Essais en Microstructure des Marchés Financiers

Jérôme Dugast

ECOLE DOCTORALE
Sciences du Management

Gestion – Organisation
Décision - Information

ECOLE DES HAUTES ETUDES COMMERCIALES DE PARIS

Ecole Doctorale « Sciences du Management/GODI » - ED 533
Gestion Organisation Décision Information

« ESSAYS IN FINANCIAL MARKET MICROSTRUCTURE »

THESE
présentée et soutenue publiquement le 19 juillet 2013
en vue de l'obtention du

DOCTORAT EN SCIENCES DE GESTION

Par

Jérôme DUGAST

JURY

Présidente de Jury :	**Madame Carole GRESSE** Professeur des Universités Université Paris Dauphine - France
Directeur de Recherche :	**Monsieur Thierry FOUCAULT** Professeur HDR HEC Paris - France
Rapporteurs :	**Monsieur Pierre-Olivier WEILL** Professeur Associé Université de Californie – Los Angeles – Etats-Unis
	Madame Sophie MOINAS Professeur des Universités Toulouse School of Economics - France
Suffragants :	**Monsieur Olivier BRANDOUY** Professeur des Universités IAE de Paris, Université Paris 1 -Panthéon Sorbonne
	Monsieur Johan HOMBERT Professeur Assistant HEC Paris - France

Ecole des Hautes Etudes Commerciales

Abstract

This dissertation is made of three distinct chapters. In the first chapter, I show that traditional liquidity measures, such as market depth, are not always relevant to measure investors' welfare. I build a limit order market model and show that a high level of liquidity supply can correspond to poor execution conditions for liquidity providers and to a relatively low welfare. In the second chapter, I model the speed of price adjustments to news arrival in limit order markets when investors have limited attention. Because of limited attention, investors imperfectly monitor news arrival. Consequently prices reflect news with delay. This delay shrinks when investors' attention capacity increases. The price adjustment delay also decreases when the frequency of news arrival increases. The third chapter presents a joint work with Thierry Foucault. We build a model to explain why high frequency trading can generate mini-flash crashes (a sudden sharp change in the price of a stock followed by a very quick reversal). Our theory is based on the idea that there is a trade-off between speed and precision in the acquisition of information. When high frequency traders implement strategies involving fast reaction to market events, they increase their risk to trade on noise and thus generate mini flash crashes. Nonetheless they increase market efficiency.

Keywords: liquidity, welfare, limit order market, news, limited attention, imperfect market monitoring, high frequency trading, mini flash crash, market efficiency.

Acknowledgements

First and foremost, I am truly grateful to Thierry Foucault for his invaluable guidance and support. I would like to thank him for introducing me to such an interesting area of finance and for teaching me how to think as an economist. My dissertation was greatly inspired by the innovative research he conducted over the last ten years. It's been both a pleasure and an honor to work under his supervision in these past five years.

I would like to thank William Fuchs, Terrence Hendershott, Johan Hombert, Stefano Lovo, Christine Parlour, Ioanid Rosu and Pierre-Olivier Weill whose advices and comments greatly stimulated my research. I particularly thank Christine for giving me the opportunity to spend a fascinating and productive year at the Haas School of Business at UC Berkeley.

I am very glad that Olivier Brandouy, Carole Gresse and Sophie Moinas accepted to take part in my thesis committee. My dissertation benefited from their work.

Over the last years, I had the chance to meet great researchers and to collaborate with some of them. I am grateful to Terrence Hendershott and Ryan Riordan for undertaking an exciting research project with me.

I very much appreciated the friendly and enthusiastic atmosphere in the Finance PhD office at HEC. I would like to thank my fellow PhD candidates for their support and helpful comments. I would also like to mention the fruitful interactions I had with PhD candidates at the Haas School of Business.

Last, but not least, my gratitude goes to my family and friends, for their unconditional support and love. Thank you all for everything.

Contents

Résumé

L'organisation des échanges sur les marchés financiers a évolué de façon extraordinaire, au cours des trois dernières décennies, avec l'avènement des nouvelles technologies de l'information et de la communication. Précédemment, les marchés financiers étaient organisés soit comme des salles de marchés, où des personnes physiques échangeaient les unes avec les autres, soit comme des marchés tenus par des intermédiaires, dans lesquels ces derniers se portent contrepartie à l'échange pour les investisseurs qui les contacteraient par téléphone. Avec l'évolution des technologies à même de générer, acheminer et exécuter des ordres, les structures de marché ont progressivement évolué vers l'informatisation des procédures d'échange, et ce à travers une organisation de *marché électronique dirigé par les ordres*. Ces nouvelles technologies ont aussi amélioré la vitesse et la capacité de traitement de l'information des acteurs de marché. Ceci a donné naissance à un nouveau type de stratégies d'échange, plus sophistiquées et totalement automatisées : le *trading algorithmique*. Le trading algorithmique s'est répandu, depuis lors, et représente désormais plus de 50% du volume des échanges sur le marché action.

Si l'arrivée de ces technologies ont été nécessaires à l'automatisation des transactions, la mise en vigueur de régulations, qui promeuvent la compétition entre bourses et autres plateformes d'échanges, ont catalysé cette évolution. De nouveaux entrants dans l'activité de cotation et de services boursiers (BATS, Chi-X,...) ont crû en «symbiose» avec le trading algorithmique. Profitant d'un environnement technologique et d'une structure de coût très favorable, les traders algorithmiques ont joué le rôle d'offreur de liquidité et, ainsi, ont aidé ces nouvelles plateformes d'échange à concurrencer les bourses historiques dans l'attrait du flux d'ordres. Ceci a généré une *fragmentation de marché*.

Les marchés électroniques dirigés par les ordres.

Les marchés électroniques dirigés par les ordres sont des plateformes d'échange boursier centralisées dans lesquelles l'offre de liquidité peut provenir de chacun des acteurs de marché. Tout agent peut effectuer une transaction en se portant offreur de liquidité avec des ordres à cours limité, qui spécifient un prix et une quantité, et sont affichés dans le carnet d'ordre électronique. Par exemple, dans la Fig.1, les ordres à cours limité à l'achat les plus compétitifs sont affichés au meilleur prix d'achat, $384.82. Le premier demande 50 parts et le deuxième, 100 parts. Les agents peuvent aussi effectuer une transaction en demandant de la liquidité avec des ordres au marché, qui sont immédiatement exécutés avec les ordres à cours limité les plus compétitifs contenus dans le carnet d'ordre. Dans la Fig.1, si un agent envoie un ordre de marché à la vente de 100 parts, cet ordre sera exécuté au prix de $384.82. Les marchés électroniques dirigés par les ordres combinent l'aspect centralisé d'une salle de marché et une large population d'investisseurs potentiels, comme les marchés tenus par des intermédiaires, et ce grâce aux moyens de communication électroniques.

LAST MATCH		TODAY'S ACTIVITY	
Price	384.9000	Orders	1,295,622
Time	15:18:56	Volume	2,791,809

BUY ORDERS		SELL ORDERS	
SHARES	PRICE	SHARES	PRICE
50	384.8200	93	384.9500
100	384.8200	100	385.0300
100	384.8100	100	385.0600
300	384.8100	100	385.0700
100	384.8000	200	385.0900
500	384.7900	100	385.1800
200	384.7700	100	385.2400
500	384.7600	25	385.2500
100	384.7100	100	385.3500
100	384.6900	15	385.5000
200	384.6800	200	385.5500
300	384.5900	200	385.6000
100	384.5000	360	385.6300
50	384.0000	100	385.6800
100	384.0000	100	385.7100
(209 more)		(283 more)	

Figure 1: Vue instantanée d'un carnet d'ordre: ordres à cours limité à la vente du côté bid (colonne de gauche), ordres à cours limité à l'achat du coté ask (colonne de droite).

Les marchés électroniques dirigés par les ordres ont commencé à se répandre au cours des années 1980, et d'abord pour les marchés action. Par exemple, la Bourse de Paris ferma sa salle de marché et devint un marché dirigé par les ordres totalement électronique en 1986.

De nos jours, la plupart des marchés action dans le monde suivent cette organisation. Cette tendance a aussi été suivie par des marchés pour d'autres type de titre financier, comme les marchés de change ou les marchés de taux d'intérêt.

La conversion massive des marchés financiers à l'organisation autour d'un carnet d'ordre électronique a motivé tout un domaine de la recherche académique en finance. Les chercheurs ont initialement voulu comprendre les dynamiques des transactions et de l'offre de liquidité dans ces marchés, ainsi que les stratégies sous-jacentes des acteurs de marché. Les premières études empiriques ont tout d'abord exposé, de façon factuelle, ces dynamiques, comme l'illustre la Fig.2 extraite de Biais, Hillion et Spatt (1995).

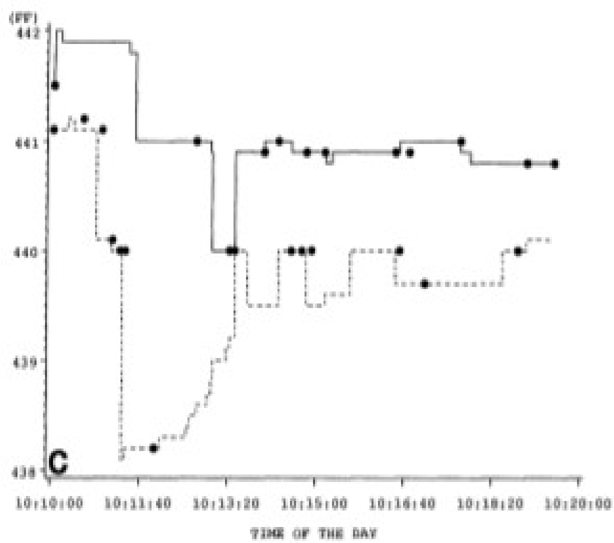


Figure 2: Prix de transactions et cotations bid, ask pour Elf-Aquitaine, 9 Novembre 1991. Source: Biais, Hillion and Spatt (1995)

La recherche théorique a étudié comment les motivations usuelles des échanges boursiers, comme le besoin de liquidité ou bien l'information privée, pouvaient être modélisées dans le cadre d'un marché dirigé par les ordres et pouvaient générer des prédictions en ligne avec les observations empiriques. Ces modèles se sont intéressés aux particularités de ces dynamiques de marché et à leurs implications pour l'efficience informationnelle ou la liquidité de marché (e.g Glosten (1994), Parlour (1998), Foucault (1999), Foucault, Kadan, Kandel (2005), Rosu(2010)).

Le Trading algorithmique.

L'avancement des technologies de l'information, de même que la conversion massive des bourses aux marchés électroniques dirigés par les ordres, ont rendu possible et accéléré le développement du trading algorithmique. Comme le montre la Fig. 3, la participation d'algorithmes dans les transactions boursières pour les actions, aux Etats-Unis, ont augmenté de façon constante depuis plus de 5 ans. Aujourd'hui les algorithmes représentent plus de 50% du volume d'échange total.

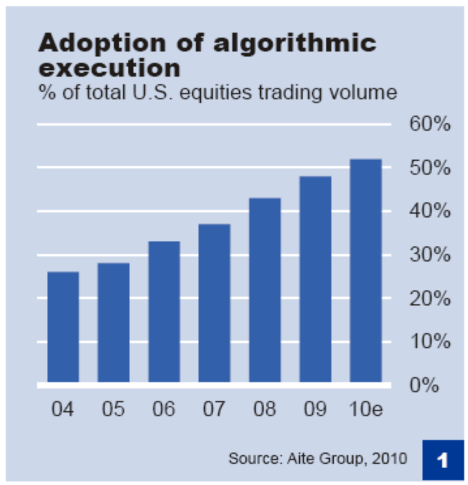


Figure 3: Part du trading algorithmique dans le volume total des échanges d'actions aux E-U. Source: Aite Group (2010).

Le trading algorithmique peut être défini comme l'ensemble des stratégies qui s'appuient sur des algorithmes pour prendre une part, ou l'intégralité, de leurs décisions. Ces stratégies automatisées conditionnent, généralement, leurs actions sur un ensemble de données de marché prédéterminées. Les stratégies de trading algorithmiques peuvent être séparées en deux catégories principales, bien que non exhaustives.

Les algorithmes pour l'execution optimal d'un ordre. Le trading algorithmique peut aider les investisseurs traditionnels et les intermédiaires, tels les gérants de fonds ou les courtiers, à optimiser l'exécution de leurs transactions. Ainsi, les courtiers utilisent couramment des robots qui découpent les ordres de leurs clients et les répartissent dans le temps et entre différentes plateformes de trading, et ce pour obtenir des coûts de transactions réduits. L'avantage principal de ces stratégies repose sur la capacité des ordinateurs, premièrement, à surveiller efficacement les fluctuations des conditions de marché et, deuxièmement, à implémenter, de façon systématique, des procédures d'exécution optimales basées sur ces conditions de marché.

Le trading à haute fréquence. La seconde, et la plus connue, catégorie de trading algorithmique, est le trading à haute fréquence (HFT à partir d'ici). Les stratégies de HFT s'appuient sur leur vitesse de réaction et des capacités très importantes de traitement computationnel pour acquèrir de grandes quantités d'information en temps réel et prendre des décisions à haute fréquence.

Le HFT affecte profondément la façon dont les marchés financiers fonctionnent, et provoquent des débats passionnés entre professionnels, académiques et régulateurs de la finance. Ainsi, dans le New-York Times, Paul Krugman écrit :

«le trading à haute fréquence dégrade, probablement, la fonction du marché financier, car c'est une sorte de taxe sur les investisseurs qui n'ont pas accès à ces ordinateurs super-rapides – ce qui signifie que l'argent que Goldman dépense pour ces ordinateurs a un effet négatif sur la richesse nationale. Comme le grand économiste de Stanford, Kenneth Arrow, l'écrivit en 1973, la spéculation basée sur de l'information privée impose une «double perte sociale» : elle consomme des ressources et affaiblit les marchés. » (P. Krugman, «Rewarding Bad Actors», NY Times, 2 août 2009)

Bien que la recherche académique ait récemment produit des analyses économiques de l'effet du HFT sur l'efficience informationnelle et la liquidité des marchés financiers, il n'existe toujours pas de consensus sur son role bénéfique ou non. Une des difficultés est que le HFT est un «mot valise» qui recouvre des activités très diverses. Certaines firmes (e.g GETCO, Timberhill, Optiver. . .) sont des teneurs de marché à haute fréquence et comptent pour une part importante de l'offre de liquidité à la fois en Europe et en Amérique. D'autres acteurs (e.g des hedge funds comme Renaissance) utilisent des ordinateurs pour prendre des positions directionnelles basées sur des «signaux» avant que d'autres investisseurs aient accès à cette information. Toutes ces activités sont clairement différentes et, de fait, peuvent avoir des conséquences différentes pour l'efficience et la liquidité des marchés.

Les études empiriques récentes (e.g Hendershott, Jones and Menkveld (2011), Hendershott and Riordan (2013), Brogaard, Hendershott and Riordan (2012), Chaboud, Chiquoine, Hjalmarsson and Vega (2009)) ont montré que le HFT avait un effet positif sur les mesures de qualité de marché. Cependant d'autres études (e.g Hasbourck (2013)) et certains évènements récents dus au HFT (i.e. le Flash Crash du 6 mai 2010) ont souligné le comportement potentiellement manipulateur et destabilisant de leurs stratégies. Cela laisse ouverte la question de savoir quel type de HFT a un impact positif pour les marchés financiers.

Le trading algorithmique et les limites cognitives. L'existence du trading algorithmique soulève, en elle-même, la question du fondement rationel de l'investissement dans ces technologies et, implicitement, interroge quelle capacité supplémentaire les ordinateurs apportent aux acteurs de marché. Lorsqu'un investisseur connait un choc de liqudité, il doit analyser ses positions et son exposition aux risques avant de prendre des décisions d'échange. Lorsque de l'information nouvelle, apportée par des nouvelles financières, est rendue publique, les acteurs de marchés doivent interpréter cette information avant de l'utiliser. De ce fait, ils doivent concentrer leur attention pour accomplir ces tâches spécifiques. Les machines peuvent obtenir ces informations puis procéder à des transactions beaucoup plus rapidement que des humains. De plus, elles peuvent surveiller plusieurs sources d'information simultanément et effectuer plusieurs tâches à la fois. Ainsi, le trading algorithmique allège la contrainte d'attention des investisseurs humains. La recherche théorique peut donc étudier le trading algorithmique en analysant les effets de l'attention imparfaite pour les marchés financiers (e.g. Foucault, Roëll et Sandas (2003), Biais, Hombert et Weill (2012), Pagnotta et Philippon (2012), Foucault, Kadan and Kandel (2013)).

Cependant le trading algorithmique ne se réduit pas à une amélioration des capacités cognitives utilisées pour les stratégies traditionnelles. Premièrement, l'utilisation d'ordinateurs, en soi, élargit le champ de l'information accessible. Par exemple, la dynamique d'un carnet d'ordre est difficilement interprétable sans une analyse quantitative informatisée, et la fréquence élevée de cette dynamique la rend à peine perceptible aux humains. Deuxièmement, le traitement de l'information par les machines diffère de celle des humains. Les machines peuvent traiter de l'information « dure » et quantifiable beaucoup plus efficacement alors qu'elles sont moins à même de gérer des scenarios non-anticipés au moment de leur conception et peuvent donc faire des erreurs le cas échéant. Une théorie globale du trading algorithmique devrait intégrer ces éléments.

La Fragmentation de marché.

Régulation et fragmentation de marché. Les marchés financiers, et spécifiquement les marchés action, sont maintenant substantiellement fragmentés. Ceci a été induit, principalement, par des actions règlementaires en Europe et aux Etats-Unis. L'Union Européenne a introduite la *Directive sur les Marchés d'Instruments Financier* (MiFID) le 1er novembre 2007, ce qui a aboli la règle de concentration dans les pays européens et a promu la concurrence pour les systèmes et les services boursiers. Les bourses traditionnelles, qui profitaient d'un pouvoir de marché dans les pays européens (London Stock Exchange en Grande-Bretagne, Euronext en France, Belgique et Pays-Bas), ont alors du affronter la concurrence des nouvelles plateformes boursières, telle Chi-X, Turquoise et BATS Europe. Aux Etats-Unis, la *Régulation du Système National de Marché* (Reg NMS) a été mise en vigueur en 2007 pour moderniser et renforcer le système national de marché des actions. Comme avec MiFID, elle a produit de la concurrence entre plateformes boursières. La Figure 4 illustre comment le NYSE aux Etats-Unis et le LSE en Europe ont perdu des parts de marché au profit des nouveaux entrants, respectivement BATS et Chi-X.

Automatisation des échanges et fragmentation de marché. Dans une revue récente pour le UK Governement Office For Science[1], Carole Gresse écrit :

«Il existe une ancienne croyance commune en théorie économique qui veut que les marchés de titre sont des monopoles naturels car le coût marginal d'une transaction décroit avec la quantité d'ordres exécutés dans un marché. Alors que cela a été longtemps dans une certaine mesure, le progrès technologique a, d'une certaine façon, changé cette réalité. Les coûts fixes et le temps nécessaire pour mettre en place un nouveau marché ont considérablement diminué et le trading assisté par ordinateur autorise des stratégies de transaction entre marchés qui connectent les multiples plateformes d'échange, comme si elles formaient un réseau consolidé de contreparties avec plusieurs entrées. Ces nouveaux outils amoindrissent l'argument de

[1] Market fragmentation in Europe : assessment and prospects for market quality, C. Gresse (2012)

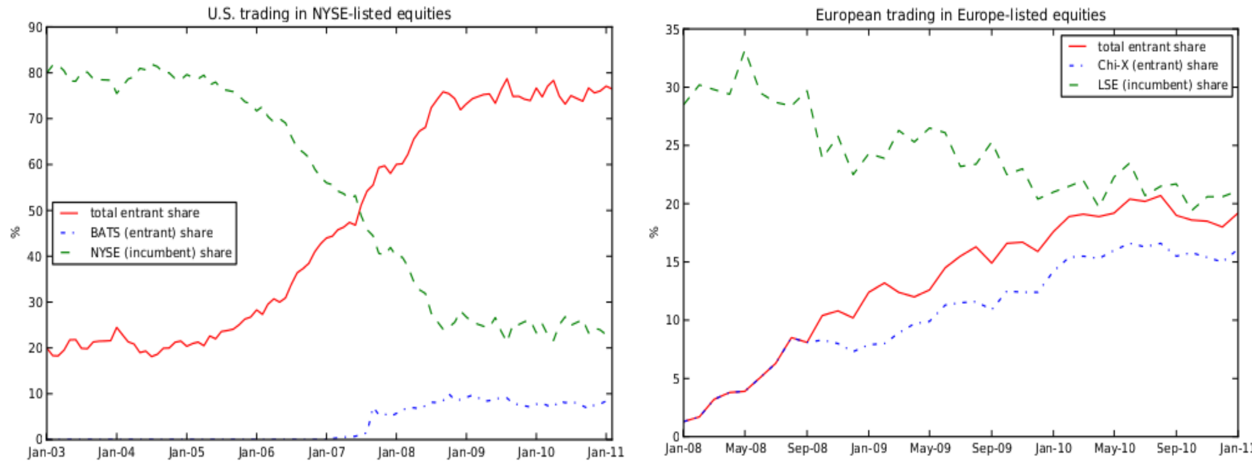


Figure 4: Fragmentation de marché en Europe et aux E-U. Le graphique de gauche représente les parts de marché de la bourse traditionnelle et de la nouvelle entrante pour les actions listées au NYSE. Le graphique de droite fait la même chose pour des actions européennes. Source: Menkveld (2012).

l'externalité d'un tel réseau. »

L'automatisation des stratégies d'échanges boursiers et la libéralisation de la concurrence pour les systèmes et services boursiers ont été concomitants car l'avancement des technologies de l'information était une condition nécessaire à ces deux évolutions. MiFID, par exemple, prévoit que les entreprises boursières doivent chercher les meilleures conditions d'exécution possibles pour les ordres de leurs clients. Dans un environnement de marchés fragmentés, cela requiert l'utilisation de systèmes informatiques de routage des ordres qui recherchent, de façon automatique, les meilleurs prix offerts parmi les différentes plateformes boursières. Ce type de tâche est difficilement réalisable par des humains.

Au-delà de leur rôle clé dans la consolidation des marchés fragmentés, les stratégies d'échanges automatisés ont probablement eu un effet important pour la croissance des nouvelles plateformes boursières. Dans un récent document de travail[2], Albert Menkveld apportent des preuves que les nouveaux marchés ont grandi dans une sorte de symbiose avec certaines entreprises de trading à haute fréquence qui se spécialisaient dans la tenue de marché à haute fréquence, telle que GETCO. Attirés par une infrastructure adéquate et des subventions aux ordres à cours limité, ces nouveaux acteurs sont devenus *de facto* teneurs de marché, ce qui a aidé les nouvelles plateformes boursières dans la compétition pour attirer le

[2]High frequency trading and the new-market maker, A. Menkveld (2012)

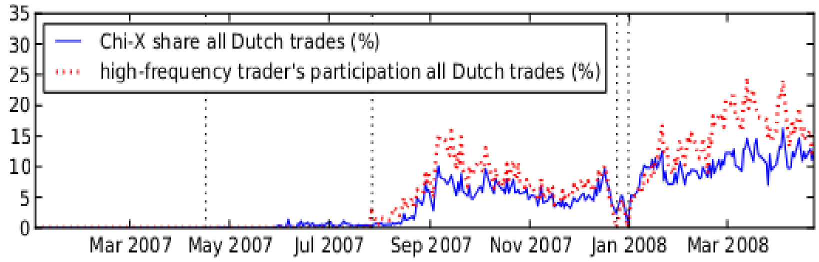


Figure 5: Le graphique représente la part de marché de l'entrant Chi-X basé sur le nombre de transactions. Le graphique représente aussi la participation des HFT aux transactions, en se basant sur leurs échanges à la fois chez Chi-X et Euronext. Source: Menkveld (2012).

flux d'ordres.

Maintenant que le trading algorithmique devient la forme dominante de transaction, les bourses entrent en concurrence pour attirer leur flux d'ordres en offrant des services attractifs. Les plateformes boursières ont réduit, de façon drastique, les latences de communication entre leurs serveurs et ceux de leurs clients. Elles ont massivement investi en bande-passante et proposent des services de co-location aux traders à haute fréquence. Aujourd'hui, le temps moyen séparant la soumission d'un ordre de son exécution est inférieur à une seconde (voir Fig. 6) et de l'ordre de quelques millisecondes pour les traders dont les ordinateurs sont co-localisés avec le serveur de la plateforme.

L'offre de services dédiés spécifiquement aux traders à haute fréquence est plus large que celle des plateformes boursières seules. Les médias financiers (Bloomberg, Thomson Reuters, Dow Jones) offrent des flux de nouvelles financières facilement déchiffrable, et presque en temps réel, qui visent explicitement les traders à haute fréquence[3].

[3]Dow Jones Newswire offre des nouvelles et des données d'évènements avec faible latence pour le trading électronique. Voir http://www.dowjones.fr/salesandtrading/low-latency-feeds.asp

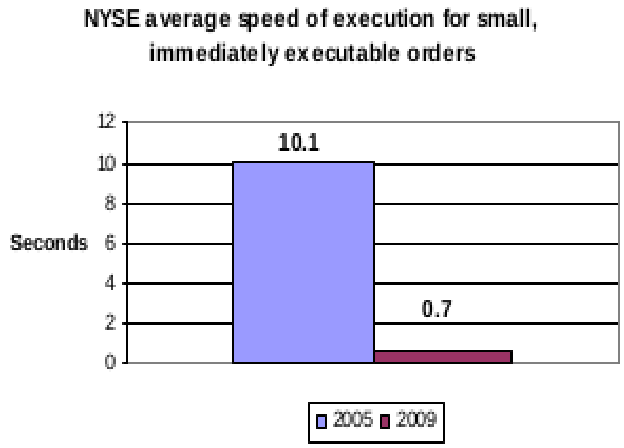


Figure 6: Vitesse moyenne d'exécution d'un ordre petit et immédiatement exécutable, pour le NYSE. Source: SEC Concept on Equity Market Structure (2010).

Objectif de la thèse

Comprendre les conséquences des changements récents de l'organisation des marchés financiers est désormais une question de premier ordre pour la recherche académique. Cette révolution technologique a notamment souligné la pertinence de l'étude des structures de marché financier au niveau microéconomique. Dans cette thèse, je traite, avec des modèles théoriques, trois questions de recherche en lien avec les évolutions récentes de ces marchés. Ainsi, j'entends contribuer à la littérature académique sur la microstructure des marchés financiers qui englobe ces questions.

Chapitre 1 - Les mesures de liquidités sont-elles pertinentes pour mesurer le bien-être des investisseurs?

Le bien-être, au sens économique, des investisseurs est un objectif majeur pour les régulateurs des marchés financiers. Le bien-être n'étant pas une quantité observable, on pense que la liquidité de marché peut être un bon concept pour approximer ce bien-être. La liquidité de marché peut se définir comme la facilité, pour un investisseur, à échanger une quantité donnée d'actif à un prix qui dévie peu en comparaison d'un prix de référence. De fait, la liquidité de marché correspond à des coûts de transactions implicites. Dans les marchés centralisés, ces coûts de transaction implicites sont traditionnellement mesurés avec la fourchette de prix et la profondeur de marché (le nombre de cotations proches des meilleurs prix proposés à l'achat et à la vente). Cette définition de la liquidité de marché est biaisée dans le sens du bien-être des consommateurs de liquidité. La plupart des marchés centralisés (actions, changes,...) sont organisés en marchés dirigés par les ordres dans lesquels tout investisseur peut échanger en cotant des prix pour offrir de la liquidité. Les mesures de liquidité précédentes prennent mal en compte le bien-être des offreurs de liquidité. Une grande profondeur de marché peut, par exemple, être due à un faible taux d'exécution des ordres à cours limité, ce qui, a priori, n'est pas signe d'un bien-être supérieur pour ceux qui utilisent des ordres à cours limité. Comment les mesures de liquidité sont déterminées pour les stratégies des investisseurs ? Comment ces mesures sont-elles reliées au bien-être des investisseurs ?

Afin de traiter ces questions, je construis un modèle dynamique de marché dirigé par les ordres. La facilité de résolution du modèle me permet d'obtenir des solutions formelles pour les variables d'équilibre telles que la profondeur de marché, le volume de transaction, le taux d'exécution des ordres à cours limité et, aussi, le bien-être des investisseurs. Lorsque j'étudie l'effet de la variation de certains paramètres du modèle, je trouve que (i) la profondeur de marché co-varie négativement avec le bien-être, (ii) dans la plupart des cas le volume de transaction co-varie positivement avec le bien-être, à l'exception d'un domaine paramétrique particulier, et (iii) le taux d'exécution des ordres à cours limité co-varie positivement avec le bien-être. Ceci montre, premièrement, que le taux d'exécution des ordres à cours limité et la profondeur de marché peuvent varier dans des directions opposées et, deuxièmement, que les conditions d'exécution des ordres à cours limité peuvent dominées pour le bien-être. Le corollaire est que des variations ou des chocs sur les mesures de liquidité, telles que la profondeur de marché ou le volume de transaction, ne correspondent pas forcément à des changements équivalents pour le bien-être des investisseurs.

La liquidité de marché est habituellement mesurée par les coûts de transaction. Les coûts de transaction explicites comprennent les commissions des courtiers, les frais de transaction,…etc. Les coûts de transaction implicites sont mesurés par l'écart entre le prix d'exécution et un prix de référence qui peut être le prix moyen entre les meilleurs prix offerts à l'achat et à la vente. La vision traditionnelle de la liquidité de marché trace un lien direct entre les coûts de transaction implicite et l'illiquidité. Dans les marchés centralisés avec intermédiation, dans lesquels l'exécution des ordres est déléguée à des teneurs de marché, les coûts de transaction implicites correspondent au surplus que ces teneurs de marché extraient des transactions. L'existence de ces coûts de transaction peut s'expliquer par des coûts d'inventaire, le risque de sélection adverse ou bien une concurrence imparfaite entre teneurs de marché. Avec des données de marchés exhaustives, les coûts implicites peuvent être directement établis à partir des prix observés et des quantités associées cotées par les teneurs de marchés. Par exemple Chordia, Roll and Subrahmanyam (2000, 2001) étudient les mouvements agrégés et les co-mouvements de liquidité pour les marchés action du NYSE qui, à l'époque, étaient tenus par des «spécialistes» (teneurs de marché). Parmi les différents proxys de liquidité, les auteurs utilisent les fourchettes de prix et les profondeurs de marché. Dans les marchés considérés, les prix et les quantités observés sont des données de transaction qui étaient annoncés par les spécialistes avant transaction. C'est pourquoi ces proxys de liquidité correspondaient effectivement aux coûts de transaction implicites auxquels les investisseurs faisaient face.

Dans les marchés dirigés par les ordres, on peut examiner, de façon similaire, la dynamique de l'offre de liquidité avec l'évolution des proxys que sont les fourchettes de prix et des profondeurs de marché. Jusqu'à récemment certains papiers (Biais, Hillion et Spatt (1995), Engle, Fleming, Ghysels et Nguyen (2011), Hasbrouck et Saar (2012)) ont étudié les dynamiques de ces marchés en utilisant des données de carnets d'ordre. Ces données comprennent typiquement l'évolution du carnet d'ordre, les soumissions, les exécutions et les annulations d'ordres. Dans ce cadre, l'offre de liquidité peut-elle être directement reliée au bien-être des investisseurs comme dans les anciens marchés action du NYSE avec des spécialistes ? Dans les marchés dirigés par les ordres, les offreurs de liquidité ne peuvent être distingués des consommateurs de liquidité comme dans les marchés précédents. Des coûts de transaction implicites élevés pour les investisseurs qui consomment de la liquidité, avec des ordres au marché, correspondent à de bonnes conditions d'exécution pour ceux qui offrent de la liquidité avec des ordres à cours limité. Ce sont des transferts monétaires des consommateurs de liquidité vers les offreurs de liquidité, à l'intérieur d'un ensemble d'investisseurs. Dans ce type de marché, le bien-être est, intuitivement, élevé lorsque la fréquence, à laquelle les gains de l'échange sont réalisés entre un offreur et un consommateur de liquidité, est elle aussi élevée (comme le montrent Colliard et Foucault (2012)). Cette fréquence de transaction semble être mieux captée par le volume de transaction, par exemple, comme c'est le cas dans mon modèle. Plus généralement, il n'est pas évident de savoir comment cette fréquence de transaction devrait être liée aux coûts de transaction implicites pour les ordres au marché, que mesurent la fourchette de prix et la profondeur de marché.

Dans mon modèle, je considère un cadre en temps continu. L'économie est constituée d'un continuum d'investisseurs qui peuvent chacun détenir 0 ou 1 unité d'un actif. Leur taux d'escompte temporel est constant. Chaque investisseur a une valeur privée, haute ou basse, pour l'actif. La valeur privée d'un agent est aléatoire et idiosyncratique. Sa dynamique est donnée par une chaine de Markov à deux états en temps continu. Elle passe de haute à basse, et inversement, avec la même intensité. La différence de valorisation de l'actif entre les agents génère des motivations pour l'échange et des gains en termes de bien-être lorsque des parts de l'actif sont transférés d'investisseurs avec une valeur privée basse vers des investisseurs avec une valeur privée haute. Les transactions ont lieu au sein d'un marché centralisé. Les investisseurs peuvent échanger soit en offrant de la liquidité avec des ordres à cours limité soit en consommant de la liquidité avec des ordres au marché.

J'étudie une classe d'équilibres stationnaires. Ces équilibres sont tels que l'état agrégé du marché dirigé par les ordres ne change pas au cours du temps. A l'équilibre, les prix sont constants au cours du temps et la fourchette de prix est égale au tick, l'écart minimum entre deux prix d'échange possibles. Tous les ordres à cours limité sont soumis aux meilleurs prix offert à l'achat et à la vente. Les ordres à cours limité à l'achat (resp. à la vente) sont soumis par des investisseurs avec une haute valeur privée qui ne détiennent pas l'actif (resp. avec une valeur privée basse et qui détiennent l'actif). Le nombre d'ordre à cours limité de part et d'autre du carnet d'ordre, c'est-à-dire la profondeur de marché, est tel que les investisseurs sont indifférents entre l'utilisation d'un ordre à cours limité, pour échanger à un prix attractif mais avec un délai, ou l'utilisation d'un ordre au marché pour une transaction immédiate mais avec un coût implicite, la fourchette de prix. Lorsque la taille du tick diminue, l'avantage comparatif d'un ordre à cours limité diminue. Le délai maximal, pour l'exécution d'un ordre à cours limité, que les investisseurs sont prêt à accepter, diminue lui aussi. Ceci implique que la profondeur de marché décroit et que les investisseurs utilisent relativement plus d'ordres au marché que d'ordres à cours limité.

Le bien-être est relié négativement à la profondeur de marché. Idéalement, tout investisseur qui attend avec un ordre à cours limité dans le carnet d'ordre devrait être apparié, et échanger, avec un investisseur similaire de l'autre côté du carnet d'ordre. Une transaction entre ces deux investisseurs transférerait un part de l'actif d'un agent à valeur privée basse vers un agent avec valeur privée haute et, ainsi, améliorerait le bien-être. Le tick permet aux investisseurs d'utiliser des ordres à cours limité pour extraire plus du surplus de l'échange que leur contrepartie utilisant un ordre au marché, et sans risquer d'être concurrencer par des ordres à cours limité plus compétitifs. Ce «pouvoir de marché» relatif qui est donné au offreur de liquidité est inefficient puisqu'il ralentit le rythme des transactions et la réalisation des surplus de l'échange associée. C'est pourquoi la taille du tick a un effet négatif sur le bien-être.

Le niveau de la valeur privée basse d'un investisseur a un effet positif sur le bien-être. Cet effet est surprenant puisque, toute chose égale par ailleurs, une réduction de l'utilité qu'un investisseur, avec une valeur privée basse, tire de l'actif devrait affecter négativement le bien-être global. L'intuition de ce résultat est qu'une diminution de cette valeur privée basse augmente le coût d'opportunité à attendre dans le carnet d'ordre avec un ordre à cours limité et de ne pas échanger immédiatement. De ce fait les investisseurs utilisent plus d'ordre au marché, la profondeur de marché baisse et le bien-être augmente.

Chapitre 2 - Attention limitée et arrivée de nouvelle

Les investisseurs ont une capacité d'attention limitée et ne peuvent donc pas surveiller continument le flux d'information arrivant sur les marchés financiers. En conséquence, ils n'ont pas la capacité d'obtenir ou d'analyser instantanément les implications des nouvelles financières au moment de leur arrivée. Et, de ce fait, le contenu de ces nouvelles ne devient pas une information commune du marché instantanément non plus. C'est pourquoi, à horizon court, de l'information publique est en fait de l'information privée pour les investisseurs qui l'observent les premiers. En raison de l'attention limitée des agents économiques, l'arrivée d'information publique génère de courtes périodes d'information asymétrique. Comment les marchés financiers réagissent autour de l'arrivée d'une nouvelle ? Et quel rôle l'attention limitée joue-t-elle dans ce processus ?

Afin de traiter ces questions, je propose un cadre théorique pour analyser le rôle de l'attention limitée dans la réaction des marchés aux nouvelles. Je conçois un modèle de marché dirigé par les ordres en présence d'incertitude sur la valeur de l'actif en raison de l'arrivée de nouvelles. Ce modèle étend, au marché dirigé par les ordres, le modèle de marché de gré à gré de Duffie, Garleanu et Pedersen (2005 ,2007). Dans Duffie et al., la principale imperfection de marché est une friction pour la recherche d'une contrepartie pour une transaction. Dans mon modèle, l'imperfection de marché vient de la capacité d'attention limitée des investisseurs. Celle-ci est équivalente à une surveillance imparfaite du marché et de l'arrivée des nouvelles. Les investisseurs ne peuvent pas, en continu, observer l'information publique et être en contact avec le marché. Ils peuvent y procéder à certains instants, aléatoires, de surveillance. Ce cadre théorique permet de générer une diffusion graduelle de l'information parmi les investisseurs à la suite de l'arrivée d'une nouvelle. La surveillance de marché imparfaite des investisseurs permet de décrire, de façon jointe, la formation de la liquidité, la découverte des prix et l'efficience de marché autour de l'arrivée d'une nouvelle.

La réaction des marchés financiers à de l'information publique a motivé toute une ligne de recherche, à la fois empirique et théorique, en particulier durant les années 1990. Étudier cette réaction des marchés nous donne entre autres, une meilleure compréhension du processus de découverte des prix dans ces mêmes marchés. La question de la réaction de marché à de l'information publique a été empiriquement traitée, par exemple, par Eredington et Lee (1995), Fleming et Remolona (1999) et Green (2004). Ces papiers s'intéressent à la réaction des marchés pour les titres du Trésor Américain aux annonces macroéconomiques attendues, c'est-à-dire dont la date et l'heure de publication sont connues. Les deux premiers papiers montrent que le marché réagit à l'annonce en deux phases successives. Au cours de la première phase, le prix décale rapidement vers une nouvelle valeur en lien avec les principaux chiffres contenus dans l'annonce. La deuxième phase de cette réaction se caractérise par une forte volatilité des prix, suggérant que l'interprétation précise de l'annonce diffère d'un investisseur à l'autre. Cette phase se termine lorsque ces différentes interprétations convergent. Le papier de Green montre que ces annonces macroéconomiques exacerbent le problème de sélection adverse, ce qui suggère que les investisseurs ayant les meilleures capacités de traitement de l'information peuvent tirer parti de ces évènements.

Mon papier contribue de manière significative à cette littérature en considérant les nouvelles inattendues. Les arrivées de nouvelles financières sont des évènements quotidiens. Elles sont publiées par les médias financiers comme Thomson Reuters ou Bloomberg. Elles délivrent de l'information souvent pertinente à l'évaluation des prix des actifs financiers. Quasiment toutes ces nouvelles arrivent sur les marchés à des moments non prévus[4]. De plus, la fréquence d'arrivée de ces nouvelles varie beaucoup d'un titre à l'autre[5]. La nature inattendue des évènements peut vraisemblablement empêcher les investisseurs d'être parfaitement attentif aux nouvelles financières. En introduisant cette attention limitée dans un modèle de marché dirigé par les ordres, je peux aborder la question de comment les nouvelles inattendues affectent les décisions de transactions boursières et, par voie de conséquence, la formation des prix et l'offre de liquidité.

L'évolution actuelle des marchés financiers vient appuyer le choix, fait dans mon modèle, de considérer l'attention limitée comme une dimension importante pour comprendre les réactions des marchés, à court terme, à de l'information publique. Ces réactions de court terme donnent lieu à des questionnements importants depuis que le trading à haute fréquence

[4]Dans un échantillon de 40 actions importantes, représentant 70% de la capitalisation boursière du FTSE100, Gross-Klussmann et Hautsch (2011) trouvent qu'une action est couverte par, en moyenne, 750 nouvelles inattendues sur 1,5 années

[5]Dans le même échantillon, la fréquence d'arrivée des nouvelles peut varier de 1 à 10 (de 200 à 2000 nouvelles).

s'est développé en utilisant des technologies de surveillance intense des marchés dans le but d'effectuer des transactions très rapides sur des nouvelles financières. Plus généralement, le boom du trading algorithmique (qui inclut le HFT) s'explique, en partie, par le besoin des investisseurs d'améliorer la dimension de surveillance des marchés de leurs stratégies. Cette tendance montre l'importance, pour les investisseurs, de la capacité d'attention qu'ils allouent à la surveillance des marchés. De plus, mon modèle considère les marchés électroniques dirigés par les ordres qui sont utilisés pour la plupart des bourses, cotant des actions et leurs dérivés, et qui ont rendu possible le développement du trading algorithmique.

Ce papier donne plusieurs implications empiriques pour l'offre de liquidité et la dynamique des prix autour des arrivées de nouvelle. Lorsque la fréquence d'arrivée de nouvelle augmente, (i) le niveau de l'offre de liquidité diminue, (ii) les prix s'ajustent plus rapidement à la suite de l'arrivée d'une nouvelle et (iii) l'importance relative de l'annulation des ordres à cours limité, dans le processus d'ajustement des prix, diminue. L'intuition pour ces résultats est liée à la période courte d'asymétrie d'information qui suit une nouvelle et qui est due à l'attention limitée. Comme la présence d'asymétrie d'information le prévoit habituellement, il existe un risque de sélection adverse pour les offreurs de liquidité et ce risque varie avec la fréquence d'arrivée des nouvelles. Les investisseurs peuvent être hésitants à offrir de la liquidité avec des ordres à cours limité puisque, à la suite de l'arrivée d'une nouvelle, l'attention limitée retarde leur réaction. Entre temps, leurs ordres à cours limité peuvent être exécutés car leur niveau de prix n'est plus en lien avec la nouvelle valeur de l'actif et ils offrent donc une opportunité de profit.

Dans le cadre théorique que je propose, les investisseurs peuvent, à la fois, offrir de la liquidité avec des ordres à cours limité et consommer de la liquidité avec des ordres au marché. Avant l'arrivée d'une nouvelle, les investisseurs échangent les uns avec les autres car leur valeur privée pour l'actif diffère, ce qui génère des gains à l'échange. Pendant cette phase, le carnet d'ordres est dans un état stationnaire. Le niveau de l'offre de liquidité reste constant et est déterminé par l'arbitrage suivant. Les ordres au marché permettent une exécution immédiate alors que les ordres à cours limité offrent un meilleur prix mais supportent un délai d'exécution et un risque de sélection adverse lorsque la valeur de l'actif change. A l'équilibre, le niveau de l'offre de liquidité s'ajuste de telle sorte que les investisseurs sont indifférents entre les deux types d'ordre.

Lorsque, à la suite de l'arrivée d'une nouvelle, la valeur de l'actif change, celle-ci est publiquement accessible mais les investisseurs n'observent pas ce changement immédiatement. Ils en prennent conscience au bout d'un certain temps qui dépend de l'intensité avec laquelle ils surveillent le marché. Ceci génère une phase de transition, à la fin de laquelle, les prix s'ajustent à la nouvelle valeur de l'actif. Ce processus de découverte du prix repose sur deux dynamiques sous-jacentes. Les investisseurs qui observent la nouvelle valeur de l'actif assez rapidement peuvent profiter d'opportunités d'arbitrage transitoires en utilisant des ordres au marché pour exécuter les ordres à cours limité « immobiles » au prix initial. Et les investisseurs, avec un ordre dans le carnet d'ordre, annulent ces ordres pour éviter la sélection adverse des ordres au marché précédents. Une fois que les ordres à cours limité, au prix initial, ont tous été annulés ou exécutés, la phase de transition prend fin et le carnet d'ordre converge vers un nouvel état stationnaire sans incertitude sur la valeur de l'actif. Ainsi le modèle offre une description, à haute fréquence, de la dynamique des prix et des ordres autour des arrivées de nouvelle. Celle-ci devrait être utile aux empiristes[6].

La décision des investisseurs d'utiliser des ordres à cours limité ou au marché pour échanger, avant l'arrivée d'une nouvelle, dépend du risque de sélection adverse durant la phase de transition. Toute chose égale par ailleurs, ce risque amplifie la perte anticipée associée avec la soumission d'un ordre à cours limité, ce qui a un effet négatif sur l'offre de liquidité. Dans ce cadre, l'effet de la fréquence d'arrivée des nouvelles est intuitif. Des nouvelles plus fréquentes augmentent la probabilité d'un évènement durant lequel un ordre à cours limité peut être sujet à la sélection adverse ce qui augmente le risque de sélection adverse. Par conséquent, le niveau de l'offre de liquidité, mesuré par la profondeur de marché (le nombre d'ordres à cours limité dans le carnet d'ordre) est relié négativement à cette fréquence. Dans un marché plus fin, la quantité d'ordres à cours limité qui doivent être annulés ou exécutés, durant la transition, est moindre, ce qui rend l'ajustement des prix plus rapide.

La capacité d'attention limitée des investisseurs influence ce risque de sélection adverse

[6]Engle et al (2009) utilisent des données haute fréquence de carnet d'ordre pour analyser la liquidité et la volatilité du marché des bons du trésors U.S.

et donc l'offre de liquidité avant l'arrivée d'une nouvelle. Une capacité d'attention plus grande a cependant un effet ambigu. Pour le comprendre, considérons une augmentation de l'intensité de surveillance des investisseurs[7]. D'un côté les investisseurs peuvent annuler leurs ordres plus rapidement après l'arrivée d'une nouvelle ce qui réduit le risque de sélection adverse et rend les ordres à cours limité plus profitables. Mais d'un autre côté les investisseurs peuvent aussi envoyer plus rapidement des ordres au marché pour exécuter des ordres à cours limité immobiles ce qui aggrave le risque de sélection adverse. Au final les ordres à cours limité peuvent devenir plus ou moins profitables après une augmentation de l'intensité de surveillance. Dans le papier, j'identifie des conditions sous lesquelles les ordres à cours limité deviennent plus profitables. Cependant la magnitude de cet effet sur l'offre de liquidité est très faible, en particulier si on la compare à l'effet de la fréquence d'arrivée de nouvelle. Ceci suggère que seul la capacité de surveillance relative, par rapport aux autres participants de marché, compte réellement pour comprendre la façon dont ce paramètre peut jouer un rôle quantitatif dans la stratégie des investisseurs autour de l'arrivée d'une nouvelle.

[7] Cette augmentation pourrait venir d'une réduction de la latence de marché

Chapitre 3 - Trading à haute fréquence, efficience de marché et « mini flash crashes »

La stabilité des marchés financiers est importante pour attirer les investisseurs. L'instabilité des prix des titres financiers peut brouiller les anticipations des investisseurs et, au final, pourrait décourager leur participation aux échanges dans les bourses traditionnelles. Au cours des dernières années, des anecdotes, provenant de l'industrie financière, rapportent l'apparition d'un nouveau type d'évènement d'instabilité de marché : le «mini flash crash» [8] (Voir l'Appendix B.1 pour une liste de mini flash crash passés). Un mini flash crash peut se définir comme un important et brusque changement de prix d'un actif suivi par un renversement très rapide (voir Figure 1). La fréquence croissante de ces évènements a été interprétée comme un symptôme de l' instabilité des marchés et a été attribuée au Trading à Haute Fréquence (HFT désormais). Entre temps, des papiers récents (e.g, Hendershott, Jones et Menkveld (2011), Hendershott et Riordan (2013), Brogaard, Hendershott et Riordan (2012) et Chaboud, Chiquoine, Hjalmarsson et Vega (2009)) suggèrent que le HFT a un effet positif sur la qualité de marché et son efficience informationnelle. Par quel canal le HFT pourrait générer des mini flash crashs ? Les marchés financiers peuvent-ils devenir à la fois plus efficients et moins stables sous l'effet du HFT ?

Afin de traiter ces questions, nous développons une théorie des mini flash crashs. Notre théorie est basée sur l'idée qu'il existe une tension entre la vitesse et la précision dans l'acquisition de l'information. Le nouvel environnement de marché permet à des participants de devenir des HFT et de réagir beaucoup plus rapidement à des nouvelles de différents types mais au détriment de la précision de ces informations. Nous introduisons cette idée dans un modèle à deux périodes dans lequel des agents stratégiques peuvent décider d'investir dans une technologie rapide, qui leur permet d'acquérir, à la période 1, un signal bruité sur la valeur fondamentale de l'actif et ensuite d'échanger, ou bien ne pas investir et d'attendre la période 2 pour échanger, en acquérant un signal parfait sur la valeur fondamentale. L'avantage des HFT en termes de vitesse d'acquisition d'information a été étudié par Foucault, Hombert

[8] En référence au Flash Crash du 6 Mai 2010. Cf. «The Flash Crash, in Miniature» in the New York Times, http://www.nytimes.com/2010/11/09/business/09flash.html. Nanex Research rapporte aussi des mini flash crashs parmi d'autres anomalies de marché, http://www.nanex.net/FlashCrash/OngoingResearch.html

et Rosu (2012) mais n'incorpore pas la possibilité d'erreur d'interprétation de l'information nouvelle. Dans Foucault, Hombert et Rosu (2012), comme dans notre papier, l'avantage de rapidité est modélisé comme une faculté à échanger une période avant les autres investisseurs. Ceci peut se voir comme une forme réduite de la capacité de surveillance de marché intense des HFT. Celle-ci pourrait-être modélisée dans un cadre où les investisseurs ont des capacités de surveillance de marché imparfaite, comme dans des papiers récents (e.g. Biais, Hombert et Weill (2013), Foucault, Kadan et Kandel (2013), Pagnotta et Philippon (2012)). La littérature sur le HFT considère que le HFT peut aussi bénéficier d'une faculté supérieure de traitement de l'information qui diffère de l'avantage de rapidité. Pour traiter cette dimension du problème, des papiers théoriques comme Biais, Foucault et Moinas (2013) modélisent le HFT comme des agents informés traditionnels (comme dans Glosten (1995)).

Nous trouvons qu'une augmentation de l'activité HFT, due à coût plus faible de la technologie rapide par exemple, accroit la vraisemblance d'un renversement de prix entre les périodes 1 et 2. Les renversements de prix se produisent quand les HFT découvrent que le signal, qu'ils ont acquis à la période 1, était faux et décident donc de corriger leurs transactions à la période 2. Ceci génère des flux de transaction opposés d'un période à l'autre et, potentiellement, des retournements de prix. L'impact sur les prix des HFT à la période 1 est proportionnel au nombre de HFT. C'est pourquoi la vraisemblance d'un renversement de prix augmente lorsque le nombre de HFT augmente. En revanche, même si plus de HFT implique plus de renversements, cela améliore aussi l'efficience informationnelle du marché. Alors que ces deux implications semblent contradictoires, la présence de HFT permet une intégration de l'information dans les prix plus rapide lorsque le signal de la période 1 est informatif, et qui fait plus que compenser le risque d'erreur.

La nouveauté de ce papier est d'introduire un arbitrage, entre la vitesse et la précision pour le traitement de l'information, et ce pour expliquer pourquoi les HFT pourraient échanger en se basant sur du bruit et générer des renversements de prix. Il existe des théories de spéculateurs de court terme qui, ex-ante, se coordonnent rationnellement pour échanger sur la base de bruit (cf. Froot, Scharfstein et Stein (1992)). Ici, nous pensons à l'échange basé sur du bruit comme à un risque qui se révèle ex-post. De notre point de vue, il provient de la compétition pour les échanges lorsque les participants au marché ont la possibilité de

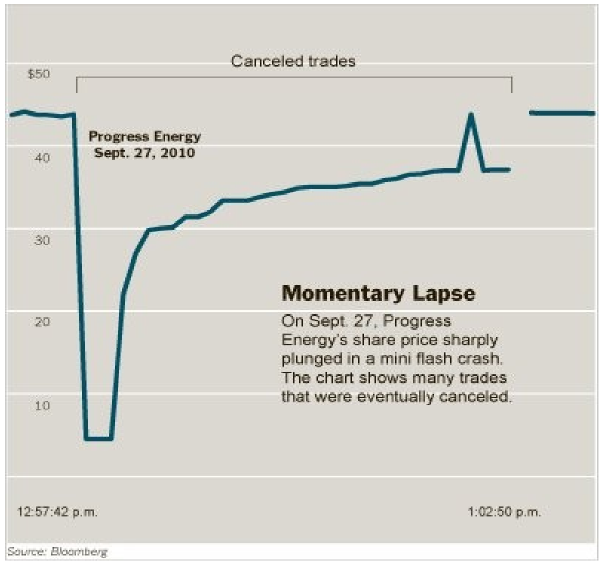


Figure 7: «Le 27 septembre 2010, l'action Progress Energy a perdu 90% en quelques secondes sans raison apparente. Cette chute brutale était la conséquence d'un mini flash crash; une version réduite du crash de mai...», http://sslinvest.com/news/mini-flash-crash-september-27th-sends-pgn-shares-down-90.

réagir à des nouvelles, ou d'autres signaux pertinents, en un court laps de temps. A première vue, l'accélération du traitement de l'information, et de son utilisation, devrait produire une intégration de l'information dans les prix plus rapide. Cependant, cette accélération accroit aussi le risque que ces participants basent leurs transactions sur des signaux moins précis. Bien entendu, pour réduire ce risque, ils pourraient décider de vérifier la précision de la nouvelle (par l'intervention humaine par exemple). Mais, ce faisant, ils prennent le risque de perdre une opportunité de profit car ils réagiraient trop tard à un signal informatif. Ainsi, cette compétition entre participants peut les pousser à réagir trop rapidement, au détriment de la précision de l'information sur la base de laquelle ils échangent, et peut donc mener à des renversements de transactions et de prix.

De façon alternative, les renversements de prix, pourraient s'expliquer par la présence d'investisseurs excessivement confiants qui sur-réagissent à des signaux privés, comme dans Daniel, Hirshleifer et Subrahmanyam (1998), ce qui génère des corrections de prix suite à la révélation publique de l'information. Dans ce contexte, les renversements de prix sont systématiques. Les retours de prix deviennent négativement auto-corrélés et prédictibles. Le marché y est inefficient contrairement à ce que nous trouvons dans notre modèle, qui ne génère pas d'auto-corrélation. De plus, notre cadre théorique implique des renversements de prix complets au sens où les prix peuvent revenir à leur niveau original lorsque les HFT réagissent à du bruit. Dans le papier cité le renversement de prix est une correction partielle d'un changement précédent excessif mais qui allait dans la bonne direction.

Les renversements de transaction, par des participants qui obtiennent de l'information plus rapidement que les autres, peuvent se produire lorsqu'un participant bénéficie d'une fuite anticipée d'information bruitée à propos d'une future annonce publique, comme dans Brunnermeier (2005). Cela lui permet, premièrement, d'acquérir seul un signal bruité sur une composante de court terme de la valeur de l'actif, et d'échanger sur cette base, et, deuxièmement, de savoir de combien son impact sur le prix était due à du bruit, après que cette composante de court terme est publiquement annoncée. Ainsi, il bénéficie encore d'un avantage informationnel après l'annonce. Il en profite en renversant partiellement la part de sa transaction initiale qui était due à la composante bruitée du signal. Cependant ce renversement de transaction est compensé par des transactions, de sens opposés, par d'autres participants stratégiques, ce qui rend l'implication pour la dynamique des prix peu claire, contrairement à notre modèle. Dans le papier de Brunnermeier, le marché est efficient puisque les prix reflètent toute l'information publique accessible. Cependant l'introduction d'une fuite d'information a des effets mitigés sur l'efficience, contrairement à notre modèle ou plus de HFT augmente l'efficience informationnelle.

Les stratégies d'échanges basés sur des signaux informatifs peuvent être aussi diverses que le spectre de l'information pertinente pour un marché particulier. Les HFT cherchent des nouvelles financières ou des tendances de marchés informatives qu'ils peuvent traiter et exploiter le plus vite possible. La source de la précision imparfaite de l'information peut être endogène ou exogène. Elle peut être endogène car les algorithmes envoient des ordres en se basant sur l'interprétation d'évènements. Toute chose égale par ailleurs, plus la réaction de l'algorithme est rapide, moins l'interprétation est précise. Par conséquent les HFT font face à cet arbitrage lorsqu'ils calibrent leur algorithme. Mais le processus de production de nouvelle information peut aussi être la source, exogène, de l'imprécision de cette information. S'il y a une chance, même faible, que certaines nouvelles soient fausses, les HFT doivent alors décider s'ils prennent le risque de réagir immédiatement à ces nouvelles ou bien s'ils attendent une correction éventuelle. L'anecdote suivante illustre, de façon assez extrême, ce problème de fausses nouvelles. Le lundi 8 septembre 2008, le prix de l'action United Airlines chuta de \$12 à \$3 en, à peu près, quinze minutes. Ensuite le prix rebondit jusque \$11 à la fin de la session de mardi. La cause de ces retournements était un vieil article de presse à propos d'une procédure de mise en faillite de United Airlines en 2002 et qui par erreur était réapparu en septembre 2008 dans les titres des pages d'information de Google.

Introduction

The organization of trading in financial markets has changed dramatically over the last three decades, along with the advent of new technologies of information and communication. Financial market structures were primarily organized as trading floors, where humans are physically trading with each other, or dealership markets, in which dealers are in charge of standing as counterparty for investors who would reach them by phone. With the evolution of technologies for generating, routing and executing orders, market structures progressively evolved towards the computerization of trading procedure through an *electronic limit order market* organization. These new technologies also improved the speed and the information processing capacity of market participants. It gave birth to a new type of trading strategies, fully automatized and more sophisticated: *algorithmic trading*. Algorithmic trading has been growing since then and now accounts for more than 50% of trading volume in equity markets. While technologies have been key to enable trading automation, regulatory actions enforcing competition among exchanges (Reg NMS in the U.S, MiFID in the E.U) have fostered its evolution. New entrants in the business of exchanges (BATS, Chi-X,...etc) have grown in «symbiosis» with algorithmic traders. Embedded in an adequate technological and cost structure environment, they have acted as liquidity providers, and, thus, helped new trading platforms to compete, with the incumbent exchange, for attracting the order flow. It ended up generating *market fragmentation*.

Understanding the consequences of the recent changes of financial markets organization has become of primary interest for academic research. It has also stressed the relevance of studying in detail financial market structures. In this dissertation, I aim to investigate research questions that help addressing the recent evolutions of financial markets and, thus, to contribute to the financial market microstructure growing literature.

In the following of the introduction, I present some features of the major changes of market structures: electronic limit order markets, algorithmic trading and market fragmentation. Then I give an overview of the three chapters of the dissertation.

Electronic limit order markets

Electronic limit order markets are centralized trading venues in which liquidity supply can be achieved by any market participants. Any agent can trade by supplying liquidity with *limit orders*, which specify a price, a quantity of shares and are posted in the electronic limit order book. For instance, in Fig.1, most competitive buy limit orders are posted at the best bid price, \$384.82, the first one is for 50 shares and the second one for 100 shares. Agents can also trade by consuming liquidity with *market orders*, which are immediately executed against most competitive limit orders displayed in the order book. In Fig.1, if a trader sends a sell market order for 100 shares, it will be executed at \$384.82. Electronic limit order markets combine the centralized trading location aspect of a trading floor with a large set of market participants, as in dealership markets, thanks to electronic communication devices.

LAST MATCH		TODAY'S ACTIVITY	
Price	384.9000	Orders	1,295,622
Time	15:18:56	Volume	2,791,809

BUY ORDERS		SELL ORDERS	
SHARES	PRICE	SHARES	PRICE
50	384.8200	93	384.9500
100	384.8200	100	385.0300
100	384.8100	100	385.0600
300	384.8100	100	385.0700
100	384.8000	200	385.0900
500	384.7900	100	385.1800
200	384.7700	100	385.2400
500	384.7600	25	385.2500
100	384.7100	100	385.3500
100	384.6900	15	385.5000
200	384.6800	200	385.5500
300	384.5900	200	385.6000
100	384.5000	360	385.6300
50	384.0000	100	385.6800
100	384.0000	100	385.7100
(209 more)		(283 more)	

Figure 8: Instant view of a limit order book: buy limit orders on the bid side (left column), sell limit orders on the ask side (right column).

Electronic limit order markets started to spread out during the 1980's, first in equity markets. For instance the Paris Bourse closed its floor and became a fully electronic limit order market in 1986. Now most of equity markets around the world are organized as limit order markets. This trend has now been followed by markets for other types of securities, as foreign exchange or fixed income markets.

The massive conversion of financial markets to the limit order market organization has motivated an extensive line of academic research in finance. Researchers have been primarily interested in understanding the dynamics of trades and liquidity supply in these markets, as well as the underlying strategies of market participants. First empirical studies have exposed the dynamics of trades and quotations in limit order markets, as in Biais, Hillion and Spatt (1995) (see Fig.2).

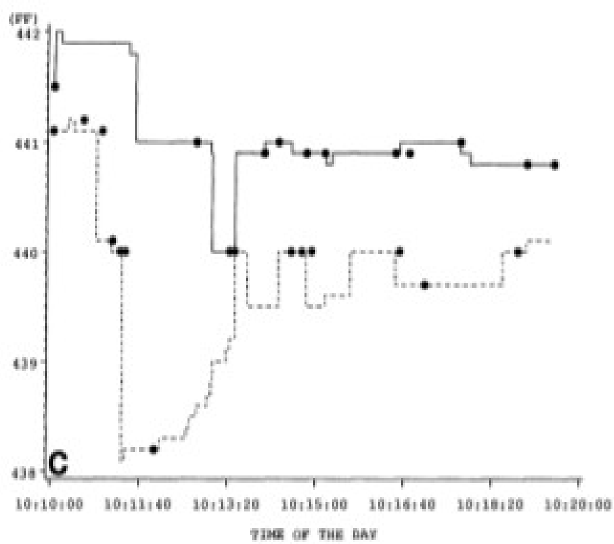


Figure 9: Transaction prices and bid and ask quotes for Elf-Aquitaine, November 9, 1991. Source: Biais, Hillion and Spatt (1995)

Theoretical research investigated how usual motives for trading, such as liquidity needs or private information, could be modelled in a limit order market setup and could generate market patterns that would match empirical findings. They studied the particularity of these market dynamics and their implications for informational efficiency or liquidity (e.g. Glosten (1994), Parlour (1998), Foucault (1999), Foucault, Kadan, Kandel (2005), Rosu (2010)).

Algorithmic trading

The advances of information technologies, as well as the massive conversion of exchanges to electronic limit order markets, have enabled and fostered the rise of automated trading. As shown in Fig.3, the participation of algorithms to trades in U.S equities transaction have been constantly growing for more than 5 years. Algorithms now intervene in more than 50% of the overall trading volume.

Figure 10: Share of algorithmic trading in the total U.S equities trading volume. Source: Aite Group (2010).

Algorithmic trading can be defined as trading strategies that relies on algorithm to perform part, or the entire, of their trading decisions. These automated strategies would usually condition their actions on a set of predetermined market outputs. Algorithmic trading strategies can be split into two main, though non-exhaustive, categories.

Optimal order execution algorithms. Algorithmic trading can help traditional investors or intermediaries, as fund managers or brokers, to optimize the execution of their trading needs. For instance brokers routinely use robots to split the orders of their clients over time and among multiple trading venues to achieve smaller trading costs. The principal advantage of these strategies relies on the computer abilities, first, to efficiently monitor fluctuations of market conditions and, second, to systematically implement optimal execution procedures that load on these market conditions.

High Frequency Trading. The second, and most famous, category of algorithmic trading, is High Frequency Trading (HFT henceforth). HFT strategies relies on intense processing capacities and reaction speed to acquire a large amount of real time information and take actions at high frequency.

HFT is profoundly affecting how financial markets work and triggered heated debates among practitioners, academics and regulators. For instance, in the New-York times, Paul Krugman writes:

«High-frequency trading probably degrades the stock market's function, because it's a kind of tax on investors who lack access to those superfast computers - which means that the money Goldman spends on those computers has a negative effect on national wealth. As the great Stanford economist Kenneth Arrow put it in 1973, speculation based on private information imposes a «double social loss»: it uses up resources and undermines markets.» (P. Krugman, «Rewarding Bad Actors», NY Times, August 2, 2009)

Even though academic research has recently produced economic analysis of the effects of HFT on informational efficiency and liquidity of financial markets, there is still no consensus on its beneficial, or detrimental, role. One difficulty is that HFT is a catchall phrase for very diverse activities. Some firms (e.g, GETCO, Timberhill, Optiver etc ...) engage in high frequency market-making and now account for a large fraction of liquidity supply both in the U.S. and Europe. Other participants (e.g., hedge funds as Renaissance) use computers to take directional positions based on «signals» before other investors get access to this information. Clearly, all these activities are different and as such may have different impact on market efficiency and market liquidity.

Recent empirical studies (e.g, Hendershott, Jones and Menkveld (2011), Hendershott and Riordan (2013), Brogaard, Hendershott and Riordan (2012) or Chaboud, Chiquoine, Hjalmarsson, and Vega (2009)) have shown that HFT had a positive effect on market quality measures. However other studies (e.g Hasbrouck (2013)) and recent market events ascribed to HFT (i.e. the Flash Crash of May 6th, 2010) have stressed the potential manipulative and destabilizing behaviour of their strategies. It leaves open the question of which type of HFT has a positive impact in financial markets

Algorithmic trading and cognitive limitations. The existence of algorithmic trading itself raises questions about the rationality of investing in these technologies and, implicitly, asks what additional capacity computers bring to market actors. When a trader is hit by a liquidity shock, he must analyze his positions and risk exposure prior to take trading decisions. When new public information, conveyed by financial news, is released, a trader must interpret this information prior to trade on it. In the previous situations, collecting and processing information takes time for humans. They must concentrate their attention to accomplish these specific tasks. Machines can access, process and trade on information much faster than humans. Moreover they can monitor simultaneously several sources of information and be multitasking. As a result, algorithmic trading alleviate the attention constraints of human traders. Consequently, theoretical research can study algorithmic trading by analyzing the effects of imperfect attention in financial markets (e.g. Foucault, Roell and Sandas (2003), Biais, Hombert and Weill (2012), Pagnotta and Philippon (2012), Foucault, Kadan and Kandel (2013)).

However algorithmic trading cannot be reduced to an improvement of cognition abilities used for traditional trading strategies. First, the use of computer, by itself, extends the field of available information. For instance, order book imbalances are difficult to interpret without quantitative computerized analysis, and their high frequency dynamics are barely perceivable by humans. Second, information processing by computers differs from humans'. While machines can process hard and quantifiable information much more efficiently, they are not able to deal with scenarios that were not anticipated at the time of their conception and can possibly make mistakes. A comprehensive theory of algorithmic trading should include these features.

Market fragmentation

Regulation and market fragmentation. Financial markets, and specifically equity markets, are now substantially fragmented. It has mainly been impulsed by regulatory actions both in Europe and in the U.S. The European Union introduced the *Markets in Financial Instruments Directive* (MiFID) on November 1, 2007, which abolished the concentration rule in European countries and promoted competition for trading systems and services. Traditional exchanges, that used to profit from some market power in European countries (London Stock Exchange in G.B, Euronext in France, Belgium and the Netherlands), have been facing competition from new trading platforms, as Chi-X, Turquoise and BATS Europe. In the U.S, the *Regulation National Market System* (Reg NMS) was promulgated in 2007 to modernize and strengthen the national market system for equity securities. As for MiFID, it fostered competition among trading platforms. Figure 4 illustrates how NYSE in the U.S and LSE in Europe lost market shares against new entrants as, respectively, BATS and Chi-X.

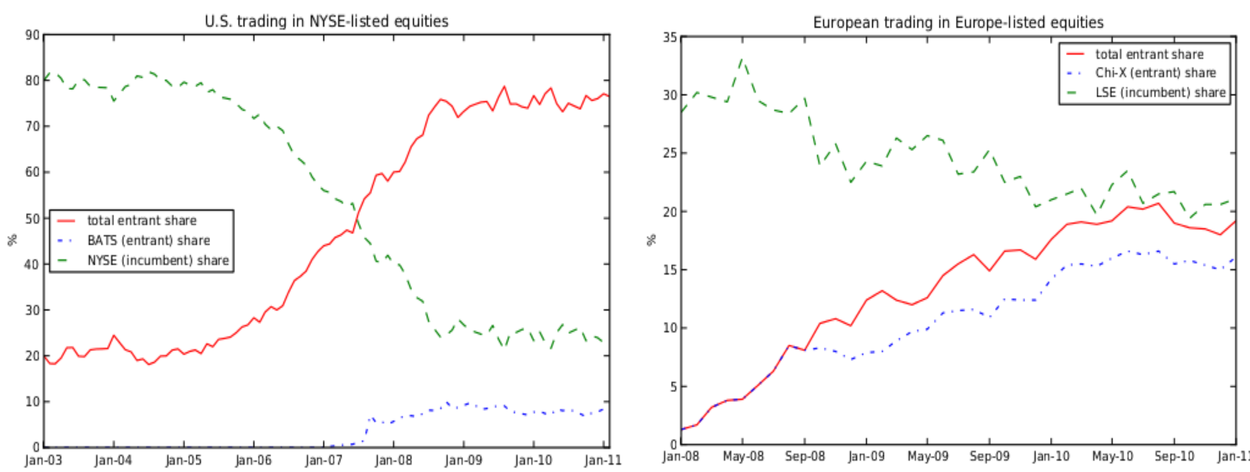


Figure 11: Market fragmentation in Europe and U.S. The left graph plots incumbent and entrant market share in NYSE-listed stocks. The right graph does the same for European listed stocks. Source: Menkveld (2012).

Trading automation and market fragmentation. In a recent review for the UK Government Office for Science[9], Carole Gresse writes:

«There is an old common belief in economic theory that security markets are natural monopolies because the marginal cost of a trade decreases with the quantity of orders executed in a market. While this has long been true to a certain extent, technological progress has somehow changed this reality. The fixed costs and time necessary to launch a new market have considerably diminished and computer trading now allows cross-market trading strategies that connect to multiple trading venues as if they were a consolidated network of counterparties

[9]Market fragmentation in Europe: assessment and prospects for market quality, C. Gresse (2012)

with several entries. Those new tools undermine the network externality argument.»

The automation of trading strategies and the decision to liberalize competition between trading systems and services happened to be contemporaneous because the advance of information technologies was a necessary condition for these two evolutions. For instance MiFID states that trading firms should aim for the best execution condition possible for their client's orders. In a fragmented markets environment, it requires to use smart order routing systems that automatically look for the best prices available across trading platforms. This kind of task is barely achievable by human traders.

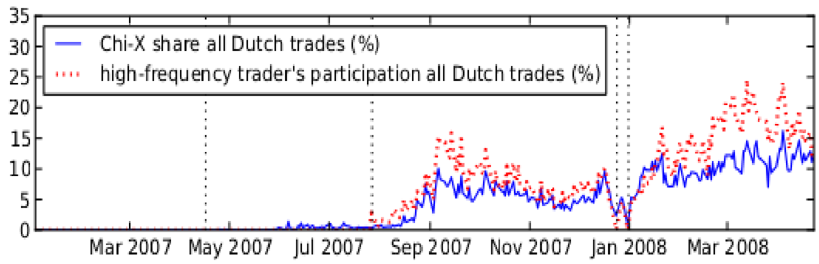


Figure 12: The graph depicts the market share of the entrant market Chi-X based on the number of trades. The graph also depicts the high-frequency trader's participation in trades, based on its trading in both the entrant (Chi-X) and in the incumbent market (Euronext). Source: Menkveld (2012).

Beyond their key role in the consolidation of fragmented markets, automated trading strategies probably had an important impact in the growth of entrant trading platforms. In a recent working paper[10], Albert Menkveld provides evidences that new entrant markets grew in a sort of symbiosis with some High Frequency Trading firms that specialized in high frequency market making, such as GETCO. Attracted by adequate infrastructure and rebates

[10] High frequency trading and the new-market maker, A. Menkveld (2012).

for limit orders, these new actors became *de facto* market makers which helped new trading platforms to compete for the order flow. Fig.5 shows the parallel trends of the market share growth of Chi-X in Dutch stocks and the growth of High Frequency Trading activity in these markets.

Now that algorithmic trading is becoming the prevalent form of trading, exchanges compete for attracting their order flows in offering attractive services. Trading platforms have drastically reduced communication latencies between their servers and those of their clients. They massively invested in bandwidth and proposed co-location to high frequency traders. Now the average duration between order submissions and executions are less than a second (see Fig.6) and of the order of milliseconds for traders whose computers are co-located with the platform's server.

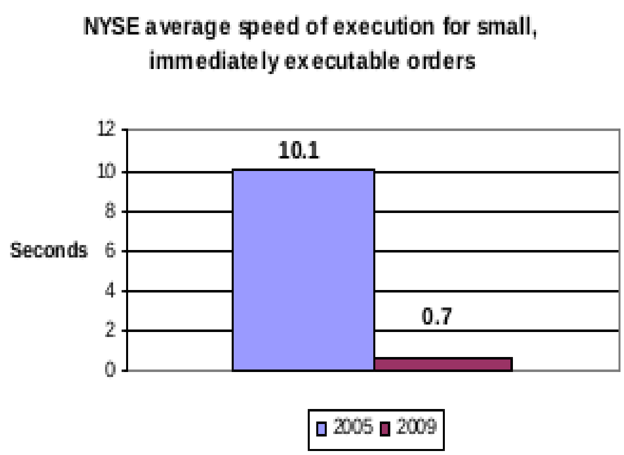


Figure 13: NYSE average speed of execution for small, immediately executable orders. Source: SEC Concept on Equity Market Structure (2010).

The supply of services specifically designed for High Frequency Trading actors have gone beyond those offered by the only trading platforms. For instance news providers (e.g. Bloomberg, Thomson Reuters, Dow Jones) propose almost real time and easily readable financial news that are explicitly aiming HFT clients[11].

[11] Dow Jones Newswire offers low-latency news and event-data for electronic trading. see http://www.dowjones.fr/salesandtrading/low-latency-feeds.asp

Dissertation overview

This dissertation tackles, with theoretical models, three important questions in financial market microstructure which cover different aspects of the new financial market structures.

The multiplication of trading venues organized as limit order markets raises questions on how financial markets currently accomplish their assigned role, that primarily is to efficiently allocate savings across investment opportunities and to allow for an efficient risk sharing among investors. In order to perform this welfare improving role, financial markets must be attractive to investors and fund seeking entities. Liquidity and informational efficiency are the usual attribute of attractive markets. It interrogates how market quality can be assessed in financial markets. For instance, now that equity markets have migrated toward a limit order market organization, are traditional liquidity measures still relevant to evaluate investors' welfare? I address this question in the first chapter of this dissertation. I show that usual liquidity measures, such as market depth, may be inversely related to the order execution quality for limit order users, and thus may not capture well the overall welfare of investors.

The reason why algorithmic trading strategies have been tremendously expanding partly lies on the fact that algorithmic trading alleviates the limited attention constraint of human investors. It helps monitoring order book activity more frequently, gathering new public information more rapidly and subsequently adapting trading strategies more efficiently. Understanding how limited attention affect trading strategies is thus key to address the effects of algorithmic trading. In the second chapter of this dissertation, I investigate how investor's limited attention affects their trading strategies and the overall market dynamic around news arrival, in a limit order market. Because of limited attention, investors imperfectly monitor news arrival. Consequently, in equilibrium, prices reflect news with delay. This delay shrinks when investors' attention capacity increases. The price adjustment delay also decreases when the frequency of news arrival increases. When news arrival frequency is higher, the picking-off risk increases for limit orders. The limit order book becomes thinner and there are fewer stale limit orders to execute or cancel after news arrival. Thus, it reduces the time it takes for market prices to reflect news content.

High Frequency Trading strategies use high market monitoring capacities to react to all kind of market events that are helpful to predict future price changes and thus generate profit opportunities. By reacting faster to such informative events, HFT can help integrating new information into prices and thus make markets more efficient. However HFT activity has come along with a new type of market instability events, mini flash crashes, that can be defined as a sudden sharp change in the price of a stock followed by a very quick reversal. In the third chapter of this dissertation, which is based on a joint work with Thierry Foucault, we investigate the effect of HFT on market efficiency and price stability when reaction speed to market events comes with a risk of trading on noise. By introducing this trade-off between reaction speed and information precision, we show that HFT activity can generate mini flash crashes. However this higher instability of prices comes along with a higher market efficiency. This finding suggests that HFT helps integrating information into prices more efficiently but in a less stable way.

Chapter 1

Are Liquidity Measures Relevant to Measure Investors' Welfare?

1.1 Introduction

Investors' welfare is one main objective of financial market regulators. As welfare is not observable, market liquidity has been thought as the right concept to approximate welfare. Market liquidity can be defined as the ease for an investor to trade a given asset quantity at a price that does not deviate much from a benchmark price. Hence market liquidity corresponds to some implicit trading costs. In centralized markets these implicit trading costs are usually measured with bid-ask spreads and market depths (the number of quotes closed to the best bid and ask prices). This definition of market liquidity and its empirical measures are biased toward the welfare of liquidity consumers. Most centralized markets (stock, FX,... etc) are organized as limit order markets in which any investor can trade by posting quotes and supplying liquidity. Previous liquidity measures do not account well for the welfare of liquidity suppliers. For instance, a high market depth may arise from a low limit order execution rate which is, a priori, not welfare improving for limit order users. How are liquidity measures determined by investors trading strategies? How can these measures be linked to investors' welfare?

To address these questions I design a dynamic model of limit order market. The tractability of the model allows me to provide closed-form solutions for equilibrium outcomes such as market depth, trading volume, limit order execution rate as well as for welfare. When I consider variations of several model parameters, I obtain that (i) market depth negatively co-varies with welfare, (ii) in most cases trading volume positively co-varies with welfare, except for a specific range of parameters, and (iii) limit order execution rate positively co-varies with welfare. It shows, first, that limit orders execution rate and market depth may vary in opposite directions and, second, that execution conditions for limit orders could dominate in the welfare outcome. The corollary is that cross-sectional variations or shocks on liquidity measures, such as market depth or trading volume, do not necessarily corresponds to equivalent changes to investors' welfare.

Market liquidity has been usually measured by trading costs. Explicit trading costs include brokerage commissions, trading fees,...etc. Implicit trading costs are measured by the wedge between the execution price and a benchmark that can be the mid-quote (the best bid and ask prices average). The traditional view on market liquidity makes a direct link between implicit trading costs and illiquidity. In intermediated centralized markets, in which trades execution is delegated to dealers or market makers, implicit trading costs correspond to the surplus that these market makers extract from trades. The existence of this trading costs can be explained by inventory costs, adverse selection or imperfect competition among dealers. With comprehensive market data, implicit costs can be directly assessed from observed prices and corresponding quantities quoted by market makers. For instance, Chordia, Roll and Subrahmanyam (2000, 2001) study aggregate movement and co-movement of liquidity for NYSE stocks that, at the time, were run by specialists (market-makers). Among different liquidity proxies, they use quoted bid-ask spreads and quoted depths. In the considered markets, quoted prices and offered quantities are transaction data. They are announced by specialists prior to a trade. Hence liquidity proxies precisely reflect implicit trading costs that investors were facing.

In limit order markets, one can similarly examines the dynamic of liquidity supply with the evolution of bid-ask spreads and market depths as proxies. Past and more recent papers (e.g., Biais, Hillion ans Spatt (1995), Engle, Fleming, Ghysels and Nguyen (2011), Hasbrouck and Saar (2012)) have investigated limit order market dynamics with order book data. These type of data usually provide the order book evolution, order submissions, executions and cancellations. Can liquidity supply be directly linked to investors welfare as in former NYSE stock markets with specialists? In order driven market, liquidity suppliers cannot be distinguished from liquidity consumers as in former markets. High implicit trading costs for investors who consume liquidity, with market orders, correspond to good execution conditions for investors who supply liquidity with limit orders. These are money transfers, from liquidity consumers to liquidity suppliers, inside the pool of investors. In this type of market, welfare is intuitively high when the frequency at which gains from trade are realized, between a liquidity supplier and a liquidity consumer, is high as well (as shown in Colliard and Foucault (2012)). This trade frequency would be better captured by trading volume for instance, which is the case in my model. Generally speaking, it is not clear how this trade frequency should be linked to implicit trading costs for market orders, measured by market depth and bid-ask spread.

In my model I consider a continuous-time framework. There is a continuum of competitive investors who can hold 0 or 1 unit of an asset. They discount time at a constant rate. Each investor has either a high or low private value for the asset. The asset private value of an agent is random and idiosyncratic. It is a two-state continuous-time Markov chain. It switches from high to low or conversely with same intensity. The difference in asset valuations across agents generates motives for trade and welfare gains when some asset shares are transferred from investors with a low private value to investors with a high private value. Trading takes place in a centralized market. Investors can trade by either supplying liquidity with limit orders or consuming liquidity with market orders.

I study a class of steady-state equilibria. They are such that the aggregate state of the limit order market does not change over time. At equilibrium, prices are constant over time and bid-ask spreads are equal to the tick size, the minimal difference between two available trading prices. All limit orders are submitted at the best bid and ask prices. Buy (resp. sell) limit orders are submitted by investors with a high private value who do not own the asset (resp. with a low private value who own the asset). The number of limit orders on each side of the book, the market depth, is such that investors are indifferent between using a limit order, to trade at a good price but with a time delay, or a market order to trade immediately with an implicit cost, the bid-ask spread. When the tick size decreases, the comparative advantage of using limit orders declines. The maximal time delay for limit order execution, that investors are willing to bear, declines as well. It implies that the market depth decreases and that investors use relatively more market orders than limit orders.

Welfare is negatively linked to the market depth. Ideally, any investors who is waiting in the book to have his limit order executed would be matched with a similar investor on the other side of the book. Trade between two of these investors would transfer the asset from a low value type to a high value type and thus would increase welfare. The tick-size of the market allows investors to use limit orders to extract more of the trading surplus than their counterpart with market orders, without risking to have their limit orders undercut by other investors. This relative «market power» that is offered to liquidity supplier is inefficient since it slows down trading and the subsequent trading surplus realization. Hence the tick-size has a negative impact on welfare.

The level of the private value of a low type investor has a positive impact on welfare. This effect is surprising since, everything else equal, a reduction of the utility that low type investors draw from the asset should negatively affect the overall welfare. The intuition for this result is that a lower asset value for low type increases the opportunity cost for waiting with a limit order in the book and not trading immediately. Hence investors use more market orders, market depth declines and welfare increases.

Chapter 1 is organized as follows. Section 1.2 presents the setup and assumptions of the model. Section 1.3 describes the model equilibrium. Section 1.4 derives model outcomes. Section 1.5 analyzes welfare implications. Section 1.6 concludes.

1.2 Model

1.2.1 Preferences and asset value

I consider a continuous time framework with an infinite horizon, $t \in [0, +\infty)$. The economy is populated with a continuum of investors $[0, 1]$. They are risk neutral and infinitely lived, with time preferences determined by a time discount rate $r > 0$. These investors can trade an asset with a common value v that is constant over time.

Preferences. As in Duffie, Garleanu and Pedersen [2005,2007], an investor is characterized by an intrinsic type, "high" or "low". A high type investor receives a utility flow v per asset unit she owns. A low type investor receives a utility flow $v - \delta$ per asset unit she owns. Between time t and time $t + dt$ an investor can switch from one type to another (high to low or low to high) with probability $\rho.dt$. Thus, a high type investor has a higher valuation for the asset than a low type.

Asset holding and supply. As in Duffie et al., investors can own either one or zero unit of the asset. The asset supply is equal to $\frac{1}{2}$. So that half of the population owns the asset.

Given the previous assumptions any investor must have a type in the set $\{ho, hn, lo, ln\}$ (h: high, l: low, o: owner, n: non-owner). And we can divide the mass of investors in 4 populations: L_{ho}, L_{hn}, L_{lo}, L_{ln}. They verify the equations

$$L_{ho} + L_{hn} + L_{lo} + L_{ln} = 1, \quad L_{ho} + L_{lo} = \frac{1}{2}.$$

It is possible to extend the number of possible types by taking into account the limit order submission status of investors. Indeed, in a limit order book, an owner can either be out of the market or have an order in the order book. As well for a non-owner. This setting can generate many subtypes of the previous types. Let's call $\mathcal{T}$ the set of all possible types. If an investor does not have any limit order submitted in the order book she is *out*. If she has a limit order submitted we have to specify at which price it is. For instance a type ln can be $ln - out$ or $ln - B$ with a buy limit order at price B. Symmetrically a type lo can be $lo - out$ or $lo - A$ with a sell limit order at price A.

1.2.2 Infrequent market monitoring

In order to impose some structure to the model, I consider that investors can trade at some random times that follow a Poisson process with a finite frequency. Based on this structure, I can consider the case where investors can continuously trade in the market by taking the Poisson process frequency to its infinite limit.

Assumption 1.1. *I assume that an investor observes contacts the market at some ran-*

dom times $\{t_i\}_{i\in\mathbb{N}}$. I call these times "market monitoring times". This sequence of market monitoring times is generated by a Poisson process of intensity $\lambda + \rho$.

More specifically between time t and $t + dt$ an investor monitors the market in two types of situation:

- *when she uses the market monitoring technology which occurs with probability $\lambda.dt$*
- *when her private value changes which occurs with probability $\rho.dt$.*

The second assumption states that investors continuously monitor their private value for the asset and contact the market whenever this private value changes. This assumption allows to reduce the anticipation problem of the investor who has to take into account the possibility of future shocks to her private value especially when facing the decision to send a limit order. Indeed she knows that when a shock occurs she has the possibility to cancel a previous limit order. Then it prevents her from being executed while it is not optimal anymore given her new private value.

1.2.3 Limit order market

Trading takes place in a limit order market. Prices at which trades can occur belong to a countable set of prices, the price grid. The minimum difference between two prices is the tick size, Δ. Investors can use limit or market orders to trade. Limit orders are orders that specify a limit price at which the order can be executed. They are stored in the order book until matched with a market order. The depth of the limit order book at price P, D_P, is the volume of all limit orders submitted at price P. Market orders do not specify a price limit. They hit the most competitive limit order and get execution immediacy.

For technical reasons I assume that the price grid is bounded. This is reasonable since trading will not occur at prices higher than a certain threshold since the asset value is bounded (the corresponding strategies would be strictly dominated by a strategy in which investors don't trade). A similar assumption is made in Parlour [1998], Foucault, Kadan and Kandel [2005] for instance.

Each time an investor monitors the market she can take any number of actions in the following list under the constraints that she cannot have more than one limit order in the book and that she can hold either 1 or 0 unit of the asset.

- As an owner she can : (i) do nothing and remain an owner; (ii) submit a sell limit order and remain an owner until her order is executed; (iii) submit a sell market order and become a non-owner; (iv) cancel a previous sell limit order.

- As a non-owner she can : (i) do nothing and remain a non-owner; (ii) send buy limit order and remain a non-owner until her order is executed; (iii) send a buy sell market order and become an owner; (iv) cancel a previous buy limit order.

This defines the action set of an investor as (with some notation abuse)

$$A = \{\text{do nothing, market order, limit orders at the different prices}\}.$$

Assumption 1.2. *In the limit order book, limit orders are executed following a "Pro-rata matching" execution rule*[1]*. In this setup all limit orders submitted at the same price have the same probability of execution at any point in time, regardless of their submission date.*

Assumption 1.3. *I assume that $\frac{\delta}{r}$ is big compared to Δ. It ensures that the gains from trade due to differences in private values, measured by $\frac{\delta}{r}$, is bigger that the implicit cost of trading, the bid-ask spread, which is measured by the tick-size, Δ. More specifically I assume that*

$$\delta > (r + 2\rho)\Delta \tag{1.1}$$

1.2.4 Value function and equilibrium concept

An investor chooses a new action at each market monitoring time. The strategy of an agent is a function σ,

$$\begin{aligned} \sigma : & H \times \Xi \times [0, \infty) \to A, \\ & (h, \xi, t) \mapsto a. \end{aligned}$$

[1]In practice there are some markets where the "Pro-rata matching" is implemented. However for the majority of stock markets Time Priority applies. The reality of the Time priority is mitigated by the fact that there are multiple trading platforms and that agents can use smart order routing technologies for achieving best trading conditions. The flow of market orders is split among different trading platforms. Then the time at which a limit order executed is randomized.

The set Ξ gathers all potential state variables. An element of this set $\xi \in \Xi$ is defined as $\xi = (\theta, v, S)$ where $\theta \in \mathcal{T}$ is the type of the investor, v is the common value of the asse,t and S is the aggregate state of the limit order book, that is to say the bid and ask prices and all the depths at these prices. H is the set of all possible histories of actions and observations of an investor:

$$H = \{h \in (a_{t_1}, \ldots, a_{t_n}, \xi_{t_1}, \ldots, \xi_{t_n}, t_1, \ldots, t_n) \in A^n \times \Xi^n \times [0, \infty)^n, t_1 < \ldots < t_n, n \in \mathbb{N}\}.$$

Her strategy, σ, and the strategies of all other investors, Σ, generate her asset holding process $\eta_t \in \{0, 1\}$ that is equal to 1 when she holds one unit of the asset, her type process $\theta_t \in \mathcal{T}$ and a process of trading prices P_t at which her orders are executed any time she changes her holding i.e. when η_t switches from 0 to 1 or conversely.

At time t the value function of an investor playing strategy σ is given by

$$\begin{aligned} V(h_t, \xi_t, t; \sigma, \Sigma) &= \mathbb{E}_t \int_t^\infty e^{-r(s-t)} dU_s, \\ s.t \;\; dU_t &= \eta_t(v - \delta \mathbb{I}_{\{\theta_t \in \, lo\}})dt - P_t d\eta_t. \end{aligned}$$

The strategy σ is a best response to the other players set of strategies Σ if and only if for all strategy γ,

$$\forall h_t \, \forall \xi_t \, \forall t \;\; V(h_t, \xi_t, t; \sigma, \Sigma) \geq V(h_t, \xi_t, t; \gamma, \Sigma).$$

In this paper I focus on **Markov perfect equilibria** where strategies depend only on state variables, (θ, v, S).

1.3 Steady state equilibrium

In this chapter, I focus on steady state equilibria in which the aggregate state of the limit order market is constant over time while trading occurs. More specifically I consider a class of steady-state equilibria in which the level of liquidity supply (i.e the number of limit orders in the book) is non zero.

Definition 1.1. *A limit order market is in a steady state when the displayed depths in the order book and the different order flows are deterministic and do not change over time.*

This steady state is possible in the model because there is a continuum of investors. Each investor faces idiosyncratic uncertainty on her type. She switches from "high" to "low" or "low" to "high" with respect to a Poisson process of intensity ρ. By the law of large numbers applied to the continuum of investors, the share of investors switching from one type to another is deterministic and equal to $\rho.dt$ at each point in time. For the same reason the share of investors monitoring the market is deterministic and equal to $\lambda.dt$. This generates a time continuous flow of investors monitoring the market.

Proposition 1.1. *For each couple of bid and ask prices (A, B) that verifies the conditions*

$$\frac{v}{r} - \frac{\delta}{r} + \frac{\rho}{r}\Delta \leq B < A \leq \frac{v}{r} - \frac{\rho}{r}\Delta,$$

$$A - B = \Delta,$$

there is a unique steady-state equilibrium in which

- *all sell limit orders are submitted at price A and all buy limit orders are submitted at price B.*
- *The market depths at these two prices are the same and equal to*

$$D_A = D_B = \alpha^0_{eq} = \frac{1}{2}\frac{\rho\Delta}{\delta - r\Delta}. \tag{1.2}$$

Proof. see Appendix A.2 □

This proposition describes trading prices and the aggregate level of liquidity supply in a specific class of steady-state equilibria. For each pair of bid and ask prices (A, B) satisfying the stated conditions there is a unique equilibrium in which all liquidity supply is concentrated at these prices. The collection of all these equilibria generates the class of these equilibria. In the following of this section, I detail the construction of such an equilibrium and its underlying trading dynamics.

Remark 1.1. *The assumption $\delta - (r+2\rho)\Delta > 0$ is necessary to ensure that the interval $\left[\frac{v}{r} - \frac{\delta}{r} + \frac{\rho}{r}\Delta, \frac{v}{r} - \frac{\rho}{r}\Delta\right]$ is non-empty and larger than Δ.*

To understand the inequality involving A and B, we can look at the subset of equilibrium prices

$$\left[\frac{v}{r} - \frac{\delta}{r}\frac{r+\rho}{r+2\rho}, \frac{v}{r} - \frac{\delta}{r}\frac{\rho}{r+2\rho}\right] \subset \left[\frac{v}{r} - \frac{\delta}{r} + \frac{\rho}{r}\Delta, \frac{v}{r} - \frac{\rho}{r}\Delta\right]$$

This inclusion is a consequence of $\delta - (r+2\rho)\Delta > 0$. $\frac{v}{r} - \frac{\delta}{r}\frac{r+\rho}{r+2\rho}$ is the value for a "low" type investor to hold the asset forever and $\frac{v}{r} - \frac{\delta}{r}\frac{\rho}{r+2\rho}$ is the value for a "high" type investor to hold the asset forever. These are the reserve values for these two types of investor when they hold the asset. In the case where a low-type owner and a high-type non-owner meet only once and leave the market afterwards then the trading price has to be in this interval for the two investors to trade.

In the steady state equilibrium of the limit order market trading also takes place between low type owners and high type non-owners. However the range of trading prices is wider than the difference between the two reserve values because investors can trade more than once.

Other equilibria. There are other equilibria than the one described above. Indeed in order to solve for the equilibrium of this game one must proceed by guess and check. The first step is to conjecture equilibrium strategies for all agents. The easiest is to assume that all agents have the same strategy. Given this strategy it is possible to determine the dynamic of the limit order book. The last step is then to check that it is not profitable to operate a one-shot deviation from the conjectured strategy for any type, at any point in time of the game while other agents are playing the conjectured strategy. Solving the problem in that way is difficult. Defining the set of all equilibria is even harder.

An example of other equilibrium is the **empty limit order book equilibrium**. In this equilibrium Investors coordinate on a trading price P where ho's and ln's send (marketable) limit orders. The buy and sell order flows due to lo's and hn's are exactly equal which implies that their limit orders are immediately executed and that the limit order book is always empty.

1.3.1 One-tick market

Proposition 1.2. *A limit order market in a steady state at equilibrium is necessarily a one-tick market. Its bid-ask spread is equal to the tick of the market, $A - B = \Delta$. Moreover liquidity supply is concentrated at best bid and ask prices:*

- *all sell limit orders are sent at the price A, generating a depth D_A, and there are no sell limit orders at higher prices than A*
- *all buy limit orders are sent at the price B, generating a depth D_B, and there are no buy limit orders at lower prices than B*

In a steady state at equilibrium limit orders and market orders are sent by investors following an equilibrium strategy. It generates flows of limit and market orders that are constant and deterministic over time so that the steady state holds. Let's consider an investor for whom it is optimal to send a buy market order at A. If there was a reachable price $A < P < B$ it would be profitable to send a limit order at P since it would be immediately hit by the flow of market orders and would get price improvement compare to A. This would contradict the optimality of the strategy.

This one-tick market result relies on the modelling approach. There is a continuum of investors and a «zero or one unit» holding constraint. Random idiosyncratic events affect a deterministic share of investors because of the law of large numbers and finally turn them into deterministic flows of orders and cancellations. This flows are finite because of the holding constraint. The key reason for this result is the market order flow that is deterministic and continuously positive which makes any limit order alone inside the bid-spread immediately executed. It is also the fact the instantaneous market order flow is infinitesimal and thus is not big enough to move prices. One might think that in a large market where trades take place quite continuously and where market orders are small enough to not push prices, for instance if robots optimize execution by slicing big orders into small ones, then the occurrence of one-tick bid-ask spreads could be high. Indeed the incentive to send a limit order inside the best quotes rather than a market order would hold because execution would be almost immediate.

1.3.2 Steady state strategy

When the limit order book is in a steady state trading takes place due to differences in private values across investors. Investors of types ho and ln do not trade because prices are between the values of owning the asset for high type and a low type. Given these prices, investors of type hn and lo are better off after changing their holding status and thus trade. If they use market orders they directly join the group of «satisfied» agents (ho and ln). If they use limit orders they become «satisfied» once their order is executed. The consequence of this strategy is that investors of type lo and hn who have once monitored the market are in the limit order book. In the steady state they all are in the limit order book.

Proposition 1.3. *The equilibrium strategy in the steady-state phase is defined as follows:*

- *ho: cancel any sell limit order and stay out of the market*
- *hn: send a buy limit or market order with respect to a mixed strategy. When she monitors the market she submits a buy market order with probability $m_A \in [0, 1]$. It is executed at the ask price A.*
- *lo: send a sell limit or market order with respect to a mixed strategy. When she monitors the market she submits a sell market order with probability $m_B \in [0, 1]$. It is executed at the bid price B.*
- *ln: cancel any buy limit order and stay out of the market*

Proof. see Appendix A.2 □

In this equilibrium the populations L_{ho} and L_{ln} are not present in the limit order book. As soon as a ho type switches to a lo type she instantaneously monitors the market: either she instantaneously switches to a ln type by sending a sell market order or remains a lo type by sending a sell limit order. Symmetrically as soon as a ln type switches to a hn type she instantaneously monitors the market: either she instantaneously switches to a ho type by sending a buy market order or remains a hn type by sending a buy limit order. Consequently we obtain $D_A = L_{lo}$ and $D_B = L_{hn}$.

1.3.3 Steady state populations

In a steady state the levels of aggregate populations do not change over time. Then the flows of population from high type to low type and from low type to high type must be equal to each other, $\rho(L_{ho} + L_{hn}).dt = \rho(L_{lo} + L_{ln}).dt$. Combined with the constraints due to the size 1 of the overall population and the asset supply 1/2 we obtain

$$L_{lo} + L_{ln} = L_{ho} + L_{hn} = \frac{1}{2},\ L_{lo} + L_{ho} = \frac{1}{2}$$

Proposition 1.4. *In a steady state there is one freedom parameter $\alpha^0 \in \mathbb{R}$ such that the different populations satisfy*

$$L_{ho} = L_{ln} = (\frac{1}{2} - \alpha^0), \quad L_{hn} = L_{lo} = \alpha^0.$$

It must satisfies the constraints of non-negativity, $\frac{1}{2} - \alpha^0 \geq 0$ and $\alpha^0 \geq 0$.

Proof. see Appendix A.2.2 □

This freedom parameter α^0 is determined at equilibrium. It is equal to the liquidity supply in the limit order book since the depths are equal to $D_A = L_{lo} = \alpha^0$ and $D_B = L_{hn} = \alpha^0$.

1.3.4 Micro-level dynamic of the limit order book

In equilibrium hn and lo investors are indifferent between limit and market orders so that they submit both market and limit orders. Flows of limit and market orders make the state of the limit order book sustainable and steady. And these flows must be steady as well.

The flows of buy market orders and buy limit orders are defined by the share m_A of hn investors monitoring the market between t and $t + dt$ who send buy market orders and the share $1 - m_A$ who send buy limit orders. On the sell side a share m_B of lo investors monitoring the market between t and $t + dt$ send sell market orders and the rest send sell limit orders.

Ask Side. At time t, on the ask side of the market the depth is constantly equal to $D_A = L_{lo}$ and the order flows going in and out of the ask side of the order book are

- **Outflow due limit order executions:** execution of buy market orders send by hn's monitoring the market, $m_A(\lambda L_{hn} + \rho L_{ln}).dt$.

- **Outflow due limit order cancellations:** investors switching from lo to ho, $\rho L_{lo}.dt$, lo's cancelling their sell limit order to send a sell market order, $m_B \lambda L_{lo}.dt$

- **Inflow due to limit order submissions:** investors switching from ho to lo submitting a sell limit order, $(1 - m_B)\rho L_{ho}.dt$

The steady state condition is : $\rho L_{lo} + m_A(\lambda L_{hn} + \rho L_{ln}) + m_B(\lambda L_{lo} + \rho L_{ho}) = \rho L_{ho}$.

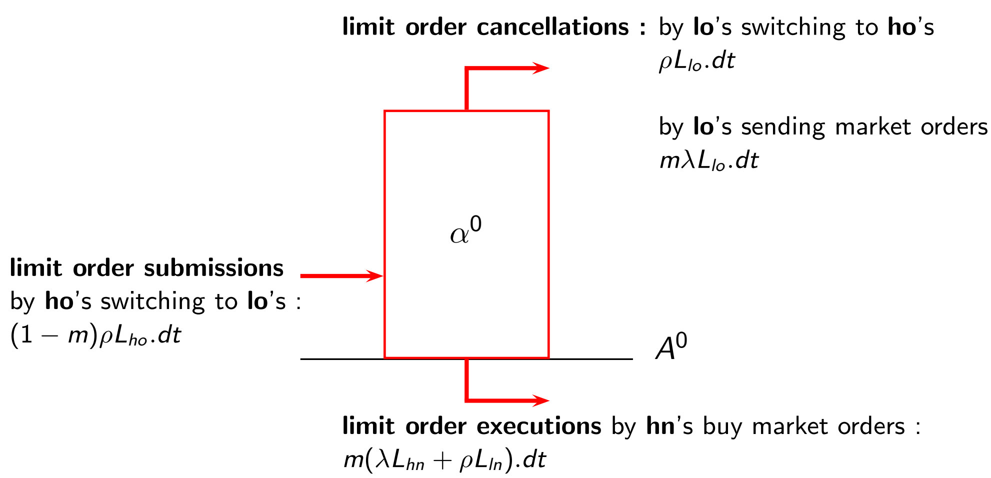


Figure 1.1: Steady-state dynamic of the market depth

Bid Side. At time t, on the ask side of the market the depth is constantly equal to $D_B = L_{hn}$ and the order flows going in and out of the bid side of the order book are

- **Outflow due limit order executions:** execution of sell market orders send by lo's monitoring the market $m_B(\lambda L_{lo} + \rho L_{ho}).dt$.

- **Outflow due limit order cancellations:** investors switching from hn to ln, $\rho L_{hn}.dt$, hn's cancelling their sell limit order to send a sell market order, $m_A \lambda L_{hn}.dt$

- **Inflow due to limit order submissions::** investors switching from ln to hn submitting a sell limit order, $(1 - m_A)\rho L_{ln}.dt$

The steady state condition is : $\rho L_{hn} + m_B(\lambda L_{lo} + \rho L_{ho}) + m_A(\lambda L_{hn} + \rho L_{ln}) = \rho L_{ln}$.

1.3.5 Execution rate and liquidity provision

At any time t in the steady state phase, the flow of market orders hits a share of limit orders in the order book. Because of the Pro-Rata execution rule, all limit orders on the same side of the book are equally likely to be executed. Between t and $t + dt$ this probability is equal to the ratio of the instantaneous flow of market orders over the market depth.

For instance on the ask side, the flow of market orders is equal to $m_A(\lambda L_{hn} + \rho L_{ln}).dt$ and the depth is equal to $D_A = L_{lo}$. Hence the instantaneous probability of execution is equal to

$$l_A.dt = \frac{m_A(\lambda L_{hn} + \rho L_{ln})}{L_{lo}}.dt. \tag{1.3}$$

l_A is the execution rate for sell limit orders. In the same way we can define the execution rate for buy limit orders, $l_B = \frac{m_B(\lambda L_{lo} + \rho L_{ho})}{L_{hn}}$.

The execution rates are more natural to handle than the mixed strategy parameters m_A and m_B as it clearly appears in the value function subsection. These quantities can be used equivalently. Indeed, once the execution rates and the state of the limit order book are defined at equilibrium, the mixed strategies are perfectly defined. For instance $m_B = \frac{L_{hn}}{\lambda L_{lo} + \rho L_{ho}} l_B = \frac{\alpha^0}{\lambda \alpha^0 + \rho\left(\frac{1}{2} - \alpha^0\right)} l_B$.

Steady state liquidity provision. By incorporating the execution rates, the two steady state equations can be rewritten as

$$\rho L_{hn} + l_B L_{hn} + l_A L_{lo} = \rho L_{ln} \tag{1.4}$$

$$\rho L_{lo} + l_A L_{lo} + l_B L_{hn} = \rho L_{ho} \tag{1.5}$$

These equations are in fact equivalent and define the value of the steady state population that is to say the value of the depth parameter α^0

$$\alpha^0 = \frac{1}{2}\frac{\rho}{2\rho + l_A + l_B} \tag{1.6}$$

The aggregate properties of the limit order market in this steady state is completely described by α and the execution rates l_A and l_B. Indeed they define the steady state populations, the depths and the aggregate order flows in the limit order book.

1.3.6 Value functions

The equilibrium strategy generates the following system of equations defining the different value functions for each investor type. Here I only provide the value function for *ho* and *hn* investors as it is very similar for *ln* and *lo*.

Type *ho*. A *ho* investor stays out of the market until she switches to the *lo* type. Her situation is affected when the common value changes. Her value function V_{ho-out} is defined as follows

$$\begin{aligned} V_{ho-out} &= v.dt + (1 - r.dt)\left[(1-\rho.dt)V_{ho-out} + \rho.dtV_{lo}\right] \\ \iff (r+\rho)V_{ho-out} &= v + \rho V_{lo}. \end{aligned}$$

Type *hn*. A *hn* investor sends a buy market order with probability m_A or limit order with probability $1 - m_A$. Sending a buy market order at price A provides her with the value function $V_{ho-out} - A$. Indeed she gets execution immediacy by trading at the ask price A and instantaneously switches to type *ho*. Sending a buy limit order at price B provides her with the value function V_{hn-B} defined as follows

$$(r + \rho + l_B + m_A\lambda)V_{hn-B} = \rho V_{ln-out} + m_A\lambda(V_{ho-out} - A) + l_B(V_{ho-out} - B).$$

Once the limit order has been submitted several events can occur: either the investor's type changes with intensity ρ and becomes *ln* or the investor monitors again the market with intensity λ and cancels her limit order to send a market order with probability m_A or the limit order is executed with intensity l_B. Each of these events correspond to a change in the utility function and define the value function of submitting a limit order. Types hn become indifferent between limit and market orders if and only if $V_{hn-B} = V_{ho} - A$. Then the value function of a type hn is $V_{hn} = V_{hn-B} = V_{ho} - A$.

The equilibrium value for the execution rate for buy limit orders, l_B, is defined by the following condition that makes a hn investor indifferent between using a limit or a market order:

$$(r + \rho + l_B)(V_{ho} - A) = \rho V_{ln} + l_B(V_{ho} - B). \tag{1.7}$$

Similarly, the equilibrium execution rate for sell limit order is defined by the following indifference condition for investors with type lo:

$$(r + \rho + l_A)(V_{ln} + B) = v - \delta + \rho V_{ho} + l_A(V_{ln} + A). \tag{1.8}$$

Proposition 1.5. *For any couple of equilibrium bid and ask prices* (A, B)*, the equilibrium limit order execution and value functions are as followed,*

$$\begin{aligned}
l_B &= \frac{v - rA - \rho\Delta}{\Delta}, \\
l_A &= \frac{rB - \rho\Delta - (v - \delta)}{\Delta}, \\
V_{ho} &= \frac{1}{r}\frac{1}{2}(v - \rho\Delta) + \frac{1}{r + 2\rho}\frac{1}{2}(v + \rho(A + B)), \\
V_{ln} &= \frac{1}{r}\frac{1}{2}(v - \rho\Delta) - \frac{1}{r + 2\rho}\frac{1}{2}(v + \rho(A + B)), \\
V_{hn} &= V_{ho} - A, \\
V_{lo} &= V_{ln} + B.
\end{aligned}$$

Proof. see Appendix A.2 □

1.4 Equilibrium outcomes

1.4.1 Limit order execution rates

The equilibrium limit order execution rates are such that investors with types *lo* and *hn* are indifferent between getting execution immediacy with a market order or having a delayed execution (for which the loss is measured by the time discount rate r) at a better price with a limit order. On the ask and on the bid side of the market these equilibrium execution rates are respectively equal to

$$l_A = \frac{rB - (v - \delta)}{\Delta} - \rho,\ l_B = \frac{v - rA}{\Delta} - \rho$$

On average a sell (resp. buy) limit order remains in the book during a time $1/l_A$ (resp. $1/l_B$) before being executed. This is the maximum average time during which an investor with type *lo* (resp. *hn*) is willing to wait with a limit order in the order book.

Limit order opportunity cost and execution rates. Depending on the equilibrium prices (A, B), with $A - B = \Delta$, the sell side or the buy side of the market extract more of the trading surplus. Indeed the higher are A and B, the smaller is l_B and the bigger is l_A. An investor with type *hn* requires a lower execution rate, is willing to wait longer in the book, because his opportunity cost for not trading immediately, $v - rA$, declines. This investor is better-off in an equilibrium with high trading prices. Conversely, an investor with type *lo* requires a higher execution rate, is not willing to wait longer in the book, because his opportunity cost for not trading immediately, $rB - (v - \delta)$, increases.

Trading surplus extraction, tick size and execution rates. One dimension of the incentive to use a limit order is the opportunity to extract more of the trade surplus compared to the option to use a market order and sell the asset at a lower price (resp. buy at a higher price). This surplus extraction component is measured by the tick size since it captures the price difference between market and limit orders. The bigger is Δ, the lower are l_A and l_B since investors can extract more of the trading surplus by using limit orders and hence are willing to wait longer in the book before being executed (cf Fig. 1.2). As we will see in the next subsection, the incentive given to investors to extract trading surplus at the expense of execution immediacy is welfare deteriorating.

Private value volatility and execution rates. The frequency ρ, at which the preference for the asset of an investor switches from high to low or low to high, has a negative effect on the equilibrium execution rates (cf Fig. 1.2). When ρ is high, investors anticipate that, if they trade immediately after a change in type, they may trade again soon after a switch back of their type. As a consequence, the incentive for trading is less. Investors suffer less from waiting with a suboptimal type, lo or hn, with a limit order in the book.

The asset holding cost δ, that low type investors suffer from, has a positive impact on l_A. This effect goes through the opportunity cost channel. A higher δ implies a higher opportunity cost for lo type investors who use a limit order.

Average execution rate. The formula for the average execution rate l^0_{eq} allows to capture more easily the effects of the model parameters on the execution rates,

$$l^0_{eq} = \frac{l_A + l_B}{2} = \frac{\delta - (r + 2\rho)\Delta}{2\Delta} \quad \text{and} \quad \frac{\partial l^0_{eq}}{\partial \Delta} < 0,\ \frac{\partial l^0_{eq}}{\partial \rho} < 0,\ \frac{\partial l^0_{eq}}{\partial \delta} > 0.$$

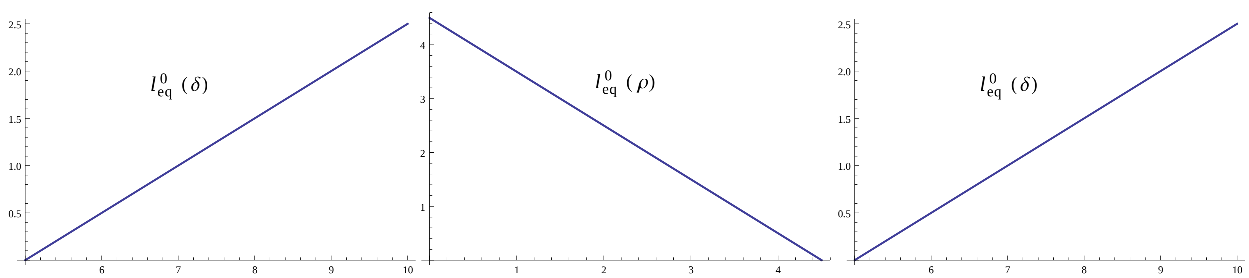


Figure 1.2: Execution rate l^0_{eq} in function of (i) $\Delta \in [0, \delta/(r + 2\rho)]$ ($\rho = 2\ \delta = 10$), (ii) $\rho \in [0, (\delta - r\Delta)/2\Delta]$ ($\Delta = 1,\ \delta = 10$) and (iii) $\delta \in [(r + 2\rho)\Delta, 2(r + 2\rho)\Delta]$ ($\Delta = 1,\ \rho = 2$) , ($r = 1$).

l^0_{eq} is also the execution rate in the symmetric equilibrium. This is the equilibrium in which the term of the trade-off, limit order vs. market order, is the same on both side of the market. The symmetric equilibrium prices are $B = \frac{1}{r}(v - \frac{\delta}{2}) - \frac{\Delta}{2}$, $A = \frac{1}{r}(v - \frac{\delta}{2}) + \frac{\Delta}{2}$ and makes the execution rates equal $l_A = l_B = l^0_{eq}$.

1.4.2 Market depth

On both bid and ask sides of the market, the market depth (i.e. the number of limit orders submitted) is equal, at each point in time, to

$$\alpha_{eq}^0 = \frac{1}{4}\frac{\rho}{\rho + l_{eq}^0} = \frac{1}{2}\frac{\rho\Delta}{\delta - r\Delta} \quad \text{and} \quad \frac{\partial \alpha_{eq}^0}{\partial \Delta} > 0, \ \frac{\partial \alpha_{eq}^0}{\partial \rho} > 0, \ \frac{\partial \alpha_{eq}^0}{\partial \delta} < 0.$$

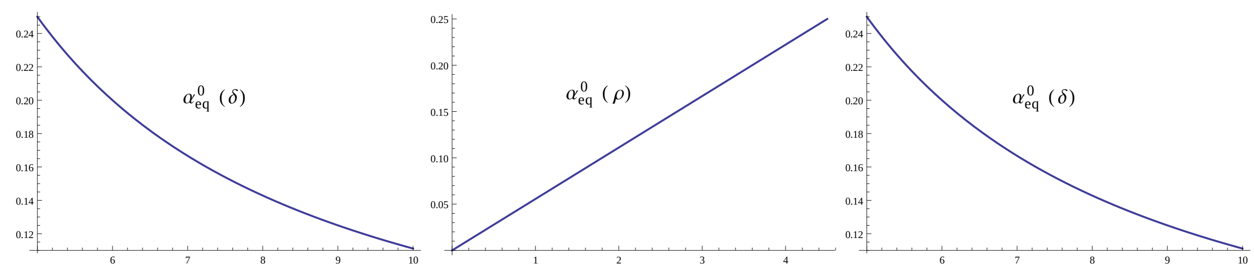


Figure 1.3: Market depth α_{eq}^0 in function of (i) $\Delta \in [0, \delta/(r + 2\rho)]$ ($\rho = 2 \ \delta = 10$), (ii) $\rho \in [0, (\delta - r\Delta)/2\Delta]$ ($\Delta = 1, \ \delta = 10$) and (iii) $\delta \in [(r + 2\rho)\Delta, 2(r + 2\rho)\Delta]$ ($\Delta = 1, \ \rho = 2$) , ($r = 1$).

The tick-size Δ has a positive effect on the market depth since, everything else equal, an increase of this parameter increases the trading surplus that one can extract with a limit order (see Fig. 1.3).

The parameter ρ has a positive effect on the market depth since a higher ρ implies that, at each time t, there is an increasing fraction of investors whose type have become suboptimal and thus have trading need, *lo* and *hn*, which has a positive effect on the number of limit order submitted. This effect is mitigated by a higher equilibrium limit order execution rate but still remains positive (see Fig. 1.3).

δ has a negative effect on the trading intensity. Through the limit order opportunity cost channel, a higher δ imposes a higher equilibrium execution rate which implies a lower market depth (see Fig. 1.3).

1.4.3 Trading intensity/volume

On the ask and the bid side of the market, the trading intensities, are respectively equal to $l_A\alpha^0_{eq}$ and $l_B\alpha^0_{eq}$. Hence the overall trading intensity is

$$(l_A + l_B) \times \alpha^0_{eq} = \frac{\delta - (r + 2\rho)\Delta}{\Delta}\alpha^0_{eq} = \frac{\rho}{2}\frac{\delta - (r + 2\rho)\Delta}{\delta - r\Delta}$$

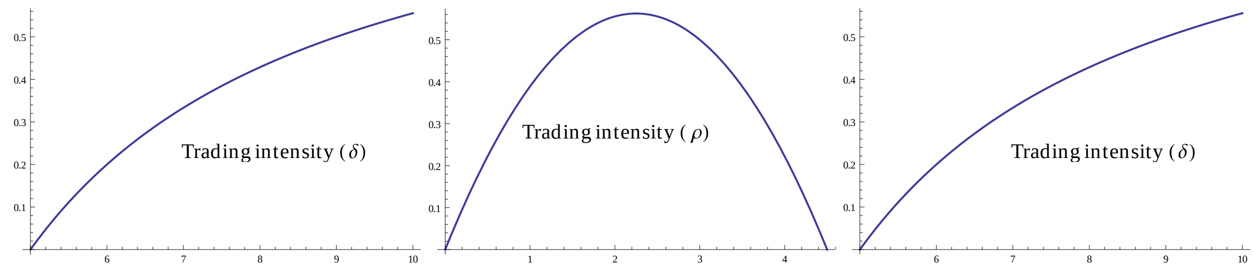


Figure 1.4: Trading intensity in function of (i) $\Delta \in [0, \delta/(r + 2\rho)]$ ($\rho = 2$ $\delta = 10$), (ii) $\rho \in [0, (\delta - r\Delta)/2\Delta]$ ($\Delta = 1,\ \delta = 10$) and (iii) $\delta \in [(r + 2\rho)\Delta, 2(r + 2\rho)\Delta]$ ($\Delta = 1,\ \rho = 2$) , ($r = 1$).

The overall trading volume can be calculated as the integral of the trading intensity over the all game period, $[0, +\infty)$, discounted at rate r. Thus trading volume is equal to trading intensity multiplied by a factor $1/r$.

The tick-size Δ has a negative effect on the trading intensity since, everything else equal, an increase of this parameter increases the trading surplus that one can extract with limit order. Thus the equilibrium execution rate declines. The number of limit order α^0_{eq} increases. The overall effect on the trading intensity is negative (see Fig. 1.4).

The effect of the parameter ρ is not monotonic. On the one hand a higher ρ implies that, at each time t, there is an increasing fraction of investors whose type have become suboptimal and thus have trading need, lo and hn, which has a positive effect. On the other hand, a higher ρ reduces the intensity of this trading need since investors anticipate that they may more likely switch back to an optimal type. As a consequence the equilibrium execution rates decline. The overall effect is positive for low ρ's and negative for high ρ's (see Fig. 1.4).

δ has a positive effect on the trading intensity. Through the limit order opportunity cost channel it imposes a higher equilibrium execution rate. This effect is mitigated by a lower implied market depth but still remain positive (see Fig. 1.4).

1.4.4 Effects of the market monitoring frequency

Monitoring intensity irrelevance. An interesting feature of this equilibrium is that aggregate outcomes, as α_{eq}^0, do not depend on λ, the monitoring intensity. This is an expected outcome of the model since trades occur because of differences in private values and because these private values are monitored continuously. This suggests that market monitoring has a limited role in a stable market. More specifically market monitoring plays a role when liquidity supply is, for instance, cyclical as in Foucault et al. [2009]. In my model there is no cycle since order flows are such that the order book is steady.

Continuous monitoring. The market monitoring rate λ does not impact the aggregate values of the equilibrium, the value functions, the population levels linked to α_{eq}^0 or execution rates l_A and l_B. Hence, we can take the model to the limit where investors are continuously monitoring the market, $\lambda = \infty$. Let's consider the ask side of the book and remind that the flow of market orders hitting the ask side at t is equal to $m_A(\lambda L_{hn} + \rho L_{ln}).dt = m_A(\lambda\alpha_{eq} + \rho(\frac{1}{2} - \alpha_{eq})).dt = l_A L_{hn}.dt = l_A\alpha_{eq}.dt$. This flow is independent of λ. When $\lambda \to \infty$ we must have $m_A \to 0$ so that this flow remains constant. At the limit, the flow of market orders is equivalent to $m_A\lambda\alpha_{eq}.dt$ which implies that $m_A\lambda \to l_A$.

For an investor of type hn, $m_A\lambda.dt$ is the probability that she submits a market order at time t. Noticing that allows to describe the investors strategy in the limit case. When an investor switches to type hn, she submits a limit order at price B with probability 1, because the probability to send a market order is m_A that is infinitesimal. At time t her order is either executed with probability $l_B.dt$, or she decides to cancel it and to send a market order with respect to a mixed strategy with probability $l_A.dt$, or she cancels it if she switches to type ln.

For the same reason, when an investor switches to type lo, she submits a limit order at price A with probability 1. At time t her order is either executed with probability $l_A.dt$, or she decides to cancel it and to send a market order with respect to a mixed strategy with probability $l_B.dt$, or she cancels it if she switches to type ln.

Taking the limit case leads to an equilibrium in which investors play a Poisson mixed strategy to choose between limit and market orders. If we were to consider directly the problem with continuous monitoring we could end up with different types of mixed strategies where, for instance, investors would submit a market order with positive probability when their type changes and then play a Poisson mixed strategy. However these strategies should be such that the flow of market orders and the execution rates are the same as the ones of the equilibrium with infrequent monitoring, since the terms of the trade-off do not change.

1.5 Welfare analysis

Proposition 1.6. *For any steady-state equilibrium with bid and ask prices* (A, B)*, the level of welfare* W *is the same and equal to*

$$W = \left(\frac{1}{2} - \alpha_{eq}^0\right) \times (V_{ho} + V_{ln}) + \alpha_{eq}^0 \times (V_{lo} + V_{hn}) = \frac{1}{r}\frac{1}{2}(v - \rho\Delta) - \alpha_{eq}^0 \Delta = \frac{v}{2r} - \alpha_{eq}^0 \frac{\delta}{r}. \quad (1.9)$$

The welfare is impacted negatively by ρ *and the tick-size* Δ*, and positively by* δ*,*

$$\frac{\partial W}{\partial \Delta} < 0, \; \frac{\partial W}{\partial \rho} < 0, \; \frac{\partial W}{\partial \delta} > 0.$$

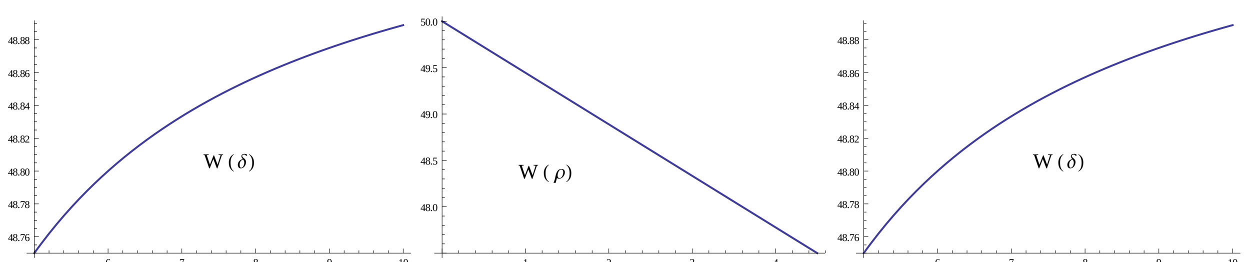


Figure 1.5: Welfare W in function of (i) $\Delta \in [0, \delta/(r + 2\rho)]$ ($\rho = 2 \;\; \delta = 10$), (ii) $\rho \in [0, (\delta - r\Delta)/2\Delta]$ ($\Delta = 1, \; \delta = 10$) and (iii) $\delta \in [(r + 2\rho)\Delta, 2(r + 2\rho)\Delta]$ ($\Delta = 1, \; \rho = 2$) , ($v = 100, \; r = 1$).

The maximum level of welfare than can be reached is equal to $\frac{v}{r}\frac{1}{2}$. It is obtained when all the asset supply $\frac{1}{2}$ is owned by high type investors whose population size is $\frac{1}{2}$ as well. In this situation each share of the asset is always offering a utility flow equal to v and then has a value equal to $\frac{v}{r}$. Reaching this optimum requires that when hn and lo investors come to the market, after they changed of type, they can trade immediately at one price that is the same for buyers and sellers.

The role of the tick size. In a steady-state equilibrium this optimum can be reached if the tick size is nil, $\Delta = 0$, as we can see in the formula for the welfare (equation 12). The tick size is the friction that prevents from reaching the optimum. Because there is a difference in the execution prices for limit and market orders, investors have an incentive to send limit orders and to wait for execution whereas it would be socially optimal that these orders get immediately executed. The corresponding loss is captured by the term $-\alpha^0_{eq}\frac{\delta}{r}$. It shows that the presence of liquidity supply, α^0_{eq} is suboptimal.

When $\Delta = 0$, the bid and the ask prices are infinitely close. Then the price improvement of submitting limit orders is nil and the execution intensities l_A and l_B must be infinite to incentivize limit order submission. Because of these infinite execution rates, limit orders are instantaneously executed and the limit order book is always empty. We can view this equilibrium as a situation where investors coordinate to trade with each other at a single price $P = A = B$ and where there is no difference between limit and market orders.

Effects of the private value volatility components. The idiosyncratic preference switching frequency, ρ, has a negative effect on the welfare (cf Fig. 1.5) through its increasing effect on α^0_{eq}. The asset holding cost δ, for low type investors, has a positive effect on the welfare.

It is interesting to compare these effects with the benchmark case where there is no trading. In this case the level of welfare is given by the value function of investors with type lo and ho, and the initial fractions of these investors type. These value function are as followed,

$$V_{lo} = \frac{v}{r} - \frac{\delta}{r}\frac{r+\rho}{r+2\rho}, \quad V_{ho} = \frac{v}{r} - \frac{\delta}{r}\frac{\rho}{r+2\rho}.$$

Depending on the initial level of populations, the effect of ρ can be positive or negative since it has a positive effect for lo type (they switch to a high private value faster) and a negative effect for *ho* type. For instance, in the steady-state case, the initial fraction of each investor would be 1/4, the welfare would be equal to $\frac{v}{2r} - \frac{\delta}{4r}$ and the parameter ρ would have no effect. In comparison, when there is trading, the welfare become sensitive to this parameter ρ even in steady-state.

The effect of δ is very clear in the situation without trading since, everything else equal, increasing this cost induces an actual or an expected utility loss for all investors who own the asset. Thus it is noticeable that δ has the opposite effect when investors can trade. It generates an opportunity cost for using limit orders which accelerate trading and make the asset allocation across investor more optimal.

Model outcomes and proxies for investor's welfare. To empirically investigate the source of welfare variations for investors in a given financial market, we need observable proxies for welfare. Usually liquidity measures are thought as positively related to investor's welfare, since higher market liquidity leads to a lower implicit trading cost. My model provides counter intuitive results in this respect.

	Limit order execution rate (l^0_{eq})	Market Depth (α^0_{eq})	Trading Intensity	Welfare
$\frac{\partial}{\partial \Delta}$	$-$	$+$	$-$	$-$
$\frac{\partial}{\partial \rho}$	$-$	$+$	$+/-$	$-$
$\frac{\partial}{\partial \delta}$	$+$	$-$	$+$	$+$

Figure 1.6: Signs of first order partial derivatives of model's outcomes with respect to model's parameters Δ, ρ and δ.

The previous results for welfare and market depth shows that any variation of parameters Δ, ρ or δ has opposite effects these model outcomes. Market depth, a traditional liquidity measure, negatively co-varies investors welfare, at least in this model. Trading intensity, which is also a usual liquidity measures, co-varies much better with the welfare (cf. Fig 1.6) except for variations of ρ when it has low values. The limit order execution rate is the only model outcome that always positively co-varies with investor's welfare.

	Limit order execution rate (l^0_{eq})	Market Depth (α^0_{eq})	Trading Intensity	Welfare
$\frac{\partial}{\partial\Delta\partial\rho}$	0	$+$	$-$	$-$
$\frac{\partial}{\partial\Delta\partial\delta}$	$-$	$-$	$+$	$+$
$\frac{\partial}{\partial\rho\partial\delta}$	0	$-$	$+$	$+$

Figure 1.7: Signs of second order cross partial derivatives of model's outcomes with respect to model's parameters Δ, ρ and δ.

One could also want to investigate the effect on liquidity and welfare of the change of one model parameter across different markets that could be sorted with respect to a second parameter. For instance one could look at the effect of decimalization, a reduction of Δ, on the cross section of security markets sorted with respect to the holding cost δ (which would imply using a proxy for such a cost though). To implement these kind of empirical analysis and draw conclusion on the cross sectional effect of such a shock on welfare, one would need a welfare proxy for which the second order cross partial derivative, with respect to the "shocked" parameter and the parameter used to sort the markets cross section, has at least the same sign as the corresponding derivative for the welfare. In our setup, trading intensity would be the best proxy (cf. Fig 1.7).

1.6 Conclusion

Market liquidity measures, as bid-ask spread and market depth, usually focus on implicit trading costs for liquidity consumers. In limit order markets, these measures do not capture the execution quality of limit orders, which are used by traders who decide to supply liquidity. Hence these liquidity measures may not be sufficient to infer investors' welfare. In this paper, I show, with a model, that market depth can be negatively related to investors welfare because high a market depth reflects a low execution rate of limit orders and a relatively low rate for the gains from trade realization. In this context, the limit order execution rate and the trading volume better capture investors' welfare.

Chapter 2

Limited Attention and News Arrival

2.1 Introduction

Investors have limited attention capacities and thus cannot monitor continuously the flow of information in financial markets. As a result they are unable to get or analyze instantaneously implications of public financial news when they arrive. And news content cannot instantaneously turn into common knowledge for the market. Consequently at short horizon public information is private information for investors who observe it first. Because of limited attention, public information release generates a short term period of information asymmetry. How do financial markets react around news arrival? And what role does limited attention play in this process?

To address these questions I propose a theoretical framework to analyze the role of limited attention on market reaction to news. I design a model of limit order market in presence of uncertainty on the asset value due to news arrival. This model extends the OTC markets framework of Duffie, Garleanu and Pedersen [2005, 2007] to limit order markets. In Duffie et al., the main market imperfection is the search friction for trading counterparty. In my model the market imperfection comes from investors' limited attention capacity. It is equivalent to an imperfect monitoring of the market and news arrival. Investors cannot continuously observe public information and contact the market. They do it at some random market monitoring times. This setup generates a gradual diffusion of new public information among investors after news arrival. Investors' imperfect market monitoring allows to jointly describe liquidity formation, price discovery and market efficiency around news arrival.

Financial markets reaction to public information has motivated an extensive line of research both empirical and theoretical especially since the 90's. Among other goals, studying the market reaction to public information allows for a better understanding of the price discovery process in financial markets. The question of market reaction to public information has been addressed empirically by Eredington and Lee [1995], Fleming and Remolona [1999] and Green [2004] for instance. These papers study the reaction of US Treasury securities markets to scheduled macroeconomic announcements. The first two papers show that the market reacts to the announcement in two successive phases. In the first phase, the price shifts quickly to a new level in line with the main figures of the announcement. The second phase of this reaction is characterized by a high volatility, suggesting that investors disagree on the precise interpretation of the announcement. This phase ends when the announcement interpretations of market participants eventually converge. Green's paper shows that these macro announcements increase the level of adverse selection, suggesting that investors with better processing abilities can take advantage of these events.

With respect to this literature, a significant contribution of this paper is to consider unscheduled news. Financial news are released everyday by news providers, as Thomson Reuters or Bloomberg, and deliver relevant information for evaluating asset prices. Virtually all these news arrive at unscheduled times[1]. In addition, the frequency of these news arrival varies a lot across stocks[2]. The unscheduled nature of these events is likely to prevent investors from perfectly paying attention to financial news. By embedding limited attention in a limit order book model, I can speak to the question as to how unscheduled news affect trading decisions and ultimately price formation and liquidity provision.

The current evolution of financial markets supports the choice of limited attention as an important determinant to address the short term dimension of market reaction to public information. These short term reactions have become an important issue since some High Frequency Trading activities have grown by using intensive monitoring technologies to trade very fast on financial news. More generally the boom of Algorithmic Trading (that includes

[1] In a subsample of 40 large stocks, representing 70% of the market cap of the FTSE100, Gross-Klussmann and Hautsch (2011) find that one stock receives on average 750 unscheduled news over 1.5 year

[2] In the same subsample, the news arrival frequency varies ten-fold, from 200 to 2000 news

HFT) partly stems from investors' needs to improve the market monitoring dimension of their trading strategies. This trend shows how important for investors is the capacity of attention that they allocate to monitor markets. Moreover my model considers electronic limit order markets which has been adopted by most equity and derivative exchanges, and which has enabled the growth of Algorithmic Trading.

The paper has several empirical implications for liquidity supply and price dynamics around news arrival. When the frequency of news arrival increases, (i) the level of liquidity supply decreases, (ii) prices adjust faster following news arrival and (iii) the relative importance of limit order cancellations in the price adjustment process declines. The intuition for these results stems from the short term period of information asymmetry around news arrival that is due to limited attention. Consistently with the presence of information asymmetry, there is a "picking-off" risk for liquidity suppliers and this risk varies with the frequency of news arrival. Investors may be reluctant to supply liquidity with limit orders since, following news arrival, limited attention delays their reaction. In the meantime, their limit orders can be "picked-off" because they are not in line with the new asset value and offer a profit opportunity.

In my framework, investors can both supply liquidity with limit orders and consume liquidity with market orders. Before news arrival, investors trade with each other because their private values for the asset are different which generates gains from trade. During this phase the limit order book is in a steady state. The level of liquidity supply is constant and determined by the following trade-off. Market orders provide execution immediacy whereas limit orders provide price improvement but bear execution delay and a picking-off risk when the asset value changes. At equilibrium the level of liquidity supply adjusts so that investors are indifferent between market and limit orders.

When, following news arrival, the asset value changes, it is publicly available but investors do not observe this change immediately. They become aware of it after a while which depends on their monitoring intensity. This generates a transition phase at the end of which prices adjust to the new asset value. This price discovery process relies on two underlying dynamics. Investors who observe the new asset value fast enough can profit from a transitory arbitrage opportunity by using market orders to "pick-off" stale limit orders at the initial price. And investors, with limit orders in the order book, cancel these orders to avoid being picked-off by previous market orders. Once limit orders at the initial price level have all been cancelled or picked-off, the transition phase stops and the limit order book converges to a new steady-state without uncertainty on the asset common value. Thus the model provides a high-frequency description of price and order dynamics around news arrival. This should prove useful for empiricists[3].

The decision for investors to use limit or market orders to trade before news arrival depends on the risk to be picked-off during the transition phase. Everything else equal this risk enhances the expected loss associated with limit order submission and has a negative impact on the liquidity supply. In this context, the effect of the frequency of news arrival is intuitive. More frequent news releases increases the likelihood of an event where a limit orders may be picked-off which turns into a higher picking-off risk. Consequently the level of liquidity supply measured by the market depth (the number of limit orders in the order book) is negatively linked to this frequency. In a thinner market, the amount of stale limit orders that must be cancelled or executed in the transition is less which makes the price adjustment faster.

Investors' limited attention capacity influences the level of this risk and thus the liquidity supply prior to news arrival. More attention however has an ambiguous effect. To see why, let's consider an increase of investors monitoring intensity[4]. On the one hand investors can cancel their limit orders faster after news arrival which reduces their risk of being picked-off and makes limit orders more profitable. However, investors can also send directional market orders faster to execute against stale limit orders which worsen the risk of being picked-off for limit orders. Overall limit orders could be more or less profitable after an increase of monitoring intensity. I identify conditions which make limit orders overall more profitable after such an increase. However the magnitude of its effect on liquidity supply appears to be small especially when compared to the effect of the news arrival frequency. This suggests that only relative monitoring abilities, with respect to other market participant, really matters to understand how this parameter can quantitatively affect investors trading strategies around

[3]Engle et al. (2009) use high-frequency limit order book data to analyze liquidity and volatility in the U.S. Treasury market.

[4]This increase could result from a reduction of latencies.

news arrival.

Chapter 2 is organized as follows. Section 2.2 reviews the related literature. Section 2.3 presents the setup and assumptions of the model. Section 2.4 gives the equilibrium and its general description. Sections 2.5, 2.6 and 2.7 describe the properties of the different phases involved in the equilibrium dynamic of the limit order market. Section 2.8 discuss empirical implications of the model. Section 2.9 concludes.

2.2 Literature review

The reaction of financial markets to public news has been and still is an active field of research. Kim and Verrecchia [1991, 1994] have proposed models to analyze the market reaction to public information release as earnings announcements. They make predictions about market liquidity or trading volume around these events. As mentioned above it has also inspired empirical studies as those of Eredington and Lee [1993, 1995], Fleming and Remolona [1999] and Green [2004]. More recently Tetlock [2010], using a large data set on all types of public news, has addressed the effect of public news release on the level of asymmetric information and how it affects returns around these events. Della Vigna and Pollet [2009] link market reaction to earnings announcements and investor's attention to explain post-earnings announcement drift.

The link between attention capacity and investors' decisions in financial markets is a fairly new research topic. Some recent works by Peng and Xiong [2006], Van Nieuwerburgh and Veldkamp [2009] or Mondria [2010] have developed theories where investors have a limited capacity of attention and allocate it across assets. The more they allocate attention to an asset the more precise is their information about its future pay-off. Through this channel these papers analyze the effect of limited attention on portfolio diversification and asset prices. My paper contributes to this literature by mapping limited attention capacity to imperfect market monitoring at the high-frequency level. It allows me to analyze its effect on trading mechanism.

Market monitoring imperfection has already been stressed as a key determinant of market dynamics by Darrell Duffie Presidential Address [2010]. Foucault, Kadan and Kandel [2009] address this problem in a limit order book framework. In their paper agents strategically choose their level of market monitoring but are exogenously considered as limit order or market order users. Biais and Weill [2009] and Biais, Hombert and Weill [2012] also consider imperfect monitoring agents. The focus of their papers differs from mine since they consider limit order market dynamics generated by aggregate liquidity shocks rather than asset value uncertainty. Moreover I model how asset value uncertainty affect trading strategies before the change in the asset common value occurs. Whereas, in Biais et al. modelling, the market dynamic starts with the liquidity shock. In my model the change in the asset common value is a publicly observable signal but it is not instantaneously observed since market monitoring is imperfect. Pagnotta and Philippon [2012] study the effect of competition between exchanges for the market monitoring intensity, or latency, they provide to their customers. They identify this competition as an incentive for investing in fast trading technologies. Biais et al. [2009,2012], Pagnotta and Philippon [2012] as well as my paper use and adapt the model of search friction in OTC markets introduced by Duffie, Garleanu and Pedersen [2005, 2007] and extended by Lagos and Rocheteau [2009], Lagos, Rocheteau and Weill [2011], Vayanos and Weill[2008] and Weill [2007,2008].

The effect of information monitoring on market liquidity provision has been studied by Foucault, Roell and Sandas [2003] in the case of a dealership market. In their model Market Makers face adverse selection by informed traders and can reduce this risk by monitoring public information and adjusting their quote. The choice of the monitoring intensity is costly. In my model monitoring intensity is an exogenous parameter but it affects both liquidity supply and demand which is more consistent with how limit order markets work. Goettler, Parlour and Rajan [2009] design a very realistic environment of limit order market that is not tractable and meant to be solved numerically. In their paper traders do not continuously monitor the market and decide ex-ante to be privately informed or not about the asset value.

This paper also builds on the dynamic limit order market literature. There are quite a few papers dealing with this problem when compared to its practical importance. One of the reasons is that limit order markets are very hard to model. Foucault [1999] and Parlour [1998] are the first models of limit order markets designed as dynamic games capturing the inter-temporal aspect of the problem. The tractability of these models is appreciable but is reached at the cost of strong assumptions. Both incorporate private and/or common value as drivers of trading and price formation processes but do not allow for strategic decision over the limit order lifetime. Foucault, Kadan and Kandel [2005] focus on the dynamic of the liquidity supply in a limit order market. In their paper investors trade for liquidity reasons and solve the market vs. limit order trade-off in function of their preference for immediacy. Rosu [2009] generalizes Foucault, Kadan and Kandel framework and design a continuous time model where traders can freely send limit orders at any price and can cancel them. Rosu [2010] add a common value environment to his previous model. The two papers by Rosu are build on the fundamental assumption that limit orders are continuously monitored by their owners. As in Rosu's models I design a framework that allow for an entire freedom of choice for investors' order management at the exception of the zero or one unit holding constraint (as in Rosu [2009,2010]). Pagnotta [2010] designs a limit order book model with insider trading where agents optimally choose their trading frequency. At equilibrium they don't trade continuously but they continuously observe the market and update their belief accordingly.

2.3 Model

2.3.1 Asset value dynamic

The model presented in chapter 2 builds on the framework introduced in chapter 1. The additional modelling block is the introduction of uncertainty for the asset value.

Preferences, Asset holding and supply. See chapter 1, section 1.2.1.

Asset value dynamic. The dynamic of the asset common value v_t is the following[5]:

[5] The modelling of the asset value dynamic is equivalent to other dynamics where the asset pays-off at some random time in the future and does not provide a continuous flow of utility. For instance the two following formulation deliver the same result.

- The asset pays off the cash-flow $V = \frac{v}{r}$ at a random time that occurs with respect to a Poisson process of intensity r. And being a low type induces a cost for holding the asset which is equal to δ per unit of time.
- Or, at a random time that occurs with respect to a Poisson process with intensity r, The asset pays off a cash-flow $\frac{v}{r}$ for high types and $\frac{v-\delta}{r}$ for low types.

- at $t = 0$, the asset common value is equal to v_0

- at date $\tau > t$, the common value of the asset changes. This time τ is random and follows a Poisson distribution of intensity μ, $\mathcal{P}(\mu)$. At date τ the common value switches to $v^u = v_0 + \omega$ or $v^d = v_0 - \omega$ with equal probabilities $\frac{1}{2}$

- for $t > \tau$ the asset value is v^u or v^d until the end of the game.

- for $0 < t < \tau$ the state of the world is $\zeta = 0$. For $\tau < t$ the state of the world is either $\zeta = u$ if $v_t = v_0 + \omega$ or $\zeta = d$ if $v_t = v_0 - \omega$.

Time τ corresponds to the news arrival event in this setup. Consequently μ can be interpreted as the news arrival frequency since it is the likelihood for such an event to occur at next period. Parameter ω measures the news surprise that is to say the innovation of the asset common value that is linked to the news content.

Assumption 2.1. *I assume that ω is big compared to δ. It ensures that, if the change in the common value, of magnitude ω, is not followed by a change in price, the profit opportunity, measured by ω, is bigger than gains from trade due to differences in private values, measured by δ. More specifically, I assume that*

$$\omega > 3\delta \times \max\left[1, \frac{2r + \rho}{2\rho}\right]. \tag{2.1}$$

2.3.2 Limited attention

Investors have a limited capacity of attention which means that they are not able to process information instantaneously. As a consequence they can neither track and interpret continuously the flow of public news nor the rest of market activity. Moreover they must allocate their attention capacity across different tasks which prevents them from being continuously in contact with the market, able to trade.

Assumption 2.2. *I assume that an investor observes the asset value, the market, her private value and contacts the market at some random times $\{t_i\}_{i\in\mathbb{N}}$. I call these times "market* *monitoring times". This sequence of market monitoring times is generated by a Poisson process of intensity $\lambda + \rho$.*

More specifically between time t and $t + dt$ an investor monitors the market in two types of situation:

- *when she uses the market monitoring technology which occurs with probability $\lambda.dt$*
- *when her private value changes which occurs with probability $\rho.dt$.*

The level of investor's attention allocated to the market monitoring is measured by λ. At the limit $\lambda = \infty$ investors continuously monitor the market. In a sense they are infinitely attentive to the market and to the flow of news.

2.3.3 Limit order market

See chapter 1, section 1.2.3.

Assumption 2.3. *As in chapter 1, I assume that $\frac{\delta}{r}$ is big compared to Δ but I need this difference to be higher. More specifically I assume that*

$$\delta > (r + 4\rho)\Delta \tag{2.2}$$

2.3.4 Value function and equilibrium concept

See chapter 1, section 1.2.4.

2.4 Equilibrium

In the first part of this section, I provide the equilibrium result of this chapter. I focus on the symmetric equilibrium. This equilibrium is the core of the chapter. I describe it in more details in section 2.5, 2.6 and 2.7. I derive empirical implications from this equilibrium in section 2.8. In the second part of this section, I show that this is not a unique equilibrium.

2.4.1 The symmetric equilibrium

Proposition 2.1. *For any intensity of news arrival, μ, there exists an equilibrium for which bid and ask prices are symmetrical with respect to the average private value of the asset, $\frac{v-\frac{\delta}{2}}{r}$,*

$$B = \frac{1}{r}(v - \frac{\delta}{2}) - \frac{\Delta}{2}, \quad A = \frac{1}{r}(v - \frac{\delta}{2}) + \frac{\Delta}{2}.$$

There are 3 pairs of these prices at which trades can take place: the one at the beginning of the game, (A^0, B^0), and the one at the ends of the game, (A^u, B^u) and (A^d, B^d). At these prices the equilibrium is unique.
Before news arrival, the depths of the limit order book at price A^0 and B^0 are both equal to the depth parameter α^0 ($D_{A^0} = D_{B^0} = \alpha^0$). Limit orders submitted at these prices executes according to a Poisson distribution of intensity l^0. These two parameters are interdependent:

- *l^0 is such that investors are indifferent between limit and market orders. It depends, among other things, on the state of the order book and particularly on α^0*
- *α^0 depends on l^0 because l^0 determines the flow of executed limit orders and consequently the level of liquidity supply in the order book.*

The equilibrium values $(\alpha^0_{eq}, l^0_{eq})$ are the fixed point solution to this problem of interdependence.

Proof. see Appendix A.5 □

Equilibrium. The equilibrium strategy and the resulting limit order book dynamic (cf figure 1) have the following features:

- In the first phase, for $0 \leq t < \tau$, when the asset common value is equal to v_0, trading occurs because of differences in **private values** across investors: *lo*'s and *hn*'s trade with each other via the limit order book, *ho*'s and *ln*'s do not trade. In particular buy limit orders are submitted by types *hn* and sell limit orders are submitted by types *lo*. During this phase the dynamic of the limit order market is the following:
  - the limit order book is in a **steady-state phase**. Liquidity supply at A^0 and B^0 is equal to α^0_{eq} and does not vary over time.

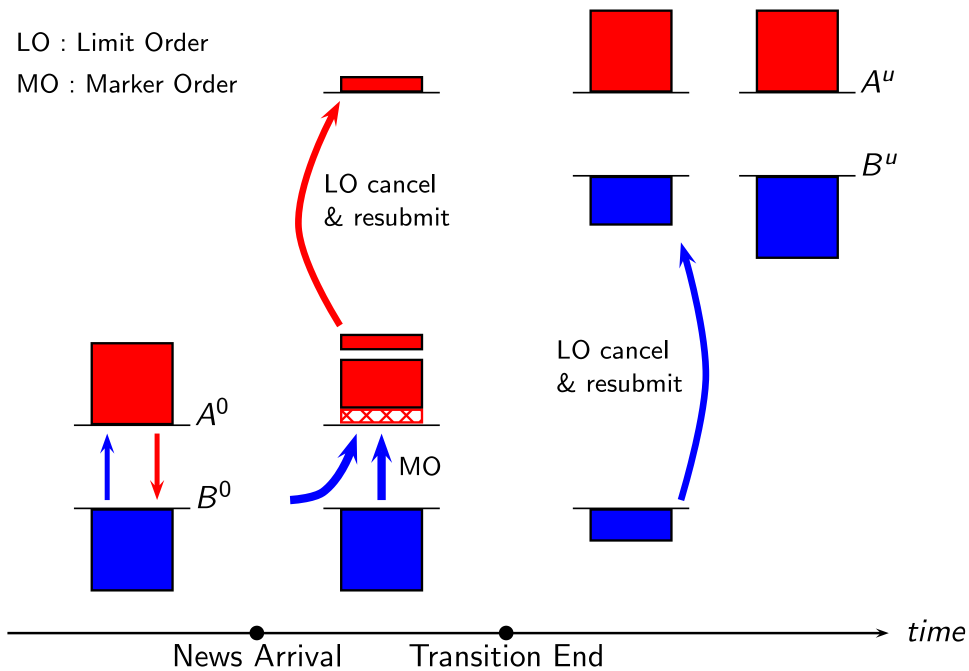


Figure 2.1: Dynamic of the limit order book following "good" news arrival

- lo type investors submit sell limit or market orders using a mixed strategy. They choose to submit a market order at price B^0 with probability m or a limit order at price A^0 with probability $1-m$. They do not submit sell limit orders at prices higher than A^0

- hn type investors submit buy limit or market orders using a mixed strategy. They choose to submit a market order at price A^0 with probability m or a limit order at price B^0 with probability $1-m$. They do not submit buy limit orders at prices lower than B^0.

- A sell or buy limit order submitted at price A^0 or B^0 executes according to a Poisson distribution of intensity l^0_{eq} such that lo's and hn's are indifferent between limit and market orders.

- Once the asset common value has changed, a **transition phase** starts for the limit order book. This transition phase lasts for a finite duration T. During this phase, for

$\tau < t < \tau + T$, trading takes place because the new **common value** of the asset is not in line with the market prices and thus generates a profit opportunity.

- if the new common value is $v_0 + \omega$, investors with type *lo* cancel their sell limit order and resubmit them at a higher price A^u, investors with types *hn* and *ln* send buy market orders to execute against stale limit orders at price A^0 and then stay out of the order book, investors with type *ho* stay out of the order book.
- if the new common value is $v_0 - \omega$, investors with type *hn* cancel their buy limit order and resubmit them at a lower price B^d, investors with types *ho* and *lo* send sell market orders to execute against stale limit orders at price B^0 and then stay out of the order book, investors with type *ln* stay out of the order book.

• When all limit orders that could potentially be picked-off have been executed or cancelled the transition phase is over. Trading occurs once again because of differences in **private values** across investors. During this last phase *ho*'s and *ln*'s do not trade, *lo*'s and *hn*'s trade with each other via the limit order book. The equilibrium strategy and the limit order book aggregate state **converge to a steady-state phase** that has the same features as the first phase except that there is no uncertainty for the future common value, $\mu = 0$. Bid and ask prices are either A^u and B^u or A^d and B^d depending on the previous change in the common value. The limit steady-state corresponds to the equilibrium studied in chapter 1.

Figure 2.1 summarizes the equilibrium dynamic of the limit order book.

2.5 Limit order book in steady state

In this section I explicit the equilibrium strategy in the first phase of the game. This first stage of the game is a steady state equilibrium similar to the equilibrium studied in chapter 1. The additional feature of this equilibrium is that investors' strategy takes into account a possible change of the asset value following news arrival. I first show how the value functions of investors are affected by the introduction of asset value uncertainty. In a second step I solve for the equilibrium in a symmetric setup.

2.5.1 Value functions

As in chapter 1, I only provide the value function for *ho* and *hn* investors as it is very similar for *ln* and *lo*.

Type *ho*. As in chapter 1, a *ho* investor stays out of the market until she switches to the *lo* type. Her situation is affected when the common value changes. Her value function V_{ho-out} is defined as follows

$$V_{ho-out} = v.dt + (1 - r.dt)\left[(1 - \rho.dt - \mu.dt)V_{ho-out} + \rho.dtV_{lo} + \mu.dt\left(\frac{1}{2}V^u_{ho-out}(0) + \frac{1}{2}V^d_{ho-out}(0)\right)\right]$$
$$\iff (r + \rho + \mu)V_{ho-out} = v + \rho V_{lo} + \frac{\mu}{2}[V^u_{ho-out}(0) + V^d_{ho-out}(0)].$$

The term $\frac{\mu}{2}[V^u_{ho-out}(0) + V^d_{ho-out}(0)]$ corresponds to the change in utility when the asset common value changes, up or down. These are the values of being a type *ho* at the beginning, the time 0, of the transition phase.

Type *hn*. As in chapter 1, a *hn* investor sends a buy market order with probability m_A or limit order with probability $1 - m_A$. Sending a buy market order at price A provides her with the value function $V_{ho-out} - A$. Indeed she gets execution immediacy by trading at the ask price A and instantaneously switches to type *ho*. The last additional term corresponds, as above, to the utility change when the asset value changes. Sending a buy limit order at price B provides her with the value function V_{hn-B} defined as follows

$$\begin{aligned}(r + \rho + l_B + m_A\lambda + \mu)V_{hn-B} &= \rho V_{ln-out} + m_A\lambda(V_{ho-out} - A) + l_B(V_{ho-out} - B)\\ &+ \frac{\mu}{2}[V^u_{hn-B}(0) + V^d_{hn-B}(0)].\end{aligned}$$

As in chapter 1, the execution rate l_B (resp. l_A) is defined by the condition that a *hn* investor (resp. *lo*) is indifferent between using a limit and a market order. It is easy to check that this value do not depend on the mixed strategy m_B. Actually this equilibrium parameter is well defined by the formula that links the mixed strategy and the execution rate. Typically l_B depends on value functions in the transition phase. These value functions depend on α^0 since the level of liquidity provision affects the duration of the transition phase among other things. Remind that α^0 corresponds to the depth of the limit order book and that the transition phase lasts until this depth has been completely executed or removed. In the end l_B depends on α^0 and conversely. Solving for the equilibrium of the game is equivalent to solve this fixed point problem in the first steady state phase. It also requires to solve for the game starting with the transition phase.

Compensation for providing liquidity. The equilibrium limit order execution rate can be seen as a compensation for risk taking. When an investor submits a limit order instead of a market order, she chooses her execution price but renounces to execution immediacy and eventually a risk of being picked-off when the asset common value changes. Providing liquidity requires therefore a compensation for risk taking. This compensation is obtained by an appropriate execution delay for limit order execution. More precisely the execution rates l_A and l_B must incentivize liquidity provision via limit orders. In equilibrium these execution rates are such that market and limit orders are equally profitable for types *hn* and *lo*. This mechanism appears clearly in agents' value functions (next subsection).

2.5.2 Steady state in the symmetric equilibrium

In the initial steady state of the symmetric equilibrium, investors coordinate to trade on the following bid and ask prices:

$$B^0 = \frac{1}{r}(v_0 - \frac{\delta}{2}) - \frac{\Delta}{2}, \quad A^0 = \frac{1}{r}(v_0 - \frac{\delta}{2}) + \frac{\Delta}{2}.$$

The symmetry of this equilibrium implies that the term of the trade-off between limit order and market order is the same on both side of the market. The execution rates that make investors indifferent between limit and market orders are the same for sell and buy orders:

$$l_{A^0} = l_{B^0} = l^0.$$

The steady state condition for the limit order book implies that the depths of the order book, measured by α^0 and the equilibrium execution rate are linked by the following formula:

$$\alpha^0 = \frac{\rho}{4(\rho + l^0)} \Longleftrightarrow l^0 = \rho\left(\frac{1}{4\alpha^0} - 1\right) = g(\alpha^0). \tag{2.3}$$

The execution rate implied by this formula is infinite for $\alpha^0 = 0$ ($g(0) = \infty$) and nil for $\alpha^0 = \frac{1}{4}$ ($g(1/4) = 0$).

Proposition 2.2. *The execution rate, l^0, that makes investors with types lo and hn indifferent between limit and market orders is a function of α^0, $l^0 = f(\alpha^0)$. For $\alpha^0 = 0$ and $\alpha^0 = \frac{1}{4}$, f is finite and positive. Moreover f is decreasing with respect to α^0,*

$$\frac{\partial f}{\partial \alpha^0} < 0. \tag{2.4}$$

There is a unique $\alpha^0_{eq} \in [0, 1/4]$ such that $f(\alpha^0_{eq}) = g(\alpha^0_{eq})$. The intersection point defines the equilibrium values α^0_{eq} and l^0_{eq} (cf Figure 2).

Proof. see Appendix A.5 and A.6 ☐

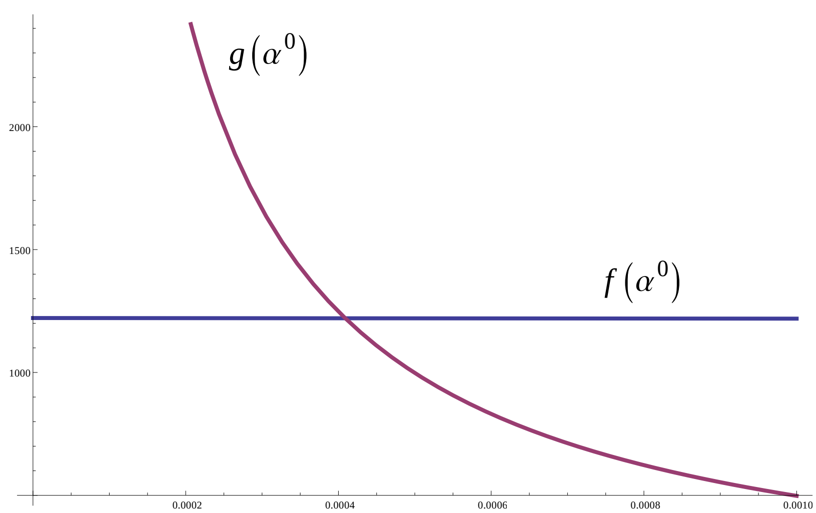


Figure 2.2: The two functions of f and g in function of α^0 ($\lambda = 100,\ \mu = 50,\ r = 1,\ \rho = 2,\ \Delta = 1,\ \delta = 10,\ \omega = 50$).

2.6 Limit order book in transition phase

When there is uncertainty, $\mu > 0$, the first steady state phase lasts a finite time (almost surely) and is followed by a transition phase. Prior to the asset value change, the world is in the state $\zeta = 0$. The transition phase starts when the asset common value changes. This corresponds to the public news arrival event. It occurs at some point in time τ that follows a Poisson distribution, $\mathcal{P}(\mu)$. For times $t > \tau$ the state of the world is either $\zeta = u$ (up) with $v = v^0 + \omega$ or $\zeta = d$ (down) with $v = v^0 - \omega$ with equal probability. I call T^u and T^d the duration of the transition phases in the different states of the world.

2.6.1 Transition phase strategy

Once the common value has switched to a higher level for instance, non-owner turn into arbitrageurs and have an incentive to buy the asset while it is tradable at a low price, A^0, and to resell it at a higher price A^u later. At equilibrium investors coordinate on the price A^u at which they will trade the asset in the future.

Proposition 2.3. *After the common value has changed, during the transition phase, the strategy is:*

- *In the case $\zeta = u$, for $\tau < t < \tau + T^u$:*
    - *Investor coordinate on a pair of future ask and bid prices (A^u, B^u)*
    - *lo's cancel any sell limit order that is not at price A^u and submit a limit order at price A^u*
    - *ho's cancel any sell limit order and stay out of the market*
    - *ln's send a buy market order and immediately behave with respect to their new type, lo*
    - *hn's send a buy market order and immediately behave with respect to their new type, ho*
- *In the case $\zeta = d$, for $\tau < t < \tau + T^d$:*

- *Investor coordinate on a pair of future ask and bid prices* (A^d, B^d)
- *hn's cancel any buy limit order that is not at price* B^d *and submit a limit order at price* B^d
- *ln's cancel any buy limit order and stay out of the market*
- *ho's send a sell market order and immediately behave with respect to their new type, hn*
- *lo's send a sell market order and immediately behave with respect to their new type, ln*

Proof. see Appendix A.4.3 □

2.6.2 Limit order book dynamics in the transition phase

Before the transition phase begins the limit order book is filled with some limit orders that supply liquidity. In particular liquidity provisions at best ask and bid prices are defined by the value of the depths of the limit order book at prices A^0 and B^0. These are equal to D_{A^0} and D_{B^0}. During the transition phase trading occurs only on one side of the order book. On this side limit orders offer a profit opportunity. On the other side investors cancel their limit order and send market orders to hit limit orders offering this opportunity.

For instance when the asset common value makes a positive jump, $\zeta = u$, sell limit orders submitted at price A^0 offer a profit opportunity to buyers. Indeed A^0 was an equilibrium price when the asset common value was equal to v_0 but is no longer once this value has moved up to $v_0 + \omega$. The liquidity supply on the ask side $D^u_{A^0}(t)$ is meant to disappear. It decreases due to two kind of mechanisms. At each time t a mass $(\lambda + \rho) D^u_{A^0}(t).dt$ of investors monitor the market and cancel their limit order at price A^0. At the same time a mass $(\lambda + \rho) \times (L_{hn}(t) + L_{ln}(t)).dt = \frac{\lambda+\rho}{2}.dt$ of investors who do not own the asset monitor the market and send buy market orders that execute at price A^0. The dynamic of $D^u_{A^0}(t)$ is given by the following proposition.

Proposition 2.4. *When* $\zeta = u$ *during the transition phase the depth at price* A^0 *is*

$$D^u_{A^0}(t) = -\frac{1}{2} + [D_{A^0} + \frac{1}{2}]e^{-(\lambda+\rho)(t-\tau)}$$

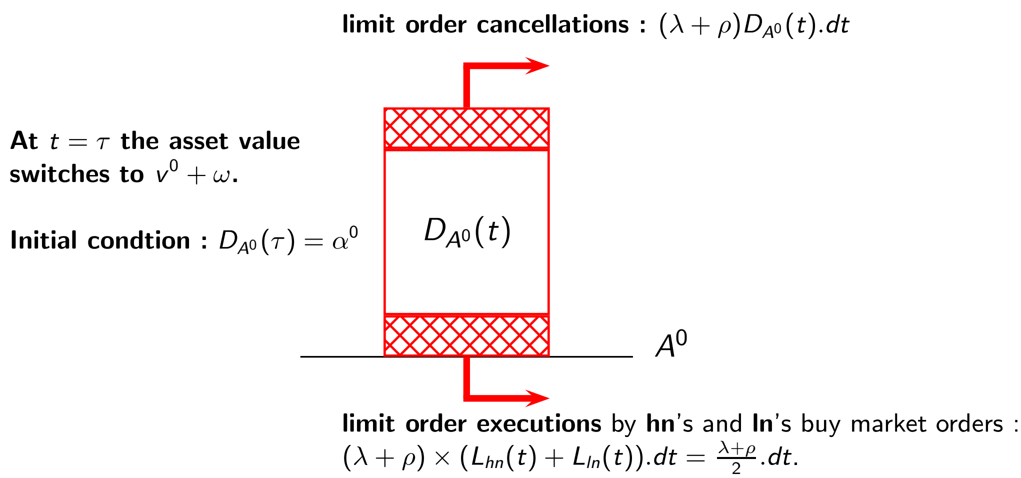


Figure 2.3: Dynamic of the market depth at A^0 following "good" news arrival

which is decreasing and reaches zero at time $t=\tau+T^u$*, defining the duration* T^u*.*

Proof. see Appendix A.4.2 □

When the asset common value moves down, the process is the same as in the "up" case but on the bid side of the order book at price B^0.

Proposition 2.5. *When* $\zeta=d$ *during the transition phase the depth at price* B^0 *is*

$$D^d_{B^0}(t)=-\frac{1}{2}+[D_{B^0}+\frac{1}{2}]e^{-(\lambda+\rho)(t-\tau)}$$

which is decreasing and reaches zero at time $t=\tau+T^d$*, defining the duration* T^d*.*

Proof. see Appendix A.4.2.4 □

The dynamic of the order book in the transition phase is given by the initial values of the depths, $D_{B^0}=D^d_{B^0}(0)=D^u_{A^0}(0)=D_{A^0}=\alpha^0$, so that the evolution of the depths in the

”up” and the ”down” cases are perfectly similar

$$\forall t \; D^d_{B^0}(t) = D^u_{A^0}(t) = D(t) = -\frac{1}{2} + [\alpha^0 + \frac{1}{2}]e^{-(\rho+\lambda)(t-\tau)}.$$

The durations of the transition phases are the same in both states u and d

$$T^u = T^d = T = \frac{1}{\rho+\lambda}\ln(1+2\alpha^0).$$

Transition phase in the symmetric equilibrium. The symmetric equilibrium is the equilibrium where investors coordinate on the following future bid and ask prices in state $\zeta = u$ and $\zeta = d$:

$$B^{u/d} = \frac{1}{r}(v_0 \pm \omega - \frac{\delta}{2}) - \frac{\Delta}{2}, \; A^{u/d} = \frac{1}{r}(v_0 \pm \omega - \frac{\delta}{2}) + \frac{\Delta}{2}$$

The analysis of this transition phase allows to understand the underlying trading mechanism for the dynamic of prices in a limit order market. As I develop it in the empirical implication section we can evaluate the impact of market monitoring or volatility on how fast prices adjust to new information. This is also possible to determine the role that limit and market orders play in this price discovery process. In particular we can quantify the effect of the market monitoring rate on the share of limit order cancellations and market order executions in the erosion of the initial liquidity supply.

2.7 After the transition phase : convergence to a steady state without uncertainty

In this section I explicit the strategy and the dynamic of the limit order book that corresponds to the last phase of the game, after the transition phase is over. In the last phase of the game the limit order book converges to a steady state without uncertainty as in chapter 1. Trading takes place at prices A^u and B^u if $v = v_0 + \omega$ or at A^d and B^d if $v = v_0 - \omega$.

Here I present the general case of this dynamic equilibrium that converges to a steady state without uncertainty. In this equilibrium the terms of the trade-off between limit and market orders do not change over time and are the same as the ones in the asymptotic steady state. I use the same notations as in the limit order book in steady state without uncertainty (α_{eq}, A, B...etc, cf subsection 5.8).

Starting at $t = 0$ from a one tick market where the depths at prices A and B are $D_A(0)$ and $D_B(0)$ constituted respectively by a share of the population $L_{lo}(0)$ and of $L_{hn}(0)$, investors follow their corresponding **steady state equilibrium** strategy described in section 5. The rates at which hn and lo types send market orders, $m_A(t)$ and $m_B(t)$, evolve so that the terms of the trade off are the same as in the steady state equilibrium. More precisely the intensities at which limit orders are executed are unchanged and equal to l_A and l_B. In this framework the dynamic of the different populations is given by the dynamic of the parameter α,

$$L_{ho}(t) = L_{ln}(t) = (\frac{1}{2} - \alpha(t))$$
$$L_{hn}(t) = L_{lo}(t) = \alpha(t)$$

and the value functions for each type are the same as in the former steady-state equilibrium.

To fully characterize the level of convergence of the limit order book we look at how its state is different from the limit steady state. First in the steady state all investors have positions in line with their optimal strategy. For instance at the limit $t = \infty$ all types lo have a limit order in the book at price A. In the dynamic game a lo type investor may have been out of the market to start with and then has to wait for her first market monitoring time to submit a limit order. The difference $L_{lo}(t) - D_A(t)$ measures the mass of types lo out of the market. At time t the mass of investors out of the market who would optimally be in the market are given by the following equations

$$L_{lo}(t) - D_A(t) = (L_{lo}(0) - D_A(0))e^{-(\lambda+\rho)t}$$
$$L_{hn}(t) - D_B(t) = (L_{hn}(0) - D_B(0))e^{-(\lambda+\rho)t}$$

The second convergence measure is the difference between the population levels at time

t and the ones in the steady state equilibrium. This difference is captured by the value $\alpha(t) - \alpha_{eq}$. The two dimensions of the convergence level actually reduce to one. Indeed, as for the steady-state case, describing the evolution of $\alpha(t)$ is enough to describe the dynamic of the order book since $L_{ho}(t)$, $L_{ln}(t)$, $L_{hn}(t)$, $L_{lo}(t)$, $D_A(t)$ and $D_B(t)$ are fully defined when $\alpha(t)$ is known.

Proposition 2.6. *The dynamics of the equilibrium populations are given by the dynamic of the parameter α,*

$$\begin{aligned}\alpha(t) &= \alpha_{eq} + (\alpha(0) - \alpha_{eq})e^{-(2\rho+l_A+l_B)t} \\ &+ l_A\kappa_A \frac{1-e^{-[\lambda-(\rho+l_A+l_B)]t}}{\lambda-(\rho+l_A+l_B)} e^{-(2\rho+l_A+l_B)t} + l_B\kappa_B \frac{1-e^{-[\lambda-(\rho+l_A+l_B)]t}}{\lambda-(\rho+l_A+l_B)} e^{-(2\rho+l_A+l_B)t}\end{aligned}$$

with $\kappa_A = L_{lo}(0) - D_A(0), \ \kappa_B = L_{hn}(0) - D_B(0)$

Proof. see Appendix A.3 □

2.8 Empirical implications

2.8.1 Determinants of the liquidity supply before news arrival

When investors decide to supply or not liquidity they anticipate that news arrival will trigger a transition phase where their limit order will bear an adverse selection risk. The model parameters influence in different ways this risk of being picked-off and thus have an effect on the size of the liquidity supply.

Proposition 2.7. *An increase of μ or ω has a negative impact on α^0_{eq} (cf. Figure 3)*

$$\frac{\partial \alpha^0_{eq}}{\partial \mu} < 0, \quad \frac{\partial \alpha^0_{eq}}{\partial \omega} < 0$$

Moreover $\lim_{\mu\to\infty} \alpha^0_{eq} = 0$.

For a value of μ not too low, an increase of the monitoring rate λ has a positive impact on α^0_{eq} (cf. Figure 3).

$$\frac{\partial \alpha^0_{eq}}{\partial \lambda} > 0$$

Proof. see Appendix A.5.4 □

Prediction 2.1. *The liquidity supply before news arrival* α^0_{eq}

- *decreases with the frequency of news arrival,* μ*, or the news surprise,* ω*.*
- *decreases with the monitoring rate* λ *when* μ *is not too low*

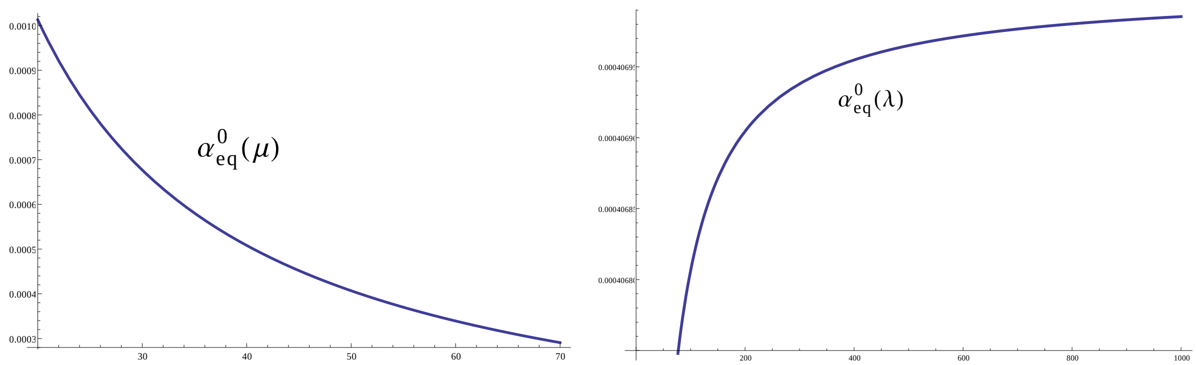


Figure 2.4: (i) Evolution of α^0_{eq} in function $\mu \in [20, 70]$ for $\lambda = 100$ and (ii) evolution of α^0_{eq} in function of $\lambda \in [0, 1000]$ for $\mu = 50$ $(r = 1,\ \rho = 2,\ \Delta = 1,\ \delta = 10,\ \omega = 50)$.

The first two comparative statics come from the fact that the execution rate that makes investors indifferent between limit and market order increases with μ and ω. Investors are less willing to use limit order when the volatility of the asset common value increases, everything else equal.

$$\frac{\partial l^0}{\partial \mu} > 0, \quad \frac{\partial l^0}{\partial \omega} > 0$$

Mechanically the depth of the order book adjusts because the relation $\alpha^0 = \frac{\rho}{4(\rho + l^0)}$ has to be satisfied.

The effect of λ on α^0_{eq} also comes from the fact that l^0 decreases with respect to λ. However this dependence of l^0 on λ comes from two different channels and has no very intuitive direction. On the one hand the increase of λ makes investors faster to cancel limit orders after news arrival. This reduces the picking-off risk and increases the incentive to use limit orders through a decrease of l^0. On the other when λ investors can send market order faster after news arrival which increases the picking-off risk and increases l^0. That is why the effect is overall ambiguous.

We can see that for different values of μ, with the same parametrization as in Figure 3, the effect be different. Figure 4 shows that it is non monotonic for $\mu = 0.1$ and decreasing for $\mu = 0.01$. The effect is changed for μ fairly small. For instance, for $\mu = 1$, α_{eq}^0 increases with λ. But overall the effect of λ is small and even negligible compared to the effect of μ.

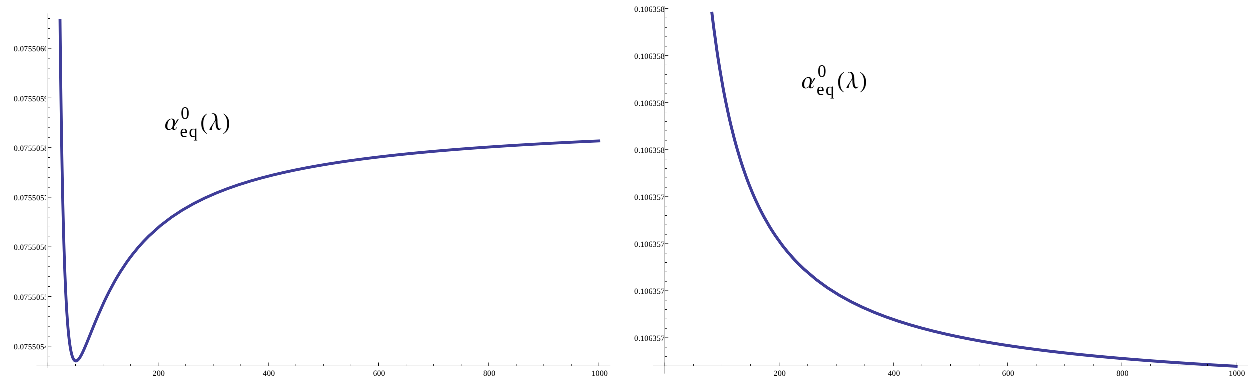


Figure 2.5: Evolution of α_{eq}^0 in function of $\lambda \in [0, 1000]$ (i) for $\mu = 0.1$ and (ii) for $\mu = 0.01$ ($r = 1,\ \rho = 2,\ \Delta = 1,\ \delta = 10,\ \omega = 50$).

The negligible effect of λ on market depth α_{eq}^0, and implicitly on investors' trading strategies, is in line with the intuition that only relative reaction speed matters in a game where faster trader will earn higher profits following the arrival of new information. In our setup an increase of λ corresponds to a reaction speed increase for all investors but without a change in their reaction speed relative to each other. Monitoring abilities are still homogenous in the investors population.

2.8.2 Duration between news arrival and price change

The duration between news arrival and the change in trading prices in the limit order book is the duration of the transition phase.

$$T = \frac{1}{\rho + \lambda} \ln(1 + 2\alpha_{eq}^0)$$

Prediction 2.2. *For values of λ and μ not too low, the duration of the transition phase T*

- *decreases with the frequency of news arrival, μ, or the news surprise, ω.*
- *decreases with the monitoring rate λ when μ is not too low*

Prices in the limit order book reflect the new common value of the asset once there is no arbitrage opportunity left, that is to say that the initial liquidity supply offering this arbitrage opportunity has disappeared. The populations of potential arbitrageurs is fixed. This is the group of non-owner if the common value goes up and the group of owner if the common value goes down. Then the instantaneous flow of directional market orders aiming to profit from the arbitrage opportunity is proportional to the rate at which this population monitors the market, $\lambda + \rho$, and it does not depend on the parameters that rule the dynamic of the common value. μ and ω only affect the initial liquidity supply α^0_{eq}. The effect of an increase of μ or ω is mechanical since it decreases the initial liquidity supply that is consumed and removed faster in the transition phase. The monitoring intensity λ affects both the liquidity supply and the flow of directional orders. Since the liquidity supply is a bounded function of λ, the monitoring rate ends up reducing the duration of the transition phase for λ "not too low".

One should notice that as soon as the asset holding constraint on investors is independent of μ or ω during the transition phase the flow of directional market orders remains independent of these parameters and the result would holds. The "zero or one unit" assumption is not key here. However there is a need for a holding constraint otherwise investors could send infinitely large orders and consume instantaneously the liquidity supply.

The fact that λ and μ or ω are independent is less obvious. As for model of limited attention allocation, investors could decide of their λ depending on the asset characteristics. This calls for further extension of the model to endogenize the choice of λ.

2.8.3 Order flow decomposition in the price discovery process

Corollary 2.1. *In the transition phase the numbers of limit orders executed and limit orders cancelled are*

$$LOE = \frac{\ln(1 + 2\alpha^0_{eq})}{2}, \quad LOC = \left[\alpha^0_{eq} - \frac{\ln(1 + 2\alpha^0_{eq})}{2}\right]$$

Moreover the ratio of limit order cancellations over executed market orders is increasing with respect to α^0:

$$\frac{\partial}{\partial \alpha^0} \frac{LOC}{LOE} > 0$$

Prediction 2.3. *In the transition phase, the ratio of limit order cancellations over limit order execution*

- *decreases with the frequency of news arrival, μ, or the news surprise, ω.*
- *increases with the monitoring rate λ when μ is not too low*

The mechanism behind this result is the following. As mentioned in the previous subsection the flow of directional market orders during the transition phase is proportional to $\lambda + \rho$ and does not depend on α^0_{eq}, μ or ω. On the side of the liquidity supply the instantaneous probability for an investor to cancel her limit order is also $(\lambda + \rho).dt$. The mass of these investors is α^0_{eq} at the beginning of the transition phase and equal to $D(t)$ afterwards. Then the flow of limit order cancellations at t during the transition phase is $(\lambda + \rho)D(t).dt$ which depends positively on α^0_{eq}. If initially α^0_{eq} is increased, at each point in time the flow of limit order cancellations is increased whereas the flow of directional market orders is the same. This explains why the share of limit order cancellation increases. However the transition phase lasts longer which explains why the number of market orders during the transition phase increases as well.

2.9 Conclusion

This paper models the effect of limited attention on market reaction to unscheduled news arrival. Investors' limited capacity of attention restricts their ability to monitor the market. This imperfect market monitoring delays price adjustments following news arrival. Because of their imperfect ability to monitor news, investors take the risk of being picked-off when they supply liquidity with limit orders. When the frequency of news arrival increases, this picking-off risk is amplified and consequently (i) the liquidity supply declines, (ii) prices adjust faster following news arrival and (iii) the share of limit orders cancellations in the price discovery process decreases.

Chapter 3

High Frequency Trading, Market Efficiency and Mini Flash Crashes

3.1 Introduction

Financial markets stability is important to attract investors. Instability of securities prices can blur investors' expectations on trading conditions and ultimately may discourage them to trade in traditional exchanges. In recent years, anecdotes from the trading industry have reported the rise of a new kind of market destabilizing event: the «mini flash crash»[1] (see Appendix B.1 for a list of past mini flash crashes). A mini flash crash can be defined as a sudden sharp change in the price of a stock followed by a very quick reversal (see Figure 1). The increasing frequency of these events has been interpreted as a symptom of market fragility and ascribed to High Frequency Trading (HFT henceforth). Meanwhile recent papers (e.g, Hendershott, Jones and Menkveld (2011), Hendershott and Riordan (2013), Brogaard, Hendershott and Riordan (2012) or Chaboud, Chiquoine, Hjalmarsson, and Vega (2009)) support that HFT has a positive effect on market quality and market informational efficiency. Through which channel HFT could generate mini flash crashes? Can financial markets become jointly more efficient and less stable under HFT actions?

To address these questions we develop a theory of mini flash crashes. Our theory is based

[1] In reference to The Flash Crash of May 6th, 2010. See «The Flash Crash, in Miniature» in the New York Times, http://www.nytimes.com/2010/11/09/business/09flash.html. Nanex Research also reports mini flash crashes among other market anomalies, http://www.nanex.net/FlashCrash/OngoingResearch.html

on the idea that there is a trade-off between speed and precision in the acquisition of information. The new market environment enables traders to become HFT and react much faster to news of any type at the cost of the precision of the information used by traders. We embed this idea in a two periods trading model in which strategic event traders can either become HFT's by investing in a fast technology, which allows them to acquire a noisy signal on the asset fundamental value and trade at period 1, or not invest and only trade at period 2 with a perfect signal on the asset fundamental value. The speed advantage of HFT in information acquisition has been studied by Foucault, Hombert and Rosu (2012) but it does not incorporate the possibility for interpretation mistakes of new information. In Foucault, Hombert and Rosu (2012), as well as in our paper, the speed advantage of HFT is modelled as an ability to trade a period ahead of other investors. It can be seen as reduced form for a high market monitoring ability of HFT. It could be modelled in a framework where investors have imperfect market monitoring capacity, as in some recent papers (e.g. Biais, Hombert and Weill (2013), Foucault, Kadan and Kandel (2013), Pagnotta and Philippon (2012)). Literature on HFT considers that HFT may also benefit from a superior information processing ability that is different from the speed advantage. To address this feature, theoretical papers, as Biais, Foucault and Moinas (2013), model HFT as traditional informed traders (as in Glosten (1985)).

We find that an increase in HFT activity, due, for instance, to a lower cost of the fast technology, increases the likelihood of a price reversal, between period 1 and 2. Price reversals arise when HFT discovered that the signal, they acquired at period 1, was wrong and then decide to correct their trade at period 2. It generates opposite trading patterns across the two periods and, possibly, price swings. The price impact of HFT's at period 1 is proportional to the number of HFT's. Hence the likelihood of a price reversal increases when the number of HFT's increases. Even though more HFT's implies more reversals, it also improves market informational efficiency. While these two implications seem to be contradictory, the presence of high frequency traders allows to faster integrate information into prices when period 1's signal is informative.

The novelty of this paper is to introduce a trade-off, between speed and precision for information processing, to explain why HFTs may trade on noise and generate price reversal.

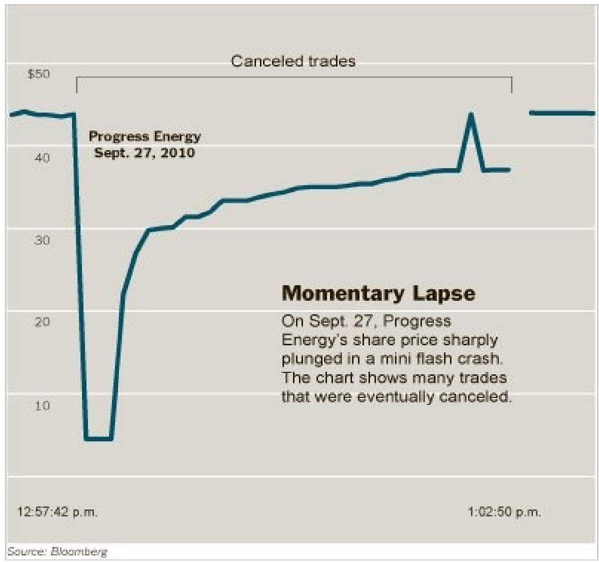


Figure 3.1: «On September 27, 2010, Progress Energy, Inc shares plummeted 90% in a matter of seconds for no apparent reason. The harrowing decline was a consequence of a mini flash crash; a much smaller version than the crash in May...», http://sslinvest.com/news/mini-flash-crash-september-27th-sends-pgn-shares-down-90.

There are theories of short-term speculators who ex-ante rationally coordinate to trade on noise (see Froot, Scharfstein and Stein (1992)). We think of trading on noise as a risk that is revealed ex-post. In our view, it stems from competition in event trading when traders have the possibility to react to news, or any other relevant signals, in a very short amount of time. At first glance, acceleration at which new information is processed, and used, should result in faster integration of information into prices. However, this acceleration also increases the risk that event traders base their trading decision on less accurate signals. Of course, to mitigate this risk, event traders may decide to check news accuracy (for instance using human intervention). But doing so, they take the risk of losing a profit opportunity by reacting too late to informative signals. Hence competition among traders may push them to react too quickly at the expense of the precision of the information on which they trade, and may lead to subsequent trade and price reversals.

Price reversals may alternatively come from overconfident investors who overreact to private signals, as in Daniel, Hirshleifer and Subrahmanyam (1998), which generates price correction following public revelation of information. In this setup, price reversals are systematic. Price returns become negatively auto-correlated and predictable. The market is inefficient as opposed to what we find with our model, which does not generate negative auto-correlation. Moreover our setup generates complete price reversals in the sense that prices can switch back to their original value when HFTs react to noise, while, here, the price reversal is a partial correction of a previous excessive price change that had the right direction nonetheless.

Trade reversal, by traders who obtain information faster than others, can arise when a trader benefit from an anticipated noisy information leakage about future public announcement, as in Brunnermeier (2005). It enables the trader, first, to privately acquire a noisy signal on a short term component of the asset value, and trade on it; and, second, to know by how much his price impact was driven by noise, after the short term component is publicly announced. Hence he benefits from an informational advantage even after the announcement. He profits from it by partially reversing the share of his past trade that happened to be driven by the noisy component of his signal. However this trade reversal is compensated by a opposite trades of other strategic traders, which makes the implications for the price dynamics unclear, contrary to our setup. In Brunnermeier's paper, the market is efficient since prices reflect all publicly available information. However the introduction of information leakage has mitigated effects on efficiency, contrary to our model in which more HFT increases market efficiency.

Event trading strategies may be as diverse as the spectrum of relevant information in a particular market. HFTs look for financial news or informative market patterns that they can process and trade on as fast as possible. The source of the imperfect information precision may be either endogenous or exogenous. On the one hand it can be endogenous because algorithms send orders based on the interpretations of these events. Every thing else equal, the faster is the algorithm reaction, the less accurate is the interpretation. Consequently High Frequency Traders face this trade-off when they calibrate their algorithm. In the other hand the process of new information release may in itself be an exogenous source of noise for new information. If there is a slight chance that some pieces of news are false, High Frequency Traders have to decide whether they take the risk to react immediately to the piece of news or wait for a correction. For instance the following anecdote illustrates, in a rather extreme way, the false news issue. On Monday Sept. 8, 2008, the stock price of

United Airlines dropped to \$3 a share from nearly \$12 in about fifteen minutes. Then the price bounced back at \$11 at the end of the Tuesday session. The cause of these swings was an old article about United Airlines' 2002 bankruptcy-court filing that mistakenly appeared on September 8, 2008 as a seemingly new headline on Google's news service.

Chapter 3 is organized as follows. Section 3.2 presents the setup and assumptions of the model. Section 3.3 gives the equilibrium trading strategies at each period. Section 3.4 gives the equilibrium population sizes and their profits. Section 3.5 describes the equilibrium price dynamics. Section 3.6 provides results on market informational efficiency. Section 3.7 concludes.

3.2 Model setup

We consider a three periods model ($t = 1,\ 2,\ 3$) of trading in a financial asset. The financial asset is traded at periods $t = 1$ and $t = 2$ and pays-off at $t = 3$. There are three types of market participants: a competitive market maker (as in Kyle (1985)), liquidity traders and event traders. To participate at period $t = 1$, event traders can pay a cost C, the cost of being fast. They observe a signal, that is either informative (equal to the asset pay-off) or noise, and trade. At period $t = 2$, all event traders participate. They learn the true nature of the previous signal, as well as its value, and trade. If the signal is informative, they can trade on this information. If the signal is noise, they learn that the expected value of the asset is still equal to its ex-ante value and they can take advantage of an eventual mis-pricing. Figure 2 summarizes the timing of market participants' decisions. Using this setting we plan to analyze whether a decrease in the cost of being fast impairs or ameliorates price discovery and price stability.

Asset Value. The pays-off of the asset is $V \in \{0, 1\}$ at period 3, with equal probabilities. Thus, prior to the beginning of the game, the asset expected value is $\mathbb{E}[V] = 1/2$.

Liquidity trading. At each trading period, some liquidity traders send orders for reasons left exogenous to our model. The order flow of liquidity traders is random and follows a uniform distribution on the interval $[-Q, Q]$. We note ϕ the density function associated

$$\phi(x) = \frac{1}{2Q} \times \mathbb{I}_{\{x \in [-Q,Q]\}}.$$

In the following of the paper, we call $\tilde{l}_1$ and $\tilde{l}_2$ the liquidity traders order flows at periods $t = 1$ and $t = 2$.

Market making. At each trading period, a risk-neutral and competitive market maker observes the aggregate order flow, that is the sum of event traders and liquidity traders orders. The aggregate order flows are noted $\tilde{Q}_1$ at period $t = 1$ and $\tilde{Q}_2$ at period $t = 2$. The market maker executes these order flows at the following trading prices at periods 1 and 2,

$$P_1 = \mathbb{E}[V|\tilde{Q}_1] = Pr[V = 1|\tilde{Q}_1]$$

and

$$P_2 = \mathbb{E}[V|\tilde{Q}_2, \tilde{Q}_1] = Pr[V = 1|\tilde{Q}_2, \tilde{Q}_1],$$

as in Glosten and Milgrom (1985) or Kyle (1985).

Event trading. There is a continuum, $[0, A]$, of competitive event traders. Ex-ante an event traders $i \in [0, A]$ chooses either to pay the cost C, so that he can trade at periods $t = 1$ and $t = 2$, or to not pay the cost, and trades only at period $t = 2$. The mass of «fast» traders who pay C is α. Traders in the interval $[0, \alpha]$ trade at periods $t = 1$ and $t = 2$ while traders in the interval $[\alpha, A]$ trade only at $t = 2$. We endogenize α in section 4.4.

Assumption 3.1. *In each trading period, trader i sends and order with size X_i that is bounded, $X_i \in [-1, 1]$.*

Assumption 3.2. *The mass of event traders is smaller than the size of liquidity trading*

$$A < Q$$

Assumptions 1 and 2 make sure that the impact of event traders is limited. Assumption 1 corresponds to a limit to arbitrage constraint for each event traders. For some reasons (cost

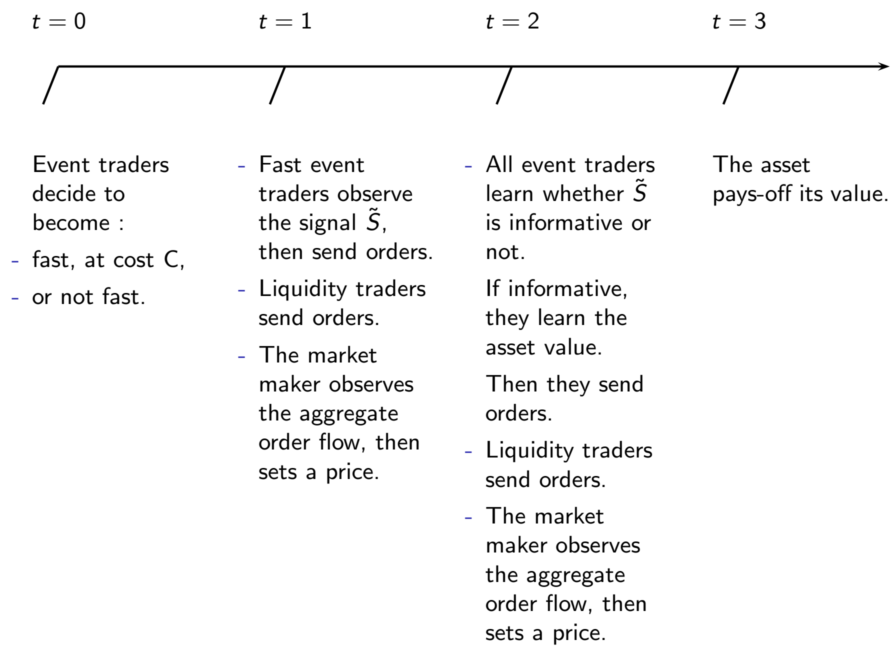


Figure 3.2: Timing of market participants' decisions.

of capital, aversion for residual risk,...etc) an event trader cannot trade unbounded quantities. Assumption 2 states that the mass of event traders is limited as well. The number of trading desks dedicated to event trading is less than some measures of global trading activity (here the volume of liquidity trading Q).

Solution concept. In this setup, we look for a Nash equilibrium of the game defined by strategies of the event traders and the pricing strategies of the market makers: $P_1(Q_1)$ and $P_2(Q_1, Q_2)$.

Given the symmetry of the model, event traders, who trade at the same period, will obtain the same profit in equilibrium. These profits are equal to π_1 at $t = 1$ and π_2 at $t = 2$.

Determination of α. At equilibrium, the endogenous value of α can be either an interior solution, $0 < \alpha < A$, or a corner solution, $\alpha = 0$ or $\alpha = A$. We reach an interior solution when there exists an $\alpha \in (0, A)$ such that each event trader is indifferent between trading and not trading at $t = 1$. In this case the equilibrium value of α is defined as the solution of the following equation,

$$\pi_1(\alpha) - C = 0.$$

If each event trader strictly prefer to trade at $t = 1$, even when all event traders participate at $t = 1$, then the equilibrium value is a corner solution, $\alpha = A$. And the following condition is satisfied,

$$\pi_1(A) - C > 0.$$

If each event trader strictly prefer to not trade at $t = 1$, even when no event trader participate at $t = 1$, then the equilibrium value is a corner solution, $\alpha = 0$. And the following condition is satisfied,

$$\pi_1(0) - C < 0.$$

The determination of α does not involve the profit of trading at $t = 2$, π_2, because each investor is infinitesimal. His isolated trades do not move prices. Hence his participation at $t = 1$ does not affect his profit at $t = 2$. That is why his decision to participate at $t = 1$ only depends on the profit of trading at $t = 1$, π_1.

Discussion of the «cost of being fast». We can interpret the «cost of being fast» as the cost of investing in the technology that allows to react fast to a market event. This technology is associated with the risk of reacting to noise. This cost can also be interpreted as an opportunity cost. Event traders must decide on which type of events or on which market they want to allocate their computing capacity. A third possibility is to interpret this cost as a technological barrier. When the technology to implement «fast event trading» strategies was not available, the cost for trading at period 1 was infinite $C = \infty$. Now that the technology is available, the cost is finite.

The rise of High Frequency Trading. With this framework we can address some of the effects High Frequency Trading activities development which stemmed from the improvement of computing technologies. In our model, it corresponds to a drop of the cost of being fast. Jointly with the development of algorithmic strategies, the number of high frequency trading desk has tremendously increased. In our model we can estimate the consequence of this expansion through an increase of the mass of event traders, A, relative to the global trading activity, Q.

3.3 Information, trading strategies, pricing policy and profits

In this section we take α as exogenously given. The mass of event traders at $t = 2$ is set equal to β. In section 4.4 we endogenize $\alpha = \alpha_{eq}$ and we set $\beta = A$ since, under our assumptions, all event traders can participate at period 2 at no cost.

3.3.1 Period 1

Event traders information at $t = 1$. At $t = 1$ an event trader $i \in [0, \alpha]$ observes a signal $\tilde{S} \in \{0, 1\}$, the same for all event traders, defined as:

$$\tilde{S} = \tilde{U}\tilde{V} + (1 - \tilde{U})\tilde{\epsilon}$$

with $\tilde{U} \in \{0, 1\}$ with respective probabilities $Pr[U = 0] = 1-\delta$ and $Pr[U = 1] = \delta$, $\tilde{\epsilon} \in \{0, 1\}$ with $Pr[\epsilon = 0] = Pr[\epsilon = 1] = 1/2$. The signal distribution conditional on the asset value is the following,

$$Pr[S = 1|V = 1] = Pr[S = 0|V = 0] = \frac{1 + \delta}{2}$$
$$Pr[S = 1|V = 0] = Pr[S = 0|V = 1] = \frac{1 - \delta}{2}$$

The unconditional distribution is symmetric $Pr[S = 1] = Pr[S = 0] = 1/2$. If we call

$$\mu(s) = Pr[V = 1|S = s] = \frac{Pr[S = s|V = 1]Pr[V = 1]}{Pr[S = s]}$$

the updated value of the asset for the event trader in period 1 is either

$$\mu(1) = \frac{1 + \delta}{2} > \frac{1}{2} \text{ or } \mu(0) = \frac{1 - \delta}{2} < \frac{1}{2}.$$

The profit of an event trader, who observed the signal and traded a quantity X in period 1, is

$$X(\mu(s) - \mathbb{E}[P_1|S = s])$$

Event traders strategy. At $t = 1$ an event trader $i \in [0, \alpha]$ sets her strategy based on the signal value, $X_i(s)$.

Proposition 3.1. *At $t = 1$, the unique equilibrium strategy for an event trader is to buy if the signal is high and to sell if the signal is low,*

$$X(s = 1) = 1, \quad X(s = 0) = -1.$$

Proof. The market maker infers the probabilities of the posterior expected asset value to be either $\mu(1)$ or $\mu(0)$, from the order flow, and set the price as the expected value of the asset. Since the market makers information set is smaller the one of event traders, the expected asset value conditional on the market maker's information set is necessarily $\mu(1) \geq \mathbb{E}[V|\tilde{Q}_1] \geq \mu(0)$. Moreover, in some state of the world the information set of the market makers is strictly inferior than the one of event traders. In these cases $\mu(1) > \mathbb{E}[V|\tilde{Q}_1] > \mu(0)$. This implies that the strategy $X(s = 1) = 1$ and $X(s = 0) = -1$ strictly dominates all other strategies. □

Depending on the realization of the event traders signal, the aggregate order flow in the first period is as follows,

$$\tilde{Q}_1 = \tilde{l}_1 + \alpha \text{ if } S = 1$$
$$\tilde{Q}_1 = \tilde{l}_1 - \alpha \text{ if } S = 0$$

Market maker's pricing policy. At $t = 1$, the market marker observes the aggregate order flow $\tilde{Q}_1$ and sets a price equal to $\mathbb{E}[V = 1|\tilde{Q}_1] = Pr[V = 1|\tilde{Q}_1]$.

Proposition 3.2. *At $t = 1$ the competitive market maker pricing policy is*

$$P_1(q) = Pr[V = 1|\tilde{Q}_1 = q] = \frac{(1 + \delta)\phi(q - \alpha) + (1 - \delta)\phi(q + \alpha)}{\phi(q - \alpha) + \phi(q + \alpha)} \times \frac{1}{2}$$

or equivalently

$$P_1(q) = \begin{cases} \frac{1-\delta}{2} & \textit{if } q \in [-Q-\alpha, -Q+\alpha) \\ \frac{1}{2} & \textit{if } q \in [-Q+\alpha, Q-\alpha] \\ \frac{1+\delta}{2} & \textit{if } q \in (Q-\alpha, Q+\alpha] \end{cases}$$

This pricing policy reflects two cases of the market maker inference problem. When the order flow belongs to the interval $[-Q+\alpha, Q-\alpha]$, the market maker cannot infer the information of event traders and then set the price equal to the ex-ante expected value of the asset. When the order flow belongs $[-Q-\alpha, -Q+\alpha)$ (resp. $(Q-\alpha, Q+\alpha]$), the market maker can infer that the event trader signal is negative (resp. positive) since the order flow takes «extreme» negative (resp. positive) values.

Event traders profit. With the event traders strategy and the market maker pricing policy, we can compute the the profit of an event trader.

Proposition 3.3. *At $t = 1$, with a mass α of event traders, the expected profit of one event trader is*

$$\pi_1(\alpha) = \frac{\delta}{2} \times \frac{Q-\alpha}{Q}.$$

3.3.2 Period 2

Event traders information. At $t = 2$, event traders observe the realization of $\tilde{S}$ and $\tilde{U}$. In other words they know if the first period signal was informative or not, and know the value of the asset, if the signal was informative. They are perfectly informed if $U = 1$. At $t = 2$, the profit of an event trader who trades a quantity X is

$$\begin{array}{ll} X(V - \mathbb{E}[P_2|V]) & \text{if } \ U = 1 \\ X\left(\frac{1}{2} - \mathbb{E}[P_2|S, U = 0]\right) & \text{if } \ U = 0 \end{array}$$

Event trader strategy. At $t = 2$, an event trader $i \in [0, \beta]$ sets her strategy conditional on the realizations of the random variables, $\tilde{S}$ and $\tilde{U}$, and on the realization of the order flow at $t = 1$, q_1 (or equivalently the previous price P_1). Her strategy is a function $X_i(s, u, q_1)$.

Proposition 3.4. *At $t = 2$, if $U = 1$, the unique equilibrium strategy is to buy if the asset value is high and to sell if the asset value low,*

$$X(1, 1, q_1) = 1, \quad X(0, 1, q_1) = -1.$$

If $U = 0$, if the $t = 1$ order flow is $q_1 \in (Q - \alpha, Q + \alpha]$, which implies that the signal was high $S = 1$ and that the asset price went up in the first trading period, then the unique equilibrium strategy is to sell,

$$X(1, 0, q_1) = -1.$$

If $U = 0$, if the $t = 1$ order flow is $q_1 \in [-Q - \alpha, -Q + \alpha)$, which implies that the signal was low $S = 0$ and that the asset price went down in the first trading period, then the unique equilibrium strategy is to buy,

$$X(0, 0, q_1) = 1.$$

If $U = 0$, if the $t = 1$ order flow is $q_1 \in (Q - \alpha, Q + \alpha]$, which implies that asset price remained equal to its the ex-ante expected value, $1/2$, in the first trading period, then any strategy $X(s, 0, q_1) \in [-1, 1]$ is an equilibrium strategy. However the only strategy that is robust to the introduction of a small trading cost is to not trade,

$$X(s, 0, q_1) = 0.$$

Hence there is a unique equilibrium strategy that is robust to the introduction of a small trading cost. In the following of the paper we consider this strategy.

The equilibrium strategy of event traders at $t = 2$ is again very intuitive. They trade in the same direction that first period event traders, if the signal is indeed informative, in order to take advantage of the remaining profit opportunity. In the other case, they trade in the opposite direction that first period event traders in order to take advantage of a previous erroneous price change. Depending on the realizations of the underlying random variables, the aggregate order flow in the second period is as follows,

- if $U = 1$ the aggregate order flow is equal to

$$\tilde{Q}_2 = \tilde{l}_2 + \beta \text{ if } V = 1,$$
$$\tilde{Q}_2 = \tilde{l}_2 - \beta \text{ if } V = 0,$$

- if $U = 0$ the aggregate order flow is equal to

$$\tilde{Q}_2 = \tilde{l}_2 + M_0(q_1) \quad \text{with} \quad M_0(q_1) = \begin{cases} \beta & \text{if } q_1 \in [-Q-\alpha, -Q+\alpha) \\ 0 & \text{if } q_1 \in [-Q+\alpha, Q-\alpha] \\ -\beta & \text{if } q_1 \in (Q-\alpha, Q+\alpha] \end{cases}$$

Market maker's pricing policy. At $t = 2$, the market marker observes the aggregate order flow $\tilde{Q}_2$, has already observed $\tilde{Q}_1$, and sets a price equal to $\mathbb{E}[V = 1|\tilde{Q}_2, \tilde{Q}_1] = Pr[V = 1|\tilde{Q}_2, \tilde{Q}_1]$.

Proposition 3.5. *At $t = 2$ the competitive market maker pricing policy is*

$$P_2(q_2, q_1) = Pr[V = 1|\tilde{Q}_2 = q_2, \tilde{Q}_1 = q_1] =$$
$$\frac{\delta\phi(q_1-\alpha)\phi(q_2-\beta) + \frac{1-\delta}{2}[\phi(q_1-\alpha)+\phi(q_1+\alpha)]\phi(q_2 - M_0(q_1))}{\delta[\phi(q_1-\alpha)\phi(q_2-\beta)+\phi(q_1+\alpha)\phi(q_2+\beta)] + (1-\delta)[\phi(q_1-\alpha)+\phi(q_1+\alpha)]\phi(q_2-M_0(q_1))}$$

or equivalently,

$$\textit{if}\ \ q_1 \in [-Q-\alpha, -Q+\alpha),\ \ P_2(q_2,q_1) = \begin{cases} 0 & \textit{if } q_2 \in [-Q-\beta, -Q+\beta) \\ \frac{1-\delta}{2} & \textit{if } q_2 \in [-Q+\beta, Q-\beta] \\ \frac{1}{2} & \textit{if } q_2 \in (Q-\beta, Q+\beta] \end{cases}$$

$$\textit{if}\ \ q_1 \in [-Q+\alpha, Q-\alpha],\ \ P_2(q_2,q_1) = \begin{cases} 0 & \textit{if } q_2 \in [-Q-\beta, -Q) \\ \frac{1-\delta}{2-\delta} & \textit{if } q_2 \in [-Q, -Q+\beta) \\ \frac{1}{2} & \textit{if } q_2 \in [-Q+\beta, Q-\beta] \\ \frac{1}{2-\delta} & \textit{if } q_2 \in (Q-\beta, Q] \\ 1 & \textit{if } q_2 \in (Q, Q+\beta] \end{cases}$$

$$\textit{if}\ \ q_1 \in (Q-\alpha, Q+\alpha],\ \ P_2(q_2,q_1) = \begin{cases} \frac{1}{2} & \textit{if } q_2 \in [-Q-\beta, -Q+\beta) \\ \frac{1+\delta}{2} & \textit{if } q_2 \in [-Q+\beta, Q-\beta] \\ 1 & \textit{if } q_2 \in (Q-\beta, Q+\beta] \end{cases}$$

Remark 3.1. *If $M_0 \neq 0$ for $q_1 \in [-Q+\alpha, Q-\alpha]$, the pricing policy is*

$$P_2(q_2,q_1) = \begin{cases} 0 & \textit{if } q_2 \in [-Q-\beta, -Q+M_0) \\ \frac{1-\delta}{2-\delta} & \textit{if } q_2 \in [-Q+M_0, -Q+\beta) \\ \frac{1}{2} & \textit{if } q_2 \in [-Q+\beta, Q-\beta] \\ \frac{1}{2-\delta} & \textit{if } q_2 \in (Q-\beta, Q+M_0] \\ 1 & \textit{if } q_2 \in (Q+M_0, Q+\beta] \end{cases}$$

Event traders profit. At $t = 2$, event traders can make profit because they trade on an informative signal that had not been completely integrated into prices in the first period either because the market maker has not inferred its value or, if he had, because the signal was noisy which has left some profit opportunity once the informative nature of the signal is known by event traders. They also make profit by correcting pricing errors due to event traders at $t = 1$ when the signal is revealed as uninformative.

Proposition 3.6. *With a mass β of event traders in the second period and a mass α of event traders in the first period, the expected profit of an event trader at $t = 2$ is*

$$\pi_2(\alpha, \beta) = \frac{\delta}{2} \times [\frac{Q-\alpha}{Q} \times (\frac{Q-\beta}{Q} + \frac{1-\delta}{2-\delta}\frac{\beta}{Q}) + (1-\delta)\frac{\alpha}{Q}\frac{Q-\beta}{Q}]$$

Remark 3.2. *Even if $M_0 \neq 0$ for $q_1 \in [-Q+\alpha, Q-\alpha]$, the event traders expected profit at $t = 2$ remains unchanged.*

3.4 Equilibrium

Equilibrium populations. With given masses, α and β, of event traders at periods $t = 1$ and $t = 2$ we obtained the equilibrium strategies of event traders, the market maker pricing policies and the event traders profits. Here we endogenize these masses of populations.

Proposition 3.7. *The masses of event traders at periods $t = 1$ and $t = 2$ are uniquely defined. Trading at $t = 2$ is costless for event traders, hence $\beta = A$. The equilibrium mass of event traders who pay the cost of being fast, C, to trade at $t = 1$ is $\alpha_{eq} \in [0, A]$, equal to*

$$\textit{if } C > \frac{\delta}{2}, \;\; \alpha_{eq} = 0,$$

$$\textit{if } \frac{\delta}{2} \geq C \geq \frac{\delta}{2}\left(1 - \frac{A}{Q}\right), \;\; \alpha_{eq} = Q\left(1 - \frac{2}{\delta}C\right)$$

$$\textit{if } C < \frac{\delta}{2}\left(1 - \frac{A}{Q}\right), \;\; \alpha_{eq} = A$$

For $C > \frac{\delta}{2}$ and $C < \frac{\delta}{2}\left(1 - \frac{A}{Q}\right)$, the equilibrium mass of event traders is a corner solution. In the first case, the fast technology is too expensive compared to the profit that a trader alone could extract with, thus no event trader decides to become fast. In the second case, the fast technology is cheap enough so that when all traders invest, they still make a positive profit.

For $\frac{\delta}{2} \geq C \geq \frac{\delta}{2}\left(1 - \frac{A}{Q}\right)$ there is an interior solution. the equilibrium mass of event traders is such that the marginal trader is indifferent between investing and not investing in the fast technology. The indifference condition is

$$\pi_1(\alpha_{eq}) = \frac{\delta}{2} \times \frac{Q - \alpha_{eq}}{Q} = C.$$

Equilibrium profit of an event trader. At equilibrium, the expected profit of an event trader, π, is equal to

$$\text{if } C > \frac{\delta}{2}, \quad \pi = \pi_2(0, A)$$

$$\text{if } \frac{\delta}{2} \geq C \geq \frac{\delta}{2}\left(1 - \frac{A}{Q}\right), \quad \pi = \pi_2(\alpha_{eq}, A)$$

$$\text{if } C < \frac{\delta}{2}\left(1 - \frac{A}{Q}\right), \quad \pi = \pi_2(A, A) + \pi_1(A) - C$$

Proposition 3.8. *The expected profit of an event trader depends on the «cost of being fast» as follows,*

$$\textit{if } C > \frac{\delta}{2}, \quad \frac{\partial \pi}{\partial C} = 0$$

$$\textit{if } \frac{\delta}{2} \geq C \geq \frac{\delta}{2}\left(1 - \frac{A}{Q}\right), \quad \frac{\partial \pi}{\partial C} > 0$$

$$\textit{if } C < \frac{\delta}{2}\left(1 - \frac{A}{Q}\right), \quad \frac{\partial \pi}{\partial C} = -1.$$

The expected profit of an event trader is decreasing with respect to the mass of event traders,

$$\frac{\partial \pi}{\partial A} < 0.$$

The effect of the mass of event traders, A, on the profit of each trader is very intuitive. When the mass of event traders increases, the market maker can infer more easily event traders' private information which reduces the profit of one of these traders.

When the cost of being fast is high enough so that only a fraction of event traders trade at $t = 1$, a drop of this cost generates a decline of the expected profit of event trader. Because of competition, her expected profit at $t = 1$ is nil and her expected profit at $t = 2$ declines. The increasing number of event traders at $t = 1$ helps the market maker to infer event traders' private information which reduces, on average, the level of information asymmetry at $t = 2$.

Aggregate profit of event traders. At equilibrium the aggregate profit of event traders, Π, is equal to

$$\text{if } C > \frac{\delta}{2}, \ \ \Pi = A\pi_2(0, A)$$

$$\text{if } \frac{\delta}{2} \geq C \geq \frac{\delta}{2}\left(1 - \frac{A}{Q}\right), \ \ \Pi = A\pi_2(\alpha_{eq}, A)$$

$$\text{if } C < \frac{\delta}{2}\left(1 - \frac{A}{Q}\right), \ \ \Pi = A\pi_2(A, A) + A\pi_1(A) - AC$$

Proposition 3.9. *The aggregate profit of event traders depends on the «cost of being fast» in the same way that for the profit of one event trader,*

$$\frac{\partial \Pi}{\partial C} = A\frac{\partial \pi}{\partial C}.$$

The effect of the «cost of being fast» on event traders aggregate profit is mitigated. A reduction of this cost has negative effect on their profit when it is too expensive so that only a fraction of event traders can invest in the fast technology. It has a positive effect when the it is cheap enough so that all event traders invest in the technology and when a reduction of its cost does not drive more competition in event trading.

To draw clearer results, we can study the variation of event traders' aggregate profit when the possibility of trading in the first period become possible for free, from $C = \infty$ to $C = 0$. This could be due to a contemporaneous development of information technologies or a change in the market structure that allows for electronic trading.

Proposition 3.10. *The variation of the aggregate profit of event traders when the «cost of being fast» drops from $C = \infty$ to $C = 0$ is equal to*

$$A\pi_2(0, A) - (A\pi_2(A, A) + A\pi_1(A)).$$

It can be either positive or negative. There is a threshold $H(\delta) \in [0, 1]$ such that

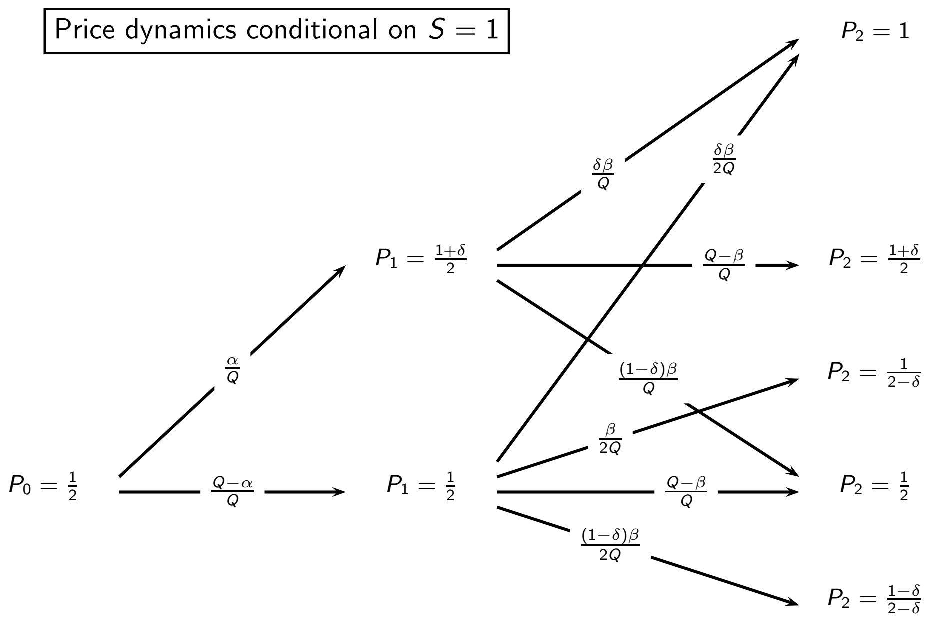
$$\textit{If } \frac{A}{Q} < H(\delta), \quad A\pi_2(0, A) - (A\pi_2(A, A) + A\pi_1(A)) < 0$$

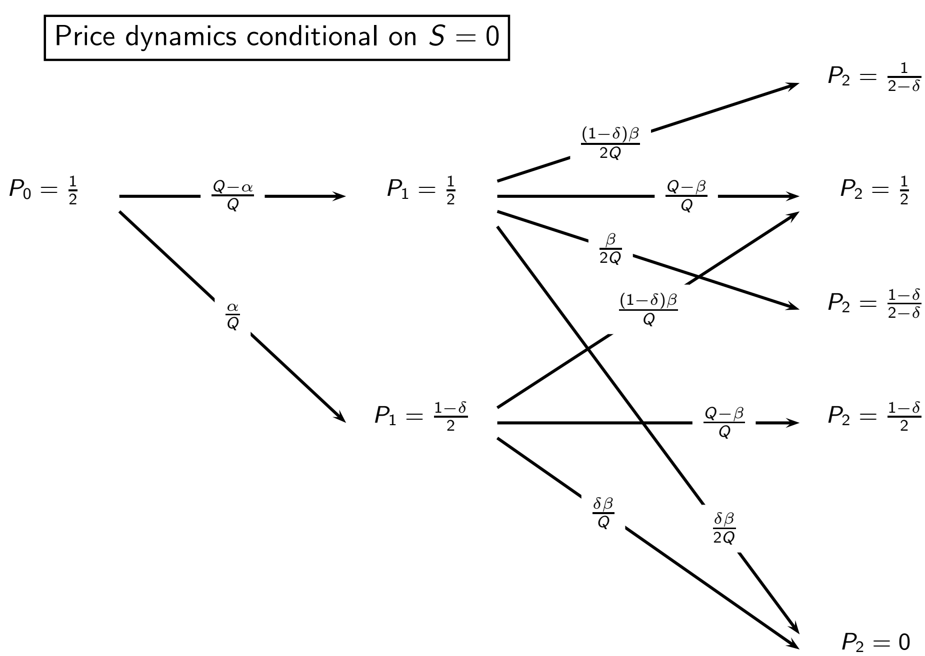
$$\textit{If } \frac{A}{Q} > H(\delta), \quad A\pi_2(0, A) - (A\pi_2(A, A) + A\pi_1(A)) > 0$$

Given that liquidity traders are not strategic and that the market maker sets competitive prices, the trading game is a fixed-sum game in which the gross aggregate profit of event traders corresponds to the implicit trading cost, for liquidity traders, due to information asymmetry. When the technology to trade fast on market events become available ($C = \infty$ to $C = 0$), these implicit trading costs decline when competition among event traders is high ($\frac{A}{Q} > H(\delta)$), which helps market prices to reflect private information. In this case, HFT is beneficial to liquidity traders.

3.5 Equilibrium price dynamics

In this section, we study the asset price equilibrium dynamics. We graphically represent, with probability trees, the different paths that the asset price can take, conditional on $S = 1$ or $S = 0$. We decide to represent conditional dynamics for the sake of graphics clarity. To obtain the unconditional equilibrium tree, one can multiply all branches' weight by 1/2 in each tree and merge the two trees by adding the weights on the matching branches. In these price dynamics, we consider an additional period, $t = 0$, in which the signal is not available yet and thus the asset is priced at its ex-ante expected value $\mathbb{E}[V] = 1/2$.





Mini flash crash. In our setup, a mini flash crash corresponds to a price path in which, at $t = 1$, the price drops, from $\frac{1}{2}$ to $\frac{1-\delta}{2}$ with probability $\frac{\alpha}{2Q}$, or jumps, from $\frac{1}{2}$ to $\frac{1-\delta}{2}$ with probability $\frac{\alpha}{2Q}$, and then, at $t = 2$, the price switches back to $\frac{1}{2}$ with probability $\frac{(1-\delta)\beta}{2Q}$ (see Figure 3.2). The magnitude of a mini flash crash, M_{crash}, is equal to difference between prices at $t = 1$ and $t = 2$,

$$M_{crash} = \frac{\delta}{2}.$$

Proposition 3.11. *The probability of a mini flash crash is equal to*

$$P_{crash} = \frac{\alpha(1-\delta)\beta}{2Q^2} = \frac{\alpha_{eq}(1-\delta)A}{4Q^2}.$$

The probability of a mini flash crash increases when the mass of event traders increases or when the «cost of being fast» declines,

$$\frac{\partial P_{crash}}{\partial A} > 0, \quad \frac{\partial P_{crash}}{\partial C} < 0.$$

The probability of a mini flash crash depends on the signal's precision δ as follows,

$$\textit{If } \frac{\delta}{2} \geq C \geq \frac{\delta^2}{2} \quad \textit{or} \quad \frac{\delta}{2} \geq C \geq \frac{\delta}{2}\left(1 - \frac{A}{Q}\right) \geq \frac{\delta^2}{2}, \qquad \frac{\partial P_{crash}}{\partial \delta} \geq 0$$

$$\textit{if } \frac{\delta}{2} \geq \frac{\delta^2}{2} \geq C \geq \frac{\delta}{2}\left(1 - \frac{A}{Q}\right), \qquad \frac{\partial P_{crash}}{\partial \delta} \leq 0,$$

$$\textit{if } \frac{\delta}{2}\left(1 - \frac{A}{Q}\right) \geq C, \qquad \frac{\partial P_{crash}}{\partial \delta} \leq 0.$$

The proposition draws a clear link between a more intense HFT activity and a greater occurrence of mini flash crashes, through the possibility of trading fast with a risk on information precision. When the mass of event traders increases or when the «cost of being fast» decreases, the number of HFTs (event traders who invest in the fast technology and trade at $t = 1$) increases. Hence their price impact at $t = 1$ is higher. The market maker infers more

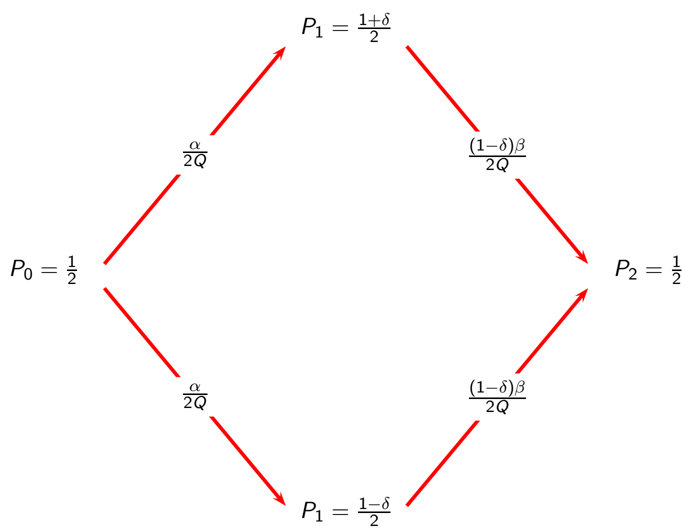


Figure 3.3: Price dynamics of mini flash crashes.

easily the signal value and sets more frequently a price different than 1/2, at $t = 1$. If the signal was noise then the price eventually reaches back its original value at $t = 2$.

The signal precision, δ, has an amplifying role on the crash magnitude, M_{crash}. When δ is high, the expected value of the asset, conditional on the signal realization, is closer from one of the possible realization of the asset value. The difference between the first period price, in the crash path, and the true expected value $\frac{1}{2}$ is thus bigger. And the price reversal is sharper.

The effect of δ on the frequency of mini flash crashes is mitigated. Intuitively when the precision of the signal increases, one would expect that the mini flash crash frequency declines. It is the case when, for instance, the fast technology has a low cost and all event traders participate at $t = 1$. In this case it reduces the probability of a price reversal between $t = 1$ and $t = 2$ while the probability of a price change between $t = 0$ and $t = 1$ is not affected by δ. However in some cases, when the «cost of being fast» is such that only a fraction of event traders invest in the fast technology, a higher δ generates a higher profit for fast traders. As a consequence the probability that P_1 reflects event traders' signal is higher, and the probability to be on the path of a crash increases.

3.6 Market informational efficiency

For a given trading period, we can measure market efficiency with the unconditional expectation of the square difference between the true value of the asset and its market price at the considered trading period, $\mathbb{E}[(\tilde{V} - P_t)^2]$. As the price is set by a competitive market maker and always reflects the expected value of the asset conditional on public information, we can rewrite this efficiency measure as

$$\begin{aligned}
\mathbb{E}[(\tilde{V} - P_1)^2] &= \mathbb{E}[\mathbb{E}[(V - P_1(\tilde{Q}_1))^2|\tilde{Q}_1]] \\
&= \mathbb{E}[\mathbb{E}[(V - \mathbb{E}[V|\tilde{Q}_1])^2]|\tilde{Q}_1] \\
&= \mathbb{E}[\mathbb{V}[V|\tilde{Q}_1]] \\
&= \mathbb{E}[P_1(\tilde{Q}_1)(1 - P_1(\tilde{Q}_1))]
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[(\tilde{V} - P_2)^2] &= \mathbb{E}[\mathbb{V}[V|\tilde{Q}_1, \tilde{Q}_2]] \\
&= \mathbb{E}[P_2(\tilde{Q}_2, \tilde{Q}_1)(1 - P_2(\tilde{Q}_2, \tilde{Q}_1))]
\end{aligned}$$

This measure corresponds to the expectation of the asset pay-off's conditional variance, at a given trading period. The smaller are these variances, the more efficient is the market.

Proposition 3.12. *Given the masses of strategic traders at periods 1 and 2, the variances at period 1 and 2 are*

$$\mathcal{V}_1 = \mathbb{E}[(\tilde{V} - P_1)^2] = \frac{1}{4} \times \left[1 - \delta^2 \frac{\alpha}{Q}\right],$$

$$\mathcal{V}_2 = \frac{1}{4}\frac{\alpha}{Q}(1-\delta)\left[(1+\delta)\frac{Q-\beta}{Q} + \frac{\beta}{Q}\right] + \frac{1}{4}\frac{Q-\alpha}{Q}\left[2\frac{1-\delta}{2-\delta}\frac{\beta}{Q} + \frac{Q-\beta}{Q}\right],$$

that can be rewritten as

$$\mathcal{V}_1 = \frac{\delta}{2}\pi_1(\alpha) + \frac{1-\delta^2}{4} \quad \textit{and} \quad \mathcal{V}_2 = \frac{1}{2}\pi_2(\alpha, \beta) + \frac{1-\delta}{4}.$$

Corollary 3.1. *At equilibrium, period 1 and 2 variances are equal to*

- *if* $C > \frac{\delta}{2}$

$$\mathcal{V}_1 = \frac{1}{4} \quad and \quad \mathcal{V}_2 = \frac{1}{2}\pi_2(0, A) + \frac{1-\delta}{4}$$

consequently

$$\frac{\partial \mathcal{V}_1}{\partial A} = 0, \quad \frac{\partial \mathcal{V}_2}{\partial A} < 0$$

.

- *if* $\frac{\delta}{2} \geq C \geq \frac{\delta}{2}\left(1 - \frac{A}{Q}\right)$

$$\mathcal{V}_1 = \frac{\delta}{2}C + \frac{1-\delta^2}{4} \quad and \quad \mathcal{V}_2 = \frac{1}{2}\pi_2(\alpha_{eq}, A) + \frac{1-\delta}{4}$$

consequently

$$\frac{\partial \mathcal{V}_1}{\partial A} = 0, \ \frac{\partial \mathcal{V}_1}{\partial C} > 0, \ \frac{\partial \mathcal{V}_2}{\partial A} < 0, \ \frac{\partial \mathcal{V}_2}{\partial C} > 0$$

- *if* $C < \frac{\delta}{2}\left(1 - \frac{A}{Q}\right)$

$$\mathcal{V}_1 = \frac{\delta}{2}\pi_1(A) + \frac{1-\delta^2}{4} \quad and \quad \mathcal{V}_2 = \frac{1}{2}\pi_2(A, A) + \frac{1-\delta}{4}$$

consequently

$$\frac{\partial \mathcal{V}_1}{\partial A} < 0, \ \frac{\partial \mathcal{V}_1}{\partial C} = 0, \ \frac{\partial \mathcal{V}_2}{\partial A} < 0, \ \frac{\partial \mathcal{V}_2}{\partial C} = 0$$

First, let's notice that the risk, for fast traders, to trade on uninformative on signals, does not affect the efficient nature of the price process. The competitive market maker sets the asset price equal to the conditional expected value of its pay-off. The asset price process is a martingale.

However informational efficiency and the existence mini flash crashes seem to contradict each other. It would be the case if the effect of HFT was to add mini flash crashes in the range of possible price dynamics, leaving the conditional distribution of other price dynamics unchanged. Then mini flash crashes would be equivalent to systematic price reversals. Said differently, HFT would generate price swings that could be anticipated. Negative auto-correlation would arise which would contradict the martingale property of an efficient price dynamic. The effect of the competitive pricing, rather than a mechanical pricing policy, is that mini flash crashes also affect the levels and likelihoods of other price dynamics.

In our model, prices remain efficient. Moreover informational efficiency improves when High Frequency Trading activity increases, whether it is triggered by a reduction of the «cost of being fast» or by an increase of the mass of event traders, because it increases their price impact at $t = 1$ and the probability that their private information is revealed. The effect is the same at $t = 2$ when the mass of event traders, A, increases. It implies that mini flash crashes are compensated by «momentum» to keep and enhance efficiency.

The effect of HFT is, ultimately, to incorporate into prices the signal they observe at $t = 1$, with probability $\frac{\alpha_{eq}}{Q}$. Following such a price change, at $t = 2$, the price will «crash» back to the initial expected value of the asset, $1/2$, with probability $\frac{(1-\delta)A}{Q}$, it will stay at its $t = 1$ level,with probability $\frac{Q-A}{Q}$, or it will keep on converging towards the true asset value, with probability $\frac{\delta A}{Q}$. The former possible dynamic, a price «momentum», can neither be anticipated. The frequencies of crashes and momentum increase under the action of HFT. The joint increases of these two types of price dynamic compensate each other so that they cannot be anticipated, and therefore it keeps market efficiency.

Our model implication is in line with empirical findings on the effect of High Frequency Trading on market quality. Moreover we find that informational efficiency improvement goes along with a higher frequency of mini flash crashes and «momentum», which suggests that HFT allows for a faster integration of new information into prices, but in a less stable way. Our theory of mini flash crashes can reconcile the empirical findings on the beneficial effect of HFT on market efficiency with the concerns raised by the increasing frequency of price instability bursts, such as mini flash crashes.

3.7 Conclusion

High Frequency Trading can enhance market efficiency by processing and incorporating faster new information into prices, as shown by several empirical works. However HFT has also triggered the emergence of mini flash crashes, punctual events of intense price instability. We show with a model that, when the high information processing speed of HFT is associated with a risk of trading on noise due to erroneous information interpretation, HFT generates price reversals unrelated to underlying change of the asset value. We also show that while HFT increases the frequency of mini flash crashes, it simultaneously improves market efficiency.

Conclusion

I briefly conclude this dissertation by emphasizing some directions in which I would like to take my research further.

There is still a lot to be done to understand the effects of algorithmic trading in financial markets. This broad question can be tackled by studying the implications of heterogeneity in attention capacities among investors, since algorithmic trading can be seen as a way to alleviate limited attention constraints. The model, I introduced in the first and second chapters of this dissertation, offers a way to theoretically pursue this investigation. In my framework, attention capacity heterogeneity could be modelled by assuming that a fraction of investors, algorithmic traders, can monitor the market more intensively than the others and thus can react faster following news arrival. This idea is already at the heart of a joint on going project, with Profs. Terrence Hendershott and Ryan Riordan, in which we plan to investigate the informational advantage of investors using Algorithmic Trading (AT) technologies over the rest of the market. We model this advantage by assuming that AT's have a higher information processing speed. Our goal is to derive empirical implications related to investors' trading decisions and to test these implications using a database provided by Deutsche Boërse which contains all AT orders in DAX stocks over 13 days.

Economic agents have cognitive limitations and, in particular, they have a limited capacity of attention. In the case of financial markets, investors must be attentive to all sort of information or market events, so that they can take trading decisions. This fact raises the fundamental question of how investors allocate their attention capacity across markets and information sources. Provided that investors make this choice rationally, solving for models of optimal attention allocation can help deriving testable implications on, for instance, the joint dynamics of asset prices, or the difference of trading strategies across markets. The theoretical framework, I introduced in first and second chapters, is a natural modelling platform to study how investors allocate their attention capacity across markets and the consequences on asset prices joint dynamics. Theoretical research on this question has been recently growing. These models usually consider limited attention as a constraint on the precision of signals on asset pay-offs. Investors allocate their attention by choosing the variances of the signals they acquire. The way I model limited attention is different and corresponds to the frequency at which investors monitor markets. To investigate the endogenous level of investors' attention, I would extend my model by allowing investors to decide on how they allocate a finite monitoring capacity across markets with different characteristics.

Appendix A

Appendix to chapters 1 and 2

In this appendix I provide all the proofs of my job market paper ”Limited Attention and News Arrival in Limit Order Markets”. I recall the model assumptions that are slightly more general than the ones of the paper.

A.1 Model Setup

A.1.1 Preferences dynamic

The economy is constituted by a continuum of investors $[0, L]$. They are all risk neutral and infinitely lived, with time preferences determined by a constant discount rate $r > 0$ which is also equal to the risk-free interest rate.

The asset supply is equal to $S = sL$ where $0 < s < 1$ that is initially distributed among investors who can hold either 1 unit or 0 unit of this asset.

As in Duffie, Garleanu and Pedersen, an investor is characterized by whether she owns the asset and by an intrinsic type that is ”high” or ”low”. A high type owner enjoys a utility flow of v by owning this asset whereas a low type owner receives a utility flow of $v - \delta$. Then she can consume in the present or save this utility flow for future consumption.

Between time t and time $t + dt$ a high type investor can ”suffer” from an idiosyncratic shock and switches to the low type with a probability $\rho^{-}.dt$. And reciprocally a low type investor switches to the high type with a probability $\rho^{+}.dt$.

Given the previous assumptions any investor must have a type in the set $\{ho, hn, lo, ln\}$ (h: high, l: low, o: owner, n: non-owner). And we can divide the mass of investors in 4 populations: L_{ho}, L_{hn}, $, L_{lo}$, L_{ln}. They verify the equations

$$L_{ho} + L_{hn} + L_{lo} + L_{ln} = L$$

$$L_{ho} + L_{lo} = sL, \quad L_{hn} + L_{ln} = (1 - s)L$$

However the number of possible types can be greater than the four aforementioned by taking into account their limit order submission status. Indeed in a limit order book an owner can either be out of the market or have an order in the order book at any price reachable. As well for a non-owner. This setting can generate many subtypes of the previous types. Let's call $\mathcal{T}$ the set of all possible types. However to precise the status of an investor in the limit order book we will precise if she is *out* or has sent a limit order. For instance a type ln can be $ln - out$ or $ln - B$ with a limit order at price B. Symmetrically a type lo can be $lo - out$ or $lo - A$ with a limit order at price A.

A.1.2 Limit order market

Trading takes place through a limit order market. Such a market is characterized by a price grid at which limit orders can be sent. We assume that this price grid is bounded which is reasonable since, if the asset utility flow is bounded, trading cannot happen at some high prices (the corresponding strategies would be strictly dominated by a strategy where investors don't trade). The investor choices are the following

- owners can : do nothing and remain an owner, send sell limit order in the order book and remain an owner until his/her order is executed, send a sell market order and become a non-owner or cancel a previous sell limit order.
- non-owners can : do nothing and remain a non-owner, send buy limit order in the order book and remain a non-owner until his/her order is executed, send a buy sell market order and become an owner or cancel a previous buy limit order.

This defines the action set of an investor,

$$A = \{\text{do nothing, market order or marketable limit orders, limit orders at all other prices}\}$$

Assumption A.1. *In the limit order book, limit order are executed following a "Pro-rata matching" execution rule. Given that all limit orders are of size 1 it means that at one price (A or B), at any time, all limit orders have the same probability of execution.*

Assumption A.2. *At time t an investor can make a trading choice any time he contacts the market*

- *if she monitors the market, which occurs with probability $\lambda.dt$*
- *if she "suffers" from an idiosyncratic shock (as described above), which occurs with probability $\rho^{+/-}.dt$*

Proposition A.1. *A limit order market is in a steady state when the displayed depth in the order book and the different order flows do not change over time.*

A steady state equilibrium, for this limit order market definition, is necessarily a one-tick market in the sense that

all sell limit orders are sent at the price A generating a depth D_A, and we don't observe any other sell limit order at higher prices than A

all buy limit orders are sent at the price B generating a depth D_B, and we don't observe any other buy limit order at lower prices than B

$A > B$ and there is no accessible exchange price between A and B. $A - B = \Delta$ is equal to the tick of the market.

Proof. In the steady state, a sell limit order that is send at a higher price than A will never be executed because the liquidity supply at the price A keep the same positive value and is never consumed. Symmetrically for buy limit orders. Then there is no incentive to send limit orders further from the best quotes A and B.

If there was a reachable price $A < P < B$ it would be profitable to send limit orders at P rather than market orders. □

A.1.3 Value function and equilibrium concept

An investor is choosing her actions at each random time when she is contacting the market. The strategy σ of an agent is a function

$$\begin{aligned} \sigma : & H \times \Xi \times [0, \infty) \to A \\ & (h, \xi, t) \mapsto a \end{aligned}$$

Where any element Ξ is the set of all potential state variables. If $\xi \in \Xi$ then $\xi = (\theta, v, S)$ where $\theta \in \mathcal{T}$ is a type, v is the fundamental value of the asset and S is the aggregate state of the limit order book, that is to say the bid and ask prices and all the depths at these prices. H is the set of all possible histories of actions and observations of an investor:

$$H = \{h \in (a_{t_1}, \ldots, a_{t_n}, \xi_{t_1}, \ldots, \xi_{t_n}, t_1, \ldots, t_n) \in A^n \times \Xi^n \times [0, \infty)^n, t_1 < \ldots < t_n, n \in \mathbb{N}\}$$

Her strategy, σ, and the strategies of all other investors, Σ, are generating an asset holding process $\eta_t^{\sigma,\Sigma} \in \{0,1\}$, a type process $\theta_t^{\sigma,\Sigma} \in \mathcal{T}$ and a trade price $P_t^{\sigma,\Sigma}$ any time $\eta_t^{\sigma,\Sigma}$ switches from 0 to 1 or conversely.

At time t the value function of an investor playing strategy σ is given by

$$V(h_t, \xi_t, t; \sigma, \Sigma) = \mathbb{E}_t \int_t^{\infty} e^{-r(s-t)} dU_s$$
$$s.t \;\; dU_t = \eta_t^{\sigma,\Sigma}(v - \delta \mathbb{I}_{\{\theta_t^{\sigma,\Sigma} \in\, lo\}})dt - P_t^{\sigma,\Sigma} d\eta_t^{\sigma,\Sigma}$$

The strategy σ is a best response to the other players set of strategies Σ if and only if for all strategy γ,

$$\forall h_t \, \forall \xi_t \, \forall t \;\; V(h_t, \xi_t, t; \sigma, \Sigma) \geq V(h_t, \xi_t, t; \gamma, \Sigma).$$

Lemma A.1. *If the following conditions are fulfilled*

- *for any contacting time this is not optimal to make more than one trade, given* Σ.
- *the sequence of contacting time with the market* $\{\tilde{T}_n\}_{n\in\mathbb{N}}$ *is such that* $\forall t \; \lim_{n\to\infty} \mathbb{P}[\tilde{T}_n < t] = 0$
- *the process* Z_t *that counts the number of contacting time is such that after for any strategy* σ, *any point in time* t *and any type* θ_t *there is a* $M > 0$ *such that*

$$\int_t^{\infty} e^{-r(s-t)} dZ_s < M$$

then a strategy σ *is a best response to* Σ *if and only if the one-shot deviation principle holds. The one-shot deviation principle is verified when the value function generated by* σ *and* Σ, *for all type and time, cannot be improved by deviating from* σ *only at one contacting time with the market and then playing* σ *at any future contacting time.*

Remark A.1. *The first condition of the lemma implies that, at equilibrium, this is not possible to generate an infinite profit. This is actually a natural feature of a limit order book where there are limit orders waiting at the best quotes which prevent anyone to buy and sell with limit orders an infinite numbers of time in an instant. The only way to trade more than once in a limit order market is to send buy and sell market orders in a row which clearly not profitable. When contacting times are "Poissonian" the second and the first point are verified.*

Proof of Lemma A.1

Let's assume σ is not improvable by a one-shot deviation. A profitable deviation γ is such that we can find an history h_t, a state ξ_t and a time t that verify

$$V(h_t, \xi_t, t; \gamma, \Sigma) > V(h_t, \xi_t, t; \sigma, \Sigma)$$

Let's call

$$dK_t^{\sigma} = \eta_s^{\sigma,\Sigma}(v - \delta \mathbb{I}_{\{\theta_s^{\sigma,\Sigma} \in\, lo\}})ds - P_s^{\sigma,\Sigma} d\eta_s^{\sigma,\Sigma}$$

Let's assume that we can find a profitable deviation γ that is different from σ over a finite number of contacting time. The set of these kind of deviations is the non-empty and we can consider a deviation with a minimal number deviation $N > 0$ by definition. It means that after N contacting time σ is played. Now let's consider γ^* the strategy of playing γ over the $N-1$ contacting time and then to play σ. Necessarily we have

$$V(h_t, \xi_t, t; \gamma^*, \Sigma) \leq V(h_t, \xi_t, t; \sigma, \Sigma)$$

Let's call $\{T_n^{\gamma}\}$ the sequence of contacting time generated by γ. By definition γ^* would generate the same distribution for the $N-1$ contacting time. And until T_{N-1}^{γ} it will also generates the same consumption path.

then we can write

$$
\begin{aligned}
V(h_t, \xi_t, t; \gamma^*, \Sigma) &= \mathbb{E}_t\left[\sum_{i=1}^{N-1}\int_{t+T_{i-1}^\gamma}^{t+T_i^\gamma} e^{-r(s-t)}dK_s^\gamma + e^{-r(t+T_{N-1}^\gamma)}V(h_{t+T_{N-1}^\gamma}^\gamma, \xi_{t+T_{N-1}^\gamma}^\gamma, t+T_{N-1}^\gamma; \sigma, \Sigma)\right] \\
\text{(one-shot unimprovability)} &\geq \mathbb{E}_t\left[\sum_{i=1}^{N-1}\int_{t+T_{i-1}^\gamma}^{t+T_i^\gamma} e^{-r(s-t)}dK_s^\gamma + e^{-r(t+T_{N-1}^\gamma)}V(h_{t+T_{N-1}^\gamma}^\gamma, \xi_{t+T_{N-1}^\gamma}^\gamma, t+T_{N-1}^\gamma; \gamma, \Sigma)\right] \\
&\geq V(h_t, \xi_t, t; \gamma, \Sigma)
\end{aligned}
$$

There is a contradiction.

Now let's consider any profitable deviation γ.

First let's show that for any strategy and any sequence of contacting time such that $\lim_{n\to\infty}\mathbb{P}[T_n < t] = 0$ we have

$$\lim_{n\to\infty}\mathbb{E}_t\left[e^{-r(t+T_n)}V(h_{t+T_n}^\gamma, \xi_{t+T_n}^\gamma, t+T_n; \gamma, \Sigma)\right] = 0$$

The assumption of the lemma gives us that $V(h_t, \xi_t, t; \gamma, \Sigma)$ is bounded whatever the strategy and that the bound does not depend on the strategy. Let's call $f(T) = e^{-r(t+T)}V(h_{t+T}^\gamma, \xi_{t+T}^\gamma, t+T; \gamma, \Sigma)$. We can find A such that $f < A$ because f is bounded. Let $\epsilon > 0$, let $T > 0$ such that $\mathbb{P}[T_n < T] < \epsilon$ and such that $f(T') < \epsilon$ for all $T' > T$. Then

$$\mathbb{E}_t[f(T_n)] = \mathbb{E}_t[f(T_n)|T < T_n]\mathbb{P}[T_n < T] + \mathbb{E}_t[f(T_n)|T_> n]\mathbb{P}[T_n > T] < A\epsilon + \epsilon$$

and we have the convergence result

let $\beta = V(h_t, \xi_t, t; \gamma, \Sigma) - V(h_t, \xi_t, t; \sigma, \Sigma) > 0$. Because of the previous convergence result we can find N_0 such that for $N > N_0$

$$\mathbb{E}_t\left[\sum_{i=1}^{N}\int_{t+T_{i-1}^\gamma}^{t+T_i^\gamma} e^{-r(s-t)}dK_s^\gamma\right] - \mathbb{E}_t\left[\sum_{i=1}^{N}\int_{t+T_{i-1}^\sigma}^{t+T_i^\sigma} e^{-r(s-t)}dK_s^\sigma\right] > \frac{\beta}{2}$$

we can then consider γ_N the strategy of playing γ until T_N and σ afterwards. Then for N big enough,

$$|\mathbb{E}_t[e^{-r(t+T_N)}V(h_{t+T_N}^{\gamma_N}, \xi_{t+T_N}^{\gamma_N}, t+T_N; \gamma_N, \Sigma)] - \mathbb{E}_t[e^{-r(t+T_N)}V(h_{t+T_N}^\sigma, \xi_{t+T_N}^\sigma, t+T_N; \sigma, \Sigma)]| < \frac{\beta}{4}$$

and then $V(h_t, \xi_t, t; \gamma_N, \Sigma) > V(h_t, \xi_t, t; \sigma, \Sigma)$ This is a contradiction with the first result of the proof.

A.2 Limit order market in steady state w/o fundamental uncertainty

The first type of equilibria I am implementing is a steady state equilibrium when the fundamental value of the asset does not change. In these equilibria the four types $\{ho, hn, lo, ln\}$ are sufficient to describe the micro dynamic of the order book. This imply that investor are pooled by type in the order book.

A.2.1 equilibrium conjecture

Conjecture A.1. *I am looking for the set of parameters such that the equilibrium strategies are*

- *ho: cancel any sell limit order and keep the asset's unit*
- *hn: send a buy limit or market order (indifferently)*
- *lo: send a sell limit or market order (indifferently)*
- *ln: cancel any buy limit order and stay without the asset*

*This conjectured equilibrium is a **Markov Perfect Equilibrium** with state variable $\xi_t = (\theta_t, v_t, S_t)$.*

In this potential equilibrium the populations L_{ho} and L_{ln} are not present in the limit order book. As soon as a ho type switches to a lo type she instantaneously contacts the market: either she instantaneously switches to a ln type by sending a sell market order or stay a lo type by sending a sell limit order. Symmetrically as soon as a ln type switches to a hn type she instantaneously contacts the market: either she instantaneously switches to a ho type by sending a buy market order or stay a hn type by sending a buy limit order. This is straightforward that in a steady state equilibrium, $D_A = L_{lo}$ and $D_B = L_{hn}$.

A.2.2 Steady state populations

In a steady state the aggregate populations stay at the same level. Then the flows of population from high type to low type and from low type to high type must be equal to each other, $\rho^-(L_{ho}+L_{hn}) = \rho^+(L_{lo}+L_{ln})$. Then we must have

$$L_{ho} + L_{hn} = \frac{\rho^+}{\rho^+ + \rho^-}L, \; L_{lo} + L_{ln} = \frac{\rho^-}{\rho^+ + \rho^-}L$$

Proposition A.2. *In a steady state equilibrium there is an $\alpha \in \mathbb{R}$ such that the different types of population verify*

$$\begin{aligned} L_{ho} &= (s-\alpha)L \\ L_{hn} &= \left(\frac{\rho^+}{\rho^+ + \rho^-} - s + \alpha\right)L \\ L_{lo} &= \alpha L \\ L_{ln} &= \left(\frac{\rho^-}{\rho^+ + \rho^-} - \alpha\right)L \end{aligned}$$

Conditions on the possible values taken by α are:

$$s - \alpha \geq 0, \; \frac{\rho^+}{\rho^+ + \rho^-} - s + \alpha \geq 0, \; \alpha \geq 0, \frac{\rho^-}{\rho^+ + \rho^-} - \alpha \geq 0$$

Assumption A.3. *The asset supply is less than the high type population in steady state and more than the low type population, $\frac{\rho^-}{\rho^+ + \rho^-} \leq s \leq \frac{\rho^+}{\rho^+ + \rho^-}$, and the high valuation population is bigger than the low valuation population, $\rho^+ > \rho^-$.*

Given this assumption the condition on α is now $0 \leq \alpha \leq \frac{\rho^-}{\rho^+ + \rho^-}$

Remark A.2. *A simplification of the problem parametrization will be the **symmetric case** where $\rho^+ = \rho^-$ and $s = \frac{1}{2}$*

Proof of proposition A.2

The steady state is defined by the system

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_{ho} \\ L_{hn} \\ L_{lo} \\ L_{ln} \end{pmatrix} = \begin{pmatrix} \frac{\rho^+}{\rho^+ + \rho^-} L \\ \frac{\rho^-}{\rho^+ + \rho^-} L \\ sL \\ (1-s)L \end{pmatrix}$$

First it is to check that $L_{ho} = sL,\ L_{hn} = (\frac{\rho^+}{\rho^+ + \rho^-} - s)L,\ L_{lo} = 0,\ L_{ln} = \frac{\rho^-}{\rho^+ + \rho^-} L$ is a particular solution of this system. The general space of solutions of this system is equal to

$$\begin{pmatrix} sL \\ (\frac{\rho^+}{\rho^+ + \rho^-} - s)L \\ 0 \\ \frac{\rho^-}{\rho^+ + \rho^-} L \end{pmatrix} + \ker \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} sL \\ (\frac{\rho^+}{\rho^+ + \rho^-} - s)L \\ 0 \\ \frac{\rho^-}{\rho^+ + \rho^-} L \end{pmatrix} + Vect\left[\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}\right]$$

A.2.3 Micro-level dynamic of the limit order book

In the equilibrium conjecture we need that hn and lo types are indifferent between limit and market order so that we observe in the same time flows of liquidity demand and supply that make the state of the limit order book sustainable and steady. Given that we look for a steady state equilibrium, the flows must be steady as well. That is why a share $0 < m_A < 1$ of the population of hn type contacting the market at t sends a buy market order and the rest of them send a buy limit order. For the same reason a share $0 < m_B < 1$ of the population of lo type contacting the market at t sends a sell market order and the rest of them send a sell limit order.

Ask Side. At time t, on the ask side of the market the depth is constantly equal to $D_A = L_{lo}$ and the order flows sustaining this steady state are

- **Outflow:** people switching from lo to ho, $\rho^+ L_{lo}.dt$, lo type cancelling their sell limit order to send a sell market order, $m_B \lambda L_{lo}.dt$, execution of buy market orders send by hn type contacting the market $m_A(\lambda L_{hn} + \rho^+ L_{ln}).dt$.
- **Inflow:** people switching from ho to lo type sending a sell limit order, $(1 - m_B)\rho^- L_{ho}.dt$

The steady state condition is then:

$$\rho^+ L_{lo} + m_A(\lambda L_{hn} + \rho^+ L_{ln}) + m_B(\lambda L_{lo} + \rho^- L_{ho}) = \rho^- L_{ho}$$

Bid Side. At time t, on the ask side of the market the depth is constantly equal to $D_B = L_{hn}$ and the order flows sustaining this steady state are

- **Outflow:** people switching from hn to ln, $\rho^- L_{hn}.dt$, hn type cancelling their sell limit order to send a sell market order, $m_A \lambda L_{hn}.dt$, execution of sell market orders send by lo type contacting the market $m_B(\lambda L_{lo} + \rho^- L_{ho}).dt$.
- **Inflow:** people switching from ln to hn type sending a sell limit order, $(1 - m_A)\rho^+ L_{ln}.dt$

The steady state condition is then:

$$\rho^+ L_{hn} + m_B(\lambda L_{lo} + \rho^- L_{ho}) + m_A(\lambda L_{hn} + \rho^+ L_{ln}) = \rho^+ L_{ln}$$

A.2.4 Value functions

The conjectured equilibrium generates the following system of equations defining the different value functions for each investor types.

type ho. An investor of type ho keeps its asset until he/she switches to the lo type.

$$V_{ho} = v.dt + (1 - r.dt)[(1 - \rho^-.dt)V_{ho} + \rho^-.dtV_{lo}] \Longleftrightarrow (r + \rho^-)V_{ho} = \rho^- V_{lo} + v$$

type ln. An investor of type ln does not sen any order until he/she switches to the hn type.

$$V_{ln} = (1 - r.dt)[(1 - \rho^+.dt)V_{ln} + \rho^+.dtV_{hn}] \Longleftrightarrow (r + \rho^+)V_{ln} = \rho^+ V_{hn}$$

type hn. An investor of type hn send a buy market or limit order indifferently. To be consistent with the fact that a share $m_A < 1$ of the hn type on the market are sending a market order, we can consider that they are playing a mixed strategy between market and limit order with probability m_A for the market order.

As we assume the steady state, being a hn type during a certain amount of time means being on the bid side of the limit order book during this time. At time t, the outflow of the bid side due to market order is $m_B(\lambda L_{lo} + \rho^- L_{ho}).dt$, then the probability for a limit order to be executed at t is $l_B.dt$ with

$$l_B = \frac{m_B(\lambda L_{lo} + \rho^- L_{ho})}{L_{hn}}$$

Indeed because the priority rule in the limit order book is ”Pro-Rata”, the instantaneous probability of execution is equal to the ratio of the market order flow over the depth of the limit order book.

Given that these investor are indifferent between sending a limit or a market order the two associated value function must be equal:

$$V_{hn} = V_{ho} - A$$

and

$$V_{hn} = (1 - r.dt)[(1 - \rho^-.dt - \lambda.dt - l_B.dt)V_{hn} + \rho^-.dtV_{ln} + \lambda.dt(m_A(V_{ho} - A) + (1 - m_A)V_{hn}) + l_B.dt(V_{ho} - B)]$$
$$\Longleftrightarrow (r + \rho^- + l_B + m_A\lambda)V_{hn} = \rho^- V_{ln} + m_A\lambda(V_{ho} - A) + l_B(V_{ho} - B)$$

leading to, given the indifference result

$$(r + \rho^- + l_B)(V_{ho} - A) = \rho^- V_{ln} + l_B(V_{ho} - B)$$

type lo. An investor of type lo send a sell market or limit order indifferently. To be consistent with the fact that a share $m_B < 1$ of the lo type on the market are sending a market order, we can consider that they are playing a mixed strategy between market and limit order with probability m_B for the market order.

For the same reason as for type hn, at time t, the outflow of the ask side due to market order is $m_A(\lambda L_{hn} + \rho^+ L_{ln}).dt$, then the probability for a limit order to be executed at t is $l_A.dt$ with

$$l_A = \frac{m_A(\lambda L_{hn} + \rho^+ L_{ln})}{L_{lo}}$$

Because these investor are indifferent we have

$$V_{lo} = V_{ln} + B$$

and

$$(r + \rho^+ + l_A + m_B\lambda)V_{lo} = v - \delta + \rho^+ V_{ho} + m_B\lambda(V_{ln} + B) + l_A(V_{ln} + A)$$

leading to

$$(r + \rho^+ + l_A)(V_{ln} + B) = v - \delta + \rho V_{ho} + l_A(V_{ln} + A)$$

Proposition A.3. *Solving the system implied by these equations gives*

$$
\begin{aligned}
l_B &= \frac{v - rA - \rho^-(A-B)}{A-B} \\
l_A &= \frac{rB - \rho^+(A-B) - (v-\delta)}{A-B} \\
V_{ho} &= \frac{1}{r}\frac{\rho^+}{\rho^+ + \rho^-}(v - \rho^-(A-B)) + \frac{1}{r+\rho^+ + \rho^-}\frac{\rho^-}{\rho^+ + \rho^-}(v + \rho^+ A + \rho^- B)) \\
V_{ln} &= \frac{1}{r}\frac{\rho^+}{\rho^+ + \rho^-}(v - \rho^-(A-B)) - \frac{1}{r+\rho^+ + \rho^-}\frac{\rho^+}{\rho^+ + \rho^-}(v + \rho^+ A + \rho^- B)) \\
V_{hn} &= V_{ho} - A \\
V_{lo} &= V_{ln} + B
\end{aligned}
$$

Corollary A.1. *The following condition is sufficient for l_A and l_B to be take positive values*

$$\frac{v}{r} - \frac{\delta}{r}\frac{r+\rho^-}{r+\rho^+ + \rho^-} \leq B < A \leq \frac{v}{r} - \frac{\delta}{r}\frac{\rho^-}{r+\rho^+ + \rho^-}$$

This implies that $\delta - (r + \rho^+ + \rho^-)\Delta > 0$.

Proof of proposition A.3

First, by replacing V_{hn} by $V_{ho} - A$ and V_{lo} by $V_{ln} + B$ this is easy to obtain that

$$
\begin{aligned}
(r+\rho^-)V_{ho} - \rho^- V_{ln} &= v + \rho^- B \\
(r+\rho^+)V_{ln} - \rho^+ V_{ho} &= -\rho^+ A
\end{aligned}
$$

and then to get the expression of V_{ho} and V_{ln}.
Replacing V_{ln} by $V_{lo} - B$ and V_{ho} by $V_{hn} + A$ in the equation of indifference between market and limit orders we obtain

$$
\begin{aligned}
(r + \rho^- + l_B)(V_{ho} - A) &= \rho^-(V_{lo} - B) + l_B(V_{ho} - B) \\
(r + \rho^+ + l_A)(V_{ln} + B) &= v - \delta + \rho^+(V_{hn} + A) + l_A(V_{ln} + A)
\end{aligned}
$$

which gives

$$
\begin{aligned}
v + \rho^- B - (r+\rho^-)A &= l_B(A-B) \\
-\rho^+ A + (r+\rho^+)B - (1-\delta) &= l_A(A-B)
\end{aligned}
$$

Proof of Corollary A.1

l_A and l_B must be positive numbers. Then the following conditions must hold:

$$A \leq \frac{v + \rho^- B}{r+\rho^-}, \quad B \geq \frac{v - \delta + \rho^+ A}{r + \rho^+}$$

Sufficient conditions for these conditions to hold is that

$$A \leq \frac{v}{r+\rho^-} + \frac{\rho^-}{r+\rho^-}\left[\frac{v-\delta}{r+\rho^+} + \frac{\rho^+}{r+\rho^+}A\right] \Leftrightarrow A \leq \frac{v}{r} - \frac{\delta}{r}\frac{\rho^-}{r+\rho^+ + \rho^-}$$

and then

$$B \geq \frac{v-\delta}{r+\rho^+} + \frac{\rho^+}{r+\rho^+}\left[\frac{v}{r+\rho^-} + \frac{\rho^-}{r+\rho^-}B\right] \Leftrightarrow B \geq \frac{v}{r} - \frac{\delta}{r}\frac{r+\rho^-}{r+\rho^+ + \rho^-}$$

A.2.5 Equilibrium outcome

The two steady state equations can be rewritten as

$$\rho^- L_{hn} + l_B L_{hn} + l_A L_{lo} = \rho^+ L_{ln}$$

$$\rho^+ L_{lo} + l_A L_{lo} + l_B L_{hn} = \rho^- L_{ho}$$

Proposition A.4. *The conjecture equilibrium is indeed an equilibrium for a set of parameters value (close enough to the symmetric case). The equilibrium populations are characterized by the value*

$$\alpha_{eq} = \frac{\rho^- s - l_B(\frac{\rho^+}{\rho^+ + \rho^-} - s)}{\rho^+ + \rho^- + l_A + l_B}$$

The aggregate properties of the limit order market in this steady state equilibrium is completely described by the equilibrium value α_{eq} since it gives the equilibrium populations, the depths and the aggregate order flow in the limit order book.

Proof of Proposition A.4

Given the equilibrium strategy σ (and Σ generated by σ) the random times at which an investor i is going to contact the market are generated by the Poisson process corresponding to her valuation of the asset (intensity ρ), the process of his market monitoring(intensity λ) and the processes of his limit order executions if she sends a limit order at A or B (intensity l_A and l_B). In this framework this is obvious that assumptions of lemma 1 can apply.

In this purpose I consider "one-shot" deviations where an investor can deviate from the assumed strategy when she contacts the market but not considering to deviate from the assumed strategy in her future decision. By backward induction, checking if this kind deviation is not profitable is also checking that a finite number of deviation in a row is not profitable as well.

type hn. A type hn has only one way to deviate which is to stay out of the market and not sending a buy market or limit order. Actually we could consider a change of the mixed strategy parameter between limit and market orders but given that the investor is infinitesimal this deviation does not change the value of these two actions. Then he/she is indifferent between all mixed strategies which makes this deviation not profitable.
The value of not trading when contacting the market is

$$\begin{aligned} V_{hn-out} &= (1 - r.dt)[(1 - \rho^-.dt - \lambda.dt)V_{hn-out} + \rho^-.dtV_{ln} + \lambda.dtV_{hn} \\ \Longleftrightarrow (r + \rho^- + \lambda)V_{hn-out} &= \rho^- V_{ln} + \lambda V_{hn} \leq (\rho^- + \lambda)V_{hn} \end{aligned}$$

if $V_{ln} \leq V_{hn}$ which is true if and only if $V_{ln} \geq 0$ because $(r + \rho^+)V_{ln} = \rho^+ V_{hn}$. In this case this deviation is not profitable. It is easy to verify that $V_{ln} \geq 0$ iff

$$A \leq \frac{1 + \rho^- B}{r + \rho^-}$$

which is verified as soon as $l_B > 0$.

type lo. As in the previous case the only deviation we have to consider is the case where a type lo in contact with the market decides to keep the asset. In this case the value function is

$$\begin{aligned} V_{lo-out} &= (v - \delta).dt + (1 - r.dt)[(1 - \rho^+.dt - \lambda.dt)V_{lo-out} + \rho^+.dtV_{ho} + \lambda.dtV_{lo} \\ \Longleftrightarrow (r + \rho^+ + \lambda)V_{lo-out} &= v - \delta + \rho^+ V_{ho} + \lambda V_{lo} \end{aligned}$$

As we know that $(r+\rho^+)(V_{lo}-B)=\rho^+(V_{ho}-A)$ we get

$$(r+\rho^+ +\lambda)V_{lo-out} = v-\delta+\rho^+(A-B)-rB+(r+\rho^+ +\lambda)V_{lo}$$

Then the deviation is not profitable iff

$$v-\delta+\rho^+(A-B)-rB \leq 0 \Longleftrightarrow B \geq \frac{1-\delta+\rho^+ A}{r+\rho^+}$$

which is verified as soon as $l_A > 0$.

type ho**.** A type ho can deviate in two different ways:

- by sending a sell market order. The value function associated is then $V = V_{hn}+B = V_{ho}-A+B < V_{ho}$. This is not profitable.
- by sending a sell limit order. In this case the value function is given by

  $$\begin{aligned} &V_{ho-A} = v.dt+(1-r.dt)[(1-\rho^-.dt-\lambda.dt-l_A.dt)V_{ho-A}+\rho^-.dtV_{lo}+\lambda.dtV_{ho}+l_A.dt(V_{hn-out}+A)] \\ \Longleftrightarrow &(r+\rho^- +\lambda+l_A)V_{ho-A} = v+\rho^- V_{lo}+(\lambda+l_A)V_{ho}+l_A(V_{hn-out}-V_{hn}) < (r+\rho^- +\lambda+l_A)V_{ho} \end{aligned}$$

  This deviation is not profitable.

type ln**.** A type ln can deviate in two different ways:

- by sending a buy market order. The value function associated is then $V = V_{lo}-A = V_{ln}+B-A < V_{ln}$. This is not profitable.
- by sending a buy limit order. In this case the value function is given by

  $$\begin{aligned} &V_{ln-B} = (1-r.dt)[(1-\rho^+.dt-\lambda.dt-l_B.dt)V_{ln-B}+\rho^+.dtV_{hn}+\lambda.dtV_{ln}+l_B.dt(V_{lo-out}-B)] \\ \Longleftrightarrow &(r+\rho^+ +\lambda+l_B)V_{ln-B} = \rho^+ V_{hn}+(\lambda+l_B)V_{ln}+l_B(V_{lo-out}-V_{lo}) < (r+\rho^+ +\lambda+l_B)V_{ln} \end{aligned}$$

  This deviation is not profitable.

The two steady state equations can be rewritten as

$$\rho^- L_{hn}+l_B L_{hn}+l_A L_{lo} = \rho^+ L_{ln}$$

$$\rho^+ L_{lo}+l_A L_{lo}+l_B L_{hn} = \rho^- L_{ho}$$

replacing by the possible value of these masses at equilibrium we obtain the equations in function of α_{eq}

$$\rho^-(\frac{\rho^+}{\rho^+ + \rho^-}-s+\alpha_{eq})+l_B(\frac{\rho^+}{\rho^+ + \rho^-}-s+\alpha_{eq})+l_A\alpha_{eq} = \rho(\frac{\rho^-}{\rho^+ + \rho^-}-\alpha_{eq})$$

$$\rho\alpha_{eq}+l_A\alpha_{eq}+l_B(\frac{\rho^+}{\rho^+ + \rho^-}-s+\alpha_{eq}) = \rho(s-\alpha_{eq})$$

that both give the same result for α_{eq}

$$\alpha_{eq} = \frac{\rho^- s - l_B(\frac{\rho^+}{\rho^+ + \rho^-}-s)}{\rho^+ + \rho^- + l_A + l_B}$$

For $s = \frac{\rho^+}{\rho^+ + \rho^-}$, $\alpha_{eq} = \frac{\rho^+}{\rho^+ + \rho^-}\frac{\rho^-}{\rho^+ + \rho^- + l_A + l_B}$ and in this case we verify that

$$0 \leq \alpha_{eq} \leq \frac{\rho^-}{\rho^+ + \rho^-}$$

For s close enough to $\frac{\rho^+}{\rho^+ + \rho^-}$ these inequalities are (strictly) verified. Equilibrium value of steady populations are given by the equations

$$L_{ho} = (s - \alpha_{eq})L, \ L_{hn} = \left(\frac{\rho^+}{\rho^+ + \rho^-} - s + \alpha_{eq}\right)L,$$
$$L_{lo} = \alpha_{eq}L, \ L_{ln} = \left(\frac{\rho^-}{\rho^+ + \rho^-} - \alpha_{eq}\right)L$$

To be sure that the equilibrium exists we need to check that

$$0 \leq m_A \leq 1, \quad 0 \leq m_B \leq 1$$

For instance

$$m_B = \frac{L_{hn}}{\lambda L_{lo} + \rho^- L_{ho}} l_B$$

Because l_B, L_{hn}, L_{lo} and L_{ho} do not depend on λ, we can always a high enough value of λ so that $m_B < 1$. As well for

$$m_A = \frac{L_{lo}}{\lambda L_{hn} + \rho^+ L_{ln}} l_A$$

We can also noticed that market order flows are independent of the monitoring parameter λ. Indeed the sell market order is equal to $m_B(\lambda L_{lo} + \rho^- L_{ho}) = L_{hn} l_B$ and the buy market order flow is $m_A(\lambda L_{hn} + \rho L_{ln}) = L_{lo} l_A$.

Remark A.3. *In the symmetric case*

$$\alpha_{eq} = \frac{1}{2} \frac{\rho}{2\rho + l_A + l_B}$$

$$m_A = \frac{\alpha_{eq}}{\lambda \alpha_{eq} + \rho(\frac{1}{2} - \alpha_{eq})} l_A < \frac{\alpha_{eq}}{\rho(\frac{1}{2} - \alpha_{eq})} l_A = \frac{l_A}{\rho + l_A + l_B} < 1$$

and for the same reason

$$m_B < \frac{l_B}{\rho + l_A + l_B} < 1$$

Whatever is the value of λ the equilibrium exits

A.3 Dynamic equilibrium converging to a steady state w/o fundamental uncertainty

To complete the former steady state class of equilibria this is possible to design a dynamic equilibrium, converging toward one of these equilibria, in which the terms of the trade-off between limit and market orders are unchanged. This will be useful for the next sections in order to define a broader class of equilibria where the limit order market switch from one steady state to another.

Starting at $t = 0$ from a one tick market where the depth at prices A and B are $D_A(0)$ and $D_B(0)$ constituted respectively by a share of the population $L_{lo}(0)$ and of $L_{hn}(0)$, agents follow their corresponding **steady state equilibrium** strategy described earlier. The rate at which hn and lo types are sending market orders, $m_A(t)$ and $m_B(t)$, are evolving so that the terms of the trade off are the same as in the steady state equilibrium. More precisely the intensity rate at which limit orders are executed are unchanged and equal to l_A and l_B. In this framework the dynamic of the different population is given by the dynamic of the parameter α,

$$\begin{aligned}
L_{ho}(t) &= (s - \alpha(t))L \\
L_{hn}(t) &= \left(\frac{\rho^+}{\rho^+ + \rho^-} - s + \alpha(t)\right) L \\
L_{lo}(t) &= \alpha(t)L \\
L_{ln}(t) &= \left(\frac{\rho^-}{\rho^+ + \rho^-} - \alpha(t)\right) L
\end{aligned}$$

and the value function for each type are the same as in the former steady-state equilibrium.

A.3.1 Micro-level dynamic of the limit order book

Ask Side. At time t, on the ask side of the market the depth is equal to $D_A(t)$ and the order flows sustaining this steady state are

- **Outflow:** people switching from lo to ho, $\rho^+ D_A(t).dt$, lo type cancelling their sell limit order to send a sell market order, $m_B(t)\lambda D_A(t).dt$, execution of buy market orders send by hn type contacting the market $m_A(t)(\lambda L_{hn}(t) + \rho^+ L_{ln}(t)).dt = l_A D_A(t).dt$.
- **Inflow:** people switching from ho to lo type sending a sell limit order, $(1 - m_B(t))\rho^- L_{ho}(t).dt$, lo type outside of the order book sending a sell limit order $(1 - m_B(t))\lambda(L_{lo}(t) - D_A(t)).dt$

Bid Side. At time t, on the ask side of the market the depth is constantly equal to $D_B(t)$ and the order flows sustaining this steady state are

- **Outflow:** people switching from hn to ln, $\rho^- D_B(t).dt$, hn type cancelling their sell limit order to send a sell market order, $m_A\lambda D_B(t).dt$, execution of sell market orders send by lo type contacting the market $m_B(t)(\lambda L_{lo}(t) + \rho^- L_{ho}(t)).dt = l_B D_B(t).dt$.
- **Inflow:** people switching from ln to hn type sending a sell limit order, $(1 - m_A(t))\rho^+ L_{ln}(t).dt$, hn type outside of the order book sending a sell limit order $(1 - m_A(t))\lambda(L_{hn}(t) - D_B(t)).dt$.

Then the dynamics of the limit order books depths are given by the first order differential equations,

$$\frac{dD_A}{dt} = \rho^- L_{ho}(t) - \rho^+ D_A(t) - l_A D_A(t) - l_B D_B(t) + \lambda(L_{lo}(t) - D_A(t))$$

$$\frac{dD_B}{dt} = \rho^+ L_{ln}(t) - \rho^- D_B(t) - l_B D_B(t) - l_A D_A(t) + \lambda(L_{hn}(t) - D_B(t))$$

A.3.2 Outcome of the dynamic equilibrium

Proposition A.5. *The dynamics of the equilibrium populations are given by the dynamic of the parameter* α*,*

$$\frac{d\alpha}{dt} = \left[\rho^- s - l_B\left(\frac{\rho^+}{\rho^+ + \rho^-} - s\right)\right] - [\rho^- + \rho^+ + l_A + l_B]\alpha(t) + l_A\kappa_A e^{-(\lambda+\rho^+)t} + l_B\kappa_B e^{-(\lambda+\rho^-)t}$$

with solution

$$\begin{aligned}\alpha(t) &= \alpha_{eq} + (\alpha(0) - \alpha_{eq})e^{-(\rho^- + \rho^+ + l_A + l_B)t} \\ &+ l_A\kappa_A \frac{1 - e^{-[\lambda-(\rho^- + l_A + l_B)]t}}{\lambda - (\rho^- + l_A + l_B)} e^{-(\rho^- + \rho^+ + l_A + l_B)t} \\ &+ l_B\kappa_B \frac{1 - e^{-[\lambda-(\rho^+ + l_A + l_B)]t}}{\lambda - (\rho^- + l_A + l_B)} e^{-(\rho^- + \rho^+ + l_A + l_B)t}\end{aligned}$$

with $\kappa_A = \frac{L_{lo}(0)-D_A(0)}{L},\ \kappa_B = \frac{L_{hn}(0)-D_B(0)}{L}$

Corollary A.2. *If* $0 \leq \alpha(0) \leq \frac{\rho^-}{\rho^- + \rho^+}$ *and* $0 \leq \alpha_{eq} \leq \frac{\rho^-}{\rho^- + \rho^+}$ *then* $\forall t \geq 0,\ 0 \leq \alpha(t) \leq \frac{\rho^-}{\rho^- + \rho^+}$

Proposition A.6. *For a set of parameter values this limit order book dynamic is an equilibrium dynamic that converges toward a steady state equilibrium at the same ask and bid prices, A and B. In this equilibrium the value functions for the different agents types are constant and equal to the value function corresponding to the limit steady state equilibrium.*

Proof of Proposition A.5

In addition the *lo* type agents who are not in the order book at $t = 0$ enter the market as soon as soon as they contact the market. The dynamic of this population is then

$$L_{lo}(t) - D_A(t) = (L_{lo}(0) - D_A(0))e^{-(\lambda+\rho^+)t}$$

For the same reason the dynamic of *hn* type not in the limit order book is given by

$$L_{hn}(t) - D_B(t) = (L_{hn}(0) - D_B(0))e^{-(\lambda+\rho^-)t}$$

As in the steady state case

$$m_B(t) = \frac{D_B(t)}{\lambda L_{lo}(t) + \rho^- L_{ho}(t)} l_B,\ m_A(t) = \frac{D_A(t)}{\lambda L_{hn}(t) + \rho^+ L_{ln}(t)} l_A$$

are well defined ($\in [0, 1]$) for all $t > 0$ for some value of λ high enough.

To obtain the differential equation that drives the dynamic of α, we use the differential equation for $D_A(t)$ and the equalities $D_A(t) = L_{lo}(t) - (L_{lo}(0) - D_A(0))e^{-(\rho^+ + \lambda)t}$. We obtain

$$\begin{aligned}\frac{dL_{lo}(t)}{dt} + (\rho^+ + \lambda)(L_{lo}(0) - D_A(0))e^{-(\rho^+ + \lambda)t} &= \rho^- L_{ho}(t) - (\rho^+ + l_A)(L_{lo}(t) - (L_{lo}(0) - D_A(0))e^{-(\rho^+ + \lambda)t}) \\ &- l_B(L_{hn}(t) - (L_{hn}(0) - D_B(0))e^{-(\rho^- + \lambda)t}) + \lambda(L_{lo}(0) - D_A(0))e^{-(\rho^+ + \lambda)t}\end{aligned}$$

which gives

$$\frac{dL_{lo}(t)}{dt} = \rho^- L_{ho}(t) - (\rho^+ + l_A)L_{lo}(t) - l_B L_{hn}(t) - l_A(L_{lo}(0) - D_A(0))e^{-(\rho^+ + \lambda)t} - l_B(L_{hn}(0) - D_B(0))e^{-(\rho^- + \lambda)t}$$

and then use that $L_{ho}(t) = (s - \alpha(t))L,\ L_{hn}(t) = \left(\frac{\rho^+}{\rho^+ + \rho^-} - s + \alpha(t)\right)L,\ L_{lo}(t) = \alpha(t)L.$ to get the final differential equation

$$\frac{d\alpha}{dt} = \left[\rho^- - l_B\left(\frac{\rho^+}{\rho^+ + \rho^-} - s\right)\right] - [\rho^- + \rho^+ + l_A + l_B]\alpha(t) + l_A\kappa_A e^{-(\lambda+\rho^+)t} + l_B\kappa_B e^{-(\lambda+\rho^-)t}$$

To obtain the general solution to this ODE, we look for the functional form $\alpha(t) = c(t)e^{-(\rho^- + \rho^+ + l_A + l_B)t}$. Then

$$e^{-(\rho^- + \rho^+ + l_A + l_B)t}\frac{dc}{dt} = \left[\rho^- - l_B\left(\frac{\rho^+}{\rho^+ + \rho^-} - s\right)\right] + l_A\kappa_A e^{-(\lambda+\rho^+)t} + l_B\kappa_B e^{-(\lambda+\rho^-)t}$$

and then

$$c(t) = c(0) + \frac{\left[\rho^- - l_B\left(\frac{\rho^+}{\rho^+ + \rho^-} - s\right)\right]}{\rho^- + \rho^+ + l_A + l_B} \times (e^{(\rho^- + \rho^+ + l_A + l_B)t} - 1) - l_A\kappa_A\frac{e^{-[\lambda-(\rho^- + l_A + l_B)]t} - 1}{\lambda - (\rho^- + l_A + l_B)}$$
$$- l_B\kappa_B\frac{e^{-[\lambda-(\rho^+ + l_A + l_B)]t} - 1}{\lambda - (\rho^+ + l_A + l_B)}$$

Proof of Corollary A.2

First we can see that $\alpha(t) > 0$ obviously. Since α is converging it can have extrema. Given the ODE that defines α these extrema must verify

$$[\rho^- + \rho^+ + l_A + l_B]\alpha(t) = \left[\rho^- s - l_B\left(\frac{\rho^+}{\rho^+ + \rho^-} - s\right)\right] + l_A\kappa_A e^{-(\lambda+\rho^+)t} + l_B\kappa_B e^{-(\lambda+\rho^-)t}$$

This gives

$$[\rho^- + \rho^+ + l_A + l_B]\alpha(t) \leq \left[\rho^- s - l_B\left(\frac{\rho^+}{\rho^+ + \rho^-} - s\right)\right] + l_A\alpha(0)e^{-(\lambda+\rho^+)t} + l_B(\alpha(0) + \frac{\rho^+}{\rho^+ + \rho^-} - s)e^{-(\lambda+\rho^-)t}$$

we can rewrite

$$[\rho^- + \rho^+ + l_A + l_B]\alpha(t) \leq (\rho^- + \rho^+)\frac{\rho^- s}{\rho^- + \rho^+} - l_B\left(\frac{\rho^+}{\rho^+ + \rho^-} - s\right)(1 - e^{-(\lambda+\rho^-)t})$$
$$+ (l_A + l_B)\alpha(0) - \alpha(0)l_A(1 - e^{-(\lambda+\rho^+)t}) - \alpha(0)l_B(1 - e^{-(\lambda+\rho^-)t})$$

And then

$$[\rho^- + \rho^+ + l_A + l_B]\alpha(t) \leq (\rho^- + \rho^+)\frac{\rho^- s}{\rho^- + \rho^+} + (l_A + l_B)\alpha(0)$$

A.4 Limit order book in transition phase

Before we turn to the description of the limit order book dynamic in the transition phase. Preceding the beginning of the transition phase the world is in the state $\zeta = \emptyset$. The transition phase starts when the asset fundamental value changes. It changes at some point in time τ. It is stochastic and follows a Poisson distribution, $\mathcal{P}(\mu)$. After τ the state of the world is either $\zeta = u$ (up) and $v = v^0 + \omega$ or $\zeta = d$ and $v = v^0 - \omega$(down) with equal probability. In this section I start time back to 0 when the fundamental value changes. Here time t corresponds to time $t + \tau$ of the overall game. I call T^u and T^d the duration of the transition phases in the different states of the world.

A.4.1 Transition phase strategy

To define properly the strategy in the transition phase it is necessary to decide what will be the steady state phase in the last phase. Different level of prices can define an equilibrium. If all investors know that a particular steady state equilibrium will be played during the last phase they coordinate on this equilibrium.

Conjecture A.2. *After the fundamental value has changed if $\zeta = u$, for $t > T^u$, investors coordinate on the steady state equilibrium over the bid-ask prices (A^u, B^u) and if $\zeta = d$, for $t > T^d$, investors coordinate on the steady state equilibrium over the bid-ask prices (A^d, B^d).*

During the transition phase, the strategy is:

- *In the case $\zeta = u$, for $t < T^u$:*
    - *lo cancel any sell limit order that is not at price A^u and submit a limit order at price A^u*
    - *ho cancel any sell limit order and stay out of the market*
    - *ln send a buy market order and immediately behave as their new type, lo*
    - *hn send a buy market order and immediately behave as their new type, ho*
- *In the case $\zeta = d$, for $t < T^d$:*
    - *hn cancel any buy limit order that is not at price B^d and submit a limit order at price B^d*
    - *ln cancel any buy limit order and stay out of the market*
    - *ho send a sell market order and immediately behave as their new type, hn*
    - *lo send a sell market order and immediately behave as their new type, ln*

The rational here is that when the fundamental value changes to become higher for instance, non-owner become arbitrageurs and have an incentive to buy the asset while it is tradable at a low price, A^0, and to resell it at a high price A^u later. This is what we are conjecturing.

A.4.2 Limit order book dynamics in the transition phase

Before the transition phase begins the limit order book is filled with some liquidity. In particular liquidity provision at best ask and bid prices is defined by the value of the depth of the limit order book at prices A^0 and B^0. These are equal $D^{\emptyset}_{A^0}$ and $D^{\emptyset}_{B^0}$. During the transition phase the limit orders at prices A^0 and B^0 are being cancelled or executed which generates a dry-out of liquidity at these prices. Here we are giving the dynamics of the liquidity supply which offers a free option opportunity after the fundamental value has changed.

In the case $S = u$

Proposition A.7. *When $S = u$ during the transition phase the depth at price A^0 is*

$$D^u_{A^0}(t) = -(1-s)L + [D^{\emptyset}_{A^0} + (1-s)L]e^{-(\lambda+\rho^+)t} + L^{\emptyset}_{hn}(e^{-(\lambda+\rho^-)t} - e^{-(\lambda+\rho^+)t})$$

which is decreasing and has a unique zero, defining the time T^u.

Proof of Proposition A.7

Following the possible equilibria conjectured, for $t < \tau$ there are potentially sell limit orders at prices A^0 and A^u and buy limit orders at prices B^0 and B^d. $D^{\emptyset}_{A^0}$, $D^{\emptyset}_{A^u}$, $D^{\emptyset}_{B^0}$ and $D^{\emptyset}_{B^d}$ are the associated depths. Given the strategy for $t < \tau$, we must have

$$D^{\emptyset}_{A^0} + D^{\emptyset}_{A^u} = L^{\emptyset}_{lo}, \quad D^{\emptyset}_{B^0} + D^{\emptyset}_{B^d} = L^{\emptyset}_{hn}$$

The limit order book dynamics is given by :

- hn and ln type agents cancel their limit orders and send a buy market order while $D^u_{A^0} > 0$

$$\frac{\partial D^u_{B^0}}{\partial t} = -(\lambda + \rho^-) D^u_{B^0}(t)$$
$$\frac{\partial D^u_{B^d}}{\partial t} = -(\lambda + \rho^-) D^u_{B^d}(t)$$

  This implies that
$$\frac{\partial L^u_{hn}}{\partial t} = -(\lambda + \rho^-) L^u_{hn}(t)$$
  because when ln types switch to hn they immediately send a market order and become ho.

- lo type with limit orders at A^0 cancel their limit orders or are executed by market orders send by hn and ln
$$\frac{\partial D^u_{A^0}}{\partial t} = -(\lambda + \rho^+) D^u_{A^0}(t) - (\lambda + \rho^-) L^u_{hn}(t) - (\lambda + \rho^+) L^u_{ln}(t)$$
  or
$$\frac{\partial D^u_{A^0}}{\partial t} = -(\lambda + \rho^+) D^u_{A^0}(t) - (\lambda + \rho^-) L^u_{hn}(t) - (\lambda + \rho^+)[(1-s)L - L^u_{hn}(t)]$$

- lo types formerly ln or in the limit order book at A^0 send sell limit orders at A^u and ho types cancel their limit orders
$$\frac{\partial D^u_{A^u}}{\partial t} = \lambda D^u_{A^0}(t) + \rho^- L^u_{hn}(t) + \lambda L^u_{ln}(t) + \rho^- L^u_{ho}(t) - \rho^+ D^u_{A^u}(t)$$

The transition phases ends at T^u such that $D^u_{A^0}(T^u) = 0$. The dynamic of $D^u_{A^0}(t)$ is given by the differential equation :

$$\frac{\partial D^u_{A^0}}{\partial t} = -(\lambda + \rho^+) D^u_{A^0}(t) - (\lambda + \rho^+)(1-s)L + (\rho^+ - \rho^-) L^{\emptyset}_{hn} e^{-(\lambda+\rho^-)t}$$

Let's find a solution of the ODE of the type $D^u_{A^0}(t) + (1-s)L = c(t) e^{-(\lambda+\rho^+)t}$. This gives

$$\begin{aligned}
&\Leftrightarrow \dot{c} e^{-(\lambda+\rho^+)t} = (\rho^+ - \rho^-) L^{\emptyset}_{hn} e^{-(\lambda+\rho^-)t} \\
&\Leftrightarrow \dot{c} = (\rho^+ - \rho^-) L^{\emptyset}_{hn} e^{(\rho^+ - \rho^-)t} \\
&\Leftrightarrow c(t) = c_0 + L^{\emptyset}_{hn} e^{(\rho^+ - \rho^-)t}
\end{aligned}$$

Then
$$D^u_{A^0}(t) + (1-s)L = c_0 e^{-(\lambda+\rho^+)t} + L^{\emptyset}_{hn} e^{-(\lambda+\rho^-)t}$$
and given the initial condition $c_0 = D^{\emptyset}_{A^0} + (1-s)L - L^{\emptyset}_{hn}$.

In the case $S = d$

Proposition A.8. *When $S = d$ during the transition phase the depth at price B^0 is*

$$D^d_{B^0}(t) = -sL + [D^{\emptyset}_{B^0} + sL] e^{-(\lambda+\rho^-)t} + L^{\emptyset}_{lo}(e^{-(\lambda+\rho^+)t} - e^{-(\lambda+\rho^-)t})$$

which is decreasing and has a unique zero, defining the time T^d.

Proof of Proposition A.8

The limit order book dynamics is given by :

- lo and lo type agents cancel their limit orders and send a sell market order while $D^d_{B^0} > 0$

$$\frac{\partial D^d_{A^0}}{\partial t} = -(\lambda + \rho^+) D^d_{A^0}(t)$$

$$\frac{\partial D^d_{A^u}}{\partial t} = -(\lambda + \rho^+) D^d_{A^u}(t)$$

  This implies that

$$\frac{\partial L^d_{lo}}{\partial t} = -(\lambda + \rho^+) L^d_{lo}(t)$$

  because when ho types switch to lo they immediately send a market order and become ln.

- hn type with limit orders at B^0 cancel their limit orders or are executed by market orders send by lo and ho

$$\frac{\partial D^d_{B^0}}{\partial t} = -(\lambda + \rho^-) D^d_{B^0}(t) - (\lambda + \rho^+) L^d_{lo}(t) - (\lambda + \rho^-) L^d_{ho}(t)$$

  or

$$\frac{\partial D^d_{B^0}}{\partial t} = -(\lambda + \rho^-) D^d_{B^0}(t) - (\lambda + \rho^+) L^d_{lo}(t) - (\lambda + \rho^-)[sL - L^d_{lo}(t)]$$

- hn types formerly ho or in the limit order book at B^0 send buy limit orders at B^d and ln types cancel their limit orders

$$\frac{\partial D^d_{B^d}}{\partial t} = \lambda D^d_{B^0}(t) + \rho^+ L^d_{lo}(t) + \lambda L^d_{ho}(t) + \rho^+ L^d_{ln}(t) - \rho^- D^d_{B^d}(t)$$

The transition phases ends at T^d such that $D^d_{B^0}(T^d) = 0$. The dynamic of $D^d_{B^0}(t)$ is given by the differential equation :

$$\frac{\partial D^d_{B^0}}{\partial t} = -(\lambda + \rho^-) D^d_{B^0}(t) - (\lambda + \rho^-) sL + (\rho^- - \rho^+) L^{\emptyset}_{lo} e^{-(\lambda+\rho^+)t}$$

The ODE solving is as for proposition D.1.

A.4.3 Equilibrium in the subgame starting after the fundamental value changed

Once the common value v has changed (after τ) we already know that after the transition phase ($t > T^{u/d}$) we are playing an equilibrium strategy solved in section III. For instance if $S = u$ at $t = T^u$ we begin to play the dynamic equilibrium with the different population at time T^u and the depth $D^u_{A^u}(T^u)$ at A^u and 0 at B^u. To obtain the equilibrium in the subgame we need to show that during the transition phase the conjecture strategy is indeed optimal.

The conjecture strategy implies that there is no possibility to send any sequence of limit order (executed) at the same point in time which show that this is not profitable to have more than on trade at one point in time. Given that contacting times are Poissonian we clearly are in the condition of the lemma 2.1. The two following propositions confirm that the strategies we proposed as equilibrium strategies candidates are indeed generating an equilibrium.

Proposition A.9. *When $S = u$ the conjecture strategy in the subgame starting at τ is an equilibrium strategy.*

Proposition A.10. *When $S = d$, the conjecture strategy in the subgame starting at τ is an equilibrium strategy.*

Proof of Proposition A.9

During the transition phase the observed types under the conjecture strategy are $lo - A^u$ (lo with a limit order at price A^u), $lo - A^0$, $ho - out$, $ln - out$, $hn - B^0$ and $hn - B^d$. But $hn - B^0$ and $hn - B^d$ can be gathered under the label $hn - out$ because their limit orders are not executed under the conjecture strategy so they would get the same outcome if they were out of the order book. In this framework we can define the system of ODE's defining the value function of these types:

- $lo - A^u$ stay in the limit order book until they switch of type

$$(r+\rho^+)V^u_{lo-A^u}(t) = v^u - \delta + \frac{\partial V^u_{lo-A^u}}{\partial t} + \rho^+ V^u_{ho-out}(t)$$

  with $V^u_{lo-A^u}(T^u) = \bar{V}^u_{lo-A^u}$ which is the value for a lo of having a limit order at price A^u once the last dynamic equilibrium is played (the optimal strategy in the last phase).

- $ho - out$ stay out until they switch of type

$$(r+\rho^-)V^u_{ho-out}(t) = v^u + \frac{\partial V^u_{ho-out}}{\partial t} + \rho^- V^u_{lo-A^u}(t), \quad V^u_{ho-out}(T^u) = \bar{V}^u_{ho-out}$$

- $ln - out$ send a buy market order and immediately behave as their new type: they send a sell limit order and become $lo - A^u$

$$(r+\rho^+ + \lambda)V^u_{ln-out}(t) = \frac{\partial V^u_{ln-out}}{\partial t} + \rho^+(V^u_{ho-out}(t) - A^0) + \lambda(V^u_{lo-A^u}(t) - A^0), \quad V^u_{ln-out}(T^u) = \bar{V}^u_{ln-out}$$

- $hn - out$ send a buy market order and immediately behave as their new type

$$(r+\rho^- + \lambda)V^u_{ln-out}(t) = \frac{\partial V^u_{hn-out}}{\partial t} + \rho^-(V^u_{lo-A^u}(t) - A^0) + \lambda(V^u_{ho-out}(t) - A^0)$$

  with $(r+\rho^- + \lambda)V^u_{hn-out}(T^u) = (r+\rho^- + \lambda)\bar{V}^u_{hn-out} = \rho^- \bar{V}^u_{ln-out} + \lambda \bar{V}^u_{ln-B^u}$ because as soon as this type contact the market being $hn - B^u$ is optimal after T^u.

- $lo - A^0$ cancel their limit order or are executed

$$(r+\rho^+ + \lambda + k_{A^0}(t))V^u_{lo-A^0}(t) = v^u - \delta + \frac{\partial V^u_{lo-A^0}}{\partial t} + \rho^+ V^u_{ho-out}(t) + \lambda V^u_{lo-A^u}(t) + k_{A^0}(t)(V^u_{ln-out}(t) + A^0)$$

  with $V^u_{lo-A^u}(T^u) = \bar{V}^u_{ln-out} + A^0$ and the intensity rate for the execution of the limit order

$$k_{A^0}(t) = \frac{(\lambda+\rho^-)L^u_{hn}(t) + (\lambda+\rho^+)L^u_{ln}(t)}{D^u_{A^0}(t)} = -(\rho^+ + \lambda) - \frac{\dot{D}^u_{A^0}}{D^u_{A^0}(t)}$$

  Let's call

$$M = \begin{pmatrix} r+\rho^- & -\rho^- & 0 & 0 \\ -\rho^+ & r+\rho^+ & 0 & 0 \\ -\rho^+ & -\lambda & r+\rho^+ + \lambda & 0 \\ -\lambda & -\rho^- & 0 & r+\rho^- + \lambda \end{pmatrix}, \quad \vec{V} = \begin{pmatrix} V^u_{ho-out} \\ V^u_{lo-A^u} \\ V^u_{ln-out} \\ V^u_{hn-out} \end{pmatrix} \text{ and } \vec{K} = \begin{pmatrix} -v^u \\ -(v^u - \delta) \\ (\rho^+ + \lambda)A^0 \\ (\rho^- + \lambda)A^u \end{pmatrix}$$

  Then the previous system of differential equations can be written as

$$\frac{\partial}{\partial t}\vec{V} = M \times \vec{V} + \vec{K}$$

plus the non-homogeneous differential equation that rules $V^u_{lo-A^u}$ in which the source term is a combination of the other value functions.

The matrix M's eigenvectors and associated eigenvalues are

$$\left[\vec{E}_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, r\right], \quad \left[\vec{E}_2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, r+\rho^+ +\lambda\right], \quad \left[\vec{E}_3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, r+\rho^- +\lambda\right], \quad \left[\vec{E}_4 = \begin{pmatrix}1\\-\frac{\rho^+}{\rho^-}\\-\frac{\rho^+}{\rho^-}\\1\end{pmatrix}, r+\rho^+ +\rho^-\right]$$

and the solution of the previous vectorial ODE is

$$\vec{V} = M^{-1} \times \vec{K} + C_1\vec{E}_1 e^{rt} + C_2\vec{E}_2 e^{(r+\rho^+ +\lambda)t} + C_3\vec{E}_3 e^{(r+\rho^- +\lambda)t} + C_4\vec{E}_4 e^{(r+\rho^+ +\rho^-)t}$$

with the constants C_i's determined by the conditions at time T^u.

Now let's check the possible deviations.

type ln. During the transition phase the only way a type ln can deviate from the conjecture strategy is to stay out until the next contacting time, and then the get the value $V^u_{ln-out}(t)$, rather than sending a buy market order and get the value $V^u_{lo-A^u}(t) - A^0$. If we call $X(t) = V^u_{lo-A^u}(t) - A^0 - V^u_{ln-out}(t)$ we obtain the ODE

$$(r+\rho^+ +\lambda)X(t) = \frac{\partial X}{\partial t} + v^u - \delta - rA^0$$

the solution of this equation is of the type

$$X(t) = C \times e^{(r+\rho^+ +\lambda)t} + \frac{v^u - \delta - rA^0}{r+\rho^+ +\lambda}$$

Given that for $t > T^u$ we have $\bar{V}^u_{lo-A^u} = \bar{V}^u_{ln-out} + B^u$ then $X(T^u) = B^u - A^0 > 0$. We also have $v^u - \delta - rA^0 > 0$. Then if $C > 0$ $X(t)$ is positive for all t, if $C < 0$ $X(t)$ is decreasing and positive in T^u and then positive over $[0, T^u]$. The deviation is not profitable.

type hn. We are in the same case than the type ln. let's call $X(t) = V^u_{ho-out}(t) - A^0 - V^u_{hn-out}(t)$ and the corresponding ODE is

$$(r+\rho^- +\lambda)X(t) = \frac{\partial X}{\partial t} + v^u - rA^0$$

with $v^u - rA^0 > 0$ and $X(T^u) = \bar{V}^u_{ho-out} - A^0 - \bar{V}^u_{hn-out} > \bar{V}^u_{ho-out} - A^0 - \bar{V}^u_{hn-B^u} = A^u - A^0 > 0$. We get the same result.

type ho.

- Instead of staying out a type ho could send a sell market order at price B^0 and get $V^u_{hn-out}(t) + B^0$. This clearly not profitable given what have been said for type hn
- Another deviation could be to send a limit order at A^0. The corresponding value function would be define by

$$\begin{aligned}(r+\rho^- +\lambda + k_{A^0}(t))V(t) &= v^u + \frac{\partial V}{\partial t} + \rho^- V^u_{lo-A^u}(t) + \lambda V^u_{ho-out}(t) + k_{A^0}(t)(V^u_{hn-out}(t) + A^0) \\ &= \frac{\partial V}{\partial t} + (r+\rho^- +\lambda + k_{A^0}(t))V^u_{ho-out}(t) - \frac{\partial V^u_{ho-out}}{\partial t} - k_{A^0}(t)(V^u_{ho-out}(t) - V^u_{hn-out}(t) - A^0\end{aligned}$$

Calling $X(t) = V^u_{ho-out} - V$ we obtain the ODE

$$(r+\rho^- +\lambda + k_{A^0}(t))X(t) = \frac{\partial X}{\partial t} + k_{A^0}(t)(V^u_{ho-out}(t) - V^u_{hn-out}(t) - A^0)$$

The solution of this ODE is of the type

$$X(t) = e^{\int_0^t (r+\rho^- +\lambda + k_{A^0}(s))ds}[C - \int_0^t k_{A^0}(s)(V^u_{ho-out}(s) - V^u_{hn-out}(s) - A^0)e^{-\int_0^s (r+\rho^- +\lambda + k_{A^0}(l))dl} ds]$$

Then $X(t) \times e^{\int_0^t (r+\rho^- + \lambda + k_{A^0}(s))ds}$ is decreasing and $X(T^u) = \bar{V}^u_{ho-out} - A^0 - \bar{V}^u_{hn-out} > A^u - A^0 > 0$. $X(t)$ is positive over $[0, T^u]$. This deviation is not profitable.

- A type ho could send a limit order at any price $A^0 < A < A^u$ The corresponding value function would be define by

$$\begin{aligned}(r + \rho^- + \lambda)V(t) &= v^u + \frac{\partial V}{\partial t} + \rho^- V^u_{lo-A^u}(t) + \lambda V^u_{ho-out}(t) \\ &= \frac{\partial V}{\partial t} + (r + \rho^- + \lambda)V^u_{ho-out}(t) - \frac{\partial V^u_{ho-out}}{\partial t}\end{aligned}$$

that would give

$$V^u_{ho-out}(t) - V(t) = C \times e^{(r+\rho^-+\lambda)t}$$

and this order would be executed at T^u, then $V^u_{ho-out}(T^u) - V(T^u) = \bar{V}^u_{ho-out} - A - \bar{V}^u_{hn-out} > A^u - A > 0$. The deviation is not profitable.

- Lastly a type ho could send a limit order at price A^u. The corresponding value function would be define by

$$\begin{aligned}(r + \rho^- + \lambda)V(t) &= v^u + \frac{\partial V}{\partial t} + \rho^- V^u_{lo-A^u}(t) + \lambda V^u_{ho-out}(t) \\ &= \frac{\partial V}{\partial t} + (r + \rho^- + \lambda)V^u_{ho-out}(t) - \frac{\partial V^u_{ho-out}}{\partial t}\end{aligned}$$

that would give again

$$V^u_{ho-out}(t) - V(t) = C \times e^{(r+\rho^-+\lambda)t}$$

And at T^u, $V^u_{ho-out}(T^u) - V(T^u) = \bar{V}^u_{ho-out} - \bar{V}^u_{ho-A^u} > 0$.

type lo.

- Instead of staying at A^u a type lo could send a sell market order at price B^0 and get $V^u_{ln-out}(t) + B^0$. This clearly not profitable given what have been said for type ln

- Another deviation could be to send a limit order at A^0. The corresponding value function is $V^u_{lo-A^0}(t)$ Calling $X(t) = V^u_{lo-A^u}(t) - V^u_{lo-A^0}(t)$ we obtain the ODE

$$(r + \rho^+ + \lambda + k_{A^0}(t))X(t) = \frac{\partial X}{\partial t} + k_{A^0}(t)(V^u_{lo-A^u}(t) - V^u_{ln-out}(t) - A^0)$$

The solution of this ODE is as before

$$X(t) = e^{\int_0^t (r+\rho^+ + \lambda + k_{A^0}(s))ds}[C - \int_0^t k_{A^0}(s)(V^u_{lo-A^u}(t) - V^u_{ln-out}(t) - A^0)e^{-\int_0^s (r+\rho^+ + \lambda + k_{A^0}(l))dl}ds]$$

Then $X(t) \times e^{\int_0^t (r+\rho^- + \lambda + k_{A^0}(s))ds}$ is decreasing and $X(T^u) = \bar{V}^u_{lo-A^u} - A^0 - \bar{V}^u_{ln-out} = B^u - A^0 > 0$. $X(t)$ is positive over $[0, T^u]$. This deviation is not profitable.

- A type lo could send a limit order at any price $A^0 < A < A^u$ The corresponding value function would be define by

$$\begin{aligned}(r + \rho^+ + \lambda)V(t) &= v^u - \delta + \frac{\partial V}{\partial t} + \rho^+ V^u_{ho-out}(t) + \lambda V^u_{lo-A^u}(t) \\ &= \frac{\partial V}{\partial t} + (r + \rho^+ + \lambda)V^u_{lo-A^u}(t) - \frac{\partial V^u_{lo-A^u}}{\partial t}\end{aligned}$$

that would give

$$V^u_{lo-A^u}(t) - V(t) = C \times e^{(r+\rho^+ + \lambda)t}$$

and this order would be executed at T^u, then $V^u_{lo-A^u}(T^u) - V(T^u) = \bar{V}^u_{lo-A^u} - A - \bar{V}^u_{ln-out} = B^u - A > 0$. The deviation is not profitable.

- Lastly a type *lo* could stay out. The corresponding value function would be define by

$$(r+\rho^+ +\lambda)V(t) = v^u - \delta + \frac{\partial V}{\partial t} + \rho^+ V^u_{ho-out}(t) + \lambda V^u_{lo-A^u}(t)$$
$$= \frac{\partial V}{\partial t} + (r+\rho^+ +\lambda)V^u_{lo-A^u}(t) - \frac{\partial V^u_{lo-A^u}}{\partial t}$$

  that would give again
$$V^u_{lo-A^u}(t) - V(t) = C \times e^{(r+\rho^+ +\lambda)t}$$
  And at T^u, $V^u_{lo-A^u}(T^u) - V(T^u) = \bar{V}^u_{lo-A^u} - \bar{V}^u_{lo-out} > 0$.

Proof of Proposition A.10

During the transition phase the observed types under the conjecture strategy are $hn-B^d$, $hn-B^0$, $ln-out$, $ho-out$, $lo-A^0$ and $lo-A^u$. But $lo-A^0$ and $lo-A^u$ can be gathered under the label $lo-out$ because their limit orders are not executed under the conjecture strategy so they would get the same outcome if they were out of the order book. In this framework we can define the system of ODE's defining the value function of these types:

- $hn-B^d$ stay in the limit order book until they switch of type

$$(r+\rho^-)V^d_{hn-B^d}(t) = \frac{\partial V^d_{hn-B^d}}{\partial t} + \rho^- V^d_{ln-out}(t)$$

  with $V^d_{hn-B^d}(T^d) = \bar{V}^d_{hn-B^d}$ which is the value for a *hn* of having a limit order at price B^d once the last dynamic equilibrium is played (the optimal strategy in the last phase).

- $ln-out$ stay out until they switch of type

$$(r+\rho^+)V^d_{ln-out}(t) = \frac{\partial V^d_{ln-out}}{\partial t} + \rho^+ V^d_{hn-B^d}(t), \quad V^d_{ln-out}(T^d) = \bar{V}^d_{ln-out}$$

- $ho-out$ send a sell market order and immediately behave as their new type: they send a buy limit order and become $hn-B^d$

$$(r+\rho^- +\lambda)V^d_{ho-out}(t) = v^d + \frac{\partial V^d_{ho-out}}{\partial t} + \rho^-(V^d_{ln-out}(t)+B^0) + \lambda(V^d_{hn-B^d}(t)+B^0), \;\; V^d_{ho-out}(T^d) = \bar{V}^d_{ho-out}$$

- $lo-out$ send a buy market order and immediately behave as their new type

$$(r+\rho^+ +\lambda)V^d_{lo-out}(t) = v^d - \delta + \frac{\partial V^d_{lo-out}}{\partial t} + \rho^+(V^d_{hn-B^d}(t)+B^0) + \lambda(V^d_{ln-out}(t)+B^0)$$

  with $(r+\rho^+ +\lambda)V^d_{lo-out}(T^d) = (r+\rho^+ +\lambda)\bar{V}^d_{lo-out} = v^d - \delta + \rho^+\bar{V}^d_{ho-out} + \lambda\bar{V}^d_{lo-A^d}$ because as soon as this type contact the market being $lo-A^d$ is optimal after T^d.

- $hn-B^0$ cancel their limit order or are executed

$$(r+\rho^- +\lambda + k_{B^0}(t))V^d_{hn-B^0}(t) = \frac{\partial V^d_{hn-B^0}}{\partial t} + \rho^- V^d_{ln-out}(t) + \lambda V^d_{hn-B^d}(t) + k_{B^0}(t)(V^d_{ho-out}(t) - B^0)$$

  with $V^d_{hn-B^0}(T^d) = \bar{V}^u_{ho-out} - B^0$ and the intensity rate for the execution of the limit order

$$k_{B^0}(t) = \frac{(\lambda+\rho^-)L^d_{ho}(t) + (\lambda+\rho^+)L^u_{lo}(t)}{D^d_{B^0}(t)} = -(\rho^- +\lambda) - \frac{\dot{D}^d_{B^0}}{D^d_{B^0}(t)}$$

Possible deviations:

type ho. During the transition phase the only way a type ho can deviate from the conjecture strategy is to stay out until the next contacting time, and then the get the value $V^d_{ho-out}(t)$, rather than sending a sell market order and get the value $V^d_{hn-B^d}(t) + B^0$. If we call $X(t) = V^d_{hn-B^d}(t) + B^0 - V^d_{ho-out}(t)$ we obtain the ODE

$$(r + \rho^- + \lambda)X(t) = \frac{\partial X}{\partial t} + rB^0 - v^d$$

the solution of this equation is of the type

$$X(t) = C \times e^{(r+\rho^+ +\lambda)t} + \frac{v^u - \delta - rA^0}{r + \rho^+ + \lambda}$$

Given that for $t > T^d$ we have $\bar{V}^d_{hn-B^d} = \bar{V}^d_{ho-out} - A^d$ then $X(T^d) = B^0 - A^d > 0$. We also have $rB^0 - v^d > 0$. Then if $C > 0$ $X(t)$ is positive for all t, if $C < 0$ $X(t)$ is decreasing and positive in T^d and then positive over $[0, T^d]$. The deviation is not profitable.

type lo. We are in the same case than the type ho. let's call $X(t) = V^d_{ln-out}(t) + B^0 - V^d_{lo-out}(t)$ and the corresponding ODE is

$$(r + \rho^+ + \lambda)X(t) = \frac{\partial X}{\partial t} + rB^0 - (v^d - \delta)$$

with $rB^0 - (v^d - \delta) > 0$ and $X(T^d) = \bar{V}^d_{ln-out} + B^0 - \bar{V}^d_{lo-out} > \bar{V}^d_{ln-out} + B^0 - \bar{V}^u_{lo-A^d} = B^0 - B^d > 0$. We get the same result.

type ln.

- Instead of staying out a type ln could send a sell market order at price A^0 and get $V^d_{lo-out}(t) - A^0$. This clearly not profitable given what have been shown for type lo
- Another deviation could be to send a limit order at B^0. The corresponding value function would be define by

$$\begin{aligned}(r + \rho^+ + \lambda + k_{B^0}(t))V(t) &= \frac{\partial V}{\partial t} + \rho^+ V^d_{hn-B^d}(t) + \lambda V^d_{ln-out}(t) + k_{B^0}(t)(V^d_{lo-out}(t) - B^0)\\ &= \frac{\partial V}{\partial t} + (r + \rho^+ + \lambda + k_{B^0}(t))V^d_{ln-out}(t) - \frac{\partial V^d_{ln-out}}{\partial t} - k_{B^0}(t)(V^d_{ln-out}(t) - V^d_{lo-out}(t) + B^0)\end{aligned}$$

  Calling $X(t) = V^d_{ln-out} - V$ we obtain the ODE

$$(r + \rho^+ + \lambda + k_{B^0}(t))X(t) = \frac{\partial X}{\partial t} + k_{B^0}(t)(V^d_{ln-out}(t) - V^d_{lo-out}(t) + B^0)$$

  The solution of this ODE is of the type

$$X(t) = e^{\int_0^t (r+\rho^+ +\lambda+k_{B^0}(s))ds}[C - \int_0^t k_{B^0}(s)(V^d_{ln-out}(t) - V^d_{lo-out}(t) + B^0)e^{-\int_0^s (r+\rho^+ +\lambda+k_{B^0}(l))dl} ds]$$

  Then $X(t) \times e^{\int_0^t (r+\rho^+ +\lambda+k_{B^0}(s))ds}$ is decreasing and $X(T^d) = \bar{V}^d_{ln-out} + B^0 - \bar{V}^d_{lo-out} > B^0 - B^d > 0$. $X(t)$ is positive over $[0, T^d]$. This deviation is not profitable.
- A type ln could send a limit order at any price $B^0 > B > B^d$ The corresponding value function would be define by

$$\begin{aligned}(r + \rho^+ + \lambda)V(t) &= \frac{\partial V}{\partial t} + \rho^+ V^d_{hn-B^d}(t) + \lambda V^d_{ln-out}(t)\\ &= \frac{\partial V}{\partial t} + (r + \rho^+ + \lambda)V^d_{ln-out}(t) - \frac{\partial V^d_{ln-out}}{\partial t}\end{aligned}$$

  that gives

$$V^d_{ln-out}(t) - V(t) = C \times e^{(r+\rho^+ +\lambda)t}$$

  and this order would be executed at T^d, then $V^d_{ln-out}(T^d) - V(T^d) = \bar{V}^d_{ln-out} + B - \bar{V}^d_{lo-out} > B - B^d > 0$. The deviation is not profitable.

- Lastly a type ln could send a limit order at price B^d. The corresponding value function would be define by

$$\begin{aligned}(r+\rho^+ +\lambda)V(t) &= \frac{\partial V}{\partial t} + \rho^+ V^d_{hn-B^d}(t) + \lambda V^d_{ln-out}(t)\\ &= \frac{\partial V}{\partial t} + (r+\rho^+ +\lambda)V^d_{ln-out}(t) - \frac{\partial V^d_{ln-out}}{\partial t}\end{aligned}$$

that would give again

$$V^d_{ln-out}(t) - V(t) = C \times e^{(r+\rho^+ +\lambda)t}$$

And at T^d, $V^d_{ln-out}(T^d) - V(T^d) = \bar{V}^d_{ln-out} - \bar{V}^d_{ln-A^d} > 0$.

type hn.

- Instead of staying at B^d a type hn could send a buy market order at price A^0 and get $V^d_{ho-out}(t) - A^0$. This clearly not profitable given what have been said for type ho
- Another deviation could be to send a limit order at B^0. The corresponding value function is $V^d_{hn-B^0}(t)$ Calling $X(t) = V^d_{hn-B^d}(t) - V^d_{hn-B^0}(t)$ we obtain the ODE

$$(r+\rho^- +\lambda + k_{B^0}(t))X(t) = \frac{\partial X}{\partial t} + k_{B^0}(t)(V^d_{hn-B^d}(t) - V^d_{ho-out}(t) + B^0)$$

The solution of this ODE is as before

$$X(t) = e^{\int_0^t (r+\rho^- +\lambda + k_{B^0}(s))ds}[C - \int_0^t k_{B^0}(s)(V^d_{hn-B^d}(t) - V^d_{ho-out}(t) + B^0)e^{-\int_0^s (r+\rho^- +\lambda + k_{B^0}(l))dl}ds]$$

Then $X(t) \times e^{\int_0^t (r+\rho^- +\lambda + k_{B^0}(s))ds}$ is decreasing and $X(T^d) = \bar{V}^d_{hn-B^d} + B^0 - \bar{V}^d_{ho-out} = B^0 - A^d > 0$. $X(t)$ is positive over $[0, T^d]$. This deviation is not profitable.

- A type hn could send a limit order at any price $B^0 > B > B^d$ The corresponding value function would be define by

$$\begin{aligned}(r+\rho^- +\lambda)V(t) &= \delta + \frac{\partial V}{\partial t} + \rho^- V^d_{ln-out}(t) + \lambda V^d_{hn-B^d}(t)\\ &= \frac{\partial V}{\partial t} + (r+\rho^- +\lambda)V^d_{hn-B^d}(t) - \frac{\partial V^d_{hn-B^d}}{\partial t}\end{aligned}$$

that would give

$$V^d_{hn-B^d}(t) - V(t) = C \times e^{(r+\rho^- +\lambda)t}$$

and this order would be executed at T^d, then $V^d_{hn-B^d}(T^d) - V(T^d) = \bar{V}^d_{hn-B^d} + B - \bar{V}^d_{ho-out} = B - A^d > 0$. The deviation is not profitable.

- Lastly a type hn could stay out. The corresponding value function would be define by

$$\begin{aligned}(r+\rho^- +\lambda)V(t) &= +\frac{\partial V}{\partial t} + \rho^- V^d_{ln-out}(t) + \lambda V^d_{hn-B^d}(t)\\ &= \frac{\partial V}{\partial t} + (r+\rho^- +\lambda)V^d_{hn-B^d}(t) - \frac{\partial V^d_{hn-B^d}}{\partial t}\end{aligned}$$

that would give again

$$V^d_{hn-B^d}(t) - V(t) = C \times e^{(r+\rho^- +\lambda)t}$$

And at T^d, $V^d_{hn-B^d}(T^d) - V(T^d) = \bar{V}^d_{hn-B^d} - \bar{V}^d_{hn-out} > 0$.

A.5 Equilibrium in the perfectly symmetric case

In order to obtain an equilibrium strategy for the entire model I am focusing on the perfectly symmetric case. The conditions for this equilibrium to hold in a more general case would probably be that the parameter's values make the model close enough to the perfectly symmetric case. This would generates model outcomes similar to the ones in the perfectly symmetric case.

The perfectly symmetric setup is defined as follows

- for investor types: $\rho^+ = \rho^- = \rho, \quad s = \frac{1}{2}$
- for the asset value and dynamic: $v^u = v^0 + \omega, \quad v^d = v^0 - \omega, \quad p = \frac{1}{2}$
- for the targeted prices: $B^* = \frac{1}{r}(v^* - \frac{\delta}{2}) - \frac{\Delta}{2}, \quad A^* = \frac{1}{r}(v^* - \frac{\delta}{2}) + \frac{\Delta}{2}$

and in term of magnitude we assume that

$$\frac{\omega}{r} >> \frac{\delta}{r} >> \Delta$$

which means that gains from arbitraging are higher than gain from trading for liquidity reason and the last one being higher that the implicit cost of trading, the bid-ask spread.

In this case the dynamics of the order book in the transition phase is given by $D^{\emptyset}_{B^0} = D^d_{B^0}(0) = D^u_{A^0}(0) = D^{\emptyset}_{A^0} = \alpha^{\emptyset} L$,

$$\forall t \ D^d_{B^0}(t) = D^u_{A^0}(t) = D(t) = -\frac{1}{2}L + [\alpha^{\emptyset} + \frac{1}{2}]Le^{-(\rho+\lambda)t}$$

and the transition phases last the same time in the states u or d

$$T^u = T^d = T = \frac{1}{\rho + \lambda} \ln(1 + 2\alpha^{\emptyset}).$$

A.5.1 equilibrium conjecture

I consider equilibria where the limit order market dynamic is in a steady-state for $t < \tau$ and converges to another steady state for $t > \tau$. The idea is to conjecture (and to solve) the class of equilibria by backward induction:

- For $S = \emptyset$ (before τ): with a model setup sufficiently symmetric, whatever the value of μ, this is a steady state equilibrium over a pair of prices (A^0, B^0) where *ho* and *ln* stay out of the market, *lo* and *hn* indifferently send limit or market orders.
- For $T + \tau > t > \tau$ this is the transition phase
- For $t > T + \tau$ if $S = u$ we are playing the equilibrium dynamic converging to a steady state over the bid-ask prices (A^u, B^u) (as described in the previous section), and if $S = d$ we are playing the equilibrium dynamic converging to a steady state over the bid-ask prices (A^d, B^d).

A.5.2 equilibrium outcome

Proposition A.11. *In the perfectly symmetric case, whatever the value of μ, there is a unique steady state equilibrium defined by the pair of prices (A^0, B^0) where lo and hn indifferently send limit or market orders and where ho and ln stay out of the market. The rate at which limit orders are executed are the same for sell and buy orders $l^{\emptyset}_{A^0} = l^{\emptyset}_{B^0} = l^{\emptyset}$. And the equilibrium populations are characterized by the value*

$$\alpha^{\emptyset}_{eq} = \frac{\rho}{4(\rho + l^{\emptyset})}$$

Moreover $\lim_{\mu \to \infty} \alpha^{\emptyset}_{eq} = 0$.

Proposition A.12. *For a value of μ high enough, an increase of the monitoring rate λ has a positive impact on $\alpha^{\emptyset}_{eq}$.*

$$\frac{\partial \alpha^{\emptyset}_{eq}}{\partial \lambda} > 0$$

An increase of the fundamental volatility, μ or ω has a negative impact on $\alpha^{\emptyset}_{eq}$

$$\frac{\partial \alpha^{\emptyset}_{eq}}{\partial \mu} < 0, \quad \frac{\partial \alpha^{\emptyset}_{eq}}{\partial \omega} < 0$$

A.5.3 Proof of Proposition A.11

Value functions in the game stage prior to the utility flow change

In the state $S = \emptyset$ when the limit order book is in the steady state, the value functions corresponding to the types in the conjecture equilibria are defined by

for the type ho, $V^{\emptyset}_{ho} = V^{\emptyset}_{ho-out}$ with

$$(r + \rho^- + \mu)V^{\emptyset}_{ho-out} = v^0 + \rho^- V^{\emptyset}_{lo} + \mu p V^u_{ho-out}(0) + \mu(1-p)V^d_{ho-out}(0)$$

for the type lo, value function of sending limit orders at A^u and A^0 are

$$(r + \rho^+ + l^{\emptyset}_{A^0} + \mu)V^{\emptyset}_{lo-A^0} = v^0 - \delta + \rho^+ V^{\emptyset}_{ho} + l^{\emptyset}_{A^0}(V^{\emptyset}_{ln} + A^0) + \mu p V^u_{lo-A^0}(0) + \mu(1-p)V^d_{lo-A^0}(0)$$

$$(r + \rho^+ + \mu)V^{\emptyset}_{lo-A^u} = v^0 - \delta + \rho^+ V^{\emptyset}_{ho} + \mu p V^u_{lo-A^u}(0) + \mu(1-p)V^d_{lo-A^u}(0)$$

If lo types only send limit orders at A^u, $V^{\emptyset}_{lo} = V^{\emptyset}_{lo-A^u}$. If lo types only send indifferently market orders and limit orders at A^0, $V^{\emptyset}_{lo} = V^{\emptyset}_{lo-A^0} = V^{\emptyset}_{ln} + B^0$. And if they are indifferent between the three actions $V^{\emptyset}_{lo} = V^{\emptyset}_{lo-A^u} = V^{\emptyset}_{lo-A^0} = V^{\emptyset}_{ln} + B^0$.

for the type ln, $V^{\emptyset}_{ln} = V^{\emptyset}_{ln-out}$ with

$$(r + \rho^+ + \mu)V^{\emptyset}_{ln-out} = \rho^+ V^{\emptyset}_{hn} + \mu p V^u_{ln-out}(0) + \mu(1-p)V^d_{ln-out}(0)$$

for the type hn, value function of sending limit orders at B^d and B^0 are

$$(r + \rho^- + l^{\emptyset}_{B^0} + \mu)V^{\emptyset}_{hn-B^0} = \rho^- V^{\emptyset}_{ln} + l^{\emptyset}_{B^0}(V^{\emptyset}_{ho} - B^0) + \mu p V^u_{hn-B^0}(0) + \mu(1-p)V^d_{hn-B^0}(0)$$

$$(r + \rho^- + \mu)V^{\emptyset}_{hn-B^d} = \rho^- V^{\emptyset}_{ln} + \mu p V^u_{hn-B^d}(0) + \mu(1-p)V^d_{hn-B^d}(0)$$

If hn types only send limit orders at B^d, $V^{\emptyset}_{hn} = V^{\emptyset}_{hn-B^d}$. If hn types only send indifferently market orders and limit orders at B^0, $V^{\emptyset}_{hn} = V^{\emptyset}_{hn-B^0} = V^{\emptyset}_{ho} - A^0$. And if they are indifferent between the three actions $V^{\emptyset}_{lo} = V^{\emptyset}_{hn-B^d} = V^{\emptyset}_{hn-B^0} = V^{\emptyset}_{ho} - A^0$.

In this case the strategy conjectured at equilibrium is such that lo and hn are indifferent between sending market orders and limit orders at A^0 and B^0 and do not send limit orders at A^u or B^d

To have indifference for lo and hn types the rates of limit orders execution must be equal to

$$l^{\emptyset}_{B^0} = \frac{1}{\Delta}[v^0 - rA^0 - \rho^-\Delta + \mu p(V^u_{ho-out}(0) - A^0 - V^u_{hn-B^0}(0)) + \mu(1-p)(V^d_{ho-out}(0) - A^0 - V^d_{hn-B^0}(0))]$$
$$l^{\emptyset}_{A^0} = \frac{1}{\Delta}[rB^0 - \rho^+\Delta - (v^0 - \delta) + \mu p(V^u_{ln-out}(0) + B^0 - V^u_{lo-A^0}(0)) + \mu(1-p)(V^d_{ln-out}(0) + B^0 - V^d_{lo-A^0}(0))]$$

To show the existence of an equilibrium we need to show that the conjecture strategy is optimal and to solve for the equilibrium (steady-state) populations in the state $S = \emptyset$ which is equivalent to prove the existence of an acceptable $\alpha^{\emptyset}$ such that:

$$\alpha^{\emptyset} = \frac{\rho^- s - l^{\emptyset}_{B^0}(\frac{\rho^+}{\rho^+ + \rho^-} - s)}{\rho^+ + \rho^- + l^{\emptyset}_{A^0} + l^{\emptyset}_{B^0}}$$

We must also verify that the rate at which market orders are send $m^{\emptyset}_{A^0}$ and $m^{\emptyset}_{B^0}$ are indeed between 0 and 1.

general case

type lo.

- A type lo can deviate by sending a limit order at price $A^u > A > A^0$ and gets the value

$$(r + \rho^+ + \lambda + \mu)V = v^0 - \delta + \rho^+ V^{\emptyset}_{ho} + \lambda V^{\emptyset}_{lo} + \mu p V^u_{lo-A}(0) + \mu(1-p)V^d_{lo-out}(0)$$

  because $V^u_{lo-A}(0) < V^u_{lo-A^u}(0)$ this deviation is less profitable than the one shot deviation of sending a limit order at A^u.

- a type lo can also deviate by staying out. For the same reason this is less profitable than the one shot deviation of sending a limit order at A^u

- when lo sends a limit order at A^u for a one shot deviation the induced value function is defined by

$$(r + \rho^+ + \lambda + \mu)V = v^0 - \delta + \rho^+ V^{\emptyset}_{ho} + \lambda V^{\emptyset}_{lo} + \mu p V^u_{lo-A^u}(0) + \mu(1-p)V^d_{lo-out}(0)$$

  and we have

$$(r + \rho^+ + \lambda + \mu)(V^{\emptyset}_{lo} - V) = l^{\emptyset}_{A^0}\Delta + \mu p(V^u_{lo-A^0}(0) - V^u_{lo-A^u}(0)) = (r + \rho^+ + \mu)(V^{\emptyset}_{lo} - V^{\emptyset}_{lo-A^u})$$

$V^u_{lo-A^0} - (0)V^u_{lo-A^u}(0) < 0$ then as soon as the strategy conjectured for the type lo is optimal then $l^{\emptyset}_{A^0}\Delta + \mu p(V^u_{lo-A^0}(0) - V^u_{lo-A^u}(0)) > 0$ and then $l^{\emptyset}_{A^0} > 0$ which a necessary condition for the equilibrium. Now we have to get the necessary condition on μ to make it hold and then study the function

$$\begin{aligned}(r + \rho^+ + \mu)(V^{\emptyset}_{lo} - V^{\emptyset}_{lo-A^u}) &= rB^0 - \rho^+\Delta - (v^0 - \delta) \\ &+ \mu p(V^u_{ln-out}(0) + B^0 - V^u_{lo-A^u}(0)) + \mu(1-p)(V^d_{ln-out}(0) + B^0 - V^d_{lo-A^0}(0))\end{aligned}$$

type hn.

- A type hn can deviate by sending a limit order at price $B^d < B < B^0$ and gets the value

$$(r + \rho^- + \lambda + \mu)V = \rho^- V^{\emptyset}_{ln} + \lambda V^{\emptyset}_{hn} + \mu p V^u_{hn-out}(0) + \mu(1-p)V^d_{hn-B}(0)$$

  because $V^d_{hn-B}(0) < V^d_{hn-B^d}(0)$ this deviation is less profitable than the one shot deviation of sending a limit order at B^d.

- a type hn can also deviate by staying out. For the same reason this is less profitable than the one shot deviation of sending a limit order at B^d

- when lo sends a limit order at B^d for a one shot deviation the induced value function is defined by

$$(r + \rho^- + \lambda + \mu)V = \rho^- V^{\emptyset}_{ln} + \lambda V^{\emptyset}_{hn} + \mu p V^d_{hn-out}(0) + \mu(1-p)V^d_{hn-B^d}(0)$$

  and we have

$$(r + \rho^- + \lambda + \mu)(V^{\emptyset}_{hn} - V) = l^{\emptyset}_{B^0}\Delta + \mu(1-p)(V^d_{hn-B^0}(0) - V^d_{hn-B^d}(0)) = (r + \rho^- + \mu)(V^{\emptyset}_{hn} - V^{\emptyset}_{hn-B^d})$$

$V^d_{hn-B^0} - (0)V^d_{hn-B^d}(0) < 0$ then as soon as the strategy conjectured for the type hn is optimal then $l^{\emptyset}_{B^0}\Delta + \mu(1-p)(V^d_{hn-B^0}(0) - V^d_{hn-B^d}(0)) > 0$ and then $l^{\emptyset}_{B^0} > 0$. Now we have to get the necessary condition on μ to make it hold and then study the function

$$\begin{aligned}(r+\rho^- + \mu)(V^{\emptyset}_{hn} - V^{\emptyset}_{hn-B^d}) &= v^0 - rA^0 - \rho^-\Delta \\ &+ \mu p(V^u_{ho-out}(0) - A^0 - V^u_{hn-B^0}(0)) + \mu(1-p)(V^d_{ho-out}(0) - A^0 - V^d_{hn-B^d}(0))\end{aligned}$$

type ho**.**

- A type ho can deviate by sending a market order at B^0. This is clearly not profitable because this is optimal for a type hn to send a market order at A^0.
- A type ho can deviate by sending a limit order at a price $A > A^0$ and gets the value defined by

$$(r+\rho^- + \lambda + \mu)V = v^0 + \rho^- V^{\emptyset}_{lo} + \lambda V^{\emptyset}_{ho} + \mu p V^u_{ho-A}(0) + \mu(1-p)V^d_{ho-out}(0)$$

  because $V^u_{ho-A}(0) < V^u_{ho-out}(0)$ (by optimality in the transition phase) we obtain $V < V^{\emptyset}_{ho}$.
- the last deviation possible for a type ho is to send a limit order at A^0 and gets the value defined by

$$(r+\rho^- + \lambda + l^{\emptyset}_{A^0} + \mu)V = v^0 + \rho^- V^{\emptyset}_{lo} + \lambda V^{\emptyset}_{ho} + l^{\emptyset}_{A^0}(V^{\emptyset}_{hn-out} + A^0) + \mu p V^u_{ho-A^0}(0) + \mu(1-p)V^d_{ho-out}(0)$$

  because $V^u_{ho-A^0}(0) < V^u_{ho-out}(0)$ and $V^{\emptyset}_{hn-out} + A^0 < V^{\emptyset}_{hn} + A^0 = V^{\emptyset}_{ho}$ we obtain $V < V^{\emptyset}_{ho}$.

type ln**.**

- A type ln can deviate by sending a market order at A^0. This is clearly not profitable because this is optimal for a type lo to send a market order at B^0.
- A type ln can deviate by sending a limit order at a price $B < B^0$ and gets the value defined by

$$(r+\rho^+ + \lambda + \mu)V = \rho^+ V^{\emptyset}_{hn} + \lambda V^{\emptyset}_{ln} + \mu p V^u_{ln-out}(0) + \mu(1-p)V^d_{ln-B}(0)$$

  because $V^d_{ln-B}(0) < V^d_{ln-out}(0)$ we obtain $V < V^{\emptyset}_{ln}$.
- the last deviation possible for a type ln is to send a limit order at B^0 and gets the value defined by

$$(r+\rho^+ + \lambda + l^{\emptyset}_{B^0} + \mu)V = \rho^+ V^{\emptyset}_{hn} + \lambda V^{\emptyset}_{ln} + l^{\emptyset}_{B^0}(V^{\emptyset}_{lo-out} - B^0) + \mu p V^u_{ln-out}(0) + \mu(1-p)V^d_{ln-B^0}(0)$$

  because $V^d_{ln-B^0}(0) < V^d_{ln-out}(0)$ and $V^{\emptyset}_{lo-out} - B^0 < V^{\emptyset}_{lo} - B^0 = V^{\emptyset}_{ln}$ we obtain $V < V^{\emptyset}_{ln}$.

Now let's show first the 4 following formulas:

$$\begin{aligned}
V^u_{ln-out}(t) + A^0 - V^u_{lo-A^0}(t) &= -[v^u - \delta - rA^0]\int_t^{T^u} \frac{D^u_{A^0}(s)e^{-rs}}{D^u_{A^0}(t)e^{-rt}}ds \\
V^d_{ln-out}(t) + B^0 - V^d_{lo-A^0}(t) &= \frac{rB^0 - (v^d - \delta)}{r+\rho^+ + \lambda} + \frac{(\lambda+\rho^+)(B^0 - B^d) - \rho^+\Delta}{r+\rho^+ + \lambda}e^{-(r+\rho^+ + \lambda)T^d} \times e^{(r+\rho^+ + \lambda)t} \\
V^d_{ho-out}(t) - B^0 - V^d_{hn-B^0}(t) &= -[rB^0 - v^d]\int_t^{T^d} \frac{D^d_{B^0}(s)e^{-rs}}{D^d_{B^0}(t)e^{-rt}}ds \\
V^u_{ho-out}(t) - A^0 - V^u_{hn-B^0}(t) &= \frac{v^u - rA^0}{r+\rho^- + \lambda} + \frac{(\lambda+\rho^-)(A^u - A^0) - \rho^-\Delta}{r+\rho^- + \lambda}e^{-(r+\rho^- + \lambda)T^u} \times e^{(r+\rho^- + \lambda)t}
\end{aligned}$$

Let's call $X_1(t) = V^u_{ln-out}(t) + A^0 - V^u_{lo-A^0}(t)$

$$(r - \frac{\dot{D}^u_{A^0}}{D^u_{A^0}(t)})X_1(t) = \frac{\partial X_1}{\partial t} - [v^u - \delta - rA^0]$$

The solution of the homogeneous ODE is $t \mapsto c\frac{D^u_{A^0}(0)}{D^u_{A^0}(t)}e^{rt}$. Then we look for a solution of the type $X_1(t) = c(t)\frac{D^u_{A^0}(0)}{D^u_{A^0}(t)}e^{rt}$ and we obtain

$$c(t) = c_0 + [v^u - \delta - rA^0]\int_0^t \frac{D^u_{A^0}(s)}{D^u_{A^0}(0)}e^{-rs}ds$$

and because $X_1(T^u) = 0$ we must have

$$c_0 = -[v^u - \delta - rA^0]\int_0^{T^u} \frac{D^u_{A^0}(s)}{D^u_{A^0}(0)}e^{-rs}ds$$

and we obtain

$$X_1(t) = -[v^u - \delta - rA^0]\int_t^{T^u} \frac{D^u_{A^0}(s)e^{-rs}}{D^u_{A^0}(t)e^{-rt}}ds$$

We also know that

$$(r - \frac{\dot{D}^d_{B^0}}{D^d_{B^0}(t)})(V^u_{ho-out}(t) - B^0 - V^u_{hn-B^0}(t)) = \frac{\partial}{\partial t}(V^u_{ho-out}(t) - B^0 - V^u_{hn-B^0}(t)) - [rA^0 - v^d]$$

which leads to the fourth equation of the lemma.

Let's call $X_2(t) = V^d_{ln-out}(t) - V^d_{lo-A^0}(t)$

$$(r + \rho^+ + \lambda)X_2(t) = \frac{\partial X_2}{\partial t} - (\lambda + \rho^+)B^0 - (v^d - \delta)$$

and $(r + \rho^+ + \lambda)X^2(T^d) = \rho^+((\bar{V}^d_{hn-B^d} - \bar{V}^d_{ho}) + \lambda(\bar{V}^d_{ln-out} - \bar{V}^d_{lo-A^d}) - (v^d - \delta) = -\rho^+A^d - \lambda B^d - (v^d - \delta)$. This can be solved easily.

And finally

$$(r + \rho^- + \lambda)(V^u_{ho-out}(t) - V^d_{hn-B^0}(t)) = \frac{\partial}{\partial t}(V^u_{ho-out}(t) - V^d_{hn-B^0}(t)) + (\lambda + \rho^-)A^0 + v^u$$

with $(r + \rho^- + \lambda)(\bar{V}^u_{ho-out} - \bar{V}^d_{hn-B^0}) = v^u + \rho^- B^u + \lambda A^u$

Symmetric parametrization

$\rho^- = \rho^+ = \rho$, $s = 1/2$
In this case the conditions on $m^{\emptyset}_{A^0}$ and $m^{\emptyset}_{B^0}$ are verified.

We can now calculate

$$\begin{aligned}
\int_0^T D(t)e^{-rt} &= \frac{1}{r}\frac{1}{2}L(e^{-rT} - 1) + \frac{1}{r + \rho + \lambda}[\alpha^{\emptyset} + \frac{1}{2}]L(1 - e^{-(r+\rho+\lambda)T}) \\
&= \frac{1}{r}\frac{1}{2}L(e^{-rT} - 1) + \frac{1}{r + \rho + \lambda}[\alpha^{\emptyset} + \frac{1}{2}]L - \frac{1}{r + \rho + \lambda}\frac{1}{2}Le^{-rT} \\
&= \frac{1}{r + \rho + \lambda}\alpha^{\emptyset}L - \frac{1}{2}\frac{\rho + \lambda}{r(r + \rho + \lambda)}L + \frac{1}{2}\frac{\rho + \lambda}{r(r + \rho + \lambda)}L\frac{1}{(1 + 2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}}}
\end{aligned}$$

Then we get

$$
\begin{aligned}
l^{\emptyset}_{B^0} + l^{\emptyset}_{A^0} + 2\rho = & \frac{\delta}{\Delta} - r \\
& + \frac{1}{\Delta}\mu p \left[-\Delta - (v^u - \delta - rA^0)\left(\frac{1}{r+\rho+\lambda} - \frac{1}{2\alpha^{\emptyset}}\frac{\rho+\lambda}{r(r+\rho+\lambda)} + \frac{1}{2\alpha^{\emptyset}}\frac{\rho+\lambda}{r(r+\rho+\lambda)}\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}}} \right) \right] \\
& + \frac{1}{\Delta}\mu(1-p)\left[\frac{rB^0 - (v^d - \delta)}{r+\rho+\lambda} + \frac{(\lambda+\rho)(B^0 - B^d) - \rho\Delta}{r+\rho+\lambda}\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r+\rho+\lambda}{\rho+\lambda}}} \right] \\
& + \frac{1}{\Delta}\mu p \left[\frac{v^u - rA^0}{r+\rho+\lambda} + \frac{(\lambda+\rho)(A^d - A^0) - \rho\Delta}{r+\rho+\lambda}\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r+\rho+\lambda}{\rho+\lambda}}} \right] \\
& + \frac{1}{\Delta}\mu(1-p)\left[-\Delta - (rB^0 - v^d)\left(\frac{1}{r+\rho+\lambda} - \frac{1}{2\alpha^{\emptyset}}\frac{\rho+\lambda}{r(r+\rho+\lambda)} + \frac{1}{2\alpha^{\emptyset}}\frac{\rho+\lambda}{r(r+\rho+\lambda)}\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}}} \right) \right]
\end{aligned}
$$

Perfectly symmetric parametrization

In the perfectly symmetric case we have

$$
\begin{aligned}
V^u_{ln-out}(t) + A^0 - V^u_{lo-A^0}(t) &= V^d_{ho-out}(t) - B^0 - V^d_{hn-B^0}(t) \\
&= -[\omega - \frac{\delta}{2} + r\frac{\Delta}{2}]\int_t^T \frac{D(s)e^{-rs}}{D(t)e^{-rt}}ds \\
V^d_{ln-out}(t) + B^0 - V^d_{lo-A^0}(t) &= V^u_{ho-out}(t) - A^0 - V^u_{hn-B^0}(t) \\
&= \frac{\omega + \frac{\delta}{2} - r\frac{\Delta}{2}}{r+\rho+\lambda} + \frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda}e^{-(r+\rho+\lambda)T} \times e^{(r+\rho+\lambda)t}
\end{aligned}
$$

which implies that $l^{\emptyset}_A = l^{\emptyset}_B = l^{\emptyset}$ and that

$$
\begin{aligned}
2l^{\emptyset} + 2\rho = & \frac{\delta}{\Delta} - r \\
& + \frac{1}{\Delta}\mu\left[-\Delta - (\omega - \frac{\delta}{2} + r\frac{\Delta}{2})\left(\frac{1}{r+\rho+\lambda} - \frac{1}{2\alpha^{\emptyset}}\frac{\rho+\lambda}{r(r+\rho+\lambda)} + \frac{1}{2\alpha^{\emptyset}}\frac{\rho+\lambda}{r(r+\rho+\lambda)}\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}}} \right) \right] \\
& + \frac{1}{\Delta}\mu\left[\frac{\omega + \frac{\delta}{2} - r\frac{\Delta}{2}}{r+\rho+\lambda} + \frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda}\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r+\rho+\lambda}{\rho+\lambda}}} \right]
\end{aligned}
$$

To prove the existence of the equilibrium we have to show that there is an $\alpha^{\emptyset}$ solution to the equation

$$G(\alpha^{\emptyset}) = \alpha^{\emptyset} \times (2l^{\emptyset} + 2\rho) = \frac{\rho}{2}$$

such that

$$(r+\rho+\mu)(V^{\emptyset}_{hn} - V^{\emptyset}_{hn-B^d}) > 0$$

$$(r+\rho+\mu)(V^{\emptyset}_{lo} - V^{\emptyset}_{lo-A^u}) > 0$$

and $l^{\emptyset} > 0$. We can notice that

$$(r+\rho+\mu)(V^{\emptyset}_{hn} - V^{\emptyset}_{hn-B^d}) = l^{\emptyset}_B\Delta - \frac{\mu}{2}(V^d_{hn-B^d}(0) - V^d_{hn-B^0}(0)) < l^{\emptyset}_B\Delta$$

so as soon as $(r+\rho+\mu)(V^{\emptyset}_{hn} - V^{\emptyset}_{hn-B^d}) > 0$, $l^{\emptyset}_B > 0$. As well for $(r+\rho+\mu)(V^{\emptyset}_{lo} - V^{\emptyset}_{lo-A^u})$ and $l^{\emptyset}_A$.

Analysis of function G. Let's prove that $G(\alpha) - \frac{\rho}{2}$ has a unique zero on $[0, 1/4]$.

$$
\begin{aligned}
G(\alpha^\emptyset) = {} & \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega-\frac{\delta}{2}+r\frac{\Delta}{2}}{2(r+\rho+\lambda)} + \alpha^\emptyset \times \left\{\frac{\delta}{\Delta} - r + \frac{\mu}{\Delta}\left[-\Delta + \frac{\delta - r\Delta}{r+\rho+\lambda}\right]\right\} \\
& - \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega-\frac{\delta}{2}+r\frac{\Delta}{2}}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha^\emptyset)^{\frac{r}{\rho+\lambda}}} + \frac{\mu}{\Delta}\frac{(\lambda+\rho)\frac{\omega}{r}-\rho\Delta}{r+\rho+\lambda}\frac{\alpha^\emptyset}{(1+2\alpha^\emptyset)^{\frac{r}{\rho+\lambda}+1}}
\end{aligned}
$$

The second derivative of G is the second derivative of

$$
f(\alpha^\emptyset) = -\frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega-\frac{\delta}{2}+r\frac{\Delta}{2}}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha^\emptyset)^{\frac{r}{\rho+\lambda}}} + \frac{\mu}{\Delta}\frac{(\lambda+\rho)\frac{\omega}{r}-\rho\Delta}{r+\rho+\lambda}\frac{\alpha^\emptyset}{(1+2\alpha^\emptyset)^{\frac{r}{\rho+\lambda}+1}}
$$

that can be rewritten

$$
\begin{aligned}
f(\alpha^\emptyset) &= -\frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega-\frac{\delta}{2}+r\frac{\Delta}{2}}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha^\emptyset)^{\frac{r}{\rho+\lambda}}} + \frac{\mu}{\Delta}\frac{(\lambda+\rho)\frac{\omega}{r}-\rho\Delta}{r+\rho+\lambda}\frac{\frac{1}{2}+\alpha^\emptyset-\frac{1}{2}}{(1+2\alpha^\emptyset)^{\frac{r}{\rho+\lambda}+1}} \\
&= \frac{\mu}{\Delta}\frac{\frac{\rho+\lambda}{r}\frac{\delta-r\Delta}{2}-\rho\Delta}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha^\emptyset)^{\frac{r}{\rho+\lambda}}} - \frac{\mu}{\Delta}\frac{(\lambda+\rho)\frac{\omega}{r}-\rho\Delta}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha^\emptyset)^{\frac{r}{\rho+\lambda}+1}}
\end{aligned}
$$

The derivatives of f are

$$
\begin{aligned}
\frac{\partial f}{\partial \alpha^\emptyset} = {} & \frac{\mu}{\Delta}\frac{1}{2(r+\rho+\lambda)}\left\{-2\frac{r}{\rho+\lambda}\left[\frac{\rho+\lambda}{r}\frac{\delta-r\Delta}{2}-\rho\Delta\right]\frac{1}{(1+2\alpha^\emptyset)^{\frac{r}{\rho+\lambda}+1}}\right\} \\
& + \frac{\mu}{\Delta}\frac{1}{2(r+\rho+\lambda)}\left\{2\left(\frac{r}{\rho+\lambda}+1\right)\left[\frac{\lambda+\rho}{r}\omega-\rho\Delta\right]\frac{1}{(1+2\alpha^\emptyset)^{\frac{r}{\rho+\lambda}+2}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial(\alpha^\emptyset)^2} = {} & \frac{\mu}{\Delta}\frac{1}{2(r+\rho+\lambda)}\left[4\frac{r}{\rho+\lambda}\left(\frac{r}{\rho+\lambda}+1\right)\left[\frac{\rho+\lambda}{r}\frac{\delta-r\Delta}{2}-\rho\Delta\right]\frac{1}{(1+2\alpha^\emptyset)^{\frac{r}{\rho+\lambda}+2}}\right] \\
& - \frac{\mu}{\Delta}\frac{1}{2(r+\rho+\lambda)}\left[4\left(\frac{r}{\rho+\lambda}+1\right)\left(\frac{r}{\rho+\lambda}+2\right)\left[\frac{\lambda+\rho}{r}\omega-\rho\Delta\right]\frac{1}{(1+2\alpha^\emptyset)^{\frac{r}{\rho+\lambda}+3}}\right]
\end{aligned}
$$

The sign of $\frac{\partial^2 f}{\partial(\alpha^\emptyset)^2}$ is the sign of

$$
s(\alpha) = \frac{r}{\rho+\lambda}\left[\frac{\rho+\lambda}{r}\frac{\delta-r\Delta}{2}-\rho\Delta\right] \times (1+2\alpha^\emptyset) - \left(\frac{r}{\rho+\lambda}+2\right)\left[\frac{\lambda+\rho}{r}\omega-\rho\Delta\right]
$$

Since $\omega > \frac{\delta-r\Delta}{2}$ we have

$$
s(\alpha) < \left[\frac{\lambda+\rho}{r}\omega-\rho\Delta\right] \times \left[\frac{r}{\rho+\lambda}\times(1+2\alpha^\emptyset) - \left(\frac{r}{\rho+\lambda}+2\right)\right]
$$

And because at equilibrium $0 \leq \alpha^\emptyset \leq 1/4 < 1$

$$
s(\alpha) < \left[\frac{\lambda+\rho}{r}\omega-\rho\Delta\right] \times \left(2\frac{r}{\rho+\lambda}-2\right) < 0
$$

On $[0, 1/4]$, $\frac{\partial^2 f}{\partial(\alpha^\emptyset)^2} < 0$ then on $[0, 1/4]$ $\frac{\partial G}{\partial \alpha^\emptyset}$ is either always positive, always negative, or positive and then negative. G can at most cross the $\rho/2$ horizontal line on $[0, 1/4]$ twice. And if $G(1/4) > \frac{\rho}{2}$ it means that it is crossed only once.

$$G(1/4) - \frac{\rho}{2} = \frac{1}{4\Delta}\left[\delta - (r+2\rho)\Delta\right]$$
$$+ \frac{\mu}{\Delta}\left[\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)} - \frac{\Delta}{4} + \frac{\delta - r\Delta}{4(r+\rho+\lambda)}\right]$$
$$+ \frac{\mu}{\Delta}\left[\frac{\frac{\rho+\lambda}{r}\frac{\delta-r\Delta}{2} - \rho\Delta}{2(r+\rho+\lambda)}\frac{1}{(3/2)^{\frac{r}{\rho+\lambda}}} - \frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{2(r+\rho+\lambda)}\frac{1}{(3/2)^{\frac{r}{\rho+\lambda}+1}}\right]$$

First $\delta - (r+2\rho)\Delta > 0$. Moreover $1 > \frac{1}{(3/2)^{\frac{r}{\rho+\lambda}}} > \frac{2}{3}$. Then

$$G(1/4) - \frac{\rho}{2} > \frac{\mu}{(r+\rho+\lambda)\Delta}\left[\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2} - \frac{(r+\rho+\lambda)\Delta}{4} + \frac{\delta - r\Delta}{4}\right]$$
$$+ \frac{\mu}{(r+\rho+\lambda)\Delta}\left[\frac{\frac{\rho+\lambda}{r}\frac{\delta-r\Delta}{2} - \rho\Delta}{3} - \frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{3}\right]$$

then

$$\frac{(r+\rho+\lambda)\Delta}{\mu}(G(1/4) - \frac{\rho}{2}) > \omega \times \left(\frac{1}{6}\frac{\rho+\lambda}{r}\right) + \frac{\delta}{2} \times \left(-\frac{1}{6}\frac{\rho+\lambda}{r} + \frac{1}{2}\right) + r\Delta \times \left(-\frac{1}{6}\frac{\rho+\lambda}{r} - \frac{1}{2}\right) > 0$$

Because $G(0) = 0$ there is unique $\alpha \in [0, 1/4]$ such that $G(\alpha^{\emptyset}) = \frac{\rho}{2}$. Moreover $\frac{\partial G}{\partial \alpha}(\alpha^{\emptyset}_{eq}) > 0$.

Other equilibrium conditions. First, because we are in the perfectly symmetric case, we have

$$(r+\rho+\mu)(V^{\emptyset}_{hn} - V^{\emptyset}_{hn-B^d}) = (r+\rho+\mu)(V^{\emptyset}_{lo} - V^{\emptyset}_{lo-A^u})$$

Indeed

$$(r+\rho+\mu)(V^{\emptyset}_{hn} - V^{\emptyset}_{hn-B^d}) = v^0 - rA^0 - \rho\Delta$$
$$+ \mu\frac{1}{2}(V^u_{ho-out}(0) - A^0 - V^u_{hn-B^0}(0)) + \mu\frac{1}{2}(V^d_{ho-out}(0) - A^0 - V^d_{hn-B^d}(0))$$

$$(r+\rho+\mu)(V^{\emptyset}_{lo} - V^{\emptyset}_{lo-A^u}) = rB^0 - \rho\Delta - (v^0 - \delta)$$
$$+ \mu\frac{1}{2}(V^u_{ln-out}(0) + B^0 - V^u_{lo-A^u}(0)) + \mu\frac{1}{2}(V^d_{ln-out}(0) + B^0 - V^d_{lo-A^0}(0))$$

with $v^0 - rA^0 - \rho\Delta = rB^0 - \rho\Delta - (v^0 - \delta) = \frac{\delta - (r+2\rho)\Delta}{2} > 0$ and $V^d_{ho-out}(t) - A^0 - V^d_{hn-B^d}(t) = V^u_{ln-out}(t) + B^0 - V^u_{lo-A^u}(t)$ because these two expressions are solutions of the same ODE

$$(r+\rho+\lambda)X = \frac{dX}{dt} + c$$

with $c = v^d - rA^0 - (\rho+\lambda)\Delta = rB^0 - (v^u - \delta) - (\rho+\lambda)\Delta = -\omega + \frac{\delta}{2} - (\frac{r}{2} + \rho + \lambda)\Delta$ and the conditions at $t = T$

$$\bar{V}^d_{ho-out} - A^0 - \bar{V}^d_{hn-B^d} = \bar{V}^u_{ln-out} + B^0 - \bar{V}^u_{lo-A^u} = -\frac{\omega}{r}$$

then we have

$$V^d_{ho-out}(t) - A^0 - V^d_{hn-B^d}(t) = V^u_{ln-out}(t) + B^0 - V^u_{lo-A^u}(t)$$
$$= -\left[\frac{\omega - \frac{\delta}{2} + (\frac{r}{2} + \rho + \lambda)\Delta}{r+\rho+\lambda} + \left(\frac{\omega}{r} - \frac{\omega - \frac{\delta}{2} + (\frac{r}{2}+\rho+\lambda)\Delta}{r+\rho+\lambda}\right)e^{-(r+\rho+\lambda)(T-t)}\right]$$

and finally

$$\begin{aligned}\frac{X(\alpha^\emptyset)}{2} &= (r+\rho+\mu)(V^\emptyset_{hn} - V^\emptyset_{hn-B^d}) = (r+\rho+\mu)(V^\emptyset_{lo} - V^\emptyset_{lo-A^u})\\ &= \frac{\delta-(r+2\rho)\Delta}{2}\\ &+\frac{\mu}{2}\left[\frac{\omega+\frac{\delta}{2}-r\frac{\Delta}{2}}{r+\rho+\lambda}+\frac{(\lambda+\rho)\frac{\omega}{r}-\rho\Delta}{r+\rho+\lambda}\frac{1}{(1+2\alpha^\emptyset)^{\frac{r+\rho+\lambda}{\rho+\lambda}}}\right]\\ &-\frac{\mu}{2}\left[\frac{\omega-\frac{\delta}{2}+(\frac{r}{2}+\rho+\lambda)\Delta}{r+\rho+\lambda}+\left(\frac{\omega}{r}-\frac{\omega-\frac{\delta}{2}+(\frac{r}{2}+\rho+\lambda)\Delta}{r+\rho+\lambda}\right)\frac{1}{(1+2\alpha^\emptyset)^{\frac{r+\rho+\lambda}{\rho+\lambda}}}\right]\end{aligned}$$

That can be rewritten

$$\begin{aligned}X(\alpha^\emptyset) = \delta-(r+2\rho)\Delta &+ \mu\frac{\delta-(r+\rho+\lambda)\Delta}{r+\rho+\lambda}\\ &+\mu\frac{(r+2\lambda)\Delta-\delta}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha^\emptyset)^{\frac{r+\rho+\lambda}{\rho+\lambda}}}\end{aligned}$$

Given that $(r+\rho+\mu)X(\alpha^\emptyset) < 2l^\emptyset$ then if we know that for all t $X(t) > 0$ then $\lim_\infty G(\alpha^\emptyset) = \infty$ and since $G(0) = 0$ there is an equilibrium.

case where $X(\alpha^\emptyset) > 0\ \forall\alpha^\emptyset$

- if $\delta > (r+2\lambda)\Delta > (r+\rho+\lambda)\Delta$ then $X(\alpha^\emptyset)$ is increasing and

$$X(0) = (\delta-(r+2\rho)\Delta)(1+\mu\frac{1}{2(r+\rho+\lambda)}) > 0$$

- if $(r+2\lambda)\Delta > \delta > (r+\rho+\lambda)\Delta$ then $X(\alpha^\emptyset)$ is decreasing and obviously $X(\alpha^\emptyset) > 0\forall\alpha^\emptyset$
- if $(r+2\lambda)\Delta > (r+\rho+\lambda)\Delta > \delta$ then $X(\alpha^\emptyset)$ is decreasing and

$$X(\infty) = \delta-(r+2\rho)\Delta-\mu\frac{(r+\rho+\lambda)\Delta-\delta}{r+\rho+\lambda}$$

which is positive iff $\mu < \frac{\delta-(r+2\rho)\Delta}{\Delta-\frac{\delta}{r+\rho+\lambda}}$.

case where $X(\alpha^\emptyset)$ can be negative

If $(r+2\lambda)\Delta > (r+\rho+\lambda)\Delta > \delta$ and $\mu > \frac{\delta-(r+2\rho)\Delta}{\Delta-\frac{\delta}{r+\rho+\lambda}}$, $X(\alpha^\emptyset)$ is decreasing and has a negative limit. Then there is a unique a_0 such that $X(\alpha^\emptyset) > 0$ for $\alpha^\emptyset < a_0$. And we have

$$\mu\frac{(r+2\lambda)\Delta-\delta}{2(r+\rho+\lambda)}\frac{1}{(1+2a_0)^{\frac{r+\rho+\lambda}{\rho+\lambda}}} = -(\delta-(r+2\rho)\Delta)+\mu\frac{(r+\rho+\lambda)\Delta-\delta}{r+\rho+\lambda}$$

then

$$\frac{1}{(1+2a_0)^{\frac{r+\rho+\lambda}{\rho+\lambda}}} < 2\frac{(r+\rho+\lambda)\Delta-\delta}{(r+2\lambda)\Delta-\delta} \Leftrightarrow a_0 > \frac{1}{2}\left[\left(\frac{(r+2\lambda)\Delta-\delta}{2[(r+\rho+\lambda)\Delta-\delta]}\right)^{\frac{\rho+\lambda}{r+\rho+\lambda}}-1\right] > 0$$

One possible sufficient conditions for the existence of an equilibrium:

$$\frac{1}{2}\left[\left(\frac{(r+2\lambda)\Delta-\delta}{2[(r+\rho+\lambda)\Delta-\delta]}\right)^{\frac{\rho+\lambda}{r+\rho+\lambda}}-1\right] > \frac{1}{4}$$

but it does not hold if λ is too big.

We can extend the domain of validity of the equilibrium by looking for conditions to verify that $G(a_0) > \frac{\rho}{2}$. Let's first notice that

$$\begin{aligned}
f(\alpha^{\emptyset}) &= \frac{\mu}{\Delta}\frac{\frac{\rho+\lambda}{r}\frac{\delta-r\Delta}{2}-\rho\Delta}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}}} - \frac{\mu}{\Delta}\frac{(\lambda+\rho)\frac{\omega}{r}-\rho\Delta}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}+1}} \\
&> \frac{\mu}{\Delta}\frac{\frac{\rho+\lambda}{r}\frac{\delta-r\Delta}{2}-\rho\Delta}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}+1}} - \frac{\mu}{\Delta}\frac{(\lambda+\rho)\frac{\omega}{r}-\rho\Delta}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}+1}} \\
&> -\frac{\mu}{\Delta}\frac{1}{2(r+\rho+\lambda)}\frac{\lambda+\rho}{r}\left(\omega-\frac{\delta-r\Delta}{2}\right)\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}+1}}
\end{aligned}$$

then

$$\begin{aligned}
G(\alpha^{\emptyset}) > {} & \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega-\frac{\delta}{2}+r\frac{\Delta}{2}}{2(r+\rho+\lambda)} + \alpha^{\emptyset}\times\left\{\frac{\delta}{\Delta}-r+\frac{\mu}{\Delta}\left[-\Delta+\frac{\delta-r\Delta}{r+\rho+\lambda}\right]\right\} \\
& -\frac{\mu}{\Delta}\frac{1}{2(r+\rho+\lambda)}\frac{\lambda+\rho}{r}\left(\omega-\frac{\delta-r\Delta}{2}\right)\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}+1}}
\end{aligned}$$

assuming that

$$\frac{\delta}{\Delta}-r+\frac{\mu}{\Delta}\left[-\Delta+\frac{\delta-r\Delta}{r+\rho+\lambda}\right] > 0$$

we obtain that

$$\begin{aligned}
G(a_0) &> \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega-\frac{\delta}{2}+r\frac{\Delta}{2}}{2(r+\rho+\lambda)} - \frac{\mu}{\Delta}\frac{1}{2(r+\rho+\lambda)}\frac{\rho+\lambda}{r}\left(\omega-\frac{\delta-r\Delta}{2}\right)\frac{1}{(1+2a_0)^{\frac{r}{\rho+\lambda}+1}} \\
&> \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega-\frac{\delta}{2}+r\frac{\Delta}{2}}{2(r+\rho+\lambda)} \\
&-\frac{\rho+\lambda}{r}\frac{\omega-\frac{\delta}{2}+r\frac{\Delta}{2}}{(r+2\lambda)\Delta-\delta}\left[-\frac{\delta-(r+2\rho)\Delta}{\Delta}+\frac{\mu}{\Delta}\frac{(r+\rho+\lambda)\Delta-\delta}{r+\rho+\lambda}\right] \\
&> \frac{\rho+\lambda}{r}\frac{\omega-\frac{\delta}{2}+r\frac{\Delta}{2}}{(r+2\lambda)\Delta-\delta}\frac{\delta-(r+2\rho)\Delta}{\Delta} \\
&+\frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega-\frac{\delta}{2}+r\frac{\Delta}{2}}{r+\rho+\lambda}\times\left[\frac{1}{2}-\frac{(r+\rho+\lambda)\Delta-\delta}{(r+2\lambda)\Delta-\delta}\right]
\end{aligned}$$

the first term decreases w.r.t λ and converges towards $\frac{\omega-\frac{\delta}{2}+r\frac{\Delta}{2}}{2r\Delta}\frac{\delta-(r+2\rho)\Delta}{\Delta} > \frac{\rho}{2}$. The second term is positive.

We can also check that for μ big enough it works without this assumption. Let's call

$$\begin{aligned}
g(\alpha^{\emptyset}) = {} & \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega-\frac{\delta}{2}+r\frac{\Delta}{2}}{2(r+\rho+\lambda)} + \alpha^{\emptyset}\times\left\{\frac{\delta}{\Delta}-r+\frac{\mu}{\Delta}\left[-\Delta+\frac{\delta-r\Delta}{r+\rho+\lambda}\right]\right\} \\
& -\frac{\mu}{\Delta}\frac{1}{2(r+\rho+\lambda)}\frac{\lambda+\rho}{r}\left(\omega-\frac{\delta-r\Delta}{2}\right)\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}+1}}
\end{aligned}$$

and look at the limit $\mu\to\infty$ with $\alpha=\frac{1}{\mu}\to 0$

$$
\begin{aligned}
g(\frac{1}{\mu}) &\approx \frac{1}{\Delta}\left[-\Delta + \frac{\delta - r\Delta}{r+\rho+\lambda}\right] + \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)} \times \left(1 - \frac{1}{(1+\frac{2}{\mu})^{\frac{r}{\rho+\lambda}+1}}\right) \\
&\cong \frac{1}{\Delta}\left[-\Delta + \frac{\delta - r\Delta}{r+\rho+\lambda}\right] + \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)} \times \frac{r+\rho+\lambda}{\rho+\lambda}\frac{2}{\mu} \\
&\cong \frac{1}{\Delta}\left[\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{r} - \Delta + \frac{\delta - r\Delta}{r+\rho+\lambda}\right] \\
&> \frac{1}{\Delta}\left[\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{r} - \Delta\right] > \frac{\rho}{2}
\end{aligned}
$$

And because $a_0 > \frac{1}{2}\left[\left(\frac{(r+2\lambda)\Delta - \delta}{2[(r+\rho+\lambda)\Delta - \delta]}\right)^{\frac{\rho+\lambda}{r+\rho+\lambda}} - 1\right] > 0$, for μ big enough the equilibrium α is less than a_0.

This also proves that $\lim_{\mu\to\infty} \alpha^{\emptyset}_{eq} = 0$

Now if

$$
\frac{\delta}{\Delta} - r + \frac{\mu}{\Delta}\left[-\Delta + \frac{\delta - r\Delta}{r+\rho+\lambda}\right] < 0
$$

Since

$$
\begin{aligned}
G(\alpha^{\emptyset}) > &\frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)} + \alpha^{\emptyset} \times \left\{\frac{\delta}{\Delta} - r + \frac{\mu}{\Delta}\left[-\Delta + \frac{\delta - r\Delta}{r+\rho+\lambda}\right]\right\} \\
&- \frac{\mu}{\Delta}\frac{1}{2(r+\rho+\lambda)}\frac{\lambda+\rho}{r}\left(\omega - \frac{\delta - r\Delta}{2}\right)\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}+1}}
\end{aligned}
$$

we obtain for the equilibrium $\alpha^{\emptyset}$

$$
\frac{\mu}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}+1}} > \frac{\mu}{2(r+\rho+\lambda)} - \frac{\rho\Delta}{2} \times \frac{1}{\frac{\lambda+\rho}{r}\left(\omega - \frac{\delta - r\Delta}{2}\right)} - \alpha^{\emptyset} \times \frac{\mu\left[\Delta - \frac{\delta - r\Delta}{r+\rho+\lambda}\right] - (\delta - r\Delta)}{\frac{\lambda+\rho}{r}\left(\omega - \frac{\delta - r\Delta}{2}\right)}
$$

then

$$
\begin{aligned}
X(\alpha^{\emptyset}) > &\ \delta - (r+2\rho)\Delta + \mu\frac{\delta - (r+\rho+\lambda)\Delta}{r+\rho+\lambda} \\
&+ [(r+2\lambda)\Delta - \delta] \times \left\{\frac{\mu}{2(r+\rho+\lambda)} - \frac{\rho\Delta}{2} \times \frac{1}{\frac{\lambda+\rho}{r}\left(\omega - \frac{\delta - r\Delta}{2}\right)} - \alpha^{\emptyset} \times \frac{\mu\left[\Delta - \frac{\delta - r\Delta}{r+\rho+\lambda}\right] - (\delta - r\Delta)}{\frac{\lambda+\rho}{r}\left(\omega - \frac{\delta - r\Delta}{2}\right)}\right\}
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
X(\alpha^{\emptyset}) > &(\delta - (r+2\rho)\Delta)(1 + \mu\frac{1}{2(r+\rho+\lambda)}) - \frac{\rho\Delta}{2} \times \frac{[(r+2\lambda)\Delta - \delta]}{\frac{\lambda+\rho}{r}\left(\omega - \frac{\delta - r\Delta}{2}\right)} \\
&- \alpha^{\emptyset} \times [(r+2\lambda)\Delta - \delta]\frac{\mu\left[\Delta - \frac{\delta - r\Delta}{r+\rho+\lambda}\right] - (\delta - r\Delta)}{\frac{\lambda+\rho}{r}\left(\omega - \frac{\delta - r\Delta}{2}\right)}
\end{aligned}
$$

Since this expression is decreasing w.r.t λ, letting $\alpha^{\emptyset}$ unchanged.

$$
\begin{aligned}
X(\alpha^{\emptyset}) &> \delta - (r+2\rho)\Delta - \frac{\rho\Delta}{2} \times \frac{2r\Delta}{\omega - \frac{\delta - r\Delta}{2}} - \alpha^{\emptyset} \times 2r\Delta \frac{\mu\Delta - (\delta - r\Delta)}{\omega - \frac{\delta - r\Delta}{2}} \\
&> \delta - (r+2\rho)\Delta + \frac{2r\Delta}{\omega - \frac{\delta - r\Delta}{2}} \left(\alpha^{\emptyset}(\delta - r\Delta) - \frac{\rho\Delta}{2} \right) - \alpha^{\emptyset} \times 2r\Delta \frac{\mu\Delta}{\omega - \frac{\delta - r\Delta}{2}} \\
&> \delta - (r+2\rho)\Delta - \frac{2r\Delta}{\omega - \frac{\delta - r\Delta}{2}} \frac{\rho\Delta}{2} - \alpha^{\emptyset} \times 2r\Delta \frac{\mu\Delta}{\omega - \frac{\delta - r\Delta}{2}}
\end{aligned}
$$

Now we need to give a bound for $\alpha^{\emptyset} \times \mu$ and show that it is small enough.

$$
\begin{aligned}
G(\alpha^{\emptyset}) &= \frac{\mu}{\Delta} \frac{\rho + \lambda}{r} \frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)} + \alpha^{\emptyset} \times \left\{ \frac{\delta}{\Delta} - r + \frac{\mu}{\Delta} \left[-\Delta + \frac{\delta - r\Delta}{r+\rho+\lambda} \right] \right\} \\
&\quad - \frac{\mu}{\Delta} \frac{\rho + \lambda}{r} \frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)} \frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}}} + \frac{\mu}{\Delta} \frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda} \frac{\alpha^{\emptyset}}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}+1}} \\
&> \alpha^{\emptyset} \frac{\mu}{\Delta} \left[-\Delta + \frac{\delta - r\Delta}{r+\rho+\lambda} \right] + \frac{\mu}{\Delta} \frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda} \frac{\alpha^{\emptyset}}{(3/2)^2} \\
&> \alpha^{\emptyset} \frac{\mu}{\Delta} \left[\frac{1}{(3/2)^2} \frac{\omega}{r} - \Delta + \frac{\omega - \rho\Delta}{r+\rho+\lambda} \frac{1}{(3/2)^2} + \frac{\delta - r\Delta}{r+\rho+\lambda} \right] \\
&> \alpha^{\emptyset} \frac{\mu}{\Delta} \left[\frac{1}{(3/2)^2} \frac{\omega}{r} - \Delta \right]
\end{aligned}
$$

Then at equilibrium

$$
\frac{\rho}{2} > \alpha^{\emptyset} \frac{\mu}{\Delta} \left[\frac{1}{(3/2)^2} \frac{\omega}{r} - \Delta \right]
$$

and

$$
X(\alpha^{\emptyset}_{eq}) > \delta - (r+2\rho)\Delta - \frac{2r\Delta}{\omega - \frac{\delta - r\Delta}{2}} \frac{\rho\Delta}{2} - 2r\Delta \frac{\Delta}{\omega - \frac{\delta - r\Delta}{2}} \times \frac{\rho\Delta}{2\left[\frac{1}{(3/2)^2} \frac{\omega}{r} - \Delta \right]}
$$

Given the difference of magnitudes we assume it implies that $X(\alpha^{\emptyset}_{eq}) > 0$.

Magnitude Assumptions. These following conditions are sufficient for the equilibrium to exist.

$$
\omega \times \left(\frac{1}{6} \frac{\rho + \lambda}{r} \right) + \frac{\delta}{2} \times \left(-\frac{1}{6} \frac{\rho + \lambda}{r} + \frac{1}{2} \right) + r\Delta \times \left(-\frac{1}{6} \frac{\rho + \lambda}{r} - \frac{1}{2} \right) > 0
$$

$$
\delta - (r+2\rho)\Delta - \frac{2r\Delta}{\omega - \frac{\delta - r\Delta}{2}} \frac{\rho\Delta}{2} - 2r\Delta \frac{\Delta}{\omega - \frac{\delta - r\Delta}{2}} \times \frac{\rho\Delta}{2\left[\frac{1}{(3/2)^2} \frac{\omega}{r} - \Delta \right]} > 0
$$

$$
\frac{1}{(3/2)^2} \frac{\omega}{r} - \Delta > 0
$$

One can check that the following conditions are also sufficient so that the conditions above hold:

$$
\omega > 3\delta \times \max\left[1, \frac{2r+\rho}{2\rho} \right] \text{ and } \delta > (r+4\rho)\Delta
$$

A.5.4 Proof of Proposition A.12

We know that $\frac{\partial G}{\partial \alpha}(\alpha^{\emptyset}_{eq}) > 0$. Since we know that when μ is big then $\alpha^{\emptyset}_{eq}$ is close to zero, we must show that $\frac{\partial G}{\partial \lambda}$ around $\alpha = 0$.

$$\begin{aligned}G(\alpha^{\emptyset}) =& \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)} + \alpha^{\emptyset} \times \left\{\frac{\delta}{\Delta} - r + \frac{\mu}{\Delta}\left[-\Delta + \frac{\delta - r\Delta}{r+\rho+\lambda}\right]\right\}\\ &- \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}}} + \frac{\mu}{\Delta}\frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda}\frac{\alpha^{\emptyset}}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}+1}}\end{aligned}$$

Then for $\alpha \to 0$

$$\begin{aligned}G(\alpha^{\emptyset}) \sim& \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)} \times \left[1 - 1 + \frac{r}{\rho+\lambda}2\alpha^{\emptyset}\right]\\ &+ \alpha^{\emptyset} \times \left\{\frac{\delta}{\Delta} - r - \mu + \frac{\mu}{\Delta}\frac{\delta - r\Delta}{r+\rho+\lambda}\right\}\\ &+ \frac{\mu}{\Delta}\frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda}\alpha^{\emptyset}\end{aligned}$$

which gives

$$G(\alpha^{\emptyset}) \sim \alpha^{\emptyset} \times \left\{\frac{\delta}{\Delta} - r - \mu + \frac{\mu}{\Delta}\frac{\omega}{r} + \frac{\mu}{\Delta}\frac{\delta - (r+2\rho)\Delta}{2(r+\rho+\lambda)}\right\}$$

which is decreasing with respect to λ.

For the two last derivative this is sufficient to show that what is in factor of μ or ω in function G is positive. For μ:

$$\frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)}\left[1 - \frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}}}\right] + \alpha^{\emptyset} \times \frac{\mu}{\Delta}\left\{\frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda}\frac{1}{(1+2\alpha^{\emptyset})^{\frac{r}{\rho+\lambda}+1}} - \Delta + \frac{\delta - r\Delta}{r+\rho+\lambda}\right\}$$

which works also for ω.

A.6 Comparative statics

The execution rate depends on 3 important parameter: the depth α, the monitoring rate λ^* of the investor and the monitoring rate λ_0 of other investors with the constraint that $\lambda^* = \lambda_0 = \lambda$. Writing l in function of these 3 parameters allows for disentangling where the effect of λ from. l depends on

$$\begin{aligned}
U_1 + \Delta = V^u_{ln-out}(0) + A^0 - V^u_{lo-A^0}(0) &= V^d_{ho-out}(0) - B^0 - V^d_{hn-B^0}(0) \\
&= -[\omega - \frac{\delta}{2} + r\frac{\Delta}{2}] \int_0^T \frac{D(t)e^{-(r+\lambda^*-\lambda_0)t}}{\alpha} ds \\
&= -[\omega - \frac{\delta}{2} + r\frac{\Delta}{2}] \int_0^T h(t)e^{-(r+\rho+\lambda^*)t} dt \\
U_2 = V^d_{ln-out}(0) + B^0 - V^d_{lo-A^0}(0) &= V^u_{ho-out}(0) - A^0 - V^u_{hn-B^0}(0) \\
&= \frac{\omega + \frac{\delta}{2} - r\frac{\Delta}{2}}{r+\rho+\lambda^*} + \frac{(\lambda^*+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda^*} e^{-(r+\rho+\lambda^*)T}
\end{aligned}$$

with $h(t) = \frac{D(t)}{\alpha e^{-(\lambda_0+\rho)t}} = 1 - \frac{1}{2}\frac{(1-e^{-(\lambda_0+\rho)t})}{\alpha e^{-(\lambda_0+\rho)t}} = 1 + \frac{1}{2\alpha} - \frac{1}{2\alpha}e^{(\lambda_0+\rho)t}$ and $T = \frac{\ln(1+2\alpha)}{\rho+\lambda_0}$, noticing that $h(T) = 0$.

$$\begin{aligned}
l(\alpha, \lambda^*, \lambda_0) = &\frac{\delta - (r+2\rho)\Delta}{2\Delta} \\
&+ \frac{1}{2\Delta}\mu \left[-\Delta - (\omega - \frac{\delta}{2} + r\frac{\Delta}{2}) \int_0^T h(t)e^{-(r+\rho+\lambda^*)t} dt \right] \\
&+ \frac{1}{2\Delta}\mu \left[\frac{\omega + \frac{\delta}{2} - r\frac{\Delta}{2}}{r+\rho+\lambda^*} + \frac{(\lambda^*+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda^*} e^{-(r+\rho+\lambda^*)T} \right]
\end{aligned}$$

The derivative of $\int_0^T h(t)e^{-(r+\rho+\lambda^*)t}dt$ with respect to α is

$$\frac{\partial T}{\partial \alpha} h(T)e^{-(r+\rho+\lambda^*)T} + \int_0^T \frac{\partial h}{\partial \alpha}(t)e^{-(r+\rho+\lambda^*)t}dt = \int_0^T \frac{\partial h}{\partial \alpha}(t)e^{-(r+\rho+\lambda^*)t}dt > 0$$

then

$$\frac{\partial l}{\partial \alpha} < 0$$

Decomposition: Obviously we have

$$\frac{\partial U_2}{\partial \lambda_0} > 0 \text{ and } \frac{\partial U_1}{\partial \lambda^*} > 0$$

Moreover

$$\frac{\partial U_1}{\partial \lambda_0} = -[\omega - \frac{\delta}{2} + r\frac{\Delta}{2}] \int_0^T \frac{\partial h}{\partial \lambda_0}(t)e^{-(r+\rho+\lambda^*)t}dt > 0$$

And since $\frac{\partial l}{\partial \lambda^*} < 0$ (appendix section J.) and $\frac{\partial U_1}{\partial \lambda^*} > 0$ we must have

$$\frac{\partial U_2}{\partial \lambda^*} < 0$$

A.7 Empirical Implications

A.7.1 Liquidity supply in the "pre-signal" phase

Before the asset fundamental value switch to a different high or low level the liquidity supply on both side of the order book is equal to $\alpha_{eq}^{\emptyset} L$.

A.7.2 Information integration speed

The duration between the change in the asset fundamental and the adjustment of transaction prices in the limit order book is the duration of the transition phase.

$$T = \frac{1}{\rho + \lambda} \ln(1 + 2\alpha^{\emptyset})$$

Corollary A.3. *For high enough values of λ and μ an increase of the monitoring rate (λ) decreases the duration of the transition phase T. An increase of the volatility of the asset fundamental (μ or ω) decreases as well this duration.*

A.7.3 Order flow decomposition of the price adjustment (transition phase)

Corollary A.4. *In the transition phase the amount of market order executed and limit order cancelled are*

$$MO = \frac{\ln(1 + 2\alpha_{eq}^{\emptyset})}{2} L, \quad LOC = \left[\alpha_{eq}^{\emptyset} - \frac{\ln(1 + 2\alpha_{eq}^{\emptyset})}{2}\right] L$$

Moreover the ratio of limit order cancellation over executed market order is increasing with respect to $\alpha_{eq}^{\emptyset}$:

$$\frac{\partial}{\partial \alpha_{eq}^{\emptyset}} \frac{LOC}{MO} > 0$$

A.7.4 Risk of being picked-off

Corollary A.5. *Conditionally on having a limit order on the "wrong side" of the limit order book when the value of the utility flow change, the risk of having this limit order executed during the transition phase is*

$$\mathbb{P}_{po} = \frac{\ln(1 + 2\alpha_{eq}^{\emptyset})}{2\alpha_{eq}^{\emptyset}}, \quad \frac{\partial \mathbb{P}_{po}}{\partial \alpha_{eq}^{\emptyset}} < 0$$

When an investor send a limit order during the initial phase the unconditional probability for this limit order to be picked-off in the transition phase is

$$\frac{\mu}{\mu + \rho + \lambda + l^{\emptyset}} \times \mathbb{P}_{po} = \frac{\mu \ln(1 + 2\alpha_{eq}^{\emptyset})}{2(\mu + \lambda)\alpha_{eq}^{\emptyset} + \frac{\rho}{2}}$$

The effect of the monitoring rate and the fundamental volatility on the unconditional probability is unclear and depends on the parameters value.

A.7.5 Proof of Corollary A.4

The amount of cancellation is given by

$$\begin{aligned}\int_0^T (\rho+\lambda)D(t)dt &= \int_0^T (\rho+\lambda)(-\frac{1}{2}L + [\alpha^\emptyset + \frac{1}{2}]Le^{-(\rho+\lambda)t})dt \\ &= (-\frac{1}{2}(\rho+\lambda)T + [\alpha^\emptyset + \frac{1}{2}](1-e^{-(\rho+\lambda)T}))L \\ &= (-\frac{1}{2}\ln(1+2\alpha) + [\alpha^\emptyset + \frac{1}{2}](1-\frac{1}{1+2\alpha}))L\end{aligned}$$

The second part of the corollary comes from the the fact that $\ln(1+x)/x$ is increasing.

A.7.6 Proof of Corollary A.5

At t a limit order is cancelled if the owner monitor the market, with probability $(\rho+\lambda)dt$, and it executed with probability $\frac{(\rho+\lambda)\frac{L}{2}}{D(t)}dt$. The probability to have the stale limit order executed between t and $t+dt$ is

$$P_t = \frac{(\rho+\lambda)\frac{L}{2}}{D(t)}dt \times \exp\left[-\int_0^t (\rho+\lambda) + \frac{(\rho+\lambda)\frac{L}{2}}{D(s)}ds\right]$$

Noticing that

$$\frac{\partial D}{\partial t} = -(\rho+\lambda)D(t) - (\rho+\lambda)\frac{L}{2}$$

this probability is equal to

$$P_t = \frac{(\rho+\lambda)\frac{L}{2}}{D(t)}dt \times \exp\left[\int_0^t + \frac{\frac{\partial D}{\partial t}}{D(s)}ds\right] = \frac{(\rho+\lambda)\frac{L}{2}}{D(0)}dt = \frac{\rho+\lambda}{2\alpha}dt$$

Then the overall probability of is

$$\int_0^T P_t dt = \frac{\rho+\lambda}{2\alpha}T = \frac{\ln(1+2\alpha)}{2\alpha}$$

A.8 Extension to the problem with 2 monitoring rates, λ_1 and λ_2, in the symmetric case, $\rho^+ = \rho^-$, $s = \frac{1}{2}$

A.8.1 steady-state populations

We consider the case where the population is heterogeneous. A mass L^1 of agents monitor the market with intensity λ_1 and a mass L^2 of agents monitor the market with intensity λ_2. We assume that the second type is the intensive monitoring type, $\lambda_2 > \lambda_1$.

Both populations have the same law of motion for their private value of the asset. So in steady state we must have

$$L^i_{ho} + L^i_{hn} = L^i_{lo} + L^i_{ln} = \frac{1}{2}L^i$$

Moreover on aggregate we must have that for the asset supply condition

$$L^1_{ho} + L^1_{lo} + L^2_{ho} + L^2_{lo} = \frac{1}{2}L$$

which imply that at aggregate there remains only one degree of liberty α as in the case with only one monitoring intensity,

$$L^1_{lo} + L^2_{lo} = L^1_{hn} + L^2_{hn} = \alpha L$$

Let's call $L^i_{lo} = \alpha^i_o L^i$ and $L^i_{hn} = \alpha^i_n L^i$. It could be the case that $\alpha^i_n \neq \alpha^i_o$ which would introduce an asymmetry in each of the populations whereas at the aggregate level $\alpha^1_n L^1 + \alpha^2_n L^2 = \alpha^1_o L^1 + \alpha^2_o L^2 = \alpha L$.

Assumption A.4. *Here we restrict ourselves to the case where $\alpha^i_n = \alpha^i_o = \alpha_i$. In the perfectly symmetric case of the game with uncertainty it will be the case because the terms of the trade-off are the same for lo's and hn's.*

A.8.2 Limit order in steady state without fundamental uncertainty

In this case the terms of the trade-off between market orders and limit orders is the same for agents in L^1 or in L^2 because the monitoring intensity does not play a role in the steady state value function.

The equilibrium value functions must be the same as in the case with a unique monitoring rate as well as the equilibrium values α, l_A and l_B.

Let's look at the micro-dynamic of the limit order book

Ask Side: At time t, on the ask side of the market the depth is constantly equal to $D_A = L^1_{lo} + L^2_{lo}$ and the order flows sustaining this steady state are

- **Outflow due to execution:** a share $l_A(L^1_{lo} + L^2_{lo}).dt$ is executed. This share is equal to the flow of buy market order

$$l_A(L^1_{lo} + L^2_{lo}).dt = m^1_A(\lambda_1 L^1_{hn} + \rho L^1_{ln}).dt + m^2_A(\lambda_2 L^2_{hn} + \rho L^2_{ln}).dt$$

- **Outflow due cancellation:** people switching from *lo* to *ho*, $\rho(L^1_{lo} + L^2_{lo}).dt$, *lo* type cancelling their sell limit order to send a sell market order, $m^1_B\lambda_1 L^1_{lo}.dt + m^2_B\lambda_2 L^2_{lo}.dt$
- **Inflow:** people switching from *ho* to *lo* type sending a sell limit order, $(1 - m^1_B)\rho L^1_{ho}.dt + (1 - m^2_B)\rho L^2_{ho}.dt$

The steady state condition is then:

$$\rho(L^1_{lo} + L^2_{lo}) + l_A(L^1_{lo} + L^2_{lo}) + l_B(L^1_{hn} + L^2_{hn}) = \rho(L^1_{lo} + L^2_{lo})$$

Which leads to following equation:

$$\alpha L = \alpha_1 L^1 + \alpha_2 L^2 = \frac{\rho}{2(2\rho + l_A + l_B)} L$$

which is the same as in the case with a unique monitoring rate.

We need to add another steady state conditions for the populations L^1 and L^2,

$$\begin{aligned}
\frac{dL^i_{lo}}{dt} &= \rho(1 - m^i_B)L^i_{ho} - l_A L^i_{lo} - \rho L^i_{lo} - m^i_B \lambda_i L^i_{lo} \\
&= \rho(1 - m^i_B)(\frac{1}{2} - \alpha_i)L^i - l_A \alpha_i L^i - \rho\alpha_i L^i - m^i_B \lambda_i \alpha_i L^i \\
&= \frac{\rho}{2} L^i - (l_A + 2\rho)\alpha_i L^i - m^i_B(\lambda_i \alpha_i L^i + \rho(\frac{1}{2} - \alpha_i))L^i
\end{aligned}$$

A possible equilibrium is the one where

$$m^i_B(\lambda_i \alpha_i L^i + \rho(\frac{1}{2} - \alpha_i))L^i = l_B \alpha_i$$

in this case

$$\alpha_i = \frac{\rho}{2(2\rho + l_A + l_B)} = \alpha$$

and we can check that $0 < m^i_B < 1$. m^i_A can be defined the same way.

With this equilibrium, this is easy to replicate the equilibrium strategies in the case where we converge to a steady state equilibrium. For the ask side, if we call D^i_A the share of the depth of the order book due to the population of L^i, we would get the dynamic

$$\frac{dD^i_A}{dt} = \rho L^i_{ho}(t) - \rho D^i_A(t) - l_A D^i_A(t) - l_B D^i_B(t) + \lambda_i(L^i_{lo}(t) - D^i_A(t))$$

and then the same dynamic for $\alpha_i(t)$ as in the case with a unique monitoring rate.

A.9 Transition phase

We assume that the strategies are the same than in the unique monitoring

Let's take the case of the transition phase when the fundamental value switches from v_0 to $v_0 + \omega$. At $t = 0$, the beginning of the transition phase the depth at price A in the limit order book is equal to $D_A(0) = D^1_A(0) + D^2_A(0)$. During the transition phase the flow of market orders that hit the limit orders at price A is equal to

$$u(t).dt = (\lambda_1 + \rho)(L^1_{hn}(t) + L^1_{ln}(t)).dt + (\lambda_2 + \rho)(L^2_{hn}(t) + L^2_{ln}(t)).dt$$

The evolution of the population L^i with limit orders at price A is given by

$$\frac{dD^i_A}{dt} = -(\lambda_i + \rho)D^i_A(t) - u(t)\frac{D^i_A(t)}{D^1_A(t) + D^2_A(t)}$$

We have to solve for a system of equation of type

$$\begin{cases} f' = -af - u(t)\frac{f}{f+g} \\ g' = -bg - u(t)\frac{g}{f+g} \end{cases}$$

We can assume that $g \geq 0$ without generality because one of the functions has to be strictly positive. It gives that

$$g(f' + af) = f(g' + bg) = -u(t)\frac{gf}{f+g}$$

and then

$$\frac{f'g - g'f}{g^2} = \left(\frac{f}{g}\right)' = (b-a)\frac{f}{g}$$

and finally

$$\frac{f}{g} = \frac{f_0}{g_0}\exp[(b-a)t]$$

or equivalently, there is a function h such that

$$f(t) = f_0 e^{-at}h(t), \text{ and } g(t) = g_0 e^{-bt}h(t)$$

In our case, we know that there is a function h_A such that $h_A(0) = 1$ and

$$\text{for } i \in \{1,2\},\ D_A^i(t) = D_A^i(0)e^{-(\lambda_i+\rho)t}h_A(t)$$

Then we obtain that

$$\begin{aligned}\frac{d}{dt}[(D_A^1(0)e^{-(\lambda_1+\rho)t} + D_A^2(0)e^{-(\lambda_2+\rho)t})h_A(t)] =& -(\lambda_1+\rho)D_A^1(0)e^{-(\lambda_1+\rho)t}h_A(t)\\ &-(\lambda_2+\rho)D_A^2(0)e^{-(\lambda_2+\rho)t}h_A(t) - u(t)\\ \iff (D_A^1(0)e^{-(\lambda_1+\rho)t} + D_A^2(0)e^{-(\lambda_2+\rho)t})\frac{dh_A}{dt} =& -u(t)\end{aligned}$$

We can use this first result to define the intensity of limit order for execution, $k_A(t)$. We know that $u(t).dt$ is the flow of market order. Then we mus have $u(t) = k_A(t)D_A(t)$.

$$D_A(t) = D_A^1(t) + D_A^2(t) = D_A^1(0)e^{-(\lambda_1+\rho)t}h_A(t) + D_A^2(0)e^{-(\lambda_2+\rho)t}h_A(t)$$

The execution intensity is equal to

$$k_A(t) = -\frac{\dot{h}_A}{h_A}$$

It is easy to see that the strategies are the same. Indeed in the value functions in the transition phase, the dynamic of the transition phase affects these value functions only through the execution intensities k_A or k_B and through the duration of the transition phase, T^u or T^d, defined here by $h_A(T^u) = 0$ and $h_B(T^d) = 0$. But to prove that deviations from the conjectured strategies are optimal we don't need to look at the particular form of k or T.

Now we can look more precisely at the function $u(t)$. The law of motion for the population of hn and ln are the following

$$\begin{aligned}\frac{dL_{hn}^i}{dt} &= -(\lambda_i+\rho)L_{hn}^i(t)\\ \frac{dL_{ln}^i}{dt} &= -(\lambda_i+\rho)L_{ln}^i(t) + u(t)\frac{D_A^i(t)}{D_A^1(t)+D_A^2(t)}\end{aligned}$$

then we obtain that

$$\frac{d}{dt}(L_{hn}^i(t) + L_{ln}^i(t) + D_A^i(t)) = -(\lambda_i+\rho)(L_{hn}^i(t) + L_{ln}^i(t) + D_A^i(t))$$

then

$$L_{hn}^i(t) + L_{ln}^i(t) = (L_{hn}^i(0) + L_{ln}^i(0) + D_A^i(0))e^{-(\lambda_i+\rho)t} - D_A^i(t)$$

The differential equation for h_A is then

$$\begin{aligned}&\frac{d}{dt}[(D_A^1(0)e^{-(\lambda_1+\rho)t}+D_A^2(0)e^{-(\lambda_2+\rho)t})h_A(t)]\\=&-(\lambda_1+\rho)(L_{hn}^1(0)+L_{ln}^1(0)+D_A^1(0))e^{-(\lambda_1+\rho)t}\\&-(\lambda_1+\rho)(L_{hn}^2(0)+L_{ln}^2(0)+D_A^2(0))e^{-(\lambda_2+\rho)t}\end{aligned}$$

which gives

$$\begin{aligned}&(D_A^1(0)e^{-(\lambda_1+\rho)t}+D_A^2(0)e^{-(\lambda_2+\rho)t})h_A(t)-(D_A^1(0)+D_A^2(0))\\=&(L_{hn}^1(0)+L_{ln}^1(0)+D_A^1(0))e^{-(\lambda_1+\rho)t}+(L_{hn}^2(0)+L_{ln}^2(0)+D_A^2(0))e^{-(\lambda_2+\rho)t}\\&-(L_{hn}^1(0)+L_{ln}^1(0)+D_A^1(0)+L_{hn}^2(0)+L_{ln}^2(0)+D_A^2(0))\end{aligned}$$

and finally

$$h_A(t)=1-\frac{(L_{hn}^1(0)+L_{ln}^1(0))(1-e^{-(\lambda_1+\rho)t})+(L_{hn}^2(0)+L_{ln}^2(0))(1-e^{-(\lambda_2+\rho)t})}{D_A^1(0)e^{-(\lambda_1+\rho)t}+D_A^2(0)e^{-(\lambda_2+\rho)t}}$$

In the transition phase, an important difference is value function is $V_{ln-out}^u(t)+A^0-V_{lo-A^0}^u(t)$ that appears in the value of the execution intensity that makes agents indifferent between limit and market orders. the formulation of this expression is very similar to the unique monitoring intensity case.

Let's call $X(t)=V_{ln-out}^u(t)+A^0-V_{lo-A^0}^u(t)$

$$(r+\rho+\lambda-\frac{\dot{h}_A}{h_A})X(t)=\frac{\partial X}{\partial t}-[v^u-\delta-rA^0]$$

and at the end we find that

$$V_{ln-out}^u(t)+A^0-V_{lo-A^0}^u(t)=-[v^u-\delta-rA^0]\int_t^{T^u}\frac{h_A(s)e^{-(r+\rho+\lambda_i)s}}{h_A(t)e^{-(r+\rho+\lambda_i)t}}ds$$

and for the same reason we would find

$$V_{ho-out}^u(t)-A^0-V_{hn-B^0}^u(t)=\frac{v^u-rA^0}{r+\rho+\lambda_i}+\frac{(\lambda_i+\rho)(A^u-A^0)-\rho\Delta}{r+\rho+\lambda_i}e^{-(r+\rho+\lambda_i)T^u}\times e^{(r+\rho+\lambda_i)t}$$

A.10 Steady-state with uncertainty in the perfectly symmetric case

In the perfectly symmetric case the populations are symmetric

$$L_{hn}^i=L_{lo}^i=\alpha_iL^i$$

$$D_A^i=D_B^i=\gamma_i\alpha_iL_i$$

We don't assume that all the *lo*'s or *hn*'s are at the best price in the limit order book. Some of them could send their order at A^u and B^d. It could be optimal with the monitoring rate heterogeneity.

In the perfectly symmetric case, the duration of the transition phase are the same as well as the execution intensity during the transition phase.

$$T^u=T^d=T,\ k_A=k_B=k,\ h_A=h_B=h$$

Then for h we have

$$h(t) = 1 - \frac{1}{2}\frac{(1-e^{-(\lambda_1+\rho)t})L^1 + (1-e^{-(\lambda_2+\rho)t})L^2}{\gamma_1\alpha_1 L^1 e^{-(\lambda_1+\rho)t} + \gamma_2\alpha_2 L^2 e^{-(\lambda_2+\rho)t}}$$

Then T is defined by

$$(1+2\gamma_1\alpha_1)L^1 e^{-(\lambda_1+\rho)T} + (1+\gamma_2\alpha_2)L^2 e^{-(\lambda_2+\rho)T} = L^1 + L^2$$

Using the calculation of the unique monitoring rate case, we obtain the value of the execution rate that makes an agent indifferent between a limit order at the best price and a market order

$$\begin{aligned} l_i = &\frac{\delta-(r+2\rho)\Delta}{2\Delta} \\ &+ \frac{1}{2\Delta}\mu\left[-\Delta - (\omega - \frac{\delta}{2} + r\frac{\Delta}{2})\int_0^T h(t)e^{-(r+\rho+\lambda_i)t}dt\right] \\ &+ \frac{1}{2\Delta}\mu\left[\frac{\omega+\frac{\delta}{2}-r\frac{\Delta}{2}}{r+\rho+\lambda_i} + \frac{(\lambda_i+\rho)\frac{\omega}{r}-\rho\Delta}{r+\rho+\lambda_i}e^{-(r+\rho+\lambda_i)T}\right] \end{aligned}$$

we can rewrite

$$\begin{aligned} l_i = &\frac{\delta-(r+2\rho)\Delta}{2\Delta} - \frac{\mu}{2} + \frac{1}{2\Delta}\mu(\omega - \frac{\delta}{2} + r\frac{\Delta}{2})\int_0^T c(t)e^{-(r+\rho+\lambda_i)t}dt \\ &+ \frac{1}{2\Delta}\mu\left[-(\omega - \frac{\delta}{2} + r\frac{\Delta}{2})\frac{1-e^{-(r+\rho+\lambda_i)T}}{r+\rho+\lambda_i} + \frac{\omega+\frac{\delta}{2}-r\frac{\Delta}{2}}{r+\rho+\lambda_i} + \frac{(\lambda_i+\rho)\frac{\omega}{r}-\rho\Delta}{r+\rho+\lambda_i}e^{-(r+\rho+\lambda_i)T}\right] \end{aligned}$$

with $c(t) = \frac{1}{2}\frac{(1-e^{-(\lambda_1+\rho)t})L^1+(1-e^{-(\lambda_2+\rho)t})L^2}{\gamma_1\alpha_1 L^1 e^{-(\lambda_1+\rho)t}+\gamma_2\alpha_2 L^2 e^{-(\lambda_2+\rho)t}}$. The first line of the expression of l_i is obviously decreasing with respect to λ_i because $c(t) > 0$ for all $t \in [0, T]$. For the second line, we can rewrite it as

$$\begin{aligned} &-(\omega - \frac{\delta}{2} + r\frac{\Delta}{2})\frac{1-e^{-(r+\rho+\lambda_i)T}}{r+\rho+\lambda_i} + \frac{\omega+\frac{\delta}{2}-r\frac{\Delta}{2}}{r+\rho+\lambda_i} + \frac{(\lambda_i+\rho)\frac{\omega}{r}-\rho\Delta}{r+\rho+\lambda_i}e^{-(r+\rho+\lambda_i)T} \\ =&\frac{\delta-r\Delta}{r+\rho+\lambda_i} + \frac{\omega}{r}e^{-(r+\rho+\lambda_i)T} + \frac{\omega-\frac{\delta}{2}+r\frac{\Delta}{2}-(\omega+\rho\Delta)}{r+\rho+\lambda_i}e^{-(r+\rho+\lambda_i)T} \\ =&\frac{\delta-r\Delta}{r+\rho+\lambda_i} + \frac{\omega}{r}e^{-(r+\rho+\lambda_i)T} - \frac{\frac{\delta-(2\rho-r)\Delta}{2}}{r+\rho+\lambda_i}e^{-(r+\rho+\lambda_i)T} \end{aligned}$$

The derivative of this expression with respect to λ_i is

$$-\frac{\delta-r\Delta}{(r+\rho+\lambda_i)^2} + \frac{\frac{\delta-(2\rho-r)\Delta}{2}}{(r+\rho+\lambda_i)^2}e^{-(r+\rho+\lambda_i)T} - Te^{-(r+\rho+\lambda_i)T}\left[\frac{\omega}{r} - \frac{\frac{\delta-(2\rho-r)\Delta}{2}}{r+\rho+\lambda_i}\right]$$

which is clearly negative. It implies that

$$l_1 > l_2$$

To be indifferent between a competitive limit order and a market order, an investor with a low monitoring intensity λ_1 requires a higher rate of limit order execution than an investor with a high monitoring intensity λ_2

Appendix B

Appendix to chapter 3

B.1 Mini flash crash events

Mini flash crashes	**Dates**
United Airlines stock, from $12 to $3 in 15 min and recovered the next day	September 8, 2008
Progress energy stock, 90% drop in a few seconds and recovered in few minutes	September 27, 2010
ACOR stock, lost 11% of it's value in under 5 seconds then quickly recovered	May 24, 2011
STBC stock, from $12.51 to $10.28 in 3 sec before quickly recovering	May 26, 2011
TLT stock, from $96.63 to $97.90 and then dropped back down in less than 1 sec	July 14, 2011
AMJ stock, from $34.90 to $32.61 and then recovered, all in just under 4 seconds	October 11, 2011
Brocade (BRCD) stock, dropped 5.5% and then recovered in about 1.5 seconds.	August 17, 2012
LTXC stock, drops 10% in 1 second, recovers in 2.	August 28, 2012
Perion Network (PERI) stock, dropped 7.6% in 1/3 of a second. 33 seconds later, the price rocketed 5.6%	January 8, 2013
GeoEye, Inc (GEOY) stock, raced 8.5% higher, stayed up for about 1/2 second, then returned near where it started a second later.	January 9, 2013
Dow Jones index, lost 200 basis points and quickly recovered, after hacked tweet: «Breaking: Two Explosions in the White House and Barack Obama is injured»	April 23, 2103

Figure B.1: sample of mini flash crashes (see http://www.nanex.net/FlashCrash/OngoingResearch.html).

B.2 Proofs

Proof of proposition 3.2.

$$\mathbb{P}[V=1|\tilde{Q}_1=q]=\frac{\mathbb{P}[\tilde{Q}_1=q|V=1]\mathbb{P}[V=1]}{\mathbb{P}[\tilde{Q}_1=q]}$$

and $\mathbb{P}[V=1]=1/2$,

$$\begin{aligned}\mathbb{P}[\tilde{Q}_1=q|V=1]&=(\mathbb{P}[U=1]+\mathbb{P}[U=0,\epsilon=1])\mathbb{P}[\tilde{l}_1=q-\alpha]+\mathbb{P}[U=0,\epsilon=0]\mathbb{P}[\tilde{l}_1=q+\alpha]\\&=(\delta+\frac{1-\delta}{2})\phi(q-\alpha)+\frac{1-\delta}{2}\phi(q+\alpha),\end{aligned}$$

$$\begin{aligned}\mathbb{P}[\tilde{Q}_1 = q] &= \mathbb{P}[S=1]\mathbb{P}[\tilde{l}_1 = q-\alpha] + \mathbb{P}[S=0]\mathbb{P}[\tilde{l}_1 = q+\alpha] \\ &= \frac{1}{2}\phi(q-\alpha) + \frac{1}{2}\phi(q+\alpha).\end{aligned}$$

Proof of proposition 3.3. At period 1 if the signal is $S=1$, the profit for a participating strategic trader is

$$\begin{aligned}\pi^1(\alpha, S=1) &= \int_{[-Q,Q]} \left[\frac{1+\delta}{2} - \frac{(1+\delta)\phi(l) + (1-\delta)\phi(l+2\alpha)}{\phi(l)+\phi(l+2\alpha)} \times \frac{1}{2}\right]\phi(l)dl \\ &= \int_{[-Q,Q]} \frac{\delta\phi(l+2\alpha)}{\phi(l)+\phi(l+2\alpha)}\phi(l)dl \\ &= \int_{[-Q+\alpha,Q+\alpha]} \frac{\delta\phi(l+\alpha)}{\phi(l-\alpha)+\phi(l+\alpha)}\phi(l-\alpha)dl \\ &= \int_{[-Q+\alpha,Q-\alpha]} \frac{\delta\phi(l+\alpha)}{\phi(l-\alpha)+\phi(l+\alpha)}\phi(l-\alpha)dl \\ &= \int_{[-Q+\alpha,Q-\alpha]} \frac{\delta}{2}dl = \frac{\delta}{2} \times \frac{Q-\alpha}{Q}\end{aligned}$$

because $\phi(l+\alpha) = 0$ for $l > Q-\alpha$.
At period 1 if the signal is $S=1$, the profit for a participating strategic trader is

$$\begin{aligned}\pi^1(\alpha, S=0) &= \int_{[-Q,Q]} \left[\frac{(1+\delta)\phi(l-2\alpha) + (1-\delta)\phi(l)}{\phi(l-2\alpha)+\phi(l)} \times \frac{1}{2} - \frac{1-\delta}{2}\right]\phi(l)dl \\ &= \int_{[-Q,Q]} \frac{\delta\phi(l-2\alpha)}{\phi(l-2\alpha)+\phi(l)}\phi(l)dl \\ &= \int_{[-Q-\alpha,Q-\alpha]} \frac{\delta\phi(l-\alpha)}{\phi(l-\alpha)+\phi(l+\alpha)}\phi(l+\alpha)dl \\ &= \int_{[-Q+\alpha,Q-\alpha]} \frac{\delta\phi(l-\alpha)}{\phi(l-\alpha)+\phi(l+\alpha)}\phi(l+\alpha)dl \\ &= \int_{[-Q+\alpha,Q-\alpha]} \frac{\delta}{2}dl = \frac{\delta}{2} \times \frac{Q-\alpha}{Q}\end{aligned}$$

because $\phi(l-\alpha) = 0$ for $l < -Q-\alpha$. Finally,

$$\pi^1(\alpha) = \frac{1}{2}\pi^1(\alpha, S=1) + \frac{1}{2}\pi^1(\alpha, S=0)$$

Proofs of propositions 3.4 and 3.5. At $t=2$, if $U=1$ the proof is similar to the period 1 case. If $U=0$, the trading decision of strategic traders is not obvious. Then we take as unknown the mass of trading

$$M_0 = \int_0^\beta X_i(U=0)di, \quad -\beta \leq M_0 \leq \beta.$$

The value of M_0 at equilibrium depends on the profit that can be achieve at period 2, knowing that the expected value of the asset is 1/2. This profit is directly linked to the average spread between 1/2 and the transaction price at period 2, when the strategy associated to the state $U=0$ is played. This spread is

$$\Sigma^{2,\epsilon}(q_1, M_0) = \int_{[-Q-\beta,Q+\beta]} [P_2(q_2,q_1) - \frac{1}{2}]\phi(q_2 - M_0)dq_2$$

Using this function we can distinguish between three types of trading outcomes at equilibrium associated to trading decisions of strategic traders

- the all-selling outcome in which all event traders sell, $M_0 = -\beta$, is an equilibrium strategy if and only if $\Sigma^{2,\epsilon}(q_1, -\beta) \geq 0$
- the all-buying outcome in which all event traders buy, $M_0 = \beta$, is an equilibrium strategy if and only if $\Sigma^{2,\epsilon}(q_1, \beta) \leq 0$
- A "mixed" outcome in which event can behave differently, which generates an order flow $-\beta < M_0 < \beta$, is an equilibrium strategy if and only if $\Sigma^{2,\epsilon}(q_1, M_0) = 0$. Otherwise there would be an incentive to unilaterally deviate to take advantage of the non zero spread $\Sigma^{2,\epsilon}$

To figure out the equilibrium strategies, we need to have the pricing policy of the market maker given a trading schedule $M_0(q_1)$.

$$\mathbb{P}[V = 1|\tilde{Q}_2 = q_2, \tilde{Q}_1 = q_1] = \frac{\mathbb{P}[\tilde{Q}_2 = q_2, V = 1|\tilde{Q}_1 = q_1]}{\mathbb{P}[\tilde{Q}_2 = q_2|\tilde{Q}_1 = q_1]]} = \frac{\mathbb{P}[\tilde{Q}_2 = q_2, V = 1, \tilde{Q}_1 = q_1]}{\mathbb{P}[\tilde{Q}_2 = q_2, \tilde{Q}_1 = q_1]]}$$

and

$$\begin{aligned}\mathbb{P}[\tilde{Q}_2 = q_2, V = 1, \tilde{Q}_1 = q_1] = \mathbb{P}[U = 1, \tilde{Q}_2 = q_2, V = 1, \tilde{Q}_1 = q_1]& \\ + \mathbb{P}[U = 0, \epsilon = 1, \tilde{Q}_2 = q_2, V = 1, \tilde{Q}_1 = q_1]& \\ + \mathbb{P}[U = 0, \epsilon = 0, \tilde{Q}_2 = q_2, V = 1, \tilde{Q}_1 = q_1]&\end{aligned}$$

with

$$\begin{aligned}\mathbb{P}[U = 1, \tilde{Q}_2 = q_2, V = 1, \tilde{Q}_1 = q_1] &= \phi(q_2 - \beta)\phi(q_1 - \alpha)\frac{1}{2}\delta \\ \mathbb{P}[U = 0, \epsilon = 1, \tilde{Q}_2 = q_2, V = 1, \tilde{Q}_1 = q_1] &= \phi(q_2 - M_0)\phi(q_1 - \alpha)\frac{1}{2}\frac{1}{2}(1 - \delta) \\ \mathbb{P}[U = 0, \epsilon = 0, \tilde{Q}_2 = q_2, V = 1, \tilde{Q}_1 = q_1] &= \phi(q_2 - M_0)\phi(q_1 + \alpha)\frac{1}{2}\frac{1}{2}(1 - \delta)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}[\tilde{Q}_2 = q_2\tilde{Q}_1 = q_1] = \mathbb{P}[U = 1, V = 1, \tilde{Q}_2 = q_2, \tilde{Q}_1 = q_1]& \\ + \mathbb{P}[U = 1, V = 0, \tilde{Q}_2 = q_2, \tilde{Q}_1 = q_1]& \\ + \mathbb{P}[U = 0, \epsilon = 1, \tilde{Q}_2 = q_2, \tilde{Q}_1 = q_1]& \\ + \mathbb{P}[U = 0, \epsilon = 0, \tilde{Q}_2 = q_2, \tilde{Q}_1 = q_1]&\end{aligned}$$

with

$$\begin{aligned}\mathbb{P}[U = 1, V = 1, \tilde{Q}_2 = q_2, \tilde{Q}_1 = q_1] &= \phi(q_2 - \beta)\phi(q_1 - \alpha)\frac{1}{2}\delta \\ \mathbb{P}[U = 1, V = 0, \tilde{Q}_2 = q_2, \tilde{Q}_1 = q_1] &= \phi(q_2 + \beta)\phi(q_1 + \alpha)\frac{1}{2}\delta \\ \mathbb{P}[U = 0, \epsilon = 1, \tilde{Q}_2 = q_2, \tilde{Q}_1 = q_1] &= \phi(q_2 - M_0)\phi(q_1 - \alpha)\frac{1}{2}(1 - \delta) \\ \mathbb{P}[U = 0, \epsilon = 0, \tilde{Q}_2 = q_2, \tilde{Q}_1 = q_1] &= \phi(q_2 - M_0)\phi(q_1 + \alpha)\frac{1}{2}(1 - \delta)\end{aligned}$$

Lemma B.1. *At period 2, when $U = 0$, the trading strategies and order flow outcome of event traders are*

- *for $q_1 \in [-Q - \alpha, -Q + \alpha)$ the all-buying outcome, $M_0 = \beta$ is the only equilibrium outcome*
- *for $q_1 \in (Q - \alpha, Q + \alpha]$ the all-selling outcome, $M_0 = -\beta$ is the only equilibrium outcome*
- *for $q_1 \in [-Q + \alpha, Q - \alpha]$ all values $M_0 \in [-\beta, \beta]$ can correspond to an equilibrium.*
  However in this case we set $M_0 = 0$ which is the only value robust to a small trading cost and the most natural because there is no coordination issue between strategic traders, they trade on the "non-information" event only if there is mispricing.

Proof of lemma B.1. knowing that

$$P_2(q_2,q_1)) = \mathbb{P}[V=1|\tilde{Q}_2=q_2,\tilde{Q}_1=q_1] =$$
$$\frac{\delta\phi(q_1-\alpha)\phi(q_2-\beta)+\frac{1-\delta}{2}[\phi(q_1-\alpha)+\phi(q_1+\alpha)]\phi(q_2-M_0)}{\delta[\phi(q_1-\alpha)\phi(q_2-\beta)+\phi(q_1+\alpha)\phi(q_2+\beta)]+(1-\delta)[\phi(q_1-\alpha)+\phi(q_1+\alpha)]\phi(q_2-M_0)}$$

The proof of this lemma corresponds to the analysis of the function

$$\Sigma^{2,\epsilon}(q_1,M_0) =$$
$$\int \frac{1}{2}\frac{\delta[\phi(q_1-\alpha)\phi(q_2-\beta)-\phi(q_1+\alpha)\phi(q_2+\beta)]\phi(q_2-M_0)}{\delta[\phi(q_1-\alpha)\phi(q_2-\beta)+\phi(q_1+\alpha)\phi(q_2+\beta)]+(1-\delta)[\phi(q_1-\alpha)+\phi(q_1+\alpha)]\phi(q_2-M_0)}dq_2$$

which is equal to

$$\Sigma^{2,\epsilon}(q_1,M_0) = \frac{1}{2Q}\int \frac{1}{2}\frac{N(q_1,q_2)}{D(q_1,q_2)}dq_2$$

with

$$N(q_1,q_2) = \delta[\mathbb{I}_{\{q_1\in[-Q+\alpha,Q+\alpha]\}}\mathbb{I}_{\{q_2\in[-Q+\beta,Q+\beta]\}} - \mathbb{I}_{\{q_1\in[-Q-\alpha,Q-\alpha]\}}\mathbb{I}_{\{q_2\in[-Q-\beta,Q-\beta]\}}]\mathbb{I}_{\{q_2\in[-Q+M_0,Q+M_0]\}}$$

and

$$\begin{aligned}D(q_1,q_2) = {} & \delta[\mathbb{I}_{\{q_1\in[-Q+\alpha,Q+\alpha]\}}\mathbb{I}_{\{q_2\in[-Q+\beta,Q+\beta]\}} + \mathbb{I}_{\{q_1\in[-Q-\alpha,Q-\alpha]\}}\mathbb{I}_{\{q_2\in[-Q-\beta,Q-\beta]\}}] \\ & + (1-\delta)[\mathbb{I}_{\{q_1\in[-Q+\alpha,Q+\alpha]\}} + \mathbb{I}_{\{q_1\in[-Q-\alpha,Q-\alpha]\}}]\mathbb{I}_{\{q_2\in[-Q+M_0,Q+M_0]\}}\end{aligned}$$

If $q_1 \in [-Q-\alpha,-Q+\alpha)$, necessarily the integration is on $q_2 \in [-Q-\beta, Q+M_0]$ because in this case the period 1 signal is negative and at period 2 strategic traders either sell because the true value of the asset is 0 or "trade M_0" to correct the mispricing. Then

$$\frac{N(q_1,q_2)}{D(q_1,q_2)} = \begin{cases} 0 & \text{if } q_2 \in [-Q-\beta,-Q+M_0) \\ -1 & \text{if } q_2 \in [-Q+M_0,Q-\beta] \\ 0 & \text{if } q_2 \in (Q-\beta,Q+M_0] \end{cases}$$

Then

$$\Sigma^{2,\epsilon}(q_1,M_0) = -\delta\frac{Q-\beta}{2Q} - \delta\frac{\beta-M_0}{4Q} < 0$$

In this case, we set $M_0=\beta$ and $\Sigma^{2,\epsilon}(q_1,\beta) = -\delta\frac{Q-\beta}{2Q}$

If $q_1 \in [-Q+\alpha, Q-\alpha]$, the integration is on $q_2 \in [-Q-\beta, Q+\beta]$. Then

$$\frac{N(q_1,q_2)}{D(q_1,q_2)} = \begin{cases} 0 & \text{if } q_2 \in [-Q-\beta,-Q+M_0[\\ \frac{1}{2-\delta} & \text{if } q_2 \in [-Q+M_0,-Q+\beta[\\ 0 & \text{if } q_2 \in [-Q+\beta,Q-\beta] \\ \frac{1}{2-\delta} & \text{if } q_2 \in]Q-\beta,Q+M_0] \\ 0 & \text{if } q_2 \in]Q+M_0,Q+\beta] \end{cases}$$

Then

$$\Sigma^{2,\epsilon}(q_1,M_0) = 0$$

In this case, M_0 can take all possible value in $[-\beta,\beta]$.

If $q_1 \in]Q-\alpha, Q+\alpha]$, necessarily the integration is on $q_2 \in [-Q+M_0, Q+\beta]$ because in this case the period 1 signal is positive and at period 2 strategic traders either buy because the true value of the asset is 1 or "trade M_0" to correct the mispricing. Then

$$\frac{N(q_1,q_2)}{D(q_1,q_2)} = \begin{cases} 0 & \text{if } q_2 \in [-Q+M_0,-Q+\beta) \\ 1 & \text{if } q_2 \in [-Q+\beta,Q+M_0] \\ 0 & \text{if } q_2 \in (Q+M_0,Q+\beta] \end{cases}$$

Then

$$\Sigma^{2,\epsilon}(q_1, M_0) = \delta\frac{Q-\beta}{2Q} + \delta\frac{\beta + M_0}{4Q} > 0$$

In this case, we set $M_0 = -\beta$ and $\Sigma^{2,\epsilon}(q_1, \beta) = \delta\frac{Q-\beta}{2Q}$

Proof of proposition 3.6. To compute the profit at period 2, we can notice that the symmetry of the problem implies that this profit conditionned on the value of the signal at period 1 is the same in the two cases $S = 0$ and $S = 1$. As these two cases are equally likely the unconditionnal value of the profit is equal to the value of one of these conditionnal profits.
Let's focus on the case where $S = 1$. Then the order flow at period 1 is necessarily such that $q_1 \in [-Q + \alpha, Q + \alpha]$.
If $q_1 \in [-Q + \alpha, Q - \alpha]$, at period 2 strategic traders buy if $U = 1$ ($q_2 \in [-Q + \beta, Q + \beta]$) and do not participate if $U = 0$. Then the corresponding share of the profit is equal to

$$\begin{aligned}
&\mathbb{P}[q_1 \in [-Q+\alpha, Q-\alpha]] \times \delta \times \\
&[\mathbb{P}[q_2 \in [-Q+\beta, Q-\beta]] \times (1 - \frac{1}{2}) + \mathbb{P}[q_2 \in [Q-\beta, Q]] \times (1 - \frac{1}{2-\delta}) + \mathbb{P}[q_2 \in [Q, Q+\beta]] \times 0] \\
=&\frac{Q-\alpha}{Q} \times \delta \times [\frac{Q-\beta}{Q} \times \frac{1}{2} + \frac{\beta}{2Q} \times \frac{1-\delta}{2-\delta}]
\end{aligned}$$

If $q_1 \in [Q - \alpha, Q + \alpha]$, at period 2 strategic traders buy if $U = 1$ ($q_2 \in [-Q + \beta, Q + \beta]$) and sell if $U = 0$ ($q_2 \in [-Q - \beta, Q - \beta]$). Then the corresponding share of the profit is equal to

$$\begin{aligned}
&\mathbb{P}[q_1 \in [Q-\alpha, Q+\alpha]] \times \\
&\{\delta \times [\mathbb{P}[q_2 \in [-Q+\beta, Q-\beta]] \times (1 - \frac{1+\delta}{2}) + \mathbb{P}[q_2 \in [Q-\beta, Q+\beta]] \times 0] + \\
&(1-\delta) \times [\mathbb{P}[q_2 \in [-Q-\beta, -Q+\beta]] \times 0 + \mathbb{P}[q_2 \in [-Q+\beta, Q-\beta]] \times (\frac{1+\delta}{2} - \frac{1}{2})]\} \\
=&\frac{\alpha}{Q} \times \{\delta \times [\frac{Q-\beta}{Q} \times \frac{1-\delta}{2}] + (1-\delta) \times [\frac{Q-\beta}{Q} \times \frac{\delta}{2}]\}
\end{aligned}$$

Proof of proposition 3.8.

$$\begin{aligned}
\frac{\partial \pi^2(\alpha, \beta)}{\partial \alpha} &= \frac{\delta}{2Q}\left[-(1 - \frac{1}{2-\delta}\frac{\beta}{Q}) + (1-\delta)(1 - \frac{\beta}{Q})\right] \\
&= -\frac{\delta}{2Q}\left[\delta + (1 - \delta - \frac{1}{2-\delta})\frac{\beta}{Q}\right]
\end{aligned}$$

If $\delta < \delta_0$ then $1 - \delta - \frac{1}{2-\delta} > 0$ and $\frac{\partial \pi^2(\alpha,\beta)}{\partial \alpha} < 0$.
If $\delta > \delta_0$

$$\frac{\partial \pi^2(\alpha, \beta)}{\partial \alpha} < \frac{\delta}{2Q}\left[-\delta + (\frac{1}{2-\delta} - (1-\delta))\right] = -\frac{\delta}{2Q}\left[\frac{1-\delta}{2-\delta}\right] < 0$$

Then, for $\beta = A$, if $C \in [(1 - \frac{A}{Q})\frac{\delta}{2}, \frac{\delta}{2}]$ we have $\frac{\partial \pi}{\partial C} > 0$.

For any fixed α, $\frac{\partial \pi^2(\alpha,\beta)}{\partial \beta} < 0$, then if $C > (1 - \frac{A}{Q})\frac{\delta}{2}$, $\frac{\partial \pi}{\partial A} < 0$ because $\alpha_e q$ does not depend on A in this interval under the assumption 2.

Because $\frac{\partial \pi^1(\alpha)}{\partial \alpha} < 0$, $\frac{\partial \pi^2(\alpha,\beta)}{\partial \beta} < 0$ and $\frac{\partial \pi^2(\alpha,\beta)}{\partial \alpha} < 0$ then, for $C < (1 - \frac{A}{Q})\frac{\delta}{2}$

$$\frac{\partial \pi}{\partial A} = \frac{\partial}{\partial A}[\pi^2(A, A) + \pi^1(A)] < 0$$

Proof of proposition 3.10. For this proof we study the sign of $\pi^2(0,A) - \pi^2(A,A) - \pi^1(A)$.

$$\pi^2(0,A)-\pi^2(A,A)-\pi^1(A) = \frac{\delta}{2}Q\times\left[1-\frac{1}{2-\delta}\frac{A}{Q}-(1-\frac{A}{Q})-(1-\frac{A}{Q})\times(1-\frac{1}{2-\delta}\frac{A}{Q})-(1-\delta)\frac{A}{Q}(1-\frac{A}{Q})\right]$$

We can rewrite it as

$$U(\frac{A}{Q}) = \left[1-\delta-\frac{1}{2-\delta}\right]\frac{A^2}{Q^2}+(1+\delta)\frac{A}{Q}-1$$

$U(0) = -1$ and $U(1) = 1 - \frac{1}{2-\delta} \geq 0$. Because this is a 2nd degree polynom there is a unique value between 0 and 1 for which $U = 0$.

Proof of proposition 3.12. To compute this measure at period 1 we need the following probabilities:

$$\begin{aligned}
\mathbb{P}[Q_1 \in [-Q-\alpha,-Q+\alpha)] &= \mathbb{P}[S=1]\times\mathbb{P}[l_1+\alpha\in[-Q-\alpha,-Q+\alpha)] \\
&+ \mathbb{P}[S=0]\times\mathbb{P}[l_1-\alpha\in[-Q-\alpha,-Q+\alpha)] \\
&= 0+\frac{1}{2}\mathbb{P}[l_1\in[-Q,-Q+2\alpha)] = \frac{1}{2}\frac{\alpha}{Q} \\
\mathbb{P}[Q_1\in(Q-\alpha,Q+\alpha]] &= \frac{1}{2}\frac{\alpha}{Q} \\
\mathbb{P}[Q_1\in[-Q+\alpha,Q-\alpha]] &= \mathbb{P}[S=1]\times\mathbb{P}[l_1+\alpha\in[-Q+\alpha,Q-\alpha]] \\
&+ \mathbb{P}[S=0]\times\mathbb{P}[l_1-\alpha\in[-Q+\alpha,Q-\alpha]] \\
&= \frac{1}{2}\times\mathbb{P}[l_1\in[-Q,Q-2\alpha]]+\frac{1}{2}\times\mathbb{P}[l_1-\alpha\in[-Q+2\alpha,Q]] \\
&= \frac{Q-\alpha}{Q}
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}[(\tilde{V}-P_1)^2] &= \frac{1}{2}\frac{\alpha}{Q}\times\frac{1-\delta}{2}\frac{1+\delta}{2}+\frac{Q-\alpha}{Q}\times\frac{1}{4}+\frac{1}{2}\frac{\alpha}{Q}\times\frac{1+\delta}{2}\frac{1-\delta}{2} \\
&= \frac{1}{4}\left[(1-\delta^2)\frac{\alpha}{Q}+\frac{Q-\alpha}{Q}\right] = \frac{1}{4}\left[1-\delta^2\frac{\alpha}{Q}\right]
\end{aligned}$$

To compute this measure at period 2 we need the following probabilities:

$$
\begin{aligned}
&\mathbb{P}[Q_2 \in [-Q-\beta,-Q+\beta), Q_1 \in [-Q-\alpha,-Q+\alpha)]\\
=&\mathbb{P}[Q_1 \in [-Q-\alpha,-Q+\alpha)|U=1,V=1] \times \mathbb{P}[U=1,V=1] \times \mathbb{P}[l_2+\beta \in [-Q-\beta,-Q+\beta)]\\
+&\mathbb{P}[Q_1 \in [-Q-\alpha,-Q+\alpha)|U=1,V=0] \times \mathbb{P}[U=1,V=0] \times \mathbb{P}[l_2-\beta \in [-Q-\beta,-Q+\beta)]\\
+&\mathbb{P}[Q_1 \in [-Q-\alpha,-Q+\alpha)|U=0] \times \mathbb{P}[U=0] \times \mathbb{P}[l_2+\beta \in [-Q-\beta,-Q+\beta)]\\
=&0+\frac{\alpha}{Q}\delta\frac{1}{2}\frac{\beta}{Q}+0=\frac{\alpha}{Q}\delta\frac{1}{2}\frac{\beta}{Q}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{P}[Q_2 \in [-Q+\beta,Q-\beta], Q_1 \in [-Q-\alpha,-Q+\alpha)]\\
=&\mathbb{P}[Q_1 \in [-Q-\alpha,-Q+\alpha)|U=1,V=1] \times \mathbb{P}[U=1,V=1] \times \mathbb{P}[l_2+\beta \in [-Q+\beta,Q-\beta]]\\
+&\mathbb{P}[Q_1 \in [-Q-\alpha,-Q+\alpha)|U=1,V=0] \times \mathbb{P}[U=1,V=0] \times \mathbb{P}[l_2-\beta \in [-Q+\beta,Q-\beta]]\\
+&\mathbb{P}[Q_1 \in [-Q-\alpha,-Q+\alpha)|U=0] \times \mathbb{P}[U=0] \times \mathbb{P}[l_2+\beta \in [-Q+\beta,Q-\beta]]\\
&=0+\frac{\alpha}{Q}\delta\frac{1}{2}\frac{Q-\beta}{Q}+\frac{1}{2}\frac{\alpha}{Q}(1-\delta)\frac{Q-\beta}{Q}\\
&=\frac{1}{2}\frac{\alpha}{Q}\frac{Q-\beta}{Q}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{P}[Q_2 \in (Q-\beta,Q+\beta], Q_1 \in [-Q-\alpha,-Q+\alpha)]\\
=&\mathbb{P}[Q_1 \in [-Q-\alpha,-Q+\alpha)|U=1,V=1] \times \mathbb{P}[U=1,V=1] \times \mathbb{P}[l_2+\beta \in (Q-\beta,Q+\beta]]\\
+&\mathbb{P}[Q_1 \in [-Q-\alpha,-Q+\alpha)|U=1,V=0] \times \mathbb{P}[U=1,V=0] \times \mathbb{P}[l_2-\beta \in (Q-\beta,Q+\beta]]\\
+&\mathbb{P}[Q_1 \in [-Q-\alpha,-Q+\alpha)|U=0] \times \mathbb{P}[U=0] \times \mathbb{P}[l_2+\beta \in (Q-\beta,Q+\beta]]\\
=&0+0+\frac{1}{2}\frac{\alpha}{Q}(1-\delta)\frac{\beta}{Q}
\end{aligned}
$$

Symmetrically,

$$
\begin{aligned}
\mathbb{P}[Q_2 \in [-Q-\beta,-Q+\beta), Q_1 \in (Q-\alpha,Q+\alpha]] &= \frac{1}{2}\frac{\alpha}{Q}(1-\delta)\frac{\beta}{Q}\\
\mathbb{P}[Q_2 \in [-Q+\beta,Q-\beta], Q_1 \in (Q-\alpha,Q+\alpha]] &= \frac{1}{2}\frac{\alpha}{Q}\frac{Q-\beta}{Q}\\
\mathbb{P}[Q_2 \in (Q-\beta,Q+\beta], Q_1 \in (Q-\alpha,Q+\alpha]] &= \frac{\alpha}{Q}\delta\frac{1}{2}\frac{\beta}{Q}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{P}[Q_2 \in [-Q-\beta, -Q), Q_1 \in [-Q+\alpha, Q-\alpha]] \\
=&\mathbb{P}[Q_1 \in [-Q+\alpha, Q-\alpha] | U=1, V=1] \times \mathbb{P}[U=1, V=1] \times \mathbb{P}[l_2+\beta \in [-Q-\beta, -Q)] \\
+&\mathbb{P}[Q_1 \in [-Q+\alpha, Q-\alpha] | U=1, V=0] \times \mathbb{P}[U=1, V=0] \times \mathbb{P}[l_2-\beta \in [-Q-\beta, -Q)] \\
+&\mathbb{P}[Q_1 \in [-Q+\alpha, Q-\alpha] | U=0] \times \mathbb{P}[U=0] \times \mathbb{P}[l_2 \in [-Q-\beta, -Q)] \\
=&0 + \frac{Q-\alpha}{Q}\delta\frac{1}{2}\frac{\beta}{2Q} + 0
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{P}[Q_2 \in [-Q, -Q+\beta), Q_1 \in [-Q+\alpha, Q-\alpha]] \\
=&\mathbb{P}[Q_1 \in [-Q+\alpha, Q-\alpha] | U=1, V=1] \times \mathbb{P}[U=1, V=1] \times \mathbb{P}[l_2+\beta \in [-Q, -Q+\beta)] \\
+&\mathbb{P}[Q_1 \in [-Q+\alpha, Q-\alpha] | U=1, V=0] \times \mathbb{P}[U=1, V=0] \times \mathbb{P}[l_2-\beta \in [-Q, -Q+\beta)] \\
+&\mathbb{P}[Q_1 \in [-Q+\alpha, Q-\alpha] | U=0] \times \mathbb{P}[U=0] \times \mathbb{P}[l_2 \in [-Q, -Q+\beta)] \\
=&0 + \frac{Q-\alpha}{Q}\delta\frac{1}{2}\frac{\beta}{2Q} + \frac{Q-\alpha}{Q}(1-\delta)\frac{\beta}{2Q} \\
=&\frac{Q-\alpha}{Q}\frac{2-\delta}{2}\frac{\beta}{2Q}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{P}[Q_2 \in [-Q+\beta, Q-\beta], Q_1 \in [-Q+\alpha, Q-\alpha]] \\
=&\mathbb{P}[Q_1 \in [-Q+\alpha, Q-\alpha] | U=1, V=1] \times \mathbb{P}[U=1, V=1] \times \mathbb{P}[l_2+\beta \in [-Q+\beta, Q-\beta]] \\
+&\mathbb{P}[Q_1 \in [-Q+\alpha, Q-\alpha] | U=1, V=0] \times \mathbb{P}[U=1, V=0] \times \mathbb{P}[l_2-\beta \in [-Q+\beta, Q-\beta]] \\
+&\mathbb{P}[Q_1 \in [-Q+\alpha, Q-\alpha] | U=0] \times \mathbb{P}[U=0] \times \mathbb{P}[l_2 \in [-Q+\beta, Q-\beta]] \\
=&\frac{Q-\alpha}{Q}\delta\frac{1}{2}\frac{Q-\beta}{Q} + \frac{Q-\alpha}{Q}\delta\frac{1}{2}\frac{Q-\beta}{Q} + \frac{Q-\alpha}{Q}(1-\delta)\frac{Q-\beta}{Q} \\
=&\frac{Q-\alpha}{Q}\frac{Q-\beta}{Q}
\end{aligned}
$$

by symmetry again

$$\mathbb{P}[Q_2 \in (Q-\beta, Q], Q_1 \in [-Q+\alpha, Q-\alpha]] = \frac{Q-\alpha}{Q}\frac{2-\delta}{2}\frac{\beta}{2Q}$$

$$\mathbb{P}[Q_2 \in (Q, Q+\beta], Q_1 \in [-Q+\alpha, Q-\alpha]] = \frac{Q-\alpha}{Q}\delta\frac{1}{2}\frac{\beta}{2Q}$$

Then

$$
\begin{aligned}
\mathbb{E}[(\tilde{V}-P_2)^2] =& \frac{\alpha}{Q}\delta\frac{1}{2}\frac{\beta}{Q} \times 0 + \frac{1}{2}\frac{\alpha}{Q}\frac{Q-\beta}{Q}\frac{1-\delta}{2}\frac{1+\delta}{2} + \frac{1}{2}\frac{\alpha}{Q}(1-\delta)\frac{\beta}{Q} \times \frac{1}{4} \\
&+ \frac{1}{2}\frac{\alpha}{Q}(1-\delta)\frac{\beta}{Q} \times \frac{1}{4} + \frac{1}{2}\frac{\alpha}{Q}\frac{Q-\beta}{Q}\frac{1-\delta}{2}\frac{1+\delta}{2} + \frac{\alpha}{Q}\delta\frac{1}{2}\frac{\beta}{Q} \times 0 \\
&+ \frac{Q-\alpha}{Q}\delta\frac{1}{2}\frac{\beta}{2Q} \times 0 + \frac{Q-\alpha}{Q}\frac{2-\delta}{2}\frac{\beta}{2Q} \times \frac{1-\delta}{2-\delta}\frac{1}{2-\delta} \\
&+ \frac{Q-\alpha}{Q}\frac{Q-\beta}{Q}\frac{1}{4} \\
&+ \frac{Q-\alpha}{Q}\frac{2-\delta}{2}\frac{\beta}{2Q} \times \frac{1-\delta}{2-\delta}\frac{1}{2-\delta} + \frac{Q-\alpha}{Q}\delta\frac{1}{2}\frac{\beta}{2Q} \times 0 \\
=& \frac{1}{4}\frac{\alpha}{Q}\left[(1-\delta)(1+\delta)\frac{Q-\beta}{Q} + (1-\delta)\frac{\beta}{Q}\right] \\
&+ \frac{1}{4}\frac{Q-\alpha}{Q}\left[2\frac{1-\delta}{2-\delta}\frac{\beta}{Q} + \frac{Q-\beta}{Q}\right]
\end{aligned}
$$

Finally the fact that $\mathbb{E}[(\tilde{V}-P_1)^2] = \frac{\delta}{2}\pi^1(\alpha) + \frac{1-\delta^2}{4}$ is obvious and to obtain the relation between $\mathbb{E}[(\tilde{V}-P_2)^2]$ and $\pi^2(\alpha,\beta)$, we just have to develop the two expressions:

$$\begin{aligned}
\pi^2(\alpha,\beta) &= \frac{\delta}{2} \times [(1-\frac{\alpha}{Q}) \times (1-\frac{1}{2-\delta}\frac{\beta}{Q}) + (1-\delta)\frac{\alpha}{Q}(1-\frac{\beta}{Q})] \\
&= \frac{\delta}{2} \times [1-\delta\frac{\alpha}{Q} - \frac{1}{2-\delta}\frac{\beta}{Q} + (\frac{1}{2-\delta} - (1-\delta))\frac{\alpha}{Q}\frac{\beta}{Q}]
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[(\tilde{V}-P_2)^2] &= \frac{1}{4}\frac{\alpha}{Q}(1-\delta)\left[(1+\delta) - \delta\frac{\beta}{Q}\right] + \frac{1}{4}(1-\frac{\alpha}{Q})\left[1 - \frac{\delta}{2-\delta}\frac{\beta}{Q}\right] \\
&= \frac{1}{4} + \frac{1}{4}[-\delta^2\frac{\alpha}{Q} - \frac{\delta}{2-\delta}\frac{\beta}{Q} + \delta(\frac{1}{2-\delta} - (1-\delta))\frac{\alpha}{Q}\frac{\beta}{Q}]
\end{aligned}$$

Bibliography

[1] Back K. and Baruch S. (2007), Working orders in limit order markets and floor exchanges, *Journal of Finance* 62, 1589-1621.

[2] Biais B., Foucault T. and Moinas S. (2013), Equilibrium Fast Trading, *Working Paper*.

[3] Biais B., Hillion P. and Spatt C. (1995), An empirical analysis of the limit order book and the order flow in the Paris Bourse, *Journal of Finance* 50, 1655-1689.

[4] Biais B., Martimort D. and Rochet J-C. (2000), Competing mechanisms in a common value environment, *Econometrica* 68, 799-838.

[5] Biais B. and Weill P.-O. (2009), Liquidity shocks and order book dynamics, *Working Paper*.

[6] Biais B., Hombert J. and Weill P.-O. (2012), Pricing and Liquidity with Sticky Trading Plans, *Working Paper*.

[7] O. Brandouy, Barneto P. and Leger L.A. (2003), Asymmetric Information, Imitative Behaviour and Communication : Price Formation in an Experimental Asset Market, *European Journal of Finance*, 9, 393-419.

[8] Brogaard J., Hendershott T. and Riordan R. (2012), High Frequency Trading and Price Discovery, *Working Paper*.

[9] Brunnermeier M. (2005), Information leakage and market efficiency, *Review of Financial Studies* 18, 417-457.

[10] Carvalho C., Klagge N. and Moench E., The Persistent effects of a false news shock, *Federal Reserve Bank of New York Staff Reports*.

[11] Chaboud A., Chiquoine B., Hjalmarsson E. and Vega C. (2009), Rise of the machines: algorithmic trading in the foreign exchange market, *Working Paper*.

[12] Chordia T., Roll R. and Subrahmanyam A. (2000), Commonality in Liquidity, *Journal of Financial Economics* 56, 3-28.

[13] Chordia T., Roll R. and Subrahmanyam A. (2001), Market Liquidity and Trading Activity, *Journal of Finance* 56, 501-530.

[14] Colliard J-E. and Foucault T. (2012), Trading fees and efficiency in limit order markets, *Review of Financial Studies* 25, 3389-3421.

[15] Daniel K., Hirshleifer D. and Subrahmanyam A. (1998), Investor psychology and security market under- and overreactions, *Journal of Finance* 53, 1839-1885.

[16] Della Vigna S. and Pollet J. M. (2009), Investor inattention and friday earnings announcements, *Journal of Finance* 64, 709-749.

[17] Dow J., Goldstein I. and Guembel A. (2011), Incentives for information production in markets where prices affect real investment, *Working Paper*.

[18] Duffie D., Garleanu N. and Pedersen L. (2005), Over-the-counter markets, *Econometrica* 73, 1815-1847.

[19] Duffie D., Garleanu N. and Pedersen L. (2007), Valuation in over-the-counter markets, *Review of Financial Studies* 20, 1865-1900.

[20] Duffie D. (2010), Presidential address: asset price dynamics and slow moving capital, *Journal of Finance* 65, 1237-1267.

[21] Ederington L.H., and Lee J. (1995), The Short-run dynamics of the price adjustment to new information, *Journal of Financial and Quantitative Analysis* 30, 117-134.

[22] Engle R., Fleming M., Ghysels E and Nguyen G. (2011), Liquidity and Volatility in the U.S. Treasury Market: Evidence From A New Class of Dynamic Order Book Models, *Working Paper.*

[23] Fleming M. J. and Remolona E. M. (1999), Price formation and liquidity in the U.S. treasury market: the response to public information, *Journal of Finance* 54, 1901-1915.

[24] Foucault T. (1999), Order flow composition and trading costs in a dynamic limit order market, *Journal of Financial Markets* 2, 99-134.

[25] Foucault T. (2010), Limit Order Markets, *Encyclopedia of Quantitative Finance, Rama Cont editor, John Wiley & Sons Limited.*

[26] Foucault T., Kadan O. and Kandel E. (2005), Limit order book as a market for liquidity, *Review of Financial Studies* 18, 1171-1217.

[27] Foucault T., Hombert J. and Rosu I. (2012), News trading and speed, *Working Paper.*

[28] Foucault T., Kadan O. and Kandel E. (2013), Liquidity cycles and make/take Fees in electronic markets, *Journal of Finance* 68, 299-341.

[29] Foucault T. and Menkveld A. (2008), Competition for order flow and smart order routing systems, *Journal of Finance* 63, 119-158.

[30] Foucault T., Roell A. and Sandas P. (2003), Market Making with Costly Monitoring: An Analysis of the SOES Controversy, *Review of Financial Studies* 16, 345-384.

[31] Foucault T., Pagano M. and Roell A. (2013), *Market Liquidity: Theory, Evidence, and Policy. Oxford University Press.*

[32] Froot K., Scharfstein D. and Stein J. (1992), Herd on the street: informational efficiencies in a market with short-term speculation, *Journal of Finance* 47, 1461-1484.

[33] Gajewski J.-F. and Gresse C. (2007), Centralised Order Books versus Hybrid Order Books: A Paired Comparison of Trading Costs on NSC (Euronext Paris) and SETS (London Stock Exchange), *Journal of Banking and Finance* 31, 2906–2924.

[34] Glosten L. and Milgrom P.(1989), Bid, ask and transaction prices in specialist market with heterogeneously informed trader, *Journal of Financial Economics* 13, 71-100.

[35] Glosten L. (1994), Is the electronic open limit order book inevitable?, *Journal of Finance* 49, 1127-1161.

[36] Goettler R., Parlour C. and Rajan U. (2005), Equilibrium in a dynamic limit order market, *Journal of Finance* 60, 2149-2192.

[37] Goettler R., Parlour C. and Rajan U. (2009), Informed traders and limit order markets, *Journal of Financial Economics* 93, 67-87.

[38] Green T. C. (2004), Economic news and the impact of trading on bond prices, *Journal of Finance* 59, 1201-1233

[39] Gresse C. (2006), The Effect of crossing-network trading on dealer market's bid-ask spreads, *European Financial Management* 12, 143-160.

[40] Gresse C. (2012), Market fragmentation in Europe: Assessment and prospects for market quality, *Foresight project on The Future of Computer Trading in Financial Markets - Driver Review 19.*

[41] Gross-Klussmann A. and Hautsch N. (2011), When machines read the news: Using automated text analytics to quantify high frequency news-implied market reactions, *Journal of Empirical Finance* 18, 321-340.

[42] Hasbrouck (2007), *Empirical Market Microstructure: The Institution, Economics, and Econometrics of Securities Trading. Oxford University Press.*

[43] Hasbrouck J. and Saar G. (2009), Technology and liquidity provision: The blurring of traditional definitions, *Journal of Financial Markets* 12, 143-172.

[44] Hasbrouck J. and Saar G. (2011), Low-Latency trading, *Working Paper.*

[45] Hasbrouck J. (2013), High frequency quoting: short-term volatility in bids and offers, *Working Paper.*

[46] Hendershott T., Jones C. and Menkveld A. (2011), Does algorithmic trading improve liquidity?, *Journal of Finance* 66, 1-33.

[47] Hollifield B., Miller R. and Sandas P. (2004), Empirical analysis of limit order markets, *Review of Economic Studies* 71, 1027-1063.

[48] Hollifield B., Miller R., Sandas P. and Slive S. (2006), Estimating the gain from trade in limit order markets, *Journal of Finance* 61, 2753-2804.

[49] Jovanovic B. and Menkveld A. (2010), Middlemen in limit-order markets, *Working Paper.*

[50] Kim O. and Verrecchia R. E. (1991), Trading volume and price reactions to public announcements, *Journal of Accounting Research* 29, 302-21.

[51] Kim O. and Verrecchia R. E. (1994), Market liquidity and volume around earnings announcements, *Journal of Accounting and Economics* 17, 41-67.

[52] Kyle A. (1985), Continuous auctions and insider trading, *Econometrica* 53, 1315-1336.

[53] Lagos R. and Rocheteau G. (2009), Liquidity in asset markets with search frictions, *Econometrica* 77, 403-426.

[54] Lagos R., Rocheteau G. and Weill P.-O. (2011), Crises and liquidity in over-the-counter markets, *Journal of Economic Theory* 146, 2169-2205.

[55] Mankiw, N. G. and Reis R. (2002), Sticky Information Versus Sticky Prices: A Proposal to Replace the New Keynesian Phillips Curve, *Quarterly Journal of Economics* 117, 1295-1328.

[56] Mondria J. (2010), Portfolio Choice, Attention Allocation, and Price Comovement, *Journal of Economic Theory* 145, 1837-1864.

[57] Pagnotta E. (2010), Information and liquidity trading at optimal frequencies, *Working Paper.*

[58] Pagnotta E. and Philippon T. (2012), Competing on speed, *Working Paper.*

[59] Parlour C. (1998), Price dynamics in a limit order market, *Review of Financial Studies* 11, 789-816.

[60] Parlour C. and Seppi D. (2003), Liquidity-based competition for order flow, *Review of Financial Studies* 16, 301-343.

[61] Parlour C. and Seppi D. (2008), Limit order markets: a survey, *Handbook of Financial Intermediation & Banking.*

[62] Peng L. and Xiong W. (2006), Investor attention, overconfidence and category learning, *Journal of Financial Economics* 80, 563-602.

[63] Rosu I. (2009), A Dynamic model of the limit order book, *Review of Financial Studies* 22, 4601-4641.

[64] Rosu I. (2010), Liquidity and information in order driven markets, *Working Paper*

[65] Securities and Exchange Commission (2010), Concept release on equity market structure, *Federal Register* 75, 3594-3614.

[66] Seppi D. (1997), Liquidity provision with limit orders and a strategic specialist, *Review of Financial Studies* 10, 103-150.

[67] Sims C. (2003), Implication of Rational Inattention, *Journal of Monetary Economics* 50, 665-690.

[68] Tetlock P. C. (2010), Does public financial news resolve asymmetric information? *Review of Financial Studies* 23, 3520-3557.

[69] Van Nieuwerburgh S. and Veldkamp L. (2009), Information immobility and the home bias puzzle, *Journal of Finance* 64, 1187-1215.

[70] Vayanos D. and Weill P.-O. (2008), A Search-based theory of the on-the-run phenomenon, *Journal of Finance* 63, 1351-1389

[71] Weill P.-O. (2007), Leaning against the wind, *Review of Economic Studies* 74, 1329-1354.

[72] Weill P.-O. (2008), Liquidity premia in dynamic bargaining markets, *Journal of Economic Theory* 140, 66-96.

Essays in Financial Market Microstructure

This dissertation is made of three distinct chapters. In the first chapter, I show that traditional liquidity measures, such as market depth, are not always relevant to measure investors' welfare. I build a limit order market model and show that a high level of liquidity supply can correspond to poor execution conditions for liquidity providers and to a relatively low welfare. In the second chapter, I model the speed of price adjustments to news arrival in limit order markets when investors have limited attention. Because of limited attention, investors imperfectly monitor news arrival. Consequently prices reflect news with delay. This delay shrinks when investors' attention capacity increases. The price adjustment delay also decreases when the frequency of news arrival increases. The third chapter presents a joint work with Thierry Foucault. We build a model to explain why high frequency trading can generate mini-flash crashes (a sudden sharp change in the price of a stock followed by a very quick reversal). Our theory is based on the idea that there is a trade-off between speed and precision in the acquisition of information. When high frequency traders implement strategies involving fast reaction to market events, they increase their risk to trade on noise and thus generate mini flash crashes. Nonetheless they increase market efficiency.

Keywords: liquidity, welfare, limit order market, news, limited attention, imperfect market monitoring, high frequency trading, mini flash crash, market efficiency.

Essais en Microstructure des Marchés Financiers

Cette thèse est composée de trois chapitres distincts. Dans le premier chapitre, je montre que les mesures de liquidité traditionnelles, tels que la profondeur du marché, ne sont pas toujours pertinents pour mesurer le bien-être des investisseurs. Je construis un modèle de marché conduits par les ordres et montrent qu'une offre de liquidité élevée peut correspondre à de mauvaises conditions d'exécution pour les fournisseurs de liquidité et à un bien-être relativement faible. Dans le deuxième chapitre, je modélise la vitesse des ajustements de prix à l'arrivée de nouvelles dans les marchés conduits par les ordres, lorsque les investisseurs ont une capacité d'attention limitées. En raison de leur attention limitée, les investisseurs suivent imparfaitement l'arrivée de nouvelles. Ainsi les prix s'ajustent aux nouvelles après un certain délai. Ce délai diminue lorsque le niveau d'attention des investisseurs augmente. Le délai d'ajustement des prix diminue également lorsque la fréquence, à laquelle les nouvelles arrivent, augmente. Le troisième chapitre présente un travail écrit en collaboration avec Thierry Foucault. Nous construisons un modèle pour expliquer en quoi le trading à haute fréquence peut générer des «mini flash crashes» (un brusque changement de prix suivie d'un retour très rapide au niveau antérieur). Notre théorie est basée sur l'idée qu'il existe une tension entre la vitesse à laquelle l'information peut être acquise et la précision de cette information. Lorsque les traders à haute fréquence mettent en oeuvre des stratégies impliquant des réactions rapides à des événements de marché, ils augmentent leur risque de réagir à du bruit et génèrent ainsi des «mini flash crashes». Néanmoins, ils augmentent l'efficience informationnelle du marché.

Mots clefs: liquidité, bien-être, marché conduit par les ordres, nouvelles, attention limitée, surveillance de marché imparfaite, trading haute fréquence, efficience informationnelle de marché.