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# Gravitational environments, flows and holographic fluids

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# Abstract

The thesis is focused on the study of various gravitational environments in four dimensions: gravitational instantons and their related geometric flows, black holes in anti-de-Sitter (AdS) both in general relativity and in N = 2 supergravity. Besides their own interest, these backgrounds offer possible holographic interpretations whose understanding provide insights both into the bulk and boundary theories.
In general relativity, the search of new exact solutions both in flat and in AdS spaces is a challenging task. A peculiar simplifying assumption in the Euclidean regime is the one of self-duality of the Riemann tensor. This condition provides a class of gravitational instantons. Their interest lies both in the bulk and in the boundary features. Asymptotically, the self-duality condition gives rise to boundaries whose effective action is proportional to the Chern-Simon action. On the gravity side, the (Euclidean) temporal evolution of the gravitational instantons is described by a geometric flow. This connection has been analyzed in full details, leading to a comprehensive understanding under the general assumptions of time foliation and homogeneity of the spatial sector. In particular, the role of the Ricci tensor and of the Yang-Mills connection within the geometric flow bas been unraveled.
In the generating techniques which aim at finding new solutions of Einstein's equations, one is often assuming the existence of Killing vectors. An important class of starting solutions is that of stationary rotating black holes. Generating methods are available in the case of vanishing cosmological constant, such as Weyl's, while in presence of the cosmological constant the system is no longer integrable. It thus remain a challenging question to exhibit new stationary axysymmetric black holes in AdS space. This question arises in the framework of holographic fluid dynamics. Rotating systems in the bulk correspond to fluids with non-trivial vorticity in the boundary: the Taub-NUT solution corresponds to a monopole-like vorticity, while the Kerr solution corresponds to a dipole-like vorticity. Regularity of the solution at the horizon puts constraints on the transport coefficient of the boundary fluid in such a way that it has the form of a perfect-fluid. The correspondence between the bulk and the boundary is usually done through a perturbative or a derivative expansion. Necessary conditions have been found such that the expansion can be resummed and exact solutions of Einstein's equations can be generated.
The development of techniques to map strongly coupled field theories to weakly-coupled gravitational systems is sustained by a better understanding of the ther-

modynamic properties of black holes, based on the accurate counting of their microscopic states. However, a microscopic counting of the entropy of black holes in AdS is not available yet. In the case of N=2 supergravity in four dimensions, a relation between rotating non-BPS extremal asymptotically flat black holes and BPS rotating asymptotically AdS black holes has been discovered via an ungauging procedure. The existence of a connection between the microscopic entropy counting of the two systems is enforced by the fact that the attractor geometries of both types of black holes fit within a common class of spaces that are supersymmetric. For both asymptotically flat and asymptotically AdS non-BPS black holes, arguments for the entropy counting have been formulated, based on extremality. The ungauging procedure indicates that, for extremal black holes, a supersymmetric conformal field theory dual can be found, thus gaining insights on the role of gaugings in the microscopic counting.

# Resumé


Différents environnements gravitationnels à quatre dimensions sont abordés dans ce travail de thése : instantons gravitationnels et leurs correspondances avec des flots géométriques et trous noirs dans des espaces anti-de-Sitter (AdS), aussi bien en relativité générale qu'en supergravité $\mathcal{N} = 2$. Au delà de leur intérêt en tant que solutions gravitationnelles, ces derniers offrent également une possible interprétation holographique, fournissant une meilleure compréhension des théories duales, tant dans l'espace ambiant que sur le bord.

De manière générale, la recherche de nouvelles solutions en relativité est un véritable défi, aussi bien en espace plat qu'en espace AdS. Cette tâche est cela dit nettement simplifiée dans l'hypothèse où l'on dispose, en régime euclidien, d'un tenseur de Riemann auto-dual. Ces solutions, dites instantons gravitationnels, portent un intérêt, tant pour leurs caractéristiques dans le bulk que pour leurs propriétés sur le bord. De façon asymptotique, la condition d'auto-dualité fournit des théories de bord dont l'action est proportionelle à celle de Chern-Simons. Du point de vue gravitationnel, l'évolution en temps des instantons est décrite par un flot géométrique. Ce lien est analysé en détail, dans l'hypothèse générale d'une foliation temporelle de la métrique et d'un espace tridimensionnel homogène. En particulier, l'attention a été focalisée sur le rôle du tensor de Ricci et de la connexion de Yang-Mills dans le flot géométrique.

Plusieurs techniques sont utilisées pour trouver de nouvelles solutions des équations d'Einstein. Ces techniques supposent souvent l'existence d'un certain nombre de vecteurs de Killing. Une classe importante étant celle des métriques stationnaires, pour lesquelles on dispose, pour une constante cosmologique nulle, de méthodes telles que la technique de Weyl. En espace de type AdS, le système n'est plus intégrable, et trouver de nouveaux trous noirs avec symétrie axiale est une question toujours ouverte. Cette même question peut être posée dans le contexte de la dynamique des fluides holographiques. Trous noirs en rotation correspondent alors sur le bord à des fluides aux vorticités particulières : le trou noir de Taub-NUT correspond à une vorticité qui se comporte comme un monopole, tandis que le trou noir de Kerr correspond à une vorticité se comportant comme un dipôle. En imposant que la solution soit régulière sur l'horizon, les coefficients de transport du fluide sur le bord sont contraints de telle façon que le fluide acquiert la forme d'un fluide parfait. La correspondance entre solution gravitationnelle et théorie hydro-

dynamique se fait usuellement par un développement perturbatif ou en dérivées. Il existe cedependant des conditions nécessaires afin que le développement puisse être ressommé et pour qu'on puisse trouver des solutions exactes de la relativité générale.
Une meilleure compréhension des propriétés thermodynamiques des trous noirs, ainsi qu'un comptage précis de leurs micro-états est nécessaire au bon développement des techniques de connexion entre théories fortement et faiblement couplées. Cependant, le comptage de l'entropie des trous noirs dans des espaces de type AdS ne fait toujours pas partie des résultats connus. Dans le cas des théories de supergravité N=2 en quatre dimensions, en considérant des solutions en rotation, une relations entre trous noirs extremaux non-BPS en espace plat et trous noirs BPS en espace AdS a été mise au point grâce à une procédure dite d'"ungauging". L'existence d'une connexion entre le comptage microscopique de l'entropie dans les deux cas est étayée par les propriétés de ces solutions sur l'horizon, auprès duquel les deux solutions entrent dans une même classe d'espaces supersymétriques. Dans les deux cas - espaces plats et espaces AdS - il existe des outils pour le calcul de l'entropie, basés sur l'extremalité des solutions. La connexion entre les solutions dans les deux types d'espace indique l'existence d'une théorie conforme, supersymétrique, duale aux trous noirs extremaux, donnant par conséquent des informations sur le rôle des champs de jauge dans le comptage microscopique.

# Chapter 1

# Introduction

The present thesis aims at exploring various aspects of the gravitational theory, related with geometrical flows, instantons and black holes, and its holographic applications.
In the first and last parts of the work we deal with solutions of classical general relativity, while in the second chapter we go beyond Einstein's gravity by adding supersymmetry to the theory. In particular, a central concept is the one of black hole solutions. The presence of a singularity in such solutions is the signal of a breakdown of classical general relativity at small scales and of the appearance of quantum phenomena. In this sense, general relativity can be viewed as the low energy (large scale) limit of a more fundamental theory of quantum gravity.

## 1.1 Quantum gravity

To describe gravity at small scales, one needs to introduce a theory conciliating quantum effects and general relativity. A consistent theory of quantum gravity should reduce to general relativity at large scales, and smooth out the singularities of the classical theory through the appearance of quantum phenomenas at small scales. In particular, such theories should be in accordance with the no-hair theorem, and thus no degrees of freedom beyond the total mass and the conserved charges should affect the description far from the singularity. Furthermore, the microscopic degrees of freedom of the quantum analysis of the black hole should be given in terms of a statistical description involving a large number of degrees of freedom, in an analogous way as the molecular description of gases.
The task of finding such quantum description of gravity is of course not completed and it remains one of the main open challenges of theoretical physics.

## 1.2 Supergravity

Supersymmetric theories relate bosonic and fermionic particles: each boson has a fermionic partner, and viceversa, forming a pair of superpartners. When more than two particles are related, the amount of supersymmetry is higher (and thus $\mathcal{N}$ is higher).
Supergravity theories are, as the name says, gravitational supersymmetric theories. The force of gravity is carried by the graviton, and its fermionic superpartner is the gravitino. The number of gravitinos is given by $\mathcal{N}$. Supergravity theories were originally introduced as genuine candidates for theories of quantum gravity. However, it was soon realized that they are power-counting non-renormalizable, *i.e.* not well behaved at high energies. Nevertheless, it is still interesting to study them as effective theories, and in particular as the limit of string theory for low energy and large distance (compared to the length of the string). In this limit, the string can be viewed as a point particle, and the effective description of the theory is given by field theories of gravity coupled to scalars and gauged fields.
Demanding a theory to be invariant under supersymmetric transformations restricts the possible interactions, which will then be described by a smaller number of functions with respect to the non-supersymmetric theory. These couplings should be derived from a more fundamental theory, like string theory, but it is possible to study the structure of supergravity theories independently, deferring the connection to a possible string origin. This is the approach we will adopt, and that can be used to constrain solutions of classical general relativity. Indeed, supersymmetric states, called “BPS”, correspond to the ground states of the theory. Since the existence of supersymmetry protects the solutions from any high energy correction, such solutions exist and keep their main properties in the full quantum solution as well.

## 1.3 AdS/CFT correspondence

The AdS/CFT duality originated from string theory, but has been mostly developed with pure gravity, and it is currently one of the main topics of research in theoretical physics. The conjecture relates two apparently different theories: gravitational $d+1$-dimensional asymptotically anti-de-Sitter theories and $d$-dimensional conformal field theories living on the boundary of the AdS spaces. This gauge/gravity duality was first studied for superstring bulk theories, but further studies showed that it is valid for many different models not involving supersymmetry. As a consequence, the correspondence can be used to obtain results in different applications to various physical systems, *i.e.* particle physics and condensed matter physics above all, where it can provide experimentally reachable results.
The key property of the correspondence is that it maps strongly-coupled regimes in one theory to weakly coupled regimes in the dual model, given the possibility

of studying various relevant field theories at strong coupling. In this context, one of the most important applications of the gauge/gravity duality is within heavy-ion physics. Experiments at the Large Hadron Collider and at the Relativistic Heavy Ion Collider produce a plasma of quarks and gluons (QGP) by heavy-ion collisions. After the collisions, the components of the QGP rapidly come into local thermal equilibrium, and their behavior can be studied by a hydrodynamic model. In particular, the QGP evolution is characterized by a set of transport coefficients, the most relevant of them being the so-called shear viscosity. For weakly-coupled theories, the transport coefficients can be computed by standard perturbative calculations. However, the temperature of the QGP is estimated to be approximately 170 MeV, which is near the confinement scale of quantum-chromodynamics (QCD). Hence, QGP is deep inside the non-perturbative regime of QCD, and thus perturbative techniques cannot be applied. Moreover, the standard numerical approach to strong interacting QCD (*i.e.* computations on lattice), which gives a precise analysis of thermodynamical quantities, is not well suited for computing transport coefficients. The gauge/gravity duality offers a theoretical framework to investigate strongly interacting systems like the one just described by considering the simpler dual weakly-coupled (super-)gravity model. Computing the transport coefficients in the near-equilibrium regime of the quantum field theory will correspond to compute deformations of black holes in the dual AdS space.

## 1.4 Fluid/gravity correspondence

The fluid/gravity correspondence is the limit of the AdS/CFT correspondence in the regime where the boundary strongly-coupled field theory is well-approximated by its long wavelength effective description. The corresponding bulk is given by black holes in classical general relativity, with the possibility of adding supersymmetries and potential string corrections. Two alternative expansions are available for finding the holographic fluid dual to an asymptotically AdS black holes.

The first expansion consists in perturbing a gravity solution, such as a black hole or a black brane, by means of isometry transformations whose infinitesimal parameters depend on the coordinates on the boundary of the AdS bulk space. The transformed expression is no longer a solution of the bulk equations of motion. In order for it to be a solution, the local boundary parameters should satisfy some differential equations which turn out to be, at first order, the linearized Navier-Stokes equations for relativistic fluids. The procedure can be made iterative and thus in principle it can be used to compute the expansion at any desired order. Details on this expansion can be found, for example, in [43] and [35].

The second expansion, called of Fefferman-Graham, is an expansion for large holographic radial coordinate, where the only independent data are the background boundary metric and the holographic stress-energy tensor satisfying the conservation equations. Within this frame, it is possible to study interesting properties of the fluids, such as the vorticity, which is important to apply the AdS/CMT (condensed matter theory) correspondence to systems such as rotating Bose or Fermi gases, turbulence or wave propagation in moving metamaterials. It is also possible to show that some of the transport coefficients are constrained by the bulk geometry. Moreover, for a particular class of boundary fluids it is possible to go beyond the perturbation level and find an exact correspondence between black holes and fluids, leading to possible interesting insights on new solutions for asymptotically AdS black holes. The procedure was studied extensively in [27], while we quote as well the original paper [36].

## 1.5 Outlook of the thesis

In the first part of the thesis we study gravitational instantons and their relation with geometric flows. Instanton solutions to Yang-Mills theory have been studied in the 70s, and later on researches have focused as well on instantonic solutions to general relativity. In a four-dimensional Euclidean space-time, a main feature is the possibility of imposing self-duality either of the Ricci or of the Weyl tensor, depending on whatever the space presents a non-vanishing cosmological constant. Within the hypothesis of foliation in one-dimensional temporal direction and three-dimensional homogeneous spatial sector, we show that the time evolution of such solution is related to a geometric flow given by a Ricci flow plus a Yang-Mills connection. Gravitational instantons can have a holographic interpretation once we turn on the cosmological constant: indeed, in the case of Lorentzian signature, we find traces of the self-duality properties of instantonic solutions in the duality between the Cotton tensor of the three-dimensional boundary background and the stress-energy tensor of the effective conformal theory. In four dimension, this corresponds to a duality between the mass and the nut charge.
Turning on the mass in a gravitational solutions generates a black hole, where a horizon is present. Such solutions can be generalized in presence of supersymmetry, *i.e.* in supergravity theories. Supergravity theories with negative cosmological constant are the natural frame for the AdS/CFT correspondence, even though the latter is often applied in a non-supersymmetric context, like in the last part of the thesis. While holography needs an asymptotically anti-de-Sitter space to be applied, we master black hole solutions and techniques to generate new solutions in a better way in the case of vanishing cosmological constant. Moreover, the microscopic counting of degrees of freedom is also better understood for asymptotically flat solutions than for asymptotically AdS ones.
The topic of the second chapter fits into this context. Indeed, it is shown that a relation is existing between asymptotically flat and asymptotically AdS solutions. For such a connection to exist, it is necessary that the cosmological constant is generated in a dynamical way. This is possible only in gauged supergravity theories. Among gauged theories, a class of solutions can be found such that the gauging is

non-vanishing but the scalar potential is. Such a theory is not equivalent to the ungauged one, which has vanishing cosmological constant, because of the different fermionic sector. This is reflected in the different supersymmetric properties of the solution, which, for example, for the case of static black holes is full BPS for the ungauged theory and 1/2 BPS for the gauged one with vanishing potential.
The topics analyzed in the first two chapters raise questions concerning holography, which is the subject of the last chapter. When considering stationary black hole solutions in four dimensions, a fluid in global equilibrium appears in the three-dimensional boundary geometry. Such fluids inherit kinematical properties, such as the vorticity, from the bulk geometry. The holographic correspondence can be used to gain information both on the boundary and on the bulk side. In one direction, the holographic stress-energy tensor turns out to be perfect-fluid like, and the combination of dynamical and kinematical variables gives information on the transport coefficient of the fluid. In the opposite direction, it is possible to define a class of three-dimensional spaces for which the correspondence is exact: that is, the holographic dual of such spaces is an exact solution of the bulk equations of motion. This "perfect" geometries are characterized by a Cotton-York tensor of the same form of the stress-energy tensor. Under this condition, the holographic expansion can be resumed and the bulk solution has a non-trivial monopolar or dipolar moment. The constant coefficients appearing in the Cotton-York tensor and in the stress-energy tensor are independent and they measure the pressure (and thus the temperature) and the vorticity of the fluid. They are related to the mass and to the nut charge of the four-dimensional gravitational environment. By imposing such coefficients to be equal to each other we recover, in the corresponding solution with Euclidean signature, self-dual solutions of the type studied in the first chapter.
Even thought the complete relationship between gravitational instantons, flows, gravitational self-duality and holography has not been fully unravelled, the work produced in the present thesis points towards a deeper connection, the most intriguing being the linear holographic realization of the non-linear gravity self-duality.

# Chapter 2

# Gravitational instantons and geometric flows

In analogy with Yang-Mills instantons, it is possible to define instanton-like solutions of Einstein's equations. The definition can be made operational by choosing space-times with the structure of a three-dimensional homogeneous foliation. In this case, the solutions show an intriguing relation with three-dimensional geometric flows.
We start the chapter by discussing the motivations for the study of gravitational instantons, followed by a brief discussion on Yang-Mills instantons. We will then give the definition of spatial-homogeneous gravitational instantons and classify them in full generality, mentioning as well some peculiar solutions which identify, after analytic continuation to Minkowski space, to the Taub-NUT solution for vanishing cosmological constant. We then discuss the relation between these solutions and three-dimensional Ricci and, in general, geometric flows.

## 2.1 Motivations

Instanton solutions for general relativity have been investigated since the 70s, soon after the discovery of Yang-Mills instantons. The first instanton-like metric discovered comes from standard solutions of black holes physics. While these solutions arise naturally in space-times with Minkowskian signature, by analytic continuation to Euclidean space-time we can produce positive-definite singularity-free metrics. The further property of being asymptotically locally flat ensures the possibility of defining asymptotic states and it is reminiscent of the intuitive idea of instantons coming from Yang-Mills theory, which should approach a pure gauge at infinity.
Early search for gravitational instanton solutions was motivated by the possible applications in cosmology or in non-perturbative quantum gravity transitions. The discovery of a relation between the dynamics of the instanton and geometric flow equations on a three-dimensional manifold has stimulated further investigations on

the topic. Geometric flows are differential equations which describe the evolution of the metric driven by geometric structure on the manifold. Within the geometric flow associated to gravitational instantons, the Ricci flow plays a crucial role in. The latter is appearing as well in other physical contexts: for example, since the Ricci-flow equations are the renormalization-group equations for two-dimensional sigma-models, the relation between gravitational instantons and Ricci flows is an indication towards a dynamical generation of time in string theory.
It is worth mentioning also that the appearance of first-order differential equations in the gravitational settings is reminiscent of holographic settings, and it can be interesting to dig more in this direction, reconstructing the bulk fields by flowing the boundary datas.

## 2.2 Yang-Mills instantons

In gauge theories, an instanton is a topologically non-trivial solution to classical field equations potentially described by a self-dual or anti-self-dual connection with finite action. For simplicity, we focus on solutions with SU(2) symmetry. The pure Yang-Mills action is given by

$$S = \int d^4x \left( \frac{1}{2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right) , \tag{2.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{2.2}$$

is the field strength. The field equations are:

$$\partial_\mu F^\mu_\nu + [A^\mu, F_{\mu\nu}] = 0. \tag{2.3}$$

We define the dual of the strength tensor as

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}. \tag{2.4}$$

The field strength $F_{\mu\nu}$ is said to be self-dual when $F_{\mu\nu} = \tilde{F}_{\mu\nu}$, and anti-self-dual when $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$. If this is the case, the Yang-Mills equations (2.3) are automatically satisfied thanks to the Bianchi identities:

$$\partial^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = \pm \left( \partial^\mu \tilde{F}_{\mu\nu} + [A^\mu, \tilde{F}_{\mu\nu}] \right) = 0. \tag{2.5}$$

We still have to impose that the gauge potential enjoys a $SU(2)$ symmetry. This leads us to choose the Ansatz:

$$A_\mu = \rho(r) g^{-1} \partial_\mu g, \tag{2.6}$$

where $\rho$ is an arbitrary function, $r^2 = t^2 + \mathbf{x}^2$ and $g = \frac{t - i\sigma\dot{x}}{r}$ , and $\sigma$ are the Pauli matrices. Inserting (2.6) in the equation for the strength tensor self-duality we find that $\rho$ obeys the equation:

$$\rho'(r) + \frac{2}{r}\rho(r)\left(\rho(r) - 1\right) = 0. \tag{2.7}$$

Thus, by imposing self-duality and SU(2) gauge symmetry we have reduced the problem from solving the second-order differential equation (2.3) to solving the first-order equation (2.7). Is is easy to find a solution of (2.7) with the further property that $A_\mu$ is a pure gauge at infinity, that is $\rho(r) \to 1$ when $r \to \infty$:

$$\rho(r) = \frac{\mathbf{r}^2}{\mathbf{r}^2 + \lambda^2}, \tag{2.8}$$

with $\lambda$ specifying the size of the instanton. By using the latter expression we find

$$A_\mu = \frac{\mathbf{r}^2}{\mathbf{r}^2 + \lambda^2} g^{-1}\partial_\mu g \tag{2.9}$$

and

$$F_{\mu\nu} = \frac{4\lambda^2}{\mathbf{r}^2 + \lambda^2}\sigma_{\mu\nu}, \tag{2.10}$$

where $\sigma_{\mu\nu}$, $\mu = \{0, i\}$ satisfy

$$\sigma_{ij} = \frac{1}{4}\left[\sigma_i, \sigma_j\right], \qquad \sigma_{i0} = \frac{1}{2}\sigma_i = -\sigma_{0i}. \tag{2.11}$$

We have thus found an Euclidean $SU(2)$ solution to Yang-Mills equations with finite action, self-dual strength tensor $F_{\mu\nu}$ localized at $r = 0$ and falling like $1/r^4$ at infinity, and with $A_\mu$ being asymptotically a pure gauge.

## 2.3 Homogeneity of the spatial sector

We would like to define gravitational instantons in the same line as we defined Yang-Mills instantons. The structure of the space time is crucial to settle an operative definition of the gravitational instanton and on its relation with geometric flows and we should pause to analyze it in details. We provide the four-dimensional space time with the structure of a foliation in three-dimensional leaves. Our manifold $\mathcal{M}_4$ becomes thus topologically equivalent to $\mathbb{R} \times \mathcal{M}_3$, with a natural separation between the one-dimensional time direction and the three-dimensional space directions. Furthermore, we consider the spatial sector to be homogeneous, that is we assume it to be invariant under an isometry group of motion of at least dimension three acting transitively on the leaves. We thus have three independent Killing vectors

$$[\xi_i, \xi_j] = c^k{}_{ij}\xi_k, \tag{2.12}$$

| Type | Structure constants |
|---|---|
| I | ${c^i}_{jk} = 0$ |
| II | ${c^1}_{23} = -{c^1}_{32} = +1$ |
| $\mathrm{VI}_{-1}$ | ${c^2}_{13} = -{c^2}_{31} = -1, \quad {c^3}_{12} = -{c^3}_{21} = -1$ |
| $\mathrm{VII}_0$ | ${c^2}_{13} = -{c^2}_{31} = -1, \quad {c^1}_{23} = -{c^1}_{32} = +1$ |
| VIII | ${c^1}_{23} = -{c^1}_{32} = -1, \quad {c^2}_{31} = -{c^2}_{13} = +1, \quad {c^3}_{12} = -{c^3}_{21} = +1$ |
| IX | ${c^1}_{23} = -{c^1}_{32} = +1, \quad {c^2}_{31} = -{c^2}_{13} = +1, \quad {c^3}_{12} = -{c^3}_{21} = +1$ |

Table 2.1: Unimodular Bianchi groups (${c^i}_{jk}$ not explicitly given are taken to be zero)

where ${c^i}_{jk}$ are the structure constants of the isometry group. Their dual vectors are the left-invariant Maurer-Cartan forms $\{\sigma^i, i = 1, 2, 3\}$, defined by $\sigma^i \rfloor \xi_j = {\delta^i}_j$ and obeying

$$d\sigma^i = \frac{1}{2}{c^i}_{jk}\sigma^j \wedge \sigma^k. \tag{2.13}$$

The structure constants can be put in the form

$$c^k{}_{ij} = \epsilon_{ijl}n^{lk} + \delta^k{}_j a_i - \delta^k{}_i a_j, \tag{2.14}$$

where $n^{lk}$ are the elements of a symmetric matrix $n$ and $a_i$ are the components of a covector $a$. The trace of the structure constants is given by ${c^j}_{ij} = 2a_i$. Unimodular groups have zero trace, while non-unimodular groups have non-vanishing trace. We refer to them as Bianchi A and Bianchi B respectively, from the name of the mathematician who first classified all three-dimensional algebras in 1897. We report the complete classification of the structure constant in tables 2.1 and 2.2, while we write the canonical structure constants in table 2.3.
For further convenience we also define the antisymmetric matrix $m$ with entries

$$m^{ij} = \epsilon^{ijk}a_k. \tag{2.15}$$

The Jacobi identity of these algebras

$$[\xi_i, [\xi_j, \xi_k]] + [\xi_j, [\xi_i, \xi_k]] + [\xi_k, [\xi_i, \xi_j]] = 0 \tag{2.16}$$

becomes then

$$\epsilon_{ijk}m^{ij}\left(n^{kl} - m^{kl}\right) = 0 \iff a_k n^{kl} = 0. \tag{2.17}$$

## 2.4 Metric Ansatz and Cartan formalism

The generic four-dimensional metric equipped with a three-dimensional homogeneous spatial sector reads

$$ds^2 = N(T)dT^2 + g_{ij}(T)\sigma^i\sigma^j. \tag{2.18}$$

| Type | Structure constants |
|---|---|
| III | $c^1{}_{13} = -c^1{}_{31} = +1$ |
| IV | $c^1{}_{13} = -c^1{}_{31} = +1, \quad c^1{}_{23} = -c^1{}_{32} = +1, \quad c^2{}_{23} = -c^2{}_{32} = +1$ |
| V | $c^1{}_{13} = -c^1{}_{31} = +1, \quad c^2{}_{23} = -c^3{}_{32} = +1$ |
| $\mathrm{VI}_{h>-1}$ | $c^1{}_{13} = -c^1{}_{31} = +1, \quad c^2{}_{23} = -c^2{}_{32} = +h$ |
| $\mathrm{VII}_{h>0}$ | $c^2{}_{13} = -c^2{}_{31} = +1, \quad c^1{}_{23} = -c^1{}_{32} = -1, \quad c^2{}_{23} = -c^2{}_{32} = +h$ |

Table 2.2: Non-unimodular Bianchi groups ($c^i{}_{jk}$ not explicitly given are taken to be zero)

| Type | a | $n^1$ | $n^2$ | $n^3$ | Group |
|---|---|---|---|---|---|
| I | (0, 0, 0) | 0 | 0 | 0 | Translations |
| II | (0, 0, 0) | 1 | 0 | 0 | Heisenberg |
| $\mathrm{VI}_0$ | (0, 0, 0) | 1 | -1 | 0 | E(1,1) |
| $\mathrm{VII}_0$ | (0, 0, 0) | 1 | 1 | 0 | E(2) |
| VIII | (0, 0, 0) | 1 | 1 | -1 | $SL(2,\mathbb{R})$ |
| IX | (0, 0, 0) | 1 | 1 | 1 | SU(2) |
| III | (1,0,0) | 1 | -1 | 0 | |
| IV | (1, 0 , 0) | 1 | 0 | 0 | |
| V | (1,0,0) | 0 | 0 | 0 | |
| $\mathrm{VI_h}$ | $(h > -1, 0, 0)$ | 1 | -1 | 0 | |
| $\mathrm{VII_h}$ | $(0, 0, h > 0)$ | 1 | 1 | 0 | |

Table 2.3: Canonical structure constants for the different Bianchi groups. The matrix $n^{ij}$ can be put in a diagonal form and we denote its diagonal elements by $n^1$, $n^2$, $n^3$.

It is convenient to introduce an orthonormal frame $\theta^a$ such that

$$ds^2 = \eta_{ab}\theta^a\theta^b, \tag{2.19}$$

where we should choose $\eta_{ab} = \mathbb{1}$ or $\eta_{ab} = \mathrm{diag}(1,1,-1,-1)$, up to permutations, to have real solutions. In particular, in order to achieve the correspondence with the geometric flow without complexifications, we need to choose $\eta_{ab} = \mathrm{diag}(1,1,-1,-1)$ for Bianchi III, VI and VIII groups, and the pure Euclidean signature for all others Bianchi groups. The vierbeins are given by:

$$\theta^0 = NdT, \qquad \theta^\alpha = \Theta^\alpha{}_j\sigma^j \text{ with } g_{ij} = \eta_{\alpha\beta}\Theta^\alpha{}_i\Theta^\beta{}_j. \tag{2.20}$$

The components of the metric, or alternatively the vierbeins, will be of course determined by imposing Einstein's equations. Note here the use of the different indices: $\alpha, \beta, \ldots$ indicate the orthonormal space frames, so that $a = (0, \alpha)$, whereas $i, j, \ldots$ correspond to our choice of the invariant forms as they follow from fixing the structure constants. Without loss of generality, we can set $N = \Theta = \sqrt{\det g}$. It is also useful to introduce a new time variable $t$ defined by $dt = NdT$.
The torsionless spin-connection one-form is uniquely defined by the Cartan structure equations

$$\omega_{ab} = -\omega_{ba} \quad d\theta^a + \omega^a_b \wedge \theta^b = 0. \tag{2.21}$$

By using the spin-connection, the Riemann curvature two-forms can be defined:

$$W^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{2.22}$$

The latter satisfies the Bianchi identity

$$W^a{}_b \wedge \theta^b = 0. \tag{2.23}$$

The Ricci one-form is given by:

$$\mathcal{R}_a = \theta^b \rfloor W_{ba} = R_{ab}\theta^b, \quad \theta^a \rfloor \theta_b = \delta^a{}_b, \tag{2.24}$$

where $R_{bc}$ are the components of the ordinary Ricci tensor and the Ricci scalar is given by $R = \theta^a \rfloor R_a$.

## 2.5 Gravitational instantons

By using the above formalism, the Einstein-Hilbert action can be expressed in terms of the curvature two-forms as

$$S = -\frac{1}{32\pi G}\int_{\mathcal{M}_4} \epsilon_{abcd} W^{ab} \wedge \theta^c \wedge \theta^d. \tag{2.25}$$

The action has an extremum when

$$W^{ab} \wedge \theta^c \epsilon_{abcd} = 0. \tag{2.26}$$

By using the self-dual curvature two-form

$$\tilde{W}^{ab} = \frac{1}{2}\epsilon^{ab}{}_{cd} W^{cd} \tag{2.27}$$

the equation of motions (2.26) can be expressed as

$$\tilde{W}^c{}_d \wedge \theta^d = 0. \tag{2.28}$$

A sufficient (but non necessary) condition for this to hold, using (2.23), is

$$\tilde{W}^{ab} = \pm W^{ab}. \tag{2.29}$$

Thus, in analogy to the Yang-Mills theory case, thanks to the Bianchi identities the (anti-)self-duality of the Riemann curvature two-form is a sufficient condition for Einstein's equations to be satisfied. The main difference is that while the equations for the self-duality of the field strength tensor in the Yang-Mills case were first order differential equations, (2.29) are still second order differential equations. First-order equations can be obtained by considering the spin-connection. Indeed, a sufficient condition for (2.29) to be solved is to impose the (anti-)self-duality of the spin connection. However, this does not exhaust all the possibilities: the necessary condition is for the spin-connection to be (anti-)self-dual up to a local SO(3) transformation, that is a gauge transformation. For the case at hand, the number of non-equivalent (anti-)self-dual spin-connections is equal to the number of homomorphisms $g \to so(3)$, where $g$ is the algebra of the group acting on the three-dimensional spatial sector. For the cases at hand, there will be two non-equivalent such homomorphisms, leading to two different branches of solutions. In order to write explicitly the conditions for the Riemann curvature and the spin-connection in an operative way, we first implement the (anti-)self-duality conditions within our choice of the metric (2.18). In order to do this, it is useful to write the spatial vielbeins as

$$\eta_{ij}\theta^j = \gamma_{ij}\sigma^j, \tag{2.30}$$

where $\gamma_{ij}(t)$ is an invertible matrix which give us the "square root" of the three-dimensional part of the metric[1]. The spin-connection and the curvature forms belong to the antisymmetric **6** representation of SO(4). In four dimensions, this group of local frame rotations factorizes into a self-dual (sd) and an anti-self-dual

[1]In order to make a connection between instantonic solutions and geometric flows, we will require $\gamma$ to be symmetric as well. This is a further non-trivial restriction for non-unimodular Bianchi groups, and it is not necessary from the gravitational instantons point of view, so we will first present the classification of the solutions for generic $\gamma$.

(asd) part as $SO(3)_{\text{sd}} \otimes SO(3)_{\text{asd}}$, and the connection and curvature SO(4)-valued forms can be reduced with respect to the $SO(3)_{\text{(a)sd}}$ as $\mathbf{6} = (\mathbf{3}_{\text{sd}}, \mathbf{3}_{\text{asd}})$:

$$\begin{aligned} \mathrm{S}_\alpha &= \frac{1}{2}\left(\omega_{0\alpha} + \frac{1}{2}\epsilon_{\alpha\beta\gamma}\omega^{\beta\gamma}\right), \\ A_\alpha &= \frac{1}{2}\left(\omega_{0\alpha} - \frac{1}{2}\epsilon_{\alpha\beta\gamma}\omega^{\beta\gamma}\right), \end{aligned} \tag{2.31}$$

for the connection and

$$\begin{aligned} \mathcal{S}_\alpha &= \frac{1}{2}\left(W_{0\alpha} + \frac{1}{2}\epsilon_{\alpha\beta\gamma}W^{\beta\gamma}\right), \\ \mathcal{A}_\alpha &= \frac{1}{2}\left(W_{0\alpha} - \frac{1}{2}\epsilon_{\alpha\beta\gamma}W^{\beta\gamma}\right) \end{aligned} \tag{2.32}$$

for the curvature. Thus, $\{\mathrm{S}_\alpha, \mathcal{S}_\alpha\}$ are vectors of $SO(3)_{\text{sd}}$ and singlets of $SO(3)_{\text{asd}}$ and viceversa for $\{A_\alpha, \mathcal{A}_\alpha\}$. The curvature now reads

$$\begin{aligned} \mathcal{S}_\alpha &= d\mathrm{S}_\alpha + \epsilon_{\alpha\beta\gamma}\ \mathrm{S}^\beta \wedge \mathrm{S}^\gamma, \\ \mathcal{A}_\alpha &= dA_\alpha + \epsilon_{\alpha\beta\gamma}A^\beta \wedge A^\gamma. \end{aligned} \tag{2.33}$$

It is sufficient to impose $\mathcal{A}_\alpha$ (or similarly $\mathcal{S}_\alpha$) to vanish to impose (anti-)self-duality and thus have a solutions of Einstein's equations. For concreteness we focus on self-dual solutions

$$\mathcal{A}_\alpha = 0. \tag{2.34}$$

In order to write explicitly the latter conditions, consider the expansion of the Levi-Civita connection along the metric:

$$\begin{aligned} \omega_{0\alpha} &= \omega_{0\alpha i}\sigma^i, \\ \omega_{\alpha\beta} &= \omega_{\alpha\beta 0}dt + \omega_{\alpha\beta i}\sigma^i. \end{aligned} \tag{2.35}$$

By introducing

$$I_{\alpha i} = \omega_{0\alpha i} - \frac{1}{2}\epsilon_{\alpha\beta\gamma}\eta^{\beta\sigma}\eta^{\gamma\rho}\omega_{\sigma\rho i} \tag{2.36}$$

we can write

$$\mathcal{A}_\alpha = \left(I'_{\alpha i} - I_{\beta i}\eta^{\beta\gamma}\omega_{\gamma\alpha 0}\right) dt \wedge \sigma^\alpha + \frac{1}{2}\left(I_{\alpha i}c^i{}_{jk} + \epsilon_{\alpha\beta\gamma}\eta^{\beta\sigma}\eta^{\gamma\rho}I_{\sigma j}I_{\rho k}\right)\sigma^j \wedge \sigma^k = 0, \tag{2.37}$$

where the prime stands for $d/dt$. In order to trivially obtain first integrals from this equation, a necessary and sufficient condition is to make the gauge choice

$$\omega_{\alpha\beta 0} = 0, \tag{2.38}$$

after which (2.37) becomes equivalent to

$$I'_{\alpha i} = 0, \quad I_{\alpha i}c^i{}_{jk} + \epsilon_{\alpha\beta\gamma}\eta^{\beta\sigma}\eta^{\gamma\rho}I_{\sigma i}I_{\rho j} = 0. \tag{2.39}$$

A sufficient condition for (2.37) to be satisfied is $A_\alpha = 0$, but this is not of course a necessary condition. Requiring the self-duality of the curvature states that the anti-self-dual part of the connection must be a pure gauge field: $dA_\alpha + \epsilon_{\alpha\beta\gamma} A^\beta \wedge A^\gamma = 0$. This condition means that $A_\alpha$ can be set to zero by performing a local $SO(4)$ gauge rotation, and it is achieved when we set

$$A_\alpha = \frac{1}{2} I_{\alpha i} \sigma^i. \tag{2.40}$$

Thus, to find a gravitational instanton solution we need to solve the first order differential equation (2.40) and the constraint equation (2.39). It is useful to introduce

$$\Omega = \det\Gamma \left( \gamma n \gamma - \gamma m \gamma - \frac{\eta}{2} \mathrm{tr}\left(\gamma n \gamma \eta\right) \right). \tag{2.41}$$

In the latter, we dropped all the indices and $\Gamma^{ij}$ is the inverse of $\gamma_{ij}$, while $\eta$ stays for both $\eta^{ij}$ and $\eta_{ij}$, which are in fact equal. Then, by using the Ansatz for the metric we can see that (2.40) is equivalent to the evolution equation

$$\gamma' = -\Omega \eta \gamma - I \tag{2.42}$$

together with the constraint equation

$$[\gamma' \eta, \eta \Gamma] = 0. \tag{2.43}$$

By using (2.42), (2.43) can be written as

$$[\Omega \eta + I \Gamma, \gamma \eta] = 0. \tag{2.44}$$

## 2.6 Classification of the solutions

Both the evolution and the constraint equations depend explicitly on the first integral $I_{ij}$. In particular, the solutions to the constraint equation in (2.39), which is equivalent to (2.44), describe the homomorphisms from the algebra of the homogeneity group $g_3$ to the algebra of the holonomy subgroup $so(3)$:

$$I : \ g_3 \to so(3). \tag{2.45}$$

It is thus natural to classify the solutions according to the rank of $I$, which will lead to inequivalent self-dual metrics. Rank 3 solutions exist only for Bianchi VIII and IX classes by taking $I = \eta$. As for rank 1 solutions, (2.39) becomes

$$I(n+m) = \eta \mathrm{Adj}(I)^T \iff I(n+m) = 0 \iff (n-m) I^T = 0. \tag{2.46}$$

Here is the complete classification:

- rank 3(maximal): VIII, IX,

- rank 2: impossible,
- rank 1: I, II, III, IV, V, VI, VII,
- rank 0: all Bianchi types

There is still an additional constraint to be satisfied coming from (2.38), which is a necessary and sufficient condition for the evolution equations to exist. By using the Ansatz on the metric we can prove that this condition is equivalent to

$$A\left(\gamma'\eta\gamma\right) = 0, \tag{2.47}$$

where we introduced the notation $A(X) = \frac{1}{2}\left(X - X^T\right)$. By using the evolution equation (2.42) we find

$$A(I\gamma) = -2\gamma m\gamma. \tag{2.48}$$

Thus when $I$ is rank zero, we must have $m = 0 \iff a_\rho = 0$, which is the case for the unimodular classes of the Bianchi algebras, while it imposes non-trivial conditions on non-unimodular classes. The classification becomes then

- rank 3(maximal): VIII, IX,
- rank 2: impossible,
- rank 1: I, II, III, $\mathrm{VI}_0$, $\mathrm{VII}_0$,
- rank 0: I, II, $\mathrm{VI}_0$, $\mathrm{VII}_0$, VIII, IX,

where here we are using the subscript to indicate the value of the parameter $h$, as defined in table 2.3. One can also further restrict the classification by imposing that the determinant of the metric does not vanish. This is the case for all unimodular classes and, within the non-unimodular classes, for the Bianchi III group. We choose here to consider both degenerate and non-degenerate solutions, since the correspondence with the geometric flows holds for both situations. We should also point out that in all unimodular classes, the metric of the manifold can be taken to be diagonal without loss of generality.
In the following sections we will present the possible singularities occurring in these metrics, as well as two peculiar examples: Bianchi IX and Bianchi III.

## 2.7 Singularities: nuts and bolts

Let us consider a generic metric, when the homogeneity group is unimodular:

$$ds^2 = dt^2 + \sum_{i=1}^{3}\left(f_i(t)\sigma^i\right)^2. \tag{2.49}$$

The metric is regular when its coefficients $f_i$ are finite for finite times. There exists singularities which are just apparent, if they can be eliminated by an appropriate change of variables. In four-dimensions, there are two types of singularities, classified as nuts and bolts. Such singularities are related to the fixed points of Killing vector fields, and they are independent of the gravitational equations. The structure of the fixed point set of a Killing vector field $\chi_\mu$ acting on a four-dimensional Riemannian manifold is determined by the rank of the $4 \times 4$ matrix $\nabla_\mu \chi_\nu$. This is an anti-symmetric matrix, since its symmetric part vanishes identically by definition of Killing vector. It can have either rank 4 or 2, while rank 0 is excluded because it would imply the vector field to vanish everywhere. The nature of the singularity depends on the rank of the matrix.

- When the matrix $\nabla_\mu \chi_\nu$ has rank 4, there are no directions left invariant at the tangent space of the fixed point, which appears thus to be isolated and in is called a nut. The geometry near $t = 0$ is given by

  $$f_i^2 \approx t^2. \tag{2.50}$$

  Note that $t = 0$ corresponds to a coordinate singularity in the flat polar coordinate system. The singularity can be removed by changing the coordinate system to a local Cartesian system near $t = 0$ and adding the point $t = 0$ to the manifold. Near $t = 0$, the manifold is then topologically $\mathbb{R}^4$.

- When the matrix $\nabla_\mu \chi_\nu$ has rank 2, only a two-dimensional subspace of the tangent space at the fixed point remains invariant under the action of the Killing vector field, whereas the two-dimensional orthogonal complement rotates into itself. Then, the fixed point set is provided by this invariant two-dimensional subspace and it is called bolt. The geometry near $t = 0$ is given by

  $$\begin{aligned} f_1^2 &= f_2^2 = \text{finite}, \\ f_3^2 &= n^2 t^2, \quad n \in \mathbb{Z}. \end{aligned} \tag{2.51}$$

  Thus, the metric will be built out of the canonical $S^2$ metric $d\theta^2 + \sin^2\theta d\phi^2$ multiplied by the constant of $f_1 = f_2$, while at constant $(\theta, \phi)$ the two remaining terms look like

  $$dt^2 + n^2 t^2 d\psi^2. \tag{2.52}$$

  Provided the range of $\frac{n\psi}{2}$ is adjusted to $[0, 2\pi]$, the apparent singularity at $t = 0$ is nothing but a coordinate singularity in the flat polar coordinate system on $\mathbb{R}^2$. Again, this singularity can be removed by using Cartesian coordinates. The topology of the manifold is locally $\mathbb{R}^2 \times S^2$, with $\mathbb{R}^2$ shrinking to a point on $S^2$ at $t \to 0$.

When $t \to \infty$, we can distinguish three different situations: the Euclidian infinity, the Taubien infinity, and the conical infinity. They are respectively given by

$$\begin{aligned} f_i &\to \frac{\xi}{2},\, \xi \to \infty \; 0 \le \psi \le 4\pi, \\ f_1, f_2 &\to \xi,\, \xi \to \infty,\; c \to \text{const}, \\ f_i &\to \frac{\xi}{2},\, \xi \to \infty,\; 0 \le \psi \le 2\pi. \end{aligned} \tag{2.53}$$

## 2.8 Bianchi IX gravitational instantons

In this case, the space-time has an $SU(2)$ symmetry. As a further condition, we consider here the axisymmetric case, where the group of symmetry is enlarged to $SU(2) \times U(1)$. This assumption simplifies the system of equations we need to solve. From the discussion of the last section, we expect two non-equivalent solutions: one when $I$ has rank three, and one when it has rank zero. Explicitly, $I_{ij}$ can be put in a diagonal form where the elements can only take values 0 and 1. The trivial homomorphism case corresponds to entries 0, while the case of entries 1 will correspond to an isomorphism. In the literature, the two cases correspond respectively to the so called Eguchi-Hanson branch and Taub-NUT branch. We remember that the metric can be chosen to be diagonal without loss of generality. The Eguchi-Hanson branch corresponds to choosing

$$ds^2 = f^2(T)dT^2 + T^2\left(\left(\sigma^1\right)^2 + \left(\sigma^2\right)^2 + \left(g(T)\sigma^3\right)^2\right), \tag{2.54}$$

while the Taub-NUT branch corresponds to choosing

$$ds^2 = f^2(T)dT^2 + T^2 g^2(T)\left(\left(\sigma^1\right)^2 + \left(\sigma^2\right)^2 + \left(T\sigma^3\right)^2\right). \tag{2.55}$$

As an example, we present the properties of both metrics, and its derivation for the case of the Eguchi-Hanson branch.

### 2.8.1 Eguchi-Hanson branch

We start from (2.54) and compute the differential of the orthonormal basis:

$$\begin{aligned} d\theta^0 &= 0, \\ d\theta^1 &= \frac{1}{Tf}\theta^0 \wedge \theta^1 + \frac{1}{2Tg}\theta^2 \wedge \theta^3, \\ d\theta^2 &= \frac{1}{Tf}\theta^0 \wedge \theta^2 + \frac{1}{2Tg}\theta^1 \wedge \theta^3, \\ d\theta^3 &= \frac{1}{Tf}\theta^0 \wedge \theta^3 + \frac{g'}{gf}\theta^0\theta^3 + \frac{1}{2}\frac{g}{T}\theta^2 \wedge \theta^2. \end{aligned} \tag{2.56}$$

The components of the spin-connection are given by

$$\begin{aligned}
{\omega^1}_0 &= \frac{1}{Tf}\theta^1, & {\omega^2}_3 &= \frac{1}{4}\frac{g}{T}\theta^1, \\
{\omega^2}_0 &= \frac{1}{Tf}\theta^2, & {\omega^3}_1 &= \frac{1}{4}\frac{g}{T}\theta^2, \\
{\omega^3}_0 &= \left(\frac{1}{Tf} + \frac{g'}{fg}\right)\theta^3, & {\omega^1}_2 &= \frac{2-g^2}{4Tg}\theta^3.
\end{aligned} \tag{2.57}$$

Imposing self-duality, (2.42), leads thus to the equations

$$\begin{aligned}
fg &= 4, \\
g + g'T &= \frac{f}{4}\left(2 - g^2\right).
\end{aligned} \tag{2.58}$$

The solution is given by

$$g^2(T) = \frac{16}{f^2(T)} = \left(1 - \left(\frac{c}{T}\right)^4\right), \tag{2.59}$$

where c in an integration constant.
The Eguchi-Hanson metric is then given by

$$ds^2 = \frac{dT^2}{16\left[1 - \left(\frac{c}{T}\right)^4\right]} + T^2\left[\left(\sigma^1\right)^2 + \left(\sigma^2\right)^2 + \left(1 - \left(\frac{c}{T}\right)^4\right)^2\left(\sigma^3\right)^2\right]. \tag{2.60}$$

For $t \approx c$ the metric behaves as

$$ds^2 \approx \frac{1}{4}du^2 + \frac{1}{4}\left(d\theta + \cos\theta d\phi\right)^2 + \frac{c^2}{4}\left(d\theta^2 + \sin^2\theta d\phi\right), \tag{2.61}$$

where

$$u^2 = T^2\left[1 - \left(\frac{c}{T}\right)^4\right]. \tag{2.62}$$

For fixed $\theta$ and $\phi$, we obtain

$$ds^2 \approx \frac{1}{4}\left(du^2 + u^2 d\psi^2\right), \tag{2.63}$$

which is the $\mathbb{R}^2$ metric written in polar coordinates. Thus, the singularity at $u = 0$ must be a removable coordinate singularity. In particular, by choosing $\psi \in [0, 2\pi]$, we can see that it is a bolt type of singularity.

### 2.8.2 Taub-NUT branch

With computations along the same lines then those done for the Eguchi-Hanson branch, it can be proven that the Taub-NUT metric is given by

$$\begin{aligned} ds^2 = & \frac{4}{\left(1+\left[1-\left(\frac{a}{T}\right)^4\right]^{\frac{1}{2}}\right)} dT^2 \\ & + \frac{T^2}{4}\left(1+\left[1-\left(\frac{a}{T}^4\right)\right]^{\frac{1}{2}}\right)\left[\left(\sigma^1\right)^2+\left(\sigma^2\right)^2\right] + T^2\left(\sigma^3\right)^2 . \end{aligned} \tag{2.64}$$

By choosing a different time-parametrization, the metric can be put in the more familiar form

$$ds^2 = \frac{1}{4}\frac{r+m}{r-m}dt^2 + \left(r^2-m^2\right)\left[\left(\sigma^1\right)^2+\left(\sigma^2\right)^2\right] + 4m^2\frac{r-m}{r+m}\left(\sigma^3\right)^2 . \tag{2.65}$$

This metric is defined in the region $m < r < \infty$. At $r = m$ we do not have a singularity, but the metric changes sign. To remove the apparent singularity at $r = m$, we fist turn $r$ to the proper distance coordinate

$$d\tau^2 = \frac{1}{4}\left(\frac{r+m}{r-m}\right)dr^2 \tag{2.66}$$

and consider the region $r = m + \epsilon$, with $\epsilon \ll m$. Then

$$\tau = \int_m^{m+\epsilon} \frac{1}{2}\left(\frac{r+m}{r-m}^{\frac{1}{2}}\right)dr \approx (2m\epsilon)^{\frac{1}{2}} . \tag{2.67}$$

The metric near $\epsilon = 0$ is thus

$$\begin{aligned} ds^2 & \approx d\tau^2 + \frac{1}{4}\left(d\theta^2+\sin^2\theta d\phi^2\right) + \frac{1}{4}\tau^2\left(d\psi+\cos\theta d\phi\right)^2 \\ & \approx d\tau^2 + \tau^2\left(\left(\sigma^1\right)^2+\left(\sigma^2\right)^2+\left(\sigma^3\right)^2\right) . \end{aligned} \tag{2.68}$$

This is the condition for a removable nut singularity. As a curiosity, we mention that the name "Taub-NUT" fits both for the property of the metric and for the names of its discoverers, Taub-Newman-Unti-Tamburino.

## 2.9 Bianchi III gravitational instantons

In this case, studied in [4], the diagonal Ansatz for the metric is not valid anymore. Furthermore, non-degenerate self-dual solutions have rank 1 for the matrix $I$, and we can distinguish two cases, depending on the values of the matrix of frame components. Note that this is in contrast with the others non-unimodular groups, for which non-degenerate solutions do not exist.

- The first solution is given by

$$
\begin{aligned}
ds^2 = & \frac{F(t)}{32\, c_0^4\, \cosh^4\Big(t-t_*\Big)} dt^2 + \frac{\Big(K\,t-1\Big)\tanh\Big(t-t_*\Big)-K}{c_0^2}(\sigma^3)^2 \\
& + \frac{g_1(t)}{F(t)}(\sigma^1)^2 + \frac{g_2(t)}{F(t)}(\sigma^2)^2 + 2\,\frac{g_3(t)}{F(t)}\,\sigma^1\,\sigma^2,
\end{aligned}
\tag{2.69}
$$

where we have introduced the functions $F(t)$, $g_1(t)$, $g_2(t)$ and $g_3(t)$ given by

$$
\begin{aligned}
F(t) &= 8\,c_0^2\,\cosh\Big(t-t_*\Big)\Bigg[\Big(K\,t-1\Big)\sinh\Big(t-t_*\Big)-K\,\cosh\Big(t-t_*\Big)\Bigg],\\
g_1(t) &= (K^2+1)\cosh\Big(2(t-t_*)\Big)+2\,K\,\sinh\Big(2(t-t_*)\Big)\\
&\quad + 2K^2\,t^2+K^2-1,\\
g_2(t) &= (K^2+1)\cosh\Big(2(t-t_*)\Big)-2\,K\,\sinh\Big(2(t-t_*)\Big)\\
&\quad + 2K^2\,t^2-8\,K\,t+K^2+7,\\
g_3(t) &= (K^2-1)\cosh\Big(2(t-t_*)\Big)+2\,K^2\,t^2-4\,K\,t+K^2+1.
\end{aligned}
$$

Denoting the zeroes of $F(t)$ as $t_i$, we find that the Kretschmann scalar diverges as $(t-t_i)^6$ as $t\to t_i$, indicating a curvature singularity at each of the two zeros of $F(t)$.

- The second solution is given by

$$
\begin{aligned}
ds^2 \;=\; & \frac{L\,(B-1)}{4}\frac{\tanh(t)}{\cosh^2(t)}\,dt^2 + L\,(B-1)\,\tanh(t)\,\Big(\sigma^3\Big)^2 \\
& + \frac{L}{4\,(B-1)}\Bigg\{\Big[2\,B^2\,\mathrm{csch}(2\,t)+\tanh(t)\Big]\Big(\sigma^1\Big)^2 + \\
& -2\,[\tanh(t)-2\,B\,(B-2)\mathrm{csch}(2\,t)]\,\sigma^1\,\sigma^2 \\
& \Big[\mathrm{csch}(2\,t)(\cosh(2\,t)+2\,(B-2)^2-1\Big]\Big(\sigma^2\Big)^2\Bigg\}.
\end{aligned}
\tag{2.70}
$$

In this case, the Kretschmann scalar is given by

$$
\mathcal{K} = \frac{384\,\coth^6(t)}{(B-1)^2L^2}
\tag{2.71}
$$

and we find at $t=0$ the metric has a curvature singularity.

## 2.10 Geometric flows

A geometric flow or geometric evolution equation is a non-linear equation that describes the deformation of a metric on Riemannian manifolds driven by their curvature in various forms. The evolution of the metric $g$ with respect to a continuous parameter $t$ is irreversible and driven by a symmetric tensor:

$$\frac{d}{dt}g_{ij} = -S_{ij}. \tag{2.72}$$

Geometric flows are naturally divided into intrinsic and extrinsic classes, depending on wether they are driven by intrinsic or extrinsic tensors of the metric. Here we will concentrate on intrinsic geometric flows.
The first and probably most known example of a geometric flow is the Ricci flow

$$\frac{d}{dt}g_{ij} = -2R_{ij}, \tag{2.73}$$

where $R_{ij}$ is the Ricci tensor and the 2 is a conventional factor that can be absorbed by rescaling $t$.
The Ricci flow was first introduced in 1982 by Hamilton as the pillar of a program designed to prove Poincaré (1904) and Thurston (1975) conjectures. This program culminated in 2002-2003 with the actual proof by Perelman. Independently and around the same time, it was also observed in physics in 1996 by Friedan, in his work studying the weak-coupling limit of the renormalization group flow for non-linear $\sigma$-models.
The Ricci flow equation resembles a heat equation: as the heat equation tends to make uniform a given temperature distribution, the Ricci flow evolves an initial metric into a homogeneous one. The short time existence of the solution is guaranteed by the parabolic form of the dynamics: that is, the minus sign on the right side of the equation ensures that the Ricci flow is well defined for sufficiently small positive times. On the contrary, there might be singularities arising at finite time, as in the case of compact manifolds with strictly positive curvature metrics. In this case, the space collapses to a point in finite time. In other words the Ricci flow does not preserve the volume. This is why it is useful to introduce a normalized version of the flow

$$\frac{d}{dt}g_{ij} = -2R_{ij} + \frac{2}{n}rg_{ij}, \tag{2.74}$$

where $n$ is the dimension of the manifold and $r$ is the average scalar curvature defined by $r = \frac{\int RdV}{\int dV}$. While the two definitions of Ricci flows are related just by a suitable time reparametrization and a rescaling of the metric by a function of time, the normalized version of the flow has better chances to admit solutions that exist for sufficiently long time.
Of course, various extensions of the Ricci flow can be considered, by adding other terms to the right side of the equation. As we will see, depending on the Bianchi group we will consider it will be indeed necessary to generalize the Ricci flow in appropriate ways. In order to make contact with the evolution equation of

the gravitational instanton, the geometric flow will not be defined on the three-dimensional homogeneous space, but rather on its “square root”, which we already defined in (2.30). This means that the flowing metric is

$$ds^2 = \gamma_{ij}\sigma^i\sigma^j, \tag{2.75}$$

rather then $g$. As an immediate consequence, we have to impose $\gamma$ to be symmetric. In the next section we will study the consequences of this additional requirement in the classification of the self-dual solutions. Then, we will set the relation between the geometric flow and the instantons, and show as explicit examples how the correspondence work for the Bianchi IX and III groups.

## 2.11 Classification of the solutions with symmetric $\gamma$.

Requiring $\gamma$ to be symmetric affects not only the constraint equation (2.44), but also the dynamics of the gravitational instanton, since we want the metric to stay symmetric during the evolution. The condition for $\gamma'$ to be symmetric is indeed not automatically satisfied and, thanks to (2.42), it is translated into

$$A\left(\Omega\eta\gamma + I\right) = 0, \tag{2.76}$$

where A stands for the anti-symmetrization operation we already introduced. Thus now the evolution of the instanton, which is governed by (2.42), is subject to two independent constraint: (2.76) and (2.44). Their combination gives

$$\gamma m\gamma = A\left(IM\eta\right), \tag{2.77}$$

where $M = \mathrm{Adj}(\gamma)$ is the adjoint matrix of $\gamma$, i.e. the matrix of the 2x2 subdeterminants of $\gamma$.
While for unimodular groups the constraints on the symmetry of $\gamma$ and of its derivative are automatically satisfied, for non-unimodular groups they do restrict the possible solutions. In particular, starting from (2.46) and using (2.76) and (2.77) we can prove that the rank-1 matrix $I$ has to be symmetric as well:

$$I = I^T. \tag{2.78}$$

Moreover, we can use the constraint (2.77) to prove that for symmetric $\gamma$ the evolution equation (2.42) can be expressed as[2]

$$\gamma' = -\det\Gamma\left(\gamma n\gamma n\gamma - \frac{1}{2}\left(\gamma m\gamma n\gamma - \gamma n\gamma m\gamma\right) - \frac{\gamma}{2}\mathrm{tr}\left(\gamma n\right)^2\right) - I. \tag{2.79}$$

[2]For IV and V Bianchi classes we need to use also the other constraint (2.76) in order to achieve this form of the equation.

Table 2.4: Basis of invariant forms and restrictions on $\gamma$ – unimodular groups

| Type | $n, \eta, a = 0$ | Restrictions from (2.76) | Restrictions from (2.44) |
|---|---|---|---|
| I | $n = 0 \quad \eta = \mathbb{1}$ | none | none rank-1 |
| II | $n = \mathrm{diag}(0,1,0)$<br>$\eta = \mathbb{1}$ | $\gamma_{12} = 0 = \gamma_{23}$ | $\gamma_{12} = 0 = \gamma_{23}$: rank-0<br>$\gamma_{13} = 0$ and/or $\gamma_{11} = \gamma_{33}$: rank-1 |
| VIII | $n = \eta = \mathrm{diag}(1,-1,-1)$ | none | none rank-0<br>none rank-3: $I = \eta$ |
| IX | $n = \eta = \mathbb{1}$ | none | none rank-0<br>none rank-3: $I = \eta$ |
| $\mathrm{VI}_{-1}$ | $n = \mathrm{diag}(0,1,-1)$<br>$\eta = \mathrm{diag}(-1,1,-1)$ | $\gamma_{12} = 0 = \gamma_{13}$ | $\gamma_{12} = 0 = \gamma_{13}$<br>rank-0,1 |
| $\mathrm{VII}_0$ | $n = \mathrm{diag}(1,1,0)$<br>$\eta = \mathrm{diag}(1,1,1)$ | $\gamma_{12} = 0 = \gamma_{23}$ | $\gamma_{13} = 0 = \gamma_{23}$<br>rank-0,1 |

This equation is valid for symmetric metrics $\gamma$, with $I$ being a symmetric rank-1 or rank-3 matrix. Of course in the latter case we have $I = \eta$ and the only algebras we can consider are Bianchi VIII and IX. At this stage we still need to impose the second constraint (2.76). This condition does not affect the Bianchi VIII and IX classes. All other classes are restricted by the condition. In particular, considering non-unimodular groups and choosing $I$ to be rank-0, the metric $\gamma$ must have vanishing determinant. The same holds for rank-1, except for the Bianchi III class, which admits non-singular self-dual gravitational instantons. We display the restrictions on the form of the metric for both unimodular and non-unimodular classes in the tables (2.4) and (2.5).

## 2.12 Correspondence between geometric flows and gravitational instantons

The last step of our analysis is to interpret the evolution of the gravitational instanton as a geometric flow. We start with the expression for the Ricci tensor on a generic homogeneous three-dimensional metric equipped with a metric $\gamma$

$$R[\gamma] = N + \det\Gamma\left(\gamma n\gamma n\gamma - \gamma m\gamma n\gamma + \gamma n\gamma m\gamma - \frac{\gamma}{2}\mathrm{tr}\,(\gamma n)^2\right) + a\otimes a - 2\gamma a\Gamma a, \quad (2.80)$$

where N is the Cartan-Killing metric of the Bianchi algebra we are considering:

$$N_{ij} = -\frac{1}{2}c^l{}_{ki}c^k{}_{lj} = -\frac{1}{2}\epsilon_{lim}\epsilon_{kjn}n^{mk}n^{nl} - a_i a_j. \quad (2.81)$$

Table 2.5: Basis of invariant forms and restrictions on $\gamma$ – non-unimodular groups

| Type | $n, \eta, a$ | Restrictions from (2.76) | Restrictions from (2.44) |
|---|---|---|---|
| III | $n = \mathrm{diag}(0,1,-1) \quad a = (1,0,0)$<br>$\eta = \mathrm{diag}(-1,1,-1)$ | $\gamma_{12} = \gamma_{13}$<br>$\gamma_{22} = \gamma_{33}$ | $\gamma_{11}, \gamma_{13}$ given in (2.102) |
| IV | $n = \mathrm{diag}(0,1,0)$<br>$\eta = \mathbb{1} \quad a = (1,0,0)$ | $\gamma_{12} = 0 = \gamma_{13}$,<br>$\gamma_{23} = \gamma_{22} + \gamma_{33}$ | singular from (2.44)<br>$\gamma_{22} = 0 = \gamma_{33}$ |
| V | $n = 0 \quad \eta = \mathbb{1}$<br>$a = (1,0,0)$ | $\gamma_{12} = 0 = \gamma_{13}$<br>$\gamma_{22} = -\gamma_{33}$ | singular from (2.44)<br>$\gamma_{22} = 0 = \gamma_{23}$ |
| $\mathrm{VI}_{h>-1}$ | $n = \mathrm{diag}(0,1,-1) \quad a = (h+1,0,0)$<br>$\eta = \mathrm{diag}(-1,1,-1)$ | $\gamma_{12} = 0 = \gamma_{13}$<br>$\gamma_{22} = \gamma_{33}$ | singular from (2.44)<br>$\gamma_{22} = \gamma_{23}$ |
| $\mathrm{VII}_{h>0}$ | $n = \mathrm{diag}(1,1,0) \quad \eta = \mathbb{1}$<br>$a = (0,0,h)$ | $\gamma_{13} = 0 = \gamma_{23}$<br>$\gamma_{11} = -\gamma_{22}$ | singular from (2.44)<br>$\gamma_{11} = 0 = \gamma_{12}$ |

### 2.12.1 Unimodular groups

For this class we have $a = 0$. The geometric flow corresponding to the evolution of the gravitational instanton is given by a Ricci flow combined with a Yang-Mills flow produced by a flat, non flowing Yang-Mills connection on $\mathcal{M}_3$. The appropriate connection to consider is SO(2,1) for Bianchi VI or VIII, and SO(3) otherwise, since it reflects the four-dimensional anti-self-dual part of Levi-Civita connection on $\mathcal{M}_4$, appearing as the first integral (2.40). The Yang-Mills connection is defined by

$$\alpha \equiv \alpha_i \sigma^i = -\lambda_{ij} \sigma^i T^j, \tag{2.82}$$

where $T^i = \eta^{ij} T_j$ are the generators of the SO(3) or SO(2,1) group:

$$[T_i, T_j] = -\epsilon_{ijk} T^k, \tag{2.83}$$

with normalization $\mathrm{tr}\,(T_i T_j) = -2\eta_{ij}$. The absence of flow for the connection, $\alpha' = 0$, is translated into $\lambda'_{ij} = 0$, while the flatness condition requires

$$F = d\alpha + [\alpha, \alpha] \equiv 0 \iff \lambda_{il} c^l{}_{jk} + \epsilon_{ijk} \eta^{jm} \eta^{kn} \lambda_{mj} \lambda_{nk} = 0. \tag{2.84}$$

The required term to achieve the correspondence between instantons and flows is given by

$$-\frac{1}{2} \mathrm{tr}\,(\alpha_i \alpha_j) = \eta^{kl} \lambda_{ki} \lambda_{lj}, \tag{2.85}$$

provided that the following relationship holds between the Levi-Civita connection $I_{ij}$ and the Yang-Mills connection $\lambda_{ij}$

$$N - I = \lambda^T \eta \lambda. \tag{2.86}$$

For unimodular groups, (2.84) is equivalent to

$$\lambda n = n \lambda^T = 0, \tag{2.87}$$

which implies that $\lambda$ should be symmetric. No matter the rank of I, it is always possible to find $\lambda$ such that is satisfies (2.86), which reads

$$\gamma' = -R[\gamma] - \frac{1}{2}\mathrm{tr}(\alpha \otimes \alpha). \tag{2.88}$$

In particular, we consider as an example Bianchi IX. In this case, we have a rank-0 solution $I = 0$ corresponding to the condition

$$\lambda = \eta, \tag{2.89}$$

and a rank-3 solution $I = \eta$, where $\lambda = 0$. We recall that the two solutions correspond, respectively, to the Eguchi-Hanson and the Taub-NUT branches. In particular, in the latter case the evolution of the instantons is a pure Ricci flow, without the need to add the Yang-Mills flow, that is $\alpha = 0$.

### 2.12.2 Non-unimodular groups

By proceeding along the same lines, we now analyze non-unimodular groups. Again we need to satisfy (2.86). For any rank-0 and rank-1 $I$ it is always possible to find a solution for the matching condition, since (2.39) and (2.84) become respectively

$$I(n+m) = (n-m)I = 0 \tag{2.90}$$

and

$$\lambda(n+m) = (n-m)\lambda^T = 0, \tag{2.91}$$

while Jacobi identities (2.17) imply that

$$N(n+m) = (n-m)N = 0. \tag{2.92}$$

More terms appear when trying to write the evolution of the instantons in terms of a geometric flow. In order to interpret these new terms, we compute the symmetric part of covariant derivative of $a$:

$$S(\nabla a) = \frac{\det\Gamma}{2}(\gamma n\gamma m\gamma - \gamma m\gamma n\gamma) + a\otimes a - \gamma a\Gamma a. \tag{2.93}$$

With this expression at hand, one can see that the evolution of the gravitational instanton is given by

$$\gamma' = -R[\gamma] + S(\nabla a) - \gamma a\Gamma a - \frac{1}{2}\mathrm{tr}(A\otimes A). \tag{2.94}$$

This describes a geometric flow driven by the Ricci tensor, combined with a Yang-Mills connection as well as a diffeomorphism generated by $a$ and by an invariant component of the scalar curvature, which indeed reads

$$S[\gamma] = \mathrm{tr}(\Gamma R[\gamma]) = \mathrm{tr}(\Gamma N) - \frac{det\Gamma}{2}\mathrm{tr}(\gamma n)^2 - 5a\Gamma a. \tag{2.95}$$

As stated, all non-unimodular Bianchi metrics but Bianchi III are degenerate due to the condition (2.44). Nevertheless, the correspondence between geometric flows and homogeneous three-dimensional metrics holds for all cases when $\gamma$ is symmetric. This of course restricts the form of the metric, but such restrictions are coming from the consistency of the instantons. We list such conditions for all Bianchi groups in tables 2.4 and 2.5. As a last step, it is interesting to consider explicitly the Bianchi III case.

### 2.12.3 Bianchi III gravitational instantons

For all non-unimodular Bianchi groups but Bianchi III we can prove that the constraint (2.44) implies

$$\gamma m \gamma = 0, \tag{2.96}$$

which eventually implies $\det \gamma = 0$. By looking at (2.77), this is trivial if $I$ is rank-0. If $I$ is rank-1, for all non unimodular groups but Bianchi III the generic solution for symmetric $I$ is

$$I_{ij} = \kappa a_i a_j, \tag{2.97}$$

where $\kappa$ is an arbitrary constant. With this form of the first integral, one can prove that the determinant has to vanish. This is not the case for Bianchi III, for which $I$ has to be of the form

$$I = \begin{pmatrix} \mu & \chi & \chi \\ \chi & -\nu & -\nu \\ \chi & -\nu & -\nu \end{pmatrix}, \qquad \chi^2 + \mu\nu = 0. \tag{2.98}$$

The generic form of $\lambda$ satisfying the flatness condition (2.84) is given by

$$\lambda = \begin{pmatrix} \rho_1\lambda & \rho_1\varsigma & \rho_1\varsigma \\ \rho_2\lambda & \rho_2\varsigma & \rho_2\varsigma \\ \rho_3\lambda & \rho_3\varsigma & \rho_3\varsigma \end{pmatrix}, \tag{2.99}$$

while requiring the matching condition (2.86) to be satisfied translates into

$$(\rho_1\lambda)^2 = 2 + \mu, \quad \left(\rho_1^2 + \rho_2^2 + \rho_3^2\right)\lambda\varsigma = \chi, \quad \left(\rho_1^2 + \rho_2^2 + \rho_3^2\right)\varsigma^2 = -\nu. \tag{2.100}$$

Demanding constraint (2.44) to be satisfied restricts the form of the metric:

$$\gamma_{11} = \frac{\chi^2 + 2\nu}{2\nu^2}\left(\gamma_{23} + \gamma_{33}\right), \quad \gamma_{13} = -\frac{\chi}{2\nu}\left(\gamma_{23} + \gamma_{33}\right). \tag{2.101}$$

The general solution to the evolution equation is thus given by

$$
\begin{aligned}
\gamma_{11}(t) &= (1 - \frac{\mu}{2})\ t + \gamma_{11}(0), \\
\gamma_{13}(t) &= \gamma_{31}(t) = -\frac{\chi}{2}\ t + \gamma_{13}(0), \\
\gamma_{33}(t) &= \gamma_{33}(0) \left(1 + \frac{\nu t}{\gamma_{33}(0) + \gamma_{23}(0)}\right), \\
\gamma_{23}(t) &= \gamma_{32}(t) = \gamma_{23}(0) \left(1 + \frac{\nu t}{\gamma_{33}(0) + \gamma_{23}(0)}\right),
\end{aligned}
\tag{2.102}
$$

where the initial condition are constrained by (2.101) and the constraint (2.76) guarantees that they will be valid for any time. These solutions are the restricted version of those analyzed in section 2.9 for the case of $\gamma$ symmetric, and they still exhibit naked singularities as in the previous case.

# Chapter 3

# Ungauging extremal black holes in four-dimensional $\mathcal{N} = 2$ supergravity

After a brief introduction to $\mathcal{N} = 2$ $D = 4$ supergravity, we devote the chapter to the study of an ungauging procedure to relate asymptotically AdS and asymptotically flat black hole solutions. We analyze in detail the horizon and global properties of the solutions, as well as the connection with existing solutions and the possibility of using our results and constructions to simplify the asymptotically AdS solutions.

## 3.1 Four-dimensional $\mathcal{N} = 2$ supergravity

### 3.1.1 Introduction

Supergravity theories are related among each others via dualities, compactifications or reductions. Their interconnection and intrinsic similarity mean that studying one particular theory can lead to a better understanding of other related theories. In this sense, among the other 4-dimensional theories the $\mathcal{N} = 2$ one is best-suited for making relations with others. They do not have so much symmetry to only allow for very restricted classes, but they possess enough symmetry to be still mathematically tractable. They possess eight conserved supercharges in four-dimensions, and present the feature of electric-magnetic duality, which we will introduce.

### 3.1.2 Bosonic abelian gauged Lagrangian

We start directly by presenting the bosonic part of the Lagrangian for abelian $U(1)$ gauged $\mathcal{N} = 2$ $D = 4$ supergravity coupled to $n_V$ vector multiplets, first studied in

[37] and [38], which reads:

$$\mathcal{L}_g = R - 2g_{i\bar{j}}\partial_\mu z^i \partial^\mu \bar{z}^{\bar{i}} - \frac{1}{2}F^I \wedge G_I + 2V. \tag{3.1}$$

We do not display here the fermionic part of the Lagrangian, since in the following we are always setting the fermionic fields to zero. The Ricci scalar is $R$, while $z^i$, $i = 1, \ldots n_V$ are the vector multiplet complex scalars. The abelian gauge fields are $F^I$ and its dual $G_I$, while $V$ is the scalar potential. We will now proceed to a more detailed analysis of each term.

The physical scalars $z^i$ parametrize a special Kähler geometry of complex dimension $n_V$, whose metric is given by $g_{i\bar{j}}$, and they naturally appear through the symplectic section $\mathcal{V}$. Choosing as a basis

$$z^i = \frac{X^i}{X^0}, \tag{3.2}$$

the section can be written in components in terms of the scalars $X^I = X^0, X^i$ as

$$\mathcal{V} = \begin{pmatrix} X^I \\ F_I \end{pmatrix} \qquad F_I = \frac{\partial F}{\partial X^I}, \tag{3.3}$$

where F is a holomorphic function of degree two called the prepotential. Note here the different uses of the indices: small indices $i$ run through the vector multiplets, while capital indices $I = \{0, i\}$ run through both the gravity and the vector multiplets. Note also that the physical scalars are defined in terms of the $X^I$ up to a local $U(1)$ transformation. In the following, we will always consider the prepotential to be cubic

$$F = -\frac{1}{6}c_{ijk}\frac{X^i X^j X^k}{X^0}, \tag{3.4}$$

where $c_{ijk}$ is a completely symmetric tensor. The section $\mathcal{V}$ is subject to the constraints

$$\langle \bar{\mathcal{V}}, \mathcal{V} \rangle = i, \qquad \langle \bar{D}_{\bar{i}}\bar{\mathcal{V}}, D_j \mathcal{V} \rangle = -i g_{\bar{i}j}, \tag{3.5}$$

where $D$ is the Kähler covariant derivative and the brackets denote the symplectic scalar product

$$\langle A, B \rangle = A^T \Omega B = A_I B^I - A^I B_I, \tag{3.6}$$

where $\Omega$ is the $Sp(2n_V + 2)$ matrix.

In the third term of the Lagrangian we find $F^I{}_{\mu\nu}$ and $G_{I\mu\nu}$, which are respectively the abelian gauge fields and their duals:

$$G^-_{\mu\nu I} := -\frac{16\pi i}{e}\frac{\partial \mathcal{L}}{\partial F^{-I}{}_{\mu\nu}} = \mathcal{N}_{IJ}F^{-J}{}_{\mu\nu}, \tag{3.7}$$

where $e$ is a constant and $\mathcal{N}_{IJ}$ is the scalar dependent periodic matrix, whose expression is not explicitly needed here. The gauge fields naturally arrange themselves into a symplectic vector of electric and magnetic gauge field strengths:

$$\mathcal{F}_{\mu\nu} = \begin{pmatrix} F^I{}_{\mu\nu} \\ G_{I\mu\nu} \end{pmatrix}. \tag{3.8}$$

Its integral over a sphere near infinity defines the associated electric and magnetic charges

$$\Gamma = \begin{pmatrix} p^I \\ q_I \end{pmatrix} = \frac{1}{2\pi}\int_{S^2} \mathcal{F}. \tag{3.9}$$

Thanks to the Bianchi identities, which take the form

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma}\partial_\mu F^I{}_{\nu\rho} &= 0, \\ \epsilon^{\mu\nu\rho\sigma}\partial_\mu G_{I\nu\rho} &= 0, \end{aligned} \tag{3.10}$$

both the charges $p^I$ and $q_I$ are conserved.
Before giving the explicit form of the last term in the Lagrangian, the scalar potential $V$, we need to introduce the Fayet-Iliopoulos (FI) parameters. They are arbitrary constant gauging parameters, and they organize themselves into the symplectic vector $G = \{g^I, g_I\}$. In general, they control the couplings of the vector fields. Since we are considering just the case where the class of gauging is abelian, these couplings occur only in the fermionic sector of the theory, in particular between the gravitino and the vector fields. Thus, the pure bosonic Lagrangian is affected by the presence of these gaugings just through the scalar potential of the theory $V$. We introduced both the electric and magnetic gaugings, but the electric-magnetic duality group will always allow to rotate $G$ in a frame where only electric gaugings are turned on, i.e. $g^I = 0$. This implies a rotation of the symplectic section and thus the choice of a preferred basis for the physical scalars $t^i$. A full $\mathcal{N}=2$ duality covariant Lagrangian has not yet been written explicitly, but as we said the bosonic part of the theory is affected only through the potential $V$, which can be written in a simple manifest covariant form. We thus decide here to consider the covariant version of the electrically gauged theory, keeping in mind that while all the expressions for bosonic backgrounds must be covariant under the electric-magnetic duality, the equation involving fermionic quantities strictly apply only to the electrically gauged theory.
We are now ready to introduce the expression for the scalar potential $V$:

$$V = Z_i(G)\bar{Z}^i(G) - 3\,|Z(G)|^2\,, \tag{3.11}$$

where $Z(G)$ and $Z_i(G)$ are scalar dependent central charges, defined respectively in terms of a generic symplectic vector A as

$$Z(A) = \langle A, \mathcal{V}\rangle \tag{3.12}$$

and

$$Z_i(A) = \langle A, D_i\mathcal{V}\rangle. \tag{3.13}$$

## 3.2 Connection between gauged and ungauged bosonic Lagrangian

As noticed, the bosonic part of the Lagrangian is modified by the presence of the gaugings just by the introduction of the scalar potential term $V$. This means that the gauged Lagrangian $\mathcal{L}_g$ can be easily written in terms of the ungauged one $\mathcal{L}_0$ as

$$\mathcal{L}_g = \mathcal{L}_0 + V. \tag{3.14}$$

This expression implies that starting from a gauged theory we can find the ungauged one just by setting the potential to zero, as far as only the bosonic sector is concerned. Thus, the ungauged theory is reached when the FI terms vanish, but, since the potential is not positive-definite, this is not the only possibility. Indeed, we can choose the FI parameters in such a way that the scalar potential is identically zero, but the theory is not reduced to an ungauged one. In this case the bosonic part of the Lagrangian for the two theories will be the same, $\mathcal{L}_g = \mathcal{L}_0$, but the FI terms will still still appear in the fermionic sector of the gauged theory. Thus, the supersymmetric vacua of the two theories will not coincide because of the different fermionic sector of the two theories.

### 3.2.1 Motivations

Such a connection between gauged and ungauged theories, and thus between asymptotically flat and asymptotically AdS solutions, is interesting because it can bring new insights especially in the gauged theory, for which in general there is a less broad knowledge. For example, most of the investigations on the microscopic description of black holes in supergravity have been carried out for asymptotically flat black holes preserving some amount of supersymmetry, which provides additional control over various aspects of these systems. The asymptotically flat and the asymptotically AdS classes of black holes are unrelated, as they arise as solutions to different supergravity theories. Consequently, the two theories are generally studied with different methods, and the problem of entropy counting is not an exception in this respect. The connection between gauged and ungauged supergravity shows that this is not always the case, and can thus bring new insights on the microscopic description of asymptotically AdS black holes.
The classification of asymptotically flat black holes includes all BPS solutions, that is solutions that remain invariant under some of the supersymmetry transformations of the theory, and, beyond the supersymmetric sector, the static and the stationary under-rotating cases are known. While the latter are more complicated than the corresponding BPS ones, they all share the feature of being described by first order differential equations. Asymptotically AdS black hole solutions are known mostly within the BPS sector, while only few examples of non-BPS solutions are known. Finding a connection between asymptotically AdS and asymptotically

flat solutions is thus interesting also in the perspective of searching for new asymptotically AdS solutions.

### 3.2.2 Very small vector

We introduce here a preliminary concept which is essential in the ungauging procedure. Within a theory with cubic prepotential (3.4), we define $R$ to be a "very small vector", as shown in [8], if it satisfies the relation

$$Z_i(R)\bar{Z}^i(R) = 3\,|Z(R)|^2\,. \tag{3.15}$$

This is just the expression we have if we demand the scalar potential to vanish in a non-trivial way, i.e. without asking the FI gaugings themselves to vanish. Thus, the connection between gauged and ungauged theories is achieved when the FI terms are described by a very small vector. By choosing the preferred duality frame in an appropriate way, we can always set the very small vector to have only one component. For example, $G$ can be rotated in such a way that the only component is the purely electric one $q_0$:

$$G = g\{0, \delta^0{}_I\}, \tag{3.16}$$

where $g$ is a constant. Anyway, we choose here to keep the vector generic.
An alternative equivalent definition of very small vector can be given by introducing a quartic form $I_4$, which is invariant under symplectic transformations:

$$I_4(R) = \frac{1}{4!}t^{MNPQ}R_M R_N R_P R_Q, \tag{3.17}$$

where $t^{MNPQ}$ is a completely symmetric tensor. The expression for the quartic invariant for the charge vector $\Gamma$ introduced in (3.9) is given by

$$I_4(\Gamma) = -\left(p^0 q_0 + p^i q_i\right)^2 + \frac{2}{3}q_0 c_{ijk}p^i p^j p^k - \frac{2}{3}p^0 c^{ijk}q_i q_j q_k + c_{ijk}p^j p^k c^{ilm}q_l q_m. \tag{3.18}$$

The absolute value of this expression determines the entropy of static black holes for any values of the charges (see for example the review [41]).
A small vector is defined by the relations

$$I_4(R) = \frac{I_4(R)}{dR} = 0, \tag{3.19}$$

while a very small vector satisfies the additional constraint

$$\frac{1}{4}I_4(R, R, Z, Z) \equiv \frac{1}{4}t^{MNPQ}R_M R_N Z_P Z_Q = -\langle R, Z\rangle^2 \tag{3.20}$$

for any vector Z. One can prove that those conditions are equivalent to (3.15).

## 3.3 Squaring of the action

Considering first asymptotically AdS solutions, the squaring procedure presented in [7] shows that it is possible to write the action (3.1) as a sum of squares of first-order differential equations and, as a consequence, a sufficient condition to obtain a solution of the equations of motion is to impose that each of the squares vanish. We consider generic asymptotically AdS static black hole configurations, whose metric ansatz is given by

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}\left(dr^2 + e^{2\psi(r)}d\Omega^2\right), \tag{3.21}$$

where $e^U$ and $e^\psi$ are two scalar functions describing the scale factor of the metric and of the three-dimensional base space.
By plugging this ansatz in the action (3.1) we obtain an effective one-dimensional theory for the scalar fields and the warp factors $U$ and $\psi$:

$$\begin{aligned} S_{1d} = \int dr \Big[e^{2\psi}\Big(\left(U' - \psi'\right)^2 + 2\psi'^2 + g_{i\bar{j}}z^i\bar{z}^{\bar{j}\prime} + e^{2U-4\psi}V_{BH} + e^{-2U}V \\ +2\psi'' - U''\big) - 1\Big], \end{aligned} \tag{3.22}$$

where the prime denotes derivative with respect to the radial coordinate and the black hole potential is given by

$$V_{BH} = Z_i(\Gamma)\bar{Z}^i(G) + |Z(\Gamma)|^2. \tag{3.23}$$

After integration by parts of (3.22), we arrive to the expression

$$\begin{aligned} S_{1d} = \int dr \left[e^{2\psi}\left(U'^2 - \psi'^2 + g_{i\bar{j}}z^i\bar{z}^{\bar{j}\prime} + e^{2U-4\psi}V_{BH} + e^{-2U}V\right) - 1\right] \\ + \int dr \frac{d}{dr}\left[2^{2\psi}\left(2\psi' - U'\right)\right], \end{aligned} \tag{3.24}$$

where the last line is just a total derivative which can be discarded. It is useful to rewrite the black hole potential (3.23) as

$$V_{BH} = -\frac{1}{2}\Gamma^T\mathcal{M}\Gamma, \tag{3.25}$$

where

$$\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.26}$$

is a symplectic matrix defined by the entries

$$\begin{aligned} A &= \mathcal{N} + \mathrm{Re}\mathcal{N}\left(\mathrm{Im}\mathcal{N}\right)^{-1}\mathrm{Re}\mathcal{N}, \\ D &= \left(\mathrm{Im}\mathcal{N}\right)^{-1}, \\ B &= C^T = \mathrm{Re}\mathcal{N}\left(\mathrm{Im}\mathcal{N}\right)^{-1}. \end{aligned} \tag{3.27}$$

In order to write the action as a sum of squares, we need to introduce various special geometry identities. A basic expression is given by

$$\frac{1}{2}\left(\mathcal{M}-i\Omega\right)=\Omega\bar{\mathcal{V}}\mathcal{V}\Omega+\Omega U_i g^{i\bar{j}}\bar{U}_{\bar{j}}\Omega, \tag{3.28}$$

which leads to

$$\mathcal{M}\mathcal{V}=i\Omega\mathcal{V} \tag{3.29}$$

and

$$\mathcal{M}U_i=-i\Omega U_i. \tag{3.30}$$

From the latter, it follows that

$$\bar{\mathcal{V}}^T\mathcal{M}\mathcal{V}=i\langle\bar{\mathcal{V}},\mathcal{V}\rangle=-1 \tag{3.31}$$

and

$$U_i^T\mathcal{M}\bar{U}_{\bar{j}}=i\langle U_i,\bar{U}_{\bar{j}}\rangle=-g_{i\bar{j}}. \tag{3.32}$$

The first step is to rewrite the kinetic term for the scalar fields and the scalar potentials $V$ and $V_{BH}$ in terms of symplectic sections using

$$-\mathcal{V}'^T\mathcal{M}\bar{\mathcal{V}}'=g_{i\bar{j}}z^{i\prime}\bar{z}^{\bar{j}\prime}+Q_r^2, \tag{3.33}$$

where

$$Q_r\equiv\frac{i}{2}\left(\bar{z}^{\bar{j}\prime}\bar{\partial}_{\bar{j}}K-z^{i\prime}\partial_iK\right) \tag{3.34}$$

is a composite connection, and $K$ is the Kähler potential.
Given the properties of the symplectic sections, we can also introduce a phase factor so that

$$-\operatorname{Im}(e^{i\alpha}\mathcal{V}'^T)\mathcal{M}\operatorname{Im}(e^{i\alpha}\mathcal{V}')=\frac{1}{2}g_{i\bar{j}}z^{i\prime}\bar{z}^{\bar{j}\prime}+\frac{1}{2}Q_r^2\ , \tag{3.35}$$

and once more obtain new identities:

$$\operatorname{Re}(e^{i\alpha}\mathcal{V})^T\mathcal{M}\operatorname{Re}(e^{i\alpha}\mathcal{V})=\operatorname{Im}(e^{i\alpha}\mathcal{V}^T)\mathcal{M}\operatorname{Im}(e^{i\alpha}\mathcal{V})=-\frac{1}{2}\ , \tag{3.36}$$

$$\operatorname{Im}(e^{i\alpha}\mathcal{V}^T)\mathcal{M}\operatorname{Re}(e^{i\alpha}\mathcal{V})=0\ , \tag{3.37}$$

$$\operatorname{Im}(e^{i\alpha}\mathcal{V}')=\operatorname{Im}(e^{i\alpha}z^{i\prime}U_i)-Q_r\operatorname{Re}(e^{i\alpha}\mathcal{V})\ , \tag{3.38}$$

$$\operatorname{Im}(e^{i\alpha}\mathcal{V}^T)\mathcal{M}\,\Gamma=\operatorname{Re}(e^{i\alpha}Z(\Gamma))\ ,\quad \operatorname{Re}(e^{i\alpha}\mathcal{V}^T)\mathcal{M}\,\Gamma=-\operatorname{Im}(e^{i\alpha}Z(\Gamma))\ , \tag{3.39}$$

$$\operatorname{Im}(e^{i\alpha}\mathcal{V}')\mathcal{M}\,\Gamma=-\operatorname{Re}(e^{i\alpha}Z(\Gamma)')+2\,Q_r\operatorname{Im}(e^{i\alpha}Z(G))\ . \tag{3.40}$$

After some long, but straightforward manipulations, the action (3.24) can then be rewritten as

$$\begin{aligned}\mathcal{S}_{1d}=\int dr\,\Big[&-\frac{1}{2}e^{2(U-\psi)}\langle\mathcal{E},J\mathcal{E}\rangle-e^{2\psi}\left[(Q_r+\alpha')+2e^{-U}\operatorname{Re}\left(e^{-i\alpha}Z(G)\right)\right]^2\\&-e^{2\psi}\left[\psi'-2e^{-U}\operatorname{Im}\left(e^{-i\alpha}Z(G)\right)\right]^2-(1+\langle G,\Gamma\rangle)\\&-2\frac{d}{dr}\left[e^{2\psi-U}\operatorname{Im}\left(e^{-i\alpha}Z(G)\right)+e^U\operatorname{Re}\left(e^{-i\alpha}Z(\Gamma)\right)\right]\Big].\end{aligned} \tag{3.41}$$

Several terms deserve a more detailed explanation. The scalar dependent complex structure $J$ is just given by

$$J = \Omega\mathcal{M}, \tag{3.42}$$

and it thus satisfies the equations

$$J\mathcal{V} = -i\mathcal{V}, \qquad JD_i\mathcal{V} = iD_i\mathcal{V}, \tag{3.43}$$

while $e^{-i\alpha}$ is just an arbitrary phase and $Q_r$ is the composite Kähler connection, and we introduced

$$\begin{aligned} \mathcal{E} \equiv\, & 2e^{2\psi}\left(e^{-U}\mathrm{Im}\left(e^{-i\alpha}\mathcal{V}\right)\right)' - e^{2(\psi-U)}JG \\ & + 4e^{-2U}\mathrm{Re}\left(e^{-i\alpha}Z(G)\right)\mathrm{Re}\left(e^{-i\alpha}\mathcal{V}\right) + \Gamma. \end{aligned} \tag{3.44}$$

The action (3.41) now indeed reads as a sum of squares of first order differential equations plus a boundary term, provided that we impose the constraint for the charges

$$\langle G, \Gamma\rangle = -1. \tag{3.45}$$

Once this is satisfied, we obtain a solution by imposing

$$\mathcal{E} = 0, \tag{3.46}$$

$$Q_r + \alpha' = -2e^{-U}\mathrm{Re}\left(e^{-i\alpha}Z(G)\right), \tag{3.47}$$

$$\psi' = 2e^{-U}\mathrm{Im}\left(e^{-i\alpha}Z(G)\right). \tag{3.48}$$

The first equation (3.46) contains the flow equations for the scalar field and the equation for $U$, the second one (3.47) gives the condition on the phase $\alpha$, while the last one (3.48) describes the evolution of $\psi$. The equations of motion following from this procedure of squaring of the action imply the equations of motion for the scalars as well as the $tt$-component of the complete Einstein equation, whereas the remaining Einstein equations are identically satisfied upon imposing the Hamiltonian constraint

$$\mathrm{e}^{2\psi}\psi'^2 - 1 - \mathrm{e}^{2\psi}U'^2 - \mathrm{e}^{2\psi}g_{i\bar{\jmath}}t^{i\,\prime}\bar{t}^{\bar{\imath}\,\prime} + \mathrm{e}^{2(U-\psi)}V_{\mathrm{BH}} + \mathrm{e}^{2(\psi-U)}V = 0\ . \tag{3.49}$$

As shown in [7], the solution (3.46)-(3.48) is BPS.
Note that we can easily recover the ungauged limit $G=0$: the constraint (3.45) would lead to an inconsistency, but the second line of (3.41) can be written as a new squared first order equation and a boundary term

$$-\left(e^{\psi}\psi - 1\right)^2 - \left(2e^{\psi}\right)'. \tag{3.50}$$

This leads to the identification $e^{\psi(r)} = r$, and hence to reducing the metric ansatz to the one for asymptotically flat configurations.

### 3.3.1 Flow equations

The writing of the effective one-dimensional action as a sum of squares can be further analyzed to find the explicit expression for the resulting flow equations for the scalar fields $z^i$. Equation (3.46), $\mathcal{E} = 0$, is a complex symplectic vector equation. To extract information, it is useful to project the equation on all possible independent sections. From the contraction

$$\langle \mathcal{E}, \mathrm{Re}\left(e^{-i\alpha}\mathcal{V}\right)\rangle = 0 \tag{3.51}$$

we obtain the flow equation for the warp factor $U(r)$:

$$U' = -e^{U-2\psi}\mathrm{Re}\left(e^{-i\alpha}Z(\Gamma)\right) + e^{-U}\mathrm{Im}\left(e^{-i\alpha}Z(G)\right). \tag{3.52}$$

The contraction

$$\langle \mathcal{E}, \mathrm{Im}\left(e^{-i\alpha}\mathcal{V}\right)\rangle = 0 \tag{3.53}$$

gives

$$\alpha' + Q_r = -e^{U-2\psi}\mathrm{Im}\left(e^{-i\alpha}Z(\Gamma)\right) - e^{-U}\mathrm{Re}\left(e^{-i\alpha}Z(G)\right). \tag{3.54}$$

The projection along the covariant derivatives of the section

$$\langle \mathcal{E}, U_i\rangle = 0 \tag{3.55}$$

leads to the scalar fields flow equation

$$z^{i'} = -e^{i\alpha}g^{i\bar{j}}\left(e^{U-2\psi}\bar{Z}_{\bar{j}}(\Gamma) + ie^{-U}\bar{Z}_{\bar{j}}(G)\right). \tag{3.56}$$

Contractions with $\Gamma$ and $G$ give identities. By using (3.47) and (3.54) we get the constraint

$$e^{U-2\psi}\mathrm{Im}\left(e^{-i\alpha}Z(\Gamma)\right) = e^{-U}\mathrm{Re}\left(e^{-i\alpha}Z(G)\right). \tag{3.57}$$

This condition arises as a consequence of the fact that in writing the action as a sum of squares we introduced an additional degree of freedom $\alpha(r)$ that was not present in the reduced action, and it can be shown to be automatically satisfied once the Hamiltonian constraint (3.49) is.

## 3.4 Ungauged limit

Our goal is now to perform the same analysis in the case where the gaugings $G$ are given by a very small vector. For this, one should take in principle the most general form of $G$ satisfying (3.15) and proceed in the same line to find the squaring of the action. However, given the homogeneity of the potential in terms of G, it is more convenient to use a different strategy. We introduce a Lagrange multiplier by performing a rescaling of the gaugings

$$G \to e^{\varphi}G, \tag{3.58}$$

where the Lagrange multiplier will be treated as an independent field whose purpose is to take into account the flatness of the potential. The presence of the Lagrange multiplier ensures that we are now dealing with a flat theory, and it should thus satisfy (3.15) without adding any new physical degree of freedom. Indeed, the choice $e^{\varphi}=0$, which corresponds to setting the gaugings to zero, will give a possible solution for the equations of motion for the ungauged theory. Of course, setting the gaugings to zero is not the only possibility, so we want to repeat the procedure of the squaring of the action by leaving explicitly the Lagrange multiplier to obtain a more generic squaring of the ungauged bosonic action. Since we are dealing with asymptotically flat solutions, we can make the choice for the metric function $e^{\psi}=r$, so that

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}\left(dt^2 + r^2 d\Omega^2\right), \tag{3.59}$$

and the three-dimensional part of the space is just flat. Note that even if here we are choosing the static ansatz for simplicity, all the results can be extended for the case of under-rotating black holes. The action can be written as a sum of squares in a similar way as above, up to an extra term originating from the partial integration involved:

$$\begin{aligned}\mathcal{S}_0 = \int dr \Big[ & -\frac{1}{2r^2}e^{2U}\langle\mathcal{E}, J\mathcal{E}\rangle - r^2\left[(Q_r+\alpha') + 2e^{-U}\mathrm{Re}\left(e^{-i\alpha}Z(G)\right)\right]^2 \\ & - \left[2re^{\varphi-U}\mathrm{Im}\left(e^{-i\alpha}Z(G)\right) - \left(1+\frac{1}{2}r\varphi'\right)\right]^2 \\ & + r^4 e^{\varphi}\left(\left(r^{-1}e^{-\varphi/2}\right)'\right)^2 - \left(2+e^{\varphi}\langle G,\Gamma\rangle\right)\Big],\end{aligned} \tag{3.60}$$

where we dropped out the total derivative. The Lagrangian is now written as a sum of squares for the physical fields, along with an extra kinetic term and a Liouville-type potential for the multiplier $\varphi$, that decouples from the rest of the action. As before, we can solve the equations of motion for the physical fields by imposing that each of the squares vanish

$$\mathcal{E}=0, \tag{3.61}$$

$$Q_r + \alpha' = -2e^{\varphi}e^{-U}\mathrm{Re}\left(e^{-i\alpha}Z(G)\right), \tag{3.62}$$

$$2re^{-U}\mathrm{Im}\left(e^{-i\alpha}Z(G)\right) = e^{-\varphi}\left(1+\frac{1}{2}r\varphi'\right). \tag{3.63}$$

Note that these equations fix as well the form of the Lagrange multiplier in terms of the physical fields. In addition to impose the squared terms to vanish, we still have to impose the equation of motions for $\varphi$:

$$\frac{d}{dr}\left(r^2u'\right) - \langle G,\Gamma\rangle r^{-2}e^{-2u} = 0, \tag{3.64}$$

where $e^u \equiv r^{-1} e^{-\phi/2}$. It remains a last equation to be imposed: the Hamiltonian constraint (3.49), giving the condition

$$\langle G, \Gamma \rangle + 4r^2 e^{-2U} e^{\varphi} \left( \mathrm{Im} \left( e^{-i\alpha} Z(G) \right) \right)^2 = 0. \tag{3.65}$$

### 3.4.1 Flow equations

The flow equations are closely related to those studied in section 3.3.1, with the differences that now the function $e^\psi$ describing the spatial part of the metric is fixed to $e^\psi = r$ and that we included the additional function $e^\varphi$. By projecting the scalar flow equation $\mathcal{E} = 0$ on the independent sections we find

$$U' = -\, r^{-2}\, \mathrm{e}^U\, \mathrm{Re}(\mathrm{e}^{-i\alpha} Z) + \mathrm{e}^{\varphi} \mathrm{e}^{-U}\, \mathrm{Im}(\mathrm{e}^{-i\alpha} W)\,, \tag{3.66}$$

$$t^{i\prime} = -\, \mathrm{e}^{i\alpha} g^{i\bar{\jmath}} \left( \mathrm{e}^{U-2\psi} \bar{Z}_{\bar{\jmath}} + i\, \mathrm{e}^{\varphi} \mathrm{e}^{-U} \bar{W}_{\bar{\jmath}} \right)\,, \tag{3.67}$$

and

$$Q_r + \alpha' = -r^{-2}\, \mathrm{e}^U\, \mathrm{Im}(\mathrm{e}^{-i\alpha} Z) - \mathrm{e}^{\varphi} \mathrm{e}^{-U}\, \mathrm{Re}(\mathrm{e}^{-i\alpha} W)\,. \tag{3.68}$$

Combining the last relation with (3.62) leads to the constraint

$$r^{-2}\, \mathrm{e}^U\, \mathrm{Im}(\mathrm{e}^{-i\alpha} Z) = \mathrm{e}^{\varphi} \mathrm{e}^{-U}\, \mathrm{Re}(\mathrm{e}^{-i\alpha} W)\,. \tag{3.69}$$

In the case of gauged supergravity, this condition was equivalent to the Hamiltonian constraint (3.49). In the ungauged case, (3.69) is not automatically satisfied once we impose (3.66)-(3.68) and the two expressions are genuinely different conditions.

## 3.5 Asymptotically flat solutions

In order to search for solutions to the above system of equations, we start by looking for explicit solutions of the equations of motion for the Lagrange multiplier (3.64). The general solution will be in terms of exponential of the type $e^{\pm 1/r}$, which are badly singular at $r = 0$ and lead to unphysical results. However, a particular enveloping solution exists:

$$e^u = \langle \Gamma, G \rangle^{-\frac{1}{2}} V, \tag{3.70}$$

where $V$ is the distinguished harmonic function

$$V = v + \frac{\langle \Gamma, G \rangle}{r}, \tag{3.71}$$

$v$ being a positive constant. We assume $\langle \Gamma, G \rangle > 0$ so that $V$ is positive definite. The expression for the Lagrange multiplier is thus given by

$$e^{\varphi} = \frac{\langle \Gamma, G \rangle}{r^2 V^2} = d \left( \frac{1}{V^2} \right). \tag{3.72}$$

The expression for the Hamiltonian constraint (3.49) becomes then

$$2e^{-U}\mathrm{Im}\left(e^{-i\alpha}Z(G)\right) = V, \tag{3.73}$$

which implies that the solution can be expressed in terms of harmonic functions. It is useful at this point to make contact between our solution and the static limit of the generic single center class of solutions for asymptotically flat black holes. Following [8], we start by defining another very small vector $R$ dual to $G$, $R \propto JG$, given by

$$R = -4\frac{e^{-2U}}{|Y|^2 V^2}\mathrm{Re}\left[Y^3 \hat{Z}(G)\mathcal{V} + |Y|^2 Y \hat{Z}^i(G) D_i \mathcal{V}\right], \tag{3.74}$$

$$Y \equiv \left(1 + ime^{2U}\right), \tag{3.75}$$

where $m$ is an arbitrary constant and

$$\hat{Z}(G) = \frac{Z(G)}{|Z(G)|}, \qquad \hat{Z}_i(G) = \frac{Z_i(G)}{|Z(G)|}. \tag{3.76}$$

By using the flow equations for the scalar fields $\mathcal{E} = 0$, it was shown in [8] that the small vector $R$ is constant. Given the definition of $R$ and by introducing a vector or harmonic functions carrying the charges,

$$\mathcal{H} = h + \frac{\Gamma}{r}, \tag{3.77}$$

the solution to the system of equation (3.61)-(3.63) is given by (see [8] for details)

$$2\mathrm{Im}\left(e^{-U}e^{-i\alpha}\mathcal{V}\right) = \mathcal{H} - 2\frac{\langle G, \mathcal{H}\rangle}{\langle G, R\rangle}R + \frac{m}{\langle G, \mathcal{H}\rangle}G. \tag{3.78}$$

Note also that the Hamiltonian constraint (3.73) fixes the distinguished harmonic function $\langle G, \mathcal{H}\rangle$ as

$$\langle G, \mathcal{H}\rangle = -V. \tag{3.79}$$

By imposing the regularity of the solution we find a constraint connecting the two very small vectors $R$ and $G$

$$\frac{1}{2}I_4'^M(\mathcal{H}, \mathcal{H}, G) = \langle G, \mathcal{H}\rangle\mathcal{H}^M - 2\frac{\langle G, \mathcal{H}\rangle}{\langle G, R\rangle}R^M, \tag{3.80}$$

where the index $M, N, \ldots$ denote both the electric and magnetic components and $I_4'^M$ is defined from the quartic invariant as

$$I_4'^M(\mathcal{H}, \mathcal{H}, G) \equiv \frac{\partial^2 I_4(\mathcal{H})}{\partial\mathcal{H}_M \partial\mathcal{H}_N}G_N = \frac{1}{2}t^{MNPQ}\mathcal{H}_N\mathcal{H}_P G_Q. \tag{3.81}$$

The constraint (3.80) is important because it selects the allowed charged $\Gamma$ for given very small vectors $G$,$R$, and viceversa. In particular, for each pole of the harmonic function $\mathcal{H}$, (3.80) constraints the charges to lie in a Lagrangian subspace that contains $R$.

## 3.6 Supersymmetric variations

In order to study supersymmetric solutions, we need to explicitly ensure that the supersymmetry variations vanish. The variations for the bosonic fields are automatically zero because of the assumption that the fermions vanish. The relevant variations are those for the fermionic fields that belong to the supermultiplets appearing in the actions, that is the gravitinos $\psi_{\mu A}$ for the gravity multiplet and the gauginos $\lambda^{iA}$ for the vector multiplets. Their supersymmetry variations, for the case of generic FI gaugings G, are given by:

$$\begin{aligned} \delta\psi_{\mu A} &= D_\mu \varepsilon_A + Z(\mathcal{F})^-_{\mu\nu}\gamma^\nu \epsilon_{AB}\varepsilon^B - \frac{1}{2} Z(G)\sigma^3_{AB}\gamma_\mu \varepsilon^B, \\ \delta\lambda^{iA} &= -i\partial\!\!\!/ t^i \varepsilon^A + \frac{i}{2}\bar{Z}(\mathcal{F})^{i-}_{\mu\nu}\,\gamma^{\mu\nu}\,\epsilon^{AB}\varepsilon_B + i\bar{Z}^i(G)\sigma^{3,AB}\varepsilon_B, \end{aligned} \tag{3.82}$$

where the covariant derivative is defined as

$$D_\mu \varepsilon_A = \Big(\partial_\mu - \frac{1}{4}\omega^{ab}_\mu \gamma_{ab} + \frac{i}{2} Q_\mu\Big)\varepsilon_A + \frac{i}{2}\langle G, A_\mu\rangle\, \sigma^{3\ B}_{A}\varepsilon_B, \tag{3.83}$$

and the central charges of the electric and magnetic field strengths are

$$Z(\mathcal{F})_{\mu\nu} = \langle \mathcal{F}_{\mu\nu}, \mathcal{V}\rangle, \qquad Z(\mathcal{F})_{i,\,\mu\nu} = \langle \mathcal{F}_{\mu\nu}, D_i\mathcal{V}\rangle. \tag{3.84}$$

Even if the bosonic sector of the flat potential gauged theory and the bosonic sector of the ungauged theory are the same, the supersymmetric variations will be different, simply because in the second case $G = 0$. This means, in particular, that the supersymmetric variations of the two theories do not overlap and form two disjoint sets. In fact, suppose we have a supersymmetric background solution of the ungauged theory and consider, for simplicity, the gaugino variation. If we want this solution to be as well a solution of the flat gauged theory, we require that it satisfies $Z^i(G) = 0$, since otherwise we cannot make the variation vanish both in the gauged and in the ungauged theory. But the flat potential condition (3.15) implies that $Z(G) = 0$ as well. Remembering the definitions of the central charges (3.12) and (3.13), we have that

$$\langle \mathcal{V}, G\rangle = 0, \qquad \langle D_i\mathcal{V}, G\rangle = 0, \tag{3.85}$$

which imply that $G = 0$ as a consequence of the properties of special geometries. Thus, the only theory for which this solution can be supersymmetric is the ungauged one. In general, BPS solutions of the asymptotically Minkowskian space-time are non supersymmetric in the flat gauged theory, and viceversa. This situation is illustrated in figure 3.1.

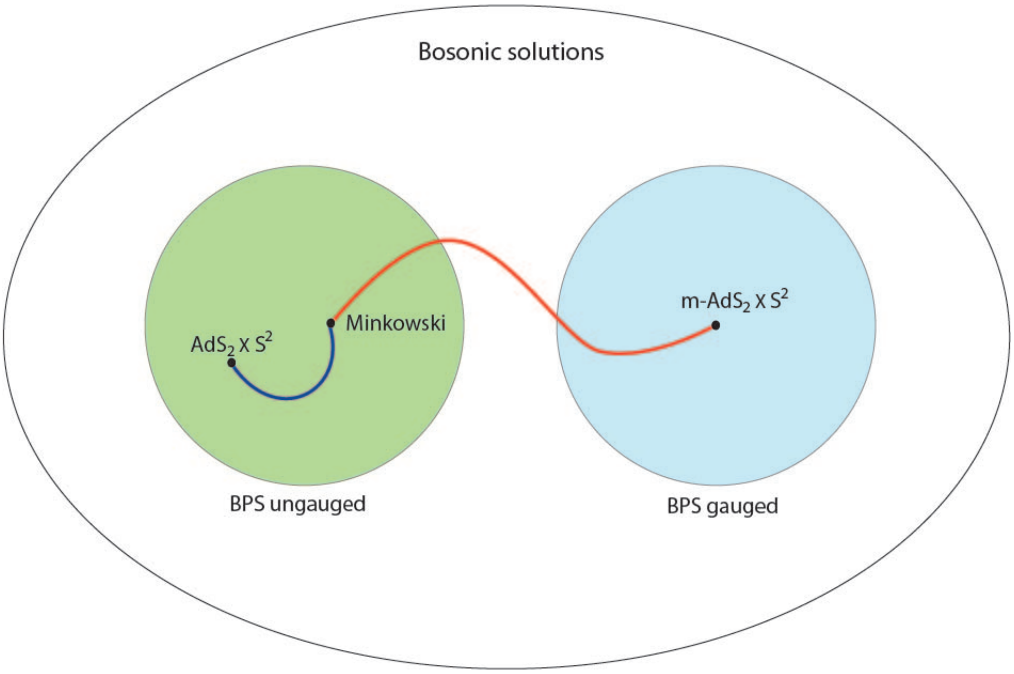


Figure 3.1: The two bubbles represent the space of BPS solutions of ungauged and abelian gauged supergravity with a flat potential, as subspaces of all bosonic solutions, common to both theories. Note the presence of two distinct $\mathrm{AdS}_2 \times S^2$ backgrounds that are supersymmetric only within one theory. The blue line represents the BPS black hole solutions, interpolating between Minkowski space and the fully BPS $\mathrm{AdS}_2 \times S^2$. It is supersymmetric with respect to the ungauged theory, but not with respect to the gauged one. The solutions described here, represented by a red line, interpolate between Minkowski and the so called magnetic $\mathrm{AdS}_2 \times S^2$ vacuum and they are globally non-supersymmetric.

## 3.7 Attractor geometry

We move to the study of the supersymmetries preserved by the near horizon geometry of the solutions above. In this limit, and considering for simplicity the static case, we expect the black hole metric to approach the metric of an $\mathrm{AdS}_2 \times S^2$ geometry

$$ds^2 = -\frac{r^2}{v_1^2}dt^2 + \frac{v_1^2}{r^2}dr^2 + v_2^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3.86}$$

where $v_1$ and $v_2$ are the radiuses of $\mathrm{AdS}_2$ and $S^2$ respectively. The corresponding vierbeins are given by

$$e^a_\mu = \mathrm{diag}\left(\frac{r}{v_1}, \frac{v_1}{r}, v_2, v_2 \sin\theta\right), \tag{3.87}$$

while the non-vanishing components of the spin-connection are

$$\omega_t^{01} = -\frac{r^2}{v_1^2}, \qquad \omega_\phi^{23} = \cos\theta. \tag{3.88}$$

To satisfy Bianchi identities and Maxwell equations in both the flat gauged and ungauged theories, it is useful to assume that the gauge field strengths are given in terms of the charges $\Gamma = \left(p^I, q_I\right)^T$ by

$$\mathcal{F}_{\mu\nu} \equiv \left(F^I_{\mu\nu}, G_{I\mu\nu}\right), \qquad F^I_{\theta\phi} = \frac{1}{2}p^I \sin\theta, \qquad G_{I\theta\phi} = \frac{1}{2}q_I \sin\theta, \tag{3.89}$$

and that the scalar fields are constant everywhere near the horizon, $\partial_\mu z = 0$. We will be using a timelike Killing spinor ansatz, ensuring that once the BPS equations are satisfied we automatically have a supersymmetric solution, thanks to the fact that the BPS equations together with Maxwell equations and Bianchi identities imply the validity of both Einstein's equations and of the equations of motion for the scalar fields.

When we consider the spherical symmetry case, it turns out that in the static limits two choices for the $AdS_2 \times S^2$ attractor geometry are possible: the one of a full BPS solution, or the one of a $\frac{1}{2}$-BPS solution. In the latter situation, the Killing spinor should satisfy the projection

$$\varepsilon_A = i\sigma^{3\,B}_A \gamma^{23}\varepsilon_B = \sigma^{3\,B}_A \gamma^{01}\varepsilon_B, \tag{3.90}$$

while in the full supersymmetric case no projection is involved.

### 3.7.1 Analysis of the BPS conditions

We can think of the Killing spinor as separating into two parts: one on $\mathrm{AdS}_2$ and one on $S^2$. The $\mathrm{AdS}_2$ part transforms in the standard way under the $SO(2,1)$ isometries of the AdS space, while the spherical part remains a scalar under rotations. The $t$ and $r$ components of the gravitino variation are therefore non-trivial due to the dependence of the spinor on these coordinates. We are however not directly interested in the explicit dependence, but only in considering the integrability condition for a solution to exist, given by $D_{[t}D_{r]}\varepsilon_A = 0$ for all $A = 1, 2$. Plugging the metric and gauge field ansatz, we obtain the equations

$$\frac{1}{2v_1^2} = |W|^2 + \frac{1}{v_2^4}|Z(\Gamma)|^2 \tag{3.91}$$

and

$$\langle G, \mathcal{F}_{tr}\rangle = 0. \tag{3.92}$$

The solution to these conditions ensure that the gravitino variation on $AdS_2$ is vanishing.
Turning now to the spherical part, we derive the two independent conditions

$$i\frac{1}{v_2^2}Z(\Gamma) = -Z(G) \tag{3.93}$$

and

$$\langle G, \Gamma \rangle = -1. \tag{3.94}$$

The latter equation is just the usual Dirac quantization condition that seems to accompany the solutions of magnetic type like the one we are dealing with. By using (3.93) we can simplify (3.91) and thus recast the equations in the more suggestive form

$$v_1^{-2} = 4\,|Z(G)|^2\,, \qquad v_2^2 = -i\frac{Z(\Gamma)}{Z(G)}. \tag{3.95}$$

Moving now to the gaugino variations, the condition for the scalars to be constant leaves us with one additional condition for each scalar on the background solution:

$$-\,iZ_i(\Gamma) = v_2^2 Z_i(G). \tag{3.96}$$

This concludes our analysis for the case of gauged supergravity, proving that we can ensure that $\mathrm{AdS}_2 \times S^2$ with radiuses $v_1$ and $v_2$ preserved half of the supersymmetries by satisfying equations (3.91)-(3.94) within the metric and gauge field ansatses chosen above. We would like now to see whether these BPS conditions admit solutions describing asymptotically flat black holes. First we notice that the condition

$$-\,4\mathrm{Im}\left(\bar{Z}(\Gamma)\mathcal{V}\right) = \Gamma + v_2^2 JG \tag{3.97}$$

is equivalent to (3.93) and (3.96), where (3.91) has been used to fix the $AdS_2$ radius. Taking the inner product of the latter expression with the gauging, and using (3.94), we can show that the radius of the sphere is given by

$$v_2^{-2} = 2g^{\bar{i}j}Z_i(G)\bar{Z}_{\bar{j}}(G) - 2\,|Z(G)|^2\,. \tag{3.98}$$

When we impose the flatness of the potential, the latter equation combined with (3.91) implies that $v_2 = v_1$, which is a necessary condition for the black hole to be asymptotically flat. The BPS attractor equation (3.97) becomes then

$$-\,4\mathrm{Im}\left(\bar{Z}(\Gamma)\mathcal{V}\right) = \Gamma + \frac{1}{2}R, \tag{3.99}$$

where we have used the definition of very small vector (3.74). This is just the generic BPS attractor equation for asymptotically flat black holes for the gauged theory (note that the corresponding ungauged theory is non-BPS). We can conclude that the near horizon geometry of static asymptotically flat extremal black holes can

| | | **Attractor** | **Global** |
|---|---|---|---|
| $\mathrm{G} = 0$ | $\mathrm{j} = 0$ | Full BPS | 1/2 BPS |
| $G \in \mathcal{S}$, **flat** | $\mathrm{j} = 0$ | 1/2 BPS | non−BPS |
| | $\mathrm{j} \neq 0$ | 1/4 BPS | non−BPS |
| $G \notin \mathcal{S}$, **AdS** | $\mathrm{j} = 0$ | 1/2 BPS | 1/4 BPS |
| | $\mathrm{j} \neq 0$ | ? | ? |

Table 3.1: An overview of supersymmetry properties of under-rotating attractors and full solutions in abelian gauged and ungauged theories, depending on whether the vector of gaugings $G$ is vanishing or lies in the very small orbit $\mathcal{S}$ or not. Th "?" for the under-rotating case in AdS signifies not only that the supersymmetry properties of these solutions are not analyzed, but also that their existence is not even certain.

be viewed as a special case of the general attractor geometry for BPS black holes in abelian gauged supergravity, upon restricting to the case where the potential is flat. In addition, when the FI terms are vanishing we immediately obtain the BPS attractor equations for ungauged supergravity, preserving the full $\mathcal{N} = 2$ supersymmetry. This provides a unifying picture, since the BPS attractor equation (3.97) appears to be universal for static extremal black holes in $\mathcal{N} = 2$ theories, independent of the asymptotic behavior (Minkowski or AdS) or on the amount of supersymmetry preserved. Note also that since the Lagrange multiplier we used to ensure that $G$ is a very small vector is a constant at the horizon, asymptotically AdS and Minkowskian solutions are undistinguishable in this region.
We present our results just for the static case, but our discussion and the squaring procedure can be generalized to stationary solutions as well (see [12] for details). The results for both the attractor and the global geometries are summarized in Tab. 3.1.

## 3.8 Asymptotically $AdS_4$ solutions

In literature, the procedure of squaring the action to obtain first-order equations was originally carried out for asymptotically AdS solutions. Given the very close similarity between the equations describing the asymptotically flat and asymptotically AdS systems, it is possible to use some of the objects we have introduced in the ungauged case to clarify the structure of asymptotically AdS static solutions. The ansatz for the metric of the black hole is given in (3.21), which allow for a non-flat three-dimensional base, and the flow equations are given by (3.46)-(3.48). We would like to discuss the properties of the gauged solutions along the same line of section 3.5, where we introduced the dual very small vector $R$ and found constraints restricting the form of the allowed harmonic functions. Of course the main difference between the asymptotically flat and the asymptotically AdS case is that an extra metric function enters in place of the Lagrange multiplier. Moreover, since the gauging field $G$ is not a very small vector anymore, $R$ is not a very small vector either. We should also note the different role of the constant $m$ defined in (3.75): in the asymptotically flat case $m$ is an arbitrary constant allowing us to get the most general solution, while the analysis of the asymptotic fall-off of the terms of the flow equation for the scalar, (3.44), shows that in the gauged case we need to choose $m = 0$, as a non-zero $m$ would spoil the behavior of the scale factor of the metric at infinity.
Using the definition (3.74) with $m = 0$ we find

$$|Z(G)|^2 R = JG. \tag{3.100}$$

This expression allows us to rewrite the flow equation for the section as

$$2e^{2\psi}\left(e^{-U}\mathrm{Im}\left(e^{-i\alpha}\mathcal{V}\right)\right)' - 2e^{2\psi}|Z(G)|^2 R + 4e^{2\psi-U}(Q_r+\alpha')\,\mathrm{Re}\left(e^{-i\alpha}\mathcal{V}\right) + \Gamma = 0. \tag{3.101}$$

It is natural to use the ansatz

$$2e^{-U}\mathrm{Im}\left(e^{-i\alpha}\mathcal{V}\right) = re^{-\psi}\mathcal{H}, \tag{3.102}$$

which implies that the Kähler connection is vanishing

$$Q_r + \alpha' = 0. \tag{3.103}$$

By choosing $e^{\psi} = r$, the ansatz (3.102) reduces to the asymptotically flat one (3.78) for $m = 0$, while in general, combined with the equation of motion for $\psi$ presented in (3.44), it gives

$$\left(e^{\psi}\right)' = r\langle G, \mathcal{H}\rangle. \tag{3.104}$$

The integration of the equation (3.101) can be done directly, but it is much simplified, using some intuition borrowed from the asymptotically flat case. In particular, we claim that the constraint (3.80)

$$\frac{1}{2} I_4'^M(\mathcal{H}, G) = \langle G, \mathcal{H} \rangle \mathcal{H}^M - 2 \frac{\langle G, \mathcal{H} \rangle}{\langle G, R \rangle} R^M \tag{3.105}$$

is relevant also for the asymptotically AdS case, where $G$ and $R$ are generic vectors. Although we cannot prove it in general, such a constraint homogeneous in $\mathcal{H}$, $G$ and $R$ is reasonable given the similarity of the flow equations and the fact that the two ansatses (3.78) and (3.102) are related by a rescaling of a function. The meaning of the constraint is still that given a vector of gauging the set of charges is fixed, but now those charges cannot be read directly from the poles of the harmonic functions $\mathcal{H}$. Its validity for asymptotically AdS solutions is important because, as in the asymptotically flat case, it is much easier to compute $R$ from the constraint, instead of performing the tedious direct computation involving the matrix $J$.
As an example, we present the STU model for both asymptotically flat and asymptotically AdS solutions, pointing out the differences between the two and the role of the constraint.

### 3.8.1 STU model

By fixing the sympletctic basis, the STU model is defined by the prepotential

$$F = \frac{X^1 X^2 X^3}{X^0}. \tag{3.106}$$

#### Asymptotically flat solutions

The very small vector $G$ can be chosen in such a way that only one of its constant components is non-vanishing. For example, we can take

$$G = (0, 0; 1, 0)^T, \tag{3.107}$$

keeping in mind that this choice is not unique. The second very small vector $R$ is then given by

$$R = (-4, 0; 0, 0)^T. \tag{3.108}$$

The charges of the solutions are given by the poles in the following choice of the harmonic functions

$$\mathcal{H} = \left(H^0, 0; 0, H_i\right), \tag{3.109}$$

where

$$H_i = h_i + \frac{q_i}{r}, \qquad H^0 = h^0 + \frac{p^0}{r}, \tag{3.110}$$

$h_i$ and $h^0$ being constants. Using (3.78), we can find the explicit expression for the physical scalars

$$t^i = \frac{M - ie^{-2U}}{2H^0 H_i}, \tag{3.111}$$

and for the real part of the section

$$2\,\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}) = \mathrm{e}^{2U}\Big(M\,H^0,\; -H^0\,|\varepsilon^{ijk}|H_jH_k\,;\; -\tfrac{M^2}{H^0} - 2H_1H_2H_3,\; M\,H_i\Big)^T\,, \tag{3.112}$$

where $M$ is a dipole harmonic function

$$M = m + \frac{j\cos\theta}{r^2}. \tag{3.113}$$

The constraint equation (3.80) is of course satisfied, and we can find the explicit form of the metric (3.59):

$$e^{-4U} = 4H^0H_1H_2H_3 - M^2, \qquad d\omega = -dM. \tag{3.114}$$

### Near horizon solution

The near horizon limit is taken by simply dropping the constants $h^0$ and $h_i$ in the harmonic functions. The scalars (3.111) become

$$z^i = \frac{j\cos\theta - ie^{-2U}}{2p^0q_i}, \tag{3.115}$$

while the near horizon metric is given by

$$\begin{aligned} ds^2 &= -e^{2U}\,r^2\,(dt+\omega)^2 + e^{-2U}\,\left(\frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta d\phi^2\right)\,, \\ \mathrm{e}^{-4U} &= 4\,p^0\,q_1q_2q_3 - j^2\cos^2\theta \qquad \omega = j\,\frac{\sin^2\theta}{r}\,d\phi\,. \end{aligned} \tag{3.116}$$

### Asymptotically AdS solutions

The FI terms are no longer given by a very small vector, but we can choose a frame where they are given by

$$G = \left(0, g^i; g_0, 0\right)^T. \tag{3.117}$$

We consider a vector of single center harmonic functions

$$\mathcal{H} = \left(H^0, 0; 0, H_i\right), \tag{3.118}$$

where

$$H^0 = \alpha^0 + \frac{\beta^0}{r}, \qquad H_i = \alpha_i + \frac{\beta_i}{r}. \tag{3.119}$$

Note that these expressions are closely similar to those for the asymptotically flat case, but here the poles of the harmonic functions do not correspond to the charges. We can use (3.102) to compute

$$e^{-U}\mathrm{Re}\left(e^{-i\alpha}\mathcal{V}\right) = re^{-\psi}e^{2U}\left(0,\ \frac{1}{2}H^0\left|\varepsilon^{ijk}\right|H_jH_k;\ H_1H_2H_3,\ 0\right)^T \tag{3.120}$$

and

$$r^{-4}e^{4\psi}e^{-4U} = 4H^0H_1H_2H_3. \tag{3.121}$$

We should also satisfy (3.104), for which a choice of the solution is given by

$$\langle G, \mathcal{H}\rangle = 2, \qquad e^{\psi} = r^2 + c, \tag{3.122}$$

where c is an arbitrary integration constant and the first expression is a simplifying condition.
Using the flow equation for the section (3.101), we find the constraints

$$\alpha^0 g_0 = \alpha^i g_i\,, \qquad p_0 = c\alpha^0 - 2(\beta^0)^2 g_0\,, \qquad -q_i = c\alpha_i - 2(\beta_i)^2 g_i, \tag{3.123}$$

where there is no implicit sum in the repeated indices and each equation is thus valid for each index $i$ separately. Thanks to (3.100), the expression for the dual of $G$, $R$, is then given by

$$R^0 = g_0\,(H^0)^2\,, \qquad R_i = g^i\,(H_i)^2\,, \tag{3.124}$$

where again there is no implicit sum.
It is now straightforward to check that the constraint (3.105) is automatically satisfied.

## 3.9 Summary

In this chapter we study a connection between asymptotically AdS and asymptotically flat solutions via an ungauging procedure. We consider the bosonic Lagrangian for abelian $U(1)$ supergravity coupled to $n_V$ vector multiplets. The ungauged and the gauged theory are simply related by the addition of the scalar potential, where the FI terms appear:

$$\mathcal{L}_g = \mathcal{L}_0 + V. \tag{3.125}$$

A so-called “squaring procedure” is available for asymptotically AdS solutions and it allows to write first-order differential equations of motions which are explicitly solvable. The asymptotically flat limit is reached for vanishing FI gaugings, but since the potential $V$ is not positive-definite, this is not the only possibility. We therefore introduced a Lagrange multiplier taking into account the flatness of the potential, and proceeded to generalize the squaring procedure for ungauged theories. The solution can be expressed in terms of harmonic functions, and it is easy to make contact with the generic single-center solution present in literature [8]. In particular, by introducing the very small vector $R$, dual to the gauging vector $G$, we show the existence of a constraint for the allowed electro-magnetic charges implying that the harmonic functions lie in a Lagrangian submanifold that includes $R$.

The next step in the analysis of the solution is the study of the supersymmetric variations. All supersymmetry variations for the bosons are automatically zero by the assumption of vanishing fermions. The variations of the fermions are different if we consider vanishing FI terms or flat potential with non-vanishing FI terms, because of the presence of the Lagrange multiplier. At the horizon the Lagrange multiplier becomes a constant, thus the asymptotically flat and the asymptotically AdS theories coincide and the both belong to the BPS class.

We moved then to analyze asymptotically AdS solutions by using the methods borrowed from the study of asymptotically flat solutions. Indeed, the equations describing the systems for the gauged and ungaged theories are very closely related: the only difference is that an extra metric function enters in place of the Lagrange multiplier. In particular, the constraint restricting the form of the allowed electromagnetic charges is still valid, even if $R$ and $G$ are not very small vectors anymore.

In the last part of the chapter we present the STU model for asymptotically flat and asymptotically AdS solutions as an example to enlighten differences and similarities between the two theories.

# Chapter 4

# Holographic perfect-like fluids, black hole uniqueness and transport coefficients

Within the AdS/CFT correspondence, an interesting limit is the one where the dynamics of the field theory reduces to that of an effective classical fluid dynamics. Under the holographic map, the dual of this long wavelenght regime is given by classic black hole solutions. After a review of relativistic fluid dynamics, we present two different techniques to find fluids dual to black hole solutions: the expansion with respect to the velocity field in the boundary and the one with respect to the holographic radial coordinate. We focus then on the latter type of expansion for known four-dimensional stationary black hole solutions. We give the properties of the boundary geometries, focusing in particular on the vorticity properties and on the analogue gravity interpretation. Then, we point out that black-hole uniqueness constraints the transport coefficients to vanish, and we derive conditions under which the expansion can be resummed to find an exact solution of Einstein's equations.

## 4.1 Introduction and motivations

The fluid/gravity correspondence states that there can be a regime such that the degrees of freedom living on the boundary of an asymptotically $\mathrm{AdS}_D$ space-time are described by the hydrodynamics of a relativistic fluid in $D-1$ dimensions. Consequently, the dynamical equations of the fluid in the boundary are encoded in the asymptotic behavior of the bulk Einstein's equations.

From the boundary point of view, a holographic mapping to classical gravity systems can in principle help gaining insights on some theoretical challenges of fluid dynamics. For example, finding globally regular solutions for non-relativistic incompressible viscous fluids, described by the Navier-Stokes equations remains an

open question, and more challenging is to reach a detailed understanding of turbulence.
From the bulk point of view, the holographic duality allows to construct in a systematic way dynamical black hole solutions with regular event horizon (see for example the review [43]).
Within the AdS/CMT correspondence, the fluid/gravity duality aims as well at finding new computational tools for strongly coupled condensed matter systems using holography. In particular, we will focus on rotating boundary fluids, to which will correspond stationary black hole solutions in the bulk. This can lead to insights in some ultra-cold-atom systems, like fast rotating gases [42], that could be studied with holography techniques. Furthermore, by studying the sound/light propagation through the boundary fluid we can give an holographic description of acoustic/luminal analogue gravity [21]. This analogue gravity interpretation opens up the possibility for applications to meta-materials with dedicated acoustic or optical properties. The crucial feature is here the sound/light propagation in supersonically/superluminally (with respect to the local velocity of sound/light) moving media.

## 4.2 Relativistic fluid dynamics

We give here a brief review of relativistic fluid dynamics [44], [45], and we focus later on conformal fluids [35], [43].
Fluid dynamics is the effective description of some interacting quantum field theories when the fluctuations are of sufficiently long wavelength. This description is intrinsically statistical in nature, since it is the collective macroscopic physics of a large number of microscopic constituents. The situation can be pictured as follows: sufficiently long-wavelenght variations are slow on the scale of the local energy/density temperature. Then, at any given point of the system we expect to encounter a domain where the local temperature is roughly constant. Different domains will be then described by different values of the intrinsic thermodynamic variables. Fluid dynamics describe how these different domains interact and exchange conserved quantities.
More formally, any interacting system is characterized by an intrinsic length scale: the mean free path length $\ell_{\mathrm{mfp}}$. In the kinetic theory context, $\ell_{\mathrm{mfp}}$ characterizes the length scale for the free motion between successive collisions. The hydrodynamic limit is defined as the regime in which the scale of the system $L$ is much larger than $\ell_{\mathrm{mfp}}$, and thus in a small region compared to $L$ the constituents of the system interact among each others several times, thermalizing locally. As a consequence, we can treat the long-distance system as a fluid described by thermodynamic macroscopic quantities.
We refer to relativist hydrodynamics when the microscopic components of the fluid are constrained by Lorentz symmetry. Consider a fluid living in a $d$-dimensional

space-time with metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{4.1}$$

Due to the presence of dissipative terms, hydrodynamics is generally formulated by the analysis of the equations of motion, rather then starting from an action principle. Under the hypothesis of uncharged fluid, the dynamical content of the hydrodynamic equation is given by the Euler equation, found by imposing the conservation of the energy:

$$\nabla_\mu T^{\mu\nu} = 0, \tag{4.2}$$

where $T^{\mu\nu}$ is the energy-momentum tensor. In $d$ dimensions, the number of independent components of $T^{\mu\nu}$ is $\frac{1}{2}d(d+1)$, while (4.2) has $d$ equations. Thus, for $d > 2$ we have more variables then equations. To close the system, we need to reduce the number of independent component of the energy-momentum tensor. Since if the perturbations are large compared to $\ell_{\mathrm{mfp}}$ the system is in local thermal equilibrium, at any given time it is determined by the temperature $T(x)$ and by the velocity field $u^\mu(x)$, describing the flow of the thermodynamic quantities. By normalizing the velocity to $-1$, *i.e.* $u_\mu u^\mu = -1$, the total number of independent fields is $d$ and we have the same number of equations and variables.
The dependence of $T^{\mu\nu}$ in terms of $T(x)$ and $u^\mu(x)$ is given by the so-called "constitutive relation". Since we assume the deviation from equilibrium to be small, we expect the contribution of the terms at higher order in derivatives of $T(x)$ and $u^\mu(x)$ to be increasingly subdominant in the hydrodynamic limit. For that, we write the constitutive relation in a derivative expansion:

$$T^{\mu\nu} = \sum_{n=0}^{\infty} T_n^{\mu\nu}, \tag{4.3}$$

where the $n$-term is the $n^{\mathrm{th}}$ order in the derivatives of the fluid fields, giving a contribution of order $\left(\frac{l_{\mathrm{lfp}}}{L}\right)^n$.

### 4.2.1 Ideal fluids

We start with the description of ideal fluids, which have no dissipation. By going to a local rest frame, where the velocity field is aligned with the direction of the energy flow, *i.e.* $u^0 = 1$, $u^i = 0$, where $i$ are the spatial coordinates, we can identify the components of the energy-momentum tensor as the energy density $\varepsilon$ and the pressure $P$:

$$(T^{\mu\nu})_{\mathrm{ideal}} = \mathrm{diag}\left[\varepsilon, p, p, \cdots\right]. \tag{4.4}$$

By acting on (4.4) with a finite boost transformation with velocity field $u^\mu$ we obtain the energy-momentum tensor in covariant formalism:

$$(T^{\mu\nu})_{\mathrm{ideal}} = \varepsilon u^\mu u^\nu + p\left(g^{\mu\nu} + u^\mu u^\nu\right). \tag{4.5}$$

It is useful to define the projector onto the spatial directions

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu. \tag{4.6}$$

It is a symmetric positive definite tensor satisfying

$$P^{\mu\nu} u_\nu = 0, \qquad P^{\mu\rho} P_{\rho\nu} = P^\mu{}_\nu = P^{\mu\rho} g_{\rho\nu}, \qquad P^\mu{}_\mu = d - 1. \tag{4.7}$$

The ideal energy-momentum tensor takes then the form

$$(T^{\mu\nu})_{\text{ideal}} = \varepsilon u^\mu u^\nu + p P^{\mu\nu}. \tag{4.8}$$

### 4.2.2 Dissipative fluids

Adding dissipation terms is necessary for a fluid system to equilibrate when perturbed away from a given equilibrium configuration. To add dissipative contributions, like viscosity and thermal conduction, it is necessary to go beyond the zeroth order in the derivative expansion. For a fluid in local thermal equilibrium, the energy-momentum tensor generalizes to

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + p\,(g^{\mu\nu} + u^\mu u^\nu) + T_n^{\mu\nu}, \tag{4.9}$$

where $T_n^{\mu\nu}$, $n > 0$, represents the potentially dissipative terms, to be determined in terms of derivatives of $T(x)$ and $u^\mu(x)$ and of the thermodynamic variables $\varepsilon$, $p$. In relativist fluids it is not possible to distinguish between mass and energy fluxes, one flux involving necessarily the other. For this reason it is convenient to fix the velocity field in an unambiguous way. Different conventions can be used, and in particular we choose here the so-called Landau frame, in which at equilibrium the components of the energy-momentum tensor which are longitudinal to the velocity are associated with the energy density:

$$u_\mu T^{\mu\nu} = -\varepsilon u^\nu. \tag{4.10}$$

As a consequence, the dissipative contributions to the energy-momentum tensor must satisfy

$$T_n^{\mu\nu} u_\mu = 0. \tag{4.11}$$

Note that within this choice of frame we have no energy flow in the local rest frame. We now consider the decomposition of the velocity gradient $\nabla^\nu u^\mu$ into irreducible representations of the Lorentz group: a mixed contribution given by the acceleration $a^\mu$ and a fully transverse part. The latter can be decomposed into a symmetric traceless tensor, the shear $\sigma^{\mu\nu}$, an antisymmetric tensor, the vorticity $\omega^{\mu\nu}$, and a pure trace part, the expansion rate $\theta$. The decomposition reads

$$\nabla^\nu u^\mu = -a^\mu u^\nu + \sigma^{\mu\nu} + \omega^{\mu\nu} + \frac{1}{d-1}\theta P^{\mu\nu}, \tag{4.12}$$

with the various components defined as

$$\theta = \nabla_\mu u^\mu = P^{\mu\nu}\nabla_\mu u_\nu, \tag{4.13}$$

$$a^\mu = u^\nu \nabla_\nu u^\mu \equiv \mathcal{D}u^\mu, \tag{4.14}$$

$$\sigma^{\mu\nu} = \nabla^{(\mu}u^{\nu)} + u^{(\mu}a^{\nu)} - \frac{1}{d-1}\theta P^{\mu\nu} = P^{\mu\rho}P^{\nu\sigma}\nabla_{(\rho}u_{\sigma)} - \frac{1}{d-1}\theta P^{\mu\nu}, \tag{4.15}$$

$$\omega^{\mu\nu} = \nabla^{[\mu}u^{\nu]} + u^{[\mu}a^{\nu]} = P^{\mu\rho}P^{\nu\sigma}\nabla_{[\rho}u_{\sigma]}, \tag{4.16}$$

where we have introduced the velocity projected covariant derivative $\mathcal{D} \equiv u^\mu \nabla_\mu$. Note also our conventions for the symmetrization and anti-symmetrization, which are respectively

$$A_{(ab)} = \frac{1}{2}\left(A_{ab} + A_{ba}\right), \qquad A_{[ab]} = \frac{1}{2}\left(A_{ab} - A_{ba}\right). \tag{4.17}$$

For further convenience, for any rank-two tensor $\mathcal{F}^{\mu\nu}$ we define the bracket tensor $\langle \mathcal{F}^{\mu\nu} \rangle$, which is symmetric, traceless and transverse:

$$\langle \mathcal{F}^{\mu\nu} \rangle = P^{\mu\rho}P^{\nu\sigma}\mathcal{F}_{(\rho\sigma)} - \frac{1}{d-1}P^{\mu\nu}P^{\rho\sigma}\mathcal{F}_{\rho\sigma}. \tag{4.18}$$

Due to the normalization of the velocity field, $u_\mu u^\mu = -1$, the following identities hold:

$$u^\mu \nabla_\nu u_\mu = 0, \qquad \nabla_\rho u^\mu \nabla_\sigma u_\mu = -u^\mu \nabla_\rho \nabla_\sigma u_\mu, \qquad u_\mu a^\mu = 0, \tag{4.19}$$

where the last one states the tracelessness of $a^\mu u^\nu$.
To determine the first order contributions to the energy momentum tensor in the gradient expansion, we write the more general symmetric two-index conserved ((4.2)) tensor satisfying the Landau frame condition (4.10) built out of first-order derivatives of the velocity field. Those contributions are given by:

$$T_1^{\mu\nu} = -2\eta\sigma^{\mu\nu} - \zeta\theta P^{\mu\nu}, \tag{4.20}$$

where we have introduced the shear viscosity $\eta$ and the bulk viscosity $\zeta$. We thus obtain a first order viscous fluid:

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + p\left(g^{\mu\nu} + u^\mu u^\nu\right) - 2\eta\sigma^{\mu\nu} - \zeta\theta P^{\mu\nu}. \tag{4.21}$$

We can explicitly show that this corresponds to Navier-Stokes equations of motion by imposing the conservation of the energy-momentum tensor. In particular, we project the conservation equation along the fluid velocity and along the transverse direction, obtaining respectively

$$u_\nu \nabla_\mu T^{\mu\nu} = -\mathcal{D}\varepsilon - (p+\varepsilon)\nabla_\mu u^\mu + u_\nu \nabla_\mu T_1^{\mu\nu} = 0, \tag{4.22}$$

and

$$P_{\rho\nu}\nabla_\mu T^{\mu\nu} = (\varepsilon + p)\,\mathcal{D}u_\rho + \nabla_\rho p + u_\rho \mathcal{D}p + P_{\rho\nu}\nabla_\mu T_1^{\mu\nu} 0. \tag{4.23}$$

The spatial components of the latter represent the generalization of the Euler equation in presence of dissipative effects, that is the Navier-Stokes equations for compressible relativistic fluids. In the non-relativistic limit $|v| << 1$ we have

$$u^\mu \approx \{1, \vec{v}\}, \qquad \mathcal{D} \approx \partial_0 + \vec{v}\cdot\vec{\partial}, \tag{4.24}$$

and the equation of state are characterized by $\varepsilon >> p$, and the energy density is dominated by matter. Assuming the incompressible condition $\partial_i v^i$, (4.23) becomes

$$\partial_0 v_i + \left(\vec{v}\cdot\vec{\partial}\right) v_i = -\frac{1}{\varepsilon}\partial_i p + \frac{\eta}{\varepsilon}\partial_j \partial^j v_i, \tag{4.25}$$

which is the incompressible non-relativistic Navier-Stokes equation.

## 4.3 Conformal fluids

Since the fluid theory appears in the boundary of a gravitational theory, it will be conformally invariant. It is thus useful to study conformal fluids in details.

### 4.3.1 Weyl transformation of the energy-momentum tensor

The energy-momentum tensor enjoys different simplifications: first, due to scale invariance, the energy-momentum tensor for conformal theories is traceless. Imposing this condition on the energy-momentum tensor with first-order dissipative effects (4.21) we get

$$p = \frac{1}{d-1}\varepsilon, \qquad \zeta = 0. \tag{4.26}$$

Hence, conformal fluids have no bulk viscosity, and the energy-momentum tensor at first order reads

$$T^{\mu\nu} = \varepsilon\left(g^{\mu\nu} + d\; u^\mu u^\nu\right) - 2\eta\sigma^{\mu\nu}. \tag{4.27}$$

The second simplification occurring is that $T^{\mu\nu}$ must transform in a covariant way under Weyl rescaling. This will restrict the possible terms appearing in the energy-momentum tensor at higher orders in the derivative expansions. Last, since there are no dimensionful scales involved, the dependence of the various hydrodynamic quantities on the temperature is determined by dimensional analysis.
To understand the conformal properties of the energy-momentum tensor, we start considering a Weyl rescaling of the background metric $g_{\mu\nu}$:

$$g_{\mu\nu} = e^{2\phi}\tilde{g}_{\mu\nu} \;\rightarrow\; g^{\mu\nu} = e^{-2\phi}\tilde{g}^{\mu\nu}. \tag{4.28}$$

The velocity field will then transform as

$$u^\mu = e^{-\phi}\tilde{u}^\mu, \tag{4.29}$$

as a direct consequence of its normalization $u^\mu u_\mu = -1$. As a consequence, the spatial projector will also transform homogeneously

$$P^{\mu\nu} = e^{-2\phi}\tilde{P}^{\mu\nu}. \tag{4.30}$$

We require that in a conformal theory the dynamical equations are invariant under conformal transformations. In general, an equation involving a field $\Psi$ is conformal invariant if there exists a number $\omega \in \mathbb{R}$ such that $\Psi$ is a solution of the equation with metric $g_{\mu\nu}$ if and only if $\tilde{\Psi} = e^{\omega\phi}\Psi$ is a solution of the equation with metric $\tilde{g}_{\mu\nu}$. The number $\omega$ is called the "conformal weight" of the field. This means that in the case of conformal fluid dynamics we should require

$$\nabla_\mu T^{\mu\nu} = e^{-k\phi}\tilde{\nabla}_\mu\tilde{T}^{\mu\nu}, \tag{4.31}$$

where $k \in \mathbb{R}$ and

$$T^{\mu\nu} = e^{-\omega\phi}\tilde{T}^{\mu\nu}. \tag{4.32}$$

The Christoffel symbols transform under Weyl rescaling as

$$\begin{aligned} \Gamma^\nu_{\lambda\mu} &= \tilde{\Gamma}^\nu_{\lambda\mu} + \tilde{\Pi}^\nu_{\lambda\mu}, \\ \tilde{\Pi}^\nu_{\lambda\mu} &= \delta^\nu_\lambda\,\partial_\mu\phi + \delta^\nu_\mu\,\partial_\lambda\phi - \tilde{g}_{\lambda\mu}\,\tilde{g}^{\nu\sigma}\,\partial_\sigma\phi\ , \end{aligned} \tag{4.33}$$

which implies that the covariant derivative of the energy-momentum tensor is given by

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \tilde{\nabla}_\mu\left(e^{\omega\phi}\tilde{T}^{\mu\nu}\right) + e^{\omega\phi}\tilde{\Pi}^\mu_{\mu\lambda}T^{\lambda\nu} + e^{\omega\phi}\tilde{\Pi}^\nu_{\mu\lambda}T^{\mu\lambda} \\ &= e^{\omega\phi}\nabla_\mu T^{\mu\nu} + (-\omega + d + 2)\,e^{(\omega-1)\phi}T^{\mu\nu}\partial_\mu e^\phi + e^{(\omega-1)\phi}T\nabla^\nu e^\phi, \end{aligned} \tag{4.34}$$

where $T = g_{\mu\nu}T^{\mu\nu}$. Since the theory is conformal, the trace of the energy-momentum tensor vanishes, thus $T = 0$. Hence, the equation of motion of fluid-dynamics is conformal invariant if the energy-momentum tensor has conformal weight $\omega = d+2$, that is it transforms as

$$T^{\mu\nu} = e^{-(d+2)\phi}\tilde{T}^{\mu\nu}. \tag{4.35}$$

To construct the fluid energy-momentum tensor at higher orders we simply need to enumerate the set of operators in our gradient expansion which transform homogeneously under (4.28). Let us consider the situation at first order in derivatives explicitly. The covariant derivative of the velocity field $u^\mu$ transforms inhomogeneously:

$$\nabla_\mu u^\nu = \partial_\mu u^\nu + \Gamma^\nu_{\mu\lambda}\,u^\lambda = e^{-\phi}\,\left[\tilde{\nabla}_\mu\,\tilde{u}^\nu + \delta^\nu_\mu\,\tilde{u}^\sigma\,\partial_\sigma\phi - \tilde{g}_{\mu\lambda}\,\tilde{u}^\lambda\,\tilde{g}^{\nu\sigma}\,\partial_\sigma\phi\right]. \tag{4.36}$$

This equation can be used to derive the transformation of various quantities of interest in fluid dynamics, such as the acceleration $a^\mu$, shear $\sigma^{\mu\nu}$, etc..

$$\begin{aligned}\theta &= \nabla_\mu u^\mu = e^{-\phi}\left(\tilde{\nabla}_\mu \tilde{u}^\mu + (d-1)\,\tilde{u}^\sigma\,\partial_\sigma\phi\right) = e^{-\phi}\left(\tilde{\theta} + (d-1)\,\tilde{\mathcal{D}}\phi\right)\,,\\ a^\nu &= \mathcal{D}u^\nu = u^\mu\nabla_\mu u^\nu = e^{-2\phi}\left(\tilde{a}^\nu + \tilde{P}^{\nu\sigma}\,\partial_\sigma\phi\right),\\ \sigma^{\mu\nu} &= P^{\lambda(\mu}\nabla_\lambda u^{\nu)} - \frac{1}{d-1}\,P^{\mu\nu}\,\nabla_\lambda u^\lambda = e^{-3\,\phi}\,\tilde{\sigma}^{\mu\nu}\end{aligned} \tag{4.37}$$

Armed with this information, we are able to write the first-order energy-momentum tensor for a conformal viscous fluid as

$$T^{\mu\nu} = \varepsilon\left(g^{\mu\nu} + d\;u^\mu u^\nu\right) - 2\eta\sigma^{\mu\nu}. \tag{4.38}$$

where the coefficient of the bulk viscosity, $\zeta$, has to vanish because the expansion $\theta$ transforms inhomogeneously. Note that this is indeed equivalent to (4.27).

### 4.3.2 Weyl covariant formulation of conformal fluid dynamics

In a straightforward way, one can carry on the same exercise and write order by order the gradient expansion of the energy-momentum tensor. However, it is useful to work in a more abstract way to take into account in a better way the Weyl transformation properties of the operators in the theory. In particular, following [34], we define a Weyl covariant derivative in terms of which the fluid mechanics can be cast into a manifestly conformal language. To do so, we first need to define the Weyl connection over $(\mathcal{M},\mathcal{C})$, where $\mathcal{M}$ is the space-time manifold and $\mathcal{C}$ denotes the conformal class of metrics on the manifold. A torsionless connection is called a Weyl connection if for every metric in the conformal class $\mathcal{C}$ there exists a one form $\mathcal{A}_\mu$ such that

$$\nabla^{\text{Weyl}}_\rho g_{\mu\nu} = 2\mathcal{A}_\rho g_{\mu\nu}. \tag{4.39}$$

We then define the Weyl covariant derivative $\nabla D^{\text{Weyl}}_\mu = \nabla_\mu + \omega\mathcal{A}_\mu$ in such a way that if a tensorial quantity $Q^{\mu\cdots}_{\nu\cdots}$ obeys $Q^{\mu\cdots}_{\nu\cdots} = e^{-\omega\phi}\tilde{Q}^{\mu\cdots}_{\nu\cdots}$, then $D_\rho Q^{\mu\cdots}_{\nu\cdots} = e^{-\omega\phi}\tilde{Q}^{\mu\cdots}_{\nu\cdots}$. Its expression is given by

$$\begin{aligned}\nabla D^{\text{Weyl}}_\rho Q^{\mu\cdots}_{\nu\cdots} \equiv{}& \nabla_\rho Q^{\mu\cdots}_{\nu\cdots} + \omega\mathcal{A}_\rho Q^{\mu\cdots}_{\nu\cdots}\\ &+ \left(g_{\rho\sigma}\mathcal{A}^\mu - \delta^\mu_\rho\mathcal{A}_\sigma - \delta^\mu_\sigma\mathcal{A}_\rho\right)Q^{\sigma\cdots}_{\nu\cdots} + \cdots\\ &- \left(g_{\rho\nu}\mathcal{A}^\sigma - \delta^\sigma_\rho\mathcal{A}_\nu - \delta^\sigma_\nu\mathcal{A}_\rho\right)Q^{\mu\cdots}_{\sigma\cdots}.\end{aligned} \tag{4.40}$$

The dots stay for the contractions with the following index of $Q^{\mu\cdots}_{\nu\cdots}$, which we don't write explicitly. From (4.39) follows immediately that the covariant derivative is metric compatible,

$$\nabla D^{\text{Weyl}}_\rho g_{\mu\nu} = 0, \tag{4.41}$$

since the metric tensor has weight $\omega = -2$. In addition, fluid mechanics requires the Weyl covariant derivative of the fluid velocity to be transverse and traceless:

$$u^\rho \mathcal{D}_\rho u^\mu = 0, \qquad \mathcal{D}_\rho u^\rho = 0. \tag{4.42}$$

These conditions enable to uniquely determine the connection one-form $\mathcal{A}_\mu$ to be the distinguished vector field

$$\mathcal{A}_\mu = u^\rho \nabla_\rho u_\mu - \frac{1}{d-1} u_\mu \nabla^\rho u_\rho \equiv a_\mu - \frac{1}{d-1}\theta u_\mu. \tag{4.43}$$

The quantities appearing in the gradient expansion of the energy-momentum tensor can be now written in this Weyl covariant notation. For example, at first order we have

$$\sigma^{\mu\nu} = \mathcal{D}^{(\mu} u^{\nu)}, \qquad \omega^{\mu\nu} = -\mathcal{D}^{[\mu} u^{\nu]}, \tag{4.44}$$

both of which have weight $\omega = 3$. The fundamental equations of fluid mechanics can be reformulated in a Weyl-covariant form. For example, the equation of energy conservation is simply recast as

$$\mathcal{D}_\mu T^{\mu\nu} = 0, \tag{4.45}$$

since

$$\begin{aligned}
\mathcal{D}_\mu T^{\mu\nu} &= \nabla_\mu T^{\mu\nu} + w\, \mathcal{A}_\mu T^{\mu\nu} + \left(g_{\mu\alpha} \mathcal{A}^\mu - \delta^\mu_\mu \mathcal{A}_\alpha - \delta^\mu_\alpha \mathcal{A}_\mu\right) T^{\alpha\nu} \qquad (4.46)\\
&+ \left(g_{\mu\alpha} \mathcal{A}^\nu - \delta^\nu_\mu \mathcal{A}_\alpha - \delta^\nu_\alpha \mathcal{A}_\mu\right) T^{\mu\alpha}\\
&= \nabla_\mu T^{\mu\nu} + (w-d-2)\, \mathcal{A}_\mu T^{\mu\nu} - \mathcal{A}^\nu T^\mu_\mu\\
&= \nabla_\mu T^{\mu\nu}\ , \qquad (4.47)
\end{aligned}$$

where we have used the conformal weight $\omega = d+2$ of the energy-momentum tensor. We mention here that for CTFs on curved manifolds in even space-time dimensions we encounter the Weyl anomaly $\mathcal{W}$. By incorporating it, the fluid dynamical equations are given by

$$D_\mu T^{\mu\nu} = \nabla_\mu T^{\mu\nu} + \mathcal{A}^\nu \left(T^\mu_\mu - \mathcal{W}\right) = 0. \tag{4.48}$$

### 4.3.3 Non-linear conformal fluids

We now discuss the general conformal fluid up to second order in the derivative expansion by using the Weyl covariant formalism. At second order, the possible operators transforming homogeneously that can appear are either two-derivative expressions acting on the dynamical degrees of freedom or terms involving squares of first derivatives. We start with the operators which can be build solely from the velocity field:

$$\begin{aligned}
&\mathcal{D}_\mu \mathcal{D}_\nu u^\lambda = \mathcal{D}_\mu \sigma_\nu{}^\rho + \mathcal{D}_\mu \omega_\nu{}^\rho = e^{-\phi} \tilde{\mathcal{D}}_\mu \tilde{\mathcal{D}}_\nu \tilde{u}^\rho\\
&\mathcal{D}_\rho \sigma_{\mu\nu} = e^\phi\, \tilde{\mathcal{D}}_\rho \tilde{\sigma}_{\mu\nu}\ , \qquad \mathcal{D}_\rho \omega_{\mu\nu} = e^\phi\, \tilde{\mathcal{D}}_\rho \tilde{\omega}_{\mu\nu}\ .
\end{aligned} \tag{4.49}$$

In addition, we have terms involving squares of first derivative operators:

$$\sigma^{\mu}_{\ \rho}\,\sigma^{\nu\rho} = e^{-4\phi}\,\tilde{\sigma}^{\mu}_{\ \rho}\,\tilde{\sigma}^{\nu\rho}\ , \qquad \omega^{\mu}_{\ \rho}\,\omega^{\nu\rho} = e^{-4\phi}\,\tilde{\omega}^{\mu}_{\ \rho}\,\tilde{\omega}^{\nu\rho}\ , \qquad \sigma^{\mu}_{\ \rho}\,\omega^{\nu\rho} = e^{-4\phi}\,\tilde{\sigma}^{\mu}_{\ \rho}\,\tilde{\omega}^{\nu\rho}\ , \tag{4.50}$$

We also have operators involving the temperature $T$, which has conformal weight $\omega = 1$. The terms involving no more than two derivatives are

$$\begin{aligned} \mathcal{D}_\mu T &= e^{-\phi}\tilde{D}_\mu \tilde{T}, \\ \mathcal{D}_\rho \mathcal{D}_\sigma T &= e^{-\phi}\tilde{\mathcal{D}}_\rho \tilde{\mathcal{D}}_\sigma \tilde{T}. \end{aligned} \tag{4.51}$$

To complete the classification, we need to study the terms appearing from commutators of covariant derivatives on the curvature tensors. All such tensors can be written in the Weyl covariant formalism, and in particular for $d > 3$ we have one symmetric traceless tensor involving two derivatives:

$$C_{\mu\nu\rho\sigma}u^\rho u^\sigma = \tilde{C}_{\mu\nu\rho\sigma}\tilde{u}^\rho \tilde{u}^\sigma, \tag{4.52}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor.

### 4.3.4 The non-linear conformal energy-momentum tensor

At first two orders in the derivative expansion, the set of symmetric traceless tensors which transform homogeneously under Weyl rescaling are:

$$\begin{aligned} \text{First order}: &\ \sigma^{\mu\nu} \\ \text{Second order}: &\ \mathcal{F}_{1\mu\nu} = 2u^\rho \mathcal{D}_\rho \sigma_{\mu\nu}, \qquad \mathcal{F}_{2\mu\nu} = C_{\mu\rho\nu\sigma}u^\rho u^\sigma, \\ &\ \mathcal{F}_{3\mu\nu} = 4\sigma^{\rho<\mu}\sigma^{\nu>}{}_{\rho}, \qquad \mathcal{F}_{4\mu\nu} = 2\sigma^{\rho<\mu}\omega^{\nu>}{}_{\rho}, \\ &\ \mathcal{F}_{5\mu\nu} = \omega^{\rho<\mu}\omega^{\nu>}{}_{\rho}. \end{aligned} \tag{4.53}$$

The general contribution to the energy-momentum tensor is thus given by

$$\begin{aligned} \Pi^{\mu\nu}_{(1)} &= -2\eta\sigma^{\mu\nu} \\ \Pi^{\mu\nu}_{(2)} &= \tau_\pi \eta \mathcal{F}_1^{\mu\nu} + \kappa \mathcal{F}_2^{\mu\nu} + \lambda_1 \mathcal{F}_3^{\mu\nu} + \lambda_2 \mathcal{F}_4^{\mu\nu} + \lambda_3 \mathcal{F}_5^{\mu\nu}. \end{aligned} \tag{4.54}$$

In general, the value of the six transport coefficients $\eta, \tau_\pi, \kappa, \lambda_i$, for $i = \{1, 2, 3\}$ is determined holographically using the bulk gravitational theory.

## 4.4 Holographic derivative expansion

We present here the first type of holographic expansion: the one in derivatives of the velocity field [26]. We first start with a static black brane solution, to which will correspond a perfect fluid configuration on the boundary. Then, by promoting the parameters of the black brane to slowly varying functions, and imposing order by order in the perturbative expansion that the metric is a solution of Einstein's equation, we find at the boundary Navier-Stokes equations and higher order corrections.

### 4.4.1 Preliminaries: Schwarzschild black holes in $\mathrm{AdS}_{\mathrm{d+1}}$

We start by considering the planar Schwarzschild-$\mathrm{AdS}_{\mathrm{d+1}}$ black hole written in Poincaré patch:

$$ds^2 = -r^2 f(br) dt^2 + \frac{dr^2}{r^2 f(br)} + r^2 \delta_{ij} dx^i dx^j, \tag{4.55}$$

where

$$f(r) = 1 - \frac{1}{r^d}. \tag{4.56}$$

The boundary energy-momentum tensor is computed by regulating the asymptotically $\mathrm{AdS}_{\mathrm{d+1}}$ metric at some cut-off hypersurface $r = \Lambda_C$ and considering the induced metric on this surface $g_{\mu\nu}$. This holographic energy-momentum tensor is given in terms of the extrinsic curvature $K_{\mu\nu}$ and of the metric data on the hyper surface. Denoting the unit outward normal to the surface by $n^\mu$ we have

$$K_{\mu\nu} = g_{\mu\rho} \nabla^\rho n_\nu. \tag{4.57}$$

For (4.55), the energy-momentum tensor is just the one of a perfect fluid in the rest frame

$$T^{\mu\nu} = \varepsilon \eta^{\mu\nu} + p u^\mu u^\nu, \tag{4.58}$$

where the conformal condition (4.26) is holding. This result is not surprising, since the solution in the bulk is static and therefore we expect it to correspond to global thermal equilibrium.
By boosting the solution (4.55) along the translationally invariant spatial directions $x^i$ we generate the $d$ parameters family of solutions

$$ds^2 = \frac{dr^2}{r^2 f(br)} + r^2 \left(-f(br) u_\mu u_\nu + P_{\mu\nu}\right) dx^\mu dx^\nu, \tag{4.59}$$

where

$$\begin{aligned} u^0 &= \frac{1}{\sqrt{1-\beta^2}}, \\ u^i &= \frac{\beta^i}{\sqrt{1-\beta^2}}, \end{aligned} \tag{4.60}$$

$\beta_i$ being the parameter associated with the boost, $\beta^2 = \beta_i \beta^i$, and $b$ the one associated with the $r$-dilation. We also remember that $P^{\mu\nu}$ is the spacial projector defined in (4.6). The boosted black brane (4.59) can be understood physically as follows. The isometry group of $\mathrm{AdS}_{\mathrm{d+1}}$ is $SO(d,2)$. The Poincaré algebra plus dilations form a distinguished subalgebra of this group. The rotations $SO(d)$ and the translations $\mathbb{R}^{1,d-1}$ that belong to this sub algebra annihilate the static black brane solution (4.59) in $\mathrm{AdS}_{\mathrm{d+1}}$ because of the symmetries of this background.

However, the remaining symmetry generators, that is the dilations and the boosts, act nontrivially on this brane, generating a $d$ parameter set of black hole solutions. The energy-momentum tensor for the boosted black brane is still the one of an ideal conformal fluid.
In order to describe hydrodynamics, we should perturb the system away from global equilibrium. It it natural to do it by promoting the parameters of the boosted black brane to be slowly varying functions of the boundary coordinates $x^\mu$. Together with this, we should also let the boundary metric vary. The boundary fluid dynamics is governed by the conservation of the energy-momentum tensor,

$$\nabla_\mu T^{\mu\nu} = 0, \tag{4.61}$$

where $T^{\mu\nu}$ is modified by adding derivatives of the parameters, that is by expressing it in a gradient expansion. The bulk metric (4.59) with local parameters is no longer a solution to Einstein's equations. Since we are assuming slowly variations of the parameters, we will be able to construct a solution of Einstein's equations starting from (4.59) by doing a derivative expansion.
Hence, the fluid/gravity correspondence gives a natural framework to derive the fluid dynamic energy-momentum tensor from the gravitational bulk. This procedure determines in principle the various order transport coefficients in an ambiguous way.

### 4.4.2 The perturbative procedure

As we said, if the parameters $\{b, \beta^i\}$ are promoted to local functions of the boundary coordinates $\{t, x^i\}$, the metric does not satisfy Einstein's equations anymore. However, it is possible to construct a new solution by modifying the starting metric and by constraining the local parameters. The boundary dual of the resulting metric can be interpreted as a fluid with dissipative contributions for $d > 2$, and the parameters $\{b, \beta^i\}$ are connected with the fluid temperature and velocity.
To implement the procedure, it is first useful to write (4.59) in Eddington-Finkelstein coordinates:

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(br) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu. \tag{4.62}$$

Note that this metric can be also casted in a Weyl covariant form as

$$ds^2 = -2u_\mu dx^\mu \left(dr + r\mathcal{A}_\mu dx^\mu\right)^2_r (1 - f(br)) u_\mu u_\nu dx^\mu dx^\nu + r^2 g_{\mu\nu} dx^\mu dx^\nu. \tag{4.63}$$

The main reason to use this chart of coordinates is that in this frame the solution is regular everywhere except at the origin $r = 0$, and in particular it is regular at the future horizon. This implies that the map from the space of distinguished solutions of fluids to solutions of Einstein's equations in the bulk is one to one. Moreover, these coordinates provide a clear physical interpretation of the locally equilibrated

fluid dynamical domains in the bulk geometry. In particular, the boundary domains where the local thermal equilibrium is attained exited along ingoing radial null geodesics into the bulk. Thus, a given boundary domain correspond to a tube of width set by the scale of variation in the boundary. By patching together these tubes we obtain a solution to Einstein's equations and, moreover, this patching can be done order by order in boundary derivatives.
The outline of the perturbation scheme is as follows. First we replace the constant parameters $b$ and $\beta_i$ in the metric (4.62) by slowing varying functions of the boundary coordinates, $b(x^\mu)$ and $\beta_i(x^\mu)$:

$$ds^2 = -2u_\mu(x^\alpha)dx^\mu dr - r^2 f(b(x^\alpha)r)u_\mu(x^\alpha)u_\nu(x^\alpha)dx^\mu dx^\nu + r^2 P_{\mu\nu}(x^\alpha)dx^\mu dx^\nu. \tag{4.64}$$

In general this metric, which we will denote as $G^{(0)}\left(b(x^\mu), \beta_i(x^\mu)\right)$, is not a solution to Einstein's equations. It is still regular everywhere but for $r = 0$, and if all the derivatives of the parameters $b(x^\mu)$ and $\beta_i(x^\mu)$ are small, it is well approximated by a boosted black brane. Thus, for slowly varying functions $b(x^\mu)$ and $\beta_i(x^\mu)$ it is intuitive to think that (4.64) is a good approximation to a true solution of Einstein's equations with regular even horizon. This is indeed the case, provided that $b(x^\mu)$ and $\beta_i(x^\mu)$ satisfy a set of equations of motion which are just the equations for boundary fluid dynamics.
In particular, in order to find a solution to Einstein's equations starting from (4.64), we need to add correction in a power series of a small parameter $\varepsilon$:

$$G = G^{(0)}\left(\beta_i, b\right) + \varepsilon G^{(1)}\left(\beta_i, b\right) + \varepsilon^2 G^{(2)}\left(\beta_i, b\right) + \mathcal{O}\left(\varepsilon^3\right), \tag{4.65}$$

where $G^{(0)}$ is the metric (4.64) and $G^{(1)}$, $G^{(2)}$, etc, are correction metrics to be determined. Note that for convenience of notation we dropped the space-time indices in $G^{(n)}$ and the dependence of the parameters $b$ and $\beta_i$ on $x^\mu$. We also need to correct the velocity and temperature fields order by order in the perturbative expansion:

$$\begin{aligned} \beta_i &= \beta_i^{(0)} + \varepsilon\beta_i^{(1)} + \mathcal{O}\left(\varepsilon^2\right) \\ b &= b^{(0)} + \varepsilon b^{(1)} + \mathcal{O}\left(\varepsilon^2\right), \end{aligned} \tag{4.66}$$

where $\beta_i^{(n)}$ and $b^{(n)}$ are functions of $\varepsilon x^\mu$. It is useful at this point to fix a gauge

$$G_{rr} = 0, \qquad G_{r\mu} = u_\mu. \tag{4.67}$$

With this gauge choice curves of $x^\mu$ = constant are affinity parameterized null geodesics in the resulting space-time, with the radial coordinate $r$ being the affine parameter.
We can now plug the Ansatz (4.65) together with (4.66) into Einstein's equations and find the solution order by order in $\varepsilon$. Let us imagine that we have solved the

perturbation theory to the $(n-1)^{\text{th}}$ order, that we we have determined $G^{(m)}$ for $m \leq n-1$ and $\beta_i^{(m)}$ and $b^{(m)}$ for $m \leq n-2$. By going one order further and imposing Einstein's equations, we get an equation of the form

$$\mathbb{H}\left[G^{(0)}\left(\beta_i^{(0)}, b^{(0)}\right)\right] G^{(n)}\left(x^\mu\right) = s_n. \tag{4.68}$$

Here $\mathbb{H}$ is a linear differential operator of second order depending only on the coordinate $r$. As $G^{(n)}$ is of order $\varepsilon^n$, a boundary derivative would produce a higher order term in $\varepsilon$. Notice also that $\mathbb{H}$ depends only on the value of the parameters at $x^\mu$ and not on their boundary derivatives. Thus, $\mathbb{H}$ in an ultra local operator in the boundary direction. Also, $\mathbb{H}$ is independent of $n$: we have the same homogeneous operator at every order in perturbation theory. This makes the perturbation expansion in $\varepsilon$ ultra-local in the boundary directions, allowing us to solve the equations point by point in the boundary. The source terms $s_n$ will however depend on the order of the expansion. They are a local expression of $n^{\text{th}}$ order in boundary derivatives o $\beta_i^{(0)}$ and $b^{(0)}$, as well as of $(n-k)^{\text{th}}$ order in $\beta_i^{(k)}$, $b^{(k)}$ for all $k \leq n-1$. The gravitational equation (4.68) form a set of $\frac{(d+1)(d+2)}{2}$ equations. It is useful to split these into two classes of equations: a class that determines the metric data we need, comprising of $\frac{d(d+1)}{2}$ equations which we call "dynamical equations", and a second set of $d$ equations which are first-order in derivatives of $r$, and which we call "constraint equations".

- **Constraint equations:** They are obtained by contracting the Einstein tensor $E_{MN}$ with the one-form normal to the boundary:

  $$E_M^{(c)} = E_{MN}\xi^N, \tag{4.69}$$

  where in our case $\xi_N = dr$ and thus $\xi^N = g^{Nr}$. Considering the boundary directions of $E_M^{(c)}$ we obtain a set of $d$ equations involving only the parameters and not the unknown metric corrections. These reproduce the equations of conservation of the boundary energy-momentum tensor at order $n-1$:

  $$\nabla_\mu T^{\mu\nu}_{(n-1)} = 0\ . \tag{4.70}$$

  We can thus interpret the parameters as fluid quantities. The constraint equations can be used to determine those parameters, namely $b^{(n-1)}$ and $\beta_i^{(n-1)}$, and are thus giving the fluid dynamics equations up to order $\mathcal{O}\left(\varepsilon^n\right)$ in the gradient expansion, assuming that the solutions at preceding orders are known.

- **Dynamical equation:** the remaining equations involve $E_r^{(C)}$ and the dynamical Einstein's equations. They depend in general on both the parameters and on $G^{(n)}$, and they can be used to solve the latter unknown function in terms of the derivatives of the parameters. Being a $n^{\text{th}}$-order contribution, $G^{(n)}$ will depend on $n$ derivatives of the parameters. It is convenient to classify the dynamical equations according to the representations of the symmetry group of the zeroth order solution. This is given by rotational symmetry in the spatial sections on the boundary, and it is thus the little group $SO(d-1)$. By using this symmetry, it is possible to decouple the system of equations into a set of first order differential operators. Having performed this diagonalization of the system of equations one has a formal solution of the form:

$$G^{(n)} = \text{particular}(s_n) + \text{homogeneous}(\mathbb{H}) \tag{4.71}$$

To determine the solution uniquely we need to prescribe boundary conditions: we impose that the solution is asymptotically $\mathrm{AdS_{d+1}}$ and also demand regularity at all $r \neq 0$. In particular, the solution should be regular at the hypersurface $b\,r = 1$. For an arbitrary non-singular and appropriately normalizable source $s_n$ it is always possible to choose such boundary conditions. Furthermore, if the solution at order $n-1$ is non-singular at all nonzero $r$, it is guaranteed to produce a non-singular source at all nonzero $r$. Consequently, the non-singularlity of $s_n$ follows inductively.

### 4.4.3 First order in the derivative expansion

We briefly illustrate how to carry out at first order the formal derivative expansion we presented in the last section. Consider the zeroth order metric $G^{(0)}$ given in (4.64). If we want to work to first order in boundary derivatives, we can pick a point on the boundary $x^\mu = x_0^\mu$, which can be chosen to be at the origin because of the symmetries of the space without loss of generality. With the technique explained previously, it is possible to construct the solution of Einstein's equations at a given order in a neighborhood of $x_0^\mu$. Note that thus the metric is not global. At $x_0^\mu$ we can use the local scaling symmetry to set $b^{(0)} = 1$ and pass to a local inertial frame so that $\beta_i^{(0)} = 0$. Expanding (4.64) up to zeroth-order at $x_0^\mu$ we get

$$\begin{aligned} ds^2_{(0)} &= 2\,dv\,dr - r^2\,f(r)\,dv^2 + r^2\,dx_i\,dx^i \\ &\quad - 2\,\delta\beta_i^{(0)}\,dx^i\,dr - 2\,\delta\beta_i^{(0)}\,r^2\,(1-f(r))\,dx^i\,dv - d\,\frac{\delta b^{(0)}}{r^{d-2}}\,dv^2\ , \end{aligned} \tag{4.72}$$

where we have introduced $\delta\beta_i^{(0)} = x^\mu\,\partial_\mu\beta_i^{(0)}$ and $\delta b^{(0)} = x^\mu\,\partial_\mu b^{(0)}$, which are the leading terms in the Taylor expansion of the velocity and temperature fields at $x_0^\mu = 0$.

We would like now to compute the first order corrections to the metric, that is $G^{(1)}$. For this, we need need to choose an Ansatz for $G^{(1)}$. As mentioned earlier, it is useful to use the $SO(d-1)$ spatial rotation symmetry at $x_0^\mu$ to decompose modes into various representations of this symmetry. Modes of $G^{(1)}$ transforming under different representations decouple from each other by symmetry. We have the following decomposition into $SO(d-1)$ irreducible representations:

$$\begin{aligned} &\text{scalars:} && G^{(1)}_{vv}\,, G^{(1)}_{vr}\,, \sum_i G^{(1)}_{ii}\,, \\ &\text{vectors:} && G^{(1)}_{vi}\,, \\ &\text{tensors:} && G^{(1)}_{ij} \end{aligned} \tag{4.73}$$

We work sector by sector and solve the constraint and the dynamical Einstein's equations.
In the scalar sector, we find that the constraint equations imply that

$$\frac{1}{d-1}\,\partial_i\beta_i^{(0)} = \partial_v b^{(0)} \;, \tag{4.74}$$

while in the vector sector we have

$$\partial_i b^{(0)} = \partial_v \beta_i^{(0)} \;. \tag{4.75}$$

These are just the Euler equations, that is the relativistic Navier-Stokes equations for zero viscosity and heat conduction terms. They are equivalent to demanding the energy momentum conservation (4.45) at a given point $x_0^\mu$.
The dynamical equations can be used to solve for the functions appearing in $G^{(1)}$, and we shall demand as well regularity for $r \neq 0$. We do not present the result here, but we just mention the form of the differential operator in the various sectors:

$$\begin{aligned} &\text{vector :} && \mathbb{H}_{\mathbf{d-1}}\mathcal{O} = \frac{d}{dr}\left(\frac{1}{r^{d-1}}\,\frac{d}{dr}\,\mathcal{O}\right) \\ &\text{tensor :} && \mathbb{H}_{\frac{\mathbf{d(d+1)}}{\mathbf{2}}}\mathcal{O} = \frac{d}{dr}\left(r^{d+1}\,f(r)\,\frac{d}{dr}\,\mathcal{O}\right) \end{aligned} \tag{4.76}$$

which, as advertised earlier, are simple differential operators in the radial variable alone.
The energy-momentum tensor is given by

$$T^{\mu\nu} = \frac{1}{b^4}\left(3u^\mu u^\nu\right) - \frac{1}{b^3}\sigma^{\mu\nu}, \tag{4.77}$$

where the last term takes into account the dissipative effects.
The calculation can in principle be carried out at any desired order in the derivative expansion. As we discussed earlier, the form of the differential operator (4.68) remains invariant in the course of the perturbation expansion. Thus, one needs to compute at any given order the source terms $s_n$ and, in addition, to ensure that the lower order energy-mometum tensor conservation equations are satisfied. The energy-momentum tensor turns out to be the one quoted in (4.54).
The essential physical points arising from the fluid-gravity correspondence in the velocity derivative expansion can be summarized as follows:

- The gravitational derivation of the relativistic Navier-Stokes equations and its higher-order generalizations confirms the basic intuition that fluid dynamics is indeed the correct long-wavelength effective description of strongly coupled field theory dynamics.
- The geometries dual to fluid dynamics turn out to be black hole spacetimes with regular event horizons. This indicates that the hydrodynamic regime is special and, in particular, that the fluid dynamical energy-momentum tensors lead to regular gravity solutions respecting cosmic censorship.
- The explicit construction of the fluid dynamical energy-momentum tensor leads to a precise determination of higher order transport coefficients for the dual field theory.

## 4.5 The Fefferman-Graham expansion

The Fefferman-Graham procedure [36], [27], [28] is an expansion in the holographic radial coordinate $r$, as opposed to the perturbative expansion we just presented which is an expansion in derivatives of the velocity field. In particular, while the Fefferman-Graham expansion is a large $r$ expansion, the expansion in derivatives of the velocity field can be seen as a tube going from the bulk to the boundary spaces, and the higher we go in the expansion the more we increase the width of the angle of the tube. While with the velocity derivative expansion we have to start from the Schwarzschild boosted black brane solution, the Fefferman-Graham expansion allows us to find the holographic fluid dual to any kind of asymptotically AdS black hole. In particular, it can be proven that any asymptotically AdS metric can be put in the form

$$ds^2 = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}\eta_{ab}\hat{E}^a(r,\mathrm{x})\hat{E}^b(r,\mathrm{x})\,, \tag{4.78}$$

where $\Lambda = -\frac{3}{L^2} = -3k^2$ is the negative cosmological constant, x design the boundary coordinates and we use the index $a, b, \dots$ for the boundary coordinates. For torsionless connections there is always a suitable gauge choice such that the metric (4.78) is fully determined by two coefficients $\hat{e}^a$ and $\hat{f}^a$ in the expansion of the coframe one-forms $\hat{E}^a(r,\mathrm{x})$ along the holographic coordinate $r \in [0, \infty)$

$$\hat{E}^a(r,\mathrm{x}) = \left[\hat{e}^a(\mathrm{x}) + \frac{L^2}{r^2}\hat{F}^a(\mathrm{x}) + \cdots\right] + \frac{L^3}{r^3}\left[\hat{f}^a(\mathrm{x}) + \cdots\right]. \tag{4.79}$$

The asymptotic boundary is at $r \to \infty$:

$$ds^2_{\mathrm{bdy}} = \lim_{r\to\infty}\frac{ds^2}{k^2r^2}. \tag{4.80}$$

The ellipses in (4.79) denote terms that are multiplied by higher negative powers of $r$. Their coefficients are not independent, but are determined in terms of $\hat{e}^a$ and

$\hat{f}^a$, and have specific geometrical interpretations[1], though this is not relevant for our discussion.
The two independent functions $\hat{e}^a(\mathrm{x})$ and $\hat{f}^a(\mathrm{x})$ can be seen as vector-valued one-forms in the boundary. They can be interpreted as proper canonical variables playing the role of boundary "coordinate" and "momentum" for the Hamiltonian evolution along $r$. For stationary backgrounds, as those we will to consider, $\hat{e}^a$ and $\hat{f}^a$ are $t$-independent.
Although the Fefferman-Graham expansion is valid in general for asymptotically AdS metrics in any dimension, we focus now on the case of our interest: that is, when the fluid dynamics is in $D = d - 1 = 2 + 1$ dimensions. The boundary "coordinate" is given by the set of one-forms $\hat{e}^a$. For this coframe we must determine the "momentum" of the boundary data. For example, when the boundary data carry zero mass, we expect the momentum to be zero. In this case $\hat{f}^a(x) = 0$ and the unique exact solution of the Einstein's equations is pure $\mathrm{AdS}_4$.
More generally, the vector-valued one-form $\hat{f}^a$ satisfies

$$\hat{f}^a \wedge \hat{e}_a = 0\,, \quad \eta_{abc}\hat{f}^a \wedge \hat{e}^b \wedge \hat{e}^c = 0\,, \quad \eta_{abc}\mathrm{D}\hat{f}^b \wedge \hat{e}^c = 0\,, \tag{4.81}$$

where the action of the generalized exterior derivative D on a vector-valued one-form $\hat{V}^a$ is defined as

$$\mathrm{D}\hat{V}^a = d\hat{V}^a + \eta^a{}_{bc}\hat{B}^b \wedge \hat{e}^c\,, \tag{4.82}$$

and $\hat{B}^a$ is the Levi-Civita spin connection associated with $\hat{e}^a$. The Levi-Civita symbols $\eta_{abc}$ should be treated as tensor densities in curved space. The objects with correct tensor transformation properties scale as metric determinants, i.e. $\eta_{\alpha\beta\gamma\delta} \propto \sqrt{-g}$, and $\eta^{\alpha\beta\gamma\delta} \propto \frac{1}{\sqrt{-g}}$.
The conditions (4.81) imply, respectively, symmetry, absence of trace and covariant conservation of the tensor $T = T^a{}_b \check{e}_a \otimes \hat{e}^b$, defined as

$$\hat{f}^a = \frac{1}{\kappa}T(\hat{e}^a) = \frac{1}{\kappa}T^a{}_b\hat{e}^b\,, \quad \kappa = \frac{3}{8\pi G_{\mathrm{N}}L}\,. \tag{4.83}$$

Hence, we can interpret the latter of (4.81) as the covariantly conserved energy-momentum tensor of a conformal field theory. As mentioned, we are interested in particular in stationary bulk solutions, for which we expect the energy-momentum tensor be reduced to the perfect relativistic form:

$$T^a{}_b = (\varepsilon + p)u^a u_b + p\delta^a_b\,. \tag{4.84}$$

Although the two necessary ingredients for the description of a relativistic perfect fluid, namely the boundary frame and the velocity one-form, are nicely packaged in the leading and subleading independent boundary data, until now we did not assume any specific relationship between them. Such a relationship is imposed from the bulk side: to any solution of the gravitational equations will correspond appropriate boundary conditions.

[1]For example, the coefficient $\hat{F}^a$ is related to the boundary Schouten tensor.

## 4.6 Fluids in 2+1 dimensional Randers-Papapetrou backgrounds

We would like to specialize to fluids appearing as the boundary of 3+1 dimensional stationary black holes in the Fefferman-Graham expansion [10], [21]. First we analyze the additional properties of the boundary fluid when the background is stationary: namely, the velocity field of such fluids is a Killing vector, and this has non-trivial consequences on the dynamics. Then, we will discuss the features of the 2+1 dimensional boundary metric, which is in general going to be of the so-called "Randers-Papapetrou" form.

### 4.6.1 2+1 dimensional fluids with additional Killing vector

When the background is stationary, the velocity field of the fluid is a Killing vector field with respect to the boundary metric. In general, a Killing vector field $\xi$ satisfies

$$\nabla_{(a}\xi_{b)} = 0. \tag{4.85}$$

They have several remarkable properties, among which we quote:

- they have vanishing expansion,
- a constant-norm[2] Killing vector field is furthermore geodesic and shearless. It can only carry vorticity.

In general, fluids exhibit dissipative phenomena as they describe media with non-zero shear viscosity. However, such fluids can be in special kinematic configurations where the effects of dissipation are ignorable[3]. In this case, their dynamics is captured by the perfect part of the energy-momentum tensor and the equations of motion read

$$\begin{cases} (\varepsilon + p)\theta + \nabla_{\check{u}}\varepsilon = 0 \\ (\varepsilon + p)\hat{a} - \nabla_{\perp}p = 0, \end{cases} \tag{4.86}$$

where $\nabla_{\perp} = \nabla + \hat{u}\nabla_{\check{u}}$ stands for the covariant derivative along the directions normal to the velocity field, and the acceleration form is $\hat{a} = a_a\hat{e}^a$. We should also take into account the conformality of the fluid, $\varepsilon = (D-1)p$. The latter implies that Eqs. (4.86) becomes

$$\begin{cases} \nabla_{\check{u}}\varepsilon = 0 \\ \hat{a} = \frac{\nabla_{\perp}p}{Dp}. \end{cases} \tag{4.87}$$

[2]This is not an empty statement since Killing vectors cannot be normalized at will. When their norm is constant, it can be consistently set to $-1$, 0 or 1.

[3]A fluid can be stationary and altogether dissipate energy provided it is not isolated. These situations are called "forced steady states". On curved boundary backgrounds, the forcing task can be met by gravity through the boundary conditions. As this feature does not appear in the backgrounds that we will consider, we don't analyze it further.

These equations are telling us that energy density is conserved along the fluid lines and that, in the absence of spatial pressure gradients, that is for energy and pressure constants in spacetime, the flow is geodesic.
We can thus summarize the properties of a fluid whose velocity field is a Killing vector:

- the flow is geodesic, shearless and expansionless.
- the internal energy density is conserved and the pressure is spatially homogeneous.
- if the fluid is conformal then $\varepsilon = (D-1)p$ is constant in spacetime.

Therefore, despite its viscosity, the kinematic state of the fluid can be steady and non-dissipative. For this to happen, however, the existence of a constant-norm time-like Killing vector is required. In other words, the background geometry must itself be stationary[4]. In this case, the constant-norm time-like Killing vector congruence allows for the definition of a global time coordinate, with associated inertial frames. The latter are comoving with the fluid. All the examples we will discuss in the following fall into this class.

### 4.6.2 Randers-Papapetrou stationary geometries

Boundary metrics appearing in the holographic analysis of stationary black holes are of the generic form:

$$ds^2 = B^2 \left( -(dt - b_i dx^i)^2 + a_{ij}(x) dx^i dx^j \right), \tag{4.88}$$

where $B, b_i, a_{ij}$ are $x$-dependent functions, where $x$ is the two-dimensional spatial part of the three-dimensional boundary metric. Note that we use $i, j, \dots$ to indicate components of the two-dimensional space, and $a, b, \dots$ as index of the three-dimensional metric. These kind of metrics are called of the "Randers-Papapetrou" type.
For later convenience, we introduce $a^{ij}, b^i$ and $\gamma$ such that

$$a^{ij} a_{jk} = \delta^i_k, \quad b^i = a^{ij} b_j, \quad \gamma^2 = \frac{1}{1 - a^{ij} b_i b_j}. \tag{4.89}$$

The metric components read:

$$g_{00} = -B^2, \quad g_{0i} = B^2 b_i, \quad g_{ij} = B^2 (a_{ij} - b_i b_j), \tag{4.90}$$

and those of the inverse metric:

$$g^{00} = -\frac{1}{\gamma^2 B^2}, \quad g^{0i} = \frac{b^i}{B^2}, \quad g^{ij} = \frac{a^{ij}}{B^2}. \tag{4.91}$$

[4]More generally, it can be shown that the velocity field $u^\mu$ of a stationary fluid flow has to be proportional to a Killing vector field of the background geometry.

Finally,

$$\sqrt{-g} = B^D \sqrt{a}, \tag{4.92}$$

where $a$ is the determinant of the symmetric matrix with entries $a_{ij}$.
In the natural frame of the above coordinate system $\{\partial_t, \partial_i\}$, any observer at rest has normalized velocity $\check{u} = \frac{1}{B}\partial_t$ and dual form $\hat{u} = -B(dt - b)$, where $b = b_i dx^i$. The normalized vector field $\check{u}$ is not in general Killing, as opposed to $\partial_t$. For this observer, the acceleration is thus non vanishing:

$$\check{a} = \nabla_{\check{u}}\check{u} = g^{ij}\partial_i \ln B \left(\partial_j + b_j \partial_t\right). \tag{4.93}$$

The motion is inertial if and only if $B$ is constant. It will be enough for our purposes to consider the case $B = 1$, and in all subsequent formulas we will assume this choice. We furthermore introduce a frame

$$\check{e}_0 = \partial_t, \quad \check{e}_\alpha = E_\alpha{}^i \left(b_i \partial_t + \partial_i\right), \quad E_\alpha{}^i E^\beta{}_i = \delta^\beta_\alpha \tag{4.94}$$

adapted to the geodesics at hand and its dual orthonormal coframe

$$\hat{e}^0 = \mathrm{d}t - b, \quad \hat{e}^\alpha = E^\alpha{}_i \mathrm{d}x^i, \quad E^\alpha{}_i E^\beta{}_i \delta_{\alpha\beta} = a_{ij}. \tag{4.95}$$

This will be referred to as the Randers-Papapetrou frame. Note that we use greek indices $\alpha, \beta, \ldots$ to indicate components in this frame.
The constant-norm Killing vector field $\check{u} = \check{e}_0 = \partial_t$ defines a geodesic congruence, that is it defines the orbits of all observers at rest in the Randers-Papapetrou frame. The velocity field is defined as

$$\hat{u} = -\hat{e}^0 = -dt + b, \tag{4.96}$$

and allows us to write the Randers-Papapetrou metric in the convenient form

$$ds^2 = -\hat{u}^2 + d\tilde{s}^2, \tag{4.97}$$

with the two-dimensional part of the metric being $\mathrm{d}\tilde{s} = a_{ij} dx^i dx^j$ (in general, we will use the tilde to indicate two-dimensional quantities). As was shown in the last section, the Randers-Papapetrou frame corresponds to zero shear and expansion, but non-trivial vorticity. From the definition of the vorticity tensor (4.16), we can define the vorticity form as

$$2\omega = \omega_{ab}\hat{e}^a \wedge \hat{e}^b = d\hat{u} + \hat{u} \wedge \hat{a}. \tag{4.98}$$

Then, in our frame we have:

$$\omega = \frac{1}{2}\mathrm{d}b \quad \Rightarrow \quad \omega_{0i} = 0, \quad \omega_{ij} = \frac{1}{2}\left(\partial_i b_j - \partial_j b_i\right). \tag{4.99}$$

The physical effect of vorticity is seen in the obstruction to the parallel transport of the spatial frame $\check{e}_\alpha$ along the congruence:

$$\nabla_{\check{e}_0}\check{e}_\alpha = \omega^{\mathrm{RP}}_{\alpha\beta}\delta^{\beta\gamma}\check{e}_\gamma \quad \Leftrightarrow \quad \nabla_{\partial_t}\partial_i = \omega_{ij}a^{jk}\left(\partial_k + b_k\partial_t\right), \tag{4.100}$$

$\omega^{\mathrm{PR}}_{\alpha\beta} = E_\alpha{}^i E_\beta{}^j \omega_{ij}$ are the components of the spatial part of the vorticity in the Randers-Papapetrou frame.
The Hodge-dual of the vorticity (4.99) has components

$$\psi^\mu = \eta^{\mu\nu\rho}\omega_{\nu\rho} \Leftrightarrow \omega_{\nu\rho} = -\frac{1}{2}\eta_{\nu\rho\mu}\psi^\mu. \tag{4.101}$$

In $2+1$ dimensions, it is aligned with the velocity field:

$$\psi^\mu = q u^\mu, \tag{4.102}$$

where

$$q(x) = -\frac{\epsilon^{ij}\partial_i b_j}{\sqrt{a}} \tag{4.103}$$

is a static scalar field. We define $\tilde{R}(x)$ as the Ricci scalar of the two-dimensional part of the Randers-Papapetrou geometry, described by the metric $a_{ij}$. The scalar field $q$ and the two-dimensional Ricci scalar $\tilde{R}(x)$ carry all relevant information about the curvature of the Randers-Papapetrou geometry. We quote for latter use the relation between the three-dimensional and the two-dimensional curvature scalars:

$$R = \tilde{R} + \frac{q^2}{2}. \tag{4.104}$$

The three-dimensional Ricci tensor can be written as

$$R_{\mu\nu}\, dx^\mu dx^\nu = \frac{q^2}{2}u^2 + \frac{\tilde{R}+q^2}{2}d\tilde{s}^2 - u\, dx^\rho u^\sigma \eta_{\rho\sigma\mu}\nabla^\mu q, \tag{4.105}$$

while the three-dimensional Cotton-York tensor takes the form (the hat here indicates two-dimensional quantities)

$$\begin{aligned} C_{\mu\nu}\, dx^\mu dx^\nu \;\; = \;\; & \frac{1}{2}\left(\tilde{\nabla}^2 q + \frac{q}{2}(\tilde{R}+2q^2)\right)\left(2u^2 + d\tilde{s}^2\right) \\ & -\frac{1}{2}\left(\tilde{\nabla}_i\hat{\nabla}_j q\, dx^i dx^j + \tilde{\nabla}^2 q\, u^2\right) \\ & -\frac{u}{2}dx^\rho u^\sigma \eta_{\rho\sigma\mu}\nabla^\mu(\tilde{R}+3q^2). \end{aligned} \tag{4.106}$$

The latter is a symmetric traceless tensor defined in general as

$$C^{\mu\nu} = \eta^{\mu\rho\sigma}\nabla_\rho\left(R^\nu_\sigma - \frac{1}{4}R\delta^\nu_\sigma\right). \tag{4.107}$$

In three-dimensional geoemtries, it replaces the role of the Weyl tensor, which is identically vanishing. In particular, conformally flat backgrounds have zero Cotton-York tensor.
The fluid on Randers-Papapetrou backgrounds in perfect equilibrium has the energy-momentum tensor

$$T^{(0)}_{\mu\nu}\, dx^\mu dx^\nu = p\left(2u^2 + d\tilde{s}^2\right), \tag{4.108}$$

with the velocity being given by (4.96) and $p =$ constant. We have also used here $\varepsilon = 2p$, which is a consequence of conformal invariance.
Randers-Papapetrou metrics do not exhibit ergoregions since $g_{00} = -1$[5]. However, regions where hyperbolicity is broken (*i.e.* where constant-$t$ surfaces become time-like) are not excluded. This happens whenever there exist regions where $b_i b_j a^{ij} > 1$. Indeed, in these regions, the spatial metric $a_{ij} - b_i b_j$ possesses a negative eigenvalue, and constant-$t$ surfaces are no longer spacelike. Therefore, the extension of the physical domain accessible to the inertial observers moving along $\check{u} = \partial_t$ is limited to spacelike disks in which $b_i b_j a^{ij} < 1$ holds.
As a last remark, following Eqs.(4.87), the shearless and expansionless geodesic congruences under consideration could describe the fluid lines of a dissipationless stationary, conformal fluid, under the assumption that the energy and the pressure are conserved and constant all over space. As we will see, this is exactly the dynamics that emerges through holography.

### 4.6.3 The Zermelo frame

There is an equivalent way to recast the Randers-Papapetrou metric: the Zermelo frame. Consider a two-dimensional Euclidean manifold $\mathcal{M}$ with metric $h_{ij}$. The so-called Zermelo navigation problem asks for the minimum-time trajectories on that manifold under the influence of a time-independent wind $W^i$. It can be proven that these minimum-time trajectories coincide with the null geodesics of the Randers-Papapetrou frame, provided that the Randers-Papapetrou data $(a_{ij}, b_i)$ are related to the Zermelo data $(h_{ij}, W^i)$ as

$$h_{ij} = \tfrac{a_{ij} - b_i b_j}{\gamma^2}, \quad h^{ik} h_{kj} = \delta^i_j, \tag{4.109}$$

$$W^i = -\gamma^2 b^i, \quad W_i = h_{ij} W^j = -\tfrac{b_i}{\gamma^2}. \tag{4.110}$$

Using the above, the Randers-Papapetrou metric (4.88) can be recast in the following form

$$\mathrm{d}s^2 = \gamma^2\left[-\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i - W^i \mathrm{d}t\right)\left(\mathrm{d}x^j - W^j \mathrm{d}t\right)\right], \tag{4.111}$$

[5]Ergoregions would require a conformal factor in (4.88) that could vanish and become negative.

which is called the "Zermelo form". This form of the metric suggests the following orthonormal coframe and its dual frame:

$$\hat{z}^0 = \gamma \mathrm{d}t\,, \quad \hat{z}^\alpha = L^\alpha{}_i(\mathrm{d}x^i - W^i\mathrm{d}t), \quad L_\alpha{}^i L^\beta{}_i = \delta^\alpha_\beta, \tag{4.112}$$

$$\check{z}_0 = \tfrac{1}{\gamma}\left(\partial_t + W^i\partial_i\right)\,, \quad \check{z}_\alpha = L_\alpha{}^i\partial_i, \quad L^\alpha{}_i L^\beta{}_j \delta_{\alpha\beta} = \gamma^2 h_{ij}. \tag{4.113}$$

We will call the latter the Zermelo frame. This frame is not inertial: indeed, its timelike vector field $\check{z}_0$ defines a congruence of accelerated lines, $\nabla_{\check{z}_0}\check{z}_0 \neq 0$. It is useful to compare the Randers-Papapetrou frame introduced previously in (4.146) and(4.95) with the Zermelo frame at hand. Being both orthonormal, they are related by a local Lorentz transformation, as we can see by combining the above formulas:

$$\check{e}_0 \;=\; \gamma\left(\check{z}_0 - W^\beta \check{z}_\beta\right), \tag{4.114}$$

$$\check{e}_\alpha \;=\; \Gamma_\alpha{}^\beta\left(\check{z}_\beta - W_\beta\check{z}_0 + \frac{\gamma^2-1}{\gamma^2}\left(\frac{W_\beta W^\gamma}{W^2} - \delta^\gamma_\beta\right)\check{z}_\gamma\right), \tag{4.115}$$

where

$$\Gamma_\alpha{}^\beta = \gamma^2 E_\alpha{}^i L^\beta{}_i, \quad W^\alpha = \tfrac{1}{\gamma}L^\alpha{}_i W^i, \quad W_\alpha = \delta_{\alpha\beta}W^\beta, \tag{4.116}$$

$$W^2 = W^\alpha W_\alpha = W^i W_i = 1 - \tfrac{1}{\gamma^2}. \tag{4.117}$$

These expressions are telling us that each spacetime point is the intersection of two lines, belonging each to the two congruences under consideration. At this point, $W^\alpha$ are the spatial velocity components of the inertial observer in the spatial frame of the accelerated observer and $\frac{1}{\gamma^2} = 1 - W^2$ is the corresponding Lorentz factor. The Randers-Papapetrou observer and the Zermelo one perceive the rotation of the fluid in different ways. The Randers-Papapetrou observers feel the rotation as described in (4.100), so through embarked gyroscope, whereas the Zermelo observers satisfies

$$\nabla_{\check{z}_0}\check{u} = \omega^{\mathrm{Z}}_{0\alpha}\delta^{\alpha\beta}\check{z}_\beta. \tag{4.118}$$

Here $\check{u} = \check{e}_0$ is the velocity of the inertial observers, while $\omega^{\mathrm{Z}}_{ab}$ are the vorticity components as observed in the Zermelo frame: $\omega^{\mathrm{Z}}_{\alpha\beta} = L_\alpha{}^i L_\beta{}^j \omega_{ij}$ and $\omega^{\mathrm{Z}}_{0\beta} = W^\alpha\omega^{\mathrm{Z}}_{\alpha\beta}$. Hence, for the accelerated observers, the inertial ones are subject to a Coriolis force: Zermelo observers are rotating themselves. The velocity vector $\check{u} = \check{e}_0$ of the inertial observers undergoes a precession around the worldline of a Zermelo observer tangent to $\check{z}_0$. Since the observer itself is accelerated, we expect a non-trivial variation of $\check{u}$, which is actually better captured as a Fermi derivative along $\check{z}_0$:

$$\mathrm{D}_{\check{z}_0}\check{u} = \left(\omega^{\mathrm{Z}}_{0\alpha} - \check{z}_\alpha(\gamma)\right)\delta^{\alpha\beta}\check{z}_\beta + W^\alpha\check{z}_\alpha(\gamma)\check{z}_0\,, \tag{4.119}$$

where $\check{z}_\alpha(\gamma) = L_\alpha{}^i\partial_i\gamma$. The extra terms result from the rotation of the Zermelo frame and contribute to the observed precession of the velocity vector $\check{u}$.

One can try to tune rotating frames so as to make the perceived angular momentum of a given congruence disappear, *i.e.* make the derivative (4.119) vanish with respect to the rotating frame. This leads to the so called zero angular momentum frames, and it is satisfied by the necessary and sufficient condition

$$W^j\omega_{ji} = \gamma\partial_i\gamma, \tag{4.120}$$

which implies that both the coefficient of $\omega^Z_{0\alpha}$ and the combination $\omega^Z_{0\alpha} - \check{z}_\alpha(\gamma)$ vanish. Equation (4.120) carries intrinsic information about the background and can indeed be recast as

$$\mathcal{L}_{\check{z}_0}\hat{e}_0 = 0. \tag{4.121}$$

When fulfilled, the Zermelo observers coincide with the locally non-rotating frames.

### 4.6.4 Analogue gravity interpretation

In the above analysis, and particularly in the change of frame from Randers-Papapetrou to Zermelo, it has been implicitly assumed that $W^2 < 1$. The velocity of Randers-Papapetrou observers with respect to the Zermelo frame is however fixed by the geometry itself, since $W^2 = b_i b^i$, and nothing a priori guarantees that $b_i b^i < 1$ everywhere. Indeed, there can exist regions where $b_i b^i > 1$, bounded by a hypersurface where $b_i b^i = 1$. We call the latter velocity-of-light hypersurface, since it is the edge where the Randers-Papapetrou observer reaches the speed of light with respect to the Zermelo frame.

The problem raised here is a manifestation of the so-called global hyperbolicity breakdown. Indeed, we have seen that in geometries of the Randers-Papapetrou form (4.88), constant-$t$ surfaces are not everywhere spacelike. The extension of the physical domain accessible to the inertial observers moving along $\check{u} = \partial_t$ is limited to spacelike disks in which $b^2 < 1$ holds, bounded by the velocity-of-light surface, where these observers become luminal.

The breaking of hyperbolicity is usually accompanied with the appearance of closed timelike curves (CTCs). These are ordinary spacelike circles, lying in constant-$t$ surfaces, which become timelike when these surfaces cease being spacelike, *i.e.* when $b^2 > 1$. CTCs can be due to the compactification of the time direction, but it is not the case for our situation, where they cannot be removed by unwrapping time. They would need an excision procedure for consistently removing, if possible, the $b^2 > 1$ domain, in order to keep a causally safe spacetime.

Although the issue of hyperbolicity is intrinsic to our stationary geometries, moving from the form (4.88) to the form (4.111) may provide alternative or complementary views. In the Zermelo form (4.111) the “trouble” is basically encapsulated in the conformal factor. However, some problems such as the original Zermelo navigation problem, are sensitive to the general conformal class[6] of (4.111), and the conformal

[6]In the Zermelo navigation problem we look for null geodesics. In that framework, going to regions where $\gamma^2 < 0$ means having a drift current faster than what the ship can overcome.

factor $\gamma^2$ can be dropped or replaced. Doing so can leave us with a geometry potentially sensible everywhere. This instance appears precisely in analogue gravity systems.
Metrics of the form (4.111) are in fact known as acoustic or optical. They are used for describing the propagation of sound/light disturbances in relativistic or non-relativistic fluids moving with velocity $W^i$ in spatial geometry $h_{ij}$, and subject to appropriate thermodynamic/hydrodynamic assumptions. In this approach, the full metric (4.111) is an analogue metric and is not the actual metric of physical spacetime. Under this perspective, peculiarities such as CTCs, potentially present in the analogue geometry, have no real physical existence. They are manifestations of other underlying physical properties such as supersonic/superluminal regimes in the flowing medium.

## 4.7 Fefferman-Graham expansion stationary black holes in four dimensions: examples

We take into account three well-known stationary asymptotically AdS black holes in four-dimensions: the Schwarzschild solution, the Kerr solution and the Taub-NUT solution. These examples will allow to get familiar with the Fefferman-Graham expansion, as well as discuss the boundary properties by using both the Randers-Papapetrou and the Zermelo frame. In the next section we will pose and solve in generality the problem of finding boundary geometries corresponding to stationary solutions and vice-versa.

### 4.7.1 Schwarzschild black hole

The first metric we consider is the Schwarzschild one. It is not only stationary but also static, and although the boundary fluid is trivial it is a good example to show the procedure to write a metric Fefferman-Graham coordinates. The solution, which we already presented in (4.55) is given by

$$ds^2 = \frac{d\rho^2}{V(\rho)} - V(\rho)dt^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.122}$$

where

$$V(\rho) = 1 + k^2\rho^2 - 2\frac{M}{\rho}. \tag{4.123}$$

In order to put the metric in the Fefferman-Graham form (4.78) we need to perform a change of coordinates involving just the radial coordinate $\rho$:

$$\frac{d\rho}{\sqrt{V(\rho)}} = \frac{L}{r}dr. \tag{4.124}$$

This integral can be solved in a polynomial expansion as

$$\rho = r - \frac{L^2}{4r} + \frac{L^2 M}{3r^2} + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{4.125}$$

We thus can write the vierbein components of the metric in Fefferman-Graham coordinates:

$$\begin{aligned} E^t &= \sqrt{V(\rho)}dt = \left(\frac{r}{L} + \frac{L}{4r} - \frac{2LM}{3r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\right) dt, \\ E^\theta &= \rho d\theta = \left(r - \frac{L^2}{4r} + \frac{L^2 M}{3r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\right) d\theta, \\ E^\phi &= \rho \sin\theta d\phi = \left(r - \frac{L^2}{4r} + \frac{L^2 M}{3r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\right) \sin\theta d\phi. \end{aligned} \tag{4.126}$$

The boundary metric in thus given by

$$ds^2_{\text{bdy}} = \lim_{r\to\infty} \frac{ds^2}{k^2 r^2} = -dt^2 + L^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.127}$$

while the stress current takes the form

$$f^t = -\frac{2Mk}{3}dt, \qquad f^\theta - \frac{M}{3}d\theta, \qquad f^\phi = \frac{M}{3}\sin\theta d\phi, \tag{4.128}$$

giving rise to the boundary energy-momentum tensor

$$T^{ab} = (\varepsilon + p)\, u^a u^b + p g^{ab}, \tag{4.129}$$

where $g$ is the boundary metric and $\varepsilon = 2p = \frac{2\kappa Mk}{3}$. We thus obtain a perfect static conformal fluid in the boundary, with no vorticity, shear or viscosity.

### 4.7.2 Kerr black hole

The next example we consider is the Kerr black hole, describing the field generating by a rotating mass. The metric is given by:

$$\begin{aligned} \mathrm{d}s^2 \quad = \quad & \frac{\mathrm{d}r^2}{V(r,\theta)} - V(r,\theta)\left[\mathrm{d}t - \frac{a}{\Xi}\sin^2\theta \mathrm{d}\phi\right]^2 \\ & + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{\sin^2\theta\Delta_\theta}{\rho^2}\left[a\mathrm{d}t - \frac{r^2+a^2}{\Xi}\mathrm{d}\phi\right]^2, \end{aligned} \tag{4.130}$$

where

$$V(r,\theta) = \frac{\Delta_r}{\rho^2} \tag{4.131}$$

and

$$\begin{aligned}
\Delta_r &= (r^2+a^2)(1+k^2r^2)-2Mr && (4.132)\\
\rho^2 &= r^2+a^2\cos^2\theta && (4.133)\\
\Delta_\theta &= 1-k^2a^2\cos^2\theta && (4.134)\\
\Xi &= 1-k^2a^2. && (4.135)
\end{aligned}$$

Let us start with a few comments on the properties of this solution. The geometry has inner ($r_-$) and outer ($r_+$) horizons, where $\Delta_r$ vanishes, as well as an ergosphere at $g_{tt}=0$. One can show that the rotating AdS black hole is stable for $a^2<k^2$, hence the asymptotically flat black hole ($k=0$) is unstable. This is a consequence of frame dragging (behind the ergosphere no static observer exists), which disappears asymptotically in the Kerr black hole, but persists in the Kerr-AdS.
On the outer horizon $\Delta_r(r_+)=0$, any fixed-$\theta$ observer has a determined angular velocity:

$$\Omega_{\mathrm{H}}=\frac{a\Xi}{r_+^2+a^2}, \tag{4.136}$$

and thus a tangent vector proportional to

$$\partial_t+\Omega_{\mathrm{H}}\partial_\phi, \tag{4.137}$$

which is light-like. The angular velocity $\Omega_{\mathrm{H}}$ *is not* the one measured at infinity by a *static* observer - contrary to what happens for the asymptotically flat plain Kerr geometry. In fact, $\Omega_{\mathrm{H}}$ is the angular velocity observed by an asymptotic observer in the natural frame of the coordinate system at hand. This observer is not static, but has an angular velocity

$$\Omega_\infty=ak^2, \tag{4.138}$$

which obviously vanishes when the cosmological constant is switched off ($k\to 0$). The angular velocity of the black hole for a static observer at infinity is thus

$$\Omega=\Omega_{\mathrm{H}}+\Omega_\infty=\frac{a(1+k^2r_+^2)}{r_+^2+a^2}. \tag{4.139}$$

When moving to the Euclidean signature, by performing the change $t=-i\tau, a=i\alpha$, the outer horizon appears as a bolt[7].
The boundary metric is given by

$$\mathrm{d}s^2_{\text{bry.}}=\lim_{r\to\infty}\frac{\mathrm{d}s^2}{k^2r^2}=-\left[\mathrm{d}t-\frac{a}{\Xi}\sin^2\theta\mathrm{d}\phi\right]^2+\frac{\mathrm{d}\theta^2}{k^2\Delta_\theta}+\frac{\Delta_\theta}{k^2\Xi^2}\sin^2\theta\mathrm{d}\phi^2. \tag{4.140}$$

It can be recasted in several ways:

$$\begin{aligned}
\mathrm{d}s^2_{\text{bry.}} &= \frac{\Delta_\theta}{\Xi}\left(-\mathrm{d}t^2+\frac{\Xi}{k^2\Delta_\theta^2}\left(\mathrm{d}\theta^2+\frac{\Delta_\theta}{\Xi}\sin^2\theta\left[\mathrm{d}\phi+\Omega_\infty\mathrm{d}t\right]^2\right)\right) && (4.141)\\
&= \frac{1}{\Delta_{\theta'}}\left(-\mathrm{d}t^2+\frac{1}{k^2}\left(\mathrm{d}\theta'^2+\sin^2\theta'\left[\mathrm{d}\phi+\Omega_\infty\mathrm{d}t\right]^2\right)\right). && (4.142)
\end{aligned}$$

[7]Bolt removable singularities where introduced in 2.7

The last expression is obtained by trading $\theta$ for $\theta'$ as

$$\Delta_\theta \Delta_{\theta'} = \Xi, \tag{4.143}$$

where $\Delta_{\theta'} = 1 - k^2 a^2 \cos^2\theta'$. It describes the boundary of Kerr-AdS as conformal to the three-dimensional Einstein universe, rotating at angular velocity $\Omega_\infty$.
From (4.140) we easily recover the component of the Randers-Papapetrou frame as

$$b = \frac{a}{\Xi}\sin^2\theta \,\mathrm{d}\phi\,, \quad a_{ij} = L^2 \mathrm{diag}\left(\frac{1}{\Delta_\theta}, \frac{\Delta_\theta}{\Xi^2}\sin^2\theta\right) \tag{4.144}$$

while the orthonormal coframes (4.95) are

$$\hat{e}^0 = \mathrm{d}t - \frac{a}{\Xi}\sin^2\theta\,\mathrm{d}\phi\,, \quad \hat{e}^1 = \frac{L}{\sqrt{\Delta_\theta}}\mathrm{d}\theta\,, \quad \hat{e}^2 = \frac{L\sqrt{\Delta_\theta}}{\Xi}\sin\theta\,\mathrm{d}\phi. \tag{4.145}$$

The dual boundary frames are of the form

$$\check{e}_0 = \partial_t\,, \quad \check{e}_\alpha = E_\alpha{}^i\left(b_i\partial_t + \partial_i\right)\,, \quad E_\alpha{}^i E^\beta{}_i = \delta^\beta_\alpha\,. \tag{4.146}$$

The boundary energy-momentum tensor is of the form (4.129), as in the Schwarzschild case, and describes a conformal fluid with constant pressure and with velocity field given by $\check{u} = \partial_t$ and $\hat{u} = -\mathrm{d}t + b$. The fluid is therefore at rest in the Randers-Papapetrou frame, and the corresponding observers are thus comoving. Furthermore, $\partial_t$ is a Killing vector with constant norm $(-1)$. Hence, its integral lines are geodesics:

$$\check{a} = \nabla_{\partial_t}\partial_t = 0\,. \tag{4.147}$$

The fluid and the comoving observers are inertial. For this geodesic congruence, the shear and expansion systematically vanish. The boundary fluid is nevertheless carrying a non-trivial vorticity:

$$\omega = \frac{1}{2}db = \frac{a\cos\theta\sin\theta}{\xi}d\theta\wedge d\phi = ak^2\cos\theta\hat{e}^1\wedge\hat{e}^2. \tag{4.148}$$

This vorticity describe a cyclonic flow, that is, for example, it could describe the motion of the atmosphere of a rotating planes as seen from the comoving frame.
In the Zermelo frame we have

$$h_{ij} = L^2\mathrm{diag}\left(\frac{\xi}{\Delta_\theta^2}, \frac{\sin^2\theta}{\Delta_\theta}\right), \tag{4.149}$$

$$W^\alpha \check{z}_\alpha = -\frac{a}{L}\frac{\sin\theta}{\sqrt{\xi}}\check{z}_2 = -\frac{a}{L^2}\partial_\phi, \tag{4.150}$$

and

$$\gamma = \sqrt{\frac{\Delta_\theta}{\xi}}. \tag{4.151}$$

As explained in the last section, the presence of CTCs is related to the value of $b^2$. For the Kerr geometry we have

$$b^2 = \frac{a^2 \sin^2\theta}{L^2 - a^2\cos^2\theta}, \tag{4.152}$$

which is bounded by 1 as long as $a < L$, and thus as long as this condition is satisfied there are no CTCs. The condition for the Fermi derivative along $\check{z}_0$ to vanish, (4.120), is fulfilled and thus the effective precession of the fluid worldline with respect to the Zerrmelo observer disappears as a consequence of the cancellation of the genuine vorticity and of the effect produced by the acceleration.

### 4.7.3 Taub-NUT

The Taub-NUT geometry is a foliation over squashed three-spheres solving Einstein's equations:

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\mathrm{d}r^2}{V(r)} + \left(r^2+n^2\right)\left((\sigma^1)^2 + \left(\sigma^2\right)^2\right) - 4n^2 V(r)\left(\sigma^3\right)^2 \\ &= \frac{\mathrm{d}r^2}{V(r)} + \left(r^2+n^2\right)\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) - 4n^2 V(r)\left(\mathrm{d}\psi + \cos\theta \mathrm{d}\phi\right)^2 \end{aligned} \tag{4.153}$$

with

$$V(r) = \frac{1}{r^2+n^2}\left[r^2 - n^2 - 2Mr + k^2\left(r^4 + 6n^2r^2 - 3n^4\right)\right], \tag{4.154}$$

where

$$\begin{cases} \sigma^1 = \sin\theta\sin\psi\,\mathrm{d}\phi + \cos\psi\,\mathrm{d}\theta \\ \sigma^2 = \sin\theta\cos\psi\,\mathrm{d}\phi - \sin\psi\,\mathrm{d}\theta \\ \sigma^3 = \cos\theta\,\mathrm{d}\phi + \mathrm{d}\psi \end{cases} \tag{4.155}$$

are the $SU(2)$ left-invariant Maurer–Cartan forms in terms of Euler angles $0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi, 0 \leq \psi \leq 4\pi$. Besides the mass $M$ and the cosmological constant $\Lambda = -3k^2$, this solution depends on an extra parameter $n$: the nut charge. It is convenient to trade $\psi$ for $t = -2n(\psi + \phi)$. With this coordinate the metric (4.153) assumes the form

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{V(r)} + \left(r^2+n^2\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) - V(r)\left[\mathrm{d}t + 4n\sin^2\frac{\theta}{2}\mathrm{d}\phi\right]^2. \tag{4.156}$$

The original Taub-NUT solution was a vacuum solution designed for cosmology. Since then, many variants have been studied, both with Lorentzian and Euclidean signature (reached by setting $\nu = in$ and $\tau = it$), with or without cosmological constant or mass. Indeed, we already encountered the Euclidean version in the absence

of mass of the Taub-NUT geometries when discussing Bianchi IX gravitational instantons. One can generalize this solution to take into account the presence of a cosmological constant by demanding the self-duality of the Weyl tensor instead of the Riemann tensor. Adding a mass opens up new possibilities according to the kind of horizons that appear, and the corresponding solutions can be either (Weyl-)self-dual or not, and they are called respectively Taub-bolt or Pedersen. Self-duality must however be abandoned in the Lorentzian framework.
Many of these properties of the Taub-NUT geometry are a consequence of its isometry group $SU(2) \times U(1)$, generated by the Killing vectors

$$\begin{cases} \xi_1 = -\sin\phi \cot\theta\, \partial_\phi + \cos\phi\, \partial_\theta - 2\nu \frac{\sin\phi}{\sin\theta}(1-\cos\theta)\, \partial_t \\ \xi_2 = \cos\phi \cot\theta\, \partial_\phi + \sin\phi\, \partial_\theta + 2\nu \frac{\cos\phi}{\sin\theta}(1-\cos\theta)\, \partial_t \\ \xi_3 = \partial_\phi - 2\nu\, \partial_t \\ e_3 = -2\nu\, \partial_t. \end{cases} \tag{4.157}$$

Two extra vectors $e_1$ and $e_2$ generate with $e_3$ the right $SU(2)$. These are not Killing, however, due to the squashing of the spherical leaves.
The solution at hand has generically two horizons ($V(r_\pm) = 0$) and is well-defined outside the outer horizon $r_+$, where $V(r) > 0$. In the Euclidean language, this horizon is a bolt[8] *i.e.* the two-dimensional fixed locus of the Killing vector $e_3$. On this surface, $\theta = \pi$ is an isolated fixed point of another Killing vector $\xi_3 + e_3$. This is a nut, carrying a net nut charge $n$.
The nut is the origin of the so-called Misner string, departing from $r = r_+$, all the way to $r \to \infty$, on this southern pole at $\theta = \pi$. The geometry is nowhere singular along the Misner string, which appears as a coordinate artifact in an analogous way as the Dirac string of a magnetic monopole is a gauge artifact. In order for this string to be invisible, coordinate transformations displacing the string must be univalued everywhere, which is achieved by requiring the periodicity condition $\psi \equiv \psi + 4\pi$ or equivalently $t \equiv t - 8\pi n$. Alternatively, one can avoid periodic time and keep the Misner string as part of the geometry. This semi-infinite spike appears then as a source of angular momentum, integrating to zero and movable at wish using the transformations generated by the above vectors. This will be our viewpoint throughout this work. However, despite the non-compact time, the AdS-Taub-NUT geometry is plagued with closed time-like curves, which disappear only in the vacuum limit. Even though this is usually an unwanted situation, it is not sufficient for rejecting the geometry, which from the holographic perspective has many interesting and novel features we are now going to discuss.

[8]By analytic continuation, the solution (4.153) with (4.154) is mapped onto the so-called AdS-Taub-bolt.

Following the Fefferman-Graham procedure, the boundary metric is given by

$$\begin{aligned} \mathrm{d}s^2_{\text{bry.}} &= \frac{1}{k^2}\left((\sigma^1)^2 + \left(\sigma^2\right)^2 - 4k^2n^2\left(\sigma^3\right)^2\right) \\ &= -\left[\mathrm{d}t + 4n\sin^2\frac{\theta}{2}\mathrm{d}\phi\right]^2 + \frac{1}{k^2}\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right). \end{aligned} \tag{4.158}$$

This is a squashed three-sphere appearing as a limiting leave of the foliation (4.153). The squashing is Lorentzian as in the bulk, and consequently the closed time-like curves survive on the boundary.

We can make contact with (4.78) by choosing

$$b = -2n(1-\cos\theta)\mathrm{d}\phi\,, \quad a_{ij} = L^2\mathrm{diag}(1, \sin^2\theta), \tag{4.159}$$

while the orthonormal coframes are given by

$$\hat{e}^0 = \mathrm{d}t + 2n(1-\cos\theta)\mathrm{d}\phi\,, \quad \hat{e}^1 = L\mathrm{d}\theta\,, \quad \hat{e}^2 = L\sin\theta\ \mathrm{d}\phi\,. \tag{4.160}$$

All results discussed in the last section are still valid: the fluid and the comoving observers are inertial, and shear and expansion vanish. The energy-momentum tensor is perfect-fluid like, and the vorticity is given by

$$\omega = \frac{1}{2}db = -n\sin\theta d\theta\wedge d\phi = -nk^2\hat{e}^1\wedge\hat{e}^2. \tag{4.161}$$

This corresponds to a Dirac-monopole-like vortex, in contrast with the Kerr case which can be considered as a dipole. In this magnetic picture, the Misner string is traded for a Dirac string. Indeed, while for the Kerr black hole we have

$$\int_{S^2}\omega = 0, \tag{4.162}$$

which describe a solid rotation, for the Taub-NUT case due to the presence of the nut charge we have

$$n = -\frac{1}{4\pi}\int_{S^2}\omega. \tag{4.163}$$

The components of the metric in the Zermelo frame are given by

$$h_{ij} = \mathrm{diag}\left(L^2 - 4n^2\tan^2\theta/2, 4\tan^2\theta/2(L^2\cos^2\theta/2 - 4n^2\sin^2\theta/2)^2\right) \tag{4.164}$$

together with

$$W = \frac{1}{\sqrt{\frac{L^2}{4n^2}\cot^2\theta/2 - 1}} \tag{4.165}$$

The necessary and sufficient condition for the Fermi derivative along $\check{z}_0$ to vanish, (4.120), is not fulfilled, and thus the Zermelo frame does not coincide with the locally non-rotating frame. Moreover, CTCs appear. Indeed, $b^2 = 1$ when $\theta$ reaches $\theta* = 2\arctan L/2n$. Hyperbolicity holds in the disk $0 < \theta < \theta*$, whereas it breaks down i the complementary disk $\pi > \theta > \theta*$ centered at the Misner string. The fluid becomes superluminal and the Misner string is interpreted as the core of the vortex.

# 4.8 Perfect-Cotton geometries

## 4.8.1 Perfect-like fluids and background geometries

As we have seen, the energy-momentum tensor of the examples we considered is perfect-fluid like. Being stationary, the fluid behaves therefore as a thermodynamic system in global equilibrium, without dissipation and entropy production. We call those geometries where the fluid can exist in perfect equilibrium "perfect geometries". In such backgrounds, the global equilibrium can be described thermodynamically. The values of the transport coefficients, which determine wether the geometry is perfect or not, is determined by the underlying microscopic dynamics of the fluid. Indeed, for neutral fluids the transport coefficients are in general functions of the temperature and of the coupling constants of the microscopic theory. Nevertheless, the boundary geometry plays a role in fixing which terms are appearing in the energy-momentum tensor, because it determines which terms can appear regardless of the value of the corresponding transport coefficient. In particular, even if the transport coefficients are non-vanishing, the tensor to which they couple may vanish because of the geometry properties, *i.e.* may vanish kinematically, and thus the corresponding term may not appear in the energy-momentum tensor.

Given a conformal boundary geometry, one can write all the possible terms in the energy-momentum tensor, as we showed in (4.53), by considering Weyl-covariant traceless transverse tensors $\mathcal{T}_{\mu\nu}$ order by order in a derivative expansion. Moreover, we should consider just non-dissipative transport coefficients, since the dissipative ones, like the shear viscosity $\eta$, lead to generation of entropy and thus to an out-of-equilbrium situation. Some of those non-dissipative Weyl-covariant traceless transverse tensors will also satisfy $\nabla^\mu \mathcal{T}_{\mu\nu} \neq 0$. We refer to them as "dangerous tensors", and we call their transport coefficients "dangerous transport coefficients". The necessary and sufficient condition for the existence of perfect equilibrium in backgrounds with a normalized Killing vector field is that the dangerous transport coefficients vanish.

Of course, transport coefficients which are dangerous in one background do not need to be dangerous for all backgrounds: for example, for certain metrics the corresponding dangerous tensor could be just vanishing.

Consider a $2+1$ dimensional background. At first order in the derivative expansion of the energy-momentum tensor we have $\sigma_{\mu\nu}$, which is associated with the shear viscosity $\eta$. As we pointed out, this is a dangerous transport coefficient, and thus we need to demand $\sigma_{\mu\nu}$ to vanish on equilibrium solutions. At second order, we still have no transport coefficients playing a role in describing equilibrium in a conformal fluid. However, at third order we can get non-dissipative transport coefficients. Their corresponding tensors can in principle appear in the energy-momentum tensor and thus contribute to describing equilibrium in a conformal theory. The third order in derivatives of the energy-momentum tensor reads:

$$T^{\mu\nu}_{(3)} = \gamma_{(3)1}\langle C^{\mu\nu}\rangle + \gamma_{(3)2}\langle \mathcal{D}^\mu V^\nu\rangle + \gamma_{(3)3}\langle \mathcal{D}^\mu W^\nu\rangle, \tag{4.166}$$

where the bracket was defined in (4.18) and

$$V^\mu = \nabla^\alpha_\perp \omega^\mu{}_\alpha + u^\mu \omega_{\alpha\beta}\omega^{\alpha\beta}, \quad W^\mu = \eta^{\mu\nu\rho} u_\nu V_\rho. \tag{4.167}$$

All other possible transport coefficients appearing at third order would contribute to dissipation. Their corresponding Weyl covariant traceless transverse tensors could appear in the form of the energy-momentum tensor, and we thus need to impose them to vanish at equilibrium.
At higher order in derivative expansion, we will get more and more non-dissipative transport coefficients, which will be required to describe the equilibrium in a stationary background geometry. At each order in derivative expansion though, we will have only finite number of such transport coefficients, even though any all-order statement involves an infinite number of them.
An equilibrium solution of conformal fluid mechanics will by definition satisfy the conservation of the energy-momentum tensor and be such that all the Weyl covariant traceless transverse tensors corresponding to non-vanishing dissipative transport coefficients become zero when evaluated on-shell. There need not be a unique equilibrium solution in a given stationary background metric. Furthermore, as we will typically have an infinite number of non-vanishing non-dissipative transport coefficients, thermodynamics is clearly not a sufficient information to describe the class of equilibrium solutions in an arbitrary stationary background.

### 4.8.2 Perfect fluids and Randers-Papapetrou geometries

As already mentioned in Sect. 4.6.2, a perfect-like fluid at equilibrium on a Randers-Papapetrou background is aligned along the geodesic congruence tangent to $\partial_t$. It has neither shear, nor expansion, but carries vorticity inherited from the fact that $\partial_t$ is not hypersurface-orthogonal[9]. As discussed in the previous section, the fluid can attain perfect equilibrium if and only if there are no dangerous terms appearing in the energy-momentum tensor. This means that if a dangerous tensor is allowed by the geometry, it's corresponding dangerous transport coefficient has to vanish. This will imply constraints on transport coefficients, as these stationary backgrounds will generically have infinitely many associated dangerous tensors. One simple example of such a tensor is $\langle \mathcal{D}_\mu V_\nu\rangle$, the second term of (4.166), which will be given in terms of covariant derivatives of $q$. In general, $\nabla^\mu \langle \mathcal{D}_\mu V_\nu\rangle \neq 0$ in these stationary backgrounds, therefore it is a dangerous tensor and the corresponding dangerous transport coefficient must vanish in order for the fluid to be

[9]For this very same reason, Randers-Papapetrou geometries may in general suffer from global hyperbolicity breakdown. This occurs whenever regions exist where $b_i b^i > 1$. There, constant-$t$ surfaces cease being space-like, and potentially exhibit closed time-like curves.

in perfect equilibrium.
However, when the stationary background has additional isometries, most Weyl-covariant traceless transverse tensors built from derivatives of $u$ will vanish. This means that we do not need to impose any condition on the corresponding transport coefficients in order to have perfect equilibrium. In particular, the more the number of isometries of the stationary background, the less the number of transport coefficients we will be able to probe by demanding perfect equilibrium to exist.

### 4.8.3 Perfect-Cotton Randers-Papapetrou geometries

The boundaries arising as the holographic duals of stationary four-dimensional Einstein spaces, as those studied in the previous sections, have two peculiar properties. We already noticed the first one, which is the existence of a time-like Killing vector field of unit norm $u$. Moreover, the Cotton tensor takes the form

$$C_{\mu\nu}dx^\mu dx^\nu = \frac{c}{2}(2u^2 + d\tilde{s}^2), \tag{4.168}$$

where c is a constant[10]. This means that the Cotton tensor has the same form as the as perfect fluid energy-momentum tensor, but with a different constant of proportionality. In such geometries the Cotton-York tensor can be interchanged with the perfect-fluid energy-momentum tensor without spoiling its property of being perfect. We refer to them as "perfect-Cotton geometries". We do not claim that this class of solutions is the only one leading to a perfect-fluid-like energy-momentum tensor, but we can prove that perfect-Cotton geometries with perfect-fluid-like energy-momentum tensor have an exact gravitational dual.
Considering (4.106), the condition (4.168) is equivalent to

$$\begin{aligned} \tilde{\nabla}^2 q + q(\delta - q^2) &= 2c && (4.169)\\ a_{ij}\left(\tilde{\nabla}^2 q + \frac{q}{2}(\delta - q^2) - c\right) &= \tilde{\nabla}_i \tilde{\nabla}_j q && (4.170)\\ \tilde{R} + 3q^2 &= \delta. && (4.171) \end{aligned}$$

Here $\delta$ is another constant and it relates the curvature of the base space with the vorticity strength.
Without loss of generality, we can choose the two-dimensional part of the metric $d\tilde{s} = a_{ij}dx^i dx^j$ to be diagonal:

$$d\tilde{s}^2 = A^2(x,y)dx^2 + B^2(x,y)\mathrm{d}y^2, \tag{4.172}$$

and make the gauge choice:

$$b = b(x,y)\, dy, \tag{4.173}$$

[10]The divergence of the Cotton-York tensor is identically zero, thus for expansionless u the form (4.168) requires $c$ constant.

for which (4.103) reads:

$$q = -\frac{\partial_x b}{AB}. \tag{4.174}$$

Further gauge fixing is possible and will be made when appropriate[11]. The explicit form of Eqs. (4.169), (4.170) and (4.171) in terms of $A(x,y), B(x,y)$ and $b(x,y)$ is not very illuminating and we will not reproduce it here.

### 4.8.4 Geometries with space-like Killing vectors

When the backgrounds have an extra Killing vector the perfect geometry condition can be solved exactly, and we can moreover make contact with the explicit examples we took into account. For now on, for simplicity we set $k = 1$.

The presence of an additional unique space-like isometry simplifies the conditions for a Papapetrou–Randers metric to be perfect-Cotton. Without loss of generality, we take the additional Killing vector to be $\partial_y$ and we chose a representation such that $A^2 = \frac{1}{G(x)}$, $B^2 = G(x)$ and $b = b(x)$. The metric takes then the form

$$\mathrm{d}s^2 = -\left(\mathrm{d}t - b(x)\,\mathrm{d}y\right)^2 + \frac{\mathrm{d}x^2}{G(x)} + G(x)\mathrm{d}y^2, \tag{4.175}$$

and we are able to solve (4.169)–(4.171) in full generality. The solution is written in terms of 6 arbitrary parameters $\tilde{c}_i$:

$$\begin{aligned} b(x) &= \tilde{c}_0 + \tilde{c}_1 x + \tilde{c}_2 x^2, & (4.176)\\ G(x) &= \tilde{c}_5 + \tilde{c}_4 x + \tilde{c}_3 x^2 + \tilde{c}_2 x^3\left(2\tilde{c}_1 + \tilde{c}_2 x\right). & (4.177)\end{aligned}$$

It follows that the vorticity strength takes the linear form

$$q(x) = -\tilde{c}_1 - 2\tilde{c}_2 x, \tag{4.178}$$

and the constants $c$ and $\delta$ are given by:

$$\begin{aligned} c &= -\tilde{c}_1^3 + \tilde{c}_1\tilde{c}_3 - \tilde{c}_2\tilde{c}_4, & (4.179)\\ \delta &= 3\tilde{c}_1^2 - 2\tilde{c}_3. & (4.180)\end{aligned}$$

Finally, the Ricci scalar of the two-dimensional base space is given by

$$\hat{R} = -2\left(\tilde{c}_3 + 6\tilde{c}_2 x(\tilde{c}_1 + \tilde{c}_2 x)\right), \tag{4.181}$$

and using (4.104) one can easily find the form of the three-dimensional scalar as well. Not all the six parameters $\tilde{c}_i$ correspond to physical quantities: some of them can be just reabsorbed by change of coordinates. In particular, we set here $\tilde{c}_0 = 0$ by performing the diffeomorphism $t \to t + py$, with constant $p$, which does not change the form of the metric.

[11] We could for example set $A = B$ since any two-dimensional space is conformally flat. We should however stress that all these choices are local, and the range of coordinates should be treated with care in order to avoid e.g. conical singularities.

### Non-vanishing $c_4$

To analyse the class with $\tilde{c}_4 \neq 0$, we first perform the further diffeomorphism $x \to x + s$, with constant $s$, which keeps the form of the metric. By tuning the value of $s$ we can set $\tilde{c}_5$ to zero. Therefore, without loss of generality we can choose:

$$
\begin{aligned}
b(x) &= \tilde{c}_1 x + \tilde{c}_2 x^2, && (4.182)\\
G(x) &= \tilde{c}_4 x + \tilde{c}_3 x^2 + \tilde{c}_2 x^3 \left(2\tilde{c}_1 + \tilde{c}_2 x\right). && (4.183)
\end{aligned}
$$

We are thus left with four arbitrary geometric parameters. For consistency we can check that $q(x)$, $c$, $\delta$, $R$ and $\hat{R}$ indeed depend only on these four parameters. Moreover, by performing the change of variables

$$
x \to \frac{x}{\tilde{c}_4}\,, \quad y \to \frac{y}{\tilde{c}_4}\,, \quad t \to \frac{t}{\tilde{c}_4}\,, \tag{4.184}
$$

and defining new variables

$$
c_1 = \frac{\tilde{c}_1}{\tilde{c}_4}\,, \quad c_2 = \frac{\tilde{c}_2}{\tilde{c}_4^2}\,, \quad c_3 = \frac{\tilde{c}_3}{\tilde{c}_4^2}\,, \quad c_4 = \tilde{c}_4 \tag{4.185}
$$

we can see that $c_4$ is an overall scaling factor of the metric. Indeed, we have

$$
\begin{aligned}
b(x) &= c_1 x + c_2 x^2, && (4.186)\\
G(x) &= x + c_3 x^2 + c_2 x^3 \left(2c_1 + c_2 x\right), && (4.187)
\end{aligned}
$$

which depend now on the three dimensionless parameters $c_1$, $c_2$ and $c_3$. Using the above variables the metric becomes $c_4^2 \mathrm{d}s^2$. Since we are dealing with a conformal theory, we can always choose appropriate units to set $c_4$ to any convenient value and deal with dimensionless quantities only.

**Monopoles: homogeneous spaces** Consider the vorticity strength (4.178). The simplest example that can be considered is the one of constant $q$, that is when $c_2 = 0$. We call the corresponding geometries *monopolar geometries*, a terminology that we will justify in the following. The two-dimensional Ricci scalar (4.181) is in this case constant: $\hat{R} = -2c_3$. This means that the parameter $c_3$ labels the curvature signature of the two-dimensional base space and that without loss of generality (by appropriately choosing $c_4$ and then dropping the overall scale factor), we can set

$$
c_3 = -\nu = 0, \pm 1\,. \tag{4.188}
$$

Thus, we are left with one continuous parameter, $c_1$, which we rename as

$$
c_1 = -2n\,. \tag{4.189}
$$

Moreover, the Cotton–York tensor is proportional to

$$
c = 2n(\nu + 4n^2)\,, \tag{4.190}
$$

hence the parameter $n$ determines whether the geometry is conformally flat or not. Note that, apart from the trivial case $n = 0$, the space is conformally flat also when $\nu = -1$ and $4n^2 = 1$ – we will briefly comment on this issue at the end of Sec. 4.8.4. The functions $b(x)$ and $G(x)$ take now the form

$$b(x) = -2nx\,, \quad G(x) = x(1+\nu x)\,, \tag{4.191}$$

The form of $G(x)$ motivates the parameterisation

$$x = f_\nu^2(\frac{\sigma}{2})\,, \quad \begin{cases} f_1(\sigma) = \sin\sigma \\ f_0(\sigma) = \sigma \\ f_{-1}(\sigma) = \sinh\sigma \end{cases}, \quad y = 2\phi\,, \quad \phi \in [0, 2\pi]\,. \tag{4.192}$$

Then, the geometries (4.175) take the form

$$\mathrm{d}s^2 = -\left(\mathrm{d}t + 4nf_\nu^2(\frac{\sigma}{2})\,\mathrm{d}\phi\right)^2 + \mathrm{d}\sigma^2 + f_\nu^2(\sigma)\mathrm{d}\phi^2\,, \tag{4.193}$$

which is that of fibrations over $S^2, \mathbb{R}^2$ and $H_2$ for $\nu = 1, 0, -1$ respectively. The two-dimensional base spaces are homogeneous with constant curvature having three Killing vectors; the three-dimensional geometry has in total four Killing vectors.

These geometries appear at the boundary of asymptotically anti-de Sitter Taub–NUT Einstein spaces with $n$ being the bulk nut charge.

We want now to discuss the presence of dangerous tensors. The velocity one-form is:

$$\mathrm{u} = -\mathrm{d}t - 4nf_\nu^2(\frac{\sigma}{2})\mathrm{d}\phi, \tag{4.194}$$

while the vorticity has constant strength:

$$q = 2n. \tag{4.195}$$

Furthermore, the geometric data ensures the following structure:

$$R_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu = \left(\nu + 4n^2\right)\mathrm{u}^2 + \left(\nu + 2n^2\right)\mathrm{d}s^2 \tag{4.196}$$

The above condition implies that all hydrodynamic scalars, vectors and tensors that can be constructed from the Riemann tensor, its covariant derivatives and the covariant derivatives of u are algebraic. More specifically

- all hydrodynamic scalars are constants,
- all hydrodynamic vectors are of the form $ku_\mu$ with constant $k$, and
- all hydrodynamic tensors are of the form $au_\mu u_\nu + bg_{\mu\nu}$ with constant $a$ and $b$.

This means that there exists no traceless transverse tensor that can correct the hydrodynamic energy–momentum tensor in perfect equilibrium. In other words, there exists no dangerous tensor. Thus, in the case of monopolar geometries it is not possible to know the value of any transport coefficient.

This above result is not surprising. Indeed, we called Papapetrou–Randers configurations given by (4.192) and (4.193) of *monopolar* type because the vorticity is constant,[12] as a consequence of the homogeneous nature of these space–times. In such a highly symmetric kinematical configuration, the fluid dynamics cannot be sensitive to any dissipative or non-dissipative coefficient. This result provides a guide for the subsequent analysis: to have access to the transport coefficients, we must perturb the geometry away from the homogeneous configuration. The above discussion suggests that this perturbation should be organised as a multipolar expansion: the higher the multipole in the geometry, the richer the spectrum of transport coefficients that can contribute, if non-vanishing, to the global equilibrium state, and that we need to set to zero for perfect fluids.

**Dipolar geometries: axisymmetric spaces** When $c_2 \neq 0$, the vorticity is not constant and hence the space ceases to be homogenous. If some symmetry remains, this must be in the form of a space-like Killing vector: therefore, these are axisymmetric spaces. We call such geometries *dipolar geometries*, as their axial symmetry connects them with the gauge potential of electric or magnetic dipoles.

For simplicity, we start considering a *pure dipolar* geometry, namely a nontrivial conformally flat metric and see how it is parameterised in terms of $c_1, c_2$ and $c_3$. We start from $\mathbb{R} \times S^2$ where we set to one the sphere's radius

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2. \tag{4.197}$$

We do then a conformal rescaling by a function $\Omega(\vartheta)$, which preserves the axial symmetry around $\varphi$ – and the conformal flatness of (4.197) *i.e.* the vanishing Cotton tensor:

$$\mathrm{d}s'^2 = \Omega^2(\vartheta)\,\mathrm{d}s^2\,. \tag{4.198}$$

The vector field $\partial_t$ is no longer of unit norm. However, if $\Omega(\vartheta)$ simply corresponds to a rotation,[13] then a new unit-norm time-like Killing vector may exist and describe trajectories of a fluid in equilibrium on the background (4.198). We will show that this is possible for

$$\Omega^2(\vartheta) = 1 - a^2 \sin^2\vartheta \tag{4.199}$$

[12]Note also that b, as given in (4.173) and (4.191), has the same form as the gauge potential of a Dirac monopole on $S^2$, $\mathbb{R}^2$ or $H_2$. This magnetic paradigm can be made more precise – see e.g. [19].

[13]Actually to a precession, hence we call this the *precession trick*.

with $a$ being a constant parameter. Consider for that the change of coordinates $(t, \vartheta, \varphi) \mapsto (t', \vartheta', \varphi')$ defined as:

$$t = t' \,, \quad \Omega^2(\vartheta) = \frac{\Delta_{\vartheta'}}{\Xi} \,, \quad \varphi = \varphi' + at' \tag{4.200}$$

with

$$\Delta_{\vartheta'} = 1 - a^2 \cos^2 \vartheta' \,, \quad \Xi = 1 - a^2 \,. \tag{4.201}$$

The metric (4.198) reads now:

$$\mathrm{d}s'^2 = -\left[\mathrm{d}t' - \frac{a}{\Xi} \sin^2 \vartheta' \mathrm{d}\varphi'\right]^2 + \frac{\mathrm{d}\vartheta'^2}{\Delta_{\vartheta'}} + \frac{\Delta_{\vartheta'}}{\Xi^2} \sin^2 \vartheta' \mathrm{d}\varphi'^2 \,. \tag{4.202}$$

Clearly the Killing vector

$$\partial_{t'} = \partial_t + a\partial_\phi \tag{4.203}$$

is of unit norm, and its vorticity

$$\omega = \frac{a}{\Xi} \cos \vartheta' \sin \vartheta' \mathrm{d}\vartheta' \wedge \mathrm{d}\varphi' \tag{4.204}$$

has strength

$$q = -2a \cos \vartheta' \,. \tag{4.205}$$

Any fluid comoving with $\partial_{t'}$ in the background metric (4.202) undergoes a cyclonic rotation on a squashed[14] $S^2$. As already stressed, this background metric is conformally flat.

Finally, by performing the change of coordinates

$$x = \frac{\sin^2 \frac{\vartheta'}{2}}{1 - a^2} \,, \quad y = 2\varphi, \tag{4.206}$$

we can bring the metric (4.202) into the form (4.175) with

$$\begin{aligned} b(x) &= 2ax\left(1 - (1 - a^2)x\right) \,, & (4.207)\\ G(x) &= x - (1 - 5a^2)x^2 - 8a^2(1 - a^2)x^3 + 4a^2(1 - a^2)^2 x^4. & (4.208) \end{aligned}$$

It is easy then to read off the parameters

$$c_1 = 2a \,, c_2 = -2a(1 - a^2) \,, c_3 = 5a^2 - 1. \tag{4.209}$$

We now move to the generalisation of the above to non-conformally flat geometries with $x$-dependent vorticity. These are the *dipolar-monopolar* metrics. In those cases the precession trick mentioned previously does not suffice and one needs to perform the appropriate parameterisations of the $c_i$s. Nevertheless, our previous

[14] The spatial metric $\mathrm{d}\ell'^2$ in (4.202) describes a squashed two-sphere.

explicit examples serve both as a guiding rule as well as a test for our results. We will present them and spare the reader from the non-illuminating technicalities. By appropriately choosing $c_4$ and dropping the overall scale factor, we can parameterise $c_1$, $c_2$ and $c_3$ by the charge $n$ and the angular momentum $a$, without loss of generality. These parameterisations will depend on the topology captured in $\nu$. As already quoted, from the holographic analysis presented in Sec. 4.10, it will become clear that $n$ is the bulk nut charge.

**Spherical ($\nu = 1$)** The relation between $a$ and $n$ and the three geometric parameters is given by:

$$\begin{aligned} c_1 &= 2(a-n), \\ c_2 &= 2a(-1+a^2-4an), \\ c_3 &= -1+5a^2-12an. \end{aligned} \tag{4.210}$$

We also perform the following coordinate transformations:

$$\begin{aligned} x &= \kappa \sin^2 \frac{\vartheta}{2}, \\ y &= \lambda\varphi, \end{aligned} \tag{4.211}$$

with

$$\kappa = \frac{1}{1+a(4n-a)}, \quad \lambda = \frac{2}{\kappa\Xi} \quad \text{and} \quad \Xi = 1-a^2. \tag{4.212}$$

The two-dimensional base space in the metric (4.175) takes then the form:

$$\mathrm{d}\ell^2 = \frac{\mathrm{d}\vartheta^2}{\Delta_\vartheta} + \frac{\sin^2\vartheta\Delta_\vartheta}{\Xi^2}\mathrm{d}\varphi^2 \tag{4.213}$$

with

$$\Delta_\vartheta = 1 + a\cos\vartheta(4n - a\cos\vartheta). \tag{4.214}$$

The coordinates range as $\vartheta \in [0,\pi]$ and $\varphi \in [0, 2\pi]$. The full $2+1$-dimensional metric is of the Papapetrou–Randers form: $\mathrm{d}s^2 = -\mathrm{u}^2 + \mathrm{d}\ell^2$. The velocity field takes the form

$$\mathrm{u} = -\mathrm{d}t + b(\vartheta)\mathrm{d}\varphi\,, \quad b(\vartheta) = \frac{2(a-2n+a\cos\vartheta)}{\Xi}\sin^2\frac{\vartheta}{2}. \tag{4.215}$$

The scalar vorticity strength is given by

$$q = 2\left(n - a\cos\vartheta\right), \tag{4.216}$$

while the constant $c$ appearing in the Cotton–York tensor is

$$c = 2n(1-a^2+4n^2). \tag{4.217}$$

The base space (4.213) is a squashed $S^2$. The vorticity (4.216) has two terms: the constant monopole and the dipole. It is maximal on the northern ($\vartheta = 0$) and southern ($\vartheta = \pi$) poles and is vanishing on the equator ($\vartheta = \frac{\pi}{2}$). Note also that in the limit $c_2$ we recovered the homogeneous metric case for $\nu = 1$.

**Flat** ($\nu = 0$) The new parameters $a$ and $n$ are now defined as follows:

$$\begin{aligned} c_1 &= 2(a-n), \\ c_2 &= 2a^2(a-4n), \\ c_3 &= a(5a-12n). \end{aligned} \tag{4.218}$$

Let us now do the following coordinate transformations:

$$\begin{aligned} x &= \kappa(\frac{\sigma}{2})^2, \\ y &= \lambda\varphi, \end{aligned} \tag{4.219}$$

with

$$\kappa = 1, \quad \lambda = 2. \tag{4.220}$$

With these transformations the two-dimensional base space in the metric (4.175) takes the form of squashed $\mathbb{R}^2$:

$$\mathrm{d}\ell^2 = \frac{\mathrm{d}\sigma^2}{\Delta_\sigma} + \sigma^2\Delta_\sigma \mathrm{d}\varphi^2 \tag{4.221}$$

with

$$\Delta_\sigma = \frac{(2+a^2\sigma^2)(8-24an\sigma^2+a^4\sigma^4-8a^3n\sigma^4+2a^2\sigma^2(3+8n^2\sigma^2))}{16}. \tag{4.222}$$

The coordinates range as $\sigma \in \mathbb{R}_+$ and $\varphi \in [0, 2\pi]$. The full 2+1-dimensional metric is $\mathrm{d}s^2 = -\mathrm{u}^2 + \mathrm{d}\ell^2$, where the velocity field takes the form

$$\mathrm{u} = -\mathrm{d}t + b(\sigma)\mathrm{d}\varphi\,, \quad b(\sigma) = \frac{\sigma^2}{4}\left(4(a-n) + a^2(a-4n)\sigma^2\right). \tag{4.223}$$

The scalar vorticity is then given by

$$q = (n-a)\left(2 + a^2\sigma^2\right), \tag{4.224}$$

while the constant $c$ appearing in Cotton–York tensor is:

$$c = 2n(-a^2 + 4n^2). \tag{4.225}$$

**Hyperbolic case** ($\nu = -1$) This case is very similar to the spherical one, with trigonometric functions traded for hyperbolic ones. We define $a$ and $n$ using:

$$\begin{aligned} c_1 &= 2(a-n), \\ c_2 &= 2a(1+a^2-4an), \\ c_3 &= 1+5a^2-12an. \end{aligned} \tag{4.226}$$

The appropriate coordinate transformations are:

$$\begin{aligned} x &= \kappa \sinh^2 \frac{\sigma}{2}, \\ y &= \lambda \varphi, \end{aligned} \tag{4.227}$$

with

$$\kappa = \frac{1}{1 - a(4n - a)}, \quad \lambda = \frac{2}{\kappa Z} \quad \text{and} \quad Z = 1 + a^2. \tag{4.228}$$

With these transformations the two-dimensional base space in the metric (4.175) takes the form of squashed $H_2$:

$$\mathrm{d}\ell^2 = \frac{\mathrm{d}\sigma^2}{\Delta_\sigma} + \frac{\sinh^2 \sigma \Delta_\sigma}{\Xi^2} \mathrm{d}\varphi^2 \tag{4.229}$$

with

$$\Delta_\sigma = 1 - a \cosh \sigma (4n - a \cosh \sigma). \tag{4.230}$$

The coordinates range as $\sigma \in \mathbb{R}_+$ and $\varphi \in [0, 2\pi]$. In the full $2+1$-dimensional metric $\mathrm{d}s^2 = -\mathrm{u}^2 + \mathrm{d}\ell^2$, the velocity field takes the form

$$\mathrm{u} = -\mathrm{d}t + b(\sigma)\mathrm{d}\varphi\,, \quad b(\sigma) = \frac{2(a - 2n + a \cosh \sigma)}{Z} \sinh^2 \frac{\sigma}{2}. \tag{4.231}$$

The scalar vorticity is

$$q = 2\left(n - a \cosh \sigma\right), \tag{4.232}$$

while the constant $c$ appearing in the Cotton–York tensor is

$$c = 2n(-1 - a^2 + 4n^2). \tag{4.233}$$

**Uniform parameterisation** It is possible to use a uniform notation to include the three different cases:

$$\begin{aligned} c_1 &= 2(a - n), \\ c_2 &= 2a(-\nu + a^2 - 4an), \\ c_3 &= -\nu + 5a^2 - 12an. \end{aligned} \tag{4.234}$$

The general coordinate transformations are:

$$\begin{aligned} x &= \kappa f_\nu^2 (\frac{\theta}{2}), \\ y &= \lambda \varphi, \end{aligned} \tag{4.235}$$

with $f_\nu$ as in (4.192), and

$$\kappa = \frac{1}{1 + \nu a(4n - a)}, \quad \lambda = \frac{2}{\kappa Z_\nu} \quad \text{and} \quad Z_\nu = 1 - \nu a^2. \tag{4.236}$$

The constant $c$ appearing in Cotton–York tensor takes the form:

$$c = 2n(\nu - a^2 + 4n^2). \tag{4.237}$$

Before moving to the general case $c_4 = 0$, a comment is in order here. One observes from (4.237) that the Cotton tensor of the monopole–dipole 2+1 geometries may vanish in two distinct instances. The first is when the charge $n$ itself vanishes, which corresponds to the absence of the monopolar component. The second occurs when

$$\nu - a^2 + 4n^2 = 0. \tag{4.238}$$

For vanishing $a$, only the case $\nu = -1$ is relevant: $n = \pm\frac{1}{2}$ and geometry $AdS_3$. For non-vanishing $a$, we obtain a conformally flat, non-homogeneous deformation of the $n$-squashed.

### Vanishing $c_4$

When the parameter $\tilde{c}_4 \equiv c_4$ is vanishing, it is not possible to perform the change of variables (4.184) and thus we have a different class of metrics. We are left with the parameters $\tilde{c}_1 \equiv c_1$, $\tilde{c}_2 \equiv c_2$, $\tilde{c}_3 \equiv c_3$ and $\tilde{c}_5 \equiv c_5$. We decide not to set to zero the latter in order to avoid a possible metric singularity (see (4.177)) when $c_2 = c_3 = 0$. The boundary metric is in this case given by

$$\begin{aligned} b(x) &= c_1 x + c_2 x^2, \\ G(x) &= c_5 + c_3 x^2 + c_2 x^3 (2c_1 + c_2 x), \end{aligned} \tag{4.239}$$

with

$$c = c_1 \left(c_3 - c_1^2\right). \tag{4.240}$$

For the flat horizon case $c_3 = 0$, this class of metrics appears as boundary of Einstein solutions studied in [46]. When $c_2 = 0$ we have a homogeneous geometry and what we concluded on transport coefficients for the case before is still valid: it is not possible to constrain any of them holographically, because the corresponding tensors vanish kinematically.

As in the previous situation, the boundary geometries at hand can be conformally flat. This occurs either when $c_1$ vanishes, or when

$$c_3 = c_1^2. \tag{4.241}$$

## 4.8.5 Geometries without space-like isometries

The perfect-Cotton geometries we have constructed in Sect. 4.8.4 possess at least one space-like Killing vector.

Our motivation for studying perfect-Cotton geometries was holographic. These particular 2+1-dimensional Randers-Papapetrou geometries turn out to be boundaries of exact $3+1$-dimensional bulk Einstein metrics, as we will discuss in detail. This property is not limited to the sole perfect-Cotton stationary geometries that admit space-like isometries: it is indeed valid in general. In absence of the additional Killing vector, the coefficients of the metric will depend on both the $x$ and the $y$ coordinates. It is thus harder to solve the perfect-Cotton conditions (4.169)-(4.171), and so to find the explicit form of the metric. Such solutions would anyway be consistent from the perspective of existence theorems of general relativity. It would be of course interesting to find those metrics, which would be new Einstein solutions, both because we would be able to go beyond the dipole geometries and because more non-vanishing dangerous tensors would appear, requiring thus to probe more transport coefficients.

## 4.9 Perfect geometries and their bulk realization

The perfect-Cotton geometries we have constructed in Sec. 4.8.4 possess at least one space-like Killing vector. Our motivation for studying such perfect-Cotton geometries was holographic, and, in particular, these metrics appear as boundaries of exact $3+1$-dimensional bulk Einstein metrics, as we will see in Sec. 4.10. This property is not limited to the sole perfect-Cotton stationary geometries that admit space-like isometries: any perfect-Cotton Papapetrou–Randers metric qualifies. It seems however very difficult to find explicit $(x, y)$-dependent solutions when the additional isometry is not present. Such solutions would play important role to go beyond the dipole and introduce more dangerous tensors, hence probe more transport coefficients. In the absence of exact solutions, we could proceed with probing further transport properties perturbatively. We leave this for the future.

## 4.10 The bulk duals of perfect equilibrium

### 4.10.1 Generic bulk reconstruction

When the boundary geometry is of the perfect-Cotton type and the boundary stress tensor is that of a fluid in perfect equilibrium, the bulk solution can be exactly determined. This is highly non-trivial because it generally involves an infinite resummation *i.e.* starting from the boundary data and working our way to the bulk.

Our main observation is that to the choice (**??**) for the boundary data corresponds the following *exact* bulk Einstein metric in Eddington–Finkelstein coordinates (where $g_{rr} = 0$ and $g_{r\mu} = -u_\mu$):

$$\begin{aligned} \mathrm{d}s^2 &= -2\mathrm{u}\left(\mathrm{d}r - \frac{1}{2}\mathrm{d}x^\rho u^\sigma \eta_{\rho\sigma\mu}\nabla^\mu q\right) + \rho^2 \mathrm{d}\ell^2 \\ &\quad - \left(r^2 + \frac{\delta}{2} - \frac{q^2}{4} - \frac{1}{\rho^2}\left(2Mr + \frac{qc}{2}\right)\right)\mathrm{u}^2, \end{aligned} \tag{4.242}$$

with

$$\rho^2 = r^2 + \frac{q^2}{4}. \tag{4.243}$$

The metric above is manifestly covariant with respect to the boundary metric. Taking the limit $r \to \infty$ it is easy to see that the boundary geometry is indeed the general stationary Papapetrou–Randers metric in (4.88) with

$$\mathrm{u} = -\mathrm{d}t + b\,\mathrm{d}y. \tag{4.244}$$

The various quantities appearing in (4.242) (like $\delta, q, c$) satisfy Eqs. (4.169), (4.170) and (4.171), and this guarantees that Einstein's equations are satisfied. Performing the Fefferman–Graham expansion of (4.242) we indeed recover the perfect form of the boundary energy–momentum tensor with

$$\varepsilon = \frac{M}{4\pi G_{\mathrm{N}}}. \tag{4.245}$$

where $G_{\mathrm{N}}$ is the four-dimensional Newton's constant. The corresponding holographic fluid has velocity field u, vorticity strength $q$ and behaves like a perfect fluid.

In the choice of gauge given by (4.172) and (4.173), the bulk metric (4.242) takes the form:

$$\begin{aligned} \mathrm{d}s^2 &= -2\mathrm{u}\left(\mathrm{d}r - \frac{1}{2}\left(\mathrm{d}y\frac{B}{A}\partial_x q - \mathrm{d}x\frac{A}{B}\partial_y q\right)\right) + \rho^2 \mathrm{d}\ell^2 \\ &\quad - \left(r^2 + \frac{\delta}{2} - \frac{q^2}{4} - \frac{1}{\rho^2}\left(2Mr + \frac{qc}{2}\right)\right)\mathrm{u}^2, \end{aligned} \tag{4.246}$$

where $q$ is as in (4.174). Note $\delta$ and $c$ can be readily obtained from $q$, $A$ and $B$ using (4.171) and (4.169) respectively.

It is clear from the explicit form of the bulk spacetime metric (4.242) that the metric has a curvature singularity when $\rho^2 = 0$. The locus of this singularity is at :

$$r = 0, \quad q(x, y) = 0. \tag{4.247}$$

However, we will find cases where $\rho^2$ never vanishes because $q^2$ never becomes zero. In such cases, the bulk geometries have no curvature singularities, but they might have regions with closed time-like curves.

Since $A$, $B$ and $b$ are functions of $x$ and $y$ the metric has only a Killing vector $\partial_t$. Although this is of unit norm at the boundary coinciding with the velocity vector of the boundary fluid, it's norm is not any more unity in the interior. The Killing vector becomes null at the ergosphere $r = R(x)$ where:

$$r^2 + \frac{\delta}{2} - \frac{q^2}{4} - \frac{1}{\rho^2}\left(2Mr + \frac{qc}{2}\right) = 0. \tag{4.248}$$

Beyond the ergosphere no observer can remain stationary, and hence he experiences frame dragging, as $\partial_t$ becomes space-like.

One could ask whether similar properties hold in higher dimensions, following the already observed exactness of the lowest term for Kerr. In particular one may wonder what replaces the perfect-Cotton geometry in higher dimensions, where there is no Cotton–York tensor. As we stressed, the bulk gravitational duality is a guiding principle that translates precisely to the boundary Cotton/energy–momentum relationship used in this paper. A similar principle is not available in every dimension and we expect only a limited number of cases where the observation made in [30, 26] about Kerr could be generalised to more general Einstein spaces.

### 4.10.2 Absence of naked singularities

We will focus here on the situation where we have an additional spatial isometry. We will show explicitly that for all perfect-Cotton geometries in this class, the bulk geometries have no naked singularities for appropriate range of values of the black hole mass. Our general solutions will be labeled by three parameters - namely the angular momentum $a$, the nut charge $n$ and the black hole mass $M$. This will cover all known solutions and also give us some new ones, as will be shown explicitly later in Appendix 4.11.

In order to analyse the bulk geometry we need to know the boundary geometry explicitly. In the previous section, we have been able to find all the perfect-Cotton geometries with at least one additional spatial Killing vector explicitly. These geometries are given by (4.175), (4.186) and (4.187), and are labelled by three continuously variable parameters $c_1$, $c_2$ and $c_3$. We have shown that without loss of generality, we can rewrite these parameters in terms of the angular momentum $a$, the nut charge $n$ and a discrete variable $\nu$ as in Eq. (4.234).

The holographic bulk dual (4.242) for perfect equilibrium in these general boundary geometries then reads:

$$\begin{aligned} \mathrm{d}s^2 &= -2\mathrm{u}\left(\mathrm{d}r - \frac{G}{2}\partial_x q\, \mathrm{d}y\right) + \rho^2\left(\frac{\mathrm{d}x^2}{G} + G\mathrm{d}y^2\right) \\ &\quad - \left(r^2 + \frac{\delta}{2} - \frac{q^2}{4} - \frac{1}{\rho^2}\left(2Mr + \frac{qc}{2}\right)\right)\mathrm{u}^2, \end{aligned} \tag{4.249}$$

where $\mathrm{u} = -\mathrm{d}t + b\mathrm{d}y$, and $b$ and $G$ are determined by three geometric $c_1$, $c_2$ and $c_3$ as in (4.186) and (4.187). Therefore $q$, $c$ and $\delta$ are as in (4.178), (4.179) and (4.180) respectively.

It is convenient for the subsequent analysis to move from Eddington–Finkelstein to Boyer–Linqvist coordinates. These Boyer–Linqvist coordinates make the location of the horizon manifest. These are the analogue of Schwarzschild coordinates in presence of an axial symmetry. In the case when the geometric parameter $c_4$ is non-vanishing, the transition to Boyer–Linqvist coordinates can be achieved via the following coordinate transformations:

$$\begin{aligned} \mathrm{d}\tilde{t} &= \mathrm{d}t - \frac{4(c_1^2+4r^2)}{3c_1^4+8c_1c_2-4c_1^2(c_3+6r^2)+16r(2M+c_3r-r^3)}\mathrm{d}r, \quad (4.250)\\ \mathrm{d}\tilde{y} &= \mathrm{d}y + \frac{16c_2}{3c_1^4+8c_1c_2-4c_1^2(c_3+6r^2)+16r(2M+c_3r-r^3)}\mathrm{d}r. \quad (4.251)\end{aligned}$$

Note even after changing $t, y$ to $\tilde{t}, \tilde{y}$, the boundary metric still remains the same - the difference between the old and new coordinates die off asymptotically.

After these transformations the bulk metric takes the form (we replace $\tilde{r}$ and $\tilde{y}$ with $r$ and $y$):

$$\mathrm{d}s^2 = \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 - \frac{\Delta_r}{\rho^2}\left(\mathrm{d}t + \beta\mathrm{d}y\right)^2 + \frac{\rho^2}{\Delta_x}\mathrm{d}x^2 + \frac{\Delta_x}{\rho^2}\left(c_2\mathrm{d}t - \alpha\mathrm{d}y\right)^2, \quad (4.252)$$

where

$$\begin{aligned} \rho^2 &= r^2 + \frac{q^2}{4} = r^2 + \frac{(c_1+2c_2x)^2}{4}, && (4.253)\\ \Delta_r &= -\frac{1}{16}\left(3c_1^4+8c_1c_2-4c_1^2(c_3+6r^2)+16r(2M+c_3r-r^3)\right), && (4.254)\\ \Delta_x &= G = x + c_3x^2 + 2c_1c_2x^3 + c_2^2x^4, && (4.255)\\ \alpha &= -\frac{1}{4}\left(c_1^2+4r^2\right), && (4.256)\\ \beta &= -b = -c_1x - c_2x^2. && (4.257)\end{aligned}$$

Note the coordinates $r$ and $x$ do not change as we transform from Eddington–Finkelstein to Boyer–Linqvist coordinates. Therefore $\rho^2$ is exactly the same as before. Also note that $\Delta_r$ and $\alpha$ are functions of $r$ only, while $\Delta_x$ and $\beta$ are functions of $x$ only.

It is easy to see that the horizons are at $r = r_*$ where:

$$\Delta_r(r = r_*) = 0, \quad \text{with} \quad r_* > 0. \quad (4.258)$$

At most we can have four horizons. These horizon(s) should clothe the curvature singularity located at $\rho^2 = 0$ or equivalently at:

$$r = 0, \quad x = -\frac{c_1}{2c_2}. \quad (4.259)$$

It is not hard to see that for fixed values of the geometric parameters $c_1$, $c_2$ and $c_3$, there exists a positive definite solution to Eq. (4.258) for an appropriate range of the black hole mass $M$. Hence the curvature singularity is not naked.

Clearly we have only two Killing vectors generically - namely $\partial_t$ and $\partial_y$. Each horizon $r = r_*$ is generated by the Killing vector:

$$\partial_t + \Omega_{\rm H}(r_*)\partial_y. \tag{4.260}$$

which is an appropriate linear combination of the two Killing vectors. $\Omega_{\rm H}(r_*)$ is a constant given by:

$$\Omega_{\rm H}(r_*) = \frac{c_2}{\alpha(r_*)} \tag{4.261}$$

and is the rigid velocity of the corresponding horizon.

The bulk geometry can have at most four ergospheres where the Killing vector $\partial_t$ becomes null. These are given by $r = R(x)$ where $R(x)$ is a solution of:

$$g_{tt} = 0, \quad \text{i.e.} \quad \Delta_r = c_2^2 G. \tag{4.262}$$

We have seen in Section 4.8.4 that the geometric structure of the boundary geometries is better revealed as fibrations over squashed $S^2$, $\mathbb{R}^2$ or $H_2$ if we do a further coordinate transformation in $x$ and $y$. We will do the same coordinate transformations given by (4.235) in the bulk metric separately for $\nu = 1, 0, -1$. We will also need to exchange parameters $c_1$, $c_2$ and $c_3$ with $a$, $n$ and $\nu$ using (4.234). Note in these coordinate transformations the radial coordinate $r$ and the time coordinate $t$ do not change, while the spatial coordinates $x$ and $y$ transform only as functions of themselves. This preserves the Boyer–Linqvist form of the metric (4.288). We can apply the same strategy to locate the horizon(s) and the ergosphere(s).

The advantage of doing these coordinate transformations is that for $\nu = 1, 0, -1$ we will see that the horizon will be a squashed $S^2$, $\mathbb{R}^2$ and $H_2$ respectively. The metrics are given explicitly in Appendix 4.11, where we will also show that we recover all known rotating black hole solutions for which the horizons will be squashed $S^2$ or $H_2$. As far as we are aware of the literature, the case of squashed $\mathbb{R}^2$ horizon (4.272) is novel.

For the case of vanishing $c_4$, we can similarly proceed to change coordinates an bring the bulk metric to Boyer–Linqvist form. The details are presented in Appendix 4.11, Eq. (4.288). Except for the special case (4.294), all such solutions in this class will be novel as far as we are aware of the literature.

Interestingly when $c_2 = 0$, $\rho^2 > c_1^2/4$, hence it never vanishes. Therefore the bulk geometry has no curvature singularity. In terms of $a$, $n$ and $\nu$, this happens when

- for $\nu = 1$: $n > a$;
- for $\nu = 0$: $n > a$ or $n < \frac{a}{4}$;

- for $\nu = -1$: $n < a$ or $|n| \leq \frac{1}{2}$.

In such cases horizon(s) may exist, but in absence of a curvature singularity, it is not necessary for the horizon to exist in order that the solution is a good solution.

### 4.10.3 The case of no spatial isometry

We comment now on the case of perfect Cotton geometries with no spatial isometry. Though we do not have explicit examples of such boundary metrics, we know that their uplift leads to the exact solutions given in (4.242). However, the coefficient of the bulk geometry are not explicitly known, and therefore we cannot analyze in detail the presence of naked singularities in the bulk.
It is also that the perfect Cotton condition itself will force the geometry to have at least an additional spatial isometry. This is consistent with the rigidity theorem in $3+1$-dimensional which requires all stationary black hole solutions in flat space to have an axial symmetry. However, as far as we are aware, it is not known if this theorem is valid for $3+1$-dimensional space-times in AdS for an arbitrary stationary boundary geometry.

## 4.11 Explicit bulk solutions

The dual of perfect-Cotton boundary geometries can be written as an exact solution of Einstein's equations. Such solutions are different depending on the value of $c_4$ and on the geometry of the horizon. We present in this section the complete classification when an extra isometry is present.

### Non-vanishing $c_4$: Kerr–Taub–NUT metrics

We start from the boundary metrics studied in Sec. 4.8.4 and uplift them using (4.288).

**Spherical ($\nu = 1$)** We set

$$\begin{aligned} c_1 &= 2(a-n), \\ c_2 &= 2a(-1+a^2-4an), \\ c_3 &= -1+5a^2-12an. \end{aligned} \tag{4.263}$$

By doing this, we recover the spherical-horizon Kerr–Taub–NUT metric [?]:

$$\mathrm{d}s^2 = \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 - \frac{\Delta_r}{\rho^2}\left(\mathrm{d}t + \beta\mathrm{d}\phi\right)^2 + \frac{\rho^2}{\Delta_\vartheta}\mathrm{d}\vartheta^2 + \frac{\sin^2\vartheta\Delta_\vartheta}{\rho^2}\left(a\mathrm{d}t + \alpha\mathrm{d}\phi\right)^2, \tag{4.264}$$

with

$$
\begin{aligned}
\rho^2 &= r^2 + (n - a\cos\vartheta)^2, && (4.265)\\
\Delta_r &= r^4 + r^2(1 + a^2 + 6n^2) - 2Mr + (a^2 - n^2)(1 + 3n^2), && (4.266)\\
\Delta_\vartheta &= 1 + a\cos\vartheta(4n - a\cos\vartheta), && (4.267)\\
\beta &= -b(\theta) = -\frac{2(a - 2n + a\cos\vartheta)}{\Xi}\sin^2\frac{\vartheta}{2}, && (4.268)\\
\alpha &= -\frac{r^2 + (n-a)^2}{\Xi}, && (4.269)\\
\Xi &= 1 - a^2. && (4.270)
\end{aligned}
$$

**Flat** $(\nu = 0)$ We set

$$
\begin{aligned}
c_1 &= 2(a - n),\\
c_2 &= 2a^2(a - 4n),\\
c_3 &= a(5a - 12n). && (4.271)
\end{aligned}
$$

and get the flat-horizon Kerr–Taub–NUT metric:

$$
\mathrm{d}s^2 = \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 - \frac{\Delta_r}{\rho^2}\left(\mathrm{d}t + \beta\mathrm{d}\phi\right)^2 + \frac{\rho^2}{\Delta_\sigma}\mathrm{d}\sigma^2 + \frac{\sigma^2\Delta_\sigma}{\rho^2}\left(a^2(a - 4n)\mathrm{d}t + \alpha\mathrm{d}\phi\right)^2, \quad (4.272)
$$

with

$$
\begin{aligned}
\rho^2 &= r^2 + \frac{1}{4}\left(2a - 2n + a^2\sigma^2(a - 4n)\right)^2, && (4.273)\\
\Delta_r &= r^4 + r^2(a^2 + 6n^2) - 2Mr + 3n^2(a^2 - n^2), && (4.274)\\
\Delta_\sigma &= \frac{(2 + a^2\sigma^2)(8 - 24an\sigma^2 + a^4\sigma^4 - 8a^3n\sigma^4 + 2a^2\sigma^2(3 + 8n^2\sigma^2))}{16}, && (4.275)\\
\beta &= -b(\theta) = \frac{\sigma^2}{4}\left(4(n - a) + a^2\sigma^2(4n - a)\right), && (4.276)\\
\alpha &= r^2 + (n - a)^2. && (4.277)
\end{aligned}
$$

It seems that this metric was never quoted in the literature. It provides the AdS generalisation of the asymptotically flat metric of [**?**].

**Hyperbolic** $(\nu = -1)$ We set

$$
\begin{aligned}
c_1 &= 2(a - n),\\
c_2 &= 2a(1 + a^2 - 4an),\\
c_3 &= 1 + 5a^2 - 12an. && (4.278)
\end{aligned}
$$

and obtain the hyperbolic-horizon Kerr–Taub–NUT metric (also mentioned in [46]):

$$
\mathrm{d}s^2 = \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 - \frac{\Delta_r}{\rho^2}\left(\mathrm{d}t + \beta\mathrm{d}\varphi\right)^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{\sinh^2\theta\Delta_\theta}{\rho^2}\left(a\mathrm{d}t + \alpha\mathrm{d}\phi\right)^2, \quad (4.279)
$$

with

$$\begin{aligned}
\rho^2 &= r^2 + (n - a\cosh\theta)^2, && (4.280)\\
\Delta_r &= r^4 + r^2(-1 + a^2 + 6n^2) - 2Mr + (a^2 - n^2)(-1 + 3n^2), && (4.281)\\
\Delta_\theta &= 1 - a\cosh\theta(4n - a\cosh\theta), && (4.282)\\
\beta &= -b(\theta) = -\frac{2(a - 2n + a\cosh\theta)}{Z}\sinh^2\frac{\theta}{2}, && (4.283)\\
\alpha &= \frac{r^2 + (n-a)^2}{Z}, && (4.284)\\
Z &= 1 + a^2. && (4.285)
\end{aligned}$$

## Vanishing $c_4$

When $c_4 = 0$, the bulk metric is obtained in Boyer–Linqvist form from (4.246) by doing the following coordinate transformations

$$\begin{aligned}
\mathrm{d}\tilde{t} &= \mathrm{d}t - \frac{4(c_1^2 + 4r^2)}{3c_1^4 - 4c_1^2(c_3 + 6r^2) - 16(c_2^2c_5 - 2Mr - c_3r^2 + r^4)}\mathrm{d}r, && (4.286)\\
\mathrm{d}\tilde{y} &= \mathrm{d}y + \frac{16c_2}{3c_1^4 - 4c_1^2(c_3 + 6r^2) - 16(c_2^2c_5 - 2Mr - c_3r^2 + r^4)}\mathrm{d}r. && (4.287)
\end{aligned}$$

The metric is explicitly given by (we replace $\tilde{r}$ and $\tilde{y}$ with $r$ and $y$):

$$\mathrm{d}s^2 = \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 - \frac{\Delta_r}{\rho^2}\left(\mathrm{d}t + \beta\mathrm{d}y\right)^2 + \frac{\rho^2}{\Delta_x}\mathrm{d}x^2 + \frac{\Delta_x}{\rho^2}\left(c_2\mathrm{d}t - \alpha\mathrm{d}y\right)^2, \qquad (4.288)$$

where

$$\begin{aligned}
\rho^2 &= r^2 + \frac{q^2}{4} = r^2 + \frac{(c_1 + 2c_2x)^2}{4}, && (4.289)\\
\Delta_r &= -\frac{1}{16}\left(3c_1^4 - 4c_1^2(c_3 + 6r^2) - 16(c_2^2c_5 - 2Mr - c_3r^2 + r^4)\right), && (4.290)\\
\Delta_x &= G = c_5 + c_3x^2 + 2c_1c_2x^3 + c_2^2x^4, && (4.291)\\
\alpha &= -\frac{1}{4}\left(c_1^2 + 4r^2\right), && (4.292)\\
\beta &= -b = -c_1x - c_2x^2. && (4.293)
\end{aligned}$$

According to our knowledge, solutions of these kind are not known in literature except for the special case where $c_1$, $c_2$, $c_3$ and $c_5$ are given by

$$c_1 = 2n, \quad c_2 = a, \quad c_3 = 0 \quad \text{and} \quad c_5 = 1, \qquad (4.294)$$

with $n$ being the nut charge and $a$ being the angular momentum. In this case we recover the flat-horizon solution of [46], which is however different from the above flat-horizon solution (4.272), or, at $n = 0$, the rotating topological black hole of [**?**].

### Geometries with conformally flat boundarties

As discussed at the end of Secs. 4.8.4 and 4.8.4, the boundary geometries become conformally flat for some specific values of the parameters: for non-vanishing $c_4$ this happens when (4.238) is satisfied, whereas for $c_4 = 0$ the requirement is (4.241).

## 4.12 Constraints on transport coefficients

In the previous section, we have shown that we can find exact black hole solutions corresponding to perfect equilibrium of the dual field theory in perfect-Cotton boundary geometries. From the perspective of the boundary fluid dynamics, we are ensured by construction that the energy-momentum tensor is exactly of the perfect type. Thus, any dangerous tensor that this deformed boundary may have, will necessarily couple to vanishing transport coefficients. This gives non-trivial information about strongly coupled holographic conformal fluids in the classical gravity approximation.
Exact black hole solutions indeed imply holographic fluids at strong coupling, and in the classical gravity approximation such fluids can have infinitely many vanishing non-dissipative transport coefficients. The condition of perfect-Cotton geometries at the boundary is not enough to constrain all non-dissipative transport coefficients. This is because many Weyl-covariant, traceless and transverse tensors which do not contribute to dissipation will vanish kinematically.
Perfect-Cotton geometries with an additional spatial isometry only can be uplifted to exact black hole solutions without naked singularities for generic values of four parameters characterizing them. Let us now examine the presence of dangerous tensors in these geometries.
For concreteness, we begin at the third order in derivative expansion. The list of possible dangerous tensors is in (4.166). We note that $\langle C_{\mu\nu} \rangle$ vanishes in any perfect-Cotton geometry, because the transverse part of $C_{\mu\nu}$ is pure trace, meaning it is proportional to $\Delta_{\mu\nu}$. Therefore, it is not a dangerous tensor in any perfect-Cotton geometry, as a result we cannot constrain the corresponding transport coefficient $\gamma_{(3)1}$.
We work in the Weyl-covariant formalism described in section 4.3.2. It is possible to show that in equilibrium the Weyl-covariant derivative $\mathcal{D}_\mu$ reduces to the covariant derivative $\nabla_\mu$. This facilitates our hunt for dangerous tensors.
For any boundary geometry of the class described in Sect. **??**, $\langle D_\mu V_\nu \rangle$ vanishes. Thus, it is not a dangerous tensor in these class of geometries and we cannot constrain the transport coefficient $\gamma_{(3)1}$. It could possibly be a dangerous tensor in perfect-Cotton geometries without any spatial isometry. Even though we know that such boundary metrics uplift to exact solutions in the bulk, we do not know these geometries explicitly. Consequently, we cannot investigate the presence of horizons in the bulk. So, we cannot use black hole solutions to constrain this

transport coefficient yet.
The first dangerous tensor we encounter is $\langle D_\mu W_\nu \rangle$. Not only such term is non-vanishing, but it is also not conserved, meaning that $\nabla^\mu \langle D_\mu W_\nu \rangle \neq 0$. Perfect equilibrium can exist only if the corresponding dangerous transport coefficient $\gamma_{(3)3}$ vanishes. Thus, this transport coefficient vanishes for all strongly coupled holographic fluids in the regime of validity of classical gravity approximation.
We can similarly show that infinite number of tensors of the form of $(C^{\alpha\beta} C_{\alpha\beta})^l \langle D_\mu W_\nu \rangle$, $(V^\alpha V_\alpha)^m \langle D_\mu W_\nu \rangle$ and $(W^\alpha W_\alpha)^n \langle D_\mu W_\nu \rangle$ for $l, m$ and $n$ being arbitrary positive integers, are dangerous tensors in geometries of Sect. **??**. We conclude that the infinitely many non-dissipative transport coefficients corresponding to these dangerous tensors should vanish.
For example, at the fourth order in the derivative expansion, we get new kind of dangerous tensors of the form $\langle V_\mu V_\nu \rangle$, $\langle W_\mu W_\nu \rangle$ and $\langle D_\mu D_\nu (\omega^{\alpha\beta} \omega_{\alpha\beta}) \rangle$ in geometries of Sect. **??**. This further implies existence of infinite number of dangerous tensors, of the form of $(C^{\alpha\beta} C_{\alpha\beta})^l \langle V_\mu V_\nu \rangle$, $(V^\alpha V_\alpha)^m \langle V_\mu V_\nu \rangle$, $(W^\alpha W_\alpha)^n \langle V_\mu V_\nu \rangle$, etc. in the geometries of section **??**. Once again this leads us to conclude that infinite number of new dangerous transport coefficients vanish.
The constraints on transport coefficients follow from requiring the bulk to be an exact solutions of Einstein's equations: infinitely many non-dissipative transport coefficients that would destroy the unique perfect-fluid solution are required to vanish in holographic systems. Not that if this was not the case, we would had get a contradiction of the black-hole uniqueness theorem.

# Chapter 5

# Conclusions

Different gravitational backgrounds in four-dimensions have been studied in this Thesis, inspired mainly by holographic motivations.

In the first chapter we studied self-dual gravitational instantons endowed with a product structure $\mathbb{R} \times \mathcal{M}_3$, $\mathcal{M}_3$ being an homogeneous three-dimensional manifold of Bianchi type. We studied the general conditions under which such solutions can be mapped into geometric flows. The temporal evolution of the instanton is then given a Ricci flow plus a Yang-Mills connection, accompanied by diffeomorphism. This correspondence holds for both unimodular and non-unimodular Bianchi classes, where in the latter case an additional term proportional to the metric itself appears in the Ricci part of the flow. The self-duality constraint does not affect some classes of solutions, namely Bianchi VIII and IX, for which the metric remains generic. It does however further restrict the metric for all other Bianchi groups, without altering the consistency flow equation. In particular, among non-unimodular groups, Bianchi III is the only group where non-singular gravitational instantons, corresponding to non-degenerate geometric flows, exist.
We should stress the specificity of four dimensions in respect to the relationship between geometric flows and gravitational instantons. Even though self-duality can be imposed in other dimensions, such as 7 or 8 thanks respectively to the $G_2$ structure and to the octonions, the holonomy group has no factorization property, which is essential for the correspondence. It seems thus more difficult, although we cannot exclude it, to find for such dimensions a class of instantons that could be interpreted as geometric flows of a lower-dimensional geometry.

In the second chapter we studied a procedure to embed the general solution for non-BPS extremal asymptotically flat static and under-rotating black-holes in abelian gauged $\mathcal{N} = 2$ supergravity, in the limit where the scalar potential vanishes but the gauging does not. The bosonic Lagrangian of the two theories is the same, but the solutions are not, due to the different fermionic sectors and thus to the different supersymmetric properties. Nevertheless, the attractor geometries, and

therefore also the microscopic counting, for BPS asymptotically AdS black holes and for asymptotically flat extremal non-BPS black holes fall within a common class of supersymmetric $AdS_2 \times S^2$ spaces, or their rotating generalizations.
An interesting extension of the abelian gauged theory is given by the possibility of including gauged hypermultiplets. An extra term enters into the scalar potential, but models with identically flat potentials are still possible, and it is an interesting subject on its own to explore the allowed gaugings. Nevertheless, the extension of the squaring procedure with inclusion of hypermultiplets is not known yet.
Also, it would be interesting to extend our procedure to the non-extremal case.

In the last chapter we studied phases of strongly coupled conformal holographic systems in various stationary backgrounds. The corresponding gravity counterpart is given by the analysis of stationary black holes in AdS space with non-trivial boundary geometries. The boundary spaces taken into account enjoy the presence of a unique time-like Killing vector field of unit norm. In this case the global equilibrium is described by a perfect fluid, that is relativistic Euler equations are satisfied. Those equations admit a unique solution describing equilibrium. If this perfect fluid configuration indeed describes the equilibrium of the three-dimensional strongly coupled conformal matter, the corresponding black hole in four-dimensional Einstein's gravity is unique. This is nothing else then the holographic version of the black hole uniqueness theorem. It is then interesting to investigate the additional properties a stationary background with a unique time-like Killing vector of unit norm should satisfy such that the equilibrium of the holographic system is described by a perfect fluid. The perfect form of the Cotton-York tensor turns out to be a sufficient condition for this to happen. This automatically implies that infinitely many non-dissipative transport coefficients, which are in principle allowed by the geometry but would destroy the unique perfect fluid solution, should vanish.
From the gravity point of view, if those transport coefficients would have been non-zero, then the energy-momentum tensor of the equilibrium state of the dual system would not have been perfect-fluid like by holographic dictionary. From the boundary point of view, it the transport coefficients were non-vanishing, then the equilibrium solution would not have been unique, since the fluid mechanics would have had higher order derivative corrections up to an arbitrary order. This would have been in contradiction with the black hole uniqueness theorem.
Moreover, when the Cotton-York tensor is perfect-fluid like, the expansion from the boundary to the bulk can be exactly resummed, leading to a classification of all possible exact black hole geometries corresponding to such fluids. It is a natural to ask whether the condition on the Cotton-York tensor can be relaxed and whether it is possible to find necessary, and not only sufficient, conditions for the resummation to be exact and for the fluid to be perfect.

# Chapter 6

# List of publications

## Self-dual gravitational instantons and geometric flows of all Bianchi types

**P.M. Petropoulos, V.P., K. Siampos**

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# Self-dual gravitational instantons and geometric flows of all Bianchi types

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**Abstract**
We investigate four-dimensional, self-dual gravitational instantons endowed with a product structure $\mathbb{R} \times \mathcal{M}_3$, where $\mathcal{M}_3$ is a homogeneous three-dimensional manifold of Bianchi type. We analyze the general conditions under which Euclidean-time evolution in the gravitational instanton can be identified with a geometric flow of a metric on $\mathcal{M}_3$. This includes both unimodular and non-unimodular groups, and the corresponding geometric flow is a general Ricci plus Yang–Mills flow accompanied by a diffeomorphism.

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An intriguing relationship between three-dimensional geometric flows and gravitational instantons in four dimensions has recently been investigated in [1]. It states that self-dual solutions of vacuum Einstein equations for Euclidean $\mathcal{M}_4 = \mathbb{R} \times \mathcal{M}_3$ foliations, with homogeneous sections $\mathcal{M}_3$ of Bianchi type, are mapped onto geometric flows on the three-dimensional manifold $\mathcal{M}_3$. The geometric flows appear as a combination of Ricci and Yang–Mills flows.

This observation, originally made in [2–4] for Bianchi IX, has been extended in [1] for all unimodular Bianchi groups, under the assumption of diagonal metrics. Although for these groups any metric can eventually be taken diagonal, such a choice obscures the reach of the correspondence, which might ultimately appear as a technical coincidence. Furthermore, it invalidates it for non-unimodular groups, where diagonal metrics are not the most general.

The aim of this paper is to show that this correspondence holds generally, without any assumption on the metric and for all Bianchi classes, including the non-unimodular groups. The latter case requires the addition of an extra term to the Ricci part of the flow, proportional to the metric and available only in the non-unimodular class, as well as a prescribed diffeomorphism. Our understanding of the phenomenon at hand is now complete and gives confidence that it might hold similarly for Einstein gravity in higher dimensional set-ups admitting self-dual solutions—much like it does for non-relativistic gravity under the detailed-balance condition [5, 6].

**Table 1.** Basis of invariant forms and restrictions on $\gamma$—unimodular groups.

| Type | $n, \eta, a = 0$ | Restrictions from (18) | Restrictions from (17) |
|---|---|---|---|
| I | $n = 0 \quad \eta = \mathbb{1}$ | None | None rank-1 |
| II | $n = \mathrm{diag}(0, 1, 0)$ <br> $\eta = \mathbb{1}$ | $\gamma_{12} = 0 = \gamma_{23}$ | $\gamma_{12} = 0 = \gamma_{23}$: rank-0 <br> $\gamma_{13} = 0$ and/or $\gamma_{11} = \gamma_{33}$: rank-1 |
| VIII | $n = \eta = \mathrm{diag}(1, -1, -1)$ | None | None rank-0 <br> None rank-3: $I = \eta$ |
| IX | $n = \eta = \mathbb{1}$ | None | None rank-0 <br> None rank-3: $I = \eta$ |
| $\mathrm{VI}_{-1}$ | $n = \mathrm{diag}(0, 1, -1)$ <br> $\eta = \mathrm{diag}(-1, 1, -1)$ | $\gamma_{12} = 0 = \gamma_{13}$ | $\gamma_{12} = 0 = \gamma_{13}$ <br> Rank-0,1 |
| $\mathrm{VII}_0$ | $n = \mathrm{diag}(1, 1, 0)$ <br> $\eta = \mathrm{diag}(1, 1, 1)$ | $\gamma_{12} = 0 = \gamma_{23}$ | $\gamma_{13} = 0 = \gamma_{23}$ <br> Rank-0,1 |

**Table 2.** Basis of invariant forms and restrictions on $\gamma$—non-unimodular groups.

| Type | $n, \eta, a$ | Restrictions from (18) | Restrictions from (17) |
|---|---|---|---|
| III | $n = \mathrm{diag}(0, 1, -1) \quad a = (1, 0, 0)$ <br> $\eta = \mathrm{diag}(-1, 1, -1)$ | $\gamma_{12} = \gamma_{13}$ <br> $\gamma_{22} = \gamma_{33}$ | $\gamma_{11}, \gamma_{13}$ given in (A.9) |
| IV | $n = \mathrm{diag}(0, 1, 0)$ <br> $\eta = \mathbb{1} \quad a = (1, 0, 0)$ | $\gamma_{12} = 0 = \gamma_{13}$, <br> $\gamma_{23} = \gamma_{22} + \gamma_{33}$ | Singular from (A.1) <br> $\gamma_{22} = 0 = \gamma_{33}$ |
| V | $n = 0 \quad \eta = \mathbb{1}$ <br> $a = (1, 0, 0)$ | $\gamma_{12} = 0 = \gamma_{13}$ <br> $\gamma_{22} = -\gamma_{33}$ | Singular from (A.1) <br> $\gamma_{22} = 0 = \gamma_{23}$ |
| $\mathrm{VI}_{h>-1}$ | $n = \mathrm{diag}(0, 1, -1) \quad a = (h + 1, 0, 0)$ <br> $\eta = \mathrm{diag}(-1, 1, -1)$ | $\gamma_{12} = 0 = \gamma_{13}$ <br> $\gamma_{22} = \gamma_{33}$ | Singular from (A.1) <br> $\gamma_{22} = \gamma_{23}$ |
| $\mathrm{VII}_{h>0}$ | $n = \mathrm{diag}(1, 1, 0) \quad \eta = \mathbb{1}$ <br> $a = (0, 0, h)$ | $\gamma_{13} = 0 = \gamma_{23}$ <br> $\gamma_{11} = -\gamma_{22}$ | Singular from (A.1) <br> $\gamma_{11} = 0 = \gamma_{12}$ |

A metric on $\mathcal{M}_4$ is generally of the following type:

$$g = \mathrm{d}t^2 + g_{ij}\sigma^i\sigma^j. \tag{1}$$

We implicitly choose a gauge with trivial shift and lapse functions. The prescribed isometry requires $g_{ij}$ be a function of $t$ only, while $\{\sigma^i, i = 1, 2, 3\}$ are the left-invariant Maurer–Cartan forms of the Bianchi group. They obey

$$\mathrm{d}\sigma^i = \tfrac{1}{2}c^i{}_{jk}\sigma^j \wedge \sigma^k. \tag{2}$$

The structure constants can be put in the form (see e.g. [7])

$$c^k{}_{ij} = -\epsilon_{ij\ell}n^{\ell k} + \delta^k_j a_i - \delta^k_i a_j, \tag{3}$$

where $n^{\ell k}$ are the elements of a symmetric matrix $n$ and $a_i$ the components of a covector $a$. We also define the antisymmetric matrix $m$ with entries

$$m^{ij} \equiv \epsilon^{ijk}a_k. \tag{4}$$

With these definitions, the Jacobi identity of the above algebra reads

$$\epsilon_{ijk}m^{ij}(n^{k\ell} - m^{k\ell}) = 0 \quad \Leftrightarrow \quad a_k n^{k\ell} = 0, \tag{5}$$

whereas the trace of the structure constants is $c^j{}_{ij} = 2a_i$. Unimodular groups have zero trace and are referred to as Bianchi A; Bianchi B are the others. Our choice for the structure constants is presented in tables 1 and 2.

Self-duality conditions are naturally implemented in an orthonormal frame, where

$$g_{ij}\sigma^i\sigma^j = \eta_{ij}\theta^i\theta^j, \tag{6}$$

and

$$\eta_{ij}\theta^j = \gamma_{ij}\sigma^j. \tag{7}$$

Two remarks are in order here, which are at the heart of the advertized correspondence. First, for real self-duality, only $\eta = \mathbb{1}$ or $\mathrm{diag}(1, -1, -1)$, up to a permutation, are allowed. The latter situation is more natural than the Euclidean one for Bianchi VIII ($SL(2, \mathbb{R})$ group) and for all other Bianchi groups obtained by or related to contractions of the latter (III and VI). With this choice, the geometric-flow correspondence can be achieved without any complexification. Secondly, we will not consider the most general vielbeins, but only those for which $\gamma_{ij}$ is *symmetric*. Although it might be restrictive[1], this choice is unavoidable because $\gamma_{ij}$ appears ultimately as the metric on $\mathcal{M}_3$,

$$\mathrm{d}s^2 = \gamma_{ij}\sigma^i\sigma^j, \tag{8}$$

whose appropriate flow coincides with the dynamics of the gravitational instanton. It should be stressed here, and kept in mind as an important—and still puzzling—feature of our analysis, that the flowing metric on $\mathcal{M}_3$ *is not* the metric on the spatial section of the corresponding gravitational instanton induced by (1), but rather its 'square root'[2].

In four dimensions, spin connection and curvature forms belong to the antisymmetric **6** representation of the group of local rotations $SO(4)$ (or $SO(2, 2)$, when $\eta \neq \mathbb{1}$). This group factorizes as $SO(3) \otimes SO(3)$ (or $SO(2, 1) \otimes SO(2, 1)$) and both the connection and the curvature forms can be reduced as $\mathbf{6} = \mathbf{3} \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathbf{3}$, referred to as self-dual and anti-self-dual components. The spin-connection one-form is defined by the torsionless and metric-compatibility equations:

$$\mathrm{d}\theta^a + \omega^a_b \wedge \theta^b = 0, \quad \omega_{ab} = -\omega_{ba}. \tag{9}$$

Its decomposition in self-dual and anti-self-dual parts is

$$\varsigma_i = \frac{1}{2}\left(\omega_{0i} + \frac{1}{2}\epsilon_{ijk}\omega^{jk}\right), \tag{10}$$

$$\alpha_i = \frac{1}{2}\left(\omega_{0i} - \frac{1}{2}\epsilon_{ijk}\omega^{jk}\right). \tag{11}$$

A similar decomposition holds for the curvature two-form.

Requiring self-duality of the curvature (see [1, 8] for details) states that the anti-self-dual Lévi–Cività $SO(3)$ (or $SO(2, 1)$) connection triplet $\{\alpha_i, i = 1, 2, 3\}$ must be a pure gauge field: $\mathrm{d}\alpha_i + \epsilon_{ijk}\alpha^j \wedge \alpha^k = 0$. This is achieved with

$$\alpha_i = \frac{1}{2}I_{ij}\sigma^j, \tag{12}$$

where the first integral $I = \{I_{ij}\}$ satisfies (the prime stands for $\mathrm{d}/\mathrm{d}t$)

$$I'_{ij} = 0, \quad I_{i\ell}c^{\ell}{}_{jk} + \epsilon_{imn}\eta^{mp}\eta^{nq}I_{pj}I_{qk} = 0. \tag{13}$$

We can compute the spin connection using equations (9) and our metric ansatz (1), (6). Inserting its expression into (11) and (12), we find the first-order dynamics of the vielbein

[1] Symmetric vielbeins are exhaustive *only for* unimodular algebras. A detailed account for this issue is available in [8], where a comprehensive and systematic analysis of all self-dual gravitational instantons of Bianchi type is presented in a purely Euclidean framework, without assuming symmetric vielbeins. Particular attention is drawn to the use of symmetries for reducing the redundant components of the vielbein. Our perspective is different here, since we want to interpret $\gamma_{ij}$ as a metric in order to translate its dynamics into a geometric flow but, within this assumption, we want to keep it as general as possible.

[2] We thank A Petkou for drawing to our attention similar properties of scalar fields in some holographic set-ups (unpublished work).

components $\gamma_{ij}$ defined in (7). These first-order equations can be put in a compact form, useful for the subsequent developments, by introducing a three-dimensional matrix

$$\Omega = \det\Gamma\left(\gamma n\gamma - \gamma m\gamma - \frac{\eta}{2}\mathrm{tr}(\gamma n\gamma\eta)\right). \tag{14}$$

The matrix notation is self-explanatory: $\gamma$ and $\Gamma$ are inverse of each other with entries $\gamma_{ij}$ and $\Gamma^{ij}$, $\eta$ stands for both $\eta_{ij}$ and $\eta^{ij}$, which are equal, $\gamma\eta = \gamma_{ij}\eta^{jk}$, .... One thus obtains two equations: a first-order evolution equation

$$\gamma' = -\Omega\eta\gamma - I \tag{15}$$

and a constraint

$$[\gamma'\eta, \eta\Gamma] = 0, \tag{16}$$

equivalent, using (15), to

$$[\Omega\eta + I\Gamma, \gamma\eta] = 0. \tag{17}$$

Since $\gamma$ is required to be symmetric, $\gamma'$ must also be, which imposes through (15) the constraint

$$\mathrm{a}(\Omega\eta\gamma + I) = 0, \tag{18}$$

with the notation $\mathrm{a}(P) = 1/2\left(P - P^{\mathrm{T}}\right)$. Constraint (18) restricts the form of $\gamma$: not all symmetric vielbeins are eligible for satisfying the self-duality conditions. The precise form of $\gamma$ depends on the Bianchi class—as well as on the basis chosen for the invariant forms (2), captured by the set of data $\{n, a\}$ entering (3) (ours are displayed in tables 1 and 2). Note finally that under (18), constraint (17) is equivalent to

$$\mathrm{a}(\Omega + I\Gamma\eta) = 0 \quad \Leftrightarrow \quad \gamma m\gamma = \mathrm{a}(IM\eta), \tag{19}$$

where $M = \mathrm{Adj}(\gamma)$ is the adjoint matrix of $\gamma$, i.e. the matrix of the $2 \times 2$ subdeterminants of $\gamma$.

For the reader's convenience, it is useful at this stage to outline our strategy for the next steps. The dynamics of self-dual gravitational instantons with symmetric vielbeins $\gamma_{ij}$ is governed by the evolution equation (15), under any two independent constraints among (17), (18) and (19). All these equations depend explicitly on the first integral $I$, which solves (13). We will therefore (i) classify the possible solutions $I$, (ii) re-express accordingly constraint (18) as well as the evolution equation (15) and (iii) interpret the resulting evolution equation as a specific geometric flow for $\gamma$ viewed as a metric on $\mathcal{M}_3$. Constraint (17) (or equivalently (19)) will be finally used to further restrict the allowed form of $\gamma$.

Any solution of equation (13) corresponds to an algebra homomorphism $G_3 \to SO(3)$, $G_3$ being a Bianchi algebra; it can be of rank 0, 1 or 3. The general solutions are as follows.

- For Bianchi VIII and IX ($n = \eta, a = 0$),
  - rank 0: $I = 0$,
  - rank 3: $I = \eta$ (Cayley–Hamilton theorem on $I\eta$).

  In these cases, constraint (18) is satisfied without restricting the form of $\gamma$.
- In all other Bianchi classes $\det n = 0$, and besides rank-0 solutions $I = 0$, only rank-1 solutions exist, for which (13) becomes[3]

  $$I(n + m) = 0 \quad \Leftrightarrow \quad (n - m)I^{\mathrm{T}} = 0, \tag{20}$$

  and they are necessarily of the form $I_{ij} = \rho_i\tau_j$. Even though $I_{ij}$ need not *a priori* be symmetric, the combination of constraints (18) and (17) as well as the integrability

[3] Equation (13) is strictly equivalent to $I(n + m) = \eta\mathrm{Adj}(I)^{\mathrm{T}}$—vanishing if $I$ is rank-1.

requirement of the latter (validity at any time, i.e. compatibility with the evolution equation (15)) implies, after a tedious computation, that

$$I = I^{\mathrm{T}}. \tag{21}$$

Under (20) and (21), constraint (18) is equivalent to

$$\eta\gamma(n+m) = (n-m)\gamma\eta \quad \Leftrightarrow \quad [n\eta, \eta\gamma] = \{m\eta, \eta\gamma\}. \tag{22}$$

This condition (also satisfied for Bianchi VIII and IX with generic $\gamma$) restricts the form a symmetric vielbein $\gamma$ must have in order to be consistent with the self-duality evolution equation (15).

In all Bianchi classes, the matrix $n$ is idempotent:

$$n\eta n = n. \tag{23}$$

Furthermore, as a consequence of the identities relating the structure constants, one obtains

$$n\eta m + m\eta n = m\,\mathrm{tr}\,n\eta. \tag{24}$$

According to this formula, the non-unimodular Bianchi algebras fall into three classes:

- III, $\mathrm{VI}_{h>-1}$, $\mathrm{VII}_{h>0}$: $n\eta m = m\eta n = m$;
- IV: $n\eta m + m\eta n = m$;
- V: $n\eta m = m\eta n = 0$.

Hence, (23) and condition (22) imply

$$n\gamma(n+m) = (n-n\eta m)\gamma\eta \quad \Leftrightarrow \quad \eta\gamma(n+m\eta n) = (n-m)\gamma n, \tag{25}$$

with, as corollary,

$$n\gamma m + m\gamma n = (m-n\eta m)\gamma\eta + \eta\gamma(m-m\eta n). \tag{26}$$

So far we have analyzed the solutions $I$ of equation (13) and processed constraint (18), with the help of various algebraic properties of the structure constants. The resulting key equations are (21) and (25), which allow for expressing the evolution equation (15) as

$$\begin{aligned}\gamma' = &-\det\Gamma\left(\gamma n\gamma n\gamma - \frac{1}{2}(\gamma m\gamma n\gamma - \gamma n\gamma m\gamma) - \frac{\gamma}{2}\mathrm{tr}(\gamma n)^2\right) - I\\ &- \frac{1}{2}\det\Gamma\left(\gamma\eta\gamma(m-m\eta n)\gamma - \gamma(m-n\eta m)\gamma\eta\gamma - \frac{\gamma}{2}\mathrm{tr}(\gamma\eta\gamma(n\eta m - m\eta n))\right),\end{aligned} \tag{27}$$

where $I$ is now symmetric. This self-duality evolution equation is valid for all symmetric metrics $\gamma$ satisfying (22) and with $I$ being a symmetric solution of (20), or $I=\eta$ for rank-3 Bianchi VIII and IX. Thanks to identity (24) we observe that the second line of equation (27) vanishes identically for all Bianchi classes but IV and V. It turns out that it also vanishes for those classes, as a consequence of condition (17), which has not yet been taken into consideration (we will elaborate on this in the appendix). In fact, self-duality constraint (17) must be fulfilled for a solution of (27) under (22) to provide a self-dual gravitational instanton.

Before analyzing the actual restrictions (17) sets on the form of $\gamma$, we would like to pause and interpret the non-vanishing part of equation (27),

$$\gamma' = -\det\Gamma\left(\gamma n\gamma n\gamma - \frac{1}{2}(\gamma m\gamma n\gamma - \gamma n\gamma m\gamma) - \frac{\gamma}{2}\mathrm{tr}(\gamma n)^2\right) - I, \tag{28}$$

as a geometric flow, which is the main purpose of this paper. This was achieved in [1] for Bianchi A (non-unimodular) under the assumption of diagonal $\gamma$. The geometric flow was shown to be a Ricci flow combined with a Yang–Mills flow produced by a flat, non-flowing and diagonal $SO(3)$ Yang–Mills connection on $\mathcal{M}_3$. The origin of the Yang–Mills connection was the flat anti-self-dual part of the Lévi–Cività connection on $\mathcal{M}_4$, appearing as the first integral (12).

For general metrics $\gamma$ (see (8)) on $G_3$-left-invariant $\mathcal{M}_3$, the Ricci tensor reads[4]

$$R[\gamma] = N + \det\Gamma\left(\gamma n\gamma n\gamma - \gamma m\gamma n\gamma + \gamma n\gamma m\gamma - \frac{\gamma}{2}\mathrm{tr}(\gamma n)^2\right) + a\otimes a - 2\gamma\, a\Gamma a. \tag{29}$$

In the last term, $a\Gamma a$ stands for $a_i\Gamma^{ij}a_j$, while $N$ is the Cartan–Killing metric of the $G_3$ algebra,

$$N_{ij} = -\tfrac{1}{2}\epsilon_{\ell im}\epsilon_{kjn}n^{mk}n^{n\ell} - a_i a_j. \tag{30}$$

The appropriate Yang–Mills connection to consider here is $SO(2,1)$ for Bianchi VI or VIII, and $SO(3)$ otherwise, since it reflects, in each case, the four-dimensional anti-self-dual Lévi–Cività connection: $A \equiv A_i\sigma^i = -\lambda_{ij}T^j\sigma^i$ with $[T_i, T_i] = -\epsilon_{ijk}T^k$. As usual $T^i = \eta^{ij}T_j$ with $\eta = \mathbb{1}$ for $SO(3)$ and $\mathrm{diag}(1,-1,-1)$ for $SO(2,1)$. The absence of flow for the Yang–Mills connection states that $A' = 0$, while flatness requires

$$F = \mathrm{d}A + [A, A] \equiv 0 \quad \Leftrightarrow \quad \lambda_{i\ell}c^{\ell}{}_{jk} + \epsilon_{ijk}\eta^{jm}\eta^{kn}\lambda_{mj}\lambda_{nk} = 0. \tag{31}$$

It is straightforward to show[5] that $-1/2\,\mathrm{tr}(A_iA_j) = \eta^{k\ell}\lambda_{ki}\lambda_{\ell j}$. Combining the latter with (29), we conclude that equation (28) is recast as[6]

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -R[\gamma] + \mathrm{s}(\nabla a) - \gamma\, a\Gamma a - \frac{1}{2}\mathrm{tr}\,(A\otimes A)\,. \tag{32}$$

This describes a geometric flow driven by the Ricci tensor, combined with Yang–Mills as well as a diffeomorphism generated by $a$ and an invariant component of the scalar curvature. The matching requires the following relationship to hold between the flat anti-self-dual Lévi–Cività connection $I_{ij}$ and the flat Yang–Mills connection $\lambda_{ij}$:

$$N - I = \lambda^{\mathrm{T}}\eta\lambda, \tag{33}$$

where $I$ (symmetric) and $\lambda$ satisfy (13) and (31), respectively. This equation is indeed consistent.

- For Bianchi VIII and IX, $N = \eta$, $I = 0$ or $\eta$ and $\lambda = \eta$ or 0. Thus, (33) translates onto $I + \lambda = \eta$.
- For all other types, equations (13) and (31) are equivalent to $I(n+m) = (n-m)I = 0$ and $\lambda(n+m) = (n-m)\lambda^{\mathrm{T}} = 0$, respectively. Since $N(n+m) = (n-m)N = 0$ (as a consequence of Jacobi identity (5)), any rank-0 or rank-1 solution $I$ provides, through (33), a solution for $\lambda$ and vice versa.

It should finally be stressed that the consistency condition (21) was instrumental in reaching (25) from (18), and therefore in rewriting (15) as (28), and further as (32) (for Bianchi IV and V, constraint (17) was also necessary to ensure the equivalence of (27) and (28)—see the appendix).

The above demonstrates the advertized general correspondence between gravitational instantons and geometric flows for metrics $\gamma$ satisfying (22) (and (17) for Bianchi IV and V), and first integral $I$ satisfying (21). Two questions are in order at this stage. The first concerns the self-consistency of the geometric flow (32). Within the framework of self-dual gravitational

[4] The scalar curvature is $S[\gamma] = \mathrm{tr}(\Gamma R[\gamma]) = \mathrm{tr}(\Gamma N) - \frac{\det\Gamma}{2}\mathrm{tr}(\gamma n)^2 - 5a\Gamma a$.

[5] Note also the normalization of the generators: $\mathrm{tr}(T_iT_j) = -2\eta_{ij}$.

[6] Here $\mathrm{s}(\nabla a)$ stands for the symmetric part of $\nabla a$: $\mathrm{s}(\nabla a) = \frac{\det\Gamma}{2}(\gamma n\gamma m\gamma - \gamma m\gamma n\gamma) + a\otimes a - \gamma\, a\Gamma a$.

instantons, the metrics $\gamma$ are forced to fulfill (22) (and (17) for Bianchi IV and V). Is the flow evolution compatible with this constraint? The answer is positive and one easily checks that $\gamma'$ satisfies (22) if $\gamma$ does. The second question is whether any consistent geometric flow of this type is eligible as a gravitational instanton. The answer in this case is negative, because only $\gamma$s further restricted to (17), the constraint that has not yet been taken explicitly into account, can be promoted to four-dimensional self-dual solutions.

The self-duality constraint (17) does not affect Bianchi VIII or IX, for which $\gamma$ remains generic. It does however further restrict $\gamma$ in the other Bianchi groups, without altering the consistency of the flow equation (28) (this follows immediately from the original formulation of the self-duality constraint (16)). For the non-unimodular Bianchi class, (17) or equivalently (19) imposes that all rank-0 ($I = 0$) metrics must have $\det \gamma = 0$. This corresponds to singular gravitational instantons or degenerate geometric flows. The same holds for rank-1 ($I \neq 0$) except for Bianchi III, which admits non-singular self-dual gravitational instantons corresponding to regular geometric flows. These properties are collected in the appendix; more details on the analysis of (17) can be found in [8] for general vielbeins (as opposed to the symmetric ones used here)[7]. For the reader's convenience, the case of Bianchi III will be presented in the appendix, whereas we summarize the results for the other classes in tables 1 and 2. There, the forms of symmetric $\gamma$s complying with (22) are displayed, together with their restriction following the self-duality constraint (17). These expressions depend on the basis of invariant forms, which are also specified.

The above developments conclude on the advertized correspondence among gravitational instantons and geometric flows. It would be interesting to provide a satisfactory geometrical interpretation of the 'square root' of $g$ as the flowing metric on $\mathcal{M}_3$, and to understand how the first-order self-duality equations (15) could emerge directly from the action—as they do for $g$ in the non-relativistic set-up under detailed balance [5, 6]—following the split formalism of [9, 10]. The analysis of general geometric flows of type (32), satisfying (22) but not (17), i.e. beyond those which are interpreted as self-dual gravitational instantons, is also an open and challenging problem. Interesting issues such as the existence of entropy functionals or universality properties in the large-time behavior deserve further investigation.

Let us finally stress the specificity of four dimensions in respect to the relationship between geometric flows and gravitational instantons. Even though self-duality can be imposed in other dimensions (such as 7 or 8 thanks to the $G_2$ structure and to the octonions [11]), the holonomy group has not the factorization property of $SO(4)$ or $SO(3, 1)$, which was instrumental here. Although not excluded, it seems more difficult to find a class of instantons that could be interpreted as geometric flows of a lower dimensional geometry.

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[7] In comparing the present developments with those of [8], the reader should bear in mind that, besides the absence of symmetry restriction for the vielbein, the signature used in that reference was Euclidean, whereas here it is Euclidean for Bianchi I, II, IV, V, VII, IX and ultrahyperbolic for III, VI and VIII.

## Appendix. On non-unimodular groups

For metrics $\gamma$ obeying (18), the self-duality evolution equation (15) translates into (27). The second line of this equation vanishes identically in all Bianchi classes for which $n\eta m = m\eta n = m$. This excludes IV and V (see (24)). For the latter classes, it turns out that the terms at hand vanish provided $\gamma$ is subject to the self-duality constraint (17) or equivalently (19). This statement is based on the simple fact that for all groups but Bianchi III,

$$\gamma m\gamma = 0, \tag{A.1}$$

as a consequence of (19) in combination with (17). Proving that the second line of (27) vanishes is thus a matter of simple algebra with the help of expressions (22)–(26).

The proof of (A.1) goes as follows. For all non-unimodular groups, a first integral $I$ solving (20) is either vanishing or rank-1. In the first case, (19) demonstrates (A.1). In the second, the general solution for symmetric $I$ is

$$I_{ij} = \kappa a_i a_j, \tag{A.2}$$

($\kappa$ is an arbitrary constant) for Bianchi IV, V, VI and VII. Constraint (19), written in Poincaré-dual form and combined with (18), reads

$$\frac{2}{\kappa} M^{i\ell} a_\ell = n^{ij}\eta_{jk} M^{k\ell} a_\ell, \tag{A.3}$$

where $M^{i\ell}$ are the entries of $\mathrm{Adj}(\gamma)$. From equation (A.3), we observe that $M^{i\ell}a_\ell$ are the components of an eigenvector of $n\eta$ with eigenvalue $2/\kappa$. Multiplying iteratively (A.3) by $n\eta$ from the left and using (23), we conclude that the eigenvalue of the eigenvector at hand is an arbitrary natural power of $2/\kappa$. This is possible only if $M^{i\ell}a_\ell$ vanish. Since

$$\epsilon^{ijk}(\gamma m\gamma)_{ij} = 2M^{k\ell}a_\ell, \tag{A.4}$$

(A.1) is proven in full generality, without reference to any particular choice of basis.

Note that following (A.1), $a_j$ is an eigenvector of $M^{ij}$ with zero eigenvalue. Therefore, $\det\gamma = 0$, as already announced, for all self-dual gravitational instantons based on non-unimodular Bianchi groups, except for III.

For Bianchi III, the above do not hold because (A.2) does not provide the most general symmetric solution of (20). The generic first integral $I$ satisfying (20) and (21) is instead

$$I = \begin{pmatrix} \mu & \chi & \chi \\ \chi & -\nu & -\nu \\ \chi & -\nu & -\nu \end{pmatrix}, \quad \chi^2 + \mu\nu = 0. \tag{A.5}$$

Bianchi III is the only non-unimodular case admitting non-degenerate $\gamma$s, once all constraints (17) and (18) are taken into account. The consistency of symmetric $\gamma$, equation (25), sets

$$\gamma_{12} = \gamma_{13}, \qquad \gamma_{22} = \gamma_{33}, \tag{A.6}$$

whereas using (A.5), the evolution equation (28) matches the geometric-flow equation (32) provided $\lambda$ satisfies (31) and (33). The general solution of (31) is

$$\lambda = \begin{pmatrix} \rho_1\lambda & \rho_1\zeta & \rho_1\zeta \\ \rho_2\lambda & \rho_2\zeta & \rho_2\zeta \\ \rho_3\lambda & \rho_3\zeta & \rho_3\zeta \end{pmatrix}. \tag{A.7}$$

Requiring (33) with (A.5) leads to the following set of equations,

$$(\rho_1\lambda)^2 = 2 + \mu, \quad (\rho_1^2 - \rho_2^2 + \rho_3^2)\lambda\zeta = \chi, \quad (\rho_1^2 - \rho_2^2 + \rho_3^2)\zeta^2 = -\nu, \tag{A.8}$$

which always admit a solution, either for first integral $I$ of rank-0 ($\mu = \nu = \chi = 0$), or of rank-1 (either $\mu$, $\nu$ or $\chi$ non-zero).

Self-duality also demands (17) to be satisfied. The further constraints on $\gamma$ are

$$\gamma_{11} = \frac{\chi^2 + 2\nu}{2\nu^2} \left(\gamma_{23} + \gamma_{33}\right), \quad \gamma_{13} = -\frac{\chi}{2\nu} \left(\gamma_{23} + \gamma_{33}\right) \tag{A.9}$$

and the general solution of the evolution equation (28) reads

$$\begin{aligned}
\gamma_{11}(t) &= \left(1 - \frac{\mu}{2}\right) t + \gamma_{11}(0), \\
\gamma_{13}(t) &= -\frac{\chi}{2} t + \gamma_{13}(0), \\
\gamma_{33}(t) &= \gamma_{33}(0) \left(1 + \frac{\nu t}{\gamma_{33}(0) + \gamma_{23}(0)}\right), \\
\gamma_{23}(t) &= \gamma_{23}(0) \left(1 + \frac{\nu t}{\gamma_{33}(0) + \gamma_{23}(0)}\right),
\end{aligned} \tag{A.10}$$

where the initial conditions are related by (A.9), and $\mu$, $\nu$, $\chi$ constrained by (A.5).

Self-dual Bianchi-III gravitational instantons of the above type were analyzed in [8], for the general case of non-symmetric vielbein. Among others, they exhibit naked singularities (points where Kretschmann's invariant becomes infinite). From the viewpoint of the geometric flow, $\gamma$ is a metric on $\mathcal{M}_3$ evolving under (32). Its components are linearly expanding or shrinking, depending on the parameters $\mu$ and $\nu$, and on the initial values $\gamma_{23}(0)$ and $\gamma_{33}(0)$. In particular, the scalar curvature depends on time as

$$S = \frac{8\nu}{\gamma_{33}(0) + \gamma_{23}(0) + \nu t}, \tag{A.11}$$

and can describe the relaxation of a singular configuration toward flatness, or the appearance of a singularity from an ancient flat space. Any attempt to go deeper in this analysis requires a more general study of flows of type (32), which stands beyond our present motivation.

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# Ungauging black holes and hidden supercharges

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# Ungauging black holes and hidden supercharges

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Abstract: We embed the general solution for non-BPS extremal asymptotically flat static and under-rotating black holes in abelian gauged $D = 4$ $\mathcal{N} = 2$ supergravity, in the limit where the scalar potential vanishes but the gauging does not. Using this result, we show explicitly that some supersymmetries are preserved in the near horizon region of all the asymptotically flat solutions above, in the gauged theory. This reveals a deep relation between microscopic entropy counting of extremal black holes in Minkowski and BPS black holes in AdS. Finally, we discuss the relevance of this construction to the structure of asymptotically $AdS_4$ black holes, as well as the possibility of including hypermultiplets.

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## Contents





## 1 Introduction

The interplay between the macroscopic description of black holes in supergravity and their corresponding microscopic description within string theory has been a source of important insights into the structure of the theory. In this respect, the most detailed investigations have been carried out for asymptotically flat black holes preserving some amount of supersymmetry, which provides additional control over various aspects of these systems. In particular, the microscopic counting of black hole entropy [1, 2] as well as the construction of the corresponding black hole geometries [3–5], depend crucially on the presence of unbroken supercharges.

Beyond the supersymmetric sector, the non-BPS class of asymptotically flat black holes in supergravity has attracted attention, based on a deeper understanding of the first order systems underlying extremal static and under-rotating solutions (i.e. rotating black holes without an ergo-region) [6–20]. While these systems are considerably more complicated than the corresponding BPS ones, they are in principle exactly solvable, since they are described by first order differential equations. In addition, various similarities to the BPS branch have been observed at the formal level, for example through the existence of a fake superpotential [21–25].

More recently, the interest in four-dimensional black holes was extended to the more general case of asymptotically anti-de Sitter (AdS) spacetimes, described as solutions to gauged supergravity theories [26–32]. Although not fully exhaustive, the existing classification of black holes in AdS shows a rich variety of possibilities with both static and rotating BPS solutions, as well as new horizon topologies. While a microscopic account of their entropy is not available yet, they provide interesting new examples in the context of the AdS/CFT correspondence. In addition, understanding phase transitions of extremal and thermal black holes in this class could lead to insight into the phase structure of physically interesting field theories at strong coupling.

A priori, the above mentioned classes of black holes in Minkowski and AdS spaces are unrelated, as they usually arise as solutions to different supergravity theories and respectively in different string theory compactifications, when these exist. Consequently, the two systems are usually clearly distinguished and studied by different methods, while the problems of microscopic entropy counting for asymptotically flat and AdS black holes are viewed independently ([1, 2] vs. [33, 34]). However, our purpose in this paper is to show that such distinction is not always present. In particular, we show that one can embed asymptotically flat non-BPS black hole solutions in certain special $D = 4$ $\mathcal{N} = 2$ gauged supergravity theories. Moreover, we show that the attractor geometries, and therefore also the microscopic counting, for BPS black holes in $AdS_4$ [28–30, 35] and asymptotically flat extremal non- BPS black holes [36–41] fall within a common class of supersymmetric $AdS_2\times S^2$ spaces,[1] or their rotating generalizations.

Let us be slightly more precise and consider the bosonic Lagrangian of abelian gauged $\mathcal{N}=2$ supergravity in four dimensions with an arbitrary number $n_v$ of vector multiplets and no hypermultiplets (i.e. we consider constant gauging Fayet-Iliopoulos (FI) parameters). Such theories are described in [45, 46] and we give more details in the following sections. For presenting the main argument we only need to know that the bosonic part of the Lagrangian is modified with respect to the one for the ungauged theory, $\mathcal{L}_0^{bos}$, by the introduction of a scalar potential term for the vector multiplet complex scalars, $t^i$, $i = 1 \dots n_v$ as

$$\mathcal{L}_g^{bos} = \mathcal{L}_0^{bos} + V(t, \bar{t}) \,, \tag{1.1}$$

with

$$V(t, \bar{t}) = Z_i(G)\, \bar{Z}^i(G) - 3\, |Z(G)|^2 \,, \tag{1.2}$$

where $G = \{g^I, g_I\}$ is a symplectic vector of arbitrary constant FI parameters and $Z(G)$, $Z_i(G)$ denote its scalar dependent central charges. An interesting possibility arises in broad classes of vector multiplet moduli spaces when the FI parameters are chosen in a way as to make the scalar potential identically zero [47], without reducing the theory to

[1]This class of horizons was called magnetic $AdS_2\times S^2$ in [42] for the reason that, just like for asymptotically magnetic $AdS_4$ spacetimes, the fermions flip their spin and become SU(2) scalars [43, 44]. We elaborate on this more in the following sections.



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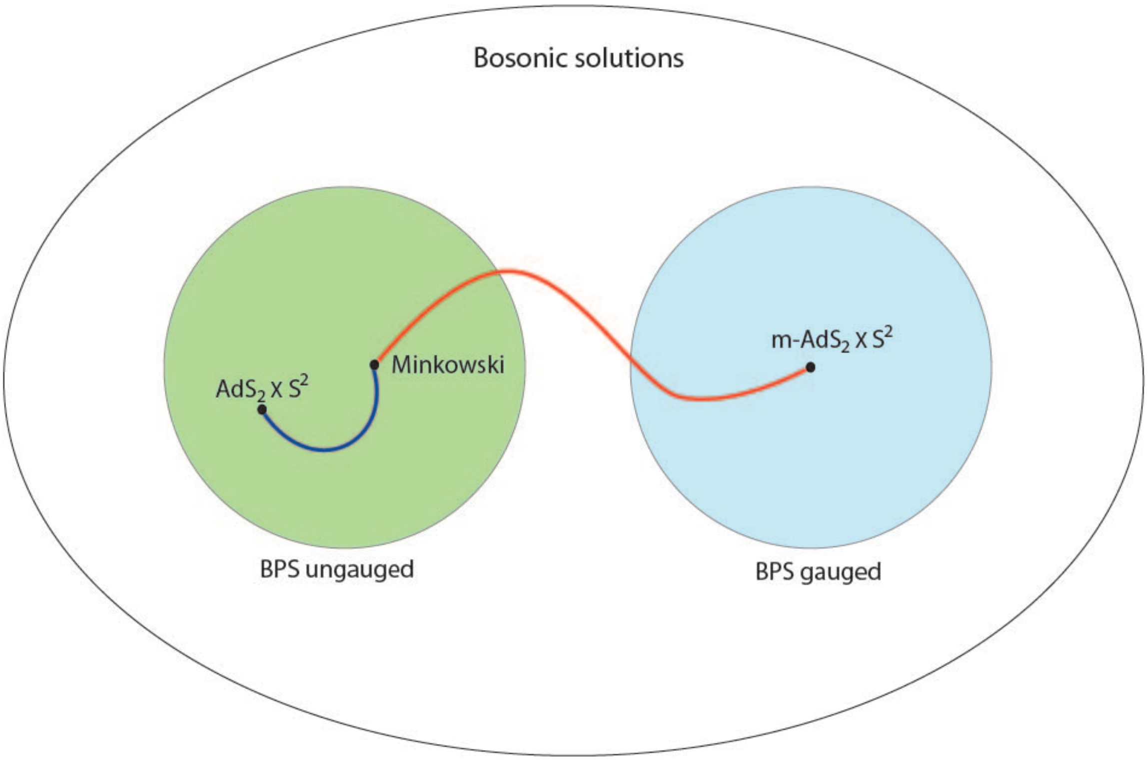


**Figure 1**. The two bubbles above represent the space of BPS solutions to ungauged and abelian gauged supergravity with a flat potential, as subspaces of all bosonic solutions, common to both theories. Note the presence of two distinct $\mathrm{AdS}_2{\times}\mathrm{S}^2$ backgrounds that are supersymmetric only within one theory. The blue line represents the BPS black hole solutions, interpolating between Minkowski space and the fully BPS $\mathrm{AdS}_2{\times}\mathrm{S}^2$. The solutions described in this paper, represented by a red line, interpolate between Minkowski and the so called magnetic $\mathrm{AdS}_2{\times}\mathrm{S}^2$ vacuum.

the ungauged one. This requires at least one FI parameter to be non zero, thus leading to a different supersymmetric completion of the same bosonic Lagrangian, since $\mathcal{L}_g^{bos} = \mathcal{L}_0^{bos}$ when the potential $V(t,\bar{t})$ vanishes, but the fermionic sector of the gauged theory still involves the vector $G$ linearly. It is then immediately obvious that all purely bosonic background solutions of the ungauged supergravity are also solutions of this "flat" gauged supergravity. However, due to their different fermionic sectors, the supersymmetric vacua of the two theories do not coincide. It is in fact easy to show that none of the BPS solutions of the ungauged theory are supersymmetric with respect to the gauged theory and vice versa (see section 3). This situation is summarised in figure 1.

Given the above, it is not surprising that some known non-BPS solutions in ungauged supergravity might be supersymmetric in these special gauged theories. Indeed, our analysis shows that all extremal under-rotating black holes[2] preserve some supersymmetry in their near horizon region. Restricting to the static solutions, we further show that these horizon solutions are part of a larger class of supersymmetric horizons based on FI terms,

[2]In what follows, we refer for simplicity to under-rotating solutions having in mind that this includes also the static case, when the rotation vanishes.



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that do not a priori satisfy the flat potential restriction and pertain to the static BPS black holes in $AdS_4$, [28–30, 35, 43]. It follows that one needs to address together the problems of microscopic entropy counting of asymptotically flat and AdS black holes in this case. We come back to this point in the concluding section of this paper, which we leave for more general discussion.

The following main sections of the paper address various aspects of the connection between solutions in gauged and ungauged theories sketched above, and are largely independent of each other. For the convenience of the reader, we give an overview of the main results presented in detail in each of these sections, as follows.

In section 2, we show that asymptotically flat extremal non-BPS black holes can be viewed as solutions to $\mathcal{N}=2$ abelian gauged supergravity, if the FI gaugings are assumed to be such that the potential is trivial. The prime example of gauged theories with an identically flat potential can be found within the interesting class of cubic prepotentials arising in the ungauged case from Calabi-Yau compactifications of string/M-theories, as first discussed in [47]. This condition is enforced by introducing a Lagrange multiplier, which allows us to write the action of the extended system as a sum of squares, similar to the 1/4-BPS squaring in [29], while demanding that the metric is asymptotically flat, as is appropriate for a theory without a potential. The result is a first order system that is otherwise identical to the corresponding one describing asymptotically $AdS_4$ BPS solutions, except for the presence of the Lagrange multiplier, which is determined independently by its own equation of motion. We finally show that the general non-BPS solutions, in the form cast in [20], are solutions to the system above, once a suitable regularity constraint is imposed. This includes the identification of the auxiliary very small vector appearing in that work as the vector of FI terms in the gauged theory.

In section 3, we consider the near horizon limit of our system, making use of the fact that the Lagrange multiplier above reduces to an irrelevant constant. It follows that the attractor equations for general asymptotically flat static black holes can be cast as a particular case of the attractor equations of gauged supergravity [29]. The latter are expected to belong to the family of attractors in [43] preserving four supersymmetries, which we show explicitly to be the case. We therefore obtain the result that all static non-BPS attractors in ungauged supergravity can be viewed as 1/2-BPS attractors once embedded in an abelian gauged supergravity with appropriately tuned FI terms. Finally, we generalise this result in section 3.2, where we show that the under-rotating attractors of all asymptotically flat black holes [48], again in the form described in [20], preserve two supercharges, i.e. they are 1/4-BPS. Note that this implies the presence of the same number of supercharges in the near horizon region of any particular center of a non-BPS multi-center solution.

In section 4 we consider the 1/4-BPS flow equations of [29] for gauged supergravity, without restricting the FI terms, and show that some of the structures found in asymptotically flat solutions are present in the more general case. Most importantly, the regularity constraint used to define the single center flow in [20] is shown to hold even for an unrestricted vector of FI terms. Since this constraint implies that only half of the charges can be present once a vector of gaugings is specified, we expect it to be of importance in understanding the moduli space of $\mathrm{AdS}_4$ solutions. In section 5 we briefly discuss the possibility of further embedding the asymptotically flat solutions above in theories with gauged hypermultiplets. In such a scenario, the additional potential induced by the hypermultiplets must also vanish, which we show to be possible in rather generic theories that result from string compactifications.

We conclude in section 6, where we comment on the implications of our results for microscopic models of black holes and on relations to recent developments in the construction of non-BPS supergravity solutions. Finally, in the appendices we present some details of our conventions, we extend the discussion of section 2 to the embedding of asymptotically flat under-rotating solutions in gauged supergravity, and we discuss an example solution in some detail for clarity.

## 2 Ungauging black holes

In this section, we present the essential argument of the ungauging procedure for black hole solutions and provide an explicit example by considering the static case for simplicity. The starting point is the bosonic action for abelian gauged supergravity [45, 46], which reads

$$S_{4\mathrm{D}} = \frac{1}{16\pi}\int_{M_4}\Big(R\star 1 - 2\,g_{i\bar{\jmath}}\,dt^i\wedge\star d\bar{t}^{\bar{\jmath}} - \frac{1}{2}F^I\wedge G_I + 2\,V_g\,\star 1\Big), \tag{2.1}$$

and describes neutral complex scalars $t^i$ (belonging to the $n_\mathrm{v}$ vector multiplets) and abelian gauge fields $F_{\mu\nu}{}^I$, $I=0$, $i=0,\dots n_\mathrm{v}$ (from both the gravity multiplet and the vector multiplets), all coupled to gravity.[3] The dual gauge fields $G_{\mu\nu I}$ are given in terms of the field strengths and the scalar dependent period matrix $\mathcal{N}_{IJ}$, by

$$G^-_{\mu\nu I} = \mathcal{N}_{IJ}F^{-\,J}_{\mu\nu}\,, \tag{2.2}$$

where the expression for the period matrix will not be needed explicitly. Finally, the scalar potential $V_g$ takes the form

$$V_g = Z_i(G)\,\bar{Z}^i(G) - 3\,|Z(G)|^2 = \langle G, \mathrm{J}\,G\rangle - 4\,|Z(G)|^2\,, \tag{2.3}$$

where we used the definition of the scalar dependent matrix J in (A.12), and the symplectic vector $G=\{g^I, g_I\}$ stands for the FI terms, which control the coupling of the vector fields. In the abelian class of gaugings we consider in this paper, these couplings occur only in the fermionic sector of the theory, through the minimal coupling of the gravitini to the gauge fields, as the kinetic term is proportional to

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu i}\gamma_\nu\,D_\rho\psi_\sigma{}^i &\equiv \epsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu i}\gamma_\nu\left(\partial_\rho + \frac{i}{2}\,\langle G, A_\rho\rangle\right)\psi_\sigma{}^i\,,\\ \langle G, A_\mu\rangle &= g_I A_\mu{}^I - g^I A_{\mu I}\,.\end{aligned} \tag{2.4}$$

This coupling is in general non-local, due to the presence of the dual gauge fields $A_{\mu I}$. However, as for any vector, $G$ can always be rotated to a frame such that it is purely electric,

[3] We refer to appendix A for some of our conventions in $\mathcal{N}=2$ supergravity.



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i.e. $g^I = 0$, leading to a local coupling of the gauge fields. More generally, one can consider couplings of magnetic vectors as well, using the embedding tensor formalism [43, 49], which requires the introduction of extra auxiliary fields.

For the theories discussed in this paper however, the bosonic action is only affected through the nontrivial potential (2.3), which can be straightforwardly written in an electric/magnetic covariant way, as above. Based on this observation, we take the pragmatic view[4] of using covariant versions of all quantities, keeping in mind that while the equations involving fermions strictly apply only to the electrically gauged theory, all results for the bosonic backgrounds must necessarily be covariant under electric/magnetic duality. We therefore employ covariant notation when dealing with the bosonic sector and covariantise the fermionic supersymmetry variations (see section 3), so that we do not have to choose a frame for the FI terms explicitly.

Given these definitions, we now discuss the connection of the gauged action above to the ungauged theory, at the bosonic level. As one would expect, ungauged supergravity is immediately recovered by putting $G = 0$ in the above Lagrangian. However, it turns out that this is not the most general choice if one is interested in the bosonic sector only, as the scalar potential is not positive definite, and one can find nonzero $G$ for which the potential is identically zero [47]. The appropriate FI terms are then described by a so-called very small vector, characterised by

$$3\,|Z(G)|^2 = Z_i(G)\,\bar{Z}^i(G)\,. \tag{2.5}$$

In the context of symmetric scalar geometries such vectors are viewed as points of the doubly critical orbit, $\mathcal{S}$, defined as the set of vectors such that (2.5) is satisfied for any value of the scalars [50, 51]. Explicitly, they can be always brought to the frame where there is only one component, e.g.

$$G_0 = g\,\{0\,,\delta_I^0\}\,, \tag{2.6}$$

but we will not impose any restriction other than (2.5). In what follows, we will be using the fact that this orbit exists for symmetric models, but the same arguments can be applied whenever (2.5) has a solution for any model. For example, (2.6) is an example solution for any cubic model, symmetric or not, and one may construct more general examples by acting with dualities.[5]

Given this special situation, it is natural to consider the possibility of finding asymptotically flat backgrounds in a gauged theory with a flat gauging as above. Indeed, a vector of parameters in a doubly critical orbit was recently encountered in [20, 52, 53], which considered the general under-rotating extremal black hole solutions in ungauged extended supergravity. As we now show, the presence of such a vector in asymptotically flat solutions can be seen to arise naturally by viewing the ungauged theory as a gauged theory with $G \in \mathcal{S}$, leading to an interpretation of the auxiliary parameters introduced in [20, 52, 53] as residual FI terms.

[4] A similar point of view was used in [29].

[5] In fact, our treatment, as well as those of [20, 29] whose results we connect, is duality covariant, so that the form of the prepotential is not fixed.

### 2.1 Squaring of the action

In order to study extremal solutions in abelian gauged supergravity with a flat potential, we consider the squaring of the action for such backgrounds [4], following closely the derivation of the known flow equations of [29] for asymptotically AdS$_4$ black holes that preserve 1/4 of the supersymmetries. The only additional ingredient we require is the introduction of a Lagrange multiplier that ensures the flatness of the potential.

As we are interested in static solutions, we consider a spherically symmetric metric ansatz of the type

$$ds^2 = -\mathrm{e}^{2U}dt^2 + \mathrm{e}^{-2U}\left(dr^2 + \mathrm{e}^{2\psi}d\theta^2 + \mathrm{e}^{2\psi}\sin^2\theta d\phi^2\right)\,, \tag{2.7}$$

as well as an analogous ansatz for the gauge field strengths

$$F^I_{\theta\varphi} = \frac{1}{2}\,p^I\,\sin\theta\,, \qquad G_{I\theta\varphi} = \frac{1}{2}\,q_I\,\sin\theta\,. \tag{2.8}$$

Here, $\mathrm{e}^U$, $\mathrm{e}^\psi$ are two scalar functions describing the scale factor of the metric and the three dimensional base space, and $\Gamma = \{p^I\,, q_I\}$ denotes the vector of electric and magnetic charges. Using these ansatze, the action (2.1) can be shown to be expressible in the form [29]

$$\begin{aligned} S_{1d} = \int dr\,\Big\{ &-\frac{1}{2}\mathrm{e}^{2(U-\psi)}\langle\mathcal{E},\mathrm{J}\,\mathcal{E}\rangle - \mathrm{e}^{2\psi}\left[(Q_r+\alpha') + 2\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\alpha}W)\right]^2 \\ &- \mathrm{e}^{2\psi}\left[\psi' - 2\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\alpha}W)\right]^2 - (1+\langle G,\Gamma\rangle) \\ &-2\frac{d}{dr}\left[\mathrm{e}^{2\psi-U}\,\mathrm{Im}(\mathrm{e}^{-i\alpha}W) + \mathrm{e}^U\,\mathrm{Re}(\mathrm{e}^{-i\alpha}Z)\right]\Big\}\,, \end{aligned} \tag{2.9}$$

where $\mathrm{e}^{-i\alpha}$ is an arbitrary phase, we defined

$$\mathcal{E} \equiv 2\mathrm{e}^{2\psi}\left(\mathrm{e}^{-U}\mathrm{Im}(\mathrm{e}^{-i\alpha}\mathcal{V})\right)' - \mathrm{e}^{2(\psi-U)}\,\mathrm{J}\,G + 4\mathrm{e}^{-2U}\,\mathrm{Re}(\mathrm{e}^{-i\alpha}W)\,\mathrm{Re}(\mathrm{e}^{-i\alpha}\mathcal{V}) + \Gamma\,, \tag{2.10}$$

and we introduced special notation for the central charges of $\Gamma$ and $G$ as

$$Z \equiv Z(\Gamma)\,, \qquad Z_i \equiv Z_i(\Gamma)\,, \qquad W \equiv Z(G)\,, \qquad W_i \equiv Z_i(G)\,, \tag{2.11}$$

for brevity. The equations of motion following from this effective action imply the equations of motion for the scalars as well as the $tt$-component of the complete Einstein equation, whereas the remaining Einstein equations are identically satisfied upon imposing the Hamiltonian constraint

$$\mathrm{e}^{2\psi}\psi'^2 - 1 - \mathrm{e}^{2\psi}U'^2 - \mathrm{e}^{2\psi}g_{i\bar{\jmath}}t^{i\,\prime}\bar{t}^{\bar{\imath}\,\prime} + \mathrm{e}^{2(U-\psi)}V_{\rm BH} + \mathrm{e}^{2(\psi-U)}V_g = 0\,. \tag{2.12}$$

Solutions of this system have been discussed in [28–30]. These works analysed in some detail the asymptotically AdS$_4$ solutions associated to generic values of the gaugings $G$, and we return to this case in section 4.

We now proceed to an analysis of the ungauged limit of the bosonic sector of the theory, by imposing that the vector of FI gaugings $G$ lies in the doubly critical orbit, $G \in \mathcal{S}$, so that the potential is identically flat for any value of the scalars. Given the homogeneity of the potential in terms of $G$, we introduce a Lagrange multiplier in the original action (2.1), through a rescaling of the type

$$G \to \mathrm{e}^{\varphi} G\,. \tag{2.13}$$

Here and henceforth $\varphi$ will be treated as an independent field, whose equation of motion is exactly (2.5), enforcing the flatness of the potential. One can then write the action as a sum of squares in a similar way as above, up to an extra term originating from the partial integration involved. The result reads

$$\begin{aligned} S_{1d} = \int dr \Big\{ & -\frac{1}{2}\mathrm{e}^{2(U-\psi)}\langle \mathcal{E}, \mathrm{J}\,\mathcal{E}\rangle - \mathrm{e}^{2\psi}\left[(Q_r+\alpha') + 2\mathrm{e}^{(\varphi-U)}\,\mathrm{Re}(\mathrm{e}^{-i\alpha}W)\right]^2 \\ & - \mathrm{e}^{2\psi}\left[\psi' - 2\mathrm{e}^{(\varphi-U)}\,\mathrm{Im}(\mathrm{e}^{-i\alpha}W)\right]^2 \\ & - (1+\mathrm{e}^{\varphi}\,\langle G,\Gamma\rangle) + 2\,\mathrm{e}^{2\psi}\,\mathrm{e}^{(\varphi-U)}\varphi'\,\mathrm{Im}(\mathrm{e}^{-i\alpha}W)\Big\}\,, \end{aligned} \tag{2.14}$$

where we discarded a total derivative. We note here that $\mathcal{E}$, originally defined in (2.10), now contains the multiplier $\mathrm{e}^{\varphi}$ due to the rescaling in (2.13) above.

Note that since the addition of the Lagrange multiplier $\mathrm{e}^{\varphi}$ in the original action leads to an ungauged theory, it is possible at this stage to proceed in solving the equations of motion by simply putting $\mathrm{e}^{\varphi}=0$, which is a consistent solution that eliminates all instances of the vector of gaugings. However, it is clear that it is not necessary to make this choice for this function a priori. Indeed, making instead a choice for the metric function $\mathrm{e}^{\psi}=r$, so that the base metric in (2.7) is flat three dimensional space, one can obtain a more general squaring of the ungauged bosonic action. In this case, the kinetic term for the function $\mathrm{e}^{\psi}$ is trivial and the action can be further rearranged into

$$\begin{aligned} S_{1d} = \int dr \Big\{ & -\frac{1}{2}\mathrm{e}^{2(U-\psi)}\langle \mathcal{E}, \mathrm{J}\,\mathcal{E}\rangle - \mathrm{e}^{2\psi}\left[(Q_r+\alpha') + 2\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\alpha}W)\right]^2 \\ & - \left[2\,r\,\mathrm{e}^{\varphi-U}\,\mathrm{Im}(\mathrm{e}^{-i\alpha}W) - \left(1+\frac{1}{2}\,r\,\varphi'\right)\right]^2 \\ & + r^4\mathrm{e}^{\varphi}\left((r^{-1}\mathrm{e}^{-\varphi/2})'\right)^2 - (2+\mathrm{e}^{\varphi}\,\langle G,\Gamma\rangle)\Big\}\,, \end{aligned} \tag{2.15}$$

which is manifestly a sum of squares for the physical fields, along with an extra kinetic term and a Liouville-type potential for the multiplier $\varphi$, that decouples from the rest of the action.

One can now solve the equations of motion for the physical fields by imposing that each of the squares vanishes, as

$$\mathcal{E} = 0\,, \tag{2.16}$$

$$Q_r + \alpha' = -2\mathrm{e}^{\varphi}\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\alpha}W)\,, \tag{2.17}$$

$$2\,r\,\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\alpha}W) = \mathrm{e}^{-\varphi}\left(1+\frac{1}{2}\,r\,\varphi'\right)\,. \tag{2.18}$$

These equations describe the flow of the scalars and the scale factor $\mathrm{e}^{U}$, as well as fix the function $\mathrm{e}^{\varphi}$ in terms of physical fields. In addition, one still has to impose the Hamiltonian



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constraint (2.12) above, as well as the equation of motion for the Lagrange multiplier $\varphi$, which reads

$$\frac{d}{dr}\left(r^2\, u'\right) - \langle G, \Gamma\rangle\, r^{-2}\, \mathrm{e}^{-2u} = 0\,, \tag{2.19}$$

where we used the variable $\mathrm{e}^u \equiv r^{-1}\mathrm{e}^{-\varphi/2}$ for convenience.

The flow equations above are closely related to the ones obtained in [29] for gauged supergravity, with the difference that the function $\mathrm{e}^{\psi}$ describing the spatial part of the metric is now fixed to $\mathrm{e}^{\psi} = r$ and that we have included the additional function $\mathrm{e}^{\varphi}$. One can decompose the scalar flow equations (2.16) in components to find

$$U' = -\, r^{-2}\, \mathrm{e}^{U}\, \mathrm{Re}(\mathrm{e}^{-i\alpha} Z) + \mathrm{e}^{\varphi}\mathrm{e}^{-U}\, \mathrm{Im}(\mathrm{e}^{-i\alpha} W)\,, \tag{2.20}$$

$$t^{i\prime} = -\, \mathrm{e}^{i\alpha} g^{i\bar{\jmath}} \left(\mathrm{e}^{U-2\psi} \bar{Z}_{\bar{\jmath}} + i\, \mathrm{e}^{\varphi}\mathrm{e}^{-U} \bar{W}_{\bar{\jmath}}\right)\,, \tag{2.21}$$

along with one more equation for the Kähler connection

$$Q_r + \alpha' = -r^{-2}\, \mathrm{e}^{U}\, \mathrm{Im}(\mathrm{e}^{-i\alpha} Z) - \mathrm{e}^{\varphi}\mathrm{e}^{-U}\, \mathrm{Re}(\mathrm{e}^{-i\alpha} W)\,. \tag{2.22}$$

Combining the last relation with (2.17) leads to the constraint

$$r^{-2}\, \mathrm{e}^{U}\, \mathrm{Im}(\mathrm{e}^{-i\alpha} Z) = \mathrm{e}^{\varphi}\mathrm{e}^{-U}\, \mathrm{Re}(\mathrm{e}^{-i\alpha} W)\,, \tag{2.23}$$

which in the case of genuinely gauged supergravity in [29], can be shown to be equivalent to the Hamiltonian constraint (2.12). However, for the theory at hand, (2.12) is not automatically satisfied upon using (2.20)–(2.23), but takes the form

$$\langle G, \Gamma\rangle + 4r^2\, \mathrm{e}^{-2U} \mathrm{e}^{\varphi}\, \left(\mathrm{Im}(\mathrm{e}^{-i\alpha} W)\right)^2 = 0\,, \tag{2.24}$$

which relates $\mathrm{e}^{\varphi}$ to the physical fields.

### 2.2 Asymptotically flat solutions

We can now look for solutions to the above system, starting with the observation that (2.19) can be solved explicitly. The general solution can be written in terms of exponentials of the type $\mathrm{e}^{\pm 1/r}$, which are badly singular at $r = 0$ and lead to unphysical results. However, this differential equation also has the particular enveloping solution

$$\mathrm{e}^u \equiv r^{-1}\mathrm{e}^{-\varphi/2} = \langle \Gamma, G\rangle^{-1/2} \left(\mathrm{v} + \frac{\langle \Gamma, G\rangle}{r}\right) = \langle \Gamma, G\rangle^{-1/2}\, V\,, \tag{2.25}$$

where v is a constant and we assumed that $\langle \Gamma, G\rangle > 0$, so that the distinguished harmonic function $V$ defined above is positive definite. From (2.24), we obtain

$$2\mathrm{e}^{-U}\, \mathrm{Im}(\mathrm{e}^{-i\alpha} W) = V\,, \tag{2.26}$$

where the positive root was chosen by imposing (2.18). The last relation implies that the solution can be expressed in terms of harmonic functions, as shown in [20].

Indeed, the flow equations (2.20) and (2.21) with the particular solution for $\varphi$ given by (2.25), can be straightforwardly shown to be identical to the static limit of the flow equations derived in [20] for the single centre class of asymptotically flat black holes, upon identifying the gaugings $G$ with the auxiliary vector $\hat{R}^*$ used to express the solution.[6] Note that in [20] the auxiliary vector $\hat{R}^*$ was required to be very small by consistency of the Einstein equations for asymptotically flat black holes. Moreover, in that work it was found that regularity requires an additional constraint on the system, which can be expressed in several equivalent ways. In terms of the scalars, this constraint takes the form of the reality condition

$$e^{-i\alpha}dt^i - i\,\mathrm{e}^{\mathcal{K}}c^{ijk}\,\hat{W}_j\,g_{k\bar{k}}d\bar{t}^{\bar{k}} + \hat{W}\,\overline{\hat{W}}^i\,\overline{\hat{W}}_{\bar{\imath}}d\bar{t}^{\bar{\imath}} = \overline{\hat{W}}^i\left(1 + e^{-i\alpha}\hat{W}\right)dU\,, \tag{2.27}$$

where we defined the following shorthand expressions for convenience

$$\hat{W} = |W|^{-1}\,W\,, \qquad \hat{W}_a = |W|^{-1}\,e_a^i W_i\,. \tag{2.28}$$

The reality condition (2.27) can be used to show the existence of a second constant very small vector throughout the flow, given by

$$R = -4\,\frac{\mathrm{e}^{-2U}}{|Y|^2\,V^2}\mathrm{Re}\Big[Y^3\,\bar{W}\,\mathcal{V} + |Y|^2\,Y\,\bar{W}^i\,D_i\mathcal{V}\Big]\,, \tag{2.29}$$

$$Y \equiv (1 + i\,\mathrm{m}\,e^{2U})\,. \tag{2.30}$$

Here, m is an arbitrary constant that is promoted to a dipole harmonic function in the rotating case (see appendix B). This vector can be shown to be mutually local with $\Gamma$, $\langle R, \Gamma\rangle = 0$, using the flow equations above, but is nonlocal with $G$, as $\langle G, R\rangle = -4$, and in simple cases it can be viewed as the magnetic dual of $G$. Alternatively, one can derive the constraint (2.27) by demanding that the vector $R$ be constant.

Given the definitions above, the solution to the system (2.20)–(2.22) is given by

$$2\,\mathrm{Im}(e^{-U}e^{-i\alpha}\mathcal{V}) = \mathcal{H} - 2\,\frac{\langle G, \mathcal{H}\rangle}{\langle G, R\rangle}\,R + \frac{\mathrm{m}}{\langle G, \mathcal{H}\rangle}\,G\,, \tag{2.31}$$

where $\mathcal{H}$ are harmonic functions carrying the charges as

$$\mathcal{H} = \mathrm{h} + \frac{\Gamma}{r}\,, \tag{2.32}$$

and the distinguished harmonic function $\langle G, \mathcal{H}\rangle$ is fixed by (2.26) as

$$\langle G, \mathcal{H}\rangle = -V\,. \tag{2.33}$$

The reality constraint (2.27) can now be recast in terms of the harmonic functions describing the solution. Using the flow equations (2.20) and (2.21) to express the derivatives of the scalars in terms of the gauge fields, one can show that an equivalent form of the same reality condition can be obtained, as

$$\frac{1}{2}\,I_4'^{\,M}(\mathcal{H}, G) = \langle G, \mathcal{H}\rangle\,\mathcal{H}^M - 2\frac{\langle G, \mathcal{H}\rangle^2}{\langle G, R\rangle}R^M\,. \tag{2.34}$$

[6] The interested reader can find an outline of this identification in appendix B, where the full rotating single center class is considered.

Here, we use the index $M, N, \dots$ to denote both electric and magnetic components and $I_4'^{M}$ is the derivative of the quartic invariant, defined in terms of a completely symmetric tensor $t^{MNPQ}$ as

$$\begin{aligned} I_4(\mathcal{H}) \equiv& \frac{1}{4!} t^{MNPQ} \mathcal{H}_M \mathcal{H}_N \mathcal{H}_P \mathcal{H}_Q \\ I_4'^{M}(\mathcal{H}, G) \equiv& \frac{\partial^2 I_4(\mathcal{H})}{\partial \mathcal{H}_M \partial \mathcal{H}_N} G_N = \frac{1}{2} t^{MNPQ} \mathcal{H}_N \mathcal{H}_P G_Q \,. \end{aligned} \tag{2.35}$$

In [20] it was shown that if $G$, and thus $R$, are very small, this constraint implies that the harmonic functions $\mathcal{H}$ lie in a Lagrangian submanifold that includes $R$. Near the horizon, one finds the same constraint for the charges, so that a particular choice of $G$ restricts the physical charges to lie in the same Lagrangian submanifold. We refer the interested reader to that work for details on the derivation of these results.

This concludes our analysis of the embedding of extremal asymptotically flat black holes in gauged supergravity for the static case. We refer to appendix B for a similar analysis in the rotating case. It turns out that the inclusion of a Lagrange multiplier in exactly the same way leads to the same equation of motion (2.19) and the same solution (2.25) as above. The result is an extension of the static embedding of this section to the most general asymptotically flat extremal under-rotating black holes, as obtained in [20]. The solution turns out to take exactly the same form as in (2.31) with the constant m replaced by a dipole harmonic function describing the rotation.

## 3 BPS attractors in abelian gauged theories

As already announced in the introduction, the embedding of asymptotically flat black holes in the flat gauged theories we consider in this paper allows to show that their near-horizon geometries are in fact supersymmetric. In the previous section we saw a close similarity between the static flow equations for asymptotically flat and 1/4-BPS black holes in AdS. Below we further establish that static horizons in both Minkowski and AdS spaces in fact belong to a common 1/2 BPS class of solutions[7] already discussed in [43]. Beyond the static class, we further analyze the near-horizon geometry of extremal under-rotating black holes [48], whose flow equations are discussed in appendix B. These turn out to preserve 1/4 of the supercharges, which completes the statement that all asymptotically flat static and under-rotating extremal black holes have BPS horizons.

In order to study supersymmetric solutions, we only need to explicitly ensure that the supersymmetry variations of the fermions vanish. All supersymmetry variations for the bosons are automatically zero by the assumption of vanishing fermions. The fermionic fields that belong to the supermultiplets appearing in the action (2.1) are the gravitini $\psi_{\mu A}$ for the gravity multiplet and the gaugini $\lambda^{iA}$ for the vector multiplets. The corresponding

[7] This is in accordance with our results in section 2. From this point of view, the crucial factor that allows for a unified discussion is that the expression for the Lagrange multiplier reduces to a constant in the near horizon region (cf. the solution in (2.25)), thus diminishing any difference between the flat and AdS case. This is the case even for under-rotating black holes, as shown in appendix B.

supersymmetry variations are:[8]

$$\delta\psi_{\mu A} = D_\mu \varepsilon_A - 2i\, X^I\, I_{IJ}\, F^{J-}_{\mu\nu}\, \gamma^\nu\, \epsilon_{AB}\varepsilon^B - \frac{1}{2}\, W\, \sigma^3_{AB}\gamma_\mu\, \varepsilon^B\,,$$
$$\delta\lambda^{iA} = -i\partial\!\!\!/ t^i\, \varepsilon^A - \bar{D}^i \bar{X}^I\, I_{IJ}\, F^{J-}_{\mu\nu}\, \gamma^{\mu\nu}\, \epsilon^{AB}\varepsilon_B + i\, \bar{W}^i\, \sigma^{3,AB}\varepsilon_B\,, \tag{3.1}$$

where the covariant derivative $D_\mu$ reads

$$D_\mu \varepsilon_A = \left(\partial_\mu - \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} + \frac{i}{2}Q_\mu\right)\varepsilon_A + \frac{i}{2}\,\langle G, A_\mu\rangle\, \sigma^3_A{}^B \varepsilon_B\,, \tag{3.2}$$

and $W$ and $W_i$ are the central charges defined in (2.11). The symplectic product $\langle G, A_\mu\rangle$ in the standard electrically gauged supergravity just involves the electric gauge fields $A^\Lambda_\mu$ [54], but has a straightforward generalization, as shown in [43, 49]. We also used the shorthand $I_{IJ} = \mathrm{Im}\mathcal{N}_{IJ}$ for the period matrix. The presence of this matrix seems to spoil duality covariance on first sight, but it is possible to rewrite the relevant terms in a form convenient for our purposes, as in [55]

$$\delta\psi_{\mu A} = D_\mu \varepsilon_A + Z(\mathcal{F})^-_{\mu\nu}\, \gamma^\nu\, \epsilon_{AB}\varepsilon^B - \frac{1}{2}\, W\, \sigma^3_{AB}\gamma_\mu\, \varepsilon^B\,,$$
$$\delta\lambda^{iA} = -i\partial\!\!\!/ t^i\, \varepsilon^A + \frac{i}{2}\, \bar{Z}(\mathcal{F})^{i\,-}_{\mu\nu}\, \gamma^{\mu\nu}\, \epsilon^{AB}\varepsilon_B + i\, \bar{W}^i\, \sigma^{3,AB}\varepsilon_B\,. \tag{3.3}$$

Here, the central charges of the electric and magnetic field strengths are computed component-wise as in (A.6), as

$$Z(\mathcal{F})_{\mu\nu} = \langle \mathcal{F}_{\mu\nu}, \mathcal{V}\rangle\,, \qquad Z(\mathcal{F})_{i,\mu\nu} = \langle \mathcal{F}_{\mu\nu}, D_i\mathcal{V}\rangle\,, \tag{3.4}$$

which are already anti-selfdual and selfdual respectively due to (A.11) and (A.10).

Note that the FI parameters $G$ above are assumed to be generic, and include the particular choice of FI parameters such that the scalar potential is identically zero. In that limit, we obtain a theory with a bosonic Lagrangian identical to ungauged supergravity, but with a different fermionic sector that involves a nonzero very small vector of FI terms explicitly. Therefore, the supersymmetry variations above are strictly valid only for gauged supergravity, even though the associated bosonic backgrounds we describe below are solutions to both gauged and ungauged supergravity. The supersymmetric solutions of the two theories however do not overlap and form two disjoint sets. This is easy to see because the ungauged supersymmetry variations are again given by (3.3) after setting $G = 0$, leading to the vanishing of $W, W^i$. Suppose now that we have a supersymmetric background solution of the ungauged theory and let us focus for simplicity on the gaugino variation. If we also want it to be a solution of the flat gauged theory, we require that it automatically satisfies $W^i = 0$, since otherwise one cannot make the variation vanish both in the gauged and in the ungauged theory. Now, using the vanishing of the scalar potential (2.3) we find

$$|W|^2 = \frac{1}{3}\, W_i \bar{W}^i\,,$$

[8]Here we choose to orient the FI terms along direction 3 of the quaternionic moment maps, as done in [30].

which means that we also need $W = 0$ for the hypothetical BPS solution in both theories. However, it is a special geometry property that

$$\begin{pmatrix} \langle \mathcal{V}, G\rangle \\ \langle D_i\mathcal{V}, G\rangle \end{pmatrix} = 0\,, \quad \Rightarrow \quad G = 0\,, \tag{3.5}$$

since one can invert the matrix multiplying $G$ in this equation. This leads us back to the ungauged case, and we find a contradiction. Therefore every BPS solution of the ungauged theory (e.g. the asymptotic Minkowski spacetime connected to the asymptotically flat black holes) is not supersymmetric in the flat gauged theory, and vice versa (e.g. the black hole attractor geometries are BPS in the gauged theory, as shown below, but break supersymmetry in the ungauged theory) as schematically illustrated by figure 1.

We now move on to the explicit analysis of the supersymmetries preserved by the various horizon geometries. In doing so, we will be using a timelike Killing spinor ansatz, ensuring that once the BPS equations hold we already have supersymmetric solutions, i.e. the BPS equations together with the Maxwell equations and Bianchi identities imply the validity of the Einstein and scalar equations of motion (see [56, 57]). This is important for the discussion of backgrounds with non-constant scalars, which are the ones relevant for rotating attractors.

In section 3.1 we verify that the attractor equations obtained as a limit of the full 1/4-BPS static solutions in $\mathrm{AdS}_4$ in [29], do exhibit supersymmetry enhancement to 4 real supercharges. We then identify the attractor equations of static asymptotically flat non-BPS black holes of [20] as a subset of the BPS attractors in gauged supergravity, in the limit of flat gauging where the FI terms are restricted to be a very small vector. Similarly, in section 3.2 we show that the general under-rotating attractor solutions of [20] preserve 1/4 of the supersymmetries.

### 3.1 Static attractors

We first concentrate on the near horizon solutions of static black holes, therefore we consider metrics of the direct product form $\mathrm{AdS}_2\times\mathrm{S}^2$ with radii $v_1$ and $v_2$ of $\mathrm{AdS}_2$ and $\mathrm{S}^2$, respectively:

$$\mathrm{d}s^2 = -\frac{r^2}{v_1^2}\,\mathrm{d}t^2 + \frac{v_1^2}{r^2}\,\mathrm{d}r^2 + v_2^2\,(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,. \tag{3.6}$$

The corresponding vielbein reads

$$e^a_\mu = \mathrm{diag}\Big(\frac{r}{v_1}, \frac{v_1}{r}, v_2, v_2\sin\theta\Big)\,, \tag{3.7}$$

whereas the non-vanishing components of the spin connection turn out to be

$$\omega_t^{01} = -\frac{r}{v_1^2}, \qquad \omega_\phi^{23} = \cos\theta\,. \tag{3.8}$$

We further assume that the gauge field strengths are given in terms of the charges $\Gamma = (p^I\,, q_I)^T$ by

$$\mathcal{F}_{\mu\nu} \equiv (F^I_{\mu\nu}; G_{I\mu\nu})\,, \qquad F^I_{\theta\varphi} = \frac{1}{2}\,p^I\,\sin\theta\,, \qquad G_{I\theta\phi} = \frac{1}{2}\,q_I\,\sin\theta\,, \tag{3.9}$$

which are needed in the BPS equations below. The scalars are assumed to be constant everywhere, $\partial_\mu z = 0$, as always on the horizon of static black holes. This ansatz for gauge fields and scalars automatically solves the Maxwell equations and Bianchi identities in full analogy to the case of ungauged supergravity.

Anticipating that the near horizon geometries of the solutions described in the previous section preserve half of the supersymmetries, we need to impose a projection on the Killing spinor. This is in accordance with the fact that these solutions cannot be fully supersymmetric once we require that not all FI terms vanish (see [58, 59] for all fully BPS solutions in $\mathcal{N}=2$ theories in $4d$). Taking into account spherical symmetry, there are only two possibilities in an $\mathrm{AdS}_2\times \mathrm{S}^2$ attractor geometry, as shown in [43]. Namely, one either has full supersymmetry, and therefore no projection is involved, or 1/2-BPS geometries satisfying the projection

$$\varepsilon_A = i\sigma^{3\,B}_A\,\gamma^{23}\,\varepsilon_B = \sigma^{3\,B}_A\,\gamma^{01}\,\varepsilon_B\,, \tag{3.10}$$

with the last equality due to the fact that spinors are chiral in the chosen conventions (these are exhaustively listed in [42, 54]). Note that a Killing spinor satisfying this projection is rather different from the standard timelike Killing spinor projection that appears in asymptotically flat 1/2-BPS solutions (shown in (3.28) below, see e.g. [3]), but is exactly the same as one of the projections appearing in asymptotically $\mathrm{AdS}_4$ 1/4-BPS solutions (see [29, 30]).

**Analysis of the BPS conditions.** Now we have all the data needed to explicitly write down the supersymmetry variations of the gravitini and gaugini. To a certain extent this analysis was carried out in section 8 of [43] and will not be exhaustively repeated here. One can essentially think of the Killing spinors as separating in two - a part on $\mathrm{AdS}_2$ and another part on $\mathrm{S}^2$. It turns out that the $\mathrm{AdS}_2$ part transforms in the standard way under the $\mathrm{SO}(2,1)$ isometries of the AdS space, while the spherical part remains a scalar under rotations. The $t$ and $r$ components of the gravitino variation are therefore non-trivial due to the dependence of the spinor on these coordinates. We are however not directly interested in the explicit dependence, but only consider the integrability condition for a solution to exist, given by $D_{[t}D_{r]}\varepsilon_A = 0$ for all $A = 1, 2$. Plugging the metric and gauge field ansatz, this results in the equations

$$\frac{1}{2v_1^2} = |W|^2 + \frac{1}{v_2^4}\,|Z|^2\,, \qquad \langle G, \mathcal{F}_{tr}\rangle = 0\,. \tag{3.11}$$

The solution of this equation therefore ensures the vanishing of the gravitino variation on $\mathrm{AdS}_2$. Turning to the spherical part, with the choice of Killing spinor ansatz it is easy to derive two independent equations that already follow trivially from the analysis of [30],

$$i\,\frac{1}{v_2^2}\,Z = -W\,, \tag{3.12}$$

and

$$\langle G, \Gamma\rangle = -1\,, \tag{3.13}$$

which is the usual Dirac quantization condition[9] that seems to accompany the solutions of "magnetic" type.[10] Note that (3.12) can be used to simplify the first of (3.11), so that we can cast the above conditions in a more suggestive form for our purposes, as

$$v_1^{-2} = 4\,|W|^2\,, \qquad v_2^2 = -i\,\frac{Z}{W}\,. \tag{3.14}$$

Moving on to the gaugino variation, the condition that the scalars remain constant leaves us with only one (for each scalar) additional condition on the background solution,

$$-\,i\,Z_i = v_2^2\,W_i\,. \tag{3.15}$$

This concludes the general part of our analysis - it turns out that in FI gauged supergravity one can ensure that $\mathrm{AdS}_2\times\mathrm{S}^2$ with radii $v_1$ and $v_2$ preserves half of the supersymmetries by satisfying equations (3.13)–(3.15) within the metric and gauge field ansatz chosen above. These equations are in agreement with the analysis of [29, 35]. Moreover, the attractors above are a realisation of the 1/2-BPS class of $\mathrm{AdS}_2\times\mathrm{S}^2$ vacua of [43], which are described by the superalgebra $\mathrm{SU}(1,1|1)\times\mathrm{SO}(3)$, as opposed to the fully BPS $\mathrm{AdS}_2\times\mathrm{S}^2$ vacua that are described by $\mathrm{SU}(1,1|2)$.

The above equations can be written in terms of symplectic vectors (e.g. as in [29]), so that they can be directly compared to [20]. To this end, one can straightforwardly see that the condition

$$-\,4\,\mathrm{Im}(\bar{Z}\mathcal{V}) = \Gamma + v_2^2\,JG\,, \tag{3.16}$$

is equivalent to (3.12) and (3.15), while the first of (3.14) has to be used to fix the $\mathrm{AdS}_2$ radius. Alternatively, one may write the attractor equations by solving for $\mathcal{V}$ in terms of the charges and gaugings. Since all the above equations are invariant under Kähler transformations, we need to introduce an a priori arbitrary local phase $\mathrm{e}^{i\alpha}$, which is defined to have unit Kähler weight. One can then combine (3.11) and (3.15) to obtain

$$2\,\frac{v_2^2}{v_1}\,\mathrm{Im}(\mathrm{e}^{-i\alpha}\mathcal{V}) = \Gamma + v_2^2\,JG\,, \tag{3.17}$$

while (3.12) has to be viewed as an additional constraint. Taking the inner product of (3.17) with $\Gamma + v_2^2\,JG$ identifies the phase $\mathrm{e}^{i\alpha}$ as the phase of the combination in (3.12), which drops out from that relation.

In order to show that these BPS conditions above do indeed admit solutions describing asymptotically flat black holes, one can consider the inner product of (3.16) with the gaugings, using (3.13), to show that the sphere radius is given by

$$v_2^{-2} = 2\,g^{i\bar{\jmath}}\,W_i\,\overline{W}_{\bar{\jmath}} - 2\,|W|^2\,. \tag{3.18}$$

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[9] From the point of view of the flow equations derived in section 2, $\langle\Gamma, G\rangle$ can be an arbitrary non-vanishing constant. This is exactly the value of the Lagrange multiplier in (2.25) at the horizon, thus rescaling the gauging vector as $G' = \langle\Gamma, G\rangle^{-1}G$ in the solution (2.31) in that limit. It follows that $\langle G', \Gamma\rangle = -1$, which is the choice made in [28–30] for the full solution and we adopt it here, dropping the primes on the gaugings, to make our notation in sections 2 and 3 consistent without any loss of generality.

[10] The solution at hand, called magnetic $\mathrm{AdS}_2\times\mathrm{S}^2$ in [42], is the near horizon geometry of asymptotically magnetic $\mathrm{AdS}_4$ black holes [44, 60].

Upon imposing triviality of the potential as in (2.5), the above expression and the first of (3.14) imply that $v_2 = v_1$, which is necessary for asymptotically flat black holes. Indeed, using the definition (2.29) in this special case, the generic BPS attractor equation in the form (3.17) can be written as

$$2\, v_1\, \mathrm{Im}(\mathrm{e}^{-i\alpha}\mathcal{V}) = \Gamma + \frac{1}{2}\, R\,. \tag{3.19}$$

These are exactly the general attractor equations for asymptotically flat black holes found in [20] for the ungauged case.[11] We conclude that the near horizon region of static asymptotically flat extremal black holes can be viewed as a special case of the general attractor geometry for BPS black holes in abelian gauged supergravity, upon restricting the FI parameters to be a very small vector, thus leading to a flat potential.

In addition, when all FI parameters are set to zero, one immediately obtains the BPS attractor equations of ungauged supergravity, preserving full $\mathcal{N} = 2$ supersymmetry [61–63]. This provides us with a unifying picture, since the BPS attractor equations (3.16) appear to be universal for static extremal black holes in $\mathcal{N} = 2$ theories, independent of the asymptotic behavior (Minkowski or AdS) or the amount of supersymmetry preserved.

One intriguing aspect of this result is that, while in the ungauged theory ($G = 0$), the attractor equation leads to a well defined metric only when the quartic invariant of the charges, $I_4(\Gamma)$, is positive, the presence of a nontrivial $G$ does not seem to allow for a charge vector $\Gamma$ with a positive quartic invariant, i.e. in all known examples $I_4 < 0$ iff $G \neq 0$, both for asymptotically flat and AdS black holes. Similarly, the explicit $AdS_4$ solutions of [28–30], also have a negative quartic invariant of the charges, contrary to the intuition one might have from the asymptotically flat case. It is natural to expect that the quartic invariant of charges allowed for asymptotically $AdS_4$ BPS solutions is negative even though this is not the only quantity that controls the horizon in that case.

In view of the above, it is interesting at this point to make some comments on the potential microscopic counting of degrees of freedom, which can be now safely discussed due to the presence of supercharges on the horizon. From a microscopic string theory perspective we know that the FI parameters are usually some particular constants corresponding to topological invariants of the compactification manifolds, see [64] for a clear overview and further references. This means that one is not free to tune the value of the vector $G$. We further know that one of the electromagnetic charges is uniquely fixed by the choice of $G$, meaning that we are not free to take the large charge limit in this particular case. We then find that the black hole entropy, S, which is proportional to the area of the horizon, scales as $\mathrm{S} \sim \Gamma^{3/2}$, a behavior that is in between the usual $\mathrm{S} \sim \Gamma^2$ of 1/2 BPS asymptotically flat black holes[12] and the $\mathrm{S} \sim \Gamma$ case of 1/4 BPS asymptotically magnetic AdS black holes [28, 30]. This is of course not a puzzle on the supergravity side, where we know that some charges are restricted, but it provides a nontrivial check on any potential microscopic descriptions of black hole states in string theory.

[11] We remind the reader that the inner product $\langle \Gamma, G \rangle$ has to be rescaled to unity in the original reference for a proper comparison with this section.

[12] Note however, that the entropy of asymptotically flat 1/2-BPS black holes in five dimensions scales exactly as $\Gamma^{3/2}$, see e.g. [1].

### 3.2 Under-rotating attractors

We now turn to the more general case of extremal under-rotating attractors corresponding to asymptotically flat solutions [48]. These are described by a more general fibration of $S^2$ over $AdS_2$ that incorporates rotation as

$$ds^2 = -e^{2U} r^2 (dt+\omega)^2 + e^{-2U}\left(\frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta d\phi^2\right),$$
$$e^{-4U} = -I_4(\Gamma) - j^2\cos^2\theta\,, \qquad \omega = j\,\frac{\sin^2\theta}{r}\,d\phi\,, \tag{3.20}$$

where j is the asymptotic angular momentum. It is easy to see that this metric reduces to (3.6) for $v_1^2 = v_2^2 = \sqrt{-I_4(\Gamma)}$ upon setting $j = 0$ above. We choose the vielbein

$$e_t^0 = re^U\,, \quad e_r^1 = \frac{e^{-U}}{r}\,, \quad e_\theta^2 = e^{-U}\,, \quad e_\phi^3 = e^{-U}\sin\theta\,, \quad e_\phi^0 = j\,e^U\sin^2\theta\,, \tag{3.21}$$

which leads to the following non-vanishing components of the spin connection:

$$\omega^{01} = -r\,e^{2U}\left(dt + \frac{1}{2}\omega\right), \qquad \omega^{23} = -e^{4U}\cos\theta\,\left(j\,r\,dt + \tilde{v}\,d\phi\right), \tag{3.22}$$

$$\omega^{02} = j\,r\,e^{6U}\sin\theta\cos\theta\,(\frac{j}{2}rdt + \tilde{v}d\phi)\,, \quad \omega^{03} = -\frac{j}{2r}\,e^{2U}\,(\sin\theta\,dt - r\,\cos\theta d\phi)\,, \tag{3.23}$$

$$\omega^{12} = -\frac{1}{2r}\,j^2\,e^{4U}\sin\theta\cos\theta\,dr\,, \qquad \omega^{13} = \frac{1}{2}\,j\,r\,e^{4U}\sin\theta(dt+\omega)\,, \tag{3.24}$$

where we defined the function

$$\tilde{v} = I_4 - \frac{1}{2}j^2(1+\cos^2\theta)\,. \tag{3.25}$$

The gauge fields for this class of solutions read [20]

$$\mathcal{F} = d\big[\zeta\,r\,(dt+\omega)\big] + \Gamma\,\sin\theta\,d\theta\wedge d\phi\,, \qquad \zeta = -2\,e^U\,\mathrm{Re}[e^{-i\alpha}\mathcal{V}] + G\,, \tag{3.26}$$

where we used the fact that the section depends on the radial coordinate by an overall $r^{-1}$ in the near horizon region, as for the static case. We refrain from giving the full solution for the scalars at this stage, since it will be derived from the BPS conditions below. Here we note that the physical scalars $t^i$ only depend on the angular coordinate $\theta$ in the near horizon region, and we give the expression for the Kähler connection

$$Q + d\alpha = \frac{1}{2}\,j\,e^{2U}\,\sin\theta\,d\theta\,, \tag{3.27}$$

for later reference. The interested reader can find an explicit example solution t the STU model in appendix C, both at the attractor and for the full flow.

As already mentioned above, the backgrounds we are interested in only preserve two supercharges, i.e. they are 1/4-BPS. The fact that we now need a second projection on the Killing spinor, in addition to (3.10), can be derived directly by considering the BPS equations, e.g. the gaugino variation. We omit details of this derivation, which is straightforward, and just give the resulting additional projection

$$\gamma^0\,\varepsilon_A = i\,e^{i\alpha}\epsilon_{AB}\,\varepsilon^B\,, \tag{3.28}$$

which is the same as the one used in e.g. [3, 29, 30, 57].

As in the static case, we make use of the complex self-duality of $\mathcal{F}$, so that we only need to use half of its components. We therefore choose for convenience $\mathcal{F}_{0\hat{a}}$ and $\mathcal{F}_{23}$, where $\hat{a} = 2, 3$ is a flat index on the sphere, given by

$$\mathcal{F}_{23} = \mathrm{e}^{2U}\,\Gamma + \zeta\, r\,(d\omega)_{23} = \mathrm{e}^{2U}\,\left(\Gamma + 2\,\zeta\,\mathrm{j}\,\cos\theta\right)\,, \qquad \mathcal{F}_{0\hat{a}} = -\partial_{\hat{a}}\zeta\,. \tag{3.29}$$

Since the central charges of these quantities appear in the BPS conditions, we note for clarity the following relations

$$Z(\zeta) = -i\,\mathrm{e}^{U}\mathrm{e}^{i\alpha} + W\,, \qquad\qquad Z_i(\zeta) = W_i\,, \tag{3.30}$$

$$Z(\partial_{\hat{a}}\zeta) = -i\,\mathrm{e}^{U}\mathrm{e}^{i\alpha}\big[\partial_{\hat{a}}U + i(Q_{\hat{a}} + \partial_{\hat{a}}\alpha)\big]\,, \qquad Z_i(\partial_{\hat{a}}\zeta) = i\,\mathrm{e}^{U}\mathrm{e}^{i\alpha}g_{i\bar{\jmath}}\partial_{\hat{a}}\bar{t}^{\bar{\jmath}}\,, \tag{3.31}$$

which can be straightforwardly derived from (3.26) using (A.4).

**Analysis of the BPS conditions.** Given the backgrounds described above, we proceed with the analysis of the conditions for unbroken supersymmetry. This is parallel to the discussion in section 3.1, but differs in that we only analyze the supersymmetry preserved by the attractors corresponding to asymptotically flat black holes as given by (3.20) rather than derive the general conditions for 1/4-BPS backgrounds. This is because there is at present no evidence that asymptotically AdS under-rotating black holes can be constructed and the near horizon properties of such hypothetical solutions is unclear. However, we note that there is no argument against the existence of such solutions in AdS and one can try to generalize our analysis by rescaling the sizes of the $\mathrm{AdS}_2$ and $\mathrm{S}^2$ also in the rotating case.

We now turn to the analysis, starting with the gravitino variation and imposing (3.10) and (3.28) on the spinor $\epsilon_A$. In the conditions below, we arrange all terms with two gamma matrices in the $0\hat{a}$ and 23 components, in order to simplify calculations. We start from the spherical components of the variation, which can be shown to vanish if the spinor $\epsilon_A$ does not depend on $\phi$ and the following conditions are imposed

$$(\partial_\theta + \frac{i}{2}\,Q_\theta)\varepsilon_A + \frac{i}{2}\,Z(\mathcal{F})^{-}_{0\theta}\,\mathrm{e}^{-i\alpha}\,\varepsilon_A = 0\,, \tag{3.32}$$

$$i\,\langle G, A_{\hat{a}}\rangle + i\,\omega_{\hat{a}}^{-\,23} + \mathrm{e}^{-i\alpha}Z(\mathcal{F})^{-\,\hat{b}}_{0}\,\varepsilon_{\hat{b}\hat{a}} = 0\,, \tag{3.33}$$

$$\frac{1}{2}\,\varepsilon^{\hat{\mu}\hat{a}}\,\omega_{\hat{\mu}}^{-\,0}{}_{\hat{a}}\mathrm{e}^{i\alpha} + i\,Z(\mathcal{F})^{-}_{23} + W = 0\,, \tag{3.34}$$

where the last relation represents a term present in both components. Using (3.20) and (3.26) for the metric and gauge fields, these are simplified as follows. The first leads to an equation that determines the angular dependence of the spinor as

$$2\,\partial_\theta\varepsilon_A = \partial_\theta U\,\varepsilon_A\,, \tag{3.35}$$

while the second relation reduces to (3.13). Finally, the third relation boils down to

$$Z - i\,\mathrm{e}^{U}\,\mathrm{j}\,\cos\theta\,\mathrm{e}^{i\alpha} + (2\,\mathrm{j}\,\cos\theta - i\,\mathrm{e}^{-2U})\,W = 0\,, \tag{3.36}$$

which generalises (3.12) in the rotating case.

Turning to the AdS$_2$ part, we analyse the time component of the Killing spinor equation, which upon assuming time independence[13] of $\epsilon_A$, implies the following constraints

$$i\,\langle G, A_t\rangle + i\,\omega_t^{-\,23} - r\mathrm{e}^U Z(\mathcal{F})^-_{23}\,\mathrm{e}^{-i\alpha} - i\,r\mathrm{e}^U\,W\,\mathrm{e}^{-i\alpha} = 0\,, \tag{3.37}$$

$$\omega_t^{-\,0}{}_{\hat{a}}\,\mathrm{e}^{i\alpha} = i\,r\mathrm{e}^U Z(\mathcal{F})^-_{0\hat{a}}\,, \tag{3.38}$$

where the second equation is identically satisfied by using (3.29) and (3.31). The first relation leads to

$$\begin{aligned} 2\,\mathrm{e}^U\;\mathrm{Re}(\mathrm{e}^{-i\alpha}\,W) &= \mathrm{e}^{4U}\,\mathrm{j}\,\cos\theta\,,\\ \mathrm{e}^{-U} + \mathrm{e}^{-i\alpha}\,Z + 2\,\mathrm{j}\,\cos\theta\,\left(\mathrm{e}^{-i\alpha}\,W - i\,\mathrm{e}^U\right) + i\,\mathrm{e}^{-2U}W\,\mathrm{e}^{-i\alpha} &= 0\,. \end{aligned}\tag{3.39}$$

Finally we consider the radial component, which leads to the constraints

$$2\,\partial_r\varepsilon_A + r^{-1}\mathrm{e}^{-U}Z(\mathcal{F})^-_{23}\,\mathrm{e}^{-i\alpha} + i\,r^{-1}\mathrm{e}^{-U}W\,\mathrm{e}^{i\alpha} = 0\,, \tag{3.40}$$

$$\omega_r^{-\,0\hat{b}}\,\varepsilon_{\hat{b}\hat{a}}\,\mathrm{e}^{i\alpha} = -r^{-1}\mathrm{e}^{-U}Z(\mathcal{F})^-_{0\hat{a}}\,. \tag{3.41}$$

These are also satisfied by using (3.29) and (3.39), for a spinor that depends on the radial coordinate according to

$$\partial_r\varepsilon_A = \frac{1}{2\,r}\,\varepsilon_A\,, \tag{3.42}$$

where we used (3.36) and (3.39) to obtain this result. Using the last equation and (3.35), find that the spacetime dependence of the Killing spinors is given by

$$\varepsilon_A(r,\theta) = e^{U/2}\,\sqrt{r}\;\varepsilon^0_A\,, \tag{3.43}$$

for arbitrary constant spinors $\varepsilon^0_A$ that obey the two projections (3.10) and (3.28) imposed above.

In addition, we need to consider the BPS conditions arising from the gaugino variation in (3.1), which in this case lead to

$$\mathrm{e}^{2U}\,Z_i + 2\,\mathrm{e}^{2U}\,\mathrm{j}\cos\theta\,Z_i(\zeta) - i\,W_i = 0\,, \qquad \partial_{\hat{a}}t^i = i\,\mathrm{e}^{i\alpha}\,\bar{Z}_i(\partial_{\hat{a}}\zeta)\,. \tag{3.44}$$

The second condition is identically satisfied upon using the $\zeta$ given in (3.26), whereas the first reads

$$\mathrm{e}^{2U}\,Z_i + 2\,\mathrm{e}^{2U}\,\mathrm{j}\,W_i\,\cos\theta - i\,W_i = 0\,. \tag{3.45}$$

This concludes our analysis of the BPS conditions for rotating attractors. The value of the scalar fields at the horizon can now be cast in terms of an attractor equation generalising (3.17) to the rotating case, as

$$2\,\mathrm{e}^{-U}\mathrm{Im}\left[(1 + 2\,i\,\mathrm{e}^{2U}\,\mathrm{j}\cos\theta)\,\mathrm{e}^{-i\alpha}\mathcal{V}\right] = \Gamma + \mathrm{e}^{-2U}\,JG + 2\,\mathrm{j}\cos\theta\,G\,, \tag{3.46}$$

where one still has to impose (3.36) as a constraint.

The BPS conditions above can be straightforwardly seen to be the horizon limit of the single center rotating black holes of [20], using the definition (2.29) to simplify the result as in the static case. Since these were shown to be the most general asymptotically flat extremal under-rotating black holes, we have thus shown that all under-rotating attractor solutions are 1/4-BPS (i.e. preserve two supercharges) when embedded in a gauged

[13] This assumption is consistent as we eventually show that all BPS equations are satisfied and we explicitly derive the spacetime dependence of the Killing spinors, which only depend on the $r$ and $\theta$ coordinates.

| | | Attractor | Global |
|---|---|---|---|
| $G \in \mathcal{S}$, **flat** | $\mathrm{j}=0$ | 1/2 BPS | non−BPS |
| | $\mathrm{j}\neq 0$ | 1/4 BPS | non−BPS |
| $G \notin \mathcal{S}$, **AdS** | $\mathrm{j}=0$ | 1/2 BPS | 1/4 BPS |
| | $\mathrm{j}\neq 0$ | ? | ? |

**Table 1**. An overview of supersymmetry properties of under-rotating attractors and full solutions in abelian gauged theories without hypermultiplets, depending on whether the vector of gaugings $G$ lies in the very small orbit $\mathcal{S}$ or not. The "?" for the under-rotating case in AdS signify that the existence of such solutions is not certain, not only that their supersymmetry properties are not analyzed.

supergravity with a flat potential.[14] Upon taking limits of vanishing angular momentum and gaugings one finds that supersymmetry is enhanced, since the static attractors in the previous section are 1/2-BPS when $G \neq 0$ and fully BPS when the gauging vanishes. In table 1 we summarise the findings of this section for all static and under-rotating attractors in abelian gauged $\mathcal{N}=2$ theories.

## 4 Asymptotically $\mathrm{AdS}_4$ BPS black holes

In section 2 we introduced a procedure to obtain first order equations for asymptotically flat non-supersymmetric black holes by mimicking the squaring of the action that leads to asymptotically $\mathrm{AdS}_4$ BPS black holes in an abelian gauged theory. Given the very close similarity between the equations describing the two systems, it is possible to clarify the structure of asymptotically $\mathrm{AdS}_4$ static black holes by recycling some of the objects used in the asymptotically flat case.

[14]The acute reader might notice that in the static case the $\mathrm{mAdS}_2\times \mathrm{S}^2$ superalgebra $\mathrm{SU}(1,1|1)\times\mathrm{SO}(3)$ can be broken to $\mathrm{SU}(1,1|1)\times\mathrm{U}(1)$ without breaking more supersymmetries. This means that one could expect the rotating attractors to also preserve half of the original supercharges. Here we explicitly showed that these attractors are 1/4 BPS by imposing (3.10) and (3.28), but this does not exclude the existence of a more general 1/2 BPS projection that also ensures the supersymmetry variations vanish. The flat rotating attractors here might also be part of a more general class of rotating attractors in gauged supergravity, such as the ones constructed in [65]. We do not pursue this subject further as our present purpose is to show that all asymptotically flat attractors are supersymmetric without focusing on the exact amount of preserved supercharges.

In this case, the appropriate form for the metric is the one in (2.7), which allows for a non-flat three dimensional base. The relevant effective action now is the one in (2.9), where no assumptions were made for the vector $G$. The flow equations that follow from this squaring are similar to (2.16)–(2.17), with vanishing Lagrange multiplier $\varphi$, together with an equation for the nontrivial $\mathrm{e}^{\psi}$, as in [29]:

$$\mathcal{E} = 0, \tag{4.1}$$

$$\psi' = 2\, e^{-U}\, \mathrm{Im}(e^{-i\alpha} W), \tag{4.2}$$

$$Q_r + \alpha' = -2e^{-U}\, \mathrm{Re}(e^{-i\alpha} W)\,, \tag{4.3}$$

and we repeat the expression for $\mathcal{E}$,

$$\mathcal{E} \equiv 2e^{2\psi} \left(e^{-U}\mathrm{Im}(e^{-i\alpha}\mathcal{V})\right)' + e^{2(\psi-U)}\, \mathrm{J}G + 4e^{2\psi-U}(Q_r + \alpha')\mathrm{Re}(e^{-i\alpha}\mathcal{V}) + \Gamma\,, \tag{4.4}$$

for the readers convenience.

Since our goal is to show the similarities between the solutions of this system to the asymptotically flat ones, we will use an ansatz and similar definitions as in section 2.2. Here however, we use the same relations restricting the constant $\mathrm{m} = 0$, as one can check by analysing the asymptotic fall-off of the terms in (4.4) that a nonzero m spoils the asymptotic behavior of the scale factor of the metric. Thus, the role of the constant m is drastically changed with respect to the asymptotically flat context, where it is "dressed" with the Lagrange multiplier and is in fact crucial to obtain the most general static solution.

The flow equations (4.4) can be simplified by defining a vector $R$ from $G$, as in the non-BPS asymptotically flat case. Using the definition (2.29) with $\mathrm{m} = 0$, we find[15]

$$-|W|^2\, R = \mathrm{J}G\,. \tag{4.5}$$

The crucial difference with the previous situation is that here $R$ is neither constant nor small, since $G$ is not. This allows to rewrite the flow equation for the section as

$$2\mathrm{e}^{2\psi} \left(\mathrm{e}^{-U}\mathrm{Im}(\mathrm{e}^{-i\alpha}\mathcal{V})\right)' - 2\,\mathrm{e}^{2\psi}\, |W|^2\, R + 4\mathrm{e}^{2\psi-U}(Q_r + \alpha')\mathrm{Re}(\mathrm{e}^{-i\alpha}\mathcal{V}) + \Gamma = 0\,. \tag{4.6}$$

It order to describe solutions, me employ the natural ansatz of [28–30], which only depends on a vector of single center harmonic functions $\mathcal{H}$ as

$$2\,\mathrm{e}^{-U}\, \mathrm{Im}(\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}) = r\, \mathrm{e}^{-\psi}\mathcal{H}\,, \tag{4.7}$$

and immediately leads to a vanishing Kähler connection, as

$$Q_r + \alpha' = 0\,. \tag{4.8}$$

Note that (4.7) reduces to the asymptotically flat solution (2.31) for $\mathrm{e}^{\psi} = r$ and $\mathrm{m} = 0$, as expected. In the more general case, equations (4.2) and (4.7) determine the function $\mathrm{e}^{\psi}$ by

$$(\mathrm{e}^{\psi})' = r\, \langle G, \mathcal{H}\rangle\,, \tag{4.9}$$

which can be easily integrated.

[15]See (B.26) for the general case including m.



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In order to integrate the flow equation (4.6) above, one can follow the direct approach of [28–30], that leads to explicit solutions (see the example below). Nevertheless, some intuition from the asymptotically flat case can be used, in order to simplify this process. In particular, we claim that the constraint (2.34), which we repeat here

$$\frac{1}{2}\, I_4'(\mathcal{H}, G) = \langle G, \mathcal{H}\rangle\, \mathcal{H} - 2\frac{\langle G, \mathcal{H}\rangle^2}{\langle G, R\rangle}\, R\,, \tag{4.10}$$

as written in the context of asymptotically flat solutions for very small vectors $G$ and $R$, is relevant also in the more general case, where these vectors are generic. Note that this might again be related to a reality constraint on the scalar flow as in (2.27), but we do not require any such assumption.

In view of the similarity in the flow equations and the fact that the ansatze in (2.31) and (4.7) are related by rescaling with a function, it is conceivable that a constraint homogeneous in all $\mathcal{H}$, $G$ and $R$ as the one in (4.10) may indeed be common in the two cases. Using the explicit examples in [28–30], one can see that this is indeed the case, as we show below.

**Example STU solution.** In order to see how the constraint above is relevant, we consider the STU model, defined by the prepotential

$$F = \frac{X^1 X^2 X^3}{X^0}\,, \tag{4.11}$$

as an example where fairly generic explicit solutions are known, and the expression of $R$ can be computed explicitly. Following [28–30], we choose a frame where the FI terms are

$$G = \left(0,\, g^i\,;\, g_0,\, 0\right)^T\,, \tag{4.12}$$

and consider a vector of single center harmonic functions

$$\mathcal{H} = \left(-H^0,\, 0\,;\, 0,\, H_i\right)^T\,, \tag{4.13}$$

where

$$H^0 = \alpha^0 + \frac{\beta^0}{r}\,, \qquad H_i = \alpha_i + \frac{\beta_i}{r}\,. \tag{4.14}$$

The corresponding asymptotically flat solution, where the gauging is only along the $g_0$ direction is given in appendix C. The reader can easily compare the expressions below with those in the appendix to appreciate the close similarity of the two systems.

With the above expressions one can compute from (4.7) that

$$\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}) = r\,\mathrm{e}^{-\psi}\,\mathrm{e}^{2U}\left(0,\, \frac{1}{2}\, H^0\,|\varepsilon^{ijk}| H_j H_k\,;\, H_1 H_2 H_3,\, 0\right)^T\,, \tag{4.15}$$

and

$$r^{-4}\,\mathrm{e}^{4\psi}\,\mathrm{e}^{-4U} = 4\, H^0 H_1 H_2 H_3\,. \tag{4.16}$$

Finally, we consider a solution to (4.9), given by

$$\langle G, \mathcal{H}\rangle = 2\,, \qquad \mathrm{e}^{\psi} = r^2 + c\,, \tag{4.17}$$

where $c$ is an arbitrary constant and the first equation is a simplifying condition.

One can now impose the flow equation (4.6) using the assumptions above, to find the constraints

$$\alpha^0 g_0 = \alpha^i g_i \,, \qquad p_0 = c\alpha^0 - 2(\beta^0)^2 g_0 \,, \qquad -q_i = c\alpha_i - 2(\beta_i)^2 g_i, \tag{4.18}$$

where all equations are valid for each value of the index $i$ separately and there is no implicit sum. The explicit expression for $R$ in (4.5) then reads

$$R^0 = g_0 \,(H^0)^2 \,, \qquad R_i = g^i \,(H_i)^2 \,, \tag{4.19}$$

where again there is no implicit sum.

One can now straightforwardly evaluate the constraint (4.10) using the harmonic functions $\mathcal{H}$ in (4.13) and the expression for $R$ in (4.19), to find that it is identically satisfied. We conclude that this constraint is also relevant for asymptotically $\mathrm{AdS}_4$ solutions, since (similar to the asymptotically flat case) one can invert the procedure above to find $R$ from (4.10) rather than performing the tedious computation of the matrix J in (A.11).

Additionally, the near horizon limit of (4.10), leads to a nontrivial constraint on the charges in terms of $G$, exactly as in the asymptotically flat non-BPS case. This is equivalent to the constraints found in [28–30] by solving the BPS flow equations in the STU model explicitly. From that point of view, (4.10) appears to be a duality covariant form of the constraints on the charges in this class of solutions, valid for other symmetric models beyond STU.

## 5 Extensions including hypermultiplets

Given the results of section 2 on the embedding of asymptotically flat black holes in gauged theories, it is natural to consider the possibility of extending the abelian gauged theory to include hypermultiplets. Indeed, the appearance of the vector of gaugings $G$ multiplied by a universal function, introduced as a Lagrange multiplier, that is determined independently from the vector multiplet scalars, is a tantalising hint towards such an embedding. In this scenario, one would require the gauging of a single U(1) factor in the hypermultiplet sector, where the overall Lagrange multiplier $\mathrm{e}^{\varphi}$ in (2.13) is now promoted to a dynamical field, identified with the corresponding moment map, and $G$ is identified with the embedding tensor [43, 49]. In this section, we explore the possibilities of constructing such a theory, without explicitly considering the embedding of the known asymptotically flat solutions.

In doing so, we consider the explicit compactifications of M-theory on Calabi-Yau manifolds fibered over a circle described in [66–68], (see [64] for a recent overview). This setting is very convenient for our purposes, as it automatically leads to a flat potential for the vector multiplets (2.3), since there is only one U(1) isometry gauged, along the vector shown in (2.6), identified with the embedding tensor. The hypermultiplet scalars, $q^u$, $u = 1 \ldots 4n_{\rm h}$, in these models parametrise target spaces in the image of the c-map, and describe a fibration of a $2n_{\rm h} + 2$ dimensional space with coordinates $(a\,,\tilde{a}\,,\xi)$, where $\xi = (\xi^A\,,\tilde{\xi}_A)$ is a $2n_{\rm h}$ dimensional symplectic vector, over a special Kähler manifold of dimension $n_{\rm h}-1$, with coordinates arranged in a complex symplectic section $\Omega = (Z^A\,,G_A)$, similar to the vector multiplets.



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Within this setting, we consider the gauging along the Killing vector

$$k_{\mathbb{U}} \;=\; (\mathbb{U}\Omega)^A \frac{\partial}{\partial Z^A} + (\mathbb{U}\bar{\Omega})^A \frac{\partial}{\partial \bar{Z}^A} + (\mathbb{U}\xi)^A \frac{\partial}{\partial \xi^A} + (\mathbb{U}\xi)_A \frac{\partial}{\partial \tilde{\xi}_A}\,, \tag{5.1}$$

where $\mathbb{U}$ is a symplectic matrix whose explicit form can be found in [64, 68], but is not of immediate importance for what follows. This leads to the standard minimal coupling term for hypermultiplets, by replacing derivatives by covariant derivatives in the kinetic term as

$$h_{uv}\, D_\mu q^u D^\mu q^v = h_{uv}\big(\partial_\mu q^u + k^u_{\mathbb{U}}\,\langle G, A_\mu\rangle\big)\big(\partial^\mu q^v + k^v_{\mathbb{U}}\,\langle G, A^\mu\rangle\big)\,, \tag{5.2}$$

where $h_{uv}$ denotes the hyper-Kähler metric. The potential of the gauged theory is now given by

$$V = V_g + V_{\text{hyp}}\,, \tag{5.3}$$

where $V_g$ is a modification of (2.3) as

$$V_g = Z_i(G)\,\bar{Z}^i(G)\,\mathcal{P}^2 - 3\,|Z(G)|^2\,\mathcal{P}^2 = \langle G, \mathrm{J}\,G\rangle\,\mathcal{P}^2 - 4\,|Z(G)|^2\,\mathcal{P}^2\,, \tag{5.4}$$

and $\mathcal{P}^2$ stands for the square of the triplet of moment maps, $\{P^x,\,P^y,\,P^z\}$ corresponding to (5.1), given by

$$\mathcal{P}^2 = (P^x)^2 + (P^y)^2 + (P^z)^2\,. \tag{5.5}$$

The second term in the scalar potential (5.3) arises from the hypermultiplet gauging and reads

$$V_{\text{hyp}} = 8\,h_{uv}\,k^u_{\mathbb{U}}\,k^v_{\mathbb{U}}\,|\langle G, \mathcal{V}\rangle|^2\,. \tag{5.6}$$

In order for the Einstein equation to allow for asymptotically flat solutions, one must impose the condition

$$h_{uv}\,k^u_{\mathbb{U}}\,k^v_{\mathbb{U}} = 0\,, \tag{5.7}$$

which eliminates both the potential $V_{\text{hyp}}$ and the term quadratic in gauge fields in the scalar kinetic term. Since the vector $G$ in (2.6) lies in the doubly critical orbit $\mathcal{S}$, the vector multiplet potential (5.4) vanishes identically, as usual. Note however that the vector of gaugings naturally appears multiplied by an overall function, the moment map $\mathcal{P}$, coming from the hypermultiplet sector. We can further simplify (5.7), using the facts that the quaternionic metric $h_{uv}$ is positive definite and only a single U(1) is gauged, to find that[16]

$$k^u_{\mathbb{U}} = 0\,. \tag{5.8}$$

This way the hypermultiplets condense to their supersymmetric constant values, a process described in detail in [57]. The resulting theory is again the abelian gauged theory of section 2, since the moment map $\mathcal{P}$, which also controls the gravitino gauging, is in general nonvanishing. Note that this is consistent despite the initial presence of a charged hypermultiplet, due to the vanishing of the quadratic term in gauge fields by (5.7), which would otherwise produce a source in the Maxwell equations.

[16]We thank Hagen Triendl for pointing out a mistake in a previous version of this paper.



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We now show more explicitly how this can be realised in a simple example involving the universal hypermultiplet [69], which is present in all Type II/M-theory compactifications to four dimensions, and is thus included in all the target spaces in the image of the c-map described above. Moreover, the possible gaugings for this multiplet are described by particular choices for $\mathbb{U}$ in the Killing vector (5.1). We use the following metric for the universal hypermultiplet, parametrised by four real scalars $\rho$, $\sigma$, $\tau$ and $\chi$ as

$$\mathrm{d}s^2_{\mathrm{hyp}} = \frac{1}{\rho^2}\left(\mathrm{d}\rho^2 + \rho\,(\mathrm{d}\chi^2 + \mathrm{d}\tau^2) + \left(\mathrm{d}\sigma + \chi\mathrm{d}\tau\right)^2\right)\,, \tag{5.9}$$

which has eight Killing vectors (see appendix D of [42]). Now, consider the particular linear combination of Killing vectors

$$k_c = -\tau\partial_\chi + \chi\partial_\tau + \frac{1}{2}(\tau^2 - \chi^2)\partial_\sigma \; - c\,\partial_\sigma\,, \tag{5.10}$$

parametrised by the arbitrary constant $c$. Now, the expression (5.7) becomes

$$h_{uv}k_c^u k_c^v = \frac{\chi^2+\tau^2}{\rho} + \frac{(\chi^2+\tau^2-2\,c)^2}{4\rho^2}\,, \tag{5.11}$$

which can vanish in two distinct situations. One is the physical minimum, corresponding to (5.8), for which

$$\chi = \tau = 0\,, \quad \rho > 0\,, \quad c = 0\,, \tag{5.12}$$

while the second solution, given by

$$\rho = -\frac{(\chi^2+\tau^2-2\,c)^2}{4(\chi^2+\tau^2)} < 0\,, \tag{5.13}$$

is unphysical, since the metric (5.9) is only positive definite when $0 < \rho < \infty$ and (5.13) implies that it is of signature $(2,2)$ instead.

The triplet of moment maps associated to the Killing vector above is given by

$$P = \left\{\frac{\tau}{\sqrt{|\rho|}}\,, \frac{\chi}{\sqrt{|\rho|}}\,, 1 - \frac{\chi^2+\tau^2-2\,c}{4\rho}\right\}\,, \tag{5.14}$$

and reduces to $P = \{0,0,1\}$ for the physical solution in (5.12), while it is a nontrivial function of $\chi$ and $\tau$ when pulled back on the hypersurface defined by (5.13). This is an explicit realisation of the scenario sketched above, since we have obtained a gauged theory with an everywhere vanishing scalar potential, but with nontrivial moment maps. From this standpoint, the embedding of the asymptotically flat solution of section 2 applies directly to these models, similar to examples discussed in [57].

From this simple discussion it follows that a single U(1) hypermultiplet gauging can only lead to asymptotically flat solutions with the hypermultiplet scalars fixed to constants, leading to a constant moment map. If the resulting value of the moment map is non-zero we are back in the case of FI term gauging that allows for a BPS horizon but non-BPS asymptotics at infinity. If on the other hand the moment maps vanish, one is in the ungauged case with BPS Minkowski vacuum and non-BPS horizon. Note however, that the existence of unphysical solutions of the type (5.13) may be helpful in constructing new solutions without hypermultiplets, where the remaining unfixed scalars may play the role of the unphysical Lagrange multiplier in section 2.



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Finally, the interesting problem of obtaining solutions with physical charged hypermultiplets remains. Given the above, it is clear that one must gauge bigger groups (at least $\mathrm{U}(1)^2$) of the hypermultiplet isometries to construct such BPS solutions, preserving supersymmetry both at the horizon and at infinity, see e.g. [59, 70]. We can then expect that such a theory, if existent, would allow for solutions preserving two supercharges in the bulk (1/4-BPS solution), given that the attractor preserves $\mathcal{N}=1$ supersymmetry, as established in section 3. Constructing such a theory seems to be a nontrivial but rather interesting task for future investigations.

## 6 Conclusion and outlook

In this paper we presented in some detail a novel connection between the solutions of ungauged supergravity and gauged supergravity with an identically flat potential in four dimensions. In particular, we identified the recently constructed general solution for under-rotating asymptotically flat black holes as special solutions to abelian gauged supergravity with a flat potential, where nontrivial gaugings are still present and are reflected on the solutions. As an application, we further showed explicitly that the attractor geometries of these black holes belong to the generic class of 1/2-BPS $\mathrm{AdS}_2\times\mathrm{S}^2$ attractor backgrounds that pertain to (generically asymptotically $\mathrm{AdS}_4$) black holes in abelian gauged theories. These results are interesting from several points of view, respectively discussed in the main sections above. In this final section, we discuss the implications of our results for possible string models of extremal black holes as well some intriguing similarities to recent results in the study of black holes in the context of supergravity.

The somewhat surprising result of obtaining hitherto hidden supercharges in the near horizon geometries of all extremal under-rotating black holes, deserves some additional attention. Firstly, the supercharges at hand only exist when appropriate FI terms are turned on for given charges, and are not present in general. This means that not all attractors characterised by a negative quartic invariant $I_4(\Gamma)$ of the charges can be made supersymmetric simultaneously, in contrast to the ones with $I_4(\Gamma)>0$, which correspond to globally BPS solutions. This situation is reminiscent of the example solutions studied recently in [71, 72], which preserve supersymmetry only when embedded in a larger theory, but appear as non-BPS in any $\mathcal{N}=2$ truncation. In combination with these examples, our results show that any supercharges preserved by a solution in a higher dimensional theory, may only be realised in more general compactifications as opposed to a naive dimensional reduction. This fits very well with the fact that asymptotically Taub-NUT non-BPS black holes in five dimensions can preserve the full $\mathcal{N}=2$ supersymmetry near their horizons [12]. Indeed, while a direct dimensional reduction along the Taub-NUT fiber breaks all supersymmetries, our results imply that the Scherk-Schwarz reduction of [68] preserves half of them. It would be interesting to develop a higher dimensional description of our alternative embedding along these lines, especially in connection to possible microscopic models.

Indeed, one of the most intriguing implications of the presence of supersymmetry for asymptotically flat under-rotating black holes is the possibility of obtaining control over the microscopic counting of the entropy, similar to globally BPS black holes. According to standard lore, one expects a dual microscopic CFT living on the worldvolume of appropriate D-branes to be the relevant description at weak coupling. Such models have been proposed in e.g. [73–77], and arguments on the extrapolation of the entropy counting for non-BPS black holes were formulated in [78, 79], based on extremality. However, our results indicate that one may be able to do better, if a supersymmetric CFT dual for extremal black holes can be found. At this point, one is tempted to conjecture that such a CFT should be a deformation of the known theories describing BPS black holes [1, 2], where half of the supercharges are broken by the presence of appropriate deformation parameters, corresponding to nonzero FI terms. Obtaining a description along these lines would be also very interesting from the point of view of black hole physics in AdS, since it would shed some light on the role of the gaugings in a microscopic setting.

A related question in this respect is the possibility of extending our embedding of asymptotically flat solutions to theories including gauged hypermultiplets. As briefly discussed in section 5, such models with identically flat potentials are possible, and exploring the various gaugings allowed is an interesting subject on its own. From a higher dimensional point of view, the particular gaugings described in [68] represent a natural choice, since they can be formulated in terms of a twisted reduction of ungauged five dimensional supergravity along a circle. A similar twist was recently used in [80] in connection to the near horizon geometry of over-rotating black holes.

Parallel to the implications on the asymptotically flat solutions, one may use the connection established here to learn more about 1/4-BPS black holes in AdS. The somehow surprising fact that the constraint on the charges defined in [20] in the case of flat gauging, is relevant in the more general setting where the gaugings are unrestricted is a hint towards a better understanding of the moduli space of these solutions. Indeed, since this constraint is relevant throughout the flow connecting two uniquely fixed vacua, the asymptotic $AdS_4$ vacuum of the theory and the BPS attractor, it may be relevant for establishing existence criteria for given charges. In addition, it would be interesting to extend our procedure to the non-extremal case, by connecting the results of [31, 32] with those of [81]. We hope to return to some of these questions in future work.

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## A Conventions on $\mathcal{N} = 2$ supergravity

In this paper we follow the notation and conventions of [20]. In this appendix we collect some basic definitions that are useful in the main text, referring to that paper for more details.

The vector fields naturally arrange in a symplectic vector of electric and magnetic gauge field strengths, whose integral over a sphere defines the associated electromagnetic charges as

$$\mathcal{F}_{\mu\nu} = \begin{pmatrix} F^I_{\mu\nu} \\ G_{I\,\mu\nu} \end{pmatrix} , \qquad \Gamma = \begin{pmatrix} p^I \\ q_I \end{pmatrix} = \frac{1}{2\pi} \int_{S^2} \mathcal{F} \,. \tag{A.1}$$

The physical scalar fields $t^i$, which parametrize a special Kähler space of complex dimension $n_{\mathrm{v}}$, appear through the so called symplectic section, $\mathcal{V}$. Choosing a basis, this section can be written in components in terms of scalars $X^I$ as

$$\mathcal{V} = \begin{pmatrix} X^I \\ F_I \end{pmatrix} , \qquad F_I = \frac{\partial F}{\partial X^I} \,, \tag{A.2}$$

where $F$ is a holomorphic function of degree two, called the prepotential, which we will always consider to be cubic

$$F = -\frac{1}{6} c_{ijk} \frac{X^i X^j X^k}{X^0} \,, \tag{A.3}$$

for completely symmetric $c_{ijk}$, $i = 1, \dots n_{\mathrm{v}}$. The section $\mathcal{V}$ is subject to the constraints

$$\langle \bar{\mathcal{V}}, \mathcal{V} \rangle = i \qquad \langle \bar{D}_{\bar{i}} \bar{\mathcal{V}}, D_j \mathcal{V} \rangle = -i\, g_{\bar{i}j} \,, \tag{A.4}$$

with all other inner products vanishing, and is uniquely determined by the physical scalar fields $t^i = \frac{X^i}{X^0}$ up to a local U(1) transformation. Here, $g_{\bar{i}j}$ is the Kähler metric and the Kähler covariant derivative $D_i \mathcal{V}$ contains the Kähler connection $Q_\mu$, defined through the Kähler potential as

$$Q = \mathrm{Im}[\partial_i \mathcal{K}\, dt^i] \,, \qquad \mathcal{K} = -\ln \left( \frac{i}{6} c_{ijk} (t - \bar{t})^i (t - \bar{t})^j (t - \bar{t})^k \right) . \tag{A.5}$$

We introduce the following notation for any symplectic vector $\Gamma$

$$Z(\Gamma) = \langle \Gamma, \mathcal{V} \rangle \,, \tag{A.6}$$
$$Z_i(\Gamma) = \langle \Gamma, D_i \mathcal{V} \rangle \,, \tag{A.7}$$

with the understanding that when an argument does not appear explicitly, the vector of charges in (A.1) should be inserted. In addition, when the argument is form valued, the operation is applied component wise. With these definitions it is possible to introduce a scalar dependent complex basis for symplectic vectors, given by $(\mathcal{V},\, D_i\mathcal{V})$, so that any vector $\Gamma$ can be expanded as

$$\Gamma = 2\, \mathrm{Im}[-\bar{Z}(\Gamma)\, \mathcal{V} + g^{\bar{i}j} \bar{Z}_{\bar{i}}(\Gamma)\, D_j \mathcal{V}] \,, \tag{A.8}$$



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whereas the symplectic inner product can be expressed as

$$\langle\Gamma_1,\Gamma_2\rangle = 2\,\mathrm{Im}[-Z(\Gamma_1)\,\bar{Z}(\Gamma_2) + g^{i\bar{\jmath}}Z_i(\Gamma_1)\,\bar{Z}_{\bar{\jmath}}(\Gamma_2)]\,. \tag{A.9}$$

In addition, we introduce the scalar dependent complex structure J, defined as

$$\mathrm{J}\mathcal{V} = -i\mathcal{V}\,, \quad \mathrm{J}D_i\mathcal{V} = iD_i\mathcal{V}\,, \tag{A.10}$$

which can be solved to determine J in terms of the period matrix $\mathcal{N}_{IJ}$ in (2.2), see e.g. [55] for more details. With this definition, we can express the complex self-duality of the gauge field strengths as

$$\mathrm{J}\,\mathcal{F} = -*\mathcal{F}\,, \tag{A.11}$$

which is the duality covariant form of the relation between electric and magnetic components. Finally, we record the important relation

$$\langle\Gamma,\mathrm{J}\,\Gamma\rangle = |Z(\Gamma)|^2 + g^{i\bar{\jmath}}Z_i(\Gamma)\,\bar{Z}_{\bar{\jmath}}(\Gamma) \equiv V_{\mathrm{BH}}(\Gamma)\,, \tag{A.12}$$

where we defined the black hole potential $V_{\mathrm{BH}}(\Gamma)$.

## B First order flows for rotating black holes

In this appendix we discuss the rewriting of the effective action as a sum of squares and the corresponding flow equations for stationary black holes in four dimensional abelian gauged $\mathcal{N}=2$ supergravity. In section B.1 we present the general case, while in section B.2 we specialise to the case of flat potential to show that the general asymptotically flat underrotating black holes are indeed solutions of the theory in this limit. We largely follow [4, 29] with respect to the method and notational conventions.

### B.1 Squaring of the action

We start with a timelike reduction to three spatial dimensions using the metric ansatz:

$$ds^2 = -e^{2U}(dt+\omega)^2 + e^{-2U}\left(dr^2 + \mathrm{e}^{2\psi}d\theta^2 + \mathrm{e}^{2\psi}\sin^2\theta d\phi^2\right)\,, \tag{B.1}$$

which generalises (2.7) by the addition of the angular momentum vector $\omega$. In this setting we allow for a dependence of the fields on all spatial coordinates, so that a timelike reduction is appropriate [82]. The effective three-dimensional action reads:

$$\begin{aligned}S_{4\mathrm{D}} = -\frac{1}{16\pi}\int dt\int_{\mathbb{M}^3}\Big[ &-2\,(d\psi\wedge\star d\psi - \star 1) + 2dU\wedge\star dU - \frac{1}{2}\mathrm{e}^{4U}d\omega\wedge\star d\omega \\ &+ 2g_{i\bar{\jmath}}\,dt^i\wedge\star d\bar{t}^{\bar{\jmath}} + (\mathcal{F},\mathcal{F}) + e^{-2U}\langle G,\diamond G\rangle\star 1 - 8e^{-2U}\,|Z(G)|^2\star 1\Big]\,,\end{aligned} \tag{B.2}$$

where $\mathcal{F}$ is the spatial projection of the four dimensional field strengths, $\star$ denotes the Hodge dual in three dimensions and we discarded a boundary term. The scalar dependent inner product denoted by $(,)$ is a generalisation of (A.12) in the rotating case that is explicitly given by [4]

$$(\mathcal{A},\mathcal{B}) = \frac{\mathrm{e}^{2U}}{1-w^2}\int\mathcal{A}\wedge\left[\star(\mathrm{J}\,\mathcal{B}) - \star(w\wedge\mathrm{J}\,\mathcal{B})\,w + \star(w\wedge\star\mathcal{B})\right]\,, \tag{B.3}$$



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for any two symplectic vectors of two-forms $\mathcal{A}$, $\mathcal{B}$, and we define $w = \mathrm{e}^{2U}\omega$, as a short-hand below. In order to stay as close as possible to the static case, we treat the gauging parameters $G$ as gauge field strengths in the three dimensional base space. We then define

$$\mathcal{G} = G \star \eta\,. \tag{B.4}$$

where $\eta$ is a one-form which we require to be invariant under the vector $\omega$, but is otherwise undetermined at this stage. The choice $\eta = dr$ is the one relevant for the static solutions.

Inspired by [29], we can use the above definitions to recombine the gauge kinetic term and the potential using the following combination ($Z(\mathcal{G}) = \langle \mathcal{G}, \mathcal{V}\rangle$)

$$\tilde{\mathcal{F}} = \mathcal{F} - e^{-2U}\left(\mathrm{J}G - w\wedge\star\mathcal{G}\right) + 4\,e^{-2U}\,\mathrm{Re}\left[\mathrm{Re}\,Z(\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{G})\mathrm{e}^{\mathrm{i}\alpha}\mathcal{V} + i\,\mathrm{Re}\,Z(\mathrm{e}^{-\mathrm{i}\alpha}\star\mathcal{G})\wedge w\mathrm{e}^{\mathrm{i}\alpha}\mathcal{V}\right]\,, \tag{B.5}$$

which is such that

$$(\tilde{\mathcal{F}},\tilde{\mathcal{F}}) = (\mathcal{F},\mathcal{F}) + e^{-2U}\langle G, \mathrm{J}G\rangle \star\eta\wedge\eta + 2\langle \mathcal{F},\star\mathcal{G}\rangle + 8\,\mathrm{Im}\,\mathrm{e}^{\mathrm{i}\alpha}Z(\mathcal{F})\wedge \mathrm{Re}\,\mathrm{e}^{-\mathrm{i}\alpha}Z(\star\mathcal{G})\,, \tag{B.6}$$

and $\mathrm{e}^{\mathrm{i}\alpha}$ is an arbitrary phase as in the static case. The scalars can be repackaged in a similar way using the standard combination [4]

$$\mathcal{W} = 2\,\mathrm{Im}\star D(\mathrm{e}^{-U}\mathrm{e}^{-i\alpha}\mathcal{V}) - 2\,\mathrm{Re}\,D(\mathrm{e}^{U}\mathrm{e}^{-i\alpha}\mathcal{V}\,\omega)\,, \tag{B.7}$$

$$D = d + i\left(Q + d\alpha + \frac{1}{2}\mathrm{e}^{2U}\star d\omega\right)\,, \tag{B.8}$$

which in turn is such that

$$\begin{aligned}2\,dU\wedge\star dU - \frac{1}{2}\mathrm{e}^{4U}d\omega\wedge\star d\omega + 2\,g_{i\bar{\jmath}}\,dt^i\wedge\star d\bar{t}^{\bar{\jmath}}& \\ = (\mathcal{W},\mathcal{W}) - 2\left(Q + d\alpha + \frac{1}{2}\mathrm{e}^{2U}\star d\omega\right)&\wedge dw + d\left[2w\wedge(Q+d\alpha)\right],\end{aligned} \tag{B.9}$$

so that the action reads

$$\begin{aligned}S_{\mathrm{4D}} = -\frac{1}{16\pi}\int d^4x\Big[&(\mathcal{W},\mathcal{W}) - 2\,(Q + d\alpha + \frac{1}{2}\mathrm{e}^{2U}\star d\omega)\wedge dw \\ &+ (\tilde{\mathcal{F}},\tilde{\mathcal{F}}) - 2\langle\mathcal{F},\star\mathcal{G}\rangle - 8e^{-2U}\,|Z(G)|^2\star 1 \\ &- 8\Big(\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\alpha}Z(\mathcal{F})) - w\wedge\mathrm{Im}(\mathrm{e}^{\mathrm{i}\alpha}Z(\star\mathcal{G}))\Big)\wedge\mathrm{Re}\,\mathrm{e}^{-\mathrm{i}\alpha}Z(\star\mathcal{G}) \\ &- 2\,\left(d\psi\wedge\star d\psi - \star 1\right)\Big]\,.\end{aligned} \tag{B.10}$$

We now proceed to write the action as a sum of squares, making use of the further definitions

$$\mathcal{E} = \tilde{\mathcal{F}} - \mathcal{W}\,, \tag{B.11}$$

$$\mathrm{Im}\langle\mathcal{E},\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}\rangle = e^{U}\,\mathrm{Im}\left(\mathrm{e}^{-\mathrm{i}\alpha}Z(\mathcal{F})\right) + e^{-U}\,\mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}\alpha}Z(\mathcal{G})\right) - e^{-U}w\wedge\mathrm{Im}\left(\mathrm{e}^{-\mathrm{i}\alpha}Z(\star\mathcal{G})\right) + \frac{1}{2}dw\,. \tag{B.12}$$

After some rearrangements one obtains the result

$$
\begin{aligned}
S_{4\mathrm{D}} = -\frac{1}{16\pi}\int dt \int_{\mathbb{R}^3} \Big[ & (\mathcal{E},\mathcal{E}) - 4\left(Q + d\alpha + 2e^{-U}\,\mathrm{Re}\,Z(\star\mathcal{G}) + \frac{1}{2}\mathrm{e}^{2U}\star d\omega\right)\wedge \mathrm{Im}\langle\mathcal{E}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}\rangle \\
& - 2\left[\langle\mathcal{F} + 2\,\mathrm{Re}\,d(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}\,\omega), \star\mathcal{G}\rangle - \star 1\right] \\
& - 2\,\left(\star d\psi - 2\,e^{-U}\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\alpha}Z(\mathcal{G}))\right)\wedge\left(d\psi - 2\,e^{-U}\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\alpha}Z(\star\mathcal{G}))\right) \\
& + 4e^{-U}e^{2\psi}\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\alpha}Z(G))\,d\left(\mathrm{e}^{-2\psi}\star\eta\right)\Big]\,. 
\end{aligned} \tag{B.13}
$$

Note that we added and subtracted a term in order to obtain the squaring of the third line, which leads to the additional factor $\mathrm{e}^{-2\psi}$ in the derivative of the last line.

The last form of the action is a sum of squares, except for the terms involving the derivative of $\eta$ and $\langle\mathcal{F},\star\mathcal{G}\rangle$, which one should demand to be a total derivative, thus constraining the one-form $\eta$. However, since an analysis of the resulting equations of motion is outside the scope of this appendix, we restrict ourselves to the case of an identically flat potential, mentioning that the general equations have the same structure as the BPS equations of [83] and may reproduce them once $\eta$ is specified.

### B.2 Asymptotically flat solutions

Turning to the asymptotically flat case, we assume that the FI terms are given by a very small vector and we choose the one-form $\eta$ in (B.4) as

$$
\eta = \mathrm{e}^{\varphi}\,dr\,, \tag{B.14}
$$

where we absorbed the Lagrange multiplier $\varphi$ of the static squaring in section 2, allowing it to depend on all spatial coordinates. Similar to the static case, we impose that the base space is flat, so that $\mathrm{e}^{\psi} = r$, which leads to a modified rewriting of the action as

$$
\begin{aligned}
S_{4\mathrm{D}} = -\frac{1}{16\pi}\int dt \int_{\mathbb{R}^3} \Big[ & (\mathcal{E},\mathcal{E}) - 4\left(Q + d\alpha + 2e^{-U}\,\mathrm{Re}\,Z(\star\mathcal{G}) + \frac{1}{2}\mathrm{e}^{2U}\star d\omega\right)\wedge \mathrm{Im}\langle\mathcal{E}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}\rangle \\
& - 2\,\langle\mathcal{F} + 2\,\mathrm{Re}\,d(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}\,\omega), \star\mathcal{G}\rangle + 2\,du\wedge\star du \\
& - 2\,\left(2\,e^{-U}\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\alpha}Z(\mathcal{G})) - \star du\right)\wedge\left(2\,e^{-U}\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\alpha}Z(\star\mathcal{G})) - du\right)\Big]\,. 
\end{aligned} \tag{B.15}
$$

where we discarded a non-dynamical term and $u$ is the scalar defined in (2.19). The equations of motion following from this action are solved by the relations

$$
\mathcal{E} = \tilde{\mathcal{F}} - \mathcal{W} = 0\,, \tag{B.16}
$$

$$
Q + d\alpha + 2e^{-U}\,\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\alpha}Z(\star\mathcal{G})) + \frac{1}{2}\mathrm{e}^{2U}\star d\omega = 0\,, \tag{B.17}
$$

$$
2\,e^{-U}\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\alpha}Z(\star\mathcal{G})) - du = 0\,, \tag{B.18}
$$

along with the equation of motion for the Lagrange multiplier, which reads

$$
d\,(\star du) - dr\wedge\langle G, \mathcal{F} + 2\,\mathrm{Re}\,d(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}\,\omega)\rangle\, r^{-2}\,\mathrm{e}^{-2u} = 0\,. \tag{B.19}
$$

Despite the apparent complication of the equations above, one can show that the rotating black holes of [20] are solutions to the equations above, in the following way. Firstly, we introduce the decomposition of the spatial field strengths in electromagnetic potentials and vector fields as

$$\mathcal{F} = d(\zeta\,\omega) + dw\,, \tag{B.20}$$

where the explicit expression for $\zeta$ follows from (B.5), (B.7) and (B.16), as

$$d\zeta = -2\,\mathrm{Re}\, d(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}) + \star\mathcal{G}\,, \tag{B.21}$$

whose integrability condition implies through (B.4) and (B.14) that $\eta$ is exact and thus $\mathrm{e}^{\varphi}$ is a total derivative with respect to the radial component. Considering a single center solution, the vector fields $dw$ define the charges $\Gamma$ through harmonic functions $\mathcal{H}$, so that the equation of motion for the Lagrange multiplier takes the form

$$d\,(\star du) - r^{-2}\,dr \wedge \langle G, \Gamma\rangle\, \mathrm{e}^{-2u} \star 1 = 0\,, \tag{B.22}$$

and thus admits the same enveloping solution (2.25), which we adopt henceforth. Note that this is indeed such that $\mathrm{e}^{\varphi}$ is a total derivative as

$$\mathrm{e}^{\varphi} = \partial_r \left(\frac{1}{V}\right)\,. \tag{B.23}$$

We are then in a position to write the linear system to be solved in the asymptotically flat case, explicitly given by

$$\zeta = -2\,\mathrm{Re}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}) + d\left(\frac{1}{V}\right)\,G\,, \tag{B.24}$$

$$dw - \mathrm{e}^{-2U}\,\hat{\mathcal{G}} + 4\,e^{-2U}\,\mathrm{Re}\,Z(\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{G})\,\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\alpha}\mathcal{V}\right) = 2\,\mathrm{Im}\star D(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}) - \frac{1}{V}\,d\omega\,G\,. \tag{B.25}$$

This can be simplified using the definition (2.29)–(2.30) for the second very small vector, and the associated decomposition

$$\mathrm{J}G = -\frac{1}{2}\,\mathrm{e}^{2U}\,V^2\,R + M\,\mathrm{e}^{2U}\,G + 4\,M\,V\,\mathrm{e}^{3U}\,\mathrm{Re}\Big(\mathrm{e}^{-i\alpha}\,\mathcal{V}\Big)\,, \tag{B.26}$$

that follows from it, as one can compute directly. Note that here we have upgraded the constant m in (2.30) to a function $M$, which will turn out to control the angular momentum. Multiplying the last relation with the Lagrange multiplier in (B.23) we find

$$\mathrm{e}^{-2U}\,\mathrm{J}\mathcal{G} = \frac{1}{2} \star dV\,R + M \star d\left(\frac{1}{V}\right) G + 4\,M\,V \star d\left(\frac{1}{V}\right)\,\mathrm{e}^{U}\,\mathrm{Re}\Big(\mathrm{e}^{-i\alpha}\,\mathcal{V}\Big)\,, \tag{B.27}$$

which can in turn be used in (B.25) to obtain

$$\begin{aligned} dw - \frac{1}{2} \star dV\,R - M \star d\left(\frac{1}{V}\right) G + \frac{1}{V}\,d\omega\,G - 2\,\mathrm{Im}\star d(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}) \\ = -4\left[2\,e^{-2U}\,\mathrm{Re}\,Z(\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{G}) - M\,V \star d\left(\frac{1}{V}\right)\mathrm{e}^{U}\right]\mathrm{Re}\Big(\mathrm{e}^{-i\alpha}\,\mathcal{V}\Big)\,. \end{aligned} \tag{B.28}$$

Here and henceforth we assume that the very small vector $R$ is a constant, which will be shown to be a consistent choice at the end. Imposing that the terms proportional to the real part of the section in (B.28) cancel, leads to the constraint

$$2\, e^{-2U}\ \mathrm{Re}\, Z(\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{G}) = M\, V \star d\left(\frac{1}{V}\right)\, \mathrm{e}^{U}\,, \tag{B.29}$$

which together with the additional condition

$$\star\, d\omega = -dM\,, \tag{B.30}$$

that implies that both $M$ and $\omega$ are harmonic, results to the system of equations

$$dw - \frac{1}{2} \star dV\, R - \star d\left(\frac{M}{V}\right)\, G = 2\,\mathrm{Im} \star d(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V})\,. \tag{B.31}$$

Integrating the last equation leads to a generalisation of (2.31), given by

$$2\,\mathrm{Im}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}) = \mathcal{H} - \frac{1}{2}\, V\, R - \frac{M}{V}\, G\,, \tag{B.32}$$

where the vector fields are given by the harmonic functions $\mathcal{H}$ as

$$dw = \star d\mathcal{H}\,. \tag{B.33}$$

One can now compare the above to the explicit equations for the general asymptotically flat under-rotating single center solutions of [20], which turn out to be described by (B.29)–(B.31) with $\omega \to -\omega$ and for $R$ being a constant very small vector, as in the static case. Indeed, one could have started from the system (B.25) to establish that the scalar flow equations are the ones of [20] up to a constraint generalising (2.27), which is again equivalent to the constancy of the vector $R$, as we did in section 2. However, we chose to present the symplectic covariant derivation of the equations both for simplicity and completeness. The analysis of [20] ensures that this is a consistent solution of the full Einstein equations, so that we do not have to consider the Hamiltonian constraint that has to be imposed on solutions to the effective action in (B.15), as in the static case (see [84] for details on this constraint).

## C Example STU solution

**Full solution.** In this appendix we present the known rotating seed solution in a specific duality frame [14], as an example to use in the main body. For comparison, we use the STU model, in (4.11) in what follows. The charges of the solution are given by poles in the following choice of harmonic functions

$$\mathcal{H} = \left(H^0,\, 0\,;0,\, H_i\right)^T\,, \tag{C.1}$$

$$H_i = \mathrm{h}_i + \frac{q_i}{r}\,, \qquad\qquad H^0 = \mathrm{h}^0 + \frac{p^0}{r}\,. \tag{C.2}$$

In this duality frame the constant small vectors can be chosen as

$$\hat{R} = (-4,\, 0\,;\, 0,\, 0)^T \,, \qquad G = (0,\, 0\,;\, 1,\, 0)^T \,, \tag{C.3}$$

but we point out that the choice is not unique, see [20] for a detailed discussion. The scalar fields are given by solving (B.32), leading to the physical scalars

$$t^i = \frac{M - ie^{-2U}}{2\, H^0\, H_i} \,, \tag{C.4}$$

as well as to the real part of the section

$$2\, \mathrm{e}^{-U}\, \mathrm{Re}(\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}) = \mathrm{e}^{2U} \left( M\, H^0,\, -H^0\, |\varepsilon^{ijk}| H_j H_k\,;\, -\frac{M^2}{H^0} - 2H_1H_2H_3,\, M\, H_i \right)^T \,. \tag{C.5}$$

The metric is given by (B.1) with

$$e^{-4U} = 4\, H^0 H_1 H_2 H_3 - M^2 \,, \qquad \star d\omega = -dM \,, \tag{C.6}$$

where $M$ is a dipole harmonic function

$$M = \mathrm{m} + \frac{\mathrm{j}\, \cos\theta}{r^2} \,. \tag{C.7}$$

Finally, the gauge fields are given by (B.20), with the $\zeta$ given by (B.24) and (C.5), while the $dw$ are given by (B.33).

**Near horizon solution.** We now take the near horizon of the solution above, which is obtained by dropping the constants in all harmonic functions. The scalars (C.4) become

$$t^i = \frac{\mathrm{j}\, \cos\theta - ie^{-2U}}{2\, p^0\, q_i} \,, \tag{C.8}$$

whereas the near horizon metric still given by

$$\begin{aligned} ds^2 &= -e^{2U}\, r^2\, (dt + \omega)^2 + e^{-2U} \left( \frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta d\phi^2 \right) \,, \\ \mathrm{e}^{-4U} &= 4\, p^0\, q_1 q_2 q_3 - \mathrm{j}^2 \cos^2\theta \qquad \omega = \mathrm{j}\, \frac{\sin^2\theta}{r}\, d\phi \,. \end{aligned} \tag{C.9}$$

For convenience we give the near horizon gauge fields, which are given by (B.20) using (B.24) and (B.33) as

$$\mathcal{F} = d\big[\zeta\, r\, (dt + \omega)\big] + \Gamma\, \sin\theta\, d\theta \wedge d\phi \,, \qquad \zeta = -2\, e^U\, \mathrm{Re}[e^{-i\alpha}\mathcal{V}] + \frac{1}{q_0}\, G \,, \tag{C.10}$$

where the real part of the section follows from (C.5) by replacing harmonic functions by their poles, as above.

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## Vorticity in holographic fluids

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# Vorticity in holographic fluids

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In view of the recent interest in reproducing holographically various properties of conformal fluids, we review the issue of vorticity in the context of AdS/CFT. Three-dimensional fluids with vorticity require four-dimensional bulk geometries with either angular momentum or nut charge, whose boundary geometries fall into the Papapetrou–Randers class. The boundary fluids emerge in stationary non-dissipative kinematic configurations, which can be cyclonic or vortex flows, evolving in compact or non-compact supports. A rich network of Einstein's solutions arises, naturally connected with three-dimensional Bianchi spaces. We use Fefferman–Graham expansion to handle holographic data from the bulk and discuss the alternative for reversing the process and reconstruct the exact bulk geometries.

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## 1. Introduction

The relationship of fluid dynamics with general relativity goes back to the work of Damour [1]. Lately, a different and perhaps more concrete shape of this relationship has been given by the so-called fluid/gravity correspondence (see for example the recent review [2]). According to the latter, the gravitational degrees of freedom that reside in the boundary of an asymptotically $\mathrm{AdS}_D$ spacetime describe the hydrodynamics of a relativistic fluid in $D-1$ dimensions. Consequently, the dynamical equations of the latter systems (e.g. Euler or Navier–Stokes) are encoded in the asymptotic behaviour of the bulk Einstein equations.

The fluid/gravity correspondence framework appears to be capable of describing different facets of relativistic fluids, such as superfluidity and dissipation. This appears to be a novel mechanism of emergence in physics whereby the low-energy effective degrees of freedom arise holographically in the boundary of a gravitational system. At a practical level, fluid/gravity correspondence currently occupies a large part of the AdS/CMT correspondence as are collectively called the efforts to find new computational tools for strongly coupled condensed matter systems using holography (see e.g. the reviews [3, 4, 5]).

Despite some important results in the study of holographic fluids, the issue of vorticity has been less well understood. This is important if one wants to extend the realm of AdS/CMT to interesting condensed matter systems such as rotating Bose or Fermi gases [6, 7], turbulence or wave propagation in moving metamaterials (e.g. [8]). With such extensions in mind, we review here our recent attempt to setup a holographic framework for the description of fluids with vorticity. Even without touching the thorny question of dissipation, *i.e.* assuming local equilibrium and non-dissipating kinematics, our studies reveal a remarkably rich structure as soon as vorticity is switched on. In particular, we note the intimate relationship of our neutral rotating holographic fluids with charged fluids in magnetic fields, as well as with the problem of wave propagation in moving media. We believe that the latter observation can lead to the holographic description of analogue gravity systems [9, 10, 11, 12].

In the present review we choose to devote most of our discussion to the gravitational side of the duality and we extensively discuss in Sec. 2 relativistic fluids, in Sec. 3 the general stationary Papapetrou–Randers and Zermelo geometries and in Sec. 4 various concrete examples of stationary geometries in $2+1$ dimensions. In Sec. 5 we review the Fefferman–Graham construction of holographic fluids in $2+1$ dimensions focusing on the Kerr–$\mathrm{AdS}_4$, the Taub–NUT–$\mathrm{AdS}_4$ and the hyperbolic NUT–$\mathrm{AdS}_4$ solutions. In Sec. 6 we give an overview of some new results, to be presented elsewhere [13], that aim to connect our approach with alternative descriptions of holographic fluids. Section 7 contains our conclusions.

## 2. Relativistic fluids

In this section, we recall the salient features of relativistic fluid dynamics (see e.g. [14, 15]). This includes aspects of vector-field congruences and properties of the energy–momentum tensor. We work in arbitrary spacetime dimension $D$.



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### 2.1 Vector-field congruences

We consider a manifold endowed with a spacetime metric of the generic form

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{ab}\hat{e}^a\hat{e}^b. \tag{2.1}$$

We will use $a,b,c,\ldots = 0,1,\ldots,D-1$ for transverse Lorentz indices along with $\alpha,\beta,\gamma = 1,\ldots,D-1$. Coordinate indices will be denoted $\mu,\nu,\rho,\ldots$ for spacetime $\mathrm{x}\equiv(t,x)$ and $i,j,k,\ldots$ for spatial $x$ directions. The dual of the orthonormal coframe $\hat{e}^a$ is the frame $\check{e}_a$, which satisfies $\hat{e}^a(\check{e}_b) = \delta^a_b$. To define parallel transport we take the Levi–Civita connection coefficients $\Gamma^a_{bc}$ defined via the spin-connection one-form $\hat{\omega}^a{}_b$ as

$$\mathrm{d}\hat{e}^a + \hat{\omega}^a{}_b \wedge \hat{e}^b = 0, \quad \hat{\omega}^a{}_b = \Gamma^a_{bc}\hat{e}^c, \quad \nabla_{\check{e}_a}\check{e}_b = \Gamma^c_{ab}\check{e}_c. \tag{2.2}$$

Consider now an arbitrary timelike vector field $\hat{u} = u_a\hat{e}^a$, normalized as $\eta_{ab}u^a u^b = -1$, later identified with the fluid velocity. Its integral curves define a congruence which is characterized by its acceleration, shear, expansion and vorticity:

$$\nabla_a u_b = -u_a a_b + \frac{1}{D-1}\Theta h_{ab} + \sigma_{ab} + \omega_{ab} \tag{2.3}$$

with[1]

$$\begin{aligned}
a_a &= u^b\nabla_b u_a, \quad \Theta = \nabla_a u^a, && (2.4)\\
\sigma_{ab} &= \frac{1}{2}h_a{}^c h_b{}^d\left(\nabla_c u_d + \nabla_d u_c\right) - \frac{1}{D-1}h_{ab}h^{cd}\nabla_c u_d && (2.5)\\
&= \nabla_{(a}u_{b)} + a_{(a}u_{b)} - \frac{1}{D-1}h_{ab}\nabla_c u^c, && (2.6)\\
\omega_{ab} &= \frac{1}{2}h_a{}^c h_b{}^d\left(\nabla_c u_d - \nabla_d u_c\right) = \nabla_{[a}u_{b]} + a_{[a}u_{b]}. && (2.7)
\end{aligned}$$

The latter allows to define the vorticity form as

$$2\omega = \omega_{ab}\,\hat{e}^a\wedge\hat{e}^b = \mathrm{d}\hat{u} + \hat{u}\wedge\hat{a}\,. \tag{2.8}$$

These tensors satisfy several simple identities:

$$u^a a_a = 0, \quad u^a\sigma_{ab} = 0, \quad u^a\omega_{ab} = 0, \quad u^a\nabla_b u_a = 0, \quad h^c{}_a\nabla_b u_c = \nabla_b u_a. \tag{2.9}$$

Killing vector fields, satisfying $\nabla_{(a}\xi_{b)} = 0$, are congruences with remarkable properties. We quote two of them, the proof of which is straightforward:

- A Killing vector field has vanishing expansion.
- A constant-norm[2] Killing vector field is furthermore geodesic and shearless. It can only carry vorticity.

[1] Our conventions are: $A_{(ab)} = 1/2\,(A_{ab} + A_{ba})$ and $A_{[ab]} = 1/2\,(A_{ab} - A_{ba})$.

[2] This is not an empty statement since Killing vectors cannot be normalized at will. When their norm is constant, it can be consistently set to $-1, 0$ or $+1$.

The timelike vector field $\check{u}$ can be used to decompose any tensor field on the manifold in transverse and longitudinal components with respect to itself. The decomposition is performed by introducing the longitudinal and transverse projectors:

$$U^a{}_b = -u^a u_b, \quad h^a{}_b = u^a u_b + \delta^a_b, \tag{2.10}$$

where $h_{ab}$ is also the induced metric on the surface orthogonal to $\check{u}$. The projectors satisfy the usual identities:

$$U^a{}_c U^c{}_b = U^a{}_b, \quad U^a{}_c h^c{}_b = 0, \quad h^a{}_c h^c{}_b = h^a{}_b, \quad U^a{}_a = 1, \quad h^a{}_a = D-1. \tag{2.11}$$

For example, any rank-two symmetric tensor $T_{ab}$ can be decomposed in longitudinal, transverse and mixed components:

$$T_{ab} = e u_a u_b + S_{ab} - u_a q_b - u_b q_a, \tag{2.12}$$

the non-longitudinal part being

$$\Sigma_{ab} = S_{ab} - u_a q_b - u_b q_a. \tag{2.13}$$

We have defined

$$e = u^a u^b T_{ab}, \quad S_{ab} = h_a{}^c h_b{}^d T_{cd}, \quad q_a = h_a{}^b T_{bc} u^c, \tag{2.14}$$

such that

$$u^a q_a = 0, \quad u^a S_{ab} = 0. \tag{2.15}$$

Finally

$$u^a T_{ab} = q_b - e u_b. \tag{2.16}$$

### 2.2 The energy–momentum tensor

The mere existence of a metric and a timelike vector-field congruence does not necessarily imply the presence of a relativistic fluid. If we wish to identify the timelike vector $\check{u}$ with the velocity of a relativistic fluid, then we should require the presence of an additional symmetric rank-two tensor field – the energy–momentum tensor $T_{ab}$ whose projection along $\check{u}$ is the (positive) energy density $\varepsilon$ of the fluid

$$T_{ab} u^a u^b = \varepsilon\,, \tag{2.17}$$

as measured in the local proper frame. The latter concept deserves a comment. In non-relativistic fluids, the velocity field is unambiguously defined as the velocity of the mass flow of the fluid. In the relativistic case, it requires a more formal definition as energy and mass cannot be distinguished, and energy flows can be the result of dissipative phenomena or thermal conduction. One way to define the velocity, which amounts to defining a specific local proper frame known as *Landau frame*, is to demand the absence of mixed terms in (2.13):

$$u^a \Sigma_{ab} = 0. \tag{2.18}$$

Let us continue applying the decomposition (2.12) to the energy–momentum tensor. Inserting (2.13) in (2.18), Eqs. (2.15) imply that $q^a$ vanishes in the Landau frame, where the energy–momentum is thus

$$T_{ab} = \varepsilon u_a u_b + S_{ab}. \tag{2.19}$$

The last piece $S_{ab}$ is the stress tensor, purely transverse.

For a perfect fluid, all information is encapsulated in a further unique piece of data: the pressure $p$ measured in the local proper frame. Hence, the stress tensor reads:

$$S^{\text{perf.}}_{ab} = p h_{ab}. \tag{2.20}$$

For a viscous fluid, the stress tensor contains friction terms:

$$S_{ab} = p h_{ab} + t_{ab}, \tag{2.21}$$

where $t_{ab}$ is usually expressed as an expansion in the derivatives of the velocity field. At lowest order one finds

$$t_{ab} = -2\eta\sigma_{ab} - \zeta h_{ab}\Theta \tag{2.22}$$

with $\eta, \zeta$ the shear and bulk viscosities. In $2+1$ dimensions there exists another term at this order, breaking the parity symmetry: $\zeta_{\mathrm{H}}\varepsilon_{cd(a}u^c\sigma^d{}_{b)}$. The coefficient $\zeta_{\mathrm{H}}$ is the *rotational Hall viscosity*. It characterizes a transport phenomenon similar to the Hall conductivity of charged fluids in magnetic fields.

The dynamical equations for the fluid (Euler, Navier–Stokes, ...) are all encoded in the covariant conservation of the energy–momentum tensor

$$\nabla^a T_{ab} = 0. \tag{2.23}$$

In the non-relativistic limit, Eq. (2.23) also delivers a matter-current conservation, which, for relativistic fluids, must be introduced separately as a consequence of charge conservation, if any.

### 2.3 Effectively perfect fluids

Relativistic fluids in the hydrodynamic regime are long wavelength approximations of finite-chemical-potential and finite-temperature states of certain (unknown) quantum field theories[3]. Quite generically all such fluids exhibit dissipative phenomena as they describe media with non-zero shear viscosity. However, *all* such fluids can be in special kinematic configurations where the effects of dissipation are ignorable[4]. In this case, their dynamics is captured by the perfect part of the stress tensor and the equations of motion read[5]:

$$\begin{cases} (\varepsilon+p)\Theta + \nabla_{\breve{u}}\varepsilon = 0 \\ (\varepsilon+p)\hat{a} - \nabla_{\perp} p = 0 \end{cases} \tag{2.24}$$

[3]A special class of such fluids, actually the one that naturally arises in holography, are conformal fluids, *i.e.* those having vanishing energy–momentum trace: $(D-1)p - \varepsilon = (D-1)\zeta\Theta$. This equation is supposed to hold for any kinematic configuration, in particular when the fluid is at rest, where $\varepsilon = (D-1)p$. The latter is therefore adopted as a thermodynamic equation of state valid *always* locally. When the fluid is not at rest, we conclude then that $\zeta\Theta = 0$, which must hold for any $\Theta$. In this scheme, the bulk viscosity for a conformal fluid is thus vanishing identically. Similar conclusions are reached for higher-order viscosity coefficients entering the traceful part of the energy–momentum tensor.

[4]A fluid can be stationary and altogether dissipate energy provided it is not isolated. These situations are better designated as *forced steady states*. On curved boundary backgrounds, the forcing task can be met by gravity through the boundary conditions. This was discussed in [16]. As this feature does not appear in the backgrounds that will be analyzed in the forthcoming sections, we will not pursue this further.

[5]The interested reader can find more information on these specific issues in e.g. [17, 18].



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($\nabla_\perp = \nabla + \hat{u}\nabla_{\breve{u}}$ stands for the covariant derivative along the directions normal to the velocity field). Under this assumption, taking also into account the conformality ($\varepsilon = (D-1)p \propto T^D$), Eqs. (2.24) lead to

$$\begin{cases} \nabla_{\breve{u}}\varepsilon = 0 \\ \hat{a} = \frac{\nabla_\perp p}{Dp}. \end{cases} \tag{2.25}$$

The energy density is conserved along the fluid lines and in the absence of spatial pressure gradients (*i.e.* for energy and pressure constants in spacetime), the flow is geodesic.

In several instances, the velocity of the fluid turns out to be a Killing vector field. Then, from the discussion of Sec. 2.1 several straightforward conclusions can be drawn:

- The flow is geodesic, shearless and expansionless.
- The internal energy density is conserved and the pressure is spatially homogeneous.
- If the fluid is conformal then $\varepsilon = (D-1)p \propto T^D$ is constant in spacetime.

Therefore, despite its viscosity, the kinematic state of the fluid can be steady and non-dissipative. For this to happen, however, the existence of a constant-norm timelike Killing vector is required. In other words, the background geometry must itself be stationary[6]. In this case, the constant-norm timelike Killing vector congruence allows for the definition of a global time coordinate, with associated inertial frames. The latter are comoving with the fluid. All the examples that we will discuss in the following fall into this class.

## 3. Papapetrou–Randers stationary geometries

Starting with appropriate time-independent bulk backgrounds[7], conformal fluids appear holographically, evolving generally on stationary but not necessarily static boundary geometries. Those fluids possess therefore non-dissipative dynamics inherited from the gravitational environment and this dynamics contains in general vorticity. We will present here some basic properties of the boundary backgrounds arising in this context and explain how they affect the fluid dynamics. We postpone to Sec. 5 the actual holographic analysis relating some of these backgrounds to exact bulk Einstein spacetimes.

### 3.1 General properties, geodesic congruences and Papapetrou–Randers frame

Stationary metrics appearing in the holographic analysis we will be presenting later on are of the generic form

$$\mathrm{d}s^2 = B^2\left(-(\mathrm{d}t - b_i \mathrm{d}x^i)^2 + a_{ij}(x)\mathrm{d}x^i \mathrm{d}x^j\right), \tag{3.1}$$

where $B, b_i, a_{ij}$ are $x$-dependent functions. These metrics were introduced by Papapetrou in [19]. They will be called hereafter *Papapetrou–Randers* because they are part of an interesting network

[6]More generally, it can be shown that the velocity field $u^\mu$ of a stationary fluid flow has to be proportional to a Killing vector field of the background geometry [17].

[7]Notice that in some instances time independence is not met in the bulk, but stationarity remains valid on the boundary as a consequence of appropriate boundary conditions [16]. These cases belong to the class of forced dissipative steady-states mentioned in footnote 4.

of relationships involving the Randers form [20], discussed in detail in [21] and more recently used in [22, 23].

For later convenience, we introduce $a^{ij}, b^i$ and $\gamma$ such that

$$a^{ij}a_{jk} = \delta^i_k, \quad b^i = a^{ij}b_j, \quad \gamma^2 = \frac{1}{1-a^{ij}b_ib_j}. \tag{3.2}$$

The metric components read:

$$g_{00} = -B^2, \quad g_{0i} = B^2 b_i, \quad g_{ij} = B^2(a_{ij} - b_ib_j), \tag{3.3}$$

and those of the inverse metric:

$$g^{00} = -\frac{1}{\gamma^2 B^2}, \quad g^{0i} = \frac{b^i}{B^2}, \quad g^{ij} = \frac{a^{ij}}{B^2}. \tag{3.4}$$

Finally,

$$\sqrt{-g} = B^D\sqrt{a}, \tag{3.5}$$

where $a$ is the determinant of the symmetric matrix with entries $a_{ij}$.

In the natural frame of the above coordinate system $\{\partial_t, \partial_i\}$, any observer at rest has normalized velocity $\check{u} = \frac{1}{B}\partial_t$ and dual form $\hat{u} = -B(\mathrm{d}t - b)$ (we set $b = b_i\mathrm{d}x^i$). The normalized vector field $\check{u}$ is not in general Killing – as opposed to $\partial_t$. For this observer, the acceleration is thus non vanishing:

$$\check{a} = \nabla_{\check{u}}\check{u} = g^{ij}\partial_i \ln B\,(\partial_j + b_j\partial_t)\,. \tag{3.6}$$

As already mentioned, the motion is inertial if and only if $B$ is constant. It will be enough for our purposes to consider the case $B = 1$, and all subsequent formulas will assume this choice. We will furthermore introduce a frame

$$\check{e}_0 = \partial_t, \quad \check{e}_\alpha = E_\alpha{}^i\,(b_i\partial_t + \partial_i)\,, \quad E_\alpha{}^i E^\beta{}_i = \delta^\beta_\alpha \tag{3.7}$$

adapted to the geodesics at hand and its dual coframe (orthonormal as in Eq. (2.1))

$$\hat{e}^0 = \mathrm{d}t - b, \quad \hat{e}^\alpha = E^\alpha{}_i\mathrm{d}x^i, \quad E^\alpha{}_i E^\beta{}_i \delta_{\alpha\beta} = a_{ij}. \tag{3.8}$$

This will be referred to as the Papapetrou–Randers frame.

The constant-norm Killing vector field $\check{u} = \check{e}_0 = \partial_t$ (with $\hat{u} = -\hat{e}^0 = -\mathrm{d}t + b$) defines a geodesic congruence (the orbits of all observers at rest in the Papapetrou–Randers frame). As was shown in Sec. 2.3, the latter has *zero shear and expansion*, but non-trivial vorticity (see Eqs. (2.7), (2.8)):

$$\omega = \frac{1}{2}\mathrm{d}b \quad \Rightarrow \quad \omega_{0i} = 0, \quad \omega_{ij} = \frac{1}{2}\left(\partial_i b_j - \partial_j b_i\right). \tag{3.9}$$

The physical effect of vorticity is seen in the obstruction to the parallel transport of the spatial frame $\check{e}_\alpha$ along the congruence:

$$\nabla_{\check{e}_0}\check{e}_\alpha = \omega^{\mathrm{PR}}_{\alpha\beta}\delta^{\beta\gamma}\check{e}_\gamma \quad \Leftrightarrow \quad \nabla_{\partial_t}\partial_i = \omega_{ij}a^{jk}\left(\partial_k + b_k\partial_t\right) \tag{3.10}$$

($\omega_{ij}$ given in (3.9) are the spacetime components of the vorticity, while $\omega^{\mathrm{PR}}_{\alpha\beta} = E_\alpha{}^i E_\beta{}^j\omega_{ij}$ are its components in the Papapetrou–Randers frame). Embarked gyroscopes undergo a rotation.

Papapetrou–Randers metrics do not exhibit ergoregions since[8] $g_{00} = -1$. However, regions where hyperbolicity is broken (*i.e.* where constant-$t$ surfaces become timelike) are not excluded. This happens whenever there exist regions where $b_i b_j a^{ij} > 1$. Indeed, in these regions, the spatial metric $a_{ij} - b_i b_j$ possesses a negative eigenvalue, and constant-$t$ surfaces are no longer spacelike. Therefore the extension of the physical domain accessible to the inertial observers moving along $\check{u} = \partial_t$ is limited to spacelike disks in which $b_i b_j a^{ij} < 1$ holds. We will come back to this important issue in Secs. 3.3 and 5.2.

Before moving to the next topic, we would like to make a last remark. Following Eqs. (2.25), the shearless and expansionless geodesic congruence under consideration could describe the fluid lines of a dissipationless stationary, conformal fluid, under the assumption that energy (and pressure) be conserved and constant all over space. As we will see in Sec. 5, this is exactly the dynamics that emerges through holography.

### 3.2 Zermelo frame

Trading the data $(a_{ij}, b_i)$ for $(h_{ij}, W^i)$ defined as

$$h_{ij} = \tfrac{a_{ij} - b_i b_j}{\gamma^2}, \quad h^{ik} h_{kj} = \delta^i_j, \tag{3.11}$$

$$W^i = -\gamma^2 b^i, \quad W_i = h_{ij} W^j = -\tfrac{b_i}{\gamma^2}, \tag{3.12}$$

the Papapetrou–Randers metric (3.1) can be recast in the following form

$$\mathrm{d}s^2 = \gamma^2 \left[ -\mathrm{d}t^2 + h_{ij} \left( \mathrm{d}x^i - W^i \mathrm{d}t \right) \left( \mathrm{d}x^j - W^j \mathrm{d}t \right) \right]. \tag{3.13}$$

The latter is called Zermelo metric because it first appeared in the framework of the Zermelo problem [24], yet another member of the relationship network mentioned above[9].

The Zermelo form of the metric suggests the following orthonormal coframe and its dual frame:

$$\hat{z}^0 = \gamma \mathrm{d}t\,, \quad \hat{z}^\alpha = L^\alpha{}_i (\mathrm{d}x^i - W^i \mathrm{d}t), \quad L_\alpha{}^i L^\beta{}_i = \delta^\alpha_\beta, \tag{3.14}$$

$$\check{z}_0 = \tfrac{1}{\gamma} \left( \partial_t + W^i \partial_i \right), \quad \check{z}_\alpha = L_\alpha{}^i \partial_i, \quad L^\alpha{}_i L^\beta{}_j \delta_{\alpha\beta} = \gamma^2 h_{ij}. \tag{3.15}$$

We will call the latter the *Zermelo frame*. Its timelike vector field $\check{z}_0$ defines a congruence of accelerated lines ($\nabla_{\check{z}_0} \check{z}_0 \neq 0$). Thus, this frame in not inertial. It is instructive to compare the Papapetrou–Randers frame introduced previously (in (3.7), (3.8)) with the Zermelo frame at hand. Being both orthonormal, they are related by a local Lorentz transformation, as one sees by combining the above formulas:

$$\check{e}_0 = \gamma \left( \check{z}_0 - W^\beta \check{z}_\beta \right), \tag{3.16}$$

$$\check{e}_\alpha = \Gamma_\alpha{}^\beta \left( \check{z}_\beta - W_\beta \check{z}_0 + \frac{\gamma^2 - 1}{\gamma^2} \left( \frac{W_\beta W^\gamma}{W^2} - \delta^\gamma_\beta \right) \check{z}_\gamma \right), \tag{3.17}$$

[8] Ergoregions would require a conformal factor in (3.1) that could vanish and become negative.

[9] The Zermelo problem is formulated as follows: find the minimal-time navigation road on a geometry $\mathrm{d}\ell^2 = h_{ij} \mathrm{d}x^i \mathrm{d}x^j$, in the presence of a moving fluid creating a drift current (wind or tide) $W = W^i \partial_i$ with a ship of fixed propelling velocity (*i.e.* fixed with respect to the frame comoving with the fluid or, put differently, of given power). The answer is reached by searching for null geodesics of (3.13).

where

$$\Gamma_\alpha{}^\beta = \gamma^2 E_\alpha{}^i L^\beta{}_i, \quad W^\alpha = \tfrac{1}{\gamma} L^\alpha{}_i W^i, \quad W_\alpha = \delta_{\alpha\beta} W^\beta, \tag{3.18}$$

$$W^2 = W^\alpha W_\alpha = W^i W_i = 1 - \tfrac{1}{\gamma^2}. \tag{3.19}$$

The interpretation of these expressions is clear. Each spacetime point is the intersection of two lines, belonging each to the two congruences under consideration. At this point $W^\alpha$ are the spatial velocity components of the inertial observer in the spatial frame of the accelerated observer and $1/\gamma^2 = 1 - W^2$ the corresponding Lorentz factor.

It is worth making several further comments. The synchronous hypersurface for the inertial observer is by definition dual to the time vector. Since $\mathrm{d}t(\partial_i) = 0$ (equivalent to $\hat{z}^0(\check{z}_\alpha) = 0$), this hypersurface is spanned by $\{\partial_i\}$, and hence is not orthogonal to the inertial congruence ($\partial_t \cdot \partial_i = b_i$). The orthogonal lines to the Papapetrou–Randers (inertial) observer's synchronous hypersurface are nothing but the accelerated congruence tangent to the vector field $\check{z}_0$ (defining the corresponding Zermelo frame) because $\check{z}_0 \cdot \partial_i = 0$, whereas the hyperplanes orthogonal to the the inertial congruence (tangent to $\check{e}_0 = \partial_t$) are spanned by $\{b_i \partial_t + \partial_i\}$. Therefore, since $(\mathrm{d}t - b)(\partial_i + b_i \partial_t) = 0$ (equivalent to $\hat{e}^0(\check{e}_\alpha) = 0$), the time $\tau$ of Zermelo observers, *i.e.* the dual of the hypersurface everywhere tangent to the latter hyperplanes, would satisfy $\mathrm{d}\tau = \mathrm{d}t - b$. Such a time cannot be defined since $\mathrm{d}b = 2\omega \neq 0$. Put differently, no hypersurface exists tangent to the hyperplanes spanned by $\{b_i \partial_t + \partial_i\}$ – Fröbenius theorem. This is a well known manifestation of vorticity.

The last statement again distinguishes the Papapetrou–Randers and Zermelo observers, which are otherwise dual to each other. As for the perception of the rotation, Papapetrou–Randers observers feel it through embarked gyroscopes (see (3.10)), whereas their inertial motion as witnessed by Zermelo observers satisfies

$$\nabla_{\check{z}_0} \check{u} = \omega^{\mathrm{Z}}_{0\alpha} \delta^{\alpha\beta} \check{z}_\beta. \tag{3.20}$$

Here $\check{u} = \check{e}_0$ is the velocity of the inertial observers, while $\omega^{\mathrm{Z}}_{ab}$ are the vorticity components as observed in the Zermelo frame: $\omega^{\mathrm{Z}}_{\alpha\beta} = L_\alpha{}^i L_\beta{}^j \omega_{ij}$ and $\omega^{\mathrm{Z}}_{0\beta} = W^\alpha \omega^{\mathrm{Z}}_{\alpha\beta}$. Hence, for the accelerated observers, the inertial ones are subject to a Coriolis force: Zermelo observers are rotating themselves. The velocity vector $\check{u} = \check{e}_0$ of the inertial observers undergoes a precession around the worldline of a Zermelo observer tangent to $\check{z}_0$. The latter being accelerated, the variation of $\check{u}$ is actually better captured as a Fermi derivative along $\check{z}_0$:

$$\mathrm{D}_{\check{z}_0} \check{u} = \left(\omega^{\mathrm{Z}}_{0\alpha} - \check{z}_\alpha(\gamma)\right) \delta^{\alpha\beta} \check{z}_\beta + W^\alpha \check{z}_\alpha(\gamma) \check{z}_0, \tag{3.21}$$

where $\check{z}_\alpha(\gamma) = L_\alpha{}^i \partial_i \gamma$. The extra terms result from the rotation of the Zermelo frame and contribute to the observed precession of the velocity vector $\check{u}$.

One can try to tune rotating frames so as to make the perceived angular momentum of a given congruence disappear, *i.e.* make its Fermi derivative vanish with respect to the rotating frame. This leads to the so called zero angular momentum frames (ZAMO [25]). In general, Zermelo frames *are not* ZAMO frames for the Papapetrou–Randers congruences, as the Fermi derivative (3.21) is generically non-zero. It can however be zero under the necessary and sufficient condition

$$W^j \omega_{ji} = \gamma \partial_i \gamma, \tag{3.22}$$

since this implies that the combination $\omega^{\mathrm{Z}}_{0\alpha} - \check{z}_\alpha(\gamma)$ as well as the coefficient of $\check{z}_0$ vanish. Equation (3.22) carries intrinsic information about the background and can indeed be recast as

$$\mathscr{L}_{\check{z}_0}\hat{e}_0 = 0. \tag{3.23}$$

When fulfilled, the Zermelo observers coincide with the locally non-rotating (or ZAMO) frames [25]. Remarkably, this occurs for a particular case that will be discussed in our subsequent developments.

### 3.3 Further properties and analogue gravity interpretation

In the above analysis and particularly in the change of frame from Papapetrou–Randers to Zermelo, it has been implicitly assumed that $W^2 < 1$. The velocity of Papapetrou–Randers observers with respect to the Zermelo frame is however dictated by the geometry itself since $W^2 = b_i b^i$, and nothing *a priori* guarantees that $b_i b^i < 1$ everywhere. There are regions in $x$-space where indeed $b_i b^i > 1$ bounded by a hypersurface where $b_i b^i = 1$. The latter was called *velocity-of-light* hypersurface in [21] since this is the edge where the Papapetrou–Randers observer reaches the speed of light with respect to the Zermelo frame.

The problem raised here is a manifestation of the global hyperbolicity breakdown. Indeed, we have seen that in geometries of the Papapetrou–Randers form (3.1), constant-$t$ surfaces are not everywhere spacelike. The extension of the physical domain accessible to the inertial observers moving along $\check{u} = \partial_t$ is limited to spacelike disks in which $b^2 < 1$ holds, bounded by the velocity-of-light surface, where these observers become luminal.

The breaking of hyperbolicity is usually accompanied with the appearance of closed timelike curves (CTCs). These are ordinary spacelike circles, lying in constant-$t$ surfaces, which become timelike when these surfaces cease being spacelike, *i.e.* when $b^2 > 1$. The CTCs at hand differ in nature from those due to compact time (as in the $SL(2,\mathbb{R})$ group manifold), and cannot be removed by unwrapping time. They require an excision procedure for consistently removing, if possible, the $b^2 > 1$ domain, in order to keep a causally safe spacetime. This is comparable to what happens in the case of the three-dimensional Bañados–Teitelboim–Zanelli black hole [26] – although in the latter case the trouble is not due to hyperbolicity issues. We will come back to the CTCs when studying the anti-de Sitter Taub–NUT geometry.

Although the issue of hyperbolicity is intrinsic to our stationary geometries, moving from the form (3.1) to the form (3.13) may provide alternative or complementary views. In the Zermelo form (3.13) the trouble is basically encapsulated in the conformal factor. However, some problems such as the original Zermelo navigation problem referred to in footnote 9 are sensitive to the general conformal class[10] of (3.13), and the conformal factor $\gamma^2$ can be dropped or replaced. Doing so can leave us with a geometry potentially sensible everywhere. This instance appears precisely in analogue gravity systems.

Metrics of the form (3.13) are in fact known as acoustic or optical (see the original works [27, 28] or e.g. [11] for an up-to-date review). They are used for describing the propagation of

[10] In the Zermelo navigation problem we look for null geodesics. In that framework, going to regions where $\gamma^2 < 0$ means having a drift current faster than what the ship can overcome.

sound/light disturbances in relativistic or non-relativistic fluids moving with velocity $W^i$ in spatial geometry $h_{ij}$, and subject to appropriate thermodynamic/hydrodynamic assumptions. In this approach, the full metric (3.13) is an *analogue* metric and is *not* the actual metric of physical spacetime. Under this perspective, peculiarities such as CTCs, potentially present in the analogue geometry, have no real, physical existence. They are manifestations of other underlying physical properties such as supersonic/superluminal regimes in the flowing medium.

In order to be concrete, we would like to quote two examples. The first is the original one, where a fluid is flowing on a geometry $\mathrm{d}\ell^2 = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ with velocity field $W = W^i\partial_i$. Assuming the fluid is non-relativistic, isolated, with mass density $\rho$ and pressure $p$, in barotropic evolution (*i.e.* such that the enthalpy variations satisfy $\mathrm{d}h = {}^{\mathrm{d}p}/_{\rho}$) and with sound velocity $c_\mathrm{s} = 1/\sqrt{\partial\rho/\partial p}$, one finds that irrotational acoustic perturbations propagate along null geodesics of the metric

$$\mathrm{d}s^2 = \frac{\rho}{c_\mathrm{s}}\left(-c_\mathrm{s}^2\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i - W^i\mathrm{d}t\right)\left(\mathrm{d}x^j - W^j\mathrm{d}t\right)\right), \tag{3.24}$$

and satisfy the corresponding scalar field equation. The metric (3.24) is of the form (3.13). It is however analogue and not the actual spacetime geometry, which is Galilean. This analogue metric is called acoustic in the case at hand. Similar results are available for light propagation, leading to optical geometries (see e.g. [12, 29]).

Similarly we can quote the case of a relativistic conformal fluid, at rest in the Papapetrou–Randers frame of a Papapetrou–Randers geometry (3.1) – with $B = 1$. In this case, fluid lines are tangent to $\check{u} = \partial_t$, the kinematics is shearless and expansionless with vorticity (3.9), and thus it is non-dissipative. It is also geodesic (see (3.6)) and Eqs. (2.25) imply that $\varepsilon, p$ and $T$ are constant everywhere. The propagation of irrotational perturbations in this set up is captured by the following acoustic metric[11]:

$$\mathrm{d}s^2 = \frac{T^{D-2}}{\sqrt{D-1}}\left(-(\mathrm{d}t - b_i\mathrm{d}x^i)^2 + (D-1)a_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j\right), \tag{3.25}$$

of the Papapetrou–Randers form, which can be recast in the Zermelo form (3.13), following (3.11) and (3.12).

## 4. Examples in $2+1$ dimensions

Examples of Papapetrou–Randers geometries are numerous, possessing diverse properties regarding their isometries, their curvature, the regularity of their Randers–Zermelo transformation, etc. Their expressions can be simple in Papapetrou–Randers form and complicated in the Zermelo representation or vice versa. All this depends in particular on the dimension[12]. We will here focus on a few three-dimensional examples that turn out to emerge as holographic duals of exact four-dimensional bulk spacetimes. Although the nature of the boundary three-dimensional spacetime regarding Einstein's equations plays little role in holography, some underlying intrinsic properties appear to be generic for the backgrounds under consideration, and would deserve further investigation. Furthermore, all examples below are homogeneous spaces[13], even though neither was

[11] This result is easy to establish, following e.g. similar reasoning as in [30].

[12] See [21] for a detailed account of properties and examples in $3+1$ dimensions.

[13] Following [31, 32], homogeneous three-manifolds include all 9 Bianchi groups plus 3 coset spaces, which are $H_3$, $H_2 \times S^1$, $S^2 \times S^1$ ($S^n$ and $H_n$ are spheres and hyperbolic spaces respectively).

homogeneity an *a priori* criterion, nor did our list provide an exhaustive classification of stationary backgrounds with a spatially homogeneous timelike Killing vector field.

### 4.1 Warped three-spheres: Bianchi IX

Warped three-spheres, earlier more appropriately called biaxially squashed three-spheres, are deformations of the standard homogeneous and isotropic (round) $S^3$. This deformation breaks the original $SU(2)\times SU(2)$ isometry group down to $SU(2)$ or $SU(2)\times U(1)$. These spaces can be endowed with the metric

$$\mathrm{d}s^2=\sum_{i=1}^{3}\left(\gamma_i\sigma^i\right)^2, \tag{4.1}$$

where $\gamma_i$ are constants and $\sigma^i$ are the left-invariant Maurer–Cartan forms of $SU(2)$. In terms of Euler angles $0\leq\theta\leq\pi, 0\leq\phi\leq 2\pi, 0\leq\psi\leq 4\pi$, these one-forms read:

$$\begin{cases}\sigma^1=\sin\theta\sin\psi\,\mathrm{d}\phi+\cos\psi\,\mathrm{d}\theta\\ \sigma^2=\sin\theta\cos\psi\,\mathrm{d}\phi-\sin\psi\,\mathrm{d}\theta\\ \sigma^3=\cos\theta\,\mathrm{d}\phi+\mathrm{d}\psi.\end{cases} \tag{4.2}$$

For reasons that will become clear in the following, we will consider situations where $\gamma_1=\gamma_2$. The spaces obtained in this way are called Berger spheres. They are axisymmetric *i.e.* have $SU(2)\times U(1)$ isometry group. Since we are interested in spaces with Lorentzian signature, we must set negative $\gamma_3^2$ and the metric finally reads:

$$\mathrm{d}s^2=L^2\left[(\sigma^1)^2+\left(\sigma^2\right)^2\right]-4n^2\left(\sigma^3\right)^2=-4n^2\left(\mathrm{d}\psi+\cos\theta\,\mathrm{d}\phi\right)^2+L^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right), \tag{4.3}$$

where $2L$ is the radius of the original undeformed $S^3$ and $2nk$ the deformation parameter ($k={}^1/_L$).

The time coordinate in (4.3) is $\psi$. In the original Euclidean sphere this was an angle, but there is no reason to keep it compact in the Lorentzian version at hand[14]. Introducing a non-compact time $t=-2n(\psi+\phi)$, the metric (4.3) assumes the form:

$$\mathrm{d}s^2=-\left(\mathrm{d}t+4n\sin^2{}^\theta/_2\,\mathrm{d}\phi\right)^2+L^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right). \tag{4.4}$$

This metric is of Papapetrou–Randers type (3.1). The base $\mathrm{d}x^2=a_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j$ is a two-sphere of radius $L$, while $b=-4n\sin^2{}^\theta/_2\,\mathrm{d}\phi$ is a Dirac-monopole-like potential. The latter creates a constant – in the Papapetrou–Randers orthonormal coframe – vorticity

$$\omega=-n\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi=-nk^2\,\hat{e}^1\wedge\hat{e}^2 \tag{4.5}$$

for the geodesic congruence tangent to $\check{u}\equiv\check{e}_0=\partial_t$ (up to a delta-function contribution of the "Misner point" at the southern pole – this name will be justified later).

As already quoted, the space at hand is homogeneous and belongs to the family of spaces invariant under a four-parameter group of motions [33, 34], here generated by the following Killing

[14] This point deserves some comments, which we postpone to the discussion on the bulk geometries leading to (4.3) as boundary.

vectors:

$$\begin{cases} \xi_1 = -\sin\phi\cot\theta\,\partial_\phi + \cos\phi\,\partial_\theta - 2n\frac{\sin\phi}{\sin\theta}(1-\cos\theta)\,\partial_t \\ \xi_2 = \cos\phi\cot\theta\,\partial_\phi + \sin\phi\,\partial_\theta + 2n\frac{\cos\phi}{\sin\theta}(1-\cos\theta)\,\partial_t \\ \xi_3 = \partial_\phi - 2n\,\partial_t \\ e_3 = -2n\,\partial_t. \end{cases} \tag{4.6}$$

The three former generate the left $SU(2)$, whereas the latter generates an extra $\mathbb{R}$ factor (instead of $U(1)$ since $t$ is non-compact).

The background (4.4) is not globally hyperbolic. Even though it is homogeneous, constant-$t$ surfaces are not and $\gamma = 1/\sqrt{1-4n^2k^2\tan^2\theta/2}$ diverges when $\theta$ reaches $\theta_* = 2\arctan L/2n$. Hyperbolicity holds in the disk $0<\theta<\theta_*$, whereas it breaks down in the complementary disk ($\theta_*<\theta<\pi$) centered at the Misner point, where $\partial_\phi$ becomes timelike. As a consequence, the circles tangent to $\partial_\phi$ become CTCs for $\theta_*<\theta<\pi$. Homogeneity implies furthermore that CTCs are present everywhere, passing through any arbitrary point of spacetime. In particular, for $0<\theta<\theta_*$ the CTCs are sections of cylinders normal to the constant-$t$ surfaces. The time coordinate $t$ evolves periodically along these elliptically shaped CTCs.

As we will see in the forthcoming sections, the situation described here is quite generic for three-dimensional homogeneous spacetimes. These include the case of Som–Raychaudhuri (Bianchi II) and the celebrated Gödel space (Bianchi VIII). They are illustrative examples of how homogeneity combined with rotation often leads to the breakdown of hyperbolicity and the emergence of CTCs. Gödel space in particular was the first to be recognized as plagued by CTCs. The CTCs present in these spaces, however, are not geodesics [34, 35, 36]. Their presence is therefore harmless for classical causality. This is why Gödel-like solutions like the case under consideration have never been truly discarded, leaving open the possibility of quantum mechanical validity[15].

Let us incidentally mention that the above three-dimensional geometry (4.4) – when lifted to four dimensions by taking the direct product with an extra flat direction – has been shown to satisfy Einstein's equations with cosmological constant and energy–momentum tensor produced by some specific charged fluid [34]. Alternatively, it also satisfies the equations of topologically massive gravity [42], subject to a Kerr–Schild [46] deformation[16]

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda^{(3)}g_{\mu\nu} = \frac{1}{4nk^2}C_{\mu\nu} + \frac{k^2\left(1+4n^2k^2\right)}{4}u_\mu u_\nu, \tag{4.7}$$

where $\hat{u} = -\hat{e}_0 = -\mathrm{d}t - 4n\sin^2\theta/2\mathrm{d}\phi$, $\Lambda^{(3)} = k^2/4$ is the cosmological constant for an undeformed $S^3$ and $C_{\mu\nu}$ are the components of the Cotton–York tensor defined as

$$C^{\mu\nu} = \frac{\varepsilon^{\mu\rho\sigma}}{\sqrt{-g}}\nabla_\rho\left(R^\nu{}_\sigma - \frac{1}{4}R\delta^\nu_\sigma\right). \tag{4.8}$$

For the background (4.4) this tensor is

$$C_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = nk^4\left(1+4n^2k^2\right)\left[2\hat{u}^2 + L^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)\right]. \tag{4.9}$$

[15]Attempts, among others in string theory within holography, were proposed a few years ago (see e.g. [36, 37, 38, 39, 40, 41] and references therein).

[16]It should be mentioned that the metric (4.4) solves also topological massive gravity equations without Kerr–Schild deformation [43, 44, 45], provided one tunes appropriately the relationship among $L^2$ and $k^2$. This holds for generic warped homogeneous spaces like $\mathrm{AdS}_3$, studied in Sec. 4.2.

For later convenience we also quote:

$$R_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = 2k^4n^2\hat{u}^2 + \left(1+2n^2k^2\right)\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right) \tag{4.10}$$

and

$$R = 2k^2\left(1+n^2k^2\right). \tag{4.11}$$

### 4.2 Warped AdS$_3$: Bianchi VIII

Following the paradigm of squashed three-spheres, studied in Sec. 4.1, we analyze here the deformations of AdS$_3$. The latter is the (universal covering of the) $SL(2,\mathbb{R})$ group manifold and has left and right $SL(2,\mathbb{R})$ isometries. Homogeneous deformations can break partially or completely one $SL(2,\mathbb{R})$ factor.

There are many realizations of the Maurer–Cartan forms of $SL(2,\mathbb{R})$. The one we choose here is convenient for the specific deformation we will consider in the following:

$$\begin{cases} \rho^0 = -\mathrm{d}\tau + \cosh\sigma\,\mathrm{d}\phi \\ \rho^1 = -\sin\tau\,\mathrm{d}\sigma - \sinh\sigma\cos\tau\,\mathrm{d}\phi \\ \rho^2 = \cos\tau\,\mathrm{d}\sigma + \sinh\sigma\sin\tau\,\mathrm{d}\phi, \end{cases} \tag{4.12}$$

where $0\leq\phi\leq 2\pi, 0\leq\sigma<+\infty$, and $\tau\in[0,2\pi]$ or better $\mathbb{R}$ if we consider the universal covering of the space. The metrics under consideration are of the form

$$\mathrm{d}s^2 = -\left(\gamma_0\rho^0\right)^2 + \left(\gamma_1\rho^1\right)^2 + \left(\gamma_2\rho^2\right)^2. \tag{4.13}$$

The $\gamma$s being constant, this geometry is homogeneous. When $\forall i, \gamma_i = L$, we recover radius-$2L$ AdS$_3$.

We will restrict ourselves here to the elliptically-squashed AdS$_3$, obtained with $\gamma_0 = 2p\in\mathbb{R}$ and $\gamma_1=\gamma_2=L$. These geometries have a 4-parameter isometry group $SL(2,\mathbb{R})\times\mathbb{R}$. The Abelian factor is the remaining one-parameter subgroup of the broken isotropy symmetry – $U(1)$ before taking the universal covering.

Elliptic deformations were studied as such in [47] and the continuous line obtained there coincides – when uplifted to four dimensions – with the family of spacetime-homogeneous Gödel-type metrics discussed in [34]. After a coordinate transformation trading $\tau$ for $t = 2p(\tau-\phi)\in\mathbb{R}$, the metric (4.13) reads:

$$\mathrm{d}s^2 = -\left(\mathrm{d}t - 4p\sinh^2{}^{\sigma}\!/\!_2\,\mathrm{d}\phi\right)^2 + L^2\left(\mathrm{d}\sigma^2+\sinh^2\sigma\,\mathrm{d}\phi^2\right). \tag{4.14}$$

This is a timelike fibration over a hyperbolic plane $H_2$.

Alternative inequivalent deformations exist in $SL(2,\mathbb{R})$, such as the hyperbolic or the parabolic ones which lead to spacelike fibrations over AdS$_2$ or plane-wave superpositions with AdS$_3$. They have been studied extensively in the framework of string theory [38, 39, 40] and more recently in holography [45]. However, these are not of the Papapetrou–Randers form (3.1), as opposed to the elliptic deformation for which $b = 4p\sinh^2{}^{\sigma}\!/\!_2\,\mathrm{d}\phi$ while the base $\mathrm{d}x^2 = a_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j$ is radius-$L$ Lobatchevsky plane. The "non-compact Dirac potential" $b$ creates for the geodesic congruence tangent to $\check{u}\equiv\check{e}_0=\partial_t$ a homogeneous vorticity, which – in the Papapetrou–Randers orthonormal coframe – reads:

$$\omega = p\sinh\sigma\,\mathrm{d}\sigma\wedge\mathrm{d}\phi = pk^2\,\hat{e}^1\wedge\hat{e}^2. \tag{4.15}$$

In the case at hand, the base is non-compact and the Misner point is rejected to infinity.

The hyperbolicity properties of the background (4.14) are richer than for (4.4). Indeed, for the space at hand, the Lorentz factor relating Zermelo and Papapetrou–Randers observers is $\gamma = 1/\sqrt{1-4p^2k^2\tanh^2\sigma/2}$. This factor remains finite for finite $\sigma$ provided $p \leq L/2$ – the limiting case $p = L/2$ corresponding to the undeformed AdS$_3$. Under this condition, the space is globally hyperbolic and constant-$t$ surfaces are spacelike everywhere. When $p > L/2$ this property breaks down for $\sigma > \sigma_* = 2\operatorname{arctanh} L/2p$. Constant-$t$ surfaces are spacelike on a disk only, centered at $\sigma = 0$ and bounded by a velocity-of-light surface located at $\sigma = \sigma_*$. The breakdown of hyperbolicity is accompanied with the appearance of CTCs, present everywhere as a consequence of homogeneity. This happens in particular for $p = L/\sqrt{2}$ corresponding to Gödel's solution.

As mentioned earlier, the absence of hyperbolicity in the backgrounds considered in our general framework is closely related to the combination of vorticity and homogeneity. This is even better illustrated in the AdS$_3$, where the non-compact nature of the base makes it possible to evade the breakdown, provided the vorticity is small enough with respect to the scale $L$ set by the curvature. The isometry group is in the present case $SL(2,\mathbb{R})\times\mathbb{R}$ generated by the following Killing vectors:

$$\begin{cases} \zeta_0 = \partial_\phi - 2p\,\partial_t \\ \zeta_1 = -\cos\phi\coth\sigma\,\partial_\phi - \sin\phi\,\partial_\sigma - 2p\frac{\cos\phi}{\sinh\sigma}(1-\cosh\sigma)\,\partial_t \\ \zeta_2 = -\sin\phi\coth\sigma\,\partial_\phi + \cos\phi\,\partial_\sigma - 2p\frac{\sin\phi}{\sinh\sigma}(1-\cosh\sigma)\,\partial_t \\ q_0 = -2p\,\partial_t. \end{cases} \tag{4.16}$$

For completeness we also quote the Levi–Civita curvature properties of the metric (4.14):

$$C_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu = pk^4\left(1-4p^2k^2\right)\left[2\hat{u}^2 + L^2\left(\mathrm{d}\sigma^2+\sinh^2\sigma\,\mathrm{d}\phi^2\right)\right], \tag{4.17}$$

$$R_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu = 2k^4p^2\hat{u}^2 - \left(1-2p^2k^2\right)\left(\mathrm{d}\sigma^2+\sinh^2\sigma\,\mathrm{d}\phi^2\right), \tag{4.18}$$

$$R = -2k^2\left(1-p^2k^2\right), \tag{4.19}$$

where $\hat{u} = -\mathrm{d}t + 4p\sinh^2\sigma/2\,\mathrm{d}\phi$.

### 4.3 Rotating Einstein universes

In the limit $n \to 0$, the geometry (4.4) becomes a homogeneous metric on $\mathbb{R}\times S^2$. This is the Einstein static universe, which is a trivial case of static Papapetrou–Randers geometry. The isometry group remains unaltered and generated by the Killing vectors (4.6). The metric reads:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + L^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{4.20}$$

Trading $(\theta,\phi)$ for $(\theta',\phi')$ defined as

$$\phi = \phi' + \Omega_\infty t, \quad D_\theta\Delta_{\theta'} = \Xi, \quad \Xi = 1 - L^2\Omega_\infty^2, \quad D_\theta = 1 - L^2\Omega_\infty^2\sin^2\theta, \quad \Delta_{\theta'} = 1 - L^2\Omega_\infty^2\cos^2\theta', \tag{4.21}$$

we obtain

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\Xi L^2}{\Delta_{\theta'}^2}\left(\mathrm{d}\theta'^2 + \frac{\Delta_{\theta'}}{\Xi}\sin^2\theta'\left[\mathrm{d}\phi' + \Omega_\infty \mathrm{d}t\right]^2\right). \tag{4.22}$$

This is the Einstein universe uniformly rotating at angular velocity $\Omega_\infty$ in spheroidal coordinates.

The metric (4.22) is in a Zermelo form (3.13). It can be brought to Papapetrou–Randers form using the general transformations (3.11) and (3.12):

$$\mathrm{d}s^2 = \frac{\Xi}{\Delta_{\theta'}}\left(-\left[\mathrm{d}t - \frac{L^2\Omega_\infty}{\Xi}\sin^2\theta'\mathrm{d}\phi'\right]^2 + \frac{L^2\mathrm{d}\theta'^2}{\Delta_{\theta'}} + \frac{L^2\Delta_{\theta'}}{\Xi^2}\sin^2\theta'\mathrm{d}\phi'^2\right). \tag{4.23}$$

In order to reach a canonical Papapetrou–Randers form (3.1) with $B=1$, a conformal transformation is required with conformal factor $\Phi = \Delta_{\theta'}/\Xi$. The resulting geometry,

$$\mathrm{d}s^2 = -\left[\mathrm{d}t - \frac{L^2\Omega_\infty}{\Xi}\sin^2\theta'\mathrm{d}\phi'\right]^2 + \frac{L^2\mathrm{d}\theta'^2}{\Delta_{\theta'}} + \frac{L^2\Delta_{\theta'}}{\Xi^2}\sin^2\theta'\mathrm{d}\phi'^2, \tag{4.24}$$

appears as the boundary of Kerr–AdS four-dimensional bulk geometry [48]. We would like to stress that although (4.23) is still invariant under $SU(2)\times\mathbb{R}$ generated by (4.6) (at $n=0$), this is no longer true for (4.24), where the isometry group is reduced to $U(1)\times\mathbb{R}$ generated by $\partial_\phi$ and $\partial_t$, while $\xi_1$ and $\xi_2$ in (4.6) become conformal Killing vectors.

Let us finally mention that we can in a similar fashion consider the limit $p\to 0$ in the elliptically deformed $\mathrm{AdS}_3$ geometries, given in (4.14). This leads to a Einstein-static-universe-like geometry of the type $\mathbb{R}\times H_2$, which can be brought into the form

$$\mathrm{d}s^2 = \frac{Z}{\Theta_{\sigma'}}\left(-\left[\mathrm{d}t - \frac{L^2\Omega_\infty}{Z}\sinh^2\sigma'\mathrm{d}\phi'\right]^2 + \frac{L^2\mathrm{d}\sigma'^2}{\Theta_{\sigma'}} + \frac{L^2\Theta_{\sigma'}}{Z^2}\sinh^2\sigma'\mathrm{d}\phi'^2\right), \tag{4.25}$$

after a coordinate transformation $(\sigma,\phi)\mapsto(\sigma',\phi')$ with

$$\phi=\phi'+\Omega_\infty t,\quad H_\sigma\Theta_{\sigma'}=Z,\quad Z=1+L^2\Omega_\infty^2,\quad H_\sigma=1-L^2\Omega_\infty^2\sinh^2\sigma,\quad \Theta_{\sigma'}=1+L^2\Omega_\infty^2\cosh^2\sigma'. \tag{4.26}$$

One can similarly perform a conformal transformation leading to

$$\mathrm{d}s^2 = -\left[\mathrm{d}t - \frac{L^2\Omega_\infty}{Z}\sinh^2\sigma'\mathrm{d}\phi'\right]^2 + \frac{L^2\mathrm{d}\sigma'^2}{\Theta_{\sigma'}} + \frac{L^2\Theta_{\sigma'}}{Z^2}\sinh^2\sigma'\mathrm{d}\phi'^2. \tag{4.27}$$

The geometry described by (4.27) is globally hyperbolic without any restriction. For the metric (4.24), global hyperbolicity holds under the condition $\Omega_\infty < k$. Again this is a bound on the magnitude of the vorticity

$$\omega = \frac{L^2\Omega_\infty\sin 2\theta'}{2\Xi}\mathrm{d}\theta'\wedge\mathrm{d}\phi' \tag{4.28}$$

carried by the geodesic congruence $\check{u}=\check{e}_0=\partial_t$ in this background – and is relaxed for the non-compact base (squashed $H_2$ in (4.27) versus squashed $S^2$ in (4.24)), where

$$\omega = \frac{L^2\Omega_\infty\sinh 2\sigma'}{2Z}\mathrm{d}\sigma'\wedge\mathrm{d}\phi'. \tag{4.29}$$

### 4.4 Zooming at the poles: Bianchi II and $\mathrm{VII}_0$

We can further analyze the geometries met so far by zooming around their poles. This exhibits new backgrounds of Papapetrou–Randers type, interesting for their own right.

**Som–Raychaudhuri and Heisenberg algebra.** The geometry around $\theta \approx 0$ in (4.4) or $\sigma \approx 0$ in (4.14) is

$$\mathrm{d}s^2 \approx -\left(\mathrm{d}t - L^2\Omega_\infty \chi^2 \mathrm{d}\phi\right)^2 + L^2\left(\mathrm{d}\chi^2 + \chi^2 \mathrm{d}\phi^2\right) \tag{4.30}$$

with $\chi = \theta$ or $\sigma$, and $\Omega_\infty = -nk^2$ or $pk^2$. This very same geometry appears also around $\theta \approx 0$ or $\pi$ in (4.24) and around $\sigma \approx 0$ in (4.27) – upon appropriate definitions of $\chi$ involving $\Xi$ or $Z$.

Metric (4.30) is the Som–Raychaudhuri space, found in [49] by solving Einstein equations with rotating, charged dust with zero Lorentz force. It belongs to the general family of three-dimensional homogeneous spaces possessing 4 isometries studied in [33, 34], which include the various metrics that we have discussed so far here. In the case of Som–Raychaudhuri (Eq. (4.30)) the isometries are generated by the following Killing vectors:

$$\begin{cases} K_x = k\frac{\sin\phi}{\chi}\,\partial_\phi - k\cos\phi\,\partial_\chi - L\Omega_\infty\chi\sin\phi\,\partial_t \\ K_y = k\frac{\cos\phi}{\chi}\,\partial_\phi + k\sin\phi\,\partial_\chi - L\Omega_\infty\chi\cos\phi\,\partial_t \\ K_0 = 2\Omega_\infty\,\partial_t \\ K = \partial_\phi . \end{cases} \tag{4.31}$$

The vectors $K_x, K_y$ and $K_0$ form a Heisenberg algebra, and indeed the Som–Raychaudhuri metric can be built as the group manifold of the Heisenberg group (Bianchi II) at an extended-symmetry (isotropy) point with an extra symmetry generator[17] $\partial_\phi$. Actually, this corresponds to a contraction of $SU(2)\times\mathbb{R}$ into a semi-direct product of the Heisenberg group with an extra $U(1)$ generated by $K_x = -k\xi_1, K_y = k\xi_2, K_0 = k^2 e_3, K = \xi_3 - e_3$ (see (4.6)). It can similarly emerge from the $SL(2,\mathbb{R})\times\mathbb{R}$ algebra (4.16).

Similarly to Gödel space, Som–Raychaudhuri space contains non-geodesic closed timelike curves. These are circles of radius $\chi$ larger than $1/\Omega_\infty$ [50].

**Flat vortex and $\mathbf{E(2)}$ geometry.** The southern pole of (4.4) is not captured in the above scheme. Indeed this is a fixed point of the transformation generated by the Killing vector $\xi_3 + e_3$ (see (4.6)), which we referred to as the Misner point. Around this point

$$\mathrm{d}s^2 \approx -\left(\mathrm{d}t + n\left(4-\chi^2\right)\mathrm{d}\phi\right)^2 + L^2\left(\mathrm{d}\chi^2 + \chi^2\mathrm{d}\phi^2\right), \tag{4.32}$$

where $\chi = \pi - \theta$. The latter is known as a flat vortex geometry, homogeneous and invariant under an $E(2)\times\mathbb{R}$ algebra ($E(2)$ is Bianchi $\mathrm{VII}_0$) generated by[18]

$$\begin{cases} L_x = k\xi_1 = k\frac{\sin\phi}{\chi}\left(\partial_\phi - 4n\,\partial_t\right) - k\cos\phi\,\partial_\chi \\ L_y = -k\xi_2 = k\frac{\cos\phi}{\chi}\left(\partial_\phi - 4n\,\partial_t\right) + k\sin\phi\,\partial_\chi \\ L_0 = \xi_3 = \partial_\phi - 2n\,\partial_t \\ L = e_3 = -2n\,\partial_t . \end{cases} \tag{4.33}$$

It also appears as a contraction of the $SU(2)\times\mathbb{R}$.

[17] $\left[K_x, K_y\right] = K_0$, $\left[K_x, K_0\right] = \left[K_y, K_0\right] = 0$, $\left[K, K_x\right] = K_y$, $\left[K, K_y\right] = -K_x$ and $\left[K, K_0\right] = 0$.
[18] $\left[L_x, L_y\right] = 0$, $\left[L_0, L_x\right] = L_y$, $\left[L_0, L_y\right] = -L_x$ and everything commutes with $L$.

## 5. Holography in the Fefferman–Graham expansion: from $3+1$ to $2+1$

In this section we study the properties of holographic fluids, as they arise in the Fefferman–Graham expansion along the holographic radial coordinate. The set of data reached in this manner contains the boundary frame *i.e.* the geometrical background hosting the fluid, as well as the energy–momentum tensor, which describes the fluid dynamics. This general method is applied to selected four-dimensional solutions of Einstein's equations, whose boundaries coincide with Papapetrou–Randers geometries studied in Sec. 4.

### 5.1 Split formalism and Fefferman–Graham in a nutshell

We find illuminating to discuss holographic fluid dynamics in $D=2+1$ dimensions starting from the $3+1$-split formalism introduced in [51, 52, 53]. We begin with the Einstein–Hilbert action in the Palatini first-order formulation

$$S=-\frac{1}{32\pi G_{\mathrm{N}}}\int \varepsilon_{ABCD}\left(R^{AB}-\frac{\Lambda}{6}E^A\wedge E^B\right)\wedge E^C\wedge E^D=\frac{1}{16\pi G_{\mathrm{N}}}\int \mathrm{d}^4x\sqrt{-g}(R-2\Lambda), \tag{5.1}$$

where $G_{\mathrm{N}}$ is Newton's constant. We also assume negative cosmological constant expressed as $\Lambda=-3/L^2=-3k^2$. We denote the orthonormal coframe $E^A$, $A=r,a$ and use for the bulk metric the signature $+-++$. The first direction $r$ is the holographic one and $\mathrm{x}\equiv(t,x^1,x^2)\equiv(t,x)$.

Bulk solutions are taken in the Fefferman–Graham form

$$\mathrm{d}s^2=\frac{L^2}{r^2}\mathrm{d}r^2+\frac{r^2}{L^2}\eta_{ab}E^a(r,\mathrm{x})E^b(r,\mathrm{x})\,. \tag{5.2}$$

For torsionless connections there is always a suitable gauge choice such that the metrics (5.2) are fully determined by two coefficients $\hat{e}^a$ and $\hat{f}^a$ in the expansion of the coframe one-forms $\hat{E}^a(r,\mathrm{x})$ along the holographic coordinate $r\in\mathbb{R}_+$

$$\hat{E}^a(r,\mathrm{x})=\left[\hat{e}^a(\mathrm{x})+\frac{L^2}{r^2}\hat{F}^a(\mathrm{x})+\cdots\right]+\frac{L^3}{r^3}\left[\hat{f}^a(\mathrm{x})+\cdots\right]. \tag{5.3}$$

The asymptotic boundary is at $r\to\infty$ and it is endowed with the geometry

$$\mathrm{d}s^2_{\text{bry.}}=\lim_{r\to\infty}\frac{\mathrm{d}s^2}{k^2r^2}. \tag{5.4}$$

The ellipses in (5.3) denote terms that are multiplied by higher negative powers of $r$. Their coefficients are determined by $\hat{e}^a$ and $\hat{f}^a$, and have specific geometrical interpretations[19], though this is not relevant for our discussion.

The $3+1$-split formalism makes clear that $\hat{e}^a(\mathrm{x})$ and $\hat{f}^a(\mathrm{x})$, being themselves vector-valued one-forms in the boundary, are the proper canonical variables playing the role of boundary "coordinate" and "momentum" for the (hyperbolic) Hamiltonian evolution along $r$. For the stationary backgrounds under consideration, describing thermally equilibrated non-dissipating boundary fluid configurations, $\hat{e}^a$ and $\hat{f}^a$ are $t$-independent.

The boundary "coordinate" is given by the set of one-forms $\hat{e}^a$. For this coframe we must determine the "momentum" of the boundary data. For example, when the boundary data carry

[19]For example, the coefficient $\hat{F}^a$ is related to the boundary Schouten tensor.

zero mass, we expect this to be zero. In this case $\hat{f}^a(x) = 0$ and the unique exact solution of the Einstein's equations is pure $AdS_4$.

More generally, the vector-valued one-form $\hat{f}^a$ satisfies

$$\hat{f}^a \wedge \hat{e}_a = 0\,, \quad \varepsilon_{abc}\hat{f}^a \wedge \hat{e}^b \wedge \hat{e}^c = 0\,, \quad \varepsilon_{abc}\mathrm{D}\hat{f}^b \wedge \hat{e}^c = 0\,, \tag{5.5}$$

where the action of the generalized exterior derivative D on a vector-valued one-form $\hat{V}^a$ is defined as

$$\mathrm{D}\hat{V}^a = \mathrm{d}\hat{V}^a + \varepsilon^a{}_{bc}\hat{B}^b \wedge \hat{e}^c\,, \tag{5.6}$$

and the "magnetic field" $\hat{B}^a$ is the Levi–Civita spin connection associated with $\hat{e}^a$ [52]. One can easily see that conditions (5.5) imply, respectively, symmetry, absence of trace and covariant conservation of the tensor $T = T^a{}_b \check{e}_a \otimes \hat{e}^b$, defined as

$$\hat{f}^a = \frac{1}{\kappa}T(\hat{e}^a) = \frac{1}{\kappa}T^a{}_b\hat{e}^b\,, \quad \kappa = \frac{3}{8\pi G_{\mathrm{N}}L}\,. \tag{5.7}$$

Hence we can interpret the latter as the covariantly conserved energy–momentum tensor of a conformal field theory. Here we are interested in particular stationary bulk solutions for which we expect the energy–momentum tensor be reduced to the perfect relativistic form, Eqs. (2.19) and (2.20):

$$T^a{}_b = (\varepsilon + p)u^a u_b + p\delta^a_b\,. \tag{5.8}$$

Although the two necessary ingredients for the description of a relativistic perfect fluid, namely the boundary frame and the velocity one-form, are nicely packaged in the leading and subleading independent boundary data, until now we did not assume any specific relationship between them. Nevertheless it is clear that such a relationship would be imposed by any *exact* solution of the bulk gravitational equations, given the interior boundary conditions. We will soon observe that the Fefferman–Graham expansion of the exact solutions of Sec. 5.2 yield the *same form* for the boundary energy–momentum tensor, namely

$$\hat{f}^0 = -\frac{2M}{3L}\hat{e}^0\,, \quad \hat{f}^\alpha = \frac{M}{3L}\hat{e}^\alpha\,. \tag{5.9}$$

The boundary frame one-forms $\hat{e}^a$ are themselves, of course, different in the three solutions. Comparing (5.7), (5.8) and (5.9), we find

$$\varepsilon = 2p = 2\kappa\frac{M}{3L}\,, \tag{5.10}$$

constant as already advertised. The solutions under consideration describe thus the *same conformal fluid* in different kinematic states. More importantly, (5.9) fixes the direction of the velocity field with respect to the boundary frame to be

$$\check{u} = \check{e}_0\,. \tag{5.11}$$

As explained in Sec. 3.1, in the Papapetrou–Randers geometry (Eq. (3.1)), this congruence is tangent to a constant-norm Killing field and has thus zero shear, expansion and acceleration (consistent, according to (2.25), with the constant pressure found in (5.10)). It also shows that the observer's frame $\check{e}_a$ is *comoving*. Therefore, in the Fefferman–Graham expansion the kinematic properties of holographic fluids are determined by the geometric properties of the boundary comoving frame.

### 5.2 Some exact geometries and their fluid interpretation

We present here three examples of holographic fluids with vorticity that reside at the boundary of two exact solutions of the bulk vacuum Einstein equations; the Kerr–AdS$_4$, the Taub–NUT–AdS$_4$ and the hyperbolic NUT–AdS$_4$ black hole solutions.

The four-dimensional Kerr solution of Einstein's equation with $\Lambda = -3k^2$ reads:

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{V(r,\theta)} - V(r,\theta)\left[\mathrm{d}t - \frac{a}{\Xi}\sin^2\theta\,\mathrm{d}\phi\right]^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{\sin^2\theta\Delta_\theta}{\rho^2}\left[a\,\mathrm{d}t - \frac{r^2+a^2}{\Xi}\mathrm{d}\phi\right]^2, \quad (5.12)$$

where

$$V(r,\theta) = \frac{\Delta_r}{\rho^2} \quad (5.13)$$

and

$$\begin{cases}\Delta_r = (r^2+a^2)(1+k^2r^2) - 2Mr \\ \rho^2 = r^2 + a^2\cos^2\theta\end{cases} \qquad \begin{cases}\Delta_\theta = 1 - k^2a^2\cos^2\theta \\ \Xi = 1 - k^2a^2.\end{cases} \quad (5.14)$$

The geometry has inner ($r_-$) and outer ($r_+$) horizons, where $\Delta_r$ vanishes, as well as an ergosphere at $g_{tt} = 0$. The solution at hand describes the field generated by a mass $M$ rotating with an angular velocity

$$\Omega = \frac{a(1+k^2r_+^2)}{r_+^2+a^2} \quad (5.15)$$

as measured by a static observer at infinity [54, 55]. Note that asymptotic observer associated with a natural frame of the coordinate system at hand is not static, but has an angular velocity

$$\Omega_\infty = ak^2. \quad (5.16)$$

The boundary metric of Eq. (5.12) is (4.24) (without primes in the angular coordinates).

The Taub–NUT–AdS$_4$ geometry is a foliation over squashed three-spheres solving Einstein's equations with negative cosmological constant (the $\sigma$s are given in (4.2)):

$$\begin{aligned}\mathrm{d}s^2 &= \frac{\mathrm{d}r^2}{V(r)} + \left(r^2+n^2\right)\left(\left(\sigma^1\right)^2 + \left(\sigma^2\right)^2\right) - 4n^2V(r)\left(\sigma^3\right)^2 && (5.17)\\ &= \frac{\mathrm{d}r^2}{V(r)} + \left(r^2+n^2\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) - V(r)\left[\mathrm{d}t + 4n\sin^2\frac{\theta}{2}\mathrm{d}\phi\right]^2, && (5.18)\end{aligned}$$

where

$$V(r) = \frac{\Delta_r}{\rho^2} \quad (5.19)$$

and

$$\begin{cases}\Delta_r = r^2 - n^2 - 2Mr + k^2\left(r^4 + 6n^2r^2 - 3n^4\right) \\ \rho^2 = r^2 + n^2.\end{cases} \quad (5.20)$$

Besides the mass $M$ and the cosmological constant $\Lambda = -3k^2$, this solution depends on an extra parameter $n$: the nut charge.

The solution at hand has generically two horizons ($V(r_\pm) = 0$) and is well-defined outside the outer horizon $r_+$, where $V(r) > 0$. The nut is the endpoint of a Misner string [56], departing from

$r = r_+$, all the way to $r \to \infty$, on the southern pole at $\theta = \pi$. The geometry is nowhere singular along the Misner string, which appears as a coordinate artifact much like the Dirac string of a magnetic monopole is a gauge artifact. In order for this string to be invisible, coordinate transformations displacing the string must be univalued everywhere, which is achieved by requiring the periodicity condition $t \equiv t - 8\pi n$.

One can avoid periodic time and keep the Misner string as part of the geometry. This semi-infinite spike appears then as a source of angular momentum, integrating to zero [57, 58], and movable at wish using the transformations generated by the above vectors. This will be our viewpoint throughout this work. However, despite the non-compact time, the Taub–NUT–AdS geometry is plagued with closed timelike curves, which disappear only in the vacuum limit $k \to 0$ [59]. Even though this is usually an unwanted situation, it is not sufficient for rejecting the geometry, which from the holographic perspective has many interesting and novel features. The boundary geometry of (5.18), reached by following (5.4), is (4.4) and this is the background where the fluid is evolving. What we called Misner point for the latter in Sec. 4.1 is the endpoint of the bulk Misner string.

On a non-compact horizon, the nut charge can be pushed to infinity. This happens in hyperbolic NUT black holes, obtained as foliations over three-dimensional anti-de Sitter spaces[20]. Using the $SL(2,\mathrm{R})$ Maurer–Cartan forms (4.12), we obtain the following solution of Einstein's equations with cosmological constant $\Lambda = -3k^2$:

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\mathrm{d}r^2}{V(r)} + \left(r^2+p^2\right)\left(\left(\rho^1\right)^2 + \left(\rho^2\right)^2\right) - 4p^2 V(r)\left(\rho^0\right)^2 && (5.21) \\ &= \frac{\mathrm{d}r^2}{V(r)} + \left(r^2+p^2\right)\left(\mathrm{d}\sigma^2 + \sinh^2\sigma\,\mathrm{d}\phi^2\right) - V(r)\left[\mathrm{d}t - 4p\sinh^2\frac{\sigma}{2}\,\mathrm{d}\phi\right]^2 && (5.22) \end{aligned}$$

with $V(r)$ given in (5.19) and

$$\begin{cases} \Delta_r = -r^2 + p^2 - 2\hat{M}r + k^2\left(r^4 + 6p^2r^2 - 3p^4\right) \\ \rho^2 = r^2 + p^2. \end{cases} \tag{5.23}$$

Here $\hat{M}$ is the mass parameter and $p$ characterizes the non-trivial $S^1$ fibration over the $H_2$ base. In this case no Misner string is however present, and the space is globally hyperbolic provided $p \leq L/2$. The boundary metric of (5.22) is (4.14), which plays the role of host for the holographic fluid. Interestingly, this family of solutions is connected to the Kerr–AdS$_4$ black hole, as we will now show.

The Kerr–AdS$_4$ black hole (5.12) has a rotation parameter $a$ restricted to $a^2 < L^2$, and is singular for $a^2 = L^2$. It has however a finite, maximally spinning limit if the $a \to L$ limit is taken keeping the horizon size finite and simultaneously zooming into the pole [61]. More explicitly, we trade the angle $\theta$ in (5.12) for a new coordinate $\sigma$ according to

$$\sin\theta = \sqrt{\Xi}\,\sinh^2\sigma/2. \tag{5.24}$$

[20] In their Euclidean section these geometries have no nut, but only a bolt [60]. We shall nevertheless conform to standard use and call them – with a slight abuse of language – *hyperbolic NUT black holes*, to stress the presence of a non-trivial $S^1$ fibration over $H_2$. Physically, they represent *rotating black hyperboloid membranes* [61].

Then, the resulting metric has a regular $a \to L$ limit,

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{V(r)} + \frac{1+k^2r^2}{4k^2}\left(\mathrm{d}\sigma^2 + \sinh^2\sigma\mathrm{d}\phi^2\right) - V(r)\left[\mathrm{d}t - \frac{1}{k}\sinh^2\frac{\sigma}{2}\mathrm{d}\phi\right]^2, \tag{5.25}$$

with

$$V(r) = 1 + k^2r^2 - \frac{2k^2Mr}{1+k^2r^2}. \tag{5.26}$$

This is again a rotating black hyperboloid membrane. Indeed, performing the rescaling $r \mapsto 2r$, $t \mapsto t/2$, this metric is cast in the form (5.22)–(5.23) with parameters $\hat{M} = M/8$ and $p = L/2$, and the boundary of (5.25) takes the form (4.14) with this value of $p$. The boundary is therefore the undeformed $\mathrm{AdS}_3$ spacetime that can equivalently be recovered by performing the coordinate transformation (5.24) directly on the Kerr–AdS boundary metric (4.24), followed by the $a \to L$ limit. As we saw, this special value corresponds to the limiting case that enjoys global hyperbolicity. Accordingly, from the Kerr–AdS point of view, it is obtained as the ultraspinning limit for which the boundary Einstein static universe rotates effectively at the speed of light [48]. In this way, we nicely connected fluids on warped $\mathrm{AdS}_3$ backgrounds to fluids living on a rotating Einstein static universe into a single, continuous family.

There are many other exact bulk four-dimensional geometries that one could study under the perspective of describing boundary fluids with vorticity. One can for instance consider the case of flat horizon reached when the trigonometric sinus in (5.18) (or the hyperbolic sinus in (5.22)) is traded for a linear function and the potential adapted by dropping the first two terms (see e.g. [59, 60]). This bulk solution, also plagued by the global hyperbolicity problem, gives rise to a boundary fluid moving on the Som–Raychaudhuri geometry (4.30). Alternatively, the flat-horizon bulk solution can be obtained as an appropriate pole-zooming of the four-dimensional Kerr–AdS, consistent with the observed relationships among the boundary geometries. Limits at $n, p \to 0$ lead to the so-called topological black holes [62, 63, 64]. These are interesting in their own right [65] even though the holographic fluid dynamics has no intrinsic vorticity – their boundaries are Einstein static universes $\mathbb{R} \times S^2$ given in (4.20) or $\mathbb{R} \times H_2$, where no fiber appears that would create vorticity. Hyperbolic Kerr–AdS or other exact bulk metrics can be found to reproduce on the boundary (4.27) or (4.32) [66]. One can also find solutions that combine nut charge and ordinary rotation [67] such as Kerr–Taub–NUT–AdS. We will neither pursue any longer the general analysis of this rich web of backgrounds exhibiting many interrelations, nor delve into a quantitative presentation, but move instead into another interesting approach to holographic fluids, which can be easily exemplified with the backgrounds at hand.

The above bulk geometries describe holographically a conformal fluid at rest without shear and expansion in the Papapetrou–Randers frame of a Papapetrou–Randers geometry (3.1). These fluids have a non-trivial kinematics, though, because of the vorticity of the geodesic congruence they fill. The vorticity is different in the various cases: it is given in Eqs. (4.28), (4.5) and (4.15), for Kerr–AdS, Taub–NUT–AdS and hyperbolic NUT–AdS. In the first case, the fluid undergoes a cyclonic motion with maximal vorticity at the poles and vanishing at the equator. In the other two backgrounds, the vorticity is constant as a consequence of the homogeneity. The velocity fields are not homogeneous, though, and behave differently in Taub–NUT–AdS and hyperbolic NUT–AdS.

Even though the boundary spacetime of Taub–NUT–AdS is homogeneous, the constant-$t$ surfaces are not. Inertial observers, comoving with the fluid have therefore a different perception depending on whether they are at $0 < \theta < \theta_*$ or in the disk $\theta_* < \theta < \pi$, surrounding the Misner string. This gives a physical existence to the $b^2 = 1$ edge, the meaning of which is better expressed in the Zermelo frame. In the latter, the fluid becomes superluminal and the Misner string is interpreted as the core of the vortex with homogeneous vorticity.

The various troublesome features which appear in Gödel-like spaces as the ones at hand are intimately related with the non-trivial rotational properties combined with the homogeneous character of these manifolds. In other words, for the Taub–NUT–AdS boundary, they are due to the existence of a monopole-like Misner vortex[21]. Although no satisfactory physical meaning has ever been given to Gödel-like spaces, the causal consistency of the latter being still questionable, they seem from our holographic perspective to admit a sensible interpretation in terms of conformal fluids evolving in homogeneous vortices (4.5)[22] or (4.15).

The case of hyperbolic NUT–AdS, Eq. (5.22), is yet of a different nature. This bulk geometry leads again to homogeneous boundary (4.14). Hence, the fluid has constant vorticity (4.15). However, in the case at hand, the spatial sections $\mathrm{d}x^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j$ are non-compact and negatively curved as opposed to the boundary of Taub–NUT–AdS. As a consequence, the combination of vorticity and homogeneity does not break global hyperbolicity, as long as $p \leq L/2$. In this regime, the velocity of the fluid is well-defined everywhere, and its Lorentz factor with respect to Zermelo observer is increasing with $\sigma$ and bounded as $\gamma \leq 1/\sqrt{1-4p^2k^2}$. For this observer, the fluid is at rest in the center (*i.e.* at the north pole) and fast rotating at infinity. When $p = L/2$, it reaches the speed of light when $\sigma \to \infty$, whereas for $p > L/2$ this happens at finite $\sigma = \sigma_*$, along the surface-of-light edge. The latter situation is similar to what happens in the Taub–NUT–AdS irrespectively of the value of the nut charge $n$. In the hyperbolic case, the major differerence is however that the vortex, together with the Misner point are sent to spatial infinity ($\sigma \to \infty$).

The above discussion holds in the perspective of interpreting the holographic data as a genuine stationary fluid. There is however an alternative viewpoint already advertised, consisting in the analogue gravity interpretation of the boundary gravitational background. From the latter, the physical data are still $(h_{ij}, W^i)$ *i.e.* a two-dimensional geometry and a velocity field. However, their combination into (3.13) is not a physical spacetime. The would be light cone, in particular, is narrowed down to the sound or light velocities in the medium under consideration – necessarily smaller than the velocity of light in vacuum. Consequently, the breaking of hyperbolicity or the appearance of CTCs are not issues of concern, and the regions where $\gamma$ becomes imaginary keep having a satisfactory physical interpretation as portions of space, where the medium is supersonic/superluminal with respect to the sound/light velocity *in the medium and not in the vacuum*. Finally, the virtual spacetime (3.13) governs the mode propagation through the fluid. This way of thinking opens up a new chapter that requires adjusting suitably the standard holographic dictionary. The latter provides indirect information on the physical system that must be retrieved.

[21] Since the bulk theory is such that the boundary does not have access to a charge current, the Misner vortex cannot be associated with a vortex in an ordinary superfluid, but is related to the spinning string of [68], the metric of which, Eq. (4.32), indeed appears when zooming in on the southern pole.

[22] As already stressed, one should add a $\delta$-function contribution to the Taub–NUT–AdS vorticity (4.5) because we keep the Misner string physical with non-compact time [22, 23].

## 6. Alternative expansion: from $2+1$ to $3+1$

The holographic fluids we have described so far emerge from exact bulk four-dimensional solutions of vacuum Einstein's equations with negative cosmological constant. They appear as a set of two pieces of boundary data – the coframe and the energy–momentum current – following the Fefferman–Graham expansion at large (appropriately chosen) radial coordinate. This is a top-down approach as opposed to the alternative bottom-up method initiated in [2]. The latter aims at reconstructing perturbatively a bulk solution starting from boundary data. The perturbative expansion of the bulk geometry obtained in this way is, however, orthogonal in spirit to that of the Fefferman–Graham series, since it is a derivative (of the velocity field) rather than a large-radius expansion. It captures therefore from the very first order the presence of a regular horizon of the black object that generates the dynamics of the boundary fluid.

As a matter of principle, one could generally follow the above procedure and perturbatively reconstruct the bulk solution corresponding to any boundary background of the Papapetrou–Randers type in terms of the data $(b_i, a_{ij})$ containing the full dynamics of the fluid. Our viewpoint is different though, and we are here interested in modestly discussing the interplay between the perturbative expansion developed in [2, 16, 69] and the exact solutions we have analyzed in Sec. 5. This is motivated by the observation made in [16, 69], according to which the proposed bulk perturbative reconstruction of the Kerr–AdS boundary fluid (in several dimensions) does coincide exactly with the original bulk geometry at second order – modulo a specific resummation, indicative of the genuinely all-order nature of the solution. This is remarkable and leads to the deeper question: why and under which conditions does this occur?

The question raised here is twofold. Given an exact bulk solution, what can make it be expressed in the form of a limited expansion in terms of its boundary data obtained via the Fefferman–Graham procedure? Given a set of arbitrary boundary data, what could ensure the corresponding bulk series be exact at finite order?

Making progress in this direction would require delving into the physics of dissipative phenomena and their holographic expressions. This analysis stands beyond our present scope. We can nevertheless make an observation that might ultimately be relevant. In all backgrounds under consideration, the bulk can be expressed exactly as a limited derivative expansion provided an extra term (with respect to the expansion proposed in [16, 69]) involving the Cotton tensor of the boundary geometry is appropriately added, and after performing a resummation similar to that of Kerr–AdS. This holds for all Papapetrou–Randers backgrounds presented in Sec. 4. To keep our presentation compact, we will only consider those for which we studied the bulk realisation in Sec. 5, namely Kerr–AdS, Taub–NUT–AdS and hyperbolic NUT–AdS.

The starting point for this analysis is the expression in Eddington–Finkelstein coordinates of the bulk metrics. For Kerr–AdS, Eq. (5.12) this is achieved by performing the following coordinate change:

$$\begin{cases} \mathrm{d}t \mapsto \mathrm{d}t - \frac{r^2+a^2}{\Delta_r}\mathrm{d}r \\ \mathrm{d}\phi \mapsto \mathrm{d}\phi - \frac{a\Xi}{\Delta_r}\mathrm{d}r \end{cases} \tag{6.1}$$

with all quantities defined in (5.20). Similarly for Taub–NUT–AdS, Eqs. (5.18) and (5.20), one performs

$$\mathrm{d}t \mapsto \mathrm{d}t - \frac{r^2+n^2}{\Delta_r}\mathrm{d}r, \tag{6.2}$$

while the same holds with $n \mapsto p$ for the hyperbolic NUT–AdS given in (5.22) and (5.23). All three bulk metrics assume then the following generic form:

$$\mathrm{d}s^2 = -2\hat{u}\mathrm{d}r + r^2k^2\mathrm{d}s^2_{\text{bry.}} + \frac{1}{k^2}\Sigma_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \frac{\hat{u}^2}{\rho^2}\left(2Mr + \frac{u^\lambda C_{\lambda\mu}\varepsilon^{\mu\nu\sigma}\omega_{\nu\sigma}}{2k^6\sqrt{-g_{\text{bry.}}}}\right), \tag{6.3}$$

where all the quantities refer to the boundary metric $\mathrm{d}s^2_{\text{bry.}}$. The latter being of the Papapetrou–Randers form (3.1), $\hat{u} = -\mathrm{d}t + b$ and $\omega = \frac{1}{2}\mathrm{d}b$. Furthermore $C_{\lambda\mu}$ are the components of the Cotton tensor (zero for Kerr–AdS, and displayed in Eqs. (4.9) and (4.17) for the other cases). Finally

$$\Sigma_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -2\hat{u}\nabla_\nu\omega^\nu{}_\mu\mathrm{d}x^\mu - \omega_\mu{}^\lambda\omega_{\lambda\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu - \hat{u}^2\frac{R}{2}, \tag{6.4}$$

$$\rho^2 = r^2 + \frac{1}{2k^4}\omega_{\mu\nu}\omega^{\mu\nu}, \tag{6.5}$$

and $\rho^2$, as computed in (6.5), coincides with the quantities defined in Eqs. (5.14), (5.20) and (5.23).

The above result (6.3) deserves a discussion. It appears as a limited derivative expansion on the velocity field of the boundary geodesic congruence $\check{u} = \partial_t$. The latter carries neither expansion, nor shear (see Secs. 2.1 and 3.1) but only vorticity given in Eqs. (4.28), (4.5) and (4.15) for the three backgrounds under investigation Kerr–AdS (5.12), Taub–NUT–AdS (5.18) and the hyperbolic NUT–AdS (5.22). It seems that at most two derivatives of the velocity field are involved, but this counting is naive. Indeed, the vorticity being ultimately an intrinsic property of the geometry[23], $R \sim \omega^2$, while $C\omega \sim \omega^4$. Furthermore, $1/\rho^2$ is a resummed power series in even powers of the vorticity. As already advertised, this resummation betrays the infinite perturbative expansion underlying the method, that would otherwise appear as limited to the fourth order.

The metric (6.3) yields an exact solution of AdS$_4$ gravity for a large class of boundary Randers data $(b_i, a_{ij})$. In addition to the cases described above, one can rewrite in this form the full Kerr–Taub–NUT–AdS$_4$ family of metrics, as well as all rotating topological black holes found in [66]: the rotating black cylinder and the rotating hyperbolic black membrane. All these metrics belong to the Plebański–Demiański type-D class of solutions [70]. It is an interesting problem to find to see if it is possible to extend this collection and find the most general set of Randers data $(b_i, a_{ij})$ generating an exact solution through (6.3).

It is remarkable that all known exact AdS$_4$ black hole solutions can be set in the above form (6.3), much like Kerr–AdS, provided an extra term (one should say an extra resummed series) based on the Cotton tensor is added. This term, of fourth order in the derivatives of the velocity field[24], was absent in the original expressions of [16, 69], only valid up to second order. In five or higher bulk dimensions, terms involving the Weyl tensor appear at the second order [16, 69], but obviously do not contribute in the four-dimensional case under consideration. Our expression

[23] Vorticity components are directly related to the connection coefficients, as e.g. $\Gamma^i_{tj} = -\omega^i{}_j$.

[24] The Cotton tensor itself is third order in the derivatives of the velocity field, but it cannot appear at this order in the fluid/gravity metric because it has the wrong parity. The fourth order is indeed the smallest order for which it can appear.

(6.3) shows that in four dimensions analogous terms, involving the Cotton tensor, appear in the derivative expansion starting from the fourth order.

## 7. Conclusions

In this review we presented an extensive discussion of the holographic description of vorticity. This is the first step in efforts to extend AdS/CMT to systems such a rotating atomic gases of analogue gravity systems. The upshot is that even the simplest setup, namely non-dissipating fluids in local equilibrium with non-zero vorticity, has an extremely rich geometric structure whose detailed analysis should lead to new and interesting physical results. One such result, presented in [23] was the calculation of the classical rotational Hall viscosity coefficient of neutral $2+1$ dimensional fluids having uniform vorticity $\Omega$

$$\zeta_{\mathrm{H}} = \frac{\varepsilon + p}{\Omega}, \tag{7.1}$$

which we were not able to find in recent works on parity broken hydrodynamics in $2+1$ dimensions.

The next steps in our program will certainly reveal interesting physical consequences. For example, the study of scalar, vector and ultimately graviton fluctuations around the above geometries should lead to the determination of various transport coefficients for rotating neutral fluids. These developments might also shed light on the thermalization processes that are expected near analogue horizons. Furthermore, we believe that our approach offers a well-defined path to study the issue of time-dependence in conjunction with irreversible, non-equilibrium dynamics as it can appear in dissipative fluid configurations or in the vicinity of analogue horizons.

In the present work we have emphasized the importance of the nut charge in the holographic description of vorticity. In superfluids this is a quantized quantity, hence one might wonder whether and how nut-charge quantization could emerge in their holographic description. The latter is in fact incomplete and the formation of vortices in rotating condensates calls for a more complete understanding, which justifies our efforts.

An issue worth mentioning is the relationship of our work with alternative approaches of fluid/gravity correspondence. We have presented some preliminary results in Sec. 6 and we plan to elaborate on that subject in a forthcoming work. Many other roads seem open for further investigation, which we have not discussed. One could for example try to describe fluids in more complicated kinematic states, with multipolar vorticity – as a generalization of the monopole-like configurations created by nut charges, or the dipoles corresponding to Kerr cyclonic motions. This would require the generalization of the Weyl multipole solutions to asymptotically AdS spaces. The magnetic paradigm of the geodesic motion in Papapetrou–Randers geometries (see e.g. [21]) might turn in a powerful tool for that task. Let us finally mention that analogue-gravity applications are very rich and diverse. Setting the bridge with holographic techniques would however require a more systematic study.

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# Holographic perfect fluidity, Cotton energy–momentum duality and transport properties

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Abstract: We investigate background metrics for $2+1$-dimensional holographic theories where the equilibrium solution behaves as a perfect fluid, and admits thus a thermodynamic description. We introduce stationary perfect-Cotton geometries, where the Cotton tensor takes the form of the energy–momentum tensor of a perfect fluid. Fluids in equilibrium in such boundary geometries have non-trivial vorticity. The corresponding bulk can be exactly reconstructed to obtain $3+1$-dimensional stationary black-hole solutions with no naked singularities for appropriate values of the black-hole mass. It follows that an infinite number of transport coefficients vanish for holographic fluids. Our results imply an intimate relationship between black-hole uniqueness and holographic perfect equilibrium. They also point towards a Cotton/energy–momentum tensor duality constraining the fluid vorticity, as an intriguing boundary manifestation of the bulk mass/nut duality.

## Contents

## 1 Introduction

It is known that thermodynamics may not be sufficient to describe the equilibrium state of systems when they are put on generic curved backgrounds. This would be possible if the system is in *perfect equilibrium*, namely its energy–momentum tensor has the perfect fluid form. This is because equilibrium is a special solution of hydrodynamics where entropy production is absent. Such a solution, if it exists at all, it need not be unique. Hence it is an important question to investigate which stationary background geometries support perfect equilibrium. We will call such geometries *perfect geometries* and their study in a holographic setup is the main issue of the present work.

Relativistic hydrodynamics provides the long wavelength[1] effective description for many-body quantum systems in local thermal equilibrium with an underlying Lorentz symmetry. In the hydrodynamical regime a neutral system (*i.e.* one without globally conserved charges) is described by a conserved energy–momentum tensor $T_{\mu\nu}$,[2] which is a function of the macroscopic hydrodynamical variables, namely the local temperature $T(x)$ and the velocity vector field $u_\mu(x)$ that determines the flow. In addition, the system's description depends on the background metric, $g_{\mu\nu}(x)$, which in the hydrodynamic limit is weakly curved, *i.e.* its curvature radius is much larger than the mean free path.

The hydrodynamic energy–momentum tensor can be constructed in well-known phenomenological expansion, the so-called derivative expansion (see for example [1] and [2]). The expansion parameter is the ratio of the mean-free path to the typical length scale of variation of hydrodynamic variables. The leading term is the perfect-fluid energy–momentum tensor constructed purely out of thermodynamic inputs:

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu + p g_{\mu\nu},$$

where $\varepsilon$ and $p$ are the energy density and the pressure respectively. Then, the conservation equation *i.e.* $\nabla^\mu T_{\mu\nu} = 0$ leads to the relativistic Euler equations.

The higher-order corrections are accompanied by transport coefficients which are independent of the background but are functions of the temperature and the couplings of the underlying microscopic theory, and can be obtained from the low-energy limit of correlation functions involving the energy–momentum tensor. These transport coefficients can be divided into dissipative and non-dissipative ones. The dissipative ones lead to entropy production, so they play a role only in non-equilibrium situations. On the other hand, non-dissipative transport coefficients do not contribute to entropy production, and therefore do play a role in determining the equilibrium description of the fluid.

[1]This means that the scale on which physical quantities vary is much larger than the mean free path of the microscopic theory, hence a derivative expansion is justifiable.

[2]The hydrodynamic energy–momentum tensor is the expectation value of the corresponding quantum operator in the vacuum state of the theory. The fact that it is non-zero means that Lorentz symmetry is broken in the vacuum and this is equivalent to the existence of the non-trivial velocity field $u_\mu$. Notice also that in quantum field theory, vacuum expectation values are usually evaluated in the absence of sources. However, in a hydrodynamic system the source is actually the background metric, hence it does not make sense to set it to zero. It does make sense, however, to start from the simplest possible case where the background metric is the Minkowski one.

Holography asserts the existence of a one-to-one correspondence between states of a conformal fluid theory on the boundary and its dual bulk geometry (see e.g. [3, 4]). The latter is required to be a solution of vacuum Einstein equations with a regular horizon, which in turn implies the existence of a relation between the boundary metric and energy–momentum tensor. We consider here $2+1$-dimensional holographic fluids in perfect equilibrium and determine the corresponding perfect geometries that can support them. This amounts to fixing the form of the energy–momentum tensor and looking for the boundary geometry that gives regular bulk solution. Such a procedure is rather unconventional in AdS/CFT, where usually one fixes the boundary metric instead. We further impose such perfect geometries to be stationary with a unique time-like Killing vector of unit norm. As a consequence, the corresponding Einstein metric will be stationary as well.

An important clue to answering this question is provided by conformally self-dual gravitational instantons of four-dimensional Euclidean Einstein's gravity with negative cosmological constant. When mapped into Lorentzian signature, the self-duality of the Weyl tensor hints at a certain duality between the Cotton–York tensor and the energy–momentum tensor of the boundary geometry [5–8]. This duality implies that the Cotton–York tensor of the boundary geometry is proportional to the energy–momentum tensor, and we call geometries with such property *perfect-Cotton geometries*. We will prove that perfect-Cotton boundary geometries correspond to resummable exact bulk solutions of Einstein's equations. This result is an important achievement of the present work. The reverse statement is an open interesting question. Its relationship with the gravitational duality is also far reaching, although putting it on firmer grounds requires more work.

Moreover, when an extra spatial isometry is present in the background boundary metric, we are able to write the explicit form of the bulk Einstein solutions, recovering well-known four-dimensional stationary black-hole metrics such as the AdS–Kerr–NUT, as well as less studied solutions.

Finally, our analysis shows as well that holography fixes to zero the value of infinitely many boundary transport coefficients that would spoil the perfect-fluid form of the energy–momentum tensor. Indeed, non-vanishing transport coefficient would give derivative corrections to the energy–momentum tensor up to arbitrary high orders, which would have been in contradiction with black-hole uniqueness.

Recently, using purely field-theoretic arguments it has been possible to put constraints on non-dissipative transport coefficients order by order in derivative expansion [9]. Such constraints are valid in any field theory and they are not tied to the existence of a holographic description. One can also derive similar constraints by requiring that the fluid mechanics should have a local entropy current with positive-definite divergence [10]. Even though this requirement does not make any use of holography, it seems to be valid in holographic theories only, as the entropy current of the boundary fluid is related to the generalized second law for dynamical black-hole horizons. It is puzzling that the equilibrium partition function approach of [9] is claimed to be sufficient to prove the existence of an entropy current in fluid mechanics. According to the analysis presented here, only a restricted class of boundary geometries can accommodate perfect fluidity for holographic theories, and this feature is provided holographically by black-hole dynamics. Hence, the consequence that infinitely many transport coefficients of a certain kind vanish in holographic theories cannot be derived entirely from either of the general approaches based on the partition function or the entropy current arguments. A detailed discussion of this important issue is beyond the scope of the present work.

The organization of the paper is as follows. In Sec. 2, we briefly review equilibrium as a solution of relativistic fluid mechanics and give necessary and sufficient condition for perfect equilibrium to exist. In Sec. 3, we discuss the stationary geometries in Papapetrou–Randers form and the kinematics of fluids in perfect equilibrium in such geometries. In Sec. 4, we study perfect-Cotton geometries and explicitly classify all of them in the special case when the boundary metric has an additional spatial isometry. Fluids in perfect equilibrium in such perfect-Cotton geometries will be studied in Sec. 5, followed by the the uplift of the corresponding boundary data to exact black-hole solutions. The validity of the rigidity theorem is also discussed. In Sec. 6, we find that an infinite number of transport coefficients should vanish for perfect-Cotton geometries to be exactly upliftable. Finally, we conclude with some comments and further directions of research.

## 2 Hydrodynamics and the equilibrium

We focus here on the $2+1$-dimensional boundary fluid system, presenting briefly its equilibrium description, and then analyzing the special case when the equilibrium is given by a perfect fluid.

### 2.1 Relativistic hydrodynamics on $2+1$-dimensional curved backgrounds

As mentioned, in the hydrodynamic limit the energy–momentum tensor $T^{\mu\nu}$ of a neutral fluid is a function of the local temperature $T(x)$, of the velocity field $u^\mu(x)$, of the background metric $g_{\mu\nu}(x)$ and of their covariant derivatives. The hydrodynamic equations are simply given by the covariant conservation of the energy–momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.1}$$

One way to define the basic thermodynamic variables is within the so-called *Landau frame*, where the non-transverse part of the energy-momentum tensor vanishes when the pressure is zero. This implies that $u^\mu$ is an eigenvector of the energy–momentum tensor with the eigenvalue being the local energy density $\varepsilon(x)$, namely $T^\mu{}_\nu u^\nu = -\varepsilon u^\mu$. If we moreover require the velocity field to be a time-like vector of unit norm, then u is uniquely defined at each point in space and time. Furthermore, we can use the equation of state for static local equilibrium[3] $\varepsilon = \varepsilon(T)$ to define the temperature $T$. Once we have defined a local temperature $T$, we can again use the equation of state to define the pressure $p(x)$. A local

[3]For the global equilibrium case, the internal energy is a function of both $T$ and the angular velocity $\Omega$, (which can be defined if the background metric has a Killing vector corresponding to an angular rotation symmetry). In the case of local equilibrium, $\varepsilon$ is a function of $T$ alone because a dependence on $\Omega$ would not be compatible with the derivative expansion. Indeed, $\Omega$ is first-order in derivatives but $\varepsilon$ is zeroth order. The global energy function $E(T, \Omega)$ can be reproduced by integrating the various components of the equilibrium form of $T_{\mu\nu}$ [12].

entropy density $s(x)$ can be also introduced. Both $p$ and $s$ can be readily obtained from the thermodynamic identities: $\varepsilon + p = Ts$ and $\mathrm{d}\varepsilon = T\mathrm{d}s$. In a conformal $2+1$-dimensional system, $\varepsilon$ and $p$ are proportional to $T^3$ while $s$ is proportional to $T^2$.

Under the assumptions above, the energy–momentum tensor of a neutral hydrodynamic system can be expanded in derivatives of the hydrodynamical variables, namely

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + T^{\mu\nu}_{(2)} + \cdots\,, \tag{2.2}$$

where the subscript denotes the number of covariant derivatives. Note that the inverse length scale introduced by the derivatives is taken to be large compared to the microscopic mean free path. The zeroth order energy–momentum tensor is the so called perfect-fluid energy–momentum tensor:

$$T^{\mu\nu}_{(0)} = \varepsilon u^\mu u^\nu + p\Delta^{\mu\nu}, \tag{2.3}$$

where $\Delta^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$ is the projector onto the space orthogonal to u. This corresponds to a fluid being locally in static equilibrium. The conservation of the perfect-fluid energy–momentum tensor leads to the relativistic Euler equations:

$$\begin{cases} \nabla_{\mathrm{u}}\varepsilon + (\varepsilon + p)\Theta = 0, \\ \nabla_\perp p - (\varepsilon + p)\mathrm{a} = 0, \end{cases} \tag{2.4}$$

where $\nabla_{\mathrm{u}} = \mathrm{u}\cdot\nabla$, $\Theta = \nabla\cdot\mathrm{u}$, $\nabla_{\perp\mu} = \Delta_\mu{}^\nu\nabla_\nu$, and $a^\mu = (\mathrm{u}\cdot\nabla)u^\mu$ (more formulas on kinematics of relativistic fluids are collected in App. A).

The higher-order corrections to the energy–momentum tensor involve the transport coefficients of the fluid. These are phenomenological parameters that encode the microscopic properties of the underlying system. In the context of field theories, they can be obtained from studying correlation functions of the energy–momentum tensor at finite temperature in the low-frequency and low-momentum regime (see for example [13]).

Transport coefficients are of two kinds: dissipative and non-dissipative ones. The former potentially contribute to the entropy production in systems evolving out of global thermodynamic equilibrium.[4] The phenomenological discussion of hydrodynamic transport is precisely based on the existence of an entropy current whose covariant divergence describes entropy production and hence must be positive-definite. This puts bounds on the dissipative transport coefficients and imposes relations between non-dissipative transport coefficients order by order in the derivative expansion [10]. A complete classification of all transport coefficients is clearly a huge task. For the purposes of this work it is sufficient to distinguish the non-dissipative transport coefficients from the dissipative ones by whether or not the tensor structures they couple to are invariant under time inversion (T-even).

A conformal fluid has traceless and Weyl-covariant energy–momentum tensor leading, in $2+1$ dimensions, to the relation $\varepsilon = 2p$. Furthermore, the Landau-frame choice requires transversality. These properties need of course to be valid order by order in the derivative expansion, *i.e.* for every term appearing in (2.2). In App. B we give details of the

[4]Local thermodynamic equilibrium will always be assumed in our discussions as it is required for the hydrodynamic description to make sense.

construction of such Weyl-covariant traceless and transverse tensors. We will here provide a few illustrative examples.

If we do not require parity invariance, at first order in $2+1$ dimensions, we can have only two such tensors, namely $\sigma^{\mu\nu}$ given in (A.4) (or (A.5)) and $\eta^{\rho\lambda(\mu}u_\rho\sigma_\lambda{}^{\nu)}$, where $\eta^{\mu\nu\rho} = \epsilon^{\mu\nu\rho}/\sqrt{-g}$ is the covariant fully antisymmetric tensor with $\epsilon^{012} = -1$. The first-order correction to the energy–momentum tensor thus reads:

$$T^{\mu\nu}_{(1)} = -2\eta\sigma^{\mu\nu} - \zeta_{\mathrm{H}}\eta^{\rho\lambda(\mu}u_\rho\sigma_\lambda{}^{\nu)}. \tag{2.5}$$

The first term in (2.5) involves the shear viscosity $\eta$, which is a dissipative transport coefficient. The second is present in systems that break parity and involves the non-dissipative rotational-Hall-viscosity coefficient $\zeta_{\mathrm{H}}$ in $2+1$ dimensions. Notice that the bulk-viscosity term $\zeta\Delta^{\mu\nu}\Theta$ cannot appear in a conformal fluid because it is tracefull, namely for conformal fluids $\zeta = 0$.

The next-order terms in (2.2) can be worked out for the fluids at hand. One can easily see that there are no T-even tensors at second order. But at third order the T-even tensors non-vanishing at equilibrium, which also do not depend on acceleration, shear and expansion are:

$$T^{\mu\nu}_{(3)} = \gamma_{(3)1}\langle C^{\mu\nu}\rangle + \gamma_{(3)2}\langle \mathcal{D}^\mu W^\nu\rangle, \tag{2.6}$$

where $C^{\mu\nu}$ is the Cotton–York tensor. For a second rank tensor $A^{\mu\nu}$ we have introduced

$$\langle A^{\mu\nu}\rangle = \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(A_{\alpha\beta} + A_{\beta\alpha}) - \frac{1}{2}\Delta^{\mu\nu}\Delta^{\alpha\beta}A_{\alpha\beta}. \tag{2.7}$$

The Weyl-covariant derivative $\mathcal{D}_\mu$ is defined in App. B, and $W^\mu$ is given by

$$W^\mu = \eta^{\mu\nu\rho}u_\nu V_\rho, \quad V^\mu = \nabla^\alpha_\perp \omega^\mu{}_\alpha + u^\mu\omega_{\alpha\beta}\omega^{\alpha\beta}, \tag{2.8}$$

with $\omega_{\alpha\beta}$ being the vorticity defined in (A.6) (or (A.7)). At the fourth order in derivative expansion, there will be non-dissipative transport coefficients corresponding to T-invariant tensors like $\langle V_\mu V_\nu\rangle$, $\langle W_\mu W_\nu\rangle$, etc.

### 2.2 Perfect equilibrium

Stationary solutions[5] of the relativistic equations of motion (2.1), when they exist, describe a fluid in global thermodynamic equilibrium.[6] The prototype example of such a situation is the one of an inertial fluid in Minkowski background with globally defined constant temperature, energy density and pressure. In this case, irrespective of whether the fluid itself is viscous, its energy–momentum tensor, evaluated at the solution, takes the zeroth-order (perfect) form (2.3) because all derivatives of the hydrodynamic variables vanish and Eqs. (2.4) are satisfied.

[5]It is admitted that a non-relativistic fluid is stationary when its velocity field is time-independent. This is of course an observer-dependent statement. For relativistic fluids, one could make this more intrinsic saying that the velocity field commutes with a globally defined time-like Killing vector, assuming that the later exists.

[6]This should not be confused with a steady state, where we have stationarity due to a balance between external driving forces and internal dissipation. Such situations will not be discussed here.

Local thermodynamic equilibrium, in general, does not require the zeroth-order equations (2.4) to be satisfied. It is however relevant to ask: are there other situations where the hydrodynamic description of a system is also perfect *i.e.* the energy–momentum tensor, in equilibrium, takes the perfect form (2.3) solving Eqs.(2.4)? As anticipated in the introduction, we call these special configurations *perfect equilibrium states*, where global thermodynamic description applies. One should stress, however, that more general equilibrium states can exist, for which only the full equations (2.1) with (2.2) are satisfied. Owing to the fact that global thermodynamic equilibrium is incompatible with entropy production, all dissipative terms in (2.2) for these more general equilibrium states necessarily vanish,[7] either because the dissipative transport coefficients are zero, or because the corresponding tensors vanish kinematically – requiring in particular a special relationship between the fluid's velocity and the background geometry. Clearly, for perfect equilibrium states, *all* higher-derivative terms in (2.2) are absent, making their realization more challenging. In this work we will provide classes of perfect equilibrium states.

Consider a hydrodynamic system with a stationary background metric, having a unique time-like, normalised Killing vector $\xi = \xi^\mu \partial_\mu$, namely

$$\nabla_{(\mu}\xi_{\nu)} = 0, \quad \xi_\mu \xi^\mu = -1. \tag{2.9}$$

Although not exhaustive, these systems are interesting in view of their intimate connection with holography, as we will see in the following. Congruences defined by $\xi_\mu$ have vanishing acceleration, shear and expansion (see App. A), but non-zero vorticity $\omega = \frac{1}{2}\mathrm{d}\xi \Leftrightarrow \omega_{\mu\nu} = \nabla_\mu \xi_\nu$. Then, it is easy to show that a special solution of the Euler equations (2.4) is:

$$\mathrm{u} = \xi, \quad T = \text{constant}, \quad \varepsilon = 2p = \text{constant}. \tag{2.10}$$

In fact this is the unique equilibrium solution, if the background has only a unique time-like Killing vector field of unit norm. Non-zero acceleration, shear and expansion can all contribute to dissipation, these vanish only when u is the Killing vector field $\xi$ itself. For the above configuration (2.10) to be a perfect equilibrium state, one must show that all higher-derivative corrections in (2.2) are actually absent. Since the congruence is shearless, the first corrections (2.5) vanish. If higher-order corrections do also vanish, the fluid indeed reaches this specific global equilibrium state, in which it aligns itself with the congruence of the Killing vector field. For observers whose worldlines are identified with the Killing congruence at hand, the fluid is at rest: the fluid and the observers are comoving. Had the higher-derivative corrections been non-zero, this comoving state with constant temperature would not have been necessarily an equilibrium state as it would not have been a solution of the equations of motion given by (2.1). Equations (2.4) would have been altered, leading in general to $\mathrm{u} = \xi + \delta\mathrm{u}(\mathrm{x})$ and $T = T_0 + \delta T(\mathrm{x})$. Such an excursion will be stationary or not depending on whether the non-vanishing corrections to the perfect energy–momentum tensor are non-dissipative or dissipative.

In order to analyze under which conditions on the transport coefficients, perfect fluid equilibrium (2.10) is realized, we must list, assuming (2.9), the Weyl-covariant, trace-

[7] See e.g. [10] for a recent discussion.

less and transverse tensors $\mathcal{T}_{\mu\nu}$ that are non-vanishing and whose divergence is also non-vanishing, when evaluated in the perfect equilibrium solution (2.10).

We call such tensors *dangerous tensors*. Their presence can destroy the existence of the perfect equilibrium solution, unless the corresponding transport coefficients are vanishing. At every order in the derivative expansion we have a finite number of linearly independent dangerous tensors and each one of them is associated with a transport coefficient, which we call *dangerous transport coefficient*. Hence, a necessary and sufficient[8] condition for the existence of perfect equilibrium in backgrounds with a normalized time-like Killing vector field is that *all dangerous transport coefficients vanish*. The vanishing of the latter is a statement about the underlying microscopic theory about which we can thus gain new non-trivial information.

We will encounter non-trivial special backgrounds (a trivial example being the Minkowski space) where no dangerous tensors are present. On the other hand, we will also consider a large class of backgrounds with a unique normalized time-like Killing vector field, which have infinitely many non-zero dangerous tensors; thus we will be able to probe that an infinite number of non-dissipative transport coefficients vanish. Nevertheless, the question of whether our analysis regarding all possible transport coefficients is exhaustive or not lies beyond the scope of the present work. It is clear that further insight on this matter can only be gained by perturbing the perfect equilibrium state.

## 3 Fluids in Papapetrou–Randers geometries

A stationary metric can be written in the generic form

$$\mathrm{d}s^2 = B^2\left(-(\mathrm{d}t - b_i \mathrm{d}x^i)^2 + a_{ij}\mathrm{d}x^i\mathrm{d}x^j\right), \tag{3.1}$$

where $B, b_i, a_{ij}$ are space-dependent but time-independent functions. These metrics were introduced by Papapetrou in [14]. They will be called hereafter *Papapetrou–Randers* because they are part of an interesting network of relationships involving the Randers form [15], discussed in detail in [16] and more recently used in [17–19].

In order for the time-like Killing vector $\partial_t$ to be normalized to $-1$, we must restrict ourselves to the case $B = 1$. Then, $\partial_t$ is identified with the generically *unique* normalized time-like Killing vector of the background and draws the geodesic congruence associated with the fluid worldlines. The normalised three-velocity one-form of the stationary perfect fluid is then

$$\mathrm{u} = -\mathrm{d}t + \mathrm{b}, \tag{3.2}$$

where $\mathrm{b} = b_i \mathrm{d}x^i$. We will often write the metric (3.1) as

$$\mathrm{d}s^2 = -\mathrm{u}^2 + \mathrm{d}\ell^2\,, \quad \mathrm{d}\ell^2 = a_{ij}\,\mathrm{d}x^i\mathrm{d}x^j. \tag{3.3}$$

[8] Although this condition is necessary if one assumes the specific perfect equilibrium with alignment of u along the Killing vector $\xi$, we cannot, on general grounds, exclude other perfect equilibrium configurations. Since this subtlety is irrelevant for our subsequent analysis, we will not make any stronger statement about the precise nature of the condition.

We will adopt the convention that hatted quantities will be referring to the two-dimensional positive-definite metric $a_{ij}$, therefore $\hat{\nabla}$ for the covariant derivative and $\hat{R}_{ij}\,\mathrm{d}x^i\mathrm{d}x^j = \frac{\hat{R}}{2}\mathrm{d}\ell^2$ for the Ricci tensor built out of $a_{ij}$. For later convenience, we introduce the inverse two-dimensional metric $a^{ij}$ and $b^i$ such that

$$a^{ij}a_{jk} = \delta^i_k, \quad b^i = a^{ij}b_j. \tag{3.4}$$

The three-dimensional metric components read:

$$g_{00} = -1, \quad g_{0i} = b_i, \quad g_{ij} = a_{ij} - b_i b_j, \tag{3.5}$$

and those of the inverse metric:

$$g^{00} = a^{ij}b_i b_j - 1, \quad g^{0i} = b^i, \quad g^{ij} = a^{ij}. \tag{3.6}$$

Finally,

$$\sqrt{-g} = \sqrt{a}, \tag{3.7}$$

where $a$ is the determinant of the symmetric matrix with entries $a_{ij}$.

A perfect fluid at equilibrium, or a fluid at perfect equilibrium (whenever this is possible, along the discussion of Sec. (2.2), on a Papapetrou–Randers background is such that the worldline of every small part of it is aligned with a representative of the congruence tangent to $\partial_t$. Since $\partial_t$ is a unit-norm Killing vector, the fluid's flow is geodesic, has neither shear, nor expansion, but does have vorticity, which is inherited from the fact that $\partial_t$ is not hypersurface-orthogonal.[9] Using (3.2) and (A.7) we find that the vorticity can be written as the following two-form

$$\omega = \frac{1}{2}\omega_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}\mathrm{db}. \tag{3.8}$$

The Hodge-dual of $\omega_{\mu\nu}$ is

$$\psi^\mu = \eta^{\mu\nu\rho}\omega_{\nu\rho} \Leftrightarrow \omega_{\nu\rho} = -\frac{1}{2}\eta_{\nu\rho\mu}\psi^\mu. \tag{3.9}$$

In $2+1$ dimensions it is aligned with the velocity field:

$$\psi^\mu = qu^\mu, \tag{3.10}$$

where

$$q(x) = -\frac{\epsilon^{ij}\partial_i b_j}{\sqrt{a}} \tag{3.11}$$

is a static scalar field that we call *t*he vorticity strength, carrying dimensions of inverse length. Together with $\hat{R}(x)$, the above scalar carries all relevant information for the curvature of the Papapetrou–Randers geometry. We quote for latter use the three-dimensional curvature scalar:

$$R = \hat{R} + \frac{q^2}{2}, \tag{3.12}$$

[9]For this very same reason, Papapetrou–Randers geometries may in general suffer from global hyperbolicity breakdown. This occurs whenever regions exist where $b_i b^i > 1$. There, constant-$t$ surfaces cease being space-like, and potentially exhibit closed time-like curves. This issues were discussed in detail in [17–19].

the three-dimensional Ricci tensor

$$R_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{q^2}{2}\mathrm{u}^2 + \frac{\hat{R}+q^2}{2}\mathrm{d}\ell^2 - \mathrm{u}\,\mathrm{d}x^\rho u^\sigma \eta_{\rho\sigma\mu}\nabla^\mu q, \tag{3.13}$$

as well as the three-dimensional Cotton–York tensor [20]:

$$\begin{aligned} C_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu \;=\; & \frac{1}{2}\left(\hat{\nabla}^2 q + \frac{q}{2}(\hat{R}+2q^2)\right)\left(2\mathrm{u}^2 + \mathrm{d}\ell^2\right) \\ & -\frac{1}{2}\left(\hat{\nabla}_i\hat{\nabla}_j q\,\mathrm{d}x^i\mathrm{d}x^j + \hat{\nabla}^2 q\,\mathrm{u}^2\right) \\ & -\frac{\mathrm{u}}{2}\mathrm{d}x^\rho u^\sigma \eta_{\rho\sigma\mu}\nabla^\mu(\hat{R}+3q^2). \end{aligned} \tag{3.14}$$

The latter is a symmetric and traceless tensor defined in general as

$$C^{\mu\nu} = \eta^{\mu\rho\sigma}\nabla_\rho\left(R^\nu{}_\sigma - \frac{1}{4}R\delta^\nu_\sigma\right). \tag{3.15}$$

In three-dimensional geometries it replaces the always vanishing Weyl tensor. In particular, conformally flat backgrounds have zero Cotton–York tensor and vice versa.

The fluid in perfect equilibrium on Papapetrou–Randers backgrounds has the energy–momentum tensor

$$T^{(0)}_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = p\left(2\mathrm{u}^2 + \mathrm{d}\ell^2\right), \tag{3.16}$$

with the velocity form being given by (3.2) and $p$ constant. We have used here $\varepsilon = 2p$. We recall that $\varepsilon$ has dimensions of energy density or equivalently $(\text{length})^{-3}$, therefore the energy–momentum tensor and the Cotton–York tensor have the same natural dimensions. This is crucial for the following.

As discussed in the previous section, the fluid can attain perfect equilibrium if and only if all the dangerous transport coefficients vanish. It is not hard to see that this will imply constraints on transport coefficients as these stationary backgrounds will generically have infinitely many associated dangerous tensors. For example, there exist non-vanishing tensor structures involving $\nabla^n \mathrm{u}, \nabla^n q$ with $n > 1$, which are traceless, transverse and Weyl-covariant, with $q$ being evaluated on the perfect fluid solution as in (3.11). One simple example of such a tensor is $\langle \mathcal{D}_\mu W_\nu \rangle$ (one of the terms in (2.6)) which, when evaluated with u as in (3.2), will be given in terms of covariant derivatives of $q$. Also generically $\nabla^\mu \langle \mathcal{D}_\mu W_\nu \rangle \neq 0$ in these stationary backgrounds. This is a dangerous tensor and the corresponding dangerous transport coefficient must vanish, in order that the fluid can attain perfect equilibrium.

When the stationary background has additional isometries, most Weyl-covariant, traceless and transverse tensors built from derivatives of u as given by (3.2) will vanish. This will make it hard for dangerous tensors to exist. In the following Sec. 4.2.1.1 and 4.2, we will find examples with (i) homogeneous and (ii) axisymmetric spaces, where indeed such a conspiracy will happen. Many possible dangerous tensors will vanish, therefore the corresponding transport coefficients need not vanish in order for perfect equilibrium to exist. As expected, the higher the symmetry of a background, the less number of transport coefficients we will be able to probe by demanding that perfect equilibrium should exist.

## 4 Perfect-Cotton geometries

The existence of perfect geometries is an issue unrelated to holography. However, in the context of the fluid/gravity correspondence a special class of Papapetrou–Randers background geometries for holographic fluids are perfect geometries. In this section we prove that this is the case if the Cotton–York tensor of the boundary metric takes the same of form of the perfect-fluid energy–momentum tensor

$$C_{\mu\nu} = \frac{c}{2}(3u_\mu u_\nu + g_{\mu\nu}), \tag{4.1}$$

where $c$ is a constant with the dimension of an energy density. We call such geometries *perfect-Cotton geometries*, and we present here their properties, as well as the complete classification when an extra spatial isometry is present. Moreover, perfect-Cotton geometries appear as boundaries of $3+1$-dimensional exact Einstein spaces, which will be studied in the next section.

### 4.1 Definition

Consider a Papapetrou–Randers metric (3.1). Requiring its Cotton–York tensor (3.14) to be of the form (4.1) is equivalent to impose the conditions:

$$\hat{\nabla}^2 q + q(\delta - q^2) = 2c \tag{4.2}$$

$$a_{ij}\left(\hat{\nabla}^2 q + \frac{q}{2}(\delta - q^2) - c\right) = \hat{\nabla}_i \hat{\nabla}_j q \tag{4.3}$$

$$\hat{R} + 3q^2 = \delta, \tag{4.4}$$

with $\delta$ being a constant relating the curvature of the two-dimensional base space $\hat{R}$ with the vorticity strength $q$.

Without loss of generality, we can choose the two-dimensional coordinates $x$ and $y$ in such a way that the base metric $a_{ij}$ is diagonal

$$\mathrm{d}\ell^2 = A^2(x,y)\mathrm{d}x^2 + B^2(x,y)\mathrm{d}y^2 \tag{4.5}$$

and that the spatial component of the velocity vector takes the form:

$$\mathrm{b} = b(x,y)\,\mathrm{d}y. \tag{4.6}$$

The vorticity strength (3.11) reads thus

$$q = -\frac{\partial_x b}{AB}. \tag{4.7}$$

Further gauge fixing is possible and will be made when appropriate[10]. The explicit form of Eqs. (4.2)–(4.4) in terms of $A(x,y), B(x,y)$ and $b(x,y)$ is not very illuminating and we do not present it here.

[10]For example, since any two-dimensional space is conformally flat it is possible to set $A = B$. We should however stress that all these choices are local, and the range of coordinates should be treated with care in order to avoid e.g. conical singularities.

### 4.2 Geometries with space-like Killing vectors

The presence of an additional unique space-like isometry simplifies the conditions for a Papapetrou–Randers metric to be perfect-Cotton. Without loss of generality, we take the additional Killing vector to be $\partial_y$ and we chose a representation such that $A^2 = {}^1\!/_{G(x)}$, $B^2 = G(x)$ and $b = b(x)$. The metric takes then the form

$$\mathrm{d}s^2 = -\left(\mathrm{d}t - b(x)\,\mathrm{d}y\right)^2 + \frac{\mathrm{d}x^2}{G(x)} + G(x)\mathrm{d}y^2, \tag{4.8}$$

and we are able to solve (4.2)–(4.4) in full generality. The solution is written in terms of 6 arbitrary parameters $c_i$:

$$b(x) = c_0 + c_1 x + c_2 x^2, \tag{4.9}$$

$$G(x) = c_5 + c_4 x + c_3 x^2 + c_2 x^3 \left(2c_1 + c_2 x\right). \tag{4.10}$$

It follows that the vorticity strength takes the linear form

$$q(x) = -c_1 - 2c_2 x, \tag{4.11}$$

and the constants c and $\delta$ are given by:

$$\mathrm{c} = -c_1^3 + c_1 c_3 - c_2 c_4, \tag{4.12}$$

$$\delta = 3c_1^2 - 2c_3. \tag{4.13}$$

Finally, the Ricci scalar of the two-dimensional base space is given by

$$\hat{R} = -2\left(c_3 + 6c_2 x(c_1 + c_2 x)\right), \tag{4.14}$$

and using (3.12) one can easily find the form of the three-dimensional scalar as well. Not all the six parameters $c_i$ correspond to physical quantities: some of them can be just reabsorbed by change of coordinates. In particular, we set here $c_0 = 0$ by performing the diffeomorphism $t \to t + py$, with constant $p$, which does not change the the form of the metric.

#### 4.2.1 Non-vanishing $c_4$

To analyze this class, we first use perform the further diffeomorphism $x \to x + s$, with constant $s$, which keeps the form of the metric. By tuning the value of $s$ we can set $c_5$ to zero. Therefore, without loss of generality we can choose:

$$b(x) = c_1 x + c_2 x^2, \tag{4.15}$$

$$G(x) = c_4 x + c_3 x^2 + c_2 x^3 \left(2c_1 + c_2 x\right). \tag{4.16}$$

We are thus left with four arbitrary geometric parameters. For consistency we can check that $q(x)$, c, $\delta$, $R$ and $\hat{R}$ indeed depend only on these four parameters. Moreover, by performing the change of variables

$$\tilde{x} = c_4 x\,, \quad \tilde{y} = c_4 y\,, \quad \tilde{t} = c_4 t\,, \tag{4.17}$$

and defining new variables

$$\tilde{c}_3 = \frac{c_3}{c_4^2}\,, \quad \tilde{c}_1 = \frac{c_1}{c_4}\,, \quad \tilde{c}_2 = \frac{c_2}{c_4^2}\,, \tag{4.18}$$

we can see that $c_4$ is an overall scaling factor of the metric. Indeed, we have

$$b(x) = \tilde{c}_1\tilde{x} + \tilde{c}_2\tilde{x}^2, \tag{4.19}$$

$$G(x) = \tilde{x} + \tilde{c}_3\tilde{x}^2 + \tilde{c}_2\tilde{x}^3\,(2\tilde{c}_1 + \tilde{c}_2\tilde{x})\,, \tag{4.20}$$

which depend now on the three parameters $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{c}_3$. Using the above variables the metric becomes $\mathrm{d}\tilde{s}^2 = c_4^2\mathrm{d}s^2$. Since we are dealing with a conformal theory, we can set $c_4 = 1$ and use dimensionless quantities from now on. For simplicity then we drop all the tildes in the following.

**4.2.1.1 Monopoles: homogeneous spaces or too-perfect geometries** Consider the vorticity strength (4.11). The simplest example that can be considered is the one of constant $q$, that is when $c_2 = 0$. We call the corresponding geometries *monopolar geometries*, a terminology that we will justify in the following. The two-dimensional Ricci scalar (4.14) is in this case constant: $\hat{R} = -2c_3$. This means that the parameter $c_3$ labels the curvature signature of the two-dimensional base space and that, without loss of generality, we can set

$$c_3 = -\nu = 0, \pm 1\,. \tag{4.21}$$

Thus, we are left with one continuous parameter, $c_1$, which we rename as

$$c_1 = -2n\,. \tag{4.22}$$

Moreover, the Cotton–York tensor is proportional to

$$c = 2n(\nu + 4n^2)\,, \tag{4.23}$$

hence the parameter $n$ determines whether the geometry is conformally flat or not. Note that, apart from the trivial case $n = 0$, the space is conformally flat also when $\nu = -1$ and $4n^2 = 1$. The functions $b(x)$ and $G(x)$ take now the form

$$b(x) = -2nx\,, \quad G(x) = x(1 + \nu x)\,, \tag{4.24}$$

The form of $G(x)$ motivates the parametrization

$$x = f_\nu^2(\sigma/2)\,, \quad \begin{cases} f_1(\sigma) = \sin\sigma \\ f_0(\sigma) = \sigma \\ f_{-1}(\sigma) = \sinh\sigma \end{cases}\,, \quad y = 2\phi\,, \quad \phi \in [0, 2\pi]\,. \tag{4.25}$$

Then, the geometries (4.8) take the form

$$\mathrm{d}s^2 = -\left(\mathrm{d}t + 4nf_\nu^2(\sigma/2)\,\mathrm{d}\phi\right)^2 + \mathrm{d}\sigma^2 + f_\nu^2(\sigma)\mathrm{d}\phi^2\,, \tag{4.26}$$

which is that of fibrations over $S^2, \mathbb{R}^2$ and $H_2$ for $\nu = 1, 0, -1$ respectively. The two-dimensional base spaces are homogeneous with constant curvature having three Killing vectors; the three-dimensional geometry has in total four Killing vectors.

These geometries appear at the boundary of asymptotically anti-de Sitter Taub–NUT Einstein spaces with $n$ being the bulk nut charge. They were analyzed in detail years ago as families of three-dimensional geometries possessing 4 isometries [21, 22]. As homogeneous space–times, they are of the Bianchi type IX (warped $S^3$, here as Gödel space), II (Heisenberg group) and VIII (elliptically warped $\mathrm{AdS}_3$). The second space is also known as Som–Raychaudhuri [23].

We want now to discuss the presence of dangerous tensors. The velocity one-form is:

$$\mathrm{u} = -\mathrm{d}t - 4n f_\nu^2(\sigma/2)\mathrm{d}\phi, \tag{4.27}$$

while the vorticity has constant strength:

$$q = 2n. \tag{4.28}$$

Furthermore, the geometric data ensures the following structure:

$$R_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(\nu + 4n^2\right)\mathrm{u}^2 + \left(\nu + 2n^2\right)\mathrm{d}s^2 \tag{4.29}$$

The above condition implies that all hydrodynamic scalars, vectors and tensors that can be constructed from the Riemann tensor, its covariant derivatives and the covariant derivatives of u are algebraic. More specifically

- all hydrodynamic scalars are constants,
- all hydrodynamic vectors are of the form $ku_\mu$ with constant $k$, and
- all hydrodynamic tensors are of the form $au_\mu u_\nu + bg_{\mu\nu}$ with constant $a$ and $b$.

This means that there exists no traceless transverse tensor that can correct the hydrodynamic energy–momentum tensor in perfect equilibrium. In other words, there exists no dangerous tensor. Thus, in the case of monopolar geometries it is not possible to know the value of any transport coefficient.

This above result is not surprising. Indeed, we called Papapetrou–Randers configurations given by (4.25) and (4.26) of *monopolar* type because the vorticity is constant,[11] as a consequence of the homogeneous nature of these space–times. In such a highly symmetric kinematical configuration, the fluid dynamics cannot be sensitive to any dissipative or non-dissipative coefficient. This result provides a guide for the subsequent analysis: to have access to the transport coefficients, we must perturb the geometry away from the homogeneous configuration. The above discussion suggests that this perturbation should be organized as a multipolar expansion: the higher the multipole in the geometry, the richer the spectrum of transport coefficients that can contribute, if non-vanishing, to the global equilibrium state, and that we need to set to zero for perfect fluids.

[11] Note also that b, as given in (4.6) and (4.24), has the same form as the gauge potential of a Dirac monopole on $S^2$, $\mathbb{R}^2$ or $H_2$. This magnetic paradigm can be made more precise – see e.g. [16].

Finally, we note that the form of the Cotton–York tensor for monopolar geometries is

$$C_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = n(\nu+4n^2)(3\mathrm{u}^2+\mathrm{d}s^2). \tag{4.30}$$

The above expression can be combined (4.29), giving:

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \lambda g_{\mu\nu} = \frac{1}{\mu}C_{\mu\nu}. \tag{4.31}$$

The latter shows that monopolar geometries solve the topologically massive gravity equations [24] for appropriate constant $\lambda$ and $\mu$. This is not surprising, as it is a known fact that, for example, squashed anti-de-Sitter or three-spheres solve topologically massive gravity equations [25]. However, what is worth stressing here is that requiring a generic Papapetrou–Randers background (3.1) to solve (4.31) leads necessarily to a monopolar geometry. The argument goes as follows. Using the expression for the Ricci tensor for Papapetrou–Randers geometries (3.13), the left side of (4.31) reads:

$$\left(\hat{R}+\frac{q^2}{2}-6\lambda\right)\frac{\mathrm{u}^2}{2}+\left(\frac{q^2}{4}+\lambda\right)\left(2\mathrm{u}^2+\mathrm{d}\ell^2\right)-\mathrm{u}\,\mathrm{d}x^\rho u^\sigma\eta_{\rho\sigma\mu}\nabla^\mu q. \tag{4.32}$$

As the right side of (4.31) is traceless, so should be the left side. This leads to:

$$\lambda = \frac{\hat{R}}{6}+\frac{q^2}{12}. \tag{4.33}$$

Equations (4.31) can now be analyzed using the expression for the Cotton–York tensor (3.14) and (4.16) together with (4.33). The off-diagonal components $\mathrm{u}\,\mathrm{d}x^\rho$ imply that $q$ must be constant. With this at hand, the rest of the equations are automatically satisfied with:

$$q = \frac{2\mu}{3}. \tag{4.34}$$

In order to provide the general form of a Papapetrou–Randers metric satisfying (4.31), we can now combine (4.33) with (4.34). These lead to the conclusion that all solutions are fibrations over a two-dimensional space with metric $\mathrm{d}\ell^2$ of constant curvature $\hat{R} = 6\lambda - 2\mu^2/9$. They are thus homogeneous spaces of either positive ($S^2$), null ($\mathbb{R}^2$) or negative curvature ($H_2$).

The reader might be puzzled by the present connection with topologically massive gravity. The $2+1$-dimensional geometries analyzed here are not supposed to carry any gravity degree of freedom since they are ultimately designed to serve as holographic boundaries. Hence, the emergence of topologically massive gravity should not be considered as a sign of dynamics, but rather as a constraint for the algebrization of the operator $\nabla$, which destroys any potential dangerous tensor. Any perfect-Cotton geometry allowing for such tensors, and thereby probing transport coefficients, will necessarily require a deviation from topologically massive gravity.

More recently, topologically massive gravity has also attracted attention from the holographic perspective [25, 26]. In these works, the homogeneous solutions appear as $2+1$-dimensional *bulk* backgrounds, whereas in the present work (see also [19]), they will turn out to be naturally leading to *boundary* geometries. Investigating the interplay between these two viewpoints might be of some relevance, beyond our scope though.

**4.2.1.2 Dipolar geometries: axisymmetric spaces** When $c_2 \neq 0$, the vorticity is not constant and hence the space ceases to be homogenous. If some symmetry remains, this must be in the form of a space-like Killing vector: therefore, these are axisymmetric spaces. When the parameter $c_1$ is vanishing, a non-zero $c_2$ introduces a global rigid rotation, and the metric is conformally flat. We call such geometries *dipolar geometries*, as their axial symmetry connects them with the gauge potential of electric or magnetic dipoles. It is of course also possible to consider the case of both $c_1$ and $c_2$ are non-vanishing, corresponding to a superposition of monopoles and dipoles.

For simplicity, we start considering a *pure dipolar* geometry, namely a nontrivial conformally flat metric and see how it is parametrized in terms of $c_2$ and $c_3$. We start from $\mathbb{R} \times S^2$ where we set to one the sphere's radius

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2. \tag{4.35}$$

We do then a conformal rescaling by a function $\Omega(\vartheta)$, which preserves the axial symmetry around $\varphi$

$$\mathrm{d}s^2 \to \Omega^{-2}(\vartheta)\mathrm{d}s^2. \tag{4.36}$$

The vector field $\partial_t$ is no longer of unit norm, hence it does not generate the congruences of a fluid in equilibrium. However, if $\Omega(\vartheta)$ simply corresponds to a rotation,[12] then by a coordinate transformation we could go to a new system where Killing vector $\partial_t$ continues to have unit norm and still describes the trajectories of the fluid's elements in equilibrium. Consider hence the change of coordinates

$$\varphi \mapsto \varphi + at, \tag{4.37}$$

where $a$ being a constant parameter. In order for $\partial_t$ to have unit norm, the following condition must be satisfied:

$$\Omega^{-2}(\vartheta) = 1 - a^2 \sin^2\vartheta. \tag{4.38}$$

The resulting metric is

$$\mathrm{d}s^2 = -\left(\mathrm{d}t - \frac{a\sin^2\vartheta}{1-a^2\sin^2\vartheta}\mathrm{d}\varphi\right)^2 + \frac{\mathrm{d}\vartheta^2}{1-a^2\sin^2\vartheta} + \frac{\sin^2\vartheta}{(1-a^2\sin^2\vartheta)^2}\mathrm{d}\varphi^2. \tag{4.39}$$

We want now to bring this metric in the form (4.8). We first set

$$\frac{\sin^2\vartheta}{1-a^2\sin^2\vartheta} = \frac{\sin^2\vartheta'}{1-a^2} \quad \Rightarrow \quad \mathrm{d}\vartheta^2 = \frac{1-a^2}{(1-a^2\cos^2\vartheta')^2}\mathrm{d}\vartheta'^2, \tag{4.40}$$

finding

$$\mathrm{d}s^2 = -\left(\mathrm{d}t - \frac{a\sin^2\vartheta'}{1-a^2}\mathrm{d}\varphi\right)^2 + \frac{\mathrm{d}\vartheta'^2}{1-a^2\cos^2\vartheta'} + \frac{\sin^2\vartheta'(1-a^2\cos^2\vartheta')}{(1-a^2)^2}\mathrm{d}\varphi^2. \tag{4.41}$$

Finally, by performing the change of coordinates

$$x = \frac{1}{1-a^2}\sin^2\vartheta'/2 = \frac{1-\cos\vartheta'}{2(1-a^2)}, \quad y = 2\varphi, \tag{4.42}$$

[12] Essentially to a precession, hence we call this *the precession trick*.

we find can bring the metric into the for (4.8) with

$$b(x) = 2ax\left(1-(1-a^2)x\right), \tag{4.43}$$
$$G(x) = x-(1-5a^2)x^2-8a^2(1-a^2)x^3+4a^2(1-a^2)^2x^4. \tag{4.44}$$

It is easy then to read the parameters

$$c_1 = 2a\,, c_2 = -2a(1-a^2)\,, c_3 = 5a^2-1. \tag{4.45}$$

The metric describes cyclonic rotation with vorticity is given

$$q = -2a\cos\vartheta' \tag{4.46}$$

while being conformally flat with $c = 0$. In this case we only have one parameter $a$ and this is consistent with the analysis of [27, 28] on conformally flat $2+1$ dimensions.

We can then generalize to non-conformally flat geometries with $x$-dependent vorticity. These are the *dipolar-monopolar* metrics. In those cases the precession trick mentioned above does not suffice and one needs to work case by case in order to find the correct parametrizations. Nevertheless, our previous explicit examples serve both as a guiding rule as well as a test for our results. We present just the results and spare the reader from the non-illuminating technicalities.

**Spherical ($\nu = 1$)** Let us define the new parameters $a$ and $n$ by:

$$\begin{aligned} c_1 &= 2(a-n),\\ c_2 &= 2a(-1+a^2-4an),\\ c_3 &= -1+5a^2-12an. \end{aligned} \tag{4.47}$$

We also perform the following coordinate transformations:

$$\begin{aligned} x &= \kappa\sin^2(\vartheta/2),\\ y &= \lambda\varphi, \end{aligned} \tag{4.48}$$

with

$$\kappa = \frac{1}{1+a(4n-a)}, \quad \lambda = \frac{2}{\kappa\Xi} \quad \text{and} \quad \Xi = 1-a^2. \tag{4.49}$$

The two-dimensional base space in the metric (4.8) takes then the form:

$$\mathrm{d}\ell^2 = \frac{\mathrm{d}\vartheta^2}{\Delta_\vartheta}+\frac{\sin^2\vartheta\Delta_\vartheta}{\Xi^2}\mathrm{d}\varphi^2 \tag{4.50}$$

with

$$\Delta_\vartheta = 1+a\cos\vartheta(4n-a\cos\vartheta). \tag{4.51}$$

The coordinates range as $\vartheta\in[0,\pi]$ and $\varphi\in[0,2\pi]$. The full $2+1$-dimensional metric is of the Papapetrou–Randers form: $\mathrm{d}s^2 = -\mathrm{u}^2+\mathrm{d}\ell^2$. The velocity field takes the form

$$\mathrm{u} = -\mathrm{d}t+\mathrm{b}(\vartheta)\mathrm{d}\varphi\,, \quad b(\theta) = \frac{2(a-2n+a\cos\vartheta)}{\Xi}\sin^2(\vartheta/2). \tag{4.52}$$

The scalar vorticity strength is given by

$$q = 2\left(n - a\cos\vartheta\right), \tag{4.53}$$

while the constant $c$ appearing in the the Cotton–York tensor is

$$c = 2n(1 - a^2 + 4n^2). \tag{4.54}$$

The base space (4.50) is a squashed $S^2$. The vorticity (4.53) has two terms: the constant monopole and the dipole. It is maximal on the northern ($\vartheta = 0$) and southern ($\vartheta = \pi$) poles and is vanishing on the equator ($\vartheta = \pi/2$). Note also that in the limit $c_2$ we recovered the homogeneous metric case for $\nu = 1$.

**Flat ($\nu = 0$)** The new parameters $a$ and $n$ are now defined as follows:

$$\begin{aligned} c_1 &= 2(a-n), \\ c_2 &= 2a^2(a-4n), \\ c_3 &= a(5a-12n). \end{aligned} \tag{4.55}$$

Let us now do the following coordinate transformations:

$$\begin{aligned} x &= \kappa(\sigma/2)^2, \\ y &= \lambda\varphi, \end{aligned} \tag{4.56}$$

with

$$\kappa = 1, \quad \lambda = 2. \tag{4.57}$$

With these transformations the two-dimensional base space in the metric (4.8) takes the form of squashed $\mathbb{R}^2$:

$$\mathrm{d}\ell^2 = \frac{\mathrm{d}\sigma^2}{\Delta_\sigma} + \sigma^2\Delta_\sigma \mathrm{d}\varphi^2 \tag{4.58}$$

with

$$\Delta_\sigma = \frac{(2 + a^2\sigma^2)(8 - 24an\sigma^2 + a^4\sigma^4 - 8a^3n\sigma^4 + 2a^2\sigma^2(3 + 8n^2\sigma^2))}{16}. \tag{4.59}$$

The coordinates range as $\sigma \in \mathbb{R}_+$ and $\varphi \in [0, 2\pi]$. The full $2+1$-dimensional metric is $\mathrm{d}s^2 = -\mathrm{u}^2 + \mathrm{d}\ell^2$, where the velocity field takes the form

$$\mathrm{u} = -\mathrm{d}t + \mathrm{b}(\sigma)\mathrm{d}\varphi\,, \qquad \mathrm{b}(\sigma) = \frac{\sigma^2}{4}\left(4(a-n) + a^2(a-4n)\sigma^2\right). \tag{4.60}$$

The scalar vorticity is then given by

$$q = (n-a)\left(2 + a^2\sigma^2\right), \tag{4.61}$$

while the constant $c$ appearing in Cotton–York tensor is:

$$c = 2n(-a^2 + 4n^2). \tag{4.62}$$

**Hyperbolic case ($\nu = -1$)** This case is very similar to the spherical one, with trigonometric functions traded for hyperbolic ones. We define $a$ and $n$ using:

$$\begin{aligned} c_1 &= 2(a-n), \\ c_2 &= 2a(1+a^2-4an), \\ c_3 &= 1+5a^2-12an. \end{aligned} \tag{4.63}$$

The appropriate coordinate transformations are:

$$\begin{aligned} x &= \kappa \sinh^2(\sigma/2), \\ y &= \lambda\varphi, \end{aligned} \tag{4.64}$$

with

$$\kappa = \frac{1}{1-a(4n-a)}, \quad \lambda = \frac{2}{\kappa Z} \quad \text{and} \quad Z = 1+a^2. \tag{4.65}$$

With these transformations the two-dimensional base space in the metric (4.8) takes the form of squashed $H_2$:

$$\mathrm{d}\ell^2 = \frac{\mathrm{d}\sigma^2}{\Delta_\sigma} + \frac{\sinh^2\sigma\Delta_\sigma}{\Xi^2}\mathrm{d}\varphi^2 \tag{4.66}$$

with

$$\Delta_\sigma = 1 - a\cosh\sigma(4n - a\cosh\sigma). \tag{4.67}$$

The coordinates range as $\sigma \in \mathbb{R}_+$ and $\varphi \in [0, 2\pi]$. In the full $2+1$-dimensional metric $\mathrm{d}s^2 = -\mathrm{u}^2 + \mathrm{d}\ell^2$, the velocity field takes the form

$$\mathrm{u} = -\mathrm{d}t + \mathrm{b}(\sigma)\mathrm{d}\varphi\,, \qquad b(\sigma) = \frac{2(a-2n+a\cosh\sigma)}{Z}\sinh^2(\sigma/2). \tag{4.68}$$

The scalar vorticity is

$$q = 2\left(n - a\cosh\sigma\right), \tag{4.69}$$

while the constant $c$ appearing in the Cotton–York tensor is

$$c = 2n(-1-a^2+4n^2). \tag{4.70}$$

**Uniform parametrization** It is possible to use a uniform notation to include the three different cases:

$$\begin{aligned} c_1 &= 2(a-n), \\ c_2 &= 2a(-\nu+a^2-4an), \\ c_3 &= -\nu+5a^2-12an. \end{aligned} \tag{4.71}$$

The general coordinate transformations are:

$$\begin{aligned} x &= \kappa f_\nu^2(\theta/2), \\ y &= \lambda\varphi, \end{aligned} \tag{4.72}$$

with $f_\nu$ as in (4.25), and

$$\kappa = \frac{1}{1+\nu a(4n-a)}, \quad \lambda = \frac{2}{\kappa Z_\nu} \quad \text{and} \quad Z_\nu = 1-\nu a^2. \tag{4.73}$$

The constant $c$ appearing in Cotton–York tensor takes the form:

$$c = 2n(\nu - a^2 + 4n^2). \tag{4.74}$$

#### 4.2.2 Vanishing $c_4$

When the parameter $c_4$ is vanishing, it is not possible to perform the change of variables (4.17) and thus we have a different class of metrics. We are left with the parameters $c_1$, $c_2$, $c_3$ and $c_5$. We decide not to set to zero the latter in order to avoid a possible metric singularity (see (4.10)) when $c_2 = c_3 = 0$. The boundary metric is in this case given by

$$\begin{aligned} b(x) &= c_1 x + c_2 x^2, \\ G(x) &= c_5 + c_3 x^2 + c_2 x^3 (2c_1 + c_2 x). \end{aligned} \tag{4.75}$$

For the flat horizon case $c_3 = 0$, this class of metrics appears as boundary of Einstein solutions studied in [29]. When $c_2 = 0$ we have a homogeneous geometry and what we concluded on transport coefficients for the case before is still valid: it is not possible to constraint any of them holographically, because the corresponding tensors vanish kinematically.

### 4.3 Geometries without space-like isometries

The perfect-Cotton geometries we have constructed in Sec. 4.2 possess at least one space-like Killing vector. Our motivation for studying such perfect-Cotton geometries was holographic, and, in particular, these metrics appear as boundaries of exact $3+1$-dimensional bulk Einstein metrics, as we will see in Sec. 5. This property is not limited to the sole perfect-Cotton stationary geometries that admit space-like isometries: any perfect-Cotton Papapetrou–Randers metric qualifies. It seems however very difficult to find explicit $(x, y)$-dependent solutions when the additional isometry is not present. Such solutions would play important role to go beyond the dipole and introduce more dangerous tensors, hence probe more transport coefficients. In the absence of exact solutions, we could proceed with probing further transport properties perturbatively. We leave this for the future.

## 5 The bulk duals of perfect equilibrium

### 5.1 Generic bulk reconstruction

When the boundary geometry is of the perfect-Cotton type and the boundary stress tensor is that of a fluid in perfect equilibrium, the bulk solution can be exactly determined. This is highly non-trivial because it generally involves an infinite resummation *i.e.* starting from the boundary data and working our way to the bulk.

The apparent resummability of the boundary data discussed above into exact bulk geometries is remarkable, but not too surprising. An early simple example was given in [30] where it was shown that setting the boundary energy–momentum tensor to zero and starting with a conformally flat boundary metric, one can find the (conformally flat) bulk solution resuming the Fefferman–Graham series. In fact, in that case the resummation involved just a few terms.

The next non-trivial example was presented in [7]. There it was shown that in Euclidean signature, imposing the condition

$$C_{\mu\nu} = \pm 8\pi G_{\mathrm{N}} T_{\mu\nu} \tag{5.1}$$

is exactly equivalent to the (anti)-self duality of the bulk Weyl tensor, hence it leads to (anti)-self dual solutions[13] However, in these cases it is not clear whether the boundary theory describes a hydrodynamical system.

Here we study a particular extension of the (anti)-self dual boundary condition of [7], which is Lorentzian and reads

$$C_{\mu\nu} = \chi T_{\mu\nu}\,, \quad \chi = \frac{c}{\varepsilon}\,, \tag{5.2}$$

with both $T_{\mu\nu}$ and $C_{\mu\nu}$ having the perfect fluid form and $\chi \neq 8\pi G_{\rm N}$ generically.

Our main observation is that to the choice (5.2) for the boundary data corresponds the following *exact* bulk Einstein metric in Eddington–Finkelstein coordinates (where $g_{rr} = 0$ and $g_{r\mu} = -u_\mu$):

$$\begin{aligned} \mathrm{d}s^2 \ =& -2\mathrm{u}\left(\mathrm{d}r - \frac{1}{2}\mathrm{d}x^\rho u^\sigma \eta_{\rho\sigma\mu}\nabla^\mu q\right) + \rho^2 \mathrm{d}\ell^2 \\ &- \left(r^2 + \frac{\delta}{2} - \frac{q^2}{4} - \frac{1}{\rho^2}\left(2Mr + \frac{qc}{2}\right)\right)\mathrm{u}^2, \end{aligned} \tag{5.3}$$

with

$$\rho^2 = r^2 + \frac{q^2}{4}. \tag{5.4}$$

The metric above is manifestly covariant with respect to the boundary metric. Taking the limit $r \to \infty$ it is easy to see that the boundary geometry is indeed the general stationary Papapetrou–Randers metric in (3.1) with

$$\mathrm{u} = -\mathrm{d}t + b\,\mathrm{d}y. \tag{5.5}$$

The various quantities appearing in (5.3) (like $\delta, q, c$) satisfy Eqs. (4.2), (4.3) and (4.4), and this guarantees that Einstein's equations are satisfied. Performing the Fefferman–Graham expansion of (5.3) we indeed recover the perfect form of the boundary energy–momentum tensor with

$$\varepsilon = \frac{M}{4\pi G_{\rm N}}. \tag{5.6}$$

where $G_{\rm N}$ is the four-dimensional Newton's constant. The corresponding holographic fluid has velocity field u, vorticity strength $q$ and behaves like a perfect fluid.

In the choice of gauge given by (4.5) and (4.6), the bulk metric (5.3) takes the form:

$$\begin{aligned} \mathrm{d}s^2 \ =& -2\mathrm{u}\left(\mathrm{d}r - \frac{1}{2}\left(\mathrm{d}y\frac{B}{A}\partial_x q - \mathrm{d}x\frac{A}{B}\partial_y q\right)\right) + \rho^2 \mathrm{d}\ell^2 \\ &- \left(r^2 + \frac{\delta}{2} - \frac{q^2}{4} - \frac{1}{\rho^2}\left(2Mr + \frac{qc}{2}\right)\right)\mathrm{u}^2, \end{aligned} \tag{5.7}$$

[13] More generally, the boundary Cotton tensor is an asymptotic component of the bulk Weyl tensor e.g. Eq. (2.8) of [6]. However, a non-vanishing Weyl does not necessarily imply a non-vanishing Cotton, as for example in the Kerr–AdS$_4$ case. A non-vanishing Cotton, on the other hand, requires the Weyl be non-zero. Non-cyclonic vorticity on the boundary requires precisely non-zero Cotton, as we discuss in Sec. 4 (see also [17, 18], as well as [11]). The structure of the perfect Cotton puts therefore constraints on the bulk Weyl tensor.

where $q$ is as in (4.7). Note $\delta$ and $c$ can be readily obtained from $q$, $A$ and $B$ using (4.4) and (4.2) respectively.

It is clear from the explicit form of the bulk spacetime metric (5.3) that the metric has a curvature singularity when $\rho^2 = 0$. The locus of this singularity is at :

$$r = 0, \quad q(x,y) = 0. \tag{5.8}$$

However, we will find cases where $\rho^2$ never vanishes because $q^2$ never becomes zero. In such cases, the bulk geometries have no curvature singularities, but they might have regions with closed time-like curves.

Since $A$, $B$ and $b$ are functions of $x$ and $y$ only the metric has a Killing vector $\partial_t$. Although this is of unit norm at the boundary coinciding with the velocity vector of the boundary fluid, it's norm is not any more unity in the interior. The Killing vector becomes null at the ergosphere $r = R(x)$ where:

$$r^2 + \frac{\delta}{2} - \frac{q^2}{4} - \frac{1}{\rho^2}\left(2Mr + \frac{qc}{2}\right) = 0. \tag{5.9}$$

Beyond the ergosphere no observer can remain stationary, and hence she experiences frame dragging, as $\partial_t$ becomes space-like.

Before closing this section, a last comment is in order, regarding the exactness of the bulk solution (5.3)–(5.4), obtained by uplifting $2+1$-dimensional perfect boundary data *i.e.* perfect energy–momentum tensor (2.3) and perfect-Cotton boundary geometry (4.1).

The Fefferman–Graham expansion, quoted previously as a way to organize the boundary (holographic) data, is controlled by the inverse of the radial coordinate $1/r$. An alternative expansion has been proposed in [3, 4]. This is a derivative expansion (long wavelength approximation) that modifies order by order the bulk geometry, all the way from the horizon to the asymptotic region. It has been investigated from various perspectives in bulk dimension greater that 4. In the course of this investigation, it was observed [31, 32] that for AdS–Kerr geometries, at least in 4 and 5 dimensions, the derivative expansion obtained with a perfect energy–momentum tensor and the Kerr boundary geometry, turns out to reproduce exactly the bulk geometry, already at first order, modulo an appropriate resummation that amounts to redefining the radial coordinate.

Lately, it has been shown [19] that the above observation holds for the Taub–NUT geometry in 4 dimensions provided the quoted derivative expansion includes a higher-order term involving the Cotton–York tensor of the boundary geometry. The derivative expansion up to that order reads:

$$\mathrm{d}s^2 = -2\mathrm{u}\mathrm{d}r + r^2\mathrm{d}s^2_{\text{bry.}} + \Sigma_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \frac{\mathrm{u}^2}{\rho^2}\left(2Mr + \frac{1}{2}u^\lambda C_{\lambda\mu}\eta^{\mu\nu\sigma}\omega_{\nu\sigma}\right), \tag{5.10}$$

where all the quantities refer to the boundary metric $\mathrm{d}s^2_{\text{bry.}}$ of the Papapetrou–Randers type (3.1), and u is the velocity field of the fluid that enters the perfect energy–momentum tensor (2.2), whose energy density is related to $M$ according to (5.6). Furthermore,

$$\Sigma_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -2\mathrm{u}\nabla_\nu\omega^\nu{}_\mu\mathrm{d}x^\mu - \omega_\mu{}^\lambda\omega_{\lambda\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu - \mathrm{u}^2\frac{R}{2}, \tag{5.11}$$

$$\rho^2 = r^2 + \frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}, \tag{5.12}$$

where, as usual $\omega_{\mu\nu}$ are the components of the vorticity and $R$ the curvature of the boundary geometry. Metric (5.10) is the expansion stopped at the fourth derivative of the velocity field (the Cotton–York counts for three derivatives).[14] It was shown to be exact for the Taub–NUT boundary in [19] – as well as for Kerr whose boundary has vanishing Cotton.

Metric (5.10) coincides precisely with (5.3) for perfect-Cotton boundary geometries. This identification explains a posteriori the observation of [31, 32] about the exactness of the limited derivative expansion (up to the redefinition $\rho(r)$), and generalizes it to all perfect-Cotton geometries with perfect-fluid energy–momentum tensor. It raises also the question whether similar properties hold in higher dimensions, following the already observed exactness of the lowest term for Kerr. In particular one may wonder what replaces the perfect-Cotton geometry in higher dimensions, where there is no Cotton–York tensor. As we stressed, the bulk gravitational duality is a guiding principle that translates precisely to the boundary Cotton/energy–momentum relationship used in this paper. A similar principle is not available in every dimension and we expect only a limited number of cases where the observation made in [31, 32] about Kerr could be generalized to more general Einstein spaces.

### 5.2 Absence of naked singularities

We will focus here on the situation where we have an additional spatial isometry. We will show explicitly that for all perfect-Cotton geometries in this class, the bulk geometries have no naked singularities for appropriate range of values of the black hole mass. Our general solutions will be labeled by three parameters - namely the angular momentum $a$, the nut charge $n$ and the black hole mass $M$. This will cover all known solutions and also give us some new ones, as will be shown explicitly later in Appendix C.

In order to analyze the bulk geometry we need to know the boundary geometry explicitly. In the previous section, we have been able to find all the perfect Cotton geometries with at least one additional spatial Killing vector explicitly. These geometries are given by (4.8), (4.19) and (4.20), and are labelled by three continuously variable parameters $c_1$, $c_2$ and $c_3$. We have shown that without loss of generality, we can rewrite these parameters in terms of the angular momentum $a$, the nut charge $n$ and a discrete variable $\nu$ as in Eq. (4.71).

The holographic bulk dual (5.3) for perfect equilibrium in these general boundary geometries then reads:

$$\begin{aligned} \mathrm{d}s^2 &= -2\mathrm{u}\left(\mathrm{d}r - \frac{G}{2}\partial_x q\,\mathrm{d}y\right) + \rho^2\Big(\frac{\mathrm{d}x^2}{G} + G\mathrm{d}y^2\Big) \\ &\quad - \left(r^2 + \frac{\delta}{2} - \frac{q^2}{4} - \frac{1}{\rho^2}\left(2Mr + \frac{qc}{2}\right)\right)\mathrm{u}^2, \end{aligned} \tag{5.13}$$

where $\mathrm{u} = -\mathrm{d}t + b\mathrm{d}y$, and $b$ and $G$ are determined by three geometric $c_1$, $c_2$ and $c_3$ as in (4.19) and (4.20). Therefore $q$, c and $\delta$ are as in (4.11), (4.12) and (4.13) respectively.

[14] Strictly speaking, the redefinition $\rho(r)$ (5.12) accounts for a full series with respect to the vorticity, *i.e.* contains terms up to infinite velocity derivatives.

It is convenient for the subsequent analysis to move from Eddington–Finkelstein to Boyer–Linqvist coordinates. These Boyer–Linqvist coordinates make the location of the horizon manifest. These are the analogue of Schwarzschild coordinates in presence of an axial symmetry. The transition to Boyer–Linqvist coordinates can be achieved via the following coordinate transformations:

$$\mathrm{d}\tilde{t} = \mathrm{d}t - \frac{4(c_1^2+4r^2)}{3c_1^4+8c_1c_2-4c_1^2(c_3+6r^2)+16r(2M+c_3r-r^3)}\mathrm{d}r, \tag{5.14}$$

$$\mathrm{d}\tilde{y} = \mathrm{d}r + \frac{16c_2}{3c_1^4-4c_1^2c_3+8c_1c_2+32Mr-24c_1^2r^2+16c_3r^2-16r^4}\mathrm{d}r. \tag{5.15}$$

Note even after changing $t, y$ to $\tilde{t}, \tilde{y}$, the boundary metric still remains the same - the difference between the old and new coordinates die off asymptotically.

After these transformations the bulk metric takes the form (we replace $\tilde{r}$ and $\tilde{y}$ with $r$ and $y$):

$$\mathrm{d}s^2 = \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 - \frac{\Delta_r}{\rho^2}\left(\mathrm{d}t+\beta\mathrm{d}y\right)^2 + \frac{\rho^2}{\Delta_x}\mathrm{d}x^2 + \frac{\Delta_x}{\rho^2}\left(c_2\mathrm{d}t-\alpha\mathrm{d}y\right)^2, \tag{5.16}$$

where

$$\rho^2 = r^2 + \frac{q^2}{4} = r^2 + \frac{(c_1+2c_2x)^2}{4}, \tag{5.17}$$

$$\Delta_r = -\frac{1}{16}\left(3c_1^4+8c_1c_2-4c_1^2(c_3+6r^2)+16r(2M+c_3r-r^3)\right), \tag{5.18}$$

$$\Delta_x = G = x + c_3x^2 + 2c_1c_2x^3 + c_2^2x^4, \tag{5.19}$$

$$\alpha = -\frac{1}{4}\left(c_1^2+4r^2\right), \tag{5.20}$$

$$\beta = -b = -c_1x - c_2x^2. \tag{5.21}$$

Note the coordinates $r$ and $x$ do not change as we transform from Eddington–Finkelstein to Boyer–Linqvist coordinates. Therefore $\rho^2$ is exactly the same as before. Also note that $\Delta_r$ and $\alpha$ are functions of $r$ only, while $\Delta_x$ and $\beta$ are functions of $x$ only.

It is easy to see that the horizons are at $r = r_*$ where:

$$\Delta_r(r = r_*) = 0, \quad \text{with} \quad r_* > 0. \tag{5.22}$$

At most we can have four horizons. These horizon(s) should clothe the curvature singularity located at $\rho^2 = 0$ or equivalently at:

$$r = 0, \quad x = -\frac{c_1}{2c_2}. \tag{5.23}$$

It is not hard to see that for fixed values of the geometric parameters $c_1$, $c_2$ and $c_3$, there exists a positive definite solution to Eq. (5.22) for an appropriate range of the black hole mass $M$. Hence the curvature singularity is not naked.

Clearly we have only two Killing vectors generically - namely $\partial_t$ and $\partial_y$. Each horizon $r = r_*$ is generated by the Killing vector:

$$\partial_t + \Omega_{\mathrm{H}}(r_*)\partial_y. \tag{5.24}$$

which is an appropriate linear combination of the two Killing vectors. $\Omega_{\rm H}(r_*)$ is a constant given by:

$$\Omega_{\rm H}(r_*) = \frac{c_2}{\alpha(r_*)} \tag{5.25}$$

and is the rigid velocity of the corresponding horizon.

The bulk geometry can have at most four ergospheres where the Killing vector $\partial_t$ becomes null. These are given by $r = R(x)$ where $R(x)$ is a solution of:

$$g_{tt} = 0, \quad \text{i.e.} \quad \Delta_r = c_2^2 G. \tag{5.26}$$

We have seen in Section 4.2.1.2 that the geometric structure of the boundary geometries is better revealed as fibrations over squashed $S^2$, $\mathbb{R}^2$ or $H_2$ if we do a further coordinate transformation in $x$ and $y$. We will do the same coordinate transformations given by (4.72) in the bulk metric separately for $\nu = 1, 0, -1$. We will also need to exchange parameters $c_1$, $c_2$ and $c_3$ with $a$, $n$ and $\nu$ using (4.71). Note in these coordinate transformations the radial coordinate $r$ and the time coordinate $t$ do not change, while the spatial coordinates $x$ and $y$ transform only as functions of themselves. This preserves the Boyer–Linqvist form of the metric (5.16). We can apply the same strategy to locate the horizon(s) and the ergosphere(s).

The advantage of doing these coordinate transformations is that for $\nu = 1, 0, -1$ we will see that the horizon will be a squashed $S^2$, $\mathbb{R}^2$ and $H_2$ respectively. The metrics are given explicitly in Appendix C, where we will also show that we recover all known rotating black hole solutions.

Interestingly when $c_2 = 0$, $\rho^2 > c_1^2/4$, hence it never vanishes. Therefore the bulk geometry has no curvature singularity. In terms of $a$, $n$ and $\nu$, this happens when

- $n > a$ for $\nu = 1$,
- $n < a/4$ and $n > a$ for $\nu = 0$ and
- $n < a$ for $\nu = -1$.

In such cases horizon(s) may exist, but in absence of a curvature singularity, it is not necessary for the horizon to exist in order that the solution is a good solution. However, if there are regions in the bulk with closed time-like curves, then these should be covered by horizons. On the other hand if this is not the case, then there is no restriction on the black hole mass $M$ as we do not need a horizon. We may ask if negative values of M is physically acceptable in such cases. We leave this question for future investigations.

## 5.3 Comments on the rigidity theorem

We now discuss the intriguing case of perfect Cotton geometries with no spatial isometry. Though we do not have explicit examples of such metrics, we do know that perfect equilibrium in such metrics should lift to exact bulk solutions given by Eq. (5.3) as discussed before. We are however unable to analyze the absence of naked singularities in the bulk geometries as we do not know the boundary geometries explicitly.

There are two possibilities:

- The perfect Cotton condition itself will force the geometry to have at least an additional spatial isometry. This is consistent with the rigidity theorem in 3+1-dimensions which requires all stationary black hole solutions in flat space to have an axial symmetry. However, as far as we are aware, it is not known if this theorem is valid for $3 + 1$-dimensional asymptotically AdS stationary black holes.

- We will be able to find explicit perfect Cotton boundary geometries without any spatial isometry, but the dual bulk solutions given by (5.3) will always have naked singularities for any value of $M$. This naked singularity need not be the curvature singularity but also regions with closed time-like curves not covered by a horizon.

In the future we will investigate which of the above is the correct possibility. If the second possibility is true, then perfect Cotton-ness will not be sufficient to ensure that perfect equilibrium can exist. We will also require an extra spatial isometry in the background geometry.

### 5.4 Black hole uniqueness from perfect fluidity

In the generic boundary geometries discussed here, there is a unique time-like Killing vector of unit norm. Physically this corresponds to the fluid velocity field of the perfect equilibrium state at the boundary.

The basic observation is that if all stationary black holes in anti-de Sitter space are dual to perfect equilibrium states in the CFT, then they are generically unique and are labeled by the mass $M$ for a fixed boundary geometry. The uniqueness is simply a consequence of the fact that there is a unique solution of fluid mechanics which is in perfect equilibrium in the boundary geometry, as given by Eq. (2.10).

For certain values of parameters we will get instances where there will be extra isometries (like boosts in flat space) which are broken by the perfect equilibrium fluid configuration. In that case we can generate new solutions by applying these isometries on the fluid configuration (like boosting u). For each such isometry, we will have an additional parameter labelling these black hole solutions (as in the case of boosted black branes).

In case of space-times with an additional spatial isometry discussed here, the black hole solutions are uniquely described by four parameters - namely $M$ and the three geometric parameters $a$, $n$ and $\nu$ for generic values. Note the perfect equilibrium solution preserves the additional spatial isometries, hence the latter cannot be used to generate any new solution.

The local equation of state is independent of the geometry and is an intrinsic property of the microscopic theory. In fact in a CFT it is simply $\varepsilon = 2p$ (which is also imposed as a constraint of Einstein's equation in the bulk). However, global thermodynamics describing the black hole geometry will depend on the choice of boundary geometry. The thermodynamic charges can be constructed by suitably integrating $T^{\mu\nu}$ over the boundary manifold [33]. In fact some of the geometric parameters will be related to conserved charges - like $a$ will be related to the angular momentum. The intrinsic variables - namely the temperature $T$ and the rotation $\Omega$ can be determined either by using thermodynamic identities or by using the properties of the outermost horizon.

## 6 Constraints on transport coefficients

In the previous section, we have shown that we can find exact black-hole solutions corresponding to perfect equilibrium of the dual field theory in perfect-Cotton boundary geometries. From the perspective of the boundary fluid dynamics, we are ensured by construction that the energy–momentum tensor is *exactly* of the perfect type. Thus any dangerous tensor that this deformed boundary may have, will necessarily couple to vanishing transport coefficients. This gives non-trivial information about strongly coupled holographic conformal fluids in the classical gravity approximation.

We will explicitly show here that exact black-hole solutions indeed imply holographic fluids at strong coupling and in the classical gravity approximation have infinitely many vanishing transport coefficients. On a cautionary note, using perfect-Cotton geometries at the boundary, we will not be able to constrain all transport coefficients. This is because many Weyl-covariant, traceless and transverse tensors will vanish kinematically. We will need to know all possible holographic perfect geometries, or equivalently all exact black-hole solutions with regular horizons, in order to know which transport coefficients vanish in three-dimensional conformal holographic fluids at strong coupling and in the classical gravity approximation. This is possibly not true and it's investigation is also beyond the scope of the present work.

We have seen in Sec. 4.2.1.1 that a class of perfect-Cotton geometries corresponding to homogeneous backgrounds have no dangerous tensors. Therefore, all conformal fluids in equilibrium in such boundary geometries are also in perfect equilibrium. In absence of dangerous tensors, we cannot use these boundary geometries to constrain transport coefficients.

Therefore we turn to perfect-Cotton geometries with an additional spatial isometry only discussed in Sec. 4.2.1.2. We have found in Sec. 5 that we can uplift these geometries to exact black-hole solutions without naked singularities for generic values of four parameters characterizing them. Let us now examine the presence of dangerous tensors in these geometries.

For concreteness, we begin at the third order in derivative expansion. The list of possible dangerous tensors is in (2.6). We note that $\langle C_{\mu\nu}\rangle$ vanishes in any perfect-Cotton geometry, because the transverse part of $C_{\mu\nu}$ is pure trace, meaning it is proportional to $\Delta_{\mu\nu}$. Therefore, it is not a dangerous tensor in any perfect-Cotton geometry, as a result we cannot constrain the corresponding transport coefficient $\gamma_{(3)1}$.

We recall from Sec. 2.2 that we need to evaluate the possible dangerous tensors on-shell, meaning we need to check if they do not vanish when $\mathrm{u} = \xi$. We have shown in App. B that in equilibrium, *i.e.* on-shell, the Weyl-covariant derivative $\mathcal{D}_\mu$ reduces to the covariant derivative $\nabla_\mu$. This facilitates our hunt for dangerous tensors.

The first dangerous tensor we encounter is $\langle \mathcal{D}_\mu W_\nu \rangle$. It is because it is non-vanishing and also it is not conserved, meaning $\nabla^\mu \langle \mathcal{D}_\mu W_\nu \rangle \neq 0$ in all geometries discussed in Sec. 4.2.1.2. Perfect equilibrium can exist only if the corresponding dangerous transport coefficient $\gamma_{(3)2}$ vanishes. Thus this transport coefficient vanishes for all strongly coupled holographic fluids in the regime of validity of classical gravity approximation.

We can similarly show that infinite number of tensors of the form of $(C^{\alpha\beta}C_{\alpha\beta})^\ell\langle\mathcal{D}_\mu W_\nu\rangle$, $(V^\alpha V_\alpha)^m\langle\mathcal{D}_\mu W_\nu\rangle$ and $(W^\alpha W_\alpha)^n\langle\mathcal{D}_\mu W_\nu\rangle$ for $\ell, m$ and $n$ being arbitrary positive integers, are dangerous tensors in geometries of Sec. 4.2.1.2. We conclude that the infinitely many non-dissipative transport coefficients corresponding to these dangerous tensors should vanish.

At the fourth order in the derivative expansion, we get new kind of dangerous tensors of the form $\langle V_\mu V_\nu\rangle$, $\langle W_\mu W_\nu\rangle$ and $\langle\mathcal{D}_\mu\mathcal{D}_\nu(\omega^{\alpha\beta}\omega_{\alpha\beta})\rangle$ in geometries of Sec. 4.2.1.2. This further implies existence of infinite number of dangerous tensors, of the form of $(C^{\alpha\beta}C_{\alpha\beta})^l\langle V_\mu V_\nu\rangle$, $(V^\alpha V_\alpha)^m\langle V_\mu V_\nu\rangle$, $(W^\alpha W_\alpha)^n\langle V_\mu V_\nu\rangle$, etc. in the geometries of Sec. 4.2.1.2. Once again this leads us to conclude that infinite number of new dangerous transport coefficients vanish.

We do not want to give an exhaustive list of all possible holographic transport coefficients we can constrain using exact black-hole solutions. Such an exhaustive list will require us to explore perfect-Cotton geometries with no spatial isometry and furthermore all possible perfect geometries which uplift to exact black-hole solutions. We leave this investigation for the future.

We want to conclude this section by arguing that the constraints on transport coefficients derived here cannot be obtained from partition-function [9] or entropy-current [10] based approaches. The latter are very general and independent of holography. On the other hand, our constraints follow from exact solutions of Einstein's equations. In particular, a certain form of duality between the Cotton–York and energy–momentum tensors at the boundary is crucial for us to find these exact solutions. This duality has no obvious direct interpretation in the dual field theory and no obvious connection with general approaches for constraining hydrodynamic transport coefficients. Unfortunately, the general approaches mentioned above have been explicitly worked out up to second order in derivative expansion only. On the other hand, the first non-trivial constraint in our approach comes at the third order in the derivative expansion. So presently we cannot give an explicit comparison of our approach with these general approaches. It will be interesting to find explicit examples where holographic constraints on transport coefficients discussed here cannot be obtained from other approaches.

## 7 Conclusions and Outlook

We end here with a discussion on possible future directions. Perhaps the most outstanding question is the classification of all possible perfect geometries for holographic systems. The difficulty in studying this question is to make a formulation which is independent of any ansatz for the metric which will sum over infinite orders in the derivative expansion. It is difficult to show that only a specific ansatz will exhaust all possibilities. In fact it is not clear whether it is necessary to have an exact solution in the bulk in order to have perfect equilibrium in the boundary. There can be derivative corrections to all orders in the bulk metric which cannot be resummed into any obvious form, though such corrections may vanish for the boundary stress tensor.

Recently an interesting technique has been realized for addressing such questions involving the idea of holographic renormalization-group flow in the fluid/gravity limit [34]. In this approach, a fluid is constructed from the renormalized energy–momentum tensor at any hypersurface in the bulk. For a unique hypersurface foliation – namely the Fefferman–Graham foliation – the radial evolution of the transport coefficients and hydrodynamic variables is first order and can be constructed without knowing the bulk spacetime metric explicitly. Once this radial evolution is solved, the bulk metric can be constructed from it for a given boundary geometry.

The advantage of this formulation is that the holographic renormalization-group flow of transport coefficients and hydrodynamic variables automatically knows about the regularity of the horizon. The renormalization-group flow terminates at the horizon and there exists a unique solution which corresponds to non-relativistic incompressible Navier–Stokes equation at the horizon. This unique solution determines the values of the transport coefficients of the boundary fluid to all orders in the derivative expansion. It is precisely these values which give solutions with regular horizons. Though it has not been proved, this agreement between the renormalization-group flow and regularity has been checked explicitly for first and second order transport coefficients.

The relevance of this approach to perfect geometries is as follows. In the special case of perfect equilibrium, we know that the boundary fluid should also flow to a fluid in perfect equilibrium at the horizon. The latter can happen only if the boundary geometry is a perfect geometry, which will impose appropriate restrictions on the fluid kinematics. The question of classification of perfect boundary geometries is thus well posed using deep connections between renormalization-group flow and horizon regularity – independently of any specific ansatz. In this approach we will also be able to know the full class of transport coefficients which should necessarily vanish such that perfect equilibrium can exist both at the boundary and the horizon.

The second immediate question involves further analysis of the black-hole solutions with at least one extra spatial isometry discussed here. This is particularly necessary for the particular values of the geometric parameters where there exists no curvature singularities in the bulk for all values of the mass. The question is what restricts the mass from being arbitrarily negative – is it possibly just the requirement that regions of space–time with closed time-like curves should be hidden by horizons? Or do we need new principles? Also we should construct the global thermodynamics of such geometries in detail and investigate if there is anything unusual.

On the same note, we should also investigate the case of perfect-Cotton boundary geometries with no spatial isometries and check the (in)validity of the rigidity theorem in the bulk dual. It will be interesting to see if perfect geometries for holographic theories need not have any spatial isometry. More generally our guiding principle in searching perfect fluidity is the mass/nut bulk duality, which is a non-linear relationship emerging *a priori* in Euclidean four-dimensional gravity. Its manifestation in Lorentzian geometries is holographic and operates linearly via the Cotton/energy–momentum duality on the $2+1$-dimensional boundary; it is a kind of duality relating the energy density with the vorticity, when the later is non-trivial *i.e.* when the Cotton–York tensor is non-vanishing. This relationship should be further investigated as it provides another perspective on gravity duality [35].

Finally, it will be interesting to find exact solutions in the bulk with matter fields corresponding to steady states in the boundary. These steady states will be sustained by non-normalizable modes of the bulk matter fields. Perhaps the simplest and the most interesting possibility is adding axion fields with standard kinetic term in the bulk which couple also to the Gauss–Bonnet term. Such bulk actions have been studied recently [36–38]. In fact, it has been shown that this leads to simple mechanism for generating vortices in the boundary spontaneously. These simple vortices describe transitions in the $\theta$ vacuum across an edge and support edge currents. It will be interesting to see if there could be non-trivial exact solutions in the bulk describing more general steady state vortex configurations in the bulk. The relevant question analogous to the one studied in this work will be which boundary geometries and axionic configurations can sustain steady vortex configurations.


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## A On vector-field congruences

We consider a manifold endowed with a space–time metric of the generic form

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{dx}^\mu\mathrm{dx}^\nu = \eta_{ab}\mathrm{e}^a\mathrm{e}^b. \tag{A.1}$$

We will use $a, b, c, \ldots = 0, 1, \ldots, D-1$ for transverse Lorentz indices along with $\alpha, \beta, \gamma = 1, \ldots, D-1$. Coordinate indices will be denoted $\mu, \nu, \rho, \ldots$ for space–time $\mathrm{x} \equiv (t, x)$ and $i, j, k, \ldots$ for spatial $x$ directions. Consider now an arbitrary time-like vector field u, normalized as $u^\mu u_\mu = -1$, later identified with the fluid velocity. Its integral curves define a congruence which is characterized by its acceleration, shear, expansion and vorticity (see e.g. [39, 40]):

$$\nabla_\mu u_\nu = -u_\mu a_\nu + \frac{1}{D-1}\Theta\Delta_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \tag{A.2}$$

with[15]

$$a_\mu = u^\nu \nabla_\nu u_\mu, \quad \Theta = \nabla_\mu u^\mu, \tag{A.3}$$

$$\sigma_{\mu\nu} = \frac{1}{2}\Delta_\mu{}^\rho \Delta_\nu{}^\sigma \left(\nabla_\rho u_\sigma + \nabla_\sigma u_\rho\right) - \frac{1}{D-1}\Delta_{\mu\nu}\Delta^{\rho\sigma}\nabla_\rho u_\sigma \tag{A.4}$$

$$= \nabla_{(\mu}u_{\nu)} + a_{(\mu}u_{\nu)} - \frac{1}{D-1}\Delta_{\mu\nu}\nabla_\rho u^\rho, \tag{A.5}$$

$$\omega_{\mu\nu} = \frac{1}{2}\Delta_\mu{}^\rho \Delta_\nu{}^\sigma \left(\nabla_\rho u_\sigma - \nabla_\sigma u_\rho\right) = \nabla_{[\mu}u_{\nu]} + u_{[\mu}a_{\nu]}. \tag{A.6}$$

The latter allows to define the vorticity form as

$$2\omega = \omega_{\mu\nu}\, \mathrm{dx}^\mu \wedge \mathrm{dx}^\nu = \mathrm{du} + \mathrm{u} \wedge \mathrm{a}\,. \tag{A.7}$$

The time-like vector field u has been used to decompose any tensor field on the manifold in transverse and longitudinal components with respect to itself. The decomposition is performed by introducing the longitudinal and transverse projectors:

$$U^\mu{}_\nu = -u^\mu u_\nu, \quad \Delta^\mu{}_\nu = u^\mu u_\nu + \delta^\mu_\nu, \tag{A.8}$$

where $\Delta_{\mu\nu}$ is also the induced metric on the surface orthogonal to u. The projectors satisfy the usual identities:

$$U^\mu{}_\rho U^\rho{}_\nu = U^\mu{}_\nu, \quad U^\mu{}_\rho \Delta^\rho{}_\nu = 0, \quad \Delta^\mu{}_\rho \Delta^\rho{}_\nu = \Delta^\mu{}_\nu, \quad U^\mu{}_\mu = 1, \quad \Delta^\mu{}_\mu = D-1, \tag{A.9}$$

and similarly:

$$u^\mu a_\mu = 0, \quad u^\mu \sigma_{\mu\nu} = 0, \quad u^\mu \omega_{\mu\nu} = 0, \quad u^\mu \nabla_\nu u_\mu = 0, \quad \Delta^\rho{}_\mu \nabla_\nu u_\rho = \nabla_\nu u_\mu. \tag{A.10}$$

## B Weyl-covariant traceless transverse tensors in hydrodynamics

The presentation here will mostly follow [4]. It is possible to express the hydrodynamics tensors in a manifest Weyl-covariant way. To do so, we first need to define a torsionless Weyl-connection $\nabla_\rho^{\text{Weyl}}$ over $(\mathcal{M}, \mathcal{C})$, where $\mathcal{M}$ is the three-dimensional manifold and $\mathcal{C}$ is the conformal class of metrics on the manifold:

$$\nabla_\rho^{\text{Weyl}} g_{\mu\nu} = 2\mathcal{A}_\rho g_{\mu\nu}. \tag{B.1}$$

In the latter, $g_{\mu\nu}$ is any metric in the conformal class $\mathcal{C}$ and $\mathcal{A}_\mu$ is a one-form. Using the Weyl-connection it is possible to define a Weyl-covariant derivative $\mathcal{D}_\mu^{\text{Weyl}} = \nabla_\mu + \omega\mathcal{A}_\mu$, where $\omega$ is the conformal weight of the tensor on which the derivative is acting. If the behavior of a tensor $\mathcal{Q}^{\mu\dots}_{\nu\dots}$ under conformal transformation is $\mathcal{Q}^{\mu\dots}_{\nu\dots} = e^{-\omega\phi}\tilde{\mathcal{Q}}^{\mu\dots}_{\nu\dots}$, then under the same transformation the derivative will transform in a covariant way, that is $\mathcal{D}_\rho^{\text{Weyl}}\mathcal{Q}^{\mu\dots}_{\nu\dots} = e^{-\omega\phi}\mathcal{D}_\rho^{\text{Weyl}}\tilde{\mathcal{Q}}^{\mu\dots}_{\nu\dots}$. The explicit expression of the Weyl-covariant derivative is given by

$$\begin{aligned} \mathcal{D}_\rho \mathcal{Q}^{\mu\dots}_{\nu\dots} \equiv{}& \nabla_\rho \mathcal{Q}^{\mu\dots}_{\nu\dots} + \omega\mathcal{A}_\rho \mathcal{Q}^{\mu\dots}_{\nu\dots} \\ &+ \left(g_{\rho\sigma}\mathcal{A}^\mu - \delta^\mu_\rho \mathcal{A}_\sigma - \delta^\mu_\sigma \mathcal{A}_\rho\right)\mathcal{Q}^{\sigma\dots}_{\nu\dots} + \cdots \\ &- \left(g_{\rho\nu}\mathcal{A}^\sigma - \delta^\sigma_\rho \mathcal{A}_\nu - \delta^\sigma_\nu \mathcal{A}_\rho\right)\mathcal{Q}^{\mu\dots}_{\sigma\dots} + \cdots \end{aligned} \tag{B.2}$$

[15]Our conventions are: $A_{(\mu\nu)} = {}^1\!/\!_2\left(A_{\mu\nu} + A_{\nu\mu}\right)$ and $A_{[\mu\nu]} = {}^1\!/\!_2\left(A_{\mu\nu} - A_{\nu\mu}\right)$.

From (B.1) it follows immediately that the Weyl-covariant derivative is metric-compatible:

$$\mathcal{D}_\rho g_{\mu\nu} = 0, \tag{B.3}$$

since the metric tensor has weight $\omega = -2$. The connection one-form $\mathcal{A}_\mu$ is uniquely determined by demanding the Weyl-covariant derivative of the velocity of the fluid to be transverse and traceless

$$u^\rho \mathcal{D}_\rho u^\nu = 0, \quad \mathcal{D}_\rho u^\rho = 0, \tag{B.4}$$

which imply

$$A_\mu = u^\rho \nabla_\rho u_\mu - \frac{1}{D-1} u_\mu \nabla^\rho u_\rho \equiv a_\mu - \frac{1}{D-1} \Theta\, u_\mu. \tag{B.5}$$

From the latter it is straightforward to see that for all the configurations we considered $\mathcal{A}_\mu = 0$, since both the acceleration and the expansion rate are vanishing, and thus the Weyl-covariant derivative reduces to the normal derivative.

## C Recovering known solutions

The dual of perfect-Cotton boundary geometries can be written as an exact solution of Einstein's equations. Such solutions are different depending on the value of $c_4$ and on the geometry of the horizon.

### Non-vanishing $c_4$: Kerr–Taub–NUT metrics

We start from the boundary metrics studied in Sec. 4.2.1.2 and uplift them using (5.16).

**Spherical ($\nu = 1$)** We set

$$\begin{aligned} c_1 &= 2(a-n), \\ c_2 &= 2a(-1+a^2-4an), \\ c_3 &= -1+5a^2-12an. \end{aligned} \tag{C.1}$$

By doing this, we recover the spherical-horizon Kerr–Taub–NUT metric [41]:

$$\mathrm{d}s^2 = \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 - \frac{\Delta_r}{\rho^2}\left(\mathrm{d}t + \beta \mathrm{d}\phi\right)^2 + \frac{\rho^2}{\Delta_\vartheta}\mathrm{d}\vartheta^2 + \frac{\sin^2\vartheta\Delta_\vartheta}{\rho^2}\left(a\mathrm{d}t + \alpha\mathrm{d}\phi\right)^2, \tag{C.2}$$

with

$$\rho^2 = r^2 + (n - a\cos\vartheta)^2, \tag{C.3}$$
$$\Delta_r = r^4 + r^2(1+a^2+6n^2) - 2Mr + (a^2-n^2)(1+3n^2), \tag{C.4}$$
$$\Delta_\vartheta = 1 + a\cos\vartheta(4n - a\cos\vartheta), \tag{C.5}$$
$$\beta = -b(\theta) = -\frac{2(a-2n+a\cos\vartheta)}{\Xi}\sin^2(\vartheta/2), \tag{C.6}$$
$$\alpha = -\frac{r^2+(n-a)^2}{\Xi}, \tag{C.7}$$
$$\Xi = 1 - a^2. \tag{C.8}$$

**Flat** ($\nu = 0$) We set

$$\begin{aligned} c_1 &= 2(a-n), \\ c_2 &= 2a^2(a-4n), \\ c_3 &= a(5a-12n). \end{aligned} \tag{C.9}$$

and get the flat-horizon Kerr–Taub–NUT metric [42]:

$$\mathrm{d}s^2 = \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 - \frac{\Delta_r}{\rho^2}\left(\mathrm{d}t + \beta\mathrm{d}\phi\right)^2 + \frac{\rho^2}{\Delta_\sigma}\mathrm{d}\sigma^2 + \frac{\sigma^2\Delta_\sigma}{\rho^2}\left(a^2(a-4n)\mathrm{d}t + \alpha\mathrm{d}\phi\right)^2, \tag{C.10}$$

with

$$\rho^2 = r^2 + \frac{1}{4}\left(2a - 2n + a^2\sigma^2(a-4n)\right)^2, \tag{C.11}$$

$$\Delta_r = r^4 + r^2(a^2+6n^2) - 2Mr + 3n^2(a^2-n^2), \tag{C.12}$$

$$\Delta_\sigma = \frac{(2+a^2\sigma^2)(8 - 24an\sigma^2 + a^4\sigma^4 - 8a^3n\sigma^4 + 2a^2\sigma^2(3+8n^2\sigma^2))}{16}, \tag{C.13}$$

$$\beta = -b(\theta) \quad = \quad \frac{\sigma^2}{4}\left(4(n-a) + a^2\sigma^2(4n-a)\right), \tag{C.14}$$

$$\alpha = r^2 + (n-a)^2. \tag{C.15}$$

**Hyperbolic** ($\nu = -1$) We set

$$\begin{aligned} c_1 &= 2(a-n), \\ c_2 &= 2a(1+a^2-4an), \\ c_3 &= 1+5a^2-12an. \end{aligned} \tag{C.16}$$

and get the hyperbolic-horizon Kerr-Taub-NUT metric:

$$\mathrm{d}s^2 = \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 - \frac{\Delta_r}{\rho^2}\left(\mathrm{d}t + \beta\mathrm{d}\varphi\right)^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{\sinh^2\theta\Delta_\theta}{\rho^2}\left(a\mathrm{d}t + \alpha\mathrm{d}\phi\right)^2, \tag{C.17}$$

with

$$\rho^2 = r^2 + (n - a\cosh\theta)^2, \tag{C.18}$$

$$\Delta_r = r^4 + r^2(-1+a^2+6n^2) - 2Mr + (a^2-n^2)(-1+3n^2), \tag{C.19}$$

$$\Delta_\theta = 1 - a\cosh\theta(4n - a\cosh\theta), \tag{C.20}$$

$$\beta = -b(\theta) \quad = \quad -\frac{2(a-2n+a\cosh\theta)}{Z}\sinh^2(\theta/2), \tag{C.21}$$

$$\alpha = \frac{r^2+(n-a)^2}{Z}, \tag{C.22}$$

$$Z = 1 + a^2. \tag{C.23}$$

### Vanishing $c_4$

In this case we recover a class of bulk metrics which for in the flat-horizon case were studied in [29]. We set:

$$\begin{aligned} c_1 &= 2n, \\ c_2 &= 2, \\ c_3 &= 0, \\ c_5 &= 1. \end{aligned} \tag{C.24}$$

and find

$$\mathrm{d}s^2 = -\frac{\Delta_r}{\rho^2}\left(\mathrm{d}t - (2nu + au^2)\mathrm{d}v\right)^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_u}\mathrm{d}u^2 + \frac{\Delta_u}{\rho^2}\left((r^2+n^2)\mathrm{d}v + a\mathrm{d}t\right)^2, \tag{C.25}$$

where

$$\begin{aligned} \Delta_u &= 1 + (a^2u^2 + 4anu)u^2, \\ \rho^2 &= r^2 + (n+au)^2, \\ \Delta_r &= a^2 + r^4 + 6n^2r^2 - 2Mr - 3n^4. \end{aligned} \tag{C.26}$$

Note that for hyperbolic and spherical horizon geometries we find back the case of non-vanishing $c_4$.

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# Chapter 7

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