi,c)\in\mathcal{X},\tau\in\mathcal{T}} \mathbb{E}\left[\int_0^\tau U_1(c_u)du + U_2(X_\tau^{x,\pi,c})\right], \tag{1.3}$$

où $\mathcal{T}$ représente un ensemble de temps d'arrêts correctement adaptés pouvant prendre des valeurs infinies. Cette problématique a été résolue via une méthode duale par Karatzas et Wang dans [52]. Si l'on suppose que la consommation et que la stratégie de portefeuille sont données et déterminées alors (1.3) correspond à un problème d'arrêt optimal. Le problème d'arrêt optimal se définit de la manière suivante :

$$v(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}\big[X_\tau\big], \tag{1.4}$$

où $X := X^{\pi,c,x}$. On peut alors envisager cette problématique comme la détermination de la meilleure date de vente pour le portefeuille X. Il est bien connu que la stratégie d'arrêt optimale est le premier temps où l'enveloppe de Snell de X, c'est à dire la plus petite surmartingale dominant X, "vient toucher" X. Beaucoup de travaux ont été effectués sur ce sujet. On mentionnera en particulier Shiryaev [80], Fakeev [33] dans le cas où X est markovien, El Karoui [28], Karatzas et Shreve , Appendice D [50] dans un cadre plus general. Ce problème est aussi fortement lié aux problèmes d'options américaines, Karatzas [46], Karatzas et Shreve [50].

D'autres types de problèmes mixtes d'investissement optimal et d'arrêt optimal peuvent être envisagés. Par exemple, nous pouvons mentionner Karatzas et Sudderth [51] qui s'intéressent à un problème de jeux où un "contrôleur" cherche à maximiser son profit tandis qu'un "stoppeur" cherche à l'arrêter. Nous pouvons aussi mentionner Henderson [41], Evans et Hobson [42] ainsi que Henderson et Hobson [42]. Cette dernière publication est à l'origine de notre étude. Je l'exposerai plus en détails dans la section suivante.

1.1.2 Le problème d'Henderson et Hobson [42]

Le problème mixte d'investissement et d'arrêt optimal décrit par Henderson et Hobson [42] peut être décrit de la manière suivante. Nous considérons un investisseur averse au risque possédant un portefeuille d'actifs ainsi qu'un actif réel et indivisible qu'il cherche à vendre. Son objectif est de maximiser l'utilité de sa richesse finale.

Le caractère indivisible de l'actif traduit le fait qu'il ne peut être divisé en actions. Par exemple, on peut considérer qu'une petite usine de production ou un morceau de terrain sont des actifs indivisibles. On suppose par ailleurs, que l'actif indivisible et le portefeuille sont indépendants. Du fait que cet actif ne puisse être répliqué par une stratégie de portefeuille adéquate, le marché est incomplet.

Soit $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ un espace de probabilité filtré et B_t un $\mathcal{F}_t$-mouvement Brownien. On modélise la dynamique du processus indivisible Y^y par :

$$dY_t^y = Y_t^y\big(\mu(Y_t^y)dt + \sigma(Y_t^y)\big)dB_t \qquad Y_0^y = y > 0.$$

où μ et σ sont telles que l'équation ci-dessus admet une unique solution forte. Notre problème est défini par :

$$V(x,y) = \sup_{\substack{X\in\mathcal{M}^{\perp}(x,y)\\ \tau\in\mathcal{T}}} \mathbb{E}\big[U(X_\tau + Y_\tau^y)\big].$$

où U est une fonction d'utilité quelconque, $\mathcal{T}$ est un ensemble de temps d'arrêt et $\mathcal{M}^{\perp}(x,y)$ est essentiellement l'ensemble des martingales càdlàg de corrélation instantanée nulle avec Y^y à toutes dates t. Ces hypothèses généralisent celles de Henderson et Hobson [42] qui considèrent une fonction d'utilité puissance et supposent que μ et σ sont constants.

Cette modélisation comporte à priori plusieurs handicaps. En effet, le portefeuille d'actif X est modélisé par une martingale alors que, avec la présence de l'actif indivisible, on est sous hypothèse de marché incomplet. De plus, il semble à première vue difficile de tirer parti de cette situation, puisque le portefeuille et le prix de l'actif indivisible sont supposés décorrélés. Par exemple, considérons le problème de gestion de portefeuille suivant :

$$i(x) = \sup_{X\in\mathcal{M}(x)} \mathbb{E}\big[U(X_\tau)\big], x \in \mathbb{R}, \tag{1.5}$$

où $\mathcal{M}(x)$ est un ensemble de martingales et τ un temps d'arrêt fixé. Par optimalité et par inégalité de Jensen, on constate que $i = U$. De plus la stratégie optimale X^* consiste à ne pas investir, c'est à dire $X_t^* = x$ sur $[0, \tau]$.
La question que nous posons est donc de savoir si l'ajout d'un actif indivisible décorrélé impacte la stratégie de portefeuille. En d'autres termes, est-il toujours optimal de ne pas investir ? Henderson et Hobson montrent que, sous certaines conditions, l'investisseur a en effet tout intérêt à gérer dynamiquement son portefeuille en accord avec la vente de l'actif indivisible.

Ce modèle, certes imparfait par sa modélisation, n'en est pas moins un bon benchmark pour des études ultérieures.

1.1.3 Résultats

Dans un premier temps, en restreignant l'ensemble des stratégies de portefeuille $\mathcal{M}^{\perp}(x,y)$ aux martingales continues, on parvient à montrer, via une équation de la programmation dynamique, que V est plus grande que l'enveloppe concave partiellement en x et partiellement en z de la fonction :

$$\bar{U}(x,z) := U(x + R(z))$$

où R est l'inverse de la fonction d'échelle S de Y^y. La construction d'une telle enveloppe concave, nommée $\bar{U}^\infty$, se construit via concavifications partielles successives. C'est à dire que $\bar{U}^\infty$ est la limite de la suite croissante suivante :

$$\begin{aligned} &\bar{U}_0 = \bar{U}, \\ &\bar{U}_{2n} = (\bar{U}_{2n-1})^{conc_x}, \\ &\bar{U}_{2n+1} = (\bar{U}_{2n})^{conc_z}. \end{aligned}$$

où $\bar{U}_{2n}$ est la plus petite fonction concave par rapport à la variable x au dessus de $\bar{U}_{2n-1}$ alors que $\bar{U}_{2n+1}$ est la plus petite fonction concave par rapport à la variable z au dessus de $\bar{U}_{2n}$. Nous devons souligner qu'à chaque étape de concavification partielle, on peut perdre le caractère concave de la précédente concavification. Cette construction est très importante pour la construction des stratégies optimales comme nous le verrons plus tard. Par ailleurs, de par la concavité partielle de $\bar{U}^\infty$ en x et en z et la définition de V, l'inégalité de Jensen nous donne :

$$V(x,y) \leq \bar{U}^\infty(x, S(y))$$

Cet argument n'est pas rigoureux, néanmoins il donne une bonne intuition du calcul de la fonction valeur V. On en donnera une argumentation rigoureuse dans le chapitre 2. Ainsi sous de bonnes hypothèses, nous pouvons évaluer la fonction valeur V comme la plus petite fonction concave partiellement en sa première variable et partiellement en sa seconde qui domine $\bar{U}$, c'est à dire :

$$V(x,y) = \bar{U}^\infty(x, S(y))$$

Reste donc maintenant à utiliser cette structure pour construire les stratégies optimales.

Les stratégies ε-optimales sont définies de la manière suivante. Pour une suite de temps de sauts correctement définies $(\tau_i^n)_{n\in\mathbb{N};1\le i\le n+1}$, on peut définir cette suite de martingales :

$X_t^0 = x \quad \forall t > 0.$

$$X_t^n = x\mathbb{1}_{t\in[0,\tau_1^n)} + \sum_{i=1}^{n-1} \eta_i^n(X_{\tau_{i-1}^n}^n, S(Y_{\tau_i^n}^y))\mathbb{1}_{t\in[\tau_i^n,\tau_{i+1}^n)} + \eta_n^n(X_{\tau_{n-1}^n}^n, S(Y_{\tau_n^n}))\mathbb{1}_{t\in[\tau_n^n,\infty)}.$$

où à chaque temps d'arrêt τ_i^n et pour v fixé, $\eta_i^n(u,v)$ saute alternativement avec respectivement une probabilité $p_i^n(u,v)$ et $1-p_i^n(u,v)$ entre :

$$a_i^n(u,v) := \inf\big\{\alpha,\ \alpha \ge u : \bar{U}^{2(n-i+1)}(\alpha,v) = \bar{U}^{2(n-i+1)-1}(\alpha,v)\big\}.$$
$$b_i^n(u,v) := \sup\big\{\alpha,\ \alpha \le u : \bar{U}^{2(n-i+1)}(\alpha,v) = \bar{U}^{2(n-i+1)-1}(\alpha,v)\big\}.$$

Cette construction est directement liée à un problème de gestion de portefeuille avec fonction d'utilité non concave et maturité aléatoire fixée comme nous le montrerons dans le chapitre 2. A chaque concavification en x, c'est à dire le fait de passer de $\bar{U}^{2(n-i+1)-1}$ à $\bar{U}^{2(n-i+1)}$, on résout un problème de gestion de portefeuille avec fonction d'utilité non concave et maturité aléatoire fixée τ_i^n. Par ailleurs, on définit la suite de temps d'arrêt $(\tau_i^n)_{n\in\mathbb{N};1\le i\le n+1}$ de la manière suivante :

$\tau_1^0 = \inf\big\{t \ge 0 : \bar{U}^1(x,Z_t) = \bar{U}^0(x,Z_t)\big\}.$

$\tau_1^n = \inf\big\{t \ge 0 : \bar{U}^{2n+1}(x,Z_t) = \bar{U}^{2n}(x,Z_t)\big\}.$

$\vdots$

$\tau_i^n = \inf\big\{t \ge \tau_{i-1}^n : \bar{U}^{2(n-i+1)+1}(X_{\tau_{i-1}^n}^n, Z_t) = \bar{U}^{2(n-i+1)}(X_{\tau_{i-1}^n}^n, Z_t)\big\}, \quad i \in \{1\dots n+1\}.$

Ainsi à chaque concavification par rapport z, on fixe la richesse $X_{\tau_{i-1}^n}^n$ et on résout un problème d'arrêt optimal. Cette structure est directement liée à la construction de l'enveloppe concave partiellement en x et en z, U^∞. Nous proposons alors le résultat suivant :

Résultat 1.1. *Sous de bonnes hypothèses, pour tout $\varepsilon > 0$, il existe $n \in \mathbb{N}$ tel que*

$$\varepsilon + \mathbb{E}\Big[\bar{U}^0\big(X_{\tau_{n+1}^n}^n, Z_{\tau_{n+1}^n}\big)\Big] \ge \bar{U}^\infty\big(x, S(y)\big). \tag{1.6}$$

où $(X^n, \tau_{n+1}^n) \in \mathcal{M}^\perp(x,y) \times \mathcal{T}$ sont des stratégies ε-optimales. De plus,

$$\bar{U}^\infty\big(x,S(y)\big) = \lim_{n\to\infty} \mathbb{E}\Big[\bar{U}^0\big(X_{\tau_{n+1}^n}^n, Z_{\tau_{n+1}^n}\big)\Big]. \tag{1.7}$$

Si μ une fonction négative sur $\mathbb{R}_*^+$, alors

$$V(x,y) = U\big(x+y\big). \tag{1.8}$$

Nous ne sommes par parvenus à obtenir de la convergence pour les suites de temps d'arrêts et de martingales que nous avons considérés. Ceci explique que nous n'ayons obtenu que des stratégies ε-optimales. Néanmoins, si la suite $(\bar{U}^n)_n$ est stationnaire, alors on est en mesure d'exhiber des stratégies optimales. Il se trouve que c'est précisement le cas pour une fonction d'utilité puissance et des coefficients constants, comme chez Henderson et Hobson [42].

De la même manière que Henderson et Hobson [42], on constate que, sous certaines conditions, on ne peut pas tirer profit de la vente de l'actif indivisible et qu'il est préférable de ne pas investir. Cependant, la plupart du temps, une gestion dynamique du portefeuille alliée à une bonne stratégie de vente permet d'accroitre son gain final.

1.1.4 Conclusion

Les résultats que nous obtenons sont consistants avec ceux obtenus par Henderson et Hobson [42]. Notre apport a consisté à utiliser une approche de type contrôle stochastique. Cette dernière nous a permis de généraliser leurs résultats au cas d'une fonction d'utilité quelconque et pour une diffusion de l'actif indivisible plus générale. Enfin, nous constatons que l'analyse de ce modèle malgré ses imperfections est un bon benchmark pour de futures études. On sait désormais que la vente d'un actif indivisible, même décorrélé du marché, peut être source de profit et a un impact sur la stratégie de portefeuille.

1.2 2EDSRs avec contraintes convexes

1.2.1 Rappels sur les EDSRs du premier et du second ordre

Soit un espace de probabilité filtré $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,1]}, \mathbb{P})$, une fonction f progressivement mesurable de $\Omega \times [0,1] \times \mathbb{R}^{d+1}$ dans $\mathbb{R}$ et une variable aléatoire $\xi \in L^2(\Omega)$, $\mathcal{F}_1$-adaptée. On peut définir la solution de l'équation rétrograde du premier ordre (EDSR) de paramètres f et ξ par un couple de processus (Y, Z) vérifiant :

$$Y_t = \xi + \int_t^1 f(\omega, s, Y_s, Z_s)ds - \int_t^1 Z_s dB_s,\ 0 \leq t \leq 1,\ \mathbb{P} - a.s. \qquad (1.9)$$

où B_t est un $\mathcal{F}_t$-mouvement Brownien de dimension d. On appelle f, le générateur de la backward et ξ, sa valeur terminale.

Les équations rétrogrades ont été introduites par Bismut [7] dans le cas où le générateur f est linéaire. Pardoux et Peng [65] en 1990 étendent ses résultas au cas où f est une application Lipschitzienne par rapport aux variables y et z et proposent un résultat d'existence et d'unicité. Par la suite, il a été montré que l'on peut affaiblir ces hypothèses. Par exemple, Lepeltier, San Martin [55] ont considéré le cas où le générateur est continu et à croissance linéaire. Kobylanski [53], quant à elle, a montré l'existence d'une solution maximale dans le cas où le générateur f est à croissance linéaire en y et à croissance quadratique en z avec une valeur terminale bornée. Ce dernier cas est très utile en finance puisqu'il permet de résoudre des problèmes de gestion de portefeuille sous contraintes, grâce à une approche backward, dans le cas où la fonction d'utilité est exponentielle, El Karoui et Rouge [79] et Hu, Imkeller et Muller [43].
On peut aussi considérer des équations différentielles rétrogrades réfléchies (REDSR). Elles ont été introduites par El Karoui, Kapoudjian, Pardoux, Peng et Quenez [30] et consistent à imposer au processus Y de rester au dessus d'un seuil donné. Ces équations rétrogrades réfléchies sont fortement liées au pricing d'options américaines, El Karoui, Peng et Quenez [31]. Par ailleurs, on peut aussi considérer le cas de système forward-backward lorsque l'on modélise la valeur terminale ξ comme la solution d'une équation différentielle stochastique forward ordinaire dépendant de Y et de Z, Ma et Young [60].
Le cas où la solution terminale est un processus de Markov, typiquement $\xi = B_1$, a d'intéressantes applications. En effet, on peut montrer que la solution de l'EDSR est Markovienne, c'est à dire qu'elle peut se représenter comme une fonction dépendant du temps t et de B_t, solution d'une équation aux dérivées partielles quasi-linéaire (c'est à dire linéaire par rapport au terme de dérivée seconde), Pardoux et Peng [66].

Les équations stochastiques du second ordre (2EDSRs) ont été introduites par Cheredito, Soner et Touzi [13]. Dans un cadre Markovien, ils montrent que la solution d'une telle équation induit une équation aux dérivées partielles non linéaire. Cependant, ils ne parviennent pas à démontrer l'existence d'une telle solution dans un cadre plus général. Le modèle à volatilité incertaine a été introduit par [1] et [59] pour prendre en compte le risque associé à l'estimation de la volatilité. Plus précisément, la valeur de la volatilité n'est spécifiée que comme appartenant à un intervalle donné. Le problème de couverture sous ce modèle est directement lié à l'équation non linéaire de Black-Scholes-Barrenblat. Denis and Martini [24] étendent le modèle à volatilité incertaine à une famille de probabilités (voir chapitre 3) grâce à la notion d'analyse quasi-sûre. Grossièrement parlant, ils proposent un cadre mathématique cohérent afin de pouvoir faire de l'analyse stochastique simultanément pour une famille de probabilités singulières. Dans ce contexte, on peut mentionner Denis, Hu, Peng [23] qui établissent une connexion entre le modèle à volatilité incertaine et la notion de G-expectation introduite par Peng [68, 67]. Soner, Touzi et Zhang [83] montrent un théorème de représentation pour une G-martingale et donnent ainsi une solution au problème de couverture dans le modèle à volatilité incertaine. Puis, ils se penchent sur l'étude des cibles stochastiques du second ordre [85] et montrent que la solution d'un tel problème est solution d'une 2EDSR. Enfin, dans [86], ils montrent l'existence et l'unicité d'une solution pour une 2EDSR à générateur Lipschitzien en y et en z, proposent une formule de représentation pour cette solution et donnent une extension non linéaire de la formule de Feynman-kac. Possamai et Zhou [75, 76] ont récemment étendu les résultats précédents au cas des équations rétrogrades du second ordre unidimensionelles dont le générateur est continu, à croissance quadratique en sa variable z et dont la valeur terminale est bornée. Cette extension a permis de considérer des équations rétrogrades du second ordre réfléchies ainsi que d'étudier le problème de gestion de portefeuille dans le modèle à volatilité incertaine, Matoussi, Possamai et Zhou [62, 61].

Pour notre étude, nous nous sommes intéressés en particulier au cas des équations différentielles rétrogrades du premier et du second ordre avec contraintes. Pour simplifier, considérons le cas des EDSRs contraintes du premier ordre :

Soit un espace de probabilité filtré $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,1]}, \mathbb{P})$, une fonction f progressivement mesurable de $\Omega \times [0,1] \times \mathbb{R}^{1+d}$ dans $\mathbb{R}$ et une variable aléatoire $\xi \in L^2(\Omega)$, $\mathcal{F}_1$-adaptée, nous nous sommes intéressés au problème suivant :

$$Y_t = \xi + \int_t^1 f_u(Y_u, Z_u)du - \int_t^1 Z_u dB_u + K_1 - K_t, \; 0 \le t \le 1, \; \mathbb{P} - a.s. \tag{1.10}$$

$$\Phi_t(t, Y_t, Z_t) = 0, \; 0 \le t \le 1, \quad \mathbb{P} - a.s. \tag{1.11}$$

où B est un mouvement Brownien d-dimensionel et Φ est une fonction de $\Omega \times [0,T] \times \mathbb{R}^{1+d}$ dans $\mathbb{R}^+$. Une solution minimale de ce problème est un triplet (Y, Z, K), où K est un processus croissant positif et càdlàg, telle que toute autre solution (Y^*, Z^*, K^*) de (1.10) et (1.11) vérifie :

$$Y_t^* \ge Y_t, \; 0 \le t \le 1, \; \mathbb{P} - a.s.$$

Ce problème a tout particulièrement été étudié par Cvitanic et Karatzas [20] dans le cas où la contrainte est de la forme :

$$\rho_c(Z_t) = 0,\ 0 \leq t \leq 1,\ \mathbb{P} - a.s. \tag{1.12}$$

où ρ_c correspond à la distance entre Z et un ensemble convexe fermé C contenant l'origine. En effet, ils montrent que le problème de couverture sous contraintes de portefeuille peut être résolu par une approche backward. En revanche, ils ne montrent pas un résultat d'existence pour ce problème. Cvitanic, Karatzas et Soner [21] résolvent ce problème par une approche de type contrôle stochastique en obtenant un théorème de représentation pour la solution minimale de (1.10) et (1.12). En revanche, ils imposent au générateur d'être convexe en z afin de pouvoir prendre sa conjuguée de Fenchel. Cette hypothèse leur permet de résonner par pénalisation et de construire la solution minimale comme limite monotone d'une suite d' EDSRs pénalisées. D'autres traveaux ont été effectués sur ce sujet, on peut mentionner Buckdahn et Hu [9, 10]. Dans ce dernier papier, ils s'intéressent au problème de couverture d'options américaines sous contraintes de portefeuilles en utilisant une EDSR contrainte unidimensionnelle comportant un mouvement Brownien, un processus de Poisson et une barrière via une méthode de pénalisation. Finalement, les travaux de Peng [67], Peng et Xu [69, 71] donnent des résultats généraux sur les EDSRs réfléchies avec contraintes. En particulier, Peng prouve l'existence d'une solution minimale pour le problème (1.10) et (1.11).

Nous nous sommes intéressés dans un premier temps à étendre les résultats de Cvitanic, Karatzas et Soner [21] en enlevant leur hypothèse de convexité sur le générateur par rapport à z. Nous proposons ainsi une formule de représentation de la solution minimale dans ce contexte. Par ailleurs, nous avons étendu ces résultats au cas des Backward du second ordre avec contraintes sur le processus Z. C'est à dire que nous avons résolu le problème (1.10) et (1.12) simultanément pour un ensemble de probabilités singulières en utilisant les travaux de Soner, Touzi et Zhang [85, 86] et de Peng [67]. Nous proposons ainsi, pour le cas d'une backward du second ordre, un théorème général d'existence ainsi qu' une formule de représentation.

1.2.2 Résultats

Soit un espace de probabilité filtré $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,1]}, \mathbb{P})$, B_t un $\mathcal{F}_t$-mouvement Brownien et C un sous ensemble convexe fermé de $\mathbb{R}^d$ contenant l'origine.

Nous nous sommes intéressés au problème suivant :

$$Y_t = \xi + \int_t^1 g_u(Y_u, Z_u)du - \int_t^1 Z_u dB_u + K_1 - K_t,\ 0 \leq t \leq 1,\ \mathbb{P} - a.s. \tag{1.13}$$

$$Z_t \in C,\ 0 \leq t \leq 1,\ \ \mathbb{P} - a.s. \tag{1.14}$$

où le générateur g est une fonction progressivement mesurable de $\Omega \times [0,1] \times \mathbb{R}^{1+d}$ dans $\mathbb{R}$ et $\xi \in L^2(\Omega)$. Une solution minimale de ce problème est un triplet (Y, Z, K) tel que (Y, Z, K) vérifie (1.13) et (1.14) et que pour toutes autres solutions (Y^*, Z^*, K^*) de ce problème, on ait :

$$Y_t^* \geq Y_t,\ 0 \leq t \leq 1,\ \mathbb{P} - a.s.$$

Nous pouvons d'ores et déjà noter, par un simple argument de comparaison, que la solution du problème contraint est toujours plus grande que celle du problème non contraint :

$$Y_t^0 = \xi + \int_t^1 g_u(Y_u^0, Z_u^0)du - \int_t^1 Z_u^0 dB_u,\ 0 \leq t \leq 1,\ \mathbb{P} - a.s. \tag{1.15}$$

Si g est une fonction linéaire de Y, indépendante de Z, alors on peut voir que résoudre (1.15) revient à effectuer un théorème de représentation des martingales (on se débarrasse du terme linéaire en escomptant). Le cas où Z est non contraint revient donc à considérer un problème de couverture lorsque le marché est complet. Le processus croissant K peut être compris comme traduisant l'incomplétude du marché.
Par la suite, nous avons supposé que :

Hypothèse 1.2. *(i) Le générateur g est une fonction Lipschitzienne par rapport à ses variables y et z.*

(ii) Le générateur g est tel que $\mathbb{E}^{\mathbb{P}}\big[\int_0^1 |g_u(0,0)|^2 du\big] < \infty$.

(iii) Il existe une solution (Y^*, Z^*, K^*) pour le problème (1.13) *et* (1.14).

Sous ces hypothèses, nous savons que par le théorème 4.2 de Peng [67], il existe une solution minimale à ce problème. Notre apport a donc consisté à trouver une formule de représentation pour ce problème dans l'esprit de ce que Cvitanic, Karatzas et Soner [21] avaient proposé pour le cas où g est convexe par rapport à sa variable z.

Soit $\mathcal{D}$ un sous-ensemble de processus bornés à valeur dans le domaine effectif de la fonction support δ de C. Pour tout ν dans $\mathcal{D}$, on définit la famille suivantes d'EDSRs :

$$Y_t^\nu = \xi + \int_t^1 \Big[g_u\big(Y_u^\nu, Z_u^\nu\big) - \big(\delta(\nu_u) - \nu_u Z_u^\nu\big)\Big] du - \int_t^1 Z_u^\nu dB_u, \ \mathbb{P} - a.s \tag{1.16}$$

Sous les hypothèses (i) et (ii) ci-dessus, on peut montrer que ces EDSRs admettent une unique solution. On peut alors définir le processus V par :

$$V_t = \operatorname*{essup}_{\nu \in \mathcal{D}_{[t,1]}} Y_t^\nu, \ \mathbb{P} - a.s.$$

où $\mathcal{D}_{[t,1]}$ est la restriction de D à $[t,1] \times \Omega$ pour tout $t \in [0,1]$. On montre alors le théorème suivant :

Résultat 1.3. *Supposons les hypothèses* 1.2 *vérifiées, alors la solution minimale* (Y, Z, K) *du problème* (1.13) *et* (1.14) *a la représentation suivante :*

$$Y_t = \operatorname*{essup}_{\nu \in \mathcal{D}_{[t,1]}} Y_t^\nu, \ \mathbb{P} - a.s.$$

pour tout $t \in [0,1]$.

La démarche utilisée pour obtenir le résultat ci-dessus utilise les notions de g-martingales et g-surmartingales introduites par El Karoui, Peng et Quenez [31]. Lorsque le générateur de la EDSR est linéaire en y et ne dépend pas de z, les notions de martingale et de g-martingale coïncident. On peut donc envisager la g-martingale et la g-surmartingale comme une généralisation des concepts classiques de martingale et de surmartingale au cas d'un générateur g non linéaire.
De fait, nous avons montré, via la programmation dynamique, que pour tout $\nu \in \mathcal{D}$, V est une càdlàg g^ν-surmartingale où g^ν est défini par :

$$g_t^\nu(y,z) := g_t(y,z) - (\delta(\nu_t) - \nu_t z).$$

et avons appliqué le théorème de décomposition de Doob-Meyer non linéaire prouvé par Peng, théorème 3.3 [67].

Considérons maintenant le problème (1.13) et (1.14) mais pour un ensemble de probabilités singulières. Nous nous plaçons sur l'espace canonique $\Omega = \big\{\omega \in C\big([0,1], \mathbb{R}^d\big) \colon \omega_0 = 0\big\}$, soit B le processus canonique, $\mathbb{P}_0$ la mesure de Wiener et $\mathbb{F} := \{\mathcal{F}_t\}_{0 \le t \le 1}$ la filtration générée par B.

On dit que $\mathbb{P}$ est une mesure martingale locale si B est une martingale locale sous $\mathbb{P}$. Par Föllmer [38], la quantité $\int_0^t B_s dB_s$ coïncide ω par ω avec l'intégrale d'Ito $\mathbb{P}-a.s.$ pour toute mesure martingale locale. On peut donc définir de manière universelle la variation quadratique de B définie ω par ω par :

$$\langle B\rangle_t := B_t B_t' - 2\int_0^t B_s dB_s \text{ et } \hat{a}_t = \overline{\lim_{\varepsilon\downarrow 0}}\frac{1}{\varepsilon}\big(\langle B\rangle_t - \langle B\rangle_{t-\varepsilon}\big)$$

L'ensemble de probabilités, $\overline{\mathcal{P}}_S$, que nous considérons peut se résumer grossièrement parlant à l'ensemble des mesures martingales locales telles que :

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1} \text{ où } X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s,\ t\in[0,1],\ \mathbb{P}_0\text{-a.s.}$$

où α est un processus $\mathbb{F}$-progressivement mesurable vérifiant $\int_0^1 |\alpha_s|ds < \infty$, $\mathbb{P}_0 - a.s.$ Par ailleurs, on considère un générateur $F : \Omega\times[0,1]\times\mathbb{R}^{1+d}$ dans $\mathbb{R}$ et un sous ensemble $\mathcal{P}_H$ de $\overline{\mathcal{P}}_S$ qui rassemble l'ensemble des mesures martingales locales telles que $\hat{a}$ soit borné et tel que l'on ait :

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^1 |\hat{F}_t(0,0)|^2 dt\right] < \infty,\ \forall \mathbb{P}\in\mathcal{P}_H$$

Suivant la définition de Denis et Martini [24], on dit qu'une propriété est vraie $\mathcal{P}_H - q.s.$, si elle est vraie pour toutes probabilités $\mathbb{P}\in\mathcal{P}_H$. On peut alors définir l'équation rétrograde du second ordre (2EDSR) avec contraintes convexes de la manière suivante :

$$Y_t = \xi - \int_t^1 F_u(Y_u, Z_u, \hat{a}_u)du - \int_t^1 Z_u dB_u + K_1 - K_t,\ 0\le t\le 1,\ \mathcal{P}_H - q.s. \tag{1.17}$$

$$Z_t \in C,\ 0\le t\le 1,\quad \mathcal{P}_H - q.s \tag{1.18}$$

Une solution minimale de ce problème est un couple de processus (Y,Z) vérifiant :

(i) $Y_1 = \xi,\ \mathcal{P}_H - q.s.$

(ii) Pour tout $\mathbb{P}\in\mathcal{P}_H$, le processus ci-dessous $K^{\mathbb{P}}$ est croissant et positif $\mathbb{P}-a.s.$ et est défini pour tout $\mathbb{P}\in\mathcal{P}_H$ par :

$$K_t^{\mathbb{P}} := Y_t - Y_0 + \int_0^t F_u(Y_u, Z_u, \hat{a}_u)du + \int_0^t Z_u dB_u,\quad 0\le t\le 1,\ \mathbb{P}-a.s. \tag{1.19}$$

(iii) De plus, si il existe une solution (Y^*, Z^*) au problème (1.17) and (1.18), nous avons :

$$Y_t^* \ge Y_t,\ 0\le t\le 1,\ \mathcal{P}_H - q.s.$$

Dans le cas des 2EDSRs non contraintes, Soner, Touzi et Zhang [86] ont montré que la solution d'une 2EDSR peut être représentée comme l'essentiel suprémum d'EDSR du premier ordre. Leur intuition vient de l'analyse des équations aux dérivées partielles associées aux EDSR. Faisons un raisonnement formel et considérons l'équation non linéaire suivante :

$$-\partial_t V(t,x) - H(t,x,V,DV,D^2V) = 0, \quad V(1,x) = x$$

où H est supposé convexe en γ et est défini par :

$$H(t,x,r,p,\gamma) := \sup_{a\geq 0} \frac{1}{2} a\gamma - F(t,x,r,p,a)$$

On voit donc que F et H sont conjuguées. Dans [82], Théorème 3.2, nous savons que l'équation ci-dessus est associée à la 2EDSR solution du problème de cible stochastique du second ordre. Soit D_f le domaine de F, par un théorème de comparaison pour les EDP paraboliques, on peut montrer que pour tout $a \in D_f$, on a $V \geq V^a$ où V^a est solution (régulière) de l'équation quasi-linéaire suivante :

$$-\partial_t V^a(t,x) - \frac{1}{2} a D^2 V^a(t,x) + f(t,x,V,DV,a) = 0, \quad V^a(1,.) = g(x).$$

Il est bien connu que cette équation est associée à l'EDSR $Y_t^a := V^a(t,B_t)$ suivante :

$$Y_t^a = X_1^a - \int_t^1 f(s,X_s^a,Y_s^a,Z_s^a)ds + \int_t^1 Z_s^a a_s^{1/2} dB_s$$

où $X_1^a := x + \int_0^1 a_r^{1/2} dB_s$. Nous avons donc montré que formellement

$$Y_t = \sup_{a\in D_f} Y_t^a$$

est un candidat naturel pour la représentation de la solution d'une EDSR du second ordre.

De manière analogue, définissons $\tilde{V}$ comme l'essentiel suprémum des solutions minimales des problèmes contraints du premier ordre pour chaque probabilités $\mathbb{P}$ de $\mathcal{P}_H$, c'est à dire :

$$\tilde{V}_t = \operatorname*{essup}_{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})}^{\mathbb{P}'} \operatorname*{essup}_{\nu\in\mathcal{D}_{[t,1]}} Y_t^\nu, \ \mathbb{P}-a.s., \ \forall \mathbb{P}\in\mathcal{P}_H,$$

où $\mathcal{P}_H(t,\mathbb{P}) := \{\mathbb{P}' \in \mathcal{P}_H : \ \mathbb{P}' = \mathbb{P} \text{ sur } \mathcal{F}_t\}$ et $\mathcal{D}_{[t,1]}$ est la restriction de $\mathcal{D}$ à $[t,1]\times\Omega$.

Par ailleurs, nous supposons principalement que :

(i) ξ et F sont uniformément continus en ω.

(ii) Il existe une solution (Y^*, Z^*) au problème du second ordre (1.17) et (1.18).

(iii) Il existe une constante $\mu > 0$ telle que :

$$|F_t(y, z, \hat{a}_t) - \hat{F}_t(y', z', \hat{a}_t)| \leq \mu\bigg(|y - y'| + \hat{a}_t^{1/2}|z - z'|\bigg),\ \mathcal{P}_H - q.s$$

Résultat 1.4. *Sous de bonnes hypothèses, il existe une solution minimale* (Y, Z) *au problème* (1.17) *et* (1.18). *De plus elle a la représentation suivante pour tout* $t \in [0, 1]$ *et* $\mathbb{P} \in \mathcal{P}_H$ *:*

$$Y_t = \underset{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})}{\text{essup}^{\mathbb{P}'}} \ \underset{\nu \in \mathcal{D}}{\text{essup}} \ Y_t^{\mathbb{P}',\nu}(1, \xi),\ \mathbb{P} - a.s.$$

La démonstration de ce résultat est similaire à celle que nous avons proposée pour obtenir le résultat (1.3) pour le cas des équations rétrogrades du premier ordre. La difficulté majeure vient du fait que contrairement au premier ordre, nous raisonnons sur un ensemble de probabilités singulières, ce qui nous oblige a être très prudent.

1.2.3 Conclusion

Dans un premier temps, il serait intéressant de déterminer si le problème contraint d'EDSRs du premier ordre (1.10) et (1.11) peut être étendu au cas des EDSRs du second ordre. Par Peng [67], nous savons que le problème (1.10) et (1.11) admet une solution minimale. Est-il possible d'étendre ses résultats au cas des EDSRs du second ordre ? En particulier, est-il possible d'obtenir un théorème d'existence et une formule de représentation pour ce problème ?

Dans un deuxième temps, on pourrait aussi s'intéresser au cas des EDSRs contraintes et réfléchies du second ordre. On sait déjà que, par Peng et Xu [69, 71], sous certaines hypothèses, il existe une solution minimale à ce problème. Dans quelle mesure peut-on l'adapter aux 2BSDEs ?

Enfin, un de mes regrets est de ne pas avoir pu étudié le cas Markovien de notre problème et ne pas avoir pu faire le lien avec les équations aux dérivées partielles. Dans le cas des EDSRs du premier ordre, l'EDP associée à ce genre de problème est une inéquation variationnelle dont la frontière libre se traduit par une contrainte sur le gradient de la solution. Qu'en est-il de ce problème dans le cadre d'EDSRs du second ordre ? Qu'est-t-il possible de faire en utilisant les résultats de Soner, Touzi et Zhang [86] et ceux de Peng et Xu [70] ?

1.3 Développement asymptotique dans un modèle à volatilité stochastique

1.3.1 Modélisation de la volatilité

La modélisation de la volatilité est un point crucial dans l'évaluation des produits dérivés. Une des lacunes du modèle de Black-Scholes vient de ce qu'il suppose une volatilité constante au cours du temps. En pratique, on s'aperçoit néanmoins que la volatilité à tendance à présenter des caractéristiques aléatoires. Plusieurs pistes s'offrent alors pour modéliser la volatilité.

La volatilité implicite traduit l'écart entre les prix théoriques donnés par la formule de Black-Scholes pour les options européennes et les prix réellement observés. Cette volatilité implicite est une fonction du strike K et de la maturité du produit T et est calculée en inversant la formule de Black-Scholes. La courbe de la volatilité implicite en fonction du strike a une forme de U pour de grands strike. Cette caractéristique célèbre appelée *smile* de volatilité est source de nombreuses analyses. Par contre, on constate que pour presque tous les strike, la volatilité implicite a tendance à décroître en fonction du strike, C'est le *skew* de volatilité. Ces phénomènes ont tendance à s'accroitre lorsque l'on considère des options avec de petites maturités. La différence entre volatilité implicite et volatilité historique vient en particulier de ce que le modèle de Black-Scholes ne prend pas en compte le risque associé à la volatilité.

Dans les modèles à volatilité stochastique, la volatilité se modélise de manière aléatoire. Les modèles à volatilités locales sont des exemples de modèles à volatilité stochastique puisque la volatilité est modélisée par une fonction positive du temps et du prix du sous-jacent. L'intérêt d'un tel modèle est qu'il permet de conserver une structure Markovienne tout en tenant compte du fait que la volatilité n'est pas constante. Pour calculer une telle quantité, Dupire [26], via l'équation du même nom, donne une méthode de calcul pour évaluer la volatilité locale en fonction des prix de différents call européens de différentes maturités et de différents strike. Mais on peut aussi mentionner d'autres célèbres modèles de volatilité comme celui de Heston où le processus de volatilité est modélisé par un processus de Cox-Ingersoll-Ross corrélé au prix du sous-jacent. Ce modèle est très utile lorsque l'on veut prendre en compte différents types de corrélation et permet d'avoir un contrôle sur la volatilité de la volatilité.

Bergomi [2, 3, 4, 5] propose de modéliser la volatilité d'un sous-jacent via la courbe de volatilité forward qui lui est associée. C'est à dire qu'à toutes dates $t \geq 0$, la volatilité de la volatilité dépend de l'ensemble des volatilités de maturités plus grandes que t. Ceci permet d'avoir un meilleur contrôle de la volatilité de la volatilité et est donc très utile pour des produits financiers mettant en jeu à la fois un sous-jacent et sa variance réalisée. Nous nous proposons dans cette étude de donner un cadre mathématique correct à ce modèle. La principale difficulté rencontrée a été de définir convenablement la courbe de volatilité forward. En d'autres termes, quel est le bon espace pour la courbe de volatilité forward ? Pour répondre à cette question, nous sommes allés chercher du côté des produits de taux.

Faisons maintenant une revue rapide des modèles principaux de taux. Certains modèles préfèrent modéliser directement le taux court. C'est la cas entre autres du modèle de Vasicek ou de Ho et Lee parmi les plus célèbres. D'autres se sont intéressés à modéliser directement la courbe des taux. Heath, Jarrow et Morton [40] ont été les premiers à proposer une modélisation non paramétrique de la courbe de taux d'intérêts. Puisque le prix d'un zéro-coupon est fonction de la maturité de l'obligation, on voit bien que le taux court dépend entièrement de la structure de la courbe des prix des zéro-coupons. En pratique, la calibration de cette courbe de taux se fait par interpolation de données de marché, elle a donc un effet régularisant sur la courbe de taux. Par ailleurs, on constate que pour de grandes maturités le prix du zéro-coupon a tendance à s'aplatir. Ces observations font, par exemple, des espaces de Sobolev d'ordre suffisamment élevé de bons candidats pour la définition de la courbe de taux. D'autres espaces peuvent cependant être envisagés. Ainsi Filipovic [34] étend HJM au Brownien de dimension infini. Cette extension l'amène à considérer des équations stochastiques à valeur dans des espaces de Hilbert qui ne sont pas des Sobolev. En revanche, Ekeland et Taflin [27] ont choisi de modéliser la courbe des prix des zéros-coupons comme appartenant à un espace de Sobolev d'ordre suffisamment élevé pour avoir des courbes régulières par rapport au temps jusqu'à maturité. Cette dernière approche a profondément influencé notre modélisation.

Notre objectif est de donner un développement asymptotique du prix d'une option européenne pour de petites variations de la volatilité. Pour cela, nous utilisons la méthode de Fleming et Souganidis [37] ainsi que Fleming et Soner, [36, 35]. Cette approche nécessite des théorèmes de stabilité sur les solutions de viscosité. Puisque le prix de notre action dépend de la courbe de volatilité forward, c'est à dire d'un objet à valeur dans un espace de dimension infinie, nous avons recherché des résultats sur les solutions de viscosité à valeur dans un Hilbert. En particulier, quel espace de fonctions test est à considérer lorsque nous travaillons dans un tel espace ? Crandall et Lions, principalement, [15, 16, 17], donnent les premiers des réponses à ces questions en étudiant des équations d'Hamilton Jacobi en dimension infinie. Cependant, ce cadre ne suffit pas pour notre étude puisque les équations aux dérivées partielles obtenues lors des problèmes de pricing comportent des termes de dérivées secondes. Nous nous sommes ainsi tout particulièrement intéressés à Lions [58, 57] qui, dans ses publications, donne une définition correcte des espaces de fonctions à test à considérer, pour des EDPs paraboliques à coefficients bornés. Swiech [88] étend ses résultats au cas d'opérateurs non bornés.

1.3.2 Résultats

Soit $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ un espace de probabilité filtré. Soit B_t un $\mathcal{F}_t$-mouvement Brownien et W_t un $\mathcal{F}_t$-mouvement Brownien de dimension infinie corrélé avec B. On s'est intéressé au problème de pricing suivant :

Soit U^ε le logarithme du prix d'un actif et $\xi^{\varepsilon,T'}$ la volatilité de maturité T'. Pour une maturité $T > 0$ fixée, ces processus vérifient sur $[0, T]$ les équations différentielles stochastiques ordinaires suivantes :

$$dU_t^\varepsilon = -\frac{1}{2}(\xi_t^{\varepsilon,t})^2 dt + \xi_t^{\varepsilon,t} dW_t^0, \tag{1.20}$$

$$d\xi_t^{\varepsilon,T} = \varepsilon^2 \hat{M}(t, T, \xi_t^{\varepsilon,\cdot}) dt + \frac{\varepsilon}{2}\hat{\Lambda}(t, T, \xi_t^{\varepsilon,\cdot}) dW_t. \tag{1.21}$$

où $\xi_t^{\varepsilon,\cdot} : [t, \infty) \ni T \mapsto \xi_t^{\varepsilon,T}$ est une courbe de volatilité forward que l'on suppose comme appartenant à un espace de Hilbert H, pour toutes les maturités plus grandes que t et $\hat{M}$ and $\hat{\Lambda}$ sont deux fonctions de $[0, T] \times \mathbb{R}$ dans H. Comment définir l'espace H ?

Pour ne pas avoir à définir un espace de Hilbert H dépendant du temps, nous adoptons la paramétrisation de Musiela et définissons le processus $Y_t^\varepsilon(x)$:

$$Y_t^\varepsilon(x) := \xi_t^{\varepsilon,t+x}, \tag{1.22}$$

pour tout $t \in [0, T]$ et $x \in \mathbb{R}^+$. Puisque les courbes de taux des zéros-coupons ont tendance à être continues et à s'aplatir vers l'infini, il est naturel de proposer pour définition de H, un espace de Sobolev d'ordre suffisamment élevé. En effet, par injection de Sobolev, on sait que ces éléments admettent un représentant régulier. Ainsi, on définit pour $\gamma > \frac{1}{2}$:

$$H(\mathbb{R}) := \mathbb{R} \oplus H^\gamma(\mathbb{R})$$

où $H^\gamma(\mathbb{R})$ est un espace de Sobolev d'ordre γ défini sur $\mathbb{R}$. On voit facilement que $H(\mathbb{R})$ est un espace de Hilbert pour la norme $\|f\|^2_{H(\mathbb{R})} = \|g\|^2_{H^\gamma(\mathbb{R})} + a^2$ pour $f = g + a$. Cet ensemble permet d'avoir des courbes de taux non nulles à l'infini.

Considérons maintenant un payoff g de $\mathbb{R}$ dans $\mathbb{R}^+$ borné et Lipschitzien. On appelle alors p^ε le prix de l'option de payoff $g(U^\varepsilon_T)$ pour une maturité $T > 0$ fixée. C'est à dire que pour tout $\varepsilon \in \mathbb{R}$, $t \in [0, T]$, $u \in \mathbb{R}$ et $y \in H$

$$p^\varepsilon(t, u, y) := \mathbb{E}\big[g\big(U^\varepsilon_T\big)|(U^\varepsilon_t, Y^\varepsilon_t) = (u, y)\big].$$

Notre objectif est de proposer un développement asymptotique autour de la volatilité pour ce prix. Par ailleurs, on constate que les hypothèses sur g nous permettent de couvrir le cas d'un put européen. Via la parité call-put, on peut aussi appliquer ce développement pour un call européen. De plus, on voit facilement que l'ordre du développement dépend de la régularité du pay-off considéré. Pour un call ou un put européen, cette dérivabilité est limitée puisque que l'on obtient un dirac lorsque l'on dérive son payoff à l'ordre 2.

La méthode que nous avons employée a été développée principalement par Fleming et Souganidis [37], Fleming et Soner, [36, 35] et utilise la stabilité des solutions de viscosité. A ce sujet, puisque la fonction p^ε comporte une variable à valeur dans un espace de Hilbert, la théorie classique des solutions de viscosité (voir [44]) n'est plus valable. Un autre ensemble de fonctions test doit être envisagé. Utilisant les résultats de Swiech [88], on montre le résultat suivant :

On appelle $BUC(\mathbb{R} \times H(\mathbb{R}))$, l'ensemble des fonctions bornées et uniformément continues sur $\mathbb{R} \times H(\mathbb{R})$ et $BUC_x\big([0, T] \times \mathbb{R} \times H(\mathbb{R})\big)$ défini par :

$$\begin{aligned} BUC_x\big([0,T] \times \mathbb{R} \times H(\mathbb{R})\big) := &\big\{u \in C^0\big([0,T] \times \mathbb{R} \times H(\mathbb{R})\big) : \\ &u(t,\cdot) \in \ BUC(\mathbb{R} \times H(\mathbb{R})), \text{ uniformément en } t\big\}. \end{aligned}$$

Résultat 1.5. *Pour tout $\varepsilon \in \mathbb{R}$, p^ε est l'unique solution de viscosité dans $BUC_x\big([0, T] \times \mathbb{R} \times B\big)$ de*

$$\begin{aligned} -\partial_t p^\varepsilon(t, u, y) - F_\varepsilon\big(t, u, y, Dp^\varepsilon(t, u, y), D^2 p^\varepsilon(t, u, y)\big) &= 0 \\ p^\varepsilon(T, u, y) &= g\big(u\big) \end{aligned}$$

où F_ε est le générateur du processus $\begin{pmatrix} U^\varepsilon \\ Y^\varepsilon \end{pmatrix}$ et B un ensemble fermé et borné de $H(\mathbb{R})$.

Supposons que g est infiniment dérivable et à dérivées bornées. Nous définissons une suite $(p_n)_n$ de fonctions de $[0,T] \times \mathbb{R} \times B$ à valeur dans $\mathbb{R}$ telle que pour tout $n \geq 1$:

$$p_n(t,u,y) = \mathbb{E}_{t,u,y}\Big[\int_t^T h_n\big(s, U_s^0, y\big)ds\Big] \tag{1.23}$$

où h_n est une suite définie récursivement à partir des coefficients du modèle et des dérivées en u et en z de p_{n-1} et p_{n-2} dont on peut montrer par récurrence qu'elle est uniformément bornée. On montre que sous de bonnes hypothèses, p_n est une solution régulière de l'EDP suivante :

$$\begin{cases} -\partial_t p_n(t,u,y) + F_0(t,u,y,Dp_n,D^2p_n) - h_n(t,u,y) = 0 \\ p_n(T,u,y) = 0 \end{cases} \tag{1.24}$$

où $F_0(t,u,y,r,p,X) = \lim\limits_{\varepsilon\to 0} F_\varepsilon(t,u,y,r,p,X)$. F_0 est un générateur de type Black-Scholes, c'est-à-dire qu'il ne comporte que des dérivées de p_n par rapport à sa composante de dimension finie u. La composante en z infinidimensionnelle est ici un paramètre.

Soit alors la fonction définie pour tout $n \geq 0$ et (t,u,y) dans $[0,T] \times \mathbb{R} \times B$

$$p^{n,\varepsilon}(t,u,y) := \frac{1}{\varepsilon^n}\big[p^\varepsilon(t,u,y) - \sum_{k=0}^{n-1} \varepsilon^k p_k(t,u,y)\big]. \tag{1.25}$$

On montre que $p^{n,\varepsilon}$ est solution de viscosité de :

$$-\partial_t p^{n,\varepsilon} - F_\varepsilon(t,u,y,Dp^{n,\varepsilon},D^2p^{n,\varepsilon}) - h_\varepsilon^n = 0.$$

où h_ε^n est une suite définie récursivement à partir des coefficients du modèle et des dérivées en u et en y de p_{n-1} et p_{n-2} et est telle que :

$$\lim_{\varepsilon\to 0} h_\varepsilon^n(t,u,y) = h_n(t,u,y),\ \forall (t,u,y) \in [0,T] \times \mathbb{R} \times B. \tag{1.26}$$

Par stabilité des solutions de viscosité, on voit alors que $\lim_{\varepsilon\to 0} p^{n,\varepsilon}(t,u,y)$ est solution de viscosité de (1.24) et on conclut grâce aux conditions de bord qu'en fait :

$$\lim_{\varepsilon\to 0} p^{n,\varepsilon}(t,u,y) = p_n(t,u,y),\ \forall (t,u,y) \in [0,T] \times \mathbb{R} \times B.$$

On obtient ainsi notre résultat principal :

Résultat 1.6. *Sous de bonnes hypothèses et pour g suffisamment dérivable, pour tout $n \in \mathbb{N}$, il existe des constantes C_n et $\varepsilon_0 > 0$ tels que pour tout $\varepsilon \in (0, \varepsilon_0]$, on ait :*

$$0 \leq p^\varepsilon(t,u,y) \leq v^{n,\varepsilon}(t,u,y) := \sum_{k=0}^{n-1} \varepsilon^k p_k(t,u,y) + C_n \varepsilon^n (T-t). \qquad (1.27)$$

De plus pour tout $n \geq 1$, $p^{n,\varepsilon}$ converge vers p_n quand $\varepsilon \downarrow 0$ uniformément sur les compacts.

La démonstration de ce résultat est complètement dépendante de la dérivabilité de g ainsi que des paramètres du modèle. Pour un payoff non régulier, tel qu'un put et un call européen, nous n'arrivons à obtenir qu'un développement à l'ordre 0. Par ailleurs, le fait de se restreindre à un ensemble fermé et borné de H découle des conditions à imposer pour avoir existence et unicité lorsque l'on veut montrer le résultat 1.5.

1.3.3 Conclusion

Je n'ai pas encore eu le temps de faire des applications numériques pour ce modèle. La calibration d'un tel modèle compte tenu de la surface de volatilité peut s'avérer délicate. Par ailleurs, il serait intéressant d'étendre l'ordre du développement asymptotique pour un call ou un put européen. Est-il possible d'augmenter cet ordre par des techniques de type régularisation de payoff ? Est-il possible de proposer un tel développement pour des options américaines ? Comment peut-on intégrer le problème à frontière libre apparaissant pour ces problèmes dans notre argumentation ?

Chapter 2

Optimal Liquidation of an Indivisible Asset with Independent Investment

2.1 Introduction

The problem we consider belongs to the class of mixed optimal stopping/optimal investment problems and was introduced by Henderson and Hobson in [42].

We consider a risk-averse agent who possesses one unit of an indivisible asset and continuously trades on some given risky assets. He wants to increase the expected utility of his total wealth. The presence of the indivisible asset can be seen as an example of real options problem. In contrast to standard options which can be hedged dynamically on the market, real options allow to make investment strategies for non financial underlyings (see [25]). For instance, one can consider a small firm, a R&D project, a piece of land etc... The addition of this indivisible asset makes the market incomplete. The general problem can thus be defined by:

$$V(x,y) = \sup_{\tau, M \in \mathcal{M}^{\perp}(Y)} \mathbb{E}\big[U(x + M_\tau, Y_\tau)\big] \tag{2.1}$$

where U is a concave function, $\mathcal{M}^{\perp}(Y)$ is a set of feasible strategies which are orthogonal to Y, Y is an exogenous Markov process and τ is a stopping time. This will be correctly defined in the next section. We assume that the indivisible asset Y is sufficiently "small" to be uncorrelated to the market, which is a caricatural assumption to say that they are independent. Indeed, our aim is to compare this problem to the case where the optimal strategy consists in not investing. If the market and the indivisible asset were correlated, we might reduce the risk exposure of the portfolio by selling the real asset. Therefore in our uncorrelated case, one can expect that there is no chance to make profit by having a dynamic management of the wealth portfolio. However Henderson and Hobson in [42] show to the contrary that the risk-averse agent can make profit of this situation.

Mixed stopping/control problems arise in many financial situations. Henderson, Evans and Hobson in [32] include correlation between the market and the asset to be sold while Karatzas and Wang in [52] propose a duality approach to reduce the mixed optimal stopping/optimal control problem to a family of pure optimal stopping problems. One can also mentioned Karatzas and Sudderth [51] for a game problem.

In the following, we focus on Henderson and Hobson's framework in [42] where they consider (2.1) with a CRRA utility function and a standard Black-Scholes diffusion. Similarly, we assume that the interest rate is zero such that the optimal portfolio is modeled by a martingale. But our study differs from [42] in two ways. Indeed, we can describe explicitly the value function for any utility function and for a more general Markov process Y. Moreover, we choose the dynamic programming approach which allows us to emphasize a concave structure and a pure jump investment strategy directly linked to the concavity of the utility function.

2.2 The problem definition

2.2.1 The framework

We consider $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space where $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$. The filtration $\mathbb{F}$ is taken such that B is a real valued $\mathbb{F}$-Brownian motion.

Let Y^y be the price process of one unit of an indivisible asset. For instance, one could think to a piece of land or a small factory. It is modelled by:

$$dY_t^y = Y_t^y\big(\mu(Y_t^y)dt + \sigma(Y_t^y)\big)dB_t \qquad Y_0^y = y > 0. \tag{2.2}$$

where μ and σ are two given functions such that:

$$\mu: \; y \in \mathbb{R}_*^+ \longrightarrow \mu(y) \; \in \mathbb{R},$$
$$\sigma: \; y \in \mathbb{R}_*^+ \longrightarrow \sigma(y) \; \in \mathbb{R},$$

which satisfy :

$$\exists K > 0 : \forall y \in \mathbb{R}_*^+ \; |\mu(y)| + |\sigma(y)| \leq K,$$

and the locally Lipschitz condition:
For all $n \in \mathbb{N}$ and $(y, y') \in (\mathbb{R}_*^+)^2$, there exists a constant $C_n > 0$ such that $|y| < n$, $|y'| < n$ and

$$|\mu(y) - \mu(y')| + |\sigma(y) - \sigma(y')| \leq C_n |y - y'|.$$

These assumptions guarantee a unique strong square integrable solution to (2.2) which is a strong positive Markov process, see for example [49].

We consider, U, a concave function from $\mathbb{R}^+$ to $\mathbb{R}$ which verifies $\lim_{x \to 0} U'(x) = +\infty$ and $\lim_{x \to \infty} U'(x) = 0$. For example, one can consider power utility function which as usual provides some interesting features in terms of calculation. We will apply our results to this example in our last section and recover the result of [42].

We denote by X the wealth process of an agent which belongs to the following set:

$$\mathcal{M}^{\perp}(x, y) = \Big\{ X \text{ càdlàg martingale with } X_0 = x \text{ such that for } t \geq 0 : \\ [X, Y^y]_t = 0 \text{ and } X_t + Y_t^y \geq 0 \Big\}$$

where $[X, Y^y]$ is the quadratic covariation between X and Y^y. Since by definition the indivisible asset Y^y can not be traded, the market is incomplete and the modelling of the portfolio X as a martingale seems not to be a good assumption. Indeed, we might have considered supermartingales instead of martingales but this assumption would completely change the structure of the problem and the calculus of the value function as we will see in the sequel. We emphasize that the results obtained in our framework are explicit and may play the role of benchmarks for a more general setting.
We suppose that there is no correlation with the real asset. This assumption translates the fact that Y^y can not be hedged dynamically by financial assets. From a mathematical point of view, this hypothesis is very important

because it fully determines the structure of the candidate solution as we will see later. Furthermore, the last condition, which imposes to the portfolio X to be such that $X_t + Y_t^y \geq 0$ at any date $t > 0$, must be understood as a solvability condition.

Our aim is to solve the following problem:

$$V(x,y) = \sup_{\substack{X \in \mathcal{M}^{\perp}(x,y) \\ \tau \in \mathcal{T}}} \mathbb{E}\Big[U(X_\tau + Y_\tau^y)\Big], \tag{2.3}$$

where

(i) $(x,y) \in D := \{\mathbb{R} \times \mathbb{R}_*^+;\ x + y \geq 0\}$

(ii) $\tau \in \mathcal{T}$ where $\mathcal{T}$ is the set of all $\mathbb{F}$-stopping times

and compare it to the case where there is no indivisible asset Y^y, that is to say for all $x \geq 0$:

$$m(x) = \sup_{X^x \in \mathcal{X}} \mathbb{E}[U(X_\tau^x)]. \tag{2.4}$$

where X^x is a process with initial value $x \geq 0$ which belongs to $\mathcal{X}$, the set of càdlàg martingales, and $\tau \in \mathcal{T}$ is a given stopping time. Since U is a concave function then by optimality and the inequality of Jensen, for all $x \geq 0$:

$$m(x) = U(x) \text{ and } X^x = x\ .$$

So it is optimal not to trade in the risky asset. Comparing (2.3) and (2.4), we would like to know whether this optimal strategy is modified by the introduction of the indivisible asset. That is to say, does the strategy that consist in not investing is still optimal when we have to sell an indivisible asset?

2.2.2 The approach of Henderson and Hobson [42]

Their framework is more restrictive than ours. Indeed they model the indivisible asset Y^y as a geometric Brownian motion. That is to say, μ and σ are assumed to be constant functions so (2.2) becomes

$$dY_t^y = Y_t^y(\mu dt + \sigma dB_t),\ Y_0^y = y > 0. \tag{2.5}$$

They consider the particular case of a power utility function with parameter $p \in (0, \infty)$. That is to say:

$$U_H(x) = \frac{x^{1-p} - 1}{1-p},\ p \neq 1.$$

At the limit, they recover the case of a logarithmic utility function ($p \to 1$). They consider the following problem which is the analogous of problem (2.3) for a power utility function:

$$V_H(x,y) = \sup_{\substack{X \in \mathcal{M}^{\perp}(x,y) \\ \tau \in \mathcal{T}}} \mathbb{E}\big[U_H(X_\tau + Y_\tau^y)\big]. \tag{2.6}$$

We can see that, when $p \geq 1$, the problem is degenerate if the total wealth $X_\tau + Y_\tau^y$ is close to 0. When $p < 1$, the condition $X_\tau + Y_\tau^y \geq 0$ gives $\frac{1}{p-1}$ as a lower bound for V_H.

We introduce the following optimal stopping problem:

$$w_H(x,y) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\big[U_H(x + Y_\tau^y)\big]. \tag{2.7}$$

We recall some results of [42] but restrict our review to the case where $p < 1$. Our goal is to focus on some interesting features and show how they have motivated our study.

Let us define $\gamma_-(p)$ as the unique solution in $(0, p \wedge 1)$ of the following equation:

$$(p-\gamma)^p(p+1-\gamma) - (2p-\gamma)^p(1-\gamma) = 0.$$

On can show that:

Theorem 2.1. *(Henderson and Hobson, [42])*
Suppose $p < 1$. For $\gamma \leq \gamma_-(p)$, we have that $V_H = w_H$. Conversely for $\gamma_-(p) < \gamma \leq p$, we have that $V_H > w_H$.

From this theorem, it follows, when $\gamma \leq \gamma_-(p)$, that the agent is indifferent to do fair investments on the market. The optimal strategy will consist in keeping its wealth constant and solving an optimal stopping time problem, that is to say w_H. To the contrary, when $\gamma_-(p) < \gamma \leq p$, the agent can take advantage of a dynamic management strategy of its portfolio. In this case, the risk-averse agent wants to make profit by doing fair gambles.

Remark 2.2. The methodology used in [42] is the following. They begin by the construction of a parametric family of stopping rules and admissible martingales. For each element of this family, they evaluate the corresponding value function and optimize it over the parameter values. Then they proceed by a verification argument. Our methodology relies on a stochastic control approach which, via a dynamic programming equation, provides a better understanding of V_H and its optimal strategies.

We recall here the main idea of the construction of the family of optimal strategies presented in [42]. Given (ξ, η) such that $-1 < \xi < \eta$ with $\eta > 0$, they describe three regions $\mathcal{S}$, $\mathcal{G}$ and $\mathcal{W}$ defined by:

(i) $\mathcal{W}$, called the waiting region, corresponds to the case where $X_t > \eta Y_t^y \geq 0$. They set X_t to be constant and wait until Y_t^y reaches the level $\frac{X_t}{\eta}$. If Y_t^y never reaches the level $\frac{X_t}{\eta}$, then the optimal stopping rule τ will be infinite.

(ii) $\mathcal{G}$, the gambling region, corresponds to the case where $\xi Y_t^y < X_t < \eta Y_t^y$. When Y_t^y reaches the level ηX_t, then X_t instantaneously jumps to the level ξY_t^y. In this case, we notice that the time and the value of the real asset are fixed so we are only interested in the amplitude of the jumps of X.

(iii) $\mathcal{S}$ corresponds to the set where $X_t \leq \xi Y_t^y$. In this region, when Y_t^y reaches the level $\frac{X_t}{\xi}$, we stop immediately.

We can highlight that the optimal strategies seem to be constructed in two steps. First, they fix the wealth of the portfolio and wait until the indivisible asset reaches a certain level. From a mathematical point of view, this corresponds to the optimal stopping problem (2.7). In the second step, they fix the time and try to optimize the jumps of the wealth portfolio. This last point seems to be unclear at this moment. We will provide in the sequel an elaborate answer to this question.

2.3 A lower bound for the value function

This section is dedicated to the use of the dynamic programming principle to characterize the value function V as a viscosity supersolution of a partial differential equation.

2.3.1 Dynamic Programming Principle

We do not know if the admissible martingales that we consider verify the basic conditions necessary for obtaining a dynamic programming principle. Let us introduce the following simplified problem:

We consider W a real valued, $\mathbb{F}$-adapted, standard Brownian motion such that $\langle B, W\rangle_t = 0$. We recall that B is the Brownian motion which drives the diffusion of Y^y (see (2.2)). We denote by $\mathcal{A}^{\perp}(x, y)$ a subset of $\mathcal{M}^{\perp}(x, y)$

defined by :

$$\mathcal{A}^{\perp}(x,y) := \Big\{ X_s^{\alpha,t,x} = x + \int_t^s \alpha_u dW_u \text{ such that } \forall T > 0 \int_0^T |\alpha_u|^2 du < \infty \ a.s.,$$
$$\mathbb{E}\Big[\sup_{0\le t\le s} |U(X_s^{\alpha,t,x} + Y_s^y)|\Big] < \infty \text{ and } \forall s \ge t \ge 0 \ X_s^{\alpha,t,x} + Y_s^y \ge 0 \Big\}.$$

We want to solve for all $(x,y) \in D$:

$$V^0(x,y) := \sup_{\substack{X\in\mathcal{A}^{\perp}(x,y)\\ \tau\in\mathcal{T}}} \mathbb{E}\big[U(X_\tau^{\alpha,t,x} + Y_\tau^y)\big]. \tag{2.8}$$

Remark 2.3. Since $\mathcal{A}^{\perp}(x,y) \subset \mathcal{M}^{\perp}(x,y)$, then for all (x,y) in D, we have:

$$V(x,y) \ge V^0(x,y). \tag{2.9}$$

Remark 2.4. Bouchard and Touzi in [8] propose a weak dynamic principle adapted for the achievement of a dynamic programming equation in the viscosity sense. However this weak framework does not work with càdlàg martingales which belong to $\mathcal{M}^{\perp}(x,y)$ and we need to work with $\mathcal{A}^{\perp}(x,y)$.

Thanks to the weak dynamic programming principle for viscosity solutions obtained in [8], Theorem 4.1, one can prove that

Proposition 2.5. *For all $(x,y) \in D$ and $\theta \in \mathcal{T}$, for any ϕ real valued upper semicontinuous function such that $V^0 \ge \phi$, we have:*

$$V^0(x,y) \ge \sup_{\substack{X\in\mathcal{A}^{\phi}(x,y)\\ \tau\in\mathcal{T}}} \mathbb{E}\big[\phi(X_\theta^{\alpha,t,x}, Y_\theta^y)\mathbf{1}_{\theta\le\tau} + U(X_\tau^{\alpha,t,x} + Y_\tau^y)\mathbf{1}_{\theta>\tau}\big] \tag{2.10}$$

where $\mathcal{A}^{\phi}(x,y) = \big\{X^{\alpha,t,x} \in \mathcal{A}^{\perp}(x,y) : \mathbb{E}[\phi(X_\theta^{\alpha,t,x}, Y_\theta^y)] < \infty\big\}$

Proof. The process $X^{\alpha,t,x}$ is a square integrable continuous stochastic process. So by classical results on the stochastic integral it is easy to check that the assumptions of Theorem 4.1 are verified. □

2.3.2 Dynamic Programming Equation

We do not know if V^0 is regular. Therefore, the appropriate tool to characterize it as a solution of a partial differential equation is the viscosity sense. For futher details on viscosity solutions, see [18].

We define the lower semicontinuous hull of V^0 by :

$$V_*^0(x,y) = \liminf_{\substack{x'\to x\\ y'\to y}} V^0(x',y'), \quad \forall (x,y) \in D$$

Lemma 2.6. *Assume that V^0 is locally bounded, then V^0_* is a viscosity supersolution of*

$$\min\Big\{-\frac{1}{2}y^2\sigma(y)^2v_{yy}(x,y)-y\mu(y)v_y(x,y);-v_{xx}(x,y);v-U(x+y)\Big\}=0 \text{ on } D \tag{2.11}$$

Proof. Take $\phi \in C^{2,2}(\mathbb{R})$ and $(x_0, y_0) \in D$ such that $\min(V^0_* - \phi) = (V^0_* - \phi)(x_0, y_0)$. After possibly adding a constant to ϕ, we can assume without loss of generality that:

$$\min(V^0_* - \phi) = (V^0_* - \phi)(x_0, y_0) = 0.$$

Let $(x_n, y_n)_{n\geq 0}$ be a sequence such that $(x_n, y_n, V^0(x_n, y_n)) \to (x_0, y_0, V^0_*(x_0, y_0))$ as n tends to infinity.
We can see that selling immediately leads to $V^0_*(x, y) \geq U(x + y)$. Indeed by the continuity of U as a concave function,

$$V^0_*(x,y) = \liminf_{(x',y')\to(x,y)} V^0(x',y') \geq \liminf_{(x',y')\to(x,y)} U(x'+y') = U(x+y).$$

Let us define $\beta_n = V^0(x_n, y_n) - \phi(x_n, y_n)$ and $(X^n, Y^n) = (x_n + \alpha W, Y^{y_n})$, where α is such that $X^n + Y^n \geq 0$, $\mathbb{P} - a.s.$ We consider the following stopping time

$$\theta_n = \inf\big\{t \geq 0 :\ (t, X^n_t - x_n, Y^{y_n}_t - y_n) \notin [0, h_n) \times \alpha r\big\},$$

where r is a positive given constant, B is the unit ball of $\mathbb{R}^2$ and

$$h_n = \sqrt{|\beta_n|}\mathbb{1}_{\beta_n\neq 0} + \frac{1}{n}\mathbb{1}_{\beta_n=0}.$$

Firstly, $\lim_{n\to\infty} \beta_n = 0$. Secondly, thanks to Proposition 2.5 and Ito's formula:

$$\begin{aligned}
&V^0(x_n,y_n) = \beta_n + \phi(x_n,y_n) > \mathbb{E}[\phi(X^n_{\theta_n}, Y^n_{\theta_n})] \\
&= \phi(x_n,y_n) + \mathbb{E}\bigg[\int_0^{\theta_n} \phi_y(X^n_u,Y^n_u)Y^n_u\mu(Y^n_u) \\
&+ \frac{1}{2}(Y^n_u)^2\sigma^2(Y^n_u)\phi_{yy}(X^n_u,Y^n_u) + \frac{1}{2}\phi_{xx}(X^n_u,Y^n_u)\alpha^2 du\bigg].
\end{aligned}$$

This leads to:

$$\beta_n > \mathbb{E}\bigg[\int_0^{\theta_n}\Big(\phi_y(X^n_u,Y^n_u)Y^n_u\mu(Y^n_u)+\frac{1}{2}(Y^n_u)^2\sigma^2(Y^n_u)\phi_{yy}(X^n_u,Y^n_u)+\frac{1}{2}\phi_{xx}(X^n_u,Y^n_u)\alpha^2\Big)du\bigg].$$

Since μ and σ are locally Lipschitz continuous and have linear growth, one can show the following classical standard estimate for all $h > 0$:

$$\mathbb{E}\bigg[\sup_{t\leq s\leq t+h}|Y^n_s - y_n|^2\bigg] \leq Ch^2(1+|y_n|^2).$$

This leads to $(X^n, Y^n) \underset{n}{\rightarrow} (x_0 + \alpha W, Y^{y_0})$ $\mathbb{P} - p.s$. For n sufficiently large and all $\omega \in \Omega$, $\theta_n(\omega) = h_n$. Moreover by definition of θ_n, the following quantity

$$\frac{1}{h_n}\int_0^{\theta_n}\left(\phi_y(X_u^n, Y_u^n)Y_u^n\mu(Y_u^n)+\frac{1}{2}\sigma^2(Y_u^n)(Y_u^n)^2\phi_{yy}(X_u^n, Y_u^n)+\frac{1}{2}\phi_{xx}(X_u^n, Y_u^n)\alpha^2\right)du$$

is bounded, uniformly in n. Therefore, by the mean value and dominated convergence theorem,

$$\mathbb{E}\left[\frac{1}{h_n}\int_0^{\theta_n} Y_u^n\mu(Y_u^n)\phi_y(X_u^n, Y_u^n) + \frac{1}{2}(Y_u^n)^2\sigma^2(Y_u^n)\phi_{yy}(X_u^n, Y_u^n) + \frac{1}{2}\phi_{xx}(X_u^n, Y_u^n)\alpha^2 du\right]$$
$$\underset{n\to\infty}{\longrightarrow} \frac{1}{2}y_0^2\sigma^2(y_0)\phi_{yy}(x_0, y_0) + y_0\mu(y_0)\phi_y(x_0, y_0) + \frac{1}{2}\alpha^2\phi_{xx}(x_0, y_0).$$

The above equation holds for every $\alpha > 0$. Therefore sending α to the infinity gives that $-\phi_{xx}(x_0, y_0) \leq 0$.
Finally, we have shown that V_*^0 is a viscosity supersolution on D of

$$\min\left\{ -\frac{1}{2}y^2\sigma^2(y)v_{yy} - y\mu(y)v_y \ ; \ -v_{xx} \ ; \ v(x,y) - U(x+y)\right\} = 0.$$

□

Remark 2.7. We have shown in the previous proof that V_*^0 must be concave in x. So, it remains to clarify the first part of the PDE which involves the first and second order derivatives in y. However, if the indivisible asset was constant in time, (2.11) would be equal to:

$$\min\left\{ -v_{xx} \ ; \ v(x,y) - U(x+y)\right\} = 0.$$

So under this assumption, the problem we consider may sum up to a classical optimal investment problem with fixed random maturity.

2.3.3 The supersolution property

We require the following assumption of non degeneracy (see [6], p339 for more details):

Assumption 2.8.

$$\forall x \in \mathbb{R}_*^+ \ \sigma^2(x) > 0 \tag{2.12}$$

(2.13)

Let us fix a constant $c > 0$ and define the scale function S of Y^y as the classical solution of

$$S'(y)y\mu(y) + \frac{1}{2}y^2\sigma^2(y)S''(y) = 0 \tag{2.14}$$

$$S'(c) = 1 \tag{2.15}$$

$$S(c) = 0 \tag{2.16}$$

Because of Assumption (2.8), S'' exists and S is a continuous one-to-one function from $\mathbb{R}_*^+$ to $dom(S) := \big(S(0), S(+\infty)\big)$. We denote by R its continuous inverse. Moreover, if we define the process $Z := S(Y)$, we have that Z is a local martingale. Indeed, by Ito's formula, it verifies the following stochastic differential equation :

$$dZ_t = \tilde{\sigma}(Z_t)dB_t \tag{2.17}$$

where $\tilde{\sigma}(z) := R(z)S'\big(R(z)\big)\sigma\big(R(z)\big)$.
Let us assume:

Assumption 2.9.

$$\int^{\infty} \frac{z}{\tilde{\sigma}^2(z)}dz = \infty.$$

Thanks to Carr, Cherny and Urusov in [11], Theorem 1, 1, we know that Z is martingale if and only if Assumption 2.9 holds.

Remark 2.10. Assumption 2.9 will be important in the construction of the optimal rules since it gives integrability to the optimal portfolio strategy. However, we can avoid this assumption in case of seeking a local martingale instead of a martingale for the optimal strategy.

From now on, we will work with the scale process Z. We have to introduce $\bar{D} = \big\{(x, z) \in \mathbb{R} \times dom(S) \ : \ x + R(z) \geq 0\big\}$ which is the analogous of D after the change of variable $z = S(y)$. Let us define for all $(x, z) \in \bar{D}$:

$$\bar{V}^0(x, z) := V^0\left(x, R(z)\right),$$

and $\bar{V}_*^0$ its associated lower semicontinuous hull on $\bar{D}$:

$$\bar{V}_*^0(x, z) = \liminf_{\substack{x' \to x \\ z' \to z}} \bar{V}^0(x', z').$$

We define:

$$\bar{U}(x, z) := U\left(x + R(z)\right).$$

Then, we have the following result:

Proposition 2.11. *Assume that V^0 is locally bounded, then $\bar{V}^0_*$ is a viscosity supersolution of :*

$$\min\Big\{ -\bar{v}_{zz}\ ;\ -\bar{v}_{xx}\ ;\ \bar{v}-\bar{U}\Big\}(x,z)=0 \ \textit{on}\ \bar{D} \tag{2.18}$$

Proof. Since R is a one-to-one, continuous function from D to $\bar{D}$, then

$$\begin{aligned}\bar{V}^0_*(x,z) &= \liminf_{\substack{x'\to x\\ z'\to z}} \bar{V}^0(x',z') \\ &= \liminf_{\substack{x'\to x\\ z'\to z}} V^0\big(x',R(z')\big) \\ &= V^0_*\big(x,R(z)\big).\end{aligned}$$

For any function ϕ in $C^{2,2}(\mathbb{R})$ and (x_0,y_0) in D such that $\min_D(V^0_*-\phi)=(V^0_*-\phi)(x_0,y_0)=0$, we have by Lemma 2.6 that:

$$\min\Big\{-\frac{1}{2}y_0^2\sigma^2(y_0)\phi_{yy}(x_0,y_0)-y_0\mu(y_0)\phi_y(x_0,y_0)\ ;\ -\phi_{xx}(x_0,y_0)\ ;\ \phi(x_0,y_0)-U(x_0+y_0)\Big\}\geq 0$$

Define $\bar{\phi}$ for all (x,y) in D by $\phi(x,y)=\bar{\phi}\big(x,S(y)\big)$, then from the above equation, we get:

$$\min\Big\{-\bar{\phi}_{yy}\big(x_0,S(y_0)\big)\ ;\ -\bar{\phi}_{xx}\big(x_0,S(y_0)\big)\ ;\ \bar{\phi}_{yy}\big(x_0,S(y_0)\big)-\bar{U}\big(x_0,S(y_0)\big)\Big\}\geq 0$$

Take the following change of variable $z_0=S(y_0)$, we get for all (x_0,z_0) in $\bar{D}$ that $\min_{\bar{D}}(\bar{V}^0_*-\bar{\phi})=(\bar{V}^0_*-\bar{\phi})(x_0,z_0)=0$ and

$$\min\Big\{ -\bar{\phi}_{zz}(x_0,z_0)\ ;\ -\bar{\phi}_{xx}(x_0,z_0)\ ;\ \bar{\phi}(x_0,z_0)-\bar{U}(x_0,z_0)\Big\}\geq 0.$$

So, $\bar{V}^0_*$ is a viscosity supersolution of (2.18). □

What we see in (2.18) is that we are seeking a function concave only in each variable which dominates $\bar{U}$. Such a function can be obtained as a sequence of concave envelopes : $(\bar{U}_n)_n$. Indeed, we define

$$\begin{aligned}\bar{U}_0 &= \bar{U}, \\ \bar{U}_{2n} &= (\bar{U}_{2n-1})^{conc_x}, \\ \bar{U}_{2n+1} &= (\bar{U}_{2n})^{conc_z}.\end{aligned}$$

where $\bar{U}_{2n}$ is the concave hull of $\bar{U}_{2n-1}$ in its first variable (the smallest concave function in x that dominates $\bar{U}_{2n-1}$) and $\bar{U}_{2n+1}$ the concave hull of

$\bar{U}_{2n}$ in the second variable (the smallest concave function in z that dominates $\bar{U}_{2n}$). Since $(\bar{U}_n)$ is a non decreasing sequence and is bounded from above by the concave hull of $\bar{U}$ jointly in (x, z), it admits a limit, denoted $\bar{U}^\infty$ which is bounded by the same upper bound. Therefore we can define $\bar{U}^\infty$ by:

$$\bar{U}^\infty = \lim_n \bar{U}_n \tag{2.19}$$

It yields us that $\bar{U}^\infty$ is the smallest concave function in x and concave in z which dominates $\bar{U}$, that is to say its concave hull in x and in z.

Remark 2.12. We insist on the fact that at each step of concavification we may lose the previous partial concave property. That is the reason why we have to construct the candidate function $\bar{U}^\infty$ as the limit of the previous sequence $(U_n)_n$.

Remark 2.13. It is well known that the concave envelope of any function can be given by its Legendre bi-conjugate. For a better understanding of what it is a concave hull, we recommend the paper of Peskir [72].

From now on, our aim is to prove that $\bar{V}^0_*$ is concave in x, concave in z and dominates $\bar{U}$.

Lemma 2.14. *Assume that V^0 is locally bounded. Then for all $(x, y) \in D$, we have the following inequality*

$$V(x, y) \geq V^0(x, y) \geq \bar{V}^0_*\big(x, S(y)\big) \geq \bar{U}^\infty\big(x, S(y)\big) \tag{2.20}$$

Proof. We fix $x_0 \in \mathbb{R}$, and define for all $z \in dom(S)$ such that $R(z) \geq -x_0$ the function g by

$$g(z) := \bar{V}^0_*(x_0, z)$$

By Proposition 2.4 in [89], we have that g is a viscosity supersolution of

$$\min\Big\{ - g_{zz}(z) \ ; \ g(z) - \bar{U}(x_0, z) \Big\} = 0 \text{ for } R(z) \geq -x_0 \tag{2.21}$$

Similarly, we fix $z_0 \in dom(S)$, and define for all $x \geq -R(z_0)$, the function h by $h(z) := \bar{V}^0_*(x, z_0)$. Then by Proposition 2.4 in [89], h is a viscosity supersolution of

$$\min\Big\{ - h_{xx}(x) \ ; \ h(x) - \bar{U}(x, z_0) \Big\} = 0 \text{ for } x \geq -R(z_0) \tag{2.22}$$

To conclude, thanks to (2.21) and (2.22) and Lemma 2.2 in [89], we can see that $\bar{V}^0_*$ is concave in x, concave in z and dominates $\bar{U}$, so it must be above the concave hull in x and in z of $\bar{U}$, that is to say by definition of $\bar{U}^\infty$ (see (2.19)) :

$$\bar{V}^0_*(x,z) \geq \bar{U}^\infty\big(x,z\big), \quad (x,z) \in \bar{D}.$$

Therefore by (2.9), we obtain the following inequality for all (x,y) in D :

$$V(x,y) \geq V^0(x,y) \geq \bar{V}^0_*\big(x,S(y)\big) \geq \bar{U}^\infty\big(x,S(y)\big).$$

□

Remark 2.15. We have shown that V is bigger than the concave envelope in x and in z of $\bar{U}$. Let us do a formal argument which avoid any considerations of measurability or independency and apply the Jensen's inequality to $\bar{U}^\infty$:

$$V(x,y) \leq \sup_{\substack{X \in \mathcal{M}^\perp(x,y) \\ \tau \in \mathcal{T}}} \mathbb{E}\big[\bar{U}^\infty(X_\tau, Z_\tau)\big] \leq \bar{U}^\infty\big(x,S(y)\big).$$

We can see that if this argument, from a mathematical point of view, was correct, we will have $V^0(x,y) = \bar{U}^\infty\big(x,S(y)\big)$. So our intuition leads us to prove that indeed this is true.

2.4 An Explicit Solution

In this section, we show that the continuous function $\bar{U}^\infty$ is the value function V of the problem (2.3). To do this, we prove that the original structure of iterate concavifications is directly linked to two different optimization problems. The resolution of these problems will give us the requested optimizers.

2.4.1 The value function

We have shown in (2.20) that V is greater than $\bar{U}^\infty$ if V^0 is locally bounded. One of our main result consists in showing the reverse inequality and obtain the following explicit solution to the optimization problem (2.3).

Proposition 2.16. *Under Assumption* 2.8 *and* 2.9*, for all* (x,y) *in* D*,*

$$V(x,y) = \bar{U}^\infty\big(x,S(y)\big)$$

In order to prove Proposition 2.16, we are going to use a regularization argument. Indeed since $\bar{U}^\infty$ is concave in x and z, it is continuous on the interior of $\bar{D}$. But it is not twice differentiable in each variable. Therefore, we regularize it by some classical convolution argument. Let $\bar{U}^\infty_\epsilon$, $\bar{U}_\epsilon$ and U_ϵ

be respectively defined for every $\epsilon \in]0,1]$ by:

$$\bar{U}_\epsilon^\infty(x,z) = \int_{\bar{D}} \bar{U}^\infty(\xi,\zeta)\rho_\epsilon(x-\xi,z-\zeta)d\xi d\zeta,\ (x,z)\in\bar{D}, \tag{2.23}$$

$$\bar{U}_\epsilon(x,z) = \int_{\bar{D}} \bar{U}(\xi,\zeta)\rho_\epsilon(x-\xi,z-\zeta)d\xi d\zeta,\ (x,z)\in\bar{D}, \tag{2.24}$$

$$U_\epsilon(x) = \int_0^\infty U(\xi)\rho_\epsilon(x-\xi,0)d\xi,\ x\in\mathbb{R}, \tag{2.25}$$

and where for all u in $\bar{D}$

$$\rho_\epsilon(u) = \epsilon^{-2}\rho(u/\epsilon) \text{ with } \rho(u) = \frac{1}{\int_{x\in B}\rho(x)dx}\exp(-\frac{1}{1-|u|^2})\mathbf{1}_{|u|<1} \tag{2.26}$$

where B is the unit ball of $\mathbb{R}^2$. We recall that the bump function ρ_ϵ defined in (2.26) has the following properties:

(i) $\int_{|u|<1}\rho(u)du = 1$.

(ii) ρ_ϵ is C^∞ and compactly supported.

(iii) $\lim_{\epsilon\to 0}\rho_\epsilon(u) = \delta(u)$ where δ is the Dirac function.

Lemma 2.17. *$\bar{U}_\epsilon^\infty$ and $\bar{U}_\epsilon$ defined in* (2.23) *and* (2.24) *have the following properties:*

(i) $\lim_{\epsilon\to 0}\bar{U}_\epsilon^\infty(x,z) = \bar{U}^\infty(x,z)$, *for all* $(x,z)\in\bar{D}$.

(ii) $\bar{U}_\epsilon^\infty \in C^\infty(\bar{D})$.

(iii) $\bar{U}_\epsilon^\infty$ *is concave in each variable.*

(iv) $\bar{U}_\epsilon(x,z) = U_\epsilon\big(x+R(z)\big)$, *for all* $(x,z)\in\bar{D}$.

(v) $\bar{U}_\epsilon^\infty(x,z) \geq \bar{U}_\epsilon(x,z)$, *for all* $(x,z)\in\bar{D}$.

Proof. (i) and (ii) are classical properties of the convolution product. For (iii), take $(x,x')\in\mathbb{R}^2$ and $\lambda\in[0,1)$, then since $\bar{U}^\infty$ is concave in x:

$$\begin{aligned}
\bar{U}_\epsilon^\infty(\lambda x+(1-\lambda)x',z) &= \int_{\mathbb{R}^2}\bar{U}^\infty\big(\lambda(x+\xi)+(1-\lambda)(x'+\xi),z+\zeta\big)\rho_\epsilon(\xi,\zeta)d\xi d\zeta\\
&\geq \lambda\int_{\mathbb{R}^2}\bar{U}^\infty\big(x+\xi,z+\zeta\big)\ \rho_\epsilon(\xi,\zeta)\ d\xi d\zeta\\
&+(1-\lambda)\int_{\mathbb{R}^2}\bar{U}^\infty\big(x'+\xi,z+\zeta\big)\ \rho_\epsilon(\xi,\zeta)\ d\xi d\zeta\\
&= \lambda\bar{U}_\epsilon^\infty(x,z)+(1-\lambda)\bar{U}_\epsilon^\infty(x',z)
\end{aligned}$$

The same argument shows the concavity in z.

For (iv), by definition of $\bar{U}$,

$$\begin{aligned}\bar{U}_\epsilon(x,z) &= \int_{\bar{D}} \bar{U}(\xi,\zeta)\rho_\epsilon(x-\xi,z-\zeta)d\xi d\zeta \\ &= \int_{\bar{D}} U(\xi+R(\zeta))\rho_\epsilon(x-\xi,z-\zeta)d\xi d\zeta \\ &= U_\epsilon\big(x+R(z)\big)\end{aligned}$$

For (v), we have by construction that $\bar{U}^\infty \geq \bar{U}$ and ρ_ϵ is non negative so

$$\bar{U}^\infty_\epsilon(x,z) - \bar{U}_\epsilon(x,z) = \int_{\bar{D}} \big(\bar{U}^\infty - \bar{U}\big)(\xi,\zeta)\rho_\epsilon(x-\xi,z-\zeta)d\xi d\zeta \geq 0$$

□

As mentioned previously, for a power utility function, Hobson and Henderson, in [42], have shown that we may have degenerate solutions when the total wealth is close to 0. An interesting feature of their solution is that the total optimal wealth process is always stopped before getting closed to zero. In our general setting, we denote $\partial\bar{D} := \big\{(x,z) \in \bar{D} \;:\; x+R(z) = 0\big\}$. Then, one can show:

Lemma 2.18. *For all $(x,z) \in \partial\bar{D}$, we have :*

$$\bar{U}^\infty(x,z) = \bar{U}(x,z). \tag{2.27}$$

Proof. For all (x,z) in $\partial\bar{D}$, by definition of $\bar{U}$,

$$\bar{U}(x,z) = U(0)$$

and since $\lim_{z\to 0} U'(z) = +\infty$ and R' is a positive function, then

$$\partial_z\bar{U}(x,z) = R'(z)U'(0) = +\infty, \quad (x,z) \in \partial\bar{D}.$$

So $\bar{U}(x,z)$ is concave in x and z and $\bar{U}^\infty = \bar{U}$ on $\partial\bar{D}$.

□

Proof of Proposition 2.16.

Let us show that $V(x,y) \leq \bar{U}^\infty\big(x,S(y)\big)$ for all (x,y) in D.

Step 1: Take a stopping time τ in $\mathcal{T}$ and a martingale X in $\mathcal{M}^{\perp}(x,y)$. Then by Ito's formula for jump processes (see Theorem 71 in [77]):

$$\begin{aligned}
&\bar{U}_\epsilon^\infty(X_{t\wedge\tau}, Z_{t\wedge\tau}) - \bar{U}_\epsilon^\infty(x,z) = \\
&\int_0^{t\wedge\tau} \frac{1}{2}\partial_{xx}\bar{U}_\epsilon^\infty(X_u, Z_u)d[X,X]_u^c + \frac{1}{2}\partial_{zz}\bar{U}_\epsilon^\infty(X_u, Z_u)\tilde{\sigma}(Z_u)du \\
&+ \int_0^{t\wedge\tau} \partial_z\bar{U}_\epsilon^\infty(X_u, Z_u)\tilde{\sigma}(Z_u)dB_u + \partial_x\bar{U}_\epsilon^\infty(X_u, Z_u)dX_u^c \\
&+ \sum_{0<u\le t\wedge\tau} \bar{U}_\epsilon^\infty(X_u, Z_u) - \bar{U}_\epsilon^\infty(X_{u-}, Z_u) - \partial_x\bar{U}_\epsilon^\infty(X_{u-}, Z_u)\Delta X_u,
\end{aligned}$$

where X^c corresponds to the continuous part of X. Since $\bar{U}_\epsilon^\infty$ is concave in x and z, then:

$$\bar{U}_\epsilon^\infty(X_{t\wedge\tau}, Z_{t\wedge\tau}) - \bar{U}_\epsilon^\infty(x,z) \le \int_0^{t\wedge\tau} \partial_z\bar{U}_\epsilon^\infty(X_u, Z_u)\tilde{\sigma}(Z_u)dB_u + \partial_x\bar{U}_\epsilon^\infty(X_u, Z_u)dX_u^c.$$

We introduce an increasing sequence of stopping times $(h_n)_n$ by

$$h_n = \inf\left\{t \ge 0 \ : X_{\tau\wedge t-} + Y_{\tau\wedge t}^y < \frac{1}{n}\right\}.$$

By the solvency condition,

$$h_\infty = \inf\left\{t \ge 0 \ : X_{\tau\wedge t-} + Y_{\tau\wedge t}^y = 0\right\}.$$

Thanks to the fact that U is non decreasing, then U_ϵ is non decreasing. Therefore by definition of h_n,

$$\begin{aligned}
U_\epsilon\big(X_{t\wedge\tau\wedge h_n} + Y_{t\wedge\tau\wedge h_n}^y\big) &\ge U_\epsilon\left(\frac{1}{n}\right) \\
&\ge \int_{\mathbb{R}^+} U(\xi + \frac{1}{n})\rho_\epsilon(\xi, 0)d\xi \\
&\ge U\left(\frac{1}{n}\right)\int_{\mathbb{R}^+} \rho_\epsilon(\xi, 0)d\xi. \\
&= U\left(\frac{1}{n}\right)
\end{aligned}$$

Since for fixed n, $|U\big(\frac{1}{n}\big)| < \infty$, then by (iv) and (v) in Lemma 2.17, the continuous local martingale

$$\left\{\int_0^{t\wedge\tau\wedge h_n} \partial_z\bar{U}_\epsilon^\infty(X_{u-}, Z_u)\tilde{\sigma}(Z_u)dB_u + \partial_x\bar{U}_\epsilon^\infty(X_{u-}, Z_u)dX_u^c\right\}_{t\ge0}$$

is bounded from below so it is a supermartingale. Then $\bar{U}^{\infty}_{\epsilon}(X_{t\wedge\tau\wedge h_n}, Z_{t\wedge\tau\wedge h_n})$ is a supermartingale too and

$$\mathbb{E}\big[\bar{U}^{\infty}_{\epsilon}(X_{t\wedge\tau\wedge h_n}, Z_{t\wedge\tau\wedge h_n})\big] \leq \bar{U}^{\infty}_{\epsilon}(x, z).$$

Step 2 : We denote for any function f, $(f)^- := \max(-f, 0)$ and $(f)^+ := \min(-f, 0)$.

Since for all (x, z) in $\bar{D}$, $\bar{U}^{\infty}_{\epsilon} \geq \bar{U}_{\epsilon}$ then $\big(\bar{U}^-\big)_{\epsilon} \geq \big(\bar{U}_{\epsilon}\big)^- \geq \big(\bar{U}^{\infty}_{\epsilon}\big)^-$ and

$$\sup_{\epsilon,t}\big(\bar{U}^{\infty}_{\epsilon}\big)^-(X_{t\wedge\tau\wedge h_n}, Z_{t\wedge\tau\wedge h_n}) \leq \sup_{\epsilon,t}\big(U^-\big)_{\epsilon}(X_{t\wedge\tau\wedge h_n} + Y^y_{t\wedge\tau\wedge h_n}).$$

U^- is non increasing so as in the previous step:

$$\big(U^-\big)_{\epsilon}\left(X_{t\wedge\tau\wedge h_n} + Y^y_{t\wedge\tau\wedge h_n}\right) \leq U^-\left(\frac{1}{n}\right) < \infty.$$

Then for fixed n, by Lemma 2.17, dominated convergence theorem and Fatou's lemma

$$\begin{aligned}
\mathbb{E}\big[\bar{U}^{\infty}(X_{\tau\wedge h_n}, Z_{\tau\wedge h_n})\big] &= \mathbb{E}\bigg[\lim_{\substack{\epsilon\to 0\\ t\to\infty}} \bar{U}^{\infty}_{\epsilon}(X_{t\wedge\tau\wedge h_n}, Z_{t\wedge\tau\wedge h_n})\bigg] \\
&\leq \liminf_{\substack{\epsilon\to 0\\ t\to\infty}} \mathbb{E}\bigg[\big(\bar{U}^{\infty}_{\epsilon}\big)^+(X_{t\wedge\tau\wedge h_n}, Z_{t\wedge\tau\wedge h_n})\bigg] \\
&- \lim_{\substack{\epsilon\to 0\\ t\to\infty}} \mathbb{E}\bigg[\big(\bar{U}^{\infty}_{\epsilon}\big)^-(X_{t\wedge\tau\wedge h_n}, Z_{t\wedge\tau\wedge h_n})\bigg] \\
&\leq \bar{U}^{\infty}(x, z).
\end{aligned}$$

We next notice that $\big(U^-(X_{\tau\wedge h_n} + Y^y_{\tau\wedge h_n})\big)_n$ is an increasing and positive sequence. So by monotonic convergence theorem and lemma 2.17 :

$$\begin{aligned}
\mathbb{E}\bigg[\lim_n \big(\bar{U}^{\infty}\big)^-(X_{\tau\wedge h_n}, Z_{\tau\wedge h_n})\mathbb{1}_{\{h_\infty\geq\tau\}}\bigg] &= \mathbb{E}\bigg[\lim_n \big(\bar{U}\big)^-(X_{\tau\wedge h_n} + Y^y_{\tau\wedge h_n})\mathbb{1}_{\{h_\infty\geq\tau\}}\bigg] \\
&= \lim_n \mathbb{E}\bigg[\big(\bar{U}\big)^-(X_{\tau\wedge h_n} + Y^y_{\tau\wedge h_n})\mathbb{1}_{\{h_\infty\geq\tau\}}\bigg] \\
&\geq \lim_n \mathbb{E}\bigg[\big(\bar{U}^{\infty}\big)^-(X_{\tau\wedge h_n}, Z_{\tau\wedge h_n})\mathbb{1}_{\{h_\infty\geq\tau\}}\bigg].
\end{aligned}$$

So, by Fatou's Lemma for the positive part:

$$\begin{aligned}\mathbb{E}\big[\bar{U}^\infty(X_\tau, Z_\tau)\big] &\leq \liminf_n \mathbb{E}\big[\big(\bar{U}^\infty\big)^+(X_{\tau\wedge h_n}, Z_{\tau\wedge h_n})\mathbb{1}_{\{h_\infty \geq \tau\}}\big] \\ &- \lim_n \big[\big(\bar{U}^\infty\big)^-(X_{\tau\wedge h_n}, Z_{\tau\wedge h_n})\mathbb{1}_{\{h_\infty \geq \tau\}}\big] \\ &\leq \liminf_n \mathbb{E}\big[\big(\bar{U}^\infty\big)(X_{\tau\wedge h_n}, Z_{\tau\wedge h_n})\mathbb{1}_{\{h_\infty \geq \tau\}}\big] \\ &\leq \bar{U}^\infty(x,z).\end{aligned}$$

By the arbitrariness of $X \in \mathcal{M}^\perp(x,y)$ and $\tau \in \mathcal{T}$, we conclude that :

$$V(x,y) \leq \bar{U}^\infty\big(x, S(y)\big), \quad (x,y) \in D. \tag{2.28}$$

From now on, we want to to show that V^0 is locally bounded in order to apply lemma 2.14 and get the reverse inequality. Indeed, we know that by optimality $V^0(x,y)$ defined in (2.11) is such that for all $(x,y) \in D$

$$V^0(x,y) \geq U(x+y).$$

Therefore by (2.9) and (2.28), we have :

$$\bar{U}^\infty\big(x, S(y)\big) \geq V^0(x,y) \geq U(x+y).$$

Since $\bar{U}^\infty$ and U are continuous, we get that $V0$ is locally bounded. So by lemma 2.14, we get that $V(x,y) \geq \bar{U}^\infty\big(x, S(y)\big)$ on D.

□

2.4.2 An example of portfolio optimization problem

Let us study the following portfolio optimization problem. On a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, we consider a random maturity τ, modeled by a $\mathbb{F}$-stopping time which may take infinite values, and G a real valued function which can be non concave. We want to solve the following portfolio optimization problem:

$$j(x) = \sup_{X \in \mathcal{M}(x)} \mathbb{E}\big[G(X_\tau)\big]. \tag{2.29}$$

where $x \in \mathbb{R}$ and $\mathcal{M}(x)$ is the set of all martingales with initial value x, adapted to $\mathbb{F}$.

We can see that taking the rule which consists in doing nothing leads to :

$$j(x) \geq G(x). \tag{2.30}$$

We denote by G^{conc} the concave envelope of G. By definition, $G^{conc} \geq G(x)$, so by Jensen inequality:

$$G^{conc}(x) \geq j(x). \tag{2.31}$$

Clearly if G is concave on $\mathbb{R}$, then $G^{conc} = G$. So combining (2.30) and (2.31) gives us that $j(x) = G(x)$ and the optimal strategy X^* is $X^* = x$.

We now consider the case where G may be not concave. Let us assume that:

Assumption 2.19. *There is a compact subset $C \subset \mathbb{R}$ such that*

$$\forall x \notin C,\ G^{conc}(x) = G(x).$$

Under this assumption, we can define for all $x \in \mathbb{R}$, $\big(a(x), b(x)\big) \in \mathbb{R}^2$ by:

$$\begin{aligned} a(x) &:= \inf\big\{\alpha,\ \alpha \geq x : G^{conc}(\alpha) = G(\alpha)\big\} \\ b(x) &:= \sup\big\{\alpha,\ \alpha \leq x : G^{conc}(\alpha) = G(\alpha)\big\} \end{aligned}$$

Roughly speaking $a(x)$ is the first point after x where G "touches" its concave envelope, that is to say $G^{conc}\big(a(x)\big) = G\big(a(x)\big)$ and $b(x)$ is the last point before x where G is equal to its concave hull ($G^{conc}\big(b(x)\big) = G\big(b(x)\big)$). Assumption 2.19 means that we get rid off any problems which may occur at infinity. For instance, if G has a convex behavior at infinity then $a(x)$ would be infinite. But this kind of problems can not happen under Assumption 2.19 since we are working on a bounded domain. Moreover G^{conc} is linear on $[b(x), a(x)]$.
Now we define the pure jump process X^* by $X_t^* = x$ for $t \in [0, \tau)$ and:

$$X_\tau^* = \begin{cases} a(x), & \text{with probability } p(x). \\ b(x), & \text{with probability } 1 - p(x). \end{cases}$$

where $p(x)$ is defined by $x = p(x)a(x) + \big(1 - p(x)\big)b(x)$. We notice that in particular, $\mathbb{E}[X_\tau^*] = x$. Then by definition of $a(x)$, $b(x)$ and $p(x)$, we can easily calculate that:

$$\begin{aligned} \mathbb{E}\big[G(X_\tau^*)\big] &= \mathbb{E}\big[G\big(a(x)\big)\mathbb{1}_{X_\tau^* = a(x)} + G\big(b(x)\big)\mathbb{1}_{X_\tau^* = b(x)}\big] \\ &= G^{conc}\big(a(x)\big)p(x) + G^{conc}\big(b(x)\big)\big(1 - p(x)\big). \end{aligned}$$

Under Assumption 2.19, the concave envelope is linear on $[b(x), a(x)]$. Thereby we get that:

$$\begin{aligned} \mathbb{E}\big[G(X_\tau^*)\big] &= G^{conc}\big(a(x)\big)p(x) + G^{conc}\big(b(x)\big)\big(1 - p(x)\big) \\ &= G^{conc}(x). \end{aligned}$$

Using (2.31), we get that $j(x) = G^{conc}(x)$ where the optimal strategy is the martingale X^* constructed above.

2.4.3 Construction of the optimal strategies

We make in this section the following assumption :

Assumption 2.20.

$\exists K$ *compact subset of the interior of* $\bar{D}$: $\forall (x,z) \notin K \ \ \bar{U}^{\infty}(x,z) = \bar{U}(x,z)$.

This is the analogous of Assumption 2.19 for the two dimensional problem. This ensures that the rules we are looking for are finite as we will see in the construction below. Moreover, this assumption is consistent with (2.18).

Let us define the sequence of stopping times $(\tau^n)_{n\geq 0}$ by

$$\tau_1^0 = \inf\{t \geq 0 : \bar{U}^1(x, Z_t) = \bar{U}^0(x, Z_t)\}.$$

and for $i \in \{1 \dots n+1\}$

$$\tau_1^n = \inf\{t \geq 0 : \bar{U}^{2n+1}(x, Z_t) = \bar{U}^{2n}(x, Z_t)\}.$$

$$\vdots$$

$$\tau_i^n = \inf\{t \geq \tau_{i-1}^n : \bar{U}^{2(n-i+1)+1}(X^n_{\tau^n_{i-1}}, Z_t) = \bar{U}^{2(n-i+1)}(X^n_{\tau^n_{i-1}}, Z_t)\}.$$

$$\vdots$$

$$\tau_{n+1}^n = \inf\{t \geq \tau_{n-1}^n : \bar{U}^1(X^n_{\tau^n_n}, Z_t) = \bar{U}^0(X^n_{\tau^n_n}, Z_t)\}.$$

Remark 2.21. For each integer n and i, we introduce the set

$$H_i^n := \left\{ z > 0 \ : \ \bar{U}^{2(n-i+1)+1}\big(X^n_{\tau^n_{i-1}}, z\big) > \bar{U}^{2(n-i+1)}\big(X^n_{\tau^n_{i-1}}, z\big) \right\}.$$

By construction of the sequence $\bar{U}^n$, we can see that outside of H_i^n, the function $\bar{U}^{2(n-i+1)}\big(X^n_{\tau^n_{i-1}}, \ .\ \big)$ is concave. With this framework τ_i^n could be understood as the first exit time of H_i^n for the process Z. Roughly speaking, we stop when $\bar{U}^{2(n-i+1)}\big(X^n_{\tau^n_{i-1}}, \ .\ \big)$ reaches its concave envelope.

Remark 2.22. By the appendix D of [49] or in the Markovian case in [33], these sequences of stopping times, for a given wealth process X, are the optimal rules of a family of optimal stopping problems.

The construction of the optimal strategy is the following and is based on the construction done in the previous section for the one dimensional case.

Indeed for given $(u,v) \in \mathbb{R} \times \mathbb{R}^+$, let us introduce for all $i \in \{1,\dots,n\}$

$$d(v) := \{x \in \mathbb{R}/(x,v) \in \bar{D}\}$$
$$a_i^n(u,v) := \inf\{\alpha \in d(v),\ \alpha \geq u : \bar{U}^{2(n-i+1)}(\alpha,v) = \bar{U}^{2(n-i+1)-1}(\alpha,v)\}.$$
$$b_i^n(u,v) := \sup\{\alpha \in d(v),\ \alpha \leq u : \bar{U}^{2(n-i+1)}(\alpha,v) = \bar{U}^{2(n-i+1)-1}(\alpha,v)\}.$$

Thereby, if $\bar{U}^{2(n-i+1)-1}(\ .\ ,\ v)$ is concave at point u, then $u = a_i^n(u,v) = b_i^n(u,v)$. Moreover, we know that $\bar{U}^{2(n-i+1)}(\ .\ ,\ v)$ is linear on $[b_i^n(u,v), a_i^n(u,v)]$.

We define the random variable $\eta_i^n(u,v)$ and the measurable function $p_i^n(u,v)$ by:

(i) If $a_i^n(u,v) = b_i^n(u,v) = u$, then $\eta_i^n(u,v) = u$ and $p_i^n(u,v) = 1$.

(ii) If $a_i^n(u,v) > b_i^n(u,v)$, then:

$$\begin{cases} p_i^n(u,v) := \mathbb{P}\big[\eta_i^n(u,v) = a_i^n(u,v)\big] = 1 - \mathbb{P}\big[\eta_i^n(u,v) = b_i^n(u,v)\big]. \\ u = p_i^n(u,v)a_i^n(u,v) + \big(1 - p_i^n(u,v)\big)b_i^n(u,v). \end{cases}$$

Under Assumption 2.20, this is correctly defined since by construction for all $i \in \{1 \dots n\}$: $|\eta_i^n(u,v)| < \infty$ and $p_i^n(u,v) = \frac{u - b_i^n(u,v)}{a_i^n(u,v) - b_i^n(u,v)} \in [0,1]$.

We define the pure jump process X^n as follows :

$$X_t^0 = x \quad \forall t > 0.$$
$$X_t^n = x\mathbb{1}_{t \in [0,\tau_1^n)} + \sum_{i=1}^{n-1} \eta_i^n(X^n_{\tau^n_{i-1}}, Z_{\tau_i^n})\mathbb{1}_{t \in [\tau_i^n, \tau_{i+1}^n)} + \eta_n^n(X^n_{\tau^n_{n-1}}, Z_{\tau_n^n})\mathbb{1}_{t \in [\tau_n^n, \infty)}.$$

Remark 2.23. We have chosen at the beginning an unspecified filtration which contains at least the natural filtration of the Brownian motion B and the filtration generated by the sequence (τ_i^n). We can see clearly that the process X_t^n is $\mathcal{F}_t$-adapted. At time τ_1^n, since x is a constant and $Z_{\tau_1^n}$ is $\mathcal{F}_{\tau_1^n}$-adapted, we have that $X^n_{\tau_1^n}$ is $\mathcal{F}_{\tau_1^n}$-adapted. So by induction, each jump occurring at τ_i^n depends on the values of the process $X^n_{\tau^n_{i-1}}$ which is $\mathcal{F}_{\tau^n_{i-1}}$-adapted and $Y_{\tau_i^n}$ which is $\mathcal{F}_{\tau_i^n}$-adapted. So, $X^n_{\tau_i^n}$ is $\mathcal{F}_{\tau_i^n}$-adapted.

Lemma 2.24. *Under Assumptions* 2.9 *and* 2.20*, for all* $n \in \mathbb{N}$*, the process* X^n *is a pure jump martingale.*

Proof. If $p_1^n(x, Z_{\tau_1^n}) = 0$ then $b_1^n(x, Z_{\tau_1^n}) = x$ and $\mathbb{E}[|X^n_{\tau_1^n}|] < \infty$.
If $p_1^n(x, Z_{\tau_1^n}) \in]0,1]$, then

$$|a_1^n(x, Z_{\tau_1^n})| \leq \frac{1}{p_1^n(x, Z_{\tau_1^n})}\big[|x| + (1 - p_1^n(x, Z_{\tau_1^n}))|b_1^n(x, Z_{\tau_1^n})|\big].$$

Moreover by construction, $|b_1^n(x, Z_{\tau_1^n})| \leq |\max(Z_{\tau_1^n}, x)|$ and thanks to Assumption 2.9, we have that

$$\mathbb{E}[|b_1^n(x, Z_{\tau_1^n})| + |a_1^n(x, Z_{\tau_1^n})|] < \infty.$$

So, we get that $\mathbb{E}[|X^n_{\tau_1^n}|] < \infty$.

Thanks to Assumption 2.20, by an easy induction, we can see that for $i \in \{1, \ldots, n\}$:

$$\mathbb{E}[|X^n_{\tau_i^n}|] < \infty. \tag{2.32}$$

Moreover, for all $i \in \{1, \ldots, n\}$:

– Take $t \in]\tau_i^n, \tau_{i+1}^n[$, then $\mathbb{E}[X_t^n|\mathcal{F}_{t-}] = X_{t-}^n$.

– Take $t = \tau_i^n$, then

$$\begin{aligned}
\mathbb{E}[X_t^n|\mathcal{F}_{t-}] &= \mathbb{E}[\eta_i^n(X^n_{\tau_{i-1}^n}, Z_{\tau_i^n})|\mathcal{F}_{t-}] \\
&= a_i^n(X^n_{\tau_{i-1}^n}, Z_{\tau_i^n})\mathbb{E}[\mathbb{1}_{\eta_i^n = a_i^n}|\mathcal{F}_{t-}] \\
&+ b_i^n(X^n_{\tau_{i-1}^n}, Z_{\tau_i^n})\mathbb{E}[1 - \mathbb{1}_{\eta_i^n = a_i^n}|\mathcal{F}_{t-}] \\
&= X^n_{\tau_{i-1}^n} \\
&= X^n_{t-}.
\end{aligned}$$

Since Z is a continuous process, the quadratic covariation between Z and the pure jump process X^n is such that $[X^n, Z] = 0$. □

The intuition behind this construction is that our mix-investment sell problem can be splitted into two optimization problems. At each stopping time τ_i^n, we have to solve an optimal stopping problem in infinite horizon with constant wealth $X^n_{\tau_{i-1}^n}$, so the value function is concave in z and its second derivative in x is equal to 0. Similarly, each concavification in x corresponds to an optimal investment problem with given random maturity and non concave utility function. The wealth strategy jumps randomly from its actual position to two possible values, while the asset price remains constant. This construction is consistant with the partial differential equation (2.18) obtained in the continuous case.

2.4.4 The ε-Optimal Strategies

The following results are crucial to solve our main problem. The first one concerns the optimization of the optimal portfolio, that is to say the construction of the optimal strategy. The second one is related to the optimal stopping theory.

Lemma 2.25. *Under Assumptions* 2.9 *and* 2.20*, for all* $i \in \{1 \dots n+1\}$*,*

$$\mathbb{E}\Big[\bar{U}^{2(n-i+1)-1}(X^n_{\tau^n_i}, Z_{\tau^n_i})\Big] = \mathbb{E}\Big[\bar{U}^{2(n-i+1)}\big(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}\big)\Big] \tag{2.33}$$

$$= \mathbb{E}\Big[\bar{U}^{2(n-i+1)+1}(X^n_{\tau^n_{i-1}}, Z_{\tau^n_{i-1}})\Big]. \tag{2.34}$$

Proof. We first prove (2.33). By definition of $X^n_{\tau^n_i}$ together with the tower property of conditional expectation, we have:

$$\begin{aligned}\mathbb{E}\Big[\bar{U}^{2(n-i+1)-1}(X^n_{\tau^n_i}, Z_{\tau^n_i})\Big] &= \mathbb{E}\Big[\bar{U}^{2(n-i+1)-1}\big(a^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}), Z_{\tau^n_i}\big)\mathbb{E}\big[\mathbb{1}_{\eta^n_i = a^n_i} | X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}\big] \\ &\quad + \bar{U}^{2(n-i+1)-1}\big(b^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}), Z_{\tau^n_i}\big)\mathbb{E}\big[\mathbb{1}_{\eta^n_i = b^n_i} | X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}\big]\Big] \\ &= \mathbb{E}\Big[\bar{U}^{2(n-i+1)-1}\big(a^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}), Z_{\tau^n_i}\big)p^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}) \\ &\quad + \bar{U}^{2(n-i+1)-1}\big(b^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}), Z_{\tau^n_i}\big)\big(1 - p^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i})\big)\Big].\end{aligned}$$

By definition of the random variables $b^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i})$ and $a^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i})$,

$$\begin{aligned}\mathbb{E}\Big[\bar{U}^{2(n-i+1)-1}(X^n_{\tau^n_i}, Z_{\tau^n_i})\Big] &= \mathbb{E}\Big[\bar{U}^{2(n-i+1)}\big(a^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}), Z_{\tau^n_i}\big)p^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}) \\ &\quad + \bar{U}^{2(n-i+1)}\big(b^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}), Z_{\tau^n_i}\big)\big(1 - p^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i})\big)\Big].\end{aligned}$$

By linearity of $\bar{U}^{2(n-i+1)}(\,.\, , Z_{\tau^n_i})$ on $\big[b^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}), a^n_i(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i})\big]$ (Lemma 2.18), we conclude that

$$\mathbb{E}\Big[\bar{U}^{2(n-i+1)-1}(X^n_{\tau^n_i}, Z_{\tau^n_i})\Big] = \mathbb{E}\Big[\bar{U}^{2(n-i+1)}\big(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}\big)\Big].$$

This ends the proof of (2.33).

Let us now prove (2.34). By construction, $X^n_{\tau^n_{i-1}}$ is $\mathcal{F}_{\tau^n_{i-1}}$-measurable and $\bar{U}^{2(n-i+1)+1}\big(X^n_{\tau^n_{i-1}}, z\big)$ is linear for z in H^n_i (see remark 2.21). By Doob's optional sampling theorem applied to the continuous local martingale Z, we get for $t \geq s \geq \tau_{i-1}^n$ that:

$$\mathbb{E}\Big[\bar{U}^{2(n-i+1)+1}\big(X^n_{\tau^n_{i-1}}, Z_{t\wedge\tau^n_i}\big)|\mathcal{F}_{s\wedge\tau^n_{i-1}}\Big] = \bar{U}^{2(n-i+1)+1}\Bigg(X^n_{\tau^n_{i-1}}, \mathbb{E}\big[Z_{t\wedge\tau^n_i}|\mathcal{F}_{s\wedge\tau^n_{i-1}}\big]\Bigg)$$
$$= \bar{U}^{2(n-i+1)+1}\Bigg(X^n_{\tau^n_{i-1}}, Z_{s\wedge\tau^n_{i-1}}\Bigg).$$

and

$$\mathbb{E}\big[\bar{U}^{2(n-i+1)+1}\big(X^n_{\tau^n_{i-1}}, Z_{t\wedge\tau^n_i}\big)\big] = \mathbb{E}\big[\bar{U}^{2(n-i+1)+1}\big(X^n_{\tau^n_{i-1}}, Z_{s\wedge\tau^n_{i-1}}\big)\big]. \tag{2.35}$$

We now need a dominated convergence argument:

$$\lim_{t\to\infty} \bar{U}^{2(n-i+1)+1}\big(X^n_{\tau^n_{i-1}}, Z_{t\wedge\tau^n_i}\big) = \bar{U}^{2(n-i+1)+1}\big(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}\big)$$
$$|\bar{U}^{2(n-i+1)+1}\big(X^n_{\tau^n_{i-1}}, Z_{t\wedge\tau^n_i}\big)| \leq \sup_{t\geq\tau^n_{i-1}} |\bar{U}^{\infty}\big(X^n_{\tau^n_{i-1}}, Z_{t\wedge\tau^n_i}\big)|.$$

By definition of τ_i^n, for all $t \geq \tau_{i-1}^n$, we have that :

$$\bar{U}^{2(n-i+1)+1}\big(X^n_{\tau^n_{i-1}}, Z_{t\wedge\tau^n_i}\big) > \bar{U}\big(X^n_{\tau^n_{i-1}}, Z_{t\wedge\tau^n_i}\big)$$

then by Assumption 2.20, $\big(X^n_{\tau^n_{i-1}}, Z_{t\wedge\tau^n_i}\big) \in K$ for all $t \geq \tau^n_{i-1}$. We recall that K is a compact set of $\mathbb{R}^2$. Since $\bar{U}^\infty$ is a continuous function, then it is bounded on K. Therefore we can find a constant $C > 0$ such that $\forall t \geq \tau^n_{i-1} \ : \ |\bar{U}^\infty\big(X^n_{\tau^n_{i-1}}, Z_{t\wedge\tau^n_i}\big)| < C$ and

$$\mathbb{E}\big[\sup_{t\geq\tau^n_{i-1}} |\bar{U}^{\infty}\big(X^n_{\tau^n_{i-1}}, Z_{t\wedge\tau^n_i}\big)|\big] < \infty.$$

The same result can be obtained with the right side of equation (2.35). So by the dominated convergence theorem,

$$\mathbb{E}\Big[\bar{U}^{2(n-i+1)+1}\big(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}\big)\Big] = \mathbb{E}\Bigg[\bar{U}^{2(n-i+1)+1}\Bigg(X^n_{\tau^n_{i-1}}, Z_{\tau^n_{i-1}}\Bigg)\Bigg].$$

and by definition of τ_i^n:

$$\mathbb{E}\Big[\bar{U}^{2(n-i+1)+1}\big(X^n_{\tau^n_{i-1}}, Z_{\tau^n_{i-1}}\big)\Big] = \mathbb{E}\Big[\bar{U}^{2(n-i+1)+1}\big(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}\big)\Big]$$
$$= \mathbb{E}\Big[\bar{U}^{2(n-i+1)}\big(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}\big)\Big].$$

□

Our main result is the following theorem.

Theorem 2.26. *Under Assumptions* 2.9 *and* 2.20, *for all* (x, y) *in* D *and for all* $\varepsilon > 0$, *there exists* $n \in \mathbb{N}$ *such that*

$$\varepsilon + \mathbb{E}\Big[\bar{U}^0\big(X^n_{\tau^n_{n+1}}, Z_{\tau^n_{n+1}}\big)\Big] \geq \bar{U}^\infty\big(x, S(y)\big), \tag{2.36}$$

where $(X^n, \tau^n_{n+1}) \in \mathcal{M}^\perp(x, y) \times \mathcal{T}$ *are* ε*-optimal strategies. Moreover,*

$$\bar{U}^\infty\big(x, S(y)\big) = \lim_{n\to\infty} \mathbb{E}\Big[\bar{U}^0\big(X^n_{\tau^n_{n+1}}, Z_{\tau^n_{n+1}}\big)\Big]. \tag{2.37}$$

Suppose that μ *is non positive on* $\mathbb{R}^+_*$, *then*

$$V(x, y) = U\big(x + y\big) \tag{2.38}$$

Proof. Let us take $(x, z) \in \bar{D}$, then we always have:

$$\begin{aligned}
\bar{U}^\infty(x, z) = {} & \bar{U}^\infty(x, z) - \bar{U}^{2n+1}(x, z) \\
& + \sum_{i=1}^{n} \mathbb{E}\Big[\bar{U}^{2(n-i+1)}(X^n_{\tau^n_{i-1}}, Z_{\tau^n_i}) - \bar{U}^{2(n-i+1)-1}(X^n_{\tau^n_i}, Z_{\tau^n_i})\Big] \\
& + \sum_{i=0}^{n} \mathbb{E}\Big[\bar{U}^{2(n-i+1)-1}(X^n_{\tau^n_i}, Z_{\tau^n_i}) - \bar{U}^{2(n-i+1)-2}(X^n_{\tau^n_i}, Z_{\tau^n_{i+1}})\Big] \\
& + \mathbb{E}\Big[\bar{U}^0(X^n_{\tau^n_n}, Z_{\tau^n_{n+1}})\Big].
\end{aligned}$$

By construction $\tau^n_{n+1} \geq \tau^n_n$, so we have that $X^n_{\tau^n_{n+1}} = X^n_{\tau^n_n}$. Thanks to Lemma 2.25, this yields

$$\bar{U}^\infty(x, z) = \bar{U}^\infty(x, z) - \bar{U}^{2n+1}(x, z) + \mathbb{E}\Big[\bar{U}^0(X^n_{\tau^n_{n+1}}, Z_{\tau^n_{n+1}})\Big].$$

By definition of $\bar{U}^\infty$, we have for all $(x, y) \in D$

$$\bar{U}^\infty\big(x, S(y)\big) = \lim_{n\to\infty} \mathbb{E}\Big[\bar{U}^0(X^n_{\tau^n_{n+1}}, Z_{\tau^n_{n+1}})\Big] = V(x, y).$$

For all $\epsilon > 0$ we can find $n > 0$ such that $\bar{U}^\infty - \bar{U}^{2n+1} \leq \varepsilon$ since $\bar{U}^\infty = \lim_n \bar{U}^n$. This leads to:

$$\varepsilon + \mathbb{E}\Big[\bar{U}^0(X^n_{\tau^n_{n+1}}, Z_{\tau^n_{n+1}})\Big] \geq \bar{U}^\infty\big(x, S(y)\big) = V(x, y).$$

An easy case that we have to take into account appears when the utility function $\bar{U}$ is concave in both variables. This situation is directly linked with the dynamics of the indivisible asset. Indeed suppose that the drift μ is non positive. We define $\bar{Y}^y$ by

$$\bar{Y}_t^y = y + \int_0^t \bar{Y}_u^y \sigma\big(\bar{Y}_u^y\big) dB_u.$$

Then by a comparison theorem, we have that:

$$\mathbb{P}\big[\bar{Y}_t^y \geq Y_t^y \ , \ \forall t \geq 0\big] = 1.$$

So we get:

$$\begin{aligned} V(x,y) &= \sup_{\substack{X \in \mathcal{M}^{\perp}(x,y) \\ \tau \in \mathcal{T}}} \mathbb{E}[U(X_\tau + Y_\tau^y)] \\ &\leq \sup_{\substack{X \in \mathcal{M}^{\perp}(x,y) \\ \tau \in \mathcal{T}}} \mathbb{E}[U(X_\tau + \bar{Y}_\tau^y)]. \end{aligned}$$

By Jensen's inequality,

$$\begin{aligned} V(x,y) &\leq \sup_{\substack{X \in \mathcal{M}^{\perp}(x,y) \\ \tau \in \mathcal{T}}} \mathbb{E}[U(X_\tau + \bar{Y}_\tau^y)]. \\ &\leq U(x+y). \end{aligned}$$

Since by optimality, V is always greater than U, we have for all (x, y) in D that $V(x, y) = U(x + y)$.

□

Remark 2.27. As in Henderson and Hobson ([42]), we can see that sometimes we can not take advantage of the real asset and the best strategy consists in doing nothing and sells the real asset at time 0. This is the case for instance when μ is negative and $\bar{U}^\infty = U$. One can decide to keep the wealth of its portfolio constant and wait for an optimal time to sell the real asset. In this case, $\bar{U}^\infty = U^1$. However, most of the time, we have to do a dynamic management of the wealth portfolio to increase the total wealth as it is shown in Theorem 2.26 and take advantage to the sell of the real indivisible asset.

Unfortunately, we were not able to obtain any convergence property for the sequence of stopping times $(\tau_i^n)_n$. The following result is a direct consequence of Theorem 2.26 and turns out to cover the power utility case of Hobson and Henderson as we will see in the next section. So an alternative way is to assume the following condition:

Assumption 2.28.

$$\exists N > 0 \textit{ such that } \forall n \geq N : \ \bar{U}^{\infty} = \bar{U}^{n}$$

In this case, we have the following Corollary of Theorem 2.26.

Corollary 2.29. *Under the Assumptions* 2.9, 2.20 *and* 2.28, *for all* (x, y) *in* D,

$$V(x,y) = \bar{U}^{N}\big(x, S(y)\big) = \mathbb{E}\Big[U\big(X^{N}_{\tau^{N}_{N+1}} + Y^{y}_{\tau^{N}_{N+1}}\big)\Big].$$

with the optimal stopping rules $(X^{N}, \tau^{N}_{N+1}) \in \mathcal{M}^{\perp}(x,y) \times \mathcal{T}$

Proof. The proof is trivial since we assume that $\bar{U}^{\infty}$ is attained in a finite number of iterations (Assumption 2.28).

□

2.5 An explicit solution with a power utility function

Let us consider a concave power utility function, that is to say, U is defined for all positive x by

$$U(x) = \frac{x^{1-p} - 1}{1-p} \quad \text{with} \quad p \geq 0, \ p \neq 1.$$

Suppose that Y is a geometric Brownian motion, such that:

$$dY_t = Y_t\big(\mu dt + \sigma dW_t\big).$$

Our goal in this section is to calculate explicitly the function $\bar{U}^{\infty}$ for the function U as it is defined previously. To achieve it, let us introduce the following notation:

$$\gamma = \frac{2\mu}{\sigma^2}.$$

The scale function S of Y is given from affine transformation by

$$S(y) = y^{1-\gamma}. \tag{2.39}$$

Let γ_* be the unique solution for $0 < \gamma \leq p \wedge 1$ of $\Psi(\gamma) = 0$ where

$$\Psi(\gamma) := (p-\gamma)^{p}(1+p-\gamma) - (1-\gamma)(2p-\gamma)^{p}. \tag{2.40}$$

We introduce the following positive parameters ξ_1 and ξ_0 defined by

$$\xi_1 = \left(\frac{p+1-\gamma}{1-\gamma}\right)^{\frac{1}{p}} \frac{\xi_0}{1+\xi_0} \tag{2.41}$$

$$\xi_0 = \frac{\xi_1}{\left(\frac{p+1-\gamma}{1-\gamma}\right)^{\frac{1}{p}} - \xi_1}. \tag{2.42}$$

Proposition 2.30. *(i) If $1 > p \geq 0$ and for (x, y) in D:*
For $\gamma \leq 0$, $V(x, y) = U(x + y)$
For $0 < \gamma \leq \gamma_*$, $V(x, y) = \bar{U}^1(x, y^{1-\gamma})$.
For $\gamma_* < \gamma \leq p$, $V(x, y) = \bar{U}^2(x, y^{1-\gamma})$.
For $p < \gamma \leq 1$, $V(x, y) = \infty$.

(ii) If $\gamma > p > 1$ and (x, y) in D , $V(x, y) = \bar{U}^1(x, y^{1-\gamma})$.

(iii) If $p > 1 > \gamma$ and (x, y) in D :
For $0 < \gamma \leq \gamma_*$, $V(x, y) = \bar{U}^1(x, y^{1-\gamma})$.
For $\gamma_* < \gamma \leq p$, $V(x, y) = \bar{U}^2(x, y^{1-\gamma})$.

Remark 2.31. These results agree with those obtained by Hobson and Henderson in [42].

Remark 2.32. In this case, the process Z obtained by applying the scale function is a martingale since it verifies

$$dZ_t = (1-\gamma)\sigma Z_t dB_t.$$

and $\int^{\infty} \frac{1}{z} = \infty$. So (2.9) holds.

Proof. **Step 1 : The easy cases**

When $\gamma \leq 0$, the first assertion is easily obtained after differentiation. Indeed:

$$\partial_{zz}\bar{U}(x, z) = \frac{z^{\frac{2\gamma-1}{1-\gamma}}}{(1-\gamma)^2}(x + z^{\frac{1}{1-\gamma}})^{-p-1}\left[\gamma x + (\gamma - p)z^{\frac{1}{1-\gamma}}\right]. \tag{2.43}$$

Then for all $(x, z) \in \bar{D}$, we get :

$$\partial_{zz}\bar{U}(x, z) \leq -p\frac{z^{\frac{\gamma}{1-\gamma}}}{(1-\gamma)^2}(x + z^{\frac{1}{1-\gamma}})^{-p-1} \leq 0.$$

So $\bar{U}^{\infty}(x, z) = \bar{U}(x, z)$ and $V(x, y) = U(x + y)$.

When $1 \geq \gamma > p$:

$$\partial_{zz}\bar{U}(x,z) \underset{\infty}{\sim} \frac{\gamma - p}{(1-\gamma)^2} z^{\frac{2\gamma-1}{1-\gamma}}(x + z^{\frac{1}{1-\gamma}})^{-p-1}. \tag{2.44}$$

(2.44) implies that $\partial_{zz}\bar{U}(x,\infty) > 0$. Since $\bar{U}$ is non decreasing in y, then $\bar{U}$ is strictly convex in z on the whole interval $\bar{D}$. Its concave hull is thus infinite.

Step 2 : Construction of $\bar{U}^1$

– Case where $1 > p \geq \gamma > 0$:

If $x \leq 0$, then by (2.43), $\bar{U}(x,\ .)$ is concave. So $V = \bar{U}$.
If $x > 0$ then by (2.44), we get that $\partial_{zz}\bar{U}(x,\infty) < 0$, so $\bar{U}(x,\ .)$ admits an inflexion point and its concave hull $\bar{U}^1(x,\ .)$ will be linear until this point. For a given x, we are from now on looking for a point z_0 such that :

$$\frac{x^{1-p}}{1-p} + \partial_z\bar{U}(x,z_0)z_0 = \bar{U}(x,z_0). \tag{2.45}$$

and a line which joins the point $\big(0,\bar{U}(x,0)\big)$ to $\big(z_0,\bar{U}(x,z_0)\big)$ with a slope equal to $\partial_z\bar{U}(x,z_0)$. This can be summarized in the following problem :

We denote ξ as : $\xi = \frac{z_0^{\frac{1}{1-\gamma}}}{x}$ and we define the function Θ by

$$\Theta(\xi) := 1 + \frac{(1+\xi)^{-p}}{1-\gamma}\big[(\gamma-p)\xi - (1-\gamma)\big]$$

Then looking for a point y_0 remains to solve the following equation:

$$\Theta(\xi) = 0. \tag{2.46}$$

After calculus, we can see that for $p > \gamma > 0$, there is a unique solution ξ_0 to (2.46) on $]0,\infty]$ such that :

$$\xi_0 > \frac{\gamma}{p-\gamma}. \tag{2.47}$$

If $\gamma = p$, the unique solution of (2.46) is 0 and $\partial_y\bar{U}(x,0) = \frac{1}{1-p}$ gives us that $\bar{U}^1(x,z) = \frac{x^{1-p}}{1-p} + \frac{z}{1-p}$.

So for $1 > p \geq \gamma > 0$, we can summarize the construction of $\bar{U}^1$ by:

For all $(x,z) \in \bar{D}$

(i) Suppose $\gamma = p$. If $x > 0$,

$$\bar{U}^1(x,z) = \frac{x^{1-p}}{1-p} + \frac{z}{1-p}.$$

Then if $x \leq 0$,

$$\bar{U}^1(x,z) = \bar{U}(x,z).$$

(ii) Suppose $p > \gamma > 0$. If $z^{\frac{1}{1-\gamma}} < \xi_0 x$

$$\bar{U}^1(x,z) = \frac{x^{1-p}}{1-p} + \frac{\xi_0^\gamma}{1-\gamma} x^{\gamma-p}(1+\xi_0)^{-p} z.$$

then if $z^{\frac{1}{1-\gamma}} \geq \xi_0 x$

$$\bar{U}^1(x,z) = \bar{U}(x,z).$$

– Case where $\gamma > p > 1$:

We remind that

$$\partial_{zz}\bar{U}(x,z) = \frac{z^{\frac{2\gamma-1}{1-\gamma}}}{(1-\gamma)^2}(x + z^{\frac{1}{1-\gamma}})^{-p-1}\Big[\gamma x + (\gamma - p) z^{\frac{1}{1-\gamma}}\Big].$$

Then for $x \geq 0$, $\partial_{zz}\bar{U}(x,z) \geq 0$. Moreover $\bar{U}(x,z) \leq \bar{U}(x,0) = 0$. So for all (x,z) in $\bar{D}$, $\bar{U}^1(x,z) = 0$.

Let us now consider the case where $x < 0$. Then, we have
$\partial_{zz}\bar{U}(x,z) \underset{z\to -|x|^{1-\gamma}}{\sim} p|x|^{1-\gamma} > 0$.
and $\partial_{zz}\bar{U}(x,z) \underset{\substack{z\to\infty \\ x\to 0 \\ x<0}}{\sim} \gamma x < 0$
In this case, there is an inflexion point. So, as it has been done previously we are looking for a point z_* such that the concave hull $\bar{U}^1(x,.)$ is linear on $\big[|x|^{1-\gamma}, z_*\big]$ with a slope equal to $\bar{U}_z\big(x, z_*\big)$. So, we are seeking a solution to this equation:

$$\bar{U}(x,0) + z_*\bar{U}_z\big(x,z_*\big) = \bar{U}(x,z_*). \tag{2.48}$$

We denote $\xi = \frac{z^{\frac{1}{1\gamma}}}{|x|}$ and (2.48) becomes :

$$\frac{\xi}{1-\gamma} = \frac{(1+\xi)}{1-p}. \tag{2.49}$$

then $\xi_* = \frac{\gamma-1}{\gamma-p}$ is the unique solution of (2.49).

So for $\gamma > p > 1$ and $(x,z) \in \bar{D}$, we have that :

If $x \geq 0$, $\bar{U}^1(x,z) = 0$.
If $x < 0$, then:

(i) For $|x| < z^{\frac{1}{1-\gamma}} < \frac{\gamma-1}{\gamma-p}|x|$:

$$\bar{U}^1(x,z) = \bar{U}(x,z).$$

(ii) For $z^{\frac{1}{1-\gamma}} \geq \frac{1-\gamma}{\gamma-p}|x|$:

$$\bar{U}^1(x,z) = -\frac{(\gamma-1)^{\gamma-1}(\gamma-p)^{p-\gamma}}{(p-1)^p}|x|^{\gamma-p}z.$$

We notice that for $\gamma > p > 1$, the function $\bar{U}^1(x,z)$ is concave in x and concave in z. So $\bar{U}^\infty(x,z) = \bar{U}^1(x,z)$.

Step 3 : Construction of $\bar{U}^2$

Consider from now on the second derivative in x of $\bar{U}^1$ for $p \geq \gamma > 0$ and $y^{\frac{1}{1-\gamma}} < \xi_0 x$.

$$\partial_{xx}\bar{U}^1(x,y) = -px^{-p-1}\left[1 - \frac{(p+1-\gamma)(p-\gamma)}{p(1-\gamma)}\xi_0^\gamma x^{\gamma-1}y(1+\xi_0)^{-p}\right].$$

We denote for ξ in $[0,\xi_0]$, the function $\Delta(\xi) := 1 - \frac{(p+1-\gamma)(p-\gamma)}{p(1-\gamma)}\xi_0^\gamma \xi^{1-\gamma}(1+\xi_0)^{-p}$. We are seeking a solution ξ_1 to the equation

$$\Delta(\xi) = 0. \tag{2.50}$$

The function Δ is non increasing with $\Delta(0) = 1$. So, we have to discuss wether $\Delta(\xi_0)$ is positive or not. To achieve it, let us introduce the function $\tilde{\Delta}$ defined by

$$\tilde{\Delta} : \mathbb{R}_*^+ \to \mathbb{R}_*^+ \tag{2.51}$$

$$x \to 1 - \frac{(p+1-\gamma)(p-\gamma)}{p(1-\gamma)}x(1+x)^{-p}. \tag{2.52}$$

This is clearly a non increasing continuous and one-to-one function on $\mathbb{R}_*^+$. And we can see that seeking the sign of $\Delta(\xi_0)$ remains to check the sign of

$\tilde{\Delta}(x)$ under the condition $\Theta(x) = 0$. So let us consider now the following non linear system of equations :

$$\tilde{\Delta}(x) = 0 \tag{2.53}$$
$$\Theta(x) = 0. \tag{2.54}$$

This is equivalent to:

$$(1+\xi_0)^{-p} = \frac{1-\gamma}{1+p-\gamma}$$
$$1 + \frac{(1+\xi_0)^{-p}}{1-\gamma}\big[(\gamma-p)\xi_0 - (1-\gamma)\big] = 0.$$

We can see after calculus that the solution of (2.53) and (2.54) is $x = \frac{p}{p-\gamma}$. Moreover, for a fixed p, we have

$$G(\gamma) = 0 \;\Leftrightarrow\; \text{there is a unique solution to (2.53) and (2.54).}$$

Since G is a non decreasing continuous and one-to-one function, its admits a unique solution γ^*. Moreover, we have that G is negative on $\gamma \leq \gamma^*$ and positive on $\gamma > \gamma^*$. This results gives us that :

For $\gamma > \gamma^*$, G positive implies $\tilde{\Delta}(x)$ negative. It means that $\Delta(\xi_0)$ is negative, so $\bar{U}^1$ is not concave in its first variable and admits an inflexion point to be determined.
For $\gamma \leq \gamma^*$, G negative implies $\tilde{\Delta}(x)$ positive. It means that $\Delta(\xi_0)$ is positive, so $\bar{U}^1$ is concave in its first variable.

More precisely for $(x,z) \in \bar{D}$:
- If $\gamma \leq \gamma_*$, then $\bar{U}^1$ is concave in x and in z and $\bar{U}^2 = \bar{U}^1$.
- If $p > \gamma > \gamma_*$, then there exists ξ_1 such that :
  For $x \geq \frac{z^{\frac{1}{1-\gamma}}}{\xi_1}$, $\bar{U}^1$ is concave in x and in z and $\bar{U}^2 = \bar{U}^1$.
  For $\frac{z^{\frac{1}{1-\gamma}}}{\xi_0} \geq x \geq \frac{z^{\frac{1}{1-\gamma}}}{\xi_i}$, $\bar{U}^1$ is not concave in x and we have to seek a straight line which joints the points $\big(\frac{z^{\frac{1}{1-\gamma}}}{\xi_0}, z\big)$ and $\big(\frac{z^{\frac{1}{1-\gamma}}}{\xi_1}, z\big)$.
  For $x \leq \frac{z^{\frac{1}{1-\gamma}}}{\xi_0}$, $\bar{U}^1$ is concave in x and in z and $\bar{U}^2 = \bar{U}^0$.

So, we have to look for a line passing by $\big(\frac{z^{\frac{1}{1-\gamma}}}{\xi_0}, z\big)$ and $\big(\frac{z^{\frac{1}{1-\gamma}}}{\xi_1}, z\big)$ such that $\bar{U}^2$ is continuous and continuously differentiable at these points. So, we must find ξ_0 and ξ_1 such that :

For $\frac{z^{\frac{1}{1-\gamma}}}{\xi_1} > x > \frac{z^{\frac{1}{1-\gamma}}}{\xi_0}$, then :

$$\bar{U}^2(x,z) = \frac{z^{\frac{-p}{1-\gamma}}\xi_1\xi_0}{(1-p)(\xi_1 - x_0)}\Bigg[\Big(x - \frac{z^{\frac{1}{1-\gamma}}}{\xi_1}\Big)\xi_0^{p-1}(1+\xi_0)^{1-p} + \Big(\frac{z^{\frac{1}{1-\gamma}}}{\xi_0} - x\Big)\xi_1^{p-1}\frac{(2p-\gamma)(1-\gamma)}{(p-\gamma)(p+1-\gamma)}\Bigg].$$

and by continuity of the first derivatives :

$$\partial_x \bar{U}^1\Big(\frac{z^{\frac{1}{1-\gamma}}}{\xi_1}, z\Big) = \partial_x \bar{U}^0\Big(\frac{z^{\frac{1}{1-\gamma}}}{\xi_0}, z\Big) = \partial_x \bar{U}^2\big(x, z\big). \tag{2.55}$$

with

$$\partial_x \bar{U}^1\big(\frac{z^{\frac{1}{1-\gamma}}}{\xi_1}, z\big) = z^{\frac{-p}{1-\gamma}}\xi_1^p \frac{1-\gamma}{p+1-\gamma} \tag{2.56}$$

$$\partial_x \bar{U}^0\big(\frac{z^{\frac{1}{1-\gamma}}}{\xi_0}, z\big) = z^{\frac{-p}{1-\gamma}}\xi_0^p(1+\xi_0)^{-p} \tag{2.57}$$

$$\partial_x \bar{U}^2\big(x,z\big) = \frac{z^{\frac{-p}{1-\gamma}}}{1-p}\frac{\xi_0\xi_1}{\xi_1 - \xi_0}\Bigg[\xi_0^{p-1}(1+\xi_0)^{1-p} - \xi_1^{p-1}\frac{(2p-\gamma)(1-\gamma)}{(p-\gamma)(p+1-\gamma)}\Bigg]. \tag{2.58}$$

So, associating (2.55), (2.56) and (2.57), we get

$$\xi_1 = \big(\frac{p+1-\gamma}{1-\gamma}\big)^{\frac{1}{p}}\frac{\xi_0}{1+\xi_0}. \tag{2.59}$$

Combining (2.55), (2.56) and (2.58), we obtain

$$\xi_1 = \frac{p}{(1-p)(p-\gamma)}\Bigg[(p+1-\gamma) - (p-\gamma)\big(\frac{p+1-\gamma}{1-\gamma}\big)^{\frac{1}{p}}\Bigg]. \tag{2.60}$$

Finally, (2.60) and (2.59) give us ξ_0.

To summarize, for all $(x,z) \in \bar{D}$, $\bar{U}^2$ is equal to :

(i) For $0 < \gamma < \gamma^*$, $\bar{U}^2(x,z) = \bar{U}^1(x,z)$ and the optimal strategy is given by ξ_0 given in (2.46).

(ii) For $p > \gamma > \gamma^*$ and $x \geq \frac{z^{\frac{1}{1-\gamma}}}{\xi_1}$

$$\bar{U}^2(x,z) = \frac{x^{1-p}}{1-p} + \frac{\xi_0^\gamma}{1-\gamma}x^{\gamma-p}z(1+\xi_0)^{-p}.$$

and for $\frac{z^{\frac{1}{1-\gamma}}}{\xi_0} < x < \frac{z^{\frac{1}{1-\gamma}}}{\xi_1}$, then

$$\bar{U}^2(x,z) = \frac{z^{\frac{-p}{1-\gamma}}\xi_1\xi_0}{(1-p)(\xi_1 - x_0)}\left[\left(x - \frac{z^{\frac{1}{1-\gamma}}}{\xi_1}\right)\xi_0^{p-1}(1+\xi_0)^{1-p} + \left(\frac{z^{\frac{1}{1-\gamma}}}{\xi_0} - x\right)\xi_1^{p-1}\frac{(2p-\gamma)(1-\gamma)}{(p-\gamma)(p+1-\gamma)}\right].$$

for $x \le \frac{z^{\frac{1}{1-\gamma}}}{\xi_0}$, then

$$\bar{U}^2(x,z) = x^{1-p}\frac{(1+\xi_0)^{1-p}}{1-p}.$$

and the optimal strategies are given by (2.60). Moreover, the function $\bar{U}^2$ is concave in x and in y. By Proposition (2.26), $\bar{U}^\infty = \bar{U}^2$.

□

2.6 Conclusion

The results obtained so far are consistent with those of Henderson and Hobson in [42]. But we provide a more general framework which gives a better understanding of the problem. We do not deal with correlated processes or with interest rate. Indeed, this additional parameters would have completely change our framework, in particular in the concave structure that we highlighted.

Chapter 3

Constrained Second Order Backward SDEs

3.1 Introduction

Second order BSDEs were introduced by Cheredito, Soner, Touzi and Victoir in [13]. In a Markovian framework, they show that there exists a connection between 2BSDEs and fully nonlinear PDEs while standard BSDEs induce quasi-linear PDEs. However, except in the case where the PDEs admits sufficiently regular solutions, they do not provide a general existence result. In [24], Denis and Martini generalized the uncertain volatility model introduced in [1] or [59] to a family of martingale measures thanks to the quasi-sure analysis. The uncertain volatility model is directly linked to the Black-Scholes-Barrenblat equation which is fully nonlinear. This problem is strongly linked to the problem of G-integration theory studied mainly by Peng (see [68], [67]) for the definition of the main properties. Denis, Hu and Peng in [23] established connections between [68] and [24] while Soner, Touzi and Zhang in [83] provide a martingale representation theorem for the G-martingale which corresponds to a hedging strategy in the uncertain volatility model. Inspired by this quasi-sure framework, Soner, Touzi and Zhang study in [85] the second order stochastic target problem whose solution solves a 2BSDE and prove existence and uniqueness for general 2BSDEs in [86] with an undominated family of mutually singular martingale measures. Recently, Possamai and Zhou extend their results for a one dimensional 2BSDE with bounded terminal condition and continuous generator with quadratic growth in the z variable (see Possamai [75], Possamai and Zhou [76]). This result allow to solve second order reflected BSDEs and utility maximization problem under volatility uncertainty as we can see in

Matoussi, Possamai and Zhou [62, 61].

Constrained BSDEs were motivated by the work of Cvitanic and Karatzas in [20] which established the link between hedging a claim with constraints on the portfolio and solving a standard BSDE. More generally, Cvitanic, Karatzas and Soner provide in [21] a stochastic control approach to solve BSDEs with convex constraints on the gain process and obtain an interesting stochastic representation theorem. We can also mention papers of Buckdahn and Hu in [9] and [10]. In this last paper, they study the problem of hedging American claims with constrained portfolio by using one dimensional constrained BSDEs driven by both a Brownian motion and a Poisson process, with a lower barrier and a penalization method. However, they do not provide a stochastic representation theorem for the solution. Peng in [67] proposes limit theorem for monotonic sequences of standard BSDEs and provides the existence of a minimal solution to BSDEs whose constraint involves both Y and Z.
Our approach follows that of Soner, Touzi and Zhang in [85] and [86] and is inspired by [21]. Firstly, we have been interested in generalizing the results of [21] to the case of a non-convex generator thanks to the strong convergence in the gain process proved by Peng in [67]. Thus, we provide the existence of a minimal solution for a general first order constrained problem as well as its corresponding stochastic representation theorem. Secondly, we extend these results to the case of a 2BSDE with convex constraints on Z. Indeed, under classical assumptions, we have proved the existence and a stochastic representation theorem for the minimal solution of a constained 2BSDE.

3.2 The setup

3.2.1 The local martingale measure

Let $\Omega = \{\omega \in C([0,1], \mathbb{R}^d) \colon \omega_0 = 0\}$ be the canonical space, B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0\leq t\leq 1}$ the filtration generated by B and the right limit of $\mathbb{F}$, $\mathbb{F}^+ := \{\mathcal{F}_t^+\}_{0\leq t\leq 1}$.
A probability measure $\mathbb{P}$ is a local martingale measure if the canonical process B is a local martingale under $\mathbb{P}$. By Föllmer [38], there exists an $\mathbb{F}$-progressively measurable process, denoted by $\int_0^t B_s dB_s$, which coincides with the Itô integral $\mathbb{P}-$a.s. for all local martingale measures $\mathbb{P}$. This provides a pathwise definition of

$$\langle B\rangle_t := B_t B_t' - 2\int_0^t B_s dB_s \text{ and } \hat{a}_t = \overline{\lim_{\varepsilon\uparrow 0}} \frac{1}{\varepsilon}\big(\langle B\rangle_t - \langle B\rangle_{t-\varepsilon}\big)$$

where $'$ denotes the transposition, and $\overline{\lim}$ is taken componentwise and pointwise in ω. Clearly, $\langle B\rangle$ coincides with the $\mathbb{P}-$quadratic variation of B, $\mathbb{P}-$a.s. for all local martingale measures $\mathbb{P}$. Let $\overline{\mathcal{P}}_W$ denotes the set of all local martingale measures $\mathbb{P}$ such that

$$\langle B\rangle_t \text{ is absolutely continuous in } t \text{ and } \hat{a} \text{ takes values in } S_d^{>0},\ \mathbb{P}-a.s. \tag{3.1}$$

where $S_d^{>0}$ denotes the set of all $d\times d$ real valued definite positive matrices. For any $\mathbb{P}\in\overline{\mathcal{P}}_W$, it follows from the Levy characterization that the Itô stochastic integral under $\mathbb{P}$

$$W_t^{\mathbb{P}} := \int_0^t \hat{a}_s^{-1/2} dB_s,\ t\in[0,1],\ \mathbb{P}-a.s. \tag{3.2}$$

defines a $\mathbb{P}$-Brownian motion. We concentrate on the subclass $\overline{\mathcal{P}}_S\subset\overline{\mathcal{P}}_W$ collecting all the probability measures which are such that

$$\mathbb{P}^\alpha := \mathbb{P}_0\circ(X^\alpha)^{-1} \text{ where } X_t^\alpha := \textstyle\int_0^t \alpha_s^{1/2} dB_s,\ t\in[0,1],\ \mathbb{P}_0\text{-a.s.}$$

and α is an $\mathbb{F}-$progressively measurable process, valued in $S_d^{>0}$ with

$$\int_0^1 |\alpha_s| ds < \infty,\ \mathbb{P}_0 - a.s.$$

We denote by $\overline{\mathbb{F}}^{\mathbb{P}}$ (resp. $\overline{\mathbb{F}^{W^{\mathbb{P}}}}^{\mathbb{P}}$) the $\mathbb{P}-$augmentation of the right-limit filtration generated by B (resp. by $W^{\mathbb{P}}$). We recall from [84] that

$$\overline{\mathcal{P}}_S = \big\{\mathbb{P}\in\overline{\mathcal{P}}_W : \overline{\mathbb{F}^{W^{\mathbb{P}}}}^{\mathbb{P}} = \overline{\mathbb{F}}^{\mathbb{P}}\big\} \tag{3.3}$$

Furthermore,

$$\text{every } \mathbb{P}\in\overline{\mathcal{P}}_S \text{ satisfies the Blumenthal zero-one law and the martingale representation property.} \tag{3.4}$$

Remark 3.1. For different $\mathbb{P}_1$ and $\mathbb{P}_2$ in $\overline{\mathcal{P}}_W$, $\mathbb{P}_1$ and $\mathbb{P}_2$ are usually mutually singular. Indeed take $\mathbb{P}_1 := \mathbb{P}_0\circ(\sqrt{2}B)^{-1}$ and $\Omega_i := \{\langle B\rangle_t = (1+i)t\}$ for $i=0,1,2$. Then we can see easily that $\mathbb{P}_0(\Omega_0)=\mathbb{P}_1(\Omega_1)=\mathbb{P}_0[\langle B_t\rangle = t]=1$ while $\mathbb{P}_0(\Omega_1)=\mathbb{P}_1(\Omega_0)=0$. So, $\mathbb{P}_0$ and $\mathbb{P}_1$ are mutually singular.

Remark 3.2. Because of (3.3), there exists an $\mathbb{F}$-progressively measurable mapping β_α such that $B=\beta_\alpha(X^\alpha),\ \mathbb{P}_0-a.s.$
Moreover since by construction the $\mathbb{P}^\alpha$-distribution of B is equal to the $\mathbb{P}_0$-distribution of X^α, $d\langle X^\alpha\rangle_t = \alpha_t(B)dt = \alpha_t\circ\beta^\alpha(X_t^\alpha)dt, \mathbb{P}_0$-a.s. So, $\hat{a}(B) = \alpha\circ\beta^\alpha(B)$, $\mathbb{P}^\alpha$-a.s.

Remark 3.3. By Lemma 2.4 of [84], we recall that for any $\mathbb{P}$ in $\overline{\mathcal{P}}^S$ and any $\overline{\mathbb{F}}^{\mathbb{P}}$-progressively measurable process X, there exists a unique $\mathbb{F}$-progressively measurable process $\tilde{X}$ such that

$$X = \tilde{X},\ dt \times d\mathbb{P} - a.s.$$

Moreover if X is càdlàg, we can choose $\tilde{X}$ to be càdlàg too, $\mathbb{P}$-a.s. In the sequel, we will only consider processes in their $\mathbb{F}$-modification.

Finally, we say that $X = \tilde{X},\ \mathbb{P} - a.s.$, if they are equal $dt \otimes d\mathbb{P} - a.s.$.

3.2.2 General Definitions

Let us introduce the following spaces. For any $\mathbb{P} \in \overline{\mathcal{P}}_S$ and Euclidean space E, we define :

(1) $L^2(\mathbb{P}, E)$ the space of all $\mathcal{F}_1$ measurable random variables ξ valued in E such that $\mathbb{E}\big[|\xi|^2\big] < \infty$.

(2) $H^2(\mathbb{P}, E)$ the space of all $\mathbb{F}^+-$progressively measurable processes Z valued in E such that $\mathbb{E}^{\mathbb{P}}\big[\int_0^1 |Z_t|^2 dt\big] < \infty$.

(3) $S^2(\mathbb{P}, E)$ the space of all $\mathbb{F}^+-$progressively measurable processes Y valued in E which have càdlàg paths, $\mathbb{P}$-a.s. such that $\mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1} |Y_t|^2\right] < \infty$.

(4) $D^2(\mathbb{P}, E)$ is a subset of $S^2(\mathbb{P}, E)$ whose elements have continuous paths.

(5) $A^2(\mathbb{P}, E)$ is a subset of $S^2(\mathbb{P}, E)$ whose elements, K, have non decreasing paths with $K_0 = 0$, $\mathbb{P}$-a.s.

We denote by H a nonlinear generator defined by:

$$H_t(\omega, y, z, \gamma) : [0, 1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}. \tag{3.5}$$

where $D_H \subset \mathbb{R}^{d\times d}$ is a given subset containing 0. The corresponding conjugate of H with respect to γ taking values in $\mathbb{R} \cup \{\infty\}$ is given by:

$$F_t(\omega, y, z, a) := \sup_{\gamma\in D_H} \left\{\frac{1}{2} a : \gamma - H_t(\omega, y, z, \gamma)\right\},\ a \in S_d^{>0}$$

We define:

$$\hat{F}_t(y, z) := F_t(\omega, y, z, \hat{a}_t). \tag{3.6}$$

Here : denotes the trace of the product matrice. We denote by $D_{F_t(y,z)}$ the domain of F in a for fixed (t, ω, y, z). That is to say for all $a \in D_{F_t(y,z)}$: $F_t(\omega, y, z, a) < \infty$.

Example 3.4. A famous example of nonlinearity is the the generator defined by $H_t(\omega, y, z, \gamma) := \sup_{a\in[\underline{a},\overline{a}]} \frac{1}{2}a : \gamma$. Such a generator is directly linked to the notion of volatility uncertainty studied in [24] and the G-expectation introduced by Peng in [68]. As an example, we notice that in our framework $F = 0$ when $a \in [\underline{a}, \overline{a}]$ and ∞ otherwise. So $D_{F_t(y,z)} = [\underline{a}, \overline{a}]$.

Example 3.5. The problem of Hedging under gamma constraints has been studied mainly in [81] and has a generator $H_t(\omega, y, z, \gamma) := \frac{1}{2}\sigma^2\gamma$ for $\gamma \in [\underline{\Gamma}, \overline{\Gamma}]$ and ∞ otherwise with $\underline{\Gamma} \le 0 \le \overline{\Gamma}$. Here, $F_t(\omega, y, z, a) = \frac{1}{2}\big[\overline{\Gamma}(a-1)^+ - \underline{\Gamma}(a-1)^-\big]$ and $D_{F_t(y,z)} = \mathbb{R}$.

We consider a nonempty, closed, convex set $C \subset \mathbb{R}^d$ which contains 0. We denote by δ its support function:

$$\delta(y) := \sup_{z\in C} yz, \; y \in \mathbb{R}^d. \tag{3.7}$$

It is finite on its effective domain:

$$\tilde{C} := \{y \in \mathbb{R}^d : \delta(y) < \infty\}. \tag{3.8}$$

Naturally δ is continuous on $\tilde{C}$ if, Theorem 10.2 p84 in [78], $\tilde{C}$ is locally simplicial. Moreover, we recall the following characterization of the closed convex set C (see [78], Theorem 13.1):

$$z \in C \Longleftrightarrow \delta(x) - xz \ge 0, \; \forall x \in \tilde{C}. \tag{3.9}$$

For all processes $\nu : \Omega \times [0, 1] \to \tilde{C}$, we define the space $\mathcal{D}$ by:

$$\mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}_n \tag{3.10}$$

with $\mathcal{D}_n := \big\{\mathbb{F} - \text{progressively measurable } \nu : \; |\nu(t, \omega)| < n, \text{ for a.e. } (t, \omega) \in [0, 1] \times \Omega\big\}$. For all $(t, s) \in [0, 1]^2$, we denote by $\mathcal{D}_{[s,t]}$ the restriction of $\mathcal{D}$ to $[s, t] \times \Omega$.

For all $\nu \in \mathcal{D}$ and $\mathbb{P} \in \overline{\mathcal{P}}_S$, we can define the $\mathbb{P}$-local martingale:

$$Z_t^{\mathbb{P},\nu} := \exp\big\{ - \int_0^t \nu_s \hat{a}_s^{-1/2} dB_s - \frac{1}{2} \int_0^t |\nu_s|^2 ds\big\} \;\; \mathbb{P} - a.s. \tag{3.11}$$

Since every $\nu \in \mathcal{D}$ is bounded, $Z^{\mathbb{P},\nu}$ is a $\mathbb{P}$-martingale and thus defines a $\mathbb{P}$-equivalent probability measure denoted $\mathbb{P}^\nu$:

$$\text{For all } A \in \mathcal{F}_1 : \; \mathbb{P}^\nu(A) := \mathbb{E}^{\mathbb{P}}[Z_1^{\mathbb{P},\nu} \mathbb{1}_A]. \tag{3.12}$$

By the Girsanov theorem, the process:

$$W_t^{\mathbb{P}^\nu} = W_t^{\mathbb{P}} + \int_0^t \nu_u du, \; 0 \le t \le 1. \tag{3.13}$$

is a $\mathbb{P}^\nu$-Brownian motion where $W^{\mathbb{P}}$ defined in (3.2) is a $\mathbb{P}$-Brownian motion. That is to say that for any $\mathbb{P} \in \overline{\mathcal{P}}_S$ and $\nu \in \mathcal{D}$, the process B^ν defined by:

$$B_t^\nu = B_t + \int_0^t \nu_u \hat{a}_u^{1/2} du, \;\; 0 \le t \le 1, \; \mathbb{P} - a.s. \tag{3.14}$$

is a local martingale under the equivalent probability measure $\mathbb{P}^\nu$.

We define a new set of measures :

Definition 3.6. *Let $\mathcal{P}_H$ denote the collection of all $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that*

$$\underline{a}_{\mathbb{P}} \le \hat{a} \le \overline{a}_{\mathbb{P}}, \; dt \times d\mathbb{P}-a.s. \; \textit{for some } \underline{a}_{\mathbb{P}}, \overline{a}_{\mathbb{P}} \in S_d^{>0}$$

and

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^1 |\hat{F}_t(0,0)|^2 dt\right] < \infty. \tag{3.15}$$

Notice that the bounds $\underline{a}_{\mathbb{P}}$ and $\overline{a}_{\mathbb{P}}$ are not uniform in $\mathbb{P}$. They are here to ensure regularity in ω to the solution of the 2BSDE. We do not need them if the nonlinearity and the terminal data are bounded.
As in [84], we introduce the following definition:

Definition 3.7. *We say that a property holds $\mathcal{P}_H-$quasi-surely ($\mathcal{P}_H-q.s$ for short) if it holds $\mathbb{P}-a.s.$ for all $\mathbb{P} \in \mathcal{P}_H$.*

Notice that this definition is different from that of Denis and Martini in [24], where a property holds quasi-surely if it holds outside a set of capacity zero (polar set).

In the sequel, we will always assume that the nonlinearity verifies:

Assumption 3.8. *$\mathcal{P}_H$ is not empty, and the domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z). Moreover, in D_{F_t}, F is $\mathbb{F}$-progressively measurable, uniformly continuous in ω under the uniform convergence norm, and there is a constant $\mu > 0$ such that:*

$$|\hat{F}_t(y,z) - \hat{F}_t(y',z')| \le \mu\Big(|y-y'| + \hat{a}_t^{1/2}|z-z'|\Big), \; \mathcal{P}_H - q.s \tag{3.16}$$

for all $t \in [0,1]$, $(y,y') \in \mathbb{R}^2$ and $(z,z') \in (\mathbb{R}^d)^2$.

Remark 3.9. One could have put assumptions on H instead of F. But if H plays an important role in the case of fully nonlinear PDEs, F remains the main tools to study the 2BSDEs, as we can see in [86]. However, one can see that assuming that H is Lipschitz continuous in (y, z), uniformly continuous in ω under the $\mathbb{L}^\infty$-norm and upper semicontinuous in γ leads to Assumption 3.8.

In a quasi-sure context, we are interested in the following spaces for any Euclidian space E:

$$\mathbb{L}^2_H(E) := \bigcap_{\mathbb{P}\in\mathcal{P}_H} L^2(\mathbb{P}, E),\ \mathbb{H}^2_H(E) := \bigcap_{\mathbb{P}\in\mathcal{P}_H} H^2(\mathbb{P}, E),$$
$$\mathbb{S}^2_H(E) := \bigcap_{\mathbb{P}\in\mathcal{P}_H} S^2(\mathbb{P}, E).$$

3.2.3 Regular conditional probability distributions

To established a dynamic programming principle, we need regular conditional probability distributions which allow to define the candidate solution on Ω without exception of any zero measure set. This notion was introduced by Stroock and Varadhan in [87].

Definition 3.10. *For any probability measure $\mathbb{P}$, $\mathbb{F}$-stopping time τ and $\omega \in \Omega$, there exists an r.c.p.d $\mathbb{P}^\omega_\tau$ such that:*

(i) For each $\omega \in \Omega$, $\mathbb{P}^\omega_\tau[.]$ is a probability measure on $(\Omega, \mathcal{F}_1)$.

(ii) For each $A \in \mathcal{F}_1$, the mapping $\omega \to \mathbb{P}^\omega_\tau[A]$ is $\mathcal{F}_\tau$-measurable.

(iii) For any bounded $\mathcal{F}_1$-measurable random variable ξ,

$$\mathbb{E}^{\mathbb{P}^\omega_\tau}(\xi) = \mathbb{E}^{\mathbb{P}}[\xi \mid \mathcal{F}_\tau](\omega),\ \mathbb{P}-a.s. \tag{3.17}$$

Moreover, for any $\omega \in \Omega$ and probability measure $\mathbb{P}$, we have:

$$\mathbb{P}^\omega_\tau\big[\omega' \in \Omega : \omega_s = \omega'_s,\ 0 \le s \le \tau(\omega)\big] = 1,\ \mathbb{P}-a.s.$$

Let us define for $0 \le t \le 1$, the shifted canonical set $\Omega^t := \{\omega \in C([t,1]) : \omega(t) = 0\}$, B^t the shifted canonical process on Ω^t, $\mathbb{P}^t_0$ the shifted Wiener measure and $\mathbb{F}^t$ the shifted filtration generated by B^t. We denote par $\omega^t \in \Omega^t$ the shifted path which is defined for every $0 \le s \le t \le 1$ and $\omega \in \Omega^s$ by:

$$\omega^t_u = \omega_{t+u} - \omega_u,\ \forall u \in [0, 1-t]$$

We next define for $0 \le s \le t \le 1$, $\omega^1 \in \Omega^s$ and $\omega^2 \in \Omega^t$, the concatenation path $\omega^1 \otimes_t \omega^2 \in \Omega^s$ by:

$$(\omega^1 \otimes_t \omega^2)(u) = \omega^1_u \mathbb{1}_{u\in[s,t)} + \big(\omega^1_t + \omega^2_u\big)\mathbb{1}_{u\in[t,1]},\ u \in [s,1] \tag{3.18}$$

and for any $\mathcal{F}_1^s$-measurable random variable ξ on Ω^s, the shifted $\mathcal{F}_1^t$-measurable random variable $\xi^{t,\omega} \in \Omega^t$:

$$\xi^{t,\omega_1}(\omega_2) = \xi(\omega_1 \otimes_t \omega_2)$$

This definition can be extended for the definition of $\mathbb{F}^t$-measurable processes which will be denoted $\{X_u^{t,\omega}, u \in [t,1]\}$.

We denote by $\mathbb{P}^{\tau,\omega}$ the rcpd of $\mathbb{P}$, that is to say the probability induced by $\mathbb{P}_\tau^\omega$ such that the $\mathbb{P}^{\tau,\omega}$-distribution of $B^{\tau(\omega)}$ is equal to the $\mathbb{P}_\tau^\omega$-distribution of $\{B_t - B_{\tau(\omega)}, t \in [\tau(\omega), 1]\}$. By the definition of $\mathbb{P}_\tau^\omega$ (see (3.17)), we get the following equality for every bounded and $\mathcal{F}_1$-measurable random variable ξ:

$$\mathbb{E}^{\mathbb{P}_\tau^\omega}[\xi] = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\xi^{\tau,\omega}] = \mathbb{E}^{\mathbb{P}}[\xi \mid \mathcal{F}_\tau](\omega), \ \mathbb{P} - a.s. \tag{3.19}$$

Remark 3.11. We notice that roughly speaking the r.c.p.d enables to "transform" conditional expectations defined $\mathbb{P}$ almost surely into expectations by working on a shifted space. One can think of the increments of a Brownian motion which are independent from the past as the basic idea of the introduction of this shifted space. The advantage of such a transformation is to define objects for every $\omega \in \Omega$ and not only in the almost sure sense.

We can now rewrite our problem in the shifted canonical space. Let us define for all $t \in [0,1]$, the set $\overline{\mathcal{P}}_S^t$ of all martingale measures $\mathbb{P}^{t,\alpha}$ where α is an $\mathbb{F}^t$-measurable process such that $\int_t^1 |\alpha_u| du < \infty, \ \mathbb{P}_0^t - a.s.$ Moreover, we denote by $\hat{a}^t$ the quadratic covariation of the shifted canonical process B^t and define:

$$\hat{F}_s^{t,\omega}(\tilde{\omega}, y, z) := F_s\big(\omega \otimes_t \tilde{\omega}, y, z, \hat{a}_s^{t,\omega}(\tilde{\omega})\big)$$

for all $s \in [t,1]$, $(\omega, \tilde{\omega}) \in \Omega \times \Omega^t$ and $(y,z) \in \mathbb{R} \times \mathbb{R}^d$. We know by Assumption 3.8 that $\hat{F}$ is uniformly continuous in ω. However since in general $\hat{a}^{t,\omega}$ does not verify this property, we can not say at this moment that $\hat{F}^{t,\omega}$ is uniformly continuous in ω.

Remark 3.12. For any $\omega \in \Omega$, $\mathbb{P} \in \overline{P}_S$ and $\mathbb{F}$-stopping time τ, by Lemma 4.1 in [85] we recall that:

$$\hat{a}_s^{\tau,\omega}(\tilde{\omega}) = \hat{a}^{\tau(\omega)}(\tilde{\omega}), \ ds \times d\mathbb{P}^{\tau,\omega} - a.e., \text{ for } (s,\tilde{\omega}) \in [\tau(\omega), 1] \times \Omega^{\tau(\omega)}. \tag{3.20}$$

where $\hat{a}^{\tau,\omega}$ corresponds to the shifted version of $\hat{a}$ on $\Omega = \Omega^0$ and $\hat{a}^\tau$ is the density process on the shifted space $\Omega^{\tau(\omega)}$.

Plugging (3.20) in the definition of $\hat{F}^{t,\omega}$ leads to the following equivalent definition :

$$\hat{F}^{t,\omega}_s(\tilde{\omega}, y, z) = F_s\big(\omega \otimes_t \tilde{\omega}, y, z, \hat{a}^t_s(\tilde{\omega})\big) \ ds \times d\mathbb{P}^{t,\omega} - a.s. \tag{3.21}$$

This equivalent definition involves $\hat{a}^t$, which does not depend on ω, instead of $\hat{a}^{t,\omega}$. Therefore Assumption 3.8 implies that $\hat{F}^{t,\omega}$ is uniformly continuous in ω. So for any $(\omega, \omega') \in \Omega^2$, $\tilde{\omega} \in \Omega^t$, $s \in [t, 1]$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, we can find a modulus of continuity ρ such that

$$|\hat{F}^{t,\omega}_s(\tilde{\omega}, y, z) - \hat{F}^{t,\omega'}_s(\tilde{\omega}, y, z)| \leq \rho\big(\|\omega - \omega'\|_t\big). \tag{3.22}$$

where $\|.\|_t$ is the norm of the uniform continuity defined by $\|\omega\|_t := \sup_{0\leq s\leq t} |\omega_s|$.

As a consequence of (3.22), for all $(t, \omega) \in [0, 1] \times \Omega$ and $\mathbb{P}^t \in \overline{\mathcal{P}}^t_S$, we have:

$$\mathbb{E}^{\mathbb{P}^t}\left[\int_t^1 |\hat{F}^{t,\omega}_s(0, 0)|^2 ds\right] < \infty. \tag{3.23}$$

So we can extend the definition 3.6 to the shifted space and define the following set of probability measures:

Definition 3.13. *For all $t \in [0, 1]$, let $\mathcal{P}^t_H$ denotes the collection of all $\mathbb{P}^t \in \overline{\mathcal{P}}^t_S$ such that:*

$$\underline{a}_{\mathbb{P}} \leq \hat{a}^t \leq \overline{a}_{\mathbb{P}}, \ dt \times d\mathbb{P}^t - a.e. \ on \ [t, 1] \times \Omega^t \ for \ some \ \underline{a}_{\mathbb{P}}, \overline{a}_{\mathbb{P}} \in S^{>0}_d, \ and$$

$$\mathbb{E}^{\mathbb{P}^t}\left[\int_t^1 |\hat{F}^{t,\omega}_s(0, 0)|^2 ds\right] < \infty \ for \ all \ \omega \in \Omega \tag{3.24}$$

Moreover, it is clear that $\hat{F}^{t,\omega}$ verifies the Lipschitz continuity Assumption (3.16). The following lemma will be usefull:

Lemma 3.14. *(see [85]) Let Assumption 3.8 hold true. Then for any $\mathbb{F}$-stopping time τ and $\mathbb{P} \in \mathcal{P}_H$, we have:*

$$\mathbb{P}^{\tau,\omega} \in \mathcal{P}^{\tau(\omega)}_H$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$.

3.3 Constrained first order BSDEs

We recall some results on limit theorems for monotonic sequences of standard BSDEs (see [67]). In particular, we apply it to solve a first order BSDE with convex constraints on the gain process. Such a problem has already been solved by Cvitanic, Karatzas and Soner in [21]. They provide a stochastic representation for this problem with at most an assumption of convexity in z for the generator. Our aim is to unify the framework of [21] and [67] and obtain the same stochastic representation without any convexity assumption on the generator.

In this section, we shall always work with a single probability $\mathbb{P}_0$, the Wiener measure, that is to say we assume that

$$\overline{\mathcal{P}}_S = \{\mathbb{P}_0\}. \tag{3.25}$$

By definition, the canonical process B is a standard d-dimensional Brownian motion $\mathbb{F}$-adapted so its quadratic variation is such that for all $t \in [0,1]$:

$$\hat{a}_t = I_d.t \tag{3.26}$$

where I_d is the identity matrix in $\mathbb{R}^d$.

3.3.1 Limit results for g-supersolutions

We recall in this subsection some definitions and results on g-supersolutions and g-supermartingales which can be found mainly in Peng [67] and [12].

Let us define $g : \Omega \times [0,1] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ a mapping such that

$$\mathbb{E}^{\mathbb{P}_0}\left[\int_0^1 g^2(\omega, s, 0, 0)ds\right] < \infty. \tag{3.27}$$

Moreover, we assume that we can find a constant $\mu > 0$ such that

$$|g(\omega,t,y,z) - g(\omega,t,y',z')| \leq \mu\big(|y-y'| + |z-z'|\big). \tag{3.28}$$

for all $(t,\omega) \in [0,1] \times \Omega$ and $(y,y',z,z') \in \mathbb{R}^{2+2d}$.

Remark 3.15. To established the connection with the framework of section 3.2, let us consider a generator H (see (3.5)) of the following form:

$$H_t(y,z,\gamma) = \frac{1}{2}I_d : \gamma - g_t(y,z).$$

In this particular case, under obvious extension of notation, we have

$$\hat{F}_t(y,z) = g_t(y,z) \text{ and } D_{F_t(y,z)} = \{I_d\}$$

In this case, we notice that we do not need to assume that g is uniformly continuous in ω. Therefore, we can see that Assumption 3.28 is the equivalent of claim (3.16) in Assumption 3.8 in our simplified framework. By definition of $\mathcal{P}_H$ (see definition 3.6), Assumption (3.37) leads to:

$$\mathcal{P}_H = \overline{\mathcal{P}}_S = \{\mathbb{P}_0\}.$$

For any $\mathcal{F}_1$-measurable random variable $\xi \in L^2(\mathbb{P}_0, \mathbb{R})$, we are interested in solving the following BSDE:

$$y_t = \xi + \int_t^1 g_s(y_s, z_s)ds - \int_t^1 z_s dB_s + K_1 - K_t, \; 0 \leq t \leq 1, \; \mathbb{P}_0 - a.s. \quad (3.29)$$

where K is a given RCLL process in $A^2(\mathbb{P}_0, \mathbb{R})$. In the sequel, we will call (g, ξ) the parameters of BSDE (3.29).

Proposition 3.16. *(Peng,[67]) Under conditions* (3.27) *and* (3.28)*, there exists a unique solution* $(y, z) \in S^2(\mathbb{P}_0, \mathbb{R}) \times H^2(\mathbb{P}_0, \mathbb{R}^d)$ *of the BSDE* (3.29) *such that* $(y_t + K_t)$ *is continuous.*

If we consider the case where for all $t \in [0, 1]$, $K_t = 0$, then the unique pair of processes (y, z) solution of (3.29) belongs to $D^2(\mathbb{P}_0, \mathbb{R}) \times H^2(\mathbb{P}_0, \mathbb{R}^d)$, that is to say y has continuous paths.

For a given stopping time τ, t in $[0, 1]$ and a $\mathcal{F}_\tau$-measurable random variable $\xi \in L^2(\mathbb{P}_0, \mathbb{R})$, we define the following BSDE:

$$y_t = \xi + \int_{\tau \wedge t}^{\tau} g_s(y_s, z_s)ds - \int_{\tau \wedge t}^{\tau} z_s dB_s + K_\tau - K_{t \wedge \tau}, \; \mathbb{P}_0 - a.s. \quad (3.30)$$

As it was introduced by Peng in [67],

Definition 3.17. *y is g-supersolution on* $[0, \tau]$ *if y is a solution of the BSDE* (3.30)*. If* $K_t = 0$ *on* $[0, \tau]$ *then we say that y is a g-solution.*

We recall the following comparison theorem for standard BSDEs which can be found for instance in [29] or in [67].

Theorem 3.18. *Let conditions* (3.27) *and* (3.28) *hold. Let* (y^1, z^1, K^1) *and* (y^2, z^2, K^2) *be the solutions of* (3.30) *with parameters* (g, ξ^1) *and* (g, ξ^2) *such that:*

$$\xi^1 \geq \xi^2 \text{ and } K^1 \geq K^2, \; \mathbb{P}_0 - a.s$$

Then for all t in $[0, 1]$*,* $y_t^1 \geq y_t^2, \; \mathbb{P}_0 - a.s.$

Remark 3.19. One can notice that this comparison theorem says that a g-supersolution is always above a g-solution.

We recall the following usefull inequality which can be found in [29]:

Lemma 3.20. *Under conditions* (3.27) *and* (3.28), *let* $(y,z) \in D^2(\mathbb{P}_0,\mathbb{R}) \times H^2(\mathbb{P}_0,\mathbb{R}^d)$ *be the unique solution of* (3.29) *with* $K=0$. *Then we can find a constant* $c>0$ *such that:*

$$\mathbb{E}^{\mathbb{P}_0}\left[\sup_{0\le t\le 1}|y_t|^2\right]+\mathbb{E}^{\mathbb{P}_0}\left[\int_0^1 |z_t|^2dt\right]\le c\mathbb{E}^{\mathbb{P}_0}\left[|\xi|^2+\int_0^1 |g_s(0,0)|^2ds\right].$$

Remark 3.21. This inequality is the key ingredient to show the stability property of BSDEs.

We recall the definition of g-martingale and g-supermartingale.

Definition 3.22. (i) *A real valued,* $\mathcal{F}_t$*-progressively measurable process* Y_t *is called a strong g-supermartingale if for each stopping times* $0\le\sigma\le\tau\le 1$, $\mathbb{E}^{\mathbb{P}_0}\left[Y_\tau^2\right]<\infty$ *and the g-solution* y_t *on* $[0,t]$ *with terminal condition* $y_\tau=Y_\tau$ *satisfies* $Y_\sigma\ge y_\sigma$.
(ii) *If* (i) *holds for deterministic times, we say that* Y *is weak g-supermartingale. Besides, a g-martingale is a g-solution.*

The case where g does not depend on (ω,y,z) corresponds to the classical notion of martingales and supermartingales. Still, this emphasizes the strong link between nonlinearity in partial differential equations and BSDEs.

We recall one of the main results in [67], Theorem 3.3. Indeed the following theorem provides a nonlinear decomposition like the one of Doob-Meyer for the g-supermartingale.

Theorem 3.23. *(Peng, [67]) Under conditions* (3.27) *and* (3.28), *let* $Y\in S^2(\mathbb{P},\mathbb{R})$ *be a right-continuous strong g-supermartingale on* $[0,1]$. *Then* Y *is a g-supersolution on* $[0,1]$, *that is to say, there exists* $(y,z,k)\in S^2(\mathbb{P}_0,\mathbb{R})\times H^2(\mathbb{P}_0,\mathbb{R}^d)\times A^2(\mathbb{P}_0,\mathbb{R})$ *such that* Y *coincides with the unique solution* y *of:*

$$y_t=Y_1+\int_t^1 g_s(y_s,z_s)ds-\int_t^1 z_sdB_s+k_1-k_s,\ \mathbb{P}_0-a.s.$$

Remark 3.24. This decomposition plays a central role in the proof of the existence of a solution for unconstrained and constrained second order BSDEs.

Peng in [67] provides an existence theorem for BSDEs with constraints on (y, z). Let us do a quick review of these results:

We consider $\Phi : [0,1] \times \mathbb{R}^{d+1} \times \Omega \to \mathbb{R}^+$ such that for every $(y, z) \in \mathbb{R} \times \mathbb{R}^d$:

$$\mathbb{E}^{\mathbb{P}_0} \int_0^1 |\Phi_t(y,z)|^2 dt < \infty.$$

Moreover Φ is globally Lipschitz continuous with respect to (y, z).

For $\xi \in L^2(\mathbb{P}_0, \mathbb{R})$, we want to find a triple (Y, Z, K) in $S^2(\mathbb{P}_0, \mathbb{R}) \times H^2(\mathbb{P}_0, \mathbb{R}^d) \times A^2(\mathbb{P}_0, \mathbb{R})$ such that the standard BSDE and the constraint:

$$Y_t = \xi + \int_t^1 g_u(Y_u, Z_u)du - \int_t^1 Z_u dB_u + K_1 - K_t, \ 0 \le t \le 1, \ \mathbb{P}_0 - a.s. \tag{3.31}$$

$$\Phi_t(Y_t, Z_t) = 0, \ 0 \le t \le 1, \ \mathbb{P}_0 - a.s. \tag{3.32}$$

hold almost surely and such that for any triple $\left(\tilde{Y}, \tilde{Z}, \tilde{K}\right) \in S^2(\mathbb{P}_0, \mathbb{R}) \times H^2(\mathbb{P}_0, \mathbb{R}^d) \times A^2(\mathbb{P}_0, \mathbb{R})$ solution to (3.31) and (3.32), we have

$$Y_t \le \tilde{Y}_t, \quad 0 \le t \le 1, \ \mathbb{P}_0 - a.s.$$

Such a triple (Y, Z, K) in $S^2(\mathbb{P}_0, \mathbb{R}) \times H^2(\mathbb{P}_0, \mathbb{R}^d) \times A^2(\mathbb{P}_0, \mathbb{R})$ is called a minimal solution for (3.31) and (3.32).

Assumption 3.25. *There exists a triple* $(Y^*, Z^*, K^*) \in S^2(\mathbb{P}_0, \mathbb{R}) \times H^2(\mathbb{P}_0, \mathbb{R}^d) \times A^2(\mathbb{P}_0, \mathbb{R})$ *which solves* (3.31) *and* (3.32)*.*

They proceed by penalization, that is to say they introduce the following sequence of BSDEs:

$$Y_t^n = \xi + \int_t^1 g_s(Y_s^n, Z_s^n)ds - \int_t^1 Z_s^n dB_s + K_1^n - K_t^n, \ 0 \le t \le 1, \ \mathbb{P}_0 - a.s. \tag{3.33}$$

where $K_t^n := n \int_0^t \Phi_s(Y_s^n, Z_s^n)ds \in A^2(\mathbb{P}_0, \mathbb{R})$.

One can show the following theorem:

Theorem 3.26. *(Peng, [67]) Let conditions* (3.27), (3.28) *and Assumption* 3.25 *hold. Then the sequence of g-supersolutions $(Y_n)_n$ is non decreasing and converges to some $Y \in D^2(\mathbb{P}_0, \mathbb{R})$. Moreover, $(Z^n)_n$ and $(K^n)_n$ converge weakly to Z and K respectively in $H^2(\mathbb{P}_0, \mathbb{R}^d)$ and in $A^2(\mathbb{P}_0, \mathbb{R})$.*
Furthermore, (Y, Z, K) is the smallest solution of BSDE (3.31) *subject to constraint* (3.32)*.*

Remark 3.27. Mainly the proof of this theorem is based on the following strong convergence for $p \in [0,2)$:

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}_0}\left[\int_0^1 |Z_s^n - Z_s|^p ds\right] = 0. \tag{3.34}$$

which is strongly linked to the fact that the increasing process K^n is continuous.

3.3.2 The first order constrained problem

We recall from Section 3.2 that $C \subset \mathbb{R}^d$ is a nonempty, closed, convex set which contains 0. We recall that δ (see (3.7)) is its support function and is finite on its effective domain $\tilde{C}$ (see (3.8)). Moreover, we recall that δ is continuous on $\tilde{C}$ and the following characterization holds:

$$z \in C \Longleftrightarrow \delta(x) - xz \geq 0, \ \forall x \in \tilde{C}. \tag{3.35}$$

We recall that $\mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}_n$ with $\mathcal{D}_n := \big\{\mathbb{F}-$ progressively measurable ν : $|\nu(t,\omega)| < n$, for a.e. $(t,\omega) \in [0,1]\times\Omega\big\}$.

Constrained BSDEs with convex constraints are related to the hedging problem with constrained portfolio in the context of Mathematical Finance, see [20]. In this context, the gain process Z can be seen as the portfolio rule such that we can hedge a contingent claim ξ. Thus, we can keep in mind the following examples of possible closed convex sets C:

(i) No short selling : $C = (\mathbb{R}^+)^d$.

(ii) Presence of $m \leq d$ untradable assets in the market: $C = \mathbb{R}^{d-m} \times \{0\}^m$

We consider a function $f : \Omega \times [0,1] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that:

Assumption 3.28. *For all $(t,\omega) \in [0,1]\times\Omega$ and $(y,y',z,z') \in \mathbb{R}^{2+2d}$, there is a constant $\mu > 0$ such that*

$$|f_t(y,z) - f_t(y',z')| \leq \mu\big(|y-y'| + |z-z'|\big). \tag{3.36}$$

and

$$\mathbb{E}^{\mathbb{P}_0}\left[\int_0^1 |f_u(0,0)|^2 du\right] < \infty. \tag{3.37}$$

As in remark 3.15, we notice that Assumption 3.28 is the equivalent of condition (3.16) (see Assumption 3.8) in the case where $\mathcal{P}_H = \overline{\mathcal{P}}_S = \{\mathbb{P}_0\}$.

The First Order Constrained Problem :

For a given $\mathcal{F}_1$-measurable process $\xi \in L^2(\mathbb{P}_0, \mathbb{R})$, our aim is to find a stochastic representation to a particular case of (3.31) and (3.32). That is to say, we impose convex constraints on the gain process:

$$Y_t = \xi + \int_t^1 f_u(Y_u, Z_u)du - \int_t^1 Z_u dB_u + K_1 - K_t,\ 0 \le t \le 1,\ \mathbb{P}_0 - a.s. \tag{3.38}$$

$$Z_t \in C \quad 0 \le t \le 1, \quad \mathbb{P}_0 - a.s. \tag{3.39}$$

Remark 3.29. This is the same problem as section 5 in [21] except that the generator f that we consider is a function not necessarily convex in z of the gain process.

We make the following Assumption:

Assumption 3.30. *There is at least one solution* $\big(\tilde{Y}, \tilde{Z}, \tilde{K}\big) \in S^2(\mathbb{P}_0, \mathbb{R}) \times H^2(\mathbb{P}_0, \mathbb{R}^d) \times A^2(\mathbb{P}_0, \mathbb{R})$ *to* (3.38) *and* (3.39).

Remark 3.31. For instance, let us assume that the generator f and the terminal condition ξ are bounded by a constant $a > 0$. Take a process $\tilde{Z} \in \mathbb{H}^2\big(\mathbb{P}_0, \mathbb{R}\big)$ such that $\tilde{Z}_t = 0, \mathbb{P} - a.s.$, $\tilde{Y}_t = a$ and a non decreasing process $\tilde{K}$ such that $\tilde{K}_t = 0$ for all $t \in [0, 1)$ and $\tilde{K}_1 = \tilde{Y}_{1-} - \xi$. Then $(\tilde{Y}, \tilde{Z}, \tilde{K})$ is a solution to (3.38) and (3.39).

For $z \in \mathbb{R}^d$, let us denote by $\rho(z)$ the distance between z and C. Then the constraint (3.39) can be rewritten as:

$$\rho(Z_t) = 0,\ 0 \le t \le 1,\ \mathbb{P}_0 - a.s.$$

Clearly ρ is 1-Lipschitz continuous. Therefore by Assumptions 3.30 and 3.28, it follows from Theorem 3.26 that there exists a minimal solution denoted

$$(Y, Z, K) \in S^2(\mathbb{P}_0, \mathbb{R}) \times H^2(\mathbb{P}_0, \mathbb{R}^d) \times A^2(\mathbb{P}_0, \mathbb{R}) \tag{3.40}$$

to (3.38) subject to constraint (3.39).

Remark 3.32. In the unconstrained case, that is to say $C = \mathbb{R}^d$ and $\tilde{C} = \{0\}$, we can perfectly hedge our contingent claim ξ since we work in a complete market. This can be represented by the following unconstrained BSDE:

$$Y_t^0 = \xi + \int_t^1 f_u(Y_u^0, Z_u^0)du - \int_t^1 Z_u^0 dB_u,\ \mathbb{P}_0 - a.s.$$

which by Assumption 3.28 admits a unique solution in $D^2(\mathbb{P}_0,\mathbb{R})\times H^2(\mathbb{P}_0,\mathbb{R}^d)$. We notice that, by a classical comparison theorem for standard BSDE (Theorem 3.18): $Y_t \geq Y^0_t, dt \times d\mathbb{P} - a.s.$ The non decreasing process K defined in (3.38) can be thus understood as the effect of the market incompleteness.

Let us introduce, for all $\nu \in \mathcal{D}$, $\eta \in L^2(\mathbb{P}_0,\mathbb{R})$ and $(t,s) \in [0,1]$ such that $s \geq t$, the following family of BSDEs:

$$Y^\nu_t(s,\eta) = \eta + \int_t^s \big[f_u\big(Y^\nu_u(s,\eta), Z^\nu_u(s,\eta)\big) - \big(\delta(\nu_u) - \nu_u Z^\nu_u(s,\eta)\big)\big]du - \int_t^s Z^\nu_u(s,\eta)dB_u,\ \mathbb{P}_0 - a.s \tag{3.41}$$

Let us denote:

$$g^\nu_t(y,z) := f_t(y,z) - \big(\delta(\nu_t) - \nu_t z\big).$$

We know that g^ν is Lipschitz continuous in y and in z by the boundedness of $\nu \in \mathcal{D}$ and Assumption 3.28 (claim (3.36)). Moreover, we have by Assumption 3.28 (claim (3.37)), the continuity of δ and the boundedness of $\nu \in \mathcal{D}$ that:

$$\mathbb{E}^{\mathbb{P}_0}\int_0^1 |g^\nu_t(0,0)|^2 dt \leq 2\mathbb{E}^{\mathbb{P}_0}\int_0^1 |f_t(0,0)|^2 dt + 2\mathbb{E}^{\mathbb{P}_0}\int_0^1 |\delta(\nu_t)|^2 dt < \infty. \tag{3.42}$$

Therefore there exists a unique solution $(Y^\nu(s,\eta), Z^\nu(s,\eta)) \in D^2(\mathbb{P}_0,\mathbb{R}) \times H^2(\mathbb{P}_0,\mathbb{R}^d)$ to (3.41).

We have the following inequality for the minimal solution Y (see (3.40)) of the first order constrained problem (3.38) and (3.39).

Lemma 3.33. *Under Assumptions* 3.30 *and* 3.28*, for all* $\nu \in \mathcal{D}$ *and* $t \in [0,1]$*, we have:*

$$\tilde{Y}_t \geq Y_t \geq Y^\nu_t(1,\xi),\ \mathbb{P}_0 - a.s. \tag{3.43}$$

where $\tilde{Y}$ *is defined in Assumption* 3.30*,* Y *in* (3.40) *and* $Y^\nu(1,\xi)$ *in* (3.41)*.*

Proof. Since by Assumption 3.30, $(\tilde{Y},\tilde{Z},\tilde{K})$ is a solution of the first order constrained problem then for all $t \in [0,1]$, we have $\tilde{Z}_t \in C,\ \mathbb{P}_0 - a.s.$ By (3.35) for all $\nu \in \mathcal{D}$ and $t \in [0,1]$, this is equivalent to :

$$\delta(\nu_t) - \tilde{Z}_t\nu_t \geq 0,\ \mathbb{P}_0 - a.s. \tag{3.44}$$

Adding and removing the quantity $\delta(\nu_t) - \tilde{Z}_t\nu_t$ in (3.38) leads to:

$$\tilde{Y}_t = \xi + \int_t^1 \big[f_u(\tilde{Y}_u,\tilde{Z}_u) - \big(\delta(\nu_u) - \tilde{Z}_u\nu_u\big)\big]du - \int_t^1 \tilde{Z}_u dB_u + \big[\tilde{K}_1 - \tilde{K}_t + \int_t^1 (\delta(\nu_u) - \tilde{Z}_u\nu_u)du\big],\ \mathbb{P}_0 - a.s.$$

Then since $\tilde{K}$ is an increasing process and (3.44) holds, by comparison theorem (see Theorem 3.18), we have for all ν and $t \in [0,1]$ that:

$$\tilde{Y}_t \geq Y_t^{\nu}(1,\xi),\ \mathbb{P}_0 - a.s. \tag{3.45}$$

Since (Y,Z,K) is the minimal solution of (3.38) and (3.39), then by definition, $\tilde{Y}$ (see Assumption 3.30) must be greater than Y. Therefore for all $\nu \in \mathcal{D}$ and $t \in [0,1]$, we have:

$$\tilde{Y}_t \geq Y_t \geq Y_t^{\nu}(1,\xi),\ \mathbb{P}_0 - a.s.$$

□

Let us define:

$$V_t = \operatorname*{essup}_{\nu \in \mathcal{D}_{[t,1]}} Y_t^{\nu}(1,\xi),\ \mathbb{P}_0 - a.s. \tag{3.46}$$

where $\mathcal{D}_{[t,1]}$ corresponds to the restriction of $\mathcal{D}$ to $[t,1]\times\Omega$. By Lemma 3.33, since (3.43) holds for all $\nu \in \mathcal{D}$, we notice that:

$$\tilde{Y}_t \geq V_t \geq Y_t^{\nu}(1,\xi)\ \mathbb{P}_0 - a.s. \tag{3.47}$$

This leads to $|V_t| \leq \max\{|\tilde{Y}_t|, |Y_t^{\nu}|\} \in S^2(\mathbb{P}_0,\mathbb{R})$. So we get:

$$\mathbb{E}^{\mathbb{P}_0}\left[\sup_{0\leq t\leq 1} |V_t|^2\right] < \infty. \tag{3.48}$$

In particular, the above claim means that $V_t \in L^2(\mathbb{P}_0,\mathbb{R})$. So for all $\nu \in \mathcal{D}$ and $\theta \in [0,1]$, there exists a unique solution $\big(Y^{\nu}(\theta,V_\theta), Z^{\nu}(\theta,V_\theta)\big) \in D^2(\mathbb{P}_0,\mathbb{R}) \times H^2(\mathbb{P}_0,\mathbb{R}^d)$ to (3.41) with terminal condition V_θ at time θ.

We have the following dynamic programming principle:

Proposition 3.34. *For every* $0 \leq t \leq \theta \leq 1$*, we have:*

$$V_t = \operatorname*{essup}_{\nu \in \mathcal{D}_{[t,\theta]}} Y_t^{\nu}(\theta,V_\theta),\ \mathbb{P}_0 - a.s. \tag{3.49}$$

where $\mathcal{D}_{[t,\theta]}$ *corresponds to the restriction of* $\mathcal{D}$ *to* $[t,\theta]\times\Omega$*.*

Proof. **Step 1:** Let us take any $\nu \in \mathcal{D}$. By definition of $\big(Y^{\nu}(1,\xi), Z^{\nu}(1,\xi)\big)$ (see (3.41)), we have for all $0 \leq t \leq \theta \leq 1$ that:

$$Y_t^{\nu}(1,\xi) = Y_\theta^{\nu}(1,\xi) + \int_t^\theta \big[f_u\big(Y_u^{\nu}(1,\xi), Z_u^{\nu}(1,\xi)\big) - \big(\delta(\nu_u) - \nu_u Z_u^{\nu}(1,\xi)\big)\big]du$$
$$- \int_t^\theta Z_u^{\nu}(1,\xi)dB_u,\ \mathbb{P}_0 - a.s$$

This result can be rewritten as:

$$Y_t^{\nu}(1,\xi) = Y_t^{\nu}\big(\theta, Y_\theta^{\nu}(1,\xi)\big), \mathbb{P}_0 - a.s. \tag{3.50}$$

By the definition (3.46) of V, we know that $V_\theta \geq Y_\theta^{\nu}(1,\xi), \mathbb{P}_0 - a.s.$. Therefore by the previous inequality and (3.50), the comparison theorem 3.18 gives us that:

$$Y_t^{\nu}\big(\theta, V_\theta\big) \geq Y_t^{\nu}(1,\xi),\ \mathbb{P}_0 - a.s.$$

Since the above equation holds for all $\nu \in \mathcal{D}$, then by definition of V, we finally get:

$$\operatorname*{essup}_{\nu\in\mathcal{D}_{[t,\theta]}} Y_t^{\nu}\big(\theta, V_\theta\big) \geq V_t,\ \mathbb{P}_0 - a.s. \tag{3.51}$$

Step 2: Let us show that there is a sequence $(\nu_n)_n$ in $\mathcal{D}_{[\theta,1]}$ such that:

$$V_\theta = \lim_n \uparrow Y_\theta^{\nu_n}(1,\xi),\ \mathbb{P}_0 - a.s.$$

To do it, we need to show that the family $\{Y_\theta^{\nu}(1,\xi)\}_{\nu\in\mathcal{D}_{[\theta,1]}}$ is directed upward. That is to say for any ν_1 and ν_2 in $\mathcal{D}_{[\theta,1]}$, we have to construct $\nu \in \mathcal{D}_{[\theta,1]}$ such that:

$$Y_\theta^{\nu}(1,\xi) \geq \max\big\{Y_\theta^{\nu_1}(1,\xi), Y_\theta^{\nu_2}(1,\xi)\big\},\ \mathbb{P}_0 - a.s. \tag{3.52}$$

For instance, one can define, A, a subset of Ω:

$$A := \big\{\omega \in \Omega :\ Y_\theta^{\nu_1}(1,\xi)(\omega) \geq Y_\theta^{\nu_2}(1,\xi)(\omega)\big\} \in \mathcal{F}_\theta.$$

We denote by ν the following process:

$$\nu(t,\omega) = \bigg(\nu_1(t,\omega)\mathbb{1}_A + \nu_2(t,\omega)\mathbb{1}_{\Omega\setminus A}\bigg)\mathbb{1}_{t\geq\theta}.$$

We can see easily that ν_t is an $\mathcal{F}_t$-measurable process. Since by construction $\mathcal{D}_{[\theta,1]}$ is stable by concatenation, the process ν belongs to $\mathcal{D}_{[\theta,1]}$. Moreover, $Y_\theta^{\nu}(1,\xi)$ is such that it verifies (3.52). So we can conclude that there exists a sequence $(\nu_n)_n$ in $\mathcal{D}_{[\theta,1]}$ such that:

$$V_\theta = \lim_n \uparrow Y_\theta^{\nu_n}(1,\xi),\ \mathbb{P}_0 - a.s. \tag{3.53}$$

Without loss of generality, for some $\mu \in \mathcal{D}_{[t,\theta]}$, this sequence (ν_n) can be taken in

$$\mathcal{M}_\mu := \big\{\nu \in \mathcal{D} : \nu = \mu \text{ on } [t,\theta)\times\Omega\big\}.$$

Therefore for all $n \geq 0$, since $\nu_n = \mu$ on $[t,\theta)\times\Omega$, we have by (3.50) that:

$$Y_t^{\mu}(1,\xi) = Y_t^{\mu}\big(\theta, Y_\theta^{\nu_n}(1,\xi)\big),\ \mathbb{P}_0 - a.s. \tag{3.54}$$

By definition of V_t (see (3.46)), this leads to:

$$V_t \geq Y_t^\mu\big(\theta, Y_\theta^{\nu_n}(1,\xi)\big),\ \mathbb{P}_0 - a.s. \tag{3.55}$$

By Lemma 3.20 and Assumption 3.28, we can find a constant $C > 0$ such that:

$$|Y_t^\mu\big(\theta, Y_\theta^{\nu_n}(1,\xi)\big) - Y_t^\mu\big(\theta, V_\theta\big)|^2 \leq C\mathbb{E}_t^{\mathbb{P}_0}\big[|Y_\theta^{\nu_n}(1,\xi) - V_\theta|^2\big] \tag{3.56}$$

By the convergence result (3.53), the monotonic convergence theorem gives us:

$$\lim_n Y_t^\mu\big(\theta, Y_\theta^{\nu_n}(1,\xi)\big) = Y^\mu(\theta, V_\theta),\ \mathbb{P}_0 - a.s. \tag{3.57}$$

Combining (3.55) and (3.57), we thus get:

$$V_t \geq Y_t^\mu\big(\theta, V_\theta\big),\ \mathbb{P}_0 - a.s. \tag{3.58}$$

Since it holds for any $\mu \in \mathcal{D}_{[t,\theta]}$, then:

$$V_t \geq \operatorname*{essup}_{\mu \in \mathcal{D}_{[t,\theta]}} Y_t^\mu\big(\theta, V_\theta\big),\ \mathbb{P}_0 - a.s. \tag{3.59}$$

□

Let us define now the right limit of V defined by:

$$V_t^+ := \limsup_{r \in \mathbb{Q} \cap (t,1], r \downarrow t} V_r,\ t \in [0,1]. \tag{3.60}$$

Clearly V^+ is defined for all $(t,\omega) \in [0,1] \times \Omega$ and is $\mathbb{F}^+$-measurable.

Lemma 3.35. *Let Assumptions* 3.30 *and* 3.28 *hold. Then:*

$$V_t^+ = \lim_{r \in \mathbb{Q} \cap (t,1], r \downarrow t} V_r,\ \mathbb{P}_0 - a.s..$$

Therefore V^+ is càdlàg, $\mathbb{P}_0 - a.s.$

Proof. For all ν in $\mathcal{D}$, let us define:

$$V^\nu = V - Y^\nu(1,\xi). \tag{3.61}$$

By definition of V (see (3.46)), we have $V^\nu \geq 0, \mathbb{P}_0 - a.s.$ Let us take $(s,t) \in [0,1]^2$ such that $0 \leq s \leq t$ and consider for all $\nu \in \mathcal{D}$:

$$\bar{y}_s^{\nu,t} := Y_s^\nu(t, V_t) - Y_s^\nu(1,\xi) \text{ and } \bar{z}_s^{\nu,t} := Z_s^\nu(t, V_t) - Z_s^\nu(1,\xi).$$

By Proposition 3.34, we have that $V_s^\nu \geq \bar{y}_s^{\nu,t},\ 0 \leq s \leq t\ \mathbb{P}_0 - a.s.$ Moreover the $\left(\bar{y}^{\nu,t}, \bar{z}^{\nu,t}\right)$ satisfies:

$$\bar{y}_s^{\nu,t} = V_t^\nu + \int_s^t h_u^\nu\left(\bar{y}_u^{\nu,t}, \bar{z}_u^{\nu,t}\right) du - \int_s^t \bar{z}_u^{\nu,t} dB_u,\ \mathbb{P}_0 - a.s. \tag{3.62}$$

where h^ν is defined by:

$$h_t^\nu(y,z) := f_t\left(y + Y_t^\nu(1,\xi), z + Z_t^\nu(1,\xi)\right) - f_t\left(Y_t^\nu(1,\xi), Z_t^\nu(1,\xi)\right) - \nu_t z. \tag{3.63}$$

We know that $V_t^\nu \in L^2(\mathbb{P}_0, \mathbb{R})$ since Y_t^ν and V_t are in $L^2(\mathbb{P}_0, \mathbb{R})$ (see (3.48)). By Assumption 3.28 and the boundedness of ν, we can see that h^ν is a Lipschitz continuous function with respect to (y,z) and verifies:

$$\mathbb{E}^{\mathbb{P}_0}\left[\int_0^t |h_s^\nu(0,0)|^2 ds\right] < \infty.$$

Therefore, there exists a unique solution $\left(\bar{y}^{\nu,t}, \bar{z}^{\nu,t}\right)$ to (3.62). This means that V^ν is a positive weak h^ν-supermartingale (see definition 3.22).

The application of the downcrossing inequality in [12], Theorem 6 shows that we can not have an infinite number of oscillations so $\lim_{r \in \mathbb{Q} \cap (t,1], r \downarrow t} V_r^\nu$ exists for all $t \in [0,1]$(see proof p14 in [49]). Since the paths of $Y^\nu(1,\xi)$ are continuous as a solution of a classical BSDE with Lipschitz continuous generator, we can see that actually

$$V_t^+ = \lim_{r \in \mathbb{Q} \cap (t,1], r \downarrow t} V_r.$$

Thus V^+ is a càdlàg process $\mathbb{P}_0$-a.s.

□

The following Lemma says that V is in fact a RCLL process.

Lemma 3.36. *Let us assume* 3.30 *and* 3.28*, then for all* $t \in [0,1]$*:*

$$V_t^+ = \operatorname*{essup}_{\nu \in \mathcal{D}_{[t+,1]}} Y_t^\nu(1,\xi),\ \mathbb{P}_0 - a.s. \tag{3.64}$$

where $\mathcal{D}_{[t+,1]}$ *is the restriction of* $\mathcal{D}$ *to* $[t+,1] \times \Omega$*. Moreover* $V^+ = V,\ \mathbb{P}_0 - a.s.$

Proof. We recall that by Lemma 3.35, we have:

$$V_t^+ = \lim_{r \in \mathbb{Q} \cap (t,1], r \downarrow t} V_r.$$

Step 1: Our aim is to prove (3.64). By definition of V (see (3.46)), for all ν in $\mathcal{D}_{[t+,1]}$, and $r \in \mathbb{Q} \cap (t,1]$, we have:

$$V_r \geq Y_r^{\nu}(1,\xi),\ \mathbb{P}_0 - a.s.$$

We recall that, by definition, $(Y^{\nu}, Z^{\nu}) \in D^2(\mathbb{P}_0,\mathbb{R}) \times H^2(\mathbb{P}_0,\mathbb{R}^d)$ is the unique solution of (3.41). So $Y^{\nu}(1,\xi)$ has continuous paths. Therefore by Lemma 3.35, sending $r \downarrow t$, we get that:

$$V_t^+ \geq Y_t^{\nu}(1,\xi),\ \mathbb{P}_0 - a.s.$$

Since the above inequality stands for all ν in $\mathcal{D}_{[t+,1]}$, then we have:

$$V_t^+ \geq \underset{\nu \in \mathcal{D}_{[t+,1]}}{\text{essup}}\ Y_t^{\nu}(1,\xi),\ \mathbb{P}_0 - a.s. \tag{3.65}$$

On the other side, for a given $r \in \mathbb{Q} \cap (t,1]$, we have shown in the proof of Proposition 3.34, claim (3.53), that there is a sequence $(\nu_n)_{n\geq 0}$ in $\mathcal{D}_{[r,1]}$ such that :

$$V_r = \lim_n \uparrow Y_r^{\nu_n}(1,\xi),\ \mathbb{P}_0 - a.s.$$

Without loss of generality, since $\mathcal{D}_{[r,1]} \subset \mathcal{D}_{[t,1]}$ then for any $\nu \in \mathcal{D}_{[t,1]}$, we can assume that for all $n \geq 0$ and $(s,\omega) \in [t,r) \times \Omega$:

$$\nu_n(s,\omega) = \nu(s,\omega). \tag{3.66}$$

By the stability of BSDEs, we obtain for all $t \in [0,r)$:

$$Y_t^{\nu}(r,V_r) = Y_t^{\nu}\big(r, \lim_n Y_r^{\nu_n}(1,\xi)\big) = \lim_n Y_t^{\nu}\big(r, Y_r^{\nu_n}(1,\xi)\big),\ \mathbb{P}_0 - a.s. \tag{3.67}$$

Plugging (3.66) in (3.67) and using (3.50), we obtain:

$$Y_t^{\nu}(r,V_r) = \lim_n Y_t^{\nu_n}\big(r, Y_r^{\nu_n}(1,\xi)\big) = \lim_n Y_t^{\nu_n}(1,\xi),\ \mathbb{P}_0 - a.s.$$

Since for all $n \geq 0$, $\nu_n \in \mathcal{D}_{[r,1]}$ is arbitrary and $\mathcal{D}_{[r,1]} \subset \mathcal{D}_{[t+,1]}$, this leads to:

$$Y_t^{\nu}(r,V_r) \leq \underset{\nu \in \mathcal{D}_{[t+,1]}}{\text{essup}}\ Y_t^{\nu}(1,\xi),\ \mathbb{P}_0 - a.s.$$

By the stability of BSDEs and Lemma 3.35, sending $r \downarrow t$ we get:

$$V_t^+ \leq \underset{\nu \in \mathcal{D}_{[t+,1]}}{\text{essup}}\ Y_t^{\nu}(1,\xi),\ \mathbb{P}_0 - a.s. \tag{3.68}$$

To conclude by (3.65) and (3.68), we have for all $t \in [0,1]$:

$$V_t^+ = \underset{\nu \in \mathcal{D}_{[t+,1]}}{\text{essup}}\ Y_t^{\nu}(1,\xi),\ \mathbb{P}_0 - a.s.$$

step 2: Let us show that $V = V^+$. Since for all $t \in [0,1]$:

$$\mathcal{D}_{[t+,1]} \subset \mathcal{D}_{[t,1]}$$

then by (3.64), $V_t \leq V_t^+, \mathbb{P}_0 - a.s.$
For all $t \in [0,1]$ and $r \in \mathbb{Q} \cap (t,1]$ let us take ν in $\mathcal{D}_{[r,1]}$, then by definition $V_r \geq Y_r^\nu(1,\xi), \mathbb{P}_0 - a.s.$ Let us send $r \downarrow t$. Then using Lemma 3.35 and the path continuity of Y^ν, we get:

$$V_t^+ \geq Y_t^\nu(1,\xi),\ \mathbb{P}_0 - a.s.$$

Since $\nu \in \mathcal{D}_{[r,1]}$ and $\mathcal{D}_{[r,1]} \subset \mathcal{D}_{[t,1]}$, we get $V_t^+ \geq V_t, \mathbb{P}_0 - a.s.$

□

The following is the main result of this section and provides the analogous of the stochastic representation given by Cvitanic, Karatzas and Soner in [21] Theorem 5.1 for a generator f not necessarily convex with respect to z.

Theorem 3.37. *Let Assumptions* 3.30 *and* 3.28 *hold. Then the minimal solution* (Y,Z,K) *defined in* (3.40) *of the first order constrained problem* (3.38) *and* (3.39) *has the following representation:*

$$Y_t = \operatorname*{essup}_{\nu \in \mathcal{D}_{[t,1]}} Y_t^\nu(1,\xi),\ \mathbb{P}_0 - a.s.$$

for all $t \in [0,1]$.

To prove it, we begin with the following Proposition which provides a non-linear Dood-Meyer decomposition for the process V.

Proposition 3.38. *Under the assumptions of Theorem* 3.37*, for every* $\nu \in \mathcal{D}$*, there exists* $(Z^\nu, K^\nu) \in H^2(\mathbb{P}_0,\mathbb{R}^d) \times A^2(\mathbb{P}_0,\mathbb{R})$ *such that:*

$$V_t = V_0 - \int_0^t \left[f_u\big(V_u, Z_u^\nu\big) - \big(\delta(\nu_u) - \nu_u Z_u^\nu\big)\right] du + \int_0^t Z_u^\nu dB_u - K_t^\nu,\ \mathbb{P}_0 - a.s. \tag{3.69}$$

for all $t \in [0,1]$.

Proof. Since by Lemma 3.35, V^+ is càdlàg, then it follows from Lemma 3.36 that V is a càdlàg process. Therefore, for all ν in $\mathcal{D}$, since $Y^\nu(1,\xi)$ has continuous paths, V^ν defined in (3.61) is a càdlàg process. Moreover for every $\nu \in \mathcal{D}$, we have shown at the end of the proof of Lemma 3.35 that V^ν is a positive weak h^ν-supermartingale (for the definition of h^ν, see (3.63)). We can then apply Theorem 7 of [12] and conclude that V^ν is a càdlàg positive strong h^ν-supermartingale. This means that for every $\mathbb{F}$-stopping times (τ_1, τ_2) and $\nu \in \mathcal{D}$, we have:

$$\bar{y}_s^{\nu,\tau_2} = V_{\tau_2}^\nu + \int_s^{\tau_2} h_u^\nu(\bar{y}_u^{\nu,\tau_2}, \bar{z}_u^{\nu,\tau_2})du - \int_s^{\tau_2} \bar{z}_u^{\nu,\tau_2} dB_u, \; s \in [\tau_1, \tau_2], \; \mathbb{P}_0 - a.s. \tag{3.70}$$

and

$$V_{\tau_1}^\nu \geq \bar{y}_{\tau_1}^{\nu,\tau_2}, \; \mathbb{P}_0 - a.s.$$

By definitions of V^ν and $\bar{y}^{\nu,\tau_2}$ (see (3.61) and (3.62)), this leads to

$$V_{\tau_1} \geq Y_{\tau_1}^\nu(\tau_2, V_{\tau_2}), \; \mathbb{P}_0 - a.s.$$

and by its definition, (3.41), $\big(Y_t^\nu(\tau_2, V_{\tau_2}), Z_t^\nu(\tau_2, V_{\tau_2})\big)$ is the unique solution of:

$$\begin{aligned} Y_s^\nu(\tau_2, V_{\tau_2}) = V_{\tau_2} + \int_s^{\tau_2} \big[f_u\big(Y_u^\nu(\tau_2, V_{\tau_2}), Z_u^\nu(\tau_2, V_{\tau_2})\big) - \big(\delta(\nu_u) - Z_u^\nu(\tau_2, V_{\tau_2})\big)\big]du \\ - \int_s^{\tau_2} Z_u^\nu(\tau_2, V_{\tau_2})dB_u, \; \mathbb{P}_0 - a.s. \end{aligned} \tag{3.71}$$

Therefore for all $\nu \in \mathcal{D}$, V is a strong g^ν-supermartingale where we recall that $g_t^\nu(y, z) := f_t(y, z) - \big(\delta(\nu_t) - \nu_t z\big)$.

Under Assumption 3.28 and the fact that V is a strong g^ν-supermartingale such that claim (3.48) holds, by the nonlinear Doob-Meyer decomposition theorem (Theorem 3.23) applied under each $\nu \in \mathcal{D}$, there exists $(Z^\nu, K^\nu) \in H^2(\mathbb{P}_0, \mathbb{R}^d) \times A^2(\mathbb{P}_0, \mathbb{R})$ such that:

$$V_t = V_0 - \int_0^t \big[f_u\big(V_u, Z_u^\nu\big) - \big(\delta(\nu_u) - \nu_u Z_u^\nu\big)\big]du + \int_0^t Z_u^\nu dB_u - K_t^\nu, \; \mathbb{P}_0 - a.s.$$

for all $t \in [0, 1]$.

□

Proof. of Theorem 3.37*:*
First, we need to show that V is a solution to the first order constrained problem (3.38) and (3.39). Following the argument of Cvitanic, Karatzas and Soner in [21], since we have obtained a nonlinear Doob-Meyer decomposition for V in Proposition 3.38, it remains to have the analogous of Proposition 2.5 in [21] for a generator which depends on z. We recall that this Proposition says that for all $\nu \in \mathcal{D}$, the process $\hat{Z}$ defined by

$$\hat{Z} := Z^\nu = Z^0. \tag{3.72}$$

does not depends on ν and neither does the process $\hat{K}$ defined for all $\nu \in \mathcal{D}$ by:

$$\hat{K}_t := K_t^0 = K_t^\nu - \int_0^t \big[\delta(\nu_u) - Z_u \nu_u\big] du.$$

Moreover $\hat{Z}$ belongs to C, $\mathbb{P}_0 - a.s.$ The proof of these above results for a generator which depends on z is the same as these of Proposition 2.5 in [21] so we do not recall it here. Therefore, combining Proposition 2.5 in [21] and Proposition 3.38, we obtain:

$$V_t = \xi + \int_t^1 f_u\big(V_u, \hat{Z}_u\big) du - \int_t^1 \hat{Z}_u dB_u + \hat{K}_1 - \hat{K}_t,\ \mathbb{P}_0 - a.s.$$

and $\hat{Z}_t \in C, \mathbb{P}_0 - a.s.$ for all $t \in [0,1]$. So $(V, \hat{Z}, \hat{K})$ is a solution to (3.38) and (3.39). Therefore, since by definition (see (3.40)) (Y, Z, K) is the minimal solution of (3.38) and (3.39), then:

$$Y_t \le V_t,\ 0 \le t \le 1,\ \mathbb{P}_0 - a.s. \tag{3.73}$$

On the other side, by Lemma 3.33, since for all $\nu \in \mathcal{D}$, we have:

$$Y_t \ge Y_t^\nu(1, \xi),\ 0 \le t \le 1,\ \mathbb{P}_0 - a.s.$$

Then by definition of V, $Y_t \ge V_t,\ 0 \le t \le 1,\ \mathbb{P}_0 - a.s.$ So by (3.73), we get that

$$Y_t = V_t,\ 0 \le t \le 1,\ \mathbb{P}_0 - a.s.$$

□

3.4 The second order BSDEs

In this section, we will provide a quick review on the unconstrained second order backward stochastic equations. For that purpose, we will consider the framework defined in subsection 3.2.2 and always consider in this section the particular case where :

$$C = \mathbb{R}^d.$$

This means that $\tilde{C} = \{0\}$, so $\mathcal{D}$ is reduced to the null process.

3.4.1 Definition

We shall consider the following unconstrained problem:

$$Y_t = \xi - \int_t^1 \hat{F}_s(Y_s, Z_s) ds - \int_t^1 Z_s dB_s + K_1 - K_t,\ 0 \le t \le 1,\ \surd_{H} - q.s. \tag{3.74}$$

We recall the definition of the solution of 2BSDEs which can be found in [86]. Throughout this subsection, we shall always assume Assumption 3.8.

Definition 3.39. *For ξ in $\mathbb{L}^2_H(\mathbb{R})$, we say that $(Y,Z) \in \mathbb{S}^2_H(\mathbb{R}) \times \mathbb{H}^2_H(\mathbb{R}^d)$ is a solution to 2BSDE* (3.74) *if*

(i) $Y_1 = \xi$, $\mathcal{P}_H$*-q.s.*

(ii) For each $\mathbb{P} \in \mathcal{P}_H$, the process $K^{\mathbb{P}}$ defined below has non decreasing paths, $\mathbb{P}-a.s.$:

$$K_t^{\mathbb{P}} := Y_0 - Y_t + \int_0^t \hat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ 0 \le t \le 1,\ \mathbb{P}-a.s.$$

(iii) The family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\}$ defined as above satisfies the following minimum condition:

$$K_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}_H(t+,\mathbb{P})}{\text{essinf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}\left[K_1^{\mathbb{P}'}\right],\ \mathbb{P}-a.s \text{ for all } \mathbb{P} \in \mathcal{P}_H,\ t \in [0,1]. \quad (3.75)$$

where for all $\mathbb{P} \in \mathcal{P}_H$ and $t \in [0,1]$, $\mathcal{P}_H(t+,\mathbb{P}) := \{\mathbb{P}' \in \mathcal{P}_H : \mathbb{P}' = \mathbb{P}$ on $\mathcal{F}_t^+\}$.

If $\{K^{\mathbb{P}},\ \mathbb{P} \in \mathcal{P}_H\}$ can be aggregated into a universal process K, we say that (Y,Z,K) is a solution to (3.74).

Remark 3.40. As it can be seen in [86], the minimum condition ensures only the uniqueness of the solution.

Let us make the following additional Assumption:

Assumption 3.41.

$$\sup_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\underset{0 \le t \le 1}{\text{essup}^{\mathbb{P}}} \underset{\mathbb{P}' \in \mathcal{P}_H(t+,\mathbb{P})}{\text{essup}^{\mathbb{P}'}} \mathbb{E}^{\mathbb{P}'}\Big[\int_0^1 |\hat{F}_t(0,0)|^2 dt\Big]\right] < \infty$$

Let us define for every $\mathbb{F}$-stopping time τ and any η in $\mathbb{L}^2_H(\mathbb{R})$, $\big(\mathcal{Y}^{\mathbb{P}}(\tau,\eta), \mathcal{Z}^{\mathbb{P}}(\tau,\eta)\big) \in D^2(\mathbb{P},\mathbb{R}) \times H^2(\mathbb{P},\mathbb{R}^d)$ the solution of the following standard BSDE:

$$\mathcal{Y}_t^{\mathbb{P}}(\tau,\eta) = \eta - \int_t^\tau \hat{F}_s\big(\mathcal{Y}_s^{\mathbb{P}}(\tau,\eta), \mathcal{Z}_s^{\mathbb{P}}(\tau,\eta)\big)ds - \int_t^\tau \mathcal{Z}_s^{\mathbb{P}}(\tau,\eta)dB_s,\ \mathbb{P}-a.s.$$

Then the following representation theorem holds for the solution of (3.74):

Theorem 3.42. *(Soner, Touzi, Zang [86])*
Let Assumptions 3.8 *and* 3.41 *hold. For any $\xi \in \mathbb{L}^2_H(\mathbb{R})$ and $(Y,Z) \in \mathbb{S}^2_H(\mathbb{R}) \times \mathbb{H}^2_H(\mathbb{R}^d)$ solution of the 2BSDE* (3.74), *we have for all $\mathbb{P} \in \mathcal{P}_H$ and $(t_1,t_2) \in [0,1]^2$:*

$$Y_{t_1} = \underset{\mathbb{P}' \in \mathcal{P}_H(t_1+,\mathbb{P})}{\text{essup}^{\mathbb{P}}} \mathcal{Y}_{t_1}^{\mathbb{P}'}\big(t_2, Y_{t_2}\big),\ \mathbb{P}-a.s.$$

Moreover the 2*BSDE* (3.74) *has at most one solution in $\mathbb{S}^2_H(\mathbb{R}) \times \mathbb{H}^2_H(\mathbb{R})$.*

Remark 3.43. This representation theorem leads in particular to a comparison theorem for 2BSDEs. Indeed for any $(\xi_1, \xi_2) \in \mathbb{L}^2_H(\mathbb{R})$ such that $\xi_1 \leq \xi_2$ and $(Y_i, Z_i)_{i=1,2}$ in $\mathbb{S}^2_H(\mathbb{R}) \times \mathbb{H}^2_H(\mathbb{R}^d)$ solution of (3.74), we have $Y_1 \leq Y_2, \mathbb{P}_H - q.s.$

We denote by $UC_b(\Omega)$ the set of all bounded and uniformly continuous maps with respect to $\|.\|_\infty$-norm. We define $\mathcal{L}^2_H(\mathbb{R})$ as the closure of $UC_b(\Omega)$ under the following norm:

$$\|\xi\|^2_{\mathcal{L}^2_H(\mathbb{R})} := \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\underset{0\leq t\leq 1}{\text{essup}}^{\mathbb{P}}\left(\underset{\mathbb{P}'\in\mathbb{P}_H(t+,\mathbb{P})}{\text{essup}^{\mathbb{P}'}} \mathbb{E}_t^{\mathbb{P}'}\left[|\xi|\right]\right)^2\right].$$

One can show the following theorem, which gives existence and uniqueness for the solution of the 2BSDE (3.74):

Theorem 3.44. *(Soner, Touzi and Zhang, [86]) Under Assumptions* 3.8 *and* 3.41*, for any ξ in $\mathcal{L}^2_H(\mathbb{R})$, the* 2*BSDE* (3.74) *has a unique solution (Y, Z) in $\mathbb{S}^2_H(\mathbb{R}) \times \mathbb{H}^2_H(\mathbb{R}^d)$.*

Remark 3.45. In contrast with the stochastic target problem [85], we do not need to assume further regularity in ω for ξ. Indeed one can prove the existence for ξ in $UC_b(\Omega)$ (proof of Theorem 4.5 in [85]) and then pass to the limit under the $\|.\|_{\mathcal{L}^2_H(\mathbb{R})}$-norm (Theorem 4.6 in [86]).

3.4.2 The second order stochastic target problem [85]

Stochastic target problems were introduced mainly in [82] as a generalization of superhedging problem in volatility uncertainty. For more accuracy, let us denote the set

$$\widehat{\mathcal{SM}}^2_H(\mathbb{R}^d) := \bigcap_{\mathbb{P}\in\mathcal{P}_H} \mathcal{SM}^2_H(\mathbb{P}, \mathbb{R}^d)$$

where $\mathcal{SM}^2_H(\mathbb{P}, \mathbb{R}^d)$ is the space of all square integrable $(\mathbb{P}, \mathbb{F})$-semimartingales Z with quadratic covariation with B, Γ. Γ is chosen to be $\mathbb{F}$-progressively measurable and such that $\frac{1}{2}\hat{a} : \Gamma - H(0, 0, \Gamma) \in H^2(\mathbb{P}, \mathbb{R}^d)$.
For any y in $\mathbb{R}$ and Z in $\widehat{\mathcal{SM}}^2_H(\mathbb{R}^d)$, we define $Y^{y,Z} := Y \in \mathbb{S}^2_H(\mathbb{R})$ solution of the following ordinary differential equation for all $t \in [0, 1]$:

$$\begin{aligned} Y_t &= y - \int_0^t H_s(Y_s, Z_s, \Gamma_s)ds + \int_0^t Z_s \circ dB_s \\ &= y + \int_0^t \left[\frac{1}{2}\hat{a}_s : \Gamma_s - H_s(Y_s, Z_s, \Gamma_s)\right] ds + \int_0^t Z_s dB_s,\ \mathcal{P}_H - q.s. \end{aligned} \quad (3.76)$$

where $\circ$ denotes the Stratonovitch stochastic integral. The second order stochastic target problem is defined for any $\mathbb{F}$-measurable random variable ξ by:

$$v(\xi) := \inf\Big\{y : Y_1^{y,Z} \geq \xi, \mathcal{P}_H - q.s., \text{ for some } Z \in \widehat{\mathcal{SM}}_H^2(\mathbb{R}^d)\Big\}.$$

Soner, Touzi and Zhang in [85] show how the problem can be relaxed in order to obtain a dual formulation. Indeed let us introduce for $y \in \mathbb{R}$, $\tilde{Z} \in \mathbb{H}_H^2(\mathbb{R}^d)$ and $\mathbb{P} \in \mathcal{P}_H$, $\tilde{Y} := \tilde{Y}^{\mathbb{P},y,\tilde{Z}} \in D^2(\mathbb{P}, \mathbb{R})$ the unique solution of:

$$\tilde{Y}_t = y + \int_0^t \hat{F}_t\big(\tilde{Y}_u, \tilde{Z}_u\big)du + \int_0^t \tilde{Z}_u dB_u, \; t \in [0,1], \; \mathbb{P} - a.s. \tag{3.77}$$

Such a unique solution exists by 3.15 and 3.16. Furthermore this is a relaxation of problem (3.76) since it does not take into account the semimartingale constraint and the dependence on Γ. We notice that in this case $\int_0^t \tilde{Z}_u dB_u$ may failed to have a $\mathcal{P}_H-$quasi-sure version. Indeed, $\int_0^t \tilde{Z}_u dB_u$ is defined $\mathbb{P}-$a.s. and there is a priori no aggregated version. Therefore (3.77) is only defined under each $\mathbb{P} \in \mathcal{P}_H$. We define the relaxed second order target problem by:

$$\tilde{V}(\xi) := \inf\big\{y : \exists \tilde{Z} \in \mathbb{H}_H^2(\mathbb{R}^d) \text{ s.t } \tilde{Y}_1^{\mathbb{P},y,\tilde{Z}} \geq \xi, \; \mathbb{P} - a.s., \text{ for all } \mathbb{P} \in \mathcal{P}_H\big\}.$$

From then on, they established the connection with the 2BSDEs on its wellposedness form (see definition 3.39 and [86]) via the following duality result:

Theorem 3.46. *(Soner, Touzi, Zhang in [85]) Under Assumption* 3.8*, for any* $\xi \in \mathbb{L}_H^2(\mathbb{R})$*,*

$$\tilde{V}(\xi) = \sup_{\mathbb{P} \in \mathcal{P}_H} \mathcal{Y}_0^{\mathbb{P}}(1, \xi)$$

To prove the dual formulation for the primal problem, that is to say when Z is a semi-martingale, they restrict the set of martingale measures to a subset of $\mathcal{P}_H$ constructed out of a countable subset. For more details, see section 5 of [85].

Remark 3.47. What is the intuition behind this theorem? To do it formally, we are going to assume a Markov structure to this problem and thus consider a function u such that $u(t, B_t) := Y_t$. Let assume that H is non-decreasing and convex in γ, then by [82], Theorem 3.2, u is a viscosity solution of a fully nonlinear PDE:

$$\begin{cases} -\partial_t u - H(t, x, u(t,x), Du(t,x), D^2 u(t,x)) = 0 \\ u(1, x) = x \end{cases}$$

For any $a \in \mathbb{S}_d^{>0} \cap D_f$, let u^a be a viscosity solution of the semilinear PDE

$$\begin{cases} -\partial_t u - \frac{1}{2} a : D^2 u^a(t,x) + F(t,x,u^a(t,x), Du^a(t,x), a) = 0 \\ u^a(1,x) = x \end{cases}$$

Formally, by classical comparison theorem for parabolic PDEs, we have: $u \geq u^a$. Assume that u^a belongs to $C^{1,2}\big([0,1]\times\mathbb{R}\big)$ and define $Y_t^a := u^a(t, X_t^a)$, $Z_t^a = Du^a(t, X_t^a)$ where $X_s^a = x + \int_s^t a_r^{1/2} dB_r$ then (Y^a, Z^a) solves the following standard BSDE:

$$Y_t^a = X_1^a - \int_t^1 F_s(X_s^a, Y_s^a, Z_s^a, a) ds - \int_t^1 Z_s^a a_s^{1/2} dB_s, \ \mathbb{P} - a.s.$$

We have show formally that $Y_t := \sup_{a \in \mathbb{S}_d^{>0} \cap D_f} Y_t^a$ is a natural candidate to solve our problem.

3.4.3 The G-integration theory [68]

Let us define $G(x) := \frac{1}{2} \sup_{\underline{a} \leq a \leq \overline{a}} a : x$ for $(\overline{a}, \underline{a})$ in $\mathbb{S}_d^{>0}$. Following Peng in [68], let u be the unique bounded viscosity solution of the following nonlinear equation:

$$\begin{cases} -\partial_t u(t,x) - G\big(D^2 u(t,x)\big) = 0. \\ u(1,x) = x. \end{cases}$$

where $(t,x) \in [0,1]\times\mathbb{R}$. Then the G-expectation of B_1 is defined by: $\mathbb{E}^G[B_1] := u(0,0)$ and the conditional G-expectation of B_1 is defined by: $\mathbb{E}_t^G[B_1] := u(t, B_t)$. By a density argument, we can extend this definition to random variable ξ in $\mathbb{L}_H^2$.
If we take $H_t(y,z,\gamma) := G(\gamma)$ then (3.74) is reduced to:

$$Y_t = \xi - \int_t^1 Z_s dB_s + K_1 - K_t = \mathbb{E}_t^G[\xi] + \int_0^t Z_s dB_s - K_t, \ \mathcal{P}_H - q.s. \tag{3.78}$$

That is to say, the solution of (3.78) corresponds to the martingale representation theorem for $G-$martingales [83], Theorem 5.1. We can see clearly that a G-martingale can be seen as a 2BSDE with a null generator. Moreover, we can see that the G-expectation can be seen as a generalization of the Feynman-Kac formula for a class of fully nonlinear second order partial differential equations.

3.5 Constrained second order BSDE

We want to extend the results obtained in the first order case, that is to say, an existence result and a stochastic representation theorem to the case of the second order BSDEs. In particular, this means that we have to generalize the results obtained for a single probability measure to a non-dominated class of singular martingale measures. At first glance, there is at least two ways to proceed to solve this kind of problem. A first idea would be to adapt the penalization argument developped by Peng in [67] to a set of non-dominated singular martingale measures. However, we do not have currently a convenient monotonic convergence theorem for such a class of martingale measures (see Possamai's thesis [74], p121-124), so we have to look for another track. Indeed, we will follow the pathwise approach of [85] and [86] and provide an existence theorem and a stochastic representation theorem (like Theorem 3.37 for the first order case) for such a problem.

In this section we will consider the framework developed in Section 3.2.

3.5.1 The problem

The Second Order Problem :
We want to find a pair of processes $(Y, Z) \in \mathbb{S}^2_H(\mathbb{R}) \times \mathbb{H}^2_H(\mathbb{R}^d)$ solution of the 2BSDE:

$$Y_t = \xi - \int_t^1 \hat{F}_u(Y_u, Z_u)du - \int_t^1 Z_u dB_u + K_1 - K_t,\ 0 \le t \le 1,\ \mathcal{P}_H - q.s. \quad (3.79)$$

Furthermore, the process Z must be such that

$$Z_t \in C,\ 0 \le t \le 1, \quad \mathcal{P}_H - q.s \quad (3.80)$$

Definition 3.48. *For $\xi \in \mathbb{L}^2_H(\mathbb{R})$, we say that $(Y, Z) \in \mathbb{S}^2_H(\mathbb{R}) \times \mathbb{H}^2_H(C)$ is a minimal solution to (3.79) and (3.80) if :*

(i) $Y_1 = \xi,\ \mathcal{P}_H - q.s.$

(ii) For each $\mathbb{P} \in \mathcal{P}_H$, the process $K^{\mathbb{P}} \in A^2(\mathbb{P}, \mathbb{R})$ defined below has non decreasing paths, $\mathbb{P} - a.s.$ and is defined by:

$$K^{\mathbb{P}}_t := Y_t - Y_0 + \int_0^t \hat{F}_u(Y_u, Z_u)du + \int_0^t Z_u dB_u, \quad 0 \le t \le 1, \quad \mathbb{P} - a.s. \quad (3.81)$$

(iii) Furthermore, for any $(\tilde{Y}, \tilde{Z}) \in \mathbb{S}^2_H(\mathbb{R}) \times \mathbb{H}^2_H(\mathbb{R}^d)$ which verify (3.79) and (3.80), we have

$$Y_t \le \tilde{Y}_t,\ 0 \le t \le 1,\ \mathcal{P}_H - q.s.$$

Remark 3.49. Comparing this definition to the one of unconstrained 2BSDEs, that is to say Definition 3.39, we do not have to verify the minimum condition (3.75). Indeed, this last condition ensures the uniqueness of the solution of the unconstrained 2BSDEs (see the proof of Theorem 4.3 in [86]) while for constrained 2BSDEs, as for constrained BSDEs, we only seek for a minimal solution.

3.5.2 Existence of a minimal solution

For all $s \in [0,1]$, $t \in [0,s]$, $\mathbb{P} \in \mathcal{P}_H^t$, $\omega \in \Omega$, η in $L^2(\mathbb{P},\mathbb{R})$ and $\nu \in \mathcal{D}$, we introduce the following family of standard BSDEs

$$\big(\mathcal{Y}^{\mathbb{P},\nu,t,\omega}(s,\eta), \mathcal{Z}^{\mathbb{P},\nu,t,\omega}(s,\eta)\big) := \big(y^{\mathbb{P},\nu,t,\omega}, z^{\mathbb{P},\nu,t,\omega}\big) \in D^2(\mathbb{P},\mathbb{R}) \times H^2(\mathbb{P},\mathbb{R}^d)$$

where $\big(y^{\mathbb{P},\nu,t,\omega}, z^{\mathbb{P},\nu,t,\omega}\big)$ is the unique solution of the following BSDE:

$$y_r^{\mathbb{P},\nu,t,\omega} = \eta^{t,\omega} - \int_r^s \Big[\hat{F}_u^{t,\omega}\big(y_u^{\mathbb{P},\nu,t,\omega}, z_u^{\mathbb{P},\nu,t,\omega}\big) + \big(\delta(\nu_u) - \nu_u z_u^{\mathbb{P},\nu,t,\omega}\big)(\hat{a}_u^t)^{1/2}\Big] du$$
$$- \int_r^s z_u^{\mathbb{P},\nu,t,\omega} dB_u^t, \quad r \in [t,s], \ \mathbb{P} - a.s. \tag{3.82}$$

Let us define:

$$\hat{G}_r^{\nu,t,\omega}(y,z) := \hat{F}_r^{t,\omega}(y,z) + \big(\delta(\nu_r) - \nu_r z\big)(\hat{a}_r^t)^{1/2}.$$

By Assumption 3.8 and the boundedness of $\nu \in \mathcal{D}$, we know that $\hat{G}^{\nu,t,\omega}$ is Lipschitz continuous in y and in z. Moreover since $\mathbb{P} \in \mathcal{P}_H^t$, (see definition 3.13), we have:

$$|\hat{a}^t| \leq \overline{a}_{\mathbb{P}}, \ dt \times d\mathbb{P}^t\text{-a.e. on } [t,1] \times \Omega^t$$

So, by the continuity of δ, the boundedness of ν and (3.24), this leads to:

$$\mathbb{E}^{\mathbb{P}}\Big[\int_t^1 \big(\hat{G}_r^{t,\nu,\omega}(0,0)\big)^2 dr\Big] < 2\mathbb{E}^{\mathbb{P}}\Big[\int_t^1 \big(\hat{F}_r^{t,\omega}(0,0)\big)^2 dr\Big] + 2\mathbb{E}^{\mathbb{P}}\Big[\int_t^1 \delta(\nu_r)^2 \hat{a}_r^t dr\Big] < \infty.$$

So we have shown that there is indeed a unique solution to (3.82).

In the sequel, we will denote by :

$$\big(Y^{\mathbb{P},\nu}(t,\eta), Z^{\mathbb{P},\nu}(t,\eta)\big) \in D^2(\mathbb{P},\mathbb{R}) \times H^2(\mathbb{P},\mathbb{R}^d).$$

the particular case where $t = 0$, that is to say for all $s \in [0,1]$, $\mathbb{P} \in \mathcal{P}_H$, η in $L^2(\mathbb{P},\mathbb{R})$ and $\nu \in \mathcal{D}$:

$$Y^{\mathbb{P},\nu}(s,\eta) := Y^{\mathbb{P},\nu,0,\omega}(s,\eta) \text{ and } Z^{\mathbb{P},\nu}(s,\eta) := Z^{\mathbb{P},\nu,0,\omega}(s,\eta). \tag{3.83}$$

At time 0, we always have $\omega_0 = 0$. This explains that we do not give a dependence in ω to the processes $\big(Y^{\mathbb{P},\nu}(t,\eta), Z^{\mathbb{P},\nu}(t,\eta)\big)$.

We make the following assumption:

Assumption 3.50. *There exists at least one solution* $(\tilde{Y}, \tilde{Z})$ *in* $\mathbb{S}^2_H(\mathbb{R}) \times \mathbb{H}^2_H(C)$ *to the second order constrained problem* (3.79) *and* (3.80).

Lemma 3.51. *Let Assumption* 3.50 *holds. Then we have:*

$$\tilde{Y}_t(\omega) \geq \mathcal{Y}_t^{\mathbb{P},\nu,t,\omega}(1,\xi), \text{ for } \mathbb{P} - a.e.\ \omega \in \Omega. \tag{3.84}$$

for every $\nu \in \mathcal{D}$, $t \in [0,1]$ *and* $\mathbb{P} \in \mathcal{P}^t_H$.

Proof. By Assumption 3.30, we know that $(\tilde{Y}, \tilde{Z})$ in $\mathbb{S}^2_H(\mathbb{R}) \times \mathbb{H}^2_H(C)$ is a solution to the second order constrained problem (3.79) and (3.80). Therefore for all $\mathbb{P} \in \mathcal{P}_H$ and $t \in [0,1]$, we have:

$$\tilde{Y}_t = \xi - \int_t^1 \hat{F}_u(\tilde{Y}_u, \tilde{Z}_u)du - \int_t^1 \tilde{Z}_u dB_u + K_1^{\mathbb{P}} - K_t^{\mathbb{P}},\ \mathbb{P} - a.s.$$

with

$$\tilde{Z}_t \in C, \mathbb{P} - a.s. \tag{3.85}$$

For all $(t,\omega) \in [0,1] \times \Omega$, we can rewrite the above equation on the shifted space Ω^t as:

$$\tilde{Y}_t^{t,\omega} = \xi^{t,\omega} - \int_t^1 \hat{F}_u^{t,\omega}(\tilde{Y}_u^{t,\omega}, \tilde{Z}_u^{t,\omega})du - \int_t^1 \tilde{Z}_u^{t,\omega} dB_u^t + K_1^{\mathbb{P},t,\omega} - K_t^{\mathbb{P},t,\omega},\ \mathbb{P}^t - a.s. \tag{3.86}$$

with $\tilde{Z}_t^{t,\omega} \in C$, $\mathbb{P}^t - a.s$ and where $\mathbb{P}^t \in \mathcal{P}^t_H$ is the martingale measure induced by $\mathbb{P} \in \mathcal{P}_H$ on Ω^t. By (3.9), for all $\nu \in \mathcal{D}$, (3.85) is equivalent to:

$$\delta(\nu_r) - \tilde{Z}_r^{t,\omega}\nu_r \geq 0,\ t \leq r \leq 1,\ \mathbb{P}^t - a.s. \tag{3.87}$$

Adding and removing the quantity $\delta(\nu) - \tilde{Z}^{t,\omega}\nu$ in (3.86) leads to:

$$\begin{aligned}\tilde{Y}_t^{t,\omega} &= \xi^{t,\omega} - \int_t^1 \left[\hat{F}_u^{t,\omega}(\tilde{Y}_u^{t,\omega}, \tilde{Z}_u^{t,\omega}) + \big(\delta(\nu_u) - \nu_u \tilde{Z}_u^{t,\omega}\big)(\hat{a}_u^t)^{1/2}\right]du - \int_t^1 \tilde{Z}_u^{t,\omega} dB_u^t \\ &+ \left[K_1^{\mathbb{P},t,\omega} - K_t^{\mathbb{P},t,\omega} + \int_t^1 \big[\delta(\nu_u) - \nu_u \tilde{Z}_u^{t,\omega}\big](\hat{a}_u^t)^{1,2}du\right],\ \mathbb{P}^t - a.s. \end{aligned} \tag{3.88}$$

We recall that $K^{\mathbb{P},t,\omega}$ is an increasing process, $\hat{a}^t$ is a positive process as the quadratic variation of B^t and (3.87) holds. Then by a standard comparison theorem (see Theorem 3.18), we have:

$$\tilde{Y}_t^{t,\omega} \geq Y_t^{\mathbb{P},\nu,t,\omega}(1,\xi), \text{ for } \mathbb{P} - a.e.\ \omega \in \Omega.$$

By definition of the concatenation paths (see (3.18)), for all $(\omega, \omega') \in \Omega \times \Omega^t$, we know that $\tilde{Y}_t^{t,\omega} = \tilde{Y}_t(\omega \otimes_t \omega') = \tilde{Y}_t(\omega)$. Therefore, we finally get:

$$\tilde{Y}_t(\omega) \geq Y_t^{\mathbb{P},\nu,t,\omega}(1,\xi), \text{ for } \mathbb{P} - a.e.\ \omega \in \Omega.$$

□

By the Blumenthal zero-one law (see (3.4)), we can see that $Y_t^{\mathbb{P},\nu,t,\omega}(1,\xi)$ is a constant for any $(t,\omega) \in [0,1] \times \Omega$, $\nu \in \mathcal{D}$ and $\mathbb{P} \in \mathcal{P}_H^t$. Therefore for all $(t,\omega) \in [0,1] \times \Omega$, the following quantity is correctly defined:

$$\tilde{V}_t(\omega) = \sup_{\mathbb{P} \in \mathcal{P}_H^t} \sup_{\nu \in \mathcal{D}_{[t,1]}} Y_t^{\mathbb{P},\nu,t,\omega}(1,\xi). \tag{3.89}$$

where $\mathcal{D}_{[t,1]}$ is the restriction of $\mathcal{D}$ to $[t,1] \times \Omega$. By Lemma 3.51, since (3.84) holds for all $\nu \in \mathcal{D}$ and $\mathbb{P} \in \mathcal{P}_H^t$, the definition of $\tilde{V}$ gives the following inequality:

$$\tilde{Y}_t(\omega) \geq \tilde{V}_t(\omega) \geq Y_t^{\mathbb{P},\nu,t,\omega}(1,\xi), \text{ for } \mathbb{P} - a.e.\ \omega \in \Omega.. \tag{3.90}$$

for all $t \in [0,1]$.

Since solutions of BSDEs can be constructed via Picard's iteration, we can easily show that:

$$Y_t^{\mathbb{P}^{t,\omega},\nu,t,\omega}(1,\xi) = Y_t^{\mathbb{P},\nu}(1,\xi)(\omega), \text{ for } \mathbb{P} - a.e.\ \omega \in \Omega. \tag{3.91}$$

for all $t \in [0,1]$, $\mathbb{P} \in \mathcal{P}_H$ and $\nu \in \mathcal{D}$. Therefore since the rcpd $\mathbb{P}^{t,\omega}$ belongs, by Lemma 3.14, to $\mathcal{P}_H^t$, (3.90) leads to:

$$\tilde{Y}_t \geq \tilde{V}_t \geq Y_t^{\mathbb{P},\nu}(1,\xi), \ 0 \leq t \leq 1, \ \mathbb{P} - a.s. \tag{3.92}$$

Then by (3.92), we have for each $\mathbb{P} \in \mathcal{P}_H$ and $\nu \in \mathcal{D}$ that:

$$|\tilde{V}_t| \leq \max\{|\tilde{Y}_t|, |Y_t^{\mathbb{P},\nu}(1,\xi)|\}, \ 0 \leq t \leq 1, \ \mathbb{P} - a.s.$$

Furthermore since $\tilde{Y} \in \mathbb{S}_H^2(\mathbb{R}) = \bigcap_{\mathbb{P} \in \mathcal{P}_H} S^2(\mathbb{P},\mathbb{R})$ and $Y^{\mathbb{P},\nu} \in D^2(\mathbb{P},\mathbb{R}) \subset S^2(\mathbb{P},\mathbb{R})$, then $\tilde{V}_t \in S^2(\mathbb{P},\mathbb{R})$ for all $\mathbb{P} \in \mathcal{P}_H$. This leads to :

$$\tilde{V}_t \in \mathbb{S}_H^2(\mathbb{R}). \tag{3.93}$$

Therefore, for any $\theta \in [0,1]$, there exists a unique solution

$$\big(Y^{\mathbb{P},\nu,t,\omega}(\theta,\tilde{V}_\theta), Z^{\mathbb{P},\nu,t,\omega}(\theta,\tilde{V}_\theta)\big) \in D^2(\mathbb{P},\mathbb{R}) \times H^2(\mathbb{P},\mathbb{R}^d)$$

to (3.82) where the terminal condition at time θ is $\tilde{V}_\theta$.

We need the following assumptions:

Assumption 3.52. *ξ is uniformly continuous in ω under the L^∞-norm.*

We define for all $\omega \in \Omega$, $\Lambda(\omega) := \sup_{0\le t\le 1} \Lambda_t(\omega)$ where for all $t \in [0,1]$:

$$\Lambda_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}_H^t} \left(\mathbb{E}^{\mathbb{P}}\big[|\xi^{t,\omega}|^2 + \int_t^1 |\hat{F}_s^{t,\omega}(0,0)|^2 ds\big]\right). \tag{3.94}$$

By the uniform continuity in ω of $\hat{F}^{t,\omega}$ (see (3.22)) and Assumption 3.52, we then have for all $\omega \in \Omega$ that:

$$\Lambda(\omega) < \infty. \tag{3.95}$$

Moreover it is uniformly continuous in ω under the $\mathbb{L}^\infty$-norm so $\mathcal{F}_1$-measurable. Let us assume:

Assumption 3.53. *For all $\mathbb{P} \in \mathcal{P}_H$,*

$$\mathbb{E}^{\mathbb{P}}[|\Lambda|^2] < \infty.$$

The following Lemma gives us some regularity in ω for the candidate solution $\tilde{V}$:

Lemma 3.54. *Let Assumptions* 3.8, 3.50 *and* 3.52 *hold. Then for all $(t,\omega,\omega') \in [0,1]\times\Omega^2$:*

$$|\tilde{V}_t(\omega) - \tilde{V}_t(\omega')| \le C\rho(\|\omega-\omega'\|_t)$$

where C is a positive constant. Moreover for all $(t,\omega) \in \Omega\times[0,1]$, we have $|\tilde{V}_t(\omega)| \le C\Lambda_t(\omega)$. Therefore $\tilde{V}_t$ is $\mathcal{F}_t$-measurable for all $t \in [0,1]$.

Proof. Under Assumption 3.52, we can find a modulus of continuity ρ such that for all $0 \le t \le 1$, $(\omega,\omega') \in \Omega^2$ and $\tilde{\omega} \in \Omega^t$:

$$|\xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega'}(\tilde{\omega})| \le \rho(\|\omega-\omega'\|_t). \tag{3.96}$$

where for $t \in [0,1]$, $\|\omega\|_t = \sup_{0\le s\le t} |\omega_s|$. Similarly, by (3.22), for all (y,z) in $\mathbb{R}$ and $0 \le s \le t \le 1$, we have:

$$|\hat{F}_s^{t,\omega}(\tilde{\omega},y,z) - \hat{F}_s^{t,\omega'}(\tilde{\omega},y,z)| \le \rho(\|\omega-\omega'\|_t) \tag{3.97}$$

Now, for (ω,ω') in Ω^2, $t \in [0,1]$, $\mathbb{P} \in \mathcal{P}_H^t$ and $\nu \in \mathcal{D}$, we denote $\tilde{\Delta}\xi = \xi^{t,\omega} - \xi^{t,\omega'}$, $\tilde{\Delta}\hat{F} = \hat{F}^{t,\omega} - \hat{F}^{t,\omega'}$ and

$$(\tilde{\Delta}y, \tilde{\Delta}z) := \big(\mathcal{Y}^{\mathbb{P},\nu,t,\omega}(1,\xi) - \mathcal{Y}^{\mathbb{P},\nu,t,\omega'}(1,\xi), \mathcal{Z}^{\mathbb{P},\nu,t,\omega}(1,\xi) - \mathcal{Z}^{\mathbb{P},\nu,t,\omega'}(1,\xi)\big).$$

By Assumption 3.8, we can find two bounded processes η and γ such that:

$$\tilde{\Delta} y_r = \tilde{\Delta}\xi - \int_r^1 \big[\tilde{\Delta}\hat{F}_s(0) + \gamma_s \tilde{\Delta} y_s + (\eta_s - \nu_s)\tilde{\Delta} z_s (\hat{a}_s^t)^{1/2}\big] ds$$
$$- \int_r^t \tilde{\Delta} z_s (\hat{a}_s^t)^{1/2} dW_s^{\mathbb{P},t}, \quad r \in [t,1], \ \mathbb{P} - a.s.$$

where $W^{\mathbb{P},t}$ is a Brownian motion under $\mathbb{P}$ on Ω^t, that is to say it is defined similarly as in (3.2) by:

$$W_u^{\mathbb{P},t} := \int_u^t (\hat{a}_s^t)^{-1/2} dB_s^t, \ u \in [t,1], \ \mathbb{P}{-}a.s. \tag{3.98}$$

Since η and ν are bounded, by the Girsanov theorem, we can define a equivalent probability measure $\bar{\mathbb{P}}$ such that the process $\overline{W}_s^{\mathbb{P},t} := W_s^{\mathbb{P},t} + \int_t^s (\eta_s - \nu_s)^{1/2} ds$ is Brownian motion under $\bar{\mathbb{P}}$ for all $s \in [t,1]$. Therefore the above equation can be rewritten as:

$$\tilde{\Delta} y_r = \tilde{\Delta}\xi - \int_r^1 \big[\tilde{\Delta}\hat{F}_s(0) + \gamma_s \tilde{\Delta} y_s\big] ds - \int_r^t \tilde{\Delta} z_s (\hat{a}_s^t)^{-1/2} d\overline{W}_s^{\mathbb{P},t}, \quad r \in [t,1], \ \mathbb{P} - a.s.$$

By Lemma 3.20, we then have:

$$|\tilde{\Delta} y_t|^2 \leq C \mathbb{E}^{\bar{\mathbb{P}}}\bigg[|\tilde{\Delta}\xi|^2 + \int_t^1 |\tilde{\Delta}\hat{F}_s(0)|^2 ds\bigg], \ \mathbb{P} - a.s. \tag{3.99}$$

where C is a positive constant. Then by (3.96), (3.97) and inequality (3.99), we can find a modulus of continuity ρ such that

$$|\tilde{\Delta} y_t| \leq C\rho\big(\|\omega - \omega'\|_t\big). \tag{3.100}$$

The above equation stands for any $\mathbb{P} \in \mathcal{P}_H^t$ and $\nu \in \mathcal{D}$, so we obtain that:

$$|\tilde{V}_t(\omega) - \tilde{V}_t(\omega')| \leq C\rho\big(\|\omega - \omega'\|_t\big). \tag{3.101}$$

Following the same argument, and using (3.1), one can show that for every $(t,\omega) \in [0,1] \times \Omega$, $\nu \in \mathcal{D}$ and $\mathbb{P} \in \mathcal{P}_H$:

$$|Y_t^{\mathbb{P},\nu,t,\omega}|^2 \leq C \mathbb{E}\bigg[|\xi^{t,\omega}|^2 + \int_t^1 |\hat{F}_s^{t,\omega}(0)|^2 ds\bigg] \leq C\Lambda_t(\omega), \ \mathbb{P} - a.s. \tag{3.102}$$

Since $\mathbb{P}$ and ν are arbitrary, we finally get for all $(t,\omega) \in [0,1] \times \Omega$:

$$|\tilde{V}_t(\omega)| \leq C\Lambda_t(\omega). \tag{3.103}$$

□

We need a dynamic programming principle.

Proposition 3.55. *Let Assumptions* 3.8*,* 3.50*,* 3.52 *and* 3.53 *hold. Then for all* $0 \le t \le s \le 1$ *and* $\omega \in \Omega$*:*

$$\tilde{V}_t(\omega) = \sup_{\mathbb{P}\in\mathcal{P}_H^t} \sup_{\nu\in\mathcal{D}_{[t,s]}} \mathcal{Y}_t^{\mathbb{P},\nu,t,\omega}\big(s, \tilde{V}_s\big). \tag{3.104}$$

where $\mathcal{D}_{[t,s]}$ *corresponds to the restriction of* $\mathcal{D}$ *to* $[t,s]\times\Omega$.

Proof. It suffices to prove that

$$\tilde{V}_0 = \sup_{\mathbb{P}\in\mathcal{P}_H} \sup_{\nu\in\mathcal{D}_{[0,t]}} Y_0^{\mathbb{P},\nu}\big(t, \tilde{V}_t\big).$$

Step 1: For every $\mathbb{P}\in\mathcal{P}_H$, $\nu\in\mathcal{D}$ and $0\le t\le 1$, we recall that :

$$\begin{aligned} Y_0^{\mathbb{P},\nu}(1,\xi) = Y_t^{\mathbb{P},\nu}(1,\xi) - &\int_0^t \big[\hat{F}_u\big(Y_u^{\mathbb{P},\nu}(1,\xi), Z_u^{\mathbb{P},\nu}(1,\xi)\big) + \big(\delta(\nu_u) - \nu_u Z_u^{\mathbb{P},\nu}(1,\xi)\big)\hat{a}_u^{1/2}\big]du \\ &- \int_0^t Z_u^{\mathbb{P},\nu}(1,\xi)dB_u, \ \mathbb{P}-a.s. \end{aligned}$$

That is to say, $Y_0^{\mathbb{P},\nu}\big(t, Y_t^{\mathbb{P},\nu}(1,\xi)\big) = Y_0^{\mathbb{P},\nu}(1,\xi)$, $\mathbb{P}-a.s.$ Therefore by (3.92) and a standard comparison theorem, we obtain for all $\mathbb{P}\in\mathcal{P}_H$, $\nu\in\mathcal{D}$ and $0\le t\le 1$:

$$Y_0^{\mathbb{P},\nu}(t,\tilde{V}_t) \ge Y_0^{\mathbb{P},\nu}(1,\xi), \ \mathbb{P}-a.s.$$

Since $\mathbb{P}\in\mathcal{P}_H$ and $\nu\in\mathcal{D}$ are arbitrary, then we obtain:

$$\sup_{\mathbb{P}\in\mathcal{P}_H} \sup_{\nu\in\mathcal{D}_{[0,t]}} Y_0^{\mathbb{P},\nu}\big(t,\tilde{V}_t\big) \ge \tilde{V}_0, \ \mathbb{P}-a.s.$$

Step 2: We fix $\mathbb{P}\in\mathcal{P}_H$ and $\varepsilon>0$. Ω is a complete separable space, so we can find a partition $\big(E_t^i\big)_{i\in\mathbb{N}} \in \mathcal{F}_t$ such that for every $i\in\mathbb{N}$, $(t,\omega,\omega')\in[0,1]\times(E_t^i)^2$, $\|\omega'-\omega'\|<\varepsilon$. For each i, we fix $\hat{\omega}_i\in E_t^i$ and consider $\mathbb{P}_t^i\in\mathcal{P}_H^t$ and $\nu_t^i\in\mathcal{D}_{[t,1]}$, two ε-optimizers of $\tilde{V}_t(\hat{\omega}_i)$:

$$\tilde{V}_t(\hat{\omega}_i) \le Y_t^{\mathbb{P}_t^i,\nu_t^i,t,\hat{\omega}_i}(1,\xi) + \varepsilon. \tag{3.105}$$

For each $n\in\mathbb{N}$ and $E\in\Omega$, we define $\mathbb{P}^n = \mathbb{P}^{n,\varepsilon}$ by:

$$\mathbb{P}^n(E) := \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^n \mathbb{E}^{\mathbb{P}_t^i}\big[(\mathbb{1}_E)^{t,\omega}\big]\mathbb{1}_{E_t^i}\right] + \mathbb{P}(E\cap\hat{E}_t^n), \quad \text{where } \hat{E}_t^n := \cup_{i>n}E_t^i. \tag{3.106}$$

According to this definition, one can see that $\mathbb{P}^n = \mathbb{P}$ on $\mathcal{F}_t$ and its r.c.p.d $(\mathbb{P}^n)^{t,\omega} = \mathbb{P}_t^i$ for $\omega\in E_i^t, 1\le i\le n$ and $(\mathbb{P}^n)^{t,\omega} = \mathbb{P}^{t,\omega}$ for $\hat{\omega}\in\hat{E}_t^n$. By the

Appendix of [85], $\mathbb{P}^n \in \mathcal{P}_H$. Moreover, for any $\nu \in \mathcal{D}_{[0,t]}$, we can define a process $\nu^n := \nu^{n,\varepsilon}$ by:

$$\nu_s^n(\omega) = \nu_s(\omega)\mathbb{1}_{s<t} + \left(\sum_{i=1}^n \nu_s^i(\omega)\mathbb{1}_{\omega\in E_t^i} + \nu_s(\omega)\mathbb{1}_{\omega\in \hat{E}_t^n}\right)\mathbb{1}_{s\geq t}. \qquad (3.107)$$

for all $(s,\omega) \in [0,1]\times\Omega$. The process ν^n belongs to $\mathcal{D}_{[0,t]}$ as the sum of bounded and $\mathbb{F}$-progressively measurable processes.

Thanks to the regularity result of Lemma 3.54, (3.105) and inequality (3.100), we get for all $i \in \{1,\dots,n\}$ and $\omega \in E_t^i$:

$$\begin{aligned}\tilde{V}_t(\omega) &\leq \tilde{V}_t(\hat{\omega}_i) + C\rho(\varepsilon)\\ &\leq Y_t^{\mathbb{P}_t^i,\nu_t^i,t,\hat{\omega}_i}(1,\xi) + \varepsilon + C\rho(\varepsilon)\\ &\leq Y_t^{\mathbb{P}_t^i,\nu_t^i,t,\omega}(1,\xi) + \varepsilon + 2C\rho(\varepsilon).\end{aligned} \qquad (3.108)$$

Since $(\mathbb{P}^n)^{t,\omega} = \mathbb{P}_t^i$ and $\nu_t^n = \nu_t^i$ for $\omega \in E_t^i$, then we have for all $\omega \in \cup_{i=1}^n E_t^i$ and $t \in [0,1]$:

$$\tilde{V}_t(\omega) \leq Y_t^{(\mathbb{P}^n)^{t,\omega},\nu^n,t,\omega}(1,\xi) + \varepsilon + 2C\rho(\varepsilon). \qquad (3.109)$$

By (3.91), we finally get:

$$\tilde{V}_t \leq Y_t^{\mathbb{P}^n,\nu^n}(1,\xi) + \varepsilon + 2C\rho(\varepsilon),\ \mathbb{P}^n - a.s. \text{ on } \cup_{i=1}^n E_t^i \qquad (3.110)$$

Let us define the pair $(y^n,z^n) \in D^2(\mathbb{P},\mathbb{R})\times H^2(\mathbb{P},\mathbb{R}^d)$ unique solution of the following BSDE:

$$\begin{aligned}y_s^n =& \big[Y_t^{\mathbb{P}^n,\nu^n}(1,\xi) + \varepsilon + 2C\rho(\varepsilon)\big]\mathbb{1}_{\cup_{i=1}^n E_t^i} + \tilde{V}_t\mathbb{1}_{\hat{E}_t^n}\\ &- \int_s^t \big[\hat{F}_u(y_u^n,z_u^n) + \big(\delta(\nu_u^n) - \nu_u z_u^n\big)\hat{a}_u^{1/2}\big]du - \int_s^t z_u^n dB_u,\ 0\leq s\leq t,\ \mathbb{P}^n - a.s.\end{aligned}$$

Since the terminal condition $\big[Y_t^{\mathbb{P}^n,\nu^n}(1,\xi)+\varepsilon+C\rho(\varepsilon)\big]\mathbb{1}_{\cup_{i=1}^n E_t^i}+\tilde{V}_t\mathbb{1}_{\hat{E}_t^n}$ belongs to $L^2(\mathbb{P}^n,\mathbb{R})$ (see (3.93)), then the above BSDE admits a unique solution (y^n,z^n).
Moreover we know that $\mathbb{P}^n = \mathbb{P}$ on $\mathcal{F}_t$ and $\nu^n = \nu$ on $[0,t)$, we have:

$$\begin{aligned}y_s^n =& \big[Y_t^{\mathbb{P}^n,\nu^n}(1,\xi) + \varepsilon + C\rho(\varepsilon)\big]\mathbb{1}_{\cup_{i=1}^n E_t^i} + \tilde{V}_t\mathbb{1}_{\hat{E}_t^n} - \int_s^t \big[\hat{F}_u(y_u^n,z_u^n) + \big(\delta(\nu_u) - \nu_u z_u^n\big)\hat{a}_u^{1/2}\big]du\\ &- \int_s^t z_u^n dB_u,\ 0\leq s< t,\ \mathbb{P} - a.s.\end{aligned}$$

Thanks to (3.110) and a comparison argument, we then have:

$$y_0^n \geq Y_0^{\mathbb{P},\nu}(t,\tilde{V}_t), \ \mathbb{P}-a.s. \tag{3.111}$$

Moreover by (3.20), we can find a constant $C>0$ such that :

$$|y_0^n - Y_0^{\mathbb{P}^n,\nu^n}(1,\xi)|^2 \leq C(\varepsilon + C\rho(\varepsilon))^2 + C\mathbb{E}^{\mathbb{P}^n}\left[|\tilde{V}_t - Y_t^{\mathbb{P}^n,\nu^n}(1,\xi)|^2 \mathbb{1}_{\hat{E}_t^n}\right]$$

By (3.102) and (3.103), since $\mathbb{P}^n \in \mathcal{P}_H$ (Appendix of [85]), we have that $|\tilde{V}_t - Y_t^{\mathbb{P}^n,\nu^n}(1,\xi)| \leq 2C\Lambda_t, \ \mathbb{P}^n - a.s.$ Since $Y_t^{\mathbb{P}^n,\nu^n}(1,\xi)$ and $\tilde{V}_t$ are $\mathcal{F}_t$-measurable and $\mathbb{P}^n = \mathbb{P}$ on $\mathcal{F}_t$, then we obtain:

$$|\tilde{V}_t - Y_t^{\mathbb{P}^n,\nu^n}(1,\xi)| \leq 2C\Lambda_t, \ \mathbb{P}-a.s.$$

which by Assumption 3.53 belongs to $L^2(\mathbb{P},\mathbb{R})$. Since Ω is separable, we have $\lim_n \hat{E}_t^n = \emptyset$. Therefore by dominated convergence theorem, sending n to infinity we obtain that:

$$|y_0^n - Y_0^{\mathbb{P}^n,\nu^n}(1,\xi)| \leq C(\varepsilon + C\rho(\varepsilon))$$

By (3.111), this leads to the following majoration:

$$Y_0^{\mathbb{P},\nu}(t,\tilde{V}_t) \leq Y_0^{\mathbb{P}^n,\nu^n}(1,\xi) + C(\varepsilon + C\rho(\varepsilon)), \ \mathbb{P}-a.s.$$

Sending $\varepsilon \to 0$, leads to :

$$Y_0^{\mathbb{P},\nu}(t,\tilde{V}_t) \leq Y_0^{\mathbb{P}^n,\nu^n}(1,\xi), \ \mathbb{P}-a.s. \tag{3.112}$$

Since $\mathbb{P}^n = \mathbb{P}$ on $\mathcal{F}_t$ and $\nu^n = \nu$ on $[0,t)$, then $Y_0^{\mathbb{P}^n,\nu^n}(1,\xi) = Y_0^{\mathbb{P},\nu}(1,\xi)$. Then we can rewrite (3.112) as :

$$Y_0^{\mathbb{P},\nu}(t,\tilde{V}_t) \leq Y_0^{\mathbb{P},\nu}(1,\xi). \tag{3.113}$$

Since $\mathbb{P} \in \mathcal{P}_H$ and $\nu \in \mathcal{D}_{[0,t]}$ are arbitrary, then :

$$\sup_{\mathbb{P}\in\mathcal{P}_H} \sup_{\nu\in\mathcal{D}_{[0,t]}} Y_0^{\mathbb{P},\nu}\big(t,\tilde{V}_t\big) \leq \tilde{V}_0, \ \mathbb{P}-a.s.$$

□

Let us recall for all $\mathbb{P} \in \mathcal{P}_H$ and $t \in [0,1]$ the definition of the following sets:

$\mathcal{P}_H(t,\mathbb{P}) := \big\{\mathbb{P}' \in \mathcal{P}_H : \ \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t\big\}$ and $\mathcal{P}_H(t+,\mathbb{P}) := \big\{\mathbb{P}' \in \mathcal{P}_H : \ \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t^+\big\}$.

We denote by $\tilde{V}^+$ the $\mathbb{F}^+$-adapted process defined by:

$$\tilde{V}_t^+ := \overline{\lim_{r\in\mathbb{Q}\cap(t,1], r\downarrow t}} \tilde{V}_r, \ t \in [0,1]. \tag{3.114}$$

Since $\tilde{V}_t$ is $\mathcal{F}_t$-adapted by Lemma 3.54, then the process $\tilde{V}_t^+$ is $\mathcal{F}_t^+$-progressively measurable (see 1.13 of [49]).

Let us show that $\tilde{V}_t^+$ is an RCLL process.

Lemma 3.56. *Let Assumptions* 3.8*,* 3.50*,* 3.52 *and* 3.53 *hold. Then:*

$$\tilde{V}_t^+ = \lim_{r\in\mathbb{Q}\cap(t,1],r\downarrow t} \tilde{V}_r, \ \mathcal{P}_H - q.s.$$

Therefore $\tilde{V}^+$ is càdlàg, $\mathcal{P}_H - q.s.$

Proof. For all $\mathbb{P}\in\mathcal{P}_H$, ν in $\mathcal{D}$, let us define:

$$\tilde{V}^{\mathbb{P},\nu} = \tilde{V} - Y^{\mathbb{P},\nu}(1,\xi). \tag{3.115}$$

By definition of $\tilde{V}$ (see (3.92)), we have $\tilde{V}^{\mathbb{P},\nu} \geq 0, \mathbb{P} - a.s.$ Let us take $(s,t)\in[0,1]^2$ such that $0\leq s\leq t$ and consider for all $\nu\in\mathcal{D}$ and $\mathbb{P}\in\mathcal{P}_H$:

$$\bar{y}_s^{\mathbb{P},\nu,t} := Y_s^{\mathbb{P},\nu}(t,V_t) - Y_s^{\mathbb{P},\nu}(1,\xi) \text{ and } \bar{z}_s^{\mathbb{P},\nu,t} := \big(Z_s^{\mathbb{P},\nu}(t,V_t) - Z_s^{\mathbb{P},\nu}(1,\xi)\big)\hat{a}_t^{-1/2}. \tag{3.116}$$

For all $\omega\in\Omega$, we recall by (3.91) that

$$\bar{y}_s^{\mathbb{P},\nu,t}(\omega) = Y_s^{\mathbb{P}^{s,\omega},\nu,s,\omega}(t,V_t) - Y_s^{\mathbb{P}^{s,\omega},\nu,s,\omega}(1,\xi), \ \mathbb{P} - a.s.$$

By Lemma 3.14, $\mathbb{P}^{s,\omega}\in\mathcal{P}_H^s$. Therefore by Proposition 3.55, we have that $\tilde{V}_s^{\mathbb{P},\nu} \geq \bar{y}_s^{\mathbb{P},\nu,t}$, $0\leq s\leq t$ $\mathbb{P} - a.s.$ Moreover the pair $\big(\bar{y}^{\mathbb{P},\nu,t}, \bar{z}^{\mathbb{P},\nu,t}\big)$ satisfies:

$$\bar{y}_s^{\mathbb{P},\nu,t} = \tilde{V}_t^\nu + \int_s^t h_u^{\mathbb{P},\nu}\big(\bar{y}_u^{\mathbb{P},\nu,t}, \bar{z}_u^{\mathbb{P},\nu,t}\big)du - \int_s^t \bar{z}_u^{\mathbb{P},\nu,t}dB_u, \ \mathbb{P} - a.s. \tag{3.117}$$

where h^ν is defined by:

$$h_t^{\mathbb{P},\nu}(y,z) := \hat{F}_t\big(y+Y_t^{\mathbb{P},\nu}(1,\xi), \big(z+Z_t^{\mathbb{P},\nu}(1,\xi)\big)\hat{a}_t^{-1/2}\big) - \hat{F}_t\big(Y_t^{\mathbb{P},\nu}(1,\xi), Z_t^{\mathbb{P},\nu}(1,\xi)\big) - \nu_t z. \tag{3.118}$$

We know that $\tilde{V}_t^\nu\in L^2(\mathbb{P},\mathbb{R})$ since $Y_t^{\mathbb{P},\nu}$ and $\tilde{V}_t$ are in $L^2(\mathbb{P},\mathbb{R})$ (see (3.93)). By Assumption 3.8 and the boundedness of ν, we can see that $h^{\mathbb{P},\nu}$ is a Lipschitz continuous function with respect to (y,z). So there exists a unique solution $\big(\bar{y}^{\mathbb{P},\nu,t}, \bar{z}^{\mathbb{P},\nu,t}\big)\in D^2(\mathbb{P},\mathbb{R})\times H^2(\mathbb{P},\mathbb{R}^d)$ to (3.117). This means that $\tilde{V}^\nu$ is a positive weak $h^{\mathbb{P},\nu}$-supermartingale under $\mathbb{P}$.(see definition 3.22).

The application of the downcrossing inequality in [12], Theorem 6 shows that $\lim_{r\in\mathbb{Q}\cap(t,1],r\downarrow t} V_r^\nu$ exists for all $t\in[0,1]$(see proof p14 in [49]). Since the paths of $Y^{\mathbb{P},\nu}(1,\xi)$ are continuous as a solution of a classical BSDE with Lipschitz continuous generator, we can can see that actually

$$\tilde{V}_t^+ = \lim_{r\in\mathbb{Q}\cap(t,1],r\downarrow t} \tilde{V}_r,\ \mathbb{P}-a.s.$$

Since $\mathbb{P}\in\mathcal{P}_H$ is arbitrary, the we can conclude that $\tilde{V}^+$ is a càdlàg process $\mathcal{P}_H$-q.s.

□

We have the following representation result:

Proposition 3.57. *Under Assumptions* 3.8*,* 3.50*,* 3.52 *and* 3.53*, for all* $\mathbb{P}\in\mathcal{P}_H$ *and* $t\in[0,1]$*, we have:*

$$\tilde{V}_t = \operatorname*{essup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})} \operatorname*{essup}^{\mathbb{P}'}_{\nu\in\mathcal{D}_{[t,1]}} Y_t^{\mathbb{P}',\nu}(1,\xi) \text{ and } \tilde{V}_t^+ = \operatorname*{essup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t+,\mathbb{P})} \operatorname*{essup}^{\mathbb{P}'}_{\nu\in\mathcal{D}_{[t+,1]}} Y_t^{\mathbb{P}',\nu}(1,\xi),\ \mathbb{P}-a.s.$$

Proof. We consider a given probability $\mathbb{P}\in\mathcal{P}_H$ and define:

$$\tilde{V}_t^{\mathbb{P}} := \operatorname*{essup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})} \operatorname*{essup}^{\mathbb{P}'}_{\nu\in\mathcal{D}_{[t,1]}} Y_t^{\mathbb{P}',\nu}(1,\xi) \text{ and } \tilde{V}_t^{\mathbb{P},+} := \operatorname*{essup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t+,\mathbb{P})} \operatorname*{essup}^{\mathbb{P}'}_{\nu\in\mathcal{D}_{[t+,1]}} Y_t^{\mathbb{P}',\nu}(1,\xi)$$

step 1: We want to prove that $\tilde{V} = \tilde{V}^{\mathbb{P}}, \mathbb{P}-a.s.$ Indeed let us take any $\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})$ and $\nu\in\mathcal{D}_{[t,1]}$. By the inequality (3.92), we have that $\tilde{V}_t \geq Y_t^{\mathbb{P}',\nu}(1,\xi), \mathbb{P}'-a.s.$ Since $\mathbb{P}'=\mathbb{P}$ on $\mathcal{F}_t$ and $\tilde{V}_t$ and $Y_t^{\mathbb{P}',\nu}(1,\xi)$ are $\mathcal{F}_t$-measurable then we get: $\tilde{V}_t \geq Y_t^{\mathbb{P}',\nu}(1,\xi), \mathbb{P}-a.s.$ This is true for any $\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})$ and $\nu\in\mathcal{D}_{[t,1]}$, so we have for all $t\in[0,1]$:

$$\tilde{V}_t \geq \tilde{V}_t^{\mathbb{P}},\ \mathbb{P}-a.s. \tag{3.119}$$

To prove the reverse inequality, we can consider the probability $\mathbb{P}^n\in\mathcal{P}_H$ and the process $\nu^n\in\mathcal{D}_{[0,t]}$ defined respectively in (3.106) and in (3.107). Since by construction $\mathbb{P}^n=\mathbb{P}$ on $\mathcal{F}_t$, we have that $\mathbb{P}^n\in\mathcal{P}_H(t,\mathbb{P})$. Moreover by (3.110), we have that:

$$\mathbb{P}\big[\tilde{V}_t \leq \tilde{V}_t^{\mathbb{P}} + \varepsilon + 2C\rho(\varepsilon)\big] \geq \mathbb{P}\big[\tilde{V}_t \leq Y_t^{\mathbb{P}^n,\nu^n}(1,\xi) + \varepsilon + C\rho(\varepsilon)\big] \geq \mathbb{P}\big[\cup_{1\leq i\leq n} E_t^i\big].$$

Since Ω is separable, by definition of E_t^i, we have that:

$$\lim_n \mathbb{P}\big[\bigcup_{1\leq i\leq n} E_t^i\big] = 1.$$

This leads to $\tilde{V}_t \leq \tilde{V}_t^{\mathbb{P}} + \varepsilon + C\rho(\varepsilon), \mathbb{P}-a.s.$ So passing to the limit in ε leads to:

$$\tilde{V}_t \leq \tilde{V}_t^{\mathbb{P}},\ \mathbb{P}-a.s. \tag{3.120}$$

Combining (3.119) and (3.120), we have that

$$\tilde{V}_t = \tilde{V}_t^{\mathbb{P}}, \ \mathbb{P} - a.s. \tag{3.121}$$

Step 2: We want to prove that $\tilde{V}_t^+ = \tilde{V}_t^{+,\mathbb{P}}, \mathbb{P} - a.s.$ First for any $r \in \mathbb{Q} \cap (t,1]$, we have by the first step of the proof that $\tilde{V}_r = \tilde{V}_r^{\mathbb{P}}$. Therefore for all $\mathbb{P}' \in \mathcal{P}_H(t+,\mathbb{P}) \subset \mathcal{P}_H(t,\mathbb{P})$ and $\nu \in \mathcal{D}_{[t+,1]} \subset \mathcal{D}_{[t,1]}$, we have by (3.92),

$$\tilde{V}_r \geq Y_r^{\mathbb{P}',\nu}(1,\xi), \mathbb{P}' - a.s.$$

By definition of $\tilde{V}^+$ (see (3.114)), Lemma 3.54 and the continuity of the paths of $Y^{\mathbb{P}',\nu}(1,\xi)$, sending $r \downarrow t$ gives that: $V_t^+ \geq Y_t^{\mathbb{P}',\nu}(1,\xi), \mathbb{P}' - a.s.$ We know that $\mathbb{P}' = \mathbb{P}$ on $\mathcal{F}_t^+$ and both V_t^+ and $Y_t^{\mathbb{P}',\nu}(1,\xi)$ are $\mathcal{F}_t^+$ measurable, so we get: $V_t^+ \geq Y_t^{\mathbb{P}',\nu}(1,\xi), \mathbb{P} - a.s.$ Since it holds for any $\mathbb{P}' \in \mathcal{P}_H(t+,\mathbb{P})$, we obtain:

$$\tilde{V}_t^+ \geq \tilde{V}_t^{+,\mathbb{P}}, \mathbb{P} - a.s. \tag{3.122}$$

Secondly, we use the definition of the essential supremum (see Neveu [64] for more details) and (3.121) to rewrite for all $r \in \mathbb{Q} \cap (t,1]$

$$\tilde{V}_r = \tilde{V}_r^{\mathbb{P}} = \sup_{n \geq 1} Y_r^{\mathbb{P}_n,\nu_n}(1,\xi), \mathbb{P} - a.s.$$

for some sequence $(\mathbb{P}_n,\nu_n) \in \mathcal{P}_H(r,\mathbb{P}) \times \mathcal{D}_{[r,1]}$. By possibly taking a subsequence of $(\mathbb{P}_n,\nu_n)$, our aim is to show that we have:

$$\lim_n \uparrow Y_r^{\mathbb{P}_n,\nu_n}(1,\xi) = \tilde{V}_r, \mathbb{P} - a.s. \tag{3.123}$$

For that purpose, we need to show that for any $r \in \mathbb{Q} \cap (t,1]$, the family $\{Y_r^{\mathbb{P},\nu}(1,\xi)\}_{\mathbb{P} \in \mathcal{P}_H(r,\mathbb{P}),\nu \in \mathcal{D}_{[t,1]}}$ is directed upward. That is to say, for any $\mathbb{P}_1$ and $\mathbb{P}_2$ in $\mathcal{P}_H(r,\mathbb{P})$ and $(\nu_1,\nu_2) \in \mathcal{D}_{[t,1]}^2$, we need to construct $\bar{\mathbb{P}}$ in $\mathcal{P}_H(r,\mathbb{P})$ and $\bar{\nu} \in \mathcal{D}_{[r,1]}$ such that:

$$Y_r^{\bar{\mathbb{P}},\bar{\nu}}(1,\xi) = \max\big\{Y_r^{\mathbb{P}_1,\nu_1}(1,\xi), Y_r^{\mathbb{P}_2,\nu_2}(1,\xi)\big\}. \tag{3.124}$$

For instance, one can consider the sets:

$$A_1 := \{\omega \in \Omega : \ Y_r^{\mathbb{P}_1,\nu_1}(1,\xi)(\omega) \geq Y_r^{\mathbb{P}_2,\nu_2}(1,\xi)(\omega)\} \in \mathcal{F}_r \text{ and } A_2 := \Omega \backslash A_1.$$

We define for all $E \in \mathcal{F}_1$:

$$\bar{\mathbb{P}}(E) = \mathbb{P}_1(E \cap A_1) + \mathbb{P}_2(E \cap A_2)$$

and the $\mathcal{F}_t$-measurable process:

$$\bar{\nu}(t,\omega) = \big(\nu_1(t,\omega)\mathbb{1}_{\omega \in A_1} + \nu_2(t,\omega)\mathbb{1}_{\omega \in A_2}\big)\mathbb{1}_{t \geq r}.$$

By the proof of claim (4.17) in [86], we can show that $\bar{\mathbb{P}} \in \mathcal{P}_H(r,\mathbb{P})$. Moreover, by the stability of $\mathcal{D}_{[r,1]}$ under concatenation, $\bar{\nu} \in \mathcal{D}_{[r,1]}$. Therefore $Y_r^{\bar{\mathbb{P}},\bar{\nu}}$ satisfies (3.124), so we can conclude that there exists a sequence $(\mathbb{P}_n, \nu_n) \in \mathcal{P}_H(r,\mathbb{P}) \times \mathcal{D}_{[r,1]}$, such that claim (3.123) holds.
Without loss of generality, for some $\nu \in \mathcal{D}_{[0,r]}$, this sequence (ν_n) can be taken in

$$\mathcal{M}_\nu := \big\{\mu \in \mathcal{D} : \mu = \nu \text{ on } [0,r) \times \Omega\big\}.$$

By the stability of BSDEs and (3.123), we have for all $\mathbb{P} \in \mathcal{P}_H(r,\mathbb{P})$, $\nu \in \mathcal{D}$ and $t \in [0,r]$:

$$Y_t^{\mathbb{P},\nu}(r,\tilde{V}_r) = Y_t^{\mathbb{P},\nu}\big(r, \lim_n Y_r^{\mathbb{P}_n,\nu_n}(1,\xi)\big) = \lim_n Y_t^{\mathbb{P},\nu}\big(r, Y_r^{\mathbb{P}_n,\nu_n}(1,\xi)\big) \ \mathbb{P}-a.s. \tag{3.125}$$

We notice $\mathbb{P}_n \in \mathcal{P}_H(r,\mathbb{P}) \subset \mathcal{P}_H(t+,\mathbb{P})$, then $\mathbb{P}_n = \mathbb{P}$ on $\mathcal{F}_t^+$. In addition, by construction, we know that $\nu_n = \nu$ on $[0,r) \times \Omega$. Since $Y_t^{\mathbb{P},\nu}(r, Y_r^{\mathbb{P}_n,\nu_n}(1,\xi))$ is $\mathcal{F}_t$-measurable and has continuous paths, (3.125) can be rewritten as:

$$Y_t^{\mathbb{P},\nu}(r,\tilde{V}_r) = \lim_n Y_t^{\mathbb{P}_n,\nu_n}\big(r, Y_r^{\mathbb{P}_n,\nu_n,}(1,\xi)\big) = \lim_n Y_t^{\mathbb{P}^n,\nu^n}(1,\xi), \ \mathbb{P}-a.s. \tag{3.126}$$

Since for all $n \geq 0$, $\mathbb{P}^n \in \mathcal{P}_H(r,\mathbb{P}) \subset \mathcal{P}_H(t+,\mathbb{P})$ and $\nu_n \in \mathcal{D}$ then by definition of $\tilde{V}^{+,\mathbb{P}}$, we have:

$$Y_t^{\mathbb{P},\nu}(r,\tilde{V}_r) \leq \tilde{V}_t^{+,\mathbb{P}} \ \mathbb{P}-a.s.$$

Moreover by stability of BSDE, definition of $\tilde{V}^+$ (see (3.114)) and Lemma 3.56, we finally get:

$$V_t^+ \leq \tilde{V}_t^{+,\mathbb{P}} \ \mathbb{P}-a.s.$$

□

We can now show that the process V is an RCLL process. Indeed, we have the following Proposition:

Proposition 3.58. *Under Assumptions* 3.8, 3.50, 3.52 *and* 3.53, *we have that:*

$$V_t = V_t^+, \ \mathcal{P}_H - q.s.$$

So V is a càdlàg process.

Proof. The proof is the same as Proposition 4.11 in [85]. □

This is our main result:

Theorem 3.59. *Under Assumptions* 3.8, 3.50, 3.52 *and* 3.53, *there exists a minimal solution* $(Y,Z) \in \mathbb{S}_H^2(\mathbb{R}) \times \mathbb{H}_H^2(C)$ *to problem* (3.79) *and* (3.80).

First let us show the following Proposition:

Proposition 3.60. *Under the assumptions of Theorem 3.59, for every $\mathbb{P} \in \mathcal{P}_H$, $\nu \in \mathcal{D}$, there exists $(Z^{\mathbb{P},\nu}, K^{\mathbb{P},\nu}) \in H^2(\mathbb{P}, \mathbb{R}^d) \times A^2(\mathbb{P}, \mathbb{R})$ such that:*

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \left[\hat{F}_u\big(\tilde{V}_u, Z_u^{\mathbb{P},\nu}\big) + \big(\delta(\nu_u) - \nu_u Z_u^{\mathbb{P},\nu}\big)\right] du + \int_0^t Z_u^{\mathbb{P},\nu} dB_u - K_t^{\mathbb{P},\nu},\ \mathbb{P}-a.s. \tag{3.127}$$

for all $t \in [0,1]$.

Proof. For every $\nu \in \mathcal{D}$, $\mathbb{P} \in \mathcal{P}_H$ and $t \in [0,1]$, we recall that $\tilde{V}^{\mathbb{P},\nu}$ is defined:

$$\tilde{V}_t^{\mathbb{P},\nu} = \tilde{V}_t - Y_t^{\mathbb{P},\nu}(1,\xi).$$

We have shown in Proposition 3.58 that $\tilde{V}$ is a positive cádlág process. Since by definition for all $\nu \in \mathcal{D}$ and $\mathbb{P} \in \mathcal{P}_H$, the process $Y^{\mathbb{P},\nu}(1,\xi) \in D^2(\mathbb{P}, \mathbb{R})$ has continuous paths, then $\tilde{V}^{\mathbb{P},\nu}$ is a positive cádlág process.

Step 1: For all $\nu \in \mathcal{D}$ and $\mathbb{P} \in \mathcal{P}_H$, let us show that $\tilde{V}^{\mathbb{P},\nu}$ is strong $h^{\mathbb{P},\nu}$-supermartinagle where $h^{\mathbb{P},\nu}$ is defined in (3.118). We already have shown in the end of Lemma 3.58 that $\tilde{V}^{\mathbb{P},\nu}$ is a positive weak $h^{\mathbb{P},\nu}$-supermartinagle. In addition, since $\tilde{V}^{\mathbb{P},\nu}$ is a cádlág process then by Theorem 7 of [12], $\tilde{V}^{\mathbb{P},\nu}$ is a positive strong $h^{\mathbb{P},\nu}$-supermartinagle. That is to say for any $\mathbb{F}$-stopping times τ_1 and τ_2, we have $\tilde{V}_{\tau_1}^{\nu} \geq \bar{y}_{\tau_1}^{\mathbb{P},\nu,\tau_2}$, $\mathbb{P}-a.s.$ where we recall that $\big(\bar{y}^{\mathbb{P},\nu,\tau_2}, \bar{z}^{\mathbb{P},\nu,\tau_2}\big)$ defined in (3.117) is the unique solution of

$$\bar{y}_s^{\mathbb{P},\nu,\tau_2} = \tilde{V}_{\tau_2}^{\mathbb{P},\nu} + \int_s^{\tau_2} h_u^{\mathbb{P},\nu}\big(\bar{y}_u^{\mathbb{P},\nu,\tau_2}, \bar{z}_u^{\mathbb{P},\nu,\tau_2}\big) du - \int_s^{\tau_2} \bar{z}_u^{\mathbb{P},\nu,\tau_2} dB_u,\ s \in [\tau_1, \tau_2],\ \mathbb{P}-a.s. \tag{3.128}$$

By definitions of $\tilde{V}^{\nu}$ and $\bar{y}^{\mathbb{P},\nu,\tau_2}$ (see (3.115) and (3.116)), this leads to

$$\tilde{V}_{\tau_1} \geq Y_{\tau_1}^{\mathbb{P},\nu}(\tau_2, \tilde{V}_{\tau_2}),\ \mathbb{P} - a.s.$$

and by its definition, (3.82), $\big(Y_t^{\nu}(\tau_2, \tilde{V}_{\tau_2}), Z_t^{\nu}(\tau_2, \tilde{V}_{\tau_2})\big)$ is the unique solution of:

$$\begin{aligned} Y_s^{\nu}(\tau_2, \tilde{V}_{\tau_2}) = \tilde{V}_{\tau_2} + \int_s^{\tau_2} \left[\hat{F}_u\big(Y_u^{\nu}(\tau_2, \tilde{V}_{\tau_2}), Z_u^{\nu}(\tau_2, \tilde{V}_{\tau_2})\big) - \big(\delta(\nu_u) - Z_u^{\nu}(\tau_2, \tilde{V}_{\tau_2})\big)\right] du \\ - \int_s^{\tau_2} Z_u^{\nu}(\tau_2, \tilde{V}_{\tau_2}) dB_u,\ \mathbb{P}\_a.s. \end{aligned} \tag{3.129}$$

Therefore for all $\nu \in \mathcal{D}$, $\tilde{V}$ is a strong $g^{\mathbb{P},\nu}$-supermartingale under every $\mathbb{P}$ where $g_t^{\mathbb{P},\nu}(y,z) := \hat{F}_t(y,z) - \big(\delta(\nu_t) - \nu_t z\big)$.

Step 2: We know that Assumption 3.8 and claim (3.15) holds for all $\mathbb{P} \in \mathcal{P}_H$. Moreover, we have shown that $\tilde{V} \in \mathbb{S}_H^2(\mathbb{R})$ (see (3.93)) is a strong cádlág

$g^{\mathbb{P},\nu}$-supermartingale for every $\nu \in \mathcal{D}$ and $\mathbb{P} \in \mathcal{P}_H$. Then by the nonlinear Doob-Meyer decomposition theorem 3.23 applied under each $\nu \in \mathcal{D}$ and $\mathbb{P} \in \mathcal{P}_H$, there exists $(Z^{\mathbb{P},\nu}, K^{\mathbb{P},\nu}) \in H^2(\mathbb{P}, \mathbb{R}^d) \times A^2(\mathbb{P}, \mathbb{R})$ such that:

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \left[\hat{F}_u\left(\tilde{V}_u, Z_u^{\mathbb{P},\nu}\right) + \left(\delta(\nu_u) - \nu_u Z_u^{\mathbb{P},\nu}\right)\right] du + \int_0^t Z_u^{\mathbb{P},\nu} dB_u - K_t^{\mathbb{P},\nu},\ \mathbb{P}-a.s.$$

for all $t \in [0,1]$.

□

Proof. of Theorem 3.59
The main arguments of this proof are taken from Proposition 2.5 of [21]. First, let us show for all $\nu \in \mathcal{D}$ and $\mathbb{P} \in \mathcal{P}_H$ that the process $\tilde{Z}$ defined by:

$$\tilde{Z} := Z^{\mathbb{P},\nu} = Z^{\mathbb{P},0},\ dt \otimes d\mathbb{P} - a.s. \tag{3.130}$$

does not depend on $\mathbb{P}$ and ν:
We consider for each $\mathbb{P} \in \mathcal{P}_H$, two processes $(\nu, \mu) \in \mathcal{D}^2$. By Proposition 3.60, we can find $\left(Z^{\mathbb{P},\nu}, K^{\mathbb{P},\nu}\right)$ such that:

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \left[\hat{F}_u\left(\tilde{V}_u, Z_u^{\mathbb{P},\nu}\right) - \left(\delta(\nu_u) - \nu_u Z_u^{\mathbb{P},\nu}\right)\right] du + \int_0^t Z_u^{\mathbb{P},\nu} dB_u - K_t^{\mathbb{P},\nu},\ \mathbb{P}-a.s. \tag{3.131}$$

By Karandikar [45], since $\tilde{V}$ is a càdlàg supermartingale under each $\mathbb{P}$ and ν, we can define uniquely a universal process $\tilde{Z} \in H_H^2(\mathbb{R}^d)$ such that $d\langle \tilde{V}, B\rangle_t = \tilde{Z}_t d\langle B\rangle_t$. Therefore, $\tilde{Z}_t = \bar{Z}_t^{\mathbb{P},\nu}, dt \times d\mathbb{P} - a.s.$ for all $\mathbb{P} \in \mathcal{P}_H$ and $\nu \in \mathcal{D}$.

Let us do the following change of probability measure, $\mathbb{P}^\mu$ defined in (3.12). Therefore, we can rewrite the above equation as :

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \left[\hat{F}_u\left(\tilde{V}_u, \tilde{Z}_u\right) + \left(\delta(\nu_u) - (\nu_u - \mu_u)\tilde{Z}_u\right)\right] du + \int_0^t \tilde{Z}_u dB_u^\mu - K_t^{\mathbb{P},\nu},\ \mathbb{P}-a.s. \tag{3.132}$$

In addition, by Proposition 3.60, we have the following equation for $\mu \in \mathcal{D}$:

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \left[\hat{F}_u\left(\tilde{V}_u, \tilde{Z}_u\right) + \left(\delta(\mu_u) - \mu_u \tilde{Z}_u\right)\right] du + \int_0^t \tilde{Z}_u dB_u^\mu - K_t^{\mathbb{P},\mu},\ \mathbb{P} - a.s. \tag{3.133}$$

Equalizing (3.132) and (3.133), we can define for all $t \in [0,1]$ a process $\tilde{K}^{\mathbb{P}}$ which does not depend on $\nu \in \mathcal{D}$:

$$\tilde{K}_t^{\mathbb{P}} := K_t^{\mathbb{P},\nu} = K_t^{\mathbb{P},0} + \int_0^t \delta(\nu_u) - \nu_u \tilde{Z}_u du. \tag{3.134}$$

Secondly, we are going to show that $\tilde{Z}_t \in C,\ 0 \leq t \leq 1,\ \mathbb{P} - a.s.$ for all $\mathbb{P} \in \mathcal{P}_H$. For that purpose, we consider for any $\mathbb{P} \in \mathcal{P}_H$ and $\nu \in \mathcal{D}$ such that for all $t \in [0,1]$, $|\nu_t| < 1$ and $|\delta(\nu_t)| \leq 1$ the set:

$$F_{\mathbb{P},\nu} := \{(t,\omega) : \delta(\nu(t,\omega)) < \tilde{Z}(t,\omega)\nu(t,\omega)\}$$

and assume that $(\lambda \otimes \mathbb{P})(F_{\mathbb{P},\nu}) > 0$. where λ corresponds to the Lebesgue measure. By definition of $\mathcal{D}$, we have that $(k\nu) \in \mathcal{D}$ for all $k > 0$. So by (3.134), we have

$$K_1^{\mathbb{P},k\nu} = K_1^{\mathbb{P},0} + k \int_0^1 \big[\delta(\nu_u) - \nu_u \tilde{Z}_u\big] du$$

Sending k to infinity gives us that $K_1^{\mathbb{P},k\nu}$ may be negative with a positive probability. Since $K^{\mathbb{P},\hat{\nu}}$ is a positive process (as an increasing process with $K_0^{\mathbb{P},k\nu} = 0$), there is a contradiction. So this leads to $(\lambda \otimes \mathbb{P})(F_{\mathbb{P},\nu}) = 0$ for all $\mathbb{P} \in \mathcal{P}_H$ and $\nu \in \mathcal{D}$, that is to say:

$$\delta(\nu_t) = \nu_t \tilde{Z}_t,\ 0 \leq t \leq 1, \mathbb{P} - a.s.$$

Therefore by Lemma 5.4.2 of [50], we can conclude that for all $\mathbb{P} \in \mathcal{P}_H$,

$$\tilde{Z}_t \in C,\ 0 \leq t \leq 1,\ \mathbb{P} - a.s.$$

Finally, we need to show that $(\tilde{V}, \tilde{Z}) \in D_H^2(\mathbb{R}) \times H_H^2(\mathbb{R}^d)$ is the minimal solution to (3.79) subject to constraint (3.80). We have shown in the claim (3.92) that for any solution (Y^*, Z^*) of the second order constrained problem (3.79) and (3.80):

$$Y_t^* \geq \tilde{V}_t,\ \mathbb{P} - a.s.,$$

for all $t \in [0,1]$ and $\mathbb{P} \in \mathcal{P}_H$. Since we have shown in the previous step of the proof that $(\tilde{V}, \tilde{Z})$ is a solution of (3.79) and (3.80), then we can conclude that this solution is indeed minimal. □

We then have the following stochastic representation result:

Proposition 3.61. *Under the assumptions of Theorem* 3.59*, the minimal solution* $(Y, Z) \in \mathbb{S}_H^2(\mathbb{R}) \times \mathbb{H}_H^2(C)$ *of* (3.79) *and* (3.80) *has the following representation for all* $t \in [0,1]$ *and* $\mathbb{P} \in \mathcal{P}_H$ *:*

$$Y_t = \operatorname*{essup}_{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})}^{\mathbb{P}} \operatorname*{essup}_{\nu \in \mathcal{D}} Y_t^{\mathbb{P}',\nu}(1,\xi),\ \mathbb{P} - a.s.$$

Proof. Under the assumptions of Theorem 3.59, by Theorem 3.59, $(\tilde{V}, \tilde{Z})$ (where $\tilde{Z}$ is defined in (3.130)) is the minimal solution of (3.79) and (3.80). Hence by Proposition 3.57, we have that :

$$\tilde{V}_t = \underset{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})}{\text{essup}^{\mathbb{P}}} \ \underset{\nu \in \mathcal{D}}{\text{essup}} \ Y^{\mathbb{P}',\nu}(1,\xi), \ \mathbb{P} - a.s.$$

□

Corollary 3.62. *Under the assumptions of Theorem* 3.59*, let us consider for all* $i \in \{1,2\}$*,* $(\xi^i) \in \mathbb{L}^2_H(\mathbb{R})$ *and* $(Y^i, Z^i) \in \mathbb{D}^2_H(\mathbb{R}) \times \mathbb{H}^2_H(C)$ *the corresponding solution of* (3.79) *and* (3.80)*. If* $\xi^1 \leq \xi^2$*,* $\mathcal{P}_H - q.s.$*, then :*

$$Y^1 \leq Y^2, \ \mathcal{P}_H - q.s.$$

Proof. It suffices to apply the comparison theorem 3.18. This shows that for all $\mathbb{P} \in \mathcal{P}_H$ and $\nu \in \mathcal{D}$,

$$Y_t^{\mathbb{P},\nu}(1,\xi^1) \leq Y_t^{\mathbb{P},\nu}(1,\xi^2), \ 0 \leq t \leq 1, \ \mathbb{P} - a.s.$$

Since $\mathbb{P}$ and ν are arbitrary, by Proposition 3.61, we finally get for all $\mathbb{P} \in \mathcal{P}_H$:

$$Y_t^1 \leq Y_t^2, \ 0 \leq t \leq 1, \ \mathbb{P} - a.s.$$

□

3.5.3 Conclusion

Let us do a quick review of future researches. Firstly, a good question is to study the Markovian case and do the connection between the constrained 2BSDEs and the partial differential equation which is associated to it. In the first order case, we already know that the PDE is a variational inequality. We would like to know if it possible to extend this result to nonlinear second order PDEs. Secondly, a another interesting question is to know if the general constrained problem (3.31) and (3.32) can be adapted to second order BSDEs. More generally, Peng and Xu [69, 71] has proved the existence of a minimal solution to the first order constrained and reflected BSDEs. Is-it possible to extend their results to 2BSDEs?

Chapter 4

Small volatility asymptotics in a stochastic volatility model

4.1 Introduction

The modelization of the volatility smile was pointed out by Bergomi in several papers [2, 3, 4, 5]. He proposed a framework where the price of an underlying relies entirely on the modelization of the forward variance curve. This modelization has the advantage of giving a better control on the term structure of the volatility of volatility. Therefore it is very useful for financial products involving both an underlying and its realized variance.

This modelization has several connections with the bond market. Heath, Jarrow, and Morton ([40]) were the first to study the term structure of interest rate in a non parametric framework. In this model, the term structure of the spot interest rate depends entirely on the forward curve of the zero-coupon bonds, that is to say an infinite dimensional process. It is well-known that in pratice the forward curve is obtained by smoothing data points and is flat for large times to maturity. Therefore the specification of the infinite dimensional space where the curve lives has to guarantee this properties. This important issue has been treated for instance in Filipovic, in [34], who provides a functional analytic framework to HJM-model. One can mention the work of Ekeland and Taflin in [27] who are interested in problems of portfolio management with stocks and interest rate. They choose to define zero-coupon bond price curve as an element of a Hilbert space, typically a Sobolev space of order sufficiently high to capture a smooth behavior.

In this chapter, our aim will be to give an expansion of the price of a vanilla option in a stochastic volatility model where the volatility of the volatility depends on the forward variance curve. As for the interest rate model, we will assume that the forward variance curve belongs to a Hilbert space. To do such an expansion, we will follow the approach of Fleming and Soner [36], [35] and Fleming and Souganidis [37] which requires stability results on viscosity solutions. Since we are working in an infinite dimensional space, the classical results on viscosity solutions for second order partial differential equations([18]) do not apply. Lions in several papers, mainly [58, 57], gives a correct setting to extend the result of viscosity solutions of real valued functions to Hilbert spaces where the state equation of the system involves bounded operators. Swiech, in [88], generalizes this results to the case of unbounded operators.

First, we will review the main tools on Sobolev spaces, semi-group and viscosity solutions in Hilbert spaces. Then, we will revisit the Bergomi model by assuming that the forward variance curve belongs to a Sobolev space and then provides a n-order expansion for the price of a vanilla option under smoothness assumptions on the payoff function.

4.2 Mathematical Preliminaries

4.2.1 Sobolev spaces

We define for $p \geq 0$, $L^p(\mathbb{R})$ the space of functions which are p-integrable, that is to say:

$$\|f\|_{L^p(\mathbb{R})} := \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{1/p} < \infty.$$

For $n \geq 0$, the Sobolev space $H^n(\mathbb{R})$ is the set of all functions f which are square-integrable together with all their derivatives of order up to n, that is to say:

$$H^n(\mathbb{R}) := \left\{ f : \frac{d^k f}{dx^k} \in L^2(\mathbb{R}) < \infty \ \forall k \in \{0, \ldots, n\} \right\}.$$

This becomes a Hilbert space endowed by the norm

$$\|f\|_{H^n(\mathbb{R})} := \left(\|f\|^2_{L^2(\mathbb{R})} + \sum_{k=1}^{n} \| \frac{d^k f}{dx^k} \|^2_{L^2(\mathbb{R})} \right)^{1/2}.$$

We define the set of tempered distributions which is the dual $\mathcal{S}'(\mathbb{R})$ of the topological vector space:

$$\mathcal{S}(\mathbb{R}) := \{f : x^\alpha D^\beta f \in L^2(\mathbb{R}) \ \ \forall\alpha\forall\beta \text{ multi-indices of size } n\}.$$

For $f \in \mathcal{S}'(\mathbb{R})$, the Fourier transform $\hat{f}$ is defined by:

$$\hat{f}(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-ixy) f(x) dx.$$

By Theorem 1.2 of [56], we can give an alternative definition of $H^n(\mathbb{R})$:

$$H^n(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R}) : (1+|y|^2)^{n/2} \hat{f} \in L^2(\mathbb{R}) \right\}.$$

with $\|f\|^2_{H^n(\mathbb{R})} = \int_{-\infty}^{\infty} (1+y^2)^n |\hat{f}(y)|^2 dy$. We can extend this definition for $\gamma \in \mathbb{R}$ and define $H^\gamma(\mathbb{R})$ the Hilbert space of tempered distributions f which are such that

$$\|f\|^2_{H^\gamma(\mathbb{R})} = \int_{-\infty}^{\infty} (1+y^2)^\gamma |\hat{f}(y)|^2 dy < \infty. \tag{4.1}$$

Clearly, for $\gamma \geq \gamma'$, we have $H^\gamma(\mathbb{R}) \subset H^{\gamma'}(\mathbb{R})$ and $H^0(\mathbb{R}) = L^2(\mathbb{R})$.

We consider three Banach sets denoted by H, G and L. We define $C^n(H,L)$ the space of the n times continuously differentiable functions from H to L and $C^n_b(H,L)$ the subspace of $C^n(H,L)$ where the n first derivatives are bounded. We denote by $C^{k,n}(G \times H, L)$ the space of functions f from $G \times H$ to L such that $(g,h) \mapsto f(g,h)$ which are jointly k times continuously differentiable with respect to g and n times continuously differentiable with respect to h. $C^{k,n}_b(G \times H, L)$ is the subspace of $C^{k,n}(G \times H, L)$ where the function together with its partial derivatives up to order n and k are bounded. When $L = \mathbb{R}$, we will use the notation $C^n(H) := C^n(H,\mathbb{R})$ and $C^{k,n}(G \times H) := C^{k,n}(G \times H, \mathbb{R})$. Moreover, we define $C^n_{b0}(H)$ as the closed subspace of $C^n_b(H)$ of functions which converge to 0 at $\pm\infty$ together with their n first derivatives. $C^n_{b0}(H)$ and $C^n_b(H)$ are Banach spaces for the norm:

$$\|f\|_{C^n_b(H)} = \max_{0 \leq k \leq n} \sup_x |f^{(k)}(x)|$$

We have the following classical results (see corollary 9.1 in [56]):

Theorem 4.1. ***(Sobolev Embedding)***
For any integer n, if $\gamma > n + \frac{1}{2}$ and if $f \in H^\gamma(\mathbb{R})$, then there is a function g in $C^n_{b0}(\mathbb{R})$ which is equal to f almost everywhere. In addition, there is a constant c_γ depending only on γ, such that:

$$\|g\|_{C^n_b(\mathbb{R})} \leq c_\gamma \|f\|_{H^\gamma(\mathbb{R})}.$$

In the sequel we will not distinguish f and g, that is to say, we will always consider the continuous bounded representative of each function in $H^\gamma(\mathbb{R})$ for $\gamma > \frac{1}{2}$. Moreover, we have the well-known theorem:

Theorem 4.2. *If $\gamma > \frac{1}{2}$, if f and g belongs to $H^\gamma(\mathbb{R})$, then fg belongs to $H^\gamma(\mathbb{R})$ and the map $(f,g) \to fg$ from $H^\gamma(\mathbb{R}) \times H^\gamma(\mathbb{R}) \to H^\gamma(\mathbb{R})$ is continuous.*

We define for $\gamma \in \mathbb{R}$ a continuous bilinear form on $H^{-\gamma}(\mathbb{R}) \times H^\gamma(\mathbb{R})$ by

$$\langle f, g\rangle \ = \int \overline{\hat{f}(x)}\hat{g}(x)dx,$$

where $\overline{x}$ denotes the complex conjugate of x. Symbolically for $\gamma \geq 0$, this continuous bilinear form could be rewritten

$$\langle f, g\rangle \ = \int f(x)g(x)dx, \tag{4.2}$$

where f is in the space of distribution $H^{-\gamma}(\mathbb{R})$ and g is in the space of test functions $H^\gamma(\mathbb{R})$. Any continuous linear form $f \mapsto u(f)$ on $H^\gamma(\mathbb{R})$ has the following representation: $u(f) \ = \langle f, g\rangle$ for some g in $H^{-\gamma}(\mathbb{R})$ such that $\|g\|_{H^{-\gamma}(\mathbb{R})} = \|u\|_{(H^\gamma(\mathbb{R}))^*}$. So we can from now on identify $H^{-\gamma}(\mathbb{R})$ as the dual of $H^\gamma(\mathbb{R})$ denoted by $(H^\gamma(\mathbb{R}))^*$.

We denote by $H^\gamma(\mathbb{R}^+)$ the quotient space:

$$H^\gamma(\mathbb{R}^+) := H^\gamma(\mathbb{R})/H^\gamma_-.$$

where H^γ_- is the closed subspace of $H^\gamma(\mathbb{R})$ which contains all functions with support in $(-\infty, 0]$. That is to say $f \in H^\gamma_-$ if $f(x) = 0$ for *a.e.* $x < 0$. Thus $H^\gamma(\mathbb{R}^+)$ is isomorphic to the set of restriction to $[0,\infty)$ of functions in $H^\gamma(\mathbb{R})$. This is a Hilbert space for the norm

$$\|f\|_{H^\gamma(\mathbb{R}^+)} = \inf\big\{\|g\|_{H^\gamma(\mathbb{R})} \mid g(x) = f(x) \text{ } a.e. \text{ on } [0,\infty)\big\}.$$

We define the continuous linear map κ from $H^\gamma(\mathbb{R})$ to $H^\gamma(\mathbb{R}^+)$ by

$$\kappa(f) = \begin{cases} f(x) & x \in [0,\infty) \\ 0 & x \in (-\infty, 0) \end{cases} \tag{4.3}$$

We denote by ι a continuous linear injection from $H^\gamma(\mathbb{R}^+)$ to $H^\gamma(\mathbb{R})$ such that $\kappa\iota = Id_{H^\gamma(\mathbb{R}^+)}$. For example, ι is such that :

$$\iota f = g \text{ where g is the unique element in } H^\gamma(\mathbb{R}) \text{such that } \kappa g = f \text{ and } \langle h, g\rangle \ = 0 \text{ for any } h \in H^\gamma_- \tag{4.4}$$

Such an injection is not unique, for instance one can see the Appendix of [27] for other possibilities.

If $\gamma \in \mathbb{R}$, 1 does not belong to $H^\gamma(\mathbb{R})$. In order to take into account constants or functions which do not converge to 0 at infinity, we need to enlarge $H^\gamma(\mathbb{R})$. For that purpose, we define

$$H(\mathbb{R}) = H^\gamma(\mathbb{R}) \oplus \mathbb{R}.$$

with a norm given by $\|f\|^2_{H(\mathbb{R})} = \|g\|^2_{H^\gamma(\mathbb{R})} + a^2$ for $f = g + a$.

The dual $(H(\mathbb{R}))^*$, is identified with $H^{-\gamma}(\mathbb{R}) \oplus \mathbb{R}$ by extending the previous continuous linear form (4.2) to

$$\langle f, f' \rangle \;=\; \langle g, g' \rangle \;+\; aa'. \tag{4.5}$$

for $f = g + a \in H(\mathbb{R})$ and $f' = g' + a' \in (H(\mathbb{R}))^*$, f in $H^\gamma(\mathbb{R})$, f' in $H^{-\gamma}(\mathbb{R})$, (a, a') in $\mathbb{R}^2$.

We denote for $\gamma \geq 0$, $H(\mathbb{R}^+) = H^\gamma(\mathbb{R}^+) \oplus \mathbb{R}$ which is similarly a Hilbert space for the norm $\|f\|^2_{H(\mathbb{R}^+)} = \|g\|^2_{H^\gamma(\mathbb{R}^+)} + a^2$ for $f = g + a$. As for $H(\mathbb{R})$, the dual of $H(\mathbb{R}^+)$ is identified with $H^{-\gamma}(\mathbb{R}^+) \oplus \mathbb{R}$.

As in the Appendix of Ekeland and Taflin in [27], we can extend κ to $H(\mathbb{R}) \to H(\mathbb{R}^+)$ by $\kappa(a + f) = a + \kappa f$ for $f \in H^\gamma(\mathbb{R})$ and $a \in \mathbb{R}$. We extend ι to $H(\mathbb{R}^+) \to H(\mathbb{R})$ by $\iota(a + f) = a + \iota f$ for $f \in H^\gamma(\mathbb{R}^+)$ and $a \in \mathbb{R}$.

4.2.2 Linear continuous maps

We recall some standard properties of linear continuous maps (see Yosida [90]). We consider $l : E \to H$ a linear map where $(E, \|.\|_E)$ and $(H, \|.\|_H)$ are two normed vector spaces.

Proposition 4.3. *The following properties are equivalent:*

(i) l is continuous at 0.

(ii) l is continuous on E.

(iii) l is uniformly continuous on E.

(iv) l is Lipschitz continuous on E.

(v) l is bounded on the unit ball of E.

We denote by $L(E,H)$ the set of all continuous linear functions from E to H. Let us define for all l in $L(E,H)$,

$$\|l\|_{L(E,H)} := \sup_{\|x\|_E \leq 1} \|l(x)\|_H.$$

We have the following theorem

Theorem 4.4. *If $(H, \|.\|_H)$ is a Banach space and $(E, \|.\|_E)$ a normed vector space, then $\big(L(E,H), \|.\|_{L(E,H)}\big)$ is a Banach space.*

4.2.3 The continuous left translation group

We want to introduce the notion of semigroup which will be usefull in the sequel. For more details, see Yosida [90].

Definition 4.5. *Let $\{T_t; t \geq 0\}$ be a one-parameter family of bounded linear operators on a Banach space H satisfying for $t, s \geq 0$*

$$T_t T_s = T_{t+s}. \tag{4.6}$$

$$T_0 = I. \tag{4.7}$$

then $\{T_t; t \geq 0\}$ is called a semi-group.

If $\{T_t; t \geq 0\}$ defined by definition 4.5 is such that for all $x \in H$:

$$\|T_t x - x\|_H \underset{t \to 0}{\longrightarrow} 0.$$

then $\{T_t; t \geq 0\}$ is called a strongly continuous semi-group. In particular, if $\|T_t\|_H \leq 1$, then it is called a strongly continuous contraction semi-group.

Remark 4.6. (i) If $\{T_t;\ t \in \mathbb{R}\}$ is a one-parameter family of bounded linear operators on a Banach space H satisfying (4.6) and (4.7) for t, s in $\mathbb{R}$ then it is a group.

(ii) If $\{T_t;\ t \geq 0\}$ is a semi-group then for all $t > 0$, T_t does not necessarily admit an inverse element.

Let $\gamma \in \mathbb{R}$, for all t in $\mathbb{R}$, we define $\mathcal{L}_t \in L\big(H^\gamma(\mathbb{R}), H^\gamma(\mathbb{R})\big)$ by

$$(\mathcal{L}_t f)(x) = f(x+t). \tag{4.8}$$

Clearly, $\{\mathcal{L}_t : t \in \mathbb{R}\}$ is a contraction group. Indeed, take a function $\phi \in H^\gamma(\mathbb{R})$ and denote its Fourier transform $\hat{\phi}$. Then for $(t, y) \in \mathbb{R}^2$, the Fourier transform of $\mathcal{L}_t\phi$ which is a tempered distribution is given by:

$$\begin{aligned}\widehat{\mathcal{L}_t\phi}(y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-ixy)\phi(x+t)dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-i(u-t)y)\phi(u)du \\ &= \exp(ity)\hat{\phi}(y)\end{aligned}$$

This gives us the following inequality:

$$\|\mathcal{L}_t\phi\|_{H^\gamma(\mathbb{R})} \le \|\phi\|_{H^\gamma(\mathbb{R})}$$

Therefore $\{\mathcal{L}_t : t \in \mathbb{R}\}$ is a contraction group on $H^\gamma(\mathbb{R})$. Similarly, we have that:

$$\|\mathcal{L}_t\phi - \phi\|^2_{H^\gamma(\mathbb{R})} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(1+y^2)^\gamma|\exp(ity)-1|^2|\hat{\phi}(y)|^2dy$$

And by dominated convergence theorem, we get :

$$\|\mathcal{L}_{t_n}\phi - \phi\|_{H^\gamma(\mathbb{R})} \underset{n}{\longrightarrow} 0.$$

So, $\{\mathcal{L}_t : t \in \mathbb{R}\}$ is a strongly continuous contraction group. We extend $\mathcal{L}_t$ to $H(\mathbb{R}) = H^\gamma(\mathbb{R}) \oplus \mathbb{R}$ ($\gamma \in \mathbb{R}$) by

$$\mathcal{L}_t(a+f)(x) = a + f(x+t),\ \forall f \in H^\gamma(\mathbb{R}),\ (t,a) \in \mathbb{R}^2. \tag{4.9}$$

Thanks to Proposition 13 of [27], we can also define a strongly continuous contraction semi-group $\mathcal{L}^+$ on $H(\mathbb{R}^+)$ by:

$$\mathcal{L}^+_t(a+f)(x) = a + f(x+t),\ \forall f \in H^\gamma(\mathbb{R}^+),\ (t,a) \in \mathbb{R}^2. \tag{4.10}$$

For $\gamma \ge 0$, the space H^γ_- is invariant under $\mathcal{L}$, so $\mathcal{L}^+$ verifies

$$\kappa\mathcal{L}_t = \mathcal{L}^+_t\kappa. \tag{4.11}$$

As in chapter IX of [90], for any Hilbert space H and for all f in H, we define the infinitesimal generator A of a semi-group $\{T_t; t \ge 0\}$ by:

$$Af = \lim_{h\to 0}\frac{1}{h}[T_hf - f]. \tag{4.12}$$

A is a linear operator with domain

$$D(A) := \{f \in H :\ \lim_{h\to 0}\frac{1}{h}[T_hf - f] \text{ exists in } H\}.$$

We have the following theorem (see [90], chapter IX, Theorem 1 (p237) and Theorem 6 of [27]):

Theorem 4.7. *Let $\{T_t; t \geq 0\}$ be a strongly continuous semigroup in a Banach space H and let A be its infinitesimal generator. Then $D(A)$ is a dense linear subspace of H and A is a closed operator in H.*

We denote by ∂ the infinitesimal generator of the group $\mathcal{L}_t$. We can see easily that $D(\partial) = \{f \in H(\mathbb{R}) : \partial f = f' \in H(\mathbb{R})\} = \mathbb{R} \oplus H^{\gamma+1}(\mathbb{R})$ by definition of the fractional Sobolev spaces. So, if $f \in \mathbb{R} \oplus H^{\gamma+1}(\mathbb{R})$, then

$$\partial f = f'. \tag{4.13}$$

where f' is the derivative of f.

4.2.4 Stochastic equations in infinite dimension

4.2.5 Infinite dimensional Brownian motion

We consider two separable Hilbert spaces $(E, \langle ., . \rangle_E)$ and $(H, \langle ., . \rangle_H)$ and an index set $\mathbb{I}$ whose cardinality equals the dimension of H. $(g_i)_{i\in\mathbb{I}}$ is an orthonormal basis of H.

To understand problems which could happen while using infinite Brownian motions, one can see that basically if we define W_t as $\sum_{i\in\mathbb{I}} \beta_i(t) g_i$, where $(\beta_i)_{i\in\mathbb{I}}$ is a family of independent real valued standard Brownian motions then

$$\mathbb{E}[\|W_t\|_H^2] = \sum_{i\in\mathbb{I}} \mathbb{E}[(\beta_i(t))^2] = \sum_{i\in\mathbb{I}} t = \infty.$$

So we have a problem of definition. To overcome it, let us denote by Q a symmetric positive operator of $L(H, H)$ such that $Tr(Q) < \infty$ and for $(\lambda_i)_{i\in\mathbb{I}}$ a positive sequence of real numbers:

$$Qg_i = \lambda_i g_i, \ i \in \mathbb{I}.$$

Definition 4.8. *A H-valued stochastic process $W_t, t \geq 0$, is called a $Q-Wiener$ process if*

(i) $W(0) = 0$.

(ii) *W has continuous trajectories.*

(iii) *W has independent increments.*

(iv) $W_t - W_s \sim \mathcal{N}\big(0, (t-s)Q\big)$, $t \geq s \geq 0$.

where $\mathcal{N}\big(0, (t-s)Q\big)$ corresponds to a Gaussian measure on Hilbert space (see [22] section 2.3.2). We have the following proposition:

Proposition 4.9. *Assume that* W *is a* $Q-Wiener$ *process, with* $Tr\ Q < \infty$. *Then for any positive* t, W_t *has the expansion*

$$W_t = \sum_{i \in \mathbb{I}} \sqrt{\lambda_i} \beta_i(t) g_i \tag{4.14}$$

where $(\beta_i)_{i \in \mathbb{I}}$ *is a family of independent real valued standard Brownian motions on a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. *For each* $i \in \mathbb{I}$, β_i *is such that:*

$$\beta_i(t) = \frac{1}{\sqrt{\lambda_i}} \langle W_t, g_i \rangle.$$

and the serie (4.14) *converges* $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. See Proposition 4.1 chapter 4 of Da Prato and Zabczyk in [22]. □

We can also consider the case where $Tr(Q) = \infty$. In this case, the process W is called a cylindrical Wiener process. In this case, we give a meaning to the expression $W_t = \sum_{i \in \mathbb{I}} g_i \beta_i(t)$ by considering its "projection" on a subspace of H. Indeed, one can show:

Proposition 4.10. *If* $W_t = \sum_{i \in \mathbb{I}} \beta_i(t) g_i$ *is a cylindrical Wiener process, then for every* $x \in H$ *such that* $\|x\|_H = 1$, *the real valued stochastic process*

$$\langle W_t, x \rangle_H = \sum_{i \in \mathbb{I}} \langle x, g_i \rangle_H \beta_i(t).$$

is a standard Brownian Motion.

Proof. For all $n, p \geq 0$, by Doob's inequality,

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |\sum_{i=n}^{n+p} \langle x, g_i \rangle_H \beta_i(t)|^2\right] \leq 4T \sum_{i=n}^{n+p} \langle x, g_i \rangle_H^2 \longrightarrow 0.$$

since g_i is an orthonormal basis. Moreover, $\sum_{i=0}^{n} \langle x, g_i \rangle \beta_i(t)$ is a finite sum of centered independent Gaussian processes and then a centered Gaussian process. Since it converges in probability, its limit is still a centered Gaussian process. Furthermore, for all $t \geq s \geq 0$,

$$\begin{aligned} \mathbb{E}\left[\langle W_t, x \rangle_H \langle W_s, x \rangle_H\right] &= s \sum_{i \in \mathbb{I}} \langle x, g_i \rangle_H^2 \\ &= s \|x\|_H^2 \\ &= s \end{aligned}$$

So, by Paul-Levy theorem, we get that $\langle W_t, x \rangle_H$ is a real valued standard Brownian motion. □

We define the subspace $E^0 := Q^{1/2}E$ of E which endowed with the inner product

$$\langle x, y\rangle_0 := \langle Q^{-1/2}x, Q^{-1/2}y\rangle_H.$$

is a Hilbert space. We denote by $L_2^0 := L_2(E^0, H)$ the separable Hilbert space of all Hilbert-Schmidt operators from E^0 to H for the norm :

$$\|A\|^2_{L_2^0} := \sum_{i\in\mathbb{I}} \|Ag_i\|^2_H.$$

We define the Hilbert space $N^2_W\big(0,T;L_2^0\big)$ of all L_2^0-valued predictable processes Φ such that for all $T>0$:

$$\mathbb{E}\bigg[\int_0^T \|\Phi_s\|^2_{L_2^0}ds\bigg] < \infty.$$

Then, we have the following theorem:

Theorem 4.11. *Let W be a cylindrical Wiener process and assume that $\Phi \in N^2_W\big(0,T;L_2^0\big)$. Then, for all $t\in[0,T]$, the stochastic integral $\int_0^t \Phi_s dW_s$ is a continuous square integrable martingale and we have the following isometry:*

$$\mathbb{E}\bigg[\|\int_0^t \Phi_s dW_s\|^2_H\bigg] = \mathbb{E}\bigg[\int_0^t \|\Phi_s\|^2_{L_2^0}ds\bigg] = \mathbb{E}\bigg[\int_0^t Tr\big[\Phi_s Q^{1/2}(\Phi_s Q^{1/2})^*\big]ds\bigg].$$

Proof. See Theorem 4.12, chapter 4 of [22]. □

4.2.6 Mild versus strong solution

We consider $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ a complete filtered probability space with $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ and a Hilbert space H. Let W be a cylindrical Wiener process. This subsection is devoted to give a meaning to the following stochastic equation for all t in $[0,T]$, $T>0$:

$$dX_t = \big(AX_t + F(t,X_t)\big)dt + B(t,X_t)dW_t,\ X_0 = x\in H \tag{4.15}$$

where $A : D(A)\to H$ is the infinitesimal generator of a strongly continuous contraction group $\{T_t; t\geq 0\}$.

(i) The mapping $F : [0,T]\times\Omega\times H \longrightarrow H$ is measurable from $\big([0,T]\times\Omega\times H, \mathcal{B}_{[0,T]}\times\mathcal{F}\times\mathcal{B}(H)\big)$ into $\big(H,\mathcal{B}(H)\big)$.

(ii) The mapping $B : [0,T]\times\Omega\times H \longrightarrow L_2^0$ is measurable from $\big([0,T]\times\Omega\times H, \mathcal{B}_{[0,T]}\times\mathcal{F}\times\mathcal{B}(H)\big)$ into $\big(L_2^0,\mathcal{B}(L_2^0)\big)$.

Definition 4.12. *Suppose that X is a H-valued predictable process satisfying*

$$\mathbb{P}\bigg[\int_0^T \big(\|X_s\|_H + \|F(s,X_s)\|_H + \|B(s,X_s)\|^2_{L_2^0}\big)ds < \infty\bigg] = 1.$$

(i) X is a mild solution of (4.15)*, if X is solution of*

$$X_t = T_t x + \int_0^t T_{t-s} F(s, X_s) ds + T_{t-s} B(s, X_s) dW_s.$$

for all $0 \leq t \leq T$.

(ii) A mild solution X is a strong solution of (4.15) *if* $X \in D(A)$, $dt \otimes d\mathbb{P}$-*a.s. and*

$$\mathbb{P}\Big[\int_0^T \|AX_s\|_H ds < \infty\Big] = 1.$$

If X is a strong solution then the integral version for all positive t,

$$X_t = x + \int_0^t \big(AX_s + F(s, X_s)\big) ds + \int_0^t B(s, X_s) dW_s.$$

of (4.15) holds $\mathbb{P}$-a.s.

In the sequel, we will need the following result which can be found in Da Prato and Zabczyk, Theorem 7.4 in [22].

Theorem 4.13. *Assume that x is a* $\mathcal{F}_0$*-measurable H-valued random variable and that F and B are such that*

$$\|F(t, \omega, y) - F(t, \omega, y')\|_H + \|B(t, \omega, y) - B(t, \omega, y')\|_{L_2^0}\| \leq C\|y - y'\|_H.$$
$$\|F(t, \omega, y)\|_H^2 + \|B(t, \omega, y)\|_{L_2^0}^2 \leq C\big(1 + \|y\|_H^2\big).$$

for (y, y') *in* H, t *in* $[0, T]$ *and* $\omega \in \Omega$.
Then there exists a mild solution X to (4.15) *unique up to equivalence among the processes satisfying*

$$\mathbb{P}\Big(\int_0^T \|X_s\|_H^2 ds < \infty\Big) = 1.$$

Moreover, it has a continuous modification and

$$\mathbb{E}\Big[\sup_{0 \leq t \leq T} \|X_s\|_H^2\Big] \leq C_T(1 + \mathbb{E}[\|x\|_H^2]).$$

If $A = 0$ in (4.15) and the conditions of Theorem 4.13 are satisfied, we notice X is a strong solution to (4.15).

4.2.7 Viscosity solutions in Hilbert space

The theory of viscosity solutions in infinite dimension is developed mainly in two papers by Lions in [57] and [58]. The notions developed in this subsection extend those of [18] which concern only the finite dimensional case.

Let us denote by H a separable Hilbert space. For any functions $\phi : \mathbb{R}^+ \times H \to \mathbb{R}$ such that $(t,x) \mapsto \phi(t,x)$, we denote by $D\phi$ and $D^2\phi$ the first and the second order Frechet derivatives in $x \in H$. We denote by $BUC_{loc}(H)$ the set of all bounded functions, uniformly continuous on bounded sets of H. $BUC(H)$ is the subset of all bounded uniformly continuous functions of $BUC_{loc}(H)$ and

$$BUC_x\big(\mathbb{R}^+ \times H\big){:=}\big\{u \in C^0\big(\mathbb{R}^+ \times H\big)\colon u(t,\cdot) \in \ BUC(H),\ \text{uniformly in } t\big\}.$$

We define $L'(H) = L_s(H \times H, \mathbb{R})$ the set of all bounded symmetric bilinear forms on H.

We are interested in solving a non linear second order degenerate parabolic equation of the following form

$$\partial_t u + F(t,x,u,Du,D^2u) = 0 \ \text{ in } \mathbb{R}^+ \times H. \tag{4.16}$$

where $(t,x) \in \mathbb{R}^+ \times H$, u, Du and D^2u are functions from $\mathbb{R}^+ \times H$ to $\mathbb{R}$, H and $L'(H)$ respectively. F is such that:

(i) F belongs to $BUC_{loc}\big(\mathbb{R}^+ \times H \times \mathbb{R} \times H \times L'(H)\big)$.

(ii) F is degenerate elliptic, that is to say,

$$F(t,x,r,p,A) \leq F(t,x,r,p,B).$$

for all $A \geq B$, for all t in $\mathbb{R}^+$, p and x in H, r in $\mathbb{R}$ where the partial ordering $A \geq B$ is defined by

$$A \geq B \text{ iff } \forall x \in H,\ (Ax,x) \geq (Bx,x).$$

As Lions in [57], we denote by $\mathcal{X}(H)$ the set of all test functions $\phi \in C^1(\mathbb{R}^+ \times H)$ such that $D\phi$ is Lipschitz on bounded sets of H and for all h and k in H,

$$\lim_{\substack{t\to 0\\ t\neq 0}} \frac{1}{t}\big(D\phi(x+tk) - D\phi(x), h\big).$$

exists and is uniformly continuous on bounded sets of H.

Remark 4.14. Suppose that H is finite dimensional, then the set $\mathcal{X}(H)$ corresponds to $C^{1,2}(\mathbb{R}^+ \times H)$. Indeed, in finite dimension, being continuous on a compact set, and thus on a closed bounded subset of H, implies the uniform continuity property on this subset by the Heine theorem.

We give the definition of viscosity solutions for F satisfying the above conditions (i) and (ii):

Definition 4.15. *We say that u in $BUC_{loc}(\mathbb{R}^+ \times H)$ is a viscosity subsolution of* (4.16) *if for each ϕ in $\mathcal{X}(H)$ and (t_0, x_0) in $\mathbb{R}^+ \times H$ such that* $\max(u-\phi)(t,x) = (u-\phi)(t_0,x_0)$*, we have*

$$\liminf_{\substack{x\to x_0\\ t\to t_0}} \partial_t u(t,x) + F(t,x,u(t,x),D\phi(t,x),D^2\phi(t,x)) \leq 0. \tag{4.17}$$

We say that u in $BUC_{loc}(\mathbb{R}^+ \times H)$ is a viscosity supersolution of (4.16) *if for each ϕ in $\mathcal{X}(H)$ and (t_0, x_0) in $\mathbb{R}^+ \times H$ such that* $\min(u-\phi)(t,x) = (u-\phi)(t_0,x_0)$*, we have*

$$\limsup_{\substack{x\to x_0\\ t\to t_0}} \partial_t u(t,x) + F(t,x,u(t,x),D\phi(t,x),D^2\phi(t,x)) \geq 0. \tag{4.18}$$

We say that u is a viscosity solution of (4.16) *if it is both a subsolution and a supersolution of* (4.16).

Remark 4.16. If (t_0, x_0) in $\mathbb{R}^+ \times H$ realize a minimum of $u-\phi$, then they realize a strict minimum of $u-\phi_\alpha$ where $\phi_\alpha(t,x) := \phi(t,x) - \alpha|x-x_0|^4 - |t-t_0|^2$ for $\alpha > 0$. Moreover, if $\phi_\alpha(t_0,x_0)$ verifies (4.17) and F is continuous in its last variable, then by sending alpha to 0 we get that ϕ is a solution to (4.17) too. So, we can replace the notion of local minimum by a strict minimum and the notion of local maximum by strict maximum.

The definition of viscosity solutions proposed by Lions in [57] are motivated by the study of the optimal control problem in infinite dimension with some small modifications compared with the finite dimensional case. Indeed one can consider the subset $\mathcal{X}'(H)$ of $\mathcal{X}(H)$ defined by:

$$\mathcal{X}'(H) := \big\{\phi \in C^{1,2}(\mathbb{R}^+ \times H):\ \phi, D\phi, D^2\phi \in BUC_{loc}(\mathbb{R}^+ \times H)\big\}. \tag{4.19}$$

Compare to the finite dimensional case, this space is more usual. The notion of sub and super differentials can still be used to define the notion of viscosity solution. That is to say, we define for $u \in BUC_{loc}(\mathbb{R}^+ \times H)$ and $(t_0, x_0) \in \mathbb{R}^+ \times H$

$$D^2_+ u(t_0,x_0) := \Bigg\{(A,p,q) \in L'(H) \times H \times \mathbb{R};\ \limsup_{y\to x_0}\big[u(t,y) - u(t_0,x_0) - (p, y-x_0)\big] - q(t-t_0) - \frac{1}{2}\langle A(y-y_0), y-y_0\rangle\big]\,\|x_0 - y\|_H^{-2} \leq 0\Bigg\}. \tag{4.20}$$

and

$$D^2_- u(t_0, x_0) := \Big\{ (A, p, q) \in L'(H) \times H \times \mathbb{R};\ \liminf_{y \to x_0} \big[u(t, y) - u(t_0, x_0) - (p, y - x_0)\big] - q(t - t_0) - \frac{1}{2} \langle A(y - y_0), y - y_0 \rangle \big] \, \|x_0 - y\|_H^{-2} \geq 0 \Big\}. \quad (4.21)$$

We can now define the concept of "classical solution" which is more usual than the concept of viscosity solution. In the sequel, F must verify conditions (i) and (ii).

Definition 4.17. *We say that u in $BUC_{loc}(\mathbb{R}^+ \times H)$ is a classical viscosity subsolution of* (4.16) *if for any ϕ in $\mathcal{X}'(H)$,* (4.17) *holds or equivalently if for any (A, p, q) in $D^2_+ u(t_0, x_0)$ and $(t_0, x_0) \in \mathbb{R}^+ \times H$, we have*

$$q + F(t_0, x_0, u(t_0, x_0), p, A) \leq 0.$$

We say that u in $BUC_{loc}(\mathbb{R}^+ \times H)$ is a classical viscosity supersolution of (4.16) *if for any ϕ in $\mathcal{X}'(H)$,* (4.18) *holds or equivalently if for any (A, p, q) in $D^2_+ u(t_0, x_0)$ and $(t_0, x_0) \in \mathbb{R}^+ \times H$, we have*

$$q + F(t_0, x_0, u(t_0, x_0), p, A) \geq 0.$$

We can see easily that a viscosity solution is always a classical viscosity solution. To have equivalence between this two notions, we need the following proposition which can be found in Lions [57]:

Proposition 4.18. *Let u in $BUC_{loc}(\mathbb{R} \times H)$ be a classical viscosity subsolution (resp. supersolution) of* (4.16)*. Then u is a viscosity subsolution (resp. supersolution) of* (4.16) *if F satisfies the following condition: there exists an increasing sequence of finite dimensional subspaces H_n of H such that $\cup_n H_n$ is dense in H and*

$$\limsup_{\delta \to 0^+} \limsup_{N} \Big[F(t, x, p, A) - F\big(t, x, p, \frac{1}{2} A P_n + \frac{1}{2} P_n A + \delta P_n + \frac{C}{\delta} Q_n\big) \Big]^+ = 0. \quad (4.22)$$

$$\limsup_{\delta \to 0^+} \limsup_{N} \Big[F(t, x, p, A) - F\big(t, x, p, \frac{1}{2} A P_n + \frac{1}{2} P_n A - \delta P_n - \frac{C}{\delta} Q_n\big) \Big]^- = 0. \quad (4.23)$$

for all x, p in H, A in $L'(H)$, $C \geq 0$ where P_n and Q_n denote respectively the orthogonal projection on H_n and $H_n^\perp$.

Lions in [58] shows that the following condition is sufficient to guarantee (4.22) and (4.23):

There exists an increasing sequence of finite-dimensional subspaces H_n of H such that $\cup_n H_n$ is dense in H for which F satisfies for all $R < \infty$:

$$\sup\Big\{ |F(t,x,r,p,X+\pi Q_n) - F(t,x,p,X)| : \ \|X\|_{L'(H)} \leq R, \\ \pi \in \mathbb{R}, |\pi| \leq R, \ X = P_n X P_n \Big\} \underset{n}{\longrightarrow} 0. \tag{4.24}$$

for all $t \in \mathbb{R}^+$, $p, x \in H$, $r \in \mathbb{R}$. For this purpose, see [58] for the case of a bounded second order partial differential equation or [88] for the unbounded case.

Remark 4.19. If (4.22) and (4.23) hold, one can show that (4.17) or (4.18) holds for all $\phi \in C^1(\mathbb{R}^+ \times H)$ such that $D\phi$ is locally Lipschitz continuous and

$$\lim_{\substack{t\to 0\\ t\neq 0}} t^{-1}\big(D\phi(x+tk) - D\phi(x), h\big),$$

exits and is continuous on H for all h, k in H. In particular, this is the case for test functions which belong to $C^{1,2}\big(\mathbb{R}^+ \times H\big)$.

Remark 4.20. Swiech [88] considers fully nonlinear PDEs of the form:

$$\partial_t u + \langle Ax, Du\rangle + F(t,x,u,Du,D^2u) = 0, \quad (0,T)\times H,$$
$$u(T,x) = g(x), \quad x \in H.$$

where A is an unbounded operator. For that purpose, he considers tests functions ϕ which are mainly such that $\phi \in C^{1,2}([0,T]\times H)$, ϕ is weakly sequentially lower semicontinuous and $A^* D\phi$ is continuous, see definition 1.1 in [88] . Therefore in our case, that is to say $A = 0$, this set of test functions is a subset of $C^{1,2}([0,T]\times H) \subset \mathcal{X}(H)$. So if u is a viscosity solution of (4.16) then u is a viscosity solution in the sense of Swiech of (4.16). Since by (4.24) and proposition 4.18, the notion of classical viscosity solutions and viscosity solutions coincide, then if u is a classical viscosity solution of (4.16) then u is a viscosity solution in the sense of Swiech for the PDE (4.16).

In the sequel, to achieve the asymptotic expansion of the price of T-maturity European call, we will use the method developed in Fleming and Soner [36], [35] and Fleming and Souganidis [37] which requires the following stability result (Lions, Proposition $II.2$, [57]):

Theorem 4.21. *For all positive n, let $u_n \in BUC_{loc}(\mathbb{R}^+ \times H)$ be a viscosity subsolution of*

$$F_n(t,x,u_n,Du_n,D^2u_n) = 0, \text{ in } H, \ n \geq 1$$

for some F_n bounded, uniformly continuous on bounded sets of $\mathbb{R}^+ \times H \times \mathbb{R} \times H \times L'(H)$. We assume that there exist $u \in BUC_{loc}\big(\mathbb{R}^+ \times H\big)$, F bounded, uniformly continuous on bounded sets of $\mathbb{R}^+ \times H \times \mathbb{R} \times H \times L'(H)$ such that

$$\begin{aligned}
&\lim_n u_n(t,x) = u(t,x) \text{ for all } (t,x) \in \mathbb{R}^+ \times H,\\
&\limsup_n u_n(t_n,x_n) \leq u(t,x)\\
&\liminf_n F_n(t_n,x_n,r_n,p_n,X_n) \geq \liminf_n F_n(t,x,r,p,X_n)
\end{aligned}$$

for $t_n \underset{n}{\to} t$ in $\mathbb{R}^+$, $x_n \underset{n}{\to} x$ in H, $r_n \underset{n}{\to} r$ in $\mathbb{R}$, $p_n \underset{n}{\to} p$ in H and X_n is bounded in $L'(H)$.

Then, u is a viscosity subsolution of (4.16).

4.3 The stochastic volatility model

The purpose of this section is to propose an extension of the Bergomi model in terms of the mathematical formulation of the bond market as in Ekeland and Taflin in [27]. This is possible due to the analogy between the forward volatility curve and zero-coupon bond price curve. More precisely, we propose here a rigorous framework for the stochastic volatility model of Bergomi. This means that we have to find a “suitable” space for the definition of the forward variance curve. We will see in the sequel that since this space is not finite dimensional, this justifies the use of viscosity solutions in Hilbert spaces developped mainly by Lions in [57, 58].

4.3.1 The framework

We consider $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a complete filtered probability space with $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$ and a family $(\beta_i)_{i\in\mathbb{N}}$ of independent standard $\mathbb{F}$-Brownian motions. Let W be a cylindrical Wiener process on the space l^2 where

$$l^2 := \Big\{ v = (v_j)_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \ \|v\|_{l^2}^2 := \sum_{j\in\mathbb{N}} |v_j|^2 < \infty \Big\}$$

We denote by $(g_j)_{j\in\mathbb{N}}$ an orthonormal basis of l^2. For example, we can take $g_1 = (1,0,\ldots)$, $g_2 = (0,1,0,\ldots)$ etc ... By Proposition 4.10, for all $x \in l^2$, the process

$$\langle W_t, x\rangle = \sum_{i\in\mathbb{N}} \langle x, g_i\rangle \beta_i(t).$$

is a real valued Brownian motion. Let us define a correlation coefficient $\rho \in l^2$ such that $\|\rho\|_{l^2} = 1$ and W^0 a real valued standard $\mathbb{F}$-Brownian motion such that for all $i \geq 0$:

$$\mathbb{E}\big[W_t^0 \beta_i(t)\big] = \rho_i t.$$

Let us fix $\varepsilon \in \mathbb{R}$. For given $T > 0$ and $t \in [0, T]$, we denote by U^ε the log-price process of an underlying and by $\xi^{\varepsilon,T}$ the forward stochastic volatility of maturity T. They verify on $[0, T]$:

$$dU_t^\varepsilon = -\frac{1}{2}(\xi_t^{\varepsilon,t})^2 dt + \xi_t^{\varepsilon,t} dW_t^0, \quad U_0^\varepsilon = u \in \mathbb{R}, \tag{4.25}$$

$$d\xi_t^{\varepsilon,T} = \varepsilon^2 \hat{M}(t, T, \xi_t^{\varepsilon,\cdot})dt + \frac{\varepsilon}{2}\hat{\Lambda}(t, T, \xi_t^{\varepsilon,\cdot})dW_t \quad \xi_0^{\varepsilon,T} = \xi^T. \tag{4.26}$$

where $\xi_t^{\varepsilon,\cdot} : [t, \infty) \ni T \mapsto \xi_t^{\varepsilon,T}$ corresponds to the forward variance curve for all the maturities greater than t, $\hat{M}$ and $\hat{\Lambda}$ will be specified in the sequel.

4.3.2 The Bergomi model revisited

We recall that for $\gamma > \frac{1}{2}$:

$$H(\mathbb{R}^+) = \mathbb{R} \oplus H^\gamma(\mathbb{R}^+).$$

For all $(t, y) \in [0, T] \times H(\mathbb{R}^+)$ and $s \in [t, T]$, we denote by $Y_s^{\varepsilon,t,y}$ a $H(\mathbb{R}^+)$-valued process such that $Y_s^{\varepsilon,t,y}(x)$ is the instantaneous forward variance curve at time s for a time to maturity $x \geq 0$ and which is such that $Y_t^{\varepsilon,t,y} = y$. That is to say, by definition, $\xi^{\varepsilon,\cdot}$ and $Y^{\varepsilon,t,y}$ are related by

$$Y_s^{\varepsilon,t,y}(x) := \xi_s^{\varepsilon,s+x}, \tag{4.27}$$

for all $(t, y) \in [0, T] \times H(\mathbb{R}^+)$, $s \in [t, T]$ and $x \in \mathbb{R}^+$.

This means that we are working with forward variance curves which are continuous functions of time to maturity. The reason why we are working with $H(\mathbb{R}^+)$ instead of $H^\gamma(\mathbb{R}^+)$ is because it allows the variance curves to have a non zero limit at infinity.

We shall suppose that for all $x \geq 0$, $y \in H(\mathbb{R}^+)$ and $0 \leq s \leq t \leq T$, the following SDE is satisfied:

$$Y_s^{\varepsilon,t,y}(x) := y(s+x) + \varepsilon^2 \int_t^s \tilde{M}_u(Y_u^{\varepsilon,t,u})(s+x-u)du + \frac{\varepsilon}{2}\int_t^s \tilde{\Lambda}_u(Y_u^{\varepsilon,t,y})(s+x-u)dW_u \tag{4.28}$$

where $\tilde{M}$ and $\tilde{\Lambda}$ are functions from $[0,T] \times H(\mathbb{R}^+)$ to $H(\mathbb{R}^+)$ which are related to $\hat{M}$ and $\hat{\Lambda}$ by :

$$\tilde{M}_t(y)(x) := \hat{M}(t, t+x, y) \text{ and } \tilde{\Lambda}_t(y)(x) := \hat{\Lambda}(t, t+x, y),$$

for all $x \in \mathbb{R}^+$.

Let $\mathcal{L}^+$ be the continuous semi-group of left translation defined in (4.10). Then we can rewrite (4.28) as

$$Y_s^{\varepsilon,t,y} = \mathcal{L}_s^+ y + \varepsilon^2 \int_t^s \mathcal{L}_{s-u}^+ \tilde{M}_u(Y_u^{\varepsilon,t,y}) du + \frac{\varepsilon}{2} \int_t^s \mathcal{L}_{s-u}^+ \tilde{\Lambda}_u(Y_u^{\varepsilon,t,y}) dW_u. \quad (4.29)$$

We can now consider mild and strong solutions of the corresponding differentiated equations:

$$dU_s^{\varepsilon,t,u,y} = -\frac{1}{2}(Y_s^{\varepsilon,t,y})^2(0)ds + Y_s^{\varepsilon,t,y}(0)dW_s^0, \quad U_t^{\varepsilon,t,u,y} = u, \quad (4.30)$$

$$dY_s^{\varepsilon,t,y} = \big(\partial^+ Y_s^{\varepsilon,t,y} + \varepsilon^2 \tilde{M}_s(Y_s^{\varepsilon,t,y})\big)ds + \frac{\varepsilon}{2}\tilde{\Lambda}_s(Y_s^{\varepsilon,t,y})dW_s, \quad Y_t^{\varepsilon,t,y} = y. \quad (4.31)$$

where ∂^+ is the infinitesimal generator of the semi-group $\mathcal{L}^+$. This is the fundamental system of SDEs for the model to be solved. To be more precise, we define $\lambda := (\lambda_i)_{i \in \mathbb{N}} : [0,T] \times H(\mathbb{R}^+) \longrightarrow L_2^0$ such that:

$$\tilde{M}_t(y) := -\tfrac{1}{8} y \sum_{i=1}^{\infty} \lambda_t^i(y)^2 \text{ and } \tilde{\Lambda}_t(y^2) := y^2 \lambda_t(y),$$

which is correctly defined by Theorem 4.2. Moreover for all $t \in [0,T]$ and (y, y') in $H(\mathbb{R}^+)^2$, λ verifies:

$$\|\lambda_t(y) - \lambda_t(y')\|_{L_2^0}^2 \le C \|y - y'\|_{H(\mathbb{R}^+)}^2. \quad (4.32)$$

$$\|\lambda_t(y)\|_{L_2^0}^2 \le C. \quad (4.33)$$

$$\lambda \text{ has bounded support.} \quad (4.34)$$

Thus, we will denote by $supp(\lambda)$ the support of λ which is a subset of $[0,T] \times B^+$ where B^+ is a bounded closed subset of $H(\mathbb{R}^+)$. We will denote by B its bounded extension to $H(\mathbb{R})$, for instance, one can choose ι (see (4.4)) such that $B = \iota B^+$ is a bounded subset of $H(\mathbb{R})$.

Remark 4.22. Let us consider the pair of processes $(S^\varepsilon, \Sigma^{\varepsilon,t})$ defined by

$$(S^\varepsilon, \Sigma^{\varepsilon,t}) := \big(\exp(U^{\varepsilon,t,u,y}), (Y^{\varepsilon,t,y})^2\big).$$

One can see easily by Ito's formula that $(S^\varepsilon, \Sigma^{\varepsilon,t})$ solves for all $s \in [t,T]$:

$$dS_s^\varepsilon = S_s^\varepsilon \sqrt{\Sigma_s^{\varepsilon,t}(0)} dW_s^0, \quad S_t^\varepsilon = \exp(u) \in \mathbb{R}^+,$$
$$d\Sigma_s^{\varepsilon,t} = \varepsilon \tilde{\Lambda}_s(\Sigma_s^{\varepsilon,t}) dW_s \quad \Sigma_t^{\varepsilon,t} = y^2.$$

which is more usual. However, we prefer considering the log-price and the square-root of the volatility because it is more interesting for the sequel.

We shall use the approach of [27] which allows to get rid off the unbounded operator ∂^+ in (4.31). To achieve it, let us work in the Hilbert space

$$H(\mathbb{R}) := \mathbb{R} \oplus H^\gamma(\mathbb{R})$$

for $\gamma > \frac{1}{2}$ where $\{\mathcal{L}_t : \ t \in \mathbb{R}\}$, defined in (4.8), is a strong continuous contraction group. In order to reformulate the stochastic partial differential equation (4.31) as a stochastic ordinary differential equation, roughly speaking to remove the term $\partial^+ Y^{\varepsilon,t,y}$, we define a $H(\mathbb{R})$-valued process $\tilde{Z}^{\varepsilon,t,z}$ such that $\kappa \tilde{Z}^{\varepsilon,t,z} = Y^{\varepsilon,t,y}$. By definition of ι and κ (see (4.4) and (4.3)) and (4.11), if such a process exists, it verifies for all $0 \le t \le s \le T$:

$$\tilde{Z}_s^{\varepsilon,t,z} = \mathcal{L}_s z + \int_t^s \mathcal{L}_{s-u}\big(\varepsilon^2 \iota \tilde{M}_u(\kappa \tilde{Z}_u^{\varepsilon,t,z}) du + \frac{\varepsilon}{2} \iota \tilde{\Lambda}_u(\kappa \tilde{Z}_u^{\varepsilon,t,z}) dW_u\big), \quad z \in H(\mathbb{R}). \tag{4.35}$$

Moreover, since $\mathcal{L}$ defines a group in $H(\mathbb{R})$ (see subsection 4.2.3), we can define for all $0 \le t \le s \le T$ the process $Z_s^{\varepsilon,t,z} := \mathcal{L}_{-s} \tilde{Z}_s^{\varepsilon,t,z}$ which solves:

$$Z_s^{\varepsilon,t,z} = z + \varepsilon^2 \int_t^s M(u, Z_u^{\varepsilon,t,z}) du + \int_t^s \frac{\varepsilon}{2} \Lambda(u, Z_u^{\varepsilon,t,z}) dW_u. \tag{4.36}$$

where the functions M and Λ are from $[0,T] \times H(\mathbb{R})$ to $H(\mathbb{R})$ and are defined by:

$$M(t,z) := \mathcal{L}_{-t} \iota \tilde{M}(t, \kappa \mathcal{L}_t z) \text{ and } \Lambda(t,z) := \mathcal{L}_{-t} \iota \tilde{\Lambda}_t(\kappa \mathcal{L}_t z). \tag{4.37}$$

We denote by $\beta : [0,T] \times H(\mathbb{R}) \to H(\mathbb{R}^+)$ the function defined by:

$$\beta(t,z) := \kappa \mathcal{L}_t z. \tag{4.38}$$

Let H be a separable Hilbert space defined by

$$H := \mathbb{R} \oplus H(\mathbb{R}),$$

for the norm $\|.\|$ defined by:

$$\|(u,z)\|^2 = |u|^2 + \|z\|^2_{H(\mathbb{R})}.$$

For $U^{\varepsilon,t,u,z}$ and $Z^{\varepsilon,t,z}$ defined respectively in (4.30) and in (4.36), let us consider the H-valued process:

$$X^{\varepsilon,t,u,z} := \begin{pmatrix} U^{\varepsilon,t,u,z} \\ Z^{\varepsilon,t,z} \end{pmatrix}.$$

Therefore for all $x := (u, z) \in H$, we have by (4.30) and (4.36)

$$dX_s^{\varepsilon,t,x} = b_\varepsilon(s, X_s^{\varepsilon,t,x})ds + \sigma_\varepsilon(s, X_s^{\varepsilon,t,x})dW_s. \tag{4.39}$$

with $b_\varepsilon : [0, T] \times H \to H$ defined by

$$b_\varepsilon(t, x) = \Big(-\frac{1}{2}\beta\big(t, \psi(x)\big)^2(0), \varepsilon^2 M\big(t, \psi(x)\big) \Big).$$

and $\sigma_\varepsilon : [0, T] \times H \to L_2^0$ defined by

$$\sigma_\varepsilon(t, x) = \Big(\rho\beta\big(t, \psi(x)\big)(0), \frac{\varepsilon}{2}\Lambda\big(t, \psi(x)\big) \Big).$$

where : $\psi : H \to H(\mathbb{R})$ is a linear continuous function defined for all $x = (u, z)$, by $\psi(x) = z$. We define $\phi : H \to \mathbb{R}$ the linear continuous function defined by $\phi(x) = u$.

Therefore if we are able to prove that there exists a unique solution $X^{\varepsilon,t,x}$ to (4.39) then it would be possible to solve our fundamental system (4.30) and (4.31). To prove the existence and the uniqueness of such a process $X^{\varepsilon,t,x}$, we shall restrict ourselves to the x which belong to $\mathbb{R} \times B$. We recall that B^+ and B are bounded subsets of $H(\mathbb{R}^+)$ and $H(\mathbb{R})$ respectively (see (4.34) and the commentary below). We prove the following Lemma.

Lemma 4.23. *For all $\varepsilon \in \mathbb{R}$, b_ε and σ_ε are uniformly bounded on $[0, T] \times \mathbb{R} \times B$. Moreover, for all $(t, x, x') \in [0, T] \times \big(\mathbb{R} \times B\big)^2$, b_ε and σ_ε verify*

$$\|b_\varepsilon(t, x) - b_\varepsilon(t, x')\| \le C\|x - x'\|. \tag{4.40}$$

and

$$\|\sigma_\varepsilon(t, x) - \sigma_\varepsilon(t, x')\|_{L_2^0} \le C\|x - x'\|. \tag{4.41}$$

where $C > 0$ is a constant.

Proof. By definition ι, κ, and $\mathcal{L}_t$ are linear continuous maps, so they are Lipschitz continuous. Then we can find a constant $C > 0$ such that for all (z, z') in B^2 and $t \in [0, T]$:

$$\|\Lambda(t, z) - \Lambda(t, z')\|_{L_2^0} \le C\|z\lambda_t(\kappa\mathcal{L}_t z) - z'\lambda_t(\kappa\mathcal{L}_t z')\|_{L_2^0}.$$

By (4.32), (4.33) and (4.34), we obtain

$$\begin{aligned}\|\Lambda(t,z)-\Lambda(t,z')\|_{L_2^0} &\leq C\|z\|_{H(\mathbb{R})}\|\lambda_t(\kappa\mathcal{L}_t z)-\lambda_t(\kappa\mathcal{L}_t z')\|_{L_2^0}\\ &\quad + C\|\lambda_t(\kappa\mathcal{L}_t z)\|_{L_2^0}\|z-z'\|_{H(\mathbb{R})}\\ &\leq C\|z-z'\|_{H(\mathbb{R})}.\end{aligned}$$

So Λ is Lipschitz continuous in z. Similarly M and β are Lipschitz continuous in z.

The map $z \mapsto \beta^2(.,z)$ is Lipschitz continuous. Indeed, β is Lipschitz with bounded support because z belongs to B, the bounded support of λ(see (4.34)). Since ψ and M are Lipschitz continuous, we can find a constant $C > 0$ such that

$$\begin{aligned}\|b_\varepsilon(t,x)-b_\varepsilon(t,x')\|^2 &\leq \frac{C}{2}|x(0)-x'(0)|^2\\ &\quad + C\varepsilon^2\|x-x'\|^2_{H(\mathbb{R})}\end{aligned}$$

By the Sobolev embedding Theorem 4.1, we can therefore find a constant $C' > 0$ such that:

$$\begin{aligned}\|b_\varepsilon(t,x)-b_\varepsilon(t,x')\|^2 &\leq \frac{CC'}{2}\|x-x'\|^2_{H(\mathbb{R})} + C\varepsilon^2\|x-x'\|^2_{H(\mathbb{R})}\\ &\leq C\|x-x'\|^2.\end{aligned}$$

where C has changed from line to line.

In the same way ψ, β and Λ Lipschitz continuous lead to σ_ε Lipschitz continuous. Therefore b_ε and σ_ε verifies (4.40) and (4.41).

Let us show that b_ε is bounded on $[0,T]\times\mathbb{R}\times B$. First, we can notice $b_\varepsilon(.,0) = 0$. Therefore for all $x := (u,z) \in \mathbb{R}\times B$ and $t \in [0,T]$, by (4.40), b_ε is bounded on $[0,T]\times\mathbb{R}\times B$. Similarly one can show that σ_ε is bounded is bounded on $[0,T]\times\mathbb{R}\times B$.

□

By Lemma 4.23 and by Theorem 4.13, there is a unique strong solution $X^{\varepsilon,t,z}$ in $\mathbb{R}\times B$ to equation (4.39) and we can find a constant $C > 0$ such that:

$$\mathbb{E}\Big[\sup_{t\leq s\leq T}\|X_s^{\varepsilon,t,x}\|^2\Big] \leq C\big(1+\|x\|^2\big). \tag{4.42}$$

Remark 4.24. Moreover, by the boundedness of λ (see (4.33)), (4.34) and a comparison argument, we have that for all $t \in [0,T]$:

$$\forall (s,z) \in [t,T] \times B : \; Z_s^{\varepsilon,t,z} \in B$$

We have the following estimates

Lemma 4.25. *For all $h \geq 0$ and (x,x') in $\mathbb{R} \times B$, we have*

$$\mathbb{E}\Big[\sup_{t \leq s \leq t+h} \|X_s^{\varepsilon,t,x} - x\|^2\Big] \leq Ch^2\big(1 + \|x\|^2\big) \tag{4.43}$$

$$\mathbb{E}\Big[\sup_{t \leq s \leq t+h} \|X_s^{\varepsilon,t,x} - X_s^{\varepsilon,t,x'}\|^2\Big] \leq C\|x - x'\|^2 \tag{4.44}$$

$$\mathbb{E}\Big[\sup_{t \leq s \leq t+h} \|X_s^{\varepsilon,t,x} - X_s^{\varepsilon,t+h,x}\|^2\Big] \leq C\big(1 + \|x\|^2\big)\sqrt{h} \tag{4.45}$$

where $C > 0$ is a constant independent of t. Moreover,

$$\mathbb{E}\Big[\sup_{t \leq s \leq T} \|\psi(X_s^{\varepsilon,t,x} - x)\|^2\Big] \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{4.46}$$

where $\psi : H \to H(\mathbb{R})$ is defined for all $x = (u,z)$, by $\psi(x) = z$.

Proof. These estimates are classical and the proof is the same as those in the finite dimensional case. In the sequel, we will focus on the dependence in ε of the constant C and then we will only prove equation (4.46).

$$\mathbb{E}\Big[\sup_{t \leq s \leq T} \|Z_s^{\varepsilon,t,z} - z\|_H^2\Big] \leq 2\mathbb{E}\Big[\sup_{t \leq s \leq T} \|\int_s^t \varepsilon^2 M(u, Z_u^{\varepsilon,t,z}) du\|_H^2\Big] + 2\mathbb{E}\Big[\sup_{t \leq s \leq T} \|\int_s^t \frac{\varepsilon}{2} \Lambda(u, Z_u^{\varepsilon,t,z}) dB_u\|_H^2\Big].$$

Then by Jensen and Burkholder-Davis-Gundy inequality, we get:

$$\mathbb{E}\Big[\sup_{t \leq s \leq T} \|Z_s^{\varepsilon,t,z} - z\|_H^2\Big] \leq 2|T-t|\mathbb{E}\Big[\int_t^T \varepsilon^4 \|M(u, Z_u^{\varepsilon,t,z})\|_H^2 du\Big] + 2\mathbb{E}\Big[\int_t^T \frac{\varepsilon^2}{4} \|\Lambda(u, Z_u^{\varepsilon,t,z})\|_H^2 du\Big].$$

By the Lipschitz property of M and Λ (see the proof of Lemma 4.23), we can find a constant $C > 0$ such that

$$\mathbb{E}\Big[\sup_{t \leq s \leq T} \|Z_s^{\varepsilon,t,z} - z\|_H^2\Big] \leq C(2\varepsilon^4|T-t| + \frac{\varepsilon^2}{2})|T-t|\mathbb{E}\Big[\sup_{t \leq s \leq T} \|Z_s^{\varepsilon,t,z})\|_H^2\Big]$$

By (4.42), we therefore get:

$$\mathbb{E}\Big[\sup_{t \leq s \leq T} \|Z_s^{\varepsilon,t,z} - z\|_H^2\Big] \leq C(2\varepsilon^4|T-t| + \frac{\varepsilon^2}{2})\big(1 + \|z\|_H^2\big).$$

From this result, we deduce (4.46). □

4.4 The pricing problem

4.4.1 The corresponding PDE

Let us consider a payoff function $g : \mathbb{R} \longrightarrow \mathbb{R}^+$. We recall that B^+, the support of λ, is a bounded subset of $H(\mathbb{R}^+)$. We make the following assumption:

Assumption 4.26. *g is a bounded Lipschitz continuous function.*

We define the measurable function $p^\varepsilon(t,x) : [0,T] \times \mathbb{R} \times B \to \mathbb{R}$ such that

$$p^\varepsilon(t,x) := \mathbb{E}\big[g\big(\phi(X_T^{\varepsilon,t,x})\big)|X_t^{\varepsilon,t,x} = x\big].$$

where $x := (u,z) \in \mathbb{R} \times B \subset H$ and $\phi : H \to \mathbb{R}$ is the linear continuous function defined by $\phi(x) = u$. Let us consider $K > 0$. For all $t \in [0,T]$, $u \in \mathbb{R}$, and $y \in B^+$, the T-maturity European put option price is defined by:

$$P^\varepsilon(t,u,y) := \mathbb{E}\big[(K - U_T^{\varepsilon,t,u,y})_+|(U_t^{\varepsilon,t,u,y}, Y_t^{\varepsilon,t,y}) = (u,y)\big]. \tag{4.47}$$

where $U^{\varepsilon,t,u,y}$ and $Y^{\varepsilon,t,y}$ solve the system (4.30) and (4.31). We denote by $B_t(T)$ the price of a Zero-coupon bond of maturity T at time t and $C^\varepsilon(t,s,y)$ the price of T-maturity European call. We remind the call-put parity equation for a T-maturity European option:

$$P^\varepsilon(t,s,y) - C^\varepsilon(t,s,y) = B_t(T)K - s \tag{4.48}$$

Thus it makes no difference to be interested in pricing the European put or the European call.

Clearly by the tower property for conditional expectations, we have the following result

Lemma 4.27. *For all $(t,x) \in [0,T] \times \mathbb{R} \times B$ and for all stopping time $\theta \geq t$, we have*

$$p^\varepsilon(t,x) = \mathbb{E}\left[p^\varepsilon\big(\theta, X_\theta^{\varepsilon,t,x}\big)\big|X_t^{\varepsilon,t,x} = x\right].$$

Proof. Clearly, under Lemma 4.23, $X^{\varepsilon,t,x}$ is a strong continuous Markov process so it verifies the flow property:

$$X^{\varepsilon,X_\theta^{\varepsilon,t,x}} = X^{\varepsilon,t,x}, \text{ on } [\theta,\infty), \ \mathbb{P} - a.s. \tag{4.49}$$

By the tower property for conditional expectations and (4.49),

$$\begin{aligned} p^\varepsilon(t,x) &= \mathbb{E}\Big[\mathbb{E}\big[g\big(\phi(X_T^{\varepsilon,X_\theta^{\varepsilon,t,x}})\big)|X_\theta^{\varepsilon,t,x}\big]\Big|X_t^{\varepsilon,t,x}=x\Big] \\ &= \mathbb{E}\big[p^\varepsilon(\theta,X_\theta^{\varepsilon,t,x})|X_t^{\varepsilon,t,x}=x\big]. \end{aligned}$$

□

We define

$$F_\varepsilon(t,x,p,X) := \langle p, b_\varepsilon(t,x)\rangle + \frac{1}{2}Tr\big(X\sigma_\varepsilon(t,x)\big(\sigma_\varepsilon(t,x)\big)^*\big). \tag{4.50}$$

where $(t,x)\in[0,T]\times H$, $p\in H$ and X in $L'(H)$. F_ε verifies the following properties.

Proposition 4.28. *For all $\varepsilon\in\mathbb{R}$, F_ε is such that*

(1) F_ε is bounded on bounded sets of $[0,T]\times\big(\mathbb{R}\times B\big)^2\times L'(\mathbb{R}\times B)$.

(2) F_ε is uniformly continuous on bounded sets of $\in[0,T]\times\big(\mathbb{R}\times B\big)^2\times L'(\mathbb{R}\times B)$.

(3) F_ε is uniformly Lipschitz continuous in $x\in\mathbb{R}\times B$ on bounded sets of $[0,T]\times\big(\mathbb{R}\times B\big)\times L'(\mathbb{R}\times B)$.

Proof. First we show (3). For all $(x,x')\in(\mathbb{R}\times B)^2$, by Cauchy-Schwarz inequality, we have:

$$\begin{aligned} |F_\varepsilon(t,x,p,X)-F_\varepsilon(t,x',p,X)| &\le |\langle p, b_\varepsilon(t,x)-b_\varepsilon(t,x')\rangle| \\ &+\frac{1}{2}\big|Tr\big[X\big(\sigma_\varepsilon(t,x)-\sigma_\varepsilon(t,x')\big)\big(\sigma_\varepsilon(t,x)-\sigma_\varepsilon(t,x')\big)^*\big]\big| \\ &\le \|p\|\|b_\varepsilon(t,x)-b_\varepsilon(t,x')\| + \|X\|_{L'(H)} \\ &\times \|\sigma_\varepsilon(t,x)-\sigma_\varepsilon(t,x')\|_{L_2^0}. \end{aligned}$$

Therefore, by Lemma 4.23, we can find a constant $C>0$ such that:

$$|F_\varepsilon(t,x,p,X)-F_\varepsilon(t,x',p,X)| \le C\bigg(\|p\|+\|X\|_{L'(H)}\bigg)\|x-x'\|. \tag{4.51}$$

This gives us (3). Moreover, since $F_\varepsilon(t,x,0,0)=0$, we have:

$$|F_\varepsilon(t,x,p,X)| \le C\|x\|\big(\|p\|+\|X\|_{L'(H)}\big). \tag{4.52}$$

So, we have (1). F_ε is linear in p and X so by (4.52), it is uniformly continuous in p and X. Moreover by (3), we have that F_ε is uniformly continuous on bounded sets of $\mathbb{R}\times B$. The uniform continuity in t follows from the continuity of F^ε in t on the compact domain $[0,T]$. So, we have (2). □

Remark 4.29. F_ε verifies the degenerate ellipticity condition. That is to say for all $X\ge Y\in L'(H)$, for all $(p,x)\in H^2$, we have

$$F_\varepsilon(t,x,p,X)\le F_\varepsilon(t,x,p,Y).$$

4.4.2 Existence and uniqueness of the viscosity solution

To prove existence and uniqueness, we shall use an approach which uses test functions in $\mathcal{X}'$ and which is developed in [58]. To prove the uniqueness, we need the following lemma:

Lemma 4.30. *F_ε defined in (4.50) verifies:*

(1) F_ε verifies condition (4.24).

(2) $F_\varepsilon(t,x,\frac{x-x'}{\delta},X) - F_\varepsilon(t,x',\frac{x-x'}{\delta},-Y) \geq -\varpi\left(\frac{\|x-x'\|^2}{\delta} + \|x-x'\|\right)$

where $|\, t \,| \leq R$, $(x,x') \in \big(\mathbb{R}\times B\big)^2$, $\delta \in (0,\delta_0)$, X, Y *satisfies*

$$-\frac{2}{\delta}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \frac{1}{\delta}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{4.53}$$

and $\delta_0 > 0$, $R \in (0,\infty)$, ϖ *is a modulus of continuity.*

(3) There is a function $\mu \in C^2(\mathbb{R}\times B)$ with bounded derivatives such that $\mu \to \infty$ as $\|x\| \to \infty$ and

$$F_\varepsilon\big(t,x,p+\alpha D\mu(x), X+\alpha D^2\mu(x)\big) \geq F_\varepsilon(t,x,p,X) - \varpi(\alpha).$$

for all $\|X\|_{L'(\mathbb{R}\times B)} \leq R$, $\|p\| \leq R$, $t \in \mathbb{R}$, $x \in \mathbb{R}\times B$, $\alpha \in (0,\alpha_0)$, *for some* $\alpha_0 > 0$ *and* ϖ *is a modulus of continuity.*

Proof. First, since $H(\mathbb{R})$ is a separable Hilbert space, there exists an increasing sequence of finite-dimensional subspaces H_n of $H(\mathbb{R})$ such that $\cup_n H_n$ is dense in $H(\mathbb{R})$. For all $t \in [0,T]$, x,p in $\mathbb{R}\times B$ and P_n and Q_n the orthogonal projections onto H_n and $H_n^\perp$, we have by Cauchy-Schwarz's inequality:

$$\begin{aligned} |F_\varepsilon(t,x,p,X+\pi Q_n) - F_\varepsilon(t,x,p,X)| &= |\frac{\pi}{2}| Tr\big[Q_n\sigma_\varepsilon(t,x)\big(\sigma_\varepsilon(t,x)\big)^*\big]| \\ &\leq |\frac{\pi}{2}| \|Q_n\sigma_\varepsilon(t,x)\|_{L_2^0} \|\sigma_\varepsilon(t,x)\|_{L_2^0}. \end{aligned}$$

Since by Lemma 4.23, σ_ε is bounded on $[0,T]\times\mathbb{R}\times B$, we can find a constant $C > 0$ such that:

$$\|\sigma_\varepsilon(t,x)\|_{L_2^0} < C$$

So this leads to

$$|F_\varepsilon(t,x,p,X+\pi Q_n) - F_\varepsilon(t,x,p,X)| \leq C|\frac{\pi}{2}| \|Q_n.\sigma_\varepsilon(t,x)\|_{L_2^0}.$$

By Lebesgue's Lemma, we have for all $x \in \mathbb{R}\times B$ and $x_n \in H_n$, such that $\|x_n - x\| \underset{n}{\to} 0$:

$$\begin{aligned} \|Q_n.\sigma_\varepsilon(t,x)\|_{L_2^0} &= \|\sigma_\varepsilon(t,x) - P_n.\sigma_\varepsilon(t,x))\|_{L_2^0} \\ &\leq (1 + \|P_n\|_{L'(H)})\|\sigma_\varepsilon(t,x) - \sigma_\varepsilon(t,x_n)\|_{L_2^0}. \end{aligned}$$

So for all $R < \infty$, by Lemma 4.23, we can find a Lipschitz constant $C > 0$ such that

$$\sup\Big\{|F_\varepsilon(t,x,p,X+\pi Q_n) - F_\varepsilon(t,x,p,X)| :\ \|X\|_{L'(H)} \leq R,$$
$$\pi \in \mathbb{R}, |\pi| \leq R,\ X = P_n X P_n\Big\}$$
$$\leq \frac{R}{2} C\big(1 + \|P_n\|_{L'(H)}\big)\|x - x_n\|$$
$$\underset{n}{\longrightarrow} 0.$$

Secondly, we have

$$F_\varepsilon\big(t,x,\frac{x-x'}{\delta},X\big) - F_\varepsilon\big(t,x',\frac{x-x'}{\delta},-Y\big)$$
$$= -Tr\big[X\big(\sigma_\varepsilon(t,x)\big)\big(\sigma_\varepsilon(t,x)\big)^* + Y\big(\sigma_\varepsilon(t,x')\big)\big(\sigma_\varepsilon(t,x')\rangle\big)^*\big] - \langle b_\varepsilon(t,x) - b_\varepsilon(t,x'), \frac{x-x'}{\delta}\rangle$$
$$= -Tr\Bigg(\begin{pmatrix} \sigma_\varepsilon(t,x)\sigma_\varepsilon(t,x)^* & \sigma_\varepsilon(t,x)\sigma_\varepsilon(t,x')^* \\ \sigma_\varepsilon(t,x')\sigma_\varepsilon(t,x)^* & \sigma_\varepsilon(t,x')\sigma_\varepsilon(t,x')^* \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\Bigg) - \langle b_\varepsilon(t,x) - b_\varepsilon(t,x'), \frac{x-x'}{\delta}\rangle$$

By (4.53) and Lemma 4.23, we have

$$F_\varepsilon\big(t,x,\frac{x-x'}{\delta},X\big) - F_\varepsilon\big(t,x',\frac{x-x'}{\delta},-Y\big)$$
$$\geq -\frac{1}{\delta}Tr\Bigg(\begin{pmatrix} \sigma_\varepsilon(t,x)\sigma_\varepsilon(t,x)^* & \sigma_\varepsilon(t,x)\sigma_\varepsilon(t,x')^* \\ \sigma_\varepsilon(t,x')\sigma_\varepsilon(t,x)^* & \sigma_\varepsilon(t,x')\sigma_\varepsilon(t,x')^* \end{pmatrix}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\Bigg) - \frac{C}{\delta}\|x-x'\|^2$$
$$\geq -\frac{1}{\delta}\|\sigma_\varepsilon(t,x) - \sigma_\varepsilon(t,x')\|_{L_2^0} - \frac{C}{\delta}\|x-x'\|^2$$
$$\geq -\frac{1}{\delta}\big[\|x-x'\| + \frac{C}{\delta}\|x-x'\|^2\big].$$

where C is the Lipschitz constant of σ_ε and I the identity matrix.

Thirdly, since by Lemma 4.23, b_ε and σ_ε are bounded then for any function $\mu \in C^2(\mathbb{R} \times B)$ with bounded derivatives such that $\mu \to \infty$ as $\|x\| \to \infty$, there is a constant $c' > 0$ such that:

$$\langle D\mu(x), b_\varepsilon(t,x)\rangle + \frac{1}{2}Tr\big(D^2\mu(x).\sigma_\varepsilon(t,x)\big(\sigma_\varepsilon(t,x)\big)^*\big) \geq -c'$$

□

Proposition 4.31. *For all $\varepsilon \in \mathbb{R}$, p^ε is the unique viscosity solution in $BUC_x\big([0,T] \times \mathbb{R} \times B\big)$ of*

$$-\partial_t p^\varepsilon(t,x) - F_\varepsilon\big(t,x,Dp^\varepsilon(t,x),D^2p^\varepsilon(t,x)\big) = 0 \qquad (4.54)$$
$$p^\varepsilon(T,x) = g\big(\phi(x)\big) \qquad (4.55)$$

where $\phi : H \to \mathbb{R}$ is defined for all $x = (u, z) \in H$ by $\phi(x) = u$.

Proof. First using the Lipschitz continuity of g (see Assumption 4.26), we can find a constant $C > 0$ such that for all (x, x') in $\mathbb{R} \times B$

$$|p^\varepsilon(t, x) - p^\varepsilon(t, x')|^2 \leq C\mathbb{E}\big[\|X_T^{\varepsilon,t,x} - X_T^{\varepsilon,t,x'}\|^2\big] \tag{4.56}$$

Thanks to (4.44),

$$|p^\varepsilon(t, x) - p^\varepsilon(t, x')| \leq C\|x - x'\| \tag{4.57}$$

To study the regularity in t, we are going to use (4.45). Indeed,

$$\begin{aligned} |p^\varepsilon(t, x) - p^\varepsilon(t', x)| &\leq C\mathbb{E}\big[\|X_T^{\varepsilon,t,x} - X_T^{\varepsilon,t',x}\|^2\big] \\ &\leq C\sqrt{t - t'}\big(1 + \|x\|^2\big) \end{aligned}$$

In addition by Assumption 4.26, g is uniformly bounded. So, we can conclude that $p^\varepsilon \in BUC_x\big([0, T] \times \mathbb{R} \times B\big)$.

The function g belongs to $BUC(\mathbb{R}^+)$ by Assumption 4.26. By Lemma 4.30, Proposition 4.28 since $F_\varepsilon(0, 0, .) = 0$, then by Theorem 3.2 in [88] and Theorem 3.3(ii) [58], p^ε is the unique solution to (4.54) and (4.55) which belongs to $BUC_x\big([0, T] \times \mathbb{R} \times B\big)$.

□

4.4.3 Zero order term

Let us introduce the following problem for all $(t, u, z) \in [0, T] \times H$:

$$p_0(t, u, z) = \mathbb{E}\big[g(U_T^{0,t,u,z})|U_t^{0,t,u,z} = u\big]. \tag{4.58}$$

where for all $t \leq s \leq T$, $U_s^{0,t,u,z} = \phi(X_s^{0,t,u,z})$. We recall that $U_s^{0,t,u,z}$ is the solution to the following stochastic equation:

$$dU_s^{0,t,u,z} = -\frac{1}{2}\beta(s, z)^2(0)ds + \beta(s, z)(0)dW_s^0, \quad U_t^{0,t,u,z} = u. \tag{4.59}$$

In this equation, z is just a parameter. The following lemma gives us the zero order therm of the asymptotic expansion.

Lemma 4.32. *For all (t, u, z) in $[0, T] \times \mathbb{R} \times B$,*

$$\lim_{\varepsilon \to 0} p^\varepsilon(t, u, z) = p_0(t, u, z).$$

Proof. By Assumption 4.26, we can find a constant $C > 0$ such that for all (t, u, z) in $[0, T] \times \mathbb{R} \times B$:

$$|p^\varepsilon(t, u, z) - p_0(t, u, z)|^2 \leq C\mathbb{E}\Big[|U_T^{\varepsilon,t,u,z} - U_T^{0,t,u,z}|^2\Big].$$

So by Jensen's inequality, we get :

$$\begin{aligned}|p^\varepsilon(t, u, z) - p_0(t, u, z)|^2 \leq 2C|T - t|\mathbb{E}\Big[\int_t^T |\beta(s, Z_s^{\varepsilon,t,u,z})(0)^2 - \beta(s, z)(0)^2|^2 ds\Big] \\ + 2C\mathbb{E}\Big[\int_t^T |\beta(s, Z_s^{\varepsilon,t,u,z})(0) - \beta(s, z)(0)|^2 ds\Big].\end{aligned}$$

We have shown in the proof of Lemma 4.23 that β is Lipschitz continuous with respect to z and has a bounded support, so β^2 is a Lipschitz continuous function. This leads to:

$$\begin{aligned}|p^\varepsilon(t, u, z) - p_0(t, u, z)|^2 &\leq 2C(|T - t| + C)\mathbb{E}\Big[\int_t^T |(Z_s^{\varepsilon,t,u,z})(0) - z(0)|^2 ds\Big] \\ &\leq 2C(|T - t| + C)|T - t|\mathbb{E}\Big[\sup_{t\leq s\leq T} |(Z_s^{\varepsilon,t,u,z})(0) - z(0)|^2 ds\Big]\end{aligned}$$

By Theorem 4.1, we can find a constant $C_b > 0$ such that:

$$|p^\varepsilon(t, u, z) - p_0(t, x)|^2 \leq 2C_bC(|T - t| + C)\mathbb{E}\Big[\sup_{t\leq s\leq T} \|X_s^{\varepsilon,t,u,z} - x\|^2 ds\Big].$$

where $x := (u, z)$. By Lemma 4.25, we can conclude. □

Remark 4.33. By the above lemma and the call-put parity equation (4.48), for all $(t, u, z) \in [0, T] \times \mathbb{R} \times B$,

$$\begin{aligned}C^\varepsilon(t, u, z) &= B_t(T)K + u - P_0(t, u, z) + o(1) \\ P^\varepsilon(t, u, z) &= P_0(t, u, z) + o(1)\end{aligned}$$

4.5 Asymptotic expansion for smooth payoffs

In this section, we will always assume that the payoff g verifies Assumption 4.26. Moreover, we suppose:

Assumption 4.34. *g belongs to $C_b^\infty(\mathbb{R})$*

The following assumption on the coefficient λ will be usefull. We recall that under (4.34), B^+, the support of λ_t is a bounded subset of $H(\mathbb{R}^+)$ and B is its extension for elements of $H(\mathbb{R})$.

Assumption 4.35.

$$\lambda \in C_b^{0,\infty}\big([0,T] \times B^+, H(\mathbb{R}^+)\big).$$

For all $(i,j) \in \mathbb{N}^2$ and function $f : [0,T] \times \mathbb{R} \times B \to \mathbb{R}$, we denote by $D^{i+j}_{u^i z^j} f(t,u,z)$ the $i+j$-derivative of f of order i in u and of order j in z. Moreover, for all $s \in [0,T]$, $(t,z) \in [0,s] \times B$, we define:

$$\delta(t,s,z) := \int_t^s \beta^2(u,z)(0)du. \tag{4.60}$$

We already know that the composition of two functions in $C^\infty(\mathbb{R})$ is in $C^\infty(\mathbb{R})$. For our purpose, we will need to show that in addition the n^{th} derivative of the composition of two functions is n-times differentiable with bounded derivatives up to order n. We will use the following well-known result:

Lemma 4.36. *For any real valued functions f and g which belong to $C^n(\mathbb{R})$, the n^{th} derivative of the composition of f and g is equal to:*

$$(f \circ g)^{(n)}(x) = \sum_{r=1}^{n} \sum_{\substack{i_1 \geq \ldots \geq i_r \\ i_1 + \ldots + i_r = n}} C_{i_1,\ldots,i_r} f^{(r)} \circ g(x). \prod_{k=1}^{r} g^{(i_k)}(x). \tag{4.61}$$

where $C_{i_1,\ldots,i_r}$ are constants. In particular, if $(f,g) \in C_b^n(\mathbb{R})^2$, then $(f \circ g)^{(n)} \in C_b^n(\mathbb{R})$.

Proof. We could not find references for this result therefore we indicate quickly its proof. We can prove (4.61) easily by an induction argument and the chain rule. Moreover, if $(f,g) \in C_b^n(\mathbb{R})^2$, that is to say the n^{th} derivatives of f and g are continuous and bounded then $(f \circ g)^{(n)}$ is continuous and bounded as a sum and product of continuous and bounded functions. $\square$

Remark 4.37. This lemma can be extended to $H(\mathbb{R})$-valued functions.

Lemma 4.38. *$\delta \in C_b^{1,\infty}([0,T] \times B)$, $\beta(t,.) \in C_b^\infty(B, H(\mathbb{R}^+))$, $M(t,.)$ and $\Lambda(t,.) \in C_b^\infty(B, H(\mathbb{R}))$ for all $t \in [0,T]$.*

Proof. For all $t \in [0,T]$, since β is linear with respect to z, then for all $z_0 \in B$, we have:

$$D_z\beta(t,z).z_0 = \beta(t,z_0)$$

Since by Lemma 4.23, $\beta(t,z_0)$ is bounded then we can conclude that for all $t \in [0,T]$:

$$z \mapsto \beta(t,z) \in C_b^\infty(B, H(\mathbb{R}^+)). \tag{4.62}$$

Therefore for all $(t,z) \in [0,T] \times B$, $D^n_{z^n}\beta(t,z)$ is continuous and bounded in $H(\mathbb{R}^+)$. By the Sobolev embedding theorem 4.1, we can find a constant $c > 0$ such that we have:

$$|D^n_{z^n}\beta(t,z)^2(0)| < c.$$

Then by differentiation under the integral sign,

$$z \mapsto D^n_{z^n}\delta(t,s,z) \text{ is continuous and bounded.} \tag{4.63}$$

Moreover since $\partial_t\delta(t,z) = -\beta^2(t,z)(0)$ which is bounded by Sobolev embedding theorem, by (4.63), we can conclude that δ belongs to $C^{1,\infty}_b([0,T] \times B)$.

Using the definition of Λ and β (see (4.37) and (4.38)), we obtain:

$$\Lambda(t,z) = \mathcal{L}_t \iota \beta(t,z) \lambda_t\big(\beta(t,z)\big)$$

Since by Assumption 4.35, for all $t \in [0,T]$, $\lambda(t,.) \in C^\infty_b\big(B^+, H(\mathbb{R}^+)\big)$ and $\beta(t,.) \in C^\infty_b(B, H(\mathbb{R}^+))$, then by Lemma 4.36 we get that $\Lambda(t,.) \in C^\infty_b(B, H(\mathbb{R}))$. Using the same argument, we can prove that $M(t,.) \in C^\infty_b(B, H(\mathbb{R}))$ for all $t \in [0,T]$.

□

Let us define for all $n \geq 1$ and $(t,u,z) \in [0,T] \times \mathbb{R} \times B$:

$$p_n(t,u,z) = \mathbb{E}_{t,u,z}\left[\int_t^T h_n\big(s, U^{0,t,u,z}_s, z\big)ds\right] \tag{4.64}$$

where p_0 is defined in (4.58), $U^{0,t,u,z}$ in (4.59) and for all $n \geq 2$:,

$$\begin{aligned} h_n(t,u,z) := &\langle M(t,z), D_z p_{n-2}(t,u,z)\rangle \\ &+ \langle D^2_{zu} p_{n-1}(t,u,z).\rho\Lambda\big(t,z\big)\rangle\beta\big(t,z\big)(0) \\ &+ \frac{1}{8} Tr\big[D^2_{zz} p_{n-2}(t,u,z)\rho\Lambda\big(t,z\big)\big(\rho\Lambda\big(t,z\big)\big)^*\big]. \end{aligned} \tag{4.65}$$

with

$$h_1(t,u,z) := \frac{1}{2}\langle D^2_{zu} p_0(t,u,z), \rho\Lambda\big(t,z\big)\rangle\beta\big(t,z\big)(0). \tag{4.66}$$

We will be interested in the sequel in showing for any $n \geq 1$ that p_n is a solution of the following partial differential equation:

$$\begin{cases} -\partial_t f(t,u,z) + \frac{1}{2}\beta(t,z)^2(0)\partial_u f(t,u,z) - \frac{1}{2}\beta(t,z)^2(0)\partial_{uu} f(t,u,z) - h_n(t,u,z) = 0 \\ f(T,u,z) = 0 \end{cases} \tag{4.67}$$

We notice that the partial derivatives concern only the finite dimensional component, u, of the solution.

Proposition 4.39. *Let Assumptions* 4.34 *and* 4.35 *hold. For all* $n \geq 0$ *:*

$$p_n \in C_b^{1,\infty,\infty}\big([0,T] \times \mathbb{R} \times B\big).$$

and there is a constant $C_n > 0$ *such that:*

$$|h_n(t,u,z)| \leq C_n, \ \forall (t,u,z) \in [0,T] \times \mathbb{R} \times B$$

Proof. We recall the definition of $p_0(t,u,z) = \mathbb{E}_t\big[g(U_T^{0,t,u,z})|U_t^{0,t,u,z} = u\big]$. Therefore by (4.59):

$$U_T^{0,t,u,z} \sim \mathcal{N}\bigg(u - \frac{1}{2}\delta(t,T,z), \delta(t,T,z)\bigg). \tag{4.68}$$

where $\mathcal{N}(a,b)$ denotes the normal law of expectation a and variance b and δ is defined in (4.60). By definition,

$$p_0(t,u,z) = \frac{1}{\sqrt{2\pi\delta(t,T,z)}} \int_{\mathbb{R}} g(v) \exp\bigg\{ - \frac{1}{2}\bigg[\frac{v - u + \frac{1}{2}\delta(t,T,z)}{\sqrt{\delta(t,T,z)}}\bigg]^2\bigg\} dv.$$

By an easy change of variable, the above equation can be rewritten as:

$$p_0(t,u,z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g\bigg(u - \frac{1}{2}\delta(t,T,z) + \delta(t,T,z)^{1/2}x\bigg)\exp\{-\frac{x^2}{2}\}dx.$$

Step 1: Our aim is to prove that for all $(i,j) \in \mathbb{N}^2$, $D^{i+j}_{u^i z^j} p_0(t,u,z)$ is continuous and bounded and that p_0 is continuously differentiable with respect to t with bounded partial derivative.

Let $l : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be defined by:

$$l(u,a,b) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g\big(u - \frac{b}{2} + \sqrt{a}x\big)\exp\{-\frac{x^2}{2}\}dx.$$

Since all the derivatives of g are continuous and bounded (see Assumption 4.34), by differentiation with respect to b under the integral sign, we can see that :

$$b \mapsto l(u,a,b) \in C_b^\infty(\mathbb{R}^+), \ \forall (u,a) \in \mathbb{R} \times \mathbb{R}^+. \tag{4.69}$$

In order to show that l is infinitely differentiable with respect to a, we need to take care of the differentiation of the term $\sqrt{a}$ which blows up in 0. To overcome this difficulty, an easy and direct iterative calculus based on successive integrations by parts shows that we have in fact:

$$D_a^n l(u,a,b) = \frac{1}{2^n\sqrt{2\pi}} \int_{\mathbb{R}} g^{(2n)}\big(u - \frac{b}{2} + \sqrt{a}x\big)\exp\{-\frac{x^2}{2}\}dx. \tag{4.70}$$

So for given $(u,b) \in \mathbb{R}\times\mathbb{R}^+$, the mapping $a \mapsto l(u,a,b)$ is infinitely differentiable. Since the derivatives of g are bounded, we have that in addition:

$$a \mapsto l(u,a,b) \in C_b^\infty(\mathbb{R}^+),\ \forall (u,b) \in \mathbb{R}\times\mathbb{R}^+. \tag{4.71}$$

Since g is infinitely differentiable with bounded derivatives, we can differentiate $D^n_{a^n} l(u,a,b)$ and $D^n_{b^n} l(u,a,b)$ at any order with respect to u. This leads to for all $(i,j,k) \in \mathbb{N}^3$:

$$D^{i+j+k}_{u^i a^j b^k} l \text{ is continuous and bounded on } \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+. \tag{4.72}$$

Combining the definition of p_0 and l, one can show that for all $(t,u,z) \in [0,T]\times\mathbb{R}\times B$:

$$p_0(t,u,z) = l(u, \delta(t,T,z), \delta(t,T,z))$$

Since by lemma 4.38, we know that $z \mapsto \delta(t,T,z) \in C_b^\infty(B,\mathbb{R}^+)$. Therefore by (4.69) and (4.71) and lemma 4.36, we have that:

$$z \mapsto p_0(t,u,z) \in C_b^\infty(B),\ \forall (t,u) \in [0,T]\times\mathbb{R}. \tag{4.73}$$

And by (4.72),

$$D^{i+j}_{u^i z^j} p_0 \text{ is continuous and bounded on} [0,T]\times\mathbb{R}\times B. \tag{4.74}$$

In particular since by lemma 4.23, β, M and Λ are bounded then by (4.74) and Sobolev embedding theorem 4.1, there exists a constant C_1 such that :

$$|h_n(t,u,z)| < C_1,\ \forall (t,u,z) \in [0,T]\times\mathbb{R}\times B.$$

In what concerns the differential in t, we know by lemma 4.38 that:

$$t \mapsto \delta(t,T,z) \in C^1([0,T]),\ \forall z \in B$$

Therefore by lemma 4.36, (4.69) and (4.71), we get that:

$$t \mapsto p_0(t,u,z) \in C_b^1([0,T]),\ \forall (u,z) \in \mathbb{R}\times B.$$

Step 2: We use an induction argument. For any $k \geq 1$, let (A_k) be defined by:

(A_k) : For any $(i,j) \in \mathbb{N}^2$ and $m \leq k$, $D^{i+j}_{u^i z^j} p_m(t,u,z)$ is continuous and bounded and p_m is continuously differentiable with respect to t with bounded partial derivative.

Our aim is to prove (A_{k+1}). Let us show that for any $(i,j) \in \mathbb{N}^2$, $D^{i+j}_{u^i z^j} p_{k+1}(t,u,z)$ is continuous and bounded. First we can see that by the law of U given in (4.68) and the definition of h_{k+1} (see (4.65)), we have:

$$p_{k+1}(t,u,z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_t^T h_{k+1}\Big(s, f(t,s,u,z,x), z\Big) \exp\{-\frac{x^2}{2}\} ds dx.$$

where for all $t \in [0,T]$, $s \in [t,T]$, $(u,x) \in (\mathbb{R})^2$ and $z \in B$, f is defined by

$$f(t,s,u,z,x) := u - \frac{1}{2}\delta(t,s,z) + \delta(t,s,z)^{1/2} x.$$

By assumption (A_k), we know that $D^{i+j}_{u^i z^j} p_k(t,u,z)$ and $D^{i+j}_{u^i z^j} p_{k-1}(t,u,z)$ are continuous and bounded for all $(i,j) \in \mathbb{N}^2$. Lemma 4.38 gives the regularity with respect to z of β, M and Λ. Therefore by definition of h_{k+1} (see (4.65)), we have that :

$$D^{i+j}_{u^i z^j} h_{k+1} \text{ is continuous and bounded on } [0,T] \times \mathbb{R} \times B. \tag{4.75}$$

Now we can use the same trick as in the first step and introduce $l' : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \times B \to \mathbb{R}$ be defined by:

$$l'(u,a,b,z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_t^T h_{k+1}\big(s, u - \frac{b}{2} + \sqrt{a}x, z\big) \exp\{-\frac{x^2}{2}\} dx.$$

Therefore doing the same argument as in Step 1 (see (4.70)), using the differentiability of h_{k+1} in its second and third variables and the boundedness of these partials derivatives(see (4.75)), we can conclude that for all $(i,j,k,y) \in \mathbb{N}^4$:

$$D^{i+j+k}_{u^i a^j b^k z^y} l(u,a,b,z) \text{ is continuous and bounded on } \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+. \tag{4.76}$$

Since by Lemma 4.38, we know that $z \mapsto \delta(t,s,z) \in C^\infty_b(B, \mathbb{R}^+)$. Therefore by (4.76) and lemma 4.61, we can differentiate under the integral sign:

$$D^{i+j}_{u^i z^j} p_{k+1}(t,u,z) \text{ is continuous and bounded on } [0,T] \times \mathbb{R} \times B. \tag{4.77}$$

In particular since by lemma 4.23, β, M and Λ are bounded then by (4.77) and the Sobolev embedding theorem 4.1, there exists a constant C_{k+1} such that :

$$|h_{k+1}(t,u,z)| < C_{k+1}, \ \forall (t,u,z) \in [0,T] \times \mathbb{R} \times B.$$

Concerning the t-derivative, we know by lemma 4.38 that for all $s \in [0,T]$:

$$t \mapsto \delta(t,s,z) \in C^1([0,s]), \ \forall z \in B \tag{4.78}$$

We recall the following result for any function f from $[0,T]^2$ to $\mathbb{R}$ such that $f \in C_b^{0,1}([0,T]^2)$:

$$\frac{d}{dt}\int_t^T f(s,t)ds = \int_t^T D_2 f(s,t)ds - f(t,t)$$

where D_2 is the first order derivative with respect to the second component. We do not know the regularity of h_{k+1} in its first component. This above result shows that, to differentiate p_{k+1} in t, we do not have to differentiate in the first component, s, of h_{k+1}. Combining this result together with (4.78) and (4.76), we get by lemma 4.36 that:

$$t \mapsto p_{k+1}(t,u,z) \in C_b^1([0,T]),\ \forall (u,z) \in \mathbb{R}\times B.$$

So, we have proved (A_{k+1}) and the proposition follows by induction. □

Lemma 4.40. *Let Assumptions* 4.34 *and* 4.35 *hold, then* p_n *is a* $C_b^{1,2,2}\big([0,T]\times \mathbb{R}\times B\big)$ *solution of* (4.67).

Proof. Let $(t_m)_m$ be defined for all $m \geq 0$ by $t_m = t + \frac{1}{m}$. Using (4.59) and proposition 4.39, we get by Ito's formula:

$$\begin{aligned} p_n\big(t_m, U_{t_m}^{0,t,u,z}, z\big) &= p_n(t,u,z) + \int_t^{t_m}\Big[\partial_t p_n(s, U_s^{0,t,u,z}, z) \\ &+ \frac{1}{2}\beta(s,z)^2(0)\partial_u p_n(s,U_s^{0,t,u,z},z) - \frac{1}{2}\beta(s,z)^2(0)\partial_{uu}p_n(s,U_s^{0,t,u,z},z)\Big]ds \\ &+ \int_t^{t_m}\Big[\frac{1}{2}\beta(s,z)^2(0)\partial_u p_n(s,U_s^{0,t,u,z},z)\Big]dB_s. \end{aligned} \tag{4.79}$$

We already know by Lemma 4.23 and the Sobolev embedding theorem 4.1 that $\beta(t,z)^2(0)$ is bounded. By proposition 4.39, the first order partial derivative with respect to u, of p_n, is bounded. Therefore the local martingale

$$\int_t^{t_m}\frac{1}{2}\beta(s,z)^2(0)\partial_u p_n(s,U_s^{0,t,u,z},z)dB_s$$

is a martingale. Moreover, by the tower property for conditional expectation:

$$p_n\big(t,u,z\big) = \mathbb{E}_{t,u,z}\Big[\int_t^{t_m} h_n\big(s,U_s^{0,t,u,z},z\big)ds + p_n\big(t_m, U_{t_m}^{0,t,u,z}, z\big)\Big] \tag{4.80}$$

Plugging (4.80) in (4.79) and taking the conditional expectation, one has:

$$\begin{aligned} &\mathbb{E}_{t,u,z}\Big[\int_t^{t_m} h_n\big(s,U_s^{0,t,u,z},z\big) - \partial_t p_n(s,U_s^{0,t,u,z},z) - \frac{1}{2}\beta(s,z)^2(0)\partial_u p_n(s,U_s^{0,t,u,z},z) \\ &+ \frac{1}{2}\beta(s,z)^2(0)\partial_{uu}p_n(s,U_s^{0,t,u,z},z)ds\Big] = 0 \end{aligned}$$

By proposition 4.39, h_n, the derivatives of p_n and $\beta(s,z)^2(0)$ are bounded, then by the mean value theorem and dominated convergence theorem, we have that p_n is a solution of the below equation on $[0,T) \times \mathbb{R} \times B$:

$$-\partial_t f(t,u,z)+\frac{1}{2}\beta(t,z)^2(0)\partial_u f(t,u,z)-\frac{1}{2}\beta(t,z)^2(0)\partial_{uu} f(t,u,z)-h_n(t,u,z)=0.$$

Since h_n is bounded and continuous, then for all $(t,u,z) \in [0,T] \times \mathbb{R} \times B$, we have:

$$|p_n(t,u,z)| \leq C_n|T-t|.$$

So if we take a sequence $(t_m,u_m)_m$ of $[0,T] \times \mathbb{R}$ such that $(t_m,u_m) \underset{m\to\infty}{\to} (T,u)$, then:

$$\lim_m p_n(t_m,u_m,z)=0.$$

So p_n is a regular solution of (4.67) □

For all $n \geq 2$, we define $h_\varepsilon^n(t,u,z) : [0,T] \times \mathbb{R} \times B \to \mathbb{R}$ such that

$$\begin{aligned} h_\varepsilon^n(t,u,z) := &\langle M(t,z), D_z p_{n-2}(t,u,z)\rangle \\ &+ \frac{\varepsilon}{8} Tr\big[D_{zz}^2 p_{n-1}(t,u,z)\big(\rho\Lambda(t,z)\big)\big(\rho\Lambda(t,z)\big)^*\big] \\ &+ \frac{1}{2}\langle D_{uz}^2 p_{n-1}(t,u,z), \rho\Lambda(t,z)\rangle\beta(t,z)(0) \\ &+ \frac{1}{8} Tr\big[D_{zz}^2 p_{n-2}(t,u,z)\big(\rho\Lambda(t,z)\big)\big(\rho\Lambda(t,z)\big)^*\big]. \end{aligned}$$

with

$$\begin{aligned} h_\varepsilon^1(t,u,z) := &\frac{\varepsilon}{8} Tr\big[D_{zz}^2 p_0(t,u,z)\big(\rho\Lambda(t,z)\big)\big(\rho\Lambda(t,z)\big)^*\big] \\ &+ \frac{1}{2}\langle D_{uz}^2 p_0(t,u,z), \rho\Lambda(t,z)\rangle\beta(t,z)(0). \end{aligned}$$

Comparing these definitions to (4.65) and (4.66), we notice that for all $n \geq 1$:

$$\lim_{\varepsilon\to 0} h_\varepsilon^n(t,u,z) = h_n(t,u,z),\ \forall (t,u,z) \in [0,T] \times \mathbb{R} \times B. \tag{4.81}$$

Let us define for all $n \geq 0$, $p^{n,\varepsilon} : [0,T] \times \mathbb{R} \times B \to \mathbb{R}$ by :

$$p^{n,\varepsilon}(t,u,z) := \frac{1}{\varepsilon^n}\big[p^\varepsilon(t,u,z) - \sum_{k=0}^{n-1}\varepsilon^k p_k(t,u,z)\big]. \tag{4.82}$$

where for all $k \leq n-1$, p_k is defined by (4.64). The following theorem is our main result. It provides an asymptotic expansion of order n for the price function p^ε.

Theorem 4.41. *Let Assumptions* 4.26*,* 4.34 *and* 4.35 *hold. For all $n \in \mathbb{N}$, there are constants C_n and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, we have:*

$$0 \leq p^{\varepsilon}(t,u,z) \leq v^{n,\varepsilon}(t,u,z) := \sum_{k=0}^{n-1} \varepsilon^k p_k(t,u,z) + C_n \varepsilon^n (T-t). \tag{4.83}$$

Moreover for every $n \geq 1$, $p^{n,\varepsilon}$ converges to p_n when $\varepsilon \downarrow 0$ uniformly on compact sets.

Proof. **Step 1:** Let us prove claim (4.83). In the proof of Proposition 4.39, we have shown that for all $n \in \mathbb{N}$ and $(i,j) \in \mathbb{N}^2$: $D^{i+j}_{u^i z^j} p_n(t,u,z)$ is continuous and bounded for all $(t,u,z) \in [0,T] \times \mathbb{R} \times B$. and $t \mapsto \partial_t p_n(t,u,z)$ is continuous and bounded (see proof of Proposition 4.39, hypothesis (A_n)). So in particular, we have that $v^{n,\varepsilon} \in C^{1,2,2}_b([0,T] \times \mathbb{R} \times B)$.

For all (t,u,z) in $[0,T] \times \mathbb{R} \times B$, by linearity of F_ε in its p and X variables (see (4.50)) and the definition of $v^{n,\varepsilon}$, we have:

$$-\partial_t v^{n,\varepsilon} - F_\varepsilon\big(t,u,z,Dv^{n,\varepsilon},D^2 v^{n,\varepsilon}\big) = C_n \varepsilon^n - \sum_{k=0}^{n-1} \varepsilon^k \Big[\partial_t p_k + F_\varepsilon\big(t,u,z,Dp_k,D^2 p_k\big)\Big] \tag{4.84}$$

We can rewrite (4.50) as:

$$\begin{aligned} F_\varepsilon(t,u,z,Dp_k,D^2p_k) = &-\frac{1}{2}\beta(t,z)^2(0) D_u p_k + \frac{1}{2}\beta(t,z)^2(0) D^2_{uu} p_k \\ &+ \varepsilon^2 \langle M(t,z), D_z p_k \rangle \\ &+ \frac{\varepsilon^2}{8} \langle D^2_{z^2} p_k, \rho\Lambda\big(t,z\big)\rangle \beta\big(t,z\big)(0)\rangle \\ &+ \frac{\varepsilon}{2} Tr\Big[D^2_{uz} p_k \big(\rho\Lambda(t,z)\big)\big(\rho\Lambda(t,z)\big)^*\Big]. \end{aligned} \tag{4.85}$$

Plugging (4.85) in (4.84), since by Lemma 4.40, $(p_k)_{1\leq k \leq n-1}$ solves (4.67), we then have:

$$\begin{aligned} -\partial_t v^{n,\varepsilon} - F_\varepsilon\big(t,x,Dv^{n,\varepsilon},D^2v^{n,\varepsilon}\big) = C_n\varepsilon^n - \sum_{k=0}^{n} \varepsilon^k \big[&\varepsilon^2 \langle M(t,z), D_z p_k\rangle \\ &+ \frac{\varepsilon^2}{8}\langle D^2_{z^2} p_k, \rho\Lambda\big(t,z\big)\rangle \beta\big(t,z\big)(0) \\ &+ \frac{\varepsilon}{2} Tr\Big[D^2_{uz} p_k \big(\rho\Lambda(t,z)\big)\big(\rho\Lambda(t,z)\big)^*\Big] \end{aligned}$$

A straightforward calculation leads to:

$$-\partial_t v^{n,\varepsilon}(t,u,z) - F_\varepsilon\big(t,u,z,Dv^{n,\varepsilon},D^2v^{n,\varepsilon}\big) + \varepsilon^n h^n_\varepsilon(t,u,z) = C_n \varepsilon^n. \tag{4.86}$$

In the proof of Proposition 4.39, we have shown, for all $n \in \mathbb{N}$, that $D_z p_{n-2}$, $D^2_{zz} p_{n-1}$, $D^2_{uz} p_{n-1}$ and $D^2_{zz} p_{n-2}$ are continuous and bounded (see the induction Assumption (A_n)). Therefore, by Lemma 4.38, we can find a constant $C_n > 0$ independent of (t, u, z) such that:

$$|h^n_\varepsilon(t, u, z)| \leq C_n \text{ and } h^n_\varepsilon \text{ is continuous on } [0, T] \times \mathbb{R} \times B. \tag{4.87}$$

Plugging (4.87) in (4.86) gives us that $v^{n,\varepsilon} \in C^{1,2,2}_b([0, T] \times \mathbb{R} \times B)$ is a classical supersolution of (4.54). Moreover, by definition of p_n (see (4.64)), we have that :

$$v^{n,\varepsilon}(T, u, z) = g(u)$$

F_ε is such that Lemma 4.30 holds. Moreover, by Proposition 4.31, we know that $p^\varepsilon \in BUC_x([0, T] \times \mathbb{R} \times B)$. Therefore, by a comparison argument given in Proposition 3.5 of [88], we get that:

$$p^\varepsilon(t, u, z) \leq v^{n,\varepsilon}(t, u, z).$$

for all $(t, u, z) \in [0, T] \times \mathbb{R} \times B$. Since by Assumption 4.34, g is a positive function then:

$$0 \leq p^\varepsilon(t, u, z) \leq v^{n,\varepsilon}(t, u, z).$$

Step 2: Now show that $p^{\varepsilon,n}$ is a viscosity solution to

$$-\partial_t p^{n,\varepsilon}(t, u, z) - F_\varepsilon(t, u, z, Dp^{n,\varepsilon}(t, u, z), D^2 p^{n,\varepsilon}(t, u, z)) - h^n_\varepsilon(t, u, z) = 0. \tag{4.88}$$

Since F_ε verifies condition (1) of Lemma 4.30, the two notions of viscosity solutions (see definitions 4.15 and 4.17) coincide. So, we can consider test functions which belongs to $\mathcal{X}'(\mathbb{R} \times B)$. Indeed take a test function ϕ in $\mathcal{X}'(\mathbb{R} \times B)$ and $(t_0, u_0, z_0) \in [0, T] \times \mathbb{R} \times B$ such that:

$$\min(p^{n,\varepsilon} - \phi) = (p^{n,\varepsilon} - \phi)(t_0, u_0, z_0) = 0.$$

Since for all $k \in \{0, \ldots, n-1\}$, p_k belongs to $C^{1,\infty,\infty}_b\big([0, T] \times \mathbb{R} \times B\big)$, in particular by the mean-value theorem, p_k has locally lipschitz continuous derivatives so belongs to $\mathcal{X}'(\mathbb{R} \times B)$. Therefore, it is equivalent to consider

$$\min\bigg(p^\varepsilon - \big[\sum_{k=0}^{n-1} \varepsilon^k p_k + \phi\varepsilon^n\big]\bigg) = \bigg(p^\varepsilon - \big[\sum_{k=0}^{n-1} \varepsilon^k p_k + \phi\varepsilon^n\big]\bigg)(t_0, u_0, z_0) = 0.$$

where the function $\eta := \sum_{k=0}^{n-1} \varepsilon^k p_k + \phi\varepsilon^n$ belongs to $\mathcal{X}'(\mathbb{R} \times B)$. By Theorem 4.31, we thus have:

$$-\partial_t \eta(t_0, u_0, z_0) - F_\varepsilon(t_0, u_0, z_0, D\eta(t_0, u_0, z_0), D^2\eta(t_0, u_0, z_0)) = 0.$$

By linearity of F_ε in p and X,

$$\varepsilon^n\Big[\partial_t\phi(t_0,x_0)+F_\varepsilon(t_0,x_0,D\phi(t_0,x_0),D^2\phi(t_0,x_0))\Big]$$
$$\sum_{k=0}^{n-1}\varepsilon^k\Big[\partial_t p_k+F_\varepsilon(t_0,x_0,Dp_k(t_0,x_0),D^2p_k(t_0,x_0))\Big]=0.$$

Using (4.85) and the fact that by Proposition 4.39 and lemma 4.40, $(p_k)_{1\leq k\leq n}$ is a solution of (4.67), a straightforward calculation gives us that :

$$-\partial_t\phi(t_0,u_0,z_0)-F_\varepsilon(t_0,u_0,z_0,D\phi(t_0,u_0,z_0),D^2\phi(t_0,u_0,z_0))-h^n_\varepsilon(t_0,u_0,z_0)=0.$$

So $p^{n,\varepsilon}$ is a viscosity solution of (4.88).

Step 3: We want to use the stability result of Proposition 4.21. By (4.87), h^n_ε is bounded and continuous on bounded sets of $[0,T]\times\mathbb{R}\times B$. Moreover, since we have shown in the induction assumption (A_n) (see the proof of Proposition 4.39) that for all $(i,j)\in\mathbb{N}^2$, $D^{i+j}_{u^iz^j}p^n(t,u,z)$ is continuous and bounded then by Lemma 4.38, $z\mapsto h^n_\varepsilon(t,u,z)$ is continuously differentiable for all $(t,u,z)\in[0,T]\times\mathbb{R}\times B$. So it is locally Lipschitz continuous and uniformly continuous on bounded sets of B. Therefore, by Proposition 4.28, we then get that $F_\varepsilon+h^n_\varepsilon$ is bounded and uniformly continuous on bounded sets of $[0,T]\times\mathbb{R}\times B$.
Let us define the upper semicontinuous and the lower semicontinuous envelope in ε, (t,u,z) of $p^{n,\varepsilon}$:

$$p^{n,*}(t,u,z)=\limsup_{\substack{(t',u',z')\to(t,u,z)\\ \varepsilon\to0}}p^{n,\varepsilon}(t',u',z')\text{ and}$$
$$p^n_*(t,u,z)=\liminf_{\substack{(t',u',z')\to(t,u,z)\\ \varepsilon\to0}}p^{n,\varepsilon}(t',u',z').$$

We consider the following sequences $(\varepsilon_m,t_m,u_m,z_m,p_m,X_m)_m$ such that:

$$\big(\varepsilon_m,t_m,u_m,z_m,p_m,X_m,p^{n,\varepsilon_m}(t_m,u_m,z_m)\big)\underset{m}{\longrightarrow}\big(0,t,u,z,p,X,p^{n,*}(t,u,z)\big)$$

Then, we have by continuity of F_{ε_m} in (t,u,z,p,X), the continuity of h_{n,ε_m} in (t,u,z) (see claim (4.87)) and (4.81) :,

$$\lim_m F_{\varepsilon_m}\big(t_m,u_m,z_m,p_m,X_m\big)+h^n_{\varepsilon_m}(t_m,u_m,z_m)=\frac{1}{2}\beta(t,z)^2(0)p_u$$
$$-\frac{1}{2}\beta(t,z)^2(0)X_{uu}+h_n(t,u,z).$$

where p_u corresponds to the first component of p and X_{uu} to the second order derivative in u. By Theorem 4.21, we get that $p^{n,*}$ is a viscosity subsolution on $[0,T]\times\mathbb{R}\times B$ of (4.67). By applying the same argument, we get that p_*^n is a supersolution of (4.67) too.

By claim (4.83) and the definition of the function $p^{\varepsilon,n}$, we know that for all $\varepsilon \in \mathbb{R}$, $(t, u', z') \in [0, T] \times \mathbb{R} \times B$ and $n \geq 1$:

$$-\sum_{k=1}^{n-1} \varepsilon^{k-n} p_k(t, u', z') \leq p^{n,\varepsilon}(t, u', z') \leq C_n(T - t). \tag{4.89}$$

Passing to the limit sup in $(t, u', z', \varepsilon) \to (T, u, z, 0)$, the right-hand part of (4.89) gives us for all $n \geq 1$ that:

$$p^{n,*}(T, u, z) \leq 0 = p_n(T, u, z).$$

Moreover by continuity at the boundary of the (p_k), the left-hand side of (4.89) gives us that

$$p^{n,*}(T, u, z) \geq \limsup_{\varepsilon \to 0} p^{n,\varepsilon}(T, u, z) = 0.$$

This leads to

$$p^{n,*}(T, u, z) = p_*^n(T, u, z) = p_n(T, u, z) = 0. \tag{4.90}$$

Since in the partial differential equation (4.67), the variable z plays the role of a parameter, we can apply comparison theorem for viscosity solutions in the finite dimension case (for instance one can use Proposition 3.4 in [88]) and deduce from (4.90) that:

$$p^{n,*}(t, u, z) \leq p_n(t, u, z) \leq p_*^n(t, u, z)$$

for all $(t, u, z) \in [0, T] \times \mathbb{R} \times B$. This leads to:

$$p^{n,*}(t, u, z) = p_*^n(t, u, z) = p_n(t, u, z). \tag{4.91}$$

Therefore $p^{\varepsilon,n}$ converges to p_n. □

4.6 Conclusion

I had not sufficient time to do some numerical results for this model. This issue is very interesting in term of calibration since we have to deal with the forward volatility curve. In addition, it would be interesting to improve the order of the asymptotic expansion for a European call or put option. Maybe, this could be done via payoff regularization methods. The extension of this problem to American option seems quite difficult. How can we deal with the variational inequality which appears for this pricing problem?

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