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► **To cite this version:**

Alexei Grinbaum. The Significance of Information in Quantum Theory. Mathematical Physics [math-ph]. Ecole Polytechnique X, 2004. English. NNT: . tel-00007634

**HAL Id: tel-00007634**

**<https://pastel.hal.science/tel-00007634>**

Submitted on 3 Dec 2004

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Thèse présentée pour obtenir le grade de  
DOCTEUR DE L'ÉCOLE POLYTECHNIQUE

Domaine : Economie et Sciences sociales

Spécialité : Sciences cognitives théoriques

par

**Alexei Grinbaum**

# LE RÔLE DE L'INFORMATION DANS LA THÉORIE QUANTIQUE

(THE SIGNIFICANCE OF INFORMATION  
IN QUANTUM THEORY)

Soutenue le 4 octobre 2004 devant le jury composé de :

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# Résumé

Les dérivations théorético-informationnelles du formalisme de la théorie quantique soulèvent un intérêt croissant depuis le début des années 1990, grâce à l'émergence de la discipline connue sous le nom d'information quantique et au retour des questions épistémologiques dans les programmes de recherche de nombreux physiciens-théoriciens. Nous proposons une axiomatique informationnelle dont nous dérivons le formalisme de la théorie quantique.

La première partie de la thèse est consacrée aux fondements philosophiques de l'approche informationnelle. Cette approche s'insère dans un cadre épistémologique que nous présentons sous la forme d'une boucle entre descriptions théoriques, ce qui nous permet de proposer une méthode nouvelle d'analyse de la frontière entre toute théorie et sa méta-théorie.

La deuxième partie de la thèse est consacrée à la dérivation du formalisme de la théorie quantique. Nous posons un système d'axiomes formulés dans le langage informationnel. En conformité avec l'argument pour la séparation entre théorie et méta-théorie, nous analysons le double rôle de l'observateur qui est à la fois un système physique et un agent informationnel. Après l'introduction des techniques de la logique quantique, les axiomes reçoivent un sens mathématique précis, ce qui nous permet d'établir une série de théorèmes montrant les étapes de la reconstruction du formalisme de la théorie quantique. L'un de ces théorèmes, celui de la reconstruction de l'espace de Hilbert, constitue un point important où la thèse innove par rapport aux travaux existants. Le double rôle de l'observateur permet de retrouver la description de la mesure par POVM, un *sine qua non* de la computation quantique.

Dans la troisième partie de la thèse, nous introduisons la théorie des  $C^*$ -algèbres et nous proposons de cette dernière une interprétation théorético-informationnelle. L'interprétation informationnelle permet ensuite d'analyser sur le plan conceptuel les questions relatives aux automorphismes modulaires et à l'hypothèse du temps thermodynamique de Connes-Rovelli, ainsi qu'à la dérivation proposée par Clifton, But et Halvorson.

Nous concluons par une liste de problèmes ouverts dans l'approche informationnelle, y compris ceux relevant des sciences cognitives, de la théorie de la décision et des technologies de l'information.

**Mots clés** : théorie quantique, information, boucle des théories, logique quantique, espace de Hilbert,  $C^*$ -algèbre, automorphismes modulaires, condition KMS, temps



# Abstract

Interest toward information-theoretic derivations of the formalism of quantum theory has been growing since early 1990s thanks to the emergence of the field of quantum computation and to the return of epistemological questions into research programs of many theoretical physicists. We propose a system of information-theoretic axioms from which we derive the formalism of quantum theory.

Part I is devoted to the conceptual foundations of the information-theoretic approach. We argue that this approach belongs to the epistemological framework depicted as a loop of existences, leading to a novel view on the place of quantum theory among other theories.

In Part II we derive the formalism of quantum theory from information-theoretic axioms. After postulating such axioms, we analyze the twofold role of the observer as physical system and as informational agent. Quantum logical techniques are then introduced, and with their help we prove a series of results reconstructing the elements of the formalism. One of these results, a reconstruction theorem giving rise to the Hilbert space of the theory, marks a highlight of the dissertation. Completing the reconstruction, the Born rule and unitary time dynamics are obtained with the help of supplementary assumptions. We show how the twofold role of the observer leads to a description of measurement by POVM, an element essential in quantum computation.

In Part III, we introduce the formalism of  $C^*$ -algebras and give it an information-theoretic interpretation. We then analyze the conceptual underpinnings of the Tomita theory of modular automorphisms and of the Connes-Rovelli thermodynamic time hypothesis. We also discuss the Clifton-Bub-Halvorson derivation program and give an information-theoretic justification for the emergence of time in the algebraic approach.

We conclude by giving a list of open questions and research directions, including topics in cognitive science, decision theory, and information technology.

**Keywords:** quantum theory, information, loop of existences, quantum logic, Hilbert space,  $C^*$ -algebra, modular automorphisms, KMS condition, time



# Acknowledgements

The warmest thanks I address to my advisor Jean-Pierre Dupuy. To him I owe my lifestyle in science and in philosophy.

I have always felt the unceasing support, both scientifically and administratively, of Jean Petitot, the director of CREA. Also at CREA, the discussions with Michel Bitbol have taught me a great deal.

I am indebted to Professors M.V. Ioffe and V.A. Franke, members of the Chair of High Energy Physics at St. Petersburg State University, for many years of undemanding support.

I have learned a lot from the valuable discussions with Carlo Rovelli. His name is quoted often in the dissertation, but indeed must be quoted on almost every its page.

Comments made by Jeffrey Bub, Chris Fuchs, Simon Saunders, and Bas van Fraassen were at the origin of some of the lines of argument.

My fellow Ph.D. students at CREA provided numerous remarks that made me spell the ideas clearer. I thank Stefano Osnaghi, Patricia Kauark, Adrien Barton, Mathieu Magnaudet, Alexandre Billon, and Manuel Bächtold.

The results of this dissertation were presented at conferences organized by Andrei Khrennikov at Växjö University and Marisa dalla Chiara in Sardinia under the auspices of the ESF Network for Philosophical and Foundational Problems of Modern Physics. I am grateful to the organizers and all the participants of those conferences who took part in the discussions.

Financial support and travel funds over the last five years were provided by the Centre de Recherche en Épistémologie Appliquée of the Ecole Polytechnique, the Fondation de l'Ecole Polytechnique, the French Embassy in Russia, and the Ministry for Education, Research and Technology of France.



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# Notation

$\mathbb{N}$ ( $\mathbb{N}_0$ )	positive (nonnegative) integers
$\mathbb{Z}$	integers
$\mathbb{R}$ ( $\mathbb{R}_+$ )	(positive) real numbers
$\mathbb{C}$	complex numbers
$\mathbb{H}$	quaternions
$\mathbb{D}$	underlying field of a vector space
$S, O$	physical system (in Part II)
$\mathfrak{M}$	fact or measurement result
$A, B$	linear operator
$P$	projection operator
$E$	positive operator (p. 104)
$H$	Hamiltonian
$\mathcal{H}$	Hilbert space (p. 60)
$\mathcal{L}$	lattice (p. 55)
$x, y, z$	lattice element
$Q_i$	yes-no questions
$W(P)$	set of questions
$\mathfrak{B}(\mathcal{H})$	algebra of all bounded linear operators on $\mathcal{H}$
$\mathcal{A}, \mathcal{B}$	$C^*$ -algebra (p. 110)
$\mathcal{M}, \mathcal{N}$	von Neumann algebra (p. 110)
$\omega, \rho, \sigma$	state over an algebra (p. 111)
$\alpha_t^\omega$	modular automorphism (p. 114)



# Note de présentation synthétique

## Résumé des résultats et plan de la thèse

Cette thèse appartient au domaine des Fondements de la physique. Cela signifie que nous mettons ensemble une analyse des concepts qui se trouvent à la base de différentes théories physiques avec des résultats formels rigoureux qui permettent d'éviter toute ambiguïté dans les conclusions. Le rôle de la preuve mathématique dans la justification des résultats est décisive.

Cette thèse mobilise également d'autres disciplines. Dans la partie III, la tâche principale consiste à donner une *interprétation* et, par conséquent, le domaine concerné est celui de la philosophie de la physique. Dans le chapitre 2, les questions soulevées sont de caractère général plutôt que spécialisé au cas de la physique, comme dans le reste du texte; ainsi, le domaine concerné est celui de la philosophie des sciences ou de l'épistémologie. Dans la Conclusion, qui présente les problèmes ouverts et les thèmes appartenant à d'autres axes de recherche, nous parlons des disciplines telles que les sciences cognitives et la théorie de la décision.

Le but de cette thèse est de développer une dérivation cohérente de l'ensemble du formalisme de la théorie quantique à partir des principes théorético-informationnels. Au cours de la dérivation, nous étudions les diverses questions conceptuelles et techniques qui se posent. La réussite du programme de dérivation dans la partie II permet d'avancer la thèse suivante :

**La théorie quantique est une théorie générale de l'information, dont la généralité est toutefois restreinte par quelques importantes contraintes théorético-informationnelles. Elle peut être**

**formellement dérivée d'une axiomatique informationnelle qui correspond à ces contraintes.**

Il y a trois manières dont nous innovons en matière des fondements de la physique :

- Nous dérivons le formalisme quantique à partir des axiomes théorético-informationnels de façon nouvelle.
- Nous donnons une formulation de l'attitude épistémologique présentée sous forme de boucle et nous montrons également son utilité pour l'analyse des théories autres que les théories physiques.
- Nous donnons une interprétation théorético-informationnelle de l'approche des  $C^*$ -algèbres et de la théorie des automorphismes modulaires Tomita.

Le premier de ces résultats est le plus important. Il est commun de considérer, même aujourd'hui, que la théorie quantique est une théorie du micromonde, ou des objets réels tels que les particules et les champs, ou d'une autre entité fondamentale qui ait nécessairement un statut ontologique. La dérivation théorético-informationnelle du formalisme quantique donne à ces questions une clarté longtemps désirée : toutes les présuppositions ontologiques sont étrangères à la théorie quantique, qui est, en soi, une pure épistémologie. La théorie quantique comme théorie de l'information doit être débarrassée des présupposés réalistes, qui ne doivent leur existence qu'aux préjugés et croyances individuelles des physiciens, sans appartenir de quelque façon que ce soit à la théorie quantique propre. Ce qui appartient à la théorie quantique, c'est exclusivement ce dont on a besoin pour sa dérivation, c'est-à-dire pour sa reconstruction dans le contexte de l'approche théorético-informationnelle. Au cours d'une telle dérivation nous montrons, pour la première fois dans la littérature, comment à partir des axiomes informationnels on peut reconstruire l'espace de Hilbert — un élément essentiel de la théorie quantique. Nous utilisons ensuite des théorèmes mathématiques puissants afin de reconstruire le reste du formalisme.

Pour séparer la théorie quantique de l'ontologie superficielle par le moyen de la dérivation théorético-informationnelle, on doit la dériver à partir de postulats dont la philosophie sous-jacente soit dénouée d'engagements de caractère ontologique. Cela

marque le deuxième point d'innovation de la thèse. Non seulement on expose la philosophie de la physique sans se référer à l'ontologie, mais on montre également comment cette philosophie peut être liée de manière cohérente au programme de dérivation formulé *dans le langage mathématique*.

Pour passer au troisième point d'innovation de la thèse, nous changeons d'attitude, passant de celle d'un scientifique qui démontre les théorèmes à celle d'un philosophe de la physique. La tâche est double : nous donnons une interprétation théorético-informationnelle du formalisme algébrique en théorie quantique et nous étudions les présupposés conceptuels de la théorie de Tomita et de l'hypothèse du temps modulaire de Connes-Rovelli. Nous continuons à suivre l'approche informationnelle, et c'est l'interprétation théorético-informationnelle du formalisme des  $C^*$ -algèbres qui est innovatrice par rapport aux travaux existants.

La thèse est composée de trois parties. Dans partie I, après quelques remarques de caractère général, le chapitre 2 s'ouvre par une section dans laquelle nous expliquons pourquoi, après plusieurs décennies d'oubli, s'est réveillé l'intérêt des physiciens pour la philosophie. Nous passons ensuite à la section centrale du chapitre où nous introduisons le concept de la boucle entre les théories. Dans la dernière section, nous montrons en quoi consiste la réponse que l'on donne du point de vue ici choisi à la question que pose tout philosophe de la physique à toute approche dite nouvelle : Comment est-ce que cela résout le problème de la mesure ? La réponse est que notre approche ne résout pas, mais dissout le problème.

Dans le chapitre 3, nous introduisons les notions de la computation quantique. Elles ne seront pas directement utilisées dans la thèse, mais elles servent à motiver l'intérêt croissant pour la notion d'information. Ce chapitre peut être omis par le lecteur intéressé exclusivement au développement de la ligne d'argumentation principale.

La partie II est consacrée à la dérivation théorético-informationnelle du formalisme de la théorie quantique. Cette dérivation est exposée en trois chapitres.

Le chapitre 4 est dédié aux fondements conceptuels de l'approche théorético-infor-

mationnelle. Il s'ouvre par une section historique où on présente un résumé des tentatives d'axiomatisation en mécanique quantique. Puis le chapitre se poursuit avec une section sur la « Mécanique quantique relationnelle » de Rovelli, qui justifie l'intuition que nous utiliserons pour le choix des axiomes. Les sections 4.3 et 4.4 sont au cœur de l'approche théorético-informationnelle en ce que nous y posons, respectivement, les notions fondamentales de la théorie et les axiomes théorético-informationnels formulés en termes de ces notions fondamentales. Le chapitre se conclut avec une section importante sur le double rôle de l'observateur, qui est à la fois un système physique et un agent informationnel.

Le chapitre 5 est consacré au formalisme de la logique quantique qui sera utilisé dans la suite. Certains résultats de ce chapitre nous appartiennent, mais la plupart sont dus à d'autres chercheurs. La dernière section du chapitre traite de la question cruciale : comment caractériser un treillis pour que l'espace dont ce treillis est le treillis de sous-espaces clos soit un espace de Hilbert ?

C'est dans le chapitre 6 que nous présentons les résultats les plus importants du programme de dérivation. Le chapitre s'ouvre par une section dans laquelle nous nous demandons quels éléments du formalisme de la théorie quantique il faut reconstruire à partir des axiomes théorético-informationnels. La section suivante expose l'idée de preuve due à Rovelli. Toutefois, la vraie preuve est développée indépendamment dans la Section 6.3 qui est le point central de la thèse. Dans cette section, partant de l'axiomatique théorético-informationnelle, nous démontrons le Théorème 6.11 qui assure que l'espace de la théorie est un espace de Hilbert. Dans les sections qui suivent, on traite les problèmes du caractère quantique de l'espace de Hilbert ; du corps sous-jacent à l'espace de Hilbert et du théorème de Solèr ; de la reconstruction de la règle de Born par le moyen du théorème de Gleason justifié par les arguments théorético-informationnels ; et de la dynamique temporelle unitaire, dérivée d'un ensemble minimal de présupposés à l'aide des théorèmes de Wigner et de Stone.

La partie III de la thèse est consacrée aux fondements conceptuels de l'approche des  $C^*$ -algèbres. Elle contient deux chapitres.

Le chapitre 7 présente le formalisme des  $C^*$ -algèbres. Sa première section est dédiée aux éléments de base de cette approche, tandis que dans les deux sections suivantes on traite de la théorie des automorphismes modulaires de Tomita, à laquelle l'intérêt contemporain est en grande partie dû aux travaux d'Alain Connes, et de la condition KMS.

Dans le chapitre 8, nous interprétons les concepts de base du formalisme présenté au chapitre précédent. Le chapitre s'ouvre par une section consacrée à la justification théorético-informationnelle des notions premières de la théorie. Nous identifions les présupposés les plus chargés philosophiquement. Cela nous mène à faire une parenthèse dans la section suivante, où nous exposons la dérivation de la théorie quantique par von Neumann. Malheureusement, von Neumann s'est trompé sur quelques points, et dans la troisième section nous développons une interprétation conceptuelle de l'approche moderne basée sur la théorie des algèbres locales. Le retour au programme de justification théorético-informationnelle suggère, dans la section suivante, la nécessité d'analyser la seule dérivation théorético-informationnelle de la théorie quantique algébrique qui existe, à savoir celle de Clifton, Bub et Halvorson. Nous montrons les points forts de leur dérivation, mais aussi ses faiblesses, qui engendrent des idées à propos de l'espace, le temps et la localité qui ne sont pas motivées du point de vue théorético-informationnel. Enfin, le chapitre se conclut avec une section sur le rôle du temps dans laquelle nous analysons le problème de justification théorético-informationnelle du temps.

La thèse se clôt par la Conclusion où nous présentons les questions ouvertes et d'autres axes de recherche concernés par les idées exposées dans la thèse, à savoir ceux des sciences cognitives et de la théorie de la décision. Dans la dernière section, nous suggérons l'hypothèse qu'avec le développement des technologies de l'information, le langage de l'information deviendra non seulement *le* langage de la physique, comme nous l'argumentons dans la thèse, mais aussi celui d'autres disciplines scientifiques.

## Partie I

Le premier et crucial présupposé philosophique fait dans la thèse est que le monde peut être décrit comme une « boucle des existences » ( Wheeler ). Cette expression est dénuée de tout engagement ontologique : l'accent est placé sur le mot « décrit » et non pas sur « monde ». Par conséquent, notre programme est celui de l'épistémologie : nous étudions la mise en jeu des descriptions sans se prononcer sur la réalité de l'objet décrit, une telle réalité pouvant exister ou ne pas exister. Quelle que soit la réponse, la question n'est pas pertinente. Afin d'être précis et d'éviter les termes dont la signification est vide, comme « monde » ou « existences », nous posons que la boucle décrit non pas les existences comme éléments de la réalité externe, mais les descriptions, c'est-à-dire les différentes théories. Ainsi, le premier présupposé devient : *L'ensemble de toutes les théories est décrit sous forme cyclique comme une boucle.*

Le deuxième présupposé philosophique consiste à dire que chaque description théorique particulière peut être obtenue à partir de la boucle par une opération consistant en sa *coupure*. Toute coupure sépare l'objet de la théorie des présupposés de la même théorie. Il est impossible de donner une description théorique de la boucle tout entière, sans la couper. Une fois la coupure donnée, certains éléments de la boucle deviennent l'objet d'étude de la théorie, d'autres restent dans la méta-théorie de cette théorie. En changeant l'endroit où est effectuée la coupure, il est possible d'échanger les rôles de ces éléments : ceux qui étaient *explanans* deviennent *explanandum* et l'inverse. Il est important de noter que la coupure a été fixée, c'est une *erreur logique* de se poser des questions qui n'ont un sens que par rapport à une autre coupure de la boucle. Le problème de la mesure se dissout ainsi comme une simple erreur logique, puisqu'il est dénué de sens dans l'approche théorético-informationnelle.

Les deux présupposés que nous avons faits forment un argument transcendantal, c'est-à-dire un argument à propos des *conditions de possibilité*. Dans notre cas, il s'agit de la possibilité de théorisation. Il n'est possible de construire une théorie que si la boucle a été coupée. L'absence de la coupure mène au cercle vicieux et à l'inconsistance logique. La théorie ne se rend possible que par la mise en évidence de

ses propres limites. La possibilité de théorisation est conditionnée par la coupure de la boucle.

La physique et l'information se trouvent dans la boucle en deux points diamétralement opposés. Il s'agit pour nous de couper la boucle de telle sorte que l'information soit à la base de la théorie physique particulière que nous considérons, à savoir la théorie quantique.

## Partie II

Dans la partie II, nous focalisons l'attention sur la coupure de la boucle qui fonde la théorie physique sur l'information. On introduit trois notions fondamentales qui ne peuvent pas être définies dans le cadre de la théorie sélectionnée : *système*, *information* et *fait*. La signification de ces notions n'est pas donnée par la théorie quantique, et par conséquent il faut les considérer comme des notions méta-théoriques.

La coupure de von Neumann entre l'observateur et le système étant mise au niveau zéro, tout peut être vu comme un système physique. La première notion fondamentale, celle de système, est ainsi universelle. La deuxième notion fondamentale, celle d'information, ne présuppose pas encore l'un des sens mathématiques précis de ce terme : les significations mathématiques n'apparaissent qu'à l'étape où les notions fondamentales seront traduites dans les termes mathématiques de l'un des formalismes de la théorie quantique. Les faits se présentent en tant qu'actes d'engendrement de l'information ou l'information indexée par le moment temporel où elle a été engendrée. La nature de la temporalité qui entre en jeu sera étudiée dans la Section 8.5. Dans une théorie physique, les faits sont habituellement introduits sous nom de résultats de la mesure. La question de la représentation mathématique de ces notions devient ainsi la question de ce qu'est la mesure. Nous y répondons selon les lignes du formalisme de la logique quantique. La mesure élémentaire est définie par une question binaire, c'est-à-dire une question qui n'admet que deux réponses : oui ou non.

Il convient maintenant de poser deux axiomes informationnels sur lesquels sera basée la reconstruction du formalisme de la théorie quantique. Axiome I : Il existe une

quantité maximale de l'information pertinente qui peut être extraite d'un système. Axiome II : Il est toujours possible d'acquérir une information nouvelle à propos d'un système. Contrairement aux apparences, il n'y a pas de contradiction entre les axiomes, en vertu de l'utilisation du terme « pertinente ». Le premier axiome parle non pas d'une information quelconque, mais de l'information pertinente, tandis que le deuxième axiome énonce qu'une information nouvelle peut toujours être engendrée, même s'il faut pour cela rendre une autre information, précédemment disponible, non-pertinente. La notion d'information pertinente est liée aux *faits*, et du fait du caractère méta-théorique de la notion fondamentale de fait, on s'attend naturellement à ce que la notion de pertinence ne puisse pas émerger de l'intérieur de la théorie, mais qu'elle nécessitera une définition externe. Ce sera le cas dans notre approche.

Chaque système étant traité comme système physique, mais aussi, potentiellement, comme observateur qui obtient l'information, il est urgent de distinguer ces deux rôles. En effet, dans chaque système, nous distinguons le P-observateur, qui est ce système vu comme un système physique, et l'I-observateur, qui est l'agent informationnel. L'I-observateur est méta-théorique par rapport à la théorie quantique dans l'approche théorético-informationnelle. La possibilité, donnée par le formalisme, d'éliminer le P-observateur de la considération d'une mesure permet d'obtenir la description de la mesure qui est essentielle pour la computation quantique, à savoir celle par une POVM, la mesure à valeurs dans la classe des opérateurs positifs. Enfin, la distinction entre P-observateur et I-observateur nous permet de poser le troisième axiome de l'approche théorético-informationnelle. Si les deux premiers axiomes témoignent de la présence de la contextualité métathéorique, le troisième installe la non-contextualité intrathéorique : si une information I a été engendrée, alors cela s'est passé sans l'engendrement de l'information J à propos du fait d'engendrement de l'information I. Cet axiome est équivalent à la demande d'absence de la méta-information.

Nous nous limitons ici à donner un seul résultat du chapitre 5 qui sera utilisé dans le théorème principal de la thèse. Ce résultat (Théorème 5.31), dû à Kalmbach, est le suivant : Soit  $\mathcal{H}$  un espace vectoriel de dimension infinie sur le corps  $\mathbb{D} = \mathbb{R}, \mathbb{C}$  ou

$\mathbb{H}$  et soit  $\mathcal{L}$  un treillis complet orthomodulaire de sous-espaces de  $\mathcal{H}$  qui satisfait aux conditions suivantes : tout sous-espace de dimension finie de  $\mathcal{H}$  appartient à  $\mathcal{L}$ , et pour tout  $U \in \mathcal{L}$  et pour tout sous-espace  $V$  de dimension finie de  $\mathcal{H}$  la somme  $U + V$  appartient à  $\mathcal{L}$ . Alors il existe le produit interne  $f$  sur  $\mathcal{H}$  tels que  $\mathcal{H}$  avec  $f$  est un espace de Hilbert, qui a  $\mathcal{L}$  pour son treillis de sous-ensembles clos.  $f$  est déterminé de façon unique à une constante positive réelle près. Un résultat analogue est démontré pour les espaces de dimension finie.

Nous procédons maintenant à la reconstruction de la théorie quantique à partir des axiomes théorético-informationnels à l'aide du formalisme de la logique quantique. Le premier élément à reconstruire est l'espace de Hilbert de la théorie. Cette reconstruction se fait en sept étapes.

À la première étape, on définit le treillis des questions binaires qui représentent la notion fondamentale d'information. La réponse à une question binaire représente la notion fondamentale de fait. On postule (Axiomes IV, V et VI) la structure requise dans la définition du treillis et, également, que le treillis est complet. À la deuxième étape, on définit la complémentation orthogonale dans le treillis et on démontre que cette notion correspond bien à toutes les conditions qui s'imposent sur le complément orthogonal. À la troisième étape, on utilise la complémentation orthogonale pour définir la pertinence d'une question par rapport à une autre. À l'aide de l'Axiome I, on prouve un lemme décisif démontrant que le treillis ainsi construit est orthomodulaire.

À la quatrième étape, on introduit un espace de Banach arbitraire dont le treillis de sous-espaces clos est isomorphe au treillis que nous avons construit. À la cinquième étape, on étudie les propriétés de cet espace et on montre, en particulier, que les conditions ci-mentionnées à propos des sous-espaces de dimension finie sont validées. À la sixième étape, on introduit axiomatiquement le type du corps sous-jacent à l'espace en question. Enfin, à la septième étape, on prouve que cet espace est un espace de Hilbert.

À l'aide de l'Axiome II, et en supposant l'absence des règles de supersélection dans l'espace de Hilbert construit, nous montrons le caractère quantique, et non pas

classique, de cet espace. Pour cela, nous prouvons que toute sous-algèbre booléenne du treillis orthomodulaire que nous avons construit est sa sous-algèbre propre. Par conséquent, le treillis lui-même est non-distributif.

Nous discutons ensuite d'une alternative à l'Axiome VII qui porte sur le type du corps numérique sous-jacent à l'espace de la théorie. Au lieu de postuler que c'est un corps simple, on pouvait utiliser le théorème de Solèr qui engendre ce résultat au prix de présupposer l'existence, dans l'espace de la théorie, d'une séquence infinie orthonormale. À cause de l'obscurité de justification théorético-informationnelle potentielle de l'existence d'une telle séquence, nous choisissons de ne pas suivre la voie alternative suggérée par le théorème de Solèr.

Une fois que l'espace de Hilbert a été construit, il est nécessaire de reconstruire les deux autres éléments du formalisme de la théorie quantique : la règle de Born avec l'espace des états et la dynamique temporelle unitaire. En utilisant le théorème de Gleason, justifié par l'Axiome III, on retrouve la règle de Born. Pour obtenir la dynamique temporelle, on postule que les ensembles de questions indexés par la variable du temps sont tous isomorphes. À l'aide des théorèmes de Wigner et de Stone, on obtient ensuite la description hamiltonienne du développement du système physique dans le temps et l'équation de Heisenberg pour l'opérateur de l'évolution.

Nous concluons la partie II par une démonstration de la description de la mesure en tant que POVM, grâce à notre argument concernant le temps et à la séparation entre I-observateur et P-observateur.

La liste complète des axiomes qui ont été utilisés pour la reconstruction du formalisme de la théorie quantique est ainsi comme suit :

**Axiome I.** Il existe une quantité maximale de l'information pertinente qui peut être extraite d'un système.

**Axiom II.** Il est toujours possible d'acquérir une information nouvelle à propos d'un système.

**Axiome III.** Si information I à propos d'un système a été engendrée, alors cela s'est passé sans l'engendrement de l'information J à propos du fait d'engendrement

de l'information I.

**Axiome IV.** Pour toute paire de questions binaires, il existe une question binaire à laquelle la réponse est positive si et seulement si la réponse à au moins une des questions initiales est positive.

**Axiome V.** Pour toute paire de questions binaires, il existe une question binaire à laquelle la réponse est positive si et seulement si la réponse à chacune des questions initiales est positive.

**Axiome VI.** Le treillis des questions binaires est complet.

**Axiome VII.** Le corps numérique sous-jacent à l'espace de la théorie est l'un des corps  $\mathbb{R}$ ,  $\mathbb{C}$  ou  $\mathbb{H}$  et l'anti-automorphisme involutif dans ce corps est continu.

De ces axiomes on déduit que, premièrement, la théorie est décrite par un espace de Hilbert qui est de caractère quantique; deuxièmement, sur cet espace de Hilbert on construit l'espace des états, puis on dérive la règle de Born et on dérive aussi, avec quelques présupposés supplémentaires, la dynamique temporelle unitaire sous la forme classique de l'évolution hamiltonienne.

## Partie III

Dans la partie II, à l'aide de l'approche de la logique quantique, nous avons dérivé le formalisme de la théorie quantique. Dans la partie III, nous considérons une approche différente, celle de la théorie des  $C^*$ -algèbres. Dans ce cadre, le programme de dérivation sera réduit au problème de l'interprétation théorético-informationnelle de l'approche algébrique. Une fois ladite interprétation sera achevée, les théorèmes des  $C^*$ -algèbres permettront de retrouver le formalisme de la théorie quantique sous la forme précise du formalisme de la théorie des algèbres locales.

Le chapitre 7 est consacré à la présentation de quelques éléments mathématiques du formalisme algébrique. Nous introduisons les notions de  $C^*$ -algèbre et d'algèbre de von Neumann concrètes et abstraites. Nous définissons ensuite ce qu'est un état sur une algèbre et nous donnons la première classification des facteurs de von Neumann.

Dans la section 7.2, les concepts de la théorie de Tomita sur les automorphismes modulaires sont introduits, ce qui mène à la deuxième classification des facteurs de von Neumann, due à Connes, et aux théorèmes montrant l'unicité des algèbres hyperfinies de type  $II_1$  et  $III_1$ .

Dans la Section 7.3, il s'agit de la théorie KMS et du lien avec la thermodynamique. Le théorème principal est celui de Tomita et Takesaki, qui dit que tout état fidèle sur une algèbre est un état KMS à la température inverse  $\beta = 1$ , par rapport à l'automorphisme modulaire qu'il génère lui-même. Ainsi, exactement de la même façon que dans le cas de la mécanique classique, un état à l'équilibre contient toute l'information sur la dynamique du système qui peut être définie par l'hamiltonien, sauf la constante  $\beta$ . Cela signifie que l'information sur la dynamique peut être entièrement remplacée par l'information sur l'état thermique. Le fait que  $\beta$  soit constante et non-modifiable de l'intérieur de la théorie quantique dans l'approche théorético-informationnelle mène à placer la thermodynamique, comme une science qui étudie les variations de la température et, par conséquent, de  $\beta$ , dans la coupure de la boucle des théories différente de celle où se trouve la théorie quantique. La thermodynamique appartient ainsi, dans l'approche théorético-informationnelle, à la méta-théorie de la théorie quantique.

C'est dans le chapitre 8 que nous donnons une interprétation théorético-informationnelle de l'approche algébrique. Les notions fondamentales sont traduites par des notions mathématiques de  $C^*$ -algèbre et d'état sur cette algèbre. Une algèbre correspond à un système, tandis que l'état, en tant que l'état informationnel, décrit l'information à propos de ce système. Cela nous mène à considérer la notion de préparation comme catalogue de *toute* l'information que l'observateur a à propos d'un système, et, à son tour, l'analyse de la notion de préparation est intrinsèquement liée à l'idée initiale de von Neumann concernant la méthode de dérivation du formalisme de la théorie quantique. Von Neumann se préoccupait de la notion d'ensemble élémentaire non-ordonné, qui lui a servi pour fonder l'Ansatz statistique – le premier jalon de la mécanique quantique. Von Neumann a utilisé son programme de dérivation, que nous

exposons dans la Section 8.2, pour argumenter le passage de la mécanique quantique basée sur l'espace de Hilbert, à la mécanique quantique basée sur un facteur de type  $II$ . Malheureusement, les facteurs de ce type, dans la théorie quantique moderne, se sont révélés inutiles, et c'est à l'interprétation des concepts de cette dernière que nous procédons.

Il s'agit dans la Section 8.3 de justifier le choix particulier qui est fait par la théorie des algèbres locales, qui donne la préférence à l'algèbre hyperfinie de type  $III_1$ . Toutefois, nous commençons par une analyse des présupposés cachés dans le choix d'une  $C^*$ -algèbre et d'un état sur elle comme représentants des notions de système et d'information. Le deuxième choix, celui d'un fonctionnel positif linéaire comme représentant de la notion d'information, est lourd de postulats implicites. En effet, toute la dérivation à l'aide de la logique quantique avait pour but l'obtention de la structure de l'espace de Hilbert, et ceci au prix d'une seule définition méta-théorique, à savoir celle de la notion d'information pertinente. Avec la traduction de la notion d'information sous forme de la notion d'état, le nombre de présupposés méta-théoriques augmente : ils sont deux – linéarité et positivité, tandis que, dans ce cadre, pour dériver l'espace de Hilbert il suffit de se référer à la construction GNS sans rentrer dans l'explicitation des détails comme on l'a fait dans le cas de la logique quantique.

Une fois que les présupposés cachés ont été dégagés, il convient de passer à l'interprétation de la théorie des algèbres locales par les Axiomes I et II. Il est suggéré et argumenté que ces deux axiomes correspondent à la demande que l'algèbre en question soit hyperfinie. L'argumentation précise est donné dans le texte de la thèse.

Ayant donné l'interprétation théorético-informationnelle de l'approche algébrique à l'aide des axiomes posés dans le chapitre 4, nous nous posons maintenant la même question que dans la Section 6.4, à savoir celle du caractère quantique vs. classique de la théorie. Il est nécessaire de se restreindre, par le moyen des présupposés théorético-informationnels, au cas quantique. La solution a été proposée par Clifton, Bub et Halvorson dans un article où ils opèrent une dérivation de la théorie quantique à

partir des théorèmes de la computation quantique. Les trois théorèmes qu'ils utilisent sont : l'absence de transfert supralumineux de l'information via la mesure ('no superluminal information transfer via measurement'), l'absence de « télédiffusion » des états ('no broadcasting') et l'impossibilité d'engager un octet de manière décisive dans un processus de transmission ('no bit commitment'). Nous analysons les détails de leur dérivation et, tout en l'endossant sur le plan formel, sauf en une seule occasion, nous la critiquons sur le plan conceptuel, en rapport avec l'utilisation d'un vocabulaire non-pertinent pour ce qui est de l'approche algébrique. Nous la reformulons ensuite pour donner un critère théorético-informationnel des systèmes physiques *distincts*. À l'aide de ce critère et en utilisant les théorèmes démontrés par Clifton, Bub et Halvorson, on retrouve le caractère quantique de l'algèbre.

L'une des critiques que nous adressons à Clifton, Bub et Halvorson consiste à mettre en question l'utilisation qu'ils font des concepts d'espace et de temps. Dans l'approche théorético-informationnelle, ces notions n'appartiennent pas à l'ensemble des notions fondamentales et elles doivent, par conséquent, être dérivées des notions fondamentales et des axiomes. Nous y consacrons la Section 8.5. En vertu de la théorie KMS, chaque état sur une algèbre acquiert son courant modulaire de Tomita, et c'est ce courant que nous appelons le temps dépendant de l'état. Il faut souligner trois conséquences importantes de la référence à la théorie KMS pour la définition du temps :

- Le temps est un concept qui dépend de l'état. Si l'état ne change pas, le temps ne change pas non plus. Un changement dans le temps signifie un changement de l'information. Ce dernier peut être engendré dans un nouveau fait. Alors, à chaque fait, le temps dépendant de l'état « redémarre ». On observe que la temporalité des faits (la variable  $t$  qui indexe les faits) n'a rien à voir avec la notion du temps qui dépend de l'état.
- La thermodynamique ne joue pas de rôle. Pour voir un état comme un état KMS à  $\beta = 1$  et pour définir le courant temporel, il n'est pas nécessaire de dire que l'état sur une  $C^*$ -algèbre est un concept thermodynamique. Par conséquent, cela

permet d'identifier la thermodynamique comme méta-théorie dans l'approche théorético-informationnelle. Pour faire ainsi, il suffit de considérer le temps modulaire et d'exécuter la rotation de Wick, en appelant température le résultat. Si l'on veut modifier la température indépendamment du temps modulaire, il est inévitable d'introduire un degré de liberté nouveau par rapport à la théorie quantique dans l'approche théorético-informationnelle.

- Dans le cadre de l'interprétation théorético-informationnelle de la théorie des algèbres locales, on justifie le caractère hyperfini de la  $C^*$ -algèbre du système. Par conséquent, s'il n'y a pas eu d'engendrement de l'information nouvelle, et si l'algèbre est un facteur de von Neumann de type  $III_1$ , le spectre du temps varie de 0 jusqu'à  $+\infty$ . Ce résultat correspond à notre intuition sur la façon dont le temps se comporte.

Le temps est une notion dépendante de l'état, mais l'on voudrait aussi avoir dans la théorie un temps qui ne dépend pas de l'état. Pourquoi ? Parce que nous sommes habitués au temps linéaire newtonien qui ne dépend pas de l'état informationnel. Pour obtenir ce temps non-dépendant de l'état, nous factorisons les automorphismes modulaires par les automorphismes internes et nous choisissons toute une classe de ces derniers qui correspond à un seul automorphisme externe. En effectuant cette opération, nous négligeons une certaine information, à savoir celle qui distinguait entre eux les automorphismes modulaires, ceux qui ont tous été projetés sur un seul automorphisme externe. Ainsi l'émergence du temps devient la question du rejet d'une certaine information comme non-pertinente. Cela évoque le mot de Bohr qui disait, « Les concepts d'espace et de temps, par leur nature même, n'acquièrent un sens que grâce à la possibilité de négliger les interactions avec les moyens de la mesure ». Nous concluons le chapitre en démontrant comment ces propos de Bohr acquièrent un sens théorético-informationnel grâce à la division entre I-observateur et P-observateur.



# Part I

## Introduction



# Chapter 1

## General remarks

### 1.1 Disciplinary identity of the dissertation

This dissertation belongs to the field of Foundations of Physics. It means that we aim at combining the analysis of concepts underlying physical theories with rigorous *formal* results that allow to avoid ambiguity in conclusions. Role of mathematical proof in the justification of conclusions is a deciding factor.

This dissertation also reaches out to other disciplines. In Part III our task is to give an interpretation and the area concerned is closer to the philosophy of physics. In Chapter 2 questions that are raised are general rather than specialized to the case of physics: the area, then, is the one of the philosophy of science or epistemology. In the Conclusion, speaking about open topics and the application of the ideas of the dissertation, we discuss disciplines such as cognitive science and decision theory.

### 1.2 Goals and results

The goal of this dissertation is to give a consistent derivation of the formalism of quantum theory from information-theoretic principles. We also study a variety of issues that arise in the process of derivation. Successful accomplishment of the derivation program in Part II allows us to advance the following thesis:

**Quantum theory is a general theory of information constrained by several important information-theoretic principles. It can be**

**formally derived from the corresponding information-theoretic axiomatic system.**

In three ways we innovate in the field of the foundations of physics:

- We derive the quantum formalism from information-theoretic axioms in a novel way.
- We formulate an epistemological attitude presented in the form of a loop and we demonstrate its utility for the analysis of theories other than physics.
- We give an information-theoretic interpretation to the  $C^*$ -algebraic approach, including the Tomita theory of modular automorphisms and the issue of time emergence.

The first of these three goals remains the most important one. It is commonplace to think, even nowadays, that quantum theory is a theory of the microworld, or of real objects like particles and fields, or of some other “first matter” that necessarily has the ontological status. Information-theoretic derivation of the quantum formalism installs the long lusted clarity: all the ontological assumptions are alien to quantum theory which is, in and of itself, a pure epistemology. Quantum theory as a theory of information must be cleared from the realist ideas which are merely brought in by the physicists working in quantum theory, with all their individual prejudices and personal beliefs, rather than belong to the quantum theory proper. What belongs to quantum theory is no more than what is needed for its derivation, i.e. for a reconstruction of the quantum theoretic formalism. In the process of such derivation we for the first time demonstrate how, from information-theoretic axioms, one can reconstruct the Hilbert space—a crucial element of quantum theory. We then use powerful mathematical results to reconstruct the remainder of the formalism.

In order to separate it from the superficial ontology by means of the information-theoretic derivation, quantum theory must be derived from such postulates of which the underlying philosophy is devoid of ontological commitments. This is the role of

the second point on which innovates this dissertation. Not only it gives an exposition of the philosophy of physics that is disconnected from ontology, but it also shows how such a philosophy can be consistently linked to the derivation program *formulated in the mathematical language*.

To move to the third point of innovation, we change the attitude from the one of the scientist proving theorems to the attitude of the philosopher of physics. The task is now to give an information-theoretic *interpretation* of the algebraic formalism in quantum theory and to study philosophical underpinnings of the Tomita theory and of the Connes-Rovelli modular time hypothesis. What links this field to the previous parts of the dissertation is that we continue to follow the information-theoretic approach; what innovates with respect to the currently existing work is that, even if there were a few specialists in the foundations of physics who worked on the conceptual basis of the  $C^*$ -algebraic approach, there is virtually no published work on the conceptual foundations of the Tomita theory of modular automorphisms in connection with the KMS condition and the modular time hypothesis. We bring together various mathematical results in an attempt to give a philosophically sound exposition of the key ideas in this field.

## 1.3 Outline

The remainder of this introduction will be devoted to two needs: presentation in Chapter 2 of the philosophy in which will be rooted the dissertation; and presentation in Chapter 3 of the few elements of quantum computation.

Chapter 2 opens with a section in which we explain why interest for philosophy has reemerged in the community of physicists after the many decades of *oubli*. We then move to the highlight of the chapter, where we introduce the philosophy of the loop of existences. In the concluding section, we explain how this point of view responds to the question that any philosopher of physics immediately asks when he hears of a new approach: How does that solve the measurement problem? Our answer is that it does not solve, but rather dissolves the problem.

Chapter 3 introduces the ideas of quantum computation. They will not be used in the thesis but serve to motivate the rising interest toward the notion of information. A reader solely interested in following the main line of the dissertation can skip this chapter.

In Part II we present the information-theoretic derivation of the formalism of quantum theory. It is exposed in three chapters.

Chapter 4 is devoted to laying out the conceptual foundations of the information-theoretic approach. It opens with a historic section about axiomatization attempts in quantum mechanics. It then continues with a section on Rovelli's Relational Quantum Mechanics that justifies the intuition which we use for selection of information-theoretic axioms. Sections 4.3 and 4.4 form the core of the information-theoretic approach by postulating, respectively, the fundamental notions of the theory and information-theoretic axioms formulated in the language of these fundamental notions. The chapter then concludes with an important section on the twofold role of the observer as physical system and as informational agent.

Chapter 5 is devoted to exposition of the quantum logical formalism that will be used in the sequel. A few results belong to us but most are taken from the literature. The last section of the chapter treats the crucial question of how to characterize a lattice so that it will force the space of which this lattice is the lattice of closed subspaces to be a Hilbert space.

It is in Chapter 6 that we present the most important results of the derivation program. The chapter opens with a section in which we ask ourselves what are the elements of the formalism of quantum theory that we have to reconstruct from information-theoretic axioms. The next section gives a sketch of Rovelli's idea of derivation. The actual proof, however, is independently developed in Section 6.3 which is the highlight of the whole dissertation. In this section, based on the information-theoretic axiomatic system, we prove Theorem 6.11 which shows that the space of the theory is a Hilbert space. Consequent sections address the problems of quantumness versus classicality of the theory; of the field underlying the Hilbert

space and the Solèr theorem; of reconstruction of the Born rule by means of Gleason's theorem justified information-theoretically; and of the unitary time dynamics derived from the allegedly minimal set of assumptions with the help of Wigner's and Stone's theorems.

Part III is devoted to the conceptual foundations of the  $C^*$ -algebraic approach. It consists of two chapters.

Chapter 7 presents the  $C^*$ -algebraic formalism. Its first section is dedicated to the basic elements of this approach, while the two subsequent sections treat of the Tomita theory of modular automorphisms, much of the contemporary interest in which is due to Alain Connes's work, and of the KMS condition.

In Chapter 8 we analyze the concepts underlying the formalism presented in the previous chapter. The opening section is devoted to information-theoretic interpretation of the basic notions of the theory. We uncover the assumptions that have a maximal philosophical weight. This leads us to a digression in the next section in which we expose von Neumann's derivation of quantum theory. Unfortunately, von Neumann was wrong on certain points, and in the third section we develop a conceptual interpretation of the modern approach based on the theory of local algebras. This return to the program of information-theoretic justification suggests, in the following section, a necessity to discuss the only available information-theoretic derivation of the algebraic quantum theory due to Clifton, Bub and Halvorson. We show the strong points of this derivation but also its weaknesses that lead to informationally unmotivated assumptions concerning space, time, and locality. Finally, we conclude with a section on the role of time where we address the problem of its information-theoretic justification.

The dissertation ends with the Conclusion in which we address questions that were left open and apply the ideas of the dissertation to theories other than physics: cognitive science and decision theory. The last section advances a hypothesis that, with the development of information technology, the language of information will become not only a language of physics, the possibility of which we demonstrate in

the dissertation, but also of other scientific disciplines.

## Chapter 2

# Philosophy of this dissertation

### 2.1 “The Return of the Queen”

The conceptual revolution brought to science by quantum theory is now almost a century old. Despite this old age, the theory’s full significance has not yet been appreciated outside a limited circle of physicists and philosophers of science. Although terms like “uncertainty principle” or “quantum jumps” have been incorporated into the everyday, common language, they are often used to convey ideas which have no relation with the physical meaning of these terms. One could say that the wider public took note of the metaphorical powers of the quantum theory, while the essence of the quantum revolution remains largely unknown, even more so because of the slow reform of the educational system.

The situation is somewhat different for another great physical revolution, the one of relativity. Ideas of relativity have much better penetrated in the mainstream culture. Terms like “black holes” and “spacetime” are a familiar occurrence in popular scientific journals. Such a relative success of relativity compared to quantum theory may be due to two reasons.

First, quantum theory’s rupture with the preceding classical paradigm, although, as we argue in Section 6.2, due to a similar shift in understanding, is more radical than the rupture of relativity with Galilean and Newtonian physics. A non-scientist can easier understand that at high velocities unusual effects occur or that black holes absorb matter and light, than that the very notions of velocity, position, particle or

wave must be questioned. Interpretation of quantum theory has always been a motive for argument even among professional physicists, leave alone the general public.

Second, the discussion of foundations of quantum theory always remained away from practical applications of the theory, and therefore away from a wider audience fascinated by the breathtaking technical development. Educational systems nowadays do little or nothing to explain that computers, mobile phones, and many other everyday devices work thanks to quantum mechanics, and even if educational systems did explain this, they would probably avoid referring explicitly to any particular interpretation of quantum theory. Working applications and problems of interpretation have long been isolated from each other.

This situation has evolved in the last ten years with the appearance of the new field of quantum information. Practical quantum information applications are perhaps around the corner, with prototypes of quantum cryptographic devices and the teleportation of structures as large as atoms already realized in laboratories [4, 153]. These applications, for the first time in history, illustrate highly counter-intuitive features of quantum theory at the level of everyday utility. One sign of the growing importance of quantum information methods and results is the increasing use of them in introductory courses of quantum mechanics. In a broader context, we see the public excitement by research in quantum information, through mass media and governmental action.

We shall see that applications of quantum theory to quantum information often suggest what is essential and what is accessory in quantum theory itself, highlighting features which may be of practical and theoretical importance. It appears that taking seriously the role of information in quantum theory might be unavoidable for the future major developments.

Yet another change in the circumstances occurred due to which the foundations of quantum mechanics receive now more attention. Echoing what we said in the discussion of the first reason, this change has to do with the ongoing effort to unite the quantum mechanical ideas with the ideas of general theory of relativity. Unlike

the founding fathers of modern physics, most of their followers of the second half of the XXth century viewed questions like “What is space? What is time? What is motion? What is being somewhere? What is the role of the observer?” as irrelevant. This view was appropriate for the problems they were facing: one does need to worry about first principles in order to solve a problem in semiconductor physics or to write down the symmetry group of strong interactions. Physicists, working pragmatically, lost interest in general issues. They kept developing the theory and adjusting it for particular tasks that they had to solve; when the basis of problem-solving is given, there is no need to worry about foundations. The period in the history of physics from 1960s till the end of 1980s was dominated by the technical attitude. However, to understand quantum spacetime and the unification of quantum mechanics with gravity, physicists need to come back to the thinking of Einstein, Bohr, Heisenberg, Boltzmann and many others: to unite the two great scientific revolutions in one, one ought to think as generally as did the great masterminds of these revolutions. The questions that we enlisted above all reemerged at the front line of the scientific interest. Queen Philosophy returned to her kingdom of physics.

## 2.2 Loop of existences

Before we start laying down the foundations of the information-theoretic approach to quantum theory, it is necessary to say what role this approach plays in our general view of the scientific venture. This section presents a philosophy in which will be rooted all of the dissertation.

A first and crucial philosophical assumption is that the world is *best described* as a *loop of existences* or, as Wheeler called it, a “self-synthesizing system of existences” [197]. This phrase is devoid of any ontological commitments; the accent is placed on the word “described” and not on “world.” The program therefore is the one of epistemology: we are studying the interplay of descriptions without saying anything on the reality of the object described, if there is any such reality. Perhaps there is none: the question is irrelevant. To be precise and to remove pure placeholders like

“world” or “existences,” we say that the loop of existences describes not the existences as elements of external reality, but the descriptions, the various theories. The first assumption then becomes: The ensemble of all theories is best described in a cyclic form as a loop.

The second philosophical assumption is that any particular theoretical description is achieved by cutting the loop at some point and thus separating the target object of the theory from the theory’s presuppositions. It is impossible to give a theoretical description of the loop of existences as a whole. Bohr said about the necessity of a cut, although from a somewhat different philosophical position, that “there must be, so to speak, a *partition* between the subject which communicates and the object which is the content of the communication”<sup>†</sup> [137]. With the position of the cut being fixed, some elements of the loop will be object of the theory, while other elements will fall into the domain of meta-theory of this theory. At another loop cut these elements may exchange roles: those that were explanans become explanandum and those that were explanandum become explanans. The reason why one cannot get rid of the loop cut and build a theory of the full loop is that the human venture of knowing needs a basis on which it can rely; at another time, this basis itself becomes the object of scientific inquiry, but then a new basis is unavoidably chosen. It is not the case that these bases form a pyramid which is reduced to yet more and more primitive elements; on the contrary, for the study of one part of the world-picture, another its part must be postulated and vice versa. Employing a notion characteristic of Wittgensteinian philosophy [202], Wheeler calls this endeavour a *mutual illumination*. Francisco Varela, in the context of phenomenology and cognitive science, spoke about *mutual constraints* [186].

Consider the loop between physical theory and information (Figure 2.1). Arrows depict possible assignment of the roles of explanans and explananda, of what falls into the meta-theory and what will be object of the theory. Physics and information mutually constrain each other, and every theory will give an account of but a part

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<sup>†</sup>Our emphasis.

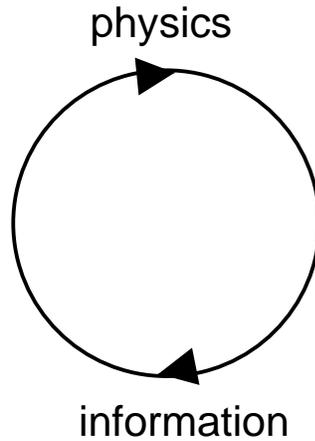


Figure 2.1: The loop of existences between physics and information

of the circle, leaving the other part for meta-theoretic assumptions. For long time physicists have lacked the understanding of this epistemological limitation. Thus historically quantum physics has been predominantly conceived as theory of non-classic waves and particles. Einstein, for instance, believed that the postulate of existence of a particle or a quantum is a basic axiom of the physics. In a letter to Born as late as 1948 he writes [20, p. 164]:

We all of us have some idea of what the basic axioms in physics will turn out to be. The quantum or the particle will surely be one amongst them; the field, in Faraday's or Maxwell's sense, could possibly be, but it is not certain.

We part radically with this view. The venture of physics is now to be seen as an attempt to produce a structured, comprehensible theory based on information. Physical theory, quantum theory including, is a general theory of information constrained by several information-theoretic principles. As Andrew Steane puts it [175],

Historically, much of fundamental physics has been concerned with discovering the fundamental particles of nature and the equations which describe their motions and interactions. It now appears that a different programme

may be equally important: to discover the ways that nature allows, and prevents, information to be expressed and manipulated, rather than particles to move.

If one removes from this quotation the reference to nature, which bears the undesired ontological flavor, what remains is the program of giving physics an information-theoretic foundation. This is what we achieve by cutting the loop: We treat quantum theory as theory of information. This is a no small change in the aim of physics. Bub [29] argues that information must be recognized as “a new sort of *physical* entity, not reducible to the motion of particles and fields”<sup>†</sup>. Although we fully endorse the second part of this phrase, we are forced into a different attitude concerning the first one. In the loop epistemology, information is not a physical entity or object of physical theory like particles or fields are. Were it physical, information would be fully reducible to the intratheoretic physical analysis. This, then, would do nothing to approach the problem of giving quantum physics a *foundation*. The only way to give an information-theoretic foundation to quantum physics is through putting information in the domain of metatheoretic concepts. When one does so consistently, conventional physical concepts such as particles and fields are reduced to information, not put along with it on equal grounds. Then the physical theory will fully and truly be a theory of information.

In the loop cut shown on Figure 2.2 information lies in the meta-theory of the physical theory, and physics is therefore based on information. The next step is to derive physics from information-theoretic postulates. In this dissertation such a derivation will be developed for the part of physics which is quantum theory.

In a different loop cut (Figure 2.3), informational agents are physical beings, and one can describe their storage of, and operation with, information by means of effective theories that are reduced, or reducible in principle, to physical theories. Cognitive science is a vast area of science dealing with this task; but informational agents can also be non-human systems such as computers. In this case, the underlying physical

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<sup>†</sup>Our emphasis.

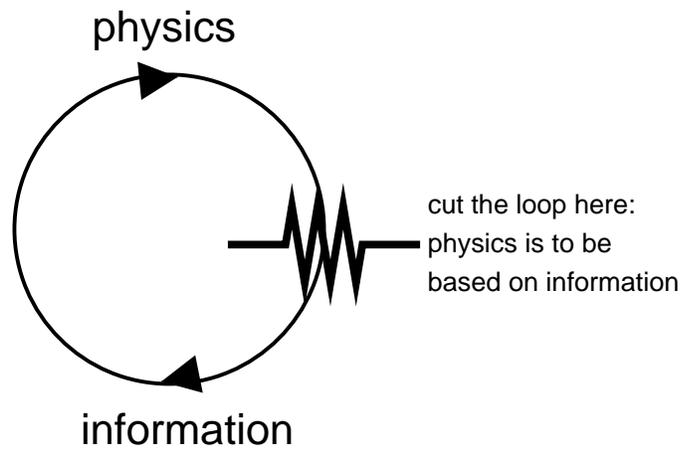


Figure 2.2: Loop cut: physics is informational

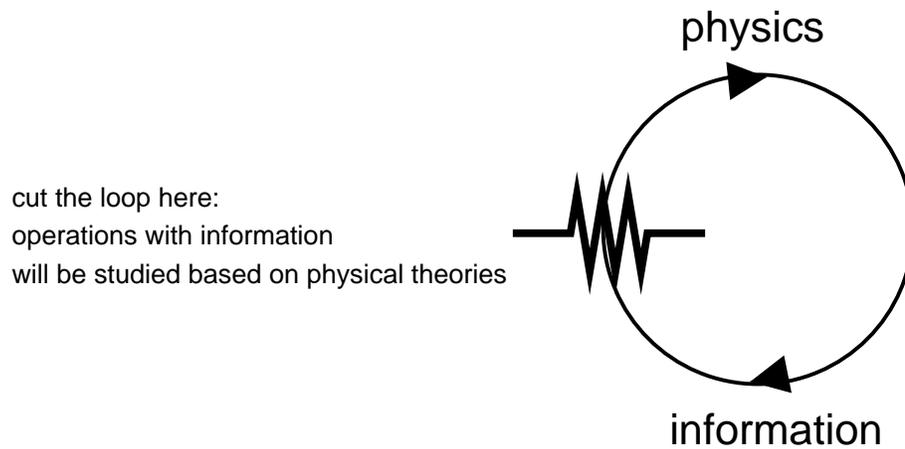


Figure 2.3: Loop cut: information is physical

theory is assumed without questioning its origin and validity. Physics now has itself the status of meta-theory and it is *postulated*, i.e. it lies in the very foundation of the theoretical effort to describe the storage of information and no result of the new theory of information can alter the physical theory. Therefore, in the loop cut of Figure 2.3, particularly in the context of cognitive science, the question of derivation or explanation of physics is meaningless. Once a particular loop cut is assumed, it is a *logical error* to ask questions that only make sense in a different loop cut. To make this last assertion clearer, let us look at the loop of existences formed by the two notions different from information and physics (Figure 2.4). We return to the study of the loop cut of Figure 2.3 in Section 10.1.

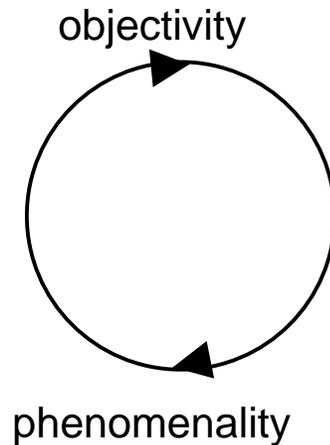


Figure 2.4: The loop of existences between objectivity and phenomenality

Loop between the phenomenal and the objective is important for understanding Husserl's phenomenology and his denunciation of science [96]. He argued that the only foundation of science is the phenomenality, and therefore no science can claim to explain the phenomenality, as, in his view, physics did. Husserl was right and wrong at the same time: if one assumes his premise about the universal primary role of phenomena, then neither physics nor any other science can explain phenomena; otherwise it would amount to a theory of the loop uncut. It then becomes a *logical error* to consider physics as explanans for phenomena. However, if one considers

Husserl's premise about phenomena not as a universal—sort of ontological—claim, but as an epistemological one: for the purposes of a *given* description it is necessary to treat the phenomenality as meta-theoretical, then nothing precludes from treating physics as meta-theoretical for the purposes of a description of phenomenality. At the very moment Husserl's premise is transferred to the sphere of epistemology, the necessity of loop cut removes the cause for Husserl's critique of physics.

Our two assumptions: viewing the ensemble of theories as a loop and postulating the necessity of loop cut for any particular theory, form a transcendental argument. Here we meet the conclusion of the paper by Michel Bitbol [16] in which what he calls “epistemological circles” also receive a transcendental treatment. By definition, a transcendental argument is an argument from the conditions of possibility. In our case, one is concerned with the conditions of possibility of theorizing, of building *a* theory, of course irrespective of the content of the theory. Theorizing is only possible if the loop is cut; uncut loop, i.e. no separation between theory and meta-theory, as in the example of Husserl's critique, is a logical error. In order to avoid the error and together with it a vicious circle, thus meeting the necessary condition of logical consistency, one must cut the loop. A theory is only possible when it knows its limits. The possibility of theorizing is conditioned by cutting the loop prior to building a theory.

## 2.3 Dissolution of the measurement problem

As a digression from the main line of development of the dissertation, in this section we address the question of how the epistemology of the loop of existences shapes the purported solution of the measurement problem. The latter is formulated as follows. In quantum mechanics a physical system can be in a superposition state, which corresponds to a certain linear combination of the eigenvectors of some observable. Temporal evolution is unitary and linear, and therefore initial superpositions of vector states are mapped onto corresponding superpositions of image vector states. Consequently, any measurement instrument will generally be entangled with the quantum

system it measures. The theory dictates that there shall be no breakdown of such entanglement. So, at the end of what we take for a measurement, neither the measuring instrument nor the system measured will have separable properties. On the other hand, our commonsense understanding of the phenomenon is that the instrument registers a definite measurement outcome. The problem is then to explain how a passage is possible, from the superposition to a definite outcome.

A classical way to tackle the measurement problem is by introducing a “wavefunction collapse.” This amounts to suspending the unitary dynamics whenever there is a measurement and saying that the quantum state collapses to one of the states in the superposition that corresponds to a definite measurement outcome. Then the final state at the end is represented as a statistical mixture of different outcomes with weights equal to probabilities defined by the entangled state. The difference between statistical mixture and entangled state is the same to which d’Espagnat refers to as proper and improper mixtures [42].

Other solutions to the problem of measurement include collapse theories that modify the unitary dynamics [69]; many-worlds interpretation [58, 195]; or subscribing to some form of modal interpretation, although it remains to be seen how this can help to solve the problem. All these theories are empirically equivalent and can be distinguished from one another on non-experimental grounds only. Apparent underdetermination of quantum theory is expressed in the fact that it allows for all the various equivalent theories to exist. We argue that this only happens if quantum theory is viewed in the usual way physical theories are looked at: namely, as a theory about physical entities that really exist, such as particles or waves, and aiming at describing these entities. Now, if one changes the stance and adopts our view of the physical theory being the theory of information, the problem of choice between various answers to the measurement problem and, indeed, the measurement problem itself are, not solved but *dissolved*. Because the loop must be cut in construction of any particular theory, the measurement problem is a mere logical error, a consequence of the failure to distinguish between theory and meta-theory.

Indeed, if we identify the measuring system with the one which stores and manipulates information, it follows from the discussion of the two possible loop cuts that the measuring system *must* remain unaccounted for by the physical theory based on information. A new, separate theory of measuring systems is possible, but in order to construe it, one ought to choose a new cut of the loop and thereby be swayed away from the theory that had information as primary notion. A purported solution of the quantum mechanical measurement problem belongs to the loop cut of Figure 2.3, while quantum mechanics as physical theory belongs to the loop cut of Figure 2.2.

The quantum mechanical measurement problem is then equivalent to the non-existence of cut in the loop, to merely confusing questions that make sense in one loop cut with questions that make sense in the opposite loop cut. Assumption of necessity of the loop cut, grounded in the transcendental argument, with its origin in the structure of the human venture of theorizing, dissolves the measurement problem: at the very moment a cut appears in the loop, the problem disappears.



# Chapter 3

## Quantum computation

This chapter is a brief introduction to the ideas of quantum computation, a domain whose rapid development in 1990s motivated the increase of interest toward the notion of information. The chapter is not essential for following the main argument of the dissertation and a reader only interested in the latter may go directly to Part II.

### 3.1 Computers and physical devices

Since ever humanity has been seeking tools to help to solve problems and tasks, and with growth of complexity of these tasks, the tools became needed for solving the problem of calculation. One needs to calculate the area of land, stress on rods in bridges, or the shortest way from one place to another. Simple calculation evolved in a complicated computation. A common feature of all these tasks, however, was that they follow the pattern:

$$\text{Input} \rightarrow \text{Computation} \rightarrow \text{Output}.$$

The computational part of the process is inevitably performed by a dynamical physical system, evolving in time. In this sense, the question of what can be computed is connected to the question of what systems can be physically realized. If one wants to perform a certain computational task, one must seek the appropriate physical system, such that the evolution of the system in time corresponds to the desired computation. If such a system is initialized according to the input, its final state will correspond to the output.

An example [2] of interconnection between physical systems and computation was invented by Gaudi, the great Spanish architect. The plan of his Sagrada Familia church in Barcelona is very complicated, with towers and arcs emerging from unexpected places, leaning to other towers and arcs, and so forth. It was practically impossible to solve the set of equations which correspond to the requirement of equilibrium of this complex. Instead of solving the equations Gaudi did the following: for each arc he took a rope, of length proportional to the length of the arc. Where arcs were supposed to lean on each other, he tied the end of one rope to the middle of another rope. Then he tied the edges of the lowest ropes, which must correspond to the lower arcs, to the ceiling. All computation was thus instantaneously done by gravity. Angles between the arcs and radii of the arcs could be easily read from this analog computer, and the whole church could be seen by simply putting a mirror on the floor under the rope construction.

Many examples of analog computers exist, which were devised to solve a specific computation task; but we do not want to build a completely different machine for each task that we have to compute. We would rather have a general purpose machine, which is “universal.” A mathematical model for such machine is Turing machine, which consists of an infinite tape, a head that reads and writes on the tape, a machine with finitely many possible states, and a transition function  $\delta$ . Given what the head reads at time  $t$  and the machine’s state at time  $t$ , function  $\delta$  determines what the head will write, to which direction it will move and what will be the new machine’s state at time  $t + 1$ . The Turing machine defines a concept of computability, according to the Church-Turing thesis in a very broad formulation:

**Church-Turing thesis:** A Turing machine can compute any function computable by a reasonable physical device.

What does “reasonable physical device” mean? The Church-Turing thesis is a statement about universal qualities of the physical world and not a formal mathematical statement; therefore it cannot be rigorously proven. However, up to now

all physical systems used for computation seem to have a simulation by a Turing machine, although often only in principle.

It is an astonishing fact that there exist families of functions which cannot be computed. In fact, most of the functions cannot be computed: there are more functions than there are ways to compute them. The reason for this is that the set of Turing machines is countable, whereas the set of families of functions is not. In spite of the simplicity of this argument (which can be formalized using the diagonal argument, as did Gödel), the observation itself came as a big surprise when it was discovered in 1930s. The subject of computability of functions is the cornerstone of *computational complexity*. Often we are interested not only in which functions can be computed but in the *cost* of such computation. The cost, or computational complexity, is measured naturally by the physical resources invested in solving the problem, such as time, energy, space, etc. A fundamental question in computational complexity is how the cost of computing a function varies as a function of the input size,  $n$ , and in particular whether it is polynomial or exponential in  $n$ . In computer science problems which can only be solved in exponential cost are regarded as intractable. The class of tractable problems consists of problems which have solutions with polynomial cost.

It is worth reconsidering what it means to *solve* a problem. An important conceptual breakthrough was the understanding [149] that sometimes it is advantageous to relax the requirements that a solution be always correct, and allow some (negligible) probability of error. This gives rise to a much more rapid solutions of different problems, which make use of random coin flips, such as an algorithm to test whether an integer is prime or not [40]. The class of tractable problems is now considered as problems solvable with a negligible probability for error in polynomial time. These solutions will be computed by a deterministic Turing machine, except that the transition function can change the configuration in one of several possible ways, randomly. The modern Church thesis refines the Church Turing thesis and asserts that the probabilistic Turing machine captures the entire concept of computational complexity:

**The Modern Church thesis:** A probabilistic Turing machine can compute any

function computable by a reasonable physical device in polynomial cost.

Again, this thesis cannot be proven because it is not a mathematical statement. It is worthwhile mentioning a few models which at the first sight might seem to contradict the modern Church thesis, such as the DNA computer [133]. Most of these models, which are currently a subject of growing interest, *do not rely on classical physics*.

## 3.2 Basics of quantum computation

In the beginning of 1980s Feynman [60, 61] and Benioff [8, 9] started to discuss the question of whether computation can be done on the scale of quantum physics. In classical computers, the elementary information unit is a bit, the value of which is either 0 or 1. The quantum analog of a bit would be a two-state particle, called a *qubit*. A two-state system is described by a unit vector in the Hilbert space isomorphic to  $\mathbb{C}^2$ . Zero state of a bit corresponds to vector  $1 \times |0\rangle + 0 \times |1\rangle = |0\rangle$ , state one of the bit corresponds to the state  $|1\rangle$ . These two states constitute a orthogonal basis in the two-dimensional Hilbert space, and the general state of a qubit is described as their normalized linear combination. To build a computer, we need to use a large number of qubits. Then the Hilbert space is a product of  $n$  spaces  $\mathbb{C}^2$ . Naturally classical strings will correspond to quantum states:

$$i_1 i_2 \dots i_n \leftrightarrow |i_1\rangle \otimes |i_2\rangle \otimes \dots \otimes |i_n\rangle \equiv |i_1 \dots i_n\rangle. \quad (3.1)$$

How to perform calculation using qubits? Suppose that we want to compute the function  $f : i_1 \dots i_n \mapsto f(i_1 \dots i_n)$ , from  $n$  bits to  $n$  bits. We would like the system to evolve according to the time evolution operator  $U$ :

$$|i_1 \dots i_n\rangle \mapsto U|i_1 \dots i_n\rangle = |f(i_1 \dots i_n)\rangle. \quad (3.2)$$

We therefore have to find a Hamiltonian  $H$  which generates this evolution. According to the Schrödinger equation, this means that we have to solve for  $H$ :

$$|\Psi_f\rangle = \exp\left(-\frac{i}{\hbar} \int H dt\right) |\Psi_0\rangle = U|\Psi_0\rangle \quad (3.3)$$

A solution for  $H$  always exists, as long as the linear operator  $U$  is unitary. Unitarity is an important restriction. Note that the quantum analog of a classical operation will be unitary only if  $f$  is one-to-one, or bijective. Hence, a reversible classical function can be implemented by a physical Hamiltonian. It turns out that any classical function can be represented as a reversible function on a larger number of bits [10], and that computation of  $f$  can be made reversible without losing much in efficiency. Moreover, if  $f$  can be computed classically by polynomially many elementary reversible steps, the corresponding  $U$  is also decomposable into a sequence of polynomially many elementary unitary operations. We see that quantum systems can imitate all computations which can be done by classical systems, and do so without losing much in efficiency.

Quantum computation is interesting not because it can imitate classical computation, but because it can probably do much more. Feynman pointed out the fact that quantum systems of  $n$  particles seem to be hard to simulate by classical devices, and this exponentially in  $n$ . In other words, quantum systems do not seem to be polynomially equivalent to classical systems, including classical computational devices, which violates the modern Church thesis. This provides an insight on why as computational devices quantum systems may be much more powerful than classical systems.

How to use “quantumness”? Consider, for example, the Greenberger-Horne-Zeilinger (GHZ) tripartite state [71]:

$$\frac{1}{\sqrt{2}} (|000\rangle + |111\rangle). \quad (3.4)$$

What is the superposition described by the first qubit? The answer is that is no such superposition. Each of the three qubits does not have a state of its own, and the state of the system is not a tensor product of states of each particle. Such states are called entangled. Entanglement is used in the Einstein-Podolski-Rosen “paradox” [56] and Bell inequalities both in the original formulation by Bell [5, 6] and the one proposed by Clauser, Holt, Horne and Shimony [33]. Because of entanglement, the state of the system can only be described as a superposition of all  $2^n$  basis states,

and, consequently,  $2^n$  coefficients are needed. This exponentiality of resources in the Hilbert space is the crucial property needed for quantum computation. To take another example, consider a uniform superposition of all basis states:

$$\frac{1}{\sqrt{2^n}} \sum_{i_1, i_2, \dots, i_n=0}^1 |i_1 i_2 \dots i_n\rangle. \quad (3.5)$$

Now apply to it the unitary operation which computes  $f$ , as in Equation 3.2. From the linearity of quantum mechanics we get:

$$\frac{1}{\sqrt{2^n}} \sum_{i_1, i_2, \dots, i_n=0}^1 |i_1, i_2, \dots, i_n\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{i_1, i_2, \dots, i_n=0}^1 |f(i_1, i_2, \dots, i_n)\rangle. \quad (3.6)$$

The conclusion is that by applying  $U$  one computes  $f$  simultaneously on all the  $2^n$  possible inputs  $i$ , which is an enormous gain in parallelism.

In fact, such an exponential gain in parallelism does not imply exponential increase in computational power. The problem lies in the question of how to extract information out of the system. In order to do this, one has to *observe* the quantum system. In a standard interpretation of quantum mechanics, after the measurement, the state is projected on one of the exponentially many possible states, and all information appears to be lost. To gain advantage, one therefore needs to combine parallelism with another feature, which is interference. The goal is to arrange the cancellation by interference so that only the interesting computations remain and all the rest cancel out. If one expresses this operation in the initial basis, rearrangement will take the form of a POVM measurement, i.e. of a measurement represented as a positive operator-valued measure [41]. This explains why POVM are an essential tool in the science of quantum information. Formal development of this idea will follow in Chapter 6.

Combination of parallelism and interference plays an important role in quantum algorithms. A quantum algorithm is a sequence of elementary unitary steps, which manipulate the initial quantum state  $|i\rangle$  for an input  $i$  so that a measurement of the final state of the system yields the correct output. The first quantum algorithm, which combines parallelism and interference to solve a problem faster than a classical

computer, was discovered by Deutsch and Jozsa [43]. The algorithm must distinguish between “constant” (all items are the same) and “balanced” databases. The quantum algorithm solves this problem exactly in polynomial cost. Classical computers cannot do better than to check all items in the database, which is exponentially long, and in polynomial cost they can only solve the problem approximately. Deutsch and Jozsa’s algorithm provides an exact solution in virtue of using the Fourier transform. A similar technique also gave rise to the most important quantum algorithm that we know today, Shor’s algorithm [171].

Shor’s algorithm is a polynomial quantum algorithm for factoring integers and for finding a logarithm over a finite field. For both problems the best known classical algorithms are exponential. However, there is no proof that a classical polynomial algorithm is impossible. Shor’s result is extremely important both theoretically and practically, due mainly to the fact that the factorization task is a cornerstone of the RSA cryptographic system, which is used almost everywhere in our life, to start with internet browsers. A cryptographic system must be secure; this means that an eavesdropper will not be able to learn in reasonable time significant information about the message that has been sent. For RSA system, to be successful in cracking the system, the eavesdropper needs to have an efficient algorithm for factoring big numbers. It is therefore understandable why Shor’s result is viewed as the first potential implication of quantum information science that will prove to be of great practical significance.

It is important to note that quantum computation does not rely on unreasonable precision of measurement, but a polynomial precision is enough. This means that the new model requires physically reachable resources, in terms of time, space, and precision; yet it is exponentially stronger than the ordinary model of a probabilistic Turing machine. Currently, quantum computer is the only model which threatens the modern Church thesis.

There are several major directions of research in the area of quantum computation. Introduction and comprehensive analysis can be found in a number of recent

monographs [21, 54, 132].

### 3.3 Why quantum theory and information?

The remarkable achievements of the science of quantum information and computation allow one to take it as the viewpoint from which to look at all of quantum physics. Still, it is obvious that quantum computation only uses a tiny fraction of the results of quantum physics, although conceptually the most profound ones. From this viewpoint one must ask, as we do in this dissertation, if information-theoretic axioms can serve as a foundation for quantum physics and if not fully, then to what extent. We close the introduction by giving four arguments why in our opinion such program deserves attention.

**Argument for a specialist in quantum computation.** A researcher in quantum computation would like to view quantum computation as an autonomous scientific area, which merits its own development from its first principles, without bringing in much from other disciplines. Such a project would try to establish “axiomatic closure” or self-sufficiency of this discipline, i.e. all information-theoretic results in quantum computation will be derived from information-theoretic axioms. With this idea in mind, a researcher in quantum computation would like to see which parts of quantum mechanics he or she needs *prima facie*, and which parts it is possible to deduce from information-theoretic axioms. The result then will show to what extent the science of quantum computation can be treated as autonomous discipline.

**Argument for a theoretical physicist.** Working physicists seldom address problems in the foundations of quantum theory and are often unprepared to talk about the role of this or that of the bricks that compose it. To understand better the structure of the theory, the origin of its first principles and their interconnections, it is challenging to attempt a reconstruction of the quantum theory from information-theoretic axioms: a reconstruction implies derivation, and the mathematical language of the derivation program is familiar to physicists. Still, one must be from the very

start aware that such a derivation will not fully replace any of the usual ways of introducing quantum theory in physics, as it would be too ambitious to expect that all of modern quantum theory, including field theory and unification attempts, can be reconstructed from the few information-theoretic axioms; additional features often need additional assumptions. In the derivation proposed in this dissertation, we only justify the algebraic structure of the theory, and with regard to issues not directly linked to algebra, such as time dependence, reconstruction from information-theoretic principles demands more assumptions (see Section 6.7).

**Argument for a laboratory physicist.** The best method to decide in which way to give a foundation to a scientific discipline lays, perhaps, in looking at how the theory is applied, i.e. at the technology that it generates. As Fuchs puts it, “If one is looking for something ‘real’ in quantum theory, what more direct tack could one take than to look to its technologies? People may argue about the objective reality of the wave function *ad infinitum*, but few would argue about the existence of quantum cryptography as a solid prediction of the theory. Why not take that or a similar effect as the grounding for what quantum mechanics is trying to tell us about nature?” [64] Some steps have already been made in the direction of studying quantum mechanics in the light of the technology to which it gave birth [131], and the program of deriving quantum mechanics from information-theoretic principles can be viewed as a development of this project.

**Argument for an educator.** The world is nowadays facing a rapid development of nanotechnology [50] and, perhaps in the near future, of the technology of quantum information. This means that the society will soon need to educate quantum engineers, whose specialization will be in quantum computers and other quantum technological devices. As any engineer, quantum engineer will not be a scientist doing fundamental research in physics, and thus will only need to be given as much of physical education as he ought to have in order to master his profession. The future educator of quantum engineers will be interested in finding out, how much of quantum physics the engineer needs to be taught and whether this much of physics can be taught by being derived

from the information-theoretic principles, which, in their turn, will be a part of the engineer's basic training.

## Part II

# Information-theoretic derivation of quantum theory



# Chapter 4

## Conceptual background

### 4.1 Axiomatic approach to quantum mechanics

In Part II of the dissertation we demonstrate a derivation of quantum theory from information-theoretic axioms. Attempts at axiomatization of quantum mechanics have been made ever since von Neumann’s early work, and we start by presenting the idea of axiomatic approach.

As such, the axiomatic method can be traced back to the Greeks. The XIXth century revolutionized this approach by bringing in the idea that an axiom can no longer be considered as an ultimate truth about reality, but a structural element—an assumption that lies in the foundation of a certain theoretical structure. Therefore “not only geometry, but many other, even very abstract, mathematical theories have been axiomatized, and the axiomatic method has become a powerful tool for mathematical research, as well as a means of organizing the immense field of mathematical knowledge which thereby can be made more surveyable” [90].

The first paper where quantum mechanics was treated as a *principle theory* appeared very shortly after the creation of quantum mechanics itself. To quote from Hilbert, von Neumann and Nordheim [91]:

The recent development of quantum mechanics, stemming on the one hand from the papers of Heisenberg, Born and Jordan and those of Schrödinger on the other hand, has put us in a position to subsume the whole domain of atomic phenomena from a single point of view... In view of the great

significance of quantum mechanics it is an urgent requirement to formulate its principles as clearly and generally as possible.

...

The route leading to the theory is the following: we make certain physical requirements of the probabilities, suggested by our previous experience and developments, and whose fulfillment entails certain relations between the probabilities. Secondly, we look for an analytical apparatus, in which quantities occur satisfying exactly the same relations. This analytical apparatus, and the arithmetic quantities occurring in it, receives now *on the basis of the physical postulates* a physical interpretation. Here, the aim is to formulate the physical requirements so completely that the analytical apparatus is just uniquely determined. Thus the route is of axiomatization.

...

The process of axiomatization indicated above is not as a rule exactly followed through in physics, but the route to the establishment of a new theory is here, as elsewhere, the following. One conjectures the analytical apparatus, before establishing a complete axiom system and only then, by interpretation of the formalism, obtains the basic physical relations. It is difficult to understand such a theory if one does not make a sharp distinction between these two things, the formalism and its physical interpretation.<sup>†</sup>

Such a standpoint led von Neumann, in collaboration with Birkhoff, to the first study of the logic of quantum mechanics [14]. Later, via the theory of projective geometries, this had led to the creation of the theory of orthomodular lattices [103]. On the way to lattices von Neumann created the algebraic theory of what was later called von Neumann algebras, which further led to the explosion of algebraic techniques in

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<sup>†</sup>Our emphasis.

quantum mechanics, field theory, and unified theories.

Since Kolmogoroff's axioms for the probability theory [108] and Birkhoff's and von Neumann's quantum logic [14] many axiomatic systems were proposed for quantum mechanics. On the side of quantum logic a partial list includes Mackey [118, 119], Zieler [204], Varadarajan [184, 185], Piron [140, 141], Kochen and Specker [107], Guenin [76], Gunson [77], Jauch [97], Pool [145, 146], Plymen [144], Marlow [121], Beltrametti and Casinelli [7], Holland [93]. We propose a quantum logical axiomatic derivation in Chapter 6. Probabilistic sets of axioms were introduced by Ludwig and his followers [117]; they will not be studied in the dissertation. Another branch of axiomatic quantum theory, the algebraic approach was first conceived by Jordan, von Neumann and Wigner [100] and developed by Segal [168, 169], Haag and Kastler [79], Plymen [143], Emch [57]. Information-theoretic interpretation of the algebraic approach will be the subject of Part III.

We close this section with an illuminating passage about axiomatization in physics due to Jean Ullmo, one of the founders of CREA [182, p. 121]:

La théorie physique moderne manifeste une tendance certaine à rechercher une présentation *axiomatique*, sur le modèle des axiomatiques mathématiques. L'idéal axiomatique, emprunté à la géométrie, revient à définir tous les « objets » initiaux d'une théorie uniquement par des *relations*, nullement par des qualités substantielles.<sup>†</sup>

Our way, thus, goes from the discussion of axioms in this section to a discussion of relations in the next one.

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<sup>†</sup>Modern physical theory shows a certain tendency for one to look for an *axiomatic* representation of the theory, modelled on axiomatic systems in mathematics. The ideal of axiomatization, borrowed from geometry, consists in defining all the initial "objects" of the theory only by *relations* and not at all by some substantial qualities.

## 4.2 Relational quantum mechanics

A quantum mechanical description of an object by means of a wave function corresponds to the relativity requirement with respect to the means of observation. This extends the concept of relativity with respect to the reference system familiar in classical physics.

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Vladimir Fock [63]

This section prepares the key two sections that follow it. It serves to explain the motivation behind the choices made in those sections, i.e. its goal is to communicate to the reader the physical intuition that the author believes to have.

Any attempt at formal derivation of quantum mechanics requires a definite conceptual background on which the derivation will further operate. It is commonplace to say that it is not easy to exhibit an axiomatic system that could supply such a background. Before one starts making judgements about plausibility of axioms, one must develop an intuition of what is plausible about quantum theory and what is not. This can be only achieved by *practicing* the quantum theory itself, i.e. by taking its prescriptions at face value, applying them, getting results, and then asking questions of what these results mean. However, it is important to notice that undertaking all actions on this list will not yet make things clear about quantum mechanics. It purely serves as a tool for developing intuition about what is a plausible claim and what must be cut off. The reasons why implausibility may arise are of various nature: from Occam's razor to direct contradiction with observation. We discussed it in Ref. [73].

Once the intuition has been developed, a scientist who wishes to follow the axiomatic approach must select axioms which he believes plausible; and then the whole remaining part of the building will be constructed "mechanically," by means of the formalism. The choice of axioms must be the only external freedom of the theory. We argue that such a program is the exclusive way to make things *clear* about quantum mechanics. To quote from Rovelli [156],

Quantum mechanics will cease to look puzzling only when we will be able to *derive* the formalism of the theory from a set of simple physical assertions (“postulates,” “principles”) about the world. Therefore, we should not try to append a reasonable interpretation to the quantum mechanical formalism, but rather to *derive* the formalism from a set of experimentally motivated postulates.

As the aforementioned experimentally motivated postulates we choose information-theoretic principles. Initially formulated by John Wheeler [197, 198], the program of deriving quantum formalism from information-theoretic principles has been receiving lately much attention. Thus, Jozsa promotes a viewpoint which “attempts to place a notion of information at a primary fundamental level in the formulation of quantum physics” [101]. Fuchs presents his program as follows: “The task is not to make sense of the quantum axioms by heaping more structure, more definitions... on top of them, but to throw them away wholesale and start afresh. We should be relentless in asking ourselves: From what deep *physical* principles might we *derive* this exquisite mathematical structure?... I myself see no alternative but to contemplate deep and hard the tasks, the techniques, and the implications of quantum information theory.” [65]

However, before we start selecting concrete information-theoretic axioms, we must say why our intuition developed so that we believe that precisely this kind of axioms, namely information-theoretic ones, are a plausible set of axioms for quantum mechanics. The intuition here is due to the relational approach to quantum mechanics [156].

The word “relational” has been used by different philosophers of quantum physics, most notably by Everett [58] and by Mermin [124]. Our sense of using this word, along the lines indicated by Rovelli, goes back to the special relativity. Special relativity is a well-understood physical theory, appropriately credited to Einstein in 1905. But it is equally well-known that the formal content of special relativity, i.e. Lorentz transformations, were written by Lorentz and Poincaré and not by Einstein, and this

several years before 1905. So what was Einstein's contribution?

Lorentz transformations were heavily debated in the years preceding 1905 and were often called “unacceptable,” “unreasonable” and so forth. Many interpretations of what the transformations mean were offered, and among them quite a plausible one about interactions between bodies and the ether. This reminds of some of the modern discussion of quantum mechanics. However, when Einstein came, things suddenly became clear and the debate stopped. This was because Einstein gave a few simple physical principles from which he *derived* Lorentz transformations, therefore closing the attempts to heap philosophy *a posteriori*, above the formal structure itself. Einstein's idea was single and ingenious: he assumed that there is no absolute notion of simultaneity. Simultaneity, said Einstein, is relative. Once the notion of absolute simultaneity has been removed, the physical meaning of Lorentz transformations stood clear, and special relativity has not raised any controversy ever since.

Vladimir Fock, as cited in the epigraph to this section, was among the first to say that quantum mechanics generalizes Einstein's principle of relativity. We argue that what becomes relative in quantum theory is the notion of state.

Consider an observer  $O$  that makes measurement of a system  $S$ . Assume that the quantity being measured, say  $x$ , takes two values, 1 and 2; and let the states of the system  $S$  be described by vectors in a two dimensional complex Hilbert space  $\mathcal{H}_S$ . Let the two eigenstates of the operator corresponding to the measurement of  $x$  be  $|1\rangle$  and  $|2\rangle$ . As follows from the standard quantum mechanics, if  $S$  is in a generic normalized state  $|\psi\rangle = \alpha|1\rangle + \beta|2\rangle$ , where  $\alpha$  and  $\beta$  are complex numbers and  $|\alpha|^2 + |\beta|^2 = 1$ , then  $O$  can measure either one of the two values 1 and 2 with respective probabilities  $|\alpha|^2$  and  $|\beta|^2$ .

Assume that in a given specific measurement at time  $t_1$  the outcome is 1. Denote this specific measurement as  $\mathfrak{M}$ . The system  $S$  is affected by the measurement, and at time  $t_1$  the state of the system is  $|1\rangle$ . In the sequence of descriptions, the states of  $S$  at some time  $t = t_0 < t_1$  and  $t = t_1$  are thus

$$\begin{array}{rcl} t_0 & \rightarrow & t_1 \\ \alpha|1\rangle + \beta|2\rangle & \rightarrow & |1\rangle \end{array} \quad (4.1)$$

Let us now consider the same fact  $\mathfrak{M}$  as described by a second observer, who we call  $O'$ .  $O'$  describes the system formed by  $S$  and  $O$ . Again, assume that  $O'$  uses the conventional quantum mechanics and assume that  $O'$  does not perform any measurement between  $t_0$  and  $t_1$  but that  $O'$  knows the initial states of  $S$  and  $O$  and is therefore able to give a quantum mechanical description of the fact  $\mathfrak{M}$ . Observer  $O'$  describes the system  $S$  by means of the Hilbert space  $\mathcal{H}_S$  and the system  $O$  by means of a Hilbert space  $\mathcal{H}_O$ . The  $S - O$  system is then described by means of the product space  $\mathcal{H}_{SO} = \mathcal{H}_S \otimes \mathcal{H}_O$ . Let us denote the vector in  $\mathcal{H}_O$  that describes the state of  $O$  at time  $t_0$  at  $|init\rangle$ . The physical process implies interaction between  $S$  and  $O$ . In the course of this interaction, the state of  $O$  changes. If the initial state of  $S$  is  $|1\rangle$  (respectively  $|2\rangle$ ), then  $|init\rangle$  evolves into a state which we denote as  $|O1\rangle$  (respectively  $|O2\rangle$ ). One can think of states  $|O1\rangle$  and  $|O2\rangle$  as states in which “the hand of the measuring apparatus points at 1” (respectively at 2). One can write down the Hamiltonian that produces evolution of this kind, and such Hamiltonian can be taken as a model for the physical interaction which produces measurement.

Let us now consider the actual case of the experiment  $\mathfrak{M}$  in which the initial state of  $S$  is  $|\psi\rangle = \alpha|1\rangle + \beta|2\rangle$ . The initial full state of the  $S - O$  system is then  $|\psi\rangle \otimes |init\rangle = (\alpha|1\rangle + \beta|2\rangle) \otimes |init\rangle$ . Linearity of quantum mechanics implies

$$\begin{array}{ccc} t_0 & \rightarrow & t_1 \\ (\alpha|1\rangle + \beta|2\rangle) \otimes |init\rangle & \rightarrow & \alpha|1\rangle \otimes |O1\rangle + \beta|2\rangle \otimes |O2\rangle \end{array} \quad (4.2)$$

Thus at  $t = t_1$  the system  $S - O$  is in the state  $\alpha|1\rangle \otimes |O1\rangle + \beta|2\rangle \otimes |O2\rangle$ . This is the conventional description of the measurement as a physical process [192].

We have described the actual physical process  $\mathfrak{M}$  taking place in the laboratory. Standard quantum mechanics requires that we distinguish the system from the observer, but it also allows us freedom in drawing the line between the two. In the above analysis this freedom has been exploited in order to describe the same temporal development in terms of two different observers. In Equation (4.1) the line that distinguished the observed system from the observer was set between  $S$  and  $O$ . In Equation (4.2) this line was set between  $S - O$  and  $O'$ . Recall that we have assumed that  $O'$  is not making any measurement between  $t_0$  and  $t_1$ . There is no physical

interaction between  $O'$  and  $S - O$  during the interval  $t_0 - t_1$ . However,  $O'$  may make a measurement at some later moment  $t_2 > t_1$ ; result of such measurement will agree with the description (4.2) that  $O'$  gave to the  $S - O$  system at time  $t_1$ . Thus, we have two different descriptions of the state at  $t_1$ : the one given by  $O$  and the one given by  $O'$ . *Both are correct.* Therefore, we conclude that

*Remark 4.1.* In quantum mechanics the state is an observer-dependent concept.

Observer-dependency is a crucial observation that marks fundamentally our intuition on how to make judgements about plausibility of postulates or principles for quantum mechanics. It is by doing research motivated by Remark 4.1 that we developed the information-theoretic derivation of quantum theory.

We now advance a thesis that the argument about relative states is in agreement with the philosophy of the loop of existences presented in Section 2.2. In relational quantum mechanics, any system is treated as physical and, consequently, the observer is a physical system as any. Therefore the special status of the observer only manifests itself in the asymmetry of the relation “ $O$  has information about  $S$ .” Physical states are then seen as a manifestation of this relation, and asymmetry of the latter makes any state a relative state: the state of  $S$  is defined with respect to  $O$ , which is a system that has information about  $S$ . Now, in some other act of bringing about information  $O$  itself can stand in the place of  $S$ , i.e. information will be *about*  $S$ . It will then be defined with respect to some other system  $O'$ . If one iterates such a chain, one will never run into a contradiction barring the question of physical nature of the systems  $S$ ,  $O$ ,  $O'$ , etc. If, for example,  $S$  are light rays,  $O$  is the retina of a human eye,  $O'$  are the visual neurons, and then come yet other brain systems and so forth, we naturally expect that such a reduction of the observing physical systems ultimately stops, as they become closer and closer to the fundamental layers of apprehension. Rovelli denies the validity, or even relevance, of this argument as having nothing to do with the formal construction of his theory.

To safeguard a sound philosophical ground for Rovelli’s point of view, we propose to treat it in the spirit of our transcendental argument as follows. Each recourse

to brain or other physical structures as observing systems (systems of type  $O$  in the above discussion) need not lead to questioning the applicability of quantum mechanics or, for that matter, of any given physical theory. This is because applications of the physical theory and the problem of its foundation lie in the different parts of the loop of existences and, in order to be theoretically analyzed, require different loop cuts. The best tactics for a partisan of the relational quantum mechanics is to loop a chain of relations “ $O$  has information about  $S$ ,” “ $O'$  has information about  $O$ ” and so on, on itself. Having become circular, the chain will fully imitate the circle of the loop of existences as it appears on Figure 2.1. This will not, however, lead to possibly contradictory questions concerning the method of storage and manipulation of information by the systems concerned, because such questions are meaningful only in a different loop cut. Therefore, with a loop cut being fixed so that it makes physics an information-based theory, physical theory will obtain a consistent foundation and at the same one will be aware of the explanatory limitations of the theory and one will know how to tackle these limitations at a future, separate stage of reflection: it will be necessary to pass to a different loop cut.

### 4.3 Fundamental notions

We now focus attention on the loop cut in which physical theory is based on information (Figure 2.2). Our task in the remainder of this chapter is to give definitions and postulates necessary for the formal development of this view. In this section we choose the language of the axiomatic system to be given in the next section.

Three notions we do not define. Their meaning is not explained by the theory and they stand in the information-based physical theory as meta-theoretic like the notion of ensemble stands in the set theory. These are: *system*, *information*, *fact*.

Without defining what these words mean, we can however explain *how* they are used in the theory. Systems are fundamental entities of the theoretic description. Any thing *distinct* from another thing can be treated as a system. It cannot be defined by means of other systems or of any functions of systems. This corresponds to the

neo-Platonic notion of thing as explained by the great Russian philosopher Alexei Fedorovich Losev [116].

The von Neumann cut between system and observer [192] that we already evoked in Section 4.2 requires that no particular system be given preferential treatment within the theory. All systems are *a priori* equivalent. In the context of conventional quantum mechanics, this means that only the descriptive purposes distinguish observers from physical systems, and any observer is a physical system as well. The von Neumann cut between observer and system is moved to position zero, i.e. everything is a system and all systems are viewed on equal grounds.

It is relatively easy to comprehend that the notion of system is chosen as meta-theoretic, whereas it is all the more difficult to accept intuitively the same choice for the notion of information. However, it is a requirement of the loop cut. Let us first say which information is *not* under consideration here: the primary notion of information does not mean the quantified, measured, calculated information that we have, for example, in the Shannon theory [170]. All these aspects of information come afterwards, when one attempts a translation of the fundamental notion of information into mathematical terms of this or that formalism. Information in question is the primary substrate, which serves the purposes of interaction or communication between systems. A neo-Leibnizian could view the system as the ontic monad and information as the epistemic monad.

Facts are acts of bringing about information, or information indexed by the temporal moment of it being brought about. In the second formulation enters the notion of time, which we did not select as fundamental. Instead of doing so, we say that facts are fundamental, and the facts that give rise to time. The latter statement will be explained in the generally covariant context of Part III. Prior to that, the theory will be non-relativistic, and time as well as facts will be treated as coming from outside the theory, therefore introduced by way of additional axioms. The understanding of facts as acts of bringing about information indexed by the moments of time brings this notion close to the notion of phenomenon, and thus the loop of

existences between physics and information is not unlike one between phenomenality and objectivity (Figure 2.4). If information that is brought about in a given fact refers to some system (it is information *about* the such and such system), then this can be viewed as an instantiation of *intentionality* of the fact. Quite naturally, once a particular philosophical system has been chosen, as in Section 2.2, then all the key conventional epistemological concepts, like intentionality for example, find their counterparts in the language of this philosophical system.

In the physical theory fundamental notions ought to be given a formal treatment. This is equivalent to saying that physical theory is always built by means of a certain mathematics, as illuminated by Wigner in the famous paper Ref. [201]. In the formalism of Chapter 6, systems will be understood as physical systems  $S$ ,  $O$ , etc., that are entities of the theory. This is a translation of the fundamental notion of system into a mathematical notion, and it comes at almost no price. Not so with information. Information will be translated in a very precise mathematical manner, namely the one introduced by Shannon. According to Shannon, information is understood as correlation between facts about systems. Correlation is a mathematical term that involves registration of statistical sequences of facts and later analyzing these sequences on the subject of finding dependencies between the facts. None of the latter things are relevant in the information-theoretic derivation program: neither registration, nor the analysis which is made backward-in-time or from an Archimedes's extra-theoretic point of view [147]. A theory of these processes requires a different loop cut. Therefore, to say that information is correlation remains a pure translation of the fundamental notion of information into mathematical terms, and not a definition of information.

Facts are presented in the physical theory under the name of measurement results. The question of their mathematical representation thus becomes the question of what is measurement and what is its result. We treat it along the lines of quantum logic. We understand an elementary measurement as a binary, or a yes-no, question. Result of the elementary measurement is a particular answer to the yes-no question. This

agrees with Rovelli’s idea in Ref. [156]. A detailed argument for the choice of yes-no questions as primitive measurements is provided in chapter 13 of Beltrametti’s and Casinelli’s seminal book [7] and will not be repeated here. We limit ourselves to postulating this choice.

To compare with a different wording that exists in the literature, take the approach proposed by Časlav Brukner and Anton Zeilinger [25]. Brukner and Zeilinger use the term “proposition.” In search of their motivation, Timpson [181] compares two formulations of Zeilinger’s fundamental principle for quantum mechanics expressed in an article different from the above [203]:

**FP1** An elementary system represents the truth value of one proposition.

**FP2** An elementary system carries one bit of information.

By referring to bits (“binary units”) and to propositions at the same time, Zeilinger implicitly suggests that, in his and Brukner’s derivation of the quantum formalism, one should treat the following phrase as the postulate of what is measurement: “Yes-no alternatives are representatives of basic fundamental units of all systems.” Although the initial wording making use of the notion of proposition which appears to be different from language of yes-no measurements, we see that this appearance is misleading: Zeilinger in fact adopts the same choice of binary questions as elementary measurements.

## 4.4 First and second axioms

Digo que no es ilógico pensar que el mundo es infinito. Quienes lo juzgan limitado, postulan que en lugares remotos los corredores y escaleras y hexágonos pueden inconcebiblemente cesar – lo cual es absurdo. Quienes lo imaginan sin límites, olvidan que los tiene el número posible de libros. Yo me atrevo a insinuar esta solución del antiguo problema: *La biblioteca es ilimitada y periódica.*

I say that it is not illogical to think that the world is infinite. Those who judge it to be limited postulate that in remote places the corridors and stairways and hexagons can conceivably come to an end—which is absurd. Those who imagine it to be without limit forget that the possible number of books does have such a limit. I venture to suggest this solution to the ancient problem: *The library is unlimited and cyclical.*

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J.L. Borges “The Library of Babel”

After the selection of the fundamental notions in the previous section that provides the language in which one can formulate the axiomatic system, the time is ripe to give the information-theoretic axioms themselves. It is the purpose of this section.

The axioms must be such as to permit a clear and unambiguous translation of themselves into formal terms, and this translation must then lead to reconstruction of the structure of quantum theory. However, first we formulate the axioms without making reference to any particular formalism.

**Axiom I.** There is a maximum amount of relevant information that can be extracted from a system.

**Axiom II.** It is always possible to acquire new information about a system.

It seems that the axioms contradict each other. Indeed, at the first sight a paradox is straightforward: Axiom I says that the quantity of information is finite, while from Axiom II follows that it must be infinite, because we can always obtain some new information. But there is no contradiction: the key is hidden in the use of the term “relevant.” There is no valuation on the set of questions that would assign to each question the amount of information that it brings about without taking into account other questions which have been asked and which create the context for the definition

of relevance. In other words, the amount of information is not a function of one argument. Let us explain this in more detail.

In the conventional quantum mechanics it is from the past or the future of a given experiment, in particular from the intentions of the experimenter, that one can learn which information about the experiment is relevant and which is not. What is relevant can either be encoded in the preparation of the experiment or selected by the experimenter later on. Both the preparation and the posterior selection require *memory*: the experimenter compares information that was brought about in facts indexed by different values of the time variable and decides which information to keep and which to throw away as irrelevant. Fact is a fundamental notion belonging to the meta-theory, and it is therefore natural to expect, because relevance is related to facts, that what is relevant and what is irrelevant cannot be deduced within the theory. In every formalization of the axioms, we need to give a separate definition of relevance and the justification of such a definition will be meta-theoretic. This is indeed what we do in the case of Definition 6.6.

Let us repeat: the experimenter, as someone who imposes a criterion of relevance, needs to be supplied with memory. In other terms, his decisions are *contextual*: the context here is the sequence of facts given to the experimenter. Because facts are meta-theoretic, we call this contextuality *meta-theoretic contextuality*. The notion of meta-theoretic contextuality must be distinguished from the notion of intratheoretic non-contextuality discussed in the next section.

Axioms I and II therefore refer, the first one to the amount of relevant information, the second one to the fact that new information as such can always be generated, perhaps at the price of rendering some other available information obsolete and thus irrelevant. In this interpretation there is no contradiction between the axioms.

To give an illustration, imagine for a moment the actual experimenter. He first makes a measurement with some fixed measuring apparatus, then changes the apparatus and make another measurement with another apparatus. Clearly, one would say that he obtained some new information about the system, so Axiom II is mean-

ingful and justified. What about Axiom I? Axiom I forbids the setting in which the experimenter could change measuring apparatus endlessly and each time get some new information, also *keeping all the old one*. Axiom I tells that the information obtained in earlier measurements must now become irrelevant. This axiom, therefore, has two implications: first, it says that in any one act of bringing about information, only a finite quantity of information can be generated; second, it says that information may “decay” with time, in such a way that one can never have infinite relevant information about the system, although one can still always learn something new about it according to Axiom II.

To conclude this section, compare our axioms with a set of two axioms proposed by Brukner and Zeilinger [25].

**Axiom I (Brukner and Zeilinger).** The information content of a quantum system is finite.

**Axiom II (Brukner and Zeilinger).** Introduce the notion of total information content of the system; state that there exist mutually complementary propositions; state that total information content of the system is invariant under a change of the set of mutually complementary propositions.

Observe a telling analogy between the first axioms, apart from Brukner’s and Zeilinger’s use of the term “information content” which suggests that they consider it as a property of the system in itself, without bringing in the relation with another system that plays the role of observer. Therefore, the term “information content” has ontological connotations, unlike our formulation that underlines the relational character of information and stays within the boundaries of epistemology. Also, in spite of the analogy between the ideas, as for the derivation of quantum mechanics which follows the choice of axioms, Brukner and Zeilinger opt for a technique different from ours. Following Rovelli, we have the ambition to derive the formalism of quantum theory from the axioms by the methods of quantum logic; and to go further than Rovelli because he does not show a way to deduce most of the structure, for instance

the superposition principle, apart from introducing it as a supplementary axiom. Embarking on where Rovelli leaves, Christopher Fuchs [64, 65] uses a decision-theoretic Bayesian approach to derive the superposition principle. He refers to Rovelli's paper in his own, and one is left free to suggest that many of his axiomatic assumptions, on which he does not clearly comment, might be similar to Rovelli's, apart from the key issue of how to define measurement. Fuchs insists on the fundamental character of positive operator-valued measures (POVM) and postulates that POVM formally describe measurement. This contradicts our choice in Section 4.3 and, indeed, may not seem intuitively evident. One is then tempted to look for ways to avoid making this assumption; thus, even if we dismiss the necessity to *define* measurement as POVM, there still remains an opportunity to introduce POVM in the theory, which in itself is a virtue since it permits to establish theorems of quantum computation following the guidelines presented in Section 3.2. POVM have a natural description as conventional von Neumann measurements on ancillary system [135], and thus to Rovelli's axiomatic derivation of the Hilbert space structure one may try to add an account of inevitability of ancillary systems and naturally obtain from this the POVM description, which, in turn, will allow to follow Fuchs's derivation. This will indeed be our plan in Section 6.7.

Brukner and Zeilinger proceed differently. If information is primary, they argue, then any formalism must deal with information and not with some other notions. We find it difficult to disagree with this. Then Brukner and Zeilinger choose not to reconstruct the physical theory, but instead to build an *information space* where they apply their axioms and use the formalism to deduce testable predictions. Brukner and Zeilinger do not refer in their derivation to the Hilbert space nor to the physical state space. In part because of their choice to build a completely new theory, Brukner and Zeilinger are forced into postulating properties of mutually complementary propositions that are hardly apprehensible in the conventional quantum mechanical language. Namely, they postulate the “homogeneity of parameter space,” while—as we shall see—in the formalism of orthomodular lattices one must postulate continuity of

a certain well-defined function. Of course, to Brukner's and Zeilinger's notions one can always find counterparts in the conventional language of numeric fields, Hilbert spaces and states; but their restriction to the terminology of abstract information space leads to the complications of language and renders the formalism less transparent in use. For the reason of clarity of language we reconstruct quantum theory in its standard form instead of giving new names to objects that are essentially the same as all the conventional ones.

## 4.5 I-observer and P-observer

At this stage we have introduced two axioms and three fundamental notions. We have also discussed the notion of relevance. A question arises: Are all the terms used in the formulation of the axioms covered by the three fundamental notions or in order to understand the axioms one needs to employ some other notions? This is a crucial stage where consistency of the theory is at stake.

Let us reread the formulations of Axioms I and II. The concept of *amount of information* refers to the mathematical representation of the fundamental notion of information as Shannon information. Consequently, this does not raise any questions due to the commonly accepted mathematical definition given by Shannon, where information is understood as a measure of the number of possibly occupied states against the total number of states. Admittedly, in our approach this latter phrase is not a definition *per se*, but it gives an unequivocal mathematical meaning to the notion of information.

Axiom I also contains a reference implicit for someone who reads (correctly) this axiom as a statement of the ordinary language rather than a mathematical statement: the reference in question is to the subject who extracts information, and it appears in the clause "can be extracted from." Note that this reading belongs to the ordinary language, and we are therefore obliged to analyze it in the context of the loop cut and separation between theory and meta-theory. The same reference is contained in Axiom II which says "it is always possible *to acquire...*" Then the question is: Who

is the one who acquires information? By one half the answer to this has already been given. As we said in the discussion of the notion of system, everything is a system (apart from the whole Universe which cannot be distinct from something). Then the “subject who acquires information” is also a system in the sense of quantum theory. With von Neumann’s cut being put at level zero, such situation is nothing else but to claim that quantum theory is universal. Next, from the point of view of the ordinary language, we still see a difference between the “subject acquiring information” and a system that this information is about. Language introduces an apparent dissymmetry between the two. Where does this dissymmetry arise from in the theory and what role does it play?

In Rovelli’s phrase quoted in Section 4.2 we stated that “there is no physical interaction between  $O'$  and  $S - O$  during the interval  $t_0 - t_1$ .” Then, if  $O'$  is a system as any and translates into the language of physics as a physical system as any, its status becomes unclear. Indeed, were  $O'$  a physical system, then it must have interacted with other systems just as all physical systems do. But there is precisely no such interaction. It means that for the purposes of description of the interval  $t_0 - t_1$  and of the physical system  $S - O$ , the system  $O'$  is not treated as physical system obeying the laws of physics. We shall say that the system  $O'$  is *effectively meta-theoretic*. It means that we have chosen to move the von Neumann cut to the position between  $S - O$  and  $O'$ , and this only for the purposes of description of the fact  $\mathfrak{M}$ . The only function of the system  $O'$  which is left after we have removed its physical function is that it is an *informational agent*, i.e. an accumulator of information or, to match the language of the axiom, the system which *acquires* information from the fact  $\mathfrak{M}$ . Because we chose  $O'$  at random among all systems, we conclude that any system can become effectively meta-theoretic for the purposes of a fixed descriptive act of bringing about information. Therefore, any system can be represented as a purely physical system plus an informational agent. By definition, this distinction does not interfere with any physical processes, because acquisition of information is, not a theoretic but a meta-theoretic concept. Thus the distinction, too, is meta-

theoretic and bears, in the case of each particular system, on one given fact only. In a description of another fact, the system which has been previously treated as an informational agent only must now again be treated as a physical system as any.

Let us give another motivation to the distinction that we have just made and then introduce the terminology. As one can expect, our observation that systems are sometimes effectively meta-theoretic will lead, not only to novel terminology, but also to tangible theoretic results that will directly bear upon the physical content of the theory. The reason why it happens so is that the way in which we construct quantum theory is based on information, and *a priori* any restriction imposed on the functioning of the concept of information must lead to constraints on the content of the theory.

In the everyday work of a physicist who uses conventional quantum mechanics, one is usually interested in information about (knowledge of) the chosen system and one disregards particular ways in which this information has been obtained. This is a manifestation of the cut of the loop that we discussed in Section 2.2. All that counts is relevant knowledge and relevant information. Because of this, one usually pays no attention to the very process of interaction between the system being measured and the measuring system, and one treats the measuring system as a meta-theoretic, i.e. non-physical, apparatus. Correspondingly, the loop cut is the one on Figure 2.2. To give an example, for some experiment a physicist may need to know the proton mass but he will not at all be interested in how this quantity had been measured (unless he is a narrow specialist whose interest is in measuring particle masses). Particular ways to gain knowledge are irrelevant, while knowledge itself is highly relevant and useful. Some of the experiments where one is interested in the measurement as a physical process, thus falling in the domain of the loop cut on Figure 2.3, are discussed in Ref. [123]. In the present derivation of quantum theory we assume a loop cut such that physics is viewed as based on information, therefore rendering the measurement details irrelevant.

An experimenter, though, always operates in both loop cuts at once, i.e. he uses

physics which is an information-based theory of the first loop cut, but he also keeps in mind that “information is physical” [114]. The last phrase means that there always is some physical support of information, some hardware. The necessity of the physical support requires that we carefully justify the division between theory and meta-theory in the selected loop cut: we first abstain from disregarding the measurement interaction and then show how one can neglect the fact that the measuring system is physical. This will allow to leave to the observer solely the role of informational agent and to formulate the physical theory only in terms of information.

The statement that any system is a physical system but also an informational agent corresponds to making a formal distinction, for each system, between these two roles. Call *any* system  $O$  an observer. Then the observer consists of an informational agent (“I-observer”) and of the physical realization of the observer (“P-observer”). In the uncut loop, there is no I-observer without P-observer. Reciprocally, there is no sense in calling P-observer an observer unless there is I-observer (otherwise P-observer is just a physical system as any). Two components of  $O$  are not in any way separate from each other; on the contrary, these are merely two viewpoints that one adopts for the needs of a given theoretical description. One has to select the viewpoint before describing any given fact  $\mathfrak{M}$ : if the selection is for I-observer, then  $O$  is treated as meta-theoretic; if for P-observer, then  $O$  is a physical system, object of study in the physical theory.

The key point of making the distinction between I-observer and P-observer is that only measurement results, or the information brought about in facts, count. We transform this principle into an axiom that will be further discussed in Section 6.6.

**Axiom III (“no metainformation”).** If information  $I$  about a system has been brought about, then it happened without bringing in information  $J$  about the fact of bringing about information  $I$ .

So formulated, Axiom III states that information, when it is brought about in a fact, is “self-sufficient,” meaning that it does not entail bringing about metainformation about how this particular fact occurred. Facts bring about information that is

clearly demarcated (*'this information'*) and thus is independent from other information that may be brought about in some other facts, but *a fortiori* not in the same one.

Looking at the same axiom from a different angle, let us reformulate it in the language of measurements. It then states that the details of measurement as physical process do not count in making this process a measurement. This is a form of non-contextuality that we call intratheoretic: information does not depend on the context that belongs to the physical theory. As we said in Section 4.4, intratheoretic non-contextuality must be distinguished from meta-theoretic contextuality, which holds in virtue of Axioms I and II. A reformulation of Axiom III then goes:

**Axiom III (“intratheoretic non-contextuality”).** If information is obtained by an observer, then it is obtained independently of *how* the measurement was conducted physically, i.e. independent of the measurement’s context internal to physical theory.



# Chapter 5

## Elements of quantum logic

In this chapter we introduce the quantum logical formalism of the theory of orthomodular lattices in a way suited for the program of deriving the formalism of quantum theory from information-theoretic principles. Most of the following exposition is based on [103]. Several results are taken from the seminal book on lattice theory [120]. Each section opens with a brief non-technical summary.

### 5.1 Orthomodular lattices

This section introduces a key concept of the orthodox quantum logic: orthomodular lattice. A lattice can be viewed as a set of logical statements such that, for any two elements of the lattice, two new elements formed by putting between the two old ones the conjunction *and* or the conjunction *or*, also belong to the lattice. Lattices can be distributive or Boolean, like in classical logic; modular, which is weaker than distributive; and orthomodular, which is yet weaker than modular. Orthomodularity is a property defined with the help of the notion of orthogonality: to each element corresponds a unique other element that “complements” it in the lattice in the sense of, roughly speaking, having all the properties opposite to the properties of the original element.

**Definition 5.1.** A lattice  $\mathcal{L}$  is a partially ordered set in which *any* two elements  $x, y$  have a supremum  $x \vee y$  and an infimum  $x \wedge y$ .

Equivalently, one can require that a set  $\mathcal{L}$  be equipped with two idempotent, commutative, and associative operations  $\vee, \wedge : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ , which satisfy  $x \vee (y \wedge x) = x$  and  $x \wedge (y \vee x) = x$ . The partial ordering is then defined by  $x \leq y$  if  $x \wedge y = x$ . The largest element in the lattice, if it exists, is denoted by 1, and the smallest one (if

exists) by 0.

**Definition 5.2.** A lattice is called **complete** when every subset of  $\mathcal{L}$  has a supremum as well as an infimum.

**Lemma 5.3.** *Complete lattice always contains elements 0 and 1.*

*Proof.* Element 0 can be defined as infimum of all elements of  $\mathcal{L}$  and element 1 as their supremum. Both are well defined in virtue of completeness of the lattice.  $\square$

**Definition 5.4.** An **atom** of lattice  $\mathcal{L}$  is an element  $a$  for which  $0 \leq x \leq a$  implies that  $x = 0$  or  $x = a$ . A lattice with 0 is called **atomic** if for every  $x \neq 0$  in  $\mathcal{L}$  there is an atom  $a \neq 0$  such that  $a \leq x$ .

**Definition 5.5.** The lattice is said to be **distributive** if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \quad (5.1)$$

One can weaken the distributivity condition by requiring (5.1) only if  $x \leq z$ . This leads to the property of modularity.

**Definition 5.6.** The lattice is said to be **modular** if

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z \quad \forall y. \quad (5.2)$$

A canonical example of a modular lattice is the collection  $L(V)$  of all linear subspaces of a vector space  $V$  over an arbitrary field  $\mathbb{D}$  [115]. The lattice operations are  $x \wedge y \equiv x \cap y$  and  $x \vee y \equiv x + y$ , where  $x + y$  is the linear span of  $x$  and  $y$  for all linear subspaces  $x, y \subset V$ . Equivalently, one can say that the partial order is given by inclusion. Evidently, lattice elements  $1 = V$  and  $0 = \emptyset$ , the empty set.

**Definition 5.7.** An **orthocomplementation** on lattice  $\mathcal{L}$  is a map  $x \mapsto x^\perp$ , satisfying for all  $x, y \in \mathcal{L}$

(i)  $x^{\perp\perp} = x$ ,

(ii)  $x \leq y \Leftrightarrow y^\perp \leq x^\perp$ ,

$$(iii) \quad x \wedge x^\perp = 0,$$

$$(iv) \quad x \vee x^\perp = 1.$$

A lattice with orthocomplementation is called an **orthocomplemented lattice**. Two lattices  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are isomorphic if there exists an isomorphism between them that preserves the lattice structure. If  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are orthocomplemented lattices, then they are isomorphic if the isomorphism respects the orthocomplementarity relation.

From the definition immediately follow de Morgan laws

$$1^\perp = 0; \quad 0^\perp = 1; \quad (x \vee y)^\perp = x^\perp \wedge y^\perp; \quad (x \wedge y)^\perp = x^\perp \vee y^\perp. \quad (5.3)$$

By imposing on an orthocomplemented lattice respectively the distributive law (5.1) and the modular law (5.2), which is weaker than the distributive law, one arrives at the following definitions.

**Definition 5.8.** A distributive orthocomplemented lattice is called a **Boolean algebra**.

**Definition 5.9.** An orthocomplemented lattice  $\mathcal{L}$  is called **orthomodular** if condition (5.2) holds for  $y = x^\perp$ , that is,

$$x \leq z \Rightarrow x \vee (x^\perp \wedge z) = z. \quad (5.4)$$

It is useful to give the following reformulation of the condition of orthomodularity.

**Lemma 5.10.** *An orthocomplemented lattice  $\mathcal{L}$  is orthomodular if and only if  $x \leq z$  and  $x^\perp \wedge z = 0$  imply  $x = z$ .*

*Proof.* If the lattice is orthomodular, i.e. Equation (5.4) holds, and if  $x^\perp \wedge z = 0$ , then  $z = x \vee 0 = x$ . To prove the converse statement, it suffices to show that if the lattice is not orthomodular then there exist elements  $x$  and  $z$  such that

$$x \leq z, \quad x^\perp \wedge z = 0, \quad x \neq z. \quad (5.5)$$

Let us use the notation  $x < z$  if  $x \leq z$  and  $x \neq z$ . We can then rewrite Equation (5.5) as

$$x < z, \quad x^\perp \wedge z = 0. \quad (5.6)$$

Assume that the lattice is not orthomodular. According to the Definition 5.9 there exist elements  $y$  and  $z$  such that

$$y \leq z, \quad y \vee (y^\perp \wedge z) \neq z. \quad (5.7)$$

Now recall that on any lattice holds [35, Chapter 2, Section 4]<sup>†</sup>

$$a \leq b \Rightarrow (c \wedge b) \vee a \leq (c \vee a) \wedge b \quad \forall c. \quad (5.8)$$

Put in (5.8)  $a = y$ ,  $b = z$ ,  $c = y^\perp$ . Follows that

$$(y^\perp \wedge z) \vee y \leq (y^\perp \vee y) \wedge z. \quad (5.9)$$

In the right-hand side replace  $y^\perp \vee y$  by 1, and  $1 \wedge z = z$ . Equation (5.9) then takes the form

$$(y^\perp \wedge z) \vee y \leq z. \quad (5.10)$$

From equations (5.10) and (5.7) one obtains that

$$(y^\perp \wedge z) \vee y < z. \quad (5.11)$$

On the other hand, from de Morgan laws (5.3) one has

$$\begin{aligned} z \wedge (y \vee (y^\perp \wedge z))^\perp &= z \wedge (y^\perp \wedge (y^\perp \wedge z)^\perp) = \\ z \wedge (y^\perp \wedge (y \vee z^\perp)) &= (z \wedge y^\perp) \wedge (y \wedge z^\perp) = \\ (z \wedge y^\perp) \wedge (z \wedge y^\perp)^\perp &= 0. \end{aligned} \quad (5.12)$$

Now put  $x = y \vee (y^\perp \wedge z)$ . Equations (5.11) and (5.12) can be rewritten as

$$x < z, \quad x^\perp z = 0. \quad (5.13)$$

This is exactly the condition (5.6) that we need to obtain. □

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<sup>†</sup>We thank Prof. V.A. Franke for this reference and the idea of proof.

We close this section with a definition of reducibility of lattices.

**Definition 5.11.** The center of an orthocomplemented lattice  $\mathcal{L}$  is

$$C(\mathcal{L}) = \{c \in \mathcal{L} \mid x = (x \wedge c) \vee (x \wedge c^\perp) \forall x \in \mathcal{L}\}.$$

**Definition 5.12.** A lattice is called **reducible** if it is (isomorphic to) a nontrivial Cartesian product  $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$  with lattice operations defined componentwise. If not, it is called **irreducible**.

**Lemma 5.13.** *The center  $C(\mathcal{L})$  of an orthomodular lattice  $\mathcal{L}$  is its Boolean subalgebra.*

**Lemma 5.14.** *An orthocomplemented lattice is irreducible if and only if its center is trivial, i.e.  $C(\mathcal{L}) = \{0, 1\}$ .*

## 5.2 Field operations and spaces

This section introduces the notion of Hilbert space. We first define automorphisms in a field, be it a numeric field like real numbers or an abstract algebraic structure with the same properties. The Hilbert space is then a space which is supplied with an internal product that behaves “rationally”, in a certain mathematically defined way, with respect to the automorphism of the underlying field.

Let  $\mathbb{D}$  be a field, i.e. a commutative ring with addition and multiplication such that, bar the unity element of the additive group, one obtains a multiplicative group. A bijective map  $\theta : \mathbb{D} \mapsto \mathbb{D}$  is an **anti-automorphism** if  $\forall a, b \in \mathbb{D}$

$$\theta(a + b) = \theta(a) + \theta(b) \text{ and } \theta(a \cdot b) = \theta(b) \cdot \theta(a). \quad (5.14)$$

The map  $\theta$  is **involutory** if  $\theta^2$  is the identity. Let  $\theta$  be an involutory anti-automorphism of the field  $\mathbb{D}$  and  $V$  a vector space over  $\mathbb{D}$ . A map  $f : V \times V \mapsto \mathbb{D}$  is called a  **$\theta$ -sesquilinear** form on  $V$  if  $\forall x, x_1, x_2, y, y_1, y_2 \in V$  and  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{D}$  one has

$$\begin{aligned} f(\alpha_1 x_1 + \alpha_2 x_2, y) &= \alpha_1 f(x_1, y) + \alpha_2 f(x_2, y), \\ f(x, \beta_1 y_1 + \beta_2 y_2) &= f(x, y_1)\theta(\beta_1) + f(x, y_2)\theta(\beta_2). \end{aligned} \quad (5.15)$$

Let  $f$  be a  $\theta$ -sesquilinear form on  $V$ . Then  $f$  is called **Hermitian** if

$$\theta(f(x, y)) = f(y, x) \quad (5.16)$$

and **definite** if  $f(x, x) = 0$  implies  $x = 0$ . A Hermitian, definite  $\theta$ -sesquilinear form is called a  **$\theta$ -product**.

Now recall the definitions of Banach and Hilbert spaces.

**Definition 5.15.** A **Banach space** is a vector space  $V$  over the field  $\mathbb{D}$  with a norm which is complete with respect to the metric  $d(x, y) \equiv \|x - y\|$  on  $V$ .

**Definition 5.16.** A **Hilbert space**  $\mathcal{H}$  over  $\mathbb{D}$  is a Banach space whose norm comes from a  $\theta$ -product  $f(x, y)$  for  $x, y \in \mathcal{H}$ , which has the following properties:

- (i)  $f(\alpha x + \beta y, z) = \alpha f(x, z) + \beta f(y, z)$ ,
- (ii)  $\theta(f(x, y)) = f(y, x)$ ,
- (iii)  $\|x\|^2 = f(x, x)$ .

**Definition 5.17.** A pre-Hilbert space is a normed linear space over  $\mathbb{D}$  with its norm satisfying the parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2). \quad (5.17)$$

A pre-Hilbert space carries a natural  $\theta$ -product  $f(x, y)$  defined as

$$f(x, y) = \frac{\|x + y\|^2 - \|x - y\|^2}{4} \quad (5.18)$$

and can be completed with respect to its norm topology up to a Hilbert space that will contain the initial pre-Hilbert space as a dense subspace. All Hilbert spaces satisfy the parallelogram law (5.17) and therefore are pre-Hilbert spaces as well.

## 5.3 From spaces to orthomodular lattices

In this section we characterize the lattice of closed subspaces of the Hilbert space. It is found to be complete, atomic and orthomodular.

The *raison d'être* of the theory of orthomodular lattices is to answer the double-way question of, firstly, how to characterize a lattice built of subspaces of the Hilbert space; secondly, how to characterize a lattice built of subspaces of a vector space  $V$  so that this space be a Hilbert space. One would also like to find such a characterization of the lattice that, the space  $V$  being built upon a coordinatizing field  $\mathbb{D}$ ,  $\mathbb{D}$  will equal either  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\mathbb{H}$ .

We start with characterizing the lattice  $\mathcal{L}(\mathcal{H})$  that one obtains given the Hilbert space  $\mathcal{H}$ . Let  $(\cdot, \cdot) : V \times V$  be a Hermitian form on a Banach space  $V$ , defined relative to an involution  $\theta$ , and  $L(V)$  the lattice of all subspaces of  $V$ . For each  $x \in L(V)$  one defines  $x^\perp \equiv \{\Psi \in V \mid (\Psi, \Phi) = 0 \forall \Phi \in x\}$ .  $x^\perp$  is an element of  $L(V)$  as well. One can easily see that  $x^{\perp\perp\perp} = x^\perp$  but in general  $x \leq x^{\perp\perp}$ , rather than the equality required for orthocomplementation in Definition 5.7. Therefore  $L(V)$  is not an orthocomplemented lattice.

As a remedy, consider the lattice  $\mathcal{L}(V)$  of orthoclosed subspaces of  $V$ , i.e.  $x \in L(V)$  lies in  $\mathcal{L}(V)$  if and only if  $x = x^{\perp\perp}$ . The lattice operation  $\wedge$  is the same as in  $L(V)$ , but  $\vee$  in  $\mathcal{L}(V)$  is defined by  $x \vee y = (x + y)^{\perp\perp}$ , which is the smallest orthoclosed subspace containing  $x$  and  $y$ . The symbol  $+$  designates a linear sum of subspaces. Lattice  $\mathcal{L}(V)$  is complete independently of the dimension of  $V$  and is modular if and only if  $V$  is finite-dimensional. Even in the finite-dimensional case,  $^\perp$  need not be an orthocomplementation on  $\mathcal{L}(V)$ . It is straightforward, however, to check the following necessary and sufficient condition.

**Proposition 5.18.** *The map  $x \mapsto x^\perp$  is an orthocomplementation on  $\mathcal{L}(V)$  if and only if  $(x + x^\perp)^\perp = 0$  for all  $x \in \mathcal{L}(V)$ , which is equivalent to the property  $(\Psi, \Psi) = 0 \Leftrightarrow \Psi = 0$  or to requiring that  $(\cdot, \cdot)$  be a  $\theta$ -product. If in addition  $x + x^\perp$  is orthoclosed (implying  $x + x^\perp = V$ ) for all  $x \in \mathcal{L}(V)$ , then  $\mathcal{L}(V)$  is orthomodular.*

*Proof.* We shall prove only the second clause of the lemma in the finite-dimensional case. For infinite dimension we prove directly Lemma 5.19.

To show that the additional assumption implies orthomodularity, note that on this assumption, for any  $x$  one has  $z = z \wedge 1 = z \wedge (x + x^\perp)$ . If  $x \leq z$ , this equals  $x + z \wedge x^\perp$  by the modular law (5.2) in  $\mathcal{L}(V)$ , with  $y = x^\perp$ . Taking the double orthocomplement of the equation  $z = x + z \wedge x^\perp$  thus found yields  $z^{\perp\perp} = z$  for the left-hand side (since  $z \in \mathcal{L}(V)$  by assumption) and  $(x + z \wedge x^\perp)^{\perp\perp} = x \vee (z \wedge x^\perp)$  by the definition of  $\vee$  in  $\mathcal{L}(V)$ . This proves the orthomodular law (5.4).  $\square$

**Lemma 5.19.** *The lattice  $\mathcal{L}(\mathcal{H})$  of all closed subspaces of a Hilbert space is complete, atomic, and orthomodular.*

*Proof.* In the finite-dimensional case proof follows directly from Proposition 5.18. We now give a general proof that can also be applied in the infinite-dimensional case.

Recall the following properties of Hilbert spaces:

- 1) Any closed subspace of the Hilbert space is itself a Hilbert space.
- 2) In every Hilbert space there exists a complete orthonormal basis.
- 3) If in a Hilbert space one is given a certain set of orthonormal vectors, it is always possible to complete it by more vectors, up to a complete orthonormal basis.
- 4) If one divides the complete orthonormal basis of space  $\mathcal{H}$  into two subsets of orthonormal vectors and then one considers linear closures of each set, one obtains two Hilbert subspaces  $V$  and  $V^\perp$  such that

$$V \cup V^\perp = \mathcal{H}, \quad V \cap V^\perp = 0, \quad (5.19)$$

where  $V \cup V^\perp$  is a linear closure of  $V$  and  $V^\perp$ , and  $V \cap V^\perp$  their intersection.

Now let  $V_1$  and  $V_2$  be two closed subspaces of the Hilbert space  $\mathcal{H}$  such that

$$V_1 \subseteq V_2, \quad V_2 \cap V_1^\perp = 0. \quad (5.20)$$

We must prove that

$$V_1 = V_2. \quad (5.21)$$

Indeed,  $V_1$  is itself a Hilbert space. Consider its complete orthonormal basis  $\mathcal{A}$ . In virtue of (5.20), all vectors of  $\mathcal{A}$  belong also to  $V_2$ . Add to  $\mathcal{A}$  a set  $\mathcal{B}$  such that it completes  $\mathcal{A}$  in  $V_2$  to a complete orthonormal basis of the latter. This full basis is now  $\mathcal{A} \cup \mathcal{B}$ . Further, add to  $\mathcal{A} \cup \mathcal{B}$  a set  $\mathcal{C}$  of orthonormal vectors which completes it to the full orthonormal basis of the Hilbert space  $\mathcal{H}$ . The basis in  $\mathcal{H}$  has the form  $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ .

Apply Property 4 of Hilbert spaces listed above. Divide the basis  $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$  into two sets, namely  $\mathcal{A}$  and  $\mathcal{B} \cup \mathcal{C}$ . Consider their linear closures, respectively  $V(\mathcal{A})$  and  $V(\mathcal{B} \cup \mathcal{C})$ . Follows that

$$V^\perp(\mathcal{A}) = V(\mathcal{B} \cup \mathcal{C}), \quad V(\mathcal{A}) \cup V(\mathcal{B} \cup \mathcal{C}) = \mathcal{H}, \quad V(\mathcal{A}) \cap V(\mathcal{B} \cup \mathcal{C}) = 0. \quad (5.22)$$

By definition  $\mathcal{A}$  is a complete orthonormal basis in subspace  $V_1$ , and consequently  $V(\mathcal{A}) = V_1$ . From this and (5.22) follows that  $V_1^\perp = V(\mathcal{B} \cup \mathcal{C})$ . Also by construction  $V_2 = V(\mathcal{A} \cup \mathcal{B})$ , where the right-hand side means linear closure of the vector set  $\mathcal{A} \cup \mathcal{B}$ . Now let

$$V_2 \cap V_1^\perp = 0, \quad (5.23)$$

that is

$$V(\mathcal{A} \cup \mathcal{B}) \cap V(\mathcal{B} \cup \mathcal{C}) = 0. \quad (5.24)$$

The latter equation means that  $\mathcal{A} \cup \mathcal{B}$  and  $\mathcal{B} \cup \mathcal{C}$  do not contain vectors in common, i.e. that  $\mathcal{B}$  is empty and

$$\mathcal{A} \cup \mathcal{B} = \mathcal{A}. \quad (5.25)$$

From equations 5.20, 5.22, and (5.25) follows that

$$V_2 = V(\mathcal{A}) = V_1. \quad (5.26)$$

Therefore, we obtained that, in the lattice notation, from  $V_1 \leq V_2$  and  $V_2 \wedge V_1^\perp = 0$  follows  $V_1 = V_2$ . By Lemma 5.10 lattice  $\mathcal{L}(\mathcal{H})$  is orthomodular. Completeness of the

lattice is trivial, and atomicity follows from the fact that 1-dimensional subspaces of the Hilbert space are atoms of the Hilbert lattice.  $\square$

The result of Lemma 5.19 states that Hilbert spaces as well as pre-Hilbert spaces are characterized among Banach spaces by the property that the lattice of closed subspaces carries an orthocomplementation. Further, the orthomodularity of  $\mathcal{L}$  differentiates between Hilbert spaces and pre-Hilbert spaces. This follows from the theorems given in the next section.

## 5.4 From orthomodular lattices to spaces

In this section we study whether with a complete, atomic and orthomodular lattice can be associated a Hilbert space. The answer is in the negative: these properties are insufficient. A different set of requirements is then given that ensures the appearance of the Hilbert space.

The much more interesting question than the one of the previous section is the reverse characteristics, i.e. a set of properties required from a lattice of closed subspaces of a vector space for this space to be a Hilbert space. Here enters a crucial property, which can manifest itself in different formulations but has always something to do with requiring continuity. First, by providing a counterexample, we explain why without requiring this additional property one cannot obtain anything like a Hilbert space. Thus, for the long time it has been the most important problem of lattice theory to find out whether the properties of being complete, atomic and orthomodular suffice for a lattice to be a lattice of closed subspaces of a real, complex or quaternionic Hilbert space. The result due to Keller [105] gives a negative answer to this. To demonstrate it, assume the following definition.

**Definition 5.20.** The space  $(\varepsilon, f)$  is called **orthomodular space** if  $\varepsilon$  is a vector space over a field  $\mathbb{K}$  with involution  $\omega$  provided with a  $\omega$ -product  $f: \varepsilon \times \varepsilon \mapsto \mathbb{K}$  such that for  $x, y \in \varepsilon$   $x \perp y$  if and only if  $f(x, y) = 0$ , and the projection theorem holds in  $\varepsilon$ :

$$\text{If } U = U^{\perp\perp} \text{ is a subspace of } \varepsilon \text{ then } \varepsilon = U + U^{\perp}. \quad (5.27)$$

One can construct a non-classical example of an orthomodular space. The ordered field  $\mathbb{K}$  is built in a special way of polynomials over real numbers in the variables  $x_1, x_2, \dots$  [75]. The elements of  $\varepsilon$  are the sequences  $(\xi_i) \in \mathbb{K}^{\mathbb{N}_0}$  such that  $\sum_0^\infty \xi_i^2 x_i$  converges. The form  $f$  is defined by  $f((\xi_i), (\eta_i)) = \sum_0^\infty \eta_i \xi_i x_i$  and  $f$  gives rise to a norm on  $\varepsilon$ . The space  $(\varepsilon, f)$  is complete in the norm-topology [104, Remark 12.3] and the projection theorem (5.27) holds in  $\varepsilon$  (*op. cit.*, Theorem 12.5). One then obtains that the lattice  $\mathcal{L}(\varepsilon)$  of all closed subspaces of  $\varepsilon$  is a complete orthomodular lattice (*op. cit.*, p. 175), and it is also atomic. Meanwhile,  $\varepsilon$  has properties quite different from the properties of Hilbert spaces. For instance, no pair of orthogonal vectors of the same length exist in  $\varepsilon$ . For the probabilities on the lattice  $\mathcal{L}(\varepsilon)$  no proof of Gleason's Theorem 6.18 can be expected. Therefore, one is driven to impose more conditions on a lattice so that non-classical cases of spaces like  $\varepsilon$  be excluded.

To start with a characterization of what is sufficient to obtain a Hilbert space, we first recall the Birkhoff-von Neumann theorem [14].

**Theorem 5.21 (Birkhoff-von Neumann).** *Consider a finite-dimensional vector space  $V$  over a field  $\mathbb{D}$  with dimension greater than 3. Let  $\mathcal{L}(V)$  be a lattice of subspaces of  $V$ . There exists a natural one-to-one correspondence between orthocomplementation on  $\mathcal{L}(V)$  and normed  $\theta$ -products  $f$  on  $V$ , where  $\theta$  is an involutory anti-automorphism on  $\mathbb{D}$ .*

The Birkhoff-von Neumann theorem associates an involutory anti-automorphism with orthocomplementation on a lattice in the finite-dimensional case only. Still, we would like to have a general characterization, both in the finite-dimensional the infinite-dimensional situations. Before doing this, we shall need to specialize from the general case of any field  $\mathbb{D}$  to real or complex numbers or quaternions. This is achieved by the following lemma.

**Lemma 5.22.** *Let  $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$  and  $V$  be a vector space over  $\mathbb{D}$  with  $\dim V \geq 2$ . Assume that  $\theta$  is an involutory anti-automorphism on  $\mathbb{D}$  and  $f$  a  $\theta$ -product on  $V$ . Then*

- (i) if  $\mathbb{D} = \mathbb{R}$  then  $\theta = \text{id}$ .
- (ii) if  $\mathbb{D} = \mathbb{C}$  then  $\theta \neq \text{id}$  and if  $\theta$  is continuous then  $\theta$  is the conjugate.
- (iii) if  $\mathbb{D} = \mathbb{H}$  then  $\theta$  is the conjugate.

In all three cases, if we assume that  $\theta$  is continuous then it is uniquely determined.

Now we are ready embark on the search for a sufficient condition for a lattice to give rise to a space  $V$  that will be a Hilbert space.

**Theorem 5.23.** *Let  $V$  be a vector space of dimension  $\geq 4$  over a field  $\mathbb{D}$ . Consider  $v_1 \in V \setminus \{0\}$  and  $\mathcal{L}$  a lattice of subspaces of  $V$  which satisfies the following conditions:*

1. *Every finite-dimensional subspace of  $V$  is in  $\mathcal{L}$ .*
2.  *$U \wedge M = U + M \in \mathcal{L}$  for  $M \in \mathcal{L}$  and  $\dim U < \infty$ .*

*If  $\perp$  is an orthocomplementation on  $\mathcal{L}$  then there exists a unique involutory anti-automorphism  $\theta$  on  $\mathbb{D}$  and a unique  $\theta$ -product  $f$  on  $V$  such that*

$$\begin{cases} f(v_1, v_1) = 1, \\ f(v, u) = 0 \Leftrightarrow v \in \Gamma(u)^\perp, \end{cases} \quad (5.28)$$

where  $\Gamma$  is a closure operator on  $V$ .

*Proof.* If  $V$  is finite-dimensional then the assertion follows from the Birkhoff-von Neumann theorem. We need to prove (a) that  $\perp$  induces an orthocomplementation ' on every finite-dimensional subspace  $M$  of  $V$ . By the Birkhoff-von Neumann theorem there exist for  $\dim M \geq 4$  an involutory anti-automorphism  $\theta_M$  of  $\mathbb{D}$  and a  $\theta_M$ -product on  $M$  which are unique if we fix an element  $v_1 \in M \setminus \{0\}$  with  $f_M(v_1, v_1) = 1$ . The pair  $(\theta_m, f_m)$  satisfies (5.28) on  $M$ . Let  $M$  be fixed and

$$f(v, u) \equiv f_N(v, u) \text{ for } N = M + \Gamma(u) + \Gamma(v). \quad (5.29)$$

Subsequently we need to prove (b) that  $f$  is well-defined and is a  $\theta$ -product on  $V$ . Finally, in (c) we show that  $\theta$  and  $f$  are uniquely determined.

(a) Let  $M$  be a finite-dimensional subspace of  $V$ . Define

$$U' = U^\perp \cap M \quad (5.30)$$

for a subspace  $U$  of  $M$ . Then  $U \subseteq W$  for a subspace  $W$  of  $M$  implies  $W' \subseteq U'$ ,  $U \cap U' = U \cap U^\perp \cap M = 0$  and  $U'' = (U^\perp \cap M)^\perp \cap M = (U \vee M^\perp) \cap M = (U + M^\perp) \cap M = U$ . Hence  $'$  is a well-defined orthocomplementation on the lattice of subspaces of  $M$ .

(b) Let  $M, W$  be finite-dimensional subspaces of  $V$  and  $M \subseteq W$  such that  $v_1 \in M$  and  $\dim M \geq 4$ . If  $U$  is a subspace of  $M$  then the orthocomplement  $U'$  defined in (5.30) for  $M$  coincides with the intersection of  $M$  with the orthocomplement of  $U$  in  $W$ . Hence

$$U' = \{v \in M \mid f_W(v, u) = 0 \forall u \in U\} \quad (5.31)$$

and  $f_W|_{M \times M}$  is a  $\theta_W$ -product on  $M$  which induces  $'$ . By the uniqueness of such a product it follows that  $\theta_W = \theta_M$  and  $f_W|_{M \times M} = f_M$ . The  $\theta$ -product  $f$  satisfies the first of the conditions (5.28) by virtue of its definition (5.29) and satisfies the second one since for  $v \in N$  the conditions  $f(v, u) = 0$ ,  $v \in \Gamma(u)^\perp \cap N$  and  $v \in \Gamma(u)^\perp$  are equivalent.

(c) Let  $\omega$  be an involutory anti-automorphism of  $\mathbb{D}$  and  $g$  a  $\omega$ -product which satisfies (5.28). Choose  $W$  as in (b). Then the restriction  $h$  of  $g$  to  $W \times W$  is a  $\omega$ -product on  $W$  which induces  $'$ . The uniqueness of  $\theta = \theta_W$  and  $f = f_W$  implies that  $\omega = \theta$  and  $h = f_W$ . By (5.29) applied to  $W = N$  we obtain  $h(v, u) = f(v, u)$  for arbitrary vectors  $v, u \in V$ .  $\square$

**Theorem 5.24.** *Let  $\mathcal{H}$  be a vector space over  $\mathbb{D} = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  of dimension  $\geq 4$  and  $\mathcal{L}$  a lattice of subspaces such that*

- (i) *Every finite-dimensional subspace of  $\mathcal{H}$  belongs to  $\mathcal{L}$ ,*
- (ii) *For every  $U \in \mathcal{L}$  and every finite-dimensional subspace  $V$  of  $\mathcal{H}$  the sum  $U + V$  belongs to  $\mathcal{L}$ .*

Assume that  $\mathcal{L}$  carries an orthocomplementation  $^\perp$ . Assume further that the associated involutory anti-automorphism  $\theta$  of Theorem 5.23 is continuous in case the field  $\mathbb{D}$  equals  $\mathbb{C}$ . Then there exists an inner product  $f$  on  $\mathcal{H}$  satisfying (5.28) which is unique up to multiplication with a positive real constant. In particular,  $\mathcal{H}$  is a pre-Hilbert space.

*Proof.* We shall apply Theorem 5.23 and Lemma 5.22. For  $v_1 \in \mathcal{H} \setminus \{0\}$  there exists a unique involutory anti-automorphism  $\theta$  on  $\mathbb{D}$  and a unique  $\theta$ -product  $f$  on  $\mathcal{H}$  which satisfies (5.28) and with  $f(v_1, v_1) = 1$ . From the assumption on  $\theta$  it follows that  $\theta$  is the conjugation for  $\mathbb{D} = \mathbb{H}$  or  $\mathbb{C}$  and it is the identity for  $\mathbb{D} = \mathbb{R}$ . Since  $f$  is normed, it is an inner product. If  $g$  is an inner product on  $\mathcal{H}$  which satisfies (5.28) then we define  $a = g(v_1, v_1)$  and  $h(v, u) = g(v, u) \cdot a^{-1}$ . Observe that  $a > 0$ . Now  $h(v_1, v_1) = 1$  implies  $h = f$  by the uniqueness of  $f$ . Therefore  $g(v, u) = af(v, u)$  holds for all  $v, u \in \mathcal{H}$ .  $\square$

We give without proof the following two propositions about properties of lattices of subspaces of Banach spaces [102].

**Proposition 5.25.** *Let  $B$  be an infinite-dimensional complex Banach space,  $\mathcal{L}(B)$  the lattice of closed subspaces of  $B$  and  $^\perp$  an orthocomplementation on  $\mathcal{L}(B)$ . Then the associated involutory anti-automorphism  $\theta$  is continuous.*

**Theorem 5.26 (Kakutani-Mackey).** *Let  $B$  be an infinite-dimensional real or complex Banach space,  $\mathcal{L}$  the lattice of closed subspaces of  $B$  and  $^\perp$  an orthocomplementation on  $\mathcal{L}$ . Then there exists an inner product on  $B$  such that for any  $U$  in  $\mathcal{L}$  its orthocomplement  $U^\perp = \{v \in B \mid f(v, u) = 0 \forall u \in U\}$ . The pair  $(B, f)$  is a Hilbert space whose topology coincides with the norm topology on  $B$ . The inner product  $f$  is unique up to multiplication with a real positive constant.*

These results are used to prove the following properties of pre-Hilbert spaces.

**Proposition 5.27.** *Let  $\mathcal{H}$  be a pre-Hilbert space and  $\mathcal{L} = \{U \subseteq \mathcal{H} \mid U = u^{\perp\perp}\}$ . The following two conditions are equivalent:*

- (i)  $\mathcal{H}$  is a Hilbert space,
- (ii)  $U + U^\perp = \mathcal{H}$  for all  $U \in \mathcal{L}$ .

*Proof.* Since every Hilbert space satisfies (ii) it is sufficient to prove that (ii) implies (i). Assume that (ii) holds. Let  $\mathcal{G}$  be the completion of  $\mathcal{H}$  and let  $x \in \mathcal{G}$ . One has to show that  $x \in \mathcal{H}$ . For this, define  $z = y - x$  where  $y \in \mathcal{H}$  such that  $x \perp (y - x)$ . The sequences  $(x_n), (z_n)$  are chosen for  $x \perp z$  so that  $x_n \perp z_m, x_n \perp z, z_n \perp x$  for all  $n, m \in \mathbb{N}$  and  $\lim x_n = x$  and  $\lim z_n = z$ . Further, let  $U = \{z_n \mid n \in \mathbb{N}\}^\perp$  and  $\text{pr} : \mathcal{G} \mapsto \Gamma(U)$  be the projection of  $\mathcal{G}$  onto the closure of  $U$  in  $\mathcal{G}$ . Then  $U = U^{\perp\perp}$  implies  $U \in \mathcal{L}$  and  $\mathcal{H} = U + U^\perp$ . The element  $y \in \mathcal{H}$  has a representation  $y = u + v$  with  $u \in U$  and  $v \in U^\perp$ . We need to prove (a) that  $U^\perp \subseteq \Gamma(U)^\perp$  and (b) that  $\text{pr}(y) = x$ . Then  $x = \text{pr}(u + v) = u \in U \subseteq \mathcal{H}$  and this shows that  $x \in \mathcal{H}$ .

(a) Let  $w \in U^\perp$ . Then  $g(w, u) = 0$  for all  $u \in U$  where  $g$  is the inner product on  $\mathcal{G}$ . Since  $g$  is continuous it follows that  $g(w, u) = 0$  holds for all  $v \in \Gamma(U)$ . Therefore  $w \in \Gamma(U)^\perp$ .

(b)  $x \in \Gamma(U)$  since  $\lim x_n = x$  and  $x_n \in U$ . Let  $v \in \Gamma(U)$  and  $v_n \in U$  with  $\lim v_n = v$ . Then  $g(z_n, v_m) = 0$  implies  $g(z, v) = 0$ . Hence  $z \in \Gamma(U)^\perp$  and  $\text{pr}(y) = \text{pr}(z) + \text{pr}(x) = 0 + x = x$ .  $\square$

**Proposition 5.28.** *Let  $\mathcal{H}$  be a pre-Hilbert space and  $\mathcal{L} = \{U \subseteq \mathcal{H} \mid U = U^{\perp\perp}\}$ . The following conditions are equivalent:*

- (i)  $\mathcal{H}$  is a Hilbert space,
- (ii)  $\mathcal{L}$  is orthomodular.

*Proof.* For proof that from (i) follows (ii) we refer to section 5.1 of [103]. Let  $\mathcal{L}$  be orthomodular. We shall demonstrate that the statement (ii) of Proposition 5.27 holds, which will be sufficient to prove that  $\mathcal{H}$  is a Hilbert space. Assume there exists  $U \in \mathcal{L}$  and  $z \in \mathcal{H}$  such that  $z \neq x + y$  holds for all  $x \in U$  and  $y \in U^\perp$ . Denote  $B = U \wedge (U^\perp \vee \Gamma(z))$  and  $C = U^\perp \wedge (U \vee \Gamma(z))$ . If  $C$  is finite-dimensional then by

virtue of properties of pre-Hilbert spaces  $B + C = B \vee C$ . We now show that  $C$  is always finite-dimensional. For every pre-Hilbert space,  $\mathcal{L}$  is an atomic, complete ortholattice which satisfies the exchange axiom, i.e. if  $a \geq a \wedge b$  then  $a \vee b \geq b$ . Since  $\mathcal{L}$  is orthomodular and  $\Gamma(z)$  is an atom in  $\mathcal{L}$  with  $\Gamma(z) \not\subseteq U$  one is in position to apply Theorem 10.9 from Ref. [103] to prove that  $C$  is an atom in  $\mathcal{L}$ . It therefore always true that  $B + C = B \vee C$ .

Further, from the orthomodularity of  $\mathcal{L}$  and the definition of  $C$  it follows that  $U \vee C = U \vee \Gamma(z)$  and

$$B \vee C = (U \vee C) \wedge (U^\perp \vee \Gamma(z) \vee C) = (U \vee \Gamma(z)) \wedge (U^\perp \vee \Gamma(z) \vee C) \geq \Gamma(z). \quad (5.32)$$

This has a consequence that

$$z \in B + C = U \wedge (U^\perp \vee \Gamma(z)) + U^\perp \wedge (U \vee \Gamma(z)) \subseteq U + U^\perp, \quad (5.33)$$

which contradicts the initial assumption on  $z$ . Therefore  $\mathcal{H}$  is a Hilbert space.  $\square$

**Corollary 5.29.** *Every finite-dimensional pre-Hilbert space  $\mathcal{H}$  is a Hilbert space.*

*Proof.* Proposition 5.28 provides for the desired outcome if we show that  $\mathcal{L} = \{U \subseteq \mathcal{H} \mid U = U^{\perp\perp}\}$  is orthomodular. In  $\mathcal{H}$  holds  $U \vee V = U + V$  for all (automatically finite-dimensional) subspaces  $U, V \subseteq \mathcal{H}$ . Let  $x \in U \wedge (V \vee W) = U \wedge (V + W)$  for some  $W \in \mathcal{L}$  such that  $W \subseteq U$ . Then  $x = x_1 + x_2 \in U$  for  $x_1 \in V$  and  $x_2 \in W \subseteq U$ . Hence  $x_1 = x - x_2 \in U$  and  $x \in (U \cap V) + W = (U \wedge V) \vee W$ . This proves that  $\mathcal{L}$  is modular by Definition 5.2. Since  $\mathcal{L}$  is also an ortholattice, it is orthomodular.  $\square$

Modularity of  $\mathcal{L}$  is characteristic of finite-dimensional Hilbert spaces. In the infinite-dimensional case  $\mathcal{L}$  is always non-modular. In application of Theorem 5.24 or Corollary 5.29 we obtain the following final lists of properties of a lattice  $\mathcal{L}$  associated with the space  $\mathcal{H}$ , which are necessary for space  $\mathcal{H}$  to be a Hilbert space. Not surprisingly, these lists differ in finite-dimensional and infinite-dimensional cases.

**Theorem 5.30.** *(finite-dimensional Hilbert space characterization)*

Let  $\mathcal{H}$  be a finite-dimensional vector space over  $\mathbb{D} = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  of dimension  $\geq 4$  and let  $\mathcal{L}$  be the lattice of subspaces of  $\mathcal{H}$ . Assume  $\mathcal{L}$  has an orthocomplementation such to which by virtue of Theorem 5.23 one associates an involutory anti-automorphism  $\theta$ , and for  $\mathbb{D} = \mathbb{C}$   $\theta$  is continuous. Then there exists an inner product  $f$  on  $\mathcal{H}$  which satisfies

$$U^\perp = \{v \in \mathcal{H} \mid f(v, u) = 0 \forall u \in U\} \quad (5.34)$$

such that  $\mathcal{H}$  together with  $f$  is a Hilbert space. The inner product  $f$  is unique up to multiplication by a positive real constant.

*Proof.* Since conditions (i) and (ii) of Theorem 5.24 hold for  $\mathcal{L}$  it follows that  $\mathcal{H}$  is a pre-Hilbert space. From Corollary 5.29 it follows that  $\mathcal{H}$  is a Hilbert space. For  $U \in \mathcal{L}$  one has

$$U^\perp = \bigwedge_{u \in U} \Gamma(u)^\perp = \bigcap_{u \in U} \{v \in \mathcal{H} \mid f(v, u) = 0\}, \quad (5.35)$$

which equals  $\{v \in \mathcal{H} \mid f(v, u) = 0\}$  by virtue of the condition (5.28) as used in Theorem 5.24.  $\square$

**Theorem 5.31. (*infinite-dimensional Hilbert space characterization*)**

Let  $\mathcal{H}$  be an infinite-dimensional vector space over  $\mathbb{D} = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  and let  $\mathcal{L}$  be a complete orthomodular lattice of subspaces of  $\mathcal{H}$  which satisfies the conditions of Theorem 5.24:

- (i) Every finite-dimensional subspace of  $\mathcal{H}$  belongs to  $\mathcal{L}$ ,
- (ii) For every  $U \in \mathcal{L}$  and every finite-dimensional subspace  $V$  of  $\mathcal{H}$  the sum  $U + V$  belongs to  $\mathcal{L}$ .

By Theorem 5.23 one associates an involutory anti-automorphism  $\theta$  and we assume that for  $\mathbb{D} = \mathbb{C}$   $\theta$  is continuous. Then there exists an inner product  $f$  on  $\mathcal{H}$  such that  $\mathcal{H}$  together with  $f$  is a Hilbert space with  $\mathcal{L}$  as its lattice of closed subspaces.  $f$  is uniquely determined up to multiplication by a positive real constant.

*Proof.* By Theorem 5.24 there exists an inner product  $f$  on  $\mathcal{H}$  which satisfies (5.28) and it is unique up to multiplication by a positive real constant.  $\mathcal{H}$  itself is a pre-Hilbert space. Let  $\mathcal{L}(\mathcal{H}) = \{U \subseteq \mathcal{H} \mid U = U''\}$  where  $U' = \{x \in \mathcal{H} \mid (x, u) = 0 \ \forall u \in U\}$ . We need to prove that  $\mathcal{L} = \mathcal{L}(\mathcal{H})$ . Then it follows by Proposition 5.28 that  $\mathcal{H}$  is a Hilbert space.

Assume  $U \in \mathcal{L}$ . Since  $\mathcal{L}$  is complete and all 1-dimensional subspaces of  $\mathcal{H}$  belong to  $\mathcal{L}$  one obtains

$$\begin{aligned} U &= \bigvee_{u \in U} \Gamma(u) = \bigwedge_{u \in U} \{v \in \mathcal{H} \mid f(v, u) = 0\} = \\ &= \{v \in \mathcal{H} \mid f(v, u) = 0 \ \forall u \in U\} = U'. \end{aligned} \quad (5.36)$$

Therefore  $U = U^{\perp\perp} = U'' \in \mathcal{L}(\mathcal{H})$ .

Assume  $U \in \mathcal{L}(\mathcal{H})$ . By (5.28) and completeness of  $\mathcal{L}$  one obtains

$$U' = \bigcap_{u \in U} \{v \in \mathcal{H} \mid f(v, u) = 0\} = \bigwedge_{u \in U} \Gamma(u)^\perp \in \mathcal{L}. \quad (5.37)$$

From the previous it follows that  $U = U'' = (U')^\perp \in \mathcal{L}$ . Hence  $\mathcal{L}(\mathcal{H})$  is a subset of  $\mathcal{L}$ . □

# Chapter 6

## Reconstruction of the quantum mechanical formalism

### 6.1 What do we have to reconstruct?

Reconstruction of the quantum mechanical formalism proceeds by building its blocks from the axioms. In this chapter we show how to achieve this; we also complete the list of axioms, which for the moment includes Axioms I and II introduced in Section 4.4 and Axiom III introduced in Section 4.5. The blocks to be reconstructed are the conventional key components of quantum theory: the Hilbert space of observables, the Born rule with the state space, and the unitary dynamics or evolution in time. Reconstruction of these blocks will be undertaken in Sections 6.3, 6.6 and 6.7 respectively.

As a preliminary exercise, we analyze the role that each of the above mentioned blocks plays in the quantum theory. We start with the last block, the unitary dynamics. Conventionally, it arises from the Schrödinger equation in the Schrödinger picture (wavefunction is time-dependent, operators are time-independent) or from the equation for the evolution operator in the Heisenberg picture (wavefunction is time-independent, operators are time-dependent). In quantum mechanics the time change does not influence the synchronic algebraic structure of the theory, and all that time evolution does is that it “shifts” this algebraic structure between different time moments. It becomes clear then, that from a mere study of the synchronic, or

better say timeless, algebraic structure of the quantum theory nothing can be inferred about unitary time evolution. Indeed, in Section 6.7 we see that one must add a new assumption from which the time dynamics will follow. More will be said about the role of time in Part III in the context of the  $C^*$ -algebraic approach.

The second block—the Born rule—is closely linked to probabilities in quantum theory. In fact, our derivation in Section 6.6 suffices for building the state space of quantum mechanics (density matrices) and for establishing usual probabilistic quantum mechanical rules. We deliberately choose not to enter into the vast domain of discussion concerning the meaning and the philosophy of probabilities.

By means of the information-theoretic reconstruction we bring some novelty to the discussion of the significance of the first block of quantum theory, i.e. the Hilbert space. The Hilbert space appeared in quantum mechanics quite *ad hoc*, following the joint work by von Neumann, Hilbert and Nordheim [91]. In 1926 nothing seemed to force physicists into accepting the Hilbert space, apart from the fact that “it was available on the market” [128]. Also, we know that von Neumann became greatly disillusioned in the Hilbert space quantum theory already in a few years after he himself created it. This will be explained and discussed in more detail in Section 8.2.

Quite naturally, this leads to a question, “Why Hilbert space?” Or, even more surprisingly, “What is Hilbert space?” The mathematical answer, as in Definition 5.16, is well-known, and yet Chris Fuchs in a recent paper [67] call this question “tough.” Why is that? The issue at stake is to justify the use of Hilbert space in quantum theory, and the most intriguing problem is to explain the dimensionality of the Hilbert space. Let us quote Fuchs further:

Associated with each quantum system is a Hilbert space. In the case of finite dimensional ones, it is commonly said that the dimension corresponds to the number of distinguishable states a system can “have.” But what are these distinguishable states? Are they potential properties a system can possess in and of itself, much like a cat’s possessing the binary value of whether it is alive or dead? If the Bell-Kochen-Specker theorem [3]

has taught us anything, it has taught us that these distinguishable states should not be thought of in that way.

From the quantum logical derivation that we propose below, the structure of the Hilbert space will follow, but not its dimension. However, this dimension will appear implicitly in Equation (6.14). The same problem of the origin of Hilbert space dimension arises in Ref. [65], where it is suggested that dimension is an “irreducible element of reality.” In Refs. [66, 68] the same author argues that dimensionality has to do something with the “sensitivity to the touch, i.e. ability of the system to be modified with respect to the external world due to the interventions of that world upon its natural course.” Fuchs then proposes a solution to a smaller problem than the problem of dimension, which is the problem of justification of quantumness of the Hilbert space. He argues that quantumness can be viewed as a characteristic of the sensitivity to eavesdropping. Dimension, on its part, plays a crucial role in the possible eavesdropping strategies.

To Fuchs's “sensitivity to the touch” we offer an alternative justification. Indeed, the way sensitivity to the touch is defined, it bears a very strong ontological connotation and a flavor of realism. The external world “intervenes upon the natural course” of the quantum system. This contradicts both our epistemological attitude and the attitude dictated by the Kochen-Specker theorem, which calls for abandoning the assignment of built-in properties to quantum systems and indeed is one of the strongest arguments against realism in quantum physics. Thus, because the realist attitude openly contradicts the philosophical position to which we stick in this dissertation, the problem of dimensionality must be given a different analysis devoid of ontological commitments. This will be attempted via the transcendental argument in Section 6.5.

## 6.2 Rovelli's sketch

Before we start the derivation of the Hilbert space structure from the information-theoretic axioms, we present in this section a conceptual sketch of such derivation due

to Rovelli. Rovelli's discussion of the results concerning the Hilbert space, however, is only a sketch, i.e. it is not rigorous. He acknowledges it when he says "I do not claim any mathematical nor philosophical rigor." [156]

Let us start with the distinction between P-observer and I-observer made in Section 4.5. P-observer interacts with the quantum system and thus provides for the physical basis of measurement. I-observer is only "interested" in the measurement result, i.e. information *per se*, and he gets information by reading it from P-observer. The act of reading or getting information is here a common linguistic expression and not a physical process, because I-observer and P-observer are not physically distinct. The concept of "being physical" only applies to P-observer, and by definition the physical content of the observer is all contained in P-observer. I-observer as informational agent is meta-theoretic, and hence the fact that its interaction with P-observer, or the act of "reading information," is unphysical. To give a mathematical meaning to this act, we assume that getting information is described as yes-no questions asked by I-observer to P-observer.

The set of these yes-no questions will be denoted  $W(P) = \{Q_i, i \in I\}$ . According to Axiom I, there is a finite number  $N$  that characterizes P-observer's capacity to supply I-observer with information. The number of questions in  $I$ , though, can be much larger than  $N$ , as some of these questions are not independent. In particular, they may be related by implication ( $Q_1 \Rightarrow Q_2$ ), union ( $Q_3 = Q_1 \vee Q_2$ ), and intersection ( $Q_3 = Q_1 \wedge Q_2$ ). One can define an always false ( $Q_0$ ) and an always true question ( $Q_\infty$ ), negation of a question ( $\neg Q$ ), and a notion of orthogonality as follows: if  $Q_1 \Rightarrow \neg Q_2$ , then  $Q_1$  and  $Q_2$  are orthogonal ( $Q_1 \perp Q_2$ ). Equipped with these structures, and under the non-trivial assumption that union and intersection are defined for every pair of questions, according to Rovelli's statement which, as we shall see, does not hold without auxiliary assumptions, " $W(P)$  is an orthomodular lattice."

Rovelli proposes a few more steps to obtain the Hilbert space structure. As follows from Axiom I, one can select in  $W(P)$  a set  $c$  of  $N$  questions that are independent from each other. In the general case, there exist many such sets  $c$ ,  $d$ , etc. If I-observer

asks the  $N$  questions in the family  $c$  then the obtained answers form a string

$$s_c = [e_1, \dots, e_N]_c. \quad (6.1)$$

This string represents the information that I-observer got from P-observer as a result of asking the questions in  $c$ . Note that it is, so to say, “raw information” meaning that it is not yet information about the quantum system  $S$  that the I-observer ultimately wants to have, but only a process due to functional separation between the P-observer and the I-observer. The string  $s_c$  can take  $2^N = K$  values. We denote them as  $s_c^{(1)}, s_c^{(2)}, \dots, s_c^{(K)}$  so that

$$\begin{aligned} s_c^{(1)} &= [0, 0, \dots, 0]_c \\ s_c^{(2)} &= [0, 0, \dots, 1]_c \\ &\dots, \\ s_c^{(K)} &= [1, 1, \dots, 1]_c \end{aligned} \quad (6.2)$$

Now define new questions  $Q_c^{(1)} \dots Q_c^{(K)}$  such that the yes answer to  $Q_c^{(i)}$  corresponds to the string of answers  $s_c^{(i)}$ :

$$\begin{aligned} Q_c^{(1)} &= [(e_1 = 0) \wedge (e_2 = 0) \wedge \dots \wedge (e_N = 0)]? = \neg Q_1 \wedge \neg Q_2 \wedge \dots \wedge \neg Q_N \\ Q_c^{(2)} &= [(e_1 = 0) \wedge (e_2 = 0) \wedge \dots \wedge (e_N = 1)]? = \neg Q_1 \wedge \neg Q_2 \wedge \dots \wedge Q_N \\ &\dots \\ Q_c^{(K)} &= [(e_1 = 1) \wedge (e_2 = 1) \wedge \dots \wedge (e_N = 1)]? = Q_1 \wedge Q_2 \wedge \dots \wedge Q_N \end{aligned} \quad (6.3)$$

To these questions we refer as to “complete questions.”

**Lemma 6.1.** *Complete questions  $Q_c^{(i)}$  are mutually exclusive*

$$Q_c^{(i)} \wedge Q_c^{(j)} = Q_0 \quad \forall i \neq j. \quad (6.4)$$

and for them holds the distributivity law (5.1):

$$Q_c^{(i)} \vee (Q_c^{(j)} \wedge Q_c^{(k)}) = (Q_c^{(i)} \vee Q_c^{(j)}) \wedge (Q_c^{(i)} \vee Q_c^{(k)}). \quad (6.5)$$

*Proof.* Equality to the always false question of the disjunction of any two different complete questions follows immediately from their definition (6.4). Because questions  $Q_1, \dots, Q_N$  in the family  $c$  are independent by construction, distributivity holds for them and, consequently, for the questions  $Q_c^{(i)}$ .  $\square$

By taking all possible unions of sets of complete questions  $Q_c^{(i)}$  of the same family  $c$  one constructs a Boolean algebra that has  $Q_c^{(i)}$  as atoms.

Alternatively, one can consider a different family  $d$  of  $N$  independent yes-no questions and obtain another Boolean algebra with different complete questions as atoms. It follows, then, from Axiom I that the set of questions  $W(P)$  that can be asked to  $P$ -observer is algebraically an orthomodular lattice containing subsets that form Boolean algebras. As Rovelli says, “This is precisely the algebraic structure formed by the family of linear subsets of Hilbert space.” This concludes his sketch.

The sketch of the Hilbert space construction is not a rigorous derivation due to two key obstacles: First, orthomodularity of the lattice was not derived and, strictly speaking, from Rovelli’s construction one cannot derive it. Second, even if one admits that the lattice is orthomodular, the fact that yes-no questions form an orthomodular lattice and that it contains as subsets Boolean algebras does not yet lead to emergence of the Hilbert space. Both these claims will now be formalized and all the assumptions needed on the way to rigorous proof will be made explicit.

### 6.3 Construction of the Hilbert space

This section is the highlight of the dissertation. We derive the structure of the Hilbert space from the information-theoretic axioms in seven steps:

1. Definition of the lattice of yes-no questions.
2. Definition of orthogonal complement.
3. Definition of relevance and proof of orthomodularity.
4. Introduction of the space structure.
5. Lemmas about properties of the space.
6. Definition of the numeric field.
7. Construction of the Hilbert space.

The fundamental notion of fact in the quantum logical formalism is represented as answer to a yes-no question. Information is then brought about by such answer, and the object that we study is the set of yes-no questions that can be asked to the system. Importantly, each such question that *can* be asked is not necessarily asked, and it means that one cannot state that the information which a question may bring about is the actual information possessed by I-observer. This possibility, but not actuality, of bringing about information is a crucial feature of our approach: only the information actually possessed by I-observer is given meta-theoretically, while there is also possible information that I-observer must take into account in building quantum theory. As it was said in an illuminating discussion of Bohr’s understanding of complementarity [142], “‘Possible information’ is the key phrase in Bohr’s formulation, indicating a crucial distinction between possible and actual events of measurement in quantum mechanics.” In this sense, we fully subscribe to Bohr’s view.

Denote the set of questions that can be asked to the system as  $W(P) = \{Q_i, i \in I\}$ . According to Axiom I, there is a finite number  $N \in \mathbb{N}$  that characterizes I-observer’s maximum amount of relevant information. The number of questions in  $W(P)$ , though, can be much larger than  $N$ , as some of these questions are not independent. Nothing stops from thinking that index set  $I$  is countably or uncountably infinite. At **step 1** of the reconstruction, for each pair of questions we postulate the existence of “or” and “and” logical operations and then define the material implication.

**Axiom IV (logical *or*).**

$$\forall Q_1, Q_2 \in I \exists Q_3 \in I \mid Q_3 = Q_1 \vee Q_2,$$

where  $Q_1 \vee Q_2$  equals *yes* if and only if any one of  $Q_1$  or  $Q_2$  equals *yes*.

**Axiom V (logical *and*).**

$$\forall Q_1, Q_2 \in I \exists Q_3 \in I \mid Q_3 = Q_1 \wedge Q_2,$$

where  $Q_1 \wedge Q_2$  equals *yes* if and only if both  $Q_1$  or  $Q_2$  equal *yes*.

When in these definitions we use the word “equals”, what we mean is not a situation in which, in one act of bringing about information, questions  $Q_1$ ,  $Q_2$  and  $Q_3$  are answered simultaneously. Indeed, we take no position at all as for the possibility of a fact in which all these questions are answered; if  $Q_1$  and  $Q_2$  are incompatible in the usual quantum mechanical sense, then such fact is certainly impossible. However, the set  $W(P)$  is a set of all questions that *can be asked* to the system, i.e. of all possible questions. In the axiomatic construction of conjunction and disjunction the values of these questions must therefore be viewed as *predictions* [45]. To be precise, a question is only answered in a fact. However, to construct a conjunction of two questions, it suffices to treat the yet ungiven answer as possible information. The conjunction will then be such a new question that the possible positive answer to it is equivalent to the positive answers to both initial questions.

**Proposition 6.2.**  $W(P)$  is a lattice.

*Proof.* Axioms IV and V define infimum and supremum for every pair of questions. The result then follows from Definition 5.1.  $\square$

As for completeness of this lattice, Definition 5.2 of complete lattice requires that lower and upper bounds be defined for any, possible infinite, set of questions. This fact is not entailed by any previous arguments and must be postulated separately. As Specker notes [173], it is sufficient to enlarge the domain of propositions so that it contains conjunctions and disjunctions of all elements. This enlargement, however, is the subject of a separate axiom.

**Axiom VI.** Lattice  $W(P)$  is complete.

By disjunction of a question and its negation one defines the always false question  $Q_0 = Q \wedge \neg Q$ . By conjunction of a question and of its negation one defines the always true question  $Q_\infty = Q \vee \neg Q$ . Questions  $Q_0$  and  $Q_\infty$  serve as lattice elements 0 and 1.

Lattice  $W(P)$  is also atomic in virtue of being constructed of yes-no questions. The answer to a yes-no question gives the indivisible 1 bit of information. Then

questions in  $W(P)$  that are not composed from other questions by conjunctions *and* or *or* are atoms of the lattice.

As **step 2** of the reconstruction we introduce orthogonal complementation in the lattice. It is important to distinguish the material implication, or entailment, which is a true or false statement about the elements of the language such as questions, from the conditional operation often referred to as implication, which is defined in the language itself. To be precise, “if  $A$  then  $B$ ” is a true or false statement and thus obeys classical logic. On the contrary,  $A \Rightarrow B$ , where  $\Rightarrow$  means the conditional operation, gives a third, new element of the language. The theory of conditionals in quantum logic was developed by Mittelstaedt [126]. For a review we refer to chapter 8 of Ref. [150]. In the following we shall only be interested in the relation of material implication expressed by the “if - then” phrase and we shall not enter in the discussion of quantum logical conditionals.

**Definition 6.3 (material implication).** Question  $Q_1$  entails question  $Q_2$ , transcribed as  $Q_1 \rightarrow Q_2$ , if in any two subsequent facts which bring about information containing answers to  $Q_1$  and  $Q_2$ , respectively, it is not the case that  $Q_1 = 1$  and  $Q_2 = 0$ , and at least one such sequence of facts is possible:

$$Q_1 \rightarrow Q_2 \Leftrightarrow \neg((Q_1|_M = 1) \wedge (Q_2|_M = 0)),$$

where  $M$  denotes a fact (or a measurement). Equivalently, one can say that I-observer never has information that  $Q_1 = 1$  and  $Q_2 = 0$ . The requirement that the facts be subsequent means that no other information is allowed to emerge between these two acts of bringing about information.

**Definition 6.4.** Questions  $Q_1$  and  $Q_2$  are **orthogonal** if

$$Q_1 \rightarrow \neg Q_2. \tag{6.6}$$

Orthocomplement  $Q^\perp$  is a union (conjunction) of all questions orthogonal to  $Q$ .

Note that according to the definition of implication, orthogonality requires validity of (6.6) *in all possible measurements*. This means that *whenever* questions  $Q_1$  and  $Q_2$  are asked to the system, it is not the case that  $Q_1 = 1$  and  $Q_2 = 1$ .

**Lemma 6.5.** *Definition 6.4 is in full accord with Definition 5.7.*

*Proof.* Indeed, (6.6) by the definition of implication is equivalent to  $Q_2 \rightarrow \neg Q_1$ , which insures that  $(Q_1^\perp)^\perp = Q_2^\perp = Q_1$ , where  $Q_2 = Q_1^\perp$ . Further, it is trivial to verify that  $Q \wedge Q^\perp = Q_0$  and  $Q \vee Q^\perp = Q_\infty$  since  $Q^\perp$  is greater or equal to  $\neg Q$ . It remains to show that property (ii) of Definition 5.7 holds. Assume that  $Q_1 \leq Q_2$ , i.e.  $Q_1 \wedge Q_2 = Q_1$ . We need to prove that  $Q_2^\perp \leq Q_1^\perp$ , i.e.  $Q_2^\perp \wedge Q_1^\perp = Q_2^\perp$ . The left-hand side of this last expression denotes such questions  $Q$  that  $Q_1 \rightarrow \neg Q$  and  $Q_2 \rightarrow \neg Q$  in all possible measurements. In its turn, these two conditions holding separately in all measurements imply that it must not be the case that  $[(Q_1 \vee Q_2) \wedge \neg Q]$ . Now insert the equality  $Q_1 \wedge Q_2 = Q_1$ . We get for the negative assumption

$$\begin{aligned} \neg[(Q_1 \vee Q_2) \wedge Q] &= (\neg Q_1 \wedge \neg Q_2) \vee Q = [(\neg Q_1 \vee \neg Q_2) \vee \neg Q_2] \vee Q = \\ &= \neg Q_2 \vee Q. \end{aligned} \quad (6.7)$$

Recall that (6.7) must *not* be the case. Then negation of the last expression in the line entails that  $\neg Q \wedge Q_2$ . Since equivalence holds everywhere in (6.7) and we started with  $Q_2^\perp \wedge Q_1^\perp$ , we conclude that  $Q_2^\perp \wedge Q_1^\perp = Q_2^\perp$ , which was the needed result. Therefore orthocomplementation as defined in  $W(P)$  fulfills the requirement for a lattice orthocomplementation.  $\square$

The notion of orthogonality as introduced in the Definition 6.4 is closely tied to the notion of relevance used in Axiom I. At this **step 3** of the reconstruction, the time is ripe to discuss the latter term. Imagine that information obtained from a question  $Q_1$  is relevant for I-observer. We are looking for ways to make it irrelevant. This can be achieved by asking some new question  $Q_2$  that will turn  $Q_1$  irrelevant. Consider  $Q_2$  such that it entails the negation of  $Q_1$ :

$$Q_2 \rightarrow \neg Q_1. \quad (6.8)$$

If I-observer asks the question  $Q_1$  and obtains an answer to  $Q_1$  but then asks a *genuine* new question  $Q_2$ , it means, by virtue of the meaning of the term “genuine,” that I-observer expects either a positive or a negative answer to  $Q_2$ . This, in turn, is only

possible if information  $Q_1$  is no more relevant; indeed, otherwise I-observer would have been bound to always obtain the negative answer to  $Q_2$ . Consequently, we conclude that, by asking  $Q_2$ , I-observer makes the question  $Q_1$  irrelevant. Note further that Equation (6.8) fully repeats the definition of orthogonality (6.6). This motivates the following interpretative definition of the notion of relevance. Remember, too, that relevance is meta-theoretic and must be defined in the physical theory independently (see page 46).

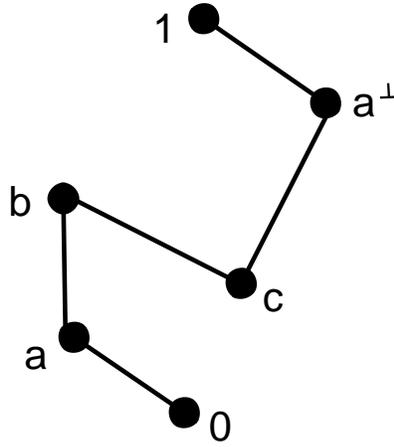


Figure 6.1: The Notion of Relevance. Order in the lattice is denoted by solid lines and grows from bottom to top, i.e.  $0 \leq a \leq b$ , etc. If there exists  $c \neq 0$  such that  $c \leq b$  and  $c \leq a^\perp$ , then question  $b$  is irrelevant with respect to question  $a$ , i.e. in  $b$  is contained a “component” of  $\neg a$ , and consequently, by genuinely asking  $b$ , one renders the question  $a$  irrelevant.

**Definition 6.6.** Question  $Q_2$  is called **irrelevant** with respect to question  $Q_1$  if  $Q_2 \wedge Q_1^\perp \neq 0$ . Otherwise question  $Q_2$  is called relevant with respect to question  $Q_1$ .

Conceptual justification of Definition 6.6 is offered on Figure 6.1. Now, the amount of information mentioned in the Axiom I is a nonnegative integer function, so 1 is its minimal nonzero value. We postulate that each atom in the lattice  $W(P)$  brings 1 bit of information. Let us now use Axiom I to demonstrate orthomodularity of the lattice  $W(P)$ .

**Proposition 6.7.**  $W(P)$  is an orthomodular lattice.

*Proof.* By Axiom I there exists a finite upper bound of the amount of relevant information. Let this be an integer  $N$ . Select an arbitrary question  $Q_1$  and consider a question  $\tilde{Q}_1$  such that

$$\{Q_1, \tilde{Q}_1\} \quad (6.9)$$

bring the maximum amount of relevant information, i.e.  $N$  bits. Notation  $\{\dots\}$  here means a sequence of questions that are asked one after another. Because all information here is relevant, we have by the definition of relevance that

$$\tilde{Q}_1 \wedge Q_1^\perp = 0 \quad (6.10)$$

.

We shall now use Lemma 5.10. It is sufficient to show that  $Q_1 \leq Q_2$  and  $Q_1^\perp \wedge Q_2 = 0$  imply  $Q_1 = Q_2$ . Note first that the second condition means, by Definition 6.6, that  $Q_2$  is relevant with respect to  $Q_1$ . Since  $Q_1 \leq Q_2$ , we obtain that

$$Q_2^\perp \leq Q_1^\perp. \quad (6.11)$$

Using this result and the result of Equation 6.10, we derive that

$$\tilde{Q}_1 \wedge Q_2^\perp = 0. \quad (6.12)$$

By definition, it means that question  $\tilde{Q}_1$  is relevant with respect to  $Q_2$ .

Now suppose, contrary to what is needed, that  $Q_2 > Q_1$  and consider the following sequence of questions:

$$\{Q_1, Q_2, \tilde{Q}_1\} \quad (6.13)$$

From Equations 6.10 and 6.12 follows that relevance is not lost in this sequence of question, i.e. all later information is relevant with respect to all earlier information. However, while relevance is preserved, this sequence, in virtue of the fact that  $Q_1 \neq Q_2$ , brings about more information than the sequence (6.9). It means that we have constructed a setting in which the amount of relevant information is strictly greater than  $N$  bits, causing a contradiction with the initial assumption. Consequently,  $Q_1 = Q_2$  and the lattice  $W(P)$  is orthomodular.  $\square$

By now, having completed steps 1 through 3 of the reconstruction, we obtained a complete, atomic and orthomodular lattice  $W(P)$ . From Section 5.4 we know that these properties do not suffice for emergence of the Hilbert space. Therefore, at this **step 4** of the reconstruction, we switch from discussing lattice  $W(P)$  alone to introducing a space of which a lattice of (certain) subspaces  $\mathcal{L}$  will be isomorphic to  $W(P)$ . Let us consider an arbitrary Banach space  $V$  satisfying this condition.

$$\mathcal{L}(V) \sim W(P) \tag{6.14}$$

Note here that the existence of space  $V$  is a relatively moderate constraint, for at this stage we require that space  $V$  be a generic Banach space. No assumption on the structure of the inner product is made. Compare this assumption with what Mackey assumes in his quantum mechanical axioms 7 and 8 [119]. Notation used in Mackey's axiom 8 will be explained in detail in Section 6.5.

Axiom 7. The partially ordered set of all questions in quantum mechanics is isomorphic to the partially ordered set of all closed subspaces of a separable, infinite dimensional Hilbert space.

Axiom 8. If  $e$  is any question different from the always false question then there exists a state  $f$  in  $\mathcal{S}$  such that  $m_f(e) = 1$ .

Unlike Mackey, we neither require that the space in question be the Hilbert space nor its infinite dimensionality. However, similar to Mackey's axiom 8, we do require that the lattice of all closed subspaces of  $V$  be isomorphic to the lattice of questions  $W(P)$ . When later we prove that  $V$  has an inner product with which it forms a Hilbert space, this requirement will be interpreted as a requirement that to every projection operator on a closed subspace of the Hilbert space corresponds a question, or alternatively that cases of product spaces with superselection rules are excluded. Indeed, had we not chosen a single vector space  $V$  "by hand," we could have considered lattices that are isomorphic to  $W(P)$  but built as direct products of several lattices  $\mathcal{L}_i, i = 1..n$ . Such cases are relevant in quantum field theories (for discussion see [148, Section 4.1]). Motivation for excluding superselection rules comes from our

search for a simpler structure; superselection can then be reintroduced as a new meta-theoretic restriction on the information acquired by I-observer. This restriction will not be general in the sense of applying to quantum theory in its most general form, but will lead to a new information-theoretic axiom in the particular case where superselection takes place. Note too that one cannot argue that allowing product spaces with superselection rules could remove quantumness by reducing the space to a product of one or two-dimensional Hilbert spaces, in which all physics can be described classically. The cause of quantumness is not linked with dimension and will be presented in Section 6.4.

Now observe that  $V$  is separable if  $W(P)$  contains countably many questions. It follows from our construction of a complete orthogonal sequence of questions in (6.4) and from the existence of an isomorphism connecting  $W(P)$  and a lattice of closed subspaces of  $V$ . One can then consider a family of projectors on these subspaces that will all commute and together form a basis in  $V$ . Then this corresponding space will be separable [152, p. 12, Theorem 2].

To summarize, at step 4 of the reconstruction we introduced the space  $V$  such that the lattice of its closed subspaces is isomorphic to  $W(P)$ . We now pass to **step 5** where we prove two lemmas concerning the space  $V$ .

**Lemma 6.8.** *Each finite-dimensional subspace of  $V$  is in  $\mathcal{L}$ .*

*Proof.* For every finite-dimensional subspace  $V_0 \subseteq V$  one can choose  $N$  being the smallest integer greater than  $\log_2 \dim V_0$ . One can then pick no more than  $N$  questions in  $W(P)$  that correspond to projections onto one-dimensional subspaces of  $V_0$ . Units and intersections of any subset of these questions are also questions and belong to  $W(P)$  by Axioms IV and V. Consequently,  $V_0$ , of which all knowledge can be exhausted by such units and intersections, belongs to  $\mathcal{L}$ .  $\square$

**Lemma 6.9.** *If  $Q$  is in  $W(P)$  with  $Q \leftrightarrow U \in \mathcal{L}$  and  $V_0$  a subspace of  $V$  such that  $\dim V_0 < \infty$  then  $U \wedge V_0 \in \mathcal{L}$ .*

*Proof.* This lemma states that to a question one can add by operations of disjunction

and conjunction any finite set of questions and obtain yet another question. The proof is analogous to the proof of Lemma 6.8. Namely, choose  $N$  being the smallest integer greater than  $\log_2 \dim V_0$ . Then pick no more than  $N$  questions in  $W(P)$  that correspond to projections onto one-dimensional subspaces of  $V_0$ . Operation  $\wedge$  taken between any subset of these questions and  $Q$  produces a question which belongs to  $W(P)$  in virtue of Axioms IV and V. By the isomorphism between  $W(P)$  and  $\mathcal{L}$ , this new question corresponds to a subset of  $\mathcal{L}$ . In virtue of the finite number of questions concerned, we obtain that  $U \wedge V_0 \in \mathcal{L}$ .  $\square$

At **step 6** of the reconstruction we study the field  $\mathbb{D}$  on which is built space  $V$ . According to Theorem 5.23 there exists an involutory anti-automorphism  $\theta$  in  $\mathbb{D}$ . We now first postulate a concrete form of  $\mathbb{D}$  and continuity of the involutory anti-automorphism and then discuss the alternatives to this postulate. Continuity will be discussed in this section, while the concrete form of  $\mathbb{D}$  will be discussed both here and in Section 6.5.

**Axiom VII.** The underlying field of the space  $V$  is one of the numeric fields  $\mathbb{R}$ ,  $\mathbb{C}$  or  $\mathbb{H}$  and the involutory anti-automorphism  $\theta$  is continuous.

*Remark 6.10.* It is commonplace to build quantum mechanics in a Hilbert space over the field  $\mathbb{C}$ . However, in one and two dimensions a complete description in a real Hilbert space is possible. The quaternionic Hilbert space can fully model all properties of the complex Hilbert space, but it will also lead to novel effects that have not been observed until now [1]. Strictly speaking, there is no theoretic argument in favor of one of the three fields only; nor shall we invent an information-theoretic argument.

Instead of directly postulating that one of the three fields is involved, real numbers, complex numbers or quaternions, we could have adopted Zieler's axiom (Co) [204] presented below in Section 6.5. In full accord with the argument about the crucial role of the continuity assumption, axiom (Co) tells that a certain function is continuous.

From this, with the help of Pontrjagin's index theorem, Zieler deduces that the field in question is one of the three fields named above.

Note that the continuity property assumed in this axiom is in direct correspondence with the continuity properties which one finds in various other proposed sets of axioms for quantum mechanics. In section 3.7 of his book [115], Landsman rephrases continuity into a "two-sphere property" which, as it is easy to expect, requires that some algebraically built structure be isomorphic to a topological continuous object, namely a sphere.

Yet a different usage of the continuity axiom can be found in Lucien Hardy's papers [84, 85]. Hardy gives five axioms from which he reconstructs quantum mechanics. They are:

**Axiom H1.** *Probabilities.* Relative frequencies (measured by taking the proportion of times a particular outcome is observed) tend to the same value (which is called probability) for any case where a given measurement is performed on an ensemble of  $n$  systems prepared by some given preparation in the limit as  $n$  becomes infinite.

**Axiom H2.** *Simplicity.* The number of the degrees of freedom of a system  $K$  is determined as a function of the dimension  $N$  (i.e.  $K = K(N)$ ) where  $N = 1, 2, \dots$  and where, for each given  $N$ ,  $K$  takes the minimum value consistent with the axioms.

**Axiom H3.** *Subspaces.* A system whose state is constrained to belong to an  $M$  dimensional subspace (i.e. have support on only  $M$  of a set of  $N$  possible distinguishable states) behaves like a system of dimension  $M$ .

**Axiom H4.** *Composite systems.* A composite system consisting of subsystems  $A$  and  $B$  satisfies  $N = N_A N_B$  and  $K = K_A K_B$ .

**Axiom H5.** *Continuity.* There exists a continuous reversible transformation on a system between any two pure states of that system.

It has been argued that one can reconstruct quantum mechanics without Axiom H1 [163]. Still, the key role is played by Axiom H5. It is this axiom which, in Hardy's construction, distinguishes quantum mechanics from classical mechanics. In our approach the latter separation will appear in Section 6.4 in virtue of Axiom II. This explains why we do not need the full machinery of Hardy's H5, but only a weaker apparatus requiring continuity of the involutory anti-automorphism of the underlying field. Unlike this choice, in his version Hardy postulates continuity of the transformation of states, which requires in turn a pre-existing notion of state of the system. Hardy's motivation that "there are generally no discontinuities in physics" appears unconvincing.

With Axiom VII and the previous results in hand, we pass to the final **step 7** of the reconstruction of the Hilbert space at which we formulate the main theorem of this section.

**Theorem 6.11 (construction of the Hilbert space).** *Let  $W(P)$  be an ensemble of all questions that can be asked to a physical system and  $V$  a vector space over  $\mathbb{D} = \mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$ , such that a lattice of its subspaces  $\mathcal{L}$  is isomorphic to  $W(P)$ . Then there exists an inner product  $f$  on  $V$  such that  $V$  together with  $f$  form a Hilbert space.*

*Proof.* If  $V$  is finite-dimensional the result follows from Theorem 5.30 and if  $V$  is infinite-dimensional it follows from Lemmas 6.8, 6.9 and Theorem 5.31. For application of both theorems the required continuity of  $\theta$  is assumed in Axiom VII.  $\square$

Space  $\mathcal{H}$  is built in Theorem 6.11 in a manner that does not allow to specify its particular elements before we know the sets of questions in  $W(P)$  that correspond to relevant information. What is relevant is reflected in the choice of questions that are asked by I-observer (note that in Definition 6.6 relevance of a question is defined only relatively to another question, i.e. contextually in the meta-theoretic sense), and it comes without surprise that the construction of tangible structure of the Hilbert space in each particular case requires knowledge of the questions which I-observer intends to, and can, ask. Theorem 6.11 is therefore non-constructive in the sense

that it makes use of the notion of relevance which is imposed on the quantum theory from its meta-theory, a circumstance that underlines the importance of the loop cut of Figure 2.2.

## 6.4 Quantumness and classicality

The Hilbert space  $\mathcal{H}$  constructed in Theorem 6.11 may happen to be decomposable into the direct product of Hilbert spaces of smaller dimension. We avoided this possibility by saying that to every question in  $W(P)$  corresponds a closed subspace of  $\mathcal{H}$  and vice versa. Indeed, were there superselection rules present, some configurations in the Hilbert space would be physically prohibited, for example subspaces that intersect with many different multipliers in the direct product. For such subspaces there would be no corresponding question in  $W(P)$ , as we assumed that  $W(P)$  does not contain questions that are conventionally called “physically prohibited.” This latter observation must be credited to the way in which we have built  $W(P)$ : it contains all questions that can be asked to the system, i.e. facts that can occur. If a fact is “physically prohibited,” it of course cannot occur. Therefore, in the philosophy of the loop of existences that motivated the selection of fundamental notions in Section 4.3, it makes no sense to speak of physically prohibited facts, and the assumption of isomorphism in Equation 6.14 only allows the appearance of Hilbert spaces without superselection rules.

However, to obtain a Hilbert space without superselection rules is not enough for building quantum theory. In 1963 Mackey [119] showed that such a logical construction fits well both the classical and the quantum cases, and one needs an additional postulate to recover either the classical formalism or the quantum one. Classical mechanics in the Hilbert space was first introduced by Koopman [109] and von Neumann [191]; for a recent discussion see Ref. [12].

Mackey formulated his additional assumption which permits to distinguish between the classical and the quantum cases as follows:

... the fundamental difference between quantum mechanics and classi-

cal mechanics is that in quantum mechanics there are non-simultaneously answerable questions, i.e. the set of all questions is not a Boolean algebra.

Axiom II in our approach plays the role of Mackey's assumption about non-simultaneously answerable questions. The Hilbert space  $\mathcal{H}$  was solely built using the consequences of Axiom I (and supplementary axioms), and indeed Axiom II remained unused through the whole discussion which preceded Theorem 6.11. It is now time for this axiom to play its role. We shall prove that Mackey's criterion of quantumness holds, i.e. that the lattice  $W(P)$  is not distributive or, equivalently, that it is not a Boolean algebra. This also meets Bub's requirement when he says that "the transition from classical to quantum mechanics involves the transition from a Boolean to a non-Boolean structure for the properties of a system." [27]

**Lemma 6.12.** *All Boolean subalgebras of  $\mathcal{L}$  are proper.*

*Proof.* If I-observer asks the  $N$  questions of family  $c$  as on page 77, i.e. a maximum number of independent questions, Axiom II requires that he still be able to ask a question the answer to which is not determined by answers to questions in the family  $c$ . Because with the help of  $c$  one can build Boolean subalgebras of the lattice  $\mathcal{L}$ , it follows that all such subalgebras are proper and the lattice  $\mathcal{L}$  itself is not Boolean. Indeed, were it not the case, one could have asked the questions  $Q_n$  of a family  $d$  such as the complete questions  $Q_d^{(i)}$  corresponding to this family  $d$ , as defined in (6.4), would form a Boolean algebra coinciding with the whole lattice  $\mathcal{L}$ . Answers to  $Q_n$  of the family  $d$  would then leave no room for a new question to which the response would have not been determined. Since this contradicts Axiom II, we conclude that all Boolean subalgebras of  $\mathcal{L}$  are proper.  $\square$

**Corollary 6.13.** *The lattice of all questions  $W(P)$  is not a Boolean algebra.*

*Proof.* Follows from Lemma 6.12 and isomorphism between the lattices  $\mathcal{L}$  and  $W(P)$ .  $\square$

## 6.5 Problem of numeric field

To complete the discussion of how to obtain the Hilbert space, we return to the problem of justification of our Axiom VII. In that axiom we postulated that the field that underlies the space  $V$  is one of  $\mathbb{R}$ ,  $\mathbb{C}$  or  $\mathbb{H}$ . Most authors also postulate this, but not all.

Let us start by looking at two attempts of justification of Mackey's axiom 7 (see page 85), one by Zierler in 1961 [204] and one by Holland in 1995 [93]. Both Zierler and Holland start with the structure which follows from Mackey's first six axioms and which is essentially the pair  $(\mathcal{L}, \mathcal{S})$  of questions and states, where  $\mathcal{L}$  and  $\mathcal{S}$  are described in the following definitions.

**Definition 6.14.**  $\mathcal{L}$  is a **countably orthocomplete orthomodular partially ordered set** if

- (1)  $\mathcal{L}$  is a partially ordered set with smallest element 0 and largest element 1;
- (2)  $\mathcal{L}$  carries a bijective map  $a \mapsto a^\perp$  that satisfies  $a^{\perp\perp} = a$  and  $a \leq b \Rightarrow a^\perp \geq b^\perp$  for all  $a, b \in \mathcal{L}$ ;
- (3) for every  $a \in \mathcal{L}$  the join  $a \vee a^\perp = 1$  and the meet  $a \wedge a^\perp = 0$  both exist and have the value indicated;
- (4) given any sequence  $a_i$ ,  $i = 1, 2, \dots$  of elements from  $\mathcal{L}$  such that  $a_i \leq a_j^\perp$  when  $i \neq j$ , the join  $\vee a_i$  exists in  $\mathcal{L}$ ;
- (5)  $\mathcal{L}$  is orthomodular:  $a \leq b \Rightarrow b = a \vee (b \wedge a^\perp)$ .

A countably orthocomplete orthomodular partially ordered set is different from a lattice with the same properties only in that join and meet are not defined for each pair of questions in  $\mathcal{L}$ .

**Definition 6.15.**  $\mathcal{S}$  is a **full, strongly convex family of probability measures on  $\mathcal{L}$**  if

- (1) each  $m \in \mathcal{S}$  is a probability measure on  $\mathcal{L}$ , i.e.  $m : \mathcal{L} \rightarrow \{s : 0 \leq s \leq 1\}$ ,  
 $m(0) = 0$ ,  $m(1) = 1$ , and  $m(\bigvee a_i) = \sum m(a_i)$  for any orthogonal family  $\{a_i : i = 1, 2, \dots\}$  of elements in  $\mathcal{L}$ ;
- (2)  $m(a) \leq m(b)$  for all  $m \in \mathcal{S}$  implies  $a \leq b$  (“full”);
- (3)  $m_i \in \mathcal{S}$ ,  $0 < t_i \in \mathbb{R}$ ,  $i = 1, 2, \dots$ , and  $\sum t_i = 1$  together imply  $\sum t_i m_i \in \mathcal{L}$  (“strongly convex”).

The structure  $(\mathcal{L}, \mathcal{S})$  is equivalent to the structure of the set of observables, states and the probability measure, which follows from Mackey’s first six axioms [119, p. 68]. Mackey himself only states this fact and a complete proof has been provided by Beltrametti and Casinelli [7].

Still, Mackey’s first six axioms, just as our axioms, do not guarantee quantumness. As we said above, the latter goal is achieved by Mackey’s axiom 7. In an early attempt to justify this axiom, Zieler proposed another list of axioms that allow one to deduce the isomorphism postulated by Mackey (we keep Zieler’s original numbering):

- (E4), (E5), (A) and (ND)  $\mathcal{L}$  is a separable atomic lattice, the center  $\mathcal{C}(\mathcal{L}) \neq \mathcal{L}$ , and element  $1 \in \mathcal{L}$  is not finite [see Definition 7.16].
- (M), (H) If  $a \in \mathcal{L}$  is finite, then  $\mathcal{L}(0, a)$  is modular; if  $a, b$  are finite elements of the same dimension, then  $\mathcal{L}(0, a)$  and  $\mathcal{L}(0, b)$  are isomorphic.
- (S2) If  $0 \neq a \in \mathcal{L}$ , then there exists  $m \in \mathcal{S}$  with  $m(a) = 1$ .
- (S3)  $m(a) = 0$  and  $m(b) = 0$  together imply  $m(a \vee b) = 0$ .
- (C’), (C) For every finite  $a \in \mathcal{L}$  and for each  $i$ ,  $0 \leq i \leq \dim a$ , the set of elements  $\{x \in \mathcal{L} : x \leq a \text{ and } \dim x = i\}$  is compact in the topology provided by the metric

$$f(x, y) = \sup\{|m(x) - m(y)| : m \in \mathcal{L}\}.$$

For each  $i = 0, 1, \dots$  the set of finite elements in  $\mathcal{L}$  of dimension  $i$  is complete with respect to the same metric.

(Co) For some finite  $b$  and real interval  $I$  there exists a nonconstant function from  $I$  to  $\mathcal{L}(0, b)$ .

One can see that axioms (C'), (C) and (Co) essentially involve non-algebraic concepts, such as topology or continuity. This comes as little surprise after we have discussed in Axiom VII the role of the continuity assumption. However, Zieler's axioms appear to import too much of "alien" terminology, and one can do better. This is mainly due to a beautiful theorem proved by Maria Pia Solèr [172].

**Theorem 6.16 (Solèr).** *Let  $\mathbb{D}$  be a field with involution,  $V$  a left vector space over  $\mathbb{D}$ , and  $f$  an orthomodular form on  $V$  that has an infinite orthonormal sequence. Then  $\mathbb{D} = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , and  $\{V, \mathbb{D}, f\}$  is the corresponding Hilbert space.*

This theorem makes use of the following definition.

**Definition 6.17.** An orthonormal sequence is a sequence

$$\{e_i : i = 1, 2, \dots\}$$

of nonzero vectors  $e_i \in V$  such that  $f(e_i, e_j) = 0$  for  $i \neq j$  and  $f(e_i, e_i) = 1$  for all  $i$ .

Solèr's theorem allowed Holland to revise Zieler's postulates, thus arriving at the following set of axioms [93].

(A1)  $\mathcal{L}$  is separable, i.e. any orthogonal family of nonzero elements in  $\mathcal{L}$  is at most countable.

(A2) If  $m(a) = m(b) = 0$  for some  $a, b \in \mathcal{L}$  and an  $m \in \mathcal{S}$ , then there exists  $c \in \mathcal{L}$ ,  $c \geq a$  and  $c \geq b$  with  $m(c) = 0$ .

(B1) Given any nonzero question  $a \in \mathcal{L}$ , there is a pure state  $m \in \mathcal{S}$  with  $m(a) = 1$ .

(B2) If  $m$  is a pure state with support  $a \in \mathcal{L}$ , then  $m$  is the only state, pure or not, with  $m(a) = 1$ .

(C) Superposition principle for pure states:

1. Given two different pure states (atoms)  $a$  and  $b$ , there is at least one other pure state  $c$ ,  $c \neq a$  and  $c \neq b$  that is a superposition of  $a$  and  $b$ .
  2. If the pure state  $c$  is a superposition of the distinct pure states  $a$  and  $b$ , then  $a$  is a superposition of  $b$  and  $c$ .
- (D) Ample unitary group: Given any two orthogonal pure states  $a, b \in \mathcal{L}$ , there is a unitary operator  $U$  such that  $U(a) = b$ .

We note that Holland's axioms (A) and (B) appear in Ref. [7]; (B) roughly states, in the ordinary language, that for every question there is a state with a yes answer, and for every pure state there is one and only question the answer to which is yes in this state and in no other.

From Solèr's theorem it follows that if a pair  $(\mathcal{L}, \mathcal{S})$  of question space and state space satisfies Holland's axioms A through D, then Mackey's axiom 7 follows as a consequence. The structure  $\mathcal{L}$ , referred to as *quantum logic*, is an orthocomplemented lattice and is isomorphic to the orthocomplemented lattice of all closed subspaces of a separable real, complex, or quaternionic Hilbert space. The beauty of Solèr's result is that it allows to weaken our Axiom VII by omitting the condition for the field to be real or complex numbers or quaternions. However, in doing so, Solèr's theorem brings to the information-theoretic approach a new complication.

The problem is that this theorem is only valid if the Hilbert space is infinite-dimensional. Theorem 6.11 uses the result of Theorem 5.30 which provided construction of a finite-dimensional Hilbert space. To obtain this, we had to postulate earlier that the underlying field is either  $\mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  and that its involutory anti-automorphism is continuous. Solèr's theorem, though elegantly avoiding assumptions about anything but the lattice structure, also avoids the finite-dimensional case. This is by itself regrettable and all the more so for the science of quantum computation: for example, to make a quantum computer work as quantum simulator, the restriction to infinite-dimensional Hilbert spaces is a major difficulty (see [122]). It is impossible to derive a finite-dimensional Hilbert space directly from lattice axioms, hence

to derive the version of quantum theory needed for quantum computation. The only option left is philosophical rather than mathematical: One must first derive the infinite-dimensional Hilbert space and then use meta-theoretic constraints to reduce the infinite-dimensional space to the finite-dimensional space of qubits. In the generic situation, information-theoretic justification of these extra meta-theoretic constraints remains an unsolved problem.

Still, and without assuming full rigor, we propose a conceptual argument that goes as follows: It is unclear why there may exist any *a priori* preferred dimensionality of the Hilbert space. The symmetry between all values of dimension is preserved, because dimensionality arises in the isomorphism between the set of questions  $W(P)$  and the lattice of closed subspaces of some space  $V$ . There are no information-theoretic constraints on the questions apart from those that enter in Axioms I, II and III. So we admit that all dimensions have *a priori* equal rights. Then, if we believe that the choice of dimension must still be justified within the theory, we are left with no particular value for the dimension and we have to seek for a case that encompasses all the values that are possible. Apparently, a candidate dimension that does not give preference to any finite value is the infinity.

In the spirit of this argument one must further say, in order to be consistent, that structure of the information-based quantum theory allows that the dimension of the Hilbert space be infinity or any reduction thereof, where each reduction is operationally (*a posteriori*) chosen. Like in the case with the transcendental deduction of probabilities (see the footnote on page 98), the structure of the theory provides a general framework but does not pick a particular value for the dimension of the Hilbert space. Like the concrete numeric values of probabilities, the value of dimension is chosen in the process of application of the theory to a concrete practical situation. Infinite-dimensional Hilbert space is then reduced to some its finite-dimensional subspace.

If we had included Solèr's theorem in our information-theoretic reconstruction of the Hilbert space, then it would have allowed us to weaken Axiom VII and only leave

the requirement that the anti-automorphism associated to the field be continuous, without making any assumption on which field this one is. The price to pay is that we would have had to postulate the existence of an infinite orthonormal sequence. By the lattice isomorphism between  $\mathcal{L}$  and  $W(P)$ , this condition means that, in  $W(P)$ , there exists an infinite sequence of orthogonal questions. Is there an information-theoretic justification for it? The answer seems to be in the negative. Axiom II says that one can always ask a new question; but this fact does not guarantee that such a question will be orthogonal to all questions that have been asked prior to this one. The word “new” does not imply orthogonality. On these grounds we believe that the assumption needed for Solèr’s theorem is not well-justified informationally and we prefer to postulate explicitly the form of the underlying field as this was done in Axiom VII.

## 6.6 States and the Born rule

In the choice of fundamental notions in Section 4.3 we stated that information and facts are fundamental. This gave rise to the Hilbert space as space of the physical theory, while subspaces of the Hilbert space correspond to yes-no questions. Nothing has been said about the notion of quantum state. Thus, state is a theoretical construction that comes after the Hilbert space and that is dependent on the Hilbert space structure. Such view is consistent with the original Heisenberg’s idea [87] and was developed with great persuasive power by van Fraassen [183]. In this section we show how the Born rule and the state space are reconstructed in the information-theoretic approach in virtue of Axiom III.

Just like the sketch of derivation of the Hilbert space presented in Section 6.2, Rovelli gives a sketch for the case of the Born rule and probabilities: From Axiom II it follows immediately that there are questions such as answers to these questions are not determined by  $s_c$ . Define, in general, as  $p(Q, Q_c^{(i)})$  the probability that a yes answer to  $Q$  will follow from the string  $s_c^{(i)}$ . Given two complete strings of answers  $s_c$

and  $s_b$ , we can then consider the probabilities

$$p^{ij} = p(Q_b^{(i)}, Q_c^{(j)}).^\ddagger$$

From the way it is defined, the  $2^N \times 2^N$  matrix  $p^{ij}$  cannot be completely arbitrary. First, we must have

$$0 \leq p^{ij} \leq 1.$$

Then, if information  $s_c^{(j)}$  is available about the system, one and only one of the outcomes  $s_b^{(i)}$  may result. Therefore

$$\sum_i p^{ij} = 1.$$

If we assume that  $p(Q_b^{(i)}, Q_c^{(j)}) = p(Q_c^{(j)}, Q_b^{(i)})$  then we also get

$$\sum_j p^{ij} = 1.$$

However, if pursued further, this introduction of probabilities encounters some difficulties. The correct approach, as it appears for example in the quantum logical derivation in Ref. [115], should address the question of the construction of a state space over the Hilbert space obtained. The Hilbert space will then be treated as space of operators acting on the state space. In this formulation, the task of building a state space vividly reminds of a similar problem in the theory of  $C^*$ -algebras, where it is solved by the Gelfand-Naimark-Segal (GNS) construction. We shall explore this similarity in greater detail in Part III. Here we limit ourselves to a less structured approach; still we avoid explicitly postulating the existence of the state space, as done for example in Holland's axioms discussed in Section 6.5.

Rovelli expresses a desire to deduce the existence of the state space and the Born rule from his third axiom, which he unofficially formulates as follows [157]:

Tentative axiom 3: Different observers hold information in a consistent way.

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<sup>‡</sup>This introduction of probabilities does not yet commit one to any particular view on what probabilities *are*. Personally, the author believes in the transcendental deduction of the structure of probabilities [138, 15] and in the subjective attribution of numeric values to probabilities [162].

Although this willingness is also expressed in Ref. [156], no development is proposed, and instead Rovelli postulates the superposition principle. We do not know how to complete the program proposed by Rovelli and we choose instead a different approach.

In Axiom III we introduced intratheoretic non-contextuality—this is the condition that will now allow to obtain more of the structure of quantum theory. For Axioms I and II we have found mathematical counterparts in the quantum logical formalism with regard to relevance and quantumness. Now time is ripe to find such a counterpart for Axiom III. It will be understood in terms of probabilities as sketched by Rovelli. The axiom can then be reformulated as a condition of independence from the physical context which has no informational share in determining the answer to a particular chosen question. This is to say that, if a question corresponds to a projection operator in the Hilbert space constructed in Theorem 6.11, then probabilities can be defined for a projector independently of the family of projectors of which it is a member, or that in  $p(Q_b^{(i)}, Q_c^{(j)})$  with fixed  $Q_b^{(i)}$  probability will be the same had the fixed question belonged not to the family  $b$  but to some other family  $d$ .

Non-contextuality remains a widely disputed assumption in the literature. There exists a multitude of its versions: in philosophy, type vs. token non-contextuality; in the foundations of quantum theory, preparation vs. transformation vs. measurement non-contextuality [174]. We discuss the general notion before returning to the intratheoretic non-contextuality that we postulated in Axiom III.

Saunders is one of those who simply reject non-contextuality because it is “too strong to have any direct operational meaning” [161]. One should also take care to avoid the Kochen-Specker paradox [106], which along with non-contextuality requires a premise of value-definiteness [88]:

All observables defined for a quantum mechanical system have definite values at all times.

Value-definiteness obviously does not hold in information-theoretic derivation programs like ours, but a deeper analysis is pending.

In the usual treatment of the Kochen-Specker paradox (for example [151]), value-definiteness is accompanied by a rule called the Functional Composition Principle, which states that  $[f(A)]^{|\varphi\rangle} = f([A]^{|\varphi\rangle})$ . Here  $A$  is a self-adjoint operator,  $[A]$  denotes the value of the corresponding observable, and  $f(A)$  denotes the observable whose associated operator is  $f(\hat{A})$ . Essentially, the latter principle states that the algebraic structure of operators should be mirrored in the algebraic structure of the possessed values of the observables. One then sees that, in our approach, the Functional Composition Principle is not justified, because the conditions of relevance imposed on a set of questions that *can* be asked do not translate into any conditions of relevance on the values of responses to these questions. Responses, in fact, are only given to a tiny fraction of the questions that can be asked. Therefore, there is no reason to think that the structure of the question lattice can be imitated by the structure on the set of ascribed values.

Let us now return to our notion of intratheoretic non-contextuality. This assumption is not trivial but in order to see its force, one must first translate it into the mathematical language of the formalism. We say that the intratheoretic context is defined by the questions surrounding some fixed question, i.e. by possible facts other than the given fact in which information was brought about. In the other words, we say that information as answer to a yes-no question is only given by the particular answer to this particular question and not by anything else, including other answers to other questions.

Remembering the correspondence between questions and subsets of the Hilbert space that form a complete, atomic and orthomodular lattice, one is now in position to prove a theorem due to Gleason [70]:

**Theorem 6.18 (Gleason).** *Let  $f$  be any function from 1-dimensional projections on a Hilbert space of dimension  $d > 2$  to the unit interval, such that for each resolution of the identity in projections  $\{P_k\}, k = 1 \dots d$*

$$\sum_{k=1}^d P_k = I, \quad \sum_{k=1}^d f(P_k) = 1. \quad (6.15)$$

Then there exists a unique density matrix  $\rho$  such that  $f(P_k) = \text{Tr}(\rho P_k)$ .

Theorem 6.18 shows how the state space is built on the Hilbert space of the Theorem 6.11 and how probabilities can be evaluated on that space by means of a trace-class operator. This justifies the Born rule. With the help of Axiom III and Gleason's theorem we have therefore constructed the second block of the formalism of the quantum theory.

## 6.7 Time and unitary dynamics

In this section we reconstruct the third and last block of the quantum formalism after the Hilbert space and the Born rule: unitary dynamics or evolution in time. As in the case of the Born rule and Gleason's theorem, we use powerful theorems to minimize the need in additional postulates. Still, additional assumptions are unavoidable. To give a reason why it is so, observe that the axioms introduced in the previous sections refer to the definition of observables, states, and the Born rule. This is the Heisenberg picture of quantum mechanics. As Rovelli says in an illuminating discussion [155, Section III.A], "In the Heisenberg picture, the time axiom can be dropped without compromising the other axioms or the probabilistic interpretation of the theory." Quantum mechanics can be represented as *timeless*. If one wishes to speak about time, then this notion has to emerge independently.

The discussion in this section will be limited to non-relativistic quantum mechanics. This is to say that we shall take into account time dynamics postulated along with the notion of *fact* in Section 4.3. If one treats only facts, and not time, as fundamental, thus not willing to assume that time is introduced axiomatically, then one has to show how time arises from the interplay of the three fundamental notions. This requires a general algebraic approach and will be further discussed in Section 8.5.

Following Rovelli's approach, every yes-no question can be labelled by the time variable  $t$  indicating the time at which it is asked. Denote as  $t \rightarrow Q(t)$  the one-parameter family of questions defined by the same procedure performed at different times. Then recall that, by Theorem 6.11, the set  $W(P)$  has the structure of a set of

linear subspaces in the Hilbert space. Assume that time evolution is a symmetry of the theory under the shift of the real variable  $t$ . From this assumption immediately follows that the set of all questions asked by I-observer to P-observer at time  $t_2$  is isomorphic to the set of all questions at time  $t_1$ . The isomorphism has some specific properties, namely it does not intermingle with the relevance of information. Because relevance is defined in connection with orthogonal complementation in the lattice, we require from the isomorphism that it commutes with orthocomplementation, thus ensuring that the relations between questions which existed at time  $t_1$  are fully transferred onto relations between the respective images of these questions at time  $t_2$ . In other words, there exists a transformation  $U(t)$  such that the inner product  $f$  is preserved

$$f(U(t_2 - t_1)Q_1(t_1), U(t_2 - t_1)Q_2(t_1)) = f(Q_1(t_1), Q_2(t_1)), \quad (6.16)$$

where  $f$  is applied to the elements of the Hilbert space of the Theorem 6.11, which isomorphically correspond to questions. We can now apply Wigner's theorem [200]. By its virtue transformation  $U$  is either unitary or antiunitary, with a possible phase factor which can be included in the norm  $f$ . Antiunitary case is excluded by considering the limit  $t_2 \rightarrow t_1$  and requiring that in this limit  $U$  becomes an identity map. Consequently,  $U$  is unitary.

Unitary matrices  $U(t_2 - t_1)$  form an Abelian group. One can write the composition law

$$U(t_1 + t_2) = U(t_1)U(t_2). \quad (6.17)$$

We require that  $t \rightarrow U(t)$  be weakly continuous and then by Stone's theorem [148, Theorem 6.1] obtain that

$$U(t_2 - t_1) = \exp[-i(t_2 - t_1)H], \quad (6.18)$$

where  $H$  is a self-adjoint operator in the Hilbert space, the Hamiltonian.

Recall the distinction between I-observer and P-observer in Section 4.5. P-observer as a physical system interacts with another physical system  $S$ , and the questions are being asked by I-observer to P-observer. In order to include the system  $S$  in the

theory, we need to make one more step, namely we need to connect the dynamics of the interaction between physical systems with the what theory says with regard to the dynamics of information acquisition by I-observer.

Interaction between P-observer and the quantum system should be viewed as physical interaction between just any two physical systems. Still, because I-observer then reads information from P-observer and because we aren't interested in what happens between P-observer and  $S$  after the act of reading information by I-observer from P-observer, we can treat P-observer as an ancillary system in course of its interaction with  $S$ . After the reading by I-observer the ancillary system “decouples.” Thus, such an ancillary system would have interacted with  $S$  and then would be subject to a standard measurement described mathematically on its Hilbert space via a set of “yes-no” orthogonal projection operators.

So far, for P-observer we have the Hilbert space and the standard Born rule. The fact that P-observer is treated as ancilla allows us to transfer some of this structure on the quantum system  $S$ . A new non-trivial assumption has to be made, that the time dynamics that has previously arisen in the context of I-observer and P-observer alone, also applies to the P-observer and  $S$ . In other words, there is only one time in the system. Time of I-observer is the one in which one can grasp the meaning of the words “past” and “future”: only what happened between P-observer and  $S$  in the past of the act of reading counts, and the future of that act has no informational impact. The unique time is thus the time in which are defined a “before the act of bringing out information” and an “after the act of bringing about information.” The hypothesis of unique time is useful for the purposes of this section and will be invalidated by the discussion in Section 8.5.

Assume now, as we proposed in Ref. [72, 74], that both the physical interaction of P-observer with  $S$  and the process of asking questions by I-observer to P-observer take place in one and the same time. Since (a) until I-observer asks the question that he chooses to ask, sets of questions at different times are isomorphic and the evolution is unitary, and (b) time at which I-observer asks the question only depends

on I-observer, one concludes that the interaction between the quantum system and P-observer must respect the unitary character all until the decoupling of the ancilla. Now write,

$$\rho_{SP} \rightarrow U \rho_{SP} U^\dagger. \quad (6.19)$$

After asking a question corresponding to a projector  $P_b$ , probability of the yes answer will be given by

$$p(b) = \text{Tr} (U(\rho_S \otimes \rho_P)U^\dagger(I \otimes P_b)). \quad (6.20)$$

Because the systems decouple, trace can be decomposed into

$$p(b) = \text{Tr}_S(\rho_S E_b), \quad (6.21)$$

where all presence of the ancilla is hidden in the operator

$$E_b = \text{Tr}_P ((I \otimes \rho_P)U(I \otimes P_b)U^\dagger), \quad (6.22)$$

which acts on the quantum system  $S$  alone. This operator is positive-semidefinite, and a family of such operators form resolution of identity. They are not, however, mutually orthogonal. Such operators form positive operator-valued measures (POVM) [135].

What we have achieved must be now described as follows: by neglecting the physical component of measurement via factoring out P-observer and treating measurement as purely informational, we made the move, from the description of measurement as yes-no questions asked by I-observer to P-observer, to the description of measurement as POVM. Information-theoretic derivation of quantum theory therefore leads to a natural introduction of POVM in virtue of the selected information-theoretic axioms and fundamental notions. Importance of this fact must not be underestimated: POVMs, we remind, are the essential tool in the science of quantum computation, and the use of this tool can now be justified based on information-theoretic principles.

## 6.8 Summary of axioms

We now bring together all axioms used in the derivation of the formalism of quantum theory. The key information-theoretic axioms are:

**Axiom I.** There is a maximum amount of relevant information that can be extracted from a system.

**Axiom II.** It is always possible to acquire new information about a system.

**Axiom III.** If information  $I$  about a system has been brought about, then it happened independently of information  $J$  about the fact of bringing about information  $I$ .

Auxiliary axioms to which no information-theoretic meaning was given are:

**Axiom IV.** For any two yes-no questions there exists a yes-no question to which the answer is positive if and only if the answer to at least one of the initial question is positive.

**Axiom V.** For any two yes-no questions there exists a yes-no question to which the answer is positive if and only if the answer to both initial questions is positive.

**Axiom VI.** The lattice of questions is complete.

**Axiom VII.** The underlying field of the space of the theory is one of the numeric fields  $\mathbb{R}$ ,  $\mathbb{C}$  or  $\mathbb{H}$  and the involutory anti-automorphism  $\theta$  in this field is continuous.

From the full set of axioms it follows that (1) the theory is described by a Hilbert space which is quantum and not classical; (2) over this Hilbert space one constructs the state space and derives the Born rule.

By way of the additional assumption of an isomorphism between the sets of questions corresponding to different time moments, unitary dynamics is introduced in the conventional form of Hamiltonian evolution.

The conceptual framework in which meta-theory is consistently separated from the theory requires that the observer be functionally separated into observer as physical system (P-observer) and observer as meta-theoretic entity or informational agent (I-observer). This, in turn, leads to a reinterpretation of the notion of measurement so that the interaction between I-observer and the physical system is formally described

via a positive operator-valued measure. Such a description meets the needs of the approach used by the science quantum information and computation.

We conclude by reiterating that, taken together, the above results allow one to reconstruct the three main blocks of the formalism of quantum theory.

## Part III

# Conceptual foundations of the $C^*$ -algebraic approach



# Chapter 7

## $C^*$ -algebraic formalism

In Part II, with the help of quantum logic, we derived the formalism of quantum theory. In Part III we consider a different approach, the one of the theory of  $C^*$ -algebras. The derivation program here will be reduced to a problem of information-theoretic *interpretation* of the algebraic approach. When such an interpretation will be given, theorems of the  $C^*$ -algebra theory will then permit to recover the formalism of quantum theory. Thus we change our attitude from the one of mathematical derivation in Part II to the attitude of conceptual justification and philosophical analysis in Part III. Although this change of attitude seems to lead to more modest results, discussion in Chapter 8 will be largely innovative: to the best of our knowledge, very little has been said in the literature concerning conceptual aspects of the Tomita theory of modular automorphisms and the Connes-Rovelli thermodynamic time hypothesis. To start the exposition, in Chapter 7 we present basic elements of the  $C^*$ -algebraic formalism.

### 7.1 Basics of the algebraic approach

Content of the algebraic quantum theoretic formalism will be exposed here following Refs. [38, 39, 78, 150].

**Definition 7.1.** In the linear space  $\mathfrak{B}(\mathcal{H})$  of bounded operators on a Hilbert space  $\mathcal{H}$  consider a system of  $\varepsilon$ -neighbourhoods of operator  $A$  defined by  $\|A - B\| < \varepsilon$ . The topology defined by this system of neighbourhoods is called the **norm** or the **uniform topology** in  $\mathfrak{B}(\mathcal{H})$ .

In quantum mechanics, a density matrix is a positive linear operator  $\omega$  with unit trace on the Hilbert space  $\mathcal{H}$  and it defines a normalized positive linear functional over  $\mathcal{A}$  via

$$\omega(A) = \text{Tr}(A\omega) \quad (7.1)$$

for every  $A \in \mathcal{A}$ . If one takes an arbitrary selection of  $\omega$  for a fixed  $A$ , this will define a system of neighbourhoods of  $A$ .

**Definition 7.2.** Topology provided by the system of seminorms  $|\text{Tr}(A\omega)|$  is called the ultraweak or weak  $*$ -topology on  $\mathfrak{B}(\mathcal{H})$  induced by the set of states  $\omega$ .

In particular, if  $\omega$  is a projection operator on a pure state  $\Psi \in \mathcal{H}$ , namely if

$$\omega = |\Psi\rangle\langle\Psi|, \quad (7.2)$$

then Equation 7.1 can be rewritten as the quantum mechanical expectation value relation

$$\omega(A) = \langle\Psi|A|\Psi\rangle. \quad (7.3)$$

With the uniform and weak  $*$ -topologies one defines two classes of algebra.

**Definition 7.3.** A **concrete  $C^*$ -algebra** is a subspace  $\mathcal{A}$  of  $\mathfrak{B}(\mathcal{H})$  closed under multiplication, adjoint conjugation (denoted as  $*$ ), and closed in the norm topology.

**Definition 7.4.** A **concrete von Neumann algebra** is a  $C^*$ -algebra closed in the weak  $*$ -topology.

From these concrete notions that have their roots in quantum mechanics one imports the intuition for definition of the following abstract algebraic notions.

**Definition 7.5.** An **abstract  $C^*$ -algebra** and an **abstract von Neumann algebra** (or a  $W^*$ -algebra) are given by a set on which addition, multiplication, adjoint conjugation, and a norm are defined, satisfying the same algebraic relation as their concrete counterparts. Namely, a  $C^*$ -algebra is closed in the norm topology and a von Neumann algebra is also closed in the weak  $*$ -topology.

**Definition 7.6.** A state  $\omega$  over an abstract  $C^*$ -algebra  $\mathcal{A}$  is a normalized positive linear functional over  $\mathcal{A}$ .

**Definition 7.7.** A state  $\omega$  is called **faithful** if, for  $A \in \mathcal{A}$ ,  $\omega(A) = 0$  implies  $A = 0$ .

**Definition 7.8.** A vector  $x$  belonging to the Hilbert space  $\mathcal{H}$  on which acts a  $C^*$ -algebra  $\mathcal{A}$  is called **separating** if  $Ax = 0$  only if  $A = 0$  for all  $A \in \mathcal{A}$ .

Given a state  $\omega$  over an abstract  $C^*$ -algebra  $\mathcal{A}$ , the Gelfand-Naimark-Segal (GNS) construction provides us with a Hilbert space  $\mathcal{H}$  with a preferred state  $|\Psi_0\rangle$  and a representation  $\pi$  of  $\mathcal{A}$  as a concrete  $C^*$ -algebra of operators on  $\mathcal{H}$ , such that

$$\omega(A) = \langle \Psi_0 | \pi(A) | \Psi_0 \rangle. \quad (7.4)$$

In the following  $\pi(A)$  will be denoted as simply  $A$ .

**Definition 7.9.** Given a state  $\omega$  on  $\mathcal{A}$  and the corresponding GNS representation of  $\mathcal{A}$  in  $\mathcal{H}$ , a **folium** determined by  $\omega$  is a set of all states  $\rho$  over  $\mathcal{A}$  that can be represented as

$$\rho(A) = \text{Tr} [A\hat{\rho}], \quad (7.5)$$

where  $\hat{\rho}$  is a positive trace-class operator in  $\mathcal{H}$ .

*Remark 7.10.* Consider an abstract  $C^*$ -algebra  $\mathcal{A}$  and a preferred state  $\omega$ . Via the GNS construction (7.4) one obtains a representation of  $\mathcal{A}$  in a Hilbert space  $\mathcal{H}$ . Definition 7.9 then introduces a folium of  $\omega$ , which determines a weak topology on  $\mathcal{A}$ . By closing  $\mathcal{A}$  under this weak topology we obtain a von Neumann algebra  $\mathcal{M}$ .

To continue the mathematical presentation, von Neumann factors can be classified into three types [129]. Assume the following series of definitions and results.

**Definition 7.11.** Commutant of a arbitrary subset  $\mathcal{M} \subseteq \mathfrak{B}(\mathcal{H})$  is a subset  $\mathcal{M}' \subseteq \mathfrak{B}(\mathcal{H})$  such that

$$B \in \mathcal{M}' \Leftrightarrow \forall A \in \mathcal{M} \quad [B, A] = 0. \quad (7.6)$$

**Theorem 7.12 (von Neumann's double commutant theorem).** *Let  $\mathcal{M}$  be a self-adjoint subset of  $\mathfrak{B}(\mathcal{H})$  that contains  $I$ . Then:*

- (i)  $\mathcal{M}'$  is a von Neumann algebra.
- (ii)  $\mathcal{M}''$  is the smallest von Neumann algebra containing  $\mathcal{M}$ .
- (iii)  $\mathcal{M}''' = \mathcal{M}$ .

**Definition 7.13.** A von Neumann algebra  $\mathcal{M}$  is called a factor if its center  $\mathcal{M} \cap \mathcal{M}'$  is trivial, i.e. it consists only of the multiples of identity.

**Theorem 7.14 ([150, Proposition 6.3]).** *The lattice of projections (self-adjoint, idempotent operators)  $P(\mathcal{M})$  of a von Neumann algebra is a complete orthomodular lattice. Furthermore, this lattice generates  $\mathcal{M}$  in the sense that  $P(\mathcal{M})'' = \mathcal{M}$ .*

Theorem 7.14 is of central importance for classification of von Neumann algebras. It shows that a classification can be achieved by investigating the lattice structure.

**Definition 7.15.** Two projections  $A$  and  $B$  in  $\mathcal{M}$  are called equivalent if there is an operator in  $\mathcal{M}$  (“partial isometry”) that takes vectors in  $A^\perp$  to zero and is an isometry between the image subspaces of  $A$  and  $B$ .

Definition 7.15 establishes an equivalence relation in  $P(\mathcal{M})$  and it allows to introduce a partial ordering of projections. Intuitively,  $A \preceq B$  means that the dimension of the image subspace of  $A$  is smaller or equal to the dimension of the image subspace of  $B$ . The order  $\preceq$  is in fact a total order on  $P(\mathcal{M})$  and, as a consequence, two von Neumann factors cannot be isomorphic if the orderings of the corresponding factorized projection lattices are different. To determine the order type, the following concept is crucial.

**Definition 7.16.** Projection  $A$  is called finite if from  $A \sim B \preceq A$  follows that  $A = B$ , i.e. if it is not equivalent to any proper subprojection of itself.

**Theorem 7.17 (classification of von Neumann factors).** *If  $\mathcal{M}$  is a von Neumann factor then there exists a map  $d$  (unique up to multiplication by a constant) defined on  $P(\mathcal{M})$  and taking its values in the closed interval  $[0, \infty]$  which has the following properties:*

Table 7.1: Classification of von Neumann factors

Range of $d$	Type of factor $\mathcal{M}$	Lattice $\mathbf{P}(\mathcal{M})$
$\{0, 1, 2, \dots, n\}$	$I_n$	modular, atomic, non-distributive if $n > 2$
$\{0, 1, 2, \dots, \infty\}$	$I_\infty$	orthomodular, non-modular, atomic
$[0, 1]$	$II_1$	modular, non-atomic
$[0, \infty]$	$II_\infty$	non-modular, non-atomic
$\{0, \infty\}$	$III$	non-modular, non-atomic

- (i)  $d(A) = 0$  if and only if  $A = 0$
- (ii) If  $A \perp B$ , then  $d(A + B) = d(A) + d(B)$
- (iii)  $d(A) \leq d(B)$  if and only if  $A \preceq B$
- (iv)  $d(A) < \infty$  if and only if  $A$  is a finite projection
- (v)  $d(A) = d(B)$  if and only if  $A \sim B$
- (vi)  $d(A) + d(B) = d(A \wedge B) + d(A \vee B)$

Types of von Neumann factors, well-defined in virtue of Theorem 7.17, are listed in Table 7.1.

## 7.2 Modular automorphisms of $C^*$ -algebras

Consider now an abstract  $C^*$ -algebra  $\mathcal{A}$  and an arbitrary faithful state  $\omega$  over it. The state  $\omega$  defines a representation of  $\mathcal{A}$  on the Hilbert space  $\mathcal{H}$  via the GNS construction with a cyclic and separating vector  $|\Psi\rangle \in \mathcal{H}$ . This, in turn, defines a von Neumann algebra  $\mathcal{M}$  with a preferred state. We are now concerned with 1-parameter groups of automorphisms of  $\mathcal{M}$ . They will be denoted  $\alpha_t^\omega : \mathcal{M} \rightarrow \mathcal{M}$ , with  $t$  real.

Consider the operator  $S$  defined by

$$SA|\Psi\rangle = A^*|\Psi\rangle. \quad (7.7)$$

One can show that  $S$  admits a polar decomposition

$$S = J\Delta_\omega^{1/2}, \quad (7.8)$$

where  $J$  is antiunitary and  $\Delta_\omega$  is a self-adjoint, positive operator. The Tomita-Takesaki theorem [178] states that the map  $\alpha_t^\omega : \mathcal{M} \rightarrow \mathcal{M}$  such as

$$\alpha_t^\omega A = \Delta_\omega^{-it} A \Delta_\omega^{it} \quad (7.9)$$

defines a 1-parameter group of automorphisms of the algebra  $\mathcal{M}$ . This group is called the group of modular automorphisms, or the modular group, of the state  $\omega$  over the algebra  $\mathcal{M}$ .

**Definition 7.18.** An automorphism  $\alpha_{inner}$  of the algebra  $\mathcal{M}$  is called an **inner automorphism** if there is a unitary element  $U$  in  $\mathcal{M}$  such that

$$\alpha_{inner} A = U^* A U. \quad (7.10)$$

Not all automorphisms are inner. We therefore consider the following equivalence relation in the family of all automorphisms of  $\mathcal{M}$ : two automorphisms are equivalent when they are related by an inner automorphism  $\alpha_{inner}$ , namely  $\alpha'' = \alpha_{inner} \alpha'$  or

$$\alpha'(A)U = U\alpha''(A), \quad (7.11)$$

for every  $A$  and some unitary  $U$  in  $\mathcal{M}$ . The resulting classes of automorphisms will be denoted as outer automorphisms, and their space as  $\text{Out } \mathcal{M}$ . In general, the modular group (7.9) is not a group of inner automorphisms. It follows that  $\alpha_t$  projects down to a non-trivial 1-parameter group in  $\text{Out } \mathcal{M}$ , which we denote as  $\tilde{\alpha}_t$ . The Cocycle Radon-Nikodym theorem [38] states that two modular automorphisms defined by two states of the von Neumann algebra are inner-equivalent. All states of the von Neumann algebra  $\mathcal{M}$ , or of the folium of the  $C^*$ -algebra  $\mathcal{A}$  that has defined  $\mathcal{M}$ , thus lead to the same 1-parameter group in  $\text{Out } \mathcal{M}$ , or in other words  $\tilde{\alpha}_t$  does not depend on the normal state  $\omega$ . This means that the von Neumann algebra possesses a canonical 1-parameter group of outer automorphisms, for which an information-theoretic interpretation will be suggested in Section 8.5.

From the Cocycle Radon-Nikodym theorem follows the intertwining property

$$(D\omega_1 : D\omega_2)(t) (\alpha_t^{\omega_2}) = (\alpha_t^{\omega_1}) (D\omega_1 : D\omega_2)(t), \quad (7.12)$$

where  $(D\omega_1 : D\omega_2)(t)$  is the Radon-Nikodym cocycle [78, Section V.2.3]. If, for a particular value of  $t$ , the modular automorphism  $\alpha_t^\omega$  is inner, then, as a consequence of Equation 7.12, it is inner for any other normal state  $\omega'$ . Therefore the set of  $t$ -values

$$\mathcal{T} = \{t : \alpha_t^\omega \text{ is inner}\} \quad (7.13)$$

is a property of the algebra  $\mathcal{M}$  independent of the choice of  $\omega$ . If  $\mathcal{M}$  is not a factor then  $\mathcal{T}$  is the intersection of the sets  $\mathcal{T}_k$  corresponding to factors  $\mathcal{M}_k$  occurring in the central decomposition of  $\mathcal{M}$ . In case  $\mathcal{M}$  is a factor, we notice that  $0 \in \mathcal{T}$  and, if  $t_1, t_2 \in \mathcal{T}$ , then  $t_1 \pm t_2 \in \mathcal{T}$ . So  $\mathcal{T}$  is a subgroup of  $\mathbb{R}$ , i.e. subgroup of the group of real numbers with addition as the group operation.

Connes [36] showed that  $\mathcal{T}$  is related to the spectrum of the modular operators  $\Delta_\omega$  that appear in Equation 7.8. He defined the spectral invariant

$$S(\mathcal{M}) = \bigcap_{\omega} \text{Spect } \Delta_\omega, \quad (7.14)$$

where  $\omega$  ranges over all normal states of  $\mathcal{M}$ , and the set

$$\Gamma(\mathcal{M}) = \{\lambda \in \mathbb{R} : e^{i\lambda t} = 1 \quad \forall t \in \mathcal{T}\}. \quad (7.15)$$

Connes's result is that

$$\Gamma(\mathcal{M}) \supset \ln(S(\mathcal{M}) \setminus 0) \quad (7.16)$$

and that  $\ln(S(\mathcal{M}) \setminus 0)$  is a closed subgroup of the multiplicative group  $\mathbb{R}_+$ . Type III von Neumann algebras are classified according to the value of  $S(\mathcal{M})$  as shown in Table 7.2.

The last notion of the von Neumann algebra theory that we introduce here is the notion of hyperfinite algebra.

**Definition 7.19.** A von Neumann algebra  $\mathcal{M}$  is called **hyperfinite** if it is the ultra-weak closure of an ascending sequence of finite dimensional von Neumann algebras.

Table 7.2: Connes's classification of von Neumann factors

Range of $S(\mathcal{M})$	Type of factor $\mathcal{M}$
$\{1\}$	$I$ and $II$
$\{0 \cup \lambda^n, n \in \mathbb{Z}\}$	$III_\lambda$ ( $0 < \lambda < 1$ )
$\mathbb{R}_+$	$III_1$
$\{0, 1\}$	$III_0$

Clearly, a type  $I_\infty$  von Neumann algebra is hyperfinite, because it is the limit of the matrix type  $I_n$  algebras of finite dimensional subspaces. Two important results can be proved about two other types of von Neumann algebras:

**Proposition 7.20 (Murray and von Neumann [130]).** *There is only one hyperfinite factor of type  $II_1$  up to isomorphism.*

**Proposition 7.21 (Haagerup [81] based on Connes [37]).** *There is only one hyperfinite factor of type  $III_1$  up to isomorphism.*

In Ref. [78, Section V.6] proof is provided using the tools of local algebraic quantum theory for the claim that algebra  $\mathcal{M}(K)$  of a diamond is isomorphic to the hyperfinite type  $III_1$  von Neumann factor. A diamond  $K$  is a spatiotemporal region defined as

$$K_r = \{x : |x^0| + |\mathbf{x}| < r\} \quad (7.17)$$

and it is characteristic of it that modular automorphisms act on a diamond geometrically (Hislop and Longo theorem [92]). Hyperfiniteness of  $\mathcal{M}(K)$  follows from the possibility to insert a type  $I$  von Neumann factor  $\mathcal{N}$  between the algebras of two concentric diamonds with radii  $r_2 > r_1$  (“split property”):

$$\mathcal{M}(K_{r_1}) \subset \mathcal{N} \subset \mathcal{M}(K_{r_2}). \quad (7.18)$$

This, in turn, was shown in Ref. [30] to be a consequence of the Buchholz-Wichmann nuclearity assumption [31], which is necessary and sufficient to ensure “normal thermodynamic properties,” namely the existence of KMS-states for all positive  $\beta$  for the

infinite system and for finitely extended parts (equivalent to absence of the Hagedorn temperature [82]). Thus, the chain of logical relations is as follows:

$$\begin{aligned} \text{KMS states at all } \beta &\Leftrightarrow \text{nuclearity} \Rightarrow \text{split property} \Rightarrow \\ &\Rightarrow \text{hyperfinite type III}_1 \text{ factor.} \end{aligned}$$

We now explain what the KMS states are and what role they play.

### 7.3 KMS condition

Let  $\mathcal{A}$  be a  $C^*$ -algebra. Consider the 1-parameter family of automorphisms of operators  $A \in \mathcal{A}$  given by

$$\gamma_t A = e^{it/H} A e^{-it/H}. \quad (7.19)$$

In the following we shall use the conventional language and say that the automorphisms are defined by the time evolution  $t$  and that  $H$  is the hamiltonian. However, equation (7.19) can be viewed purely formally, as the definition of a group of automorphisms, without giving any physical meaning to symbols  $t$  and  $H$ . We now look at the system from the thermodynamical point of view.

**Definition 7.22.** A state  $\omega$  over  $\mathcal{A}$  is called a Kubo-Martin-Schwinger (or KMS) state at inverse temperature  $\beta = 1/k_b T$  ( $k_b$  being the Boltzmann constant and  $T$  the absolute temperature), with respect to  $\gamma_t$ , if, for all  $A, B \in \mathcal{A}$ , the function

$$f(t) = \omega(B(\gamma_t A)) \quad (7.20)$$

is analytic in the strip

$$0 < \text{Im } t < \beta \quad (7.21)$$

and

$$\omega((\gamma_t A)B) = \omega(B(\gamma_{t+i\beta} A)). \quad (7.22)$$

The most important element of this definition is that, in the right-hand side of Equation 7.22, to the parameter  $t$  with a conventional meaning of time variable is

added the product of the imaginary unit  $i$  by the inverse temperature  $\beta$ . One can therefore view the KMS condition as a generalized Wick rotation, imposing a certain relation between dynamical and thermodynamical quantities. Justification given to the particular form (7.22) of the KMS condition is always *a posteriori*: it so happens that, with this specific choice of the relation between statistics and dynamics, one obtains correct predictions, including such ones as for example the Unruh effect. The working success of the prediction-making procedure justifies the form of the equation. It remains an open problem in the foundations of physics to uncover the principles that give rise to the fact that a certain mathematical relation between physical quantities on the complex plane (multiplication by  $i$ ) receives clearly preferential treatment over all other possible relations. As it is the case with the Wick rotation in quantum field theory, KMS condition at the imaginary time can be seen as a consequence of locality and of the spin-statistics connection. Conversely, more fundamentally and undoubtedly more interestingly for philosophers, one can view the spin-statistics connection and locality as consequences of the KMS condition.

In the case of systems with a finite number of the degrees of freedom, KMS condition reduces to Gibbs condition [78, Section V.1.2]

$$\omega = N e^{-\beta H}. \quad (7.23)$$

Following Ref. [80], one can postulate that the KMS condition represents a correct physical extension of the Gibbs postulate (7.23) to infinite dimensional systems. It is interesting to note that the authors who introduced the KMS condition in quantum statistical mechanics were led to this condition by the way starting from the Gibbs postulate. We refer to the review paper [13] for a description of this point of view. However, we shall see that, for the information-theoretic justification of the algebraic approach, the fact that the KMS condition is a generalized form of the Wick rotation is more significant than the fact that it is a generalization of the Gibbs postulate. The two lines of development can be brought together in speaking of the *twofold meaning* of the KMS condition.

The following link between the KMS condition and the Tomita-Takesaki theorem

(7.9) was established in Ref. [178]. It is arguably one of the most important and profound theorems in all physics of the second half of the XXth century.

**Theorem 7.23.** *Any faithful state is a KMS state at the inverse temperature  $\beta = 1$  with respect to the modular automorphism  $\gamma_t$  it itself generates.*

Thus, exactly as it is in the context of classical mechanics, an equilibrium state contains all information on the dynamics which is defined by the hamiltonian, apart from the constant  $\beta$ . This means that the information about dynamics can be fully replaced by the information about the thermal state. Indeed, imagine that the statistical state  $\rho$  is known. Then, remembering that  $\beta = 1$ , take the quantity  $H = -\ln \rho$ , treat it as the hamiltonian, and take its one-parameter flow [159, Sect. 3.4]. This will supply full information about dynamics, where  $t$  is none but the parameter of the hamiltonian flow.

We close this section by discussing the role of thermodynamics in the information-theoretic approach rooted in the philosophy of the loop of existences. As we have seen, quantum theory based on a  $C^*$ -algebra and a state over it contains all information that is needed for the theory, including dynamics; what it does not contain is the possibility to modify  $\beta$ , i.e. to modify the temperature. When at the end of Section 7.2 we required the existence of KMS states at all  $\beta$ , it was implicitly assumed that modification of the value of  $\beta$  does not have its origin inside the theory and must be motivated somehow else. Recall now the distinction between theory and meta-theory made by cutting the loop on Figure 2.2. One obtains that the theory describing modification of temperature, which we call thermodynamics, does not belong to this loop cut, as the loop cut with its information-theoretic view of quantum theory provides only for a fixed value of  $\beta$ . Therefore, thermodynamics, insofar as it describes the change in temperature, belongs to meta-theory of the information-based quantum theory. Is such a position surprising?

The answer is that the place of thermodynamics in the loop cut of Figure 2.3 is to be expected. This is due to the conceptual link between such terms as information and entropy, and also the link between entropy and temperature that is described by

thermodynamics. Because information is a meta-theoretic concept in the information-based quantum theory, any theory having information for its object of study falls necessarily into the domain of meta-theory. The conceptual link between information and entropy consists in the definition of information in statistical physics as relative entropy. In the physical theory, facts, seen as acts of bringing-about information, are measurement results. Szilard [177] argued that the measurement procedure is fundamentally associated with the production of entropy, and Landauer [113] and Bennett [11], refuting Szilard's argument, showed that entropy increase comes from the erasure of information, say, in the preparation of the system. To erase information means to render it irrelevant in the sense of Axiom I. We discussed the concept of relevant information in Definition 6.6 and explained on page 46 that any such definition must originate in meta-theory; it can now be seen that the concept of relevance is tied to thermodynamics.

The Szilard-Landauer-Bennett debate still continues [52, 53, 28] and we do not take a particular side in it in this dissertation. Another debate into which we do not enter is the one concerning applicability of Shannon's vs. von Neumann's entropy [24, 181]. But the very existence of these two debates shows that thermodynamics has its say in the information-theoretic approach, which is instantiated, at least, in the definition of relevant information and in the temporality of facts. To justify this last claim, we shall return to questions connected with thermodynamics and the KMS formalism in the discussion of time in Section 8.3.

# Chapter 8

## Information-theoretic view on the $C^*$ -algebraic approach

### 8.1 Justification of the fundamentals

In this section we show how the algebraic approach arises in the context of fundamental notions of system, information, and fact introduced in Chapter 4. But before doing that, we pay homage to an early attempt to justify the algebraic approach to quantum mechanics that was made in the seminal book by Gérard Emch [57].

The *raison d'être* of the algebraic approach, for Emch, is that, besides the standard quantum effects, it successfully describes phase transitions and nonperturbative phenomena which the Hilbert space formalism fails to incorporate. Needless to say, this is very far from our information-theoretic point of view.

Emch gives a set of ten axioms that provide for the whole of quantum mechanics. He postulates that a physical system is given by the set of observables and proposes the first five axioms that structure this set of observables. Axiom 6 then aims at establishing that this set is a Jordan-Banach algebra, a direct generalization of the notion of  $C^*$ -algebra. Axioms 7 and 8 install a topology on the set of observables, axiom 9 introduces the GNS construction, and axiom 10 provides for the uncertainty principle. At no place in the whole axiomatic construction, however, is anything said about time or about the dynamic aspect of the theory. But Emch's quantum theory is not timeless: time evolution is further defined as a group of automorphisms [57,

pp. 163, 300] connected with the KMS condition [57, p. 205]. This last suggestion, together with the view that a quantum system is a set of operators, are the only elements that we shall borrow from Emch.

Emch's axioms 1 through 5 establish the structure of the set of observables. Note that at this stage there is no space nor time assumed, so one cannot use the geometric intuition in determining the structure of what one observes. Instead, one can only employ the abstract intuition about the algebraic structure of observables. It is in these circumstances that Emch postulates that observables form a vector space and possess certain other non-trivial properties. We must add to this that it remains to be seen how a selection of axioms that installs a great deal of *a priori* mathematical structure on the set of observables could be justified. What is needed is an *interpretation* of the algebraic approach. Our interpretation will be given along the lines of the information-theoretic approach, and we now start laying it out. As it was argued in Section 4.3, the first step always consists in giving a translation into the mathematical language of each of the fundamental notions of the information-theoretic approach.

A  $C^*$ -algebra is interpreted as a mathematical counterpart of the fundamental notion of system. We have said that, in the quantum logical approach, system is represented as physical system, to which refers information obtained in elementary measurements in the form of answers to yes-no questions. Imagine for a moment the inverse optics: one could postulate that a large family of elementary propositions defines what a physical system *is*. We employ the inverse optics here only in the formal sense: instead of saying that the mathematical counterpart of the notion of system is the physical system of the quantum logical approach, we now formally represent the system as a  $C^*$ -algebra.

Further, as stated in Section 4.3, facts are acts of bringing about information and, in the physical theory, they are represented as measurement results. Usually we characterize a system not separately, but together with the information about it. Indeed, the system is mathematically described by a family of operators that form a  $C^*$ -algebra. These operators have the potential to frame an act of bringing-about

information and, consequently, to give rise to a fact. One observes that operations such as to characterize a system by a family of operators and to be given some information about the system come closely connected, both conceptually and formally.

Therefore, let us now consider a system and a fact. The fact is an act of bringing-about information, so there is some information available about the system. While the system is mathematically represented as a  $C^*$ -algebra of observables, we postulate that the information that was brought about in the chosen fact is represented as a state over this  $C^*$ -algebra in the sense of Definition 7.6. The notion of state as a positive linear functional is a translation of the concept of information into mathematical terms. This definition also falls in line with a recent observation by Duvenhage that “we can define information as being the state on the observable algebra” [51].

Let us look at how our terminological translation corresponds to the conventional one, where information is correlation between measurement results. In the conventional quantum mechanics, measurement results receive theoretical treatment due to introduction in the theory of the concept of *preparation*. In almost any textbook on quantum mechanics one will find a phrase, “The system is prepared in a such-and-such state.” Now, when we prepare a system, we make a catalogue of *all our knowledge* about this system. Indeed, to prepare a system means to set it up in accordance with our requirements to the system. These requirements are nothing but information about the system or our current knowledge thereof. Quantum mechanical preparation thus means that we make a list of, or exhibit, all knowledge about the system. Once the list has been compiled, the system has been prepared in a state corresponding to information on this list. An important element here is to accept that it is *all* our knowledge. Indeed, if an observer *genuinely* wants to learn something, it means that at present, as of the time before learning a new fact, the observer does not know it and does not possess information contained in that fact. What is going to be measured in a specially prepared setting is yet completely unknown at the preparation stage, and the catalogue of information that corresponds to preparation bears no trace of the particular information that is yet to be brought about. The

argument here can be regarded as an equivalent of the condition of intratheoretic non-contextuality discussed in Section 4.5.

Recall now that the “what is to be measured” is just a collection of operators in a  $C^*$ -algebra according to our definition of system. “Completely unknown” with respect to these operators means that the genuine state over the algebra, in the sense of information state, corresponds to no *a priori* information or no *a priori* knowledge. To say the same phrase in the language of thermodynamics amounts to requiring that the prepared state over the algebra of observables correspond to infinite temperature or, in the terminology of the KMS formalism, to  $\beta = 0$ .

It so happened historically that von Neumann’s original idea about how to derive quantum mechanics was related to the conclusion that the prepared state over the algebra of observables corresponds to infinite temperature. To illustrate the analogy, we open a parenthesis where we give a sketch of von Neumann’s derivation.

## 8.2 Von Neumann’s derivation of quantum mechanics

This historic section falls out of the main development of the dissertation. It offers a perspective on how were born the key ideas of quantum theory, like the use of the Hilbert space or the algebraic approach, and a well-informed reader may skip it.

Bub [26] and Rédei [150] give a concise exposition of von Neumann’s attempt to derive the probabilistic structure of quantum mechanics. In a 1927 paper on the mathematical foundations of quantum mechanics [188], the heart of the whole theory is the “statistical Ansatz.” It states that the relative probability that the values of the pairwise commuting quantities  $S_i$  lie in the intervals  $I_i$  if the values of the pairwise commuting quantities  $R_j$  lie in the intervals  $J_j$  is given by

$$\mathrm{Tr} [E_1(I_1)E_2(I_2) \dots E_n(I_n)F_1(J_1)F_2(J_2) \dots F_m(J_m)], \quad (8.1)$$

where  $E_i(I_i)$  and  $F_j(J_j)$  are the spectral projections of the corresponding operators  $S_i$  and  $R_j$  belonging to the respective intervals. Note that we are using here not the

von Neumann's original notation, but Rédei's account of it coined out in the modern terms.

In Ref. [190] von Neumann made an attempt to “work out inductively,” a phrase that meant, for von Neumann, a requirement that the statistical Ansatz (8.1) be derived from the basic principles of the theory. The starting point of the derivation is the assumption of an *elementary unordered ensemble* (“elementar ungeordnete Gesamtheit”). Von Neumann also calls this ensemble a fundamental ensemble in Ref. [189] and in the same paper appears a characterization “ensemble corresponding to ‘infinite temperature’”. For von Neumann this is an *a priori* ensemble  $E$  of which one does not have any specific knowledge. Every system of which one knows more is obtained from this ensemble by selection: one checks the presence of a certain property  $P$ , e.g. that quantity  $S$  has its value in the set  $I$ , and one collects into a new ensemble those elements of the *a priori* ensemble that have the property  $P$ . This new ensemble  $E'$  is therefore derived from  $E$ . On  $E'$  one can compute the relative probability defined in the Ansatz (8.1). Relative here means relative to the condition  $P$ . Computation of the probability is done via checking again the presence or absence of a certain property and collecting those elements that have this property. Because von Neumann was a partisan of the von Mises frequency interpretation of probabilities [187], he believed that one must simply calculate the frequency of occurrence of the selected elements in ensemble  $E$ . Identifying ensembles with expectation value assignments and assuming the formalism of quantum mechanics, von Neumann then showed that each ensemble can be described by a positive operator  $U$ , such that the description in question is given by

$$\text{Tr}(UQ). \tag{8.2}$$

Statistical operator  $U$  of the *a priori* ensemble  $E$  is the identity operator  $I$ .

Importance of the *a priori* ensemble can be seen as follows. The formula  $\text{Tr}(UQ)$  is not yet what von Neumann wants to achieve, for the goal is to obtain the statistical Ansatz (8.1). Suppose that we only know of the system  $S$  that the values of the pairwise commuting quantities  $R_j$  lie in the intervals  $J_j$ . “What statistical operator for

this ensemble should be inferred from this knowledge?” asks von Neumann. Assuming that it was the *a priori* ensemble on which we checked that the quantities  $R_j$  lie in the intervals  $J_j$ , and that we have collected those members of  $E$  on which this property was found present into a new ensemble  $E'$ , von Neumann proved that the statistical operator is indeed  $F_1(J_1)F_2(J_2) \dots F_m(J_m)$  needed for Equation 8.1.

In this derivation the *a priori* ensemble plays a distinguished role. Its statistical operator is the identity  $I$ , so it can be viewed as completely unselected, primary ensemble from which all other ensembles, carrying particular properties, are obtained. In our discussion in Section 8.1, this corresponds to saying that at the preparation stage one creates a catalogue of all knowledge, the genuine state is a state at infinite temperature or at  $\beta = 0$ , which has the significance of not yet knowing the information that will be brought about by the new facts. The quantum mechanical theory, so to say, starts at the point of the observer not knowing anything, at the price of collecting all his previous knowledge in the definitions of algebra and a state on it.

An expected but telling analogy arises from the fact that von Neumann himself used thermodynamical language and thermodynamical considerations to speak about the *a priori* ensemble, which immediately brings to mind the thermodynamical origin of the KMS condition. In the sequel of his work, von Neumann, who had to stick to the frequency interpretation of probability, was forced to remove some important assumptions about the *a priori* ensemble. Thus, already in Ref. [192] he drops a phrase which in Ref. [190] reads,

The basis of a statistical investigation is always that one has an “elementary unordered ensemble”  $\{S_1, S_2, \dots\}$ , in which “all conceivable states of the system  $S$  occur with equal relative frequency;” one must associate the distribution of values on this ensemble with those systems  $S$ , on the states of which one does not have any specific knowledge.

As Rédei argues, von Neumann was moved to reject this language because of its inconsistency with his view on probabilities as relative frequencies (in the theory appear infinite probabilities that cannot be interpreted as frequencies). Meanwhile,

nothing precludes from safeguarding the original reasoning if one chooses some other philosophy of probability, e.g. subjective probabilities [162].

To clarify the parallel, let us now give the main consequence of the existence of the *a priori* ensemble in von Neumann's derivation of the statistical Ansatz. Facing the clash between the necessary but infinite *a priori* probability and the frequency interpretation, von Neumann was left with one option only, which was to consider the appearance of infinite, non-normalizable *a priori* probabilities as a pathology of the Hilbert space quantum mechanics and to try to work out a well-behaving non-commutative probability theory, one in which there exists normalized *a priori* probability or, as says von Neumann, "a priori thermodynamic weight of states." This program was successfully completed by classification of factors and the discovery of the type  $II_1$  factor. Indeed, on the lattice of a type  $II_1$  factor the needed probability exists and is given by the trace.

How deeply von Neumann became disillusioned in the Hilbert space quantum mechanics is especially clear from his 1935 letter to Birkhoff [150, p. 112]:

I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more. After all Hilbert space (as far as quantum mechanical things are concerned) was obtained by generalizing Euclidean space, footing on the principle of "conserving the validity of all formal rules"... Now we begin to believe that it is not the *vectors* which matter, but the lattice of all linear (closed) subspaces. Because: 1) The vectors ought to represent the physical *states*, but they do it redundantly, up to a complex factor only, 2) and besides, the states are merely a derived notion, the primitive (phenomenologically given) notion being the qualities which correspond to the *linear closed subspaces*. But if we wish to generalize the lattice of all linear closed subspaces from a Euclidean space to infinitely many dimensions, then one does not obtain Hilbert space, but that configuration which Murray and I called "case  $II_1$ ." (The lattice of all linear closed subspaces of Hilbert space is our

“case  $I_\infty$ .”)

Von Neumann’s repetitive reference to the “a priori thermodynamic weight of states” now gets a clear meaning: the usual trace on an infinite dimensional Hilbert space gives a thermodynamic weight via the *a priori* unordered ensemble, but this trace does not exist as a finite quantity. To have a finite *a priori* thermodynamic weight of states, von Neumann proposes to switch from the type  $I_\infty$  factor algebras, which are just collections of all closed linear subspaces of an infinite dimensional Hilbert space, to type  $II_1$  factor algebras. Note that, as Rédei notices, “a priori” in the context of type  $II_1$  factors acquires a new meaning: it reflects the symmetry of the system. Indeed, Equation 8.2 arises from the fact that the trace is a unique positive linear normalized functional on a type  $II_1$  factor that is invariant with respect to all unitary transformations. The meaning of “a priori” as reflecting symmetries of the system immediately reminds of the transcendental view of quantum physics [15, 138].

Unfortunately, having made the first step right, von Neumann made a wrong second step: type  $II_1$  algebras do not make things easier in quantum theory. We now explain the modern alternative von Neumann’s views.

### 8.3 An interpretation of the local algebra theory

Development of the algebraic quantum theory that followed the early work by von Neumann showed that quantum theory as type  $II_1$  von Neumann algebra is not a viable solution. Algebras in the quantum theory of infinite systems, i.e. quantum field theory, involve factors of type  $III$  and, further, of subtype  $III_1$  (see Table 7.2); an extended argument for this was given by Haag [78]. For our approach this means that some of the assumptions that have led, following von Neumann’s path, to favoring type  $II_1$  factors must be rejected as biased. It is now time to change the attitude: in this section we assume the formal results of the local algebra theory briefly presented on page 116 and we give them an information-theoretic interpretation. Such an interpretation will then allow to treat these results as theorems deriving the formalism of quantum theory in the context of the information-theoretic approach. To state

clearly the goal of this section, it is to discuss the theory of local algebras and to give to the algebraic approach a novel justification, but without presenting any novel mathematical results.

The most natural critique of the chain of assumptions that have led to von Neumann's erroneous preference for type  $II$  algebras is of course to say that, while the selection of a  $C^*$ -algebra with a state over it as formal counterparts of the notions of system and information was perhaps justified, the point about no *a priori* knowledge is questionable. This is indeed Rédei's position. We now show that the former selection itself contains no fewer built-in assumptions than the latter one.

When one starts to build a theory by choosing a  $C^*$ -algebra and by saying that a linear positive functional on it corresponds to the notion of state, one commits himself to a great deal of presupposed structure. This is manifest in the fact that, with the help of the GNS construction, a  $C^*$ -algebra and a faithful state on it give rise to the representation in a Hilbert space. To compare, the whole quantum logical enterprise of Section 6.3 aimed at obtaining the Hilbert space. In the  $C^*$ -algebraic approach, as a consequence of the postulated linearity and positivity, it is given for free.

What are the essential inputs that one adheres to in choosing a  $C^*$ -algebra and a state over it? The first such input is the structure of the  $C^*$ -algebra itself. This can be weakened to Jordan-Banach or to Segal algebras, which then leads to losing much of the deductive power of the theory. The second input is more peculiar and often overlooked. As hinted above, it lies in saying that physical states are states over the algebra, while states are defined as linear positive functionals. Both these properties of states: linearity and positivity, are to be justified from the general information-theoretic principles. It appears that there are no arguments coming from within the theory that could be used to this purpose. Furthermore, in the spirit of Section 4.2, one would like to justify why no such arguments are available. States, as argued in that section, are relative states and require a reference to the observing system. In Schrödinger's language [164], the quantum state is the most compact representative of expectation catalogues that give lists of results the observer may obtain for the specific

observable he may choose to measure. We say, using our terminology, that it is just a catalogue of all relevant information available to I-observer. Consequently, linearity or any other property of states can only arise from the consideration of particular properties of the I-observer. The theory of I-observer belongs to meta-theory of the information-based physical theory, and therefore one needs a different loop cut (Figure 2.3) to justify linearity or positivity of states. If one looks at information as being based on some physical support, then one will possibly deduce the necessary properties of information states; but such a point of view is complementary to the one that had been chosen throughout all of the previous discussion, i.e. to treating physics as based on information.

As argued above, linearity and positivity of states cannot be justified in the loop cut of Figure 2.2. In the quantum logical approach there was only one notion that could not be so justified: relevance of information. Algebraic approach, by treating states on the algebra as information states, uses at least two properties that remain unjustified from within the theory. In this sense, quantum logical approach goes somewhat deeper into the structure of quantum theory, because it assumes less: it aims at explaining, not only why the theory on the Hilbert space is quantum rather than classical, but also why the Hilbert space itself emerges based on only one meta-theoretic definition of relevance. In the algebraic approach, if one postulates linearity and positivity, one then immediately obtains the Hilbert space in virtue of the GNS construction 7.4.

Let us now return to the information-theoretic justification of the theory of local algebras. We have seen how the fundamental notions of system, information and fact receive their respective mathematical meanings. It is now time to ask how one can make sense of the information-theoretic Axioms I and II of Section 4.4 and of Axiom III of Section 4.5.

To start the discussion, before going to the first axiom, we observe that our interpretation of the fundamental notions already justifies the passage from a  $C^*$ -algebra to a von Neumann algebra in case I-observer has some (or none) information about

the system. Information is represented as a state over the algebra, and via the GNS construction one obtains a representation of  $\mathcal{A}$  in a Hilbert space  $\mathcal{H}$ . Definition 7.9 then introduces a folium of  $\omega$ , which determines a weak topology on  $\mathcal{A}$ . By closing  $\mathcal{A}$  under this weak topology, as explained in Remark 7.10, we obtain a von Neumann algebra  $\mathcal{M}$ . Therefore, with each state over a  $C^*$ -algebra one associates a von Neumann algebra. In the theory of local algebras the algebra in question is normally a von Neumann, and not a  $C^*$ -, algebra, and we wish to remove the state-dependence of the definition of a von Neumann algebra by a  $C^*$ -algebra. This can be achieved, for example, by considering the universal enveloping von Neumann algebra of a  $C^*$ -algebra [179, p. 120]. However, although we were able to give information-theoretic justification of the passage from a  $C^*$ -algebra to the state-dependent von Neumann algebra, we do not know whether such a justification exists for replacing  $C^*$ -algebra with a von Neumann algebra with regard to representation of the notion of system; and, on the other hand, this is exactly what is required if one considers a von Neumann algebra in a manner independent of the state. All we can say at this stage is that, in the same *fiat* way in which we postulated that the fundamental notion of system is formally represented by a  $C^*$ -algebra, one may postulate that it is represented by a von Neumann algebra.

As a consequence of the above discussion, where necessary we shall take the algebra to be a von Neumann algebra. Let us now give sense in the algebraic formalism to Axiom I. We have the freedom to choose an algebraic meaning for the phrase “amount of relevant information is finite.” If one recalls that information is associated with states on a  $C^*$ -algebra, an immediate suggestion would be to treat the amount of information as some measure on the state space and to require that this measure be finite. Note that such a proposal ignores the presence of the adjective “relevant” before the term “information.” Now, if one follows the named path, then a seemingly natural candidate is the function  $d$  used in Theorem 7.17 for classification of von Neumann factors. However, this function is defined on projections, and in our current framework information and facts correspond not to a particular kind of operators within the  $C^*$ -

algebra, but to the states on the algebra. Also, to require that  $d$  take finite values would mean a restriction to type  $I_n$  or type  $II_1$  algebras and would exclude quantum field theories, as it was previously the case with von Neumann's derivation of quantum mechanics. We need something else.

Our choice of translation of Axiom I into the algebraic terms is to require that the von Neumann algebra representing the system be hyperfinite. Fell [59] showed that a folium of the faithful representation  $\pi_\omega$  of a  $C^*$ -algebra  $\mathcal{A}$  is weakly dense in the set of all states over  $\mathcal{A}$ . Therefore, in the context of the  $C^*$ -algebraic approach, with only a finite amount of relevant information, we can never find out if the state belongs to the given folium. This, in turn, means that the theory, generically, cannot tell us which information states are the possible states, once a particular von Neumann algebra had been chosen. However, we want to preserve this capacity of the theory as it is an essential component of its predictive power. To do so, we extend the theory beyond finite amounts of information and consider "infinite amounts" of information, the quotation marks meaning that some of this information will necessarily be irrelevant for I-observer. Let us reiterate that it is crucial to be in position to respond to the above discussed question, i.e. to determine if the state belongs to the folium of another state. This is because it is only by comparing the previously possessed with the incoming information that one can decide if representation of the system as a given von Neumann algebra holds or if the folium on the  $C^*$ -algebra has changed and the corresponding weak closure, giving a von Neumann algebra, has changed too. To compare information means to compare the states, and one is then forced not limit the  $C^*$ -algebraic approach to only one equivalence class of representations.

Now, once we have decided to take into consideration the full variety of the representations of  $\mathcal{A}$ , we must make sure that, by the acts of bringing about more information, we shall be able to approach this theoretic idealization with a sufficiently high precision; or otherwise the theory would contain a surplus that could be removed from it without damaging its information-theoretic content. Compare this idea with the requirement of absence of superselection rules in the quantum logical

approach (see pages 85 and 90). Absence of superselection rules was postulated, in order to guarantee that to every projector on a closed subspace of the Hilbert space corresponds a question in  $W(P)$  and that there are no such subspaces about which information can never be brought about. In other words, only such elements are considered that fall in the domain of *possible information*, in the spirit of the quotation from Bohr given on page 79. Similarly with the algebraic formalism: only that now the surplus to be avoided are those states which cannot be approached with a finite amount of information. We require that, in the weak  $*$ -topology, the precision of state detection shall tend to infinitely high in the limit of the infinite number of acts of bringing-about information. This, in turn, means that we require that  $\mathcal{A}$  be a limit of finite dimensional algebras, i.e. a hyperfinite algebra. If one only considers type *III* algebras, as dictated by the local algebras' theory, one can say that the algebra must be *the* hyperfinite algebra, in virtue of Theorem 7.21.

At the same time, the requirement of hyperfiniteness will guarantee that we have fully observed Axiom II. To satisfy the constraint of this axiom, and because information is mathematically represented as a state over the algebra, we ought to make sure that, by the acts of bringing about information, one can always change folium and thus switch to a representation of the  $C^*$ -algebra that is not equivalent to the previous one. Hyperfiniteness supplies precisely what is needed: the algebra is sufficiently rich so that one can always change folium and bring in novel information, but at the same time, because there is only one hyperfinite algebra of each of the types *II* and *III*<sub>1</sub>, the algebra will remain the same, and, in accordance with Axiom I, one will be able to come infinitely close to it by pursuing a chain of finite dimensional algebras. Thus hyperfiniteness is a unique balance between two constraints: that there be non-equivalent representations defining different folia and that one could get information with any degree of precision from a finite sequence of facts.

To move now to the discussion of Axiom III, its meaning is not significantly different from what we have had in the quantum logical reconstruction. In virtue of presence of  $\sigma$ -additivity in von Neumann algebras, Gleason's Theorem 6.18 is

applicable so as to justify the probabilistic interpretation and the Born rule. In the same sense as in the quantum logical formalism, Gleason's theorem gives rise to the state space with the Born rule.

Now that the choice of the hyperfinite von Neumann algebra in the theory of local algebras has been given an information-theoretic interpretation, we explore in the next section the question that was studied in Section 6.4 in the context of the quantum logical approach; namely, the problem of quantumness of the algebra. For this, we analyze the only existing, as of today, attempt at information-theoretic *derivation* of quantum theory by means of the algebraic formalism.

## 8.4 CBH derivation program

Clifton, Bub and Halvorson (CBH) [34] and Halvorson [83] proved a series of results, gathered under the title “CBH theorem,” showing equivalence between certain information-theoretic constraints and the algebraic properties possessed by quantum  $C^*$ -algebras. CBH show, for a composite system,  $\mathcal{A} + \mathcal{B}$ , consisting of two component subsystems,  $\mathcal{A}$  and  $\mathcal{B}$ , that (i) the requirement of ‘no superluminal information transfer via measurement’<sup>†</sup> entails that the  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  whose self-adjoint elements are the observables  $A$  and  $B$ , commute with each other (i.e. all  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$  commute; this is also called the condition of kinematic independence), and (ii) the condition of ‘no broadcasting’ of a quantum state entails that  $\mathcal{A}$  and  $\mathcal{B}$  separately are noncommutative. Then, adding an independence condition for the algebras, they show the existence of nonlocal entangled states on the  $C^*$ -algebra  $\mathcal{A} \vee \mathcal{B}$  that  $\mathcal{A}$  and  $\mathcal{B}$  jointly generate. This guarantees the presence of nonlocal entangled states in the mathematical formalism used in the theory, but does not yet guarantee that these states, a resource available mathematically, are actually instantiated. In his second paper Halvorson shows that the third information-theoretic constraint, ‘no bit commitment’, delivers this missing component, thus completing the proof of the CBH

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<sup>†</sup>We use single quotes instead of double quotes as elsewhere in the text to preserve the original choice by the authors of the CBH article, for whom this phrase clearly has more of a literal, i.e. empirical, and not simply a metaphoric, sense.

theorem.

We first discuss the significance of information-theoretic constraints used in the CBH theorem. The sense of the ‘no superluminal information transfer’ constraint, the term being chosen by CBH, is that when Alice and Bob (conventional names for physical systems) perform local measurements, Alice’s measurements can have no influence on the statistics for the outcomes of Bob’s measurements, and vice versa. CBH go on to say that “otherwise this would mean instantaneous information transfer between Alice and Bob” and “the mere performance of a local measurement (in the nonselective sense) cannot, in and of itself, transfer information to a physically distinct system.” Upon reading these statements, one has a feeling that for CBH *distinct* and *distant* are synonyms, and it is this very issue that we shall explore. CBH explain to their reader that the  $C^*$ -algebraic framework includes not only the conventional quantum mechanics, but also quantum field theories; we add that it also includes generally covariant settings, i.e. theory on a manifold. In all of these, one has to deal with  $C^*$ -algebras. However, neither in quantum mechanics or quantum field theory formulated as timeless theories [159], nor in the generally covariant formalism, there exist space and time that play any special role. If one wishes to give information-theoretic axioms from which to derive the quantum  $C^*$ -algebraic framework, one must not assume the spatiotemporal structure; indeed, only in some particular cases of hand-picked  $C^*$ -algebras will one be able to single out the preferred notion of time.

We shall offer several critical points concerning the CBH theorem. For this, let us have a closer look at how the authors’ language is reflected in their mathematical formalism. They give the following definition:

**Definition 8.1** ([34, Section 3.2]). Operation  $T$  on algebra  $\mathcal{A} \vee \mathcal{B}$  conveys no information to Bob if

$$(T^*\rho)|_{\mathcal{B}} = \rho|_{\mathcal{B}} \text{ for all states } \rho \text{ of } \mathcal{B}. \quad (8.3)$$

An operation here is understood as a completely positive linear map on algebra

$\mathcal{A}$  and  $T^*\rho$  is a state over the algebra defined for every state  $\rho$  on the *same* algebra  $\mathcal{A}$  as

$$(T^*\rho)(A) = \frac{\rho(T(A))}{\rho(T(I))} \quad (8.4)$$

at the condition that  $\rho(T(I)) \neq 0$ . Nonselective measurements  $T$  are the ones that have  $T(I) = I$ , and then  $\rho(T(I)) = \rho(I) = \|\rho\| = 1$ . CBH explain that, in their view, Definition 8.1 entails

$$T(B) = B \text{ for all } B \in \mathcal{B}. \quad (8.5)$$

CBH then assert that if the condition (8.5) holds for all self-adjoint  $B \in \mathcal{B}$  and for all  $T$  of the form

$$T = T_E(A) = E^{1/2}AE^{1/2} + (I - E)^{1/2}A(I - E)^{1/2}, \quad (8.6)$$

where  $A \in \mathcal{A} \vee \mathcal{B}$  and  $E$  is a positive operator in  $\mathcal{A}$ , then algebras  $\mathcal{A}$  and  $\mathcal{B}$  are kinematically independent [34, Theorem 1]. CBH seek for kinematic independence of algebras in order to show that the algebras of two distinct systems commute, and this is derived from the assumption of  $C^*$ -independence and from the condition (8.3), where  $C^*$ -independence is brought into the discussion to grasp the meaning of the fact that systems  $\mathcal{A}$  and  $\mathcal{B}$  are distinct. Mathematically,  $C^*$ -independence means that for any state  $\rho_1$  over  $\mathcal{A}$  and for any state  $\rho_2$  over  $\mathcal{B}$  there is a state  $\rho$  over  $\mathcal{A} \vee \mathcal{B}$  such that  $\rho|_{\mathcal{A}} = \rho_1$  and  $\rho|_{\mathcal{B}} = \rho_2$ .  $C^*$ -independence does not follow from and does not entail kinematic independence. In the CBH paper, Definition 8.1 is equated with the ‘no superluminal information transfer by measurement’ constraint. The term “superluminal” is an evident spatiotemporal concept designating velocities that exceed the speed of light. In the discussion of this constraint, however, no light quanta or any other carriers that actually transfer information are considered and indeed no space-time at all is necessary: the mathematics involved is purely algebraic. Then, the question is whether one could give a different meaning to this condition, without bringing in spatiotemporal concepts that do not naturally belong to the language of the algebraic approach. Before suggesting an answer to this question, we stop to

present two critical points concerning Definition 8.1 and its discussion in the CBH paper.

Our first critique is connected with the phrasing of Definition 8.1 itself. If, following the CBH authors, in this definition  $\rho$  is to be taken as a state over  $\mathcal{B}$ , then the definition does not make sense: operation  $T$  is defined on  $\mathcal{A} \vee \mathcal{B}$  and consequently, in accordance with (8.4),  $T^*\rho$  is defined for the states  $\rho$  over  $\mathcal{A} \vee \mathcal{B}$ . If one follows the CBH definition with a state  $\rho$  over  $\mathcal{B}$ , then there would be no need to write  $\rho|_{\mathcal{B}}$  as CBH do, for a simple reason that  $\rho|_{\mathcal{B}} = \rho$ . To suggest a remedy, we extend the reasoning behind this definition and reformulate it in three alternative ways.

- The first one is to require that in Definition 8.1 the state  $\rho$  be a state over the algebra  $\mathcal{A} \vee \mathcal{B}$ .
- The second alternative is to consider states  $\rho$  on  $\mathcal{B}$  but to require a different formula, namely that  $(T|_{\mathcal{B}})^*\rho = \rho$  as states over  $\mathcal{B}$ .
- Finally, the third alternative proceeds as follows: Take arbitrary states  $\rho_1$  over  $\mathcal{A}$  and  $\rho_2$  over  $\mathcal{B}$  and, in virtue of  $C^*$ -independence, consider the state  $\rho$  over  $\mathcal{A} \vee \mathcal{B}$  such that its marginal states are  $\rho_1$  and  $\rho_2$  respectively. Then  $T^*\rho$  is also a state over  $\mathcal{A} \vee \mathcal{B}$ . If its restriction  $(T^*\rho)|_{\mathcal{B}}$  is equal to  $\rho_2$ , then  $T$  is said to convey no information to Bob.

With the original formulation of Definition 8.1, proof of Equation 8.5 is problematic. We show how to prove this equation with each of the three alternative definitions. First observe the following remark.

*Remark 8.2.* Each  $C^*$ -algebra has sufficient states to discriminate between any two observables (i.e., if  $\rho(A) = \rho(B)$  for all states  $\rho$ , then  $A = B$ ).

To justify (8.5), the CBH authors then say:

$(T^*\rho)|_{\mathcal{B}} = \rho|_{\mathcal{B}}$  if and only if  $\rho(T(B)) = \rho(B)$  for all  $B \in \mathcal{B}$  and for all states  $\rho$  on  $\mathcal{A} \vee \mathcal{B}$ . Since all states of  $\mathcal{B}$  are restrictions of states on  $\mathcal{A} \vee \mathcal{B}$ , it follows that  $(T^*\rho)|_{\mathcal{B}} = \rho|_{\mathcal{B}}$  if and only if  $\omega(T(B)) = \omega(B)$  for all states  $\omega$  of  $\mathcal{B}$ , i.e., if and only if  $T(B) = B$  for all  $B \in \mathcal{B}$ .

Let us examine this derivation under each of the three alternative definitions of conveying no information. By the definition of  $T^*$ , we have  $(T^*\rho)(B) = \rho(T(B))$  for all states  $\rho$  over  $\mathcal{A} \vee \mathcal{B}$ . To obtain from this that  $\rho(T(B)) = \rho(B)$ , one must show that  $(T^*\rho)(B) = \rho(B)$ , and this is equivalent to saying that  $(T^*\rho)|_{\mathcal{B}} = \rho|_{\mathcal{B}}$  for all states  $\rho$  over  $\mathcal{A} \vee \mathcal{B}$ . Now, according to CBH, one would need to show that  $\rho(T(B)) = \rho(B)$  if and only if  $\omega(T(B)) = \omega(B)$  with states  $\rho$  over  $\mathcal{A} \vee \mathcal{B}$  and  $\omega$  over  $\mathcal{B}$ . The latter formula, however, is not well-defined: operator  $T(B)$ , generally speaking, is not in  $\mathcal{B}$ . Fortunately, we are salvaged by the first alternative reformulation of Definition 8.1: because  $\rho(T(B)) = \rho(B)$  is true for all states  $\rho$  over  $\mathcal{A} \vee \mathcal{B}$ , we obtain directly that  $T(B) = B$  in virtue of Remark 8.2.

The second alternative definition of conveying no information makes use of an object such as  $(T|_{\mathcal{B}})^*\rho$ . To give it a meaning in the algebra  $\mathcal{B}$ , one needs to impose a closure condition on the action of  $T$  on operators  $B \in \mathcal{B}$ : namely, that  $T$  must not take operators out of  $\mathcal{B}$ . The problem here is the same as the one we encountered in the discussion of the previous alternative, and it is only by assuming the closure condition that one is able to obtain that  $T(B) = B$ .

In the third alternative, for the state  $\rho$  over  $\mathcal{A} \vee \mathcal{B}$ , write from the definition of  $T^*$  that  $(T^*\rho)(B) = \rho(T(B))$ . The result  $(T^*\rho)(B)$  is the same as  $(T^*\rho)|_{\mathcal{B}}(B)$ , and this is equal to  $\rho_2(B)$ . Consequently,  $\rho(T(B)) = \rho_2(B) = \rho(B)$ . Can we now say that this holds for all states  $\rho$  over  $\mathcal{A} \vee \mathcal{B}$ ? The answer is obviously yes, and this is because each state over  $\mathcal{A} \vee \mathcal{B}$  can be seen as an extension of its own restriction to  $\mathcal{B}$ . Therefore, one has to modify Definition 8.1 for it to be formally correct, and this entails a modification in the proof of Equation 8.5.

The second critique of the CBH program has to do with postulating  $C^*$ -independence. Notions of independence of algebras are a legion [62]; why, then, take  $C^*$ -independence as a mathematical representation of the distinction between the systems? For this we must look back at the origins of the notion of  $C^*$ -independence. It was first introduced in Ref. [79] under the name of statistical independence; this was due to the fact that Haag and Kastler wanted to give a mathematical meaning to the

ability to prepare any states on two algebras by the same preparation procedure. As Florig and Summers importantly note, if one has an entangled pair, then it generates  $C^*$ -independent algebras that are not kinematically independent. Now read again the phrase from the CBH article that we have already quoted: The sense of the ‘no superluminal information transfer’ constraint is that “when Alice and Bob perform local measurements, Alice’s measurements can have no influence on the statistics for the outcomes of Bob’s measurements.” So which is the statistical independence:  $C^*$ -independence or the ‘no superluminal information transfer’ constraint? This is where we have to look at the meaning of the mysterious term “superluminal” that in the CBH case has nothing to do with faster-than-light transfer of information. In fact, conveying no information as defined in 8.1 does not prohibit only superluminal communication; it prohibits all information transfer whatsoever. The real meaning of the CBH condition is thus that nonselective POV measurements can convey no information to Bob at all. As for selective measurements, the authors themselves grant that they “trivially change the statistics of observables measured at a distance, simply in virtue of the fact that the ensemble relative to which one computes the statistics has changed.”

Now, if the operation  $T$  is nonselective, the most important thing that does not change is that the identity operator remains in the image of  $T$ . Presence of the identity is a *sine qua non* for all algebras in the CBH paper. However, if the identity is present in the algebra, the latter becomes quite special; for instance, according to Theorem 7.12, requiring that the algebra be unital is a first step on the way to von Neumann algebras. More seriously, which operators are included in  $\mathcal{B}$  determines Bob’s observational capacities. Consider, for example, Alice and Bob as two entangled particles; then the identity will generally not be a part of their algebras. In an example from Ref. [62], the following operators on the 6-dimensional complex Hilbert space

are considered:

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}. \quad (8.7)$$

Each of these operators generates a  $C^*$ -algebra. These algebras  $\mathcal{E}$  and  $\mathcal{F}$  are  $C^*$ -independent but evidently do not commute. They also do not contain the identity. According to the CBH view, the entangled systems  $\mathcal{E}$  and  $\mathcal{F}$  are distinct, but the transfer of information by measurement is possible between them. The general form of operation  $T$  acting on operators from  $\mathcal{E}$  and  $\mathcal{F}$  is to leave the diagonal elements untouched and to nullify all others, so it does not preserve the form of  $B$ . One can now see that the notion of system in the CBH understanding is quite peculiar: by requiring kinematic independence, they for example contradict Rovelli's requirement (see Section 4.2) that everything be equally treated as physical system. They indeed see a  $C^*$ -algebra as a collection of operators "sitting" in some place, that includes the identity as the operator that corresponds to *doing nothing* on the part of the observer. In other words, to be  $C^*$ -independent is not enough for being distinct: there has to be a supplementary intuitive assumption of the local identity of systems made along the way. In Rovelli's sense, state on an algebra and the information that it reflects are observer-dependent concepts; then the point of the first CBH constraint is to say that the information obtained in measurement can either be possessed exclusively by Alice or exclusively by Bob, i.e. the observer who performs the measurement in question and who obtains the new fact in which information is brought about.

In an attempt to escape from the above identified intuitive assumption, let us reformulate the CBH mathematical results, which we fully endorse, in a different language. As a possible additional assumption to  $C^*$ -independence, one can directly *postulate* that to be physically distinct means to be kinematically independent. Then, to derive kinematic independence would amount to explaining what it means to be physically distinct, based on the statistical independence; and this will be the meaning

of Definition 8.1. A methodological argument for this latter choice goes as follows:  $C^*$ -independence is a notion that relies on the notion of state. In the conceptual framework of Section 8.1, the notion of state represents information that I-observer has about the system, while the notion of operator, which is an element of a  $C^*$ -algebra, contributes to the definition of the system as such. As we have seen, for the CBH, too, the choice of operators that are included in the  $C^*$ -algebra is crucial for comprehension of the concept of observer. It is then natural to require that the fact that two systems are distinct be expressed, first of all, in the same language as used to define what a system is; i.e. in the language of the  $C^*$ -algebraic constituent operators and not the one of the states.

Only after one had postulated what it means for two physical systems represented as  $C^*$ -algebras to be distinct, it comes without surprise that in order to establish this difference between the two systems practically, one will appeal to constraints on how information about one system relates to information about the other. Further, because the notion of information has so reemerged and because information is represented by states on the algebra, one expects a definition in terms of states; and indeed Definition 8.1 speaks the language of states.

Let us now clarify what we formally mean by *distinct physical systems*.

**Definition 8.3.** Two systems represented as  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  are distinct if  $\forall A \in \mathcal{A}, B \in \mathcal{B} [A, B] = 0$ . In the standard terminology, we say that, by definition, systems are physically distinct if they are kinematically independent.

The meaning of the notion of distinct physical systems here becomes operational. This is due to the following theorem which rephrases the first theorem by CBH:

**Theorem 8.4 (information-theoretic criterion for two systems to be physically distinct).** *If all POV measurements on system  $\mathcal{A}$  provide no information on system  $\mathcal{B}$  (in the sense of Definition 8.1), then systems  $\mathcal{A}$  and  $\mathcal{B}$  are physically distinct.*

With the reformulations 8.3 and 8.4 of the CBH result, we have liberated the

discussion from the spatiotemporal language that appeared in the usage of terms like “superluminal” or “local” and that does not belong to the natural language of algebra. The term “locality” was introduced in the theory of algebraic independence conditions by Kraus [110, 111], who formulated the condition of strict locality for  $W^*$ -algebras that we do not present here to avoid heaping too many definitions. Under the assumption of kinematic independence, strict locality is equivalent to  $C^*$ -independence [62, Proposition 9]. In our language, this means that if two systems are distinct, then strict locality would be equivalent to statistical independence: a strange condition that links together words belonging to different vocabularies. Indeed, algebra is the mathematical science of structure, and that “ $A$  is distinct from  $B$ ” is a perfectly structural claim that need not refer to spacetime concepts like locality. One then sees that the strangeness arises from the use of the term “locality,” and it is this use that must be questioned.

The second CBH information-theoretic constraint is the ‘no broadcasting’ condition whose aim is to establish that algebras  $\mathcal{A}$  and  $\mathcal{B}$ , taken separately, are non-Abelian. Broadcasting is defined as follows:

**Definition 8.5** ([34, Section 3.3]). Given two isomorphic, kinematically independent  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ , a pair  $\{\rho_1, \rho_2\}$  of states over  $\mathcal{A}$  can be broadcast in case there is a standard state  $\sigma$  over  $\mathcal{B}$  and a dynamical evolution represented by an operation  $T$  on  $\mathcal{A} \vee \mathcal{B}$  such that  $T^*(\rho_i \otimes \sigma)|_{\mathcal{A}} = T^*(\rho_i \otimes \sigma)|_{\mathcal{B}} = \rho_i$ , for  $i = 0, 1$ . A pair  $\{\rho_1, \rho_2\}$  of states over  $\mathcal{A}$  can be cloned just in case  $T^*(\rho_i \otimes \sigma) = \rho_i \otimes \rho_i$  ( $i = 0, 1$ ).

Equivalence between the ‘no broadcasting’ condition and non-Abelianness of the  $C^*$ -algebra is then derived from the following theorem:

**Theorem 8.6.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two kinematically independent  $C^*$ -algebras. Then:*

- (i) *If  $\mathcal{A}$  and  $\mathcal{B}$  are Abelian then there is an operation  $T$  on  $\mathcal{A} \vee \mathcal{B}$  that broadcasts all states over  $\mathcal{A}$ .*
- (ii) *If for each pair  $\{\rho_1, \rho_2\}$  of states over  $\mathcal{A}$ , there is an operation  $T$  on  $\mathcal{A} \vee \mathcal{B}$  that broadcasts  $\{\rho_1, \rho_2\}$ , then  $\mathcal{A}$  is Abelian.*

It is an interesting fact that in the section where broadcasting is discussed, although it, too, is a term with explicit spatiotemporal connotations, the authors never refer to broadcasting as actually transferring information in space. Such is not the case with the two other information-theoretic constraints. It is perhaps due to the fact that initial intention was to use the ‘no cloning’ condition, with the word “cloning” being free of spatial connotations. However, one fact deserves closer attention: that non-Abelianness of the algebras  $\mathcal{A}$  and  $\mathcal{B}$ , taken one by one, is proved by assuming that they are kinematically independent. It means that quantumness, of which non-Abelianness is a necessary ingredient, is not a property of any given system taken separately, as if it were the only physical system in the Universe, but in order to derive the quantum behaviour, one must consider the system in the context of at least one other system that is physically distinct from the first one. As a consequence, for example, this forbids the possibility of treating the whole Universe as a quantum system, echoing our remark on page 50. For the remainder of the discussion of the second constraint we agree with the conclusions made by the CBH authors.

The third, ‘no bit commitment’ constraint is discussed in Section 3.4 of Ref. [34]. The section opens with the following claim:

We show that the impossibility of unconditionally secure bit commitment between systems  $\mathcal{A}$  and  $\mathcal{B}$ , in the presence of kinematic independence and noncommutativity of their algebras of observables, entails nonlocality: spacelike separated systems must at least sometimes occupy entangled states. Specifically, we show that if Alice and Bob have spacelike separated quantum systems, but cannot prepare any entangled state, then Alice and Bob can devise an unconditionally secure bit commitment protocol.

This citation essentially involves spatiotemporal terms. One is then tempted to analyze the CBH proof so as to enlist the occurrences of formal space-time considerations in it. The derivation starts by showing that quantum systems are characterized by the existence of non-uniquely decomposable mixed states: a  $C^*$ -algebra  $\mathcal{A}$  is non-Abelian if and only if there are distinct pure states  $\omega_{1,2}$  and  $\omega_{\pm}$  over  $\mathcal{A}$  such that

$\frac{1}{2}(\omega_1 + \omega_2) = \frac{1}{2}(\omega_+ + \omega_-)$ . This result is used to prove a theorem showing that a certain proposed bit commitment protocol is secure if Alice and Bob have access only to classically correlated states (i.e. convex combinations of product states).

**Theorem 8.7 (the CBH ‘no bit commitment’ theorem).** *If  $\mathcal{A}$  and  $\mathcal{B}$  are non-Abelian then there is a pair  $\{\rho_0, \rho_1\}$  of states over  $\mathcal{A} \vee \mathcal{B}$  such that:*

1.  $\rho_0|_{\mathcal{B}} = \rho_1|_{\mathcal{B}}$ .
2. *There is no classically correlated state  $\sigma$  over  $\mathcal{A} \vee \mathcal{B}$  and operations  $T_0$  and  $T_1$  performable by Alice such that  $T_0^*\sigma = \rho_0$  and  $T_1^*\sigma = \rho_1$ .*

From this theorem the authors deduce that the impossibility of unconditionally secure bit commitment entails that “if each of the pair of *separated*<sup>†</sup> physical systems  $\mathcal{A}$  and  $\mathcal{B}$  has a non-uniquely decomposable mixed state, so that  $\mathcal{A} \vee \mathcal{B}$  has a pair  $\{\rho_0, \rho_1\}$  of distinct classically correlated states whose marginals relative to  $\mathcal{A}$  and  $\mathcal{B}$  are identical, then  $\mathcal{A}$  and  $\mathcal{B}$  must be able to occupy an entangled state that can be transformed to  $\rho_0$  or  $\rho_1$  at will by a local operation.” The term “separated” is essential and, nevertheless, its precise meaning is not defined in the CBH article. In Theorem 8.7 one requires that algebras  $\mathcal{A}$  and  $\mathcal{B}$  be non-Abelian. This latter fact is taken as a consequence of Theorem 8.6, which, in turn, requires that algebras  $\mathcal{A}$  and  $\mathcal{B}$  be kinematically independent. So the meaning of “separated” must be no more than to say that the systems are distinct in the sense of the Definition 8.3. There are no mathematical reasons to claim, as the authors do in the above cited passage, that they have taken into account the case when Alice and Bob have “*spacelike* separated systems.” Theorem 8.7 means that if systems  $\mathcal{A}$  and  $\mathcal{B}$  are distinct and unconditionally secure bit commitment is impossible, then these systems can actually be in an entangled state. To be in an entangled state here means that information about systems  $\mathcal{A}$  and  $\mathcal{B}$  is such that any act of bringing it about will necessarily provide one with the information about the system  $\mathcal{A}$  and, *logically* linked to it, with the information about the system  $\mathcal{B}$ . At no place here enters any spatiotemporal language.

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<sup>†</sup>Our emphasis.

Note the importance of the word “actually”: in fact, presence of entangled states in the mathematical formalism has long been guaranteed by non-Abelianness and the kinematic and the  $C^*$ -independencies of algebras [176]. The CBH authors devise the whole argument in order to demonstrate that the entangled states, mathematically allowed, are *actually*—or shall we say *necessarily*—non-locally instantiated.

The authors of the CBH article then discuss a result converse to Theorem 8.7 which is arguably more interesting: namely, in their terminology, that nonlocality—“the fact that spacelike separated systems occupy entangled states”—entails the impossibility of unconditionally secure bit commitment. We have already seen that the term “nonlocality” is superfluous in the algebraic context, although for this converse result it is not an issue of first importance. The derivation relies on the availability of the Hughston-Jozsa-Wootters (HJW) theorem [95] for arbitrary  $C^*$ -algebras. The most general proof up-to-date was given by Halvorson [83]; it covers the cases of type I von Neumann factors, type I von Neumann algebras with Abelian superselection rules and the case of a  $C^*$ -algebra whose commutant is a hyperfinite von Neumann algebra. Let us stress the term *hyperfinite*. Halvorson claims that it remains an open question whether an analogue of the HJW theorem holds for general  $C^*$ -algebras that are not necessarily nuclear. Recall that nuclearity, mentioned in Section 7.1, is the cause of hyperfiniteness of the type  $III_1$  von Neumann factors, and it is equivalent to the requirement for the system to have normal thermodynamic properties. Halvorson’s desire to establish the analogue of the HJW theorem in absence of nuclearity may therefore be prevented from realization by the theory itself. The phrase “normal thermodynamic properties” means that KMS states exist for all positive  $\beta$  for the system and its finitely extended parts, and this is intimately linked to information-theoretic interpretation of the formalism of local algebras. There may exist no information-theoretic approach as such beyond the limits of applicability of the KMS condition.

We have given in Section 8.3 an information-theoretic interpretation in which hyperfiniteness is justified based on Axioms I and II. In this section we offered

critique of the extensive use of spatiotemporal notions in the CBH articles. We must now explain how space and time, instead of being postulated, can arise in the algebraic information-theoretic framework. This, in turn, will involve the KMS formalism, and hyperfiniteness as the condition of well-definedness of the KMS states will be required.

## 8.5 Non-fundamental role of spacetime

... the concepts of space and time by their very nature acquire a meaning only because of the possibility of neglecting the interactions with the means of measurement.

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Bohr [18, p. 99]

At many occasions in the history of quantum theory it has been noticed that time and the ordering of wavefunction collapses are unrelated, of which we cite two: First was the point emphasized by Dirac [44] and later discussed by Hartle [86] and Rovelli [155, 154]. The argument here is very general: The formalism of quantum mechanics allows a sequence of measurements not ordered in the time in which the system evolves. Thus, we can measure  $B(t)$  and *later* measure  $A(t')$ , with  $t' < t$ . In the standard Copenhagen interpretation we then say that the wavefunction is projected twice: *first* on the eigenstate of  $B(t)$  and *then* on the eigenstate of  $A(t')$ . This sequence of projections describes the conditional probability of finding at  $A(t')$  the system that will have been detected at  $B(t)$ . Such a probability can be understood either as subjective or as objective in terms of frequencies: none of this changes the inverse order of detection events with respect to the time in which the system evolves. In an illuminating passage following this example, Rovelli writes:

The example suggests that the ordering of the collapses is not determined by  $t$ . Rather, the ordering depends on the *question* that we want to formulate. The ordering is usually related to  $t$  only because we are more interested in calculating the future than the past.

The idea that the ordering depends on the question that we want to formulate is in full accord with the conceptual approach that we have chosen in Chapter 4, where

questions correspond to facts as acts of bringing about information. Facts, in turn, belong to fundamental notions on which rests the physical theory. Thus time ordering is secondary, and it comes without surprise that quantum theory can be formulated as *timeless* quantum theory [159, Chapter 5].

The same idea is echoed in the thought of Peres who studies the second occasion when scientists realized how little the conventional linear time means to a quantum system. Discussing quantum teleportation, Peres writes:

Alice and Bob are not real people. They are inanimate objects. They know nothing. What is teleported instantaneously from one system (Alice) to another system (Bob) is the applicability of the preparer's knowledge to the state of a particular qubit in these systems. [136]

Applicability of the preparer's knowledge is the same thing as Rovelli's "question that we want to formulate." In our approach, it corresponds to the concept of relevance of information for I-observer. Indeed, by saying that "they know nothing" Peres places Alice and Bob in the domain of purely physical, i.e. intratheoretic, and the metatheoretic function of informational agent, or I-observer, is transferred to an external "preparer." If one now returns to the fundamental view in which the von Neumann cut is put to position zero, and all systems are treated on equal grounds, then the metatheoretic function of I-observer can as well belong to Alice or to Bob, but this will not change Peres's argument: what is "teleported" is relevant information. Quotation marks mean that no information is actually instantaneously transferred, because information states, as we have emphasized, are relational, and information in question is always possessed by one I-observer only, i.e. exclusively Alice or exclusively Bob. Communication of information from Alice to Bob via a classical channel falls out of the field of interest of the information-based quantum theory with a given observer, as any other theory of communication of information between distinct informational agents requires a loop cut of Figure 2.3.

The above mentioned second occasion has to do with the long-lasting debate that was originally started by Einstein and Bohr who discussed the double-slit experiment

[56, 19], later continued by Wheeler in the form of the “delayed-choice” experiment [196], and that we present here in the version having to do with quantum information, which is called “entanglement swapping” [98, 160, 99] (Figure 8.1).

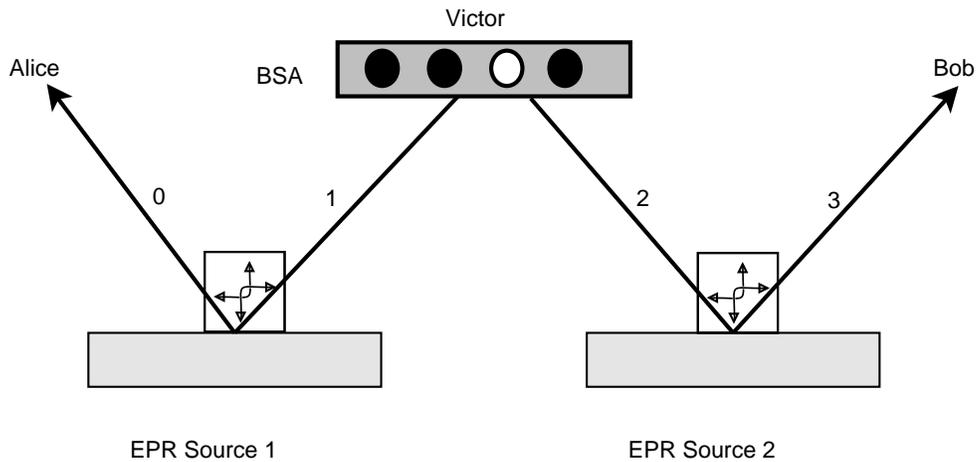


Figure 8.1: Scheme of entanglement swapping, as adopted from [23]. Two pairs of entangled particles 0-1 and 2-3 are produced by two Einstein-Podolsky-Rosen (EPR) sources. One particle from each of the pairs is sent to two different observers, say particle 0 is sent to Alice and particle 3 to Bob. The other particles 1 and 2 from each pair are sent to Victor who subjects them to a Bell-state analyzer (BSA), by which particles 0 and 3 become entangled although they may have never interacted in the past.

Contrary to the CBH paper discussed in Section 8.4, here the authors, who also employ the quantum computational language of Alice and Bob, state very clearly that their usage of terms like “locality” has nothing to do with spacetime separation. The only important factor is that Alice, Bob and Victor be distinct physical systems. Irrelevance of the temporal ordering may even give rise to seemingly paradoxical situations, like in the following passage:

It is now important to analyze what we mean by “prediction.” As the relative time ordering of Alice’s and Bob’s events is irrelevant, “prediction” cannot refer to the time order of the measurements. It is helpful to remember that the quantum state is just an expectation catalogue. Its

purpose is to make predictions about possible measurement results a specific observer does not know yet. Thus which state is to be used depends on which information Alice and Bob have, and “prediction” means prediction about measurement results they will learn in the future independent of whether these measurements have already been performed by someone or not... It is irrelevant whether Alice performs her measurement earlier in any reference frame than Bob’s or later or even if they are spacelike separated when the seemingly paradoxical situation arises that different observers are spacelike separated. [99]

It is clear from the discussion of the entanglement swapping and from Dirac’s argument given above that the concept of two *distinct* physical systems (e.g. observers) in the information-based quantum theory has very little to do with the spacetime separation between the systems. What role do then space and time play? In our program of the foundation of quantum theory, there is no place for space and time among the fundamental notions of the theory. They are, consequently, non-fundamental and need to be derived from the fundamental notions and the axioms. We propose a way to achieve this for the notion of time. As for space, we can only say that the allegedly very important role of the spatial notion of locality has been overestimated, as we intended to show in Section 8.4. In the information-theoretic approach, locality as the criterion of distinction between systems can be replaced by a different, properly information-theoretic criterion. Perhaps, a consistent mathematical approach to reconstructing space in the context of the information-theoretic approach will proceed by the methods of loop quantum gravity [159].

To return to the problem of time, the intuition here is to use the ideas from thermodynamics. Indeed, if quantum mechanics can be formulated as timeless theory, then one has to look elsewhere for reasons why time is so special a parameter. An interesting possibility [159, p. 100] is that it is the statistical mechanics, and therefore thermodynamics, that singles out  $t$  and gives it special properties.

In the algebraic approach, we have a  $C^*$ -algebra with a preferred state, giving

rise to the Hilbert space representation. One then defines a von Neumann algebra as explained in Remark 7.10, and, in a von Neumann algebra, Gleason's Theorem 6.18 is applicable so as to justify the probabilistic interpretation and the Born rule. This construction allows to build all elements of the quantum theory except unitary dynamics. As discussed in Section 6.7, in a non-generally covariant setting it is impossible to derive spacetime without introducing additional assumptions. We also know from Equation 7.19 and Proposition 7.23 that, in a non-generally covariant theory, an equilibrium state is the one whose modular group is the time translation group.

Now consider generally covariant theories. The theory is given by the hyperfinite  $C^*$ -algebra  $\mathcal{A}$  of generally covariant physical operators, states  $\omega$  over  $\mathcal{A}$  and no additional information about dynamics. Each state  $\omega$  that represents information about the system is generically impure, for it cannot but approach—recall that the amount of information is finite—the large number of the degrees of freedom allowed in a hyperfinite  $C^*$ -algebra. The hypothesis in Ref. [39] (see also [89]) is that to define time in such a case, one must look at the thermodynamics of the system. In one phrase, “time is a side effect of our ignorance of the microstate” [158]; we should like to shorten this assertion even further: time is ignorance; or yet in a third way: time is not knowing. When I-observer chooses to throw away some previously available information as irrelevant, it gives rise to time. To translate this idea into formal terms, we say that time is a state-dependent notion and is given by the modular group  $\alpha_t^\omega$  of  $\omega$  as defined in Equation (7.9). This time flow will be denoted as *thermal time*. Connes's and Rovelli's thermal time hypothesis reads:

In nature, there is no preferred physical time variable  $t$ . There are no equilibrium states  $\rho_0$  preferred *a priori*. Rather, all variables are equivalent; we can find the system in an arbitrary state  $\rho$ ; if the system is in state  $\rho$ , then a preferred variable is singled out by the state of the system.

This variable is what we call time. [159, p. 101]

The fact that time is determined by the KMS state, and therefore the system is always in thermodynamic equilibrium with respect to the thermal time flow, does not

imply that its evolution is frozen. In a quantum system with an infinite number of the degrees of freedom, what we generally measure is the effect of small perturbations around a thermal state. In other words, facts bring about new information and thereby define new states, but on the scale of the  $C^*$ -algebra of the system, each new state does not drastically differ from the old state. In a generally covariant setting, given the algebra of observables  $\mathcal{A}$  and a state  $\omega$ , the modular group gives a time flow  $\alpha_t^\omega$ . Then, the theory describes physical evolution in the state-dependent thermal time in terms of amplitudes of the form

$$F_{A,B}(t) = \omega(\alpha_t(B)A), \quad (8.8)$$

where  $A$  and  $B$  are operators in  $\mathcal{A}$ . The quantity  $F_{A,B}(t)$  is related to the probability amplitude for obtaining information pertaining to  $B$  in a fact that will be established after “waiting” for time  $t$  following a preparation  $\mathfrak{M}$ , i.e. departing from a state  $\omega_A$  that describes information about the complete knowledge of  $\mathfrak{M}$ . Time  $t$  here is the thermal time determined by the state  $\omega_A$  of the system. In a generally covariant setting the thermal time is the only definition of time available. The essence of the definition is then that the quotation marks around the word *waiting* must be removed.

In a theory in which a geometrical definition of time is assumed independently from the thermal time (as in Section 6.7), arises a problem of relating the two times. From the study of the non-relativistic limit of generally covariant theories with thermal time one obtains that the latter is proportional to geometrical time, and the temperature can be interpreted as a ratio between the two. Connes and Rovelli [39] study the non-relativistic limit, where modular time is preserved but the conventional time also becomes meaningful, and show that the modular group of Equation (7.9) and the time evolution group in the non-relativistic limit introduced in Equation (7.19) are linked:

$$\alpha_t^\omega = \gamma_{\beta t}. \quad (8.9)$$

In the spirit of Bohr’s quotation put in the epigraph to this section we must now show how from the state-dependent notion of time one can, by way of neglecting

certain information, make sense of the state-independent notion of time. It is the time of the state-independent notion of time that indexes acts of bringing about information and turns them into facts, an assumption that we made for the non-generally covariant theory in Section 4.3. Note that Bohr's words are also closely tied with our discussion in Section 4.5 of the necessity to distinguish between I-observer and P-observer. If one places himself in a world-picture in which there is no cut (Figure 2.1), then one would have to accept simultaneously that time (and space) can be derived within a physical theory, but it also determines the possibility of meta-theory of that physical theory. Both aspects of the concept of time cannot be described in a single theory, for otherwise that would render it logically circular. What Bohr says one must neglect for space and time to arise is that measurement is physical, i.e. the existence of P-observer. It corresponds to cutting the loop (Figure 2.2) and neglecting the fact the information is physical, i.e. that it has some physical support like, for instance, a human body, and thereby one will render the concept of time a topic open for a theoretic justification. We base the theory on information and we are thus uninterested, as it was the case with factoring out P-observer, in the loop cut of Figure 2.3. However, we must justify why, by neglecting information, I-observer, or the informational agent, acquires the possibility to observe a single state-independent flow of time instead of the variety of different state-dependent notions of time.

In the covariant setting, in general, the modular flow is not an inner automorphism of the algebra, namely, there is no hamiltonian in  $\mathcal{M}$  that generates it. However, as shown in Section 7.1, the difference between two modular flows is always an inner automorphism and, therefore, any modular flow projects on the same 1-parameter group of elements in  $\text{Out } \mathcal{M}$ . Consequently, the flow  $\tilde{\alpha}_t$  defined after Equation (7.11) is canonical: it depends only on the algebra itself. To factorize the states into classes of states of which modular automorphisms are inner-equivalent means to neglect information: only that information is kept which is characteristic of the class, and information that distinguishes states within the class is lost. The passage from the state-dependent modular time flow to the flow  $\tilde{\alpha}_t$  is therefore achieved via neglecting

information, in full accord with Bohr's idea.

As follows from Table 7.2, in type *I* and type *II* von Neumann algebras the canonical modular flow is frozen at modular time  $t = 0$ : indeed, evolution is unitary and no information can be brought about by the no-collapse Schrödinger dynamics. In type  $III_1$  von Neumann algebra, which corresponds to the theory of local algebras which we interpreted information-theoretically in Section 8.3, the modular time flow covers all  $\mathbb{R}_+$ , thus coinciding with the intuition of infinite linear time; but it is now the algebra that determines the “intuitive” time flow. Therefore, a von Neumann algebra contains an intrinsic dynamics, and the time needs no more to be externally postulated: indeed, it can be derived intratheoretically in the context of the information-theoretic approach, with the conceptual help of thermodynamics that belongs to meta-theory of this approach, but without any interference of thermodynamics in the actual formalism of the theory.

To conclude, let us briefly summarize the key ideas of this section. In an information-theoretic framework we start with the fundamental notions of system, information and fact. In the algebraic formalism a system is interpreted as a  $C^*$ -algebra and information is interpreted as state over this algebra. There is no space and no time yet, for we have not postulated anything like space or time. Via the KMS formalism every state gets its flow, so each information state has its own flow; we call it state-dependent time. What are the consequences?

- Time is a state-dependent concept. Unless the state is changed time does not change. A change in the state means a change in information. A change in information can be brought about in a new fact. At each fact state-dependent time “restarts.” We see that the temporality of facts (variable  $t$  that indexes facts) has nothing to do with the state-dependent notion of time.
- Thermodynamics has not played any role so far. To view a state as a KMS state at  $\beta = 1$  and to define the flow, we need not say that a state over  $C^*$ -algebra is a thermodynamical concept. Therefore, this allows to separate thermodynamics as meta-theory in the information-theoretic approach. To achieve this,

take the modular time of the state, perform the Wick rotation, and call the result temperature. If we now change the temperature independently of the modular time, we shall thus have added a new degree of freedom with respect to the information-theoretic approach. Evidently, this degree of freedom may not come from within the approach; so it must be meta-theoretic and related to the notions that were merely postulated in the information-theoretic approach. Such notions are information and fact, but also relevance. This is how, at least at the conceptual level, one explains the origin of the link between information and thermodynamics.

- Assume the information-theoretic interpretation of the local algebra theory in which Axioms I and II justify why the  $C^*$ -algebra of the system is hyperfinite. Then, if no new information is brought about, and if the algebra is a type  $III_1$  factor, the spectrum of  $t$  is from 0 to  $+\infty$ . It is a satisfactory result that the internal, state-dependent time behaves as one would think the time must behave: it is a real positive one-dimensional parameter.

Time is a state-dependent notion but one would wish to have also a state-independent time. Why would one wish that? Because we are accustomed to the linear time that does not depend on the information state. The word “accustomed” translates as a requirement to obtain Newtonian time in the limit. Now, to obtain this state-independent notion we factorize by inner automorphisms and pick up the whole class of these that will correspond to one outer automorphism. What have we done in information-theoretic terms? To each modular automorphism corresponds a state that defines it; by factorizing over modular automorphisms we neglect the difference between these states and therefore neglect the differences in information that we have in these different states. Thus state-independent time becomes an issue of rendering some information irrelevant.

We have said that time is ignorance. In fact, the word “ignorance” is perhaps not the best pick; the problem is that ignorance has a strong flavor of being able to, but not knowing something. In fact, there is no “being able to.” The state as information

state is given from meta-theory, and there is nothing inside the theory that tells one how to pass from one state to another (i.e. the measurement problem is not solved, but *dissolved*, see Section 2.3). So if we “were able” to know more, that would have defined another state over the algebra and another state-dependent time, which is not the case.

To formulate the main idea even shorter, let us come back to Bohr’s words in the epigraph: “The concepts of space and time by their very nature acquire a meaning only because of the possibility of neglecting the interactions with the means of measurement.” We explained that if we functionally separate the observer into meta-theoretical informational agent I and physical system P, we are then able to define facts as answers to yes-no questions posed by I to P and, in the course of interaction of P with a physical system S, by chasing P out of the formalism, these yes-no questions translate into POV measurement of I on S. P-observer is the ancilla. So we see that POV measurements emerge as an act of neglecting that the observer is a physical system. By themselves, POV measurements are just positive operators that span a  $C^*$ -algebra, and, as we said, a  $C^*$ -algebra corresponds to the notion of system. Consequently, to determine the system, i.e. a  $C^*$ -algebra, one must “put oneself” on the metalevel with respect to that system by leaving the informational agent and factoring out P-observer.

Now, each von Neumann algebra has a unique state-independent time. Put the two together: by “neglecting the interactions with the means of measurement” (Bohr) and therefore getting rid of P-observer in the formalism, we define the algebra and its state-independent time. This is how time acquires a meaning exactly as Bohr wanted it.

As Einstein said, “time and space are modes by which we think and not conditions in which we live” [55]. Let us rephrase Einstein and reconcile him with Bohr: time and space are the modes by which information is operated with and are not the unjustified postulates in the information-based physical theory.



**Part IV**  
**Conclusion**



# Chapter 9

## Summary of information-theoretic approach

### 9.1 Results

John von Neumann was a great, and the only, scientist of the first 70 years of the XXth century who made major contributions to both quantum theory and the theory of information, and in quantum theory he contributed to both quantum logic and algebraic quantum theory. Although von Neumann's interest dates back to late 1920s, it was in 1940s that he and his collaborators, taking inspiration from physical sciences, taught their colleagues in biology, psychology, and social science to speak the language of information. The new language proved so successful that over time it became possible to take it back to physics and to teach physics itself a new language. Furthermore, time has been ripe since 1970s for the world-picture as a whole, i.e. the philosophy of the human theoretical inquiry into nature, to be built around the notion of information.

The new world-picture is not akin to many its predecessors. The attempts proved futile to reduce the full enterprise of theoretical inquiry to relying upon information as the first notion. Such a reductionist point of view cannot be defended because of its circularity. Here, the futility and the circularity are due to the fact that information, too, can be taken as object of study, but this in a separate theory, which, obviously, will no more be able to have information as the first notion. The theories, then, are

mutually connected by what they choose as their basis and as their object of study, and there exists no set of primary concepts common to each and every theory. Such a situation amounts to a picture of the theoretical inquiry as a loop of existences. Consistent exposition of the epistemological attitude of the loop of existences, with its consequences for distinguishing theory from meta-theory, is the **first highlight** of this dissertation.

Theories have flourished since 1940s studying information by the means and tools of physics. To give just one result, computers are the greatest achievement of this current of human thought. Areas like artificial intelligence strive to demystify operations with information, its storage and communication, and cognitive science aims at giving a theory of mind. On the other part of the loop, information itself has been put in the very foundation of physics, and so since the appearance of the science of quantum information in 1980s. Questions have been raised: Can physics be derived from information-theoretic postulates? What are these postulates? What other assumptions must be added to them? As the **second highlight** of this dissertation, we have given one possible answer for a part of physics which is the quantum theory.

Two key axioms: that the amount of relevant information is finite and that it is always possible to acquire new information, suffice to grasp the essence of the quantum-theoretic structure. Mathematically, they need to be formulated in one of the formalisms of quantum theory and properly adjusted to the needs of this formalism; thus, being supplied with additional assumptions, they give rise to the conventional quantum theory. By means of the quantum logical formalism, we have shown how to achieve the goal of derivation of the Hilbert space and other blocks of which consists the formalism of quantum theory. Also, all along the derivation we have studied the role that play the additional assumptions and have compared our system of axioms with the existing alternatives.

Reconstruction by means of the quantum logical formalism has not met the need for an information-theoretic justification of the notions of space and time. To give such a justification along the lines of the algebraic formalism, we have first interpreted

this formalism in information-theoretic terms. As the **third highlight** of the dissertation, this interpretation together with the argument for non-fundamental role of time belong to a field seldom ploughed of the conceptual analysis of the  $C^*$ -algebraic formalism in the theory of local algebras.

The importance of the information-theoretic approach to quantum theory must not be underestimated. Apart from being an integral part of the world-picture that implies the loop of existences, this approach allows to view quantum theory as *a theory of knowledge*, i.e. a particular epistemology. From the general epistemology it differs in imposing two axiomatic constraints on the kind of knowledge that will be studied: that the amount of information must be finite and that it must always be possible to acquire new information. While the first constraint appears plausible even for the most general theory of knowledge, the second one clearly distinguishes quantum theory as theory of knowledge from, say, classical physics as theory of knowledge, for which no such axiom can be formulated. Indeed, the significance of Axiom II lies in non-Abelianness of the structure of observables such as lattice or  $C^*$ -algebra. Let us repeat once again: quantum theory is a theory of knowledge; it is not a theory of micro-objects nor of the physical reality. Its two key axioms, perhaps with a different set of supplementary axioms than that of Chapter 6, will allow to apply the essentially quantum theoretic approach to areas of human theoretical inquiry other than the theory of micro-objects. As one of the areas of potential interest we cite the application of the quantum mechanical ideas to cognitive psychology and economics [112].

The importance of the information-theoretic approach to quantum theory must not be overestimated. This approach responds to the need of giving a sound foundation to quantum physics, but it does not bring any added value to the way in which quantum theory is applied in the daily work of an ordinary physicist. Information-theoretic approach to the foundations of physics belongs to the area of theory, as opposed to application, and even to the philosophy of science, although its development was inspired by the purportedly practical field of quantum information. Thus

the information-theoretic approach cannot, for instance, help to make the world economy grow faster or poor people live a happier life, at least in the short run. Like poetry in W.H. Auden's words, it makes nothing happen; but it creates a new language for science and by doing so imposes on the human thought a novel pattern.

## 9.2 Open questions

Many questions that are raised in the context of the information-theoretic approach to reconstructing quantum theory were left open in this dissertation. These questions are listed below, and despite our effort the list is most probably incomplete.

1. Although they install the structure of a complete lattice, Axioms IV, V and VI have not been given an information-theoretic justification. One such justification could be based on the capacities offered to human beings by their language: namely, in the language any two questions can be concatenated or united in a longer question by a conjunction. But to reason so would mean to assume that I-observer is a human agent possessing a language, something that we have tried to avoid in Section 4.2. Even if to carry on with this assumption, it will still be necessary to decide whether human language has the complexification capacity *de facto* or only *in abstracto*, especially when applied to very large or countably infinite sets of questions, as requires Axiom VI. Information-theoretic approach, in the choice of Axioms I and II, aims explicitly at eliminating all abstract structure never to be exemplified. It would be a pity if the justification of Axioms IV, V and VI had to be at odds with this aim.
2. Information-theoretic meaning of Axiom VII is unclear and so is the one of its replacement offered by the Solèr theorem 6.16. We discussed this question in Section 6.5.
3. The appeal to Gleason's theorem 6.18 is not completely justified by Axiom III of intra-theoretic non-contextuality. The condition of Gleason's theorem involves a function  $f$  but nothing is said about the origin and meaning of this function.

It is easy to see that to justify the appearance of  $f$  amounts to explaining the origin of probabilities in quantum theory. Although the Born rule fulfils in part this task, information-theoretic meaning of the function  $f$  remains to be uncovered.

4. A series of assumptions about time evolution were made in Section 6.7. Although we have said that these assumptions cannot be properly justified on the information-theoretic grounds without exploring the other cut of the loop, it remains to be seen how, in this other cut of the loop (Figure 2.3), emerge these very assumptions. Partially this task has been carried out by the demonstration of classical limit of the modular time hypothesis by Connes and Rovelli.
5. We deliberately postulated the absence of superselection rules in the Hilbert space and gave an argument for this choice of ours (see pages 85 and 90). We are however ready to acknowledge a decisive weakness of this argument: in Hilbert spaces of the quantum theory as it is conventionally used, superselection rules are usually present. One needs to find a way out of this dilemma.
6. Section 8.5 treats of the problem of time in algebraic quantum theory, but only a few lines are consecrated to the problem of space. More research is needed that will perhaps go in the direction described on page 149.
7. Reaching out both to the conceptual foundations of the information-theoretic approach laid in Part I and to the concrete mathematical problems described in Part III, the question of justification of the link between thermodynamics and quantum theory (or equivalently, of the Wick rotation) remains unanswered. Indeed, it would be too ambitious to pretend to have found an answer to this question. What is clear, though, is that the answer may only come from a meta-theoretic analysis in which the two theories concerned will be somehow intertwined in one context. To close the chapter, we suggest as a joke that a mathematical formalization of the loop of existences may play the role of such context: indeed, the imaginary unit  $i$  is encoded in the equation of a circle, and,

as we argued, thermodynamics and quantum theory lie in different cuts of the circle which is the loop of existences. So to connect them would mean to pass from one part of the circle to another, i.e. make a rotation, and this requires a reference to  $i$ . We are of course fully aware of the non-scientific (as of today) character of this proposal but we end with a proverb which goes, “In every joke there is a grain of truth.”

# Chapter 10

## Other research directions

### 10.1 Physics and information in cognitive science

In this closing chapter of the Conclusion, we discuss questions pertaining to other research directions that arise in the context of the ideas explored in the dissertation. The first such question concerns the theory that emerges if the loop of Section 2.2 is cut as on Figure 2.3; this is to say that we analyze a theory which is based on physics as datum and has information for the object of inquiry, thus aiming at giving a theoretic account of how to operate with, store, represent, and communicate information. These areas fall into the large domain of cognitive science, i.e. the scientific study of mind. The Oxford English Dictionary defines the word *cognitive* as “pertaining to the action or process of knowing.” In a science of information that is based on physics, the concept of information is to be viewed as the means by which biological or even social questions from the study of mind could be reduced to problems of physics. This was Norbert Wiener’s view [47, p. 114], and we start by explaining the philosophy that underlies it.

Two main currents of thought in cognitive science are connectionism and cognitivism. Connectionism (Figure 10.1), with its roots in the first cybernetics of Macy conferences, asserts that meaning and mind are associated with matter because they *arise* from it. The matter in question is a neuronal network in the brain, and thinking is an algorithm operating on the neuronal machine. Meaning then has no essence, or rather its essence is just its appearance. Neural network is a complex system,

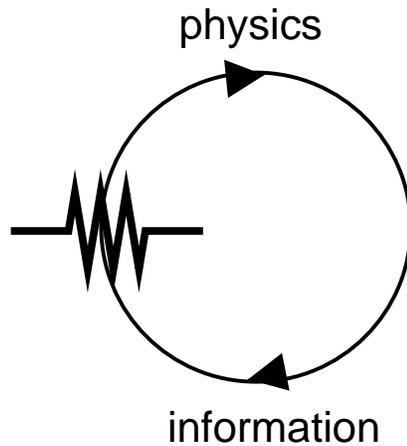


Figure 10.1: Connectionism: With its roots in the first cybernetics, connectionism asserts that objects have no symbolic value. Meaning and mind arise from matter, and in the theory there is no intermediate level of concepts between physics and information.

and the mind is “perfectly susceptible to a physicalist approach provided that we rely upon the qualitative macrophysics of complex systems and no longer upon the microphysics of elementary systems” [139]. No argument is however given that would allow one to reject a particular physical theory, and indeed in 1986 Roger Penrose, coming from a domain initially very remote from cognitive science, that of quantum gravity, proposed [134] that consciousness, which is one of the main objects of study in cognitive science, be seen as linked to the deep microphysics, and this without abandoning complexity. The contradistinction in views leaves open the question of which physical theory in the physicalist doctrine must be taken as the basis on which relies the theory of mind.

In our world-picture of Section 2.2 connectionism and its physicalist paradigm correspond to the loop cut so that the theory of information is based on physics as datum. However, besides the two configurations of Figures 2.2 and 2.3 that only use one cut in the whole loop, one can think of theories that arise in two or more loop cuts. One such theory, and indeed a major current of thought in the philosophy of cognitive science, is known under the name of *cognitivism* (Figure 10.2).

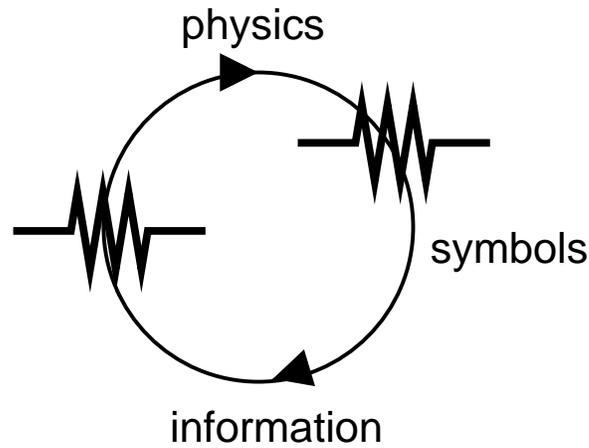


Figure 10.2: Cognitivism: What is essential for the emergence of mind is not a concrete causal structure but an abstract symbolic organization, which remains invariant when one passes from one physical system to another.

Cognitivism asserts that if the mind arises as a result of implementing a certain algorithm, or a program, in the physical world, then any implementation of the same program on a different hardware, no matter what it may be, would produce a mind endowed with the same properties. Therefore, what is essential for emergence of the mind is not the concrete physical causal organization of the material system possessing a mind; what is essential is the abstract organization, which remains invariant under the change of the material system. This abstract organization is symbolic, meaning that the level on which it operates is the level of symbols. On the cognitivist view, symbols have three aspects: physical, syntactic and semantic. Syntactic computations are rooted in the physical processes, but “syntax by itself is neither constitutive of nor sufficient for semantics” [165]. Thus a cognitivist theory of mind is directly grounded in the symbolic and only indirectly in the physical, in virtue of the fact that a theory of symbols, physicalist in itself, requires a different loop cut (Figure 10.3) than the cognitivist cognitive science of Figure 10.2.

A theory that is urged on the cognitivist approach by the necessity to consider, not only the loop cut of Figure 10.2, but also the one of Figure 10.3, is a *grand oubli* of the proponents of cognitivism. They tend to forget the second of the loop cuts altogether

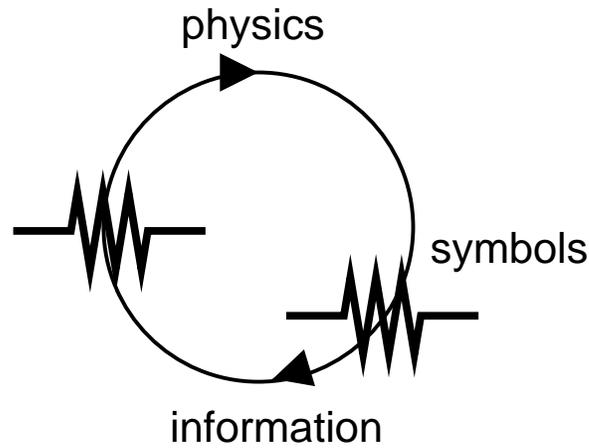


Figure 10.3: A cognitivist needs a theory of how the symbolic level arises from physics.

and focus their research on the symbolic level as if it were the only fundamental level; those cognitivists who call themselves physicalists are in fact no more than scientists whose reflection went deep enough to recognize the necessity of the second theory, but without ever achieving practical results. The physics of cognitivists is a physics of philosophers that is unconnected with the actual physics of physicists. When a scientist seriously addresses the need for a theory of which the schema is drawn on Figure 10.3, he is at once inclined to pass in the camp of connectionists and to remove the second loop cut thereby obtaining a theory of Figure 10.1.

Let us now return to the choice of physical theory on which a theory of mind may rely. We are going to give an argument showing that if one adopts the connectionist view of Figure 10.1, then the theory of consciousness cannot rely on classical physics, although it still can rely on quantum physics. Two assumptions that we make are as follows:

- Consciousness is an object of theoretical inquiry, i.e. there exists a *theory* of consciousness.
- Assumption of strong physicalism, i.e. every proposition of the theory of consciousness can be translated into a proposition of physical theory, even though this latter proposition may be quite complex.

Both these assumptions are far from being consensual among cognitive scientists and philosophers. Concerning the first one, we deliberately abstain from discussing whether consciousness is a phenomenon [17, 125] and if it has a place in the loop of existences. Perhaps it does not, and then consciousness is purely epiphenomenal. For example, such is nowadays the case with the notion of life, although some 150 years ago a rare scientist would call life epiphenomenal. We simply assume that consciousness is a legitimate object of theoretical inquiry.

Regarding the second assumption, its proponents are a few but include such philosophers as John Searle, who asserts that all mental phenomena must be reduced, at the last instance, to the level of physical fields and fundamental interactions [166, 167]. Although we do not endorse Searle's *ontological* physicalism and instead propose the loop *epistemology*, both lead to the assumption of strong physicalism that we make in the sequel.

In order to find out which physical theory can serve as foundation for the theory of consciousness, we follow a filtering procedure. This procedure consists in taking a particular property of consciousness that must be explained by the theory of consciousness and checking which physical theories are capable of giving an account of that property. In fact, we shall only be concerned with one such property: self-referentiality. The requirement of taking into account self-referentiality of consciousness will lead to a situation when only some, and not other, physical theories, which can be a foundation for the theory of consciousness, will survive filtering. Filtering criteria, including the one of self-referentiality, are non-constructive in the sense that they allow to eliminate candidate theories but they do not tell one how the theory of consciousness can be built using physical theories that will have survived filtering.

We start by treating observation as a semantic concept. Generic statement of a physical theory has the form, "The state of the system has such and such properties." Irrespectively of the meaning of the term *state* which, as we argued in Section 4.2, must be relational, this generic form of the physical statement permits, instead of speaking about the validity statements of the theory, to speak about sets of states:

to every statement corresponds a set of states in which the statement is valid. To verify a statement about the system means to make an observation of the system and to check if the observed state falls into the expected set of states. In this sense observations contribute to set up semantics of the theory.

Largely avoiding some crucial philosophical aspects of the discussion in Chalmers's illuminating book [32], we assume that "I am aware that" is a predicate of the theory of consciousness. In light of the semantic role of observations, "I am aware that" is at the same time an observation in the theory of consciousness and a semantic statement belonging to the theory of consciousness. For the reason of simplicity, in the following argument we take the theory of consciousness to contain only the predicate "I am aware that."

Let us now give several definitions. A theory is semantically complete if and only if objects and processes that are necessary for testing and interpreting the theory are themselves included among the phenomena described by the theory [127, p. 4]. Meta-theory of a given theory is a theory that contains predicates about the predicates of the theory. Follows that if a theory is semantically complete, then its meta-theory is a subset of the theory.

A theoretical statement is self-referential if it refers to the states of the system which, in their turn, refer to this very statement (i.e. the set of states) [180, 22]. In every semantically complete theory one necessarily finds self-referential statements. The converse does not hold: presence of a self-referential statement in a theory does not make the theory semantically complete.

The concept of self-reference leads to introducing the concept of self-referential inconsistency (Figure 10.4). In self-referential statements observation of the system (which is a semantic proposition) is made from inside the system, and this observation provides information not only about the system as such, but also about the measuring apparatus which is a part of the system. The latter information must be consistent with the fact that this measurement apparatus is indeed a *measurement* apparatus: for instance, the information obtained must not preclude the apparatus from existing.

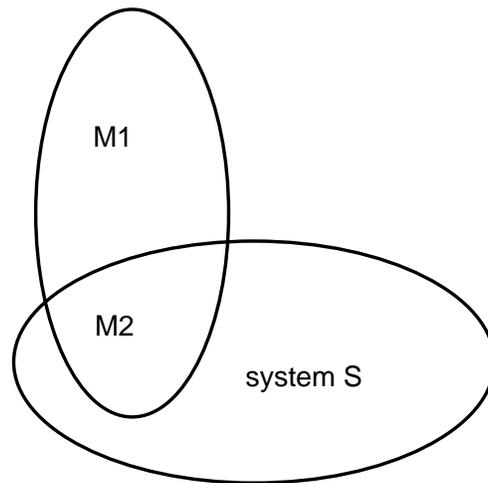


Figure 10.4: Self-referential consistency: Observation of  $S$  by  $M = M_1 + M_2$  provides information about the state of  $S$ , including certain information about  $M_2$ . This information must be compatible with the fact that  $M_2$  is a part of the measuring apparatus.

Self-referential consistency is a necessary requirement for any self-referential theory, because self-referential inconsistency leads to logical paradoxes. From this we learn an important lesson: If in a theory there are self-referential propositions then one must impose the condition of self-referential consistency.

Petersen writes, “To define the phenomenon of consciousness, Bohr used a phrase somewhat like this: a behaviour so complex that an adequate account would require references to the organism’s self-awareness.” [137] Somewhat in the spirit of Bohr’s idea, we now show that self-referentiality of consciousness implies self-referentiality of the theory of consciousness, which in turn implies self-referentiality of the physical theory on which relies the theory of consciousness.

“I am aware that I am aware”: this statement, viewed as a linguistic statement about the state of the system, reports a valid observation and thus belongs to meta-theory of the theory of consciousness. On the other hand, “I am aware that I am aware” is a statement of the type “I am aware that” and is itself a state of consciousness, so it belongs to the theory of consciousness. Every act of observation in the theory of consciousness, which we agreed to limit to “I am aware that” statements,

is therefore self-referential.

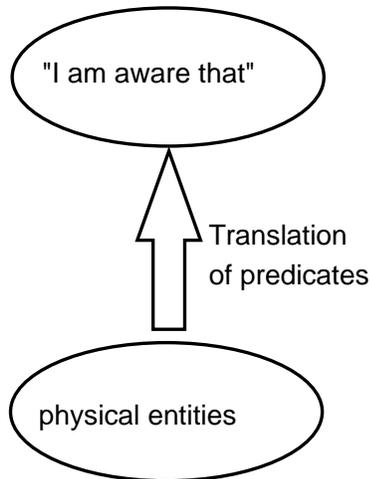


Figure 10.5: Translation of theoretic predicates in virtue of the assumption of strong physicalism.

Let us now show that self-referentiality of the theory of consciousness implies self-referentiality of the physical theory that serves as a foundation to the theory of consciousness. According to the assumption of strong physicalism, every predicate of the theory of consciousness can be translated into a predicate of the physical theory (Figure 10.5). Consider the predicate “I am aware that I am aware.” Put in the place of each of the two clauses “I am aware” its physical counterpart. We obtain a predicate of the physical theory which at the same time belongs to the theory and to meta-theory. This proof works if translation of the predicate “I am aware” into the language of physics does not depend on the content of the referring part of the predicate: evidently, referents of the two clauses “I am aware” are different, and their translations may therefore differ.

Consider now the opposite: namely, predicate translation depends on the referent. Translation is possible for any referent, so let us take as referent an arbitrary semantic statement of the form “such and such properties are true,” which belongs to the meta-theory of the physical theory. Add to it “I am aware that;” appears a statement that belongs to the theory of consciousness. Now translate this statement into the language

of physical theory in virtue of the assumption of strong physicalism. Starting with a meta-theoretical physical statement, we have thus obtained a statement of the physical theory itself. This confirms that physical theory on which relies the theory of consciousness is self-referential.

In short, what this procedure allows to achieve can be called “a new gödelization” fully analogous to the original idea of Gödel’s: “The language of the formal system used by Gödel ... does not contain any expressions referring explicitly to meta-theoretical concepts. But after assigning numbers to the propositions, these numbers can be interpreted as expressions of the language referring to its own propositions.” [22] Instead of assigning to every proposition a number, as did Gödel, we add to it a clause “I am aware that” that allows to put in correspondence with each semantic statement over the physical theory (i.e. observation) a physical state.

Having established that the physical theory must be self-referential, we would like to use this result to complete the filtering procedure. For this, we return to the notion of self-referential inconsistency and show that classical physics viewed as self-referential theory is inconsistent.

Key intuition comes from Einstein’s words that measuring instruments which we use to interpret theoretical expressions must be really existing physical objects. Skip the word “really” and focus on the word “existing”: this will lead to the check by self-referential consistency. In a theory of consciousness, measuring instrument is the human brain. If the theory runs into a contradiction when the brain elements are considered as measuring instruments, then the theory is inconsistent. One sort of such brain elements are hydrogen atoms. Consider a human observer  $O$  who observes hydrogen atoms in his own brain and assume that the theory of consciousness relies on classical physics. Result of this observation can be represented as “I am aware that hydrogen atoms in my brain have property  $P$  predicted by classical physics.” This observation, according to the new gödelization procedure, is itself a predicate of classical physics. Now, because predictions of classical physics about hydrogen atoms do not allow the existence of hydrogen atoms, being projected within the domain of

classical physical on  $M_2$  of Figure 10.4, they prevent the very existence of observer  $O$ . Consequently, classical physics is self-referentially inconsistent. It cannot serve as a foundation for the theory of consciousness.

As for quantum theory as basis of the theory of consciousness, it passes filtering by the criterion of self-referentiality: Mittelstaedt [127] in the discussion of the objectification postulate gives a classification of situations where quantum theory might appear to be self-referentially inconsistent and then, based on Breuer's result [22], proves the impossibility of such situations. This, however, does not guarantee that there exist no other reasons why the theory of consciousness may not rely on quantum physics as an underlying physical theory. So if for classical physics this question is settled in the negative, for quantum physics it remains open to future investigation.

## 10.2 Two temporalities in decision theory

We have seen in Section 8.5 that in the algebraic quantum theory interpreted in information-theoretic terms there arise two temporalities:

- (a) a state-dependent notion of time which is characterized by the I-observer's information state, and
- (b) a state-independent notion of time which is obtained by neglecting certain information and therefore factoring over whole classes of state-dependent temporalities.

It is the second, state-independent time that indexes facts as acts of bringing about information. If for the first, state-dependent time one can say that its range of values, in the hyperfinite type  $III_1$  von Neumann algebra, covers all positive real numbers, nothing at this level of precision can be said about the state-independent time. So there is no obvious reason why one would think that the state-independent time is "linear" in the usual sense and covers all  $\mathbb{R}_+$ . Still, it is this very time that the informational agent perceives as indexing facts in which information is brought about.

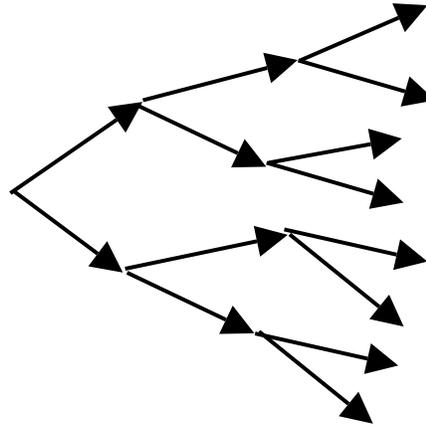


Figure 10.6: Occurring time.

A similar situation arises in decision theory [46, 48, 49]. The familiar commonsense temporality is encoded in a decision tree which we call *occurring time* (Figure 10.6). Occurring time is the linear time that embodies the commonsense understanding that the future is open and the past is fixed. The agent has no causal power over the past, but also no counterfactual power; on the contrary, with regard to the future the agent has both causal and counterfactual power. Decision theory employing this temporality leads to many paradoxes, i.e. such cases where action prescribed by the theory as the rational choice seems to be completely bizarre and is practically never chosen by the real human decision makers. Such paradoxes arise in a variety of settings, from simple Take-or-Leave games to the nuclear deterrence problem and the Newcomb paradox.

To avoid the paradoxes of decision theory in the occurring time, Dupuy proposed a different temporality that he called *projected time*. Projected time is the time in which reasoning of the agent takes place, and it is very different from the linear occurring time: in fact, it takes the form of the loop (Figure 10.7). In the projected time future has counterfactual power over the past, while the only causal power is, as before, the power of the past over the future. To find a decision-theoretic equilibrium in projected time, it is necessary to seek a fixed point of the loop, where an expectation (on the

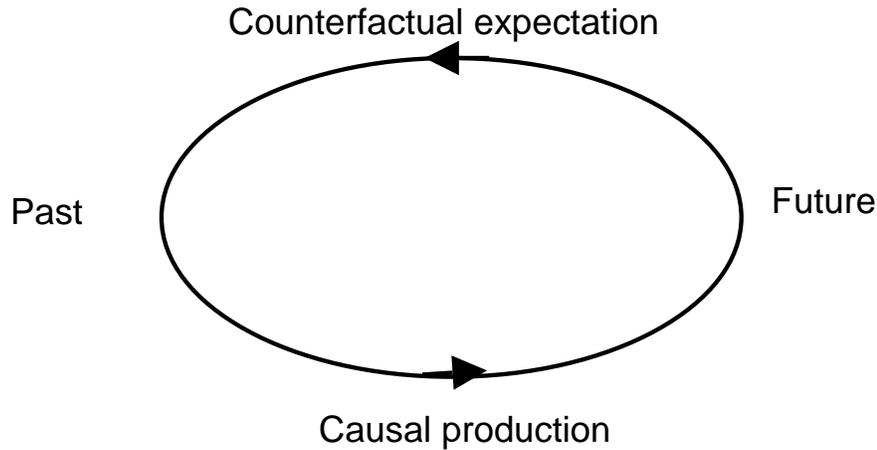


Figure 10.7: Projected time.

part of the past with regard to the future) and causal production (of the future by the past) coincide. The agent, knowing that his prediction is going to produce causal effects in the world, must take account of this fact if he wants the future to confirm what he has foretold.

Circular temporality of the projected time gives rise to a full new decision theory drastically different from the old decision theory that made use of the occurring time. Indeed, *decision* belongs in the kind of temporality in which *reasoning* is done, and this temporality is the one of the circular projected time. Linear time, so to say, ceased to be the interesting time. Projected time, which is not linear, raised to become an upfront temporal decision-theoretic notion.

Whether there are or there are not good grounds to claim a parallel between the two temporalities in the information-based physical theory and in the decision theory, as of now we are not yet ready to say. It is certainly tempting to seek an analogy between the two: in the information-theoretic approach one speaks about the temporality of facts being externally given to the physical theory in the loop cut of Figure 2.2, and this is not far from the temporality of reasoning in the decision-theoretic context. After all, facts are acts of bringing about information, and reasoning is just the analysis of information. So does the non-necessarily linear state-independent notion of

time have anything to do with the circular (i.e. non-linear) temporality of projected time? To answer in the affirmative would amount to an ambitious hypothesis that we can only leave as subject to a future investigation.

### 10.3 Philosophy and information technology

As we repeatedly said in this dissertation, foundations of the modern theory of information were laid out by von Neumann and other scientists whose work initially belonged in the theoretical, rather than applied, science. But these very people were also among the pioneers of the construction of computers and what was later called the field of information technology. Nowadays information technology is a vast domain causing public excitement and fascination and in which are employed thousands of professionals most of whom have never given any attention to the problems that interested the founding fathers of their discipline. A software engineer does not need to think about thermodynamics and its link with information. Chip maker does not need to worry about advanced programming languages or web browsers that will be run on computers using his chips. As many others, the field of information technology is divided into numerous cells to each of which are assigned hundreds of narrow specialists. Such is also the situation in physics since 1970s, and today this situation seems to be slowly changing: Queen Philosophy is coming back to her kingdom of physics. Will information technology sooner or later undergo a similar return to the fundamental questions? Probably yes.

One prospective direction that information technology may take if it decides to look back at the notions that lie in its foundation is the route shown by Clifton, Bub and Halvorson, whose results we discussed in Section 8.4. Quantum information developed powerful and beautiful theorems that are now used to serve as foundation of the physical theory itself. Metaphorically, the situation is like the one when a man for the first time looks in the binoculars in the wrong direction: before this man used to believe uncritically that the road is one-way only and that it leads from quantum physics to quantum information, until one day, out of curiosity, he looked

in the binoculars from the wrong end, and the view of the world has changed. It will never be the same: we now know that quantum theory can be viewed as based on information. Will information technology take the challenge to produce for the world a new philosophy based on its values and its fundamental notions? Will information technology, with the development of the field of quantum information, install a clear demarkation line between the superfluous ontological and the efficient epistemological arguments? We are still living in the days when articles by important information scientists speak about “ontic states” [174]. Perhaps it is with the future return of the interest toward its own fundamental concepts that information technology will consistently and insistingly teach other disciplines the language of information.

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