Dynamique algébrique des applications rationnelles de surfaces

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献给我亲爱的父母

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Résumé

Cette thèse se compose de trois parties. La première partie est consacrée à l'étude des points périodiques des applications birationnelles des surfaces projectives. Nous montrons que toute application birationnelle de surface dont la croissance des degrés est exponentielle admet un ensemble de points périodiques Zariski dense.

Dans la seconde partie, nous démontrons la conjecture de Mordell-Lang dynamique pour toute application polynômiale birationnelle du plan affine définie sur un corps de caractéristique nulle. Notre approche donne une nouvelle démonstration de cette conjecture pour les automorphismes polynomiaux du plan.

Enfin la troisième partie porte sur un problème de géométrie affine inspiré par la généralisation au cas de toutes les applications polynomiales du plan affine de la conjecture de Mordell-Lang dynamique. Etant donné un ensemble fini S de valuations sur l'anneau de polynômes $k[x,y]$ sur un corps algébriquement clos k triviales sur k, nous donnons une condition nécessaire et suffisante pour que le corps des fractions de l'intersection des anneaux de valuations de S avec $k[x,y]$ soit de degré de transcendance 2 sur k.

Abstract

This thesis contains three parts. The first one is devoted to the study of the set of periodic points for birational surface maps. We prove that any birational transformation of a smooth projective surface whose degree growth is exponential admits a Zariski-dense set of periodic orbits.

In the second part, we prove the dynamical Mordell-Lang conjecture for all polynomial birational transformations of the affine plane defined over a field of characteristic zero. Our approach gives a new proof of this conjecture for polynomial automorphisms of the affine plane.

The last part is concerned with a problem in affine geometry that was inspired by the generalization to any polynomial map of the dynamical Mordell-Lang conjecture. Given any finite set S of valuations that are defined on the polynomial ring $k[x, y]$ over an algebraically closed field k, trivial on k, we give a necessary and sufficient condition so that the field of fractions of the intersection of the valuation rings of S with $k[x, y]$ has transcendence degree 2 over k.

Contents

Introduction

Foreword

This manuscript is divided into three parts that can be read independently. The precise statements of our main results are explained thoroughly in the introduction of each part. In this introduction we shall present the general framework in which all the results of this thesis take place, and try to convey the common ideas lying behind all of them.

We do not pretend to give a survey of the rapidly growing literature on all aspects of the subject of algebraic dynamics. In particular, we have chosen to leave aside many interesting problems (including the dynamics on non-archimedean Berkovich spaces, equidistributions results, and the geometry of moduli spaces of dynamical systems) since they do not appear in the results presented in this thesis.

1. What is algebraic dynamics?

The subject of this thesis, algebraic dynamics, can be defined as the study of iterations of (rational) endomorphisms on algebraic varieties endowed with their Zariski topology.

More precisely, an *algebraic dynamical system* is a pair (X, f) where X is a quasi-projective variety defined over a field $\mathbf{k}$ and $f : X \dashrightarrow X$ is a dominant rational (or regular) self-map on X defined over $\mathbf{k}$.

Let us fix some notations. For any closed point $p \in X(\mathbf{k})$, we denote by $O_f(p) := \{f^n(p) | \ n \geq 0\}$ the orbit of p. We say that p is *preperiodic* if $O_f(p)$ is a finite set. When there exists an integer $n \geq 1$ such that $f^n(p) = p$, then we say p is *periodic* of period n.

1.1. Basic questions in algebraic dynamics.

Let us begin by presenting some representative questions from this field, referring to [**19**] for more open problems.

Suppose (X, f) is an algebraic dynamical system defined over a projective variety and $\mathbf{k}$ is an algebraically closed field.

1.1.1. *Periodic orbits.* The first natural question we may raise about periodic points concerns their existence. There are several ways to give a precise formulation of this problem. Here is one possibility.

QUESTION 1.1. Give necessary and sufficient conditions for the map f to admit a Zariski dense set of isolated periodic orbits.

When the number of periodic orbits is infinite, one can also ask for a count of the number of isolated periodic points of a fixed period and its asymptotic value when the period goes to infinity.

We are far from understanding this simple problem in complete generality. Let us observe though that the set of periodic points of a rational endomorphism may not be Zariski dense. For example when f is an automorphism of a projective space of infinite order, the set of periodic points of f is a finite union of finitely many proper linear subspaces. However there are many cases where a positive answer to the previous question is known to be true.

Recall that an endomorphism f on a projective variety X is said to be *polarized* if there exists an ample line bundle L on X satisfying $f^*L = L^{\otimes d}$ for some integer $d \geq 2$.

Suppose f is a polarized endomorphism. One can show that all its periodic points of are isolated. Using methods from complex geometry and complex analysis Briend-Duval [**10**] and subsequently Dinh-Sibony [**17**] have proved that the number of isolated periodic points of f of period m is equivalent to $d^{m \dim X}$, and that the set of periodic points is Zariski dense in X. By the Lefschetz principle, these results hold true whenever $\mathbf{k}$ has characteristic zero.

More recently, Hrushovski and Fakhruddin [**19**] gave a purely algebraic proof of the Zariski density of periodic points over any algebraically closed field. We shall return to their approach later in this introduction.

The complex analytic methods alluded to above have been used to give a positive answer to Question 1.1 for several other classes of maps. Building on the work of Guedj [**28**], Dinh, Nguyên and Truong [**44**] proved it when the topological degree (i.e. the number of preimages of a general point) is large in the sense that it dominates the action of f on the cohomology of X. This class of maps includes polarized endomorphisms.

On the other hand, Dinh and Sibony [**16**] proved the Zariski density of periodic points for automorphisms $f : \mathbb{A}^m \to \mathbb{A}^m$ on complex affine spaces that are regular (i.e. the indeterminacy loci of f and its inverse f^{-1} are disjoint on the hyperplane at infinity in $\mathbb{P}^m$). This result extends former works by Bedford-Lyubich-Smillie [**5**] in the case of Hénon maps in dimension 2.

In the first part of my thesis, I will give an essentially complete answer to the question above in the case when f is a birational transformation on a projective surface over arbitrary algebraically closed field of characteristic different from 2 and 3.

1.1.2. *Zariski dense orbits.* Recall that we are given an algebraic dynamical system (X, f) on a projective variety. Another natural and basic question is to ask for conditions ensuring the existence of a Zariski dense orbit.

Amerik, Bogomolov and Rovinsky [**1**] have formulated the following precise conjecture in the case of regular self-maps.

CONJECTURE 1.2. Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. Let X be a quasi-projective variety defined over $\mathbf{k}$ and $f : X \to X$ be a dominant endomorphism defined over $\mathbf{k}$. We suppose that for all $m \geq 1$ there exists no nonconstant rational function g on X satisfying $g \circ f^m = g$. Then there exists a point $p \in X(\mathbf{k})$ such that the set $\{f^n(p)|\ n \geq 0\}$ is Zariski dense in $X(\mathbf{k})$.

Since a polarized endomorphism cannot preserve a fibration fiberwise, the previous conjecture reduces in this case to a conjecture of Zhang [**49**] asserting the existence of a closed point $p \in X(\mathbf{k})$ whose iterates form a Zariski dense subset of $X(\mathbf{k})$.

When $\mathbf{k}$ is uncountable, Conjecture 1.2 was proved by Amerik and Campana [**3**]. However the conjecture is wide open, even when $\mathbf{k} = \bar{\mathbb{Q}}$.

In [**18**], Fakhruddin proved Conjecture 1.2 for *generic* endomorphisms[1] on projective spaces over arbitrary fields $\mathbf{k}$ of characteristic zero.

Medvedev and Scanlon proved Conjecture 1.2 for arbitrary fields $\mathbf{k}$ of characteristic zero in the case when $f := (f_1(x_1), \dots, f_N(x_N))$ is an endomorphism of $\mathbb{A}^N_{\mathbf{k}}$ where f_i's are one-variable polynomials defined over $\mathbf{k}$. Their proof relied on model theoretic methods.

In [**6**], Bell, Ghioca and Tucker proved a weaker version of this conjecture when f is an automorphism. Namely they showed the existence of a subvariety V of codimension of at least two defined over $\mathbf{k}$ such that the set $\cup_{n=1}^{\infty} f^n(V)$ is Zariski dense in X. This result is based on [**2**, Corollary 9] of Amerik, which proves that there exists a nonpreperiodic algebraic point when f is of infinite order.

A proof of Conjecture 1.2 for (most) birational surface selfmaps over any field of characteristic 0 is given *in the first part of my thesis.*

1.2. Algebraic and arithmetic methods in the case of polarized endomorphisms. Let us explain in more detail how one can use purely algebraic and arithmetic tools to tackle the problems and conjectures stated in the previous sections in the case of polarized endomorphisms. These tools will be crucial in all parts of this thesis.

1.2.1. *Zariski density of the set of periodic points.* Recall that one wants to show that the set of periodic points of a polarized dynamical system $f : X \to X$ is Zariski dense.

The strategy we briefly explain below was designed by Hrushovski, Fakhruddin and Poonen.

Step 1. One first treats the case when $\mathbf{k}$ is the algebraic closure of a finite field. Let U by any Zariski open subset of X. There exists a finite subfield $\mathbb{F}_q$ of $\mathbf{k}$ such that X, f, L and U are defined over $\mathbb{F}_q$. Hrushovski's twisted Lang-Weil estimates [**29**] imply that the intersection $(U \times U) \cap \Gamma_f \cap \Gamma_{\Phi_{q^m}}$ is not empty when m is large enough, where Γ_f is the graph of f in $X \times X$ and $\Gamma_{\Phi_{q^m}}$ is the graph of the q^m-Frobenius map. It follows that there exists a point $x \in U(\mathbf{k})$ satisfying $f(x) = \Phi_{q^m}(x)$. Since f is defined over $\mathbb{F}_q$, we have $f^n(x) = \Phi^n_{q^m}(x)$ for all $n \geq 0$. In particular, x is a periodic point of f contained in U, we conclude that the set of periodic points of f is Zariski dense in X.

In fact, the argument above holds for all dominant rational endomorphisms.

Step 2. To deal with the general case of a characteristic zero field $\mathbf{k}$, one proves that one can assume $\mathbf{k}$ is a number field, say $\mathbf{k} = \mathbb{Q}$, and one uses a reduction argument. A key property makes it possible to lift periodic points from positive

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[1] An endomorphism $f : \mathbb{P}^N_{\mathbf{k}} \to \mathbb{P}^N_{\mathbf{k}}$ satisfying $f^* O_{\mathbb{P}^N_{\mathbf{k}}}(1) = O_{\mathbb{P}^N_{\mathbf{k}}}(d)$ is said to be generic if it conjugates by a suitable linear automorphism on $\mathbb{P}^N_{\mathbf{k}}$ to an endomorphism $[x_0 : \cdots : x_N] \mapsto [\sum_{|I|=d} a_{0,I} x^I : \cdots : \sum_{|I|=d} a_{N,I} x^I]$ where the set $\{a_{i,I}\}_{0 \leq i \leq N, |I|=d}$ is algebraically independent over $\bar{\mathbb{Q}}$.

characteristic to zero characteristic: the fact that a polarized endomorphism over *any* field has only *isolated* periodic points.

In the case of birational maps this property is no longer satisfied. However under a suitable assumption on the algebraic complexity of the map, Cantat [**12**] has proved that a birational surface map can only have finitely many curves of periodic points. Everything then boils down to proving that the algebraic complexity assumption is satisfied for the reduction of a birational surface map when it is true for the original map. This constitutes the core of our analysis and relies on the natural linear action of birational surface maps on a suitable hyperbolic space of infinite dimension. This space is constructed as a set of cohomology classes in the Riemann-Zariski space of X and was introduced by Cantat in [**13**]. See also [**9, 14, 21, 34**].

1.2.2. *Existence of a Zariski dense orbit.* Conjecture 1.2 is open in this case except in dimension 1. However it is possible to prove the existence of a point with infinite orbit using the notion of *canonical height*. For simplicity, we suppose that $\mathbf{k} = \overline{\mathbb{Q}}$ in our discussion.

Recall that to an ample line bundle $L \to X$ is attached a real-valued function $h_L : X(\overline{\mathbb{Q}}) \to \mathbb{R}_+$ that "measures" the arithmetic complexity of a given point. It was observed by Call and Silverman [**11**] that the sequence $\frac{1}{d^n} h_L(f^n(p))$ converges for any $p \in X(\overline{\mathbb{Q}})$. The limit $\hat{h}_f(p)$ is referred to as the canonical height and satisfies the relation

$$\hat{h}_f(f(p)) = d\hat{h}_f(p)$$

so that any preperiodic point has height 0. It is now a fact that there exists a point $q \in X(\overline{\mathbb{Q}})$ satisfying $\hat{h}_f(q) > 0$ whence this point has an infinite orbit.

The canonical height turned out to be very important in the study of algebraic and arithmetic properties of polarized endomorphisms. It was thus a natural question whether one could extend its definition for more general classes of rational endomorphisms. Silverman [**42**] proposed a program to construct canonical heights and actually constructed it for monomial endomorphisms on projective spaces and for automorphisms on some special K3 surfaces in [**40**]. Kawaguchi and Lee constructed it independently for regular automorphisms on $\mathbb{A}^N$ [**30, 32**]. Jonsson and Wulcan constructed it for plane polynomial maps of small topological degree [**33**].

2. Dynamical versions of classical arithmetic problems

In [**41**], Silverman promoted a "dictionary" between the theory of abelian varieties and algebraic dynamics and translated many classical questions about abelian varieties into a more dynamical framework.

We gather here three problems that we feel particularly appealing and that are directly related to this dictionary and refer to [**4, 19, 41, 43, 42**] for more open problems.

2.1. Dynamical Uniform Boundedness Conjecture. Assume $\mathbf{k}$ is a number field. If $f : \mathbb{P}^N_{\mathbf{k}} \to \mathbb{P}^N_{\mathbf{k}}$ is a dominant endomorphism defined over $\mathbf{k}$ then it is naturally polarized since $f^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}_{\mathbb{P}^N}(d)$ where d is the common degree of the homogeneous polynomials defining f in homogeneous coordinates.

Denote by $\hat{h}_f$ the canonical height for f. We have seen above that $\hat{h}_f(q) = 0$ whenever q is preperiodic. It is actually a theorem that the converse holds true. Denote by $\mathrm{PrePer}(f, \mathbf{k})$ the set of preperiodic points that are defined over $\mathbf{k}$. Then by the so-called Northcott's property, one can even show that $\mathrm{PrePer}(f, \mathbf{k})$ is a finite set.

Building on these remarks, the following conjecture was proposed by Morton and Silverman [**41**].

Dynamical Uniform Boundedness Conjecture. Fix integers $d \geq 2$, $N \geq 1$ and $D \geq 1$. There is a constant $C(d, N, D)$ such that for all number fields $\mathbf{k}/\mathbb{Q}$ of degree at most D and all dominant endomorphisms $f : \mathbb{P}^N_{\mathbf{k}} \to \mathbb{P}^N_{\mathbf{k}}$ of degree d defined over $\mathbf{k}$, we have $\# \mathrm{PrePer}(f, \mathbf{k}) \leq C(d, N, D)$.

Fakhruddin [**19**] has shown that this conjecture generalizes the *strong torsion conjecture for abelian varieties* which states that the order of the torsion group of an abelian variety defined over a number field can be bounded in terms of the dimension of the variety and the degree of the number field. We shall mention that the strong torsion conjecture for abelian varieties is proved in dimension one by Merel [**36**].

Dynamical Uniform Boundedness Conjecture is wide open. In fact it not known even in the case $(d, N, D) = (2, 1, 1)$.

2.2. Dynamical Manin-Mumford. The Manin-Mumford conjecture was proved by Raynaud [**38, 39**] and concerns the geometry of subvarieties in abelian varieties.

More precisely, suppose V is an irreducible subvariety inside an abelian variety A over an algebraically closed field $\mathbf{k}$ of characteristic zero such that the intersection of the set of torsion points of A and V is Zariski dense in V. Then the conjecture asserts that there exists an abelian subvariety V_0 of A and a torsion point $a \in A(\mathbf{k})$ such that $V = V_0 + a$.

Observe that the set of torsion points of A is exactly the set of preperiodic points of the endomorphism $[m]$ of multiplication by m for any integer $m \geq 2$. The conclusion of Raynaud' theorem thus amounts to the following statement. If the set of preperiodic points of the endomorphism $[m]$ is Zariski dense inside V, then V is itself preperiodic under $[m]$.

Inspired by this result a first conjecture was proposed by Zhang [**48**]. It asserted that given any polarized endomorphism $f : X \to X$ on a projective variety defined over $\mathbf{k}$, then any subvariety V containing a Zariski dense subset of preperiodic points for f is itself preperiodic.

This optimistic conjecture was later disproved in [**26, 37**]. This motivated the proposal of several modified versions of this conjecture [**26, 47**]. The conjecture is now known in some very special cases [**18**, Theorem 1.3], [**35**, Theorem 7.33], see also [**26**, Theorem 3.1].

2.3. Dynamical Mordell-Lang. The Mordell-Lang conjecture for abelian varieties was proved by Faltings [**20**] and Vojta [**45**]. It says that if V is a subvariety of a semiabelian variety G defined over an algebraically closed field $\mathbf{k}$ of characteristic 0 and Γ is a finitely generated subgroup of $G(\mathbf{k})$, then $V(\mathbf{k})\bigcap\Gamma$ is a union of at most finitely many translates of subgroups of Γ.

According to the dictionary, the following dynamical analogue of the Mordell-Lang conjecture was proposed by Ghioca and Tucker.

Dynamical Mordell-Lang Conjecture ([**25**]). Let X be a quasi-projective variety defined over $\mathbb{C}$, let $f: X\to X$ be an endomorphism, and V be any subvariety of X. For any point $p\in X(\mathbb{C})$ the set $\{n\in\mathbb{N}|\ f^n(p)\in V(\mathbb{C})\}$ is a union of at most finitely many arithmetic progressions[2].

Observe that this conjecture implies the classical Mordell-Lang conjecture in the case $\Gamma\simeq(\mathbb{Z},+)$.

Another motivation for this conjecture comes from the Skolem-Mahler-Lech Theorem [**31**] on linear recurrence sequences.

More precisely, suppose $\{A_n\}_{n\geq 0}$ is any recurrence sequence satisfying $A_{n+l}=F(A_n,\cdots,A_{n+l-1})$ for all $n\geq 0$, where $l\geq 1$ and $F(x_0,\cdots,x_l)=\sum_{i=0}^{l-1}a_ix_i$ is a linear form on $\mathbf{k}^l$. The Skolem-Mahler-Lech Theorem asserts that the set $\{n\geq 0|\ A_n=0\}$ is a union of at most finitely many arithmetic progressions. This statement is equivalent to the dynamical Mordell-Lang conjecture for the linear map $f:(x_0,\cdots,x_{l-1})\mapsto(x_1,\cdots,x_{l-1},F(x_0,\cdots,x_l))$ and the hyperplane $V=\{x_0=0\}$.

It is thus natural to ask whether this theorem still holds when F is an arbitrary polynomial in $\mathbf{k}[x_0,\cdots,x_{l-1}]$, that is if the Skolem-Mahler-Lech Theorem holds for non-linear recurrence sequences.

Let us mention that Wibmer [**46**] has conjectured a generalization of Skolem-Mahler-Lech Theorem in yet another direction. He also showed that such a generalization is equivalent to a certain special case of the Dynamical Mordell-Lang conjecture. Moreover he indicted the connection of the Dynamical Mordell-Lang conjecture to the Galois theory of linear difference equations.

The Dynamical Mordell-Lang conjecture is already known in quite a few cases. It is known for all étale maps of quasi-projective varieties by Bell, Ghioca and Tucker [**7**]. In the earlier paper [**15, 31**], the Dynamical Mordell-Lang conjecture was shown for some automorphisms taking special forms and in particular, in [**15**], Cutkosky and Srinivas solved a problem of Zariski by applying a such result.

It is also known in the case f is a generic endomorphism on a projective space [**18**], in the case when $f=(F(x_1),G(x_2)):\mathbb{A}^2_{\mathbb{C}}\to\mathbb{A}^2_{\mathbb{C}}$ where F,G are polynomials and the subvariety V is a line ([**27**]), and in the case when $f=(F(x_1),\dots,F(x_n)):\mathbb{A}^n_K\to\mathbb{A}^n_K$ where $F\in K[t]$ is an indecomposable polynomial defined over a number field K which has no periodic critical points other than the points at infinity and V is a curve ([**8**]).

[2] an arithmetic progression is a set of the form $\{an+b|\ n\in\mathbb{N}\}$ with $a,b\in\mathbb{N}$.

Let us briefly explain the main ingredient in the proof of the dynamical Mordell-Lang conjecture for étale maps. It turns out that this ingredient appears in all other proofs of the conjecture with the important exception of [**27**]. The key is to find a $\mathfrak{q}$-adic parametrization of the orbit $O_f(p)$ for a suitable prime $\mathfrak{q}$ using (non-archimedean analytic) dynamical arguments. More precisely, one proves that there exists a $\mathfrak{q}$-adic analytic function $\Phi : \mathbb{Z}_{\mathfrak{q}} \to X$ satisfying $\Phi(n) = f^n(p)$ for all $n \geq 0$. Since $\mathbb{Z}_{\mathfrak{q}}$ is compact, we have either $\mathrm{Im}(\Phi) \cap V$ is finite or $\mathrm{Im}(\Phi) \subseteq V$ which concludes the proof.

3. Polynomial mappings of the affine plane

3.1. Dynamical Mordell-Lang conjecture for birational polynomial morphisms on $\mathbb{A}^2$. The second part of this thesis is devoted to the proof of the dynamical Mordell-Lang Conjecture mentioned above for birational polynomial maps of the affine space.

This class of maps is of course quite restrictive but the strategy of proof that we will follow is completely new. It even gives a new proof of the dynamical Mordell-Lang conjecture for polynomial automorphisms of Hénon type that does not rely on a $\mathfrak{q}$-adic parametrization of an orbit as explained above.

Let us summarize our approach in the case the data f, V, p are all defined over $\mathbb{Q}$.

We use the work of Favre and Jonsson [**24**] to construct a suitable compactification of $\mathbb{A}^2$ in which the dynamics of f at infinity has the following property. A suitable iterate of the map f contracts all curves at infinity to a super attracting fixed point q. We then show that the intersection of $f^n(V)$ with the line at infinity is reduced to that point q for all n large enough. Finally we use a height argument to show that either p is periodic, or that there exists a place at which $f^n(p) \to q$. In the later case, one argues in a neighborhood of q to show that $f^n(p)$ belongs to V for only finitely many n's.

3.2. Intersection of valuation rings in $k[x, y]$. Let us describe in broad terms the content of the last part of the thesis.

Denote by $\mathbf{k}$ an algebraically closed field. We pick any finite set S of valuations that are defined on the polynomial ring $\mathbf{k}[x, y]$, are trivial on $\mathbf{k}$, and takes at least one negative value on a non-constant polynomial. We shall then give a necessary and sufficient condition so that the field of fractions of the intersection of the valuation rings of S with $\mathbf{k}[x, y]$ has transcendence degree 2 over $\mathbf{k}$. Let us say that S is *rich* when the later condition is satisfied.

The heuristic lying behind our result is the following. When S contains the valuation $-\deg$ it is not difficult to check that the intersection of the valuation rings of S with $\mathbf{k}[x, y]$ is reduced to the constant hence S is not rich. Our condition states that S is rich iff all valuations in S are "far" from the valuation $-\deg$. The precise statement is quite technical and builds on the description of the space of valuations done by Favre and Jonsson in [**22, 23, 24**].

This part does not look to be connected to any dynamical problem. However it was largely inspired by our approach to the dynamical Mordell-Lang conjecture for any polynomial maps that we are currently working on.

Let us explain in rough terms how valuations play a role in this problem. For simplicity, we shall assume that $f : \mathbb{A}^2 \to \mathbb{A}^2$ is defined over $\mathbb{Q}$, that its extension to $\mathbb{P}^2$ contracts the line at infinity to a super-attracting point q, and that the curve V has only one place at infinity. The assumptions are satisfied in most cases by using [**24**] and Siegel's theorem.

Two situations may appear. Either $f^n(V)$ contains q for some n large enough. Then we may argue in a very similar way as in the case of birational polynomial maps. Or $f^n(V)$ goes through a point of indeterminacy of f for all n. It is exactly in that case that we apply the technique developed in Part 3. The set of valuations associated to the branches of $f^n(V)$ at infinity is rich, and we show that there exists a polynomial P whose restriction to the branch of $f^n(V)$ at infinity is vanishing for all $n \geq 0$. This implies V and all its images by f^n are components of $\{P = 0\}$ whence V is periodic.

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Part 1

Periodic points of birational transformations on projective surfaces

4. Introduction

Hrushovski and Fakhruddin [21] recently proved by purely algebraic methods that the set of periodic points of a *polarized* endomorphism of a projective variety over any algebraically closed field is Zariski dense. Recall that a morphism $f : X \to X$ of a projective variety is said to be polarized if there is an ample line bundle $L \to X$ such that $f^*L = dL$ for some $d \geq 2$. In this article, we give a complete classification of birational surface maps whose periodic points are Zariski dense.

In order to state our main result, we first review some basic notions related to birational transformations of surfaces. Let X be a projective surface, L be an ample line bundle on X and $f : X \dashrightarrow X$ be any birational transformation. We set $\deg_L(f) = (f^*L \cdot L)$ and call it the degree of f with respect to L. One can show (see e.g. [4]) that $\deg_L(f^{m+n}) \leq 2\deg_L(f^m)\deg_L(f^n)$ for all $n, m \geq 0$, so that the limit

$$\lambda_1(f) := \lim_{n\to\infty} \deg_L(f^n)^{1/n} \geq 1$$

is well defined. It is independent on the choice of L and it is called the first dynamical degree of f. It is also constant on the conjugacy class of f in the group of birational transformations of X. It is a fact ([17, 26]) that when $\lambda_1(f) = 1$ and $\deg_L(f^n)$ is unbounded, f preserves either an elliptic or a rational fibration and this invariant fibration is unique.

A point p is said to be periodic *non critical* if its orbit under f meets neither the indeterminacy set of f nor its critical set and is finite. Since our map is birational, if some iteration $f^n(p)$, $n \geq 0$ of a point p is a non-critical periodic point, then p itself is periodic non-critical.

Theorem 4.1. *Let X be a smooth projective surface over an algebraically closed field of characteristic different from 2 and 3. Let $L \to X$ be an ample line bundle and $f : X \dashrightarrow X$ be a birational transformation of X. Denote by $\mathcal{P}$ the set of non-critical periodic points of f. Then we are in one of the following three cases.*

- (i) *If $\lambda_1(f) > 1$, then $\mathcal{P}$ is Zariski dense.*
- (ii) *If $\lambda_1(f) = 1$ and $\deg_L(f^n)$ is unbounded then $\mathcal{P}$ is Zariski dense if and only if the action of f on the base of its invariant fibration is periodic.*
- (iii) *If $\lambda_1(f) = 1$, and $\deg_L(f^n)$ is bounded, then $\mathcal{P}$ is Zariski dense if and only if there is an integer $N > 0$ such that $f^N = \mathrm{id}$.*

The most interesting case in the previous theorem is case (i). We actually prove this result over a field of arbitrary characteristic.

Theorem 4.2. *Let X be a projective surface over an algebraically closed field $\mathbf{k}$, and $f : X \dashrightarrow X$ be a birational transformation. If $\lambda_1(f) > 1$ then the set of non-critical periodic points is Zariski dense in X.*

In the case $\mathbf{k} = \mathbb{C}$, this theorem has been proved in many cases using analytic methods. In [3, 20, 16] Diller, Dujardin and Guedj proved it for birational polynomial maps, or more generally for any birational transformation such that the points of indeterminacy of f^{-1} do not cluster too much near the points of indeterminacy of f.

It follows from [17] that any birational transformation on a non-rational surface with $\lambda_1 > 1$ is birationally equivalent to an automorphism i.e. there exists a birational map $\pi : X' \dashrightarrow X$ between surfaces and an automorphism $f' : X \to X$ satisfying $\pi \circ f' = f \circ \pi$. When f is an automorphism with $\lambda_1 > 1$, then it is possible to get a more precise count on the number of isolated periodic points based on Lefschetz-Saito's fixed point formula, see [7, 29, 34].

Theorem 4.3. *Let X be a smooth projective surface over an algebraically closed field of characteristic 0, and $f : X \to X$ be an automorphism with $\lambda_1(f) > 1$. We denote by $\#\mathrm{Per}_n(f)$ the number of isolated periodic points of period n counted with multiplicities. Then we have*

$$\#\mathrm{Per}_n(f) = \begin{cases} \lambda_1(f)^n + O(1), & \text{if } X \text{ is not an abelian surface;} \\ \lambda_1(f)^n + O(\lambda_1(f)^{n/2}), & \text{if } X \text{ is an abelian surface.} \end{cases}$$

For completeness, we also deduce from the work of Amerik [2] the following result.

Theorem 4.4 ([2])**.** *Let X be a projective surface over an algebraically closed field k of characteristic 0, and $f : X \dashrightarrow X$ be a birational transformation with $\lambda_1(f) > 1$. Then there exists a k-point $x \in X(k)$ such that $f^n(x) \in X \setminus I(f)$ for any $n \in \mathbb{Z}$ and $\{f^n(x) | n \in \mathbb{Z}\}$ is Zariski dense.*

This Theorem is closely related to a question of S.-W.Zhang who asked in [36] whether for any polarized endomorphism on a projective variety defined over an algebraically closed field k of characteristic 0 there exists a k-point with a Zariski dense orbit.

Let us explain now our strategy to prove Theorem 4.2. We follow the original method of Hrushovski and Fakhruddin by reducing our result to the case of finite fields.

For the sake of simplicity, we shall assume that $X = \mathbb{P}^2$ and $f = [f_0 : f_1 : f_2]$ is a birational transformation with $\lambda_1(f) > 1$ and has integral coefficients.

First assume we can find a prime $p > 0$ such that the reduction f_p modulo p of f satisfies $\lambda_1(f_p) > 1$. Then $\mathcal{P}(f_p)$ is Zariski dense in $\mathbb{P}^2(\overline{\mathbb{F}_p})$ by a direct application of Hrushovski's arguments. One then lifts these periodic points to $\mathbb{P}^2(\overline{\mathbb{Q}})$ by combining a result of Cantat [8] proving that most periodic points are isolated together with a simple dimensional argument borrowed from Fakhruddin [21].

The main difficulty thus lies in proving that $\lambda_1(f_p) > 1$ for at least one prime p. Recall that a birational transformation of the projective plane is defined over the integers if it can be represented in homogeneous coordinates by polynomials with integral coefficients.

Theorem 4.5. *Let f be any birational transformation of the projective plane defined over $\mathbb{Z}$. Then for any prime p sufficiently large, f induces a birational transformation $f_p : \mathbb{P}^2_{\mathbb{F}_p} \dashrightarrow \mathbb{P}^2_{\mathbb{F}_p}$ on the special fiber at p, and*

$$\lim_{p\to\infty} \lambda_1(f_p) = \lambda_1(f).$$

We also give an example of a birational transformation f such that $\lambda_1(f_p) < \lambda_1(f)$ for all p (see Section 7.3).

In fact we prove a quite general version of Theorem 4.5 for families of birational transformations of surfaces over integral schemes. It allows us to prove:

Theorem 4.6. *Let* $\mathbf{k}$ *be an algebraically closed field and* $d \geq 2$ *be an integer. Denote by* Bir_d *the space of birational transformations of* $\mathbb{P}^2_{\mathbf{k}}$ *of degree* d*. Then for any* $\lambda < d$*, the set* $U_\lambda = \{f \in \mathrm{Bir}_d | \lambda_1(f) > \lambda\}$ *is open and Zariski dense in* Bir_d*.*

In particular, for a general birational transformation f *of degree* $d > 1$*, we have* $\lambda_1(f) > 1$*.*

In order to prove Theorem 4.5, we need to control $\lambda_1(f)$ in terms of the degree of a fixed iterate of f. This control is given by our Key Lemma.

Key Lemma. *Let* X *be a projective surface over an algebraically closed field, let* L *be an ample bundle on* X *and let* $f : X \dashrightarrow X$ *be a birational transformation. If* $q = \frac{\deg_L(f^2)}{3^{18}\sqrt{2}\deg_L(f)}$ *is greater than one, then we have*

$$\lambda_1(f) > q \geq 1.$$

In particular if $\deg_L(f^2) \geq 3^{18}\sqrt{2}\deg_L(f)$ then $\lambda_1(f) > 1$. This result has been stated in [22] by Favre without proof. To prove this lemma we rely on the natural linear action of f on a suitable hyperbolic space of infinite dimension. This space is constructed as a set of cohomology classes in the Riemann-Zariski space of X and was introduced by Cantat in [9]. See also [4, 10, 22, 33].

The article is organized in 7 sections. In Section 5 we give background informations on intersection theory on surfaces and Riemann-Zariski spaces. In Section 6 we prove our Key Lemma. We apply it in Section 7 to study the behavior of the first dynamical degree in families of birational transformations on surfaces. In Section 7.3 we give an example of a birational transformation f on $\mathbb{P}^2_{\mathbb{Z}}$ such that $\lambda_1(f_p) < \lambda_1(f)$ for all prime p. In Section 8 we prove Theorem 4.2 and Theorem 4.4. In Section 9, we use Lefschetz-Saito's formula to study isolated periodic points of automorphisms and prove Theorem 4.3. In Section 10, we study the Zariski density of periodic points in the case $\lambda_1 = 1$. Finally we combine the results that we obtain in Section 8 and Section 10 to get Theorem 4.1.

5. Background and Notation

In this paper, a variety is always defined over an algebraically closed field and we use the notation $\mathbf{k}$ to denote an algebraically closed field of arbitrary characteristic except in Subsection 8.4, Section 9 and Section 10. In Subsection 8.4 and Section 9, $\mathbf{k}$ denotes an algebraically closed field of characteristic 0. In Section 10, $\mathbf{k}$ denotes an algebraically closed field of characteristic different from 2 and 3.

5.1. **Néron-Severi group.** Let us recall the definition and some properties of the Néron-Severi group [15, 31].

Let X be a projective variety over $\mathbf{k}$. We denote by $\mathrm{Pic}(X)$ the Picard group of X. The Néron-Severi group of X is defined as the group of numerical equivalence classes of divisors on X. We denote it by $N^1(X)$, and write $N^1(X)_\mathbb{R} = N^1(X) \otimes_\mathbb{Z} \mathbb{R}$. The group $N^1(X)$ is a free abelian group of finite rank (see [30]). Let $\phi : X \to Y$ be a morphism of projective varieties. It induces a natural map $\phi^* : N^1(Y)_\mathbb{R} \to N^1(X)_\mathbb{R}$.

We denote by $N_1(X)_\mathbb{R}$ the space of numerical equivalence classes of real one-cycles of X. One has a perfect pairing

$$N^1(X)_\mathbb{R} \times N_1(X)_\mathbb{R} \to \mathbb{R}, \ \ (\delta, \gamma) \to (\delta \cdot \gamma) \in \mathbb{R}$$

induced by the intersection form for which $N_1(X)_\mathbb{R}$ is dual to $N^1(X)_\mathbb{R}$. We denote by $\phi_* : N_1(X)_\mathbb{R} \to N_1(Y)_\mathbb{R}$ the dual operator of ϕ^*.

If X is a projective surface defined over $\mathbf{k}$, for any classes $\alpha_1, \alpha_2 \in N^1(Y)$, we denote by $(\alpha_1 \cdot \alpha_2)$ their intersection number. We recall the following

Proposition 5.1 (Pull-back formula, see [24])**.** *Let $\pi : X \to Y$ be a surjective morphism between two projective surfaces defined over $\mathbf{k}$. For any classes $\alpha_1, \alpha_2 \in N^1(Y)$, we have*

$$(\pi^*\alpha_1 \cdot \pi^*\alpha_2) = \deg(\pi)(\alpha_1 \cdot \alpha_2).$$

When X is a smooth projective surface, we can (and will) identify $N^1(X)_\mathbb{R}$ and $N_1(X)_\mathbb{R}$. In particular we get a natural bilinear form on $N^1(X)_\mathbb{R}$.

A class $\alpha \in N^1(X)_\mathbb{R}$ is said to be nef if and only if $(\alpha \cdot [C]) \geq 0$ for any curve C.

Theorem 5.2 (Hodge index theorem)**.** *Let L and M be two $\mathbb{R}$-divisors on a smooth projective surface, such that $(L^2) \geq 0$ and $(L \cdot M) = 0$. Then we have $(M^2) \leq 0$ and $(M^2) = 0$ if and only if $(L^2) = 0$ and M is numerically equivalent to a multiple of L.*

In other words the signature of the intersection form on $N^1(X)_\mathbb{R}$ is equal to $(1, \dim N^1(X)_\mathbb{R} - 1)$.

5.2. **Basics on birational maps on surfaces.** Recall that the resolution of singularities of surfaces over any algebraically closed field exists (see [1]).

Let X, Y be two smooth projective surfaces. A birational map $f : X \dashrightarrow Y$ is defined by its graph $\Gamma(f) \subseteq X \times Y$, which is an irreducible subvariety for which the projections $\pi_1 : \Gamma(f) \to X$ and $\pi_2 : \Gamma(f) \to Y$ are birational morphisms. We denote by $I(f) \subseteq X$ the finite set of points where π_1 does not admit a local inverse and call it the *indeterminacy* set of f. We set $\mathcal{E}(f) = \pi_1\pi_2^{-1}(I(f^{-1}))$. For any algebraic subset $V \subset X$, we write $f(V) := \overline{f(V \setminus I(f))}$.

If $g : Y \dashrightarrow Z$ is another birational map, the graph $\Gamma(g \circ f)$ of the composite map is the closure of the set

$$\{(x, g(f(x))) \in X \times Z |\ x \in X \setminus I(f), f(x) \in Y \setminus I(g)\}.$$

This is included in the set

$$\Gamma(g) \circ \Gamma(f) = \{(x, z) \in X \times Z | \ (x, y) \in \Gamma(f), (y, z) \in \Gamma(g) \text{ for some } y \in Y\}$$

with equality, if and only if there is no component $V \subseteq \mathcal{E}(f)$ such that $f(V) \subseteq I(g)$.

Let $f : X \dashrightarrow Y$ be a birational map between smooth projective surfaces, and Γ be a desingularization of its graph. Denote by $\pi_1 : \Gamma \to X, \pi_2 : \Gamma \to Y$ the natural projections. Then we define the following linear maps

$$f^* = \pi_{1*}\pi_2^* : N^1(Y)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}},$$

and

$$f_* = \pi_{2*}\pi_1^* : N^1(X)_{\mathbb{R}} \to N^1(Y)_{\mathbb{R}}.$$

Observe that $f_* = f^{-1*}$.

Proposition 5.3 (see [17]). *Let $f : X \dashrightarrow Y$ be a birational map between smooth projective surfaces.*

(i) *The linear map f^* (resp. f_*) is integral in the sense that it maps $N^1(Y)$ (resp. $N^1(X)$) to $N^1(X)$ (resp. $N^1(Y)$).*
(ii) *If $\alpha \in N^1(Y)_{\mathbb{R}}$ is nef, then $f^*\alpha \in N^1(X)_{\mathbb{R}}$ is nef.*
(iii) *The maps f^* and f_* are adjoint for the intersection form, i.e.*

$$(f^*\alpha \cdot \beta) = (\alpha \cdot f_*\beta),$$

for any $\alpha \in N^1(Y)_{\mathbb{R}}$ and $\beta \in N^1(X)_{\mathbb{R}}$.

It is important to observe that $f \mapsto f^*$ is not functorial in general. In fact, let X, Y, Z be smooth projective surfaces, and let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be two birational maps. For any given ample class $\alpha \in N^1(Z)_{\mathbb{R}}$, $f^*g^*\alpha = (f \circ g)^*\alpha$ if and only if $I(\mathcal{E}(f) \bigcap I(g)) = \emptyset$.

Fix any euclidean norm $\| \cdot \|$ on $N^1(X)_{\mathbb{R}}$. It follows from [17, 19] that the sequence of rescaled operator norms $C \, \|(f^n)^*\|$ is sub-multiplicative for a suitable constant $C > 0$. We may thus define the first dynamical degree

$$\lambda_1(f) := \lim_{n\to\infty} \|f^{n*}\|^{1/n}.$$

It is not difficult to check that $\lambda_1(f) \geq 1$ and that it only depends on the conjugacy class of f in the group of all birational transformations of X.

For any class $\omega \in N^1_{\mathbb{R}}(X)$, we set

$$\deg_\omega(f) := (f^*\omega \cdot \omega).$$

If L is an ample line bundle on X, we also write $\deg_L(f)$ for $\deg_{[L]}(f)$. It is possible to compute the dynamical degree of a map in terms of the degree growth of its iterates as follows.

Proposition 5.4 ([17, 19]). *Let $f : X \dashrightarrow X$ be a birational transformation on a projective smooth surface. Then we have $\lambda_1(f) = \lim_{n\to\infty} \deg_\omega(f^n)^{1/n}$, for any big and nef class $\omega \in N^1(X)$.*

Proposition-Definition 5.5 (see [17, 23])**.** Let $f : X \dashrightarrow X$ be a birational transformation on a projective smooth surface, and fix any ample class $\omega \in N^1(X)_{\mathbb{R}}$. Then f is said to be *algebraically stable* if and only if one of the following holds:

(i) for every $\alpha \in N^1(X)_{\mathbb{R}}$ and every $n \in \mathbb{N}$, one has $(f^*)^n\alpha = (f^n)^*\alpha$;
(ii) there is no curve $V \subseteq X$ such that $f^n(V) \subseteq I(f)$ for some integer $n \geq 0$;
(iii) for all $n \geq 0$ one has $(f^*)^n\omega = (f^n)^*\omega$.

Observe that in the case $X = \mathbb{P}^2$, f is algebraically stable if and only if $\deg_L(f^n) = (\deg_L f)^n$ for any $n \in \mathbb{N}$ where L is the hyperplane line bundle.

Theorem 5.6 ([17])**.** *Let $f : X \dashrightarrow X$ be a birational transformation of a projective smooth surface, then there is a proper modification $\pi : \widehat{X} \to X$ such that the lift of f to $\widehat{X}$ is algebraically stable.*

5.3. **Classes on the Riemann-Zariski space.** All facts in this subsection can be found in [4, 9, 10, 33]. Let X be a smooth projective surface over $\mathbf{k}$.

Given any two birational morphisms $\pi : X_\pi \to X$ and $\pi' : X_{\pi'} \to X$, we say that π' dominates π and write $\pi' \geq \pi$ if there exists a birational morphism $\mu : X_{\pi'} \to X_\pi$ such that $\pi' = \pi \circ \mu$. The *Riemann − Zariski space* of X is defined to be the projective limit

$$\mathfrak{X} := \varprojlim_{\pi} X_\pi.$$

Definition 5.7. The space of Weil classes of $\mathfrak{X}$ is defined to be the projective limit

$$W(\mathfrak{X}) := \varprojlim_{\pi} N^1(X_\pi)_{\mathbb{R}}$$

with respect to pushforward arrows. The space of Cartier classes on $\mathfrak{X}$ is defined to be the inductive limit

$$C(\mathfrak{X}) := \varinjlim_{\pi} N^1(X_\pi)_{\mathbb{R}}$$

with respect to pullback arrows.

Concretely, a Weil class $\alpha \in W(\mathfrak{X})$ is given by its *incarnations* $\alpha_\pi \in N^1(X_\pi)_{\mathbb{R}}$, compatible with pushforwards; that is, $\mu_*\alpha_{\pi'} = \alpha_\pi$ as soon as $\pi' = \pi \circ \mu$.

The projection formula shows that there is an embedding $C(\mathfrak{X}) \subseteq W(\mathfrak{X})$, so that a Cartier class is a Weil class.

For each π, the intersection pairing $N^1(X_\pi)_{\mathbb{R}} \times N^1(X_\pi)_{\mathbb{R}} \to \mathbb{R}$ is denoted by $(\alpha\cdot\beta)_{X_\pi}$. By the pull-back formula, it induces a pairing $W(\mathfrak{X})\times C(\mathfrak{X}) \to \mathbb{R}$ which is denoted by $(\alpha \cdot \beta)$.

We define the space

$$\mathbb{L}^2(\mathfrak{X}) := \{\alpha \in W(\mathfrak{X})| \inf_{\pi}\{(\alpha_\pi \cdot \alpha_\pi)\} > -\infty\}.$$

It is an infinite dimensional subspace of $W(\mathfrak{X})$ that contains $C(\mathfrak{X})$. It is endowed with a natural intersection product extending the one on Cartier classes and that is of Minkowski's type. Since this fact is crucial to our proof of Theorem 1.2 we state it as a

Proposition 5.8 ([9]). *If α, β are any two non zero classes in $\mathbb{L}^2(\mathfrak{X})$ such that $(\alpha^2) > 0$ and $(\alpha \cdot \beta) = 0$, then we have $(\beta^2) < 0$.*

Definition 5.9. We define $\mathbb{H}(\mathfrak{X})$ to be the unique connected component of $\{\alpha \in \mathbb{L}^2(\mathfrak{X}) | \alpha^2 = 1\}$ that contains all Cartier nef classes of self intersection $+1$.

For any $\alpha, \beta \in \mathbb{H}(\mathfrak{X})$, we define

$$d_{\mathbb{H}(\mathfrak{X})}(\alpha, \beta) = (\cosh)^{-1}(\alpha \cdot \beta),$$

where $\cosh(x) = (e^x + e^{-x})/2$. Recall that the function $d_{\mathbb{H}(\mathfrak{X})}$ induces a distance on the space $\mathbb{H}(\mathfrak{X})$, see [9, 10].

Let $f : X \dashrightarrow Y$ be a birational map between two smooth projective surfaces. For each blowup Y_ϖ of Y, there is a blowup X_π of X such that the induced map $X_\pi \to Y_\varpi$ is regular. The associated pushforward map $N^1(X_\pi)_\mathbb{R} \to N^1(Y_\varpi)_\mathbb{R}$ and pullback map $N^1(Y_\varpi)_\mathbb{R} \to N^1(X_\pi)_\mathbb{R}$ are compatible with the projective and injective systems defined by pushforwards and pullbacks that define Weil and Cartier classes respectively.

Definition 5.10. Let $f : X \dashrightarrow X$ be a birational transformation on a smooth projective surface. We denote by $f_* : W(\mathfrak{X}) \to W(\mathfrak{X})$ the induced pushforward operator and by $f^* : C(\mathfrak{X}) \to C(\mathfrak{X})$ the induced pullback operator.

Proposition 5.11 ([9, 10]). *The pullback $f^* : C(\mathfrak{X}) \to C(\mathfrak{X})$ extends to a linear map $f^* : \mathbb{L}^2(\mathfrak{X}) \to \mathbb{L}^2(\mathfrak{X})$, such that*

$$((f^*\alpha)^2) = (\alpha^2)$$

for any $\alpha \in \mathbb{L}^2(\mathfrak{X})$. In particular f^* induces an isometry on $(\mathbb{H}(\mathfrak{X}), d_{\mathbb{H}(\mathfrak{X})})$.

Observe that since f is birational $f_* = (f^{-1})^*$ and the pushforward f_* also induces an isometry on $\mathbb{H}(\mathfrak{X})$. For any $\alpha, \beta \in \mathbb{L}^2(\mathfrak{X})$ we have

$$(f^*\alpha \cdot \beta) = (\alpha \cdot f_*\beta).$$

5.4. **Hyperbolic spaces.** In this subsection, we review some properties of hyperbolic spaces in the sense of Gromov.

Recall that a metric space (M, d) is geodesic if and only if for any two points $x, y \in X$, there exists at least one isometric immersion of a segment of $\mathbb{R}$ with boundary x and y. For any given number $\delta \geq 0$, a metric space (M, d) satisfies the Rips condition of constant δ if it is geodesic, and for any geodesic triangle $\Delta = [x, y] \bigcup [y, z] \bigcup [z, x]$ of M, and any $u \in [y, z]$, we have $d(u, [x, y] \bigcup [z, x]) \leq \delta$. A space M is called *hyperbolic in the sense of Gromov* if there is a number $\delta \geq 0$ such that M satisfies the Rips condition of constant δ.

Lemma 5.12 ([13]). *The hyperbolic plane $\mathbb{H}^2$ satisfies the Rips condition of constant* $\log 3$.

Since the Rips condition only needs to be tested on geodesic triangles, we have the following

Lemma 5.13. *The space $(\mathbb{H}(\mathfrak{X}), d_{\mathbb{H}(\mathfrak{X})})$ satisfies the Rips condition of constant* $\log 3$.

Recall that a topological space is separable if and only if it admits a countable dense subset.

Theorem 5.14 ([25]). *Let (M,d) be a separable geodesic and hyperbolic metric space which satisfies the Rips condition of constant δ. If $(x_i)_{0\leq i\leq n}$ is a sequence of points such that*

$$d(x_{i+1},x_{i-1}) \geq \max(d(x_{i+1},x_i), d(x_i,x_{i-1})) + 18\delta + \kappa$$

for some constant $\kappa > 0$ and $i = 1,\cdots,n-1$. Then

$$d(x_n,x_0) \geq \kappa n.$$

6. Effective bounds on λ_1

We begin with proving our

Key Lemma. *Let X be a smooth projective surface over $\mathbf{k}$, L be an ample line bundle on X, and $f : X \dashrightarrow X$ be a birational transformation. If $q = \frac{\deg_L(f^2)}{3^{18}\sqrt{2}\deg_L(f)} \geq 1$, then we have*

$$\lambda_1(f) > q \geq 1.$$

Proof. For any $n > 0$, set $\mathcal{L}_n = f^{*n}L \in \mathbb{H}(\mathfrak{X})$. Since f^* is an isometry of $\mathbb{H}(\mathfrak{X})$, we have

$$d_{\mathbb{H}(\mathfrak{X})}(\mathcal{L}_{n+1},\mathcal{L}_{n-1}) = d_{\mathbb{H}(\mathfrak{X})}(\mathcal{L}_2,\mathcal{L}) = \cosh^{-1}(\deg_L(f^2)) = \cosh^{-1}(3^{18}\sqrt{2}\deg_L(f)q)$$

for any $n \geq 1$. We claim that for any $u,q \geq 1$,

$$\cosh^{-1}(3^{18}\sqrt{2}uq) > \cosh^{-1}(u) + 18\log 3 + \log(q) \qquad (*).$$

Taking this claim for granted we conclude the proof. First we have

$$\cosh^{-1}(3^{18}\sqrt{2}\deg_L(f)q) > \cosh^{-1}(\deg_L(f)) + 18\log 3 + \log(q)$$

Pick $\kappa > \log(q) \geq 0$ such that

$$\cosh^{-1}(3^{18}\sqrt{2}\deg_L(f)q) > \cosh^{-1}(\deg_L(f)) + 18\log 3 + \kappa.$$

Then we get

$$d_{\mathbb{H}(\mathfrak{X})}(\mathcal{L}_{n+1},\mathcal{L}_{n-1}) > \cosh^{-1}(\deg_L(f)) + 18\log 3 + \kappa$$

for every $n \geq 1$. Since $d_{\mathbb{H}(\mathfrak{X})}(\mathcal{L}_{n+1},\mathcal{L}_n) = \cosh^{-1}(\deg_L(f))$, we obtain

$$d_{\mathbb{H}(\mathfrak{X})}(\mathcal{L}_{n+1},\mathcal{L}_n) > \max(d_{\mathbb{H}(\mathfrak{X})}(\mathcal{L}_{n+1},\mathcal{L}_n), d_{\mathbb{H}(\mathfrak{X})}(\mathcal{L}_n,\mathcal{L}_{n-1})) + 18\log 3 + \kappa.$$

Let W be the subspace of $\mathbb{H}(\mathfrak{X})$ spanned by $\{\mathcal{L}_n\}$. Then W is separable and for any $x,y \in W$, the geodesic segment $[x,y]$ is included in W. It follows that $(W, d_{\mathbb{H}(\mathfrak{X})}|_W)$ is a separated geodesic and hyperbolic metric space which satisfies the Rips condition of constant $\log 3$. By Theorem 5.14, we get for $n > 0$

$$\cosh^{-1}(\deg_L(f^n)) = d_{\mathbb{H}(\mathfrak{Y})}(\mathcal{L}_n,\mathcal{L}) > \kappa n,$$

which is equivalent to

$$\deg_L(f^n) > (e^{\kappa n} + e^{-\kappa n})/2 > e^{\kappa n}/2.$$

We conclude that $\lambda_1(f) \geq e^{\kappa} > q$.

Let us prove $(*)$. For any $u \geq 1$, $q \geq 1$, we have

$$\cosh^{-1}(3^{18}\sqrt{2}u) = \log(3^{18}\sqrt{2}u+\sqrt{2\times 3^{36}u^2-1}) > \log(3^{18}u+1+\sqrt{2\times 3^{36}u^2-1})$$
$$> \log(3^{18}u+\sqrt{2\times 3^{36}u^2}) = 18\log 3+\log(u+\sqrt{2u^2}) \geq 18\log 3+\log(u+\sqrt{u^2+1})$$
$$= \cosh^{-1}(u)+18\log 3,$$

and

$$\cosh^{-1}(3^{18}\sqrt{2}uq)-\cosh^{-1}(3^{18}\sqrt{2}u) = \log\Big(\frac{3^{18}\sqrt{2}uq+\sqrt{2\times 3^{36}u^2q^2-1}}{3^{18}\sqrt{2}u+\sqrt{2\times 3^{36}u^2-1}}\Big).$$

It follows that

$$\frac{3^{18}\sqrt{2}uq+\sqrt{2\times 3^{36}u^2q^2-1}}{3^{18}\sqrt{2}u+\sqrt{2\times 3^{36}u^2-1}} = q-\frac{\sqrt{2\times 3^{36}u^2q^2-q^2}-\sqrt{2\times 3^{36}u^2q^2-1}}{3^{18}\sqrt{2}u+\sqrt{2\times 3^{36}u^2-1}} \geq q,$$

which concludes the proof. □

Our Key Lemma implies the following estimate on $\lambda_1(f)$ knowing $\deg_L(f^n)$ and $\deg_L(f^{2n})$ for some n sufficiently large.

Corollary 6.1. *Let f be a birational transformation of a smooth projective surface X over $\mathbf{k}$. For any integer $n > 0$, we set $q_n := \frac{\deg_L(f^{2n})}{3^{18}\sqrt{2}\deg_L(f^n)}$. If $q_n \geq 1$, we have*

$$q_n^{1/n} < \lambda_1(f) \text{ and } \lim_{n\to\infty} q_n^{1/n} = \lambda_1(f).$$

Proof. Our Key Lemma implies $\lambda_1(f)^n = \lambda_1(f^n) > q_n$. By the definition of $\lambda_1(f)$, we have

$$\lim_{n\to\infty} q_n^{1/n} = \frac{\lim_{n\to\infty}(\deg_L(f^{2n}))^{1/n}}{\lim_{n\to\infty}(3^{18}\sqrt{2}\deg_L(f^n))^{1/n}} = \lambda_1(f).$$

□

7. The behavior of λ_1 in family

7.1. **Lower semi-continuity of λ_1.** In this subsection we use our Key Lemma to study the behavior of the first dynamical degree in families. We aim at proving a version of Theorem 4.5 in the general context of integral schemes, see Theorem 7.3 below. We shall rely on the following

Lemma 7.1. *Let S be a smooth integral scheme, and $\pi : X \to S$ be a smooth projective and surjective morphism such that $\dim_S X = 2$. Let $L \to X$ be a line bundle which is nef over S, and $f : X \dashrightarrow X$ be a birational transformation over S such that for any point $p \in S$, f induces a birational transformation f_p of the special fiber X_p. Set $L_p := L|X_p$.*

Then $p \mapsto \deg_{L_p}(f_p)$ is a lower semi-continuous function on S.

Observe that $p \mapsto \deg_{L_p}(f_p)$ is not continuous in general as the following example shows.

Example 7.2. The map

$$f[x:y:z] = [xz : yz + 2xy : z^2]$$

is a birational transformation of $\mathbb{P}^2$ over $\operatorname{Spec}\mathbb{Z}$. Denote by L the hyperplane line bundle on $\mathbb{P}^2_{\mathbb{Z}}$. Then f_p is birational for any prime p, $\deg_{L_p}(f_p) = 1$ for $p = 2$ and $\deg_{L_p}(f_p) = 2$ for any odd prime.

Proof of Lemma 7.1. Denote by κ the generic point of S. We claim that on any integral scheme S we have $\deg_{L_p}(f_p) \leq \deg_{L_\kappa}(f_\kappa)$ on S with equality on a Zariski open subset of S.

The lower semicontinuity then follows. Indeed pick any $\lambda \in \mathbb{R}$ and define $R = \{x \in S, \deg_{L_x}(f_x) \leq \lambda\}$. Pick any irreducible component Z of the Zariski closure of R. Our claim applied to Z implies $\deg_{L_\kappa}(f_\kappa) = \deg_{L_p}(f_p)$ for some p hence $\deg_{L_\kappa}(f_\kappa) \leq \lambda$. And it follows that $\deg_{L_p}(f_p) \leq \deg_{L_\kappa}(f_\kappa)$ for all $p \in Z$ so that $R \bigcap Z = Z$ is Zariski closed.

We now prove the claim. Let $\Gamma \subseteq X \times_S X$ be the graph of f, and $\pi_1, \pi_2 : \Gamma \to X$ be the natural projections such that $\pi_2 \circ \pi_1^{-1} = f$. For any point $p \in S$, let Γ_p be the fiber of Γ above p, and π_{1p}, π_{2p} be the restrictions of π_1 and π_2 respectively to Γ_p. Denote by κ the generic point of S, then the function

$$\int_{\Gamma_p} \pi_{1p}^* L_p \cdot \pi_{2p}^* L_p = \int_{\Gamma_\kappa} \pi_{1\kappa}^* L_\kappa \cdot \pi_{2\kappa}^* L_\kappa = \deg_{L_\kappa}(f_\kappa)$$

is constant on S by [24, Proposition 10.2].

For $p \in S$, Γ_p may have several irreducible components, but there is only one component Γ'_p that satisfies $\pi_{1p}(\Gamma'_p) = X$, hence $\deg_{L_p}(f_p) = \int_{\Gamma'_p} \pi_{1p}^* L_p \cdot \pi_{2p}^* L_p \leq \int_{\Gamma_p} \pi_{1p}^* L_p \cdot \pi_{2p}^* L_p$. Since there is a nonempty open set U of S such that for any point $x \in U$, Γ_x is irreducible, it follows that $\deg_{L_x}(f_x) = \int_{\Gamma_x} \pi_{1x}^* L_x \cdot \pi_{2p}^* L_p = \deg_{L_\kappa}(f_\kappa)$ for $p \in U$. □

Theorem 7.3. *Let S be an integral scheme, $\pi : X \to S$ be a smooth projective and surjective morphism where the relative dimension $\dim_S X = 2$. Let $f : X \dashrightarrow X$ be a birational transformation over S such that for any $p \in S$, the reduction f_p is a birational transformation. Then the function $p \in S \mapsto \lambda_1(f_p)$ is lower semi-continuous.*

Proof. As in the proof of Lemma 7.1, it is sufficient to check that for any integral scheme S then $\lambda_1(f_p) \leq \lambda_1(f_\kappa)$ for all $p \in S$ and that and for any $\lambda < \lambda_1(f_\kappa)$, there is a nonempty open set U of S, such that for every point $p \in U$, $\lambda_1(f_\kappa) \geq \lambda_1(f_p) > \lambda$.

For any $p \in S$, and any integer $n > 0$, we have $\deg_{L_p}(f_p^n) \leq \deg_{L_\kappa}(f_\kappa^n)$ hence

$$\lambda_1(f_p) \leq \lambda_1(f_\kappa).$$

The theorem trivially holds in the case $\lambda \leq 1$, so we may assume that $\lambda_{1,\kappa} > \lambda > 1$. For every $\lambda_{1,\kappa} > \lambda > 1$, there is an integer $n > 0$ such that

$$\Big(\frac{\deg_{L_\kappa}(f_\kappa^{2n})}{3^{18}\sqrt{2}\deg_{L_\kappa}(f_\kappa^n)}\Big)^{1/n} > \lambda > 1$$

by Corollary 6.1. By Lemma 7.1, there is an nonempty open set $U \subseteq S$ such that for any $q \in U$, $\deg_{L_q}(f_q^n) = \deg_{L_\kappa}(f_\kappa^n)$ and $\deg_{L_q}(f_q^{2n}) = \deg_{L_\kappa}(f_\kappa^{2n})$, hence

$$\Big(\frac{\deg_{L_q}(f_q^{2n})}{3^{18}\sqrt{2}\deg_{L_q}(f_q^n)}\Big)^{1/n} > \lambda.$$

By Corollary 6.1, we conclude that for any $q \in U$,

$$\lambda_1(f_q) \geq \Big(\frac{\deg_{L_q}(f_q^{2n})}{3^{18}\sqrt{2}\deg_{L_q}(f_q^n)}\Big)^{1/n} > \lambda,$$

as required. □

Corollary 7.4. *Let S be an integral scheme, $\pi : X \to S$ be a smooth projective and surjective morphism such that $\dim_S X = 2$. Let $f : X \dashrightarrow X$ be a birational transformation over S such that for any $p \in S$, f_p is birational. Then there is an integer $M > 0$ such that for every $p \in S$, $\lambda_1(f_p) = 1$ if and only if*

$$\frac{\deg_{L_p}(f_p^{2n})}{3^{18}\sqrt{2}\deg_{L_p}(f_p^n)} < 1$$

for $n = 1, 2, \cdots, M$.

Proof. Fix any integer $m > 0$, we set

$$Z_m := \{p \in S | \frac{\deg_{L_p}(f_p^{2n})}{3^{18}\sqrt{2}\deg_{L_p}(f_p^n)} < 1, \text{ for any } 0 < n \leq m\}$$

and $Z = \{p \in S | \lambda_1(f_p) = 1\}$. By Theorem 7.3 and Corollary 6.1, Z is closed and we have $Z = \bigcap_{m\geq 1} Z_m$. Since $\overline{Z_m}$ is a decreasing sequence of Zariski closed subsets then $\overline{Z_M} = \bigcap_{m\geq 1} \overline{Z_m}$ for some integer M and $Z \subset \overline{Z_M}$.

Suppose by contradiction $Z \neq \overline{Z_M}$, and pick a point $x \in \overline{Z_M} \setminus Z$. Let Y be an irreducible component of $\overline{Z_M}$ containing x and κ be the generic point of Y. Then $Y = \overline{Y \bigcap Z_N}$ for every $N \geq M$. Since $\lambda_1(f_x) > 1$, we have $\lambda_1(f_\kappa) > 1$ by Lemma 7.1. There exists $N \geq M$ such that κ is not in Z_N, and we have

$$\frac{\deg_{L_\kappa}(f_\kappa^{2N})}{3^{18}\sqrt{2}\deg_{L_\kappa}(f_\kappa^N)} \geq 1.$$

By lemma 7.1, there is an open subset U of Y such that for any point $y \in U$ and $n = 1, 2, \cdots, N$, we have $\deg_{L_y}(f_y^n) = \deg_{L_\kappa}(f_\kappa^n)$ and $\deg_{L_y}(f_y^{2n}) = \deg_{L_\kappa}(f_\kappa^{2n})$. In particular $U \bigcap Z_N = \emptyset$, which contradicts the fact that $Y = \overline{Y \bigcap Z_N}$.

We get $Z = \overline{Z_M}$, and $\overline{Z_M} \supseteq Z_M \supseteq Z = \overline{Z_M}$, so that $Z = Z_M$ as required. □

7.2. **Proof of Theorem 4.6.** If $f : \mathbb{P}^2_{\mathbf{k}} \to \mathbb{P}^2_{\mathbf{k}}$ is a birational transformation defined over an algebraically closed field $\mathbf{k}$, we set alg.deg$(f) := \deg_{\mathcal{O}(1)}(f)$ and call it the degree of f. We denote by Bir_d the space of birational transformations of $\mathbb{P}^2_{\mathbf{k}}$ of degree d. It has a natural algebraic structure which makes it a quasi-projective space, see [11] for details.

By Theorem 7.3, it is easy to see that if a component of Bir_d contains a point with $\lambda_1 > 1$, then $\lambda_1 > 1$ for a general point in this component. However since there are few informations on the geometry of the components of Bir_d for $d \geq 3$ [12, 14], it is a priori non obvious to decide which component contains such a point.

The rest of the section is devoted to the proof of Theorem 4.6 stated in the introduction.

Theorem 4.6. *Let* $\mathbf{k}$ *be an algebraically closed field and* $d \geq 2$ *be an integer. Denote by* Bir_d *the space of birational transformations of* $\mathbb{P}^2_{\mathbf{k}}$ *of degree* d*. Then for any* $\lambda < d$*, the set* $U_\lambda = \{f \in \mathrm{Bir}_d | \lambda_1(f) > \lambda\}$ *is open and Zariski dense in* Bir_d*.*

In particular, for a general birational transformation f *of degree* $d > 1$*, we have* $\lambda_1(f) > 1$*.*

Remark 7.5. If the base field is uncountable, then the set $\{f \in \mathrm{Bir}_d | \lambda_1(f) = d\} = \bigcap_{n=1}^{\infty} U_{d-1/n}$ is dense in Bir_d. So for very general point $f \in \mathrm{Bir}_d$ we have $\lambda_1(f) = d$.

Remark 7.6. Our proof actually shows that for any $f \in \mathrm{Bir}_d$, the set $\{A \in \mathrm{PGL}_3(\mathbf{k}) |\ \lambda_1(A \circ f) > \lambda\}$ is dense in $\mathrm{PGL}_3(\mathbf{k})$.

Proof of Theorem 4.6. We claim that for any irreducible component S of Bir_d, there is a point $f \in S$ such that $\lambda_1(f) > \lambda$.

Since the function $f \mapsto \lambda_1(f)$ is lower semi-continuous by Theorem 7.3, the claim immediately implies that the set $\{f \in S |\ \lambda_1(f) > \lambda\}$ is Zariski open and dense which concludes the proof of the Theorem 4.6.

It thus remains to prove the claim. For that purpose, choose $f \in \mathrm{Bir}_d$, and consider the map

$$T_f : \mathrm{PGL}_3(\mathbf{k}) \to \mathrm{Bir}_d$$

sending A to $A \circ f$. Let $I(f) = \{z_1, z_2, \cdots, z_m\}$, and $I(f^{-1}) = \{x_1, x_2, \ldots, x_n\}$ be the indeterminacy sets of f and f^{-1} respectively. For any $i = 1, 2, \cdots, n$, there is a curve C_i such that $f(C_i) = x_i$. Let y_i be any point in $C_i \setminus (I(f) \bigcup I(f^{-1}))$, and pick a point $A_i \in \mathrm{PGL}_3(\mathbf{k})$ such that $A_i(x_i) = y_i$. Then for any $n \geq 0$, we have $(A_i \circ f)^n \circ A_i(x_i) = y_i$.

For any $i = 1, 2, \cdots, n$, we define the map

$$V_{1,i} : \mathrm{PGL}_3(\mathbf{k}) \to \mathbb{P}^2$$

by $V_{1,i}(A) := A(x_i)$. Let $U_{0,i} = \mathrm{PGL}_3(\mathbf{k})$ and set

$$U_{1,i} = V_{1,i}^{-1}(\mathbb{P}^2 \setminus I(f)).$$

Then $U_{1,i}$ is an open subset of $\mathrm{PGL}_3(\mathbf{k})$. Since $A_i(x_i) = y_i$ does not belong to $I(f)$, A_i lies in $U_{1,i}$, so $U_{1,i}$ is not empty. Now we define the map

$$V_{2,i} : U_{1,i} \to \mathbb{P}^2,$$

by

$$V_{2,i}(A) = A \circ f \circ A(x_i)$$

and set

$$U_{2,i} = V_{2,i}^{-1}(\mathbb{P}^2 \setminus I(f)).$$

Since $A \circ f \circ A_i(x_i) = y_i$, $U_{2,i}$ is as before an open set containing A_i.

By induction, for any i we build a sequence of non empty open subsets $U_{l,i} \subseteq U_{l-1,i}$ of $\mathrm{PGL}_3(\mathbf{k})$, and maps $V_{l+1,i} : U_{l,i} \to \mathbb{P}^2$ sending A to $(A \circ f)^l \circ A(x_i)$ such that for any $A \in U_{l,i}$, $(A \circ f)^t \circ A(x_i)$ is not in $I(f)$ for $t = 0, \cdots, l-1$. Let $U_l = \bigcap_{i=1}^n U_{l,i}$. Since the $U_{l,i}$'s are nonempty and $\mathrm{PGL}_3(\mathbf{k})$ is irreducible, then U_l is also non empty and Zariski dense. For any A in U_l, we have $\mathrm{alg.deg}((A \circ f)^s) = (\mathrm{alg.deg}(A \circ f))^s = d^s$ for $s = 0, 1, \cdots, l+1$.

Pick l sufficiently large such that $(d^l/3^{18}\sqrt{2})^2 > \lambda$. For any point $A \in U_{2l-1}$, we have $\mathrm{alg.deg}(A \circ f)^{2l} = d^{2l}$ and $\mathrm{alg.deg}(A \circ f)^l = d^l$, and by our Key Lemma, we conclude $\lambda_1(A \circ f) > \lambda$.

Pick any irreducible component S of Bir_d. There is a birational transformation

$$G : \mathbb{P}^2 \times S \dashrightarrow \mathbb{P}^2 \times S$$

over S by $G(x, f) = (f(x), f)$. Then the map G_f on the fiber at $f \in S$ induced by G is exactly f. For any $f \in S$ that is not lying in any other component of Bir_d, and for any $A \in \mathrm{PGL}_3(\mathbf{k})$, we have $A \circ f \in S$ since $\mathrm{PGL}_3(\mathbf{k})$ is irreducible. By the discussion of the previous paragraph, there is a point $A \in \mathrm{PGL}_3(\mathbf{k})$ such that $\lambda_1(A \circ f) > \lambda$. Let κ be the generic point of S. Then by Theorem 7.3, we get

$$\lambda_1(G_\kappa) \geq \lambda_1(G_{A\circ f}) = \lambda_1(A \circ f) > \lambda.$$

Applying again Theorem 7.3, we conclude that $\lambda_1(f) > \lambda$ for a general point f in S. □

7.3. **An example.** In this section we provide an example of a birational transformation f on $\mathbb{P}^2$ over $\mathbb{Z}$ such that $\lambda_1(f_p) < \lambda_1(f)$ for any prime number $p > 2$ and f_p is not dominant when $p = 2$. Let us introduce the following two birational transformations $g = [xy : xy + yz : z^2]$ and $h = [x : x - 2z : -x + y + 3z]$.

Proposition 7.7. *The map* $f = h \circ g = [xy : xy - 2z^2 : yz + 3z^2]$ *is algebraically stable.*

Proof. A direct computation shows that

$$f^{-1} = [2x^2 - 2xy : (-3x + 3y + 2z)^2 : (x - y)(-3x + 3y + 2z)],$$

$$I(f) = \{[1 : 0 : 0], [0 : 1 : 0]\} \text{ and } I(f^{-1}) = \{[1 : 1 : 0], [0 : -2 : 3]\}.$$

Observe that the line $C := \{x = 0\}$ is f-invariant, and that $f([0 : y : z]) = [0 : -2z : y + 3z]$. Let us compute the orbits of the points in $I(f^{-1})$. Since $[1 : 1 : 0]$ is a fixed point of f, its orbit does not meet $I(f)$. Let i be the automorphism of C sending $[0 : y : z]$ to $[0 : y - 2z : -y + z]$, then

$$l := i^{-1} \circ f_{|C} \circ i([0 : y : z]) = [0 : 2y : z],$$

$$i^{-1}([0 : 1 : 0]) = [0 : 1 : 1] \text{ and } i^{-1}([0 : -2 : 3]) = [0 : 4 : 1].$$

In particular the orbit of $[0 : -2 : 3]$ is equal to $i(\{[0 : 2^{l+2} : 1] | l = 0, 1, 2 \cdots\})$ which does not meet $I(f)$. We conclude that f is algebraically stable. □

Since f is algebraically stable, we have $\lambda_1(f) = \mathrm{alg.deg}(f) = 2$.

Proposition 7.8. *For any prime* $p > 2$, f_p *is a birational transformation of* $\mathbb{P}^2_{\overline{\mathbb{F}_p}}$ *and* $\lambda_1(f) < 2$.

Proof. Observe that $f^{-1} \circ f = [4xyz^2 : 4y^2z^2 : 4yz^3]$ so that f_p is birational as soon as $p \geq 3$. Let $n_p \geq 2$ be the order of 2 in the multiplicative group $\mathbb{F}_p^\times$. We have

$$l_p^{n_p-2}([0:4:1]) = [0:1:1]$$

over $\overline{\mathbb{F}_p}$, which implies

$$f^{n_p-2}([0:-2:3]) = [0:1:0] \in I(f_p).$$

In particular f_p is not algebraically stable. It follows that there is a number n such that $\text{alg.deg}(f_p^n) < \text{alg.deg}(f_p)^n = 2^n$ (in fact the least number having this property is $n_p - 1$), and $\lambda_1(f_p) \leq \text{alg.deg}(f_p^n)^{1/n} < 2$ by Corollary 6.1. □

Remark 7.9. We can compute $\lambda_1(f_p)$ by explicitly constructing an algebraically stable model dominating $\mathbb{P}^2_{\overline{\mathbb{F}_p}}$. For $p > 2$, we find that $\lambda_1(f)$ is the greatest real root of the polynomial

$$x_p^{n_p} - 2x_p^{n_p-1} + 1 = 0.$$

Define $F_n(x) = (x-2)x^{n-1} + 1$. When $n > 2$, observe that $F_n(3/2) < 0$, and $F_n(2) = 1 > 0$, so that the largest root x of $F_n(x) = 0$ satisfies $2 > x > 3/2$. Since $2^{n_p-1}(x_p - 2) + 1 < 0 = (x_p-2)x_p^{n_p-1} + 1 < (x_p-2)(3/2)^{n_p-1} + 1$, we get

$$(1/2)^{n_p-1} < 2 - \lambda_1(f) < (2/3)^{n_p-1}.$$

8. The case $\lambda_1 > 1$

The purpose of this section is to prove Theorem 4.2.

8.1. **The case of finite fields.**

First we recall the following theorem of Hrushovski.

Theorem 8.1 ([28])**.** *Let $g : X \to \operatorname{Spec} k$ be an irreducible affine variety of dimension r over an algebraically closed field k of characteristic p, and let q be a power of p. We denote by ϕ_q the q-Frobenius map of k, and by X^{ϕ_q} the same scheme as X with g replaced by $g \circ \phi_q^{-1}$. Let $V \subseteq X \times X^{\phi_q}$ be an irreducible subvariety of dimension r such that both projections*

$$\pi_1 : V \to X \text{ and } \pi_2 : V \to X^{\phi_q}$$

are dominant and the second one is quasi-finite. Let $\Phi_q \subseteq X \times X^{\phi_q}$ be the graph of the q-Frobenius map ϕ_q. Set

$$u = \frac{\deg \pi_1}{\deg_{insep} \pi_2},$$

where $\deg \pi_1$ denotes the degree of field extension $K(V)/K(X)$ and $\deg_{insep} \pi_2$ is the purely inseparable degree of the field extension $K(V)/K(X)$.

Then there is a constant C that does not depend on q, such that

$$|\#(V \bigcap \Phi_q) - uq^r| \leq Cq^{r-1/2}.$$

Building on [21, Proposition 5.5], we show that the set of periodic points of a birational transformation is Zariski dense over the algebraic closure of a finite field.

Proposition 8.2. *Pick any prime $p > 0$, and let X be an algebraic variety over $\overline{\mathbb{F}_p}$. Let $f : X \dashrightarrow X$ be a birational transformation. Then the subset of $X(\overline{\mathbb{F}_p})$ consisting of non-critical periodic points of f is Zariski dense in X.*

Recall that a non critical periodic point is a point whose orbit under f meets neither the indeterminacy set of f nor its critical set and is finite.

Proof of Proposition 8.2. Let Z be the Zariski closure of the set of non-critical periodic points of f in $X(\overline{\mathbb{F}_p})$ and suppose by contradiction that $Z \neq X$. Set $Y := Z \bigcup I(f) \bigcup \mathcal{E}(f)$ and then Y is a proper closed subset of X. Let $q = p^n$ be such that X and f are defined over the subfield $\mathbb{F}_q$ of $\overline{\mathbb{F}_p}$ having exactly q elements. Let ϕ_q denote the Frobenius morphism acting on X and let Γ_f(resp. Γ_m) denote the graph of f (resp. ϕ_q^m) in $X \times X$. Let U be an irreducible affine open subset of $X \setminus Y$ that is also defined over $\mathbb{F}_q$ and such that f is an open embedding from U to X. Set $V = \Gamma_f \bigcap (U \times U)$. By Theorem 8.1 there exists an integer $m > 0$ such that $(V\bigcap\Gamma_m)(\overline{\mathbb{F}_p})\neq\emptyset$ i.e. there exists $u \in U(\overline{\mathbb{F}_p})$ such that $f(u) = \phi_q^m(u) \in U$. Since f is defined over $\mathbb{F}_q$, it follows that $f^l(u) = \phi_q^{lm}(u) \in U$ for all $l \geq 0$. In particular $f(u)$ is a non-critical periodic point of f. This contradicts the definition of Y and U, and the proof is complete. □

For the convenience of the reader, we repeat the arguments of [21, Theorem 5.1] which allows us to lift any isolated periodic point from the special fiber to the generic fiber.

Lemma 8.3. *Let $\mathbf{X}$ be a projective scheme, flat over a discrete valuation ring R with fraction field K and residue field k. Let F be a birational map $\mathbf{X} \dashrightarrow \mathbf{X}$ over R which is well defined at least at one point on the special fiber. Let X be the special fiber of $\mathbf{X}$ and X' be the generic fiber of $\mathbf{X}$, f be the restriction of F to X, and f' be the restriction of F to X'.*

If the set of periodic $\overline{k}$-points of f is Zariski dense in X, and moreover there are only finitely many curves of periodic points in X, then the set consisting of periodic $\overline{K}$-points of f' is Zariski dense in the generic fiber of X'.

Proof. The set of periodic $\overline{k}$-points of f of period dividing n can be viewed as the set of $\overline{k}$-points in $\Delta_X \bigcap \Gamma_{f^n}$, where Δ_X is the diagonal and Γ_{f^n} is the graph of f^n in $X \times X$.

For any positive integer n, consider the subscheme $\Delta_{\mathbf{X}} \bigcap \Gamma_{F^n}$ of $\mathbf{X} \times_R \mathbf{X}$, where $\Delta_{\mathbf{X}}$ is the diagonal and Γ_{F^n} is the graph of F^n in $\mathbf{X} \times_R \mathbf{X}$. If $x \in X \setminus \mathrm{Sing}X$ is a periodic point of f that does not lie in any curve of periodic points, then (x, x) is contained in a closed subscheme of $\Delta_{\mathbf{X}} \bigcap \Gamma_{F^n}$ of dimension one. Since x is not in any curve of periodic points, the generic point x' of this subscheme is in $\Delta_{X'} \bigcap \Gamma_{f'^n}$ the generic fiber of $\Delta_{\mathbf{X}} \bigcap \Gamma_{F^n}$, so that x' is a periodic point. Since x is non critical, F^k is a local isomorphism on a neighborhood $\mathbf{U}^k$ of x in $\mathbf{X}$. Since x' is the generic point of a curve containing x, we get $x' \in \mathbf{U}^k$ hence it is non-critical.

We identify $\mathbf{X}$ with $\Delta_{\mathbf{X}}$. For any open subset U' of X', let Z' be a Weil-divisor of X' containing $X' \setminus U'$. Let $\mathbf{Z}$ be the closure of Z' in $\mathbf{X}$, then $\mathrm{codim}(\mathbf{Z}) = 1$ and each component of $\mathbf{Z}$ meets Z'. Each component of X is of codimension 1.

If $X \subseteq \mathbf{Z}$, each component of X is a component of $\mathbf{Z}$. Since $X' \bigcap X = \emptyset$, we get $X \nsubseteq \mathbf{Z}$. Let $\mathbf{U} = \mathbf{X} \setminus \mathbf{Z}$ and $U = \mathbf{U} \bigcap X$, then $\mathbf{U} \bigcap X' = U'$ and $U \neq \emptyset$. There is a periodic point x of f not in any curve of periodic points which is a smooth point in U, then x' is in U' and is a non-critical periodic point of f'. □

8.2. **Invariant curves.** From the previous subsection, we see that curves of periodic points are the main obstructions to lift periodic points from finite fields. The following theorem which essentially is [8, Corollary 3.3] of Cantat tells us that if $\lambda_1 > 1$ on the special fiber, then this obstruction can be removed.

Theorem 8.4 ([8, 18])**.** *Let X be a smooth projective surface defined over* $\mathbf{k}$*. Then a birational transformation $f : X \dashrightarrow X$ with $\lambda_1(f) > 1$ admits only finitely many periodic curves.*

In particular, there are only finitely many curves of periodic points.

Observe that [8, Corollary 3.3] is stated over the field of complex numbers. However the proof relies mainly on the fact that $\dim H^i(X, \Omega^1_X)$, $i = 0, 1$ is finite and thus works over any projective varieties defined over an algebraically closed field of any characteristic endowed with its Zariski topology.

Proof. Assume by contradiction that there exists infinitely many $f-$invariant curves. By [8, Corollary 3.3], there is a rational function $\Phi : X \dashrightarrow \mathbb{P}^1$ and a non-zero constant α such that $\Phi \circ f = \alpha\Phi$. This implies $\lambda_1(f) = 1$. For the convenience of the reader, we give a proof of this fact. By Theorem 5.6, we may assume that f is algebraically stable on X. Then there is a nef class $\omega \in N^1(X)_{\mathbb{R}} \setminus \{0\}$, such that $f^*(\omega) = \lambda_1(f)\omega$. Let $[F] \in N^1(X)_{\mathbb{R}}$ be the class of a fiber of the invariant fibration. Since f is birational, we have $f^*[F] = f_*[F] = [F]$, and

$$(\omega \cdot [F]) = (\omega \cdot f_*[F]) = (f^*\omega \cdot [F]) = \lambda_1(f)(\omega \cdot [F]).$$

If $(\omega \cdot [F]) \neq 0$, we are done. Otherwise, since $(F^2) = 0$ and L is nef, then $L = lF$ for some $l \in \mathbb{R}$ by Theorem 5.2. In this case we also have that $\lambda_1(f) = 1$, because $f^*[F] = [F]$. □

8.3. **Proof of Theorem 4.2.** Let L be any very ample line bundle on X. We may assume that the transcendence degree of $\mathbf{k}$ over its prime field F is finite, since we can find a subfield of $\mathbf{k}$ which is finitely generated over F such that X, f and L are all defined over this subfield. We complete the proof by induction on the transcendence degree of $\mathbf{k}$ over F.

If $\mathbf{k}$ is the closure of a finite field, then the theorem holds by Proposition 8.2.

If $\mathbf{k} = \overline{\mathbb{Q}}$, there is a regular subring R of $\overline{\mathbb{Q}}$ which is finitely generated over $\mathbb{Z}$, such that X, L, f are defined over R. By Theorem 7.3, there is a maximal ideal $\mathfrak{m}$ of R such that the fiber $X_{\mathfrak{m}}$ is smooth and the restriction $f_{\mathfrak{m}}$ of f on this fiber is a birational transformation with $\lambda_1(f_{\mathfrak{m}}) > 1$. Since R is regular and finitely generated over $\mathbb{Z}$, the localization $R_{\mathfrak{m}}$ of R at $\mathfrak{m}$ is a discrete valuation ring such that $\overline{\mathrm{Frac}\,(R_{\mathfrak{m}})} = \overline{\mathbb{Q}}$ and $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}} = R/\mathfrak{m}$. Then, by Proposition 8.2 the set of non-critical periodic points of $f_{\mathfrak{m}}$ is Zariski dense in the special fiber. Since $\lambda_1(f_{\mathfrak{m}}) > 1$, Theorem 8.4 shows that the number of curves of periodic points is finite. We are thus in position to apply Lemma 8.3: the set of non-critical periodic points of f forms a Zariski dense subset of X, and the theorem holds in this case.

If the transcendence degree of $\mathbf{k}$ over F is greater than 1, we pick an algebraically closed subfield K of $\mathbf{k}$ such that the transcendence degree of K over F equals the transcendence degree of $\mathbf{k}$ over F minus 1. Then we pick a subring R of $\mathbf{k}$ which is finitely generated over K, such that X, L and f are all defined over R. Since $\operatorname{Spec} R$ is regular on an open set, we may assume that R is regular by adding finitely many inverses of elements in R. We may repeat the same arguments as in the case $\mathbf{k} = \overline{\mathbb{Q}}$.

8.4. **Existence of Zariski dense orbits.** In this subsection, we denote by $\mathbf{k}$ an algebraically closed field of characteristic 0. Our aim is to show the following result from the introduction:

Theorem 4.4. *Let X be a projective surface over an algebraically closed field $\mathbf{k}$ of characteristic 0. Let $f : X \dashrightarrow X$ be a birational transformation with $\lambda_1(f) > 1$. Then there is a point $x \in X$ such that $f^n(x) \in X \setminus I(f)$ for any $n \in \mathbb{Z}$ and $\{f^n(x) | n \in \mathbb{Z}\}$ is Zariski dense.*

As a first intermediate step, we extend E. Amerik's results [2] to arbitrary algebraically closed field of characteristic 0. Namely we prove

Theorem 8.5. *Let X be a variety over an algebraically closed field $\mathbf{k}$ of characteristic 0, and $f : X \dashrightarrow X$ be a birational transformation, then there is a point $x \in X$ such that $f^n(x) \in X \setminus I(f)$ for all $n \in \mathbb{Z}$ and $\{f^n(x) | n \in \mathbb{Z}\}$ is infinite.*

Remark 8.6. Proposition 8.5 can not be true over $\overline{\mathbb{F}_p}$, since for any $q = p^n$, $\#X(\mathbb{F}_q)$ is finite.

To do so we shall rely on the following Lemma which is completely standard and whose proof is left to the reader.

Lemma 8.7. *Let $\pi : X \to Y$ be a dominant morphism between two irreducible varieties defined over an algebraically closed field. For every point $x \in X$, there is an irreducible subvariety S through x of X, such that $\dim S = \dim Y$, and the restriction of π on S is dominant to Y.*

We are now in position to prove

Proof of Theorem 8.5. In the case $\mathbf{k} = \overline{\mathbb{Q}}$, the theorem is due to E. Amerik in [2].

In the general case, there is a subring R of $\mathbf{k}$ which is finitely generated over $\overline{\mathbb{Q}}$ on which X and f are defined, and we may assume that $\mathbf{k}$ is the algebraic closure of the fraction field of R. We then pick a scheme $\pi : X_R \to \operatorname{Spec} R$ and a birational transformation f_R of X_R over R such that the geometric generic fiber of X_R is X and the restriction of f_R on X is f. Pick any closed point $\mathfrak{m} \in \operatorname{Spec} R$ such that the restriction $f_{\mathfrak{m}}$ of f on the special fiber $X_{\mathfrak{m}}$ at $\mathfrak{m}$ is birational. Since $R/\mathfrak{m} = \overline{\mathbb{Q}}$, there is a point $y \in X_{\mathfrak{m}}$ such that $f_{\mathfrak{m}}^n(y) \in X_{\mathfrak{m}} \setminus I(f_{\mathfrak{m}})$ for any $n \in \mathbb{Z}$ and $\{f_{\mathfrak{m}}^n(y) | n \in \mathbb{Z}\}$ is infinite. By Lemma 8.7, there is an irreducible subvariety S of X containing y and such that $\dim S = \dim R$, and the restriction of π to S is dominant to $\operatorname{Spec} R$. Let x be the generic point of S, then we have

$x \in X_R(\mathbf{k}) = X$. We thus get $f^n(x) \in X \setminus I(f)$ for any $n \in \mathbb{Z}$ and $\{f^n(x)|\ n \in \mathbb{Z}\}$ is infinite. □

Proof of Theorem 4.4. By Theorem 8.4, there are only finitely many invariant curves for f. Let C be the union of these curves and set $U = X \setminus C$. By Theorem 8.5, there is a non-preperiodic point $x \in U$ such that $f^n(x) \in U \setminus I(f)$ for any $n \in \mathbb{Z}$. Let O be the Zariski closure of $\{f^n(x)|n \in \mathbb{Z}\}$ in X. If $O \neq X$, then $\dim O = 1$ since it is infinite. The union of all one-dimensional irreducible components of O is then invariant and intersects U which is a contradiction. □

9. Automorphisms and Lefschetz-Saito's fixed point formula

In this section, $\mathbf{k}$ is any algebraically closed field of characteristic 0.

9.1. **Proof of Theorem 4.3.** Since there are only finitely many coefficients in the definition of X and f, we may always assume that the transcendence degree of $\mathbf{k}$ over $\overline{\mathbb{Q}}$ is finite, and embed $\mathbf{k}$ in $\mathbb{C}$. We may thus suppose that $\mathbf{k} = \mathbb{C}$.

Introduce the Lefschetz number

$$L(f^n) := \sum_i (-1)^i \mathrm{Tr}[f^{n*} : H^i(X) \to H^i(X)].$$

Recall from [29, Lemma 7.8] that

$$|L(f^n) - \lambda_1(f)^n| = 4(\lambda_1(f))^{n/2} + O(1)$$

if X is an abelian surface, and

$$|L(f^n) - \lambda_1(f)^n| = O(1)$$

otherwise.

The theorem now follows from the next result whose proof will occupy Subsection 9.2 below.

Theorem 9.1. *Let X be a smooth projective surface over $\mathbb{C}$ and f be an automorphism of X with $\lambda_1(f) > 1$. We have*

$$|\#\mathrm{Per}_n(X) - L(f^n)| = O(1).$$

Remark 9.2. Theorem 4.3 gives an alternative to the methods of Hrushovski and Fakrhuddin to prove the Zariski density of periodic points for automorphisms. Indeed, the Zariski closure of the set of all periodic points is f-invariant. Since it is infinite, it is either X or a curve. But the latter case is impossible since an automorphism of a projective curve admits only finitely many isolated periodic points.

9.2. **Local invariants associated to fixed points.** The proof of Theorem 9.1 relies on a version of Lefschetz-Saito's formula previously used by Iwasaki and Uehara. Let us recall the ingredients appearing in this formula. We refer to [29] for detail.

We fix $f : X \to X$ an automorphism of a projective smooth surface, and denote by $\mathrm{Fix}(f)$ the set of fixed points.

Pick any $x \in \mathrm{Fix}(f)$, and write $\mathfrak{m} \subset \hat{\mathcal{O}}_{X,x}$ for the maximal ideal in the completion of the local ring at x. Since X is smooth, we have the isomorphism $\hat{\mathcal{O}}_{X,x} \simeq \mathbb{C}[[z_1, z_2]]$, and $\mathfrak{m}$ is the ideal generated by z_1, z_2. We may then write

$$f(z_1, z_2) = (z_1 + gh_1, z_2 + gh_2)$$

for some elements $g, h_1, h_2 \in \mathfrak{m}$, where $g \neq 0$ and h_1, h_2 are relatively prime. The first important invariant is

$$\delta(f, x) := \dim_{\mathbb{C}} \mathbb{C}[[z_1, z_2]]/(h_1, h_2). \tag{9.1}$$

Since h_1, h_2 are relatively prime, δ is finite.

Next denote by $\Lambda(f, x)$ the set of irreducible components of the fixed point locus of f at x, and pick $C \in \Lambda(f, x)$ any irreducible component. Then we set

$$v_C(f) = \mathrm{ord}_C(g) \tag{9.2}$$

Note that given any *reduced* equation $h \in \mathfrak{m}$ of C, we have $v_C(f) = \max\{m \in \mathbb{N} |\, h^m \text{ divides } g\} = \min\{\mathrm{ord}_C(\phi \circ f - \phi),\, \phi \in \mathfrak{m}\}$. In particular this quantity does not depend on the choice of coordinates.

Let us introduce the holomorphic 1-form

$$\omega_{f,x} := h_2 dz_1 - h_1 dz_2.$$

A smooth curve $C \in \Lambda(f, x)$ is said to be:

- of type I if the restriction $\omega_{f,x}|C \in \Omega^1_C$ is non zero;
- of type II if $\omega_{f,x}|C$ vanishes identically.

Observe that the form $\omega_{f,x}$ depends on the choice of coordinates, but the type of a curve does not. It is also independent on the choice of a point x on the curve (see [29]). We shall denote by $X_{\mathrm{I}}(f)$ (resp. $X_{\mathrm{II}}(f)$) the set of irreducible curves that are fixed by f and of type I (resp. of type II). Suppose $C \in \Lambda(f, x)$ is *smooth*. If it is of type I, then we set

$$\mu_{C,x}(f) := \mathrm{ord}_x(\omega_{f,x}|C). \tag{9.3}$$

If $s : \mathbb{C}[[t]] \to C$ is any local parametrization of C at x, then we have $\mu_{C,x}(f) = \mathrm{ord}_t(s^*\omega_{f,x})$. If C is of type II, define

$$\mu_{C,x}(f) := \mathrm{ord}_x(\partial_{f,x}|C), \tag{9.4}$$

where $\partial_{f,x} = h_1 \partial_{z_1} + h_2 \partial_{z_2}$. In terms of the 1-form, $\omega_{f,x}$ this multiplicity can be interpreted as follows. By assumption there exists $a \in \mathfrak{m}$ such that $a|C \neq 0$ and $\omega_{f,x} - a dh$ is divisible by h. Then we have $\mu_{C,x}(f) := \mathrm{ord}_0(a)$. We leave to the reader to check that these quantities are independent on the choice of coordinates.

Proposition 9.3. *Suppose $f : X \to X$ fixes a smooth curve C pointwise. Pick $x \in C$ and assume $df(x)$ has one eigenvalue λ which is not a root of unity. Then C is of type I, and we have*

$$v_C(f^n) = 1, \text{ and } \delta(f^n, x) = \mu_{x,C}(f^n) = 0 \tag{9.5}$$

for all $n \in \mathbb{Z} \setminus \{0\}$.

Proof. Locally we may choose coordinates z_1, z_2 such that $C = \{z_1 = 0\}$, and then we have $f(z_1, z_2) = (z_1 + z_1(\lambda - 1 + o(1)), z_2 + z_1 h_2)$ for some power series h_2. The computations of $v_C(f), \mu_{x,C}(f)$ and $\delta(f^n, x)$ then follow immediately from the definitions. □

Proposition 9.4. *Suppose $f : X \to X$ fixes a point x, and assume $df(x) = \mathrm{id}$. If the set of all periodic points has simple normal crossing singularities at x, then we have*

$$\delta(f^n, x) = \delta(f, x),\ v_C(f^n) = v_C(f),\ \text{ and } \mu_{x,C}(f^n) = \mu_{x,C}(f)\ , \tag{9.6}$$

for all $n \in \mathbb{Z} \setminus \{0\}$.

Proof. Since f is tangent to the identity then there exists a unique formal vector field ∂ vanishing up to order 2 and such that $\exp(\partial) = f$. Let us recall how this vector field is constructed, see for instance [5].

Choose coordinates z_1, z_2 and write

$$f = (z_1 + gh_1, z_2 + gh_2) = (z_1 + \sum_{n \geq 2} p_n(z_1, z_2), z_2 + \sum_{n \geq 2} q_n(z_1, z_2))$$

where p_n, q_n are homogeneous polynomials of degree n. Similarly write $\partial = \sum_{n \geq 2} a_n(z_1, z_2)\partial_{z_1} + \sum_{n \geq 2} b_n(z_1, z_2)\partial_{z_2}$ with a_n, b_n homogeneous of degree n. For each $m \geq 2$, set $\partial_m = \sum_{n \leq m} a_n(z_1, z_2)\partial_{z_1} + \sum_{n \leq m} b_n(z_1, z_2)\partial_{z_2}$, and define recursively $\partial^j_m(\phi) = \partial_m(\partial^{j-1}_m(\phi))$ for any ϕ. Then we have

$$p_{m+1} = a_{m+1} + HT_{m+1}\left(\sum_{j=2}^{m} \frac{1}{j!}\, \partial^j_m(z_1)\right) \tag{9.7}$$

$$q_{m+1} = b_{m+1} + HT_{m+1}\left(\sum_{j=2}^{m} \frac{1}{j!}\, \partial^j_m(z_2)\right)\ , \tag{9.8}$$

where $HT_{m+1}(\phi)$ denotes the homogeneous part of degree $m+1$ of the power series expansion of ϕ in z_1, z_2.

Since the fixed point locus of f is assumed to have simple normal crossing singularities at x we may choose coordinates such that $g(z_1, z_2) = z_1^{n_1} z_2^{n_2}$ for some $n_1, n_2 \geq 0$ with $n_1 + n_2 \geq 1$.

We first claim that $\partial = z_1^{n_1} z_2^{n_2}\, \widetilde{\partial}$ for some reduced formal vector field $\widetilde{\partial}$, i.e. whose zero locus is zero dimensional.

Indeed by assumption $z_1^{n_1} z_2^{n_2}$ divides p_n and q_n for all $n \geq 2$. Let us prove by induction that $z_1^{n_1} z_2^{n_2}$ divides a_n and b_n for all n. This is true for $n = 2$ since $p_2 = a_2$ and $q_2 = b_2$. Suppose it is true for all $m \leq n$. Then $z_1^{n_1} z_2^{n_2}$ divides ∂_m for $2 \leq m \leq n$, hence $\partial^j_m(z_1)$ and $\partial^j_m(z_2)$ for all j, and it follows from (9.7) and (9.8) that $z_1^{n_1} z_2^{n_2}$ also divides a_{n+1} and b_{n+1} as required. Conversely if for some m_1, m_2 the monomial $z_1^{m_1} z_2^{m_2}$ divides a_n, b_n for all n, the same argument shows it divides p_n and q_n as well for all n. This proves the claim.

The claim implies by definition that $v_C(f) = \mathrm{ord}_C(\partial)$ for any curve of fixed point C of f. Since $f^n = \exp(n\partial)$ by construction, it follows that

$$v_C(f^n) = \mathrm{ord}_C(n\partial) = \mathrm{ord}_C(f) = v_C(f)\ .$$

Assume now $C = \{z_1 = 0\}$ is a curve a fixed point so that $n_1 \geq 1$. Write $\widetilde{\partial} = \widetilde{a}\partial_{z_1} + \widetilde{b}\partial_{z_2}$ with $\widetilde{a}, \widetilde{b}$ having no common factors.

Suppose first that $\widetilde{\partial}$ is generically transverse to C, i.e. $\widetilde{a}(0, z_2) \not\equiv 0$. Let us compute $\exp(\partial)(z_1) = z_1 + \sum_{j\geq 1} \frac{1}{j!}\, \partial^j z_1$. Write $\partial^j z_1 = g\widetilde{a}_j$ so that

$$\widetilde{a}_1 = \widetilde{a} \text{ and } \widetilde{a}_{j+1} = \widetilde{\partial} g\, \widetilde{a}_j + g\, \widetilde{\partial}\widetilde{a}_j. \tag{9.9}$$

Then for any $j \geq 1$, we get

$$\mathrm{ord}_0(\widetilde{a}_{j+1}(0, z_2)) = \mathrm{ord}_0(\widetilde{\partial} g(0, z_2)) + \mathrm{ord}_0(\widetilde{a}_j(0, z_2)) \geq 1 + \mathrm{ord}_0(\widetilde{a})\ .$$

Since $f(z_1, z_2) = (z_1 + g \sum \frac{1}{j!}\, \widetilde{a}_j, \star)$, we conclude first that C is of type I and then that $\mu_{x,C}(f) = \mathrm{ord}_0(\sum \frac{1}{j!}\, \widetilde{a}_j(0, z_2)) = \mathrm{ord}_0(\widetilde{a}(0, z_2))$. This proves that

$$\mu_{x,C}(f^n) = \mathrm{ord}_0(n\widetilde{a}(0, z_2)) = \mathrm{ord}_0(\widetilde{a}(0, z_2)) = \mu_{x,C}(f)\ .$$

Suppose next that C is $\widetilde{\partial}$-invariant, i.e. $\widetilde{a}(0, z_2) \equiv 0$ but $\widetilde{b}(0, z_2) \not\equiv 0$. We are now interested in $\exp(\partial)(z_2) = z_2 + \sum_{j\geq 1} \frac{1}{j!}\, \partial^j z_2$. Write $\partial^j z_2 = g\widetilde{b}_j$ so that as before we have

$$\widetilde{b}_1 = \widetilde{b} \text{ and } \widetilde{b}_{j+1} = \widetilde{\partial} g\, \widetilde{b}_j + g\, \widetilde{\partial}\widetilde{b}_j. \tag{9.10}$$

Then it is not difficult to see that C is of type II, and $\mathrm{ord}_0(\widetilde{b}_{j+1}(0, z_2)) \geq 1 + \mathrm{ord}_0(\widetilde{b})$ for all $j \geq 1$, so that

$$\mu_{x,C}(f^n) = \mathrm{ord}_0(n\widetilde{b}(0, z_2)) = \mathrm{ord}_0(\widetilde{b}(0, z_2)) = \mu_{x,C}(f)\ .$$

Finally let $I = \langle \widetilde{a}, \widetilde{b} \rangle \subset \hat{\mathcal{O}}_{X,x}$ be the ideal generated by $\widetilde{a}$ and $\widetilde{b}$. Since $\widetilde{\partial}\phi \in I$ for any ϕ, by induction on j we see that $\widetilde{b}_{j+1}, \widetilde{a}_{j+1} \in I^2 + (g)I \subset \mathfrak{m}\cdot I \subset I$ for all $j \geq 1$. From the identities $h_1 = \sum \frac{1}{j!}\, \widetilde{a}_j$ and $h_2 = \sum \frac{1}{j!}\, \widetilde{b}_j$, we infer $J := \langle h_1, h_2 \rangle \subset I$.

We claim that the integral closures of I and J are equal. Grant this claim. For any $\mathfrak{m}$-primary ideal $\mathfrak{a} \subset \hat{\mathcal{O}}_{X,x}$, we let $e(\mathfrak{a}) = \lim_{n\to\infty} \frac{1}{2n^2} \dim_{\{\mathbb{C}\}}(\hat{\mathcal{O}}_{X,x}/\mathfrak{a}^n)$ be the (Hilbert-Samuel) multiplicity of $\mathfrak{a}$. Two ideals having the same integral closure have the same multiplicity, see [32]. We thus have

$$\delta(x, f) := e(J) = e(I) = e\langle \widetilde{a}, \widetilde{b} \rangle = e\langle n\widetilde{a}, n\widetilde{b} \rangle = \delta(f^n, x)$$

for all $n \neq 0$, which concludes the proof of the proposition.

To prove the claim, pick any sequence of point blow-ups $\pi : \hat{X} \to X$ centered above x and such that the ideal sheaf $J\cdot\mathcal{O}_{\hat{X}}$ is locally principal so that we can write $J\cdot\mathcal{O}_{\hat{X}} = \mathcal{O}_{\hat{X}}(-\sum m_i E_i)$ where E_i are exceptional and $m_i = \mathrm{ord}_{E_i}(\pi^* J) \geq 1$. Now recall that $h_1 - \widetilde{a}$ and $h_2 - \widetilde{b}$ lie in $\mathfrak{m}\cdot I$. Pick any exceptional curve E_i. By definition $\mathrm{ord}_{E_i}(\pi^* I) = \min\{\mathrm{ord}_{E_i}(\pi^*\widetilde{a}), \mathrm{ord}_{E_i}(\pi^*\widetilde{b})\}$. Say $\mathrm{ord}_{E_i}(\pi^* I) = \mathrm{ord}_{E_i}(\pi^*\widetilde{a})$. Then we have

$$\mathrm{ord}_{E_i}(\widetilde{a} - h_1) \geq \mathrm{ord}_{E_i}(\pi^*\mathfrak{m}) + \mathrm{ord}_{E_i}(\pi^* I) > \mathrm{ord}_{E_i}(\pi^*\widetilde{a})$$

hence $\mathrm{ord}_{E_i}(\pi^* h_1) = \mathrm{ord}_{E_i}(\pi^*\widetilde{a}) = \mathrm{ord}_{E_i}(\pi^* I)$. On the other hand, $\mathrm{ord}_{E_i}(\pi^* h_2) \geq \mathrm{ord}_{E_i}(\pi^* I)$, hence we get $\mathrm{ord}_{E_i}(\pi^* J) = \mathrm{ord}_{E_i}(\pi^* I)$. It follows from [32, Théorème 2.1 (iv)] that I is included in the integral closure $\bar{J}$ of J, and $J \subset I \subset \bar{J}$ implies $\bar{I} = \bar{J}$ as was to be shown. □

Remark 9.5. When f is tangent to the identity, the proof shows that we have the following geometrical interpretations. Let $\mathcal{F}$ be the (formal) foliation associated to the formal vector field ∂ vanishing up to order 2 at 0 and satisfying $f = \exp(\partial)$. Let $\widetilde{\partial}$ be the reduced vector field associated to ∂.

Then a curve C of fixed points is of type I, if it is generically transversal to $\mathcal{F}$, and of type II if it is a leaf of $\mathcal{F}$. The multiplicity $v_C(f)$ is the generic order of vanishing of ∂ along C; $\delta(f)$ is the Hilbert-Samuel multiplicity of the ideal generated by $\widetilde{\partial}\mathfrak{m}$. When C is smooth, then $\mu_{x,C}(f)$ is the order of vanishing of $\widetilde{\partial}|C$ (when C is of type II), or of its dual 1-form (when C is of type I).

Finally we define

$$v_{(C,x)}(f) := \delta(f,x) + \sum_{C\in\Lambda(f,x)} v_C(f)\,\mu_{C,x}(f). \tag{9.11}$$

9.3. **Proof of Theorem 9.1.** By Theorem 8.4, there are only finitely many curves of periodic points on X. In particular, one can find an integer M such that any curve of periodic points is included in $\mathrm{Fix}(f^M)$. In the sequel we shall assume that $M = 1$. Pick a curve $C \subseteq \mathrm{Fix}(f)$. At any point $x \in C$, the differential df_x has two eigenvalues, one equal to 1 and the other one to $\lambda(x) = \det df_x$. Since C is compact, it follows that $\lambda(x) \equiv \lambda(C)$ is a constant. Replacing f by a suitable iterate, we may also assume that either $\lambda(C) = 1$ or $\lambda(C)^n \neq 1$ for all $n \geq 1$.

Step 1: Suppose that $\mathrm{Fix}(f)$ has only simple normal crossing singularities. We apply Lefschetz-Saito's formula, see [29, Theorem 1.2], and use results from the preceding section.

Theorem 9.6. *Assume $f : X \to X$ is an automorphism such that all irreducible components of $\mathrm{Fix}(f^n)$ are smooth. Then we have*

$$L(f^n) = \sum_{x\in\mathrm{Fix}(f^n)} v_x(f^n) + \sum_{C\in X_{\mathrm{I}}(f^n)} \chi(C)\,v_C(f^n) + \sum_{C\in X_{\mathrm{II}}(f^n)} (C^2)\,v_C(f^n). \tag{9.12}$$

Here $\chi(C)$ denotes the Euler characteristic of C, and C^2 its self-intersection. It follows easily from our standing assumptions and from Propositions 9.3 and 9.4 that $|L(f^n) - \#\mathrm{Per}_n|$ is actually independent on n.

Step 2: Let $S_1 \subseteq X$ be the set of singular points of curves of periodic points. This set is finite and f-invariant so that f lifts as an automorphism f_1 to the blowup $\pi_1 : X_1 \to X$ of X at all points in S_1. The exceptional components of π_1 are permuted by f_1, and we have

$$|\mathrm{Tr}(f_1^{*n})_{H^{1,1}} - \mathrm{Tr}(f^{*n})_{H^{1,1}}| \leq \#S_1.$$

On the other hand, $\pi_* : H^{i,j}(X_1) \to H^{i,j}(X)$ is an isomorphism for $(i,j) \neq (1,1)$, so that

$$|L(f_1^n) - L(f^n)| = O(1).$$

There are at most 2 isolated fixed points of f^n on each exceptional component, hence $|\#\mathrm{Per}_n(X_1) - \#\mathrm{Per}_n(X)| = O(1)$. Repeating the argument finitely many times, we end up with an automorphism for which the union of all curves of periodic points has only simple normal crossing singularities and all curves are smooth. This concludes the proof.

10. The case $\lambda_1 = 1$

In this section, we denote by $\mathbf{k}$ an algebraically closed field of characteristic different from 2 and 3. Our aim is to prove Theorem 4.1 in the remaining case $\lambda_1 = 1$.

Recall the following structure theorem for this class of maps.

Theorem 10.1. *([17, 26]) Let X be a smooth projective surface over $\mathbf{k}$, let $L \to X$ be an ample line bundle, and let f be a birational transformation of X. Assume $\lambda_1(f) = 1$. Then up to a birational conjugacy, we are in one of the following three cases:*

- (i) *the sequence $\deg_L(f^n)$ is bounded, f is an automorphism and some positive iterate of f acts on $N^1(X)$ as the identity;*
- (ii) *the sequence $\deg_L(f^n)$ is equivalent to cn for some $c > 0$, and f preserves a rational fibration;*
- (iii) *the sequence $\deg_L(f^n)$ is equivalent to cn^2, for some $c > 0$ and f is an automorphism preserving an elliptic fibration.*

We shall argue case by case.

10.1. The case when $\deg(f^n)$ is bounded.

Proposition 10.2. *Let X be a projective variety, f be an automorphism of X which acts on $N^1(X)$ as the identity. If the periodic points of f are Zariski dense, then there is an integer $n > 0$ such that $f^n = \mathrm{id}$.*

Remark 10.3. When k is the algebraic closure of a finite field, any automorphism f acting trivially on $N^1(X)$ satisfies $f^n = \mathrm{id}$ for some $n \geq 1$ since all points in X are periodic. However in any other field there are some automorphisms acting trivially on $N^1(X)$ of infinite order such as $[x : y : z] \to [x : ty : z]$ on $X = \mathbb{P}^2$ where $t \in k$ is not a root of unity.

Remark 10.4. When the action of f on the Picard group is the identity, then the arguments of [21, Proposition 2.1] apply directly. We can thus find an embedding of X in $\mathbb{P}^N$ and an automorphism $A \in \mathrm{PGL}_{N+1}$ such that $f = A|_X$. The result then follows easily.

Proof of Proposition 10.2. We denote by $\mathrm{Aut}(X)$ the automorphism group of X. Pick a line bundle $L \to X$ and let $\mathrm{Aut}_{[L]}$ be the subgroup of all automorphisms fixing the class $[L] \in N^1(X)$. We denote by $\mathrm{Aut}_0(X)$ the irreducible component of the identity. For any $g \in \mathrm{Aut}_{[L]}(X)$, let $\Gamma_g \subseteq X \times X$ be the graph of g. We denote by π_1, π_2 the projections onto the first and second factors. Since $\pi_1^* L \otimes \pi_2^* L$ is ample on $X \times X$, we may consider the Hilbert polynomial $P_g(m)$ of Γ_g:

$$P_g(m) = \chi(\Gamma_g, (\pi_1^* L \otimes \pi_2^* L)^{\otimes m}) = \chi(X, (L \otimes g^* L)^{\otimes m}).$$

By the Hirzebruch-Riemann-Roch theorem, we see that $\chi(X, (L \otimes g^* L)^{\otimes m})$ is a polynomial function of m whose coefficients only depend on the numerical class

$[L \otimes g^*L] = 2[L] \in N^1(X)$, it follows that $P := P_g$ is independent of g. Let Y be the Hilbert scheme parameterizing closed subschemes of $X \times X$ with Hilbert polynomial P: it is a scheme that admits finitely many irreducible components and $\mathrm{Aut}_{[L]}(X)$ is an open subvariety of Y.

Since f acts on $N^1(X)$ as the identity, it follows that $f \in \mathrm{Aut}_{[L]}(X)$, hence $f^M \in \mathrm{Aut}_0(X)$ for some $M \geq 1$. We may thus assume that $f \in \mathrm{Aut}_0(X)$. Let S_m be the set of fixed points of $f^{m!}$ for $m \geq 1$, so that $S_m \subseteq S_{m+1}$ for any $m \geq 1$. Let $F_m = \{g \in \mathrm{Aut}_0(X) |\ g|_{S_m} = \mathrm{id}\}$. Then F_m is a closed set and $F_{m+1} \subseteq F_m$ for any $m \geq 1$. By noetherianity there is an integer l such that $F_l = \bigcap_{m\geq 1} F_m$, and it follows that then $f^n \in F_l = \bigcap_{m\geq 1} F_m$ for $n = l!$. In particular, we have $f^n|S_m = \mathrm{id}$ for any $m \geq 1$. Since the Zariski closure of $\bigcup_{m\geq 1} S_m$ is X by assumption, we conclude that $f^n = \mathrm{id}$. □

Proposition 10.5. *Let X be a smooth projective surface over* $\mathbf{k}$*, and $L \to X$ be an ample line bundle. Let f be a birational transformation of X, such that the sequence $\deg_L(f^n)$ is bounded. If the set of non critical periodic points of f is Zariski dense, then there is an integer $n > 0$ such that $f^n = \mathrm{id}$.*

Proof. By Theorem 10.1, we may assume that f is an automorphism and acts on $N^1(X)$ as the identity and we conclude by Proposition 10.2. □

10.2. **The linear growth case.**

Proposition 10.6. *Let X be a projective smooth surface over* $\mathbf{k}$*, and $L \to X$ be an ample line bundle. Let f be a birational transformation of X, such that $\deg_L(f^n) \sim cn$ for some $c > 0$. Then the set of non-critical periodic points of f is Zariski dense if and only if its action on the base of its invariant rational fibration is periodic.*

Proof. Suppose first that the set of non-critical periodic points is Zariski dense. By Theorem 10.1, we may assume that $X = C \times \mathbb{P}^1_{\mathbf{k}}$ where C is a smooth projective curve defined over $\mathbf{k}$ and f is written under the form

$$f(x, y) = \Big(g(x), \frac{A_1(x)y + B_1(x)}{A_2(x)y + B_2(x)}\Big)$$

where g is an automorphism of C and $A_1(x), B_1(x), A_2(x), B_2(x)$ are rational functions on C such that $A_1(x)B_2(x) - A_2(x)B_1(x) \neq 0$. Since the set non-critical periodic points of f is Zariski dense, the set of all periodic points of g is also Zariski dense, hence $g^n = \mathrm{id}$ for some $n \geq 0$. Replacing f by a suitable iterate, we may thus assume that $g = \mathrm{id}$, and

$$f = \Big(x, \frac{A_1(x)y + B_1(x)}{A_2(x)y + B_2(x)}\Big). \qquad (**)$$

Conversely suppose that $\deg_L(f^n) \to \infty$ and that f can be written under the form $(**)$. We denote the function field of C by K. Let

$$T(x) = (A_1(x) + B_2(x))^2/(A_1(x)B_2(x) - A_2(x)B_1(x))$$

and let $t_1, t_2 \in \overline{K}$ be the two eigenvalues of the matrix

$$\begin{pmatrix} A_1(x) & B_1(x) \\ A_2(x) & B_2(x) \end{pmatrix}.$$

If $(A_1(x)+B_2(x))^2/(A_1(x)B_2(x)-A_2(x)B_1(x)) \in \mathbf{k}$, then

$$t_1/t_2 + t_2/t_1 + 2 = (A_1(x)+B_2(x))^2/(A_1(x)B_2(x)-A_2(x)B_1(x)) \in \mathbf{k},$$

which implies $t_1/t_2 \in \mathbf{k}$, since $\mathbf{k}$ is algebraically closed.

If $t_1 = t_2$, then $t_1 = t_2 = (A_1(x)+B_2(x))/2 \in K$. We may replace $A_i(x)$ (resp. $B_i(x)$) by $2A_i(x)/(A_1(x)+B_2(x))$ (resp. $2B_i(x)/(A_1(x)+B_2(x))$), so that we may assume that $t_1 = t_2 = 1$. Changing coordinates if necessary, f can be the written as $(x, y + B(x))$ where $B(x) \in K$. It follows that $\deg_L f^n$ is bounded, which is a contradiction.

If $t_1 \neq t_2$, then $K(t_1)$ is a finite extension over K. There is a curve $\pi : B \to C$ corresponding to this field extension. Since f acts on C trivially, it induces a map $\widetilde{f}$ on $\mathbb{P}^1 \times_C B$. We set $\widetilde{L} = (\mathrm{id} \times_C \pi)^* L$. Since t_1, t_2 are rational functions on B, $\widetilde{f}$ is under the form $(x, (t_1/t_2)y)$, this implies that $\deg_{\widetilde{L}} \widetilde{f}^n$ is bounded. Since $\deg_{\widetilde{L}} \widetilde{f}^n = \deg \pi \times \deg_L f^n$, we get a contradiction.

We have shown that $T(x)$ is a non-constant rational function on B. For any $n > 0$, pick a primitive $n-$th root r_n of unity, there is at least one point $x \in C$ such that $T(x) = 2 + r_n + 1/r_n$. Changing coordinates if necessary, f acts on this fiber as $y \mapsto r_n y$ has finite order. It follows that periodic points of f are Zariski dense. □

10.3. **The quadratic growth case.** The proof of the following theorem is similar to the proof of [6, Proposition 7.4].

Proposition 10.7. *Let X be a smooth projective surface over $\mathbf{k}$, and $L \to X$ be an ample line bundle. Let $f : X \dashrightarrow X$ be an automorphism of X such that $\deg_L(f^n) \to \infty$. Then the set of periodic points of f is Zariski dense if and only if its action on the base of its invariant elliptic fibration is periodic.*

Proof. Let $\pi : X \to C$ be the invariant elliptic fibration. Denote by g the automorphism on the base curve C induced by f.

Suppose first that the set of periodic points of f is Zariski dense. Then the set of periodic points of g is Zariski dense too. So there is an integer $N > 0$, such that $g^N = \mathrm{id}$.

Conversely suppose that $\deg_L(f^n) \to \infty$ and $g = \mathrm{id}$. Since f is an elliptic fibration, then all but finitely many fibers are elliptic curves.

By [35, Theorem 10.1 III], the order of the automorphism group of an elliptic curve (as an algebraic group) is a divisor of 24. We may replace f by f^{24}, so that the restriction of f to each smooth fiber is a translation. Observe that $f|_{E_x}$ admits a fixed point on a smooth fiber E_x if and only if $f|_{E_x}$ is the identity. Assume by contradiction that the set of periodic points of f is not Zariski dense. Then there is a set $T = \{x_1, \cdots, x_m\} \in C$, such that for any $x \in C \setminus T$, E_x is a smooth elliptic curve and f has no periodic points in E_x.

By replacing L by a sufficiently large power, we may assume it is very ample. By Bertini's Theorem (see [27]), we can find a general section S of L such that for any $x \in T$, the intersection of S and F_x is transverse, and these intersection points are smooth in S and in F_x.

Let H be the set of periodic points which lies in S. Then $H \subseteq S \bigcap (\bigcup_{x\in T} E_x)$ is a finite set. By replacing f by f^l for some $l > 0$, we may assume that all points in H are fixed. Observe that for any $i \neq j$, we have $f^i(S) \bigcap f^j(S) = H$. Let x be any point in H and write $\pi(x) = y$. In some local coordinates (z_1, z_2) at x, since both S and F_y are smooth at x and the intersection of S and E_y at x is transverse, we may assume that $S = (z_2 = 0)$, $E_y = (z_1 = 0)$ and π depends only on z_1. Then f^{-1} can be written as

$$f^{-1} = (z_1, z_2 + h(z_1, z_2))$$

where $h(0, 0) = 0$. Since the fixed point locus is given by $h = 0$ and this set lie in $z_1 = 0$ in this chart, it follows that we can write $h = z_1^l(a + b(z_1, z_2))$ where $l \geq 1$, $a \neq 0$ and $b(0, 0) = 0$.

Lemma 10.8. *One can find local coordinates (z_1, z_2) such that for any integer n, one has*

$$f^{-n} = (z_1, z_2 + z_1^l(na + b_n(z_1, z_2)))$$

where $b_1 = b$ and $b_n(0, 0) = 0$.

It follows that $f^n(S)$ is defined by the equation $z_2 + z_1^l(na + b_n(z_1, z_2)) = 0$. When n is not divisible by the characteristic of $\mathbf{k}$, it follows that the intersection product $(S \cdot f^n(S))_x$ is equal to l independently on n. It follows that $(f^*L \cdot L) = \Sigma_{x\in H}(S \cdot f^n(S))_x$ is bounded which gives a contradiction. □

Proof of Lemma 10.8. We proceed by induction on n. If it is true for n, then we have

$$f^{-(n+1)} = f^{-1} \circ f^{-n},$$

so that

$$\begin{aligned} f^{-n-1} &= (z_1, z_2 + z_1^l(na + b_n(z_1, z_2)) + z_1^l(a + b(z_1, z_2 + z_1^l(na + b_n(z_1, z_2)))))) \\ &= (z_1, z_2 + z_1^l((n + 1)a + b_{n+1}(z_1, z_2))) \end{aligned}$$

where $b_{n+1} = b_n(z_1, z_2) + b(z_1, z_2 + z_1^l(na + b_n(z_1, z_2)))$. In particular, we have $b_{n+1}(0, 0) = 0$. □

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Part 2

Dynamical Mordell-Lang conjecture for birational polynomial morphisms on $\mathbb{A}^2$

11. Introduction

The Mordell-Lang conjecture proved by Faltings [9] and Vojta [21] says that if V is a subvariety of a semiabelian variety G defined over $\mathbb{C}$ and Γ is a finitely generated subgroup of $G(\mathbb{C})$, then $V(\mathbb{C}) \cap \Gamma$ is a union of at most finitely many translates of subgroups of Γ.

The following dynamical analogue of the Mordell-Lang conjecture was proposed by Ghioca and Tucker.

Dynamical Mordell-Lang Conjecture ([13])**.** Let X be a quasiprojective variety defined over $\mathbb{C}$, let $f : X \to X$ be an endomorphism, and V be any subvariety of X. For any point $p \in X(\mathbb{C})$ the set $\{n \in \mathbb{N} | \ f^n(p) \in V(\mathbb{C})\}$ is a union of at most finitely many arithmetic progressions.

An arithmetic progression is a set of the form $\{an + b| \ n \in \mathbb{N}\}$ with $a, b \in \mathbb{N}$ possibly with $a = 0$.

Observe that this conjecture implies the classical Mordell-Lang conjecture in the case $\Gamma \simeq (\mathbb{Z}, +)$.

The Dynamical Mordell-Lang conjecture has been proved by Denis [6] for automorphisms of projective spaces and was later generalized by Bell [2] to the case of automorphisms of affine varieties. In [3], Bell, Ghioca and Tucker proved it for étale maps of quasiprojective varieties. The conjecture is also known in the case where $f = (F(x_1), G(x_2)) : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ where F, G are polynomials and the subvariety V is a line ([14]), and in the case $f = (F(x_1), \cdots, F(x_n)) : \mathbb{A}^n_K \to \mathbb{A}^n_K$ where $F \in K[t]$ is an indecomposable polynomial defined over a number field K which has no periodic critical points other than the point at infinity and V is a curve ([4]).

Our main result can be stated as follows.

Theorem A. *Let K be any algebraically closed field of characteristic 0, and $f : \mathbb{A}^2_K \to \mathbb{A}^2_K$ be any birational polynomial morphism defined over K. Let C be any curve in $\mathbb{A}^2_K$, and p be any point in $\mathbb{A}^2(K)$. Then the set $\{n \in \mathbb{N} | \ f^n(p) \in C\}$ is a union of at most finitely many arithmetic progressions.*

Observe that if p is preperiodic or C is periodic, then $\{n \in \mathbb{N} | \ f^n(p) \in C\}$ is infinite if and only if there exists one $n \geq 0$ such that $f^n(p) \in C$. It is easy to see that the conclusion of Theorem A is equivalent to say that this is the only one possibility.

In the case the map is an automorphism of $\mathbb{A}^2_K$ of Hénon type (see [11]) then this result follows from [3]. Our proof provides however an alternative approach and does not rely on the construction of p-adic invariant curves.

Recall that the *algebraic degree* of a polynomial transformation $f(x, y) = (f_1(x, y), f_2(x, y))$ is defined by $\deg f := \max\{\deg f_1, \deg f_2\}$. The limit $\lambda(f) := \lim_{n\to\infty}(\deg f^n)^{1/n}$ exists and we refer to it as the *dynamical degree* of f (see [7, 8]). Our proof shows that when $\lambda(f) > 1$, then Theorem A holds for fields of arbitrary characteristic.

Note however that our Theorem A does not hold when char$K > 0$ and $\lambda(f) = 1$ (see [2, Proposition 6.1] for a counter-example).

To explain our strategy, we fix a birational polynomial morphism $f : \mathbb{A}^2_K \to \mathbb{A}^2_K$. By some reduction arguments, we may assume that $K = \overline{\mathbb{Q}}$.

We may compactify $\mathbb{A}^2$ by [10] to a smooth projective surface, such that f extends to a birational transformation on X fixing a point Q in $X \setminus \mathbb{A}^2$, and f contracts all curves at infinity to Q (see [10] and Section 16.1).

The key idea of our proof is to take advantage of this attracting fixed point and to apply the following local version of the Dynamical Mordell-Lang conjecture.

Theorem 11.1. *Let X be a smooth projective surface over an arbitrary valued field $(K, |\cdot|)$ and $f : X \dashrightarrow X$ be a birational transformation defined over K. Let C be any curve in X. Pick any K-point p such that $f^n(p) \in X \setminus I(f)$ for all integers $n \geq 0$, and $f^n(p)$ tends to a fixed K-point $Q \in I(f^{-1}) \setminus I(f)$ with respect to a projective metric induced by $|\cdot|$ on X.*

If the set

$$\{n \in \mathbb{N} |\ f^n(p) \in C\}$$

is infinite, then either $f^n(p) = Q$ for some $n \geq 0$ or C is fixed.

To complete the proof of Theorem A we now rely on a global argument. When the curve C is passing through the fixed point Q in X, we cover the $\overline{\mathbb{Q}}$-points of the curve C by the basin of attraction of Q with respect to all absolute values on $\overline{\mathbb{Q}}$. If the point p belongs to one of these attracting basins, then the local dynamical Mordell-Lang applies and we are done. Otherwise it is possible to bound the height of p and Northcott theorem shows that it is periodic.

Finally when neither the curve C nor its iterates contain the fixed point Q, we are in position to apply the next result which allows us to conclude.

Theorem 11.2. *Let X be a smooth projective surface over an algebraically closed field, $f : X \dashrightarrow X$ be an algebraically stable birational transformation and C be an irreducible curve in X such that f^n does not contract C for any $n \geq 0$.*

If $f^n(C) \cap I(f) \neq \emptyset$ for all $n \geq 0$, then C is periodic.

We should mention that it seems that it would be difficult to deal with arbitrary endomorphisms of surfaces using our approach. The key point of our proof is to take advantage of an attracting fixed point in some suitable model. But such a point does not exist for a general surface endomorphism.

The article is organized in 8 sections. In Section 12 we give background information on birational surface maps and metrics on projective varieties defined over a valued field. In Section 13 we prove Theorem 11.2, which is a criterion for a curve to be periodic. In Section 14 we prove some basic properties of the maps satisfying the conclusion of dynamical Mordell-Lang conjecture. In Section 15 we prove Theorem 11.1. In Section 16 we prove Theorem A in the case the dynamical degree $\lambda(f) = 1$. In Section 17 we prove a technical lemma which gives a upper bound on height when $\lambda(f) > 1$. In Section 18 we prove Theorem A.

12. Notations and basics

12.1. **Basics on birational maps on surfaces.** See [5, 7, 10] for details.

In this section a variety is defined over an algebraically closed field k. Recall that the resolution of singularities exists for surfaces over any algebraically closed field (see [1]).

Let X be a smooth projective surface. We denote by $N^1(X)$ the Néron-Severi group of X i.e. the group of numerical equivalence classes of divisors on X and write $N^1(X)_{\mathbb{R}} := N^1(X) \otimes \mathbb{R}$. Let $\phi : X \to Y$ be a morphism of smooth projective surfaces. It induces a natural map $\phi^* : N^1(Y)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$. Since $\dim X = 2$, one has a perfect pairing

$$N^1(X)_{\mathbb{R}} \times N^1(X)_{\mathbb{R}} \to \mathbb{R}, \ \ (\delta, \gamma) \to (\delta \cdot \gamma) \in \mathbb{R}$$

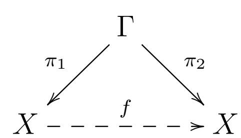
induced by the intersection form. We denote by $\phi_* : N^1(X)_{\mathbb{R}} \to N^1(Y)_{\mathbb{R}}$ the dual operator of ϕ^*.

Let X, Y be two smooth projective surfaces and $f : X \dashrightarrow Y$ be a birational map. We denote by $I(f) \subseteq X$ the indeterminacy set of f. For any curve $C \subset X$, we write

$$f(C) := \overline{f(C \setminus I(f))}$$

the strict transform of C.

Let $f : X \dashrightarrow X$ be a birational transformation and Γ be a desingularization of its graph. Denote by $\pi_1 : \Gamma \to X, \pi_2 : \Gamma \to X$ the natural projections. Then the diagram



(*)

is commutative and we call it a *resolution* of f.

Proposition 12.1 ([15])**.** *We have the following properties.*

(i) *The morphisms π_1, π_2 are compositions of point blowups.*
(ii) *For any point $p \notin I(f)$, there is a Zariski open neighborhood U of p in X and an injective morphism $\sigma : U \to \Gamma$ such that $\pi_1 \circ \sigma = \mathrm{id}$.*

Then we define the following linear maps

$$f^* = \pi_{1*}\pi_2^* : N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}},$$

and

$$f_* = \pi_{2*}\pi_1^* : N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}.$$

Observe that this definition is independence on the choice of the resolution and $f_* = f^{-1*}$. Note that in general we have $(f \circ g)^* \neq g^* f^*$.

For any big and nef class $\omega \in N^1_{\mathbb{R}}(X)$, we set

$$\deg_\omega(f) := (f^*\omega \cdot \omega),$$

the limit $\lim_{n\to\infty} \deg_\omega(f^n)^{1/n}$ exists and does not depend on the choice of ω (see [7, 8]). We denote this limit by $\lambda(f)$ and call it the *dynamical degree* of f.

Definition 12.2 (see [7])**.** Let $f : X \dashrightarrow X$ be a birational transformation on a smooth projective surface. Then f is said to be *algebraically stable* if and only if there is no curve $V \subseteq X$ such that $f^n(V) \subseteq I(f)$ for some integer $n \geq 0$.

In the case $X = \mathbb{P}^2$, f is algebraically stable if and only if $\deg(f^n) = (\deg f)^n$ for any $n \in \mathbb{N}$.

Theorem 12.3 ([7])**.** *Let $f : X \dashrightarrow X$ be a birational transformation of a smooth projective surface. Then there exists a smooth projective surface $\widehat{X}$, and a proper modification $\pi : \widehat{X} \to X$ such that the lift of f to $\widehat{X}$ is an algebraically stable map.*

By a *compactification* of $\mathbb{A}^2$, we mean a smooth projective surface X admitting a birational morphism $\pi : X \dashrightarrow \mathbb{P}^2$ that is an isomorphism above $\mathbb{A}^2 \subseteq \mathbb{P}^2$, see [10].

The theorem follows from [10, Proposition 2.6] and [10, Theorem 3.1], and provides us with a good compactification of $\mathbb{A}^2$.

Theorem 12.4 ([10])**.** *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a birational polynomial transformation with $\lambda(f) > 1$. Then there exists a compactification X of $\mathbb{A}^2$ satisfying the following properties.*

(i) *The map f extends to an algebraically stable map $\widetilde{f}$ on X.*
(ii) *There exists an $\widetilde{f}$-fixed point $Q \in X \setminus \mathbb{A}^2$ such that $d\widetilde{f}^2(Q) = 0$.*
(iii) *There exists an integer $n \geq 1$ such that $\widetilde{f}^n(X \setminus \mathbb{A}^2) = Q$.*

12.2. **Branches of curves on surfaces.** [12, 16] Let X be a smooth projective surface over an algebraically closed field k. Let C be an irreducible curve in X and p be a point in C.

Definition 12.5. A *branch* of C at p is a point in the normalization of C whose image is p.

Let $I_{C,p}$ be the prime ideal associated to C in the local function ring $O_{X,p}$ at p and $\widehat{I_{C,p}}$ be the completion of $I_{C,p}$ in the completion of local function ring $\widehat{O_{X,p}}$.

Let $i : \widetilde{C} \to C$ is a normalization of C and $\widetilde{p}$ a point in $i^{-1}(p)$. Let s be the branch of C at p defined by the point $\widetilde{p}$. The morphism $i : \widetilde{C} \to C$ induces a morphism $i^* : \widehat{\mathcal{O}_{X,p}} \to \widehat{\mathcal{O}_{\widetilde{C},\widetilde{p}}}$ between the completions of local function rings. The map $s \mapsto \mathfrak{p}_s := \ker i^*$ gives us a one to one correspondence between the set of branches of C at p and the set of prime ideals of $\widehat{O_{X,p}}$ with height 1 which contains $\widehat{I_{C,p}}$.

Given any two different branches s_1 and s_2 at a point $p \in X$, the intersection number is denoted by

$$(s_1 \cdot s_2) := \dim_k \widehat{\mathcal{O}_{X,p}}/(\mathfrak{p}_{s_1} + \mathfrak{p}_{s_2}).$$

For convenience, we set $(s_1 \cdot s_2) := 0$ if s_1 and s_2 are branches at different points.

Let Z be a smooth projective surface and $f : X \dashrightarrow Z$ be a birational map. If f does not contract C then we denote by $f(s)$ the branch of $f(s)$ defined by the point $\widetilde{p}$ in the normalization $f \circ i : \widetilde{C} \to f(C)$ and call it the strict transform of s. Observe that $f(s)$ is a branch of $f(C)$ and when $p \notin I(f)$, we have that $f(s)$ is a branch of curve at $f(p)$.

If f is regular at p, we write

$$f_*s = \begin{cases} f(s), & \text{when } f \text{ does not contract } C; \\ 0, & \text{otherwise.} \end{cases}$$

Let Y be another smooth projective surface and $\pi : Y \to X$ be a birational morphism. Denote by $\pi^\# s := \pi^{-1}(s)$ the strict transform of s. Let E_i, $i = 1, \cdots, m$ be the exceptional curves of π. There is a unique sequence of non negative integers $(a_i)_{0 \le i \le m}$ such that for any irreducible curve D in Y different from $\pi^\# C$, we have $(s \cdot \pi_* D) = (\pi^\# s + \sum_{i=1}^m a_i E_i \cdot D)$. Denote by $\pi^* s := \pi^\# s + \sum_{i=1}^m a_i E_i$ and call it the pull back of s.

Proposition 12.6. *We have the following properties.*

(i) *We have $\pi_* \pi^* s = s$.*

(ii) *For any irreducible curve (resp. any branch of curve) D in Y different from $\pi^\# C$ (resp. $\pi^\# s$), we have*

$$(\pi^* s \cdot D) = (s \cdot \pi_* D).$$

(iii) *For any curve (resp. any branch of curve) D in X different from C (resp. s) then we have*

$$(s \cdot D) = (\pi^\# s \cdot \pi^* D).$$

12.3. **Metrics on projective varieties defined over a valued field.** A field with an absolute value is called a valued field[1].

Definition 12.7. Let $(K, |\cdot|_v)$ be a valued field. For any integer $n \geq 1$, we define a metric d_v on the projective space $\mathbb{P}^n(K)$ by

$$d_v([x_0 : \cdots : x_n], [y_0 : \cdots : y_n]) = \frac{\max_{0 \le i,j \le n} |x_i y_j - x_j y_i|_v}{\max_{0 \le i \le n} |x_i|_v \max_{0 \le j \le n} |y_j|_v}$$

for any two points $[x_0 : \cdots : x_n], [y_0 : \cdots : y_n] \in \mathbb{P}^n(K)$.

Observe that when $|\cdot|_v$ is archimedean, then the metric d_v is not induced by a smooth riemannian metric. However it is equivalent to the restriction of the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$ or $\mathbb{P}^n(\mathbb{R})$ to $\mathbb{P}^n(K)$ induced by σ_v.

More generally, for a projective variety X defined over K, if we fix an embedding $\iota : X \hookrightarrow \mathbb{P}^n$, we may restrict the metric d_v on $\mathbb{P}^n(K)$ to a metric $d_{v,\iota}$ on $X(K)$. This metric depends on the choice of embedding ι in general, but for different embeddings ι_1 and ι_2, the metrics d_{v,ι_1} and d_{v,ι_2} are equivalent. Since we are mostly intersecting in the topology induced by these metrics we shall usually write d_v instead of $d_{v,\iota}$ for simplicity.

[1]This definition of valued field is not universally adopted. In some literatures, a valued field means a field with a *nonarchimedean* absolute value.

13. A criterion for a curve to be periodic

Our aim in this section is to prove Theorem 11.2 from the introduction. Let us recall the setting:

(i) X is a smooth projective surface over an algebraically closed field;
(ii) $f : X \dashrightarrow X$ is an algebraically stable birational transformation;
(iii) C is an irreducible curve in X such that f^n does not contract C and $f^n(C) \cap I(f) \neq \emptyset$ for all n.

Our aim is to show that C periodic. Let us begin with the following special case.

Lemma 13.1. *Let x be a point in $I(f) \cap C$. If there exists a branch s of C at x such that $f^n(s)$ is again a branch at x for all $n \geq 0$, then C is fixed by f.*

Proof of Lemma 13.1. Since f is birational, we may chose a resolution of f as in the diagram $(*)$ in Section 12.1.

If C is not fixed, we have $f(s) \neq s$ so that $A := (s \cdot f(s))_x < \infty$. By Proposition 12.1, π_2 is invertible on a Zariski neighbourhood of x. Let F_x be the fiber of π_1 over x.

For any $m \geq 0$, we have,

$$\begin{aligned}((f^m(s) \cdot f^{m+1}(s))_x =& \sum_{y \in F_x} (\pi_1^{\#} f^m(s) \cdot \pi_1^{*} f^{m+1}(s))_y \\ \geq& (\pi_1^{\#} f^m(s) \cdot \pi_1^{*} f^{m+1}(s))_{\pi_2^{-1}(x)} \\ =& (\pi_1^{\#} f^m(s) \cdot \pi_1^{\#} f^{m+1}(s))_{\pi_2^{-1}(x)} + (\pi_1^{\#} f^m(s) \cdot F_x)_{\pi_2^{-1}(x)} \\ =& (f^{m+1}(s) \cdot f^{m+2}(s))_x + (\pi_1^{\#} f^m(s) \cdot F_x)_{\pi_2^{-1}(x)} \\ \geq& (f^{m+1}(s) \cdot f^{m+2}(s))_x + 1.\end{aligned}$$

It follows that $A = (s \cdot f(s))_x \geq (f^m(s) \cdot f^{m+1}(s))_x + m \geq m$ for all $m \geq 0$ which yields a contradiction. □

We now treat the general case.

Proof of Theorem 11.2. Recall that f^n does not contract C and $f^n(C) \cap I(f) \neq \emptyset$ for all n. By Lemma 13.1, it is sufficient to find a point $x \in I(f) \cap C$ such that the image by f^n of the branch of C at x is again a branch of a curve at x for all $n \geq 0$. By contradiction we suppose that C is not periodic.

To do so, we introduce the set

$$P(f) = \{x \in I(f) |\ \text{ there is } n_1 > n_2 \geq 0 \text{ such that } f^{-n_1}(x) = f^{-n_2}(x)\}$$

and the set

$$O(f) = \{f^{-n}(x) |\ x \in P(f) \text{ and } n \geq 0\}.$$

By definition, $O(f)$ is finite. Since f is algebraically stable, $O(f) = O(f^n)$ for all $n \geq 1$. Replacing f by f^l for a suitable $l \geq 1$, we may assume that $O(f) = P(f)$. Set $N(f) = I(f) \setminus P(f)$.

First, we prove

Lemma 13.2. *For all $n \geq 0$, $f^n(C) \cap O(f) \neq \emptyset$.*

Proof of Lemma 13.2. We assume that $I(f) = \{p_1, \cdots, p_m\}$ and define the map

$$F = (f^{-1}, \cdots, f^{-1}) : X^m \dashrightarrow X^m.$$

Denote by π_i the projection onto the i-th factor and set

$$D = \cup_{i=1}^m \pi_i^{-1}(C).$$

Pick a point $q = (p_1, \cdots, p_m) \in X^m$. Since $f^n(C) \cap I(f) \neq \emptyset$ for all $n \geq 0$ by assumption, we have $F^n(q) \in D$ for all $n \geq 0$. Let Z' be the Zariski closure of $\{F^n(q)|n \geq 0\}$. Then we have $Z' \subseteq D$. Let Z be the union of all irreducible components of Z' of positive dimension. If Z is empty, then p_i is f^{-1}-preperiodic for all i and we conclude.

Otherwise since $\{F^n(q)|n \geq 0\} \cap I(F) = \emptyset$, the proper transformation of Z by F is well defined and satisfies $F(Z) = Z$, hence all irreducible components of Z are periodic. Let l be a common period for all components of Z. Observe that any irreducible component of Z is included in some $\pi_i^{-1}(C)$ for $i = 1, \cdots, m$. In other words, there exists $k \geq 0$ and $i \in \{1, \cdots, m\}$ such that $f^{-ln-k}(p_i) \in C$ for all $n \geq 0$. If p_i is not f^{-1}-preperiodic, then C is the Zariski closure of $\{f^{-ln-k}(p_i)|n \geq 0\}$ which is f^{-l}-invariant. This implies C to be periodic which contradicts to our hypothesis. It follows that p_i is f^{-1}-preperiodic.

Repeating the same argument for $f^n(C)$, we have $f^n(C) \cap O(f) \neq \emptyset$ for all $n \geq 0$. □

Denote by $D(n)$ the number of branches of $f^n(C)$ at points of $O(f)$. Since $f^{-1}(O(f)) \subseteq O(f)$, we have $D(n)$ is decrease and by Lemma 13.2, we have $D(n) \geq 1$. Replace C by $f^M(C)$ for some $M \geq 0$, we may assume that $D(n)$ is constant for $n \geq 0$. It follows that for any branch of curve of $f^n(C)$ at a point in $O(f)$, its image by f is again a branch of $f^{n+1}(C)$ at a point of $O(f)$. Set

$$S = \{x \in O(f)| \text{ there are infinitely many } n \geq 0 \text{ such that } x \in f^n(C)\}.$$

By the finiteness of $O(f)$, we may suppose that

$$f^n(C) \cap O(f) = f^n(C) \cap S$$

for all integer $n \geq 0$.

We claim that

Lemma 13.3. *Replacing f by a positive iterate, there exists a point $x \in C \cap S$ for which there is a branch s of C at x such that $f^n(s)$ is again a branch of curve at x for all $n \geq 0$.*

According to Lemma 13.1, we conclude. □

Proof of Lemma 13.3. Pick a resolution of f as in the diagram $(*)$ in Section 12.1. For any point $x \in S$, denote by F_x the fibre of π_1 over x and $E_x = \pi_2(F_x) \cap S$.

We have $E_x \neq \emptyset$. Otherwise, there exists $n \geq 0$ for which $x \in f^n(C)$ and a branch s of $f^n(C)$ at x. The assumption $E_x = \emptyset$ implies that $f(s)$ is not a branch at any point in S. This shows that $D(n+1) < D(n)$ and we get a contradiction.

On the other hand, let x_1, x_2 be two different points in S. If $E_{x_1} \cap E_{x_1} \neq \emptyset$, there exists $y \in S$ such that $y \in \pi_2(F_{x_1}) \cap \pi_2(F_{x_2})$. By Zariski's main theorem,

$\pi_2^{-1}(y)$ is a connected curve meeting F_{x_1} and F_{x_2}. So $\pi_1(\pi_2^{-1}(y))$ is a curve and it is contracted by f to $y \in S \subseteq I(f)$. This contradicts the fact that f is algebraically stable. So we have

$$E_{x_1} \cap E_{x_1} = \pi_2(F_{x_1}) \cap \pi_2(F_{x_2}) \cap S = \emptyset.$$

Set $T = \coprod_{x \in S} E_x \subseteq S$. Since $\#E_x \geq 1$ for all x, we have $\#T \geq \#S$. It follows that $T = S$ and $\#E_x = 1$ for all $x \in S$. This allows us to define a map $G : S \to S$ sending $x \in S$ to the unique point in E_x. Then G is an one to one map. For all $n \geq M$, f sends a branch of $f^n(C)$ at a point $x \in S$ to a branch of $f^{n+1}(C)$ at the point $G(x)$. By replacing f by $f^{(\#S)!}$, we may assume that $G = \mathrm{id}$. Then for any $x \in S \cap C$ and s a branch of C at x, we have $f^n(s)$ is again a branch at x for all $n \geq 0$. □

14. The DML property

For convenience, we introduce the following

Definition 14.1. Let X be a smooth surface defined over an algebraically closed field, and $f : X \dashrightarrow X$ be a rational transformation. We say that the pair (X, f) satisfies the DML property if for any irreducible curve C on X and for any closed point $p \in X$ such that $f^n(p) \notin I(f)$ for all $n \geq 0$, the set $\{n \in \mathbb{N} |\ f^n(p) \in C\}$ is a union of at most finitely many arithmetic progressions.

In our setting the DML property is equivalent to the following seemingly stronger property.

Proposition 14.2. *Let X be a smooth surface defined over an algebraically closed field, and $f : X \dashrightarrow X$ be a rational transformation. The following statements are equivalent.*

1. *The pair (X, f) satisfies the DML property.*
2. *For any curve C on X and any closed point $p \in X$ such that $f^n(p) \notin I(f)$ for all $n \geq 0$ and the set $\{n \in \mathbb{N} | f^n(p) \in C\}$ is infinite, then p is preperiodic or C is periodic.*

Proof. Suppose (1) holds. Let C be any curve in X and p be a closed point in X such that $f^n(p) \notin I(f)$ for all $n \geq 0$. Assume that the set $\{n \in \mathbb{N} |\ f^n(p) \in C\}$ is infinite. The DML property of (X, f) implies that there are integers $a > 0$ and $b \geq 0$ such that $f^{an+b}(p) \in C$ for all $n \geq 0$. If p is not preperiodic, the set $O_{a,b} := \{f^{an+b}(p) |\ n \geq 0\}$ is Zariski dense in C and $f^a(O_{a,b}) \subseteq O_{a,b}$. It follows that $f^a(C) \subseteq C$, hence C is periodic.

Suppose (2) holds. If the set $S := \{n \in \mathbb{N} |\ f^n(p) \in C\}$ is finite or p is preperiodic, then there is nothing to prove. We may assume that S is infinite and p is not preperiodic. The property (2) implies that C is periodic. There exists an integer $a > 0$ such that $f^a(C) \subseteq C$. We may suppose that $f^i(C) \nsubseteq C$ for $1 \leq i \leq a - 1$. Since p is not preperiodic, there exists $N \geq 0$, such that $f^n(p) \notin (\cup_{1 \leq i \leq a-1} f^i(C)) \cap C$ for all $n \geq N$. So $S \setminus \{1, \cdots, N-1\}$ takes form $\{an + b |\ n \geq 0\}$ where $b \geq 0$ is an integer, and it follows that (X, f) satisfies the DML property. □

Theorem 14.3. *Let X be a smooth surface defined over an algebraically closed field, and $f : X \dashrightarrow X$ be a rational transformation, then the following properties hold.*

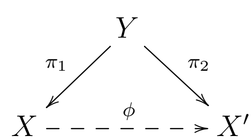
(i) *For any $m \geq 1$, (X, f) satisfies the DML property if and only if (X, f^m) satisfies the DML property.*
(ii) *Suppose U is an open subset of X such that the restriction $f_{|U} : U \to U$ is a morphism. Then (X, f) satisfies the DML property, if and only if $(U, f_{|U})$ satisfies the DML property.*
(iii) *Suppose $\pi : X \to X'$ is a birational morphism between smooth projective surfaces, and $f : X \dashrightarrow X$, $f' : X' \dashrightarrow X'$ are rational maps such that $\pi \circ f = f' \circ \pi$. If the pair (X, f) satisfies the DML property, then (X', f') satisfies the DML property.*
(iv) *Suppose $\pi : X \to X'$ is a birational morphism between smooth projective surfaces, and $f : X \dashrightarrow X$, $f' : X' \dashrightarrow X'$ are* birational *transformations such that $\pi \circ f = f' \circ \pi$. If f' is algebraically stable and the pair (X', f') satisfies the DML property, then (X, f) satisfies the DML property.*

Definition 14.4. Let X be a smooth projective surface defined over an algebraically closed field and $f : X \dashrightarrow X$ be a birational transformation. We say that (X', f') is a birational model of (X, f) if there is a birational map $\pi : X' \dashrightarrow X$ such that

$$f' = \pi^{-1} \circ f \circ \pi.$$

Corollary 14.5. *Let X be a smooth projective surface defined over an algebraically closed field and $f : X \dashrightarrow X$ be an algebraically stable birational transformation such that (X, f) satisfies the DML property. Then all birational models (X', f') of (X, f) satisfy the DML property.*

Proof of Corollary 14.5. Pick Y a desingularization of the graph of f and set π_1, π_2 the projections which make the diagram



to be commutative. Since f is algebraically stable, its lift to Y satisfies the DML property by Theorem 14.3 (iv). We conclude that (X', f') satisfies the DML property by Theorem 14.3 (iii). ☐

Proof of Theorem 14.3. (i). The "only if" part is trivial, so that we only have to deal with the "if" part. We assume that (X, f^m) satisfies the DML property. Let C be a curve in X and p be a point in X such that $f^n(p) \notin I(f)$ for all $n \geq 0$. Suppose that the set $\{n \in \mathbb{N} |\ f^n(p) \in C\}$ is infinite. Since

$$\{n \in \mathbb{N} |\ f^n(p) \in C\} = \cup_{i=0}^{m-1} \{n \in \mathbb{N} |\ f^{nm}(f^i(p)) \in C\},$$

then for some i, the set $\{n \in \mathbb{N} |\ f^{nm}(f^i(p)) \in C\}$ is also infinite. Since (X, f^m) satisfies the DML property, C is periodic or $f^i(p)$ is preperiodic. It follows that C is periodic or p is preperiodic.

(ii). If (X, f) satisfies the DML property, since $f_{|U} : U \to U$ is a morphism, $(U, f_{|U})$ satisfies the DML property.

Conversely suppose that $(U, f_{|U})$ satisfies the DML property. Let C be an irreducible curve in X, p be a closed point in X such that $f^n(p) \notin I(f)$ for all $n \geq 0$ and the set $\{n \in \mathbb{N} | f^n(p) \in C\}$ is infinite. The set $E = X - U$ is a proper closed subvariety of X. If $p \in U$, then we have that $C \not\subseteq E$. Since $(U, f_{|U})$ satisfies the DML property, we have either p is preperiodic or C is periodic. Otherwise, we may assume that for all $n \geq 0$, $f^n(p) \in E$, then the Zariski closure D of $\{f^n(p) |\ n \geq 0\}$, is contained in E. We assume that p is not preperiodic, then $C \subseteq D$. Since D is fixed, we have that C is periodic.

(iii). It is sufficient to treat the case when π is the blowup at a point $q \in X'$. Let C' be a curve in X', p' be a point in X' such that $(f')^n(p') \notin I(f')$ for all $n \geq 0$ and the set $\{n \in \mathbb{N} |\ (f')^n(p') \in C'\}$ is infinite. We assume that p' is not a periodic point, so that for n large enough, $f'^n(p') \neq q$. Replacing p by $f'^m(p')$ for some m large enough, we may assume that $f'^n(p') \neq q$ for all $n \geq 0$. Set $p = \pi^{-1}(p')$ and $C = \pi^{-1}(C')$, then we have $f^n(p) \notin I(f)$ for all $n \geq 0$ and the set

$$\{n \in \mathbb{N} |\ f^n(p) \in C\}$$

is infinite. This implies C and then C' to be periodic.

(iv). Let $C \subseteq X$ be a curve, p be a point in X such that $f^n(p) \notin I(f)$ for all $n \geq 0$ and the set $\{n \in \mathbb{N} |\ f^n(p) \in C\}$ is infinite. We may assume that C is irreducible. Let E be the exceptional locus of π.

Lemma 14.6. *If $C \subseteq E$ and $\pi(C)$ is a point in $I(f')$, then (iv) holds.*

Proof of Lemma 14.6. Set $q := \pi(C) \in I(f')$. Since f' is algebraically stable, we have $q \notin I((f')^{-n})$ and

$$\pi(f^{-n}(C)) = (f')^{-n}(q)$$

for all $n \geq 1$. It follows that $f^{-n}(C)$ is a point or an exceptional curve of π for $n \geq 1$.

If there exists $l \geq 1$ such that $f^{-l}(C)$ is a point, we pick two integers $n_1 > n_2 \geq l$ such that $f^{n_1}(p), f^{n_2}(p) \in C$. Then $f^{n_1 - l}(p) = f^{n_2 - l}(p)$, which implies p to be preperiodic.

Otherwise $f^{-n}(C)$ is an exceptional curve of π, for all $n \geq 0$. Since there are only finitely many irreducible components of E, we have that C is periodic. □

Denote by $K = \pi^{-1}(I(f'))$.

Lemma 14.7. *If there are infinitely many $n \geq 0$ such that $f^n(p) \in K$, then (iv) holds.*

Proof of 14.7. There is an irreducible component F of K such that the set $\{n \geq 0 |\ f^n(p) \in F\}$ is infinite.

If F is a point, then p is preperiodic.

Otherwise F is a curve, then $F \subseteq E$ and $\pi(F) \subseteq I(f')$. Suppose that p is not preperiodic, Lemma 14.6 shows that F is periodic. Then $F' = \cup_{k \geq 0} f^k(F)$ is a curve and $f^n(p) \subseteq F'$ for all $n \geq 0$. If $C \subseteq F'$, then C is periodic. If $C \not\subseteq F'$, then $C \cap F'$ is finite, and this shows that p is preperiodic. □

Lemma 14.8. *If $C \subseteq E$, then (iv) holds.*

Proof. By Lemma 14.7, we may assume that there exists an integer $N \geq 0$, such that $f^n(p) \notin K$ for all $n \geq N$.

Set $q := \pi(C)$. By Lemma 14.6, we assume that $q \notin I(f')$. Then we have

$$\pi(f^{N+l}(p)) = f'^l(\pi(f^N(p)))$$

for $l \geq 0$. It follows that there are infinitely many $l \geq 0$, such that $f'^l(\pi(f^N(p))) = q$. Then q is preperiodic and the obit of $f'^N(q)$ does not meet $I(f')$. Since $\pi(f^n(C)) = f'^n(q)$ for all $n \geq 0$, we have $f^n(C) \subseteq \cup_{k \geq N} \pi^{-1}(f^k(q))$ for all $n \geq N$. Hence ether C is periodic or for some $n \geq 1$, $f^n(C)$ is a point. In the second case, we conclude that p is preperiodic. □

Let $L = K \cup E$.

Lemma 14.9. *If there are infinitely many $n \geq 0$ such that $f^n(p) \in L$, then (iv) holds.*

Proof of Lemma 14.9. There is an irreducible component F of L such that $\{n \geq 0 |\ f^n(p) \in F\}$ is infinite.

If F is a point, then p is preperiodic.

Otherwise F is a curve, then $F \subseteq E$. Suppose that p is not preperiodic, Lemma 14.8 shows that F is periodic. Then $F' = \cup_{k \geq 0} f^k(F)$ is a curve and $f^n(p) \subseteq F'$ for all $n \geq 0$. If $C \subseteq F'$, then C is periodic. Otherwise $C \not\subseteq F'$, we have that $C \cap F'$ is finite and then p is preperiodic. □

We may assume that there is an integer $M \geq 0$, such that $f^n(p) \notin L$ for all $n \geq M$.

If $C \not\subseteq E$, $\pi(C)$ is a curve. For all $l \geq 0$ we have

$$\pi(f^{M+l}(p)) = f'^l(\pi(f^M(p))) \notin I(f').$$

Since (X', f') satisfies the DML property, either $\pi(C)$ is periodic or $\pi(p)$ is preperiodic. When $\pi(C)$ is periodic, we have C is periodic. Otherwise $\pi(p)$ is preperiodic. For any $l \geq 0$, π is invertible on some Zariski neighborhood of the point $f'^l(\pi(f^M(p)))$ and then we conclude that p is peperiodic. □

15. Local dynamical Mordell Lang theorem

The aim of this section is to prove Theorem 11.1. We are in the following situation:

(i) X is a smooth projective surface defined over an arbitrary valued field $(K, |\cdot|)$.
(ii) $f : X \dashrightarrow X$ is a birational transformation defined over K;
(iii) Q is K-point of X such that $Q \in I(f^{-1}) \setminus I(f)$ and $f(Q) = Q$;
(iv) p is K-point of X such that $f^n(p) \notin I(f)$ for all $n \geq 0$;
(v) $f^n(p) \to Q$ as $n \to \infty$ with respect to the topology induced by $|\cdot|$;
(vi) C is a curve in X such that the set $\{n \in \mathbb{N} |\ f^n(p) \in C\}$ is infinite;
(vii) for all $n \geq 0$, $f^n(p) \neq Q$.

We want to prove that C is fixed by f.

Proof of Theorem 11.1. Since $f^n(p) \to Q$ as $n \to \infty$, we may suppose that $f^n(p) \notin I(f^{-1})$ for all $n \geq 0$. If $f^{-1}(C)$ is one point, then $f^n(p) \notin C$ for all $n \geq 0$ which is a contradiction. Then $f^{-1}(C)$ is not a point. By the same reason, $f^{-n}(C)$ is not a point for all $n \geq 0$.

Pick a resolution of f as in the diagram $(*)$ in Section 12.1. Recall Proposition 12.1. There is an infinite sequence $\{n_k\}_{k\geq 0}$ such that $f^{n_k}(p) \in C \setminus \{Q\}$. It follows that $f^{n_k-m}(p) \in f^{-m}(C)$ for k large enough. Setting $k \to \infty$, we get $Q \in f^{-m}(C)$ for all $m \geq 0$.

If $C \neq f^{-1}(C)$, then we have $f^{-m}(C) \neq f^{-m-1}(C)$ for all $m \geq 0$. By computing local intersection at Q, we get

$$(15.1) \quad (f^{-m}(C) \cdot f^{-m-1}(C))_Q = \sum_{x \in \pi_2^{-1}(Q)} (\pi_2^* f^{-m}(C) \cdot \pi_2^{\#} f^{-m-1}(C))_x$$
$$= \sum_{x \in \pi_2^{-1}(Q)} \left(\left(\pi_2^{\#} f^{-m}(C) + \sum_{i=1}^{s} v_{E_i}(f^{-m}(C)) E_i \right) \cdot \pi_2^{\#} f^{-m-1}(C) \right)_x$$

where $E_i, 1 \leq i \leq s$ are irreducible exceptional curves for π_2. Since

$$\mathrm{Supp}(\sum_{i=1}^{s} v_{E_i}(f^{-m}(C)) E_i) = \cup_{1 \leq i \leq s} E_i = \pi_2^{-1}(Q),$$

we have

$$(5.1) = \sum_{x \in \pi_2^{-1}(Q)} (\pi_2^{\#} f^{-m}(C) \cdot \pi_2^{\#} f^{-m-1}(C))_x + \left(\left(\sum_{i=1}^{s} v_{E_i}(f^{-m}(C)) E_i \right) \cdot \pi_2^{\#} f^{-m-1}(C) \right)$$
$$\geq \sum_{x \in \pi_2^{-1}(Q)} (\pi_2^{\#} f^{-m}(C) \cdot \pi_2^{\#} f^{-m-1}(C))_x + 1$$
$$= \big(\sigma(f^{-m-1}(C)) \cdot \sigma(f^{-m-2}(C))\big)_{\sigma(Q)} + 1$$
$$= (f^{-m-1}(C) \cdot f^{-m-2}(C))_Q + 1.$$

It follows that

$$0 < (f^{-m}(C) \cdot f^{-m-1}(C))_Q \leq (f^{-m+1}(C) \cdot f^{-m}(C))_Q - 1 \leq \cdots \leq (C \cdot f^{-1}(C))_Q - m$$

for all $m \geq 0$, which yields a contradiction. So we have $C = f^{-1}(C)$ and then $f(C) = C$. □

Observe that our proof of Theorem 11.1 actually gives

Proposition 15.1. *Let X be a projective surface over an algebraically closed field and $f : X \dashrightarrow X$ be a birational map with a fixed point $Q \in I(f^{-1}) \setminus I(f)$. Then all periodic curves passing through Q are fixed.*

16. The case $\lambda(f) = 1$

In this section, we prove Theorem A in the case $\lambda(f) = 1$. Denote by K an algebraically closed field of characteristic 0.

Recall from [7] and [10], that if $\lambda(f) = 1$, then we are in one of the following two cases:

(1) there exists a smooth projective surface X and an automorphism f' on X such that the pair (X, f') is birationally conjugated to $(\mathbb{A}^2, f)$;
(2) in suitable affine coordinates, $f(x, y) = (ax + b, A(x)y + B(x))$ where A and B are polynomials with $A \neq 0$ and $a \in K^*$, $b \in K$.

The case of automorphism has been treated by Bell, Ghioca and Tucker. Theorem A thus follows from [3, Theorem 1.3] in case (1) and in case (2) where $\deg A = 0$. So in this section we suppose that f takes form

$$f(x, y) = (ax + b, A(x)y + B(x)) \qquad (**)$$

with $A, B \in K[x]$, $\deg A \geq 1$, $a \in K^*$ and $b \in K$.

16.1. **Algebraically stable models.** Any map of the form $(**)$ can be made algebraically stable in a suitable Hirzebruch surface $\mathbb{F}_n$ for some $n \geq 0$. It is convenient to work with the presentation of these surfaces as a quotient by $(\mathbb{G}_m)^2$, as in [17]. By definition, the set of closed points $\mathbb{F}_n(K)$ is the quotient of $\mathbb{A}^4(K) \setminus (\{x_1 = 0 \text{ and } x_2 = 0\} \cup \{x_3 = 0 \text{ and } x_4 = 0\})$ by the equivalence relation generated by

$$(x_1, x_2, x_3, x_4) \sim (\lambda x_1, \lambda x_2, \mu x_3, \mu/\lambda^n x_4)$$

for $\lambda, \mu \in K^*$. We denote by $[x_1, x_2, x_3, x_4]$ the equivalence class of (x_1, x_2, x_3, x_4). We have a natural morphism $\pi_n : \mathbb{F}_n \to \mathbb{P}^1$ given by $\pi_n([x_1, x_2, x_3, x_4]) = [x_1 : x_2]$ which makes $\mathbb{F}_n$ into a locally trivial $\mathbb{P}^1$ fibration.

We shall look at the embedding

$$i_n : \mathbb{A}^2 \hookrightarrow \mathbb{F}_n : (x, y) \mapsto [x, 1, y, 1].$$

Then $\mathbb{F}_n \setminus \mathbb{A}^2$ is union of two lines: one is the fiber at infinity F_∞ of π_n, and the other one is a section of π_n which we denote by L_∞.

Recall that f has the form $(**)$. For each $n \geq 0$, set $d = \max\{\deg A, \deg B - n\}$. By the embedding i_n, the map f extends to a birational transformation

$$f_n : [x_1, x_2, x_3, x_4] \mapsto [ax_1 + bx_2, x_2, A(x_1/x_2)x_2^d x_3 + B(x_1/x_2)x_2^{d+n} x_4, x_2^d x_4]$$

on $\mathbb{F}_n$. For any $n \geq \deg B - \deg A + 1$, we have $d = \deg A$ and

$$I(f_n) = \{[x_1, x_2, x_3, x_4] \in \mathbb{F}_n | x_2 = x_3 = 0\}.$$

The unique curve which is contracted by f_n is $F_\infty = \{x_2 = 0\}$ and its image is $f_n(F_\infty) = [1, 0, 1, 0]$. It implies the following:

Proposition 16.1. *For any integer $n \geq \deg B - \deg A + 1$, f_n is algebraically stable on $\mathbb{F}_n$ and contracts the curve F_∞ to the point* $[1, 0, 1, 0]$.

16.2. **The attracting case.** In the remaining of this section, we fix an integer m such that the extension of f to $\mathbb{F}_m$ is algebraically stable. For simplicity, we write f for the map f_m induced by f on $\mathbb{F}_m$.

Proposition 16.2. *Let $|\cdot|$ be an absolute value on K such that $|a| > 1$. Then $(\mathbb{F}_m, f)$ satisfies the DML property.*

Proof. Since $a \neq 1$, by changing coordinates, we may assume that $f = (ax, A(x)y + B(x))$. Since f contracts the fiber F_∞ to $O := L_\infty \cap F_\infty$, the point O is fixed and the two eigenvalues of df at O are $1/a$ and 0. Since $|a| > 1$, there is a neighbourhood U of O, such that $U \cap I(f) = \emptyset$, $f(U) \subseteq U$ and $f^n \to O$ uniformly on U.

Let C be an irreducible curve in $\mathbb{P}^2_K$ and p be a point in $\mathbb{A}^2_K$ such that the set $\{n \in \mathbb{N} |\ f^n(p) \in C\}$ is infinite. By Lemma 14.3, we may assume that $p \in \mathbb{A}^2_K$ and $C \not\subseteq L_\infty \cup F_\infty$.

If $C \cap F_\infty = \{O\}$, there is an open set V of $\mathbb{P}^1_K$, such that $[1:0] \in V$ and $\pi_m^{-1}(V) \cap C \subseteq U$. Since $|a| > 1$, for n large enough, $f^n(p) \in \pi_m^{-1}(V)$. So there is an integer $n_1 > 0$ such that $f^{n_1}(p) \in U$. Theorem 11.1 implies that the curve C is fixed.

We may assume now that $f^n(C) \cap F_\infty \neq \{O\}$ for all $n \geq 0$.

If $C \cap F_\infty = \emptyset$, then C is a fiber of the rational fibration $\pi_m : \mathbb{F}_m \to \mathbb{P}^1$. Since $\{n \in \mathbb{N} |\ f^n(p) \in C\}$ is infinite, the curve C is fixed.

Finally assume that $f^n(C) \cap F_\infty \neq \emptyset$ for all $n \geq 0$. Since f contracts F_∞ to O, we have

$$f^n(C) \cap I(f) \neq \emptyset,$$

and we conclude by Theorem 11.2 that C is periodic in this case. □

16.3. **The general case.**

Proposition 16.3. *The pair $(\mathbb{F}_m, f)$ satisfies the DML property.*

Proof. Let C be a curve in $\mathbb{F}_m$, and p be a point in $\mathbb{A}^2_K$ such that the set $\{n \geq 0 | f^n(p) \in C\}$ is infinite. We may assume that the transcendence degree of K is finite, since we can find a subfield of K such that it has finite transcendence degree and f, C and p are all defined over this subfield.

In the case f acts on the base as the identity, the proposition holds trivially. Assume that it is not that case. Let $O = L_\infty \cap F_\infty$. As in the proof of Proposition 16.2, we only have to consider the case $C \cap F_\infty = O$.

If a is a root of unity, we may replace f by f^n for some integer $n > 0$ and assume that $a = 1$ and $b = 1$. Since the transcendence degree of K is finite, we may embed K in the field of complex numbers $\mathbb{C}$. Let $|\cdot|$ be the standard absolute value on $\mathbb{C}$. Since f contracts F_∞ to O, there is a neighborhood U of O with respect to the usual euclidian topology such that for all point $q \in U \cap \{(x, y) \in \mathbb{C}^2 |\ \mathrm{Re}(x) > 0\}$, we have $\lim_{n\to\infty} f^n(q) = O$. Since $C \cap F_\infty = O$, there exists $M > 0$, such that $C \cap \{(x, y) |\ \mathrm{Re}(x) > M\} \subseteq U$ and we conclude by using Theorem 11.1 in this case.

If a is an algebraic number over $\mathbb{Q}$ and is not a root of unity, by [20, Theorem 3.8] there exists an absolute value $|\cdot|_v$ (either archimedean or non-archimedean) on $\overline{\mathbb{Q}}$ such that $|a|_v > 1$. This shows that $(\mathbb{F}_m, f)$ satisfies the DML property by Proposition 16.2.

If a is not an algebraic number over $\mathbb{Q}$, we claim that there exists a field embedding $\iota : K \hookrightarrow \mathbb{C}$ such that $|\iota(a)| > 1$, and we may conclude again by using Proposition 16.2.

It thus remains to prove the claim. There is a subring R of K which is finitely generated over $\overline{\mathbb{Q}}$, such that f, C and p are all defined over R. There is an integer $l > 0$, such that $R = \overline{\mathbb{Q}}[t_1, \cdots, t_l]/I$, where I is a prime ideal of $\overline{\mathbb{Q}}[t_1, \cdots, t_l]$. It induces an embedding $\operatorname{Spec} R := V \subseteq \mathbb{A}^l_{\overline{\mathbb{Q}}}$. We set

$$V_{\mathbb{C}} := V \times_{\operatorname{Spec} \overline{\mathbb{Q}}} \operatorname{Spec} \mathbb{C} \subseteq \mathbb{A}^l_{\mathbb{C}}.$$

For any polynomial $F \in \overline{\mathbb{Q}}[t_1, \cdots, t_l] \setminus I$, we also define $V_F := \{F = 0\}$. Then $V_{\mathbb{C}} \setminus V_F$ is a dense open set in the usual euclidian topology. Since $\overline{\mathbb{Q}}[t_1, \cdots, t_l] \setminus I$ is countable, the set $V_{\mathbb{C}} \setminus (\cup_{F \in \overline{\mathbb{Q}}[t_1, \cdots, t_l] \setminus I} V_F)$ is dense. Interpreting a a nonconstant holomorphic function on $V_{\mathbb{C}}$, we see that there exists an open set $W \subseteq V_{\mathbb{C}}$ such that $|a| > 1$ on W.

Pick a closed point $(s_1, \cdots, s_l) \in W \setminus (\cup_{F \in \overline{\mathbb{Q}}[t_1, \cdots, t_l] \setminus I} V_F)$ and consider the unique morphism $\iota : R = \mathbb{Q}[t_1, \cdots, t_l]/I \to \mathbb{C}$ sending t_i to s_i. This morphism is in fact an embedding. We may extend it to an embedding of K as required. $\square$

17. Upper bound on heights when $\lambda(f) > 1$

17.1. **Absolute values on fields.** ([20]) Set $\mathcal{M}_{\mathbb{Q}} := \{|\cdot|_\infty$ and $|\cdot|_p$ for all prime p$\}$ where $|\cdot|_\infty$ is the usual absolute value and $|\cdot|_p$ is the p-adic absolute value defined by $|x| := p^{-\operatorname{ord}_p(x)}$ for $x \in \mathbb{Q}$.

Let $K/\mathbb{Q}$ be a number field. The *set of places* on K is denoted by $\mathcal{M}_K$ and consists of all absolute values on K whose restriction to $\mathbb{Q}$ is one of the places in $\mathcal{M}_{\mathbb{Q}}$. Further we denote by $\mathcal{M}_K^\infty$ the set of archimedean places; and by $\mathcal{M}_K^0$ the set of nonarchimedean places.

When v is archimedean, there exists an embedding $\sigma_v : K \hookrightarrow \mathbb{C}$ (or $\mathbb{R}$) such that $|\cdot|_v$ is the restriction to K of the usual absolute value on $\mathbb{C}$ (or $\mathbb{R}$).

Similarly, we introduce the set of places on function fields.

Let C be a a smooth projective curve defined over an algebraically closed field k and $L := k(C)$ be the function field of C. The set of places on L, denoted by $\mathcal{M}_L$ consists of all absolute values of the form:

$$|\cdot|_p : x \mapsto e^{\operatorname{ord}_p(x)}$$

for any $x \in L$ and any closed point $p \in C$.

Let K/L be a finite field extension. The set of places on K is denoted by $\mathcal{M}_K$ and consists of all absolute values on K whose restriction to L is one of the places in $\mathcal{M}_L$. In this case, all the places in $\mathcal{M}_K$ are nonarchimedean. Set $\mathcal{M}_K^0 = \mathcal{M}_K$ and $\mathcal{M}_K^\infty = \emptyset$ for convenience.

Let K/L be a finite field extension where $L = \mathbb{Q}$ or a function field $k(C)$ of a curve C. For any place $v \in \mathcal{M}_K$, denote by $n_v := [K_v : L_v]$ the local degree of v then we have the product formula

$$\prod_{v\in\mathcal{M}_K} |x|_v^{n_v} = 1$$

for all $x \in K^*$.

For any $v \in \mathcal{M}_K$, denote by $O_v := \{x \in K|\ |x|_v \leq 1\}$ the ring of v-integers. In the number field case, we also denote by $O_K := \{x \in K|\ |x|_v \leq 1 \text{ for all } v \in \mathcal{M}_K^0\}$ the ring of integers.

17.2. **Basics on Heights.** We recall some basic properties of heights that are needed in the proof of Theorem A, see [18] or [19] for detail.

In this section, we set $L = \mathbb{Q}$ or $k(C)$ the function field of a curve C defined over an algebraically closed field k. Denote by $\overline{L}$ its algebraic closure.

Proposition-Definition 17.1. Let K/L be a finite field extension. Let $p \in \mathbb{P}^n(K)$ be a point with homogeneous coordinate $p = [x_0 : \cdots : x_n]$ where $x_0, \cdots, x_n \in K$. The *height* of p is the quantity

$$H_{\mathbb{P}^n}(p) := (\prod_{v\in\mathcal{M}_K} \max\{|x_0|_v, \cdots, |x_n|_v\}^{n_v})^{1/[K:L]}.$$

The height $H_{\mathbb{P}^n}(p)$ depends neither on the choice of homogeneous coordinates of p, nor on the choice of a field extension K which contains p.

When $L = k(C)$, we have a geometric interpretation of the height $H_{\mathbb{P}^n}(p)$. Observe that $\mathbb{P}^n_L$ is the generic fiber of the trivial fibration $\pi : \mathbb{P}^n_C := \mathbb{P}^n \times C \to C$. We set $s_p : D \to \mathbb{P}^n_C$ the normalization of the Zariski closure of p in $\mathbb{P}^n_C$. Then we have

$$H_{\mathbb{P}^n}(p) = e^{\deg(s_p^* O_{\mathbb{P}^n_C}(1))/\deg(\pi\circ s_p)}.$$

Proposition 17.2. *Let $f : \mathbb{P}^n_{\overline{L}} \dashrightarrow \mathbb{P}^m_{\overline{L}}$ be a rational map and X be a subvariety of $\mathbb{P}^n_{\overline{L}}$ such that $I(f) \cap X$ is empty and the restriction $f|_X$ is finite of degree d onto its image $f(X)$.*

Then there exist $A > 0$ such that for all point $p \in X(\overline{L})$, we have

$$\frac{1}{A} H_{\mathbb{P}^n}(p)^d \leq H_{\mathbb{P}^m}(f(p)) \leq A H_{\mathbb{P}^n}(p)^d.$$

Proposition 17.3 (Northcott Property)**.** *Let $K/\mathbb{Q}$ be a number field, and $B > 0$ be any constant. Then the set*

$$\{p \in \mathbb{P}^n(K)|\ H_{\mathbb{P}^n}(p) \leq B\}$$

is finite.

Remark 17.4. The Northcott Property does not hold in the case $K = k(C)$ when k is not a finite field. For example, the set

$$\{p \in \mathbb{P}^n(k(t))|\ H_{\mathbb{P}^n}(p) = 0\} = \{[x : y]|\ (x, y) \in k^2 \setminus \{(0, 0)\}\}$$

is infinite.

17.3. **Upper bounds on heights.** Let K be a number field or a function field of a smooth curve over an algebraically closed field k'. Let $f : \mathbb{A}^2_{\overline{K}} \to \mathbb{A}^2_{\overline{K}}$ be any birational polynomial morphism defined over $\overline{K}$ and assume that $\lambda(f) > 1$.

According to Theorem 12.4, we may suppose that there exists a compactification X of $\mathbb{A}^2_{\overline{K}}$, a closed point $Q \in X \setminus \mathbb{A}^2_{\overline{K}}$ such that f extends to a birational transformation $\widetilde{f}$ on X which satisfies the following properties:

(i) $\widetilde{f}$ is algebraically stable on X;
(ii) there exists a closed point $Q \in X \setminus \mathbb{A}^2$ fixed by $\widetilde{f}$, such that $d\widetilde{f}(Q) = 0$;
(iii) $\widetilde{f}(X \setminus \mathbb{A}^2) = Q$.

To simplify, we write $f = \widetilde{f}$ in the rest of the paper. We fix an embedding $X \subseteq \mathbb{P}^N_{\overline{K}}$. Let C be an irreducible curve in X whose intersection with $\mathbb{A}^2_{\overline{K}}$ is non empty.

Proposition 17.5. *Suppose that C is not periodic and $C \setminus \mathbb{A}^2_{\overline{K}} = \{Q\}$. Then there exists a number $B > 0$ such that for any point $p \in C(K)$ for which the set $\{n \in \mathbb{N} |\ f^n(p) \in C\}$ is infinite, we have $H_{\mathbb{P}^N}(p) \leq B$.*

Proof. Assume that X, f, C and Q are all defined over K and $Q = [1 : 0 : \cdots : 0] \in \mathbb{P}^N_K$. We can extend f to a rational morphism on $\mathbb{P}^N$ which is regular at Q. Then there exists an element $a \in K^*$ and $F_i \in (x_1, \cdots , x_N)K[x_0, \cdots , x_N]$ for $i = 0, \cdots , N$ such that

$$f([1 : x_1 : \cdots : x_N]) = [a + F_0 : F_1 : \cdots : F_N]$$

for any $[1 : x_1 : \cdots : x_N] \in X$. Since f is regular at Q and $a \neq 0$, there is a finite set $S \subseteq \mathcal{M}^0_K$ such that for any $v \in \mathcal{M}^0_K \setminus S$, we have $|a|_v = 1$ and all coefficients of f are defined in O_v. Recall that we may endow X with a metric d_v, see Section 12.3.

For any $v \in \mathcal{M}^0_K \setminus S$, set $r_v := 1$ and $U_v := \{x \in X(K) |\ d_v(x, Q) < 1\}$. Since $df(Q) = 0$, we see that for all $x \in U_v$, $d_v(f(x), Q) \leq d_v(x, Q)^2$, hence

$$\lim_{n\to\infty} f^n(x) = Q.$$

For any $v \in S$, set $r_v := |a|_v$ and $U_v := \{x \in X(K) |\ d_v(x, q) < r_v\}$. We see that for all $x \in U_v$, $d_v(f(x), Q) \leq d_v(x, Q)^2/r_v$, and again it follows that

$$\lim_{n\to\infty} f^n(x) = Q.$$

For any $v \in \mathcal{M}^\infty_K$, since $df(Q) = 0$, there is $r_v > 0$ such that for any $x \in U_v := \{x \in X(K) |\ d_v(x, q) < r_v\}$ we have $f(x) \subseteq U_v$ and

$$\lim_{n\to\infty} f^n(x) = Q.$$

If $p \in \cup_{v\in\mathcal{M}_K} U_v$, Theorem 11.1 shows that C is periodic and this contradicts our assumption. In other words, we need to estimate the height of a given point

$$p \in C(K) \setminus \cup_{v\in\mathcal{M}_K} U_v.$$

If C intersects the line at infinity only at the point Q, then we may directly estimate the height of p given by the embedding of C into $\mathbb{P}^N_K$. Since we do not assume that this is the case, we shall work first with a height induced by a divisor on C given by the divisor Q, and then estimate $h_{\mathbb{P}^N}(p)$ using Proposition 17.2. To do so, let $i : \widetilde{C} \to C \subseteq X$ be the normalization of C and pick a point $Q' \in i^{-1}(Q)$. There is a positive integer l such that lQ' is a very ample divisor of $\widetilde{C}$. So there is an embedding $j : \widetilde{C} \hookrightarrow \mathbb{P}^M$ for some $M > 0$ such that

$$Q' = [1 : 0 : \cdots : 0] = H_\infty \cap \widetilde{C}$$

where $H_\infty = \{x_M = 0\}$ is the hyperplane at infinity. Let

$$g : \widetilde{C} \to \mathbb{P}^1$$

be a morphism sending $[x_0 : \cdots : x_M] \in \widetilde{C}$ to $[x_0 : x_M] \in \mathbb{P}^1$. It is well defined since $\{x_0 = 0\} \cap H_\infty \cap \widetilde{C} = \emptyset$. Then g is finite and

$$g^{-1}([1 : 0]) = H_\infty \cap \widetilde{C} = [1 : 0 \cdots : 0].$$

By base change, we may assume that $\widetilde{C}, i, j, g$ are all defined over K.

In the function field case, there is a smooth projective curve D such that $K = k'(D)$; and in the number field case, we set $D = \operatorname{Spec} O_K$.

We consider the irreducible scheme $\widetilde{\mathcal{C}} \subseteq \mathbb{P}^M_D$ over D whose generic fiber is $\widetilde{C}$ and the irreducible scheme $\mathcal{X} \subseteq \mathbb{P}^N_D$ over D whose generic fiber is X. Then i extends to a map $\iota : \widetilde{\mathcal{C}} \dashrightarrow \mathcal{X}$ over D birationally to its image. For any $v \in \mathcal{M}_K^0$, let

$$\mathfrak{p}_v = \{x \in O_v |\ v(x) > 0\}$$

be a prime ideal in O_v. There is a finite set T consisting of those places $v \in \mathcal{M}_K^0$ such that ι is not regular along the special fibre $\widetilde{C}_{O_v/\mathfrak{p}_v}$ at $\mathfrak{p}_v \in D$ or $\widetilde{C}_{O_v/\mathfrak{p}_k} \cap H_{\infty, O_v/\mathfrak{p}_v} \neq \{[1 : 0 : \cdots : 0]\}$.

For any $v \in \mathcal{M}_K^0 \setminus T \cup S$, observe that we have

$$\begin{aligned} V_v :=& \{[1 : x_1 : \cdots : x_M] \in \widetilde{C}(K) |\ |x_i|_v < 1, i = 1, \cdots, M\} \\ =& \{[1 : x_1 : \cdots : x_M] \in \widetilde{C}(K) |\ |x_M|_v < 1\} = g^{-1}(\Omega_v) \cap \widetilde{C}(K) \end{aligned}$$

with $\Omega_v := \{[1 : x] \in \mathbb{P}^1(K) |\ |x|_v < t_v\}$ and $t_v := 1$.

For any $v \in T \cup S \cup \mathcal{M}_K^\infty$, by the continuity of i, there is $s_v > 0$ such that

$$i(V_v) \in U_v$$

where $V_v = \{[1 : x_1 : \cdots : x_M] \in \widetilde{C}(K) |\ |x_i|_v < s_v, i = 1, \cdots M\}$. Since $g^{-1}([1 : 0]) = \{[1 : 0 : \cdots : 0]\}$, there exists $t_v > 0$, such that

$$g^{-1}(\Omega_v) \cap \widetilde{C} \subseteq V_v$$

where $\Omega_v = \{[1 : x] \in \mathbb{P}^1(K) |\ |x|_v < t_v\}$.

We need to find an upper bound for the height of points in $C(K) \setminus \cup_{v \in \mathcal{M}_K} U_v$. Since the set $\operatorname{Sing}(C)$ of singular points of C is finite, we only have to bound the height of points in $C(K) \setminus (\operatorname{Sing}(C) \cup_{v \in \mathcal{M}_K} U_v)$.

Let p be a point in $C(K) \setminus (\operatorname{Sing}(C) \cup_{v \in \mathcal{M}_K} U_v)$. Observe that $i^{-1}(p) \in \widetilde{C}(K)$ and $x := j(i^{-1}(p))$ is also defined over K. We have $x \notin V_v$ hence $y := g(x) \notin \Omega_v$ for all $v \in \mathcal{M}_K$.

For any $y = [y_0 : y_1] \in \mathbb{P}^1(K) \setminus (\cup_{v\in\mathcal{M}_K}\Omega_v)$, we have $|y_1/y_0|_v \geq t_v$ for all v. We get the following upper bound

$$\begin{aligned} H_{\mathbb{P}^1}(y)^{[K:\mathbb{Q}]} &= \prod_{v\in\mathcal{M}_K} \max\{|y_0|_v, |y_1|_v\}^{n_v} \\ &\leq \prod_{v\in\mathcal{M}_K} \max\{|y_1|_v/t_v, |y_1|_v\}^{n_v} \\ &= \prod_{v\in\mathcal{M}_K} |y_1|_v^{n_v} \prod_{v\in\mathcal{M}_K} \max\{1, 1/t_v\}^{n_v} \\ &= \prod_{v\in\mathcal{M}_K} \max\{1, 1/t_v\}^{n_v} =: B' < \infty. \end{aligned}$$

By Proposition 17.2 applied to $g : \widetilde{C} \hookrightarrow \mathbb{P}^M$ and $i : \widetilde{C} \to \mathbb{P}^N$, we get $H_{\mathbb{P}^N}(p) \leq B$ for some constant B independent on the choice of p as we require. □

18. Proof of Theorem A

Let C be a curve in $\mathbb{A}^2_K$. We want to show that for any point $p \in \mathbb{A}^2(K)$ such that the set

$$\{n \in \mathbb{N}|\ f^n(p) \in C\}$$

is infinite, then either p is preperiodic or C is periodic.

According to Section 16, we may assume that $\lambda(f) > 1$. As in Section 17.3, we use Theorem 12.4 to get a compactification X of $\mathbb{A}^2_K$. For simplicity, we also denote by f the map induced by f on X. There exists $n \geq 1$ such that f^n contracts $X \setminus \mathbb{A}^2_K$ to a superattracting fixed point $Q \in X \setminus \mathbb{A}^2_K$. We extend C to a curve in X. Suppose that C is not periodic. By Theorem 11.2, we may assume that $C(K) \setminus \mathbb{A}^2(K) = \{Q\}$. Finally we fix an embedding $X \hookrightarrow \mathbb{P}^N_K$ for some $N \geq 1$.

We first treat the case $K = \overline{\mathbb{Q}}$.

There is a number field K' such that both f and p are defined over K'. Then $f^n(p) \in \mathbb{A}^2(K')$ for all $n \geq 0$.

Proposition 17.5 and the Northcott Property imply that the set $\{f^n(p)|\ n \geq 0\} \cap C$ is finite. Since the set $\{n \in \mathbb{N}|\ f^n(p) \in C\}$ is infinite, there exists $n_1 > n_2 > 0$ such that $f^{n_1}(p) = f^{n_2}(p)$. We conclude that p is preperiodic.

Next we consider the general case of an algebraically closed field K of characteristic 0.

By replacing K by an algebraically closed subfield over which p, C and f are all defined, we may suppose that the transcendence degree tr.d.$K/\mathbb{Q}$ of K over $\mathbb{Q}$ is finite. We argue by induction on tr.d.$K/\mathbb{Q}$.

If tr.d.$K/\mathbb{Q} = 0$, then $K = \overline{\mathbb{Q}}$ and we are due by what precedes.

If tr.d.$K/\mathbb{Q} \geq 1$, then there is an algebraically closed subfield k of K such that tr.d.$k/\mathbb{Q} =$ tr.d.$K/\mathbb{Q} - 1$.

There is a smooth projective curve D over k, such that X, f, p, Q and C are defined over the function field $k(D)$ of D. Observe that $K = \overline{k(D)}$.

We consider the irreducible scheme

$$\pi : \mathcal{X} \subseteq \mathbb{P}^N_D \to D$$

over D whose generic fiber is X and $\mathcal{C} \subseteq \mathbb{P}^N_D$ the Zariski closure of C in $\mathcal{X}$.

The map f extends to a birational map $f' : \mathcal{X} \dashrightarrow \mathcal{X}$ over D. For any $x \in D$, denote by X_x and C_x the fiber of $\mathcal{X}$ and $\mathcal{C}$ at x respectively, and denote by f_x the restriction of map f' to the fiber X_x.

Proposition 17.5 implies that there is a number $M \geq 0$ such that for all $n \geq 0$ either $f^n(p) \notin C$ or $H_{\mathbb{P}^N}(f^n(p)) \leq M$.

A point $s \in X(k(D))$ is associated to its Zariski closure in $\mathcal{X}$ which is a section of $\pi : \mathcal{X} \to D$. For simplicity, we also write s for this section. Then the height of s is

$$H_{\mathbb{P}^N}(s) = e^{(s \cdot L)}$$

where $L := O_{\mathbb{P}^N_D}(1)$.

For any section s, observe that π induces an isomorphism from s to the curve D. We may consider the Hilbert polynomial

$$\chi(L^{\otimes n}, s) = 1 - g(s) + n(s \cdot L) = 1 - g(D) + n \log H(s).$$

It follows that there is a quasi-projective k-variety M_H that parameterizes the sections s of π such that $H_{\mathbb{P}^N}(s) \leq M$ (see [5]).

Let T_1 be the set of points $x \in D$ such that f_x is birational and $I(f_x^{-1}) \cap I(f_x) \neq \emptyset$. Observe that T_1 is finite. Let T_2 be the set of the points $x \in D \setminus T_1$, such that C_x is fixed. Since C is not fixed, T_2 is finite. Because k is algebraically closed, $D \setminus (T_1 \cup T_2)$ is infinite. For any point $x \in D$, denote by $p_x : M_H \to X_x$ the map sending s to $s(x)$. Pick a sequence of distinct points $\{x_i\}_{i \geq 0} \subseteq D \setminus (T_1 \cup T_2)$. For any $l \geq 1$, let

$$p_l = \prod_{i=1}^{l} p_{x_i} : M_H \to \prod_{i=1}^{l} X_{x_i}.$$

Observe that any two points $s_1, s_2 \in M_H$ are equal if and only if $p_i(s_1) = p_i(s_2)$ for all $i \geq 0$.

We claim the following lemma, and prove it later.

Lemma 18.1. *Let X be any reduced quasi-projective variety over an algebraically closed field k. For any $i \geq 1$, let $\pi_i : X \to Y_i$ be a morphism. If for any difference points $x_1, x_2 \in X$, there exists $i \geq 0$, such that $\pi_i(x_1) \neq \pi_i(x_2)$, then for l large enough the map*

$$p_l = \prod_{i=1}^{l} \pi_i : X \to \prod_{i=1}^{l} Y_i$$

is finite.

By Lemma 18.1, there is an integer L large enough, such that the map p_L is finite. By Proposition 15.1, C_{x_i} is not periodic for all $i \geq 1$. The set $\mathsf{N} := \{n \geq 0 | f^n(p) \in C\}$ is infinite, enumerate $\mathsf{N} = \{n_1 < n_2 < \cdots < n_i < n_{i+1} < \cdots\}$. For any $i \geq 0$, there exists $s_i \in M_H$ such that $s_i = f^{n_i}(p)$. By the induction hypothesis, we know that $s_{n_0}(x_i) = f^{n_0}(p)(x_i)$ is a preperiodic point of f_{x_i} for any $1 \leq i \leq L$. Then the orbit G_i of $p(x_i)$ in X_{x_i} is finite. So the set

$$p_L(\{s_i\}_{i\geq 0}) \subseteq \prod_{i=0}^{L} G_i$$

is finite. Since p_L is finite, then we have $\{s_i\}_{i\geq 0}$ is finite. Then there is $i_1 > i_2$ such that $s_{i_1} = s_{i_2}$, and $f^{n_{i_1}}(p) = f^{n_{i_2}}(p)$. Then p is preperiodic. □

Proof of Lemma 18.1. We prove this lemma by induction on the dimension of X.

If $\dim X = 0$, then the result is trivial.

If $\dim X > 0$, we may assume that X is irreducible. We pick any point $x \in X$, and let F_l be the fiber of p_l which contains x. Observe that

$$F_{l+1} \subseteq F_l,$$

so that there is an integer $L' \geq 1$, such that for any $L \geq L'$,

$$F_L = \cap_{l\geq 0} F_l.$$

Since for any point $x_1 \in X - \{x\}$, there exists $i \geq 0$, such that $\pi_i(x_1) \neq \pi_i(x)$, we have

$$F_L = \cap_{l\geq 0} F_l = \{x\},$$

so that

$$\dim X - \dim p_L(X) \leq \dim F_L = 0.$$

In particular p_L is generically finite. It means that there exists an open set U of $p_L(X)$, such that $p_L : p_L^{-1}(U) \to U$ is finite. Set $X' = X - p_L^{-1}(U)$, then we have $\dim X' \leq \dim X - 1$.

By the induction hypothesis, there is $L'' \geq L'$, such that for any $L \geq L''$, $p_L|_{X'}$ is finite and then p_L is finite. □

Remark 18.2. With a little modification, this proof of Theorem A still works for birational polynomial endomorphisms f on the affine plan defined over fields of arbitrary characteristic satisfying $\lambda_1 f > 1$.

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Part 3

Intersection of valuation rings in $k[x, y]$

19. Introduction

Let $R := k[x,y]$ denote the ring of polynomials in two variables over an algebraically closed field k. Given any finite set of valuations S on R that are trivial on k, we define $R_S = \cap_{v\in S}\{P \in R,\, v(P) \geq 0\}$ as the intersection of the valuation rings of the elements in S with R. We obtain in this way a k-subalgebra of R, and it is a natural question to ask for the transcendence degree of the fraction field of R_S over k which is an integer $\delta(S) \in \{0,1,2\}$.

Our main result is the construction of a symmetric matrix $M(S)$ whose signature characterizes the case when $\delta(S) = 2$. We should mention that when all valuations in S are divisorial, this matrix $M(S)$ is the same as the matrix $\widetilde{\mathcal{M}}$ in [7, Corollary 4.9].

As we shall see below, this construction is based on the analysis developped by C. Favre and M. Jonsson [4] on the tree of normalized rank 1 valuations centered at infinity on R. In the case S consists only of divisorial valuations, $M(S)$ can however be defined using classical intersection theory on an appropriate projective compactification of the affine plane, and we shall explain that one can recover in this way recent results by Schroer [11] and Mondal [9].

To get some insight into the problem, let us now describe a couple of examples. We first observe that if S_1, S_2 are two finite sets of valuations satisfying $S_1 \subseteq S_2$, then we have $R_{S_2} \subseteq R_{S_1}$. Also it is only necessary to consider valuations v that are centered at infinity in the sense that R is not contained in the valuation ring of v.

We first recall the definition of a monomial valuation. Given $(s,t) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we denote by $v_{s,t} : R \to \mathbb{R}$ the rank 1 valuation defined by

$$v_{s,t}\left(\sum_{i,j\geq 0} a_{i,j}x^iy^j\right) := \min\{si+tj |\ a_{i,j} \neq 0\}\ . \tag{19.1}$$

The valuation $v_{s,t}$ is centered at infinity iff $\min\{s,t\} < 0$, and one immediately checks that $R_{\{v_{s,t}\}} = k$ when $\max\{s,t\} < 0$ so that $\delta(\{v_{s,t}\}) = 0$ in this case. This happens in particular when $(s,t) = (-1,-1)$ that is $\delta(\{-\deg\}) = 0$.

Fix the compactification $\mathbb{A}^2_k \subset \mathbb{P}^2_k$, and write $L_\infty = \mathbb{P}^2_k \setminus \mathbb{A}^2_k$ for the line at infinity. Recall that a polynomial $P \in R$ is said to have one place at infinity, if the closure of $P = 0$ intersects L_∞ at a single point and the germ of curve it defines at that point is analytically irreducible. If P has one place at infinity, it follows from a theorem of Moh [13] that all curves $\{P = \lambda\}$ have one place at infinity. This pencil thus defines a rank 1 (divisorial) valuation $v_{|P|}$ sending $Q \in R$ to $v_{|P|}(Q) := -\#\{P^{-1}(\lambda) \cap Q^{-1}(0)\}/\deg P$ for λ generic. One has in this case $R_{\{v_{|P|}\}} = k[P]$, hence $\delta(\{v_{|P|}\}) = 1$.

To get examples of a finite family valuations such that $\delta = 2$, it is necessary to choose valuations that are far enough from $-\deg$. A first construction arises as follows. Pick $s,t \in \mathbb{R}^2$ such that $s < 0 < t$ and let m be any integer larger than $|s|/t$. Since $k[xy^m, y] \subset R_{v_{s,t}}$ it follows that $\delta(\{v_{s,t}\}) = 2$.

Next choose $\{s_i\}_{1\le i\le m}$ any finite set of branches based at points lying on L_∞ of algebraic curves defined in $\mathbb{A}^2$ by equations $\{P_i = 0\}$. Let v_i be the rank 2 valuation on R associated to the branch s_i. Then one checks that $(P_1 \cdots P_m).R \subset R_{\{v_1,\cdots,v_m\}}$ so that $\delta(\{v_1,\cdots,v_m\}) = 2$.

A first (simple) characterization of the case $\delta(S) = 2$ is as follows.

Theorem 19.1. *Let S be any finite set of rank one valuations on $R = k[x,y]$ that are trivial on k^*. Then the transcendence degree $\delta(S)$ of the fraction field of the intersection of R with the valuation rings of the valuations in S is equal to 2 iff there exists a polynomial $P \in R$ satisfying $v(P) > 0$ for all $v \in S$.*

We now describe more precisely our main result. Since the construction of our matrix $M(S)$ relies on the fine tree structure of the space of normalized rank 1 valuations centered at infinity (see Section 20), we first explain our main theorem in the simplified (yet important) situation when all valuations are divisorial.

Now pick any proper modification $\pi : X \to \mathbb{P}^2$ that is an isomorphism above the affine plane with X a smooth projective surface. Let $\{E_0, E_1, \cdots, E_m\}$ be the set of all irreducible components of $X \setminus \mathbb{A}^2_k$ with E_0 the strict transform of L_∞, and S be a subset of $\{\mathrm{ord}_{E_0}, \mathrm{ord}_{E_1}, \cdots, \mathrm{ord}_{E_m}\}$.

Since the intersection form on the divisors E_i's is non-degenerate, for each i, there exists a unique divisor $\check{E}_i$ supported at infinity such that $(\check{E}_i \cdot E_j) = \delta_{i,j}$ for all i, j. Observe that $(\check{E}_0 \cdot \check{E}_0) = +1 > 0$.

Finally we define $M(S)$ to be the symmetric matrix whose entries are given by $[(\check{E}_i \cdot \check{E}_j)]_{1\le i,j\le m}$.

Our main theorem in the case of divisorial valuations reads as follows.

Theorem 19.2. *Given any finite set of divisorial valuations S on R that are centered at infinity, we have $\delta(S) = 2$ if and only if the matrix $M(S)$ is negative definite.*

By the Hodge index theorem, the matrix $M(S)$ is negative definite if and only if $\chi(S) := (-1)^m \det M(S) > 0$.

When S is reduced to a singleton, Theorem 19.2 is due to P. Mondal, see [9, Theorem 1.4].

To treat the case of not necessarily divisorial valuations we need to briefly recall some facts on the valuation tree as defined by C. Favre and M. Jonsson (see Section 20 for details).

We denote by V_∞ the set of functions

$$v : k[x,y] \to \mathbb{R} \cup \{+\infty\}$$

that satisfy the axiom of valuations $v(PQ) = v(P) + v(Q)$, and $v(P+Q) \ge \min\{v(P), v(Q)\}$ and normalized by $\min\{v(x), v(y)\} = -1$. However, we allow v to take the value $+\infty$ on a non-constant polynomial. The set V_∞ is a compact topological space when equipped with the topology of the pointwise convergence. It can be also endowed with a natural partial order relation given by $v \le v'$ if and only if $v(P) \le v'(P)$ for all $P \in R$. The unique minimal point for that order relation is $-\deg$, and V_∞ carries a tree structure in the sense that for any v' the set $\{v \in V_\infty | -\deg \le v \le v'\}$ is isomorphic as a poset to a segment in $\mathbb{R}$ with its standard order relation. In particular, one may define the minimum $v \wedge v'$ of any two valuations $v, v' \in V_\infty$.

There is a canonical way to associate an element $\bar{v} \in V_\infty$ to a given valuation v on R that is trivial on k. When v has rank 1, we may assume it takes its values in $\mathbb{R}$, and $\bar{v}$ is the unique valuation that is proportional to v and normalized by $\min\{\bar{v}(x), \bar{v}(y)\} = -1$. For instance when E is an irreducible component of $\pi^{-1}(L)$ for some proper modification $\pi : X \to \mathbb{P}^2$ as above, then we define $b_E := -\min\{\mathrm{ord}_E(x), \mathrm{ord}_E(y)\}$, and we have $v_E = \frac{1}{b_E}\mathrm{ord}_E \in V_\infty$. When v has rank 2 and is associated to a branch s at infinity of an irreducible curve at infinity C in $\mathbb{A}^2$, then $\bar{v}(P)$ is the local intersection number of s with the divisor of P with the convention that $\bar{v}(P) = +\infty$ when P vanishes on C. Finally when v has rank 2 and its valuation ring contains the valuation ring of a divisorial valuation centered at infinity, we set $\bar{v}$ to be this divisorial valuation.

The skewness function $\alpha : V_\infty \to [-\infty, 1]$ is the unique upper semicontinuous function on V_∞ that is decreasing along any segment starting from $-\deg$, and that satisfies $\alpha(v_E) = b_E^{-2}(\check{E} \cdot \check{E})$ for any divisorial valuation (in the notation introduced above). On the other hand, $\alpha(v) = -\infty$ when v is associated to a branch at infinity of an algebraic curve in $\mathbb{A}^2$.

Now given any finite subset $S = \{v_1, \cdots, v_m\}$ of valuations centered at infinity and trivial on k, we let $\bar{S} = \{\bar{v},\, v \in S\} \subset V_\infty$ and define

$$M(\bar{S}) := [\alpha(\bar{v}_i \wedge \bar{v}_j)]_{1 \le i,j \le m}. \tag{19.2}$$

This is a symmetric matrix with entries in $\mathbb{R} \cup \{-\infty\}$.

As above, we then have

Main Theorem. *Given any finite set of valuations S on R that are trivial on k and centered at infinity, we have $\delta(S) = 2$ if and only if $M(\bar{S})$ is negative definite.*

When one entry of the matrix $\alpha(\bar{v}_i \wedge \bar{v}_j)$ is equal to $-\infty$, we say that $M(\bar{S})$ is negative definite if and only if the matrix $[(\max\{\alpha(\bar{v}_i \wedge \bar{v}_j), -t\}]_{1 \le i,j \le m}$ is negative definite for t large enough.

Observe that one can use the Hodge index theorem to characterize the case when $M(\bar{S})$ is negative definite by a *numerical invariant* $\chi(\bar{S}) := (-1)^l \det M(\bar{S})$. Here l denotes the cardinality of $\bar{S}$ and $\det(M(\bar{S})) := \lim_{t \to -\infty} \det(\max\{\alpha(\bar{v}_i \wedge \bar{v}_j), t\})_{1 \le i,j \le m}$ when one entry of the matrix $\alpha(\bar{v}_i \wedge \bar{v}_j)$ is equal to $-\infty$. Observe that the limit exists because the quantity $\det(\max\{\alpha(\bar{v}_i \wedge \bar{v}_j), t\})_{1 \le i,j \le m}$ is a polynomial for t large enough.

Indeed our Main Theorem can be phrased by saying that $\delta(S) = 2$ if and only if $\chi(\bar{S}) > 0$.

When S contains only one point v, we get $M(S) = \alpha(v)$ and Theorem 19.1 together with our Main Theorem imply the following result of P. Mondal.

Theorem 19.3 ([9])**.** *For a valuation $v \in V_\infty$, the existence of a non constant polynomial $P \in k[x, y]$ such that $v(P) > 0$ is equivalent to $\alpha(v) < 0$.*

Our Main Theorem also implies the following

Corollary 19.4. *Let $s_1, \cdots, s_m$ be a finite set of formal branches of curves centered at infinity. Then there exists a polynomial $P \in k[x,y]$ such that* $\mathrm{ord}_\infty(P|_{s_i}) > 0$ *for all $i = 1, \cdots, m$.*

In a sequel to this paper [14], we shall use these results to get a proof of the dynamical Mordell-Lang conjecture for polynomial endomorphisms on $\mathbb{A}^2_{\overline{\mathbb{Q}}}$.

We conclude this introduction by giving a criterion of arithmetic nature for an analytic branch at infinity to be algebraic.

The setting is as follows. Let K be a number field. For any finite set S of places of K containing all archimedean places, denote by $O_{K,S}$ the ring of S-integers in K. For any place v on K, denote by K_v the completion of K w.r.t. v. We cover the line at infinity L_∞ of the compactification of $\mathbb{A}^2_K = \mathrm{Spec}\, K[x,y]$ by $\mathbb{P}^2_K$ by charts $U_q = \mathrm{Spec}\, K[x_q, y_q]$ centered at $q \in L_\infty(K)$ so that $q = \{(x_q, y_q) = (0,0)\}$, $L_\infty \cap U_i = \{x_i = 0\}$, and $x_q = 1/x$, $y_q = y/x + c$ for some $c \in K$ (or $x_q = 1/y$, $y_q = x/y$).

We shall say that s is an *adelic* branch defined over K at infinity if it is given by the following data.

(i) s is a formal branch based at a point $q \in L_\infty(K)$ given in coordinates x_q, y_q as above by a formal Puiseux series $y_q = \sum_{j \geq 1} a_j x_q^{j/m} \in O_{K,S}[[x_q^{1/m}]]$ for some positive integer m and some finite set S of places of K containing all archimedean places.

(ii) for each place $v \in S$, the radius of convergence of the Puiseux series determining s is positive, i.e. $\limsup_{j\to\infty} |a_j|_v^{-m/j} > 0$.

Observe that for any other place $v \notin S$, then the radius of convergence is a least 1. In the sequel, we set $r_{C,v}$ to be the minimum between 1 and the radius of convergence over K_v of this Puiseux series.

Any adelic branch s at infinity thus defines an analytic curve

$$C^v(s) := \{(x_i, y_i) \in U_i(K_v) |\ y_i = \sum_{j=1}^{\infty} a_{ij} x^{\frac{j}{m_i}}, |x_i|_v < \min\{r_{C_i,v}, 1\}\}.$$

Theorem 19.5. *Suppose $s_1, \cdots, s_l$, $l \geq 1$ is a finite set of adelic branches at infinity. Let $\{B_v\}_{v \in M_K}$ be a set of positive real numbers such that $B_v = 1$ for all but finitely many places.*

Finally let $p_n = (x^{(n)}, y^{(n)})$, $n \geq 0$ be an infinite collection of K-points in $\mathbb{A}^2(K)$ such that for each place $v \in M_K$ then either $\max\{|x^{(n)}|_v, |y^{(n)}|_v\} \leq B_v$ *or $p_n \in \cup_{i=1}^l C^v(s_i)$.*

Then there exists an algebraic curve C in $\mathbb{A}^2_K$ such that any branch of C at infinity is contained in the set $\{s_1, \cdots, s_l\}$ and p_n belongs to $C(K)$ for all n large enough.

In particular, by the theorem of Faltings [1]*, the geometric genus of C is at most one.*

The article is organized in five sections. Section 20 contains background informations on the valuation tree V_∞. Section 21 is entirely devoted to the description of a potential theory in V_∞. Especially important for us are the notion of subharmonic functions and the definition of a Dirichlet energy. The proof of our main theorem can be found in Section 22. Section 23 contains various remarks in the case $\delta = 0$ or 1. Finally Section 24 contains the proof of Theorem 19.5.

20. The valuation tree

Let k be any algebraically closed field. In this section, we recall some basic facts on the space of normalized valuations centered at infinity in the affine plane and its tree structure following [2, 3, 4, 5].

20.1. **Definition.** The set V_∞ is defined as the set of functions $v : k[x,y] \to (-\infty, +\infty]$ satisfying:

(i) $v(P_1P_2) = v(P_1) + v(P_2)$ for all $P_1, P_2 \in k[x,y]$;
(ii) $v(P_1 + P_2) \geq \min\{v(P_1), v(P_2)\}$;
(iii) $v(0) = +\infty$, $v|_{k^*} = 0$ and $\min\{v(x), v(y)\} = -1$.

We endow V_∞ with the topology of the pointwise convergence, for which it is a compact space.

Given $v \in V_\infty$, the set $\mathfrak{P}_v := \{P, v(P) = +\infty\}$ is a prime ideal. When it is reduced to (0) then v is a rank 1 valuation on $k[x,y]$. Otherwise it is generated by an irreducible polynomial Q, and for any $P \in k[x,y]$ the quantity $v(P)$ is the order of vanishing of $P|_Q$ at a branch of the curve $Q^{-1}(0)$ at infinity with the convention $v(P) = +\infty$ when $P \in \mathfrak{P}_v$.

Let s be a formal branch of curve centered at infinity. We may associate to s a valuation $v_s \in V_\infty$ defined by $P \mapsto -\min\{\mathrm{ord}_\infty(x|_s), \mathrm{ord}_\infty(y|_s)\}^{-1}\mathrm{ord}_\infty(P|_s)$. Such a valuation is called a curve valuation.

Suppose X is a smooth projective compactification of $\mathbb{A}^2_k$. The center of $v \in V_\infty$ in X is the unique scheme-theoretic point on X such that its associated valuation is strictly positive on the maximal ideal of its local ring. A divisorial valuation is an element $v \in V_\infty$ whose center has codimension 1 for at least one compactification X as above.

More precisely, let E be an irreducible divisor of $X \setminus \mathbb{A}^2_k$. Then the order of vanishing ord_E along E determines a divisorial valuation on $k[x,y]$, and $v_E := (b_E)^{-1}\mathrm{ord}_E \in V_\infty$ where $b_E := -\min\{\mathrm{ord}_E(x), \mathrm{ord}_E(y)\}$.

Warning. In the sequel, we shall refer to elements in V_∞ as *valuations* even when the prime ideal $\mathfrak{P}_v$ is non trivial.

20.2. **The canonical ordering and the tree structure.** The space V_∞ of normalized valuations is equipped with a partial ordering defined by $v \leq w$ if and only if $v(P) \leq w(P)$ for all $P \in k[x,y]$ for which $-\deg$ is the unique minimal element.

All curve valuations are maximal and no divisorial valuation is maximal.

It is a theorem that given any valuation $v \in V_\infty$ the set $\{w \in V_\infty, -\deg \leq w \leq v\}$ is isomorphic as a poset to the real segment $[0,1]$ endowed with the standard ordering. In other words, $(V_\infty, \leq)$ is a rooted tree in the sense of [2, 5].

It follows that given any two valuations $v_1, v_2 \in V_\infty$, there is a unique valuation in V_∞ which is maximal in the set $\{v \in V_\infty | \ v \leq v_1 \text{ and } v \leq v_2\}$. We denote it by $v_1 \wedge v_2$.

The segment $[v_1, v_2]$ is by definition the union of $\{w,\, v_1 \wedge v_2 \leq w \leq v_1\}$ and $\{w,\, v_1 \wedge v_2 \leq w \leq v_2\}$.

Pick any valuation $v \in V_\infty$. We say that two points v_1, v_2 distinct with v lie in the same direction at v if the segment $[v_1, v_2]$ does not contain v. A direction (or a tangent vector) at v is an equivalence class for this relation. We write Tan_v for the set of directions at v.

When Tan_v is a singleton, then v is called an endpoint. In V_∞, the set of endpoints is exactly the set of all maximal valuations. This set is dense in V_∞.

When Tan_v contains exactly two directions, then v is said to be regular. In V_∞, regular points are given by monomial rank 1 valuations as in (19.1) for which the weights are rationally independent, see [2, 5] for details.

When Tan_v has more than three directions, then v is a branched point. In V_∞, branched points are exactly the divisorial valuations. Given any smooth projective compactification X in which v has codimension 1 center E, one proves that the map sending an element V_∞ to its center in X induces a map $\mathrm{Tan}_v \to E$ that is a bijection.

Pick any $v \in V_\infty$. For any tangent vector $\vec{v} \in \mathrm{Tan}_v$, we denote by $U(\vec{v})$ the subset of those elements in V_∞ that determine $\vec{v}$ (i.e. we say w determines $\vec{v}$ if $\vec{v}$ is the equivalence class of w). This is an open set whose boundary is reduced to the singleton $\{v\}$. The complement of $\{w \in V_\infty,\, w \geq v\}$ is equal to $U(\vec{v}_0)$ where $\vec{v}_0$ is the tangent vector determined by $-\deg$.

It is a fact that finite intersections of open sets of the form $U(\vec{v})$ form a basis for the topology of V_∞.

Finally recall that the *convex hull* of any subset $S \subset V_\infty$ is defined the set of valuations $v \in V_\infty$ such that there exists a pair $v_1, v_2 \in S$ with $v \in [v_1, v_2]$.

A *finite subtree* of V_∞ is by definition the convex hull of a finite collection of points in V_∞. A point in a finite subtree $T \subseteq V_\infty$ is said to be an end point if it is maximal in T.

20.3. **The valuation space as the universal dual graph.** One can understand the tree structure of V_∞ from the geometry of compactifications of $\mathbb{A}^2_k$ as follows.

Pick any smooth projective compactification X of $\mathbb{A}^2_k$. The divisor at infinity $X \setminus \mathbb{A}^2_k$ has simple normal crossings, and we denote by Γ_X its dual graph: vertices are in bijection with irreducible components of the divisor at infinity, and vertices are joined by an edge when their corresponding component intersect at a point.

The choice of coordinates x, y on $\mathbb{A}^2_k$ determines a privileged compactification $\mathbb{P}^2_k$ for which the divisor at infinity is a rational curve L_∞ and $\mathrm{ord}_{L_\infty} = -\deg$. In this case, the dual graph is reduced to a singleton.

For a general compactification X, we may look at the convex hull (in V_∞) of the finite set of valuations v_E where E ranges over all irreducible components of $X \setminus \mathbb{A}^2_k$. It is a fact that the finite subtree that we obtain in this way is a geometric realization of the dual graph Γ_X. To simplify notation, we shall identify Γ_X with its realization in V_∞. Observe that the dual graph Γ_X inherits a partial order relation from its inclusion in V_∞.

There is also a canonical retraction map $r_X : V_\infty \to \Gamma_X$ sending a valuation $v \in V_\infty$ to the unique $r_X(v) \in \Gamma_X$ such that $[r_X(v), v] \cap \Gamma_X = \{r_X(v)\}$.

Say that a compactification X' dominates another one X when the canonical birational map $X' \dashrightarrow X$ induced by the identity map on $\mathbb{A}^2_k$ is regular. The category $\mathcal{C}$ of all smooth projective compactifications of $\mathbb{A}^2_k$ is an inductive set for this domination relation, and one can form the projective limit $\Gamma_{\mathcal{C}} := \varprojlim_{X\in\mathcal{C}} \Gamma_X$ using the retraction maps. In other words, a point in $\Gamma_{\mathcal{C}}$ is a collection of points $v_X \in \Gamma_X$ such that $r_X(v_{X'}) = v_X$ as soon as X' dominates X.

It is a theorem that $\Gamma_{\mathcal{C}}$ endowed with the product topology is homeomorphic to V_∞.

Warning. In the sequel, we shall mostly consider smooth projective compactifications that *dominates* $\mathbb{P}^2_k$, and refer to them as *admissible* compactifications of the affine plane.

Observe that Γ_X contains $-\deg$ when X is an admissible compactification.

20.4. **Parameterization.** The *skewness* function $\alpha : V_\infty \to [-\infty, 1]$ is the function on V_∞ that is strictly decreasing (for the order relation of V_∞) satisfying $\alpha(-\deg) = 1$ and

$$|\alpha(v_E) - \alpha(v_{E'})| = \frac{1}{b_E b_{E'}} \ .$$

whenever E and E' are two irreducible components of $X \setminus \mathbb{A}^2_k$ that intersect at a point in some admissible compactification X of the affine plane.

Since divisorial valuations are dense in any segment $[-\deg, v]$ it follows that α is uniquely determined by the conditions above. One knows that $\alpha(v) \in \mathbb{Q}$ for any divisorial valuation, that $\alpha(v) \in \mathbb{R} \setminus \mathbb{Q}$ for any valuation that is a regular point of V_∞, and that $\alpha(v) = -\infty$ for any curve valuation. However there are endpoints of V_∞ with finite skewness.

There is a geometric interpretation of the skewness of a divisorial valuation as follows. Let X be an admissible compactification of $\mathbb{A}^2_k$, and E be an irreducible component of $X \setminus \mathbb{A}^2_k$. Let $\check{E}$ be the unique divisor supported on the divisor at infinity such that $(\check{E} \cdot E) = 1$ and $(\check{E} \cdot F) = 0$ for all components F lying at infinity. Then we have

$$\alpha(v_E) = \frac{1}{b_E^2} (\check{E} \cdot \check{E}) \ .$$

Since the skewness function is strictly decreasing, it induces a metric d_{V_∞} on V_∞ by setting

$$d_{V_\infty}(v_1, v_2) := 2\alpha(v_1 \wedge v_2) - \alpha(v_1) - \alpha(v_2)$$

for all $v_1, v_2 \in V_\infty$. In particular, any segment in V_∞ carries a canonical metric for which it becomes isometric to a real segment.

21. Potential theory on V_∞

As in the previous section k is any algebraically closed field. We recall the basic principles of a potential theory on V_∞ including the definition of subharmonic functions, and their associated Laplacian. We then construct a Dirichlet pairing on subharmonic functions and study its main properties.

We refer to [5] for details.

21.1. **Subharmonic functions on V_∞.** To any $v \in V_\infty$ we attach its Green function

$$g_v(w) := \alpha(v \wedge w) \ .$$

This is a decreasing continuous function taking values in $[-\infty, 1]$, satisfying $g_v(-\deg) = 1$. Moreover pick any $v' \in V_\infty$ and define the function $g(t) : [\alpha(v'), 1] \to [-\infty, 1]$ by sending t to $g_v(v_t)$ where v_t is the unique valuation in $[-\deg, v']$ with skewness t. Then g is a piecewise affine increasing and convex function with slope in $\{0, 1\}$.

Denote by $M^+(V_\infty)$ the set of positive Radon measures on V_∞ that is the set of positive linear functionals on the space of continuous functions on V_∞. We endow $M^+(V_\infty)$ with the weak topology.

Lemma 21.1. *For any positive Radon measures ρ on V_∞, there exists a sequence of compactification $X_n \in \mathcal{C}$, $n \geq 0$ such that X_{n+1} dominates X_n for all $n \geq 0$, and ρ is supported on the closure of $\cup_{n\geq 0}\Gamma_{X_n}$.*

Proof. Observe that V_∞ is complete rooted nonmetric tree and weakly compact (See [2, Section 3.2]), thus [2, Lemma 7.14] applies. By [2, Lemma 7.14], there exists a sequence of finite subtrees T_n $n \geq 0$ satisfying $T_n \subseteq T_{n+1}$ for $n \geq 0$ such that ρ is supported on the closure T of $\cup_{n\geq 0}T_n$. Since T_n is a finite tree and the divisorial valuations are dense in T_n, there exists a sequence of subtrees T_n^m such that

- all vertices in T_n^m are divisorial;
- $T_n^m \subseteq T_n^{m+1}$ for $m \geq 0$;
- T_n is the closure of $\cup_{m\geq 0}T_n^m$.

Set $Y_n := \cup_{1\leq i,j\leq n}T_i^j$, then we have

- Y_n is a finite tree;
- all vertices in Y_n are divisorial;
- $Y_n \subseteq Y_{n+1}$ for $n \geq 0$;
- T is the closure of $\cup_{n\geq 0}Y_n$.

To conclude, we pick by induction a sequence of increasing compactification $X_n \in \mathcal{C}$ such that $Y_n \subseteq \Gamma_{X_n}$. □

Lemma 21.2. *Let ρ be any positive Radon measures on V_∞ and T_n be a sequence of finite subtree of V_∞ such that $T_n \subseteq T_{n+1}$ for $n \geq 0$ and ρ is supported on the closure of $\cup_{n\geq 0}T_n$. Then we have $r_{T_n *}\rho \to \rho$ weakly.*

Proof. Let T be the closure of $\cup_{n\geq 0}T_n$ and f be any continuous function on V_∞. For any $\varepsilon > 0$ and any point $v \in T$, there exists a neighborhood U_v of v

such that $\sup_{U_v} |f - f(v)| \le \varepsilon/2$. We may moreover choose it such that either $U_v = \{w, w > w_1\}$ or $U_v = \{w, w_1 < (w \wedge w_2) < w_2\}$. Since T is compact, it is covered by finitely many such open sets $U_{v_1}, \dots, U_{v_m}$. Since $\cup_{n\ge1} T_n$ is dense in T, for any $i = 1, \cdots, m$, there exists $w_i \in U_{v_i} \cap (\cup_{n\ge1} T_n)$. There exists $N \ge 0$, such that T_N contains $\{w_1, \cdots, w_m\}$. For any $n \ge N$, if v is a point in U_{v_i}, we have $r_{T_n} v \in U_{v_i}$. It follows that for all points $v \in T$, we have $|f(v) - f(r_{T_n})(v)| \le \varepsilon$ and

$$\Big|\int_{V_\infty} f(v) d\rho(v) - \int_{V_\infty} f(v) d r_{T_n *}\rho(v)\Big| = \Big|\int_T f(v) - f(r_{T_n}(v)) d\rho(v)\Big| \le \varepsilon \rho(V_\infty)$$

which concludes the proof. □

Given any positive Radon measure ρ on V_∞ we define

$$g_\rho(w) := \int_{V_\infty} g_v(w)\, d\rho(v) \ .$$

Observe that $g_v(w)$ is always well-defined in $[-\infty, 1]$ since $g_v \le 1$ for all v. Since the Green function g_v is decreasing for all $v \in V_\infty$, we get

Proposition 21.3. *For any any positive Radon measure ρ on V_∞, g_ρ is decreasing.*

The next result is

Theorem 21.4. *The map $\rho \mapsto g_\rho$ is injective.*

To prove this theorem, we first need the following

Lemma 21.5. *For any continuous function $f : V_\infty \to \mathbb{R}$ and any $\varepsilon > 0$, there exists $X \in \mathcal{C}$ such that $|f - f \circ r_X| \le \varepsilon$.*

Proof of Lemma 21.5. For any v we may find a neighborhood U_v such that

$$\sup_{U_v} |f - f(v)| \le \varepsilon/2.$$

We may moreover choose it such that $U_v = \{w, w > w_1\}$ or $U_v = \{w, w_1 < (w \wedge w_2) < w_2\}$ where w_1, w_2 are divisorial. Since V_∞ is compact it is covered by finitely many such open sets $U_{v_1}, \dots, U_{v_m}$. Choose X to be an admissible compactification such that the boundary valuations of U_{v_i} all have codimension 1 center in X. For any $v \in V_\infty$ pick an index i such that $v \in U_{v_i}$. Then we have $|f(v) - f \circ r_X(v)| \le |f(v) - f(v_i)| + |f(r_X(v)) - f(v_i)| < \varepsilon$. This concludes the proof. □

Proof of Theorem 21.4. By contradiction, suppose that $\rho_1 \ne \rho_2$ in $M^+(V_\infty)$ but $g_{\rho_1} = g_{\rho_2}$. There exists a continuous function $f : V_\infty \to \mathbb{R}$ satisfying

$$\int_{V_\infty} f(v) d\rho_1(v) \ne \int_{V_\infty} f(v) d\rho_2(v).$$

Set $M := \max\{\rho_1(V_\infty), \rho_2(V_\infty)\}$.

By Lemma 21.5, for any $\varepsilon > 0$, there exists $X \in \mathcal{C}$ such that $|f \circ r_X - f| \le \varepsilon/2$. There exists a piecewise linear function h on Γ_X such that $|f \circ r_X - h \circ r_X| \le \varepsilon/2$.

Since Γ_X is a finite graph, there exists $v_1, \cdots, v_m \in \Gamma_X$ such that $h \circ r_X = \sum_{i=1}^m r_i g_{v_i}$ where $r_1, \cdots, r_m \in \mathbb{R}$.
Since $g_{\rho_1}(v_i) = g_{\rho_2}(v_i)$ for $i = 1, \cdots, m$, we have

$$\int_{V_\infty} h \circ r_X(v) d\rho_1(v) = \int_{V_\infty} \sum_{i=1}^m r_i g_{v_i}(v) d\rho_1(v) = \sum_{i=1}^m r_i \int_{V_\infty} g_v(v_i) d\rho_1(v)$$

$$= \sum_{i=1}^m r_i g_{\rho_1}(v_i) = \sum_{i=1}^m r_i g_{\rho_2}(v_i) = \int_{V_\infty} h \circ r_X(v) d\rho_2(v).$$

It follows that

$$|\int_{V_\infty} f(v) d\rho_1(v) - \int_{V_\infty} f(v) d\rho_2(v)| \leq 2\varepsilon M.$$

We obtain a contradiction by letting $\epsilon \to 0$. □

One can thus make the following definition.

Definition 21.6. A function $\phi : V_\infty \to \mathbb{R} \cup \{-\infty\}$ is said to be *subharmonic* if there exists a positive Radon measure ρ such that $\phi = g_\rho$. In this case, we write $\rho = \Delta\phi$ and call it the *Laplacian* of ϕ.

Denote by SH (resp. $\mathrm{SH}^+(\mathrm{V}_\infty)$) the space of subharmonic functions on V_∞ (resp. of non-negative subharmonic functions on V_∞).

Proposition 21.7. *For any subharmonic function ϕ on V_∞, there exists a sequence of compactifications $X_n \in \mathcal{C}$, $n \geq 0$ such that X_{n+1} dominates X_n for all $n \geq 0$, and $\phi = \lim_{n\to\infty} \phi \circ r_{X_n}$ pointwise.*

Proof. Write ρ for $\Delta\phi$. 21.3 Pick X_n as in Lemma 21.1. By Lemma 21.2, $r_{X*}\rho \to \rho$ weakly. For any $w \in V_\infty$, pick a sequence $w_n \in [-\deg, w]$ satisfying $w_n \to w$ when $n \to \infty$.

$$g_\rho(w) = \int_{V_\infty} g_v(w)\, d\rho(v) = \lim_{m\to\infty} \int_{V_\infty} g_v(w_m)\, d\rho(v)$$

$$= \lim_{m\to\infty} \lim_{n\to\infty} \int_{V_\infty} g_v(w_m)\, dr_{X_n*}\rho(v). \quad (1)$$

Observe that $\int_{V_\infty} g_v(w_m)\, dr_{X_n*}\rho(v) = \int_{V_\infty} g_v(r_{X_n*}(w_m))\, d\rho(v)$ which is decreasing in n and m. We have

$$g_\rho(w) = \lim_{n\to\infty} \lim_{m\to\infty} \int_{V_\infty} g_v(w_m)\, dr_{X_n*}\rho(v) = \lim_{n\to\infty} \int_{V_\infty} g_v(w)\, dr_{X_n*}\rho(v)$$

$$= \lim_{n\to\infty} \int_{V_\infty} g_v(r_{X_n} w)\, d\rho(v) = \lim_{n\to\infty} g_\rho \circ r_X(w).$$

□

21.2. **Reduction to finite trees.** Let T be any finite subtree of V_∞ containing $-\deg$. Denote by $r_T : V_\infty \to T$ the canonical retraction defined by sending v to the unique valuation $r_T(v) \in T$ such that $[r_T(v), v] \cap T = \{r_T(v)\}$.

For any function ϕ, set $R_T\phi := \phi \circ r_T$. Observe that $R_T\phi|_T = \phi|_T$ and that $R_T\phi$ is locally constant outside T.

Moreover we have the following

Proposition 21.8. *Pick any subharmonic function ϕ Then for any finite subtree T, $R_T\phi$ is subharmonic, $R_T\phi \geq \phi$ and $\Delta(R_T\phi) = (r_T)_*\Delta\phi$.*

Proof. Set $\Delta\phi = \rho$. Then we have

$$R_T\phi(w) = \int_{V_\infty} g_v(r_T(w))d\rho$$
$$= \int_{V_\infty} g_{r_T(v)}(w)d\rho = \int_{V_\infty} g_v(w)dr_{T*}\rho = g_{r_{T*}\rho}$$

which concludes our proposition. □

Let T be a finite tree containing $\{-\deg\}$ such that for all points $v \in T$, we have $\alpha(v) > -\infty$. Let $\phi = g_\rho$ be a subharmonic function satisfying $\mathrm{Supp}\,\rho \subseteq T$. Set $t(v) := -\alpha(v)$. Let E be the set of all edges of T. For each edge $I = [w_1, w_2] \in E$, this function $t(v)$ parameterizes I. Denote by $\frac{d^2\phi|_I}{dt^2}dt$ the usual real Laplacian of $\phi|_I$ on the segment I i.e. the unique measure on I such that

(i) For any segment $(v_1, v_2) \subseteq I$, we have $\int_{[v_1,v_2]} \frac{d^2\phi|_I}{dt^2}dt = D_{\vec{v_1}}\phi(v_1) + D_{\vec{v_2}}\phi(v_2)$ where $\vec{v_i}$ is the direction at v_i in (v_1, v_2) for $i = 1, 2$.

(ii) $\frac{d^2\phi|_I}{dt^2}dt\{w_i\} = -D_{\vec{w_i}}\phi$ where $\vec{w_i}$ is direction at w_i in I for $i = 1, 2$.

Proposition 21.9. *We have*

(i)
$$\Delta\phi = \phi(-\deg)\delta_{-\deg} + \sum_{I\in E} \frac{d^2\phi|_I}{dt^2}dt;$$

(ii) *the mass of $\Delta\phi$ at a point $v \in T$ is given by $\phi(-\deg)\delta_{-\deg}\{x\} + \sum D_{\overrightarrow{v}}\phi$ the sum is over all tangent directions $\overrightarrow{v}$ in T at v;*

(iii) *for any segment I contained in T, $\phi|_I$ is convex and for any point $v \in T$, we have*
$$\phi(-\deg)\delta_{-\deg}\{v\} + \sum D_{\overrightarrow{v}}\phi \geq 0$$
where $\delta_{-\deg}$ is the dirac measure at $-\deg$ and the sum is over all tangent directions $\overrightarrow{v}$ in T at v.

Sketch of the proof. First check that our proposition holds when $\phi = g_v$ for any $v \in T$. Since all the conclusions in our proposition are linear, they hold for $g_\rho(w) = \int_{V_\infty} g_v(w)d\rho = \int_T g_v(w)d\rho$ also. □

Theorem 21.10. *Let $X_n \in \mathcal{C}$, $n \geq 0$ be a sequence of compactifications such that X_{n+1} dominates X_n for all $n \geq 0$ and let T be the closure of $\cup_{n\geq 0}\Gamma_{X_n}$. Suppose that we are given a sequence ϕ_n of subharmonic functions satisfying* $\mathrm{Supp}\Delta\phi_n \subseteq \Gamma_{X_n}$ *and $R_{\Gamma_{X_n}}\phi_m = \phi_n$ when $m \geq n$.*

Then there exists a unique subharmonic function $\phi \in \mathrm{SH}(V_\infty)$ *satisfying* $\mathrm{Supp}\Delta\phi \subseteq T$*,* $R_{\Gamma_{X_n}}\phi = \phi_n$ *and* $\phi = \lim_{n\to\infty}\phi_n$*.*

Proof. Set $\rho_n := \Delta\phi_n$. For any $m \geq n$, we have $r_{X_n}\rho_m = \rho_n$. It follows that $\rho_n(V_\infty)$ is independent on n and we may suppose that $\rho_n(V_\infty) = 1$ for all $n \geq 0$. Given a continuous function f on V_∞ and a real number $\varepsilon > 0$, by Lemma 21.5, there exists $N \geq 0$ such that $|f \circ r_{X_n} - f \circ r_{X_m}| \leq \varepsilon$ for all $n, m \geq N$. It follows that $|\int_{V_\infty} f d\rho_n - \int_{V_\infty} f d\rho_m| \leq \varepsilon$ for all $n, m \geq N$. It follows that $\lim_{n\to\infty}\int_{V_\infty} f d\rho_n$ exists.

The functional $f \mapsto \lim_{n\to\infty}\int_{V_\infty} f d\rho_n$ is continuous, linear and positive, and thus defines a positive Radon measure ρ. Observe that $r_{\Gamma_{X_n}}\rho = \rho_n$ for all $n \geq 0$ and $\rho_n \to \rho$ when $n \to \infty$. Set $\phi := g_\rho$. We have $R_{\Gamma_{X_n}}\phi = \phi_n$. By Proposition 21.7, we get $\phi = \lim_{n\to\infty}\phi_n$. □

21.3. **Main properties of subharmonic functions.** The next result collects some properties of subharmonic functions.

Theorem 21.11. *Pick any subharmonic function* ϕ *on* V_∞*. Then*

(i) ϕ *is decreasing and* $\phi(-\deg) = \Delta\phi(V_\infty) > 0$ *if* $\phi \neq 0$*;*
(ii) ϕ *is upper semicontinuous;*
(iii) *for any valuation* $v \in V_\infty$ *the function* $t \mapsto \phi(v_t)$ *is convex, where* v_t *is the unique valuation in* $[-\deg, v]$ *of skewness* t*.*

Proof. The first statement follows from Proposition 21.3 and the equality

$$\phi(-\deg) = \int_{V_\infty} g_v(-\deg) d\rho(v) = \rho(V_\infty).$$

The second statement is a consequence of Proposition 21.7 and Proposition 21.9 that implies that $\phi \circ r_X$ is continuous on V_∞ for any $X \in \mathcal{C}$. The last statement follows from Proposition 21.9. □

Now pick any direction $\vec{v}$ at a valuation $v \in V_\infty$. One may define the directional derivative $D_{\vec{v}}\phi$ of any subharmonic function as follows. If $\alpha(v) \neq -\infty$, pick any map $t \in [0, \epsilon) \mapsto v_t$ such that $v_0 = v$, $|\alpha(v_t) - \alpha(v_0)| = t$ and v_t determines $\vec{v}$ for all $t > 0$. By property (iii) above, the function $t \mapsto \phi(v_t)$ is convex and continuous at 0, so that its right derivative is well-defined. We set

$$D_{\vec{v}}\phi := \left.\frac{d}{dt}\right|_{t=0} \phi(v_t)\ .$$

This definition does not depend on the choice of map $t \mapsto v_t$. If $\alpha(v) = -\infty$, then v is an endpoint in V_∞ and there exists a unique direction $\vec{v}$ at v. For any $w < v$, denote by $\vec{w}$ the direction at w determined by v. Then we define

$$D_{\vec{v}}\phi := -\lim_{w\to v} D_{\vec{w}}\phi$$

which exists since $\phi|_{[-\deg, v]}$ is convex.

Given any direction $\vec{v}$ at a valuation in V_∞, recall that $U(\vec{v})$ is the open set of valuations determining $\vec{v}$.

Theorem 21.12. *Pick any subharmonic function ϕ on V_∞. Then one has*

$$\Delta\phi(U(\vec{v})) = -D_{\vec{v}}\phi$$

for any direction $\vec{v}$ that is not determined by $-\deg$. In particular, one has

$$\begin{aligned}\Delta\phi\{-\deg\} &= \sum_{\vec{v}\in \mathrm{Tan}_{-\deg}} D_{\vec{v}}\phi + \phi(-\deg)\ ;\ \textit{and}\\ \Delta\phi\{v\} &= \sum_{\vec{v}\in \mathrm{Tan}_{v}} D_{\vec{v}}\phi\end{aligned}$$

if $v \neq -\deg$.

Proof. Since $\vec{v}$ is not determined by $-\deg$, v is not an endpoint of V_∞. Pick $w \in U(\vec{v})$, we have $w > v$. Set $I := [-\deg, w]$. We have $\Delta R_I\phi(U(\vec{v})) = \int_{V_\infty} \frac{d^2\phi}{dt^2}dt = \int_{V_\infty} d\frac{d\phi}{dt} = -D_{\vec{v}}R_I\phi$. Since $R_I\phi|_I = \phi|_I$ and $\Delta R_I = r_{I*}\Delta\phi$, we have $\Delta R_I\phi(U(\vec{v})) = \Delta\phi(U(\vec{v}))$ and $D_{\vec{v}}R_I\phi = D_{\vec{v}}\phi$. It follows that $\Delta\phi(U(\vec{v})) = -D_{\vec{v}}\phi$.

If $v = -\deg$, then we have

$$\begin{aligned}\phi(-\deg) = \Delta\phi(V_\infty) &= \Delta\phi\{-\deg\} + \sum_{\vec{v}\in \mathrm{Tan}_{-\deg}} \Delta\phi(U(\vec{v}))\\ &= \Delta\phi\{-\deg\} - \sum_{\vec{v}\in \mathrm{Tan}_{-\deg}} D_{\vec{v}}\phi.\end{aligned}$$

It follows that

$$\Delta\phi\{-\deg\} = \sum_{\vec{v}\in \mathrm{Tan}_{-\deg}} D_{\vec{v}}\phi + \phi(-\deg).$$

If $v \neq -\deg$, let w_n be a sequence of valuations in $[-\deg, v)$. Denote by $\vec{w_n}$ the direction at w_n determined by v and $\vec{v_0}$ the direction at v determined by $-\deg$. Observe that

$$-\lim_{n\to\infty} D_{\vec{w_n}}\phi = \lim_{n\to\infty} \Delta\phi(U(\vec{w_n})) = \Delta\phi\{v\} + \sum_{\vec{v}\in \mathrm{Tan}_v\setminus\{\vec{v_0}\}} \Delta\phi(U(\vec{w_n})).$$

It follows that $D_{\vec{v_0}}\phi = \Delta\phi\{v\} - \sum_{\vec{v}\in \mathrm{Tan}_v} D_{\vec{v}}\phi$ and then

$$\Delta\phi\{v\} = \sum_{\vec{v}\in \mathrm{Tan}_v\setminus\{\vec{v_0}\}} D_{\vec{v}}\phi.$$

□

Theorem 21.13. *Suppose $\phi : V_\infty \to [-\infty, +\infty)$ is a function such that*

(i) *for any valuation $v \in V_\infty$ the function $[\alpha(v), 1] \ni t \mapsto \phi(v_t)$ is continuous and convex, where v_t is the unique valuation in $[-\deg, v]$ of skewness t;*

(ii) *the inequalities*

$$\sum_{\vec{v}\in \mathrm{Tan}_{-\deg}} D_{\vec{v}}\phi + \phi(-\deg) \geq 0\ ;\ \textit{and}\ \sum_{\vec{v}\in \mathrm{Tan}_{v}} D_{\vec{v}}\phi \geq 0 \tag{21.1}$$

are satisfied for all valuations $v \neq -\deg$.

Then ϕ is subharmonic.

Proof. Let $v_1, v_2 \in V_\infty$ be two valuations satisfying $v_1 < v_2$. There exists an end point $w \in V_\infty$ satisfying $v_1, v_2 \in [-\deg, w]$. Denote by $\vec{w}$ the unique direction in Tan_w. By (ii), we have $D_{\vec{w}}\phi \geq 0$. Since ϕ is convex on $[-\deg, w]$, it is decreasing on $[-\deg, w]$. It follows that $\phi(v_1) \geq \phi(v_2)$ and then ϕ is decreasing.

For any $v \in V_\infty \setminus \{-\deg\}$, denote by $\vec{v}$ the direction at v determined by $-\deg$. For any $n \geq 1$, set $T_n := \{v \in V_\infty \setminus \{-\deg\} |\ D_{\vec{v}}\phi \geq 1/n\}$. Since the map $v \mapsto D_{\vec{v}}\phi$ is non negative and decreasing, it follows that T_n is a tree.

We claim that T_n is a finite tree. If $T_n = \{-\deg\}$, there is nothing to prove.

For convenience, we define $D_{\overrightarrow{-\deg}}\phi := -\sum_{\vec{v} \in \mathrm{Tan}_{-\deg}} D_{\vec{v}}\phi = \phi(-\deg)$. Let w be a valuation in T_n and $v_1, \cdots, v_m$, $m \geq 1$ be valuations in T_n satisfying $v_i \wedge v_j = w$ for all $i \neq j$. Denote by $\vec{w_i}$ the direction at w determined by v_i. Then we have

$$\sum_{i=1}^{m} D_{\vec{v_i}}\phi \leq \sum_{i=1}^{m} -D_{\vec{w_i}}\phi \leq D_{\vec{w}}\phi.$$

Pick m valuations $v_1, \cdots, v_m \in T_n$ such that any two valuations v_i, v_j $i \neq j$ are not comparable. Let S be the set of maximal elements in the set $\{v_i \wedge v_j |\ 1 \leq i < j \leq m\}$ and write $S = \{w_1, \cdots, w_l\}$. Observe that $l \leq m - 1$ if $m \geq 2$. Let S_w be the set of v_i satisfying $v_i > w$. Then we have $\sum_{v \in S_w} D_{\vec{v}}\phi \leq D_{\vec{w}}\phi$ and $\{v_1, \cdots, v_m\} = \coprod_{w \in S} S_w$. It follows that $\sum_{i=1}^{m} D_{\vec{v_i}}\phi \leq \sum_{w \in S} D_{\vec{w}}\phi$. By induction, we have

$$\sum_{i=1}^{m} D_{\vec{v_i}}\phi \leq D_{\overrightarrow{\wedge_{i=1}^{m} v_i}}\phi \leq D_{\overrightarrow{-\deg}}\phi = \phi(-\deg).$$

Since $D_{\vec{v_i}}\phi \geq 1/n$, we conclude that $m \leq n\phi(-\deg)$. This fact implies that T_n is a finite tree with at most $n\phi(-\deg)$ end points.

As in the proof of Lemma 21.1, we an now show that there exists a sequence of admissible compactification $X_n \in \mathcal{C}$, $n \geq 0$ such that X_{n+1} dominates X_n for all $n \geq 0$ and $\cup_{n \geq 0} T_n$ is contained in the closure of $\cup_{n \geq 0} \Gamma_{X_n}$. Set $\phi_n := R_{\Gamma_{X_n}}\phi$.

Let v be a point in V_∞. Set $I := [-\deg, v]$ and $I_n := I \cap \Gamma_{X_n} = [-\deg, v_n]$. Observe that v_n is increasing and define $v' := \lim_{n\to\infty} v_n$. Observe that for all $(v', v] \subseteq V_\infty \setminus (\cup_{n \geq 1} T_n)$, and then $D_{\vec{w}} = 0$ for all $w \in (v', v]$. It follows that

$$\phi(v) = \phi(w) = \lim_{n\to\infty} \phi(v_n) = \lim_{n\to\infty} \phi_n(v).$$

Denote by $\rho_n := \phi_n(-\deg)\delta_{-\deg}\{x\} + \sum \frac{d^2\phi|_I}{dt^2} dt$ where the sum is over all edges of Γ_{X_n}. It is a Radon measure supported on Γ_{X_n}. It follows that $\phi_n = g_{\rho_n}$ which is subharmonic and $\phi_n = R_{\Gamma_{X_n}}\phi_m$ for any $m \geq n$. Then we conclude by applying Theorem 21.10. □

The next result collects the main properties of the space of subharmonic functions.

Theorem 21.14. *The sets* $\mathrm{SH}(V_\infty)$ *and* $\mathrm{SH}^+(\mathrm{V}_\infty)$ *are convex cones that are stable by* max. *In other words, given any* $c > 0$, *and any* $\phi, \phi' \in \mathrm{SH}(V_\infty)$ *(resp. in* $\mathrm{SH}^+(\mathrm{V}_\infty)$*), then* $c\phi, \phi + \phi'$ *and* $\max\{\phi, \phi'\}$ *all belong to* $\mathrm{SH}(V_\infty)$ *(resp. to* $\mathrm{SH}^+(\mathrm{V}_\infty)$*).*

Proof. By Theorem 21.13, it is easy to check that $c\phi$ and $\phi+\phi'$ all belong to $\mathrm{SH}(V_\infty)$ (resp. to $\mathrm{SH}^+(\mathrm{V}_\infty)$) when $c>0$, and $\phi,\phi'\in\mathrm{SH}(V_\infty)$ (resp. in $\mathrm{SH}^+(\mathrm{V}_\infty)$).

We only have to check that $\max\{\phi,\phi'\}$ belongs to $\mathrm{SH}(V_\infty)$ when $\phi,\phi'\in\mathrm{SH}(V_\infty)$. It is easy to see that the condition (i) in Theorem 21.13 holds. For any point $v\in V_\infty$ and any direction $\vec{v}$ at v, if $\phi(v)>\phi'(v)$ (resp. $\phi(v)<\phi'(v)$), then $D_{\vec{v}}\max\{\phi,\phi'\}=D_{\vec{v}}\phi$ (resp. $D_{\vec{v}}\max\{\phi,\phi'\}=D_{\vec{v}}\phi'$). It follows that the condition (ii) in Theorem 21.13 holds when $\phi(v)\neq\phi'(v)$. Otherwise, if $\phi(v)=\phi'(v)$, we have $D_{\vec{v}}\max\{\phi,\phi'\}=\max\{D_{\vec{v}}\phi,D_{\vec{v}}\phi'\}$ and then the condition (ii) in Theorem 21.13 holds. Now we conclude by applying Theorem 21.13. □

21.4. **Examples of subharmonic functions.** For any nonconstant polynomial $Q\in k[x,y]$, we define the function

$$\log|Q|(v):=-v(Q)\ ,$$

which takes values in $[-\infty,\infty)$.

Proposition 21.15. *The function* $\log|Q|$ *is subharmonic, and*

$$\Delta(\log|Q|)=\sum_i m_i\delta_{v_{s_i}}$$

where s_i are the branches of the curve $\{Q=0\}$ at infinity, and m_i is the intersection number of s_i with the line at infinity in $\mathbb{P}^2_k$.

Sketch of proof. Let $g=\sum_i m_i g_{v_{s_i}}$. One has to prove that $\log|Q|=g$. To that end, we pick any admissible compactification X of $\mathbb{A}^2_k$ and prove that $\log|Q|(v_E)=g(v_E)$ for any irreducible component of $X_\infty:=X\setminus\mathbb{A}^2_k$. The proof then goes by induction on the number of irreducible component of X_∞ and observing that this number is 1 only if $X=\mathbb{P}^2_k$. □

Proposition 21.16. *The function* $\log^+|Q|:=\max\{0,\log|Q|\}$ *belongs to* $\mathrm{SH}^+(\mathrm{V}_\infty)$.

Denote by $s_1,\cdots,s_l$ the branches of $\{Q=0\}$ at infinity and by T the convex hull of $\{-\deg,v_{s_1},\cdots,v_{s_l}\}$. Then the support of $\Delta(\log^+|Q|)$ is the set of points $v\in T$ satisfying $v(Q)=0$ and $w(Q)<0$ for all $w\in(v,-\deg]$.

In particular, $\mathrm{Supp}\Delta(\log^+|Q|)$ is finite.

Proof. By Theorem 21.14 we have $\log^+|Q|\in\mathrm{SH}(V_\infty)$. Observe that $\log^+|Q|$ is locally constat on $V_\infty\setminus T$ so that the support of $\Delta\log^+|Q|$ is included in T. Let $\{v_1,\cdots,v_m\}$ be the set of points $v\in T$ satisfying $v(Q)=0$ and $w(Q)<0$ for all $w\in(v,-\deg]$. For any $v\in V_\infty$, we have $\log|Q|\geq\deg(Q)\alpha(v)$. It follows that $\alpha(v_i)\leq 0$ and then $v_i\neq-\deg$. Denote by $m_i's$ the intersection number of s_i with the line at infinity in $\mathbb{P}^2_k$. For any $i=1,\cdots,m$, denote by S_i the set of branches of the curve s_j satisfying $v_{s_j}>v_i$. Observe that $S_i\neq\emptyset$ and $\{s_1,\cdots,s_l\}=\coprod_{i=1}^m S_i$. By Theorem 21.12, we have $\Delta\log^+|Q|\{v_i\}=\sum_{s_j\in S_i}m_j>0$. Then we have $\sum_{i=1}^m\Delta\log^+|Q|\{v_i\}=\sum_{j=1}m_j=\deg(Q)=\log^+|Q|(-\deg)=\Delta(\log^+|Q|)(V_\infty)$. It follows that

$$\Delta(\log^+|Q|)=\sum_{i=1}^m(\sum_{s_j\in S_i}m_j)\delta_{v_i}.$$

It follows that $\operatorname{Supp}\Delta(\log^+|Q|) = \{v_1, \cdots, v_m\}$ and moreover we have $m \leq \deg(Q)$. □

21.5. **The Dirichlet pairing.** Let ϕ, ψ be any two subharmonic functions on V_∞. Since ϕ is bounded from above one can define the Dirichlet pairing

$$\langle \phi, \psi \rangle := \int_{V_\infty^2} \alpha(v \wedge w) \Delta\phi(v) \Delta\psi(w) \in [-\infty, +\infty).$$

Observe that $\langle \phi, \psi \rangle = \langle \psi, \phi \rangle$.

Proposition 21.17. *The Dirichlet pairing induces a symmetric bilinear form on* $\mathrm{SH}(V_\infty)$ *that satisfies*

$$\langle \phi, \psi \rangle = \int_{V_\infty} \phi \, \Delta\psi \qquad (*).$$

Proof. The linearity and the symmetry are obvious from the definition. Equation (*) follows from Fubini's Theorem. □

We shall prove

Theorem 21.18 (Hodge inequality)**.** *For any two subharmonic functions ϕ, ψ, we have*

$$(\phi(-\deg)\psi(-\deg) - \langle \phi, \psi \rangle)^2 \leq (\phi(-\deg)^2 - \langle \phi, \phi \rangle)(\psi(-\deg)^2 - \langle \psi, \psi \rangle).$$

Proof of the Theorem 21.18. We first need the following

Proposition 21.19. *Let ϕ, ψ be two subharmonic functions in* $\mathrm{SH}(V_\infty)$. *Then there exists a sequence of compactifications $X_n \in \mathcal{C}$, $n \geq 0$ such that X_{n+1} dominates X_n for $n \geq 0$ and $\langle \phi, \psi \rangle = \lim_{n \to \infty} \langle R_{\Gamma_{X_n}} \phi, R_{\Gamma_{X_n}} \psi \rangle$.*

We only have to prove our theorem in the case $\Delta\phi$ and $\Delta\psi$ are supported on a finite subtree T of V_∞. Set $t(v) := -\alpha(v)$ for $v \in T$. Denote by E the set of all edges of T, v_1^I, v_2^I the two endpoints of I and $\vec{v}_1^I, \vec{v}_2^I$ the two direction at v_1^I and v_2^I. Denote by $\{v_1, \cdots, v_l\}$ the set of all endpoints and branch points in T and T_v the set of direction at v in T.

By integration by parts, we have

$$\int_I \phi \frac{d^2\psi}{dt^2} = \int_{V_\infty} \phi_I \frac{d^2\psi_I}{dt^2} = -\int_I \frac{d\phi}{dt}\frac{d\psi}{dt} dt$$

for all $I \in E$. Then we have

$$\langle \phi, \psi \rangle = \int_{\mathbb{V}_\infty} \phi(v)\psi(-\deg)\delta_{-\deg}(v) + \sum_{I \in E} \int_I \phi \frac{d^2\psi}{dt^2} = \phi(-\deg)\psi(-\deg) - \int_T \frac{d\phi}{dt}\frac{d\psi}{dt} dt.$$

It follows that $\langle \phi, \psi \rangle = \langle \psi, \phi \rangle$, and by Cauchy inequality, we get

$$(\phi(-\deg)\psi(-\deg) - \langle \phi, \psi \rangle)^2 \leq (\phi(-\deg)^2 - \langle \phi, \phi \rangle)(\psi(-\deg)^2 - \langle \psi, \psi \rangle).$$

□

Proof of Proposition 21.19. By Proposition 21.7, there exists a sequence of compactifications $X_n \in \mathcal{C}$ $n \geq 0$ such that X_{n+1} dominates X_n for $n \geq 0$ and $R_{\Gamma_{X_n}}\phi$ (resp. $R_{\Gamma_{X_n}}\psi$) decreases pointwise to ϕ (resp. ψ).

We have

$$|\langle\phi,\psi\rangle - \langle R_{\Gamma_{X_n}}\phi, R_{\Gamma_{X_n}}\psi\rangle| \leq \Big|\int_{V_\infty} R_{\Gamma_{X_n}}(\phi)\Delta R_{\Gamma_{X_n}}(\psi) - \int_{V_\infty}\phi\Delta R_{\Gamma_{X_n}}\psi\Big|$$
$$+\Big|\int_{V_\infty}\phi\Delta R_{\Gamma_{X_n}}\psi - \int_{V_\infty}\phi\Delta\psi\Big|.$$

Observe that

$$\Big|\int_{V_\infty} R_{\Gamma_{X_n}}(\phi)\Delta R_{\Gamma_{X_n}}(\psi) - \int_{V_\infty}\phi\Delta R_{\Gamma_{X_n}}\psi\Big| \to 0$$

and

$$\Big|\int_{V_\infty}\phi\Delta R_{\Gamma_{X_n}}\psi - \int_{V_\infty}\phi\Delta\psi\Big| \to 0$$

by monotone convergence. It follows that

$$|\langle\phi,\psi\rangle - \langle R_{\Gamma_{X_n}}\phi, R_{\Gamma_{X_n}}\psi\rangle| \to \infty$$

as $n \to \infty$.

□

Finally, we collect two useful results.

Proposition 21.20. *Pick any two subharmonic functions $\phi,\psi \in \mathrm{SH}(V_\infty)$. For any finite subtree $T \subset V_\infty$ one has*

$$\langle R_T\phi, R_T\psi\rangle \geq \langle\phi,\psi\rangle\ .$$

Proof. Since $R_T\phi \geq \phi$, for any $\psi \in \mathrm{SH}(V_\infty)$ we have $\langle R_T\phi,\psi\rangle = \int_{V_\infty} R_T\phi\Delta\psi \geq \int_{V_\infty}\phi\Delta\psi = \langle\phi,\psi\rangle$. It follows that

$$\langle R_T\phi, R_T\psi\rangle \geq \langle\phi, R_T\psi\rangle \geq \langle\phi,\psi\rangle.$$

□

Proposition 21.21. *Pick any subharmonic function $\phi \in \mathrm{SH}(V_\infty)$. For any finite subtree $T \subset V_\infty$ one has*

$$\langle R_T\phi, R_T\phi\rangle \geq \langle\phi,\phi\rangle$$

and the equality holds if and only if $\Delta\phi$ is supported on T.

Proof. By Proposition 21.20, we only have to show that $\langle R_T\phi, R_T\phi\rangle > \langle\phi,\phi\rangle$ when $\Delta\phi$ is not supported on T.

Suppose that $\Delta\phi$ is not supported on T. It follows that $\Delta\phi(V_\infty\setminus T) > 0$. Pick $X \in \mathcal{C}$ such that $r_{X*}\Delta\phi(V_\infty\setminus T) > 0$, and set $Y := T\cup\Gamma_X$, so that Y is a finite tree.

Since $\langle R_T(\phi), R_T(\phi)\rangle \geq \langle R_Y(\phi), R_Y(\phi)\rangle \geq \langle\phi,\phi\rangle$, by replacing ϕ by $R_Y\phi$, we may suppose that $\Delta\phi$ is supposed by Y. There exists a connected component U of $Y\setminus T$ satisfying $\int_U\Delta\phi > 0$. There exists a unique point $y_0 \in \overline{U}\cap T$ where $\overline{U}$ is the closure of U in Y. It follows that $\phi(y) < \phi(y_0) = R_T\phi(y)$ for all $y \in U$. Then we conclude that

$$\langle\phi,\phi\rangle = \int_Y \phi\Delta\phi = \int_{T\setminus U}\phi\Delta\phi + \int_U \phi\Delta\phi$$
$$< \int_{T\setminus U}\phi\Delta\phi + \int_U R_T\phi\Delta\phi \le \int_{T\setminus U} R_T\phi\Delta\phi + \int_U R_T\phi\Delta\phi$$
$$= \int_Y R_T\phi\Delta\phi = \int_Y \phi\Delta R_T(\phi)$$
$$= \int_Y R_T(\phi)\Delta R_T(\phi) = \langle R_T(\phi), R_T(\phi)\rangle.$$

□

21.6. **Positive subharmonic functions.** We prove here a technical result that will play an important role in the next section.

For any set $S \subset V_\infty$ we define $B(S) := \cup_{v\in S}\{w,\, w \ge v\}$.

Proposition 21.22. *Let ϕ be a function in $\mathrm{SH}^+(\mathrm{V}_\infty)$ such that $\langle\phi,\phi\rangle = 0$ and* $\mathrm{Supp}\Delta\phi = \{v_1, \cdots, v_s\}$ *where s is a positive integer.*

Then for any finite set $S \subseteq B(\{v_1,\cdots,v_s\})$ satisfying $\{v_1,\cdots,v_s\} \not\subseteq S$, there exists a function $\psi \in \mathrm{SH}^+(\mathrm{V}_\infty)$ such that

- *$\psi(v) = 0$ for all $v \in B(S)$;*
- *$\langle\psi,\psi\rangle > 0$.*

Example 21.23. Let $Q \in k[x,y]$ be any nonconstant polynomial. Proposition 21.16 implies that $\log^+|Q| \in \mathrm{SH}^+(\mathrm{V}_\infty)$, $\langle\log^+|Q|, \log^+|Q|\rangle = 0$ and $\#\mathrm{Supp}\Delta\log^+|Q| < \infty$ so that the preceding proposition applies to $\phi = \log^+|Q|$.

Proof. Write $\Delta\phi = \sum_{i=1}^s r_i\delta_{v_i}$ with $r_i > 0$. Since $\langle\phi,\phi\rangle = 0$ we have $\phi(v_i) = 0$ for all i. Observe now that the restriction of ϕ to any segment $[-\deg, v_i]$ is not locally constant. It follows that the sets $B(\{v_i\})$ are disjoint, or in other words that $v_i \wedge v_j < v_i$ for any $i \neq j$.

Suppose first that there exists an index $i \in \{1,\cdots,s\}$ such that $S\cap B(\{v_i\}) = \emptyset$, and denote by T the convex hull of $\{-\deg, v_1,\cdots,v_s\} \setminus \{v_i\}$. Then $\psi := R_T\phi$ satisfies all the required conditions.

Otherwise we may suppose that $v_1 \notin S$ and pick $w_1 \in S$ satisfying $w_1 > v_1$.

Choose any $v_1' < v_1$ such that $(\mathrm{Supp}\,\Delta\phi) \cap B(\{v_1'\}) = \{v_1\}$, and $w^1 \in (v_1', v_1)$, $w^2 \in (v_1, w_1)$ such that $\alpha(w^1) - \alpha(v_1) = \alpha(v_1) - \alpha(w^2)$. The subharmonic function $\psi := \sum_{i=2}^s r_i g_{v_i} + \frac{r_1}{2}(g_{w^1} + g_{w^2})$ satisfies all required conditions. □

21.7. **The class of $\mathbb{L}^2$ functions.** We define $\mathbb{L}^2(V_\infty)$ to be the set of functions $\phi : \{v \in V_\infty |\ \alpha(v) > -\infty\} \to \mathbb{R}$ such that $\phi = \phi_1 - \phi_2$ on $\{v \in V_\infty |\ \alpha(v) > -\infty\}$ with $\phi_i \in \mathrm{SH}(V_\infty)$ and $\langle\phi_i,\phi_i\rangle > -\infty$ for $i = 1, 2$. Then $\mathbb{L}^2(V_\infty)$ is a vector space.

For sake of convenience, we shall always extend ϕ to V_∞ by setting $\phi(v)$ to be an arbitrary number in $\phi(v) \in [\liminf_{w<v}\phi(w), \limsup_{w<v}\phi(w)]$ when $\alpha(v) = -\infty$.

Observe that by Proposition 21.19 (iii), we have $\langle \phi_1, \phi_2 \rangle > -\infty$ so that the pairing $\langle \cdot, \cdot \rangle$ extends to $\mathbb{L}^2(V_\infty)$ as a symmetric bilinear form and the Hodge inequality 21.18 is still valid.

All bounded subharmonic functions are contained in $\mathbb{L}^2(V_\infty)$. In particular, $g_v \in \mathbb{L}^2(V_\infty)$ if $\alpha(v) > -\infty$ and $\mathrm{SH}^+(\mathrm{V}_\infty) \subseteq \mathbb{L}^2(V_\infty)$.

22. Proof of the Main Theorem

22.1. **First reductions.** Let us recall the setting from the introduction. Let $R := k[x, y]$ denote the ring of polynomials in two variables over an algebraically closed field k. Let S be a finite set of valuations on R that are trivial on k. We define $R_S = \cap_{v \in S}\{P \in R,\, v(P) \geq 0\}$. This is a k-subalgebra of $k[x, y]$ and we denote by $\delta(S) \in \{0, 1, 2\}$ the transcendence degree of its field of fraction over k.

We first do the following reduction.

Lemma 22.1. *Given any finite set of valuations S on R that are trivial on k and centered at infinity, we have $\delta(S) = 2$ if and only if $\delta(\bar{S}) = 2$.*

Proof. Since $R_S \subset R_{\bar{S}}$ it follows that $\delta(S) = 2$ implies $\delta(\bar{S}) = 2$.

Conversely suppose that $\delta(\bar{S}) = 2$. Let $v_1, \ldots, v_s$ be the rank 2 valuations in S whose associated valuations $\bar{v}_1, \ldots, \bar{v}_s$ in V_∞ are divisorial. Observe that when $v \in S \setminus \{v_1, \ldots, v_s\}$ then $R_{\{v\}} = R_{\{\bar{v}\}}$.

By Theorem 22.7 (ii), there is a nonzero polynomial $P \in R$ such that $v(P) > 0$ for all $v \in \bar{S}$. Pick any polynomial Q. Then for m large enough, we have $v(P^m Q) > 0$ for all $v \in \bar{S}$. In particular, we get $\bar{v}_i(P^m Q) > 0$ which implies $v_i(P^m Q) > 0$. We conclude that $P^m Q$ also belongs to R_S so that the fraction field of R_S is equal to $k(x, y)$ and $\delta(S) = 2$. □

In the rest of this section, let $S = \{v_1, \cdots, v_l\} \subset V_\infty$ be a finite set. It will be convenient to use the following terminology.

Definition 22.2. A subset of valuations $S \subset V_\infty$ is said to be *rich* when $\delta(S) = 2$.

We shall also write:

- $S^{\min} \subset S$ for the set of valuations that are minimal for the order relation restricted to S;
- $S_+ \subset S$ for the subset of valuations in S with finite skewness;
- $S_+^{\min} \subset S^{\min}$ for the subset of valuations in $S^{\min}$ with finite skewness;
- $B(S)$ for the set of all valuations $v \in V_\infty$ such that $v \geq w$ for some $w \in S$;
- $B(S)^\circ$ for the interior of $B(S)$;
- $M(S)$ for the symmetric matrix whose entries are given by $[\alpha(v_i \wedge v_j)]_{1 \leq i,j \leq l}$.

The set $B(S)$ is compact and has as many connected components as there are elements of $S^{\min}$. In fact, the boundary of any connected component of $B(S)$ is a singleton, and this point lies in $S^{\min}$. Observe that $R_{S^{\min}} = R_S$.

The next result follows directly from the Hodge index theorem in the case of divisorial valuations and by a continuity argument in the general case.

Lemma 22.3. *Let S be a finite subset of V_∞ such that $\alpha(v) > -\infty$ for all $v \in S$. Then the symmetric matrix $M(S)$ has at most one non-negative eigenvalue.*

Definition 22.4. Let S be a finite subset of V_∞. The symmetric matrix $M(S)$ is said to be negative definite if and only if the matrix $[(\max\{\alpha(v_i \wedge v_j), -t\}]_{1 \leq i,j \leq m}$ is negative definite for t large enough.

Observe that for $-t$ large enough the function $t \mapsto \det(\max\{\alpha(v_i \wedge v_j), t\})_{1\le i,j\le l}$ is a polynomial, and that we defined

$$\chi(S) = \lim_{t\to-\infty} (-1)^{\#S} \det(\max\{\alpha(v_i \wedge v_j), t\})_{1\le i,j\le l} \in \mathbb{R} \cup \{\pm\infty\}$$

with the convention $\chi(\emptyset) := 1$. When $S = S_+$ we simply have $\chi(S) := (-1)^{\#S} \det((\alpha(v_i \wedge v_j))_{1\le i,j\le l})$.

With this definition, lemma 22.3 implies immediately

Lemma 22.5. *Let S be a finite subset of V_∞. The symmetric matrix $M(S)$ is negative definite if and only if $\chi(S) > 0$.*

Finally we make the following reduction

Lemma 22.6. *Let S be a finite subset of V_∞. We have $\chi(S) > 0$ if and only if $\chi(S_+^{\min}) > 0$.*

Proof. Suppose that $S = \{v_1, \cdots, v_l\}$ and $S_+ = \{v_1, \cdots, v_{l'}\}$ where $l' \le l$. When $-t$ is large enough the function $t \mapsto \det(\max\{\alpha(v_i \wedge v_j), t\})_{1\le i,j\le l}$ is a polynomial with leading term $\chi(S_+)t^{l-l'}$. It follows that $\chi(S) > 0$ if and only if $\chi(S_+) > 0$. Now, we may suppose that $S = S_+$.

Since $S^{\min}$ is a subset of S, if $M(S)$ is negative definite then $M(S^{\min})$ is negative definite. By Lemma 22.5, we conclude the " only if" part.

To prove the " if" part, we suppose that $\chi(S^{\min}) > 0$. For any $w \in S^{\min}$, set $S_w := \{v \in S | \ v \ge w\}$. It follows that $S = \coprod_{w\in S^{\min}} S_w$. For any $w \in S^{\min}$, denote by $C(S_w)$ the set of valuations taking forms $\wedge_{v\in S'_w} v$ where S'_w is a subset of S_w. Set $C(S) := \coprod_{w\in S^{\min}} C(S_w)$. We complete the proof of our theorem by induction on the number $\#C(S) - \#S^{\min}$.

If $\#C(S) - \#S^{\min} = 0$, then $S = C(S) = S^{\min}$. Our theorem trivially holds.

If $\#C(S) - \#S^{\min} \ge 1$, there exists $w \in S^{\min}$ satisfying $C(S_w) \ge 2$. Let w_0 be a maximal element in $C(S_w)$ then $w_0 > w$. Let w_1 be the maximal element in $[w, w_0) \cap S_w$ and set $S_1 := C(S) \setminus \{w_0\}$. For any valuation $v \in C(S) \setminus \{w_0\}$, we have $v \wedge w_0 = v \wedge w_1$. Then we have

$$M(C(S)) = \begin{pmatrix} \alpha(w_0) & \dots & \alpha(w_0 \wedge v) & \dots & \alpha(w_0 \wedge w_1) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha(v \wedge w_0) & \dots & \alpha(v) & \dots & \alpha(v \wedge w_1) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha(w_1 \wedge w_0) & \dots & \alpha(w_1 \wedge v) & \dots & \alpha(w_1) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(w_0) & \dots & \alpha(w_1 \wedge v) & \dots & \alpha(w_1) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha(v \wedge w_1) & \dots & \alpha(v) & \dots & \alpha(v \wedge w_1) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha(w_1) & \dots & \alpha(w_1 \wedge v) & \dots & \alpha(w_1) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

It follows that

$$M(C(S)) = \begin{pmatrix} \alpha(w_0) & \dots & \alpha(w_1 \wedge v) & \dots & \alpha(w_1) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha(v \wedge w_1) & \dots & \alpha(v) & \dots & \alpha(v \wedge w_1) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha(w_1) & \dots & \alpha(w_1 \wedge v) & \dots & \alpha(w_1) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & 0 & \dots & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \alpha(w_0) - \alpha(w_1) & \dots & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \alpha(v) & \dots & \alpha(v \wedge w_1) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \alpha(w_1 \wedge v) & \dots & \alpha(w_1) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$\begin{pmatrix} 1 & \dots & 0 & \dots & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

It follows that $\chi(C(S)) = (\alpha(w_1) - \alpha(w_0))\,\chi(S_1)$. Since $C(S_1) = S_1 = C(S)\setminus\{w_0\}$ and $S_1^{\min} = S^{\min}$, we have $\chi(S_1) > 0$ by induction hypotheses. Since $\alpha(w_1) - \alpha(w_0) > 0$, we have $\chi(C(S)) > 0$ and $M(C(S))$ is negative definite. Since $M(S)$ is a principal submatrix of $M(C(S))$, it is also negative definite. It follows that $\chi(S) > 0$. □

22.2. **Characterization of rich sets using potential theory on V_∞.** As an important intermediate step towards our Main Theorem we shall prove the following characterization of rich subsets of V_∞ in terms of the existence of adapted functions in $\mathbb{L}^2(V_\infty)$.

Theorem 22.7. *Let S be a finite set of valuations in V_∞. Then the following statements are equivalent.*

(i) *The set S is rich, i.e. $\delta(S) = 2$.*
(ii) *There exists a nonzero polynomial $P \in R_S$ such that $v(P) > 0$ for all $v \in S$.*
(iii) *There exists a valuation $v \in S$ and a nonzero polynomial $P \in R_S$ such that $v(P) > 0$.*
(iv) *There exists a function $\phi \in \mathrm{SH}^+(\mathrm{V}_\infty)$ such that $\phi(v) = 0$ for all $v \in B(S)$ and $\langle \phi, \phi \rangle > 0$.*
(v) *There exists a function $\phi \in \mathbb{L}^2(V_\infty)$ such that $\phi(v) = 0$ for all $v \in B(S)$ and $\langle \phi, \phi \rangle > 0$.*
(vi) *There exists a finite set $S' \subseteq V_\infty$ such that $S \subseteq B(S')^\circ$ and S' is rich.*

Moreover when these conditions are satisfied, then the fraction field of R_S is equal to $k(x, y)$.

Proof. Observe first that when (ii) is satisfied, then for any polynomial Q there exists an integer n such that QP^n belongs to R_S. This implies that $k[x,y]$ is included in the fraction field of R_S hence the latter is equal to $k(x,y)$.

We now prove the equivalence between the six statements. The three implications (ii)⇒(iii), (iv)⇒(v) and (vi)⇒(i) are immediate.

(i)⇒(ii). Replacing S by $S^{\min}$, we may suppose that $S = S^{\min}$. By contradiction, we suppose that $v(P) = 0$ for all $v \in S$ and all $P \in R_S \setminus \{0\}$.

For every $v \in S$, we have $\min\{v(x), v(y)\} = -1$. Since k is infinite, for a general linear polynomial $Q \in k[x,y]$, we have $v(Q) < 0$ for all $v \in S$. Since the transcendence degree of $\mathrm{Frac}\,(R_S)$ over k is 2, we have

$$\sum_{i=0}^{m} a_i Q^i = 0$$

where $m \geq 1$, $a_i \in R_S$. We may suppose that $a_m \neq 0$. Let v be a valuation in S. It follows that $v(a_i Q^i) = iv(Q) + v(a_i) \geq iv(Q) > mv(Q)$ for $i = 1, \cdots, m-1$. If $v(a_m) = 0$ for some v, we have $v(\sum_{i=0}^{m} a_i Q^i) = mv(Q) < 0$ which is a contradiction. It follows that $v(a_m) > 0$ for all $v \in S$.

(iii)⇒(iv). By assumption there exists a polynomial $P \in R_S$ and a valuation $v_0 \in S$ for which $v_0(P) > 0$. It follows that $\mathrm{Supp}(\Delta \log^+ |P|) \not\subseteq S$. Since we have $S \subset B(\mathrm{Supp}\, \Delta \log^+ |P|)$, Proposition 21.22 implies the existence of $\phi \in \mathrm{SH}^+(\mathrm{V}_\infty)$ such that $\phi(v) = 0$ for all $v \in B(S)$. And we get $\langle \phi, \phi \rangle > 0$ as required.

The proof of the implication (v)⇒(vi) is the core of our Theorem 22.7. We state it as a separate Proposition 22.8 and prove it below. □

Proposition 22.8. *Let S be a finite subset of V_∞. Suppose that there exists a function $\phi \in \mathbb{L}^2(V_\infty)$ such that $\phi(v) = 0$ for all $v \in B(S)$, and $\langle \phi, \phi \rangle > 0$.*

Then there exists a finite set S' of divisorial valuations such that $S \subseteq B(S')^\circ$ and $\mathrm{Frac}\,(R_{S'}) = k(x,y)$.

The proof relies on the following lemma that is a corollary of [11, Proposition 3.2]. For the convenience of the reader, we give a simplified proof of it at the end of this section.

Lemma 22.9. *Let X be any smooth projective compactification of $\mathbb{A}^2_k$. Let C be a reduced curve contained in $X \setminus \mathbb{A}^2_k$, and set $U := X \setminus C$.*

If there exists a $\mathbb{R}$-divisor A supported on C such that $A^2 > 0$, then the fraction field of the ring of regular functions on U is equal to $k(x,y)$.

Proof of Proposition 22.8. We may assume $S = S^{\min}$. Let T_S be the convex hull of $S \cup \{-\deg\}$. This is a finite tree. Write $\phi = \phi_1 - \phi_2$ where both functions ϕ_i lie in $\mathrm{SH}(V_\infty)$ and satisfy $\langle \phi_i, \phi_i \rangle > -\infty$ for $i = 1, 2$. By Proposition 21.19 and Proposition 21.20, there exists a finite tree T containing T_S such that

$$\langle R_T(\phi_1), R_T(\phi_2) \rangle \leq \langle \phi_1, \phi_2 \rangle + \frac{1}{2} \langle \phi, \phi \rangle.$$

Using Proposition 21.20, we get

$$\begin{aligned}\langle R_T(\phi_1)-R_T(\phi_2),R_T(\phi_1)-R_T(\phi_2)\rangle &\geq \langle\phi_1,\phi_1\rangle+\langle\phi_2,\phi_2\rangle-2\langle R_T(\phi_2),R_T(\phi_1)\rangle\\ &\geq \langle\phi_1,\phi_1\rangle+\langle\phi_2,\phi_2\rangle-2\langle\phi_1,\phi_2\rangle-\frac{1}{2}\langle\phi,\phi\rangle\\ &= \frac{1}{2}\langle\phi,\phi\rangle>0\ .\end{aligned}$$

Replacing ϕ by $R_T(\phi_1)-R_T(\phi_2)$, we may thus assume that ϕ is the difference of two functions $\phi_1,\phi_2\in\mathrm{SH}(V_\infty)$ such that $\Delta\phi_1$ and $\Delta\phi_2$ are supported on a finite tree T whose set of vertices is the union of S and a finite set of divisorial valuations.

Proposition 22.10. *Let T be any finite subtree of V_∞ containing $-\deg$, and T' be any dense subset of T. Suppose $\phi\in\mathbb{L}^2(V_\infty)$ is a function such that $\Delta\phi$ is supported on T and $\phi(v)\in\mathbb{R}$ for any end point v of T.*

Then for any $\epsilon>0$ there exists a piecewise linear function ϕ' such that

(1) *the support of $\Delta\phi'$ is a finite collection of valuations that belong to T';*
(2) *$\phi=\phi'$ at any endpoint of T;*
(3) *$|\langle\phi,\phi\rangle-\langle\phi',\phi'\rangle|\leq\epsilon$.*

Applying this lemma to $\epsilon=\frac{1}{2}\langle\phi,\phi\rangle$, and to the set T' consisting of all divisorial valuations lying in $T\setminus S$, we obtain a piecewise linear function ϕ' such that $\langle\phi',\phi'\rangle>0$ and the properties (1) – (3) above are satisfied.

Let S' be the set of extremal points of the support of $\Delta\phi'$. Observe that thanks to our choice of T' and the fact that $\phi|_S=0$, we have $S\subset B(S')^\circ$ and $\phi'|_S=0$.

Now pick any smooth projective compactification X of $\mathbb{A}^2_k$ such that any valuation in $\operatorname{Supp}\Delta\phi'\cup S'$ has codimension 1 center in X. Denote by $E_1,\cdots,E_s$ the centers of valuations in S', and by $E_{s+1},\cdots,E_l$ the other irreducible components of $X\setminus\mathbb{A}^2$. Introduce now the $\mathbb{R}$-divisor

$$A':=\sum_{i=1}^{l}b_{E_i}\phi'(v_{E_i})E_i\ .$$

By [4, Lemma A.2.],

$$(\sum_{j=1}^{l}b_{E_j}g_{v_{E_i}}(v_{E_j})E_j\cdot E_k)=0$$

when $k\neq i$, and

$$(\sum_{j=1}^{l}b_{E_j}g_{v_{E_i}}(v_{E_j})E_j\cdot E_k)=b_{E_i}^{-1}$$

when $k=i$. It follows that $\check{E}_i=b_{E_i}\sum_{j=1}^{l}b_{E_j}g_{v_{E_i}}(v_{E_j})E_j$ for all $i=1,\cdots,l$.

Write $\phi'=\sum_{i=1}^{l}c_ig_{v_{E_i}}$. Then we have

$$A'=\sum_{i=1}^{l}b_{E_i}\phi'(v_{E_i})E_i=\sum_{i=1}^{l}b_{E_i}\left(\sum_{j=1}^{l}c_jg_{v_{E_j}}(v_{E_i})\right)E_i$$

$$= \sum_{j=1}^{l} b_{E_j}^{-1} c_j \left(b_{E_j} \sum_{i=1}^{l} b_{E_i} g_{v_{E_j}}(v_{E_i}) E_i \right) = \sum_{j=1}^{l} b_{E_j}^{-1} c_j \check{E}_j.$$

It follows that

$$(A')^2 = \left((\sum_{i=1}^{l} b_{E_i} \phi'(v_{E_i}) E_i) \cdot (\sum_{i=1}^{l} b_{E_i}^{-1} c_i \check{E}_i) \right)$$

$$= \sum_{i=1}^{l} c_i \phi'(E_i) = \langle \phi', \phi' \rangle > 0.$$

Since $\phi'|_{S'} = 0$ and S' is the set of extremal points of the support of $\Delta\phi'$ it follows that $\phi'(v_{E_i}) = 0$ for any $v_{E_i} \in B(S')$. In other words, the support C of A' contains no component C_i such that $v_{C_i} \in B(S')$. Now pick $P \in \Gamma(X \setminus C, O_X)$. Then $v_{E_j}(P) \geq 0$ for all $j = 1, \cdots, s$ hence $v(P) \geq 0$ for all $v \in B(S')$ and we conclude that

$$\Gamma(X \setminus C, O_X) \subset R_{S'} = \cap_j \{P \in k[x,y] |\ v_{E_j}(P) \geq 0\} \ .$$

One completes the proof using Lemma 22.9. ☐

Proof of Proposition 22.10. Write $\phi = \phi_1 - \phi_2$ where both functions ϕ_i lie in $\mathrm{SH}(V_\infty)$ and satisfy $\langle \phi_i, \phi_i \rangle > -\infty$ for $i = 1, 2$.

Step 1. We first suppose that all end points of T are contained in T'.

For any $n \geq 0$, let T_n be a subset of T' such that

- all end points of T are contained in T_n;
- for any end point w of T and any point $v \in [-\deg, w]$, there exists a point $v' \in [-\deg, w] \cap T_n$ such that $|\alpha(v) - \alpha(v')| \leq 1/2^{n+1}$.

For $i = 1, 2$, let ϕ_i^n be the unique piecewise linear function on T such that $\phi_i^n(v) = \phi_i(v)$ for all $v \in T_n$. We extend ϕ_i^n to a function on V_∞ by $\phi_i^n(v) := \phi_i^n(r_T(v))$ for all $v \in V_\infty$. We see that

(i) $\phi_i^n \in \mathrm{SH}(V_\infty)$;
(ii) $\Delta\phi_i^n$ is supported on T;
(iii) $\int_T \Delta\phi_i^n = \int_T \Delta\phi_i$;
(iv) $0 \leq \phi_i^n(v) - \phi_i(v) \leq \int_T \Delta\phi_i / 2^n$ for all $v \in V_\infty$.

Set $\phi^n = \phi_1^n - \phi_2^n$. We have

$$\langle \phi^n, \phi^n \rangle = \sum_{i=1,2; j=1,2} (-1)^{i+j} \int_T \phi_i^n \Delta\phi_j^n$$

$$= \sum_{i=1,2; j=1,2} (-1)^{i+j} (\int_T \phi_i \Delta\phi_j + \int_T (\phi_i^n - \phi_i) \Delta\phi_j^n + \int_T (\phi_j^n - \phi_j) \Delta\phi_i)$$

$$\geq \langle \phi, \phi \rangle - 2(\int_T (\phi_1^n - \phi_1) \Delta\phi_2^n + \int_T (\phi_2^n - \phi_2) \Delta\phi_1)$$

$$\geq \langle \phi, \phi \rangle - 4 \int_T \Delta\phi_1 \int_T \Delta\phi_2 / 2^n.$$

Then we have $\langle \phi^n, \phi^n \rangle > 0$ for n large enough. Set $\phi' := \phi^n$, then we conclude our Proposition.

Step 2. We complete the proof by induction on the number n_T of end points of T not contained in T'.

When $n_T = 0$, by Step 1, our Proposition holds.

When $n_T \geq 1$, there exists an end point w' of T not contained in T'. There exists an increasing sequence $v_n \in [-\deg, w]$ tending to w satisfying $\phi(v_n) \to \lim_{v<w, v\to w} \phi(v) = \phi(w)$. Since w is an end point, we may suppose that $T_n := T \setminus (v_n, w]$ is a finite tree. There exists a function $g \in \mathrm{SH}^+(\mathrm{V}_\infty)$ such that $\mathrm{Supp}\,\Delta g \subseteq [-\deg, w]$ and it is strict decreasing on $[-\deg, w]$. By replacing ϕ_i by $\phi_i + g$ for $i = 1, 2$, we may suppose that ϕ_i's are strict decreasing on $[-\deg, w]$.

When $\phi(v_n) = \phi(w)$, set $\psi_n := R_{T_n}\phi$.

When $\phi(v_n) > \phi(w)$, the function $\phi_1(v) - \phi_2(v_n)$ is decreasing. Observe that $\phi_1(v_m) - \phi_2(v_n) = \phi(v_m) - \phi_2(v_n) + \phi_2(v_m) \to \phi(w) - \phi_2(v_n) + \phi_2(w)$ when $m \to \infty$. Since ϕ_2 is strict decreasing on $[-\deg, w]$, we have $\phi_2(v_n) > \phi_2(w)$ and then there exists $v' \in (v_n, w)$ such that $\phi_1(v') - \phi_2(v_n) = \phi(w)$, set $\psi_n := R_{T\setminus(v',w]}\phi_1 - R_{T_n}\phi_2$.

When $\phi(v_n) < \phi(w)$, by the previous argument for $-\phi$, there exists $v' \in (v_n, w)$ such that $\phi_1(v_n) - \phi_2(v') = \phi(w)$, set $\psi_n := R_{T_n}\phi_1 - R_{T\setminus(v',w]}\phi_2$.

By Proposition 21.19 and Proposition 21.20, there exists $n \geq 0$ such that $|\langle\psi_n, \psi_n\rangle - \langle\phi, \phi\rangle| \leq \varepsilon/2$. Since T' is dense in T, there exists $w' \in (v_n, w) \cap T'$ such that $\mathrm{Supp}\,\Delta\psi_n \subseteq T \setminus (v_n, w]$. Apply the induction hypotheses to ψ_n, there exists a piecewise linear function ϕ' such that

- the support of $\Delta\phi'$ is a finite collection of valuations that belong to T';
- $\phi' = \psi_n = \phi$ at any endpoint of T;
- $|\langle\psi_n, \psi_n\rangle - \langle\phi', \phi'\rangle| \leq \epsilon/2$.

It follows that $|\langle\phi, \phi\rangle - \langle\phi', \phi'\rangle| \leq \epsilon$ which concludes our Proposition. □

Proof of Lemma 22.9. Decompose $A = A^+ - A^-$ into its positive and negative parts. Since $(A^+)^2 + (A^-)^2 - 2A^+A^- = A^2 > 0$, and $A^+A^- \geq 0$, we have $(A^+)^2 > 0$ or $(A^-)^2 > 0$. Replacing A by A^+ or A^-, we may thus suppose that A is effective.

Pertubing slightly the coefficients of A, we can also impose that A is a $\mathbb{Q}$-divisor. Let $A = P + N$ be the Zariski decomposition of A, see [6, Theorem 2.3.19]. Here P is a nef and effective $\mathbb{Q}-$divisor, N is an effective $\mathbb{Q}-$divisors, and they satisfy $P \cdot N = 0$ and $N^2 < 0$. It follows that $P^2 \geq P^2 + N^2 = A^2$. Replacing A by a suitable multiple of P we may thus assume that A is an effective nef integral divisor with $A^2 > 0$. Now pick any effective integral divisor D whose support is equal to the union of all components of $X \setminus \mathbb{A}^2_k$ that are not contained in C. For n large enough $nA - D$ is big, hence $H^0(nA - D, X) \neq 0$. Since

$$H^0(nA - D, X) = \{P \in k(x, y) |\ \mathrm{div}(P) + nA \geq D\}\ ,$$

we may find $P \in k(x, y)$ such that $\mathrm{div}(P) + nA \geq D$. Since A is supported on $X \setminus U$ and D is effective, P is a regular function on U. Now pick any polynomial $Q \in k[x, y]$. For m large enough, $v_E(P^mQ) \geq 0$ for any component E of the support of D, which implies P^mQ to be regular on U. This shows that Q is included in the fraction field of $\Gamma(U, O_X)$ hence the latter is equal to $k(x, y)$. □

22.3. **Reduction to the case of finite skewness.** Recall that given a finite set $S \subset V_\infty$, we let $S_+^{\min}$ be the subset of S consisting of valuations that are minimal in S and of finite skewness.

Our aim is to prove

Theorem 22.11. *Let S be a finite subset of V_∞. Then S is rich if and only if $S_+^{\min}$ is rich.*

The proof relies on the following result of independent interest.

Theorem 22.12. *Let S be a finite set of valuations in V_∞. Suppose that there exists a function $\phi \in \mathrm{SH}(V_\infty)$ such that $\langle \phi, \phi \rangle > 0$ and $\phi(v) = 0$ for all $v \in B(S)$.*

For any integer $l \geq 0$, there exists a real number $M_l \leq 1$ such that for any set S' of valuations such that

(1) $S' \setminus B(S)$ has at most l elements and,
(2) $S' \setminus B(S) \subset \{v \in V_\infty | \ \alpha(v) \leq M_l\}$,

then there exists a function $\phi' \in \mathbb{L}^2(V_\infty)$ satisfying $\phi'(v) = 0$ for all $v \in B(S')$ and $\langle \phi', \phi' \rangle > 0$.

In the particular case where $S = \emptyset$, the previous result says the following.

Corollary 22.13. *For any positive integer $l > 0$, there exists a real number $M_l \leq 1$ such that given any valuations $v_1, \cdots, v_l$ satisfying $\alpha(v_i) \leq M_l$, there exists a function $\phi \in \mathbb{L}^2(V_\infty)$ satisfying $\phi'(v) = 0$ for all $v \in B(\{v_1, \cdots, v_l\})$ and $\langle \phi', \phi' \rangle > 0$.*

Proof of Theorem 22.11. As before, we may suppose that $S = S^{\min}$.

Since $S_+^{\min} \subseteq S$, we only have to show the "if" part. Suppose that $S_+^{\min}$ is rich, and set $l = \#(S \setminus S_+^{\min})$. Since $S_+^{\min}$ is rich, Theorem 22.7 implies the existence of a function $\phi \in \mathrm{SH}^+(\mathrm{V}_\infty)$ such that $\langle \phi, \phi \rangle > 0$ and $\phi(v) = 0$ for all $v \in B(S_+^{\min})$. Since $S \setminus B(S_+^{\min}) \subset \{\alpha = -\infty\}$ Theorem 22.12 then implies the existence of $\phi' \in \mathbb{L}^2(V_\infty)$ satisfying $\phi'(v) = 0$ for all $v \in B(S)$ and $\langle \phi', \phi' \rangle > 0$.

We conclude that S is rich by applying Theorem 22.7 once again. □

Proof of Theorem 22.12. We first make a couple of reductions. Let T_S be the convex hull of S. Replacing ϕ by $R_{T_S}(\phi)$, we may suppose that $\Delta\phi$ is supported on T_S. We can also scale ϕ so that $\phi(-\deg) = 1$ which implies $0 \leq \phi(v) \leq 1$ for all $v \in V_\infty$ since $\phi(v) = 0$ for all $v \in B(S)$.

Further, we may apply Theorem 22.7 (vi) and suppose $M_0 := \inf_S \alpha > -\infty$.

To simplify notation, set $r := \langle \phi, \phi \rangle > 0$.

We prove the theorem by induction on l. In the case $l = 0$, there is nothing to prove. Suppose that the result holds for $(l-1) \geq 0$ with $M_{l-1} \leq M_0$, and set $M_l := M_{l-1} - 2l/r$.

Suppose S' is a set of valuations satisfying the conditions (1) and (2) of the theorem. When $\#(S' \setminus B(S)) \leq l - 1$, we are done since $M_l < M_{l-1}$. So we have $\#(S' \setminus B(S)) = l$, and we write $S' \setminus B(S) = \{v_1, \cdots, v_l\}$. If there exist a pair of valuations v_i, v_j such that $\alpha(v_i \wedge v_j) \leq M_{l-1}$, then we may conclude by replacing S' by $(S' \setminus \{v_i, v_j\}) \cup \{v_i \wedge v_j\}$ and using the induction hypothesis.

Whence $\alpha(v_i\wedge v_j) > M_{l-1}$ when $i \neq j$. For each i, let v_i^0 be the unique valuation in V_∞ such that $v_i^0 \leq v_i$ and $\alpha(v_i^0) = M_{l-1}$, so that $v_i^0 \neq v_j^0$ when $i \neq j$. Define

$$\Phi_i = x_i(g_{v_i} - g_{v_i^0}) \in \mathbb{L}^2(V_\infty) \text{ with } x_i := \phi(v_i)/(M_{l-1} - \alpha(v_i)) \ .$$

Observe that $\Phi_i(-\deg) = 0$, $\Delta\Phi_i = x_i(\delta_{v_i} - \delta_{v_i^0})$; $1 \geq |\Phi_i| \geq 0$, and that $\Phi_i(v) = -\phi(v_i)$ when $v \geq v_i$. It follows that $\langle \Phi_i, \Phi_i\rangle = -x_i\phi(v_i)$ and $\langle \Phi_i, \Phi_j\rangle = 0$ when $i \neq j$.

Set

$$\phi' := \phi + \sum_{i=1}^{l} \Phi_i \ .$$

Then $\phi' \in \mathbb{L}^2(V_\infty)$, and it is not difficult to check that $\phi'(v) = 0$ for all $v \in B(S')$. Finally we have

$$\begin{aligned}
\langle \phi', \phi'\rangle &= \langle \phi', \phi\rangle + \sum_{i=1}^{l}\langle \phi', \Phi_i\rangle = \langle \phi', \phi\rangle - \sum_{i=1}^{l} x_i\phi'(v_i^0) \\
&= \langle \phi, \phi\rangle - \sum_{i=1}^{l} x_i\phi(v_i) \geq r - \sum_{i=1}^{l} \phi(v_i)^2/(M_{l-1} - \alpha(v_i)) \\
&\geq r - \sum_{i=1}^{l} 1/(M_{l-1} - \alpha(v_i)) \geq r/2 > 0 \ ,
\end{aligned}$$

which concludes the proof. □

22.4. **Proof of the Main Theorem.**

By Lemma 22.1, Lemma 22.6 and Theorem 22.11 we may suppose that $S = S_+^{\min}$.

Denote by T the convex hull of $S \cup \{-\deg\}$. To simplify notation, set $S = \{v_1, \cdots, v_l\}$ and $v_0 := -\deg$. Since $\alpha(v_0 \wedge v_0) = 1 > 0$, by Lemma 22.3, we have the following

Lemma 22.14. *The matrix $[\alpha(v_i \wedge v_j)]_{0\leq i,j\leq l}$ is invertible, and its determinant has the same sign as $(-1)^l$.*

We may thus find real numbers $a_0, \ldots, a_l$ such that

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & \alpha(v_1) & \ldots & \alpha(v_1 \wedge v_l) \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \alpha(v_1 \wedge v_l) & \ldots & \alpha(v_l) \end{pmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_l \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (*).$$

Lemma 22.15. *The subset S is rich if and only if a_0 is positive.*

Now observe that

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \alpha(v_1) & \dots & \alpha(v_1 \wedge v_l) \\ \dots & \dots & \dots & \dots \\ 1 & \alpha(v_1 \wedge v_l) & \dots & \alpha(v_l) \end{pmatrix} \begin{pmatrix} a_0 & 0 & \dots & 0 \\ a_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_l & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & \alpha(v_1) & \dots & \alpha(v_1 \wedge v_l) \\ \dots & \dots & \dots & \dots \\ 0 & \alpha(v_1 \wedge v_l) & \dots & \alpha(v_l) \end{pmatrix},$$

hence $a_0 > 0$ iff $\chi(S) := (-1)^l \det(\alpha(v_i \wedge v_j)_{1 \leq i,j \leq l}) > 0$ as required.

Proof of Lemma 22.15. Set $\phi^* := \sum_0^l a_i g_{v_i} \in \mathbb{L}^2(V_\infty)$. By (*), we have $\phi^*(-\deg) = 1$, $\phi^*(v) = 0$ for all $v \in B(S)$ and $\langle \phi^*, \phi^* \rangle = a_0$.

Suppose first that $a_0 > 0$ $\langle \phi^*, \phi^* \rangle = a_0 > 0$. It follows from Theorem 22.7 that S is rich.

Conversely if S is rich, then again by Theorem 22.7 there exists $\phi \in \mathrm{SH}^+(\mathrm{V}_\infty)$ such that $\phi(v) = 0$ for all $v \in B(S)$ and $\langle \phi, \phi \rangle > 0$. By replacing ϕ by $R_T(\phi)$, we may suppose that $\Delta\phi$ is supported on T, and by scaling, that $\phi(-\deg) = 1$.

Observe that on each connected component of $T \setminus (S \cup \{-\deg\})$, we have $\Delta(\phi - \phi^*) = \Delta(\phi - \phi^*) = \Delta\phi \geq 0$. The following lemma is basically the maximum principle for subharmonic functions on finite trees.

Lemma 22.16. *Let T be a finite subtree in V_∞ and S be the set of end points of T. Suppose that all points in S are with finite skewness. Let ϕ subharmonic function on $T \setminus S$ i.e. $\Delta\phi$ is a positive measure on $T \setminus S$. Then if there exists a point $w \in T \setminus S$ satisfying $\phi(w) = \sup\{\phi(v) | \, v \in T \setminus S\}$ then ϕ is constant in the connected component containing w.*

Since $\phi - \phi^*(v_i) = 0$ for all $i = 0, \cdots, l$, Lemma 22.16 implies that $\phi - \phi^* \leq 0$ on T. Then we conclude that

$$a_0 = \int \phi^* \Delta\phi^* \geq \int \phi\Delta\phi^* = \int \phi^*\Delta\phi \geq \int \phi\Delta\phi > 0.$$

□

Proof of Lemma 22.16. We suppose that there exists a point $w \in T \setminus S$ satisfying $\phi(w) = \sup\{\phi(v) | \, v \in T \setminus S\}$.

If w is not a branch point, then there exists open segment I in T containing w such that there are no branch points in I. Since $\Delta\phi|_I = \frac{d^2\phi}{dt^2}$, we get that $\phi|_I$ is convex. It follows that ϕ is constant on I.

If w is a branch point, we have $0 \leq \Delta\phi\{w\} = \sum_{\vec{w}} D_{\vec{w}}\phi$ where the sum is over all tangent directions $\overrightarrow{w}$ in T at w. Then there exists a direction $\vec{v}$ satisfying $D_{\vec{v}}\phi = \max\{D_{\vec{w}}\phi\}$ where the max is over all tangent directions $\overrightarrow{w}$ in T at w. Then we have $D_{\vec{v}}\phi \geq 0$. There exists a segment $[w, v')$ determining $\vec{v}$ and containing no branch points except w. Since ϕ is convex on $[w, v')$ and $D_{\vec{v}}\phi \geq 0$, it follows that ϕ is constant on $[w, v')$ and then $D_{\vec{w}}\phi = 0$ for all tangent directions $\overrightarrow{w}$ in T

at w. We conclude that there exists an open set U in T containing w such that ϕ is constant on U.

So the set $\{w|\ \sup\{\phi(v)|\ v \in T \setminus S\}\}$ is both open and closed. It is thus a union of connected components of $Y \setminus S$ which concludes our lemma. $\square$

23. Further remarks in the case $\chi(S) = 0$

In this section, we discuss the case when $\chi(S) = 0$ for some finite subset S of valuations in V_∞, and explore its relations with the condition $\delta(S) = 1$.

As before, k is any algebraically closed field. To simplify the discussion we shall always assume that $S = S^{\min}$, that is no two different valuations in S are comparable.

23.1. Characterization of finite sets with $\chi(S) = 0$.

Theorem 23.1. *If any valuation in S has finite skewness, the following conditions are equivalent:*

1. $\chi(S) = 0$*;*
2. *there exists* $\phi \in \mathrm{SH}^+(\mathrm{V}_\infty)$ *such that* $\phi|_S = 0$*, the support of* $\Delta\phi$ *is equal to* S*, and* $\langle \phi, \phi \rangle = 0$*.*

Moreover when either one of these conditions are satisfied, the function ϕ as in (2) is unique up to a scalar factor. If all valuations in S are divisorial and we normalize ϕ such that $\phi(-\deg) = +1$ then the mass of $\Delta\phi$ at any point is a rational number.

Remark 23.2. When $S = S_+$, $\chi(S) = 0$ if and only if the matrix $M(S)$ has a one-dimensional kernel by Lemma 22.3.

Definition 23.3. When $\chi(S) = 0$ and $S = S_+^{\min}$, let ϕ_S be the unique function in $\mathrm{SH}^+(\mathrm{V}_\infty)$ such that $\phi_S(-\deg) = +1$, $\phi_S|_S = 0$, the support of $\Delta\phi_S$ is equal to S, and $\langle \phi_S, \phi_S \rangle = 0$ as above.

Proof. Denote by T the convex hull of $S \cup \{-\deg\}$. To simplify notation, set $S = \{v_1, \cdots, v_l\}$ and $v_0 := -\deg$.

(1) ⇒ (2). By Lemma 22.14, we may thus find real numbers $a_0, \ldots, a_l$ such that

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & \alpha(v_1) & \ldots & \alpha(v_1 \wedge v_l) \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \alpha(v_1 \wedge v_l) & \ldots & \alpha(v_l) \end{pmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_l \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} .$$

As in the proof of the Main theorem, the sign of a_0 is the same as $\chi(S)$. It follows that $a_0 = 0$. Consider the function $\phi := \sum_{i=1}^{l} a_i g_{v_i}$. Observe that $\phi(-\deg) = 1$, $\phi|_S = 0$ and $\operatorname{Supp} \Delta\phi \subseteq S$. Lemma 22.16 implies that $\phi > 0$ on T. Since ϕ is piecewise linear on T and $\phi = 0$ on $B(S)$, $a_i = \Delta\phi(v_i) > 0$ for $i = 1, \cdots, l$. It follows that $\phi \in \mathrm{SH}^+(\mathrm{V}_\infty)$, $\operatorname{Supp} \Delta\phi = S$, $\phi|_S = 0$ and $\langle \phi, \phi \rangle = \sum_{i=1}^{l} a_i \phi(v_i) = 0$.

(2) ⇒ (1). Write $\phi = \sum_{i=1}^{l} a_i g_{v_i}$ where $a_i \in \mathbb{R}^+$, $i = 1, \cdots, l$. Since $\phi|_S = 0$, we have

$$\begin{pmatrix} \alpha(v_1) & \ldots & \alpha(v_1 \wedge v_l) \\ \ldots & \ldots & \ldots \\ \alpha(v_1 \wedge v_l) & \ldots & \alpha(v_l) \end{pmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_l \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} .$$

It follows that $\chi(S) = (-1)^l \det(\alpha(v_i \wedge v_j)_{1 \le i,j \le l}) = 0$.

Further, Lemma 22.3 implies that the rank of the $l \times l$ matrix $[\alpha(v_i \wedge v_j)]_{1 \le i,j \le l}$ is $l-1$. It follows that the function ϕ is unique up to a scalar factor. When all v_i, $i = 1, \cdots, l$ are divisorial, then all $\alpha(v_i \wedge v_j)$, $1 \le i, j \le l$ are rational. If we normalize ϕ such that $\phi(-\deg) = +1$ then the mass of $\Delta\phi$ at any point is a rational number. □

23.2. **The relation between $\chi(S) = 0$ and $\delta(S) = 1$.** Let us begin with the following simple consequence of the Main Theorem.

Proposition 23.4. *If $\delta(S) = 1$ then $\chi(S) = 0$ and v is divisorial for all $v \in S$.*

Remark 23.5. The converse of Proposition 23.4 is not true. Let L_∞ be the line at infinity of $\mathbb{P}^2_{\mathbb{C}}$. Let O be a point in L_∞ and (u, v) be a local coordinate at O such that locally $L_\infty = \{u = 0\}$ and $\{v = 0\}$ is a line in $\mathbb{P}^2_{\mathbb{C}}$. Let C be a branch of curve at O defined by $(v - u^2)^5 - u^3 = 0$. We blow up 14 times at the center of (the strict transform of) C and denote by E the last exceptional curve. One can check that $\alpha(v_E) = 0$. By [8, Example 1.3, Example 2.5], we have $\delta(\{v_E\}) = 0$.

Proof of Proposition 23.4. Write $S = \{v_1, \cdots, v_l\}$. Pick any non constant polynomial $Q \in R_S$, and define $\phi := \log^+ |Q| \in \mathrm{SH}^+(\mathrm{V}_\infty)$. Since $\delta(S) \neq 2$ it follows from Theorem 22.7 (iv) that $\langle \phi, \phi \rangle \le 0$ hence $\langle \phi, \phi \rangle = 0$, and $\phi(v) = 0$ for all $v \in S$.

Suppose $v_1 \in S$ is not divisorial, then there exists $w_1 < v_1$ such that $\phi(w_1) = \phi(v_1) = 0$. By Proposition 21.22 and Proposition 22.8, we have S is rich which contradicts to our assumption. It follows that v is divisorial for all $v \in S$.

For every $v_1' > v_1$, By Proposition 21.22, the set $S' := \{v_1', v_2, \cdots, v_l\}$ is rich. It follows that $\chi(S') > 0$. Let $v_1' \to v_1$, we have $\chi(S) \ge 0$. Since S is not rich, we have $\chi(S) \le 0$ and then $\chi(S) = 0$. □

Our aim is to state a partial converse to the preceding result. To do so we need to introduce an important invariant that is referred to as the *thinness* of a valuation in [4]. Recall that this is unique function $A : V_\infty \to [-2, \infty]$ that is increasing and lower semicontinuous function on V_∞ and such that

$$A(v_E) = \frac{1}{b_E}\ (1 + \mathrm{ord}_E(dx \wedge dy))$$

for any irreducible component E of $X \setminus \mathbb{A}^2_k$ in any admissible compactification.

By the very definition we have $A(-\deg) = -2$ and the thinness of any divisorial valuation is a rational number whereas the thinness of any valuation associated to a branch of an algebraic curve is $+\infty$.

We can now state the main result of the section.

Proposition 23.6. *Suppose $\chi(S) = 0$, v is divisorial for all $v \in S$ and $\int A\, \Delta\phi_S \le 0$. Then $\delta(S) = 1$.*

Proof. Write $S = \{v_1, \cdots, v_l\}$ and $v_i := v_{E_i}$ for $E_i \in \mathcal{E}$. Write $\phi_S = \sum_{i=1}^{l} r_i g_{v_i}$ where $r_i \in \mathbb{Q}^+$. Let X be a compactification of $\mathbb{A}^2_k$ such that E_i can be realized as an irreducible component of $X \setminus \mathbb{A}^2_k$. Let E_X be the set of all irreducible components of $X \setminus \mathbb{A}_k^2$. Set $\theta := \sum_{E \in E_X} b_E \phi_S(v_E) E = \sum_i^l r_i b_{E_i}^{-1} \check{E}_i$. Then we have

$$(\theta \cdot K_X) = \sum_{i=1}^{l} r_i b_{E_i}^{-1} (\check{E}_i \cdot K_X) = \sum_{i=1}^{l} r_i b_{E_i}^{-1} \mathrm{ord}_{E_i} K_X$$

$$= \sum_{i=1}^{s} r_i b_{E_i}^{-1} (-1 + b_{E_i} A(v_i)) = -\sum_{i=1}^{s} r_i b_{E_i}^{-1} + \int A \, \Delta\phi_S < 0.$$

There exists $m \in \mathbb{Z}^+$ such that $D := m\theta$ is a $\mathbb{Z}$ divisor supported by X_∞. Then we have that D is effective, $D^2 = 0$ and $(D \cdot K) \leq -1$. Recall the Riemann-Roch theorem we have

$$h_X^0(D) - h_X^1(D) + h_X^2(D) = \chi(O_X) + (D \cdot (D - K))/2 = \chi(O_X) - (D \cdot K)/2.$$

Since X is rational, we have $\chi(O_X) = 1$. Since D is effective, we have $h_X^2(D) = h^0(K_X - D) \leq h^0(K_X) = 0$. It follows that

$$h_X^0(D) \geq 1 - (D \cdot K)/2 > 1.$$

Then there exists an element $P \in k(x, y) \setminus k$ such that $\mathrm{div}(P) + D$ is effective. Since D is supposed by $X \setminus \mathbb{A}_k^2$, we have $P \in k[x, y] \setminus k$. It follows that

$$v_i(P) = (b_{E_i})^{-1} \mathrm{ord}_{E_i}(P) \geq -(b_{E_i})^{-1} \mathrm{ord}_{E_i}(D) = -m\phi_S(v_i) = 0$$

for all $i = 1, \cdots, l$. □

Remark 23.7. The condition $\sum_{i=1}^{l} r_i A(v_i) \leq 0$ is not necessary. Set $P := y^2 - x^3 \in \mathbb{C}[x, y]$. Consider the pencil C_λ consisting of the affine curves $C_\lambda := \{P = \lambda\} \subseteq \mathbb{C}^2$ for $\lambda \in \mathbb{C}$. We see that C_λ has one branch at infinity for every $\lambda \in \mathbb{C}$. Let $v_{|C|}$ be the normalized valuation defined by $Q \mapsto 3^{-1} \mathrm{ord}_\infty(Q|_{C_\lambda})$ for λ generic. We see that $\alpha(v_{|C|}) = 0$, $A(v_{|C|}) = 1/3 > 0$ and $P \in R_S$.

23.3. The structure of R_S when $\delta(S) = 1$.

Proposition 23.8. *Suppose that $\delta(S) = 1$. Then there exists a polynomial $P \in k[x, y] \setminus k$ such that $R_S = k[P]$.*

Proof of Proposition 23.8. Set $S = \{v_1, \cdots, v_l\}$ and suppose that $S = S^{\min}$.

If there exists $Q \in k[x, y]$ such that $Q \in \overline{\mathrm{Frac}\,(R_S)} \setminus R_S$, then we have $\sum_{i=1}^{d} a_i Q^i = 0$ where $d \geq 1$, $a_i \in R_S$ and $a_d \neq 0$. Since S is not rich, we have $v(a_i) = 0$ for all $v \in S$ and $i = 1, \cdots, d$. Since $Q \neq R_S$, there exists $v \in S$ satisfying $v(Q) < 0$. Then we have $v(a_i Q^i) = iv(Q) < 0$ for $i = 1, \cdots, d$. It follows that $v(a_i Q^i) = iv(Q) > dv(Q) = v(a_d Q^d)$ for $i = 1, \cdots, m-1$. Then we have $v(\sum_{i=0}^{d} a_i Q^i) = dv(Q) < 0$ which is a contradiction. Then we have

$$\overline{\mathrm{Frac}\,(R_S)} \bigcap k[x, y] = R_S.$$

Pick a polynomial $P \in R_S \setminus k$ with minimal degree. If there are infinitely many $r \in k$ such that $P - r$ is not irreducible, then by [10, Théorème fondamental], there exists a polynomial $Q \in k[x, y]$ and $R \in k[t]$ of degree at least two satisfying $P = R \circ Q$. Then we have $Q \in \overline{\mathrm{Frac}\,(R_S)} \cap k[x, y] = \mathrm{Frac}\,(R_S)$ and $\deg(Q) < \deg(P)$ which contradicts the minimality of $\deg(P)$. It follows that there are infinitely many $r \in k$ such that $P - r$ is irreducible.

If $R_S \neq k[P]$, there exists $R \in R_S \setminus k[P]$ with minimal degree. Since $R \in \overline{\mathrm{Frac}\,(R_S)} = \overline{k(P)}$, we have

$$\sum_{i=0}^{m} a_i(P)R^i = 0$$

where $m \geq 1$, $a_i \in k[t]$ and $a_m \neq 0$ in $k[t]$. There exists $r \in k$ such that the polynomial $P - r$ is irreducible and $a_m(r) \neq 0$. We have

$$0 = (\sum_{i=0}^{m} a_i(P)R^i)|_{\{P-r=0\}} = \sum_{i=0}^{m} a_i(r)(R|_{\{P-r=0\}})^i.$$

It follows that $r_1 := R|_{\{P-r=0\}}$ is a constant in k. Since $P-r$ is irreducible, there exists $R_1 \in k[x,y]$ such that $R - r_1 = (P-r)R_1$. It follows that

$$R_1 \in k(R,P) \bigcap k[x,y] \subseteq \mathrm{Frac}\,(R_S) \bigcap k[x,y] = R_S$$

and $\deg R_1 < \deg R$. Since the degree of R is minimal in $R_S \setminus k[P]$, we have $R_1 \in k[P]$. Then we have $R = (P-r)R_1 + r_1 \in k[P]$ which contradicts to our hypotheses. It follows that $R_S = k[P]$. ∎

24. An application to the algebraization problem of analytic curves

The aim of this section is to prove Theorem 19.5.

24.1. **K-rational points on plane curves.** Let K be a number field, $\mathcal{M}_K^\infty$ the set of its archimedean places, $\mathcal{M}_K^0$ the set of its non-archimedean places, and $\mathcal{M}_K = \mathcal{M}_K^\infty \cup \mathcal{M}_K^0$. For any $v \in \mathcal{M}_K$, denote by $O_v := \{x \in K | \, |x|_v \leq 1\}$ the ring of v-integers and define $O_K := \{x \in K | \, |x|_v \leq 1 \text{ for all } v \in \mathcal{M}_K^0\}$.

Let S be a finite set of places of K containing all archimedean places. We define the ring of S-integers to be

$$O_{K,S} = \{x \in K | \, |x|_v \leq 1 \text{ for all } v \in M_K \setminus S\}.$$

Let X be a compactification of $\mathbb{A}_K^2$. We fix an embedding $\mathbb{A}_K^2 \hookrightarrow X$. Fix a projective embedding $X \hookrightarrow \mathbb{P}^N$ defined over K. For each place $v \in \mathcal{M}_K$, there exists a distance function d_v on X, defined by

$$d_v([x_0 : \cdots : x_N], [y_0 : \cdots : y_N]) = \frac{\max_{0\leq i,j \leq N} |x_i y_j - x_j y_i|_v}{\max_{0 \leq i \leq N} |x_i|_v \max_{0 \leq j \leq n} |y_j|_v}$$

for any two points $[x_0 : \cdots : x_N], [y_0 : \cdots : y_N] \in X(K) \subseteq \mathbb{P}^N(K)$. Let C be an irreducible curve in X which is not contained in $X_\infty := X \setminus \mathbb{A}_K^2$.

Proposition 24.1. *Pick any point $q \in C(K) \bigcap X_\infty$. For every place $v \in \mathcal{M}_K$, let r_v be a positive real number and set $U_v := \{p \in \mathbb{A}^2(K_v) | \, d_v(q,p) < r_v\}$. Suppose moreover that $r_v = 1$ for all places v outside a finite subset S of $\mathcal{M}_K$. Then the set $C(K) \setminus \cup_{v \in \mathcal{M}_K} U_v$ is finite.*

Proof. We shall prove that $C(K) \setminus \cup_{v \in \mathcal{M}_K} U_v$ is a set of points of bounded heights for a suitable height.

Let $i : \widetilde{C} \to C$ be the normalization of C and pick a point $Q \in i^{-1}(q)$.

There exists a positive integer l such that lQ is a very ample divisor of $\widetilde{C}$. Choose an embedding $j : \widetilde{C} \hookrightarrow \mathbb{P}^M$ such that

$$O = [1 : 0 : \cdots : 0] = H_\infty \bigcap \widetilde{C}$$

where $H_\infty = \{x_M = 0\}$ is the hyperplane at infinity. Let $g : \widetilde{C} \to \mathbb{P}^1$ be the rational map sending $[x_0 : \cdots : x_M] \in \widetilde{C}$ to $[x_0 : x_M] \in \mathbb{P}^1$. It is a morphism since $\{x_0 = 0\} \bigcap H_\infty \bigcap \widetilde{C} = \emptyset$. It is also finite and satisfying

$$g^{-1}([1 : 0]) = H_\infty \bigcap \widetilde{C} = [1 : 0 \cdots : 0].$$

By base change, we may assume that $\widetilde{C}, i, j, g$ are all defined over K.

Set $D = \operatorname{Spec} O_K$. We consider the irreducible scheme $\widetilde{\mathcal{C}} \subseteq \mathbb{P}_D^M$ over D whose generic fiber is $\widetilde{C}$ and the irreducible scheme $\mathcal{X} \subseteq \mathbb{P}_D^N$ over D whose generic fiber is X. Then i extends to a map $\iota : \widetilde{\mathcal{C}} \dashrightarrow \mathcal{X}$ over D that is birational onto its image.

For any $v \in \mathcal{M}_K^0$, let

$$\mathfrak{p}_v = \{x \in O_v | \, v(x) > 0\}$$

be a prime ideal in O_v. There is a finite set T consisting of those places $v \in \mathcal{M}_K^0$ such that ι is not regular along the special fibre $\overline{C}_{O_v/\mathfrak{p}_v}$ at $\mathfrak{p}_v \in D$ or $\overline{C}_{O_v/\mathfrak{p}_k} \bigcap H_{\infty,O_v/\mathfrak{p}_v} \neq \{[1:0:\cdots:0]\}$.

Pick any place $v \in \mathcal{M}_K^0 \setminus (S \bigcup T)$, and define

$$\begin{aligned} V_v =& \{[1:x_1:\cdots:x_M] \in \widetilde{C}(K) |\ |x_i|_v < 1, i = 1, \cdots, M\} \\ =& \{[1:x_1:\cdots:x_M] \in \widetilde{C}(K) |\ |x_M|_v < 1\}. \end{aligned}$$

Since $r_v = 1$, for such a place we set $\Omega_v := \{[1:x] \in \mathbb{P}^1(K) |\ |x|_v < 1\}$. We have $V_v = g^{-1}(\Omega_v) \bigcap \widetilde{C}(K)$, so that $i^{-1}(U_v \bigcap C(K)) \supseteq V_v$ for all $v \in \mathcal{M}_K^0 \setminus (S \bigcup T)$.

Now choose a place $v \in S \bigcup T$. Since $g^{-1}([1:0]) = Q$, we may supose that $r_v > 0$ satisfying $i^{-1}(U_v \bigcap C(K)) \supseteq g^{-1}(\{[1:x] \in \mathbb{P}^1(K) |\ |x|_v < r_v\})$.

By contradiction, we suppose that there exists a sequence $\{p_n = (x_n, y_n)\}_{n \geq 0}$ of distinct K-points in $C(K) \bigcap \mathbb{A}^2(K)$. Since there are only finitely many singular points in C, we may suppose that for all $n \geq 0$, C is regular at p_n. Set $q_n := i^{-1}(p_n)$, and $y_n := g(q_n)$. Since g is finite, we may suppose that the y_n's are distinct. Write $y_n := [x_n : 1]$ so that $|x_n|_v < r_v$ for all $v \in \mathcal{M}_K$.

We now observe that

$$\begin{aligned} [K:\mathbb{Q}] h_{\mathbb{P}^1}(y_n) &= \sum_{v \in \mathcal{M}_K} n_v \log(\max\{|x_n|_v, 1\}) \\ &\leq \sum_{v \in \mathcal{M}_K \setminus \{v \in \mathcal{M}_K\}} n_v \log(\max\{r_v, 1\}) \\ &= \sum_{v \in S \bigcup T} n_v \log(\max\{r_v, 1\}) \end{aligned}$$

where $h_{\mathbb{P}^1}$ denotes the naive height on $\mathbb{P}^1$. We get a contradiction by Northcott property (see [12]). ☐

We also have a version of Proposition 24.1 for S-integral points.

Given any finite set of places containing $\mathcal{M}_K^\infty$, we say that $(x, y) \in \mathbb{A}^2(K) \subseteq X$ is S-integral if $x, y \in O_{K,S}$.

Proposition 24.2. *Let $\{p_n = (x_n, y_n)\}_{n \geq 0}$ be an infinite set of S-integral points lying in $C \bigcap \mathbb{A}^2$. Then for any point $q \in X_\infty \bigcap C(K)$, there exists a place $v \in \mathcal{M}_K$ such that there exists an infinite subsequence $\{p_{n_i}\}_{i \geq 1}$ satisfying $p_{n_i} \to q$ with respect to d_v as $i \to \infty$.*

Proof of Proposition 24.2. We define $\widetilde{C}$, i,j,g and T as in the proof of Proposition 24.1.

We may suppose that for all $n \geq 0$, p_n is regular in C. The K-points $q_n := i^{-1}(p_n)$ are distinct K-points in $\widetilde{C}$.

For any $v \in \mathcal{M}_K^0 \setminus (S \bigcup T)$, Set

$$\begin{aligned} V_v =& \{[1:x_1:\cdots:x_M] \in \widetilde{C}(K) |\ |x_i|_v < 1, i = 1, \cdots, M\} \\ =& \{[1:x_1:\cdots:x_M] \in \widetilde{C}(K) |\ |x_M|_v < 1\}. \end{aligned}$$

We set $\Omega_v := \{[1:x] \in \mathbb{P}^1(K) | \, |x|_v < 1\}$, then $V_v = g^{-1}(\Omega_v) \bigcap \widetilde{C}(K)$. It follows that $q_n \notin V_v$. Set $[x_n : 1] := g(q_n)$. Then we have $|x_n|_v < 1$ for all $v \in \mathcal{M}_K \setminus \{S \bigcup T\}$

Since g is finite, we may suppose that $g(q_n)$'s are distinct. By Northcott property, we have $h_{\mathbb{P}^1}(g(q_n)) \to \infty$ as $n \to \infty$. Observe that

$$[K:\mathbb{Q}]h_{\mathbb{P}^1}(g(q_n)) = \sum_{v \in \mathcal{M}_K} n_v \log(\max\{|x_n|_v, 1\})$$

$$= \sum_{v \in \mathcal{M}_K \setminus \{S \bigcup T\}} n_v \log(\max\{|x_n|_v, 1\}) + \sum_{v \in S \bigcup T} n_v \log(\max\{|x_n|_v, 1\})$$

$$= \sum_{v \in S \bigcup T} n_v \log(\max\{|x_n|_v, 1\})$$

Since $S \bigcup T$ is finite, there exists $v \in S \bigcup T$, such that there exists a subsequence n_i such that $\log(\max\{|x_{n_i}|_v, 0\}) \to \infty$ as $i \to \infty$. Then $g(q_{n_i}) \to [1:0]$ with respect to d_v as $i \to \infty$. Since $g^{-1}([1:0]) = \{Q\}$, we have $q_{n_i} \to Q$ and then $p_{n_i} = i(q_{n_i}) \to q$ respect to d_v as $i \to \infty$. □

24.2. **The adelic analytic condition in Theorem 19.5.** Let K be a number field. Recall that s is an adelic branch at infinity defined over K if it is given by the following data.

(i) s is a formal branch based at a point $q \in L_\infty(K)$ given in coordinates x_q, y_q as in the introduction by a formal Puiseux series $y_q = \sum_{j \geq 1} a_j x_q^{j/m} \in O_{K,S}[[x_q^{1/m}]]$ for some positive integer m and a finite set S of places of K containing all archimedean places.

(ii) for each place $v \in S$, the radius of convergence of the Puiseux series determining s is positive, i.e. $\limsup_{j \to \infty} |a_j|_v^{-m/j} > 0$.

Further, we say s is a adelic branch at infinity if it is a adelic branch defined over some number field.

Remark 24.3. The definition of adelic branch at infinity does not depend on the choice of affine coordinate in $\mathbb{A}^2_{\overline{\mathbb{Q}}}$.

Remark 24.4. If C is a branch of an algebraic curve at infinity defined over $\overline{\mathbb{Q}}$, then C is adelic.

An adelic branch need not to be algebraic. Pick a formal Puiseux series $y_q = \sum_{i=1}^{\infty} a_i x_q^{\frac{i}{m}} \in K[[x_q^{\frac{1}{m}}]]$ which comes from a branch at $q \in L_\infty(K)$ of an algebraic curve such that all a_i's are non zero. For example $y_q = \sum_{i=1}^{\infty} x_q^i = \frac{x_q}{1-x_q}$. To each subset T of $\mathbb{Z}^+$, we attach a formal Puiseux series $y_q = \sum_{i \in T} a_i x_q^{\frac{i}{m}} \in K[[x_q^{\frac{1}{m}}]]$ which defines a formal curve C_T. It is easy to check that all C_T's are adelic-analytic curves and $C_T \neq C_{T'}$ if $T \neq T'$. So the cardinality of set $\{C_T\}_{T \subseteq \mathbb{Z}^+}$ is $2^{\aleph_0}$. On the other hand, since $\overline{\mathbb{Q}}$ is countable, the set of all branches of algebraic curves at O is countable. Then there exists an adelic-analytic curve C_T for some $T \subseteq \mathbb{Z}^+$ which is not algebraic.

24.3. **Proof of Theorem 19.5.** Let S be a finite set of places of K containing all archimedean places. We may suppose that $s_1, \cdots, s_l$, $l \geq 1$ are adelic branches defined over K. Denote by q_i the center of s_i. Write U_i for U_{q_i}, x_i (resp. y_i) for x_{q_i} (resp. y_{q_i}). By changing coordinates, we may suppose that $x_i = 1/x$, $y_i = y/x + c_i$ for some $c_i \in O_{K,S}$. Suppose that s_i is defined by $y_i = \sum_{j=1} a_{ij} x^{\frac{j}{m_i}} \in O_{K,S}[[x^{\frac{1}{m_i}}]]$ where m_i is a positive integer. Observe that $C^v(s_i)$ is contained in the ball $\{p \in \mathbb{P}^2(K_v)|\ d_v(p, q_i) < 1\}$ for $v \in \mathcal{M}_K \setminus S$. We may suppose that $B_v = 1$ for $v \in M_K \setminus S$.

Since $\alpha(v_{s_i}) = -\infty$, by Theorem 22.12 and Theorem 22.7, there exists a polynomial $P \in \overline{\mathbb{Q}}[x, y]$ such that $v_i(P) > 0$ for all $i = 1, \cdots, l$. Replacing K by a larger number field and S by a larger set, we may suppose that $P \in O_{K,S}[x, y]$.

Observe that $P(x, y) = P(x_i^{-1}, (y_i - c_i)x_i^{-1})$ in U_i, so that

$$P|_{s_i} = P\left(x_i^{-1}, (\sum_{j=1} a_{ij} x^{\frac{j}{m_i}} - c_i)x_i^{-1}\right)$$

is a formal Puiseux series. We may write it as $\sum_j^{\infty} b_{i,j} x_i^{\frac{j}{m_i}} \in K((x_i^{\frac{1}{m_i}}))$. It is easy to see that $b_{i,j} \in O_{K,S}$. Observe that q_i is not a pole of $P|_{C_i}$. It follows that $b_{i,j} = 0$ for $j \leq 0$ and then $P|_{C_i} \in K[[x^{\frac{1}{m_i}}]]$. There exists a real number $M_v \geq 0$ satisfying $|P(p)|_v \leq M_v$ for all $p \in C^v(s_i)$, $i = 1, \cdots, l$ and $v \in \mathcal{M}_K$. Observe that we may chose $M_v = 1$ for $v \in M_K \setminus S$.

There exists a number R_v satisfying $|P(x, y)|_v \leq R_v$ for all $(x, y) \in K^2$ satisfying $|x|_v \leq B_v, |y|_v \leq B_v$. We may chose $R_v = 1$ for all $v \in M_K \setminus S$. Set $A_v := \max\{B_v, M_v\}$, we have $A_v = 1$ for $v \in \mathcal{M}_K \setminus S$.

The height of $P(p_n)$ is

$$h(P(p_n)) = \sum_{v \in \mathcal{M}_K} \log\{1, |P(p_n)|_v\}$$

$$\leq \sum_{v \in \mathcal{M}_K} \log\{1, A_v\} = \sum_{v \in S} \log\{1, A_v\} < \infty.$$

By Northcott property, the set $T := \{P(p_n)|\ n \geq 0\}$ is finite. We denote by D the curve defined by the equation $\prod_{t \in T}(P(x, y) - t) = 0$. Then D contains the set $\{p_n\}_{n \geq 0}$. Let C be the union of all irreducible components of D which contains infinitely many p_n. Then for n large enough, we have $p_n \in C$.

We only have to show that all branches of C at infinity are contained in the set $\{s_1, \cdots, s_l\}$. By contradiction, we suppose that there exists a branch Z_1 of C at infinity which is not contained in $\{s_1, \cdots, s_l\}$. Let Z be the irreducible component containing Z_1. Set $R_Z := \{p_n\}_{n \geq 0} \bigcap Z$. Then R_Z is an infinite set. Pick a compactification X of $\mathbb{A}^2_K$ such that all centers q_i' of the strict transforms of s_i's are difference from the center z of the strict transform of Z_1. For every $v \in \mathcal{M}_K$ there exists $r_v > 0$ such that the ball $D_v := \{p \in \mathbb{P}^2(K_v)|\ d_v(p, z) < r_v\}$ does not intersect $C^v(s_i)$ for all $i = 1, \cdots, l$ and does not in intersect the set $\{(x, y) \in \mathbb{A}^2(K_v)|\ \max\{|x|_v, |y|_v\} \leq B_v\}$. Moreover we may suppose that $r_v = 1$ for all v outside a finite set F of $\mathcal{M}_K$. Let $U_v := D_v \bigcap Z(K_v)$. By Proposition

24.1, we have the set $Z(K) \setminus (\cup_{v\in\mathcal{M}_K} U_v)$ is finite. Then there exists a point $p_n \in R_Z$ and a place $v \in \mathcal{M}_K$ such that $p_n = (x_n, y_n) \in U_v$. Then we have $\max\{|x_n|_v, |y_n|_v\} > B_v$ and $p \notin C^v(s_i)$ for all $i = 1, \cdots, l$, which contradicts to our hypotheses.

Remark 24.5. In fact, we can prove a stronger version of Theorem 19.5. Our proof actually shows that it is only necessary to assume that p_n is a sequence of $\bar{\mathbb{Q}}$ points having bounded degree over $\mathbb{Q}$ (instead of assuming it to belong to the same number field).

We also have an analogue of Theorem 19.5 for S-integer points.

Theorem 24.6. *Let K be a number field and S be a finite subset of places in $\mathcal{M}_K$ containing $\mathcal{M}_K^\infty$.*

Let $s_1, \cdots, s_l$ where $l \geq 1$ be a finite set of formal curves in $\mathbb{P}^2_{\bar{\mathbb{Q}}}$ define over K whose centers q_i's are K-points in the line L_∞ at infinity. Suppose that for all place $v \in S$, s_i is convergence to a v-analytic curve $C^v(s_i)$ in a neighbourhood at q_i w.r.t. v for $i = 1, \cdots, l$.

Finally let $p_n = (x^{(n)}, y^{(n)})$, $n \geq 0$ be an infinite collection of S-integer points in $\mathbb{A}^2(K)$ such that for each place $v \in M_K$ then either $\max\{|x^{(n)}|_v, |y^{(n)}|_v\} \leq B_v$ *or $p_n \in \cup_{i=1}^l C^v(s_i)$.*

Then there exists an algebraic curve C in $\mathbb{A}^2_K$ such that any branch of C at infinity is contained in the set $\{s_1, \cdots, s_l\}$ and p_n belongs to C for all n large enough.

The proof of Theorem 24.6 is very similar to the proof of Theorem 19.5. We leave it to the reader.

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