THÈSE

présentée pour l'obtention du titre de

DOCTEUR DE L'ÉCOLE NATIONALE DES PONTS ET CHAUSSÉES

Spécialité : Mathématiques et Informatique

par

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Sujet : *Contribution à l'analyse d'équations aux dérivées partielles décrivant le mouvement de fronts avec applications à la dynamique des dislocations.*

Soutenue le 02 juillet 2007 devant le jury composé de :

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Remerciements

Mes remerciements s'adressent en tout premier lieu à Régis Monneau qui a su guider mes travaux avec compétence, dynamisme et gentillesse. Travailler avec lui fut un grand honneur et une occasion de profiter de sa grande culture scientifique, de son intuition, de sa rigueur mathématique, mais aussi de ses qualités humaines.

Olivier Alvarez m'a introduit à la théorie des solutions de viscosité avec beaucoup de pédagogie. Il m'a également permis de rencontrer Régis et a suivi mes travaux durant ces trois années. Il me fait également le grand honneur de participer à mon jury. Pour tout cela, je le remercie vivement.

Jean-Michel Roquejoffre a accepté la lourde tâche de rapporter sur ce travail. Ses remarques très judicieuses et constructives ont permis d'améliorer ce travail et je l'en remercie.

Mete Soner a également accepté la tâche ingrate de rapporteur et je le remercie de s'être intéressé à mon travail.

Je tiens à exprimer ma plus profonde reconnaissance à Guy Barles et Pierre Cardaliaguet qui ont accepté d'être membre du jury. Je tiens également à les remercier pour les conseils judicieux qu'ils ont su me donner et pour la disponibilité dont ils ont fait preuve tout au long de ma thèse.

Henri Berestycki et Luis Caffarelli me font le grand honneur de faire partie de ce jury. Je leur en suis profondément reconnaissant.

Je salue Francesca Da Lio, Maurizio Falcone et Emiliano Cristiani avec qui j'ai eu beaucoup de plaisir à travailler.

J'adresse mon amitié à Elisabetta Carlini, Ahmad El Hajj, Cyril Imbert, Christian Gout, Carole Le Guyader et Aurélien Monteillet avec qui j'ai noué des collaborations fructueuses et qui, je le souhaite, se poursuivront.

Participer aux ACI "dynamique des dislocations" et "Mouvements d'interfaces avec termes non-locaux" a été un très grand plaisir et m'a permis de rencontrer de nombreuses personnes qui ont su me conseiller dans mes recherches. Je tiens à remercier tous les membres de ces deux ACI, Nathael Alibaud, Guy Barles, Pierre Cardaliaguet, Sébastien Collin, Christine Georgelin, Philippe Hoch, Cyril Imbert, Olivier Ley, Aurélien Monteillet, Elisabeth Rouy et Ali Srour.

Je tiens également à remercier les phycisiens Yann Le Bouar, Benoît Devincre, Alphonse Finel et Mathis Plapp pour les nombreuses discussions que nous avons pu avoir et qui m'ont permis de mieux comprendre la théorie des dislocations.

Le CERMICS est un endroit exceptionnel pour mener à bien un travail de thèse et je tiens à remercier tous les membres pour la bonne ambiance qu'ils créent. Un merci particulier aux membres (et anciens membres) de l'équipe "EDP et Matériaux" : Adrien Blanchet, Ariela Briani, Elisabtta Carlini, Ahmad El Hajj, Amin Ghorbel, Hassan Ibrahim, Cyril Imbert et Mohammed El Rhabi.

Vient maintenant le tour de ma famille qui a joué un rôle essentiel durant ces trois années. Merci à mes parents pour m'avoir toujours laissé le choix dans mes décisions et m'avoir soutenu dans toutes les entreprises auxquelles j'ai voulu me risquer. Merci également à mon frère et à sa femme. Je finis par celle qui a une place toute particulière : ma femme, Marjorie. Merci de m'avoir toujours soutenu et d'avoir été là quand il le fallait. Merci également pour tout ce que tu m'apportes.

Résumé : Ce travail porte sur la modélisation, l'analyse et l'analyse numérique de la dynamique des dislocations ainsi que sur les liens très forts qui existent avec les mouvements de type mouvement par courbure moyenne. Les dislocations sont des défauts linéaires qui se déplacent dans les cristaux lorsque ceux-ci sont soumis à des contraintes extérieures. D'une manière générale, la dynamique d'une ligne de dislocation est décrite par une équation eikonale où la vitesse dépend de manière non locale de l'ensemble de la ligne. Il est également possible d'ajouter un terme de courbure moyenne dans la modélisation.

La première partie de ce mémoire est consacrée aux propriétés qualitatives de la dynamique d'une ligne de dislocation (existence, unicité, comportement asymptotique...). Cette étude repose en grande partie sur la théorie des solutions de viscosité. On propose également plusieurs schémas numériques pour cette dynamique et on montre leur convergence ainsi que des estimations d'erreurs entre la solution et son approximation numérique.

Dans une seconde partie nous faisons le lien entre la dynamique d'un nombre fini de dislocations et la dynamique de densité de dislocations en montrant des résultats d'homogénéisation. Nous étudions également, de manière théorique et numérique, un modèle pour la dynamique de densité de dislocations.

Abstract : This work deals with the modeling, the analysis and the numerical analysis of the dislocation dynamics and with the very strong links which exists with mean curvature type motion. Dislocations are linear defects which move in crystals when those are subjected to exterior stress. More precisely, the dynamics of a dislocation line is described by an eikonal equation where the speed depends in a nonlocal way on the whole line. In the modeling, it is also possible to add a mean curvature term.

The first part of this work is devoted to the study of the qualitative properties of dynamics of a dislocation line (existence, uniqueness, asymptotic behaviour...). This study relies essentially on the theory of viscosity solutions. We also propose several numerical scheme for this dynamics and we show their convergence as well as error estimates.

In a second part, we establish the link between the dynamics of a finite number of dislocations and the dynamics of dislocation density by showing homogenization results. We also study, in a theoretical and numerical way, a model for the dynamics of dislocation density.

Publications issues de la thèse

Articles acceptés

- (avec F. Da Lio et R. Monneau) *Convergence of a non-local eikonal equation to anisotropic mean curvature motion. Application to dislocations dynamics*, accepté au Journal of the European Mathematical Society.

- (avec A. El Hajj) *A convergent scheme for a non-local coupled system modelling dislocations densities dynamics*, accepté à Mathematics of Computation.

Articles soumis et Preprint

- *Dislocations dynamics with a mean curvature term : short time existence and uniqueness*, soumis à Differential and Integral Equations. Rapport de recherche du Cermics [2005-273].

- (avec E. Carlini, M. Falcone et R. Monneau) *Convergence of a Generalized Fast Marching Method for a non-convex eikonal equation*, soumis à SIAM journal on Numerical Analysis. Rapport de recherche du Cermics [2006-325].

- *An error estimate for a new scheme for mean curvature motion*, soumis à SIAM journal on Numerical Analysis. Rapport de recherche du Cermics [2006-334].

- (avec C. Imbert et R. Monneau) *Homogenization of the dynamics of dislocations and of some systems of particules with two-body interactions*, soumis à Duke Mathematical journal. Preprint déposé sur HAL.

- (avec A. Monteillet) *Minimizing movements for dislocation dynamics with a mean curvature term*, soumis à Control, Optimisation and Calculus of Variations.. Preprint déposé sur HAL.

Proceedings

- (avec P. Cardaliaguet, F. Da Lio et R. Monneau) *Dislocation dynamics : a non-local moving boundary*, proceedings du congres FBP 2005, Coimbra, Portugal, International Series of Numerical Mathematics, Vol. 154, Birkhaüser Verlag Basel/Switzerland, 125-135, (2006).

- (avec E. Carlini et E. Cristiani) *A non-monotone Fast Marching scheme for a Hamilton-Jacobi equation modeling dislocation dynamics*, Numerical Mathematics and Advanced Applications Proceedings of ENUMATH 2005 the 6th European Conference on Numerical Mathematics and Advanced Applications, Santiago de Compostela, Spain, July 2005 Bermdez de Castro, A. ; Gmez, D. ; Quintela, P. ; Salgado, P. (Eds.).

Article en cours de rédaction

- (avec R. Monneau) *Existence of solutions for a model describing the dynamics of junctions between dislocations.*

Sommaire

Introduction générale

Cette thèse porte sur la modélisation et l'étude mathématique de la *dynamique des dislocations* ansi que sur les liens très forts qui existent avec les mouvements de type *mouvement par courbure moyenne.*

1 Motivations physiques

Une dislocation est un défaut linéaire qui correspond à une discontinuité dans l'organisation de la structure cristalline. Les dislocations, dont l'ordre de longueur typique dans les matériaux est $10^{-6}m$ et l'épaisseur $10^{-9}m$, ont été introduites par Orowan [150], Polanyi [155] et Taylor [176] dans les années 1930 comme l'une des principales explications à l'échelle microscopique des déformations plastiques macroscopiques des cristaux. Ce concept fut confirmé dans les années 1950 par les premières observations directes de dislocations par Hirsh, Horne, Whean [105] et Bollman [38] grâce à la microscopie électronique (voir Figure 1 pour un exemple d'observation de dislocations). Dans la structure à face cubique centrée, les dislocations se déplacent, au moins à basse température par rapport à la température de fusion, dans des plans cristallographiques bien définis (les plans de glissement) à des vitesses de l'ordre de 10 ms^{-1} (les vitesses peuvent être variables suivant les matériaux). Nous renvoyons à Hirth, Lothe [106] pour une description plus détaillée des dislocations.

L'étude théorique des dislocations ainsi que le développement de moyens d'investigation comme la microscopie éléctronique ont permis de mieux comprendre les mécanismes élémentaires à l'origine de la déformation plastique des matériaux cristallins. Cependant, compte tenu de la complexité et du grand nombre de phénomènes physiques mis en jeu, la relation entre les propriétés microscopiques des dislocations et le comportement plastique des matériaux est encore à mieux comprendre.

Depuis le début des années 1990, la recherche dans le domaine des dislocations est en plein essor, notamment grâce à la puissance des ordinateurs qui permet désormais de simuler un grand nombre de dislocations dans un domaine 3D. Plus récemment,

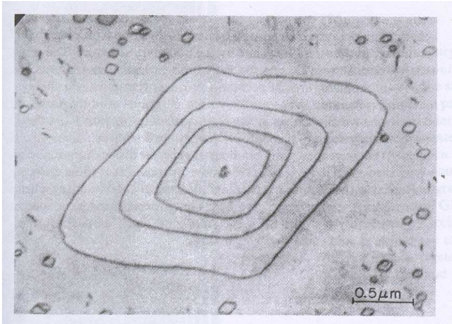

FIG. 1 – Dislocations dans un matériau Al-Mg (issu de [182])

une nouvelle approche a été introduite : *modèle de champs de phase pour les dislocations* (voir par exemple Rodney, Le Bouar, Finel [158]). Un des avantages de cette méthode est qu'elle permet de gérer de manière automatique les changements de topologie durant le déplacement. Dans ce modèle, la ligne de dislocation se déplace dans son plan de glissement avec une vitesse normale qui est proportionnelle à la force de Peach-Koeller résolue s'exerçant sur la ligne. Dans le cas où il n'y a pas de contraintes extérieures, cette force est simplement la force créée par le champ élastique généré par la ligne de dislocation elle-même.

Dans [10], Alvarez, Hoch, Le Bouar et Monneau ont proposé de réécrire ce modèle comme une équation de Hamilton-Jacobi. Cette équation est simplement une équation eikonale où la vitesse normale est non-locale et dépend de l'ensemble de la ligne de dislocation. Nous rappelons cette équation ainsi que sa modélisation de manière succinte dans la section suivante et de manière plus détaillée dans le chapitre 1.

2 Enoncés des résultats

Les hypothèses utilisées dans les théorèmes qui suivent ne sont pas optimales. Nous renvoyons aux chapitres correspondants pour des résultats plus généraux, en particulier en dimension N quelconque et avec des hypothèses de régularité moins restrictives.

2.1 Dynamique d'une ligne de dislocation

2.1.1 Modélisation succinte

Dans cette sous-section, nous expliquons de manière très simple la modélisation de la dynamique d'une ligne de dislocation et nous donnons les équations que nous allons étudier par la suite. Nous renvoyons à la Section 2 du chapitre 1 pour une dérivation plus détaillée des équations présentées ici.

Une idéalisation d'une ligne de dislocation consiste à considérer que l'épaisseur de la ligne est nulle et, dans le cas d'une seule ligne, à supposer qu'elle est contenue et qu'elle se déplace dans le plan (x_1, x_2). La dislocation peut alors être représentée par le bord d'un domaine Ω_t (où l'indice t représente le temps) que l'on note Γ_t. Le déplacement de la ligne Γ_t est alors simplement donné par la vitesse normale c.

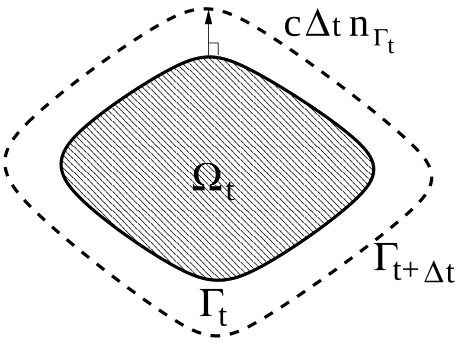

FIG. 2 – Evolution schématique d'une ligne de dislocation Γ_t à la vitesse normale c entre les temps t et $t + \Delta t$ avec la normale unitaire n_{Γ_t}.

La vitesse normale c dépend du champ élastique généré par la ligne de dislocation (plus précisemment, elle est proportionnelle à la force de Peach-Koeller résolue calculée à partir du champ élastique). Autrement dit, l'équation du mouvement est donnée par

$$\frac{d\Gamma_t}{dt} = cn_{\Gamma_t}$$

où $c = c(\Gamma)$ est la force de Peach-Koeller résolue. Dans le cas d'une seule ligne de dislocation, Alvarez, Hoch, Le Bouar et Monneau ont montré dans [10] que la vitesse c pouvait se réécrire comme une quantité non locale dépendant de la forme complète de la ligne de dislocation :

$$c(x,t) = (c_0 \star \rho(\cdot, t))(x) + c_1(x,t) \tag{1}$$

où $c_0(x)$ est un noyau donné, symétrique et dépendant du matériau et ρ est la fonction caractéristique de l'ouvert $\Omega_t \subset \mathbb{R}^2$:

$$\rho(x,t) = 1_{\Omega_t} := \begin{cases} 1 & \text{si} \quad x \in \Omega_t \\ 0 & \text{si} \quad x \in \mathbb{R}^2 \backslash \Omega_t. \end{cases} \tag{2}$$

Ici, la convolution a lieu en espace uniquement. Dans l'expression (1), le terme de convolution $c_0 \star \rho$ représente la force crée par le champ élastique généré par la ligne de dislocation alors que la fonction c_1 représente les contraintes extérieures.

On vérifie alors facilement (au moins formellement) que l'évolution sur l'intervalle de temps $(0,T)$ de la ligne de dislocation Γ_t est décrite par l'équation de la dynamique des dislocations :

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = (c_0 \star \rho + c_1)|D\rho| \quad \text{dans} \quad \mathbb{R}^2 \times (0,T) \\ \\ \rho(\cdot,0) = \rho^0(\cdot) = 1_{\Omega_0} \qquad \text{sur} \quad \mathbb{R}^2 \end{cases} \tag{3}$$

où $D\rho$ représente le gradient de ρ en espace et Ω_0 est un ensemble ouvert dont la frontière $\Gamma_0 = \partial\Omega_0$ représente la position de la ligne de dislocation au temps initial $t=0$.

Le problème (3) est associé formellement à l'énergie

$$\mathcal{E}(\rho) = \int_{\mathbb{R}^2} -\frac{1}{2}(c_0 \star \rho + c_1)\rho. \tag{4}$$

Il est également possible dans la modélisation d'ajouter un terme de tension de ligne de la forme $\int_{\Gamma_t} \gamma(n_{\Gamma_t})$. Dans ce cas, la force de Peach-Koeller résolue est alors donnée par

$$c = (c_0 \star \rho) + c_1 + \lambda(n_{\Gamma_t})H_{x,t} \tag{5}$$

où $H_{x,t}$ est la courbure moyenne et $\lambda = \gamma + \gamma''$. On peut alors reformuler le problème par une équation "level set" sur l'ensemble de niveau $\{u \leq 0\}$ d'une fonction régulière u qui vérifie alors :

$$\frac{\partial u}{\partial t} = (c_0 \star [u] + \lambda(n_{\Gamma_t})H_{x,t})|Du| \tag{6}$$

avec

$$\begin{cases} \rho = [u] = \begin{cases} 1 & \text{si} \quad u > 0 \\ 0 & \text{si} \quad u \leq 0 \end{cases} \\ n_{\Gamma_t} = \frac{Du}{|Du|} \\ H_{x,t} = \operatorname{div}\left(\frac{Du}{|Du|}\right) \end{cases} \tag{7}$$

2.1.2 Résultats d'existence et d'unicité en temps court

Les équations que nous considérons ((3) et (6)-(7)) sont des équations de Hamilton-Jacobi non-locales. La première partie de ce travail est destinée à étudier ces équations (existence, unicité, comportement asymptotique...). Le cadre le mieux adapté pour les résoudre est la théorie des solutions de viscosité introduite par Crandall et Lions [63] (pour une bonne introduction à cette théorie, nous renvoyons à Barles [18], [19], Bardi, Capuzzo-Dolcetta [17] et Crandall, Ishii, Lions [61] et pour une introduction aux solutions de viscosité pour l'évolution de fronts, nous renvoyons à Ambrosio [13], Barles, Soner, Souganidis [27], Chen, Giga, Goto [58], Evans [80], Evans, Spruck [83] et Souganidis [174]).

Une remarque importante est que, d'un point de vue physique, s'il n'y a pas de contraintes extérieures ($i.e$ $c_1 \equiv 0$), on s'attend à ce que les droites ne bougent pas. Ceci implique en particulier que c_0 est à moyenne nulle et donc que c_0 change de signe. A cause de cela, on peut montrer qu'il n'y a pas de principe d'inclusion, c'est à dire que si l'on considère deux ensembles inclus l'un dans l'autre au temps initial, alors cette inclusion ne perdure pas durant l'évolution. Or, le principe de comparaison est un élément essentiel dans la théorie des solutions de viscosité. Ceci explique en particulier la difficulté pour obtenir des résultats d'existence et d'unicité en temps long.

Le premier résultat d'existence et d'unicité concernant les équations de la dynamique des dislocations a été obtenu par Alvarez *et al.* dans [9, 10]. Il s'agit d'un résultat d'existence et d'unicité en temps court pour la solution de viscosité discontinue de l'équation (3). Un résultat similaire dans le cas où la donnée initiale est un graphe a été obtenu par Alvarez, Carlini, Monneau et Rouy dans [8] pour la formulation level set de (3).

Le premier résultat de ce travail est un résultat similaire pour l'équation (6)-(7) :

Théorème 2.1. (Existence et unicité en temps court pour (6)-(7)**, [87, Théorème 3.1])**
Soit $u_0 : \mathbb{R}^2 \to \mathbb{R}$ *une fonction lipschitzienne continue sur* $\mathbb{R}^2$ *telle que*

$$|Du_0| \leq B_0 \quad dans\ \mathbb{R}^2 \tag{8}$$

et

$$\frac{\partial u_0}{\partial x_2} \geq b_0 > 0 \quad dans\ \mathbb{R}^2. \tag{9}$$

On suppose que $c_0 \in C_c^\infty(\mathbb{R}^2)$ *(infiniment différentiable et à support compact) et que* $c_1 \equiv 0$*. Alors, il existe un temps* $T^* > 0$ *tel qu'il existe une unique solution de viscosité au problème* (6)-(7) *dans* $\mathbb{R}^2 \times [0, T^*)$*. De plus, la solution est uniformément continue en temps et vérifie également :*

$$|Du(x,t)| \leq 2B_0 \quad sur \quad \mathbb{R}^2 \times [0, T^*), \tag{10}$$

$$\frac{\partial u}{\partial x_2}(x,t) \geq b_0/2 > 0 \quad sur \quad \mathbb{R}^2 \times [0, T^*). \tag{11}$$

L'hypothèse $c_1 \equiv 0$ est utilisée seulement pour simplifier l'écriture des preuves mais elle peut être facilement remplacée par c_1 Lipsctictz uniformément en (x,t). L'idée générale de la démonstration est de geler la vitesse non-locale dans l'équation et d'utiliser un argument de type point fixe. La difficulté majeure est d'obtenir des bonnes estimations sur le gradient en espace et sur le module de continuité en temps de la solution du problème local (celui où la vitesse est gelée).

Dans le cas de la loi d'évolution (5), si on suppose que γ est constant ($\gamma \equiv 1$ pour simplifier), alors, en utilisant l'effet régularisant de la courbure moyenne, on peut en fait obtenir de meilleurs résultats. De manière plus précise, si l'ensemble initial est suffisament régulier, alors nous avons existence et unicité d'une évolution régulière en temps court. Avant d'énoncer ce résultat, nous avons besoin de quelques notations et définitions.

Définition 2.2. (Evolution régulière)
• *On note $\mathcal{P}$ l'ensemble de tous les sous-ensembles bornés de $\mathbb{R}^2$ ayant un périmètre fini.*

• *Pour un sous-ensemble E de $[0,T] \times \mathbb{R}^2$, on pose $E(t) = \{x \in \mathbb{R}^2; (t,x) \in E\}$. A l'opposé, une application $t \in [0,T] \mapsto E(t) \in \mathcal{P}$ peut être vue comme un sous-ensemble de $[0,T] \times \mathbb{R}^2$ en identifiant E avec son graphe $\cup_{t\in[0,T]}\{t\} \times E(t)$.*

• *On appelle tube un sous-ensemble E de $[0,T] \times \mathbb{R}^2$ tel que $E(t)$ est borné pour tout $t \in [0,T]$.*
On appelle tube régulier un tube E dont le bord est C^1 dans $(0,T) \times \mathbb{R}^2$ tel que pour tout $(x,t) \in \partial E$, la normale extérieure (ν_t, ν_x) à E au point (t,x) satisfait $\nu_x \neq 0$.

• *Finalement, une application $t \in [0,T] \mapsto E_r(t)$ est appelée une évolution régulière dont le bord est $C^{2+\alpha}$ si E_r est un tube régulier compact tel que $E_r(t)$ a un bord $C^{2+\alpha}$ pour tout $t \in [0,T]$.*

Théorème 2.3. (Existence d'une solution régulière, [91, Théorème 1.6])
Soit Ω_0 (dont le bord représente la ligne de dislocation au temps initial) un domaine compact dont le bord est uniformément $C^{3+\alpha}$. On suppose que $c_0 \in C_c^\infty(\mathbb{R}^2)$ et $c_1 \in C_c^\infty(\mathbb{R}^2 \times [0,\infty))$. Alors, il existe un temps $t_0 > 0$ et une évolution régulière $\{\Omega_r(t)\}_{0\le t\le t_0}$ dont le bord est $C^{2+\alpha}$, partant de Ω_0 avec une vitesse normale

$$V_{x,t} = H_{x,t} + c_0 \star 1_{\Omega_r(t)}(x) + c_1(x,t), \tag{12}$$

où $H_{x,t}$ est la courbure moyenne de $\partial\Omega_r(t)$ à x.

L'idée de la preuve est inspirée de Evans et Spruck [84] pour le mouvement par courbure moyenne (voir aussi Giga [99], Lunardi [140], Maekawa [141]). Il s'agit de supposer que l'on a une évolution régulière et de considérer la fonction distance signée au bord. On peut alors montrer que cette fonction vérifie une équation uniformément parabolique. Le but est ensuite de construire directement une solution de cette équation et de vérifier à posteriori que la ligne de niveau zéro de la solution trouvée définit bien une évolution régulière avec la bonne vitesse normale.

2.1.3 Résultats d'existence et d'unicité en temps long

Comme nous l'avons déjà mentionné, les théorèmes en temps long pour les équations de la dynamique d'une ligne de dislocation sont beaucoup plus difficiles à obtenir. Jusqu'à présent, l'existence et l'unicité en temps long avec des hypothèses générales est encore une question ouverte. Il existe cependant des résultats partiels. Tout d'abord, sous certaines hypothèses de monotonie sur la vitesse, Alvarez, Cardaliaguet, Monneau [6] puis Barles, Ley [25] ont montré, avec des méthodes différentes, qu'il existait une unique solution de viscosité de (3) pour tout temps (nous renvoyons également à Cardaliaguet, Marchi [52]). Barles, Cardaliaguet, Ley, Monneau [21] ont également montré l'existence globale en temps d'une solution faible. Cette notion de solution faible repose sur les solutions de visosité L^1 introduites par Ishii [113] (voir aussi Nunziante [147, 148], Bourgoing [41, 42]) et sur le résultat de stabilité de Barles [20].

Une autre hypothèse qui sera souvent utilisée dans la suite est que $c_0 = J \geq 0$. Dans ce cas, pour que les droites ne bougent pas s'il n'y a pas de contrainte extérieure, il faut choisir $c_1 = -\frac{1}{2}\int J$. Cela revient de manière formelle à concentrer la partie négative du noyau à l'origine. Pour ce type d'équation, Slepčev [169] a montré que la formulation level set à utiliser est la suivante

$$\begin{cases} u_t(x,t) = \left((J \star 1_{\{u(\cdot,t) > u(x,t)\}})(x) - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^2} J \right) |Du(x,t)| & \text{dans} \quad \mathbb{R}^2 \times (0,T), \\ u(\cdot,0) = u_0(\cdot) & \text{sur} \quad \mathbb{R}^2 \end{cases} \tag{13}$$

Cette formulation est en fait beaucoup plus stable et permet d'obtenir un résultat d'existence et d'unicité en temps long. Ici la ligne de dislocation est représentée par n'importe quelle ligne de niveau de la fonction u. Avant d'énoncer notre théorème d'existence, nous rappelons la définition de solution de viscosité que nous utilisons et qui n'est pas standard. Cette définition a été proposée par Slepčev [169] (voir aussi Da Lio, Kim, Slepčev [69]) :

Définition 2.4. (Sous/sur/solution de viscosité pour (13)**)**
Une fonction $u : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$ semi-continue supérieurement (resp. semi-continue inférieurement) est une sous-solution de viscosité (resp. sur-solution) de (13) *si $u(0,x) \leq u_0(x)$ dans $\mathbb{R}^2$ (resp. $u(0,x) \geq u_0(x)$) et pour tout $(x,t) \in \mathbb{R}^2 \times (0,\infty)$ et pour toute fonction test $\phi \in C^2(\mathbb{R}^2 \times \mathbb{R}^+)$ telle que $u - \phi$ atteint un maximum (resp. un minimum) au point (x,t), alors on a*

$$\phi_t(x,t) \leq \left((J \star 1_{\{u(\cdot,t_0) \geq u(x,t)\}})(x) - \frac{1}{2}\int_{\mathbb{R}^2} J \right) |D\phi(x,t)|$$

$$\left(\text{resp.} \quad \phi_t(x,t) \geq \left((J \star 1_{\{u(\cdot,t_0)>u(x,t)\}})(x) - \frac{1}{2}\int_{\mathbb{R}^2} J \right) |D\phi(x,t)| \right).$$

Une fonction continue est une solution de viscosité de (13) *si et seulement si c'est une sous et une sur-solution de viscosité.*

Théorème 2.5. (Existence et unicité en temps long pour (13), **[67, Théorème 1.1])**
On suppose que $u_0 \in \mathrm{Lip}(\mathbb{R}^2)$ *et que* $J \in W^{1,1}(\mathbb{R}^2)$. *Alors, il existe une unique solution de viscosité de* (13).

Dans le cas de l'équation d'évolution (5) (sans condition de signe sur c_0), il est encore possible d'utiliser l'effet régularisant de la courbure moyenne pour construire une solution faible en temps long. La construction de cette solution utilise les mouvements minimisants introduits par Almgrem, Taylor et Wang dans [3] pour l'équation par courbure moyenne (nous renvoyons également à Ambrosio [12] pour une présentation simplifiée).

La notion de solution faible que nous utilisons est la même que celle définie par Barles, Cardaliaguet, Ley et Monneau [21] pour l'équation level set associée à (3). Un concept similaire de solution apparaît également dans Soravia, Souganidis [171] pour des systèmes de type Fitzhugh-Nagumo.

Une solution faible est définie de la manière suivante :

Définition 2.6. *On suppose que* $c_0 \in C_c^\infty(\mathbb{R}^2)$ *et* $c_1 \in C_c^\infty(\mathbb{R}^2 \times [0,T))$. *Soit* $\Omega : [0,T] \to \mathcal{P}(\mathbb{R}^2)$ *une application telle que* $t \mapsto 1_{\Omega(t)}$ *appartient à* $C^0([0,T], L^1(\mathbb{R}^2))$. *Soit* u *l'unique solution de viscosité uniformément continue de*

$$\begin{cases} u_t(x,t) = \left[div\left(\frac{Du(x,t)}{|Du(x,t)|}\right) + c_0(\cdot,t) \star 1_{\Omega(t)}(x) + c_1(x,t) \right] |Du(x,t)| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{dans } \mathbb{R}^2 \times (0,T) \\ u(x,0) = u_0(x) \quad \textit{sur } \mathbb{R}^2, \end{cases} \tag{14}$$

où u_0 *est une fonction uniformément continue vérifiant*

$$\overline{\Omega}_0 = \{u_0 \geq 0\} \quad \text{et} \quad \overset{\circ}{\Omega}_0 = \{u_0 > 0\}.$$

On dit que Ω *est une solution faible de la loi d'évolution donnée par la vitesse normale*

$$V_{x,t} = H_{x,t} + c_0 \star 1_{\Omega(t)}(x) + c_1(x,t) \tag{15}$$

si pour tout $t \in [0,T]$ *et presque partout dans* $\mathbb{R}^2$, *on a*

$$\{u(\cdot,t) > 0\} \subset \Omega(t) \subset \{u(\cdot,t) \geq 0\}.$$

Comme nous l'avons indiqué précédemment, la construction d'une solution faible repose sur la notion de mouvements minimisants. De manière plus précise, nous montrerons que tout mouvement minimisant est une solution faible. Nous commençons par introduire de manière détaillée la notion de mouvement minimisant. On fixe $T > 0$ et pour $h > 0$ (le pas de temps), $k \in \mathbb{N}$ tels que $kh \leq T$, E et F dans $\mathcal{P}$, on définit la fonctionnelle

$$\mathcal{F}(h,k,E,F) = P(E) + \frac{1}{h}\int_{E\Delta F} d_{\partial F}(x)\,dx - \int_E \left(\frac{1}{2}c_0 \star 1_E(x) + c_1(x,kh)\right) dx, \quad (16)$$

où d_C est la fonction distance à l'ensemble fermé C et $P(E)$ est le périmètre de l'ensemble E.

Étant donné un pas de temps h, le principe des mouvements minimisants est de construire itérativement une suite $\Omega_h(k)$ en minimisant la fonctionnelle $E \mapsto \mathcal{F}(h,k,E,\Omega_h(k))$. Le but est ensuite de passer à la limite $h \to 0$.

Nous donnons maintenant la définition rigoureuse de mouvement minimisant :

Définition 2.7. (Mouvement minimisant [3])
Soit $T > 0$ et $\Omega_0 \in \mathcal{P}$. On dit que $\Omega : [0,T] \to \mathcal{P}$ est un mouvement minimisant associé à la fonctionnelle $\mathcal{F}$ avec Ω_0 comme condition initiale s'il existe une suite (h_n), $h_n \to 0^+$ et des ensembles $\Omega_{h_n}(k) \in \mathcal{P}$ pour tout $k \in \mathbb{N}$ avec $kh_n \leq T$, tels que :

1. $\Omega_{h_n}(0) = \Omega_0$,
2. *Pour tout $n, k \in \mathbb{N}$ avec $(k+1)h_n \leq T$,*

$$\Omega_{h_n}(k+1) \text{ minimise la fonctionnelle } E \to \mathcal{F}(h_n,k,E,\Omega_{h_n}(k)) \quad (17)$$

   parmi tous les $E \in \mathcal{P}$,

3. *Pour tout $t \in [0,T]$, $\Omega_{h_n}([t/h_n]) \to \Omega(t)$ dans $L^1(\mathbb{R}^N)$ quand $n \to +\infty$, c'est à dire que $1_{\Omega_{h_n}([t/h_n])} \to 1_\Omega$ dans $L^1(\mathbb{R}^2)$.*

où $[\cdot]$ est la fonction partie entière.

Le premier résultat concernant les mouvements minimisants est un résultat d'existence :

Théorème 2.8. (Existence de mouvements minimisants, [91, Théorème 1.3])
Soit $\Omega_0 \in \mathcal{P}$ tel que $\mathcal{L}^2(\partial\Omega_0) = 0$. On suppose que $c_0 \in C_c^\infty(\mathbb{R}^2)$ et $c_1 \in C_c^\infty(\mathbb{R}^2 \times [0,T))$. Alors, il existe un mouvement minimisant Ω associé à $\mathcal{F}$avec Ω_0 comme condition initiale.

Le résultat suivant justifie le lien entre ces mouvements minimisants et l'équation d'évolution donnée par (5). Il montre que tout mouvement minimisant correspond en fait à l'évolution régulière donnée par le Théorème 2.3 tant que cette dernière existe.

Théorème 2.9. (Correspondance avec le flot régulier, [91, Théorème 1.5]) *Soit Ω_0 un domaine compact dont le bord est $C^{3+\alpha}$. On suppose que c_0 est symétrique, que $c_0 \in C_c^\infty(\mathbb{R}^2)$ et que $c_1 \in C_c^\infty(\mathbb{R}^2 \times [0,T))$. Soit Ω_r le flot régulier donné par le Théorème 2.3. Alors, tout mouvement minimisant associé à $\mathcal{F}$ avec Ω_0 comme condition initiale vérifie $\Omega(t) = \Omega_r(t)$ pour tout $t \in [0,t_0)$ et presque partout dans $\mathbb{R}^2$.*

La preuve de ce théorème repose sur les sous/sur paires de solutions introduites par Cardaliaguet et Pasquignon [53].

Nous allons maintenant expliquer pourquoi les mouvements minimisants sont des solutions faibles. Pour toute suite $(h_n)_n$ tendant vers 0 et telle que $\Omega_{h_n}([\cdot/h_n])$ converge vers un mouvement minimisant Ω, on peut considérer les sous et sur limites des $\Omega_{h_n}(k)$ quand $n \to \infty$, Ω_* et Ω^*. Nous serons alors capables, grâce à l'équation d'Euler correspondante à la minimisation de la fonctionnelle $\mathcal{F}$, de calculer la vitesse (au sens de viscosité) de Ω_* et Ω^* en fonction de Ω. Cela permet de comparer Ω_* et Ω^* avec l'ensemble de niveau 0 de la solution de viscosité u utilisée dans la Définition 2.6. Comme $\Omega_* \subset \Omega \subset \Omega^*$, on pourra alors déduire que Ω est une solution faible de (15) :

Théorème 2.10. (Les mouvements minimisants sont des solutions faibles, [91, Théorème 1.4]))
Soit $\Omega_0 \in \mathcal{P}$ tel que $\mathcal{L}^2(\partial\Omega_0) = 0$. Soit Ω un mouvement minimisant associé à $\mathcal{F}$ avec Ω_0 comme condition initiale. On suppose que c_0 est symétrique, que $c_0 \in C_c^\infty(\mathbb{R}^2)$ et que $c_1 \in C_c^\infty(\mathbb{R}^2 \times [0,T))$.

Alors Ω est une solution faible de (15). *En particulier, s'il n'y a pas de fattening, i.e. si $\{u(\cdot,t) = 0\}$ est de mesure de Lebesgue nulle, alors u est une solution de viscosité de* (6)-(7) *avec u_0 comme condition initiale.*

2.1.4 Comportement à grande échelle

Nous allons maintenant présenter un résultat important concernant la dynamique d'une ligne de dislocation. Il s'agit de montrer la convergence à grande échelle de cette dynamique vers des mouvements de type mouvement par courbure moyenne. Ce résultat justifie de manière rigoureuse les calculs et les approximations faits par les physiciens.

Comme nous voulons regarder la dynamique à grande échelle, nous avons besoin de regarder ce qui se passe en temps long. Nous allons donc utiliser l'équation (13) dans laquelle le noyau J est positif et l'on considère les noyaux vérifiant également

$$\begin{cases} J(x) = \dfrac{1}{|x|^3}\, g\left(\dfrac{x}{|x|}\right) \text{ si } |x| \geq 1, \\ \\ J(-x) = J(x) \geq 0 \quad \forall\, x \in \mathbb{R}^2 \end{cases} \tag{18}$$

où $g \in C^\infty(\mathbf{S}^1)$ est une fonction définie sur le cercle $\mathbf{S}^1$ de $\mathbb{R}^2$.

On s'intéresse à la dynamique de lignes de dislocations de taille de l'ordre $1/\varepsilon$ et à la limite quand $\varepsilon \to 0$. Pour ce faire, on définit pour $\varepsilon > 0$ la fonction "rescallée" suivante :

$$u^\varepsilon(x,t) = u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2|\ln\varepsilon|}\right)$$

qui satisfait l'équation

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \left(\left(J^\varepsilon \star 1_{\{u^\varepsilon(\cdot,\cdot,t)\geq u^\varepsilon(x,t)\}}\right)(x) - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^2} J^\varepsilon\right)|Du^\varepsilon| & \text{dans} \quad \mathbb{R}^2 \times (0,T) \\ \\ u^\varepsilon(\cdot,0) = u_0^\varepsilon & \text{sur} \quad \mathbb{R}^2 \end{cases} \tag{19}$$

où J^ε est le noyau "rescallé" définit par

$$J^\varepsilon(x) = \frac{1}{\varepsilon^3|\ln\varepsilon|} J\left(\frac{x}{\varepsilon}\right). \tag{20}$$

Le scaling utilisé pour u^ε (et donc pour J^ε) est presque le scaling parabolique. Le terme $|\ln\varepsilon|$ est un facteur bien connu en physique (nous renvoyons par exemple à Barnett, Gavazza [31], Brown [45] et Hirth, Lothe [106]). D'un point de vue mathématique, il vient de la mauvaise décroissance à l'infini du noyau (en $\frac{1}{|x|^3}$) qui ne vérifie pas l'hypothèse classique (voir Barles, Georgelin [22], Ishii, Pires, Souganidis [120] ou Ishii [116])

$$\int_{\{(x,p)=0\}} c_0(x)|x|^2 < \infty \quad \forall\, p \in \mathbf{S}^1.$$

Nous renvoyons à la Section 4.1 du Chapitre 4 pour une discussion détaillée sur cette condition.

Dans la limite $\varepsilon \to 0$, cette dynamique est bien approchée par le mouvement par courbure moyenne anisotrope suivant :

$$\begin{cases} \dfrac{\partial u^0}{\partial t} + F(D^2u^0, Du^0) = 0 & \text{dans} \quad \mathbb{R}^2 \times (0,T) \\ \\ u^0(\cdot,0) = u_0 & \text{sur} \quad \mathbb{R}^2 \end{cases} \tag{21}$$

avec

$$F(M,p) = -g\left(\frac{p^\perp}{|p|}\right) \text{trace}\left(M \cdot \left(Id - \frac{p}{|p|} \otimes \frac{p}{|p|}\right)\right) \tag{22}$$

où $p^\perp$ est l'image du vecteur p par une rotation d'angle $\pi/2$. En particulier, on voit que l'équation (21) décrit le mouvement par courbure moyenne anisotrope avec la vitesse

$$g(\tau)\kappa$$

où κ est la courbure de la ligne de niveau de u^0 et τ est le vecteur unitaire tangent à la ligne de niveau de u^0.

Le résultat principal de cette sous-section est le suivant :

Théorème 2.11. (Convergence de la dynamique des dislocations vers le mouvement par courbure moyenne, [67, Théorème 1.4] et [88, Théorème 2.1])
On suppose que $u_0 \in Lip(\mathbb{R}^2)$, et $J \in W^{1,1}(\mathbb{R}^2)$. Alors la solution u^ε de la dynamique des dislocations (19) *converge localement uniformément sur les compacts de $\mathbb{R}^2 \times [0,\infty)$ vers l'unique solution u^0 du mouvement par courbure moyenne anisotrope* (21). *De plus, il existe une constante K_1 dépendant seulement de* $\sup_{\mathbb{R}^2} J$, $|Dg|_{L^\infty(\mathbf{S}^1)}$ *et $|Du_0|_{L^\infty(\mathbb{R}^2)}$ telle que la différence entre u^ε et u^0 est donnée par*

$$\sup_{\mathbb{R}^2\times(0,T)} |u^\varepsilon - u^0| \le K_1 \left(\frac{T}{|\ln \varepsilon|}\right)^{\frac{1}{6}} + \sup_{\mathbb{R}^2} |u_0^\varepsilon - u_0|$$

pour $T \le 1$.

Remarque 2.12. *Des résultats similaires (sans estimation d'erreur) ont été prouvés pour des noyaux généraux en relation avec l'algorithme de Merriman, Bence, Oscher pour la courbure moyenne [143]. Nous renvoyons par exemple à Barles, Georgelin [22], Evans [79], Ishii [116] et Ishii, Pires, Souganidis [120] pour de tels résultats. Néanmoins, comme nous l'avons indiqué précédemment, notre noyau ne satisfait pas les hypothèses de ces papiers.*

À partir de l'équation (22), on ne voit pas si le mouvement par courbure moyenne anisotrope (21) est de type variationnel ou non. Le Théorème 2.14 ci-dessous va répondre à cette question en montrant qu'il est effectivement de type variationnel. Pour énoncer ce résultat de manière précise, nous avons besoin de la définition suivante :

Définition 2.13. *Soit $g \in C^\infty(\mathbb{R}^2\backslash\{0\})$ tel que $g(\lambda p) = \dfrac{g(p)}{|\lambda|^3}$, $\forall \lambda \in \mathbb{R}\backslash\{0\}$ and $p \in \mathbb{R}^2\backslash\{0\}$. On associe alors à g une distribution tempérée L_g définie par*

$$\langle L_g, \varphi\rangle = \int_{\mathbb{R}^2} dx\; g(x)\left(\varphi(x) - \varphi(0) - x\cdot D\varphi(0)\; 1_{B_1(0)}(x)\right)$$

pour $\varphi \in \mathcal{S}(\mathbb{R}^2)$*, où* $\mathcal{S}(\mathbb{R}^2)$ *est l'espace des fonctions tests de Schwartz et* $B_1(0)$ *est la boule unité centrée en zéro.*

On définit également la transformée de Fourier de $\varphi \in \mathcal{S}(\mathbb{R}^2)$ *par*

$$\mathcal{F}(\varphi)(\xi) = \int_{\mathbb{R}^2} dx\ \varphi(x) e^{-i\xi\cdot x}.$$

On a alors le théorème suivant :

Théorème 2.14. (Origine variationnelle du mouvement par courbure moyenne anisotrope, [67, Théorème 1.7])
Soit $g \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ *tel que* $g(\lambda p) = \dfrac{g(p)}{|\lambda|^3}$, $\forall \lambda \in \mathbb{R}\setminus\{0\}$, $\forall p \in \mathbb{R}^2$*. Soit*

$$G := -\frac{1}{2\pi}\mathcal{F}(L_g) \tag{23}$$

où $\mathcal{F}(L_g)$ *est la transformée de Fourier de* L_g*. Alors* $G(\lambda p) = |\lambda| G(p)$*,* $\forall \lambda \in \mathbb{R}\setminus\{0\}$*,* $\forall p \in \mathbb{R}^2$*, et*

$$g\left(\frac{p^\perp}{|p|}\right) \frac{p^\perp}{|p|} \otimes \frac{p^\perp}{|p|} = D^2 G\left(\frac{p}{|p|}\right) \tag{24}$$

En particulier, on voit que G est convexe si et seulement si $g \geq 0$. De plus, (24) signifie que dans l'équation (21), on a

$$-F(D^2u^0, Du^0) = \text{div}\ \left(\nabla G\left(\frac{Du^0}{|Du^0|}\right)\right)\ |Du^0|,$$

c'est à dire que le mouvement par courbure moyenne anisotrope dérive d'une énergie convexe $\int G\,(Du^0)$.

Remarque 2.15. *Physiquement, la quantité* $\mathcal{F}(L_g)$ *est naturellement donnée et la fonction* g *peut alors être calculée en utilisant (23)-(24) où l'on peut également vérifier si* g *est positive ou non.*

Dans le cas le plus simple d'application pour la dynamique des dislocations, le cristal est décrit par l'élasticité isotrope. Quand le vecteur de Burgers est parallèle à l'axe x_1*, on a alors*

$$G(p) = \frac{p_2^2 + \frac{1}{1-\nu}\ p_1^2}{|p|} \quad \textit{avec} \quad \nu \in (-1, \frac{1}{2})$$

où ν *est le coefficient de Poisson du matériau et*

$$g(\theta) = \frac{(2\gamma-1)(\theta_1)^2 + (2-\gamma)(\theta_2)^2}{|\theta|^5} \geq 0 \quad \textit{avec} \quad \gamma = \frac{1}{1-\nu} \in (\frac{1}{2}, 2).$$

Notre Théorème 2.14 peut également être généralisé à des noyaux plus généraux. Il a été démontré que l'on pouvait approcher des mouvements par courbure moyenne généraux en utilisant le schéma de Merriman Bence Osher [143] avec des noyaux généraux K_0 satisfaisant $K_0(-x) = K_0(x)$ et pour tout $p \in \mathbf{S}^1$

$$\int_{\{x,\ \langle x,p\rangle=0\}} K_0(x)|x|^2 < \infty \tag{25}$$

et avec le "scaling parabolique" $K_0^\varepsilon = \dfrac{1}{\varepsilon^3} K_0\left(\dfrac{x}{\varepsilon}\right)$. Nous renvoyons par exemple à Barles Georgelin [22], Evans [79], Ishii [116] et Ishii, Pires, Souganidis [120] (Nous renvoyons également à la Section 4.1 du Chapitre 4 pour une preuve formelle).

De manière plus précise, sous l'hypothèse supplémentaire que $\int_{\{x,\ \langle x,p\rangle=0\}} K_0(x) = 1$ pour tout $p \in \mathbf{S}^1$, le mouvement limite trouvé dans [120], Section 3 avec le seuil $\theta = 1/2$, est (21)-(22), avec g remplacé par

$$g(\theta) = \int_{(0,+\infty)} dr\ r^2 K_0(r\theta) \tag{26}$$

À notre connaissance, dans ce cadre général, on ne savait pas si le mouvement par courbure moyenne limite était de type variationnel ou non. Il s'avère que cela est juste une conséquence de notre Théorème 2.14.

2.1.5 Analyse numérique de la dynamique d'une ligne de dislocation

Dans cette sous-section, nous allons donner quelques schémas numériques pour la dynamique d'une ligne de dislocation. Nous commençons par le cas où le noyau est positif, *i.e.* l'équation (13).

Schéma numérique pour (13)
Etant donné une taille de discrétisation $\Delta x, \Delta t$ et une grille

$$Q_T^\Delta = Q^\Delta \times \{0, ..., (\Delta t)N_T\}$$

où $Q^\Delta = \{(i_1\Delta x, i_2\Delta x),\ I = (i_1, i_2) \in \mathbb{Z}^2\}$ et N_T est la partie entière de $T/\Delta t$, on note $(x_{i_1}, x_{i_2}, t_n) = (x_I, t_n)$ le noeud $(i_1\Delta x, i_2\Delta x, n\Delta t)$ et v_I^n la valeur de l'approximation numérique de la solution exacte $u(x_I, t_n)$ de (13).

La solution discrète v est calculée de manière itérative en résolvant le schéma implicite suivant

$$v_I^0 = \tilde{u}_0(x_I), \qquad \frac{v_I^{n+1} - v_I^n}{\Delta t} = c^\Delta[v]_I^{n+1} G(v^{n+1})_I \tag{27}$$

où $\tilde{u}_0$ est une approximation de u_0 et $G(v^{n+1})_I$ est une approximation appropriée du gradient de v^{n+1} pris au point x_I. La vitesse non-locale est donnée par convolution discrète

$$c^\Delta[v]_I^{n+1} = c^\Delta[v](x_I, t_{n+1}) = \sum_{K\in\mathbb{Z}^2} \bar{J}_{I-K} 1_{\{v_K^{n+1} \geq v_I^{n+1}\}} (\Delta x)^2 - \frac{1}{2} \sum_{K\in\mathbb{Z}^2} \bar{J}_K (\Delta x)^2 \tag{28}$$

avec

$$\bar{J}_I = \frac{1}{|Q_I|} \int_{Q_I} J(x)\, dx \tag{29}$$

où Q_I est le cube unité centré en x_I

$$Q_I = [x_{i_1} - \Delta x/2, x_{i_1} + \Delta x/2] \times [x_{i_2} - \Delta x/2, x_{i_2} + \Delta x/2]. \tag{30}$$

Pour finir, on définit

$$v_\#(y, t_n) = \sum_I v(x_I, t_n) \chi_{Q_I}(y) \tag{31}$$

où χ_{Q_I} est la fonction indicatrice de Q_I.

L'approximation du gradient est obtenue en utilisant le schéma d'Osher, Sethian [154] (On peut également utiliser celui proposé par Rouy, Tourin [159]). Elle est monotone, consistante et dépend du signe de la vitesse non-locale. Nous renvoyons à la Section 5.1 du Chapitre 5 pour une définition précise.

Comme la vitesse c^Δ est non-locale et même non continue, on doit donner un sens à l'égalité dans le schéma (27). En fait, nous allons utiliser l'analogue de le formulation Slepčev [169] en utilisant la notion de sous et sur-solution discrète et nous utiliserons une version discrète de la méthode de Perron pour construire une solution discrète. Pour donner une définition précise des solutions discrètes, nous avons besoin de la notation suivante :

$$\tilde{c}^\Delta[v]_I^{n+1} = \sum_{K\in\mathbb{Z}^2} \bar{J}_{I-K} 1_{\{v_K^{n+1} > v_I^{n+1}\}} (\Delta x)^2 - \frac{1}{2} \sum_{K\in\mathbb{Z}^2} \bar{J}_K (\Delta x)^2, \tag{32}$$

où $\bar{J}$ est défini par (29). On note également $\tilde{G}$ l'approximation du gradient qui dépend du signe de la vitesse $\tilde{c}$.

Definition 2.16. (Sous/sur/solution discrète du schéma)
On dit que v est une sous-solution discrète (resp sur-solution) du schéma (27) si pour tout $I \in \mathbb{Z}^2$, $n \in \mathbb{N}$, on a

$$v_I^{n+1} \leq v_I^n + \Delta t\, c^\Delta[v]_I^{n+1} G(v^{n+1})_I$$

$$\left(\text{resp. } v_I^{n+1} \geq v_I^n + \Delta t\, \tilde{c}^\Delta[v]_I^{n+1} \tilde{G}(v^{n+1})_I\right).$$

Finalement, on dit que v est une solution discrète si et seulement si c'est une sous et une sur-solution.

Une des propriétés importantes de ce schéma est qu'il n'est pas monotone (à cause de la vitesse non-locale, voir Proposition 5.8 du Chapitre 5). Néanmoins, la monotonie de l'approximation du gradient G suffit pour montrer que pour toutes les solutions du schéma, nous avons une estimation d'erreur de type Crandall-Lions [63] :

Théorème 2.17. (Estimation d'erreur discret-continue pour (13), [88, Théorème 2.2])
Soit $T \leq 1$. On suppose que $u_0, \tilde{u}_0 \in W^{1,\infty}(\mathbb{R}^2)$ et que J donné par (18) satisfait $J \in W^{1,1}(\mathbb{R}^2)$. On suppose que $\Delta x + \Delta t \leq 1$. Alors, il existe une constante $K_2 > 0$ dépendant seulement de $|J|_{W^{1,1}(\mathbb{R}^2)}$, $|Du_0|_{L^\infty(\mathbb{R}^2)}$ et $|D\tilde{u}_0|_{L^\infty(\mathbb{R}^2)}$ telle que l'estimation d'erreur entre la solution continue u de (13) et n'importe quelle solution discrète v du schéma aux différences finies (27) est donnée par

$$\sup_{\mathbb{R}^2\times\{0,\dots,t_{N_T}\}} |u - v_\#| \leq K_2\sqrt{T}\,(\Delta x + \Delta t)^{1/2} + \sup_{\mathbb{R}^2} |u_0 - (\tilde{u}_0)_\#|$$

sous l'hypothèse complémentaire $\Delta x + \Delta t \leq \dfrac{1}{K_2^2}$.

Remarque 2.18. *Il est également possible d'expliciter le calcul du gradient, c'est à dire de remplacer le terme $G(v^{n+1})_I$ par $G(v^n)_I$ dans le schéma (27) et ainsi de considérer la solution v de*

$$v_I^0 = \tilde{u}_0(x_I), \qquad \frac{v_I^{n+1} - v_I^n}{\Delta t} = c^\Delta[v]_I^{n+1} G(v^n)_I. \tag{33}$$

Dans ce cas, comme d'habitude, on doit rajouter une condition CFL, par exemple de la forme

$$\Delta t \leq \frac{\Delta x}{2|J|_{L^1(\mathbb{R}^2)}}$$

pour la discrétisation du gradient proposée par Osher et Sethian. Sous cette hypothèse supplémentaire, le Théorème 2.17 reste vrai avec v la solution du schéma (33).

Il est également possible d'expliciter le calcul de la vitesse dans le schéma (27) et de considérer

$$v_I^0 = \tilde{u}_0(x_I), \qquad \frac{v_I^{n+1} - v_I^n}{\Delta t} = c^\Delta[v]_I^n G(v^n)_I. \tag{34}$$

Dans ce cas, il est possible de montrer la convergence du schéma (voir le Théorème 2.19 ci-dessous) sous la condition CFL suivante

$$\Delta t \leq \frac{\Delta x}{2|J|_{L^1(\mathbb{R}^2)}}. \tag{35}$$

Par contre, dans ce cas, nous ne sommes pas capables de montrer une estimation d'erreur. La différence vient du fait que lorsque l'on implicite la vitesse, cela la "gèle" et on peut alors utiliser l'erreur de consistance du schéma où la vitesse est donnée. Au contraire, pour le schéma explicite, nous devons contrôler l'estimation de consistance de la vitesse, ce qui n'est pas possible comme nous le montrons dans la Proposition 5.6 du Chapitre 5.

Pour énoncer le théorème de convergence du schéma (34), on définit

$$v^\delta(x,t) = v(x_I, t_n) \quad \text{si} \quad x \in Q_I,\ t \in [t_n, t_{n+1})$$

où $\delta = (\Delta x, \Delta t)$ et v est la solution du schéma (34)-(28)-(29). Le résultat de convergence est alors le suivant :

Théorème 2.19. (Convergence du schéma explicite (34), [88, Théorème 2.7])
On suppose que la condition CFL (35) est vérifiée. Alors, sous les hypothèses du Théorème 2.17, la fonction v^δ converge localement uniformément sur les compacts quand $\delta \to 0$ vers la solution u de (13).

Remarque 2.20. *Dans le schéma (34), il est également possible d'impliciter le calcul du gradient et ainsi d'enlever la condition CFL (35) pour obtenir le même résultat de convergence que le Théorème 2.19.*

Schéma numérique pour le mouvement par courbure moyenne
Le résultat de convergence de la dynamique des dislocations vers le mouvement par courbure moyenne suggère en fait un algorithme pour approximer la solution du mouvement par courbure moyenne anisotrope. Pour cela, il suffit de considérer la solution v^ε du schéma numérique (27)-(28)-(29) avec J^ε à la place de J dans la définition de $\bar{J}$. En combinant l'estimation d'erreur du Théorème 2.11 et celle du Théorème 2.17, on pourra alors montrer une estimation d'erreur entre la solution du mouvement par courbure moyenne (21) et son approximation numérique.

Ce genre de résultat est important d'un point de vue analyse numérique. En effet, il y a un grand nombre d'applications pour les mouvements par courbure moyenne, comme la propagation de fronts, le traitement d'image, la dynamique des fluides (nous renvoyons à Sethian [163] et Osher, Paragios [153] pour des exemples d'applications). D'un point de vue numérique, l'analyse de ce genre d'équation est assez difficile. Il y a cependant plusieurs travaux sur cette question. Le premier est celui d'Osher, Sethian [154]. Leur méthode numérique est très utilisée en pratique mais il n'existe pas, à notre connaissance, de preuve de convergence. Un autre algorithme est celui de Merriman, Bence, Osher [143] dans lequel le mouvement par courbure moyenne est vu comme la limite singulière d'une équation de diffusion. La convergence de ce schéma a été montré par Barles, Georgelin [22] et Evans [79] (voir aussi Ishii [116], Ishii, Pires, Souganidis [120] et Chambolle, Novaga [56]). Crandall et Lions [65] ont également proposé une classe de schéma numérique convergent pour des équations paraboliques non-linéaires incluant le mouvement par courbure moyenne. Un autre algorithme convergent a été proposé par Oberman [149]. Dans ces deux derniers travaux, les auteurs utilisent deux échelles différentes. La première est bien sur le pas d'espace Δx. La seconde est la taille du stencil sur lequel on va faire des calculs. Comme Crandall et Lions l'ont indiqué dans [65], ces deux échelles sont très importantes pour approximer des équations dégénérées comme l'équation par courbure moyenne.

Dans notre schéma, nous retrouvons ces deux échelles, ε étant ici lié à la taille du stencil sur lequel on calcule la convolution. L'estimation d'erreur pour le mouvement par courbure moyenne que nous allons énoncer n'est pas optimale (il est par exemple possible de changer le noyau, voir Section 3.3 du Chapitre 5), mais à notre connaissance, c'est la première estimation pour un schéma complètement discrétisé.

Théorème 2.21. (Estimation d'erreur discret-continue pour le mouvement par courbure moyenne, [88, Théorème 2.4])
Soit $T \leq 1$. On note v^ε la solution de (27)-(28)-(29) avec J^ε à la place de J. On suppose que $u_0, \tilde{u}_0 \in W^{1,\infty}(\mathbb{R}^2)$, $g \in C^\infty(\mathbf{S}^1)$ et que J donnée par (18) satisfait $J \in W^{1,1}(\mathbb{R}^2)$. On suppose également que $\Delta x + \Delta t \leq 1$. Alors il existe une constante $K_3 > 0$ dépendant seulement de $\sup_{\mathbb{R}^2} J$, $|Dg|_{L^\infty(\mathbf{S}^1)}$, $|J|_{W^{1,1}(\mathbb{R}^2)}$, $|Du_0|_{L^\infty(\mathbb{R}^2)}$ et $|D\tilde{u}_0|_{L^\infty(\mathbb{R}^2)}$ telle que l'estimation d'erreur entre la solution continue u^0 du mouvement par courbure moyenne anisotrope (21) *et son approximation numérique v^ε est donnée par*

$$\sup_{\mathbb{R}^2 \times \{0,..,t_{N_T}\}} |u^0 - v^\varepsilon_\#| \leq K_3 \left(\frac{T}{|\ln \varepsilon|} \right)^{1/6} + \sup_{\mathbb{R}^2} |u_0 - (\tilde{u}^\varepsilon_0)_\#|$$

où $\varepsilon \geq K_3(\Delta x + \sqrt{\Delta t})$.

Remarque 2.22. *Comme pour l'équation* (13)*, il est également possible d'expliciter le schéma en ajoutant une condition CFL. Dans ce cas, nous sommes toujours capables de montrer la convergence du schéma. Par contre l'estimation d'erreur est encore un problème ouvert.*

Remarque 2.23. *Il est possible de tronquer le noyau J à l'infini et de considérer*

$$\tilde{J}^R(x) = \begin{cases} J(x) & \text{si } |x| \leq R, \\ 0 & \text{sinon} \end{cases}$$

Dans ce cas, nous faisons une erreur de l'ordre de $\displaystyle\int_{\mathbb{R}^2 \setminus B_R(0)} J \leq \frac{K}{R}$ et il est possible de faire le calcul de la convolution sur un stencil de taille fini, même si Δx tend *vers zéro. Cela est possible si on choisit ε du même ordre que Δx. La condition $\varepsilon \geq K(\Delta x + \sqrt{\Delta t})$ dans le Théorème 2.21 implique alors que l'on doit imposer une condition CFL $\Delta t \leq K \Delta x^2$ (ce qui est classique pour les équations du second ordre).*

A l'inverse, si l'on impose aucune condition CFL, on peut choisir Δt beaucoup plus grand que Δx^2, mais on doit choisir ε du même ordre que $\sqrt{\Delta t}$ et donc, on doit faire la convolution sur des stencils de plus en plus grands quand Δx tend vers zéro.

Schéma numérique pour (3)
Nous considérons maintenant l'équation (3). Un schéma numérique pour l'équation level set associé à cette équation a été proposé par Alvarez, Carlini, Monneau et Rouy dans [8, 7]. Une estimation d'erreur a également été montrée en temps court (tant que la solution de viscosité continue existe). Le problème de cette approche est que la dislocation est représentée par la ligne de niveau zéro d'une fonction continue u. Numériquement, pour reconstruire cette ligne, il faut donc que le gradient de u ne s'annule pas. Le problème est que ce gradient décroit très vite sur la ligne de dislocation et numériquement, il faut donc relever le gradient très souvent. On peut alors se demander quel sens a vraiment la solution calculée et si on résoud toujours bien l'équation considérée.

Pour éviter ce genre de difficultés, nous allons essayer d'utiliser une méthode de type “Fast Marching Method” (FMM, voir Sethian [161] et Tsitsiklis [179]). L'avantage de ce type de méthode est qu'elle suit vraiment le front et il n'y a donc pas de problème pour le localiser. Par contre, la FMM classique ne marche que pour les vitesses données $c(x)$ qui sont strictement positives. Une version plus récente a été proposée par Vladimirsky [180] pour les vitesses positives mais dépendantes également du temps. La première étape est donc de généraliser cette méthode pour des vitesses données $c(x,t)$ qui peuvent s'annuler et changer de signe. La seconde étape (qui est encore un travail en cours) sera alors d'adapter cette méthode au cas de la dynamique d'une ligne de dislocation.

Nous commençons par rappeler la méthode Fast Marching proposée par Sethian en 1996 [161] dans le cas classique, c'est à dire quand la vitesse c est positive et ne dépend pas du temps. Cette méthode repose sur l'équation level set pour l'évolution d'un front avec une vitesse normale $c(x)$. Le front va alors être représenté par la ligne de niveau zéro d'une fonction continue u solution de l'équation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = c(x)|Du(x,t)| & \text{dans} \quad \mathbb{R}^2 \times (0,T) \\ u(\cdot,t) = u_0 & \text{sur} \quad \mathbb{R}^2 \end{cases}$$

La fonction u_0 est choisie de telle sorte que le front initial est représenté par la ligne de niveau zéro de u_0. On peut alors chercher des solutions de la forme $u(x,t) = t - T(x)$

où T est solution de

$$|DT(x)| = \frac{1}{c(x)}. \tag{36}$$

On se ramène donc à un problème stationnaire qui est en fait un problème de temps minimal. Dans cette fromulation, le front à l'instant t est représenté par la ligne de niveau t de T. L'équation (36) peut être discrétisée en utilisant le schéma de Rouy, Tourin [159]. On obtient alors le schéma suivant pour l'équation (36)

$$\max(T_{i,j} - T_{i-1,j}, T_{i,j} - T_{i+1,j}, 0)^2 + \max(T_{i,j} - T_{i,j-1}, T_{i,j} - T_{i,j+1}, 0)^2 = \left(\frac{\Delta x}{c_{i,j}}\right)^2 \tag{37}$$

où Δx est le pas d'espace, $T_{i,j}$ est l'approximation de T au point du maillage $(x_i, x_j) = (i\Delta x, j\Delta x)$ et $c_{i,j} = c(i\Delta x, j\Delta x)$.

L'idée introduite par Sethian dans [161] est très simple. Il s'agit de résoudre le schéma (37) seulement dans un voisinage du front. Il définit alors trois zones (voir Figure 3) :

1. Les points *acceptés* qui sont ceux sur lesquels le front est déjà passé et pour lesquels on connait la valeur de T.
2. Les points de la *Narrow Band* (NB) qui sont ceux qui ont un voisin accepté. Ce sont les prochains points potentiellement atteints par le front. Ce sont les seuls points pour lesquels on va résoudre le schéma (37).
3. Les points *Far Away* qui sont les autres points. Ils représentent en fait les points qui ne peuvent pas être atteint par le front dans l'immédiat.

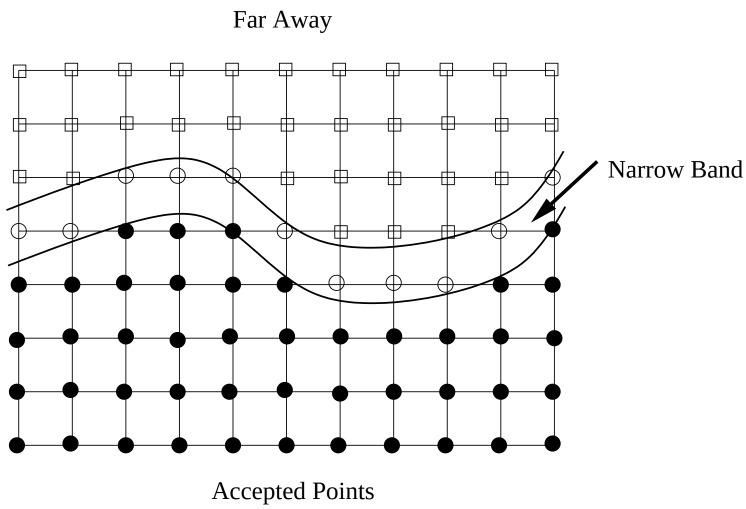

FIG. 3 – Schéma de la Narrow Band

Parmi les points de la Narrow Band, afin de résoudre (37), l'idée est de prendre pour les valeurs de T uniquement des points voisins qui sont déjà acceptés. L'algorithme est alors très simple :
Initialisation :
$T_{i,j} = 0 \; \forall (i,j)$ tels que $(i\Delta x, j\Delta x) \in \Omega_0$.
Boucle :

1. Pour tous les points (i,j) de la Narrow Band, on calcule $T_{i,j}$ en résolvant (37).
2. Le(s) point(s) (i,j) de la Narrow Band ayant la valeur minimale pour $T_{i,j}$ est(sont) accepté(s).
3. La Narrow Band est redéfinie comme le bord de la nouvelle région acceptée.

En utilisant cette formulation, le temps est donné directement par l'algorithme. À l'étape n de l'algorithme, on définit le temps t_n comme la valeur minimale pour $T_{i,j}$ des points de la Narrow Band.

Nous allons maintenant énoncer notre résultat de convergence pour le cas de la Fast Marching classique. Pour cela, nous supposons que le front initial est le bord d'un domaine ouvert Ω_0 qui est représenté par la fonction caractéristique $1_{\Omega_0} - 1_{\Omega_0^c}$ qui vaut 1 dans Ω_0 et -1 dans son complémentaire. On considère alors la solution de viscosité discontinue $\theta(x,t)$ de l'équation

$$\begin{cases} \dfrac{\partial \theta}{\partial t}(x,t) = c(x)|D\theta(x,t)| & \text{dans} \quad \mathbb{R}^2 \times (0,T) \\ \\ \theta(\cdot,0) = 1_{\Omega_0} - 1_{\Omega_0^c} & \text{sur} \quad \mathbb{R}^2. \end{cases} \tag{38}$$

Ici le front est représenté par le support des discontinuités de la fonction θ.

On définit alors $\theta_{i,j}^n$ qui vaut 1 si le point (i,j) est accepté au temps t_n et -1 sinon et on étend cette fonction sur $\mathbb{R}^2 \times (0,T)$ en une fonction continue par morceaux de la manière suivante :

$$\theta^\varepsilon(x,t) = \theta_{i,j}^n \quad \text{si } t \in [t_n, t_{n+1}) \text{ et } x \in [x_i, x_i + \Delta x) \times [x_j, x_j + \Delta x),$$

où $\varepsilon = (\Delta x, \Delta t)$.

Notre résultat de convergence s'énonce alors sur les semi-limites relaxées de la fonction θ^ε :

$$\overline{\theta}^0(x,t) = \limsup_{\varepsilon \to 0, y \to x, s \to t} \theta^\varepsilon(y,s), \quad \underline{\theta}^0(x,t) = \liminf_{\varepsilon \to 0, y \to x, s \to t} \theta^\varepsilon(y,s). \tag{39}$$

Théorème 2.24. (Résultat de convergence, [55, Théorème 2.5])
On suppose que $c \in C^\infty(\mathbb{R}^2)$ et que Ω_0 est un ensemble de classe C^2 avec un bord $\partial\Omega_0$ borné. Alors $\overline{\theta}^0$ (resp. $\underline{\theta}^0$) est une sous solution (resp. une sur solution) de viscosité de (38).

Remarque 2.25. *Ce théorème est en fait une preuve de convergence alternative à celle proposée par Cristiani et Falcone [66] (qui utilise le lien avec les Level Set). Ici, nous proposons une preuve directe qui utilise la théorie des solutions de viscosité et qui pourra s'adapter au cas où la vitesse change de signe et dépend du temps.*

Nous allons expliquer maintenant les différences que nous avons dû introduire pour généraliser cet algorithme dans le cas de vitesses générales $c(x,t)$. Il y a en fait deux difficultés assez différentes.

La première est la dépendance en temps. Comme nous l'avons indiqué pour la FMM, le temps est donné par la valeur de T du dernier point accepté. A l'étape n de l'algorithme, il suffit donc de résoudre le schéma (37) en remplaçant $c_{i,j}$ par $c_{i,j}^{n-1} = c(x_i, x_j, t_{n-1})$. Il y a par contre quelques précautions à prendre. Tout d'abord, le minimun des $T_{i,j}$ calculés à l'étape n peut être plus petit que le temps t_{n-1} (par exemple si la vitesse croît en temps). Dans ce cas, pour garder une suite de temps croissante, il ne faut pas définir t_n comme le minimum des $T_{i,j}$ mais poser $t_n := t_{n-1}$.
Ensuite, le minimum des $T_{i,j}$ calculés peut être très grand (par exemple si la vitesse est très petite). Dans ce cas, si on ne fait rien, l'algorithme avancerait beaucoup trop vite et ne verrait pas les changements de la vitesse au cours du temps. Pour cette raison, nous devons introduire un pas de temps Δt assez petit. Si le temps candidat calculé (c'est à dire le minimun des $T_{i,j}$) est plus grand que $t_{n-1} + \Delta t$ alors il ne faut accepter aucun point et seulement faire évoluer le temps en posant $t_n := t_{n-1} + \Delta t$.

La deuxième difficulté vient du fait que la vitesse peut changer de signe en espace temps. Le front peut donc avancer dans un sens ou dans l'autre et nous avons donc besoin de connaître le temps de chaque côté du front. La première chose à faire est de régulariser la vitesse numérique en espace en ajoutant une bande de zéro pour séparer les régions où elle est positive de celles où elle est négative. On définit également un champ θ qui vaudra 1 dans Ω et -1 dans son complémentaire. L'idée est alors, à chaque étape, de résoudre deux Fast Marching différentes suivant le signe de la vitesse et ensuite de prendre le minimum des $T_{i,j}$ sur les deux zones. Dans la zone où la vitesse est positive, c'est en fait l'ensemble $\{\theta = 1\}$ qui va grossir alors que dans la zone où la vitesse est négative c'est la zone $\{\theta = -1\}$ qui va grossir.

Pour éviter de rendre cette introduction trop technique, nous ne donnons pas l'algorithme en détail et nous renvoyons à la Section 2.1 du Chapitre 6 pour plus de précisions.

Nous allons maintenant énoncer notre résultat de convergence. Tout d'abord, la suite $\{t_n, n \in \mathbb{N}\}$, définie par l'algorithme est seulement croissante. On peut alors extraire une sous-suite $\{t_{n_k}, k \in \mathbb{N}\}$ strictement croissante telle que

$$t_{n_k} = t_{n_k+1} = ... = t_{n_{k+1}-1} < t_{n_{k+1}}.$$

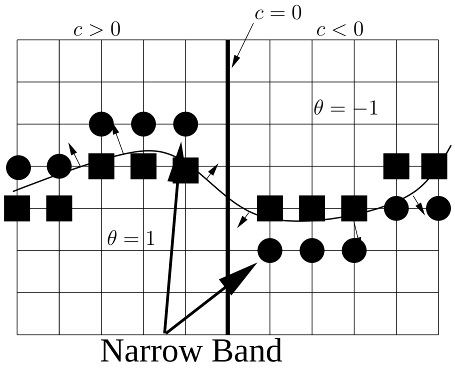

FIG. 4 – Schéma de la Fast Marching Gérnéralisée

On définit alors la fonction θ^ε par

$$\theta^\varepsilon(x,t) = \theta_I^{n_{k+1}-1} \text{ si } (x,t) \in [x_{i_1}, x_{i_1} + \Delta x) \times [x_{i_2}, x_{i_2} + \Delta x) \times [t_{n_k}, t_{n_{k+1}}[\tag{40}$$

et on considère ses semi-limites relaxées définies par (39).

Théorème 2.26. (Résultat de convergence, [55, Théorème 2.5]) *On suppose que $c \in C^\infty(\mathbb{R}^2 \times [0,T))$ et que Ω_0 est un ensemble de classe C^2 avec un bord $\partial\Omega_0$ borné. Alors $\overline{\theta}^0$ (resp. $\underline{\theta}^0$) est une sous-solution de viscosité (resp. sur-solution) de (38) (avec $c(x,t)$ à la place de $c(x)$). En particulier, si (38) satisfait un principe de comparaison, alors $\overline{\theta}^0 = (\underline{\theta}^0)^*$ et $(\overline{\theta}^0)_* = \underline{\theta}^0$ est l'unique solution de viscosité discontinue de (38).*

Remarque 2.27. *Quand l'unicité a lieu, c'est aux enveloppes semi-continues près.*

Le but est ensuite d'adapter cet algorithme au cas des dislocations. L'idée est juste de remplacer la vitesse $c(x_I, t_n)$ par une approximation de la vitesse de la ligne de dislocation donnée par convolution. La difficulté majeure vient du fait que la vitesse non-locale dépend fortement de la géométrie de l'ensemble Ω et donc quand on accepte un seul point, la géométrie locale change beaucoup et influe sur la vitesse. Il faut donc introduire un pas de temps $\tilde{\Delta t}$ et fixer la vitesse pendant un temps $\tilde{\Delta t}$. Ce travail est en cours et nous présentons seulement quelques simulations réalisées mais sans aucun résultat théorique.

Nous présentons une simulation concernant la propagation d'une ligne au travers d'un obstacle. La vitesse est donnée par $c(x,t) = c^0 \star 1_\theta(x,t) + c^1(x,t)$ avec

$$c^1(x,t) = \begin{cases} -0.5 & \text{si } |x|^2 < 0.3 \\ 2 & \text{sinon.} \end{cases}$$

Le calcul de la solution discrète est faite sur le domaine numérique $[-3,3]\times[-3,3]$, avec 129 noeuds sur chaque face et la vitesse est remise à jour tous les $\Delta t = 0.01$.
Comme nous avons rajouté une contrainte extérieure, la ligne doit commencer à se déplacer. La partie négative de la contrainte extérieure peut être interprétée comme un obstacle dans le domaine. Quand la ligne rencontre l'obstacle, une partie de la ligne va être capturée par l'obstacle. Il y a alors un changement de topologie et une partie de la ligne continue son déplacement en ayant passé l'obstacle.
Dans la Figure 5 nous montrons l'évolution de la ligne en rouge et l'obstacle en bleu aux itérations $0, 40, 90, 120, 160, 200$.

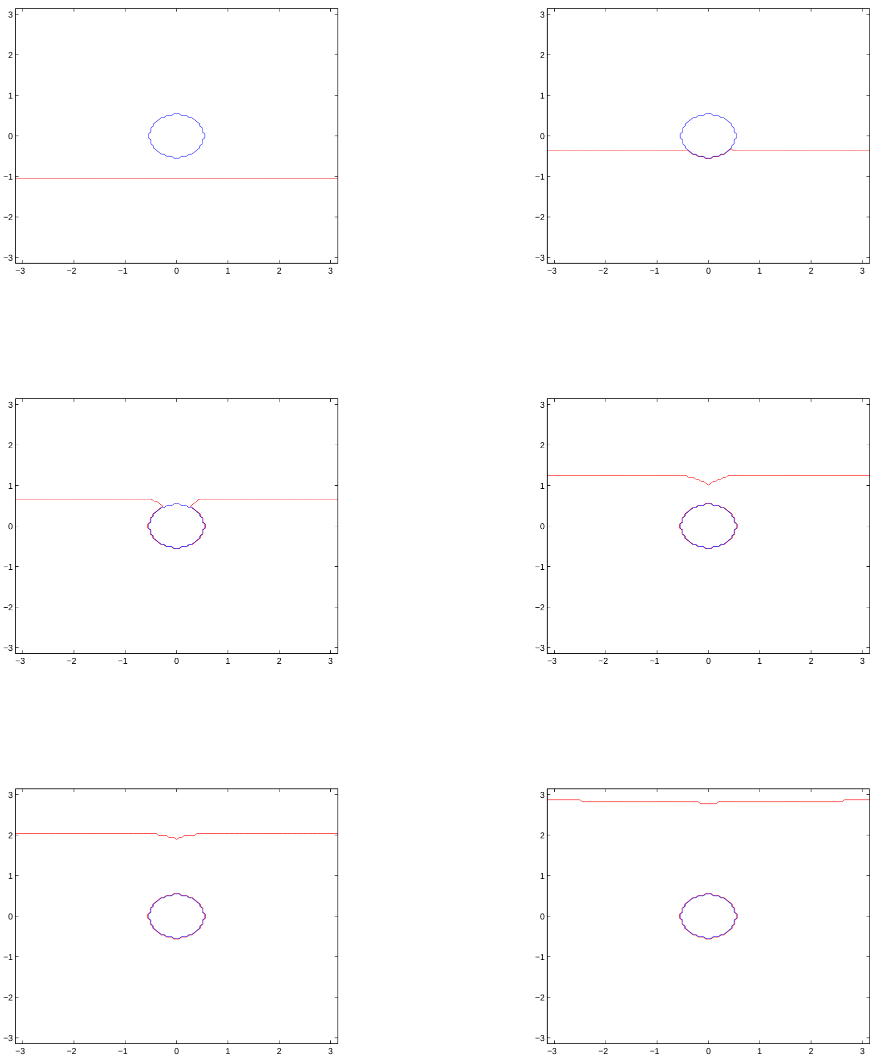

FIG. 5 – Evolution d'une ligne de dislocation à travers un obstacle.

2.2 Dynamique de jonctions entre dislocations

Dans la section précédente, nous avons étudié des modèles simplifiés où l'on était en présence d'une seule ligne de dislocation. La réalité est bien sûr beaucoup plus compliquée. On observe par exemple dans les cristaux des structures auto-organisées, comme des réseaux de Frank, c'est à dire des réseaux de dislocations reliées par des jonctions (voir Hull, Bacon [109, p188-190] pour de telles observations). Dans cette section, nous allons considérer un cas particulier où le réseau est contenu dans un seul plan de glissement, où les dislocations peuvent bouger. Nous allons étudier une version dynamique d'un modèle de champs de phase de Koslowski et Ortiz [127] pour les réseaux de dislocations. On va s'intéresser en particulier au déplacement des jonctions entre dislocations, qui reste un problème assez ouvert, autant du point de vue de la modélisation que de l'analyse mathématique (nous renvoyons à Garroni [92] pour le cas stationnaire). Notre but est de proposer, dans un cas simple, un modèle pour la dynamique de jonctions de dislocations et de l'étudier mathématiquement.

La question de l'étude mathématique des jonctions a beaucoup d'autres applications en physiques et il y a beaucoup de littérature sur ce sujet. Nous mentionnons, par exemple, le problème de croissance de cristal (voir Taylor [177, 178] et Bronsard, Reitich [44]). Nous renvoyons également à Bonnet [40] pour des problèmes concernant la minimisation de la fonctionnelle de Mumford-Shah.

2.2.1 Modélisation de la dynamique de jonctions entre dislocations

Nous allons considérer un modèle de champs de phase où la dislocation est représentée par la transition de phase d'un paramètre de phase $\rho(x) = \rho_1(x)e^1 + \rho_2(x)e^2 \in \mathbb{R}^2$ défini pour $x = x_1e^1 + x_2e^2$ dans le plan $\mathbb{R}^2$ avec (e^1, e^2) une base orthonormale. Koslowski et Ortiz [127] ont proposé un modèle où l'énergie élastique des dislocations, en présence d'une contrainte appliquée extérieure constante $\sigma^0 \in \mathbb{R}^2$, est donnée par

$$\mathcal{E}(\rho) = \int_{\mathbb{R}^2} -\frac{1}{2}(C^0 \star \rho) \cdot \rho - \sigma^0 \cdot \rho + W(\rho).$$

Le noyau $C^0(x)$ est une matrice symétrique 2×2 qui prend en compte les interactions longue distance entre les dislocations et

$$(C^0 \star \rho)_i = \sum_{j=1,2} C^0_{ij} \star \rho_j, \quad \text{pour} \quad i = 1, 2$$

où $\star$ est la convolution usuelle.

Pour chaque transition de phase entre deux états A et B, la différence $B - A$ doit être, physiquement, le vecteur de Burger de la dislocation, c'est à dire un vecteur du réseau cristallin $\Lambda = \mathbb{Z}a^1 + \mathbb{Z}a^2$ que nous considérons, avec une base générale

(a^1, a^2). Cette information est contenue dans le potentiel $W : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ qui va être minimal sur Λ et qui va avoir la périodicité du réseau cristallin Λ :

$$W(\rho + a) = W(\rho) \quad \text{pour tout} \quad a \in \Lambda. \tag{41}$$

Dans ce modèle, une jonction entre trois dislocations de vecteur de Burger $b^1, b^2, b^3 \in \Lambda$ avec $b^1 + b^2 + b^3 = 0$ est possible, et peut être représentée, par exemple, par les transitions de phases entre les états $0, b^1, -b^3$ (voir Figure 6).

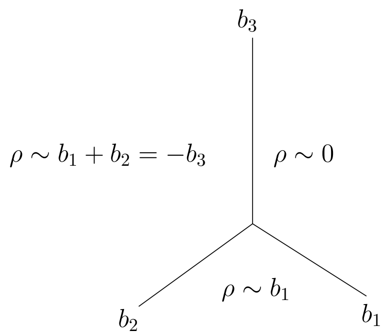

FIG. 6 – Une jonction entre trois dislocations comme les transitions de phases de ρ.

La contrainte résolue $\sigma[\rho]$ créée par toutes les dislocations est donnée, de manière formelle, par l'opposé du gradient de l'énergie $-\mathcal{E}'(\rho)$ et peut être exprimée par la quantité non locale suivante :

$$\sigma[\rho] = \sigma^0 + C^0 \star \rho - W'_\rho(\rho).$$

Le paramètre de phase $\rho(t, x) \in \mathbb{R}^2$ est alors supposé satisfaire l'équation suivante pour $k = 1, 2$

$$\begin{cases} (\rho_k)_t = |\nabla\rho|^{-1} \displaystyle\sum_{i=1,2} \sum_{j=1,2} (\sigma[\rho])_i \nabla_j \rho_i \nabla_j \rho_k + \varepsilon \Delta \rho_k, & \text{dans} \quad (0, T) \times \mathbb{T}^2, \\ \rho(0, x) = \rho^0(x) & \text{sur} \quad \mathbb{T}^2, \end{cases} \tag{42}$$

où

$$|\nabla\rho|^2 = \sum_{i=1,2} \sum_{j=1,2} |\nabla_j \rho_i|^2.$$

Le paramètre $0 < \varepsilon < 1$ est une petite viscosité introduite dans le modèle pour régulariser l'équation mais n'a pas vraiment de sens physique. L'équation (42) est uniquement posée sur le tore $\mathbb{T}^2 = \mathbb{R}^2 \backslash \mathbb{Z}^2$ pour simplifier l'analyse. Ce genre de conditions périodiques a également un sens physique si on veut décrire des réseaux de dislocations périodiques.

Finalement, nous mentionnons également que notre modèle a quelques similitudes avec le modèle d'Allen, Cahn [2] dans lequel ils considèrent des flots de gradients associés à une fonctionnelle d'énergie libre. Cela conduit à l'étude d'équation de diffusion scalaire de type Ginzburg-Landau de la forme

$$u_t = \Delta u - W'(u).$$

2.2.2 Résultat d'existence pour la dynamique de jonctions entre dislocations

Nous allons présenter un résultat d'existence pour la dynamique de jonctions entre dislocations. Pour cela, nous avons besoin de l'hypothèse suivante sur le noyau $C^0 : \mathbb{T}^2 \longrightarrow \mathbb{R}^{2\times 2}_{sym}$

(A) On suppose qu'il existe une constante $m > 0$, telle que pour tout $k \in \mathbb{R}^2$, les coefficients de Fourier du noyau $\widehat{C}^0(k) = \int_{\mathbb{T}^2} dx\ e^{-2i\pi k\cdot x} C^0(x)$ satisfassent $\widehat{C}^0(k) = M(k)$, pour tout $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ et pour tout $p = (p_1, p_2) \in \mathbb{R}^2$

$$\left\{ \begin{array}{l} M \in C^\infty(\mathbb{R}^2 \setminus \{0\}\,; \mathbb{R}^{2\times 2}_{sym}), \quad M(-\xi) = M(\xi), \quad M(\xi) = |\xi| M\left(\dfrac{\xi}{|\xi|}\right) \\ \\ \dfrac{|\xi|\ |p|^2}{m} \geq -\displaystyle\sum_{i=1,2}\sum_{j=1,2} p_i \cdot M_{ij}(\xi) \cdot p_j \geq m|\xi|\ |p|^2 \quad \text{avec} \quad |p|^2 = \sum_{i=1,2} |p_i|^2 \end{array} \right. \tag{43}$$

Cette hypothèse est satisfaite naturellement par le noyau physique (voir Section 2 du Chapître 7).

On a alors le théorème d'existence suivant :

Théorème 2.28. (Existence d'une solution pour la dynamique de jonctions entre dislocations, [90, Théorème 1.1])
On suppose que C^0 vérifie l'hypothèse (A) et que $W \in C^2(\mathbb{T}^2, \mathbb{R}_+)$ vérifie (41). On suppose également que $\rho^0 \in (H^2(\mathbb{T}^2))^2$. Alors pour toute contrainte constante $\sigma^0 \in \mathbb{R}^2$ et pour tout temps $T > 0$, il existe une solution ρ of (42) avec $\rho \in C^0\left([0,T); \left(L^{\frac{4}{3}}(\mathbb{T}^2)\right)^2\right)$.

L'unicité de la solution n'est pas connue, ni l'existence d'une solution quand $\varepsilon = 0$. Nous mentionnons également que l'équation (42) est un système non local d'équations scalaires et peut être comparée à l'équation suivante

$$v_t = |\nabla v|^2 + \Delta v \tag{44}$$

En effet, cela vient de l'hypothèse (A) qui implique que la convolution avec le noyau C^0 se comporte comme un opérateur du premier ordre. Beaucoup de travaux ont été faits sur des équations similaires à l'équation (44). Nous renvoyons par exemple aux travaux de Boccardo, Murat, Puel [35, 36, 37] dans lesquels ils ont montré l'existence de solutions pour des équations générales incluant l'équation (44).

L'équation (44) est également similaire à l'équation de Navier-Stokes écrite pour le potentiel A tel que la vitesse du fluide est donnée par $u = \text{curl } A$ (voir Leray [136]).

2.3 Homogénéisation de la dynamique de dislocations

Dans cette section, nous allons présenter des résultats d'homogénéisation pour la dynamique de lignes de dislocations en interactions. L'objectif est de faire le lien entre la dynamique d'un nombre fini de dislocations et la dynamique de densité de dislocations.

Nous allons commencer par un résultat qui peut paraître surprenant et qui est l'homogénéisation d'un système dynamique de particules (représentant les lignes de dislocations). De manière plus précise, on veut décrire la dynamique effective pour le mouvement collectif de lignes de dislocations ayant toutes le même vecteur de Burgers et toutes contenues dans le même plan de glissement et se déplaçant dans un milieu périodique (voir Figure 7).

On suppose également que toutes les lignes de dislocations sont parallèles. On est donc ramené à un problème en dimension $N = 1$ où les lignes de dislocations sont représentées par des points. Il s'agit alors d'étudier le mouvement d'un nombre fini N_ε de particules $y_i(\tau)$ (représentant la position de la dislocation i au temps τ) qui satisfont l'équation différentielle ordinaire suivante :

$$\dot{y}_i = F - V_0'(y_i) - \sum_{j \in \{1,\dots,N_\varepsilon\} \setminus \{i\}} V'(y_i - y_j) \quad \text{pour } i = 1, \dots, N_\varepsilon \tag{45}$$

où F est une force constante donnée, V_0 est un potentiel $1-$périodique et

$$V(z) = -\ln |z|$$

représente les interactions par paires (voir Figure 8 pour une représentation schématique des interactions ; pour être plus proche de la réalité, nous devrions ajouter un ressort entre chaque paire de dislocations).

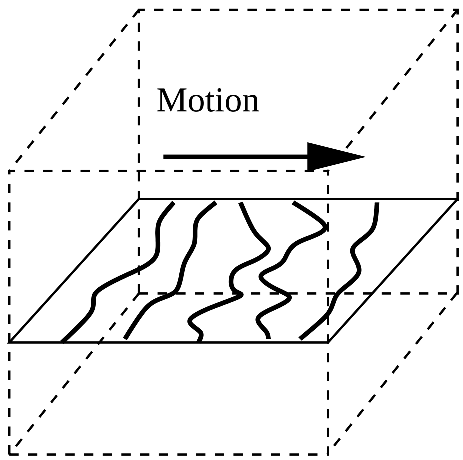

FIG. 7 – Les dislocations dans un plan de glissement.

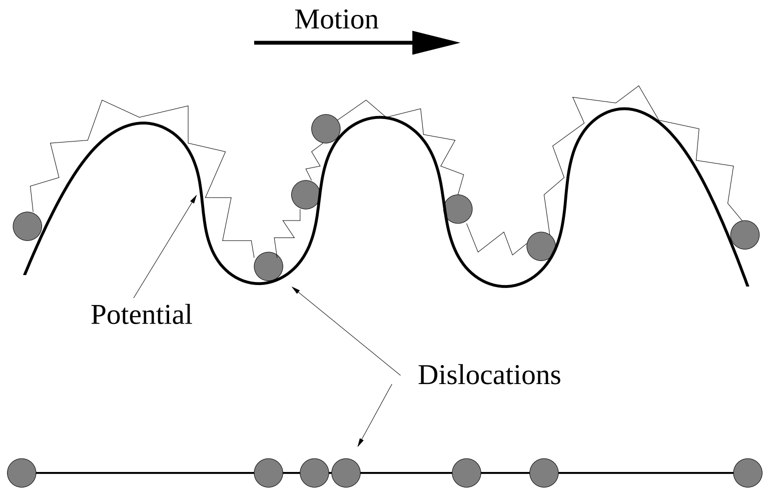

FIG. 8 – Représentation schématique d'un modèle 1D pour des dislocations reliées par des ressorts

Ce système d'EDO a des similitudes avec le modèle de Frenkel-Kontorova suramorti [126], sauf que dans le modèle de Frenkel-Kontorova classique, seules les interactions aux plus proches voisins sont considérées (voir Hu, Qin, Zheng [108] et aussi les travaux reliés de Aubry [15], Aubry, Le Daeron [16] pour les solutions stationnaires).

Pour énoncer notre résultat d'homogénéisation, nous avons besoin de définir la fonction de répartition constante par morceaux ρ de la manière suivante :

$$\rho(y,\tau) = -\frac{1}{2} + \sum_{i=1}^{N_\varepsilon} H(y - y_i(\tau)) \tag{46}$$

où H est la fonction de Heaviside définie par

$$H(r) = \begin{cases} 1 & \text{si} \quad r \geq 0, \\ 0 & \text{si} \quad r < 0. \end{cases}$$

On définit également

$$\rho^\varepsilon(y, \tau) = \varepsilon \rho \left(\frac{y}{\varepsilon}, \frac{\tau}{\varepsilon} \right). \tag{47}$$

Le but est de montrer un résultat d'homogénéisation, c'est à dire de montrer que la limite u^0 de ρ^ε quand $\varepsilon \to 0$ existe et qu'elle est l'unique solution d'une équation homogénéisée. A la limite, on s'attend à obtenir une équation satisfaite par la "fonction de répartition de dislocations" u^0 dont le gradient représente la densité de dislocations. Plus précisement, l'équation effective a la forme suivante :

$$\begin{cases} \dfrac{\partial u^0}{\partial t} = \overline{H}(\mathcal{I}_1[u^0(\cdot, t)], Du_0) & \text{dans} \quad \mathbb{R} \times (0, \infty) \\ \\ u^0(\cdot, 0) = u_0 & \text{sur} \quad \mathbb{R} \end{cases} \tag{48}$$

où $\overline{H}$ est une fonction continue qui sera définie plus loin (voir (53)), $u_0 \in W^{2,\infty}(\mathbb{R})$ et $\mathcal{I}_1$ est un opérateur de Lévy d'ordre 1 défini pour une fonction $U \in C_b^2(\mathbb{R})$ par

$$\begin{aligned} \mathcal{I}_1[U](x) = & \int_{|z| \leq r} (U(x+z) - U(x) - \nabla_x U(x) \cdot z) \, \frac{1}{|z|^2} dz \\ & + \int_{|z| \geq r} (U(x+z) - U(x)) \, \frac{1}{|z|^2} dz, \end{aligned}$$

pour tout $r > 0$ et où l'expression est indépendante de r.

Le résultat d'homogénéisation pour le système de particules est alors le suivant :

Théorème 2.29. (Homogénéisation du système de particules, [89, Théorème 2.14])
On suppose que V_0 est $1-$périodique et que V_0' est Lipschitz. On suppose également que $y_1(0) < \ldots < y_{N_\varepsilon}(0)$ sont données par les discontinuités d'une fonction $\rho_0^\varepsilon = \varepsilon E \left(\dfrac{u_0(x)}{\varepsilon} \right)$ avec $u_0 \in W^{2,\infty}(\mathbb{R})$, u_0 croissante sur $\mathbb{R}$ et E une modification de la partie entière définie par

$$E(\alpha) = k + \frac{1}{2} \quad \text{si } k \leq \alpha < k+1. \tag{49}$$

Alors ρ^ε définie par (47) converge vers la solution u^0 de (48).

Dans l'équation (48), l'opérateur intégral garde en mémoire les intéractions longue distance alors que la non-linéarité garde en mémoire les interactions à courte distance. Nous renvoyons à Imbert, Monneau, Rouy [112] pour une interprétation mécanique du problème homogénéisé.

Nous avons également les propriétés qualitatives suivantes pour l'Hamiltonien effectif $\overline{H}$, avec les notations

$$c(y) = V_0'(y) - F \qquad \text{et} \qquad J = V'' \text{ sur } \mathbb{R}\backslash\{0\}. \tag{50}$$

Théorème 2.30. (Propriétés qualitatives de $\overline{H}$, [89, Théorème 2.6]) *Sous l'hypothèse $\int_{(0,1)} c = 0$, la fonction $\overline{H}^0(L,p)$ est continue et satisfait les propriétés suivantes :*

1. *Si $c \equiv 0$, alors $\overline{H}(L,p) = L|p|$.*
2. **(Borne)** *Il existe une constante C telle que :*

$$\left|\frac{\overline{H}(L,p)}{|p|} - L\right| \leq C \quad \textit{pour} \quad (L,p) \in \mathbb{R}\times\mathbb{R}.$$

3. **(Signe de l'Hamiltonien)**

$$\overline{H}(L,p)L \geq 0 \quad \textit{pour} \quad (L,p) \in \mathbb{R}\times\mathbb{R}.$$

4. **(Module de continuité en L)** *Il existe une constante $C = |p|C_1$ avec C_1 dépendant seulement de $\|\nabla c\|_\infty$ telle que*

$$0 \leq \overline{H}(L+L',p) - \overline{H}(L,p) \leq \frac{C}{|\ln L'|} \quad \text{pour } 0 < L' \leq \frac{1}{2}.$$

5. **(Antisymétrie en L)** *S'il existe $a \in \mathbb{R}$ tel que $-c(y) = c(y+a)$, alors :*

$$\overline{H}(-L,-p) = -\overline{H}(L,p).$$

6. **(Symétrie en p)** *S'il existe $a \in \mathbb{R}$ tel que $c(-y) = c(y+a)$, alors :*

$$\overline{H}(L,-p) = \overline{H}(L,p).$$

7. **(0-plateau)** *Si $c \not\equiv 0$, alors il existe $r_0 > 0$ (dépendant seulement de $\|c\|_\infty$ et de $J_{|\mathbb{R}\backslash[-1,1]}$) tel que :*

$$\overline{H}(L,p) = 0 \quad \textit{pour} \quad (L,p) \in B_{r_0}(0) \subset \mathbb{R}^2.$$

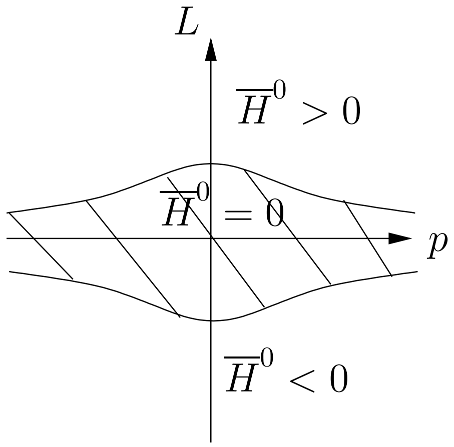

FIG. 9 – Représentation schématique de l'Hamiltonien effectif

Dans la Figure 9, nous avons représenté le profil typique de $\overline{H}^0$. Nous renvoyons également à Ghorbel [96] et Ghorbel, Hoch, Monneau [97] pour des simulations.

L'idée qui fait marcher ces deux résultats est que l'on peut injecter notre système de particules dans une équation aux dérivées partielles. De manière plus précise, on peut montrer que ρ^ε est solution de

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \left(c\left(\dfrac{x}{\varepsilon}\right) + M^\varepsilon\left[\dfrac{u^\varepsilon(\cdot,t)}{\varepsilon}\right](x)\right)|Du^\varepsilon| & \text{dans} \quad (0,+\infty)\times\mathbb{R}, \\ u^\varepsilon(\cdot,0) = u_0 & \text{sur} \quad \mathbb{R} \end{cases} \tag{51}$$

où M^ε est un opérateur non local d'ordre 0 défini par

$$M^\varepsilon\left[U\right](x) = \int_{\mathbb{R}} dz\, J(z) E\left(U(x+\varepsilon z) - U(x)\right) \tag{52}$$

avec E définie par (49).

L'Hamiltonien effectif est alors déterminé par l'unique constante λ telle que pour $L, p \in \mathbb{R}$, il existe une solution $1-$périodique v du problème dans la cellule suivant

$$\lambda + \partial_\tau v = \Big(c(\tau, y) + L + M_p[v(\tau,\cdot)](y)\Big)|p + \nabla v| \text{ dans} \quad (0,+\infty)\times\mathbb{R}^N \tag{53}$$

où

$$M_p[U](y) = \int dz\, J(z)\left\{E\left(U(y+z) - U(y) + p\cdot z\right) - p\cdot z\right\}.$$

De plus, on a le résultat de convergence suivant (ainsi que le Théorème 2.30) :

Théorème 2.31. (Convergence, [89, Théorème 2.5])
On suppose que c est Lipschitz et $1-$périodique et que $u_0 \in W^{2,\infty}(\mathbb{R})$. Alors l'unique solution de viscosité bornée u^ε de (51) converge quand $\varepsilon \to 0$ localement uniformément en (t,x) vers l'unique solution de viscosité bornée u^0 de (48).

Ce résultat suit les travaux de Imbert, Monneau [111] et Imbert, Monneau et Rouy [112] (nous renvoyons également à Alvarez, Bardi [4, 5], Evans [77, 78] et Lions, Papanicolaou et Varadhan [138] pour une bonne introduction à l'homogénéisation des équations de Hamilton-Jacobi). La différence majeure est qu'ici on propose un modèle qui décrit mieux la dynamique d'un nombre fini de lignes de dislocations en interactions. La difficulté technique est d'arriver à traiter la partie entière E qui est discontinue. Dans [112], les auteurs ont considéré un modèle différent où la partie entière a été en quelque sorte régularisée. Ici, au contraire, on veut garder le modèle avec la partie entière. Pour surmonter la difficulté créée par la discontinuité, il faut utiliser la formulation level set proposée par Slepčev [169] pour les équations non-locales. L'avantage de cette formulation est qu'elle permet d'avoir de la stabilité dans la définition des solutions de viscosité, ce qui est crucial pour cette théorie.

2.4 Dynamique de densités de dislocations

Après avoir étudié la dynamique d'une ligne de dislocation et fait le lien entre la dynamique de plusieurs lignes et la dynamique de densités de dislocations, nous nous intéressons maintenant à un modèle de dynamique de densités de dislocations. Nous présentons un modèle simplifié pour lequel nous allons montrer que le problème est bien posé. Nous étudierons également l'aspect numérique de ce problème.

De manière plus précise, nous nous intéressons à des dislocations coins, en particulier le vecteur de Burgers et les dislocations sont contenus dans le même plan. On considère deux types de dislocations bougeant respectivement avec le vecteur de Burgers $\pm\vec{b}$. Ce modèle a été introduit par Groma et Balogh dans [103] comme un système couplé, à savoir un problème de transport dans lequel la vitesse est donnée par les équations de l'élasticité dans le cas 2-D.

En fait, dans un cas particulier, ce modèle peut être simplifié. En effet, si l'on suppose que le domaine 2-D est $1-$périodique en x_1 et x_2 (ce qui revient à considérer un domaine infini et évite les problèmes dus aux conditions aux bords) et si les densités de dislocations dépendent seulement de la variable $x = x_1 + x_2$ (où (x_1, x_2) sont les coordonnées d'un point de $\mathbb{R}^2$) alors quand $\vec{b} = (1,0)$, le système 2-D de [103] peut se réduire à un système couplé d'équations de Hamilton-Jacobi 1-D non-locales (on renvoit à la Section 3 du Chapitre 9 pour une modélisation physique)

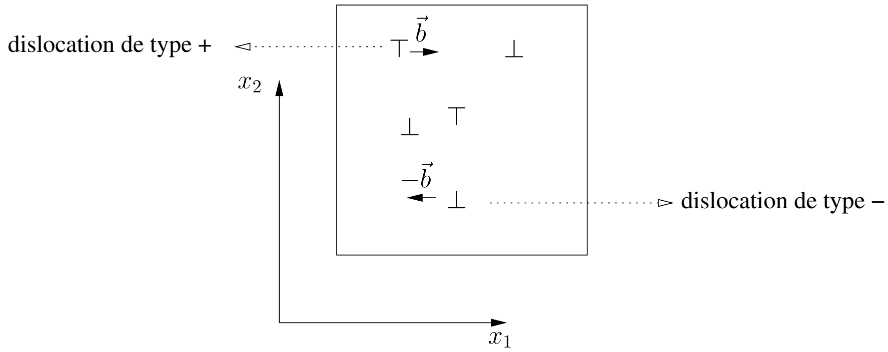

FIG. 10 – Section transversale des lignes de dislocations.

$$\left\{\begin{array}{l} (\rho_+)_t = -\left(\rho_+ - \rho_- + \displaystyle\int_0^1 (\rho_+(x,t) - \rho_-(x,t))\,dx + L(t)\right)|D\rho_+| \\ \hfill \text{dans} \quad \mathbb{R}\times(0,T) \\ (\rho_-)_t = \left(\rho_+ - \rho_- + \displaystyle\int_0^1 (\rho_+(x,t) - \rho_-(x,t))\,dx + L(t)\right)|D\rho_-| \\ \hfill \text{dans} \quad \mathbb{R}\times(0,T) \end{array}\right. \tag{54}$$

où $L(t)$ représente le champ de contraintes de cisaillement, ρ_+, ρ_- sont les inconnues scalaires telles que $(\rho_+ - \rho_-)$ représente la déformation plastique et leurs dérivées en espace $D\rho_\pm := \dfrac{\partial \rho_\pm}{\partial x}$ sont les densités de dislocations de type $+$ et $-$. Ce système est complété par les conditions initiales suivantes :

$$\rho_\pm(x,0) = \rho^0_\pm(x) = P^0_\pm(x) + L_0 x \text{ sur } \mathbb{R} \tag{55}$$

où $P^0_\pm$ sont périodiques de période 1 et Lipschitz et L_0 est une constante donnée qui représente la densité totale de dislocations de type $\pm$, c'est à dire qu'à l'instant initial, on suppose que l'on a la même densité de type $+$ et $-$.

Le premier résultat concernant ce système est un résultat d'existence et d'unicité. Un cadre naturel pour cette étude est la théorie des solutions de viscosité. Cette théorie a été formalisée pour les systèmes par Ishii, Koike [117] et Ishii [115]. Nous renvoyons également à Engler, Lenhart [76], Ishii, Koike [118], Lenhart [133], Lenhart, Belbas [134], Lenhart, Yamada [135] et Yamada [184] pour d'autres applications.

Théorème 2.32. (Existence et unicité pour le système (54)-(55)**, [75, Théorème 1.1])**
Soit $T \geq 0$ et $L_0 \in \mathbb{R}$. On suppose que $\rho^0_\pm \in Lip(\mathbb{R})$ et que $L \in W^{1,\infty}(\mathbb{R}^+)$. Alors

le système (54)-(55) admet une unique solution de viscosité $\rho = (\rho^+, \rho^-)$. De plus, cette solution est uniformément Lipschitz en espace-temps.

La difficulté principale pour montrer ce résultat vient du fait qu'il n'y a pas de principe de comparaison. Ceci vient de la présence du terme non-local. Pour résoudre cette difficulté, l'idée est d'utiliser, de manière classique, un théorème de point fixe. Quand le terme non-local est gelé, le système devient quasi-monotone au sens de [117] et nous avons donc existence et unicité de la solution. Il suffit ensuite d'utiliser une estimation sur la norme du gradient en espace pour obtenir le résultat pour le système non-local.

Un résultat similaire a été montré par El Hajj [74] pour des solutions faibles dans H^1_{Loc}. Nous renvoyons également à Canone, El Hajj, Monneau, Ribaud [49] pour l'étude du cas général en 2D.

On s'intéresse maintenant à l'approximation numérique de la solution du système (54)-(55). Etant donné une taille de discrétisation Δx, Δt et une grille $Q_T^\Delta = Q^\Delta \times \{0, ..., (\Delta t)N_T\}$ où $Q^\Delta = \{i\Delta x,\ i \in \mathbb{Z}\}$ et N_T est la partie entière de $T/\Delta t$, on note $v_{k,i}^n$ la valeur de l'approximation numérique de la solution exacte ρ_k au point (x_i, t_n).

Nous allons maintenant introduire le schéma numérique. La difficulté principale vient du terme non-local qui nécessite la connaissance de la solution que l'on est en train de calculer. Pour résoudre ce problème, on fixe la solution $v_i^n = (v_{+,i}^n, v_{-,i}^n)$ sur chaque intervalle de temps $[t_n, t_{n+1})$ et on applique le schéma monotone suivant :

$$v_i^0 = \tilde{\rho}^0(x_i), \qquad \frac{v_{k,i}^{n+1} - v_{k,i}^n}{\Delta t} = c_k^\Delta[v]_i^n G(v_k^n)_i, \quad \forall k \in \{+,-\} \tag{56}$$

où

$$c_k^\Delta[v]_i^n = -k(v_{+,i}^n - v_{-,i}^n + a^\Delta[v](t_n))$$

et le terme non-local $a^\Delta[v](t_n)$ est donné par

$$a^\Delta[v](t_n) = \sum_{i=0}^{N_x - 1} \Delta x\, (v_+(x_i, t_n) - v_-(x_i, t_n)) + L(t_n)$$

où N_x est la partie entière de $1/\Delta x$. Comme dans la Section 2.1.5, le terme $G(v_k^n)_i$ est une approximation appropriée du gradient de v_k^n pris au point x_i. Cette approximation est par exemple celle donnée par Osher et Sethian [154] et dépend du signe de la vitesse.

Finalement, on suppose que la condition CFL suivante est vérifiée

$$\Delta t \leq \frac{1}{2M} \Delta x \tag{57}$$

où
$$M = 2\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})} + 2.$$

On a alors l'estimation d'erreur suivante

Théorème 2.33. (Estimation d'erreur numérique discret-continue, [75, Théorème 1.3])
Soit $T \geq 0$. On suppose que $\Delta x + \Delta t \leq 1$, $L \in W^{1,\infty}([0,T))$ et que la condition CFL (57) est vérifiée.

Alors, il existe une constante $K_4 > 0$ dépendant seulement de $\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$, $\|L\|_{W^{1,\infty}(0,T)}$ et $\max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ telle que l'estimation d'erreur entre la solution continue ρ du système (54) *et son approximation numérique v, solution du schéma aux différences finies* (56) *est donnée par*

$$\max_{k\in\{+,-\}} \sup_{Q_T^\Delta} |\rho_k - v_k| \leq K_4 \left((T + \sqrt{T})\,(\Delta x + \Delta t)^{1/2} + \max_{k\in\{+,-\}} \sup_{Q^\Delta} |\rho_k^0 - v_k^0| \right)$$

sous l'hypothèse complémentaire

$$K_4 \left((T + \sqrt{T})(\Delta x + \Delta t)^{\frac{1}{2}} + \max_{k\in\{+,-\}} \sup_{Q^\Delta} (\rho_k^0 - v_k^0) \right) \leq 1.$$

Pour montrer ce théorème, on utilise la même méthode que pour le cas continu, c'est à dire que l'on considère la solution approximée du système (54) comme un point fixe d'un système local. Cette preuve est inspirée de la preuve de Alvarez *et al.* [7] pour montrer une estimation d'erreur de type Crandall-Lions [63] entre la solution continue et son approximation numérique.

Chapitre 1

Modélisation de la dynamiqe d'une ligne de dislocation

1 Propriétés des dislocations

Nous commençons ce chapitre en rappelant les propriétés classiques des dislocations. Cette présentation est largement inspirée de Xiang [183] et Hirth, Lothe [106]. Pour plus de références sur les dislocations, nous renvoyons également à Nabarro [146], Hull, Bacon [109], Weertman, Weertman [181], Lardner [132] et Landau, Lifshitz [131].

Comme nous l'avons déjà indiqué, les dislocations sont des défauts linéaires dans les cristaux. Il existe deux types de dislocations droites : les *dislocations coins* et les *dislocations vis.*

Une façon d'imaginer une dislocation coin dans une structure cubique simple est d'insérer un demi-plan supplémentaire d'atomes dans un cristal parfait (voir Figure 1.1(b)). La dislocation coin est alors la frontière du demi-plan supplémentaire, là où la structure cristalline est fortement déformée. La géométrie d'une dislocation coin est représenté dans la Figure 1.1(a), où la distribution des atomes est uniforme dans la direction perpendiculaire au papier.

Le second type de dislocation est la dislocation vis dont la géométrie est représentée pour un cristal cubique simple dans la Figure 1.2. Une manière d'imaginer une dislocation vis est de faire une coupure sur un demi-plan d'un cristal parfait et de translater le quart d'espace supérieur dans une direction parallèle au bord du demi-plan de coupure et le demi-espace inférieur dans la direction opposée. La dislocation vis est alors représentée par le bord du demi-plan de coupure.

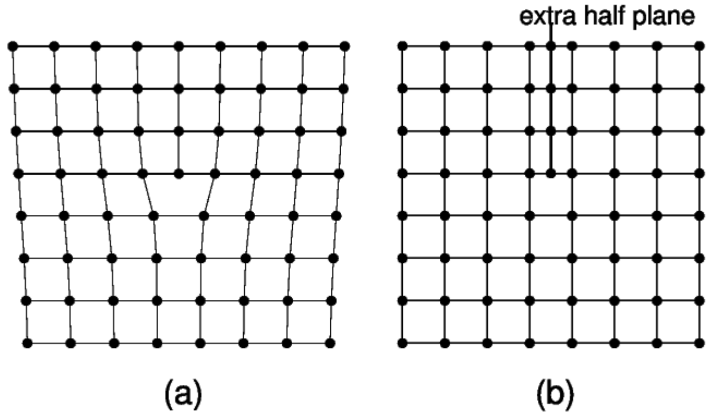

FIG. 1.1 – Géométrie d'une dislocation coin

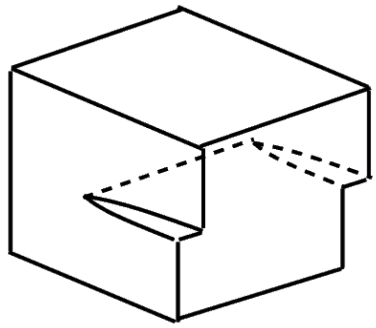

FIG. 1.2 – Géométrie d'une dislocation vis

Nous allons maintenant expliquer comment le mouvement de dislocations peut entrainer des déformations plastiques dans un cristal. Quand un cristal est soumis à une contrainte extérieure suffisament forte, il peut subir des déformations plastiques. Néanmoins, la contrainte critique dans un matériau réel est beaucoup moins élevée que la contraintre théorique pour cisailler un cristal parfait. Ce sont Taylor [176], Orowan [150] et Polanyi [155] les premiers à avoir mis en évidence que les dislocations étaient responsables de cette faible contrainte critique. Pour mieux comprendre ce fait, il suffit de regarder de plus près le déplacement d'une dislocation coin dans un cristal parfait, comme indiqué dans la Figure 1.3. Dans ce cas, la dislocation coin est dans le plan perpendiculaire au papier et la distribution des atomes est uniforme dans cette direction. Loin de la dislocation, la distribution des atomes est proche de celle d'un cristal parfait alors que près de la dislocation, la structure atomique est fortement déformée. Le déplacement de la dislocation se fait alors de manière très simple. La liaison atomique entre les points B et C va se briser alors qu'une liaison va se créer entre A et C (voir Figure 1.3(a)-(b)). La dislocation s'est alors déplacée vers la droite d'une distance atomique. Sous l'application d'une contrainte, ce processus va se répéter et la dislocation va continuer de se déplacer. Quand elle aura atteint le bord du cristal, celui-ci aura subi une déformation plastique permanente (voir Figure 1.3(c)). En fait, la contrainte de cisaillement nécessaire pour ce processus est beaucoup plus faible que celle pour cisailler un cristal parfait.

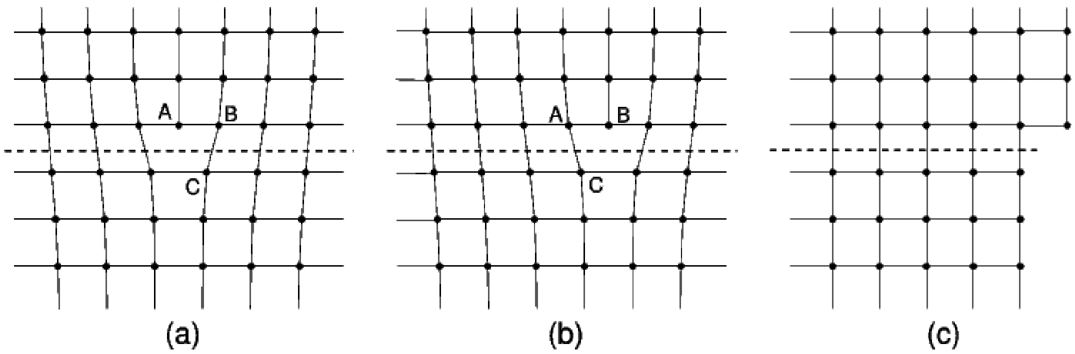

FIG. 1.3 – Déplacement d'une dislocation coin

Il y a essentiellement deux invariants pour les dislocations. Le premier élément qui caractérise une dislocation est le vecteur de Burgers [47]. Pour le définir, il faut choisir l'orientation de la dislocation qui est donnée par la direction du vecteur unitaire tangent à la dislocation. On considère alors un chemin fermé entourant la dislocation, comme indiqué sur la Figure 1.4. Ce chemin, appelé circuit de Burgers, est orienté dans le sens "right-hand". Par exemple, si le vecteur tangent d'une dislocation coin pointe en dehors du papier vers le lecteur, alors le circuit est orienté dans le sens inverse des aiguilles d'une montre. Dans l'autre cas, il est orienté dans le sens des aiguilles d'une montre. Pour obtenir le vecteur de Burgers, il suffit alors de reproduire le même chemin dans un cristal parfait. Le point de départ (A) est alors différent du point d'arrivée (B). Le vecteur de Burgers est alors le vecteur $b = \vec{BA}$ (voir Figure 1.4(b)). On peut remarquer que si l'on change le signe de l'orientation de la dislocation coin, cela change la direction du vecteur de Burgers.

Comme nous l'avons vu dans la Figure 1.4, pour une dislocation coin, le vecteur de Burgers est perpendiculaire à la ligne de dislocation. Dans le cas des dislocations vis, il est parallèle à la ligne de dislocation. Dans le cas général, la ligne de dislocation forme un angle arbitraire avec le vecteur de Burgers et nous avons donc une dislocation mixte. La Figure 1.5 montre une boucle de dislocation circulaire dans un plan ainsi que son vecteur de Burgers. Aux points A et B, la dislocation est pu-

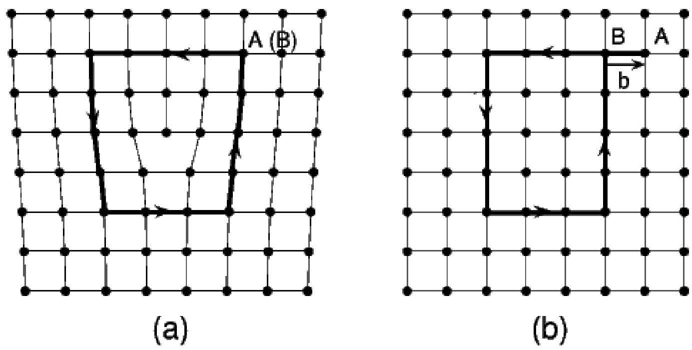

FIG. 1.4 – Circuit de Burgers

rement coin, aux points C et D la dislocation est purement vis et à tous les autres points (E par exemple) elle est mixte.

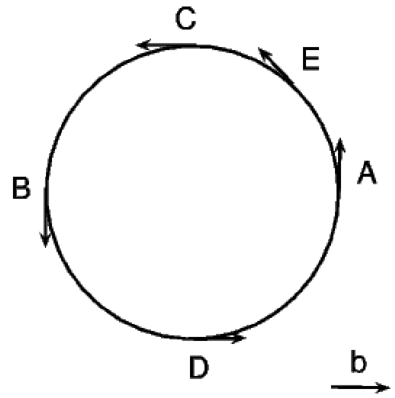

FIG. 1.5 – Boucle de dislocation circulaire

Le deuxième élément qui caractérise une dislocation est le vecteur normal à son plan de glissement. En effet, au moins à faible température, une dislocation se déplace dans des plans cristallins bien définis. Par exemple, dans le cas de la dislocation coin, le plan de glissement est celui qui contient le vecteur de Burgers b et le vecteur tangent à la dislocation. Sur la Figure 1.3, il est représenté par la ligne en pointillé et est perpendiculaire au plan du papier.

2 Modélisation de la dynamique d'une ligne de dislocation

Nous allons maintenant décrire la dynamique d'une ligne de dislocation. Cette partie est inspirée de Alvarez *et al.* [10], mais la présentation de la modélisation est faite de manière un peu différente. Le but de cette section est de modéliser la dynamique d'une ligne de dislocation dans le cas où les contraintes extérieures sont nulles. Nous renvoyons à Alvarez *et al.* [10, Section 2.6] pour voir comment prendre en compte les contraintes extérieures.

On se donne une base (e_1, e_2, e_3) de $\mathbb{R}^3$. Une idéalisation pour une ligne de dislocation consiste à supposer que l'épaisseur de la ligne est nulle et qu'elle est représentée par le bord d'un domaine ouvert Ω_t contenu dans le plan (x_1, x_2) (où l'indice t représente le temps) et on note Γ_t le bord de Ω_t (c'est à dire la ligne de dislocation). De plus, au moins à faible température, les dislocations bougent dans des plans bien définis (car c'est très coûteux énergétiquement pour une dislocation de changer de plan cristallin). On suppose alors, pour simplifier, que la ligne de dislocation est contenue et se déplace dans le plan horizontal et on note $n = e_3$ le vecteur normal à ce plan. On définit également la matrice de cisaillement

$$e^0 = \frac{1}{2}(b \otimes n + n \otimes b) \tag{1.1}$$

où b est le vecteur de Burgers de la dislocation.

La dynamique d'une ligne de dislocation la plus simple pour une courbe $t \mapsto \Gamma_t$ est de supposer que la vitesse normale à la courbe est proportionnelle à la force de Peach-Koeller résolue c :

$$\frac{d\Gamma}{dt} = \frac{1}{B} c(\Gamma) n_\Gamma$$

où n_Γ est la normale extérieure au domaine Ω_t et B est une constante, appelée "viscous drag coefficient" (voir Hirth et Lothe [106, page208]). Cette constante dépend des propriétés physiques du matériau mais, quitte à rescaler l'équation en temps, on peut supposer que $B = 1$.

Nous allons maintenant définir l'énergie élastique associée à la ligne de dislocation. Pour cela on considère un matériau représenté par l'espace tout entier, soumis à l'élasticité linéaire et dont les coefficients d'élasticité sont donnés par $\Lambda = (\Lambda_{ijkl}) \in \mathbb{R}^{3\times3\times3\times3}$. On suppose que ces coefficients satisfont la propriété de symétrie suivante

$$\Lambda_{ijkl} = \Lambda_{jikl} = \Lambda_{ijlk} = \Lambda_{klij}$$

et l'hypothèse de coercitivité suivante pour un $m > 0$

$$\sum_{i,j,k,l=1}^{3} \Lambda_{ijkl} e_{ij} e_{kl} \geq m \sum_{i,j=1}^{3} (e_{ij})^2$$

pour toutes les matrices constantes $e = (e_{ij}) \in \mathbb{R}^{3\times 3}_{\text{sym}}$, c'est à dire telles que $e_{ij} = e_{ji}$ pour $i, j = 1, 2, 3$.

On note $u : \mathbb{R}^3 \mapsto \mathbb{R}^3$ le déplacement et on définit la déformation par

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial}{\partial x_i} u_j + \frac{\partial}{\partial x_j} u_i\right).$$

L'énergie élastique est alors définie par

$$\mathcal{E}^{el}(e^{el}) = \int_{\mathbb{R}^3} \frac{1}{2} \Lambda\ e^{el}\ e^{el}$$

où $e(u) = e^{el} + e^{pl}$ avec e^{el} la déformation élastique et e^{pl} la déformation plastique donnée par

$$e^{pl} = e^0 \delta_\Omega = e^0 \rho \delta_0(x_3)$$

où e^0 est défini par (1.1) et $\rho(x) = \rho(x_1, x_2)$ est la fonction caractéristisque de Ω qui vaut 1 dans Ω et 0 dans son complémentaire. Finallement, on note $\sigma_{ij}(x)$ $i, j = 1, 2, 3;$ $x \in \mathbb{R}^3$, les contraintes élastiques qui sont données par

$$\sigma_{ij} = (\Lambda\ e^{el})_{ij} = \sum_{kl} \Lambda_{ijkl} e^{el}_{kl}.$$

D'après les équations de Euler-Lagrange, on a donc (par minimisation de l'énergie élastique sur u) que

$$\text{div}(\sigma) = 0. \tag{1.2}$$

Nous allons maintenant calculer la force de Peach-Koeller résolue en calculant la première variation de l'énergie. On note s l'abscisse curviligne de Γ et $\Gamma(s)$ le point courant sur Γ. Soit $n_\Gamma(s)$ le vecteur normal extérieur au domaine ouvert Ω_0 dans le plan (e_1, e_2). Pour une fonction donnée $h(s)$, on définit alors la perturbation planaire Γ_ε de Γ par

$$\Gamma_\varepsilon(s) = \Gamma(s) + \varepsilon h(s) n_\Gamma(s).$$

Le but est maintenant de calculer $\frac{d}{d\varepsilon}\mathcal{E}(\Gamma_\varepsilon)_{|\varepsilon=0}$ où $\mathcal{E}(\Gamma) = \mathcal{E}^{el}(e^{el}(\Gamma))$. On montrera alors que

$$-\frac{d}{d\varepsilon}\mathcal{E}(\Gamma_\varepsilon)_{|\varepsilon=0} = \int_\Gamma ds\ h(s) c(\Gamma(s))$$

où la fonction c est la force de Peach-Koeller résolue. De manière formelle, on a

$$\begin{aligned}\frac{d\mathcal{E}(\Gamma_\varepsilon)}{d\varepsilon} &= \int_{\mathbb{R}^3} \sigma \left(e\left(\frac{du_{\Gamma_\varepsilon}}{d\varepsilon}\right) - e^0 \frac{d}{d\varepsilon}\delta_{\Omega_{\Gamma_\varepsilon}} \right) \\ &= -\int_{\mathbb{R}^3} (\sigma e^0) \frac{d}{d\varepsilon}\delta_{\Omega_{\Gamma_\varepsilon}} \\ &= \int_\Gamma -(\sigma e^0) h\end{aligned}$$

où l'on a utilisé le fait que $\mathrm{div}(\sigma) = 0$ pour la deuxième ligne. La force Peach-Koeller est donc donnée par

$$c = \sigma e^0 = (\sigma b) \cdot n$$

où $\sigma = \Lambda e(u) - \Lambda e^0 \rho \delta_0(x_3)$ avec

$$\begin{cases} \mathrm{div}(\Lambda e(u)) = \mathrm{div}(\Lambda e^0 \rho \delta_0(x_3)) \\ e(u) = \frac{1}{2}\left(\frac{\partial}{\partial x_j} u_i + \frac{\partial}{\partial x_i} u_j\right) \end{cases}$$

En utilisant la transformée de Fourier et un calcul explicite de la fonction de Green pour l'élasticité linéaire, on peut voir qu'il existe un tenseur d'ordre quatre R tel que

$$\sigma = (\Lambda R e^0) \star \rho \delta_0(x_3).$$

On peut donc réécrire la force c de la manière suivante :

$$c = c_0 \star \rho$$

avec $c_0(x_1, x_2) = C_0(x_1, x_2, 0)$ où C_0 dépend de Λ, R, e^0. Nous renvoyons à Alvarez *et al.* [10, Section 2.6] pour plus de détails sur le calcul du noyau C_0 et sur ses propriétés.

Remarque 2.1. *Les calculs présentés ici sont faits de manière formelle. En particulier, sur le coeur de la dislocation, l'énergie élastique n'a pas de sens et est même infinie car la contrainte élastique se comporte en $1/r$ où r est la distance à la dislocation. Bien sûr, cela n'a pas de sens physique mais vient du fait que la théorie de l'élasticité linéaire n'est plus valide proche de la dislocation. Pour contourner cette difficulté, il faut régulariser les contraintes en utilisant un tenseur de coeur régularisant. Pour plus de détails, nous renvoyons à Alvarez et al. [10]*

Dans [87], nous avons proposé de rajouter un terme de tension de ligne à l'énergie élastique pour mieux modéliser ce qui se passe près de la dislocation. Ce terme de tension ligne apparait en particulier dans les travaux de Brown [45] et Barnett, Gavazza [31]. L'énergie est donc donnée par

$$\mathcal{E}^{el}(e^{el}) = \int_{\mathbb{R}^3} \frac{1}{2}\Lambda\, e^{el}\, e^{el} + \eta \int_{\Gamma} \gamma(n_\Gamma)$$

où γ est une énergie de tension de ligne et η est un préfacteur. En calculant la première variation de l'énergie, on obtient, comme précedemment, une vitesse normale donnée par

$$c = c_0 \star \rho + c_1 + \lambda(n_{\Gamma_t}) H_{x,t}$$

où $H_{x,t}$ est la courbure moyenne et $\lambda = \gamma + \gamma''$.

Chapitre 2

Dynamique des dislocations avec terme de courbure moyenne : existence et unicité en temps court

Ce chapitre est une version rallongée de [87].
Dans ce travail, nous étudions un nouveau modèle pour la dynamique de lignes de dislocations avec un terme de courbure moyenne. Le modèle est une équation d'Hamilton-Jacobi non-locale. Nous montrons l'existence et l'unicité d'une solution de viscosité en temps court pour cette équation. Nous obtenons également une estimation Lipschitz en espace et une estimation du module de continuité en temps pour la solution.

Dislocation dynamics with a mean curvature term : short time existence and uniqueness

N. Forcadel

Abstract In this paper, we study a new model for dislocation dynamics with a mean curvature term. The model is a non-local Hamilton-Jacobi equation. We prove a short time existence and uniqueness result for this equation. We also prove a Lipschitz estimate in space and an estimate of the the modulus of continuity in time for the solution.
AMS Classification : 35D05, 35D10, 35G25, 35Q99, 35G25, 49L25.
Keywords : Dislocation dynamics, non-local equations, mean curvature motion, viscosity solutions.

1 Introduction

Plastic deformation is mainly due to the movement of linear defects called dislocations, whose typical length in metallic alloys is of the order of $10^{-6}m$ and thickness of the order of $10^{-9}m$. Since the beginning of the 90's, the research field of dislocation is enjoying a new boom, in particular thanks to the power of computers which now makes possible to simulate dislocations in a 3D domain. The concept of dislocations in crystals was put forward in the XXth century as the main microscopic explanation of the macroscopic behaviour of metallic crystals (see the physical monograph Hirth and Lothe [106]).

More recently, a new approach has been introduced : *phase field model of dislocations* (see for example Rodney, Le Bouar, Finel [158]). One of the advantage of this method is that the possible topological changes during the dislocation movement are automatically taken into account. In the face centered cubic structure, the dislocation line in the crystal moves in it slip plane with a normal velocity which is proportional to the Peach-Koeller force acting on the line. This force is the self force created by the elastic field generated by the dislocation itself. In [10] and [9], using a level set formulation, Alvarez, Hoch, Le Bouar and Monneau proposed a non-local Hamilton Jacobi equation to model this approach. Having such an equation allows in particular to study mathematically and numerically dislocations dynamics. Since the equation is nonlinear, a natural framework to study this kind of equation is the theory of viscosity solutions (see for instance the monographs of Barles [18] and Bardi and Capuzzo-Dolcetta [17] for a presentation of first order equations and the article of Crandall, Ishii and Lions [61] for the second order case). The theory of viscosity solutions has been first introduced by Crandall and Lions [62]. The main difficulty of this equations is that the comparison principle, which is a crucial argument in this theory, does not hold. Nevertheless, using the geometric property of the equation (we refer to Barles, Soner, Souganidis [27] for a detailed presentation of this theory) Alvarez *et al.* prove short time existence and uniqueness for this equation. Then Alvarez, Cardaliaguet and Monneau [6] and Barles and Ley [25] prove a long time existence and uniqueness result under certain assumptions on the monotonicity of the velocity. This model was also numerically studied by Alvarez, Carlini, Monneau and Rouy [7, 8].

Here, we consider a new model in which the energy also contains a line tension term which approximates better what happen near the dislocation. That amounts to adding a mean curvature term in the equation. This line tension term appears in a lot of physical models (see for instance Gavazza, Barnett [31] and Brown [45]). We also refer to Garroni, Müller [93] for a mathematical reference. Non-local equation with mean curvature term have also been studied by Chen, Hilhorst and Logak [57].

For this model we show a short time existence and uniqueness result. Since the comparison principle does not hold, the strategy of the proof is the same as the one used by Alvarez *et al.* in [10], *i.e.* is to use a fix point method by freezing the nonlocal term. Here, the main difficulties come from the fact that the equation is a second order one and so the regularity estimates are really more difficult to obtain. Indeed, for the fix point to work, we need fines estimates on the Lipschitz constant in space and on the modulus of continuity in time of the solution. To get round these difficulties, we use a regularisation of the initial condition to obtain an estimate of the modulus of continuity in time of the solution for a class of operators. Using the geometric property of the equation, we also prove that the estimate we get is like $\sqrt{t}$. We also prove a Lipschitz estimate in space for the solution. The estimate we obtain is the same as in the first order case (*i.e.*, without the mean curvature term).

The long time existence and uniqueness result for this non-local equation is really more difficult to obtain. Indeed, due to the non validity of the comparison principle, defining a large time solution is rather difficult and this problem is still open. Moreover, even in the the first order case, the problem is still open for general velocities. We refer to Alvarez, Cardaliaguet and Monneau [6] and Barles and Ley [25] for such results with monotonicity assumptions on the velocity.

Let us now explain how this paper is organised : in Section 2, we present the model. The main result is state in Section 3. In Section 4, we give some preliminary results on a local problem. First, in Subsection 4.1, we recall the definition of viscosity solutions and we give an existence and uniqueness result for the local problem. Then, in Subsection 4.2, we give some results on the regularity of the solution of the local problem. In section 5, we prove Theorem 3.2 for the non-local problem. Finally, we give, in Appendix, the proof of the parabolic version of Ishii's *Lemma* used in section 4.

2 Presentation of the model

We describe the model in an heuristic way. Let us consider an orthonormal basis (e_1, e_2, e_3) of $\mathbb{R}^3$ and denote by $x = (x_1, x_2, x_3)$ the coordinates. The energy of the dislocation along the line is singular. To solve this problem, Brown [45], [46] then Barnett [30] and Gavazza, Barnett [31] propose to surround the dislocation Γ by a tube T_ϵ of size ϵ and to consider the energy of the form :

$$\mathcal{E} = \int_{\mathbb{R}^3 \setminus T_\epsilon} \frac{1}{2} \Lambda e^{class}.e^{class} + \int_{T_\epsilon} \gamma_0(\vec{n}), \tag{2.1}$$

where Λ represents the elastic coefficients, γ_0 is an energy of tension line, $\vec{n}$ is the outward normal to the curve and e^{class} is a deformation and is solution of :

$$\begin{cases} \mathrm{div}(\Lambda e^{class}) = 0, \\ \mathrm{inc}(e^{class}) = -\mathrm{inc}(\rho \delta_0(x_3) e^0) \text{ where } e^0 = \frac{1}{2}(b \otimes e_3 + e_3 \otimes b), \\ e^{class}(x) \to 0 \text{ when } |x| \to \infty. \end{cases} \tag{2.2}$$

Here, Γ belongs to the plane (e_1, e_2), the vector e_3 is the vector normal to the plane, $b \in \mathbb{R}^3$ is a constant vector (called Burgers' vector) associated with the dislocation line and the operator of incompatibility "inc" is defined for a field $e = (e_{ij}) \in S^3$, the set of symmetric 3×3 matrix, by :

$$(\mathrm{inc}(e))_{ij} = \sum_{i_1, j_1 = 1}^{3} \varepsilon_{i i_1 i_2} \varepsilon_{j j_1 j_2} \partial_{i_1} \partial_{j_1} e_{i_2 j_2}, \quad i, j = 1, 2, 3$$

where we note as usual

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if} \quad (ijk) \quad \text{is a positive permutation of} \quad (123), \\ -1 & \text{if} \quad (ijk) \quad \text{is a negative permutation of} \quad (123), \\ 0 & \text{if} \quad \text{two indices are the same.} \end{cases}$$

The solution e^{class} of (2.2) satisfies $e^{class} \sim \frac{1}{r}$, for r small, where r is the distance to the dislocation (cf Alvarez *et al.* [10] for a description of the physical model). Finally, the cutoff tube is represented by Figure 2.1. We consider an approximate model of this one where the field e is given by $e = \chi \star e^{class}$, with χ a regularising core function (connected to ϵ) to be adjusted, and the energy (2.1) is replaced by the following one :

$$\mathcal{E} = \int_{\mathbb{R}^3} \frac{1}{2} \Lambda e.e + \eta \int_{\Gamma} \gamma_0(\vec{n}),$$

where η is to be adjusted and connected to ϵ. We set $\gamma(\vec{n}) = \eta \gamma_0(\vec{n})$. In order to model the movement of a dislocation Γ in its crystallographic plane, we assume that

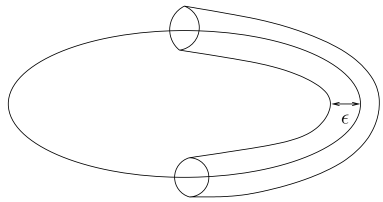

FIG. 2.1 – The cutoff tube of radius ϵ.

Γ is the edge of a smooth bounded set $\Omega \subset \mathbb{R}^2$ and we compute the first variation of the energy (see Alvarez *et al.* [10]). We define $\Gamma_\delta(s) = \Gamma(s) + \delta h(s).\vec{n}_\Gamma(s)$. Then, the following holds

$$-\left.\frac{d\mathcal{E}(\Gamma_\delta)}{d\delta}\right|_{\delta=0} = \int_\Gamma c.h,$$

with $c = c_0 \star \rho + \lambda(\vec{n})\kappa$, where $\lambda(\vec{n}) = (\gamma(\vec{n}) + \gamma''(\vec{n}))$ the kernel $c_0 = c_0(x_1, x_2)$ only depends on Λ and χ, $\star$ denotes the convolution in space and ρ is defined as follows :

$$\rho = \begin{cases} 1 & \text{in} \quad \Omega, \\ 0 & \text{in} \quad \mathbb{R}^2/\Omega. \end{cases} \tag{2.3}$$

Thus, the evolution is postulated to be $\frac{\partial \rho}{\partial t} = c|D\rho|$. We can then reformulate the problem by a "level set" equation on the set $\{u \leq 0\}$ of a smooth function u which then satisfies :

$$u_t = (c_0 \star [u] + \lambda(\vec{n})\kappa)|Du|, \tag{2.4}$$

with :

$$\begin{cases} \rho = [u] = \begin{cases} 1 & \text{if} \quad u > 0 \\ 0 & \text{if} \quad u \leq 0 \end{cases}, \\ \vec{n} = \frac{Du}{|Du|}, \\ \kappa = \operatorname{div}\left(\frac{Du}{|Du|}\right). \end{cases} \tag{2.5}$$

Remark 2.1. (Explicit example for c_0 and γ)
If $b = |b|e_1$, then, for the isotropic elasticity, one can give the value of c_0 :

$$\widehat{c_0}(\xi) = \frac{-\mu b^2}{2} e^{-\zeta\sqrt{\xi_1^2+\xi_2^2}} \left(\frac{\xi_1^2 + \frac{1}{1-\nu}\xi_2^2}{\sqrt{\xi_1^2 + \xi_2^2}}\right)$$

and the form of γ :

$$\gamma(\vec{n}) = C\left(n_1^2 + \frac{1}{1-\nu} n_2^2\right),$$

where $\vec{n} = (n_1, n_2)$ is the normal vector to the curve, C is a prefactor (which depends on the Burgers vector and elasticity coefficients), $\zeta > 0$ is a physical parameter *(depending on the material), ν is the Poisson ratio and μ is the Lamé coefficient. We refer to Alvarez et al. [10, Section 6] for the expression of c_0 and to Hirth, Lothe [106, Chapter 6 and 7] for the form of γ.*

Remark 2.2. *Formally, we have :*

$$\frac{d\mathcal{E}}{dt} = \int_\Gamma -c^2 \leq 0.$$

3 Main result

The goal of the paper is to prove short time existence and uniqueness for the problem (2.4). Here we consider more general second order term and we study the n-dimensional case (see (2.6)). Since the Hamiltonian intervening in the equation is not continuous and singular, a natural framework for the study is the theory of viscosity solutions (for a good introduction to this theory, we refer to Barles [18], [19], Crandall, Ishii, Lions [61], Crandall, Lions [63], [64], Ishii [114] and Ishii, Lions [119] and for an introduction to viscosity solution for evolving fronts, we refer to Ambrosio [13], Barles, Soner, Souganidis [27], Chen, Giga, Goto [58], Evans [80], Evans, Spruck [83] and Souganidis [174]). We consider the following problem : find $u(x,t)$ solution of

$$\begin{cases} u_t = (c_0 \star [u])|Du| - F(Du, D^2u) \text{ in } \mathbb{R}^n \times (0,T), \\ u(x,t=0) = u_0(x) \text{ in } \mathbb{R}^n, \end{cases} \tag{2.6}$$

where $[u]$ is the characteristic function of the set $\{u > 0\}$ (see (2.5)). Moreover, we assume that

$$c_0 \in L^\infty_{\text{int}}(\mathbb{R}^n) \cap BV(\mathbb{R}^n), \tag{2.7}$$

where $BV(\mathbb{R}^n)$ is the space of bounded variations functions and

$$L^\infty_{\text{int}}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} : \|f\|_{L^\infty_{\text{int}}(\mathbb{R}^n)} < \infty\}$$

with

$$\|f\|_{L^\infty_{\text{int}}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \|f\|_{L^\infty(Q(x))}$$

and $Q(x)$ is the unit square centered at x :

$$Q(x) = \left\{x' \in \mathbb{R}^n : |x_i - x'_i| \leq \frac{1}{2}\right\}. \tag{2.8}$$

The assumptions (HF) on the operator F are the following ones :

(i) The operator F is elliptic, *i.e.*, $\forall X,\ Y \in S^n$, $\forall p \in \mathbb{R}^n$,

$$\text{if } X \leq Y \text{ then } F(p, X) \geq F(p, Y), \tag{2.9}$$

where S^n (the set of symmetric $n \times n$ matrices) is equipped with its natural partial order.

(ii) F is locally bounded on $\mathbb{R}^n \times S^n$, continuous on $\mathbb{R}^n \backslash \{0\} \times S^n$ and

$$F^*(0, 0) = F_*(0, 0) = 0, \tag{2.10}$$

where F^* (resp. F_*) is the upper-semicontinous (usc) envelope (resp. lower semicontinuous (lsc) envelope) of F, *i.e.* the smallest usc function greater than F (resp. the greatest lsc function smaller than F).

(iii) F is geometric, *i.e.*

$$F(\nu p, \nu A + \mu p \otimes p) = \nu F(p, A), \qquad \forall \nu > 0, \mu \in \mathbb{R}, A \in S^n. \tag{2.11}$$

Let us recall the definition of viscosity solution for this problem :

Definition 3.1. (Viscosity subsolution, supersolution and solution)
An upper semi-continuous (resp. lower semi-continuous) function $u : \mathbb{R}^n \times [0, T) \mapsto \mathbb{R}$ *is a viscosity sub-solution (resp. super-solution) of (2.6) if it satisfies :*

(i) $u(x, t = 0) \leq u_0(x)$ *(resp.* $u(x, t = 0) \geq u_0(x)$*) in* $\mathbb{R}^n$,

(ii) for every $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ *and for every test function* $\Phi : (\mathbb{R}^n \times (0, T)) \to \mathbb{R}$, C^1 *in time and* C^2 *in space, that is tangent from above (resp. below) to* u *at* (x_0, t_0)*, the following holds :*

$$\frac{\partial \Phi}{\partial t}(x_0, t_0) + G_*\left(x_0, t_0, D\Phi, D^2\Phi\right) \leq 0$$

$$\left(resp. \frac{\partial \Phi}{\partial t}(x_0, t_0) + G^*\left(x_0, t_0, D\Phi, D^2\Phi\right) \geq 0\right).$$

A function $u \in C^0(\mathbb{R}^n \times [0, T))$ *is a viscosity solution of (2.6) if, and only if, it is a sub and a super-solution of (2.6).*

The main result is :

Theorem 3.2. (Short time existence and uniqueness)
Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ *be a Lipschitz continuous function on* $\mathbb{R}^n$ *such that*

$$|Du_0| < B_0 \quad in\ \mathbb{R}^n \tag{2.12}$$

and

$$\frac{\partial u_0}{\partial x_n} > b_0 > 0 \quad in\ \mathbb{R}^n. \tag{2.13}$$

Let c_0 satisfying (2.7). Then, under assumptions (HF), there exists a unique viscosity solution of the problem (2.6) in $\mathbb{R}^n \times [0, T^*)$ with

$$T^* = \inf\left\{ \frac{1}{|c_0|_{BV(\mathbb{R}^n)}} \ln\left(1 + \frac{b_0}{2B_0}\right), \quad \frac{b_0}{B_0} \frac{1}{16\|c_0\|_{L^\infty_{\text{int}}(\mathbb{R}^n)}}, \right.$$

$$\left. \frac{1}{|c_0|_{BV(\mathbb{R}^n)}} \ln\left(1 + \frac{b_0}{B_0} \frac{|c_0|_{BV(\mathbb{R}^n)}}{8\|c_0\|_{L^\infty_{\text{int}}}}\right) \right\}.$$

Moreover, the solution satisfies

$$|Du(x,t)| < 2B_0 \quad on \quad \mathbb{R}^n \times [0, T^*), \tag{2.14}$$

$$\frac{\partial u}{\partial x_n}(x,t) > b_0/2 > 0 \quad on \quad \mathbb{R}^n \times [0, T^*). \tag{2.15}$$

and u is uniformly continuous in time and its modulus of continuity behaves like $\sqrt{t}$.

Remark 3.3. *This theorem gives, in particular, in the two dimensional case, and for*

$$F(p, X) = -tr\left(\left(\left(I - \frac{p \otimes p}{|p|^2}\right) X\right) \left(\lambda\left(\frac{p}{|p|}\right)\right)\right), \tag{2.16}$$

with $\lambda > 0$ and smooth, short time existence and uniqueness for dislocation dynamics with a mean curvature term.

Remark 3.4. *Due to the non validity of the comparison principle, defining a large time solution is rather difficult and this problem is still open. Even in the one order case this problem is still open for general velocity.*

4 Preliminary results for a local problem

Given $T > 0$, we consider the following problem :

$$\begin{cases} u_t + G(x,t,Du,D^2u) = 0 \text{ in } (0,T) \times \mathbb{R}^n, \\ u(x, t=0) = u_0(x) \text{ in } \mathbb{R}^n, \end{cases} \tag{2.17}$$

with the following assumptions (H_0) :

(i) $G(x,t,p,X) = -c(x,t)|p| + F(p,X)$ and F satisfies the assumptions (HF),

(ii) $c : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ is bounded, Lipschitz continuous in space (we denote by L_c its Lipschitz constant) and uniformly continuous in time (we denote by ω_c its modulus of continuity, defined by : $\forall x \in \mathbb{R}^n,\ \forall s,\ t \in [0,T),\ |c(x,t) - c(x,s)| \leq \omega_c(|t-s|)$),

(iii) u_0 is Lipschitz continuous (we denote by B_0 its Lipschitz constant).

4.1 Existence and uniqueness for the problem (2.17)

We define the following sets :

$$USC(\mathbb{R}^n \times [0,T)) = \{u : \mathbb{R}^n \times [0,T) \to \mathbb{R} \text{ locally bounded, upper semicontinuous}\}$$

$$LSC(\mathbb{R}^n \times [0,T)) = \{u : \mathbb{R}^n \times [0,T) \to \mathbb{R} \text{ locally bounded, lower semicontinuous}\}$$

We then define the solutions of (2.17) in the following way :

Definition 4.1. (Viscosity subsolution, supersolution and solution) *A function $u \in USC(\mathbb{R}^n \times [0,T))$ is a viscosity subsolution of (2.17) if it satisfies :*

(i) $u(x,t=0) \le u_0(x)$ in $\mathbb{R}^n$,

(ii) for every $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and for every test function $\Phi : (\mathbb{R}^n \times (0,T)) \to \mathbb{R}$, C^1 in time and C^2 in space, that is tangent from above to u at (x_0,t_0), the following holds :

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) + G_*\left(x_0,t_0,D\Phi,D^2\Phi\right) \le 0.$$

A function $v \in LSC(\mathbb{R}^n \times [0,T))$ is a viscosity supersolution of (2.17) if it satisfies :

(i) $v(x,t=0) \ge u_0(x)$ in $\mathbb{R}^n$,

(ii) for every $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and for every test function $\Phi : (\mathbb{R}^n \times [0,T)) \to \mathbb{R}$, C^1 in time and C^2 in space, that is tangent from below to v at (x_0,t_0), the following holds :

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) + G^*\left(x_0,t_0,D\Phi,D^2\Phi\right) \ge 0.$$

A function $u \in C^0(\mathbb{R}^n \times [0,T))$ is a viscosity solution of (2.17) if, and only if, it is a sub and a supersolution of (2.17).

Remark 4.2. *The condition ϕ is C^1 in time and C^2 in space means that ϕ is differentiable in time, twice differentiable in space and ϕ, $\phi_t, D_x\phi, D_x^2\phi$ are continuous in space and time.*

We give another equivalent definition. In order to do that, we define parabolic sub and superdifferentials of semi-continuous functions. If $u : \mathbb{R}^n \times (0,T) \to \mathbb{R}$, then $\mathcal{P}^+u$ is defined by $(a,p,X) \in \mathbb{R} \times \mathbb{R}^n \times S^n$ belongs to $\mathcal{P}^+u(x,t)$ if $(x,t) \in \mathbb{R} \times (0,T)$ and

$$u(y,s) \le u(x,t) + a(s-t) + \langle p, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + o\left(|s-t| + |y-x|^2\right)$$

as $\mathbb{R} \times (0,T) \ni (y,s) \to (x,t)$. Similarly, $\mathcal{P}^- u = -\mathcal{P}^+(-u)$. We also define the two following sets :

$$\bar{\mathcal{P}}^+ u(x,t) = \left\{ \begin{array}{l} (a,p,X) \in \mathbb{R} \times \mathbb{R}^n \times S^n, \\ \exists\, (x_n,t_n,a_n,p_n,X_n) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times S^n \\ \text{ such that } (a_n,p_n,X_n) \in \mathcal{P}^+ u(x_n,t_n) \\ \text{ and } (x_n,t_n,u(x_n,t_n),a_n,p_n,X_n) \to (x,t,u(x,t),a,p,X) \end{array} \right\}$$

The set $\bar{\mathcal{P}}^- u(x,t)$ is defined in a similar way. We then have the following definition for the solutions of (2.17), which is equivalent to Definition 4.1 (see Crandall *et al.* [61]) :

Definition 4.3. (Equivalent definition for viscosity solutions)
A function $u \in USC(\mathbb{R}^n \times [0,T))$ is a viscosity subsolution of (2.17) if it satisfies :

(i) $u(x,t=0) \le u_0(x)$ in $\mathbb{R}^n$,

(ii) for every $(x,t) \in \mathbb{R}^n \times (0,T)$ and for every $(a,p,X) \in \mathcal{P}^+ u(x,t)$, we have :

$$a + G_*(x,t,p,X) \le 0.$$

A function $v \in LSC(\mathbb{R}^n \times [0,T))$ is a viscosity supersolution of (2.17) if it satisfies :

(i) $v(x,t=0) \ge u_0(x)$ in $\mathbb{R}^n$,

(ii) for every $(x,t) \in \mathbb{R}^n \times (0,T)$ and for every $(a,p,X) \in \mathcal{P}^- v(x,t)$, we have :

$$a + G_*(x,t,p,X) \ge 0.$$

A function $u \in C^0(\mathbb{R}^n \times [0,T))$ is a viscosity solution of (2.17) if, and only if, it is a sub and a supersolution of (2.17).

Assumption (C) We say that a usc function w satisfies the compactness assumption (C) if for every $(z,s) \in \mathbb{R}^n \times \mathbb{R}^+_*$, there exists $r > 0$ such that, for every $M > 0$, there exists C such that,

$$\left. \begin{array}{c} |(x,t) - (z,s)| \le r \\ (\tau,p,X) \in \bar{\mathcal{P}}^+ w(x,t) \\ |w(x,t)| + |p| + |X| \le M \end{array} \right\} \Rightarrow \tau \le C.$$

We recall the parabolic version of Ishii's *lemma*, proved in Appendix (Section 4) :

Lemma 4.4. (Parabolic version of Ishii's *lemma*)
Let U and V be open sets of $\mathbb{R}^n$, $u \in USC(U \times \mathbb{R}^+)$ and $v \in LSC(V \times \mathbb{R}^+)$. Let $\phi : U \times V \times \mathbb{R}^+ \to \mathbb{R}$ of class C^2. Assume that $(x,y,t) \mapsto u(x,t) - v(y,t) - \phi(x,y,t)$ reaches a local maximum in $(\bar{x},\bar{y},\bar{t}) \in U \times V \times \mathbb{R}^+_*$. We set $\tau = \partial_t \phi(\bar{x},\bar{y},\bar{t})$, $p_1 = D_x\phi(\bar{x},\bar{y},\bar{t})$, $p_2 = -D_y\phi(\bar{x},\bar{y},\bar{t})$ and $A = D^2\phi(\bar{x},\bar{y},\bar{t})$. Assume also that u and $-v$

satisfy assumption (C)*. Then, for every* $\alpha > 0$ *such that* $\alpha A < I$*, there exists* τ_1, $\tau_2 \in \mathbb{R}$ *and* X, $Y \in S^n$ *such that :*

$$\begin{array}{c} \tau = \tau_1 - \tau_2, \\ (\tau_1, p_1, X) \in \bar{\mathcal{P}}^+ u(\bar{x}, \bar{t}),\ (\tau_2, p_2, Y) \in \bar{\mathcal{P}}^- v(\bar{y}, \bar{t}), \\ \frac{-1}{\alpha}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq (I - \alpha A)^{-1} A. \end{array}$$

Remark 4.5. *The assumption* (C) *is satisfied by* u *and* $-v$ *as soon as* u *is a subsolution and* v *is a supersolution of a parabolic equation.*

We also recall the fundamental property of geometric equations :

Lemma 4.6. (Fundamental property of geometric equations)
Let $\theta : \mathbb{R} \to \mathbb{R}$ *be a continuous, non decreasing scalar function and* u *be a subsolution (respectively a supersolution) of (2.17), then* $\theta(u)$ *is also a subsolution (resp. a supersolution).*

For the proof of this *Lemma*, we refer to Soner [170] (Theorem 1.11).
We have the following comparison principle :

Theorem 4.7. (Comparison principle)
Let u*, a locally bounded usc function, be a subsolution and* v*, a locally bounded lsc function, be a supersolution of (2.17). Assume that* $u_0(x) = u(0, x) \leq v(0, x) = v_0(x)$ *in* $\mathbb{R}^n$*, then, under the assumptions* (H_0)*,* $u \leq v$ *in* $\mathbb{R}^n \times [0, T)$*.*

Proof of theorem 4.7
The proof of this theorem is rather classical when the functions u and v are bounded (see for instance Chen, Giga, Goto [58]), but for the reader's convenience, we give the detailed proof. Let us consider for the moment that u and v are bounded. Moreover, we remark that for $\gamma > 0$, $\tilde{u} = u - \frac{\gamma}{T-t}$ is a subsolution of (2.17) and satisfies, in the viscosity sense :

$$\tilde{u}_t + G_*(x, t, D\tilde{u}, D^2\tilde{u}) \leq \frac{-\gamma}{(T-t)^2} \leq \frac{-\gamma}{T^2}.$$

Indeed, $D\tilde{u} = Du$, $D^2\tilde{u} = D^2 u$ and $\tilde{u}_t = u_t - \frac{\gamma}{(T-t)^2}$. So :

$$\tilde{u}_t + G_*(x, t, D\tilde{u}, D^2\tilde{u}) = -\frac{\gamma}{(T-t)^2} + u_t + G_*(x, t, Du, D^2u) \leq \frac{-\gamma}{(T-t)^2}.$$

This is written in a formal way but it is easy to show it by using test functions. Since $u \leq v$ follows from $\tilde{u} \leq v$ in the limit $\gamma \to 0$, it will simply suffice to prove the comparison principle under the additional assumptions :

$$\begin{cases} u_t + G_*(x, t, Du, D^2u) \leq \dfrac{-\gamma}{T^2} < 0, \\ \lim\limits_{t \to T} u(t, x) = -\infty. \end{cases} \tag{2.18}$$

We set :

$$M = \sup_{(x,t)\in\mathbb{R}^n\times[0,T)} \{u(x,t) - v(x,t)\}.$$

We want to show that $M \le 0$. We argue by contradiction. Assume $M > 0$. So, there exists (x^*, t^*) such that $u(x^*, t^*) - v(x^*, t^*) > 0$. Then, we duplicate the variables in space by considering :

$$\bar{M} = \sup_{(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,T)} \left\{ u(x,t) - v(y,t) - \frac{|x-y|^4}{4\epsilon} - \frac{\alpha}{2}\left(|x|^2 + |y|^2\right) \right\}.$$

We remark that $\bar{M} \ge u(x^*, t^*) - v(x^*, t^*) - \alpha|x^*|^2$, and so $\bar{M} > 0$ for α small enough. Thanks to the term $\frac{\alpha}{2}(|x|^2 + |y|^2)$, this supremum is reached (because u and v are bounded). We then denote by $(\bar{x}, \bar{y}, \bar{t})$ a point of maximum. We will use the following *lemma* :

Lemma 4.8. (Passing to the limit in α and ϵ)
We set $M' = \lim_{h\to 0} \sup_{|y-x|\le h,\ t\in[0,T)} (u(x,t) - v(y,t))$. Then, the following holds :

1. $\lim_{\alpha\to 0} \alpha\bar{x} = \lim_{\alpha\to 0} \alpha\bar{y} = 0$,
2. $\lim_{\epsilon\to 0} |\bar{x} - \bar{y}|^4 = 0$,
3. $\lim_{\epsilon\to 0}\lim_{\alpha\to 0} \bar{M} = M'$,
4. $\lim_{\epsilon\to 0}\lim_{\alpha\to 0} \frac{1}{\epsilon}|\bar{x} - \bar{y}|^4 = 0$,
5. $\lim_{\epsilon\to 0}\lim_{\alpha\to 0} \alpha(|\bar{x}|^2 + |\bar{y}|^2) = 0$.

The proof of the *lemma* is postponed.
We then distinguish two cases :

1$^{\text{st}}$ case : $\forall\epsilon > 0, \exists\alpha \in (0,\epsilon)$ such that $\bar{t} = 0$. Then, there exists sequences $\epsilon_n \to 0$ and $\alpha_n \to 0$ such that $\bar{t}_n = 0$ and :

$$0 < \bar{M} \le u(\bar{x}_n, 0) - v(\bar{y}_n, 0) \le u_0(\bar{x}_n) - u_0(\bar{y}_n) \le B_0(|\bar{x}_n - \bar{y}_n|),$$

where B_0 is the Lipschitz constant of u_0. We then obtain a contradiction because $|\bar{x}_n - \bar{y}_n| \to 0$ (see *Lemma* 4.8).

2$^{\text{nd}}$ case : Assume that there exists $\epsilon > 0$ such that for every $\alpha \in (0,\epsilon)$, we have $\bar{t} > 0$. We can choose ϵ small enough (else, we apply the argument of the first case), *i.e.*, so that :

$$\frac{|\bar{x} - \bar{y}|^4}{\epsilon} \le \frac{\gamma}{2T^2 L_c}. \tag{2.19}$$

We set $\tilde{u}(x,t) = u(x,t) - \frac{\alpha}{2}|x|^2$ and $\tilde{v}(x,t) = v(x,t) + \frac{\alpha}{2}|x|^2$. Then, we have :

$$\bar{M} = \sup_{(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,T)} \left\{ \tilde{u}(x,t) - \tilde{v}(y,t) - \frac{|x-y|^4}{4\epsilon} \right\}.$$

Then, we consider the test function $\psi(x,y,t) = \frac{|x-y|^4}{4\epsilon}$ and we set $\bar{p} = \bar{x} - \bar{y}$. We use the parabolic version of Ishii's *Lemma*. With the same notations, the following holds :

$$\tau = 0,$$
$$p_1 = \frac{|\bar{p}|^2\bar{p}}{\epsilon} = p_2,$$
$$A = \frac{2}{\epsilon}|\bar{p}|^2 \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix}, \text{ with } Z = \frac{I}{2} + \frac{\bar{p}\otimes\bar{p}}{|\bar{p}|^2}.$$

and for every β such that $\beta A < I$, there exists X, $Y \in S^n$ and two reals τ_1 and τ_2 such that :

$$\tau_1 - \tau_2 = 0,$$
$$\left(\tau_1, \frac{|\bar{p}|^2\bar{p}}{\epsilon}, X\right) \in \bar{\mathcal{P}}^+\tilde{u}(\bar{x},\bar{t}),$$
$$\left(\tau_2, \frac{|\bar{p}|^2\bar{p}}{\epsilon}, Y\right) \in \bar{\mathcal{P}}^-\tilde{v}(\bar{y},\bar{t}),$$
$$\frac{-1}{\beta}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le (I-\beta A)^{-1}A.$$

The last inequality implies that $X \le Y$. We use the following *lemma* :

Lemma 4.9. (Matrix estimates)

We have the following estimates on the matrix A

$$\begin{cases} \dfrac{1}{2\|A\|}A < I, \\ \|A\| \le \dfrac{6|\bar{p}|^2}{\epsilon}, \\ \textit{if } \delta = \dfrac{1}{2\|A\|}, \textit{ then } (I-\delta A)^{-1}A \le 2\|A\|I \le \dfrac{12}{\epsilon}|\bar{p}|^2 I, \end{cases}$$

where $\|A\| = \sup_{\zeta\in\mathbb{R}^{2n}} \dfrac{|A\zeta.\zeta|}{\zeta.\zeta}$.

The proof of the *lemma* is postponed.

Using *Lemma* 4.9 with $\beta = \frac{1}{2\|A\|}$, we can rewrite the matrix inequality in the following form :

$$\frac{-12|\bar{p}|^2}{\epsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{12|\bar{p}|^2}{\epsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Thus X and Y are bounded independently of α (it is already the case for $\bar{p}$ according to (2.19)). In particular, $\frac{-12|\bar{p}|^2}{\epsilon} I \leq X \leq \frac{12|\bar{p}|^2}{\epsilon} I$. So there exists $\alpha_n \to 0$ such that $\bar{t} \to t_\infty$, $\bar{p} \to p_\infty$ and $(X, Y) \to (X_\infty, Y_\infty)$. Moreover, since u is a subsolution and v is a supersolution, by using Condition (2.18), one finds :

$$\tau_1 + G_* \left(\bar{x}, \bar{t}, \frac{|\bar{p}|^2 \bar{p}}{\epsilon} + \alpha_n \bar{x}, X + \alpha_n I \right) \leq \frac{-\gamma}{T^2},$$

$$\tau_2 + G^* \left(\bar{y}, \bar{t}, \frac{|\bar{p}|^2 \bar{p}}{\epsilon} - \alpha_n \bar{y}, Y - \alpha_n I \right) \geq 0.$$

Then, by using the ellipticity of the equation and the matrix inequality $X \leq Y$, we obtain :

$$\tau_2 + G^* \left(\bar{y}, \bar{t}, \frac{|\bar{p}|^2 \bar{p}}{\epsilon} - \alpha_n \bar{y}, X - \alpha_n I \right) \geq 0.$$

Subtracting the two previous inequalities, yields :

$$\frac{\gamma}{T^2} + G_*(\bar{x}, \bar{t}, \frac{|\bar{p}|^2 \bar{p}}{\epsilon} + \alpha_n \bar{x}, X + \alpha_n I) \leq G^*(\bar{y}, \bar{t}, \frac{|\bar{p}|^2 \bar{p}}{\epsilon} - \alpha_n \bar{y}, X - \alpha_n I).$$

i.e. :

$$\begin{aligned} & c(\bar{x}, \bar{t}) \left| \frac{|\bar{p}|^2 \bar{p}}{\epsilon} + \alpha_n \bar{x} \right| - c(\bar{y}, \bar{t}) \left| \frac{|\bar{p}|^2 \bar{p}}{\epsilon} - \alpha_n \bar{y} \right| \\ & - F_* \left(\frac{|\bar{p}|^2 \bar{p}}{\epsilon} + \alpha_n \bar{x}, X + \alpha_n I \right) + F^* \left(\frac{|\bar{p}|^2 \bar{p}}{\epsilon} - \alpha_n \bar{y}, X - \alpha_n I \right) \geq \frac{\gamma}{T^2}. \end{aligned}$$

We send α to 0. By using item (i) of *Lemma* 4.8 and the fact that c is bounded, we obtain :

$$(c(\bar{x}, \bar{t}) - c(\bar{y}, \bar{t})) \frac{|p_\infty|^3}{\epsilon} - F_* \left(\frac{|p_\infty|^2 p_\infty}{\epsilon}, X_\infty \right) + F^* \left(\frac{|p_\infty|^2 p_\infty}{\epsilon}, X_\infty \right) \geq \frac{\gamma}{T^2}.$$

Moreover,

$$(c(\bar{x}, \bar{t}) - c(\bar{y}, \bar{t})) \frac{|p_\infty|^3}{\epsilon} \leq \frac{L_c |p_\infty|^4}{\epsilon},$$

where L_c is the Lipschitz constant of c. This implies :

$$\frac{L_c |p_\infty|^4}{\epsilon} - F_* \left(\frac{|p_\infty|^2 p_\infty}{\epsilon}, X_\infty \right) + F^* \left(\frac{|p_\infty|^2 p_\infty}{\epsilon}, X_\infty \right) \geq \frac{\gamma}{T^2}.$$

By using (2.19), we obtain :

$$\frac{\gamma}{2T^2} - F_* \left(\frac{|p_\infty|^2 p_\infty}{\epsilon}, X_\infty \right) + F^* \left(\frac{|p_\infty|^2 p_\infty}{\epsilon}, X_\infty \right) \geq \frac{\gamma}{T^2}.$$

We then distinguish two cases :

1st case : $p_\infty = 0$. In this case, we have $X_\infty = 0$ and $F_*(0,0) = F^*(0,0) = 0$. So, we obtain :

$$0 \geq \frac{\gamma}{2T^2}.$$

This is absurd.

2nd case : $p_\infty \neq 0$. In this case $F_* = F^* = F$. So, we have :

$$0 \geq \frac{\gamma}{2T^2}.$$

This is absurd.

To achieve the proof in the case where the functions are not bounded, it suffices to use the fundamental property of geometric equations. We then consider the truncature functions T_k (see Figure 2.2). For every k, we then have $T_k(u) \leq T_k(v)$ and

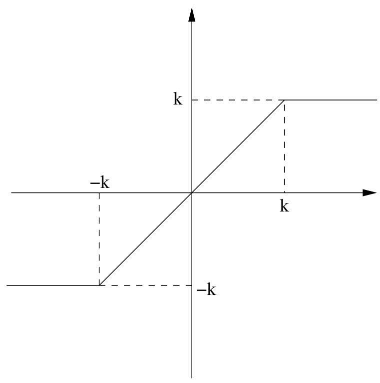

FIG. 2.2 – A truncature function T_k.

by sending k to infinity, we obtain the result. □

We now prove *Lemma* 4.8.

Proof of *Lemma* 4.8

The function $(x,y) \mapsto u(x,t) - v(y,t)$ is bounded (because u and v are bounded). Moreover, we have assumed that $\bar{M} > 0$. We then have :

$$\frac{|\bar{x}-\bar{y}|^4}{4\epsilon} + \frac{\alpha}{2}(|\bar{x}|^2 + |\bar{y}|^2) \leq C.$$

From which, we get

$$\frac{\alpha}{2}(|\bar{x}|^2 + |\bar{y}|^2) \leq C,$$

and

$$|\bar{x}-\bar{y}|^4 \leq C\epsilon,$$

and so $\lim\limits_{\epsilon\to 0} |\bar{x}-\bar{y}| = 0$ and $\lim\limits_{\alpha\to 0} \alpha\bar{x} = \lim\limits_{\alpha\to 0} \alpha\bar{y} = 0$.

We set

$$M_h = \sup_{|x-y|\le h,\ t\in[0,T)} (u(x,t) - v(y,t)).$$

Let (x_n^h, y_n^h, t_n^h) be such that $u(x_n^h, t_n^h) - v(y_n^h, t_n^h) \ge M_h - \frac{1}{n}$ with $|x_n^h - y_n^h| \le h$ (x_n^h and y_n^h do not depend on α). We then have

$$\begin{aligned}
&M_h - \frac{1}{n} - \frac{h^4}{4\epsilon} - \frac{\alpha}{2}\left(|x_n^h|^2 + |y_n^h|^2\right)\\
\le& u(x_n^h, t_n^h) - v(y_n^h, t_n^h) - \frac{|x_n^h - y_n^h|^4}{4\epsilon} - \frac{\alpha}{2}\left(|x_n^h|^2 + |y_n^h|^2\right)\\
\le& \bar{M}\\
\le& u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}).
\end{aligned}$$

Sending $\alpha \to 0$, we get

$$\begin{aligned}
M_h - \frac{1}{n} - \frac{h^4}{4\epsilon} &\le \liminf_{\alpha\to 0}(u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}))\\
&\le \limsup_{\alpha\to 0}(u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}))
\end{aligned}$$

(we have used the fact that x_n^h and y_n^h do not depend on α). We send h to 0 and we obtain :

$$\begin{aligned}
M' - \frac{1}{n} &\le \liminf_{\alpha\to 0}(u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}))\\
&\le \limsup_{\alpha\to 0}(u(\bar{x},\bar{t}) - v(\bar{y},\bar{t})).
\end{aligned}$$

Sending $\epsilon \to 0$, we get

$$\begin{aligned}
M' - \frac{1}{n} &\le \liminf_{\epsilon\to 0}\liminf_{\alpha\to 0}(u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}))\\
&\le \limsup_{\epsilon\to 0}\limsup_{\alpha\to 0}(u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}))\\
&\le \limsup_{\epsilon\to 0}\limsup_{\alpha\to 0}\left(\sup_{|x-y|\le c\epsilon^{\frac{1}{4}},\ t\in[0,T]}(u(x,t) - v(y,t))\right)\\
&\le \limsup_{h\to 0}\sup_{|x-y|\le h,\ t\in[0,T]}(u(x,t) - v(y,t))\\
&= M'
\end{aligned}$$

so $\lim_{\epsilon\to 0}\lim_{\alpha\to 0} u(\bar{x},\bar{y}) - v(\bar{y},\bar{t}) = M'$. In the same way, we have $\lim_{\epsilon\to 0}\lim_{\alpha\to 0} \bar{M} = M'$. Therefore,

$$\lim_{\epsilon\to 0}\lim_{\alpha\to 0}\left(\frac{|\bar{x}-\bar{y}|^4}{4\epsilon} + \frac{\alpha}{2}(|\bar{x}|^2 + |\bar{y}|^2)\right) = \lim_{\epsilon\to 0}\lim_{\alpha\to 0}\left(u(\bar{x},\bar{y}) - v(\bar{y},\bar{t}) - \bar{M}\right) = 0.$$

So,

$$\begin{cases} \lim\limits_{\epsilon\to 0}\lim\limits_{\alpha\to 0}\dfrac{|\bar{x}-\bar{y}|^4}{4\epsilon}=0, \\ \lim\limits_{\epsilon\to 0}\lim\limits_{\alpha\to 0}\dfrac{\alpha}{2}(|\bar{x}|^2+|\bar{y}|^2)=0 \end{cases}$$

what achieves the proof of *Lemma* 4.8. □

We now prove *Lemma* 4.9.

Proof of *Lemma* 4.9

By definition of the norm of A, we have : $\frac{A\zeta.\zeta}{\zeta.\zeta}\le \|A\|$, so $\frac{A\zeta.\zeta}{\|A\|}\le I\zeta.\zeta$, what gives the first result of the *lemma*.

Let $\zeta=\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}\in\mathbb{R}^{2n}$. We then have :

$$\begin{aligned} A\zeta.\zeta =&\frac{2}{\epsilon}|\bar{p}|^2 Z(\zeta_1-\zeta_2).(\zeta_1-\zeta_2)\\ \le&\frac{4}{\epsilon}|\bar{p}|^2 Z(\zeta_1.\zeta_1+\zeta_2.\zeta_2)\\ \le&\frac{4}{\epsilon}|\bar{p}|^2\|Z\||\zeta|^2. \end{aligned}$$

So, it suffices to show that $\|Z\|\le\frac{3}{2}$:

$$\begin{aligned} Z\xi.\xi =&\frac{\xi^2}{2}+\frac{\bar{p}\xi\otimes\bar{p}\xi}{|\bar{p}|^2}\\ \le&\frac{\xi^2}{2}+\xi^2\\ \le&\frac{3}{2}\xi^2. \end{aligned}$$

For the last item, it suffices to notice that if $B\ge 0$ and $C\ge 0$ with $B,C\in S^n$ such that $BC=CB$, then $CB\ge 0$. Indeed, $CB\xi.\xi=CB^{\frac{1}{2}}\xi.B^{\frac{1}{2}}\xi\ge 0$ (we have used the fact that B is positive, then symmetric and finally that C is positive). So, if $B\ge C$ and $D\ge 0$ with $D(B-C)=(B-C)D$, then $DB\ge DC$. So, it suffices to show that $A\le 2\|A\|(I-\delta A)=2\|A\|I-A$, *i.e.* $A\le\|A\|I$ what is true by definition of the norm of A.

This achieves the proof of *Lemma* 4.9. □

We now give an existence result by using the classical Perron's method (for the proof, we refer to Crandall, Ishii, Lions [61]) :

Theorem 4.10. (Existence and uniqueness)
Assume that there exists a subsolution $U^-\in USC(\mathbb{R}^n\times[0,T))$ *and a supersolution* $U^+\in LSC(\mathbb{R}^n\times[0,T))$ *of (2.17) such that* $U^-(x,0)\le u_0(x)\le U^+(x,0)$*, then, there exists a unique continuous solution of (2.17).*

We now construct a sub and a supersolution which satisfy the initial condition. We begin with studying the problem $u_t + F(Du, D^2u) = 0$. We have the following *Lemma* :

Lemma 4.11. (Existence and uniqueness, case $c = 0$)
There exists a unique solution u of the problem

$$\begin{cases} u_t + F(Du, D^2u) = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{2.20}$$

Moreover, u is uniformly continuous in time and its modulus of continuity, ω_F, depends only on the Lipschitz constant of u_0, B_0 and behaves like $\sqrt{t}$. So, u satisfies :

$$\forall t, s \in [0,T),\ \forall x \in \mathbb{R}^n \quad |u(x,t) - u(x,s)| \le \omega_F(|t-s|).$$

Proof of *Lemma* 4.11
We assume, in a first time, that $u_0 \in C_b^2 = \{u\ C^2, \exists\ C,\ ||Du||_{L^\infty},\ ||D^2u||_{L^\infty} \le C\}$. We set $u^\pm = u_0 \pm C_1 t$ with $C_1 = \inf_{x\in R^n}\{-F^*(Du_0, D^2u_0), F_*(Du_0, D^2u_0)\}$ (C_1 depends only on the bounds of Du_0 and D^2u_0). It then easy to check that u^+ is a supersolution and u^- is a subsolution. Then, there exists a unique solution of (2.20) (Theorem 4.10) and, by the comparison principle, the following holds :

$$\forall t \in [0,T),\ \forall x \in \mathbb{R}^n,\ |u(x,t) - u_0(x)| \le C_1 t. \tag{2.21}$$

We then set $v(x,t) = u(x,t+h)$. So v is solution of (2.20) and, by the comparison principle, we obtain :

$$u(x,t+h) - u(x,t) \le \sup(u(\cdot,h) - u_0) \le C_1 h.$$

Similarly, we have :

$$u(x,t) - u(x,t+h) \le \sup(u(\cdot,h) - u_0) \le C_1 h,$$

and so :

$$|u(x,t+h) - u(x,t)| \le \sup(u(\cdot,h) - u_0) \le C_1 h.$$

We now assume that u_0 is only Lipschitz continuous. We set $u_\epsilon^0 = u_0 \star \rho_\epsilon$ where ρ_ϵ is a regularising sequence, *i.e.* $\rho_\epsilon = \frac{1}{\epsilon^n}\rho(\frac{\cdot}{\epsilon})$ where $\rho \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ and satisfies :

$$\rho \ge 0, \quad \mathrm{supp}(\rho) \subset \bar{B}(0,1), \quad \int_{\mathbb{R}^n} \rho(x)dx = 1.$$

Then $u_\epsilon^0 \in C_b^2$ and $\|Du_\epsilon^0\|_{L^\infty(\mathbb{R}^n)},\ \|D^2u_\epsilon^0\|_{L^\infty(\mathbb{R}^n)} \leq \frac{B_0C_2}{\epsilon}$. Indeed :

$$\begin{aligned}|Du_\epsilon^0(x)| =& |Du_0 \star \rho_\epsilon(x)|\\ =& \left|\int_{\mathbb{R}^n} Du_0(x-y)\rho_\epsilon(y)dy\right|\\ \leq& \int_{\mathbb{R}^n} |Du_0(x-y)|\,\rho_\epsilon(y)dy\\ \leq& B_0\int_{\mathbb{R}^n}\rho_\epsilon(y)dy\\ =& B_0\end{aligned}$$

and

$$\begin{aligned}|D^2u_\epsilon^0(x)| =& |Du_0 \star D\rho_\epsilon(x)|\\ =& \int_{\mathbb{R}^n} |Du_0(x-y)|\,|D\rho_\epsilon(y)|dy\\ \leq& B_0\frac{1}{\epsilon}\int_{\mathbb{R}^n}\frac{1}{\epsilon^n}\left|D\rho\left(\frac{y}{\epsilon}\right)\right|dy\\ =& B_0\frac{1}{\epsilon}\|D\rho\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Moreover, $\|u_0 - u_\epsilon^0\|_{L^\infty(\mathbb{R}^n)} \leq B_0\epsilon$. Indeed, since $\int_{\mathbb{R}^n}\rho_\epsilon(x)dx = 1$

$$\begin{aligned}|u_0 - u_\epsilon^0(x)| \leq& \int_{\mathbb{R}^n} |u_0(x) - u_0(x-y)|\rho_\epsilon(y)dy\\ \leq& B_0\int_{\bar{B}(0,\epsilon)} |y|\rho_\epsilon(y)dy\\ \leq& \epsilon B_0\int_{\bar{B}(0,\epsilon)}\rho_\epsilon(y)dy = \epsilon B_0.\end{aligned}$$

We note u_ϵ the solution with initial condition u_ϵ^0. Then, by the comparison principle, $\|u_{\epsilon'}(\cdot,t) - u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u_{\epsilon'}^0 - u_\epsilon^0\|_{L^\infty(\mathbb{R}^n)}$, and so u_ϵ converge uniformly (since u_ϵ^0 converge uniformly) to u which is, by stability (see for instance Theorem 2.3 of Barles [18]), the solution of (2.20) with initial condition u_0. We then have, by the comparison principle, $\|u_\epsilon(\cdot,t) - u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u_\epsilon^0 - u_0\|_{L^\infty(\mathbb{R}^n)}$. We then deduce :

$$\begin{aligned}\|u(\cdot,t+h) - u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq& 2\|u_0 - u_\epsilon^0\|_{L^\infty(\mathbb{R}^n)} + \|u_\epsilon(\cdot,t+h) - u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\\ \leq& 2B_0\epsilon + C_1\left(B_0, \frac{B_0C_2}{\epsilon}\right)h.\end{aligned}$$

By taking the minimum on ϵ, we obtain the modulus of continuity of u, ω_F, which depends only on B_0. Moreover, using the geometric property of the equation, one deduces that $C_1\left(B_0, \frac{B_0C_2}{\epsilon}\right) \sim \frac{1}{\epsilon}$ and so $\omega_F(h)$ behaves like $\sqrt{h}$. $\square$

Remark 4.12. *In the case of dislocation dynamics, i.e. with the function F given by (2.16), an alternative proof can be found in Chen, Giga, Goto [58], based on self-similar solutions (Wulff Shape) of the mean curvature motion.*

Lemma 4.13. (Existence of sub and supersolutions, general case)
There exists U^+ (respectively U^-) supersolution (respectively subsolution) of (2.17) such that :

$$\begin{aligned} u_0(x) - \omega_F(t) - \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))}B_0 t \leq & U^-(x,t) \\ \leq & U^+(x,t) \leq u_0(x) + \omega_F(t) + \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))}B_0 t. \end{aligned}$$

for every $(x,t) \in \mathbb{R}^n \times (0,T)$.

Proof of *Lemma* 4.13
First, we study the problem (2.20). According to *Lemma* 4.11, this problem has a unique continuous solution u which satisfies :

$$u_0(x) - \omega_F(t) \leq u(t,x) \leq u_0(x) + \omega_F(t).$$

We prove that $\|Du\|_{L^\infty(\mathbb{R}^n\times(0,T))} \leq \|Du_0\|_{L^\infty(\mathbb{R}^n)}$. Indeed, let us consider the function $u^h(x,t) = u(x+h,t) + \|Du_0\|_{L^\infty(\mathbb{R}^n)}\,|h|$. Then u^h is still solution of (2.20). Therefore, $u^h(x,0) = u_0(x+h) + \|Du_0\|_{L^\infty(\mathbb{R}^n)}\,|h| \geq u_0(x)$. So, by the comparison principle, we have $u^h \geq u$. We deduce that, for every $h \in \mathbb{R}^n$,

$$u(x,t) - u(x+h,t) \leq \|Du_0\|_{L^\infty(\mathbb{R}^n)}\,|h|,$$

and so

$$\|Du\|_{L^\infty(\mathbb{R}^n\times(0,T))} \leq \|Du_0\|_{L^\infty(\mathbb{R}^n)}$$

We then set $U^+(x,t) = u(x,t) + \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))}B_0 t$. Then, $\|DU^+\|_{L^\infty(\mathbb{R}^n\times(0,T))} \leq \|Du\|_{L^\infty(\mathbb{R}^n\times(0,T))} \leq B_0$ and U^+ is solution of :

$$\begin{cases} v_t - \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))}B_0 + F(Dv, D^2v) = 0, \\ v(x,0) = u_0, \end{cases}$$

and so U^+ is supersolution of (2.17) and satisfies :

$$\begin{aligned} U^+(x,t) =& u(x,t) + \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))}B_0 t \\ \leq & u_0(x) + \omega_F(t) + \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))}B_0 t. \end{aligned}$$

Similarly, we construct a subsolution U^- such that $U^-(x,t) \geq u_0(x) - \omega_F(t) - \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))}B_0 t$ by setting $U^-(x,t) = u(x,t) - \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))}B_0 t$. To achieve the proof, it suffices to apply the comparison principle to U^- and U^+. □

Finally, we proved the following Theorem :

Theorem 4.14. (Existence and uniqueness for the local problem)
Let $T > 0$. Then, under the assumptions (H_0), there exists a unique viscosity solution of the problem (2.17) in $\mathbb{R}^n \times [0,T)$. Moreover, the solution satisfies, for every $(x,t) \in \mathbb{R}^n \times (0,T)$:

$$u_0(x) - \omega_F(t) - \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))} B_0 t \le u(x,t) \le u_0(x) + \omega_F(t) + \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))} B_0 t.$$

where ω_F is the modulus of continuity of the solution of (2.20) and behaves like $\sqrt{t}$.

4.2 Regularity results for the local problem

Lemma 4.15. (Regularity results for the local problem)
Assume that $\|Du_0\|_{L^\infty(\mathbb{R}^n)} \le B_0$ and $\frac{\partial u_0}{\partial x_n} \ge b_0$, with $B_0 > 0$ and $b_0 > 0$. Then, the solution of (2.17) given by Theorem 4.14 satisfies

$$\|Du(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le B(t) \text{ and } \frac{\partial u}{\partial x_n} \ge b(t),$$

with $B(t) = B_0 e^{L_c t}$ and $b(t) = b_0 - B_0(e^{L_c t} - 1)$. Moreover, u is uniformly continuous in time and its modulus of continuity in time ω_u, defined by :

$$\forall x \in \mathbb{R}^n,\ \forall s,\ t \in [0,T),\ |u(x,t) - u(x,s)| \le \omega_u(|t-s|),$$

satisfies :

$$\omega_u(\delta) \le \omega_F(\delta) + \|c\|_{L^\infty} B_0 \delta + \omega_c(\delta) \int_0^T B(s)ds,$$

where ω_c is the modulus of continuity in time of c, and ω_F is the modulus of continuity in time of the solution of (2.20) and behaves like $\sqrt{t}$.

Proof of *Lemma* 4.15 For the proof of the Lipschitz estimate in space, we assume in a first time that u is bounded. We set $\phi^\epsilon(x,y,t) = B(t)\left(|x-y|^2 + \epsilon^2\right)^{1/2}$. We prove that $u(x,t) - u(y,t) \le \phi^\epsilon$. We set :

$$M = \sup_{(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,T)} \left\{u(x,t) - u(y,t) - \phi^\epsilon(x,y,t)\right\},$$

Assume that $M > 0$. Then we set :

$$\bar{M} = \sup_{(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,T)} \left\{u(x,t) - u(y,t) - \phi^\epsilon(x,y,t) - \frac{\alpha}{2}(|x|^2 + |y|^2) - \frac{\gamma}{T-t}\right\}.$$

For $\alpha > 0,\ \gamma > 0$ small enough, we have $\bar{M} > 0$. Moreover u is bounded, so the supremum is reached in $(\bar{x},\bar{y},\bar{t})$ (with $\bar{x} \ne \bar{y}$) and

$$\frac{\alpha}{2}(|\bar{x}|^2 + |\bar{y}|^2) \le C$$

and so $\alpha\bar{x} \to 0$ and $\alpha\bar{y} \to 0$. We prove that $\bar{t} > 0$. Indeed, assume the contrary. Then, we have

$$u_0(\bar{x}) - u_0(\bar{y}) - \phi^\epsilon(\bar{x}, \bar{y}, 0) > 0,$$

i.e.

$$u_0(\bar{x}) - u_0(\bar{y}) > B_0\left(|\bar{x} - \bar{y}|^2 + \epsilon^2\right)^{\frac{1}{2}} > B_0|\bar{x} - \bar{y}|,$$

what is absurd since $\|Du_0\|_{L^\infty(\mathbb{R}^n)} \le B_0$. We set

$$\bar{p} = D_x\phi^\epsilon = \left(|\bar{x} - \bar{y}|^2 + \epsilon^2\right)^{-1/2}(\bar{x} - \bar{y})B(t) = -D_y\phi^\epsilon \neq 0 \text{ (because } \bar{x} \neq \bar{y}),$$

$$\begin{aligned} Z =& D_x^2\phi^\epsilon = \left(\left(|\bar{x} - \bar{y}|^2 + \epsilon^2\right)^{-1/2} I - \left(|\bar{x} - \bar{y}|^2 + \epsilon^2\right)^{-3/2}(\bar{x} - \bar{y}) \otimes (\bar{x} - \bar{y})\right) B(t) \\ =& D_y^2\phi^\epsilon, \end{aligned}$$

$$A = D^2\phi^\epsilon = \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix}.$$

Then, by parabolic version of Ishii's *Lemma* applied to $\tilde{u} = u(x,t) - \frac{\alpha}{2}|x|^2$, $\tilde{v}(y,t) = v(y,t) + \frac{\alpha}{2}|y|^2$ and $\phi(x,y,t) = \phi^\epsilon(x,y,t) + \frac{\gamma}{T-t}$, for every β such that $\beta A < I$, there exists $\tau_1,\ \tau_2 \in \mathbb{R}$ and $X,\ Y \in S^n$ such that :

$$\tau_1 - \tau_2 = \frac{\gamma}{(T - \bar{t})^2} + L_c B(t)\left(|\bar{x} - \bar{y}|^2 + \epsilon^2\right)^{\frac{1}{2}},$$

$$(\tau_1, p + \alpha\bar{x}, X + \alpha I) \in \bar{\mathcal{P}}^+ u(\bar{x}, \bar{t}),$$
$$(\tau_2, p - \alpha\bar{y}, Y - \alpha I) \in \bar{\mathcal{P}}^- v(\bar{y}, \bar{t}),$$
$$\frac{-1}{\beta}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le (I - \beta A)^{-1} A.$$

So, the following holds

$$\tau_1 - c(\bar{x}, \bar{t})|\bar{p} + \alpha\bar{x}| + F_*(\bar{p} + \alpha\bar{x}, X + \alpha I) \le 0,$$

$$\tau_2 - c(\bar{y}, \bar{t})|\bar{p} - \alpha\bar{y}| + F^*(\bar{p} - \alpha\bar{y}, Y - \alpha I) \ge 0.$$

The matrix inequality implies in particular that $X \le Y$, so by using the ellipticity of F, we deduce :

$$\tau_2 - c(\bar{y}, \bar{t})|\bar{p} - \alpha\bar{y}| + F^*(\bar{p} - \alpha\bar{y}, X - \alpha I) \ge 0.$$

From that, by subtracting :

$$\begin{aligned} &\frac{\gamma}{(T - \bar{t})^2} + L_c B(t)\left(|\bar{x} - \bar{y}|^2 + \epsilon^2\right)^{\frac{1}{2}} - c(\bar{x}, \bar{t})|\bar{p} + \alpha\bar{x}| + c(\bar{y}, \bar{t})|\bar{p} - \alpha\bar{y}| \\ &+ F_*(\bar{p} + \alpha\bar{x}, X + \alpha I) - F^*(\bar{p} - \alpha\bar{y}, X - \alpha I) \le 0. \end{aligned}$$

We let α go to 0 ($\bar{p}$ and X are bounded so we can extract a converging subsequence and we still note $\bar{p}$ and X their limit) :

$$\begin{aligned}&\frac{\gamma}{(T-\bar{t})^2}+\lim_{\alpha\to 0}\left(L_cB(t)\left(|\bar{x}-\bar{y}|^2+\epsilon^2\right)^{\frac{1}{2}}+\left(-c(\bar{x},\bar{t})+c(\bar{y},\bar{t})\right)|\bar{p}|\right)\\&+F_*(\bar{p},X)-F^*(\bar{p},X)\leq 0.\end{aligned}$$

Now, $\bar{p}\neq 0$, therefore $F_*(\bar{p},X)=F^*(\bar{p},X)$. Moreover,

$$\begin{aligned}&L_cB(t)\left(|x-y|^2+\epsilon^2\right)^{1/2}-c(x,t)|\bar{p}|+c(y,t)|\bar{p}|\\
=&\left(|x-y|^2+\epsilon^2\right)^{1/2}\left(L_cB(t)-\frac{|x-y|B(t)}{|x-y|^2+\epsilon^2}\left(c(x,t)-c(y,t)\right)\right)\\
\geq&\left(|x-y|^2+\epsilon^2\right)^{1/2}\left(L_cB(t)-\frac{|x-y|^2L_cB(t)}{|x-y|^2+\epsilon^2}\right)\\
\geq&\left(|x-y|^2+\epsilon^2\right)^{1/2}\left(L_cB(t)-L_cB(t)\right)\\
\geq&0,\end{aligned}$$

so

$$\frac{\gamma}{(T-\bar{t})^2}\leq 0,$$

what is absurd. So $u(x,t)-u(y,t)\leq\phi^{\epsilon}$. By letting ϵ go to 0, we obtain :

$$u(x,t)-u(y,t)\leq B(t)|x-y|.$$

Exchanging x and y, yields

$$|u(x,t)-u(y,t)|\leq B(t)|x-y|,$$

what gives the first result in the case where u is bounded. If u is not bounded, we consider the truncature functions $T_k=\max(\min(x,k),-k)$. Then $T_k(u)$ is bounded and solution of the problem, and so :

$$|T_k(u(x,t))-T_k(u(y,t))|\leq B(t)|x-y|.$$

Letting k go to infinity, yields :

$$|u(x,t)-u(y,t)|\leq B(t)|x-y|,$$

and we obtain the first estimate.

For the second estimate, we set, for $x=(x',x_n)$, $u^{\lambda}(x,t)=u(x',x_n+\lambda,t)-\lambda b(t)$. We have

$$\begin{aligned}u^{\lambda}(x',x_n,0)=&u(x',x_n+\lambda)-\lambda b_0\\\geq&u(x',x_n,0).\end{aligned}$$

Moreover,

$$\begin{aligned}
&u_t^\lambda + G^*\left(x', x_n, t, Du^\lambda, D^2u^\lambda\right)\\
=&u_t - \lambda b'(t) - c(x', x_n, t)|Du| + F^*\left(Du, D^2u\right)\\
=&u_t + \lambda B_0 L_c e^{L_c t} - c(x', x_n, t)|Du| + F^*\left(Du, D^2u\right)\\
\geq &u_t + \lambda B_0 L_c e^{L_c t} - \left(c(x', x_n+\lambda, t) + \lambda L_c\right)|Du| + F^*\left(Du, D^2u\right)\\
\geq &\lambda B_0 L_c e^{L_c t} - \lambda B_0 L_c e^{L_c t} + u_t + G^*\left(x', x_n+\lambda, t, Du, D^2u\right)\\
\geq &0,
\end{aligned}$$

where u_t, Du, D^2u are taken at the point (x', x_n, t). This is written in a formal way and it can be justified by using a test function. So, we obtain that u^λ is a supersolution. By the comparison principle, we deduce $u^\lambda \geq u$, and so

$$u(x', x_n+\lambda, t) - u(x', x_n, t) \geq \lambda b(t).$$

what proves the second estimate.

It thus remains to be shown that u is uniformly continuous in time. We set $\delta > 0$. For every $(x,t) \in \mathbb{R}^n \times (0,T)$ such that $t+\delta \leq T$, we set $v(x,t) = u(x,t+\delta)$. Then, v is a subsolution of

$$w_t - \omega_c(\delta)B(t+\delta) - c(x,t)|Dw| + F\left(Dw, D^2w\right) = 0$$

on $\mathbb{R}^n \times (0, T-\delta)$ in the sense of definition 4.1 (ii). Indeed, we have

$$v_t - c\left(x, t+\delta\right)|Dv| + F\left(Dv, D^2v\right) = 0,$$

and

$$-c\left(x, t+\delta\right)|Dv| \geq -\omega_c(\delta)B(t+\delta) - c(x,t)|Dv|,$$

what gives in a formal way :

$$v_t - \omega_c(\delta)B(t+\delta) - c(x,t)|Dv| + F\left(Dv, D^2v\right) \leq 0.$$

Moreover, $u + \omega_c(\delta)\displaystyle\int_0^{t+\delta} B(s)ds$ is solution of the same problem. We then deduce that $\tilde{u} = u + \sup_{x\in\mathbb{R}^n}\left(u(x,\delta) - u_0(x)\right)^+ + \omega_c(\delta)\int_0^{t+\delta} B(s)ds$ is a supersolution and $v(x,0) \leq \tilde{u}(x,0)$. By Theorem 4.14 and the comparison principle, we then have :

$$\begin{aligned}
u(x,t+\delta) - u(x,t) &\leq \sup_{x\in\mathbb{R}^n}\left(u(x,\delta) - u_0(x)\right)^+ + \omega_c(\delta)\int_0^{t+\delta} B(s)ds\\
&\leq \omega_F(\delta) + \|c\|_{L^\infty} B_0 \delta + \omega_c(\delta)\int_0^T B(s)ds.
\end{aligned}$$

Similarly, v is a supersolution of

$$w_t + \omega_c(\delta) B(t+\delta) - c(x,t)|Dw| + F\left(Dw, D^2 w\right) = 0$$

and $\tilde{u} = u - \sup_{x\in\mathbb{R}^n} \left(u(x,\delta) - u_0(x)\right)^- - \omega_c(\delta)\int_0^{t+\delta} B(s)ds$ is subsolution. So, by the comparison principle, we have

$$\begin{aligned} u(x,t) - u(x,t+\delta) \leq & \omega_F(\delta) + \|c\|_{L^\infty} B\delta + \omega_c(\delta)\int_0^{t+\delta} B(s)ds \\ \leq & \omega_F(\delta) + \|c\|_{L^\infty} B_0\delta + \omega_c(\delta)\int_0^{T} B(s)ds, \end{aligned}$$

i.e.

$$|u(x,t) - u(x,t+\delta)| \leq \omega_F(\delta) + \|c\|_{L^\infty} B_0\delta + \omega_c(\delta)\int_0^{T} B(s)ds,$$

what achieves the proof of the *lemma*. □

5 The non local problem : proof of Theorem 3.2

For the proof of Theorem 3.2, we will need the three following *lemmata* and we have to intoduce the following space :

$$L^1_{\text{unif}}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} : \|f\|_{L^1_{\text{unif}}(\mathbb{R}^n)} < \infty\}$$

where

$$\|f\|_{L^1_{\text{unif}}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \int_{Q(x)} |u|$$

and Q is defined in (2.8).

Lemma 5.1. (Estimate on the characteristic functions)
Let $u^1 \in C(\mathbb{R}^n)$ satisfying

$$\frac{\partial u^1}{\partial x_n} \geq b$$

in the distributions sense for some $b > 0$ and $u^2 \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ satisfying the same condition. Then, we have the following estimate :

$$\left\|[u^2] - [u^1]\right\|_{L^1_{\text{unif}}} \leq \frac{2}{b}\left\|u^2 - u^1\right\|_{L^\infty}. \tag{2.22}$$

For the proof of this *Lemma*, we refer to the proof of Alvarez *et al.* [8] in the case $n = 2$, which adapts without difficulty to the case of any dimension.

Lemma 5.2. (Convolution inequality)
For every $f \in L^1_{\text{unif}}(\mathbb{R}^n)$ and $g \in L^\infty_{\text{int}}(\mathbb{R}^n)$, the convolution product $f \star g$ is bounded and satisfies

$$\|f \star g\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1_{\text{unif}}(\mathbb{R}^n)} \|g\|_{L^\infty_{\text{int}}(\mathbb{R}^n)}.$$

For the proof, we refer to Alvarez *et al.* [10].

Lemma 5.3. (Stability of the solution with respect to the velocity)
Let $T > 0$. We consider for $i = 1, 2$ two different equations :

$$\begin{cases} u^i_t = c^i(x,t)|Du^i| - F\left(Du^i, D^2u^i\right) \text{ in } \mathbb{R}^n \times (0,T), \\ u^i(x,0) = u_0(x), \end{cases} \tag{2.23}$$

where c^i satisfy the assumption $(H_0)(ii)$, u_0 satisfies $(H_0)(iii)$ and F satisfies the assumptions (HF). Then, for every $t \in [0,T)$, we have

$$\|u^1(\cdot,t) - u^2(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|c^1 - c^2\|_{L^\infty(\mathbb{R}^n \times (0,T))} \int_0^t B(s)ds,$$

where u^i are the solutions of (2.23) (see Theorem 4.14), $B(t) = B_0 e^{L_c t}$ with $L_c = \sup_i L_{c^i}$ (L_{c^i} is the Lipcshitz constant of c^i).

Proof of *Lemma* 5.3
We set $K = \|c^1 - c^2\|_{L^\infty(\mathbb{R}^n \times (0,T))}$. We remark that u^1 is subsolution of

$$u_t - c^2(x,t)|Du| + F\left(Du, D^2u\right) - KB(t) = 0.$$

Indeed, we have :

$$\begin{aligned} u^1_t - c^2(x,t)|Du^1| + F\left(Du^1, D^2u^1\right) \leq & c^1(x,t)|Du^1| - F\left(Du^1, D^2u^1\right) \\ & - c^2(x,t)|Du^1| + F\left(Du^1, D^2u^1\right) \\ \leq & \|c^1 - c^2\|_{L^\infty(\mathbb{R}^n \times (0,T))} B(t) \\ \leq & KB(t). \end{aligned}$$

It is a routine exercise to check that the differential inequality actually holds in the viscosity sense. Moreover, $u^2 + K \int_0^t B(s)ds$ is solution of the same problem. By the comparison principle (Theorem 4.7), we deduce

$$u^1 \leq u^2 + K \int_0^t B(s)ds.$$

From what

$$\|u^1(\cdot,t) - u^2(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|c^1 - c^2\|_{L^\infty(\mathbb{R}^n \times (0,T))} \int_0^t B(s)ds.$$

□

We now prove Theorem 3.2.

Proof of Theorem 3.2

We set $\omega(\delta) = \omega_F(\delta) + \|c_0\|_{L^1} B_0 \delta$, where ω_F is the modulus of continuity of (2.20) and behaves like $\sqrt{t}$. We define the space

$$E = \left\{ u \in L^\infty_{loc}(\mathbb{R}^n \times [0, T^*)), s.t. \quad \middle| \begin{array}{l} |Du(x,t)| \leq 2B_0, \\ \dfrac{\partial u}{\partial x_n}(x,t) \geq \dfrac{b_0}{2} \\ u \text{ is uniformly continuous in time} \\ \text{ and } \omega_u(\delta) \leq 2\omega(\delta) \end{array} \right\}$$

where ω_u is the modulus of continuity in time of u.

For $u \in E$, we set $c(x,t) = (c_0 \star [u(\cdot,t)])(x)$. We see that c is bounded, Lipschitz continuous in space (with $L_c = |c_0|_{BV}$ as Lipschitz constant) and uniformly continuous in time. Indeed,

$$\begin{aligned} \|c\|_{L^\infty(\mathbb{R}^n \times [0,T^*))} \leq & \sup_{t \in \mathbb{R}} \|c_0\|_{L^1} \|[u(\cdot,t)]\|_{L^\infty(\mathbb{R}^n)} \\ \leq & \|c_0\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

Moreover, for every t

$$\begin{aligned} \|Dc(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = & \|Dc_0 \star [u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \\ \leq & |c_0|_{BV} \|[u(\cdot,t)]\|_{L^\infty(\mathbb{R}^n)} \\ \leq & |c_0|_{BV}. \end{aligned}$$

Finally, for $0 < t, s < T^*$:

$$\begin{aligned} |c(x,t) - c(x,s)| = & |(c_0 \star [u(\cdot,t)])(x) - (c_0 \star [u(\cdot,s)])(x)| \\ = & |c_0 \star ([u(\cdot,t)] - [u(\cdot,s)])(x)| \\ \leq & \|c_0\|_{L^\infty_{\text{int}}} \|[u(\cdot,t)] - [u(\cdot,s)]\|_{L^1_{\text{unif}}(\mathbb{R}^n)} \\ \leq & \frac{4\|c_0\|_{L^\infty_{\text{int}}}}{b_0} \|u(\cdot,t) - u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} \\ \leq & \frac{4\|c_0\|_{L^\infty_{\text{int}}}}{b_0} \omega_u(|t-s|) \\ \leq & \frac{8\|c_0\|_{L^\infty_{\text{int}}}}{b_0} \omega(|t-s|), \end{aligned}$$

so c is uniformly continuous in time and $\omega_c(\delta) \leq \frac{8\|c_0\|_{L^\infty_{\text{int}}}}{b_0} \omega(\delta)$.

For $u \in E$, we then define $v = \Phi(u)$ as the unique viscosity solution (see Theorem 4.14) of

$$\begin{cases} v_t = (c_0 \star [u])|Dv| - F(Dv, D^2 v) \text{ in } \mathbb{R}^n \times (0, T^*), \\ v(x, t = 0) = u_0(x) \text{ in } \mathbb{R}^n. \end{cases} \tag{2.24}$$

We show that $\Phi : E \to E$ is a contraction. First, we show that Φ is well defined. We have $\|Dv(\cdot,t)\| \le B(t) \le B_0 e^{L_c T^*} \le 2B_0$, by definition of T^* (see *Lemma* 4.15). Moreover, $\frac{\partial v}{\partial x_n} \ge b(t) = b_0 - B_0(e^{L_c t} - 1)$ (see *Lemma* 4.15), and we want $\frac{\partial v}{\partial x_n} \ge \frac{b_0}{2}$, so it suffices to ensure that

$$B_0(e^{L_c t} - 1) \le \frac{b_0}{2}$$

$$e^{L_c t} \le \frac{b_0}{2B_0} + 1$$

$$t \le \frac{\ln\left(\frac{b_0}{2B_0} + 1\right)}{L_c},$$

which is true according to the choice of T^*. It thus remains to be shown that v is uniformly continuous with $\omega_v(\delta) \le 2\omega(\delta)$. Now, by the estimate of *Lemma* 4.15 on the modulus of continuity in time of the solution, we have :

$$\omega_v(\delta) \le \omega_F(\delta) + \|c\|_{L^\infty} B_0 \delta + \omega_c(\delta) \int_0^{T^*} B(s)ds.$$

Since $\|c\|_{L^\infty(\mathbb{R}^n \times [0,T^*))} \le \|c_0\|_{L^1}$, it suffices to show that $\omega_c(\delta) \int_0^{T^*} B(s)ds \le \omega(\delta)$, *i.e.*

$$\frac{8\|c_0\|_{L^\infty_{\text{int}}}}{b_0} \omega(\delta) \int_0^{T^*} B(s)ds \le \omega(\delta)$$

$$\int_0^{T^*} B(s)ds \le \frac{b_0}{8\|c_0\|_{L^\infty_{\text{int}}}}$$

$$\frac{1}{L_c}\left(e^{L_c T^*} - 1\right) \le \frac{b_0}{8B_0\|c_0\|_{L^\infty_{\text{int}}}}$$

$$T^* \le \frac{\ln\left(\frac{L_c b_0}{8B_0\|c_0\|_{L^\infty_{\text{int}}}} + 1\right)}{L_c},$$

which is true according to the choice of T^* and so $v \in E$.

It thus remains to be shown that Φ is a contraction. For $v^i = \Phi(u^i)$, according to the *Lemmata* 5.3, 5.2 and 5.1, we have

$$\begin{aligned}
\|v^2 - v^1\|_{L^\infty(\mathbb{R}^n\times(0,T^*))} \leq & 2B_0T^*\|c_0 \star [u^2] - c_0 \star [u^1]\|_{L^\infty(\mathbb{R}^n\times(0,T^*))} \\
\leq & 2B_0T^*\|c_0\|_{L^\infty_{\text{int}}(\mathbb{R}^n)} \sup_{t\in(0,T^*)} \|[u^2(\cdot,t)] - [u^1(\cdot,t)]\|_{L^1_{\text{unif}}(\mathbb{R}^n)} \\
\leq & \frac{8B_0T^*}{b_0}\|c_0\|_{L^\infty_{\text{int}}(\mathbb{R}^n)}\|u^2 - u^1\|_{L^\infty(\mathbb{R}^n\times(0,T^*))} \\
\leq & \frac{1}{2}\|u^2 - u^1\|_{L^\infty(\mathbb{R}^n\times(0,T^*))}.
\end{aligned}$$

And so Φ is a contraction on E which is a closed set for the L^∞ topology. So, there exists a unique viscosity solution of (2.6) in E on $(0,T^*)$. $\square$

Remark 5.4. *To be rigourous, we should consider the intersection of E with a ball of center u_0 and write the elements of E as $u = \tilde{u} + u_0$ with $\tilde{u}$ bounded. Then we could make the same computations on $\tilde{u}$ and we will obtain the same result.*

6 Appendix : proof of the parabolic version of Ishii's *Lemma*

We are going to prove parabolic version of Ishii's *Lemma* (see Crandall Ishii [60]). The result is classic, but we give the proof for reader's convenience. To do that, we will use an elliptic Ishii *Lemma*. First, we give some definitions :

Definition 6.1. (Sub and superdifferential of order two)
If $u : \mathbb{R}^n \to \mathbb{R}$, then the superdifferential of order two of u, $\mathcal{D}^{2,+}u$, is defined by $(p,X) \in \mathbb{R}^n \times S^n$ belongs to $\mathcal{D}^{2,+}u(x,t)$ if $x \in \mathbb{R}$ and

$$u(y) \leq u(x) + \langle p, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + o\left(|y-x|^2\right)$$

as $\mathbb{R} \ni y \to x$. In a similar way, we defined the subdifferential of order two by $\mathcal{D}^{2,-}u = -\mathcal{D}^{2,+}(-u)$. We also defined the two following sets :

$$\bar{\mathcal{D}}^{2,+}u(x,t) = \left\{ \begin{array}{l} (p,X) \in \mathbb{R}^n \times S^n,\ \exists\,(x_n,p_n,X_n) \in \mathbb{R}^n \times \mathbb{R}^n \times S^n \\ \text{such that } (p_n,X_n) \in \mathcal{D}^{2,+}u(x_n) \\ \text{and } (x_n,u(x_n),p_n,X_n) \to (x,u(x),p,X) \end{array} \right\}.$$

The set $\bar{\mathcal{D}}^{2,-}u(x,t)$ is defined in a similar way. Lastly, we defined

$$\bar{\mathcal{D}}^2u(x) = \bar{\mathcal{D}}^{2,+}u(x) \cap \bar{\mathcal{D}}^{2,-}u(x).$$

Lemma 6.2. (elliptic version of Ishii's *Lemma*)
Let U and V be two open sets of $\mathbb{R}^n$, $u : U \to \mathbb{R}$ usc and $v : V \to \mathbb{R}$ lsc. Let $\phi : U \times V \to \mathbb{R}$ be of class C^2. Assume that $(x,y) \to u(x) - v(y) - \phi(x,y)$ reaches a local maximum in $(\bar{x},\bar{y}) \in U \times V$. We note $p_1 = D_x\phi(\bar{x},\bar{y})$, $p_2 = -D_y\phi(\bar{x},\bar{y})$ and $A = D^2\phi(\bar{x},\bar{y})$. Then, for every $\alpha > 0$ such that $\alpha A < I$, there exists X, $Y \in S^n$ such that :

$$(p_1, X) \in \bar{\mathcal{D}}^{2,+}u(\bar{x}), (p_2, Y) \in \bar{\mathcal{D}}^{2,-}v(\bar{y}),$$
$$\frac{-1}{\alpha}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le (I - \alpha A)^{-1}A.$$

For the proof, we refer to Droniou, Imbert [73]. We now prove parabolic version of Ishii's *Lemma* :

Proof of *Lemma* 4.4
The principle of this proof is to duplicate the variables in time and then to pass to the limit by using the compactness assumption (C). The key point is to regularise by sup-convolution using two different parameters for space and time.

We duplicate the variables by considering :

$$u(x,t) - v(y,s) - \phi(x,y,t) - \frac{|t-s|^2}{2\epsilon}.$$

This function admits a local maximum in $(x_\epsilon, y_\epsilon, t_\epsilon, s_\epsilon)$ with

$$(x_\epsilon, y_\epsilon, t_\epsilon, s_\epsilon) \to (\bar{x}, \bar{y}, \bar{t}, \bar{t})$$

(because u and v are locally bounded). We note $\tilde{A}_\epsilon = D^2\left(\phi(x,y,t) + \frac{|t-s|^2}{2\epsilon}\right)$. By elliptic Ishii *Lemma*, we have : for every α and α' such that :

$$\tilde{A}_\epsilon < \begin{pmatrix} \frac{I}{\alpha} & & & \\ & \frac{1}{\alpha'} & & \\ & & \frac{I}{\alpha} & \\ & & & \frac{1}{\alpha'} \end{pmatrix},$$

(This assumption holds for α such that $\alpha A < I$ and α' small enough) there exists X_ϵ, $Y_\epsilon \in S^n$, C, $D \in \mathbb{R}^n$ and λ, $\sigma \in \mathbb{R}$ such that :

$$\left(\left(p_1^\epsilon, \phi_t + \frac{t-s}{\epsilon}\right), \begin{pmatrix} X_\epsilon & C \\ {}^tC & \lambda \end{pmatrix}\right) \in \bar{\mathcal{D}}^{2,+}u(x_\epsilon, t_\epsilon), \tag{2.25}$$

$$\left(\left(p_2^\epsilon, \frac{t-s}{\epsilon}\right), \begin{pmatrix} Y_\epsilon & D \\ {}^tD & \sigma \end{pmatrix}\right) \in \bar{\mathcal{D}}^{2,-}v(y_\epsilon, s_\epsilon), \tag{2.26}$$

and

$$\begin{pmatrix} \frac{-I}{\alpha} & 0 & 0 & 0 \\ 0 & \frac{-1}{\alpha'} & 0 & 0 \\ 0 & 0 & \frac{-I}{\alpha} & 0 \\ 0 & 0 & 0 & \frac{-1}{\alpha'} \end{pmatrix} \le \left(\begin{array}{cc|cc} X_\epsilon & C & \multicolumn{2}{c}{0} \\ {}^tC & \lambda & & \\ \hline \multicolumn{2}{c|}{0} & -Y_\epsilon & -D \\ & & -{}^tD & -\sigma \end{array}\right) \le B_\epsilon^{\alpha,\alpha'}, \tag{2.27}$$

where $p_1^\epsilon = D_x\phi(\cdot)$, $p_2^\epsilon = D_y\phi(\cdot)$ and $B_\epsilon^{\alpha,\alpha'}$ is the regularised by sup-convolution in the sense of quadratic form of parameters α and α' of :

$$\tilde{A}_\epsilon = \begin{pmatrix} D_x^2\phi & v_1 & D_{xy}\phi & 0 \\ {}^t v_1 & \phi_{tt} + \frac{1}{\epsilon} & {}^t v_2 & -\frac{1}{\epsilon} \\ D_{xy}\phi & v_2 & D_y^2\phi & 0 \\ 0 & -\frac{1}{\epsilon} & 0 & \frac{1}{\epsilon} \end{pmatrix},$$

where $v_1 = \phi_{xt}(\cdot)$ and $v_2 = \phi_{yt}(\cdot)$. So, for $\xi = (\xi', \xi_{n+1})$, $\eta = (\eta',\ \eta_{n+1}) \in \mathbb{R}^{n+1}$, we have :

$$B_\epsilon^{\alpha,\alpha'}(\xi,\eta).(\xi,\eta) = \sup_{(\zeta,\Gamma)\in\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}} \left\{ \begin{array}{l} \tilde{A}_\epsilon(\zeta,\Gamma).(\zeta,\Gamma) - \frac{1}{\alpha}\left(|\xi'-\zeta'|^2 + |\Gamma'-\eta'|^2\right) \\ -\frac{1}{\alpha'}\left(|\xi_{n+1}-\zeta_{n+1}|^2 + |\Gamma_{n+1}-\eta_{n+1}|^2\right) \end{array} \right\}.$$

Moreover, we can see $B_\epsilon^{\alpha,\alpha'}$ as the regularised by supconvolution of the regularised by supconvolution of a quadratic form, so $B_\epsilon^{\alpha,\alpha'}$ is still a quadratic form (see the proof of Elliptic Ishii in Droniou, Imbert [73])). Equations (2.25) and (2.26) imply in particular :

$$\left(\phi_t(x_\epsilon, y_\epsilon, t_\epsilon) + \frac{t_\epsilon - s_\epsilon}{\epsilon}, p_1^\epsilon, X_\epsilon\right) \in \bar{\mathcal{P}}^+ u(x_\epsilon, t_\epsilon),$$

$$\left(\frac{t_\epsilon - s_\epsilon}{\epsilon}, p_2^\epsilon, Y_\epsilon\right) \in \bar{\mathcal{P}}^- v(y_\epsilon, s_\epsilon),$$

We note $\rho_\epsilon = \phi_t(x_\epsilon, y_\epsilon, t_\epsilon) + \frac{t_\epsilon - s_\epsilon}{\epsilon}$ and $\gamma_\epsilon = \frac{t_\epsilon - s_\epsilon}{\epsilon}$. We remark that $\phi_t(x_\epsilon, y_\epsilon, t_\epsilon) = \rho_\epsilon - \gamma_\epsilon$. Applying the vector $(\xi', 0, \eta', 0)$ to the matrix inequality (2.27), yields :

$$\begin{aligned} \frac{-1}{\alpha}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}(\xi',\eta').(\xi',\eta') \leq & \begin{pmatrix} X_\epsilon & 0 \\ 0 & -Y_\epsilon \end{pmatrix}(\xi',\eta').(\xi',\eta') \\ \leq & B_\epsilon^{\alpha,\alpha'}(\xi',0,\eta',0).(\xi',0,\eta',0). \end{aligned} \tag{2.28}$$

We show that the right hand go to $A^\alpha(\xi',\eta').(\xi',\eta')$ as $\alpha' \to 0$, where A^α is the regularised by sup-convolution of $A = D^2\phi$. Indeed, $B_\epsilon^{\alpha,\alpha'} = (B_\epsilon^\alpha)^{\alpha'}$. So

$$B_\epsilon^{\alpha,\alpha'}(\xi',0,\eta',0).(\xi',0,\eta',0) \to B_\epsilon^{\alpha}(\xi',0,\eta',0).(\xi',0,\eta',0) \text{ when } \alpha' \to 0$$

and

$$\begin{aligned} & B_\epsilon^{\alpha}(\xi',0,\eta',0).(\xi',0,\eta',0) \\ = & \sup_{(\zeta',\Gamma')\in\mathbb{R}^n\times\mathbb{R}^n} \left\{ \tilde{A}_\epsilon(\zeta',0,\Gamma',0).(\zeta',0,\Gamma',0) - \frac{1}{\alpha}\left(|\xi'-\zeta'|^2 + |\eta'-\Gamma'|^2\right) \right\} \\ = & \sup_{(\zeta',\Gamma')\in\mathbb{R}^n\times\mathbb{R}^n} \left\{ A(\zeta',\Gamma').(\zeta',\Gamma') - \frac{1}{\alpha}\left(|\xi'-\zeta'|^2 + |\eta'-\Gamma'|^2\right) \right\} \\ = & A^\alpha(\xi',\eta').(\xi',\eta'), \end{aligned}$$

what gives

$$\frac{-1}{\alpha}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_\epsilon & 0 \\ 0 & -Y_\epsilon \end{pmatrix} \leq A^\alpha = (I-\alpha A)^{-1}A.$$

We then have, for a subsequence, $p_1^\epsilon \to p_1$, $p_2^\epsilon \to p_2$ and $(X_\epsilon, Y_\epsilon) \to (X,Y)$ (because p_1^ϵ, p_2^ϵ, X_ϵ and Y_ϵ are bounded) . We choose ϵ small enough such that $|(\bar{x},\bar{t})-(x_\epsilon,t_\epsilon)| \leq r$ where r is defined in the assumption (C). Then $(\rho_\epsilon, p_1^\epsilon, X_\epsilon) \in \bar{\mathcal{P}}^{2,+}u(x_\epsilon,t_\epsilon)$. Moreover, $u(x_\epsilon,t_\epsilon)$, p_1^ϵ and X_ϵ are bounded (because u is locally bounded and by the last matrix inequality for X_ϵ), so, by the compactness assumption (C), ρ_ϵ is bounded from above. Similarly, by using the fact that $(-\gamma_\epsilon, -p_2^\epsilon, -Y_\epsilon) \in \bar{\mathcal{P}}^{+}(-v)(y_\epsilon,s_\epsilon)$, γ_ϵ is bounded from below. So, $\rho_\epsilon = \phi_t + \gamma_\epsilon$ and γ_ϵ are bounded. So, for a subsequence, we have : $\rho_\epsilon \to \rho$ and $\gamma_\epsilon \to \gamma$. Passing to the limit, yields :

$$\begin{aligned} \tau &= \rho - \gamma, \\ (\rho, p_1, X) &\in \bar{\mathcal{P}}^{+}u(\bar{x},\bar{t}), \\ (\gamma, p_2, Y) &\in \bar{\mathcal{P}}^{-}v(\bar{y},\bar{t}), \\ \frac{-1}{\alpha}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} &\leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A^\alpha = (I-\alpha A)^{-1}A, \end{aligned}$$

what achieves the proof. □

Acknowledgements

The author would like to thank G. Barles, P. Cardaliaguet, B. Devincre, C. Imbert and O. Ley for fruitful discussions in the preparation of this article. The author also would like to thank O. Alvarez for introducing him to the theory of viscosity solutions and for many enlighting discussions and R. Monneau for stimulating and enriching discussions. This work was supported by the contract JC 1025 called "ACI jeunes chercheuses et jeunes chercheurs" of the French Ministry of Research (2003-2005).

Chapitre 3

Mouvements minimisants pour la dynamique des dislocations avec terme de courbure moyenne

Ce chapitre est un travail en collaboration avec A. Monteillet [91].
Nous montrons l'existence de mouvements minimisants pour la loi d'évolution de la dynamique des dislocations avec un terme de courbure moyenne. On montre qu'un mouvement minimisant est une solution faible de cette loi d'évolution, dans un sens relié à la solution de viscosité de l'équation level-set correspondante. On prouve également la consistance de l'approche, en montrant que tout mouvement minimisant coïncide avec l'évolution régulière tant que cette dernière existe. En relation avec cela, on montre finalement l'existence en temps court d'une évolution régulière, à condition que la forme initiale soit suffisament régulière.

Minimizing movements for dislocation dynamics with a mean curvature term

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Abstract

We prove existence of minimizing movements for the dislocation dynamics evolution law with a mean curvature term. We prove that any such minimizing movement is a weak solution of this evolution law, in a sense related to viscosity solutions of the corresponding level-set equation. We also prove the consistency of this approach, by showing that any minimizing movement coincides with the smooth evolution as long as the latter exists. In relation with this, we finally prove short time existence of a smooth front evolving according to our law, provided the initial shape is smooth enough.

Key words and phrases : Front propagation, non-local equations, minimizing movements, sets of finite perimeter, currents, viscosity solutions, dislocation dynamics.

1 Introduction

In this paper, we investigate the existence of minimizing movements (see Almgrem, Taylor, Wang [3] and Ambrosio [12]) for a non-local geometric law governing the movement of a family $\{K(t)\}_{0\leq t\leq T}$ of compact subsets of $\mathbb{R}^N$:

$$V_{x,t} = H_{x,t} + c_0(\cdot,t) \star 1_{K(t)}(x) + c_1(x,t), \tag{3.1}$$

where $V_{x,t}$ denotes the normal velocity at time t of a point x of $\partial K(t)$, $H_{x,t}$ the mean curvature of $\partial K(t)$ at x (with negative sign for convex sets), $\star$ is the convolution in space, $1_{K(t)}$ is the indicator function of the set $K(t)$ and $c_0, c_1 : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ are given functions.

The non-local dependence $c_0(\cdot,t) \star 1_{K(t)}$ in the expression of $V_{x,t}$ is typical of models for dislocation dynamics (see Alvarez, Hoch, Le Bouar and Monneau [10]). Moreover we think of the term c_1 as a prescribed driving force. Equation (3.1) with only these two terms (and without a mean curvature term) is currently also a center of interest : in the context of viscosity solutions, its level-set formulation, namely

$$u_t(x,t) = [c_0(\cdot,t) \star 1_{\{u(\cdot,t)\geq 0\}}(x) + c_1(x,t)]|Du(x,t)|, \tag{3.2}$$

has been initially investigated by Alvarez, Hoch, Le Bouar and Monneau [10], who proved short time existence and uniqueness of a viscosity solution of (3.2), and then by Alvarez, Cardaliaguet and Monneau [6], and by Barles and Ley [25], who proved, by different methods, long time existence and uniqueness under suitable monotonicity assumptions. In (3.2) and throughout the paper, u_t denotes the time derivative of u, Du denotes the space gradient of u, and $|\cdot|$ is the standard Euclidean norm. The mean curvature term in (3.1) corresponds to an additional line tension term in the elastic energy of the dislocation which better approximates what happens near the dislocation (see the introduction of [87] for a discussion on the model). The level-set formulation of the geometric law (3.1),

$$u_t(x,t) = \left[\operatorname{div}\left(\frac{Du(x,t)}{|Du(x,t)|}\right) + c_0(\cdot,t) \star 1_{\{u(\cdot,t)\geq 0\}}(x) + c_1(x,t)\right] |Du(x,t)|, \quad (3.3)$$

has been studied by the first author in [87]. He proved short time existence and uniqueness of a viscosity solution to (3.3).

In both cases, the source of major difficulties is the non-local dependence of the velocity, $c_0(\cdot,t)\star 1_{K(t)}$, which prevents comparison principle to hold. Indeed, c_0 is not necessarily non-negative, a situation that can not be avoided, as shown by physical models. The problem of existence and uniqueness of a viscosity solution to the level-set equations (3.2) and (3.3) for general kernels c_0 is therefore still open. The long time existence and uniqueness results mentioned above were obtained under the assumption that $c_0(\cdot,t) \star 1_E + c_1(x,t) \geq 0$ for any set E, which guarantees that the dislocation is expanding, and a regularity assumption on the initial shape $K(0)$. The short time existence and uniqueness for (3.3) was obtained in the case where the initial shape is a graph or a Lipschitz curve, without assumption on the sign of the non-local term. It is worth mentioning however that this equation benefits from the regularising effect of the mean curvature term.

To overcome this difficulty, Barles, Cardaliaguet, Ley and Monneau defined in [21] a notion of weak solution for (3.2), and proved existence of such weak solutions under general assumptions on c_0 and c_1. A similar concept of solution already appears in [171] for Fitzhugh-Nagumo systems. In this work, we wish to provide such weak solutions for (3.1). We will work with set-valued mappings $E : [0,T] \to \mathcal{P}(\mathbb{R}^N)$ with bounded images which are continuous in the L^1 topology, that is to say, $t \mapsto \mathbf{1}_{E(t)}$ belongs to $C^0([0,T], L^1(\mathbb{R}^N))$. We assume that c_0 and c_1 satisfy some regularity assumptions (to be precised later), which guarantee that $(x,t) \mapsto c_0(\cdot,t) \star 1_{E(t)}(x) + c_1(x,t)$ is smooth enough for such a mapping E. Let us now explain what we call a weak solution of (3.1) :

Definition 1.1. *Let $E : [0,T] \to \mathcal{P}(\mathbb{R}^N)$ be a set-valued mapping such that $t \mapsto \mathbf{1}_{E(t)}$ belongs to $C^0([0,T], L^1(\mathbb{R}^N))$. Let u be the unique uniformly continuous viscosity* *solution of*

$$\left\{\begin{array}{l} u_t(x,t) = \left[div\left(\frac{Du(x,t)}{|Du(x,t)|}\right) + c_0(\cdot,t) \star 1_{E(t)}(x) + c_1(x,t)\right] |Du(x,t)| \\ \hfill \textit{for } (x,t) \in \mathbb{R}^N \times (0,T) \\ u(x,0) = u_0(x) \quad \textit{for } x \in \mathbb{R}^N, \end{array}\right. \tag{3.4}$$

where u_0 *is a uniformly continuous function such that* $\overline{E_0} = \{u_0 \geq 0\}$, $\overset{\circ}{E_0} = \{u_0 > 0\}$.

We say that E *is a weak solution of* (3.1) *if we have, for all* $t \in [0,T]$, *and almost everywhere in* $\mathbb{R}^N$,

$$\{u(\cdot,t) > 0\} \subset E(t) \subset \{u(\cdot,t) \geq 0\}.$$

The goal of this paper is to construct a weak solution to the geometric law (3.1). To do this, we wish to adapt the approach of Almgrem, Taylor and Wang [3] (also discovered independently by Luckhaus and Sturzenhecker [139])- initially proposed for the mean curvature motion - to the geometric law (3.1) with its additional non-local term and driving force. The key point in this approach is that the mean curvature term is the dominating term and the non-local term is just a perturbation and so we can adapt the proof of Almgrem, Taylor, Wang [3]. The idea of minimizing movements is, for a given initial set E_0, to select a sequence of sets $E_h(k)$ associated with time-steps of size h by minimizing a suitable functional, so that the corresponding Euler equation is a discretization of our evolution law. A compactness result for sets of finite perimeter guarantees the existence of a subsequence (h_n) and of a set-valued mapping $E : [0,T] \mapsto \mathcal{P}(\mathbb{R}^N)$ such that $E_{h_n}([t/h_n])$ converges to $E(t)$ in $L^1(\mathbb{R}^N)$ for all t. Such a E is called a minimizing movement associated to the geometric law. Moreover, we prove *a priori* estimates for the discrete evolution E_h, which imply the Hölder continuity of the limit E in the appropriate metric. This guarantees that the sets $E(t)$ cannot vary in a wildly discontinuous way.

Let us now explain the interest of this approach in the perspective of proving existence of weak solutions. For any sequence (h_n) going to 0 and such that $E_{h_n}([\cdot/h_n])$ converges to a minimizing movement E, we are able, thanks to the Euler equation corresponding to our minimization procedure, to compute the velocity (in the viscosity sense) of the upper and lower limit of the $E_{h_n}(k)$'s as $n \to \infty$, E^* and E_*, in function of E. This enables us to compare E_* and E^* with the 0 level set of the viscosity solution u appearing in Definition 1.1. Since $E_* \subset E \subset E^*$, we will deduce that E is a weak solution of (3.1). In case no fattening occurs for u, we remark that u is a viscosity solution of (3.3).

We further show that if ∂E_0 is a smooth hypersurface, then there is a smooth solution for small times of the evolution law (3.1) and that any minimizing movement

E coincides with the smooth evolution as long as the latter exists. This uses the notions of lower/upper limits mentioned above and of sub/super pairs of solutions of Cardaliaguet and Pasquignon [53].

To state our results in more details below, we first need to fix a few notation and assumptions that will be used throughout the paper.

Notations

• In the sequel, $B_r^k(x)$ (resp. $\overline{B}_r^k(x)$) denotes the open (resp. closed) ball of radius r centred at $x \in \mathbb{R}^k$, and $\mathcal{L}^k$ is the Lebesgue measure on $\mathbb{R}^k$, $k \in \mathbb{N}$. If k is not specified, we mean that $k = N$. We set $\omega_k = \mathcal{L}^k(B_1^k(0))$. The Hausdorff measure of dimension k on $\mathbb{R}^N$ is denoted by $\mathcal{H}^k$.

• The notation Sym_N represents the set of real square symmetric matrices of size N.

• We say that a sequence (E_n) of subsets of $\mathbb{R}^N$ converges to E in $L^1(\mathbb{R}^N)$ if $1_{E_n} \to 1_E$ in $L^1(\mathbb{R}^N)$ as $n \to +\infty$.

• Let $\mathcal{P}$ be the set of all bounded subsets of $\mathbb{R}^N$ having finite perimeter (see [82] for the definition and properties of sets of finite perimeter). We denote by $P(E)$ the perimeter of $E \in \mathcal{P}$, by $P(E, U)$ the perimeter of E in U subset of $\mathbb{R}^N$, and we endow $\mathcal{P}$ with the metric

$$\delta(E, F) = \|1_E - 1_F\|_{L^1(\mathbb{R}^N)} = \mathcal{L}^N(E\Delta F),$$

where $E\Delta F$ is the symmetric difference of E and F, *i.e.*, $E\Delta F = (E \cup F) \setminus (E \cap F)$. Moreover $\partial^* E$ denotes the reduced boundary of $E \in \mathcal{P}$. We also define a notion of boundary for $E \in \mathcal{P}$ that is invariant in the class of E formed by the sets differing from E by a zero-measure set :

$$\partial E = \{x \in \mathbb{R}^N;\, 0 < \mathcal{L}^N(E \cap B_r(x)) < \mathcal{L}^N(B_r(x)) \text{ for all } r > 0\}.$$

Then ∂E is closed, and in fact $\partial E = \overline{\partial^* E}$.

Definitions of tubes (see [50])

• For any subset E of $[0, T] \times \mathbb{R}^N$, we set $E(t) = \{x \in \mathbb{R}^N; (t, x) \in E\}$. Conversely a mapping $t \in [0, T] \mapsto E(t) \in \mathcal{P}(\mathbb{R}^N)$ can be seen as a subset of $[0, T] \times \mathbb{R}^N$ by identifying E with its graph $\cup_{t \in [0,T]}\{t\} \times E(t)$.

• We call *tube* a bounded subset E of $[0, T] \times \mathbb{R}^N$.
We call *regular tube* a tube E with C^1 boundary except possibly at points of $\partial E(0)$ or $\partial E(T)$ such that for any regular point $(t, x) \in \partial E$, the unit outer normal (ν_t, ν_x) to E at (t, x) satisfies $\nu_x \neq 0$. In this case, the *normal velocity* of E at (t, x) is $-\nu_t/|\nu_x|$.

• Finally a mapping $t \in [0,T] \mapsto E_r(t)$ is said to be a smooth evolution with $C^{2+\alpha}$ boundary if E_r is a compact regular tube such that $E_r(t)$ has $C^{2+\alpha}$ boundary for all $t \in [0,T]$.

Assumptions on c_0 and c_1

Throughout the paper, c_0 and c_1 are assumed to satisfy the following regularity assumption :

(A) $c_0 \in Lip\left([0,T], L^1(\mathbb{R}^N)\right), \quad c_1 \in Lip\left([0,T], L^\infty(\mathbb{R}^N)\right)$.

In particular, we set $K_0 = Lip(c_0)$, and $K_1 = Lip(c_1)$, so that for all $t, s \in [0,T]$,

$$\|c_0(\cdot,t) - c_0(\cdot,s)\|_1 \le K_0|t-s| \quad \text{and} \quad \|c_1(\cdot,t) - c_1(\cdot,s)\|_\infty \le K_1|t-s|.$$

We finally set

$$L_0 = \|c_0\|_{L^\infty([0,T],L^1(\mathbb{R}^N))},\ L_1 = \|c_1\|_{L^\infty([0,T],L^\infty(\mathbb{R}^N))} \text{ and } L = L_0 + L_1. \tag{3.5}$$

We will sometimes need additional regularity for c_0 and c_1. When this happens, we will specify which assumptions are made in each of the statements of theorems. In particular we will sometimes need to require that c_0 be symmetric, so that the gradient flow of our functional is, at least formally, a solution of (3.1) :

(Symmetry of c_0) We say that c_0 is symmetric if $c_0(-(\cdot),t) = c_0(\cdot,t)$ for all $t \in [0,T]$.

Main results

For $h > 0$ (the time step), $k \in \mathbb{N}$ such that $kh \le T$, E and F in $\mathcal{P}$, we define, following the original idea of Almgren, Taylor and Wang [3], the functional

$$\mathcal{F}(h,k,E,F) = P(E) + \frac{1}{h}\int_{E\Delta F} d_{\partial F}(x)\,dx - \int_E \left(\frac{1}{2}c_0(\cdot,kh) \star 1_E(x) + c_1(x,kh)\right) dx, \tag{3.6}$$

where d_C is the distance function to a closed set C.

Let us now define a minimizing movement :

Definition 1.2 (Minimizing movement [3]). *Let $T > 0$ and $E_0 \in \mathcal{P}$. We say that $E : [0,T] \to \mathcal{P}$ is a minimizing movement associated to $\mathcal{F}$ with initial condition E_0 if there exist a sequence (h_n), $h_n \to 0^+$ and sets $E_{h_n}(k) \in \mathcal{P}$ for all $k \in \mathbb{N}$ verifying $kh_n \le T$, such that :*

1. $E_{h_n}(0) = E_0$.

2. *For any* $n, k \in \mathbb{N}$ *with* $(k+1)h_n \leq T$,

$$E_{h_n}(k+1) \text{ minimizes the functional } E \to \mathcal{F}(h_n, k, E, E_{h_n}(k)) \tag{3.7}$$

among all E'*s in* $\mathcal{P}$.

3. *For any* $t \in [0, T]$, $E_{h_n}([t/h_n]) \to E(t)$ *in* $L^1(\mathbb{R}^N)$ *as* $n \to +\infty$.

The first result of the paper is the existence of minimizing movements associated to our functional $\mathcal{F}$:

Theorem 1.3 (Existence of minimizing movements). *Assume that* c_0 *and* c_1 *satisfy* ***(A)***. *Let* $E_0 \in \mathcal{P}$ *with* $\mathcal{L}^N(\partial E_0) = 0$. *Then, there exists a minimizing movement* E *associated to* $\mathcal{F}$ *with initial condition* E_0 *such that for all* t, s *verifying* $t \leq T$ *and* $0 \leq s \leq t < s+1$, *we have*

$$\delta(E(t), E(s)) \leq \gamma\,(t-s)^{\frac{1}{N+1}}, \tag{3.8}$$

where $\gamma = \gamma(N, T, E_0, K_0, K_1, L_0, L_1)$ *is a constant.*

We then prove that any such minimizing movement is a weak solution of (3.1) :

Theorem 1.4 (Minimizing movements are weak solutions). *Assume that* c_0 *is symmetric, that* c_0 *and* c_1 *satisfy* ***(A)*** *and are Lipschitz continuous on* $\mathbb{R}^N \times [0, T]$. *Let* $E_0 \in \mathcal{P}$ *with* $\mathcal{L}^N(\partial E_0) = 0$. *Let* E *be any minimizing movement associated to* $\mathcal{F}$ *with initial condition* E_0.

Then E *is a weak solution of* (3.1) *in the sense of Definition 1.1. In particular if no fattening occurs, i.e. if* $\{u(\cdot, t) = 0\}$ *has zero* $\mathcal{L}^N$ *measure, then* u *is a viscosity solution of* (3.3) *with initial data* u_0.

Our third result states that any minimizing movement E coincides with the smooth evolution E_r as long as the latter exists :

Theorem 1.5 (Agreement with any smooth flow). *Assume that* c_0 *is symmetric, that* c_0 *and* c_1 *satisfy* ***(A)*** *and are Lipschitz continuous on* $\mathbb{R}^N \times [0, T]$. *Let* E_0 *be a compact domain with uniformly* $C^{3+\alpha}$ *boundary. If* E_r *is a smooth evolution with* $C^{2+\alpha}$ *boundary defined on* $[0, T]$, *starting from* E_0 *with normal velocity given by* (3.11), *then any minimizing movement* E *associated to* $\mathcal{F}$ *with initial condition* E_0 *verifies* $E(t) = E_r(t)$ *for all* $t \in [0, T]$ *and almost everywhere in* $\mathbb{R}^N$.

In relation with this, we finally prove short time existence of a smooth solution E_r to (3.1), when E_0 is sufficiently smooth. The regularity assumptions on c_0 and c_1 are the following ones :

$$c_0 \in L^\infty([0, T]; W^{2,\infty}(\mathbb{R}^N)) \cap W^{1,\infty}([0, T], L^\infty(\mathbb{R}^N)) \tag{3.9}$$

and

$$c_1 \in W^{2,1;\infty}(\mathbb{R}^N \times [0,T]), \tag{3.10}$$

where $f \in W^{1,\infty}([0,T], L^\infty(\mathbb{R}^N))$ means that f is Lipschitz continuous with respect to $t \in [0,T]$, uniformly with respect to $x \in \mathbb{R}^N$, and

$$W^{n,1;\infty}(\mathbb{R}^N \times (0,T)) = \left\{ \begin{array}{ll} f \in L^\infty(\mathbb{R}^N \times (0,T)); & f_t,\ \frac{\partial^\alpha f}{\partial x^\alpha} \in L^\infty(\mathbb{R}^N \times (0,T)) \\ & \text{for } \alpha \in \mathbb{N}^N \ s.t. \ \sum_{i=0}^N \alpha_i \leq n \end{array} \right\}.$$

Theorem 1.6 (Existence of a smooth solution)**.** *Assume the regularity* (3.9)-(3.10). *Let E_0 be a compact domain with uniformly $C^{3+\alpha}$ boundary. Then there exists a small time $t_0 > 0$ and a smooth evolution $\{E_r(t)\}_{0 \leq t \leq t_0}$ with $C^{2+\alpha}$ boundary, starting from E_0 with normal velocity*

$$V_{x,t} = H_{x,t} + c_0(\cdot, t) \star 1_{E_r(t)}(x) + c_1(x,t), \tag{3.11}$$

where $H_{x,t}$ is the mean curvature of $\partial E_r(t)$ at x.

Let us now explain how this paper is organised. First, in Section 2, we prove the existence of minimizing movements and the Hölder estimate Theorem 1.3. Section 3 is devoted to proving a regularity result for $\mathcal{F}$-minimizers that we use in Section 4 to prepare the proofs of Theorems 1.4 and 1.5, respectively given in Sections 5 and 6. Finally, in Section 7, we prove Theorem 1.6.

2 Existence of minimizing movements

This section is concerned with the existence of minimizing movements associated to $\mathcal{F}$ (Theorem 1.3). Let us start with existence and basic properties of $\mathcal{F}$-minimizers.

2.1 $\mathcal{F}$-minimizers

The first point to check is the existence of $\mathcal{F}$-minimizers :

Proposition 2.1 (Existence of $\mathcal{F}$-minimizers)**.** *For all $h > 0$, $k \in \mathbb{N}$ with $kn \leq T$, and $F \in \mathcal{P}$, there exists a minimizer of $E \mapsto \mathcal{F}(h,k,E,F)$ on $\mathcal{P}$.*
Moreover, if L is defined by (3.5)*, then*

$$F \subset B_R(0) \Rightarrow E \subset B_{R+Lh}(0)$$

whenever E is a minimizer.

Proof. Let $F \in \mathcal{P}$, $F \subset B_R(0)$, and $B = B_{R+Lh}(0)$. Let (E_n) be a minimizing sequence for $\mathcal{F}(h, k, \cdot, F)$. We want to prove that for all $n \in \mathbb{N}$,

$$\mathcal{F}(h, k, E_n \cap B, F) \le \mathcal{F}(h, k, E_n, F). \tag{3.12}$$

First, since B is open and convex, we know that

$$P(E_n \cap B) \le P(E_n). \tag{3.13}$$

Let us compare $\int_{E_n} c_0(\cdot, kh) \star 1_{E_n}(x)\, dx$ and $\int_{E_n \cap B} c_0(\cdot, kh) \star 1_{E_n \cap B}(x)\, dx$: for all $x \in \mathbb{R}^N$,

$$\begin{aligned} c_0(\cdot, kh) \star 1_{E_n}(x) &= \int_{E_n} c_0(x - y, kh)\, dy \\ &= c_0(\cdot, kh) \star 1_{E_n \cap B}(x) + \int_{E_n \setminus B} c_0(x - y, kh)\, dy. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{E_n} c_0(\cdot, kh) \star 1_{E_n}(x)\, dx &= \int_{E_n \cap B} c_0(\cdot, kh) \star 1_{E_n \cap B}(x)\, dx \\ &+ \int_{E_n \setminus B} c_0(\cdot, kh) \star 1_{E_n \cap B}(x)\, dx + \int_{E_n} \int_{E_n \setminus B} c_0(x - y, kh)\, dy dx. \end{aligned}$$

Since $|c_0(\cdot, kh) \star 1_A(x)| \le L_0$ for all measurable set A, it follows that

$$\begin{aligned} &\int_{E_n} \left(\frac{1}{2} c_0(\cdot, kh) \star 1_{E_n}(x) + c_1(x, kh) \right) dx \\ &\ge \int_{E_n \cap B} \left(\frac{1}{2} c_0(\cdot, kh) \star 1_{E_n \cap B}(x) + c_1(x, kh) \right) dx - L\mathcal{L}^N(E_n \setminus B) \end{aligned} \tag{3.14}$$

thanks to the definition of L. Moreover, $F \subset B$, so that

$$E_n \Delta F = (E_n \cap B)\Delta F \ \cup \ (E_n \setminus B),$$

whence

$$\begin{aligned} \frac{1}{h} \int_{E_n \Delta F} d_{\partial F}(x)\, dx &= \frac{1}{h} \int_{(E_n \cap B)\Delta F} d_{\partial F}(x)\, dx + \frac{1}{h} \int_{E_n \setminus B} d_{\partial F}(x)\, dx \\ &\ge \frac{1}{h} \int_{(E_n \cap B)\Delta F} d_{\partial F}(x)\, dx + L\mathcal{L}^N(E_n \setminus B), \end{aligned} \tag{3.15}$$

since $d_{\partial F}(x) \ge Lh$ for all $x \in E_n \setminus B$ by definition of B. Putting (3.13), (3.14) and (3.15) together proves (3.12). Therefore we can replace (E_n) by $(E_n \cap B)$ as a minimizing sequence, and in particular we can assume that $E_n \subset B$ for all n. Then

$$\begin{aligned}\mathcal{F}(h,k,E_n,F) &\geq -\int_{E_n}\left(\frac{1}{2}c_0(\cdot,kh)\star 1_{E_n}(x)+c_1(x,kh)\right)dx\\ &\geq -\left(\frac{1}{2}L_0+L_1\right)\mathcal{L}^N(B),\end{aligned}$$

so that $\inf_{E\in\mathcal{P}}\mathcal{F}(h,k,E,F)>-\infty$. Besides, for n large enough,

$$\mathcal{F}(h,k,E_n,F)\leq \inf_{E\in\mathcal{P}}\mathcal{F}(h,k,E,F)+1.$$

This implies that

$$P(E_n)\leq \inf_{E\in\mathcal{P}}\mathcal{F}(h,k,E,F)+1+\left(\frac{1}{2}L_0+L_1\right)\mathcal{L}^N(B),$$

and gives a uniform bound on the perimeter of the E_n's. Since they are also uniformly bounded, it follows from the compactness theorem for sets of finite perimeter [82, section 5.2.3] that we can extract a converging subsequence (E_{n_k}) of (E_n) in the sense that there exists $E_\infty\in\mathcal{P}$, $E_\infty\subset B$, such that $E_{n_k}\to E_\infty$ in $L^1(\mathbb{R}^N)$. Therefore

$$\mathcal{F}(h,k,E_\infty,F)\leq\liminf_{k\to\infty}\mathcal{F}(h,k,E_{n_k},F)=\inf_{E\in\mathcal{P}}\mathcal{F}(h,k,E,F), \tag{3.16}$$

because all terms in the expression of $\mathcal{F}$ are at least lower semi-continuous in the E variable for the L^1 topology. Thus E_∞ is a minimizer of $E\mapsto\mathcal{F}(h,k,E,F)$ on $\mathcal{P}$. Finally, if E is any other minimizer, then the previous comparisons show that $P(E\cap B)=P(E)$, whence $E\subset B$ almost everywhere (see the comparison theorem [12, p. 216]). □

Remark 2.2. *This lemma shows that the $E_h(k)$'s are uniformly bounded for all h and k, if $E_0\in\mathcal{P}$: more precisely, if $E_0\subset B_R(0)$, then since $kh\leq T$, we have $E_h(k)\subset B_{R+LT}(0)$ independently of h, k. Therefore we can choose $\Omega=B_{R+LT+1}(0)$ so that $E_h(k)\Subset\Omega$ for all k, h. We will always do so in the sequel, and set $D=R+LT+1$.*

Remark 2.2 gives a uniform bound Ω for $E_h(k)$, independently of h,k, provided that E_0 is bounded. In order to have compactness in $\mathcal{P}$, so as to construct our minimizing movement, we also want a uniform bound on the perimeter of $E_h(k)$.

Proposition 2.3 (Uniform bound on the perimeter)**.** *Let $E_0\in\mathcal{P}$ with $E_0\subset B_R(0)$. Then, there exists a constant $c=c(E_0,\Omega,T,K_0,K_1,L_0,L_1)>0$ independent of h and k such that if E_h is defined by the procedure* (3.7)*, we have*

$$P(E_h(k))\leq c\quad\forall\, h,k \text{ such that } kh\leq T.$$

Proof. By definition of E_h, we have for all j such that $jh \leq T$,

$$\mathcal{F}(h, j, E_h(j), E_h(j-1)) \leq \mathcal{F}(h, j, E_h(j-1), E_h(j-1)),$$

and in particular,

$$\begin{aligned} &P(E_h(j)) - \int_{E_h(j)} \left(\frac{1}{2}c_0(\cdot, jh) \star 1_{E_h(j)}(x) + c_1(x, jh)\right) dx \\ &\leq P(E_h(j-1)) - \int_{E_h(j-1)} \left(\frac{1}{2}c_0(\cdot, jh) \star 1_{E_h(j-1)}(x) + c_1(x, jh)\right) dx \end{aligned}$$

Adding these inequalities for $j = 1, \dots, k$ with $kh \leq T$, we find, with the notation

$$J_h(i, j) = \int_{E_h(i)} \left(\frac{1}{2}c_0(\cdot, jh) \star 1_{E_h(i)}(x) + c_1(x, jh)\right) dx, \tag{3.17}$$

$$\begin{aligned} P(E_h(k)) - P(E_0) &\leq \sum_{j=1}^{k} J_h(j, j) - J_h(j-1, j) \\ &= \sum_{j=1}^{k} \int_\Omega c_1(\cdot, jh) 1_{E_h(j)} - c_1(\cdot, jh) 1_{E_h(j-1)} \\ &+ \frac{1}{2} \sum_{j=1}^{k} \int_\Omega (c_0(\cdot, jh) \star 1_{E_h(j)}) 1_{E_h(j)} - (c_0(\cdot, jh) \star 1_{E_h(j-1)}) 1_{E_h(j-1)}. \end{aligned} \tag{3.18}$$

Doing an Abel transformation on the first sum of the last member of (3.18) yields

$$\begin{aligned} &\sum_{j=1}^{k} \int_\Omega c_1(\cdot, jh) 1_{E_h(j)} - c_1(\cdot, jh) 1_{E_h(j-1)} \\ &= \int_\Omega c_1(\cdot, kh) 1_{E_h(k)} - \int_\Omega c_1(\cdot, h) 1_{E_0} + \sum_{j=1}^{k-1} \int_\Omega [c_1(\cdot, jh) - c_1(\cdot, (j+1)h)] 1_{E_h(j)} \\ &\leq \; 2L_1 \, \mathcal{L}^N(\Omega) + (k-1) K_1 \, h \, \mathcal{L}^N(\Omega) \\ &\leq \; (2L_1 + K_1 \, T) \, \mathcal{L}^N(\Omega). \end{aligned}$$

The same manipulation with the second sum gives

$$\begin{aligned} &\sum_{j=1}^{k} \int_\Omega (c_0(\cdot, jh) \star 1_{E_h(j)}) 1_{E_h(j)} - (c_0(\cdot, jh) \star 1_{E_h(j-1)}) 1_{E_h(j-1)} \\ &\leq (2L_0 + K_0 \, T) \, \mathcal{L}^N(\Omega). \end{aligned}$$

This proves that for all k such that $kh \leq T$,

$$\sum_{j=1}^{k} J_h(j,j) - J_h(j-1,j) \leq (2L_1 + L_0 + \frac{1}{2} K_0\, T + K_1\, T)\, \mathcal{L}^N(\Omega) \tag{3.19}$$

and gives the result, with $c = P(E_0) + (2L_1 + L_0 + \frac{1}{2} K_0\, T + K_1\, T)\, \mathcal{L}^N(\Omega)$. $\square$

2.2 Minimizing movements

We are now ready to address the problem of the existence of minimizing movements. Proofs in this section closely follow the ideas of Almgren, Taylor and Wang [3], and are adaptations of Ambrosio's simplified presentation of these ideas (see [12]).

The main result in the perspective of the proof of existence of minimizing movements is the following theorem on the behaviour of the solutions of procedure (3.7) :

Theorem 2.4 (Discrete Hölder estimate)**.** *Let $E_0 \in \mathcal{P}$ with $E_0 \subset B_R(0)$. There exists a constant $\gamma = \gamma(N, D) > 0$ (where D is defined in remark 2.2) and $h_0 > 0$ such that for all $h \in (0, h_0)$, for all $m, l \in \mathbb{N}$ verifying $mh \leq T$ and $0 < l < m < l + \frac{1}{h}$, we have :*

$$\delta(E_h(m), E_h(l)) \leq \gamma\, c\, [h(m-l)]^{\frac{1}{N+1}}, \tag{3.20}$$

where c is the uniform bound on $P(E_h(k))$ given by Proposition 2.3.

Theorem 1.3 is a corollary of this result, and the proof of this fact can be found in [12, pp. 231-232]. However the arguments of [3, Theorem 4.4] or [12, Theorem 3.3] for the proof of Theorem 2.4 need a few adaptations due to the particular form of $\mathcal{F}$. This is what the rest of this section is devoted to. We begin by giving some preliminary results which will be necessary in the proof of Theorem 2.4.

Lower density bound for $\mathcal{F}$-minimizers

Theorem 2.5 (Density bound for $\mathcal{F}$-minimizers)**.** *There exist two constants α and β (depending only on N) and $h_0 > 0$ such that if $E \in \mathcal{P}$ is a minimizer of $\mathcal{F}(h, k, \cdot, F)$ with $F \in \mathcal{P}$, $E \cup F \subset B_{D-1}(0)$, and $h \in (0, h_0)$, then*

$$\forall x \in \partial E,\ \forall \rho \in (0, \frac{\alpha h}{D}), \quad P(E, B_\rho(x)) \geq \beta \rho^{N-1}. \tag{3.21}$$

Proof. The proof relies on the following lemma relating the perimeter of $E \in \mathcal{P}$ and the perimeter of E replaced by a cone in a small ball :

Lemma 2.6 ([12], Lemma 3.5)**.** *Let $E \in \mathcal{P}$, $x \in \mathbb{R}^N$ and $f(\rho) = P(E, B_\rho(x))$. Set*

$$E_\rho = (E \cap (\mathbb{R}^N \setminus B_\rho(x))) \cup \left\{ y \in B_\rho(x);\ x + \rho \frac{y-x}{|y-x|} \in E \right\}.$$

Then for almost all $\rho > 0$ (all ρ such that f is differentiable at ρ), we have

$$P(E_\rho, \overline{B}_\rho(x)) \leq \rho \frac{f'(\rho)}{N-1}.$$

Let us now prove Theorem 2.5. Fix $x \in \partial^* E$ and $\rho > 0$ such that f is differentiable at ρ. By definition of E, we know that $\mathcal{F}(h, k, E, F) \leq \mathcal{F}(h, k, E_\rho, F)$, that is to say

$$\begin{aligned} P(E) \leq & P(E_\rho) + \frac{1}{h} \left\{ \int_{E_\rho \Delta F} d_{\partial F}(x)\, dx - \int_{E \Delta F} d_{\partial F}(x)\, dx \right\} \\ & + \int_E \left(\frac{1}{2} c_0(\cdot, kh) \star 1_E(x) + c_1(x, kh) \right) dx \\ & - \int_{E_\rho} \left(\frac{1}{2} c_0(\cdot, kh) \star 1_{E_\rho}(x) + c_1(x, kh) \right) dx. \end{aligned} \tag{3.22}$$

But since E coincides with E_ρ in $\mathbb{R}^N \setminus \overline{B}_\rho$, we have

$$P(E, \mathbb{R}^N \setminus \overline{B}_\rho(x)) = P(E_\rho, \mathbb{R}^N \setminus \overline{B}_\rho(x)).$$

Moreover f is continuous at ρ, which together with (3.22) implies that

$$P(E, B_\rho(x)) = P(E, \overline{B}_\rho(x)) \leq P(E_\rho, \overline{B}_\rho(x)) + \frac{2D}{h} \omega_N \rho^N + 2L\, \omega_N \rho^N,$$

due to the fact that $d_{\partial F}(y) \leq 2D$ for all $y \in B_\rho(x)$, provided $\rho < 1$. Now Lemma 2.6 implies that for almost all $\rho \in (0, 1)$,

$$f(\rho) \leq \rho \frac{f'(\rho)}{N-1} + \left(\frac{2D}{h} + 2L \right) \omega_N \rho^N. \tag{3.23}$$

Therefore, the function

$$g : \rho \mapsto \frac{f(\rho)}{\rho^{N-1}} + \left(\frac{2D}{h} + 2L \right) (N-1)\, \omega_N\, \rho$$

is nondecreasing on $(0, 1)$. In particular if $x \in \partial^* E$ and $\rho \in (0, 1)$,

$$g(\rho) \geq \liminf_{\overline{\rho} \to 0^+} g(\overline{\rho}) \geq \omega_{N-1} \tag{3.24}$$

because of [82, corollary 1 (ii) page 203]. As a consequence, for all $\rho \in (0,1)$,

$$f(\rho) \geq \omega_{N-1}\rho^{N-1} - \left(\frac{2D}{h} + 2L\right)(N-1)\,\omega_N\,\rho^N. \tag{3.25}$$

Let us set $\alpha = \frac{\omega_{N-1}}{8(N-1)\omega_N}$ and $\beta = \frac{\omega_{N-1}}{2}$. Then, provided $h < \min\{\frac{D}{L}, \frac{D}{\alpha}\} =: h_0$, we deduce from (3.25) that for all $\rho \in (0, \frac{\alpha h}{D})$,

$$P(E, B_\rho(x)) = f(\rho) \geq \beta\,\rho^{N-1}.$$

Since $\partial^* E$ is dense in ∂E, this also holds for all $x \in \partial E$. $\square$

Corollary 2.7 ([12], Corollary 3.6). *Let $E \in \mathcal{P}$ be a minimizer of $\mathcal{F}(h, k, \cdot, F)$ with $F \in \mathcal{P}$, $E \cup F \subset B_{D-1}(0)$, and $h \in (0, h_0)$. Then*

$$\mathcal{H}^{N-1}(\partial E \setminus \partial^* E) = 0.$$

Distance-Volume comparison

We recall here a general result which makes it possible to compare $\mathcal{L}^N(A \setminus C)$ and $\int_A d_{\partial C}$ under conditions of density of C similar to (3.21). Such comparison will be essential for proving Theorem 2.4.

Theorem 2.8 (Distance-Volume comparison, [12], p. 230). *Let C be a compact subset of $\mathbb{R}^N$ such that there exists $\beta > 0$, $\tau > 0$ with*

$$\mathcal{H}^{N-1}(C \cap B_\rho(x)) \geq \beta\rho^{N-1} \quad \forall\, x \in \partial C,\ \forall\, \rho \in (0,\tau).$$

Then there exists a constant $\Gamma = \Gamma(N) > 0$ such that for all $R > \tau$, for all Borel set $A \subset \mathbb{R}^N$, we have

$$\mathcal{L}^N(A \setminus C) \leq \left[2\Gamma\left(\frac{R}{\tau}\right)^{N-1}\mathcal{H}^{N-1}(C)\right]^{\frac{1}{2}}\left[\int_A d_C(x)\,dx\right]^{\frac{1}{2}} + \frac{1}{R}\int_A d_C(x)\,dx. \tag{3.26}$$

We are now able to prove Theorem 2.4.

Proof of Theorem 2.4
Let us fix $h \in (0, h_0)$. By definition of E_h, we have for all j such that $jh \leq T$,

$$\mathcal{F}(h, j, E_h(j), E_h(j-1)) \leq \mathcal{F}(h, j, E_h(j-1), E_h(j-1)),$$

that is to say,

$$\begin{aligned}&\int_{E_h(j)\Delta E_h(j-1)} d_{\partial E_h(j-1)}(x)\,dx \\ &\leq h\left[P(E_h(j-1)) - P(E_h(j))\right] + h\left[J_h(j,j) - J_h(j-1,j)\right],\end{aligned}$$

where $J_h(i,j)$ is defined by (3.17). Let us set

$$I_h(j) = \{[P(E_h(j-1)) - P(E_h(j))] + [J_h(j,j) - J_h(j-1,j)]\}^{\frac{1}{2}}.$$

We now use Theorem 2.8 with $C = \partial E_h(j-1)$, $A = E_h(j)\Delta E_h(j-1)$, $\tau = \frac{\alpha h}{D}$, which is justified for $j \geq 2$ because of the density estimate (3.21). Thanks to Corollary 2.7, we know that $\mathcal{L}^N(C) = 0$, so that for all $R > \frac{\alpha h}{D}$,

$$\mathcal{L}^N(E_h(j)\Delta E_h(j-1)) \leq \left[2\Gamma\left(\frac{R}{\tau}\right)^{N-1}\mathcal{H}^{N-1}(\partial E_h(j-1))\right]^{\frac{1}{2}}\sqrt{h}\,I_h(j) + \frac{1}{R}h\,I_h(j)^2. \tag{3.27}$$

Recall that Proposition 2.3 gives a uniform bound c on the perimeter of $\mathcal{F}$-minimizers, so that $\mathcal{H}^{N-1}(\partial E_h(j-1)) \leq c$.

Let $m, l \in \mathbb{N}$ verify $mh \leq T$ and $0 < l < m < l + \frac{1}{h}$. We take

$$R = \frac{\alpha h}{D}\,[h(m-l)]^{\frac{-1}{N+1}} > \frac{\alpha h}{D},$$

and add up inequalities (3.27) for $j = l+1, \ldots, m$. Recall that (3.18) and (3.19) show that

$$\begin{aligned}\sum_{j=l+1}^{m} I_h(j)^2 &\leq P(E_h(l)) + \sum_{j=l+1}^{m} J_h(j,j) - J_h(j-1,j)\\ &\leq P(E_0) + \sum_{j=1}^{m} J_h(j,j) - J_h(j-1,j) \leq c,\end{aligned}$$

Moreover, the Cauchy-Schwartz inequality shows that

$$\sum_{j=l+1}^{m} I_h(j) \leq \sqrt{m-l}\left\{\sum_{j=l+1}^{m} I_h(j)^2\right\}^{\frac{1}{2}} \leq \sqrt{m-l}\,\sqrt{c}.$$

Finally, we find that

$$\begin{aligned}&\mathcal{L}^N(E_h(m)\Delta E_h(l))\\ &\leq \left[2\Gamma[h(m-l)]^{-\frac{N-1}{N+1}}c\right]^{\frac{1}{2}}\sqrt{h(m-l)}\,\sqrt{c} + \frac{D}{\alpha h}\,[h(m-l)]^{\frac{1}{N+1}}\,h\,c\\ &= \left(\sqrt{2\Gamma} + \frac{D}{\alpha}\right)c\,[h(m-l)]^{\frac{1}{N+1}}.\end{aligned}$$

3 Regularity for $\mathcal{F}$-minimizers

One interest of the variational approach used in [3] is that it enables to use the regularity theory for area-minimizing currents described for instance in [39, 86, 144, 168]. This is the idea we follow in this section. We use the notations of [3]. In particular, the notation $\mathbf{M}$ and $\mathbf{S}$ stand respectively for the mass and size of an integral current.

3.1 Existence of tangent cones

We begin by recalling the definition of a current :

Definition 3.1 (Currents). *Let us denote by $\mathcal{D}^m$ the set of C^∞ differential m-form with compact support. Then, the dual space, $\mathcal{D}_m$, is the space of m-dimensional currents.*

Definition 3.2. *Let $f_{p,R} : x \mapsto R(x-p)$, for $p \in \mathbb{R}^N$, $R > 0$. A locally integral current $[J]$ is called a tangent current to ∂E at p if there exists a sequence $(R_i) \to +\infty$ such that if we set $E(R) = f_{p,R}(E)$, then $[E(R_i)] \to [J]$ as $i \to \infty$, in the sense that $\mathcal{L}^N((J \Delta E(R_i)) \cap B_r(x)) \to 0$ for each $x \in \mathbb{R}^N$ and $r > 0$.*

Lemma 3.3 (Existence of tangent cones). *Let $F \in \mathcal{P}$ and let E be a minimizer for $\mathcal{F}(h,k,\cdot,F)$ on $\mathcal{P}$. For each $p \in \partial E$, there exists a tangent current $[J]$ to ∂E at p. Each such tangent current $[J]$ is a cone and locally minimises the perimeter P. Moreover $0 \in \partial J$.*

Proof. The proof is inspired by that of [3, Theorem 3.9]. We easily check that for all $R > 0$,

$$
\begin{gathered}
P(E(R)) = R^{N-1} P(E), \\
\frac{1}{h} \int_{E(R)\Delta F(R)} d_{\partial F(R)}(x)\, dx = R^{N+1} \frac{1}{h} \int_{E\Delta F} d_{\partial F}(x)\, dx, \\
\int_{E(R)} \frac{1}{2} c_0(\cdot, kh) \star 1_{E(R)}(x)\, dx = R^{2N} \int_E \frac{1}{2} c_0(R(\cdot), kh) \star 1_E(x)\, dx \\
\int_{E(R)} c_1(x, kh)\, dx = R^N \int_E c_1(R(x-p), kh)\, dx
\end{gathered}
$$

so that $E(R)$ minimises

$$
\begin{aligned}
E \mapsto & P(E) + \frac{1}{R^2 h} \int_{E\Delta F(R)} d_{\partial F(R)}(x)\, dx - \frac{1}{R^{N+1}} \int_E \frac{1}{2} c_0^R(\cdot, kh) \star 1_E(x)\, dx \\
& - \frac{1}{R} \int_E c_1^R(x, kh)\, dx, \qquad (3.28)
\end{aligned}
$$

where we have set $c_0^R(x,t) = c_0(x/R,t)$, $c_1^R(x,t) = c_1(p + x/R, t)$. Let us therefore compare $E(R)$ and $E(R) \setminus B(q,r)$ for fixed $q \in \mathbb{R}^N$ and $r > 0$, with respect to this last functional. It follows from manipulations similar to those of the proof of Proposition 2.1 that for almost all $r > 0$,

$$P(E(R), B(q,r)) \leq P(B(q,r)) + \frac{1}{R^2 h} \int_{B(q,r)} d_{\partial F(R)}(x)\, dx + \frac{L}{R}\mathcal{L}^N(B(q,r)).$$

where L is defined by (3.5). But $diam\, F(R) = R\, diam\, F$, so that

$$\frac{1}{R^2 h} \int_{B(q,r)} d_{\partial F(R)}(x)\, dx$$

is bounded as a function of R (and even tends to 0 as R goes to infinity). This provides the bound on the perimeter of $E(R)$ in balls sufficient to infer the existence of a tangent current $[J]$ (using the compactness result [160, Theorem 1.1 p. 225]).

Let us prove that $[J]$ locally minimises the perimeter. This means that for all $x \in \mathbb{R}^N$, all $r > 0$, and all $(N-1)$ integral current X with $\partial X = 0$ and having support in $C = \overline{B}_r(x)$, then $\mathbf{M}(\partial[J]\lfloor_C) \leq \mathbf{M}(\partial[J]\lfloor_C + X)$. We first recall from [3, Section 3.1.6] that there exists an N integral current Q with compact support in C such that $\partial Q = X$ and

$$\mathbf{S}(Q) \leq \mathbf{M}(Q) \leq \frac{r}{N}\mathbf{M}(X).$$

Then according to [86, Section 4.5.17], we can write Q as

$$Q = \sum_{i=0}^{+\infty}[Q_i] - \sum_{i=0}^{+\infty}[P_i],$$

where $Q_i, P_i \in \mathcal{P}$ and (Q_i), (P_i) are nested families such that $P_1 \cup Q_1 \subset Supp(Q)$ and $P_1 \cap Q_1 = \emptyset$. Let us set $K = (E(R) \setminus P_1) \cup Q_1$ and compare $E(R)$ and K with respect to the functional defined by (3.28) :

$$\begin{aligned} P(E(R)) &\leq P(K) + \frac{1}{R^2 h}\int_{P_1 \cup Q_1} d_{\partial F(R)}(x)\, dx + \frac{L}{R}\mathcal{L}^N(P_1 \cup Q_1) \\ &\leq \mathbf{M}(\partial[E(R)] + \partial Q) + \frac{1}{R^2 h}\int_C d_{\partial F(R)}(x)\, dx + \frac{L}{R}\mathbf{S}(Q). \end{aligned}$$

Since Q and $\partial Q = X$ have compact support in C, and since

$$P(E(R), C) = \mathbf{M}(\partial[E(R)]\lfloor_C),$$

we deduce that

$$\mathbf{M}(\partial[E(R)]\lfloor_C) \leq \mathbf{M}(\partial[E(R)]\lfloor_C + X) + \frac{1}{R^2 h}\int_C d_{\partial F(R)}(x)\, dx + \frac{L}{R}\mathcal{L}^N(C).$$

Knowing this, we can adapt [168, Theorem 34.5] to show that $[J]$ locally minimises the perimeter and also that $P(E(R_i), B_\rho(x)) \to P(J, B_\rho(x))$ as $i \to \infty$, for all x and almost all $\rho > 0$, where R_i is such that $[E(R_i)] \to [J]$ as $i \to \infty$.

Finally we check that $[J]$ is a cone, *i.e.* that J is invariant under all homothetic expansions $x \mapsto \lambda x$ for $\lambda > 0$. To see this we recall from (3.23) and (3.24) that for all $x \in \partial E$, the function

$$g : \rho \mapsto \frac{P(E, B_\rho(x))}{\rho^{N-1}} + c\rho$$

is nondecreasing on $(0,1)$, where c is a constant, and that for all $\rho \in (0,1)$,

$$\frac{P(E, B_\rho(x))}{\rho^{N-1}} + c\rho \geq \omega_{N-1}.$$

It follows that ∂E has a density $\theta(\partial E, x)$ at x with $\theta(\partial E, x) \geq 1$. For all $\rho > 0$,

$$\frac{P(E(R), B_\rho(0))}{\rho^{N-1}} = \frac{P(E, B_{\rho/R}(p))}{(\rho/R)^{N-1}} \underset{R\to\infty}{\longrightarrow} \theta(\partial E, p)\, \omega_{N-1}.$$

Moreover for almost all $\rho > 0$,

$$\frac{P(E(R_i), B_\rho(0))}{\rho^{N-1}} \underset{i\to\infty}{\longrightarrow} \frac{P(J, B_\rho(0))}{\rho^{N-1}}.$$

This shows that the ratio $\rho^{1-N} P(J, B_\rho(0))$ is independent of ρ, which is known to imply that J is a cone (see [100, proof of Theorem 9.3]). Moreover $\rho^{1-N} P(J, B_\rho(0)) = \theta(\partial E, p)\, \omega_{N-1} > 0$, so that $0 \in \partial J$. We finally observe that $\theta(\partial J, 0) = \theta(\partial E, p)$. □

3.2 Regularity of $\mathcal{F}$-minimizers

The existence of tangent cones enables to prove regularity results for $\mathcal{F}$-minimizers, as in [3, Section 3.5 and 3.7].

Theorem 3.4 (C^1-regularity for $\mathcal{F}$-minimizers)**.** *Let $F \in \mathcal{P}$, $F \subset B_{D-1}(0)$, and let $E \subset B_{D-1}(0)$ be a minimizer for $\mathcal{F}(h, k, \cdot, F)$ on $\mathcal{P}$. Then ∂E is a C^1-hypersurface, except for a set of Hausdorff dimension less than $N-8$ (empty if $N \leq 7$).*

Proof. We verify that E is an almost minimal current in the sense of Bombieri, that is, for some $\delta > 0$, for all $(N-1)$ integral current X with $\partial X = 0$ and having compact support in C, $diam(C) = r \leq \delta$, then

$$\mathbf{M}(\partial[E]\lfloor_C) \leq (1 + \omega(r))\, \mathbf{M}(\partial[E]\lfloor_C + X) \tag{3.29}$$

for some function ω such that $\omega(r) \to 0$ as $r \to 0^+$. To do so we proceed as in the previous proof, write $X = \partial Q$ with

$$Q = \sum_{i=0}^{+\infty} [Q_i] - \sum_{i=0}^{+\infty} [P_i],$$

set $K = (E \setminus P_1) \cup Q_1$ and compare E and K with respect to $\mathcal{F}$:

$$P(E, C) \leq P(K) + \frac{1}{h} \int_{P_1 \cup Q_1} d_{\partial F}(x)\, dx + L\mathcal{L}^N(P_1 \cup Q_1),$$

which yields if $B_{D-1}(0) \cap C \neq \emptyset$ (otherwise (3.29) is obvious), and $\delta \leq 1$:

$$\begin{aligned}
\mathbf{M}(\partial[E]\lfloor_C) &\leq \mathbf{M}(\partial[E]\lfloor_C + \partial Q) + \left(\frac{2D}{h} + L\right) \mathbf{S}(Q) \\
&\leq \mathbf{M}(\partial[E]\lfloor_C + X) + \left(\frac{2D}{h} + L\right) \frac{r}{N} \mathbf{M}(X) \\
&\leq \mathbf{M}(\partial[E]\lfloor_C + X) + \left(\frac{2D}{h} + L\right) \frac{r}{N} (\mathbf{M}(\partial[E]\lfloor_C + X) + \mathbf{M}(\partial[E]\lfloor_C)).
\end{aligned}$$

This easily implies the result with $\omega(r) = 3\left(\frac{2D}{h} + L\right)\frac{r}{N}$ and $\delta = \frac{N}{3}\left(\frac{2D}{h} + L\right)^{-1}$.

In addition, at any point p of ∂E there exists a tangent cone $[J]$ that minimises the perimeter (Lemma 3.3). Such a cone must be a hyper-plane for $N \leq 7$ ([168, Appendix B]), so that in particular $\theta(E, p) = \theta(J, 0) = 1$. We then deduce the result from the final remark in [39]. In case $N \geq 8$, we use the dimension reduction argument of Federer ([100, Theorem 11.8]). □

Now, we prove that minimizers are smooth at contact points with smooth hypersurfaces :

Theorem 3.5. *Let $F \in \mathcal{P}$, $F \subset B_{D-1}(0)$, and let $E \subset B_{D-1}(0)$ be a minimizer for $\mathcal{F}(h, k, \cdot, F)$ on $\mathcal{P}$. Assume that there exists $K \subset \mathbb{R}^N$ closed such that ∂K is a C^1 hypersurface and $\partial E \cap K = \{p\}$. Then ∂E is a C^1 hypersurface near p.*

Proof. Let $[J]$ be any tangent cone to ∂E at p. The assumption that $\partial E \cap K = \{p\}$ guarantees that ∂J is contained in the open half-space orthogonal to the outer unit normal $\mathbf{n}$ to K at p and containing $\mathbf{n}$. Since $0 \in \partial J$, [100, Theorem 15.5 p. 174] implies that ∂J is regular at 0, and therefore is a hyperplane. The result follows as in the proof of Theorem 3.4. □

Actually, we can deduce higher regularity for $\mathcal{F}$-minimizers at each point where they are C^1 hypersurfaces :

Theorem 3.6 (Higher regularity for $\mathcal{F}$-minimizers)**.** *Assume that c_0 is symmetric, that c_0 and c_1 satisfy **(A)** and are Lipschitz continuous in space. Let $F \in \mathcal{P}$, $F \subset B_{D-1}(0)$, and let $E \subset B_{D-1}(0)$ be a minimizer of $\mathcal{F}(h, k, \cdot, F)$ on $\mathcal{P}$. Set $g(p) = \pm d_{\partial F}(p)$, where we take the $-$ sign if $p \in F$, and the $+$ sign otherwise.*

Let $p \in \partial E$ such that ∂E is a C^1 hypersurface near p : there exist $R > 0$, $M > 0$ and a C^1 function $f : B_R^{N-1}(p) \to (-M, M)$ such that up to rotating and relabelling, we have

$$E \cap (B_R^{N-1}(p) \times (-M, M)) = \{(x, y);\ x \in B_R^{N-1}(p), -M < y < f(x)\}.$$

Then f is $C^{2,\alpha}$ in $B_R^{N-1}(p)$ for some $0 < \alpha < 1$ and satisfies

$$\frac{1}{h} g((x, f(x))) = \Delta f(x) + c_0(\cdot, kh) \star 1_E((x, f(x))) + c_1((x, f(x)), kh). \tag{3.30}$$

Therefore the mean curvature H_p of a point p of ∂E verifies

$$\frac{1}{h} g(p) = H_p + c_0(\cdot, kh) \star 1_E(p) + c_1(p, kh). \tag{3.31}$$

Proof. We begin by proving that f satisfies (3.30) in the sense of distributions. This is simply the Euler-Lagrange equation for $\mathcal{F}$, and the proof is the same as that of Ambrosio ([12], after statement of Theorem 3.3), with the additional observation that the first variation of

$$K \mapsto \frac{1}{2} \int_K c_0(\cdot, kh) \star 1_K(x)\, dx, \quad K \mapsto \int_K c_1(x, kh)\, dx$$

in the direction of a C^2 vector field Φ is respectively

$$K \mapsto \int_{\partial K} c_0(\cdot, kh) \star 1_K(x) \langle \Phi(x), \nu_x \rangle\, d\mathcal{H}^{N-1}(x),$$

$$K \mapsto \int_{\partial K} c_1(x, kh) \langle \Phi(x), \nu_x \rangle\, d\mathcal{H}^{N-1}(x),$$

where ν_x is the outer unit normal to K at $x \in \partial K$. The symmetry of c_0 is used here, along with the continuity of c_1 and $c_0 \star 1_K$ in space. For the reader's convenience, let us make formally the computation for the non-local term. To do this, let us define $K_t := (Id + t\Phi)(K)$ for t small enough such that $x \mapsto (x + t\Phi(x))$ is a diffeomorphism and let us denote

$$\mathcal{G}(K) = \frac{1}{2} \int_K c_0(\cdot, kh) \star 1_K(x)\, dx.$$

Then, the following holds

$$
\begin{aligned}
\mathcal{G}(K_t) &= \int_{K_t}\left(\int_{K_t} c_0(x-y,kh)\,dy\right)dx\\
&= \int_{K_t}\left(\int_{K} c_0(z+t\Phi(z)-x,kh)|det(Id-t\Phi'(z))|\,dz\right)dx\\
&= \int_{K_t}\Bigg(\int_{K} c_0(z-x,kh)+t\left(c_0(z-x)\mathrm{div}\Phi(z)+\langle Dc_0(z-x),\Phi(z)\rangle\right)\\
&\qquad\qquad + o(t)\,dz\Bigg)dx\\
&= \int_{K_t}\left(\int_{K} c_0(z-x,kh)+t\left(\mathrm{div}(c_0(z-x)\Phi(z)\right)+o(t)\,dz\right)dx\\
&= \int_{K_t}\Bigg(\int_{K} c_0(z-x,kh)\,dz\\
&\qquad\qquad + t\int_{\partial K} c_0(z-x,kh)\langle\Phi(z),\nu_z\rangle\,d\mathcal{H}^{N-1}(z)+o(t)\Bigg)dx
\end{aligned}
$$

where we have used the fact that

$$|det(Id-t\Phi'(z))| = 1+t\mathrm{div}\Phi(z)+o(t)$$

and

$$c_0(z+t\Phi(z)-x,kh) = c_0(z-x)+t\langle Dc_0(z-x),\Phi(z)\rangle+o(t)$$

for the third line and the Stoke's Theorem for the last line. With the same kind of computation, we finally get

$$\mathcal{G}(K_t) = \int_K c_0(\cdot,kh)\star 1_K(x)\,dx + t\int_{\partial K} c_0(\cdot,kh)\star 1_K(x)\,\langle\Phi(x),\nu_x\rangle\,d\mathcal{H}^{N-1}(x)+o(t)$$

which gives the result.

Knowing this, we apply [98, Theorem 1.2 page 219] to f and to each of the $\frac{\partial f}{\partial x_i}$, to conclude that f is $C^{2,\alpha}$ in $B_R^{N-1}(p)$. This last assertion uses the Lipschitz continuity of c_1 and $c_0\star 1_K$ in space. Both conclusions immediately follow. □

4 The upper and lower limits

In this section we are going to prepare the proofs of Theorems 1.4 and 1.5. Let E be a minimizing movement with initial condition E_0 and let (h_n) be a sequence such that $E_{h_n}([t/h_n])$ converges to $E(t)$ in $L^1(\mathbb{R}^N)$ for all $t\in[0,T]$ as n goes to infinity.

We define the upper and lower limits of the sets $E_{h_n}(k)$ for $n \to \infty$ and $k \in \mathbb{N}$ as follows :

$$E^*(t) = \left\{ \begin{array}{l} x \in \mathbb{R}^N; \exists (h_{n'}) \subset (h_n), k_{n'} \to +\infty \text{ and } x_{n'} \in E_{h_{n'}}(k_{n'}) \\ \text{with } k_{n'}h_{n'} \to t \text{ and } x_{n'} \to x \end{array} \right\},$$

$$E_*(t) = \mathbb{R}^N \setminus \left\{ \begin{array}{l} x \in \mathbb{R}^N; \exists (h_{n'}) \subset (h_n), k_{n'} \to +\infty \text{ and } x_{n'} \notin E_{h_{n'}}(k_{n'}) \\ \text{with } k_{n'}h_{n'} \to t \text{ and } x_{n'} \to x \end{array} \right\}.$$

By construction, E^* is closed while E_* is open, and $E_*(t) \subset E(t) \subset E^*(t)$ for all $t \in [0,T]$ and almost everywhere in $\mathbb{R}^N$. Indeed $E_*(t)$ and $E^*(t)$ were defined respectively as the sets of cluster points of sets $E_{h_n}(k)$ and $\mathbb{R}^N \setminus E_{h_n}(k)$ for *all* $k \to +\infty$ such that $kh_n \to t$, and, up to a subsequence and a set of 0 $\mathcal{L}^N$ measure, our minimizing movement $E(t)$ was constructed as the pointwise limit of sets $E_{h_n}(k)$ for *some* such $k = [t/h_n]$.

We will use the regularity result Theorem 3.6 to compute the normal velocity of the evolutions $t \mapsto E^*(t)$ and $t \mapsto E_*(t)$. Then we will prove a regularity result for E^* and E_*, and compare the initial sets $E_*(0)$, $E^*(0)$ and E_0.

In order that our minimizing procedure be consistent with the evolution law (3.1) as ensured by Theorem 3.6, we will assume in particular throughout this section that c_0 is symmetric.

4.1 Velocity of E^* and E_*

Here we are going to prove a rigorous version of the heuristic fact that E^* moves with velocity

$$V_{x,t} \le H_{x,t} + c_0(\cdot,t) \star 1_{E(t)}(x) + c_1(x,t),$$

while E_* moves with velocity

$$V_{x,t} \ge H_{x,t} + c_0(\cdot,t) \star 1_{E(t)}(x) + c_1(x,t).$$

Following Cardaliaguet [50], we formulate this statement in terms of test functions : let us first define the classical mean curvature operator

$$h(p,X) = Trace(X) - \frac{\langle Xp, p\rangle}{|p|^2},$$

for $X \in Sym_N$ and $p \in \mathbb{R}^N$, and let us define, for any subset A of $\mathbb{R}^N$, $\widehat{A} = \overline{\mathbb{R}^N \setminus A}$, and for any subset B of $\mathbb{R}^N \times [0,T]$, $\widehat{B} = \overline{(\mathbb{R}^N \times [0,T]) \setminus B}$.

Proposition 4.1. *Assume that c_0 and c_1 are Lipschitz continuous on $\mathbb{R}^N \times [0,T]$. Then :*

1. For any $t \in (0,T)$, if a test function ϕ of class C^2 has a local maximum on E^* at some point $(x,t) \in \partial E^*$, with $D\phi(x,t) \neq 0$, then

$$\phi_t(x,t) \geq h(D\phi(x,t), D^2\phi(x,t)) - \left[c_0(\cdot,t) \star 1_{E(t)}(x) + c_1(x,t)\right] |D\phi(x,t)|.$$

2. For any $t \in (0,T)$, if a test function ϕ of class C^2 has a local minimum on $\widehat{E_*}$ at some point $(x,t) \in \partial\widehat{E_*}$, with $D\phi(x,t) \neq 0$, then

$$\phi_t(x,t) \leq h(D\phi(x,t), D^2\phi(x,t)) - \left[c_0(\cdot,t) \star 1_{E(t)}(x) + c_1(x,t)\right] |D\phi(x,t)|.$$

Proof. We only prove the first point, the proof of the second being similar. Let $t \in (0,T)$ and ϕ of class C^2 have a local maximum on E^* at some point $(x,t) \in \partial E^*$, with $D\phi(x,t) \neq 0$. We can assume without loss of generality that it is a strict maximum. By definition of E^*, there exist sequences $k_n \to \infty$ and $x_n \in \partial E_{h_n}(k_n)$ with $k_n h_n \to t$ and $x_n \to x$, such that ϕ has a local maximum (that we can assume to be strict) on $E_{h_n} = \cup_{k_n} E_{h_n}(k_n) \times \{k_n h_n\}$ at $(x_n, k_n h_n)$, with $D\phi(x_n, k_n h_n) \neq 0$. It follows that $\Gamma_{h_n}(k_n) = \{x \in \mathbb{R}^N; \phi(x, k_n h_n) = \phi(x_n, k_n h_n)\}$ is a smooth exterior contact surface to $E_{h_n}(k_n)$ at x_n, and therefore Theorems 3.5 and 3.6 imply that $E_{h_n}(k_n)$ is $C^{2,\alpha}$ near x_n. We now infer from the local relative position of Γ and $\partial E_{h_n}(k_n)$ that the curvature of $\partial E_{h_n}(k_n)$ at x_n, $H^n_{x_n}$, is less than the curvature of Γ at x_n :

$$H^k_{x_n} \leq -\frac{1}{|D\phi(x_n, k_n h_n)|} h(D\phi(x_n, k_n h_n), D^2\phi(x_n, k_n h_n)).$$

Now (3.31) implies, if $k_n \geq 1$, that

$$\pm\frac{1}{h_n} d_{\partial E_{h_n}(k_n-1)}(x_n) = H^k_{x_n} + c_0(\cdot, k_n h_n) \star 1_{E_{h_n}(k_n)}(x_n) + c_1(x_n, k_n h_n),$$

where we take the $-$ sign if $x_n \in E_{h_n}(k_n - 1)$, and the $+$ sign otherwise. With this convention,

$$\pm\frac{1}{h_n} d_{\partial E_{h_n}(k_n-1)}(x_n) \geq \pm\frac{1}{h_n} d_{\Gamma_{h_n}(k_n-1)}(x_n) = -\frac{\phi_t(x_n, k_n h_n)}{|D\phi(x_n, k_n h_n)|} + o(1).$$

Putting together the last three equations yields

$$\begin{aligned}\phi_t(x_n, k_n h_n) + o(1) \geq\ & h(D\phi(x_n, k_n h_n), D^2\phi(x_n, k_n h_n)) \\ & - \left[c_0(\cdot, k_n h_n) \star 1_{E_{h_n}(k_n)}(x_n) + c_1(x_n, k_n h_n)\right] |D\phi(x_n, k_n h_n)|.\end{aligned} \tag{3.32}$$

Thanks to the discrete Hölder estimate Theorem 2.4, we know, since $k_n h_n \to t$, that $E_{h_n}(k_n) \to E(t)$ in $L^1(\mathbb{R}^N)$. Up to a subsequence, we can assume that $E_{h_n}(k_n) \to E(t)$ almost everywhere. As a consequence, sending n to $+\infty$, we get the result, namely :

$$\phi_t(x,t) \geq h(D\phi(x,t), D^2\phi(x,t)) - \left[c_0(\cdot,t) \star 1_{E(t)}(x) + c_1(x,t)\right] |D\phi(x,t)|.$$

□

4.2 Regularity of E^* and E_*

Now we are going to prove a regularity result for the tubes E^* and E_* which allows in particular to treat the degenerate case $D\phi(x,t) = 0$ in Proposition 4.1 :

Proposition 4.2. *For all x in $\mathbb{R}^N$, the maps $t \mapsto d_{E^*(t)}(x)$ and $t \mapsto d_{\widehat{E_*(t)}}(x)$ are left-continuous on $(0,T]$.*

To prove this we first need to estimate in a finer way than what we have done in Section 2 how $E_h(k)$ can expand or shrink at most at each iteration. This is the equivalent of [3, Theorem 5.4]. Let us first define for simplicity of forthcoming estimates the scaled ball $W_R = \overline{B}_{R/(\omega_N)^{1/N}}(0) = \overline{B}_{R/\omega_*}(0)$, so that $\mathcal{L}^N(W_R) = R^N$. Then W_R minimises the perimeter among all sets $E \in \mathcal{P}$ such that $\mathcal{L}^N(E) = R^N$. This property will provide the necessary estimates.

Let us also define, for any subsets A and B of $\mathbb{R}^N$, $A - B = \mathbb{R}^N \setminus ((\mathbb{R}^N \setminus A) + B)$.

Lemma 4.3. *Let $F \in \mathcal{P}$ and let E be a minimizer for $\mathcal{F}(h,k,\cdot,F)$ on $\mathcal{P}$. Let L be defined as in* (3.5). *Let $R(h) = 2L\omega_* h + 2\sqrt{L^2\omega_*^2 h^2 + 2\omega_* h P(W_1)}$. Then*

$$F - W_{R(h)} \subset E \subset F + W_{R(h)}.$$

Proof. We begin by proving the left-hand side inclusion, and we will see that the other inclusion immediately follows. We adapt the proofs of [3, Section 5].

Step 1 : Let us first prove that if $0 < R < S$, $W_S \subset F$ and $0 < 2\mathcal{L}^N(W_R \setminus E) \leq R^N$, then

$$\frac{S-R}{\omega_* h} R - 2LR \leq \frac{N-1}{N} P(W_1) + \frac{2^{1/N}(N-1)}{N^2} P(W_1) \frac{\mathcal{L}^N(W_R \setminus E)}{R^N}.$$

We compare E and $E \cup W_R$ with respect to the functional $\mathcal{F}(h,k,\cdot,F)$:

$$\begin{aligned} &P(E) + \frac{1}{h}\int_{E\Delta F} d_{\partial F}(x)\,dx - \int_E \left(\frac{1}{2}c_0(\cdot,kh) \star 1_E(x) + c_1(x,kh)\right) dx \\ &\leq P(E \cup W_R) + \frac{1}{h}\int_{(E\cup W_R)\Delta F} d_{\partial F}(x)\,dx \\ &\qquad - \int_{E\cup W_R} \left(\frac{1}{2}c_0(\cdot,kh) \star 1_{E\cup W_R}(x) + c_1(x,kh)\right) dx. \end{aligned}$$

Since $W_R \subset F$, we check that $E\Delta F = ((E \cup W_R)\Delta F) \cup (W_R \setminus E)$. This, together with manipulations similar to those of previous proofs, implies that

$$\begin{aligned} P(E \cup W_R) - P(E) &\geq \frac{1}{h}\int_{W_R \setminus E} d_{\partial F}(x)\,dx - 2L\mathcal{L}^N(W_R \setminus E) \\ &\geq \left(\frac{S-R}{\omega_* h} - 2L\right)\mathcal{L}^N(W_R \setminus E), \end{aligned} \tag{3.33}$$

since the inclusion $W_S \subset F$ implies that $d_{\partial F}(x) \geq (S-R)/\omega_*$ for each $x \in W_R$. But conclusion (4) of [3, Proposition 5.)] implies that

$$\begin{aligned}
&P(E \cup W_R) - P(E) \\
&\leq R^{N-1} P(W_1) \left\{ \frac{N-1}{N} \frac{\mathcal{L}^N(W_R \setminus E)}{R^N} + \frac{2^{1/N}(N-1)}{N^2} \left(\frac{\mathcal{L}^N(W_R \setminus E)}{R^N} \right)^2 \right\},
\end{aligned}$$

and the result follows from the last two inequalities.

Step 2 : Now let us assume that the conclusion of the lemma does not hold, *i.e.* that $(F - W_{R(h)}) \setminus E$ contains some point p. We can assume up to applying a translation that $p = 0$. Therefore $W_{R(h)} \subset F$ and $\mathcal{L}^N(W_{R(h)/2} \setminus E) > 0$. Moreover we also have $2\mathcal{L}^N(W_{R(h)/2} \setminus E) \leq \left(\frac{R(h)}{2}\right)^N$, otherwise we would obtain as in Step 1 with $S = R(h)$ and $R = R(h)/2$ that

$$\begin{aligned}
P(E \cup W_{R(h)/2}) - P(E) &\geq \left(\frac{R(h)}{2\omega_* h} - 2L \right) \mathcal{L}^N(W_{R(h)/2} \setminus E) \\
&> \left(\frac{R(h)}{2\omega_* h} - 2L \right) \frac{1}{2} \left(\frac{R(h)}{2} \right)^N,
\end{aligned}$$

because $\frac{R(h)}{2\omega_* h} - 2L > 0$. But $P(E \cup W_{R(h)/2}) \leq P(E) + P(W_{R(h)/2})$, whence

$$\left(\frac{R(h)}{2h} - 2L \right) \frac{1}{2} \left(\frac{R(h)}{2} \right)^N < P(W_{R(h)/2}) = \left(\frac{R(h)}{2} \right)^{N-1} P(W_1),$$

or equivalently

$$\frac{1}{h} \left(\frac{R(h)}{2} \right)^2 - LR(h) < 2P(W_1),$$

which is contradictory with the choice of $R(h)$, since equality should hold instead of the last inequality. Then we can apply Step 1 with $S = R(h)$ and $R = R(h)/2$ to infer that

$$\frac{1}{\omega_* h} \left(\frac{R(h)}{2} \right)^2 - LR(h) \leq \frac{N-1}{N} P(W_1) + \frac{2^{1/N}(N-1)}{N^2} P(W_1) \frac{\mathcal{L}^N(W_{R(h)/2} \setminus E)}{(R(h)/2)^N},$$

or thanks to the choice of $R(h)$:

$$2 \leq \frac{N-1}{N} + \frac{2^{1/N}(N-1)}{N^2} \frac{1}{2},$$

which is false. This proves the left-hand side inclusion of Lemma 4.3.

Step 3 : Let us now explain why the left-hand side inclusion is sufficient to deduce the right-hand side one. Let $B = B_D(0)$ be a large ball. It is easy to check that if $F \in \mathcal{P}$ with $F \subset B_{D-1}(0)$, and if $E \in \mathcal{P}$ with $E \subset B_{D-1}(0)$ is a minimizer of $\mathcal{F}(h, k, \cdot, F)$ on $\mathcal{P}$, then $B \setminus E$ is a minimizer of

$$E \mapsto P(E) + \frac{1}{h} \int_{E \Delta (B \setminus F)} d_{\partial F}(x)\, dx - \int_E \left(\frac{1}{2} c_0(\cdot, kh) \star 1_E(x) + \overline{c_1}(x, kh) \right) dx$$

among all sets in $\mathcal{P}$ and included in B, where $\overline{c_1}(x, kh) = -c_1(x, kh) + c_0(\cdot, kh) \star 1_B(x)$. Therefore, up to taking h small enough so that $R < 1$, the arguments on E and F in the previous steps transform into the same arguments for $B \setminus E$ and $B \setminus F$, since in particular the term $2L$ appearing in (3.33) was taken so large (with the *a priori* useless factor 2) as to get the lower bound there also with $\overline{c_1}$ in place of c_1. The conclusion $F - W_R \subset E$ transforms into $(B \setminus F) - W_R \subset B \setminus E$, that is exactly $E \subset F + W_R$. □

The last lemma provides a bound on the growth of $\mathcal{F}$-minimizers at each iteration equal to $2L\omega_* h + 2\sqrt{L^2 \omega_*^2 h^2 + 2\omega_* h P(W_1)}$, and of the order of $\sqrt{h}$. This is not fine enough to conclude the left continuity, mainly because if $kh \to s$, then $k\sqrt{h} \to +\infty$ and the bound is lost in the limit movement. The following lemma refines the bound to the order h.

Lemma 4.4. *Let us set* $\delta = 2 \frac{N-1}{N} \omega_* P(W_1)$ *and*

$$R(h) = 2L\omega_* h + 2\sqrt{L^2 \omega_*^2 h^2 + 2\omega_* h P(W_1)}.$$

1. Assume that $p + W_S \subset E_h(k)$ *for some* $p \in \mathbb{R}^N$ *and* k, h *such that* $kh \leq T$. *If* h *and* j *are small enough so that* $R(h) < \frac{S}{4}$ *and* $jh \leq \min\{\frac{S^2}{4(\delta + 2LS)}, T - kh\}$, *then*

$$p + W_{S - (\frac{\delta}{S} + 2L)jh} \subset E_h(k + j).$$

2. Assume that $p + W_S \subset \mathbb{R}^N \setminus E_h(k)$ *for some* $p \in \mathbb{R}^N$ *and* k, h *such that* $kh \leq T$. *If* h *and* j *are small enough so that* $R(h) < \frac{S}{4}$ *and* $jh \leq \min\{\frac{S^2}{4(\delta + 2LS)}, T - kh\}$, *then*

$$p + W_{S - (\frac{\delta}{S} + 2L)jh} \subset \mathbb{R}^N \setminus E_h(k + j).$$

Proof. Let us prove the first assertion. For simplicity we assume without loss of generality that $p = 0$. We prove the result by induction on j. The result for $j = 0$ is the assumption. Let us assume that the result holds for some j such that $(j+1)h < \min\{\frac{S^2}{4(\delta + 2LS)}, T - kh\}$. We know thanks to Lemma 4.3 that

$$E_h(k + j) - W_{R(h)} \subset E_h(k + j + 1). \tag{3.34}$$

Since the induction assumption states that $W_{S - (\frac{\delta}{S} + 2L)jh} \subset E_h(k + j)$, and since the assumptions on j and h imply that $R(h) < \frac{S}{2} - (\frac{\delta}{S} + 2L)jh$, we deduce from (3.34) that $W_{S/2} \subset E_h(k + j + 1)$. Let us set

$$r_{max} = \sup\{r; W_r \subset E_h(k + j + 1)\} \geq \frac{S}{2}.$$

Step 1 of Lemma 4.3, by sending R to r_{max}^{+}, shows that

$$\frac{1}{\omega_* h}\left(\left\{S-\left(\frac{c\delta}{S}+2L\right)jh\right\}-r_{max}\right)r_{max}-2Lr_{max}\leq\frac{N-1}{N}P(W_1)=\frac{\delta}{2\omega_*},$$

from which we infer that

$$\left\{S-\left(\frac{\delta}{S}+2L\right)jh\right\}-r_{max}\leq\left(\frac{\delta}{2r_{max}}+2L\right)h\leq\left(\frac{\delta}{S}+2L\right)h,$$

and the result for $E_h(k+j+1)$ follows, so that the proof by induction is complete. The proof of the second point is entirely identical, according to the remark in Step 3 of the proof of Lemma 4.3. □

We are now ready to prove Proposition 4.2. This proof is inspired by the proof of [51, Lemma 4.7].

Proof of Proposition 4.2. Let us start with E^*. Assume on the contrary of our claim that there exist $x \in \mathbb{R}^N$ and $t \in (0,T]$ such that $s \mapsto d_{E^*(t)}(x)$ is not left continuous at t. Since this map is lower semi-continuous thanks to the closedness of E^*, we deduce that there exist $\varepsilon > 0$ and a sequence $t_p \to t^-$ such that for all $p \in \mathbb{N}$,

$$d_{E^*(t_p)}(x) > d_{E^*(t)}(x) + \varepsilon.$$

Let $S = \varepsilon\omega_*$, so that W_S is the closed ball of radius ε centred at 0. Up to considering a projection of x on $E^*(t)$, we can assume that $x \in E^*(t)$ and for all $p \in \mathbb{N}$,

$$d_{E^*(t_p)}(x) > \varepsilon.$$

Set for a fixed p, $k_n = [t_p/h_n]$, so that $k_n h_n \to t_p^-$. By definition of $E^*(t_p)$, there exists n_0 large enough depending on p so that for all $n \geq n_0$, $d_{E_{h_n}(k_n)}(x) > \varepsilon$. Let us set

$$M = \left(\frac{\delta}{S}+2L\right)/\omega_*.$$

Then we can apply assertion 2 of Lemma 4.4 to deduce that for all $n \geq n_0$ such that $R(h_n) < \frac{S}{4}$ and for all j such that $jh_n \leq \min\{\frac{S^2}{4(\delta+2LS)}, T-k_n h_n\}$

$$d_{E_{h_n}(k_n+j)}(x) \geq \varepsilon - Mjh_n. \tag{3.35}$$

Indeed we have $W_{S-(\frac{\delta}{S}+2L)jh_n}(x) \subset \mathbb{R}^N \setminus E_{h_n}(k_n+j)$. Let us set

$$\tau = \min\{\frac{\varepsilon}{2M}, \frac{S^2}{4(\delta+2LS)}\}$$

and fix $s \in (0, \tau)$ with $s \le T - t_p$. We set $j_n = [s/h_n]$ so that $j_n h_n \to s^-$ as $n \to +\infty$. Then $j_n h_n \le \min\{\frac{S^2}{4(\delta+2LS)}, T - k_n h_n\}$ for n large enough, so that sending n to $+\infty$ in (3.35) yields, by definition of $E^*(t_p + s)$,

$$d_{E^*(t_p+s)}(x) \ge \varepsilon - Ms \ge \frac{\varepsilon}{2}.$$

Taking $s = t - t_p$ for p big enough so that $0 < s < \tau$, we get $d_{E^*(t)}(x) \ge \frac{\varepsilon}{2}$, which contradicts the fact that $x \in E^*(t)$.
The proof for $d_{\widehat{E_*}}$ is obtained in the same way by using assertion 1 of Lemma 4.4. □

4.3 Comparison at initial time

We finish by giving a consequence of previous growth results on the comparison of initial sets $E_*(0)$ and $E^*(0)$ with E_0. This result will be essential for comparison at later times :

Lemma 4.5. *We have* $\overset{\circ}{E_0} \subset E_*(0) \subset E^*(0) \subset \overline{E_0}$.

Proof. We only prove that $E^*(0) \subset \overline{E_0}$, the left-hand side inclusion is obtained by similar arguments. Suppose on the contrary that there exists $x \in E^*(0) \setminus \overline{E_0}$. Then we can find some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \mathbb{R}^N \setminus \overline{E_0}$. By definition of $E^*(0)$, there exist sequences $k_n \to +\infty$ and $x_n \to x$ with $k_n h_n \to 0$ and $x_n \in E_{h_n}(k_n)$. Thanks to Lemma 4.4 and the facts that $E_{h_n}(0) = E_0$ and $k_n h_n \to 0$, we know that there exists $M > 0$ depending only on ε, L and N such that if k is large enough, then

$$B_{\varepsilon - M k_n h_n}(x) \subset \mathbb{R}^N \setminus E_{h_n}(k_n).$$

But $x_n \to x$ and $\varepsilon - M k_n h_n \to \varepsilon$, so that $x_n \in B_{\varepsilon - M k_n h_n}(x)$ for n large enough. This is a contradiction since $x_n \in E_{h_n}(k_n)$, and this proves the lemma. □

5 Minimizing movements and weak solutions

We are now ready to prove Theorem 1.4. Since $E_*(t) \subset E(t) \subset E^*(t)$ for all $t \in [0, T]$ and almost everywhere in $\mathbb{R}^N$, it suffices to prove that for all $t \in [0, T]$,

$$\{u(\cdot, t) > 0\} \subset E_*(t) \quad \text{and} \quad E^*(t) \subset \{u(\cdot, t) \ge 0\}.$$

To this end, we will use a comparison principle for discontinuous viscosity solutions. We therefore start by giving equations satisfied by 1_{E_*} and 1_{E^*} in the viscosity sense, in relation with Theorem 4.1 :

Theorem 5.1. *Assume that* c_0 *and* c_1 *are Lipschitz continuous on* $\mathbb{R}^N \times [0,T]$. *Then :*

1. For any $(x,t) \in \mathbb{R}^N \times (0,T)$, *if a test function* ϕ *of class* C^2 *is such that* $1_{E^*} - \phi$ *has a local maximum at* (x,t), *then :*

- *if* $D\phi(x,t) \neq 0$, *we have*
$$\phi_t(x,t) \leq h(D\phi(x,t), D^2\phi(x,t)) + \left[c_0(\cdot,t) \star 1_{E(t)}(x) + c_1(x,t)\right] |D\phi(x,t)|.$$
- *if* $D\phi(x,t) = 0$ *and* $D^2\phi(x,t) = 0$, *we have*
$$\phi_t(x,t) \leq 0.$$

2. For any $(x,t) \in \mathbb{R}^N \times (0,T)$, *if a test function* ϕ *of class* C^2 *is such that* $1_{E_*} - \phi$ *has a local minimum at* (x,t), *then :*

- *if* $D\phi(x,t) \neq 0$, *we have*
$$\phi_t(x,t) \geq h(D\phi(x,t), D^2\phi(x,t)) + \left[c_0(\cdot,t) \star 1_{E(t)}(x) + c_1(x,t)\right] |D\phi(x,t)|.$$
- *if* $D\phi(x,t) = 0$ *and* $D^2\phi(x,t) = 0$, *we have*
$$\phi_t(x,t) \geq 0.$$

Proof. We only prove the first point, since the second point uses the same arguments. We only need to consider the case where $(x,t) \in \partial E^*$, since otherwise all derivatives of ϕ at (x,t) vanish and the equation is trivially satisfied.

First case : $D\phi(x,t) \neq 0$. In this case it is straightforward to check that $-\phi$ has a local maximum on E^* at (x,t). Therefore, the first point of Proposition 4.1 gives the result.

Second case : $D\phi(x,t) = 0$ and $D^2\phi(x,t) = 0$. We can always assume that our maximum is equal to 0, *i.e.* $\phi(x,t) = 1_{E^*}(x,t) = 1$. Let us also assume that $\phi_t(x,t) > 0$. Then a Taylor expansion of ϕ at (x,t) shows that there exist $\delta > 0$ and $k > 0$ such that for all (y,s) verifying $s \in (t-\delta, t)$ and $|y-x| < 2k(t-s)^{1/3}$, $1_{E^*}(y,s) \leq \phi(y,s) < \phi(x,t) = 1$, hence $y \notin E^*(s)$. As a consequence for all $s \in (t-\delta,t)$,
$$d_{E^*(s)}(x) > k(t-s)^{1/3}.$$

Now we can proceed as in the proof of Proposition 4.2, using the growth control given by Lemma 4.4, to prove that there are positive constants k_1 and k_2 such that for all $s < t$ close enough to t,
$$d_{E^*(t)}(x) > k(t-s)^{1/3} - \left(\frac{k_1}{(t-s)^{1/3}} + k_2\right)(t-s) > 0,$$

which contradicts the fact that $x \in E^*(t)$. □

Proof of Theorem 1.4. The previous theorem shows that 1_{E^*} is a subsolution of the level-set equation (3.4), while 1_{E_*} is a supersolution. Indeed, an argument of Barles and Georgelin [22, Proposition 1] shows that under the conclusions of Theorem 5.1 there is no property to check when the test function satisfies $D\phi(x,t) = 0$ and $D^2\phi(x,t) \neq 0$. To conclude we use a method initiated by Barles, Soner and Souganidis [27, Theorem 2.1] : let (Φ_n) be a sequence of smooth functions such that $\Phi_n \equiv 1$ on $[0,+\infty)$, $\Phi_n' \geq 0$ in $\mathbb{R}$, $\Phi_n(\mathbb{R}) \subset [0,1]$ and $\inf_n \Phi_n = 0$ on $(-\infty, 0)$. Thanks to Lemma 4.5, we know that $1_{E^*(0)} \leq \Phi_n(u_0)$ in $\mathbb{R}^N$. Since (3.4) is a geometric equation, $\Phi_n(u)$ is a uniformly continuous solution of this equation. The comparison principle [27, Theorem 1.3] implies that for all $t \in [0,T)$,

$$1_{E^*(t)} \leq \Phi_n(u(\cdot,t)).$$

If $x \in \{u(\cdot,t) < 0\}$, we therefore have

$$1_{E^*(t)}(x) \leq \inf_n \Phi_n(u(x,t)) = 0,$$

which means that $x \notin E^*(t)$. As a consequence $E^*(t) \subset \{u(\cdot,t) \geq 0\}$ for all $t \in [0,T)$, which also holds for $t = T$ by continuity of u and thanks to Proposition 4.2. The argument to prove that $\{u(\cdot,t) > 0\} \subset E_*(t)$ is similar.

In case there is no fattening, we deduce that for all $t \in [0,T]$, $E(t) = \{u(\cdot,t) \geq 0\}$ almost everywhere, and we can replace $\{u(\cdot,t) \geq 0\}$ by $E(t)$ in (3.4) to deduce that u is a viscosity solution of (3.4). □

6 Comparison with the smooth flow

We now turn to the proof of Theorem 1.5. Following Cardaliaguet and Pasquignon [53], we define a sub/super pair of solutions for our non-local motion. Roughly speaking, it is a pair $(\mathcal{K}_1, \mathcal{K}_2)$ of tubes, where $\mathcal{K}_1$ moves with velocity

$$V_{x,t} \leq H_{x,t} + \inf_{\mathcal{K}_1(t) \subset K \subset \mathcal{K}_2(t)} \{c_0(\cdot,t) \star 1_K(x)\} + c_1(x,t),$$

while $\mathcal{K}_2$ moves with velocity

$$V_{x,t} \geq H_{x,t} + \sup_{\mathcal{K}_1(t) \subset K \subset \mathcal{K}_2(t)} \{c_0(\cdot,t) \star 1_K(x)\} + c_1(x,t).$$

As at the beginning of section 4.1, we formulate this in terms of test functions :

Definition 6.1 ([53], Definition 2.5). *Let K_1 and K_2 be compact subsets of $\mathbb{R}^N$ such that $K_1 \subset \overset{\circ}{K_2}$. A sub/super pair of solutions with initial data (K_1, K_2) is a pair $(\mathcal{K}_1, \mathcal{K}_2)$ of tubes such that*

1. $\mathcal{K}_1 \subset \mathcal{K}_2$.
2. $\mathcal{K}_1(0) = K_1$ *and* $\widehat{\mathcal{K}_2}(0) \subset \widehat{K_2}$.
3. For any $t \in (0,T)$*, if a test function* ϕ *of class* C^2 *has a local maximum on* $\mathcal{K}_1$ *at some point* $(x,t) \in \partial\mathcal{K}_1$*, then*

$$\begin{aligned}&\phi_t(x,t)\\ \geq& h(D\phi(x,t), D^2\phi(x,t)) - \left[\inf_{\mathcal{K}_1(t)\subset K\subset\mathcal{K}_2(t)}\{c_0(\cdot,t)\star 1_K(x)\} + c_1(x,t)\right]|D\phi(x,t)|.\end{aligned}$$

4. For any $t \in (0,T)$*, if a test function* ϕ *of class* C^2 *has a local minimum on* $\widehat{\mathcal{K}_2}$ *at some point* $(x,t) \in \partial\widehat{\mathcal{K}_2}$*, then*

$$\begin{aligned}&\phi_t(x,t)\\ \leq& h(D\phi(x,t), D^2\phi(x,t)) - \left[\sup_{\mathcal{K}_1(t)\subset K\subset\mathcal{K}_2(t)}\{c_0(\cdot,t)\star 1_K(x)\} + c_1(x,t)\right]|D\phi(x,t)|.\end{aligned}$$

Such sub/super pairs of solutions exist and we can define, following Cardaliaguet and Pasquignon, *extremal* sub/super pairs of solutions $(\mathcal{K}_1^\varepsilon, \mathcal{K}_2^\varepsilon)$ with initial data $(E_0 - \varepsilon B_1(0), E_0 + \varepsilon\overline{B}_1(0))$. The extremality holds with respect to the inclusion. Moreover, if E_0 is compact with uniformly $C^{2+\alpha}$ boundary, and if E_r is a smooth evolution with $C^{2+\alpha}$ boundary, starting from E_0 with normal velocity given by (3.11), then $\mathcal{K}_1^\varepsilon \subset E_r \subset \mathcal{K}_2^\varepsilon$ and both $\mathcal{K}_1^\varepsilon$ and $\mathcal{K}_2^\varepsilon$ converge to E_r in the Hausdorff distance as $\varepsilon \to 0$, as proved by Cardaliaguet [50].

Now, owing to the respective velocities of $\mathcal{K}_1^\varepsilon$, E_*, E^* and $\mathcal{K}_2^\varepsilon$, we want to compare these sets. Going through the corresponding proofs in [53] and [50], we check that the estimation on the velocities of E^* and E_* (Proposition 4.1), their regularity property (Proposition 4.2) and their initial position relatively to E_0 (Lemma 4.5) give the following result :

Theorem 6.2 ([53], Theorem 2.11)**.** *Under the assumptions of Theorem 1.5, let* $(\mathcal{K}_1^\varepsilon, \mathcal{K}_2^\varepsilon)$ *be an extremal sub/super pair of solutions with initial data*

$$(E_0 - \varepsilon B_1(0), E_0 + \varepsilon\overline{B}_1(0)).$$

If $\mathcal{K}_1^\varepsilon(t)$ *and* $\mathcal{K}_2^\varepsilon(t)$ *are non-empty for all* $t \in [0,T]$*, then*

$$\mathcal{K}_1^\varepsilon(t) \subset E_*(t) \subset E^*(t) \subset \mathcal{K}_2^\varepsilon(t) \quad \forall t \in [0,T).$$

We are finally ready to prove Theorem 1.5.

Proof of Theorem 1.5. Since $\mathcal{K}_1^\varepsilon$ and $\mathcal{K}_2^\varepsilon$ converge to the smooth evolution E_r starting from E_0 in the Hausdorff distance if the latter exists, we deduce that for all $t \in [0,T)$, $E_*(t) = E^*(t) = E_r(t)$. This also holds for $t = T$ thanks to Proposition 4.2. Moreover we know that for all $t \in [0,T]$ and almost everywhere in $\mathbb{R}^N$, $E_*(t) \subset E(t) \subset E^*(t)$, so the result follows. $\square$

7 Existence of smooth solutions

To prove Theorem 1.6, we will use a fixed point method. We begin by constructing a smooth solution for the local problem (with prescribed velocity).

7.1 Existence of smooth solutions for the local problem

Theorem 7.1 (Existence of a smooth solution for the local problem)**.** *Assume that E_0 is a compact domain with uniformly $C^{3+\alpha}$ boundary and that $c \in W^{2,1;\infty}(\mathbb{R}^N \times [0,T])$. Then there exist a small time $t_0 > 0$ depending only on E_0 and on an upper bound on $\|c\|_{W^{2,1;\infty}(\mathbb{R}^N \times [0,T])}$ and a smooth evolution $\{E_r(t)\}_{0 \le t \le t_0}$ with $C^{2+\alpha}$ boundary, starting from E_0 with normal velocity*

$$V_{x,t} = H_{x,t} + c(x,t), \tag{3.36}$$

where $H_{x,t}$ is the mean curvature of $\Gamma(t) = \partial E_r(t)$ at x.

The proof is an adaptation of the one proposed by Evans and Spruck [84] for the classical mean curvature motion (see also Giga, Goto [99] and Maekawa [141] for more general equations). For the reader's convenience, we give the steps of the proof to explain how to treat the dependence in the space variable of the velocity c.

Assume we are given the smooth hypersurface $\Gamma_0 = \partial E_0$, a time $t_0 > 0$ and a smooth evolution $\{\Gamma(t)\}_{0 \le t \le t_0} = \{\partial E(t)\}_{0 \le t \le t_0}$ of surfaces developing from Γ_0 with normal velocity $V_{x,t}$. Heuristically, one can show (see [84]) that the signed distance function d to $\Gamma(t)$ defined by

$$d(x,t) = \begin{cases} \operatorname{dist}(x,\Gamma(t)) & x \in \mathbb{R}^N \setminus E(t) \\ -\operatorname{dist}(x,\Gamma(t)) & x \in E(t) \end{cases}$$

is a solution of

$$v_t = F(D^2 v, v) + c(x - v(x,t)Dv(x,t), t) \tag{3.37}$$

with

$$F(R,z) = f(\lambda_1(R), ..., \lambda_n(R), z) = \sum_{i=1}^{n} \frac{\lambda_i(R)}{1 - \lambda_i(R) z}, \tag{3.38}$$

where $\lambda_1(R) \le \lambda_2(R) \le ... \le \lambda_n(R)$ are the eigenvalues of R. F is *a priori* defined and smooth for $|R|$ and $|z|$ small enough, but we extend it to be smooth on all of $Sym_N \times \mathbb{R}$ with $|F|$, $|DF|$ and $|D^2 F|$ bounded as in [84].

The idea is to study directly the PDE (3.37). For this, set $\Gamma_0 = \partial E_0$ and let

$$g(x) = \begin{cases} \operatorname{dist}(x,\Gamma_0) & x \in \mathbb{R}^N \setminus E_0 \\ -\operatorname{dist}(x,\Gamma_0) & x \in E_0 \end{cases} \tag{3.39}$$

be the signed distance function to Γ_0. We fix δ_0 so small that g is smooth within

$$V = \{x \in \mathbb{R}^N,\ -\delta_0 < g(x) < \delta_0\}$$

and we set, for $t_0 > 0$ to be determined,

$$Q = V \times (0, t_0), \quad \Sigma = \partial V \times [0, t_0].$$

The plan is to consider a solution to the PDE

$$\begin{cases} v_t = F(D^2 v, v) + c(x - vDv, t) & \text{in} \quad Q \\ |Dv|^2 = 1 & \text{on} \quad \Sigma \\ v = g & \text{on} \quad V \times \{t = 0\} \end{cases} \tag{3.40}$$

and to prove that the zero level sets of $v(\cdot, t)$ are smooth hypersurfaces evolving with normal velocity given by (3.36).

First, we have the following existence result for this non-linear PDE (see Lunardi [140, Theorem 8.5.4 and Proposition 8.5.6] :

Theorem 7.2 (Existence for the non-linear PDE)**.**

There exist δ_0, $t_0 > 0$ depending only on E_0 and on an upper bound on $\|c\|_{W^{2,1;\infty}(\mathbb{R}^N \times [0,T])}$ such that there exists a unique solution $v \in C^{2+\alpha, \frac{2+\alpha}{2}}(\overline{Q})$ of the PDE (3.40)*. Moreover, if $g \in C^{3+\alpha}(\overline{V})$, then the first order space derivatives v_{x_k}, for $1 \leq k \leq N$, belong to $C^{2+\alpha, \frac{2+\alpha}{2}}(Q)$.*

Evolution of the zero level set of v

The rest of the proof is devoted to proving that the sets

$$\Gamma(t) = \{x \in V,\ v(x,t) = 0\} \quad (\text{for } t \in [0, t_0]) \tag{3.41}$$

are in fact smooth hypersurfaces evolving with normal velocity

$$V_{x,t} = H_{x,t} + c(x,t),$$

where $H_{x,t}$ is the mean curvature of $\Gamma(t)$ at x.

Theorem 7.3 (Distance property of v)**.**

Let v be the solution of (3.40) *given by Theorem 7.2. Then we have*

$$|Dv|^2 = 1 \quad \text{in } Q. \tag{3.42}$$

Proof. We adapt the proof of Evans and Spruck [84, Theorem 3.1].

1. Let $w = |Dv|^2 - 1$. By the regularity assumption on E_0, we have that $w \in C^{2+\alpha,\frac{2+\alpha}{2}}(Q)$. Moreover, using the PDE (3.40) and the definition of g given by (3.39), we get that

$$w = 0 \quad \text{on } \Sigma, \tag{3.43}$$

and

$$w = 0 \quad \text{on } V \times \{t = 0\}. \tag{3.44}$$

2. Differentiating (3.40), we compute (the summations are implicit)

$$v_{tx_k} = \frac{\partial F}{\partial r_{ij}}(D^2v, v)v_{x_ix_jx_k} + \frac{\partial F}{\partial z}(D^2v, v)v_{x_k} + \frac{\partial}{\partial x_k}c(x - vDv, t).$$

Therefore

$$\begin{aligned}
w_t =&2v_{x_k}v_{x_kt}\\
=&2\frac{\partial F}{\partial r_{ij}}(D^2v, v)v_{x_k}v_{x_kx_ix_j} + 2\frac{\partial F}{\partial z}(D^2v, v)|Dv|^2 + 2\frac{\partial}{\partial x_k}(c(x - vDv, t))v_{x_k}\\
=&\frac{\partial F}{\partial r_{ij}}(D^2v, v)w_{x_ix_j} - 2\frac{\partial F}{\partial r_{ij}}(D^2v, v)v_{x_kx_i}v_{x_kx_j} + 2\frac{\partial F}{\partial z}(D^2v, v)|Dv|^2 \qquad (3.45)\\
&+ 2\frac{\partial}{\partial x_k}(c(x - vDv, t))v_{x_k}. \qquad (3.46)
\end{aligned}$$

Now

$$\begin{aligned}
2\frac{\partial}{\partial x_k}(c(x - vDv, t))v_{x_k} =&2\sum_{i,k=1}^{N}\frac{\partial c}{\partial x_i}(x - vDv, t)(\delta_{ik} - v_{x_k}v_{x_i} - vv_{x_kx_i})\ v_{x_k}\\
=&-2(|Dv|^2 - 1)\sum_{i=1}^{N}\frac{\partial c}{\partial x_i}(x - vDv, t)\ v_{x_i}\\
&-\sum_{i=1}^{N}\frac{\partial c}{\partial x_i}(x - vDv, t)\ v\ w_{x_i}\\
=&-w\ l_1(x, t) - w_{x_i}\ l_{2,i}(x, t),
\end{aligned}$$

where

$$l_1(t, x) = 2\sum_{i=1}^{N}\frac{\partial c}{\partial x_i}(x - vDv, t)\ v_{x_i}$$

and

$$l_{2,i}(x, t) = \frac{\partial c}{\partial x_i}(x - vDv)\ v.$$

Moreover
$$F(D^2v,v)=f(...,\lambda_i(D^2v),...,v) \quad \text{in } Q.$$
Thus
$$\frac{\partial F}{\partial z}(D^2v,v)=\frac{\partial f}{\partial z}(...,\lambda_i(D^2v),...,v)=\sum_{i=1}^{n}\frac{\lambda_i(D^2v)^2}{(1-\lambda_i(D^2v)v)^2}.$$
On the other hand
$$\begin{aligned}\frac{\partial F}{\partial r_{ij}}(D^2v)v_{x_kx_i}v_{x_kx_j}&=\sum_{i=1}^{n}\frac{\partial f}{\partial \lambda_i}(...,\lambda_i(D^2v),...,v)\lambda_i(D^2v)^2\\&=\sum_{i=1}^{n}\frac{\lambda_i(D^2v)^2}{(1-\lambda_i(D^2v)v)^2}=\frac{\partial F}{\partial z}(D^2v,v).\end{aligned}$$
As a consequence (3.45) becomes
$$w_t=\frac{\partial F}{\partial r_{ij}}(D^2v,v)w_{x_ix_j}+\left(2\frac{\partial F}{\partial z}(D^2v,v)-l_1(x,t)\right)w-l_{2,i}(x,t)\,w_{x_i}.$$

In view of the uniform ellipticity of F (see [84, Lemma 2.1]), we get that this is a uniformly parabolic equation. Using the fact that $w=0$ on the parabolic boundary of Q, we deduce that $w=0$ in Q. This ends the proof of the theorem. □

Now, using (3.42), we get that
$$\Gamma=\{(x,t)\in\overline{Q},\ v=0\}$$
is a smooth hypersurface in $\mathbb{R}^{n+1}\cap Q$ and each slice $\Gamma(t)=\{x\in V,\ v(x,t)=0\}$ is a smooth hypersurface in V. Moreover we have the equivalent of [84, Theorem 3.2] :

Theorem 7.4 (Existence of a classical evolution).

The surfaces $\{\Gamma(t)\}_{0\le t\le t_0}$ comprise a classical motion starting from Γ_0 with normal velocity
$$V_{x,t}=H_{x,t}+c(x,t).$$

7.2 Existence of smooth solution for the non-local problem

This subsection is devoted to proving Theorem 1.6. To this purpose, we will use a fixed point method. We use the same notations as in the previous section, in particular F, Q, Σ and V, for some t_0 to be determined. Using the same method as in Section 7.1, our goal is to construct a solution to the PDE
$$\left\{\begin{array}{l}v_t=F(D^2v,v)+(c_0(\cdot,t)\star_V 1_{\{v(\cdot,t)\le 0\}})(x-vDv,t)+\tilde{c}(x-vDv,t)\quad\text{in}\quad Q\\ |Dv|^2=1\quad\text{on}\quad\Sigma\\ v=g\quad\text{on}\quad V\times\{t=0\}\end{array}\right.\tag{3.47}$$

where $\star_V$ denotes the convolution restricted to V, *i.e.*

$$c_0(\cdot,t) \star_V 1_{\{v(\cdot,t)\leq 0\}}(x) = \int_V c_0(x-y,t)1_{\{v(\cdot,t)\leq 0\}}(y)\,dy$$

and

$$\tilde{c}(x,t) = \int_{E_0\setminus V} c_0(x-y,t)dy + c_1(x,t).$$

We define the set

$$E = \left\{ v \in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q}) \;\middle|\; \begin{array}{l} ||v-g||_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})} \leq R_0 \\ |Dv|^2 = 1 \text{ in } Q \\ v = g \text{ on } V\times\{t=0\} \\ v_{t|_{t=0}} = h_0 \text{ on } V\times\{t=0\} \end{array} \right\}$$

where g is defined by (3.39), R_0 is a small constant which will be precised later and

$$h_0 = F(D^2g,g) + c_0 \star 1_{E_0}(x-gDg,0) + c_1(x-gDg,0).$$

For $w\in E$, we set

$$c_w(x,t) = c_0(\cdot,t)\star_V 1_{\{w(\cdot,t)\leq 0\}}(x) + \tilde{c}(x,t).$$

Under the assumptions on c_0 and c_1 it is easy to check that $c_w \in W^{2,1;\infty}(\mathbb{R}^N\times[0,T])$ (see the definition of $W^{2,1;\infty}(\mathbb{R}^N\times[0,T])$ after (3.10)). Indeed, the only difficulty is to check that c_w is Lipschitz in time. To do this, let us state the following lemma :

Lemma 7.5 (Estimate on characteristic functions)**.**
There exists a constant C which does not depend on t_0, such that if $u_1,\ u_2\in C^1(V)$ satisfy $Du_i\cdot Dg\geq \frac{1}{2}$ in V for $i=1,2$, then

$$\|1_{\{u_1\leq 0\}} - 1_{\{u_2\leq 0\}}\|_{L^1(\overline{V})} \leq C\|u_1-u_2\|_{L^\infty(\overline{V})}.$$

The proof is an easy adaptation of [8, Lemma 42] (using local cards and a partition of unity), so we skip it.

For any $u\in E$, Du satisfies $Du(\cdot,0)=Dg$ and is Hölder in time. As a consequence, for t_0 small enough depending only on an upper bound on $\|u\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})} \leq R_0 + \|g\|_{C^{2+\alpha}(\overline{V})}$, we have $Du(\cdot,t)\cdot Dg \geq \frac{1}{2}$ in V for any $u\in E$ and $t\in[0,t_0]$. Therefore, using the previous lemma, we can compute

$$\begin{aligned} |c_w(x,t)-c_w(x,s)| =& |c_0(\cdot,t)\star_V 1_{\{w(\cdot,t)\leq 0\}}(x) - c_0(\cdot,s)\star_V 1_{\{w(\cdot,s)\leq 0\}}(x) \\ &+ \tilde{c}(x,t)-\tilde{c}(x,s)| \\ \leq& |c_0(\cdot,t)\star_V 1_{\{w(\cdot,t)\leq 0\}}(x) - c_0(\cdot,t)\star_V 1_{\{w(\cdot,s)\leq 0\}}(x)| \\ &+ |c_0(\cdot,t)\star_V 1_{\{w(\cdot,s)\leq 0\}}(x) - c_0(\cdot,s)\star_V 1_{\{w(\cdot,s)\leq 0\}}(x)| \\ &+ |\tilde{c}(x,t)-\tilde{c}(x,s)| \\ \leq& C_w|t-s|, \end{aligned}$$

where

$$\begin{aligned} C_w =& C\|c_0\|_{L^\infty(\mathbb{R}^N\times[0,T])}\|w\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})} + 2\|c_0\|_{W^{1,\infty}([0,T];L^\infty(\mathbb{R}^N))}\mathcal{L}^N(E_0) \\ &+ \|c_1\|_{W^{2,1;\infty}(\mathbb{R}^N\times(0,T))}. \end{aligned}$$

The factor 2 appears if we assume that $\mathcal{L}^N(V \setminus E_0) \leq \mathcal{L}^N(E_0)$, which is always possible. We remark that this constant C_w can be chosen independently of w since we have $\|w\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})} \leq R_0 + \|g\|_{C^{2+\alpha}(\overline{V})}$. This, together with similar estimates on space derivatives, implies that for any $w \in E$,

$$\|c_w\|_{W^{2,1;\infty}(\overline{Q})} \leq C(1+R_0),$$

where the constant C does not depend on t_0, R_0.

As a consequence, for t_0 small enough (depending only on R_0), we can therefore define for any $w \in E$, $v = \Phi(w)$ as the unique solution of

$$\begin{cases} v_t = F(D^2v, v) + c_w(x - vDv, t) & \text{in} \quad Q \\ |Dv|^2 = 1 & \text{on} \quad \Sigma \\ v = g & \text{on} \quad V \times \{t=0\}. \end{cases}$$

Moreover the proof of Theorem 7.2 shows that provided t_0 is small enough (depending only on R_0), then $v \in E$ for any $w \in E$. Let us now prove that Φ is a contraction.

Let $w_1,\ w_2 \in E$, $v_1 = \Phi(w_1)$, $v_2 = \Phi(w_2)$ and $v = v_2 - v_1$. Then v is a solution of

$$\begin{cases} v_t - a_{ij}v_{x_ix_j} + f_iv_{x_i} + ev = \delta + A(D^2v, Dv, v, x, t) & \text{in} \quad Q \\ \dfrac{\partial v}{\partial \nu} = a(Dv, x, t) & \text{on} \quad \Sigma \\ v = 0 & \text{on} \quad V \times \{t=0\}, \end{cases}$$

where

$$a_{ij} = \frac{\partial F}{\partial r_{ij}}(D^2v_1, v_1)v_{ij}, \quad f_i = \frac{\partial c}{\partial x_i}v_1, \quad e = Dc_{w_1}\cdot Dv_1 - \frac{\partial F}{\partial z}(D^2v_1, v_1),$$

$$\delta = c_{w_2}(x - v_2Dv_2, t) - c_{w_1}(x - v_2Dv_2, t),$$

$$\begin{aligned} A(R,p,z,x,t) =& F(D^2v_1 + R, v_1 + z) - F(D^2v_1, v_1) \\ &- \frac{\partial F}{\partial z}(D^2v_1, v_1)z - \frac{\partial F}{\partial r_{ij}}(D^2v_1, v_1)r_{ij} \\ &+ c_{w_1}(x - (v_1+z)(Dv_1+p), t) - c_{w_1}(x - v_1Dv_1, t) \\ &+ (Dc_{w_1}(x - v_1Dv_1, t)\cdot Dv_1)z + \frac{\partial c_{w_1}(x - v_1Dv_1, t)}{\partial x_i}v_1p_i \end{aligned}$$

and

$$a(p,x,t) = \begin{cases} -\dfrac{1}{2}\left(2p\cdot(Dv_1 - Dg) + |p|^2\right) & \text{on } \{g=\delta_0\} \\ \dfrac{1}{2}\left(2p\cdot(Dv_1 - Dg) + |p|^2\right) & \text{on } \{g=-\delta_0\}, \end{cases}$$

where we have used the fact that Dg is normal to ∂V. Using the same arguments as those of Evans and Spruck [84, Lemma 5.3] (*i.e.* a Taylor expansion) and the fact that $||v||_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})} \leq 2R_0$, we get that

$$||A||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})}, ||a||_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Sigma)} \leq CR_0||v||_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})},$$

where C does not depend on t_0, R_0. Using [84, Lemma 2.2], we then deduce that :

$$\begin{aligned} \|v_1 - v_2\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})} =& ||v||_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})} \\ \leq& C\left(\|\delta\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})} + \|A\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})} + \|a\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Sigma)}\right), \end{aligned}$$

from which we deduce that

$$\|v\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})} \leq C\|\delta\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})}$$

for R_0 small enough. We now use the following lemma, the proof of which is postponed :

Lemma 7.6 (Estimate on the velocities)**.**
With the previous notation, we have for t_0 small enough (depending only on R_0)

$$||\delta||_{W^{1,1;\infty}(\overline{Q})} \leq C||w||_{W^{1,1;\infty}(\overline{Q})}$$

where $w = w_1 - w_2$ and C does not depend on t_0.

This implies in particular, also using the Hölder regularity of w and the fact that $w_{t|t=0} = 0 = Dw(\cdot,0)$, that

$$\|\delta\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})} \leq ||\delta||_{W^{1,1;\infty}(\overline{Q})} \leq C||w_1 - w_2||_{W^{1,1;\infty}(\overline{Q})} \leq Ct_0^{\frac{\alpha}{2}}\|w_1 - w_2\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})}$$

so that for t_0 small enough

$$\|v_1 - v_2\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})} \leq \frac{1}{2}\|w_1 - w_2\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})}.$$

This implies that Φ is a contraction and so that, using the Banach fixed point Theorem, we deduce that there exists a unique solution v of (3.47).

Using Theorem 7.4, we deduce that $E_r(t) = \{x \in V,\ v(x,t) \le 0\}$ defines a smooth evolution with $C^{2+\alpha}$ boundary starting from E_0 with normal velocity

$$V_{x,t} = H_{x,t} + c_0(\cdot,t) \star 1_{E_r(t)}(x) + c_1(x,t).$$

This ends the proof of Theorem 1.6.

We now prove Lemma 7.6 :

Proof of Lemma 7.6. We begin by computing the derivative of δ in time. We have, using Hadamard's formula :

$$\frac{\partial \delta}{\partial t}(x + v_2 D v_2, t) = \int_V (c_0)_t(x-y,t)(1_{\{w_1(\cdot,t)\le 0\}}(y) - 1_{\{w_2(\cdot,t)\le 0\}})(y)\,dy \tag{3.48}$$

$$- \int_{\{w_1(\cdot,t)=0\}} (w_1)_t(y,t) c_0(x-y,t) d\mathcal{H}^{n-1} \tag{3.49}$$

$$+ \int_{\{w_2(\cdot,t)=0\}} (w_2)_t(y,t) c_0(x-y,t) d\mathcal{H}^{n-1}$$

First, using Lemma 7.5, we have that

$$\left| \int_V (c_0)_t(x-y,t)(1_{\{w_1(y,t)\le 0\}} - 1_{\{w_2(y,t)\le 0\}}) dy \right| \le C \|c_0\|_{W^{1,\infty}([0,T];L^\infty(\mathbb{R}^N))} \|w\|_{L^\infty(\overline{Q})}. \tag{3.50}$$

For the second term, we write

$$\int_{\{w_2(\cdot,t)=0\}} (w_2)_t(y,t) c_0(x-y,t) d\mathcal{H}^{n-1} - \int_{\{w_1(\cdot,t)=0\}} (w_1)_t(y,t) c_0(x-y,t) d\mathcal{H}^{n-1}$$
$$= \mathcal{I}_1 + \mathcal{I}_2$$

where

$$\mathcal{I}_1 = \int_{\{w_2(\cdot,t)=0\}} (w_2)_t(y,t) c_0(x-y,t) d\mathcal{H}^{n-1} - \int_{\{w_2(\cdot,t)=0\}} (w_1)_t(y,t) c_0(x-y,t) d\mathcal{H}^{n-1}$$

and

$$\mathcal{I}_2 = \int_{\{w_2(\cdot,t)=0\}} (w_1)_t(y,t) c_0(x-y,t) d\mathcal{H}^{n-1} - \int_{\{w_1(\cdot,t)=0\}} (w_1)_t(y,t) c_0(x-y,t) d\mathcal{H}^{n-1}$$

We remark that

$$|\mathcal{I}_1| \le C \|c_0\|_{L^\infty(\overline{Q})} \|w_t\|_{L^\infty(\overline{Q})} \tag{3.51}$$

where the constant C is a bound on the perimeter of $\{u(\cdot,t) = 0\}$, uniform for $u \in E$ and $t \in [0, t_0]$.

We now treat $\mathcal{I}_2$, and to this aim we use a local parametrisation. We choose local coordinates and r small enough such that $\dfrac{\partial g}{\partial x_n} \geq \dfrac{3}{4}$ in $B_r \subset \mathbb{R}^{N-1}$. Now, for t_0 small enough, recalling that $w_i(\cdot,0) = g$ and $\|w_i\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})} \leq R_0 + \|g\|_{C^{2+\alpha}(\overline{V})}$, we get that $\dfrac{\partial w_i}{\partial x_n} \geq \dfrac{1}{2}$ in B_r. We fix $t \leq t_0$ and we assume that $\{w_i(\cdot,t) = 0\} = \{(x', f_i(x')),\ x' \in B_r\}$. Using a partition of unity, we will then recover the complete estimate. We define $\varepsilon(x') = f_2(x') - f_1(x')$. For t_0 small enough (depending only on R_0 and g) we can assume that

$$|\varepsilon(x')| \leq \frac{1}{2(R_0 + \|g\|_{C^{2+\alpha}(\overline{V})})} \tag{3.52}$$

We then have

$$\begin{aligned}|\mathcal{I}_2| \leq C \int_{x'\in B_r} \Big| & \sqrt{1+|Df_1|^2}\, c_0(x'-y', x_n - f_1(y'), t) \\ & -\sqrt{1+|Df_1 + D\varepsilon|^2}\, c_0(x'-y', x_n - f_1(y') - \varepsilon(y'), t)\Big|\, dy' \\ \leq & C\|\varepsilon\|_{W^{1,\infty}(B_r)}\end{aligned}$$

where we have used the fact that $c_0 \in L^\infty([0,T]; W^{1,\infty}(\mathbb{R}^N))$ and where the constant C depends only on R_0, g and c_0.

Our goal now is just to estimate $\|\varepsilon\|_{W^{1,\infty}(B_r)}$ with respect to $\|w\|_{L^\infty([0,t_0],W^{1,\infty}(\overline{V}))}$. For simplicity of notation, we forget the dependence in time of w, w_1 and w_2. We recall that

$$\begin{aligned} w_1(x', f_1(x')) = 0 = & w_2(x', f_1(x') + \varepsilon(x')) \\ = & w_1(x', f_1(x') + \varepsilon(x')) - w(x', f_1(x') + \varepsilon(x')). \end{aligned} \tag{3.53}$$

Using a Taylor expansion, we get that

$$w_1(x', f_1(x') + \varepsilon(x')) = w_1(x', f(x')) + \frac{\partial w_1}{\partial x_n}(x', f(x')) \cdot \varepsilon(x') + o(\varepsilon).$$

Moreover,

$$\begin{aligned}\|o(\varepsilon)\|_{L^\infty} \leq & \frac{1}{2}\left|\frac{\partial^2 w_1}{\partial x_n^2}\right| \|\varepsilon\|^2_{L^\infty} \\ \leq & \frac{1}{4}\|\varepsilon\|_{L^\infty}\end{aligned}$$

where we have used (3.52) and the fact that $\left|\dfrac{\partial^2 w_1}{\partial x_n^2}\right| \leq R_0 + \|g\|_{C^{2+\alpha}(\overline{V})}$. Injecting this in (3.53), we get

$$\|\varepsilon\|_{L^\infty} \leq 4\|w\|_{L^\infty(\overline{Q})}. \tag{3.54}$$

where we have used that $|\frac{\partial w_1}{\partial x_n}| \geq \frac{1}{2}$. Differentiating (3.53) with respect to x_i and using a Taylor expansion, we get as above

$$\|\varepsilon_{x_i}\|_{L^\infty} \leq C\frac{\|w\|_{L^\infty([0,t_0],W^{1,\infty}(\overline{V}))}}{|\frac{\partial w_2}{\partial x_n}|} \leq C\|w\|_{L^\infty([0,t_0],W^{1,\infty}(\overline{V}))}.$$

Combining the last inequality with (3.54), we get

$$|\mathcal{I}_2| \leq C\|w\|_{L^\infty((0,t_0);W^{1,\infty}(\overline{V}))}. \tag{3.55}$$

Using (3.50), (3.51) and (3.55), we get that

$$\|\frac{\partial \delta}{\partial t}\|_{L^\infty(\overline{Q})} \leq C\|w\|_{W^{1,1,\infty}(\overline{Q})}.$$

The estimates on $\|\delta\|_{L^\infty(\overline{Q})}$ and $\|D\delta\|_{L^\infty(\overline{Q})}$ are easier (they use the regularity of c_0), so we skip their proofs. This ends the proof of the Lemma. □

Acknowledgements
The authors would like to thank P. Cardaliaguet and R. Monneau for fruitful discussions during the preparation of this paper, as well as Y. Giga for sending us some references. The first author was supported by the contract JC 1025 called "ACI jeunes chercheuses et jeunes chercheurs" of the French Ministry of Research (2003-2005).

Chapitre 4

Convergence d'une équation eikonale non-locale vers le mouvement par courbure moyenne anisotrope. Applications à la dynamique des dislocations

Ce chapitre est un travail en collaboration avec F. Da Lio et R. Monneau [67]. Dans ce travail, on montre la convergence à grande échelle d'une équation du premier ordre non locale vers un mouvement par courbure moyenne anisotrope. Cette équation du premier ordre, qui est une équation de type eikonale avec une vitesse dépendant de façon non locale de la solution elle-même, intervient dans la théorie de la dynamique des dislocations. On montre que si un mouvement par courbure moyenne anisotrope est approximé par ce genre d'équation alors il est de type variationnel alors que la réciproque est vraie seulement en dimension 2.

Convergence of a non-local eikonal equation to anisotropic mean curvature motion. Application to dislocations dynamics.

F. Da Lio , N. Forcadel, R. Monneau

Abstract

In this paper we prove the convergence at a large scale of a non-local first order equation to an anisotropic mean curvature motion. This first order equation is an eikonal-type equation with a velocity depending in a non-local way on the solution itself, that arises in the theory of dislocations dynamics. We show that if an anisotropic mean curvature motion is approximated by this type of equations then it is always of variational type, whereas the converse is true only in dimension two.
AMS Classification : 35F25, 35D05, 35Q99, 35B40, 35G25, 49L25.
Keywords : Dislocations dynamics, asymptotic behaviour, non-local equations, eikonal equation, mean curvature motion, viscosity solutions.

1 Introduction

1.1 Physical motivation

In this paper, we study the asymptotic behaviour of an equation modelling dislocations dynamics. More precisely, we show that, on a large scale, dislocations dynamics is given by a mean curvature motion (we refer to Subsection 1.3 for the exact setting of the result). Dislocations are line defects in crystals whose typical length in metallic alloys is of the order of $10^{-6}m$ and thickness of the order of $10^{-9}m$. The concept of dislocations in crystals was put forward in the XXth century, as the main microscopic explanation of the macroscopic plastic behaviour of metallic crystals (see the physical monograph Hirth, Lothe [106]). Since the beginning of the 90's, the research field of dislocations is enjoying a new development, in particular thanks to the power of computers which allows simulations with a large number of dislocations.

Recently Rodney, Le Bouar, Finel introduced in [158] a new model called *the phase field model of dislocation*. In this model, the dislocation line in the crystal moves in its slip plane with a normal velocity which is proportional to the Peach-Koeller force acting on this line. In the case where there are no exterior stress, this force is simply the self-force created by the elastic field generated by the dislocation line itself. In [10], [9], Alvarez, Hoch, Le Bouar and Monneau proposed to rewrite this model as a non-local Hamilton-Jacobi equation. Using viscosity solutions (we refer to the monographs of Barles [18] and Bardi and Capuzzo-Dolcetta [17] and to the paper of Crandall, Ishii and Lions [61] for a good introduction to this theory), Alvarez *et al.* [10], [9] proved a short time existence and uniqueness result. Then, Alvarez, Cardaliaguet and Monneau [6] and Barles and Ley [25] proved a long time result under certain assumptions. We also refer to Forcadel [87] for a uniqueness and existence result for dislocations dynamics with a mean curvature term. This equation was also numerically studied by Alvarez, Carlini, Monneau, Rouy [7], [8].

Mathematically, a dislocation line is represented by the boundary of a bounded domain $\Omega \subset \mathbb{R}^2$ which moves with normal speed given by

$$V_n = \overline{c_0} \star \rho$$

where the kernel $\overline{c_0} = \overline{c_0}(x)$ depends only on the space variables, $\star$ denotes the convolution in space and ρ is the characteristic function of the set Ω, *i.e.*

$$\rho(x) = \begin{cases} 1 & \text{if} \quad x \in \Omega \\ 0 & \text{if} \quad x \notin \Omega. \end{cases}$$

In this paper, we consider a simplified model of the one proposed by Alvarez *et al.* [10], [9]. Here, we assume that the negative part of the kernel $\overline{c_0}$ is concentrated on one point, *i.e.*, $\overline{c_0} = c_0 - \left(\int_{\mathbb{R}^2} c_0\right) \delta_0$ where c_0 is now a positive kernel. Because of the formal half contribution of the Dirac mass to $\overline{c_0} \star \rho$ on the dislocation line $\partial\Omega$, we can rewrite (formally on the dislocation line)

$$V_n = c_0 \star \rho - \frac{1}{2} \int_{\mathbb{R}^2} c_0.$$

For this model, we will be able to prove, in the framework of the Slepčev level set formulation (see [169]), a long time existence and uniqueness result for the solution of this equation (see Section 2).

Physically, the kernel c_0 is assumed to behave like $\frac{1}{|x|^3}$ at infinity. We refer to Section 4.1 for more details and note that the decay of our kernel misses the natural integrability condition (4.34). For this reason, we can rescale the characteristic function ρ, defining

$$\rho^\varepsilon(x,t) = \rho\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2 |\ln \varepsilon|}\right).$$

This is almost the parabolic scaling. Here the presence of the logarithm is a well-known factor in physics (see for instance Barnet Gavazza [31], Brown [45] or Hirth and Lothe[106]). We then show that in a large scale (*i.e.* $\varepsilon \to 0$), the normal speed of the dislocation line associated to ρ^ε is given by anisotropic mean curvature of the line. More precisely, we show that the solution of the non-local Hamilton-Jacobi equation modelling dislocations dynamics converges, at a large scale, to the solution of a mean curvature motion. We also study the link between the energy of dislocations and the energy associated to the mean curvature motion and we prove a formal convergence of the energies. We show that the mean curvature motion we can approach with this type of non-local eikonal equations is always of variational type. Finally, we show that in the two dimensional case, essentially all mean curvature motion of variational type can be approximated, which is not true in higher dimensions.

This result is very natural for dislocation dynamics. Indeed, in many references in physics, the authors describes dislocations dynamics by line tension terms deriving from an energy associated to the dislocation line. See for instance Brown [45] and Barnet Gavazza [31] for physical references and Garroni, Müller [93], [94] for a variational approach. As far as we know, our result is the first rigorous proof for the convergence of dislocation dynamics to mean curvature motion.

Similar results have already been proved for general kernels in relation with the Merriman, Bence, Osher algorithm for computing mean curvature motion [143]. We refer to Barles, Georgelin [22] Evans [79], Ishii [116] and Ishii, Pires, Souganidis [120] for such kind of results. We also refer to Souganidis [172] for example where the kernels are fractional laplacian. Nevertheless, our kernel does not satisfy the assumptions of these papers. We refer to Subsection 4.1 for a comparison with other related works. Moreover, we show in Section 7 that the limit mean curvature motion obtained by convolution is of variational type.

1.2 Mathematical setting of the problem

Given a function g defined on the unit sphere $\mathbf{S}^{n-1}$of $\mathbb{R}^n$ by

$$g \in C^0(\mathbf{S}^{n-1}), \quad g(-\theta) = g(\theta) \geq 0, \quad \forall \theta \in \mathbf{S}^{n-1} \tag{4.1}$$

we consider kernels $c_0 \in L^\infty(\mathbb{R}^n)$ satisfying

$$\begin{cases} c_0(x) = \dfrac{1}{|x|^{n+1}} \; g\left(\dfrac{x}{|x|}\right) \quad \text{if} \quad |x| \geq 1, \\ \\ c_0(-x) = c_0(x) \geq 0, \quad \forall x \in \mathbb{R}^n. \end{cases} \tag{4.2}$$

We want to look what happen for large dislocation, *i.e.*, in a large scale. Up to a change of variable, this is equivalent to concentrate the kernel. Since c_0 behaves like $\frac{1}{|x|^{n+1}}$ at infinity (see (4.2)), the "natural scaling" is then the following one for $0 < \varepsilon < 1$

$$c_0^\varepsilon(x) = \frac{1}{\varepsilon^{n+1}|\ln \varepsilon|} \; c_0\left(\frac{x}{\varepsilon}\right). \tag{4.3}$$

The presence of the logarithm comes out naturally in the proofs (see Subsection 4.1) but is also expected from a physical point of view.

We will use the level set formulation in the sense that the dislocation line (here in any dimension $n \geq 1$) is represented by any level set of a continuous function u^ε, solving the following equation (in the sense of Definition 2.1)

$$\begin{cases} u^\varepsilon_t(x,t) = \left((c^\varepsilon_0 \star 1_{\{u^\varepsilon(\cdot,t) > u^\varepsilon(x,t)\}})(x) - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^n} c^\varepsilon_0 \right) |Du^\varepsilon(x,t)| & \text{in} \quad \mathbb{R}^n \times (0,T), \\ u^\varepsilon(\cdot,0) = u_0(\cdot) & \text{in} \quad \mathbb{R}^n \end{cases} \tag{4.4}$$

where Du^ε indicates the gradient of u^ε with respect to the space variables, the convolution is done in space only and $1_{\{u^\varepsilon(\cdot,t) > u^\varepsilon(x,t)\}}$ is the characteristic function of the set $\{u^\varepsilon(\cdot,t) > u^\varepsilon(x,t)\}$. Here, we consider the simultaneous evolutions of all the level sets of the function u^ε. This approach has been introduced by Slepčev [169] (see also Da Lio, Kim, Slepčev [69]).

We will prove that the unique viscosity solution of (4.4) converges to the unique solution of a mean curvature-type equation.

1.3 Main results

We denote by $C^{1,1/2}_{x,t}(\mathbb{R}^n \times [0,T])$ the set of continuous functions satisfying a Lipschitz condition in x and a Hölder condition in t of exponent $1/2$ and by $\mathrm{Lip}(\mathbb{R}^n)$ the set of Lipschitz continuous functions.

Theorem 1.1. (Existence, uniqueness and regularity for the ε-problem) *Let $n \geq 1$. Assume that the initial data $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$ and that $c_0 \in W^{1,1}(\mathbb{R}^n)$. Then for all $\varepsilon \in (0,1)$, there exists a unique viscosity solution u^ε of (4.4) in the sense of Definition 2.1. Moreover, u^ε is $C^{1,1/2}_{x,t}(\mathbb{R}^n \times [0,T])$ uniformly in ε for $\varepsilon \in (0,\frac{1}{2})$. Namely, we have the following estimates for $\varepsilon \in (0,\frac{1}{2})$:*

$$|Du^\varepsilon(\cdot,t)|_{L^\infty(\mathbb{R}^n)} \leq |Du_0|_{L^\infty(\mathbb{R}^n)}, \quad \forall t \geq 0$$

and

$$|u^\varepsilon(x,t+h) - u^\varepsilon(x,t)| \leq C|Du_0|_{L^\infty(\mathbb{R}^n)}\sqrt{h}, \quad \forall x \in \mathbb{R}^n,\ \forall t \geq 0,\ h \in [0,1],$$

where the constant C depends only on n and $\sup\limits_{\mathbb{R}^n} c_0$.

We are interested in the limit problem satisfied by the limit u^0 of u^ε as ε goes to zero. To this purpose, we consider the following problem

$$\begin{cases} u^0_t(x,t) + F(D^2u^0, Du^0) = 0 & \text{in} \quad \mathbb{R}^n \times (0,T) \\ u^0(\cdot,0) = u_0(\cdot) & \text{in} \quad \mathbb{R}^n \end{cases} \tag{4.5}$$

with

$$F(M,p) = -\text{trace }\left(M \cdot A\left(\frac{p}{|p|}\right)\right) \tag{4.6}$$

with

$$A\left(\frac{p}{|p|}\right) = \int_{\theta\in \mathbf{S}^{n-2} = \mathbf{S}^{n-1}\cap\left\{\langle x, \frac{p}{|p|}\rangle = 0\right\}} \left(\frac{1}{2}g(\theta)\theta\otimes\theta\right)\, d\theta \tag{4.7}$$

Hereafter $M \cdot A$ and $\langle \cdot,\cdot\rangle$ denote respectively the product between the two matrices and the usual scalar product.

Remark 1.2. *In particular F is geometric (see Barles, Soner, Souganidis [27]) because $M \mapsto F(M,p)$ is linear and*

$$F(M,p) = F\left(\left(Id - \frac{p}{|p|}\otimes\frac{p}{|p|}\right)\cdot M, \frac{p}{|p|}\right)$$

Remark 1.3. *In the particular case where $g \equiv 1$, we get $A = \dfrac{|\mathbf{S}^{n-2}|}{2(n-1)} Id_{\{x,\ <x,p>=0\}}$ where $|\mathbf{S}^{n-2}|$ is the Lebesgue measure of $\mathbf{S}^{n-2}$, and then*

$$F(M,p) = \frac{-|\mathbf{S}^{n-2}|}{2(n-1)}\text{ trace}\left(\left(Id - \frac{p}{|p|}\otimes\frac{p}{|p|}\right)\cdot M\right)$$

We recover the classical mean curvature motion up to the factor $|\mathbf{S}^{n-2}|/2(n-1)$.

We prove the following result

Theorem 1.4. (Convergence of dislocations dynamics to mean curvature motion)
Let $n \geq 1$. Given $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$ and $c_0 \in W^{1,1}(\mathbb{R}^n)$, we consider the solution u^ε of problem (4.4) with the kernel c_0^ε defined in (4.1)-(4.2)-(4.3). Then the solution u^ε converges locally uniformly on compact sets of $\mathbb{R}^n \times [0,+\infty)$ to the unique viscosity solution u^0 of (4.5)-(4.6)-(4.7).

Remark 1.5. *This result also suggests a natural scheme to compute numerically mean curvature motion. This is the subject of a paper in preparation [68].*

From expression (4.6)-(4.7) it is not clear if the anisotropic mean curvature motion (4.5) is of a variational type or not. Theorem 1.7 below will show that this mean curvature motion is indeed of variational type. Before to state Theorem 1.7, we need the following definition :

Definition 1.6. *Let* $g \in C^0(\mathbb{R}^n \backslash \{0\})$ *satisfy* $g(\lambda p) = \dfrac{g(p)}{|\lambda|^{n+1}}$, $\forall \lambda \in \mathbb{R}\backslash\{0\}$ *and* $p \in \mathbb{R}^n\backslash\{0\}$. *We then associate to* g *a temperate distribution* L_g *defined by*

$$\langle L_g, \varphi \rangle = \int_{\mathbb{R}^n} dx\ g(x) \left(\varphi(x) - \varphi(0) - x \cdot D\varphi(0)\ 1_{B_1(0)}(x) \right)$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, *where* $\mathcal{S}(\mathbb{R}^n)$ *is the Schwartz space of test functions, and* $B_1(0)$ *denotes the unit ball centered in zero.*

We define the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *as*

$$\mathcal{F}(\varphi)(\xi) = \int_{\mathbb{R}^n} dx\ \varphi(x) e^{-i\xi\cdot x}.$$

We have the following theorem :

Theorem 1.7. (Variational origin of the anisotropic mean curvature motion)
Let $n \geq 2$. *Let* $g \in C^0(\mathbb{R}^n\backslash\{0\})$ *satisfy* $g(\lambda p) = \dfrac{g(p)}{|\lambda|^{n+1}}$, $\forall \lambda \in \mathbb{R}\backslash\{0\}$, $\forall p \in \mathbb{R}^n\backslash\{0\}$. *We have*

$$\int_{S^{n-1}\cap\left\{\langle x, \frac{p}{|p|}\rangle = 0\right\}} \frac{1}{2} g(\theta)\theta \otimes \theta d\theta = D^2 G\left(\frac{p}{|p|}\right) \quad \text{with} \quad G := -\frac{1}{2\pi}\mathcal{F}(L_g) \tag{4.8}$$

where $\mathcal{F}(L_g)$ *is the Fourier transform of* L_g. *Moreover* $G(\lambda p) = |\lambda| G(p)$, $\forall \lambda \in \mathbb{R}\backslash\{0\}$, $\forall p \in \mathbb{R}^n$ *and, with* A *defined in (4.7), if* $u^0 \in C^2(\mathbb{R}^n)$ *with* $|Du^0| \neq 0$, *then the following holds :*

$$\frac{1}{|Du^0|} trace \left(A\left(\frac{Du^0}{|Du^0|}\right) \cdot D^2 u^0 \right) = div\left(\nabla G\left(\frac{Du^0}{|Du^0|}\right)\right), \tag{4.9}$$

which means that the mean curvature motion derives from the following energy : $\int G(Du^0)$.

Moreover, if $g \geq 0$, *then* G *is convex.*

The converse is true in the two dimensional case, namely, if $G \in C^0(\mathbb{R}^2) \cap C^2(\mathbb{R}^2\backslash\{0\})$ *is convex and satisfies* $G(\lambda p) = |\lambda| G(p)$ $\forall \lambda \in \mathbb{R}\backslash\{0\}$, $p \in \mathbb{R}^2$, *then there exists a non-negative function* g *such that* $L_g := -2\pi\mathcal{F}(G)$.

A different non-local equation for a mean field model describing a spin flip dynamics has been studied in De Masi, Orlandi, Presutti, Triolo [71], Katsoulakis, Souganidis [124] and Barles, Souganidis [29]. In [32], Bellettini, Buttà and Presutti have proved that the limit dynamics is related to the Hessian of an energy.

Proposition 1.8. (Counter-example)
The converse of Theorem 1.7 is false in dimension $n \geq 3$, i.e., there exists g which changes its sign such that $A(p) = D^2G(p) \geq 0$.

Remark 1.9. *If g is a positive measure, we can formally approximate crystalline curvature by our non-local eikonal equation.*

Remark 1.10. *Physically, only $\mathcal{F}(L_g)$ is known. We see that formula (4.8) allows easily to compute g in dimension $n = 2$ and then to check that $g \geq 0$ or not. See Hirth and Lothe [106] Chapter 13-8 for an example where g is not non-negative, and Head [104] for examples in cubic elasticity.*

In the simplest case of applications for dislocations dynamics, the crystal is described by isotropic elasticity (see [106]). When the Burgers vector is along the x_1 direction, we have

$$G(p) = \frac{p_2^2 + \frac{1}{1-\nu}\ p_1^2}{|p|} \quad \textit{with} \quad \nu \in (-1, \frac{1}{2})$$

where ν is the Poisson ratio of the material, and

$$g(\theta) = \frac{(2\gamma - 1)(\theta_1)^2 + (2-\gamma)(\theta_2)^2}{|\theta|^5} \geq 0 \quad \textit{with} \quad \gamma = \frac{1}{1-\nu} \in (\frac{1}{2}, 2).$$

It is well-known that we can approach generalized mean curvature motion with Merriman Bence Osher [143] construction with a general kernel K_0 satisfying

$$K_0(-x) = K_0(x)$$

and for every $p \in \mathbf{S}^{n-1}$

$$\int_{p^\perp} K_0(x)|x|^2 < \infty \tag{4.10}$$

where $p^\perp = \left\{x,\ \langle x, \frac{p}{|p|}\rangle = 0\right\}$ and with the "parabolic scaling" $K_0^\varepsilon = \dfrac{1}{\varepsilon^{n+1}} K_0\left(\dfrac{x}{\varepsilon}\right)$. We refer, for instance to Barles Georgelin [22], Evans [79], Ishii [116] and Ishii, Pires, Souganidis [120] (we also refer to Subsection 4.1 for a formal proof).

More precisely, under the additional assumption that $\int_{p^\perp} K_0(x) = 1$ for any $p \in \mathbf{S}^{n-1}$, the limit motion found in [120], Section 3 with the threshold $\theta = 1/2$, is (4.5)-(4.6), with (4.7) replaced by

$$A\left(\frac{p}{|p|}\right) = \int_{x \in \mathbb{R}^{n-1} = \mathbb{R}^n \cap \left\{x,\ \langle x, \frac{p}{|p|}\rangle = 0\right\}} \left(\frac{1}{2} K_0(x) \cdot x \otimes x\right)\ dx \tag{4.11}$$

Up to our knowledge, it was not known in this general setting if the limit mean curvature motion associated to (4.11) is of variational type (*cf* (4.9)). It turns out that this is a simple consequence of our Theorem 1.7 :

Theorem 1.11. (Variational property of the limit motion)
Every mean curvature motion of the form of (4.5)-(4.6) with A defined in (4.11) is of variational type.

The problem we consider is formally associated to the following energy :

$$\mathcal{E}^\varepsilon(u^\varepsilon) = \int_\lambda \overline{\mathcal{E}^\varepsilon}(\lambda)d\lambda \tag{4.12}$$

where

$$\overline{\mathcal{E}^\varepsilon}(\lambda) = \int_{\mathbb{R}^n} -\frac{1}{2}\left(\overline{c_0}^\varepsilon \star \rho_\lambda^\varepsilon\right)\rho_\lambda^\varepsilon$$

with

$$\rho_\lambda^\varepsilon = 1_{\{u^\varepsilon>\lambda\}}, \quad \overline{c_0}^\varepsilon = c_0^\varepsilon - \left(\int_{\mathbb{R}^n} c_0^\varepsilon\right)\delta_0.$$

We will show formally in Section 8 that this energy is non increasing in time and that there is a convex function G such that $\mathcal{E}^\varepsilon(u^\varepsilon) \to \int G(Du^0)$ which is the energy associated to a mean curvature motion of the limit solution u^0.

1.4 Organisation of the paper

Let us now explain how this paper is organised : Section 2 is devoted to the study of the ε-problem. In Section 3, we recall some known results on the limit problem. Then, we give, in Section 4, a result on the convergence of the velocity for a test function. The regularity result of Theorem 1.1 is proved in Section 5 (see Corollary 5.3) as well as estimates at initial time. The convergence result Theorem 1.4 is proved in Section 6. The variational property of the limit motion Theorem 1.7 and Theorem 1.11 and the counter-example Proposition 1.8 are proved in Section 7. In Section 8, we study very formally the link between energy and mean curvature motion. Finally, in an appendix, we give some technical *lemmata* on Fourier transform.

2 Existence and uniqueness for the ε-problem

In the sequel we will denote by $B_{\text{loc}}USC(\mathbb{R}^n \times [0,T])$ and $B_{\text{loc}}LSC(\mathbb{R}^n \times [0,T])$ respectively the set of locally bounded upper semicontinuous and lower semicontinuous functions in $\mathbb{R}^n \times [0,T]$.

Definition 2.1. (Viscosity sub/super/solution for the non-local eikonal equation)
A function $u^\varepsilon \in B_{\text{loc}}USC(\mathbb{R}^n \times [0,T])$ is a viscosity subsolution of (4.4) if it satisfies :
(i) $u^\varepsilon(x,0) \le u_0(x)$ in $\mathbb{R}^n$,

(ii) for every $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ *and for every test function* $\Phi \in C^\infty(\mathbb{R}^n \times [0, T))$ *such that* $u^\varepsilon - \Phi$ *has a maximum at* (x_0, t_0)*, the following holds :*

$$\Phi_t(x_0, t_0) \leq \left((c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot, t_0) \geq u^\varepsilon(x_0, t_0)\}})(x_0) - \frac{1}{2} \int_{\mathbb{R}^n} c_0^\varepsilon \right) |D\Phi(x_0, t_0)|. \quad (4.13)$$

A function $u^\varepsilon \in B_{\text{loc}}LSC(\mathbb{R}^n \times [0, T])$ *is a viscosity supersolution of (4.4) if it satisfies :*

(i) $u^\varepsilon(x, 0) \geq u_0(x)$ *in* $\mathbb{R}^n$*,*

(ii) for every $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ *and for every test function* $\Phi \in C^\infty(\mathbb{R}^n \times [0, T))$ *such that* $u^\varepsilon - \Phi$ *has a minimum at* (x_0, t_0)*, the following holds :*

$$\Phi_t(x_0, t_0) \geq \left((c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot, t_0) > u^\varepsilon(x_0, t_0)\}})(x_0) - \frac{1}{2} \int_{\mathbb{R}^n} c_0^\varepsilon \right) |D\Phi(x_0, t_0)|. \quad (4.14)$$

A continuous function u^ε *is a viscosity solution of (4.4) if, and only if, it is a sub and a supersolution of (4.4).*

This definition comes from the definition of viscosity solution for nonlocal equation given by Slepčev [169] (see also Da Lio, Kim, Slepčev [69]) and it permits to extend to non-local equations all properties enjoyed by viscosity solutions of local equations.

Note the difference in the choice of the set in the indicatrice function in the definition of a subsolution and a supersolution. This is crucial to extend all the properties of viscosity solutions to nonlocal, geometric parabolic equations (see Slepčev [169]), in particular for the stability of the solution, *i.e.*, the $\limsup$ of subsolution is a subsolution (and so the existence by Perron's method).

Next we prove a comparison result between locally bounded semicontinuous viscosity sub and supersolutions to the equation (4.4).

Theorem 2.2. (Comparison principle for the ε-problem)
Assume $c_0 \in W^{1,1}(\mathbb{R}^n)$*. Let* $u \in B_{\text{loc}}USC(\mathbb{R}^n \times [0, T])$*,* $v \in B_{\text{loc}}LSC(\mathbb{R}^n \times [0, T])$ *be respectively viscosity sub and supersolution of (4.4). If* $u(x, 0) \leq u_0(x) \leq v(x, 0)$ *for all* $x \in \mathbb{R}^n$ *then* $u(x, t) \leq v(x, t)$ *for all* $(x, t) \in \mathbb{R}^n \times [0, T]$*.*

To prove this result, we need the analogous of the Ishii's *Lemma* for non-local equations. We first recall the definition of the limit sub and super-differentials :

$$\bar{\mathcal{P}}^+ u(x, t) = \left\{ \begin{array}{l} (p, a) \in \mathbb{R}^n \times \mathbb{R}, \ \exists\, (x_n, t_n, p_n, a_n) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \\ \text{such that } (p_n, a_n) \in \mathcal{P}^+ u(x_n, t_n) \\ \text{and } (x_n, t_n, u(x_n, t_n), p_n, a_n) \to (x, t, u(x, t), p, a) \end{array} \right\}$$

where $\mathcal{P}^+$ is the classical super-differentials. The set $\bar{\mathcal{P}}^- u(x,t)$ is defined in a similar way. It is well known that we have an equivalent definition for viscosity solution by using sub and super-differentials (cf Crandall, Ishii, Lions [61]). We claim that the definition remains equivalent if we replace the classical sub and super-differentials by the limit ones. Indeed, let $u \in B_{\mathrm{loc}}USC(\mathbb{R}^n \times [0,T])$ be a viscosity subsolution of (4.4). We will show that

$$(p,a) \in \bar{\mathcal{P}}^+ u(x,t) \Rightarrow a \leq \left(c_0^\varepsilon \star 1_{\{u(\cdot,t) \geq u(x,t)\}}(x) - \frac{1}{2}\int c_0^\varepsilon \right) |p|. \tag{4.15}$$

Let $(x_n, t_n, p_n, a_n) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ such that $(p_n, a_n) \in \mathcal{P}^+ u(x_n, t_n)$ and such that $(x_n, t_n, u(x_n,t_n), p_n, a_n) \to (x, t, u(x,t), p, a)$. We then have, by definition,

$$\begin{aligned} a_n &\leq \left(c_0^\varepsilon \star 1_{\{u(\cdot,t_n) \geq u(x_n,t_n)\}}(x_n) - \frac{1}{2}\int c_0^\varepsilon \right) |p_n| \\ &\leq \left(c_0^\varepsilon \star 1_{\{u(\cdot,t_n) \geq u(x_n,t_n)\} \cup \{u(\cdot,t) \geq u(x,t)\}}(x_n) - \frac{1}{2}\int c_0^\varepsilon \right) |p_n|. \end{aligned}$$

We just have to show that

$$c_0^\varepsilon \star 1_{\{u(\cdot,t_n) \geq u(x_n,t_n)\} \cup \{u(\cdot,t) \geq u(x,t)\}}(x_n) \to c_0^\varepsilon \star 1_{\{u(\cdot,t) \geq u(x,t)\}}(x).$$

To do this, we use the following decomposition :

$$\begin{aligned} & c_0^\varepsilon \star 1_{\{u(\cdot,t_n) \geq u(x_n,t_n)\} \cup \{u(\cdot,t) \geq u(x,t)\}}(x_n) - c_0^\varepsilon \star 1_{\{u(\cdot,t) \geq u(x,t)\}}(x) \\ =& c_0^\varepsilon \star 1_{\{u(\cdot,t_n) \geq u(x_n,t_n)\} \cup \{u(\cdot,t) \geq u(x,t)\}}(x_n) - c_0^\varepsilon \star 1_{\{u(\cdot,t_n) \geq u(x_n,t_n)\} \cup \{u(\cdot,t) \geq u(x,t)\}}(x) \\ & + c_0^\varepsilon \star 1_{\{u(\cdot,t_n) \geq u(x_n,t_n)\} \cup \{u(\cdot,t) \geq u(x,t)\} \setminus \{u(\cdot,t) \geq u(x,t)\}}(x). \end{aligned}$$

The first part clearly goes to zero as n goes to infinity. For the second part, we need the following *lemma* :

Lemma 2.3. *Let f_n be a sequence of measurable functions on $\mathbb{R}^n$ and*

$$f \geq \limsup{}^* f_n(x) := \sup \left\{ \limsup_{n \to 0} f_n(y) : y \to x \right\}.$$

Let a_n be a sequence converging to zero. Then

$$\mathcal{L}\left(\{f_n \geq a_n\} \setminus \{f \geq 0\} \right) \to 0 \quad \text{as n} \to \infty.$$

where, for any measurable set A, $\mathcal{L}(A)$ denotes the Lebesgue measure of A.

For the proof of this *lemma*, we refer to Slepčev [169].

Applying this *lemma* with $f_n = u(\cdot,t_n) - u(x,t)$, $a_n = u(x_n,t_n) - u(x,t)$ and $f = u(\cdot,t) - u(x,t)$ yields the result. The proof for supersolution is analogous.

Using (4.15), we can rewrite the Ishii's *Lemma* (see Crandall, Ishii, Lions [61] *Lemma* 8.3) for non-local equations :

Lemma 2.4. (Ishii's *Lemma* for non-local equations)
Let U and V be open sets of $\mathbb{R}^n$, and for $T > 0$, $u \in B_{\mathrm{loc}}USC(U \times (0,T))$ and $v \in B_{\mathrm{loc}}LSC(V \times (0,T))$ be respectively subsolution and supersolution of (4.4). Let $\phi : U \times V \times (0,T) \to (0,\infty)$ of class C^∞. Assume that $(x,y,t) \mapsto u(x,t) - v(y,t) - \phi(x,y,t)$ reaches a local maximum in $(\bar{x},\bar{y},\bar{t}) \in U \times V \times (0,T)$. We set $\tau = \partial_t \phi(\bar{x},\bar{y},\bar{t})$, $p_1 = D_x\phi(\bar{x},\bar{y},\bar{t})$, and $p_2 = -D_y\phi(\bar{x},\bar{y},\bar{t})$ Then, there exists τ_1, $\tau_2 \in \mathbb{R}$ such that :

$$\tau = \tau_1 - \tau_2,$$
$$(p_1,\tau_1) \in \bar{\mathcal{P}}^+ u(\bar{x},\bar{t}),\ (p_2,\tau_2) \in \bar{\mathcal{P}}^- v(\bar{y},\bar{t}),$$

and then

$$\tau_1 \leq \left(c_0^\varepsilon \star 1_{\{u(\cdot,\bar{t}) \geq u(\bar{x},\bar{t})\}}(\bar{x}) - \frac{1}{2}\int c_0^\varepsilon \right) |p_1|$$

and

$$\tau_2 \geq \left(c_0^\varepsilon \star 1_{\{v(\cdot,\bar{t}) > v(\bar{y},\bar{t})\}}(\bar{y}) - \frac{1}{2}\int c_0^\varepsilon \right) |p_2|.$$

Proof of Theorem 2.2
The proof of this theorem is inspired by Barles, Cardaliaguet, Ley and Monneau [21].
Let $u \in B_{\mathrm{loc}}USC(\mathbb{R}^n \times [0,T])$, $v \in B_{\mathrm{loc}}LSC(\mathbb{R}^n \times [0,T])$ be respectively viscosity sub and supersolution of (4.4). Since the equation is geometric we may assume without loss of generality that u and v are bounded (see Slepčev [169], property (P1)). Suppose by contradiction that $M = \sup_{\mathbb{R}^n \times [0,T]}(u(x,t) - v(x,t)) > 0$. Then for $\eta \in (0,1)$ small enough we have $M_\eta = \sup_{t \in [0,T]} \limsup_{|x-y| \to 0}(u(x,t) - v(y,t) - \eta t) > 0$ as well.

For all $\gamma > 0$ and $\alpha > 0$ with $\alpha \ll \gamma$, we introduce the auxiliary function $\Phi_{\gamma,\alpha} \colon \mathbb{R}^n \times \mathbb{R}^n \times [0,T] \to \mathbb{R}$ defined by

$$\Phi_{\gamma,\alpha}(x,y,t) = u(x,t) - v(y,t) - \eta t - \frac{|x-y|^2}{\gamma^2} - \alpha(|x|^2 + |y|^2). \tag{4.16}$$

We observe that $\limsup_{|x|,|y| \to +\infty} \Phi_{\gamma,\alpha}(x,y,t) = -\infty$, thus $\Phi_{\gamma,\alpha}(x,y,t)$ reaches its maximum at a point $(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) \in \mathbb{R}^n \times \mathbb{R}^n \times [0,T]$. Standard arguments show that

$$\alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2),\ \frac{|x_{\gamma,\alpha} - y_{\gamma,\alpha}|^2}{\gamma^2} \leq C_0, \tag{4.17}$$

with $C_0 > 0$ depending on $||u||_\infty, ||v||_\infty$. In particular we get that

$$\lim_{\gamma \to 0} \limsup_{\alpha \to 0} |x_{\gamma,\alpha} - y_{\gamma,\alpha}| = 0\,.$$

Then, the following estimate holds

$$\begin{aligned}
&\limsup_{\gamma\to 0}\limsup_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha})\\
\leq&\limsup_{\gamma\to 0}\limsup_{\alpha\to 0}(u(x_{\gamma,\alpha},t_{\gamma,\alpha})-v(y_{\gamma,\alpha},t_{\gamma,\alpha})-\eta t_{\gamma,\alpha})\\
\leq& M_\eta\,. \qquad (4.18)
\end{aligned}$$

We also have

$$\liminf_{\gamma\to 0}\liminf_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha})\geq M_\eta\,. \qquad (4.19)$$

Indeed, by definition, we have for all $(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,T]$

$$\begin{aligned}
u(x,t)-v(y,t)-\eta t-\frac{|x-y|^2}{\gamma^2}-\alpha(|x|^2+|y|^2)\leq&\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha})\\
\leq& u(x_{\gamma,\alpha},t_{\gamma,\alpha})-v(y_{\gamma,\alpha},t_{\gamma,\alpha})-\eta t_{\gamma,\alpha}.
\end{aligned}$$

We first take $\liminf\limits_{\alpha\to 0}$. We get

$$\begin{aligned}
u(x,t)-v(y,t)-\eta t-\frac{|x-y|^2}{\gamma^2}\leq&\liminf_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha})\\
\leq&\liminf_{\alpha\to 0}(u(x_{\gamma,\alpha},t_{\gamma,\alpha})-v(y_{\gamma,\alpha},t_{\gamma,\alpha})-\eta t_{\gamma,\alpha}).
\end{aligned} \qquad (4.20)$$

We then take $\limsup\limits_{|x-y|\to 0}$ and get

$$\sup_{t\in[0,T]}\limsup_{|x-y|\to 0}(u(x,t)-v(y,t)-\eta t)\leq\liminf_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha}),$$

and finally take $\liminf\limits_{\gamma\to 0}$ and get (4.19).

By combining (4.19) and (4.18) we get

$$\begin{aligned}
M_\eta\leq&\liminf_{\gamma\to 0}\liminf_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha})\\
\leq&\limsup_{\gamma\to 0}\limsup_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha})\\
\leq& M_\eta\,.
\end{aligned}$$

Therefore

$$\lim_{\gamma\to 0}\liminf_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha})=\lim_{\gamma\to 0}\limsup_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha})=M_\eta.$$

In a analogous way, we can deduce that (using (4.18) and (4.20))

$$\begin{aligned} M_\eta &= \lim_{\gamma\to 0}\liminf_{\alpha\to 0}\left(u(x_{\gamma,\alpha},t_{\gamma,\alpha}) - v(y_{\gamma,\alpha},t_{\gamma,\alpha}) - \eta t_{\gamma,\alpha}\right) \\ &= \lim_{\gamma\to 0}\limsup_{\alpha\to 0}\left(u(x_{\gamma,\alpha},t_{\gamma,\alpha}) - v(y_{\gamma,\alpha},t_{\gamma,\alpha}) - \eta t_{\gamma,\alpha}\right). \end{aligned}$$

We then get

$$\lim_{\gamma\to 0}\limsup_{\alpha\to 0}\left(\frac{|x_{\gamma,\alpha}-y_{\gamma,\alpha}|^2}{\gamma^2} + \alpha(|x_{\gamma,\alpha}|^2+|y_{\gamma,\alpha}|^2)\right) = 0\,. \tag{4.21}$$

Let us fix $\gamma_0 > 0$ such that for all $\gamma \le \gamma_0$, and for all α small enough we have

$$M_{\gamma,\alpha} = \Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha}) > \frac{M_\eta}{2}$$

and

$$\begin{aligned} &\limsup_{\alpha\to 0}\left(\|Dc_0^\varepsilon\|_1\left(2\frac{|x_{\gamma,\alpha}-y_{\gamma,\alpha}|^2}{\gamma^2} + \alpha|y_{\gamma,\alpha}|^2|x_{\gamma,\alpha}-y_{\gamma,\alpha}| + \alpha|x_{\gamma,\alpha}-y_{\gamma,\alpha}|\right)\right. \qquad (4.22)\\ &\left. + \frac{3}{2}\|c_0\|_1\alpha\left(2+|x_{\gamma,\alpha}|^2+|y_{\gamma,\alpha}|^2\right)\right) \le \frac{\eta}{3}. \end{aligned}$$

We claim that there is $\gamma \le \gamma_0$ such that for all α small enough $t_{\gamma,\alpha} > 0$. Indeed if, for all $\gamma \le \gamma_0$, there is $\alpha \in (0,\gamma)$ such that $t_{\gamma,\alpha} = 0$, then the following estimate holds

$$\begin{aligned} \frac{M_\eta}{2} < M_{\gamma,\alpha} &\le u(x_{\gamma,\alpha},0) - v(y_{\gamma,\alpha},0) \\ &\le u_0(x_{\gamma,\alpha}) - u_0(y_{\gamma,\alpha}) \\ &\le \|Du_0\||x_{\gamma,\alpha}-y_{\gamma,\alpha}| \\ &\le C\|Du_0\|\gamma, \end{aligned}$$

where we have use (4.17). Thus we get a contradiction if γ is small enough and we prove the claim. Hence, by *Lemma* 2.4 (if $t_{\gamma,\alpha} = T$, we use the fact that u (resp. v) is subsolution (resp. supersolution) in $(0,T]$, see *Lemma* 2.8 of Barles [18]), there are $(a,p) \in \bar{D}^+u(x_{\gamma,\alpha},t_{\gamma,\alpha})$ and $(b,q) \in \bar{D}^-v(y_{\gamma,\alpha},t_{\gamma,\alpha})$ such that

$$a - b = \eta\,;$$

$$p = 2\frac{(x_{\gamma,\alpha}-y_{\gamma,\alpha})}{\gamma^2} + 2\alpha x_{\gamma,\alpha}\,;$$

$$q = 2\frac{(x_{\gamma,\alpha}-y_{\gamma,\alpha})}{\gamma^2} - 2\alpha y_{\gamma,\alpha}\,;$$

$$a - \left((c_0^\varepsilon \star 1_{\{u(\cdot,t_{\gamma,\alpha})\ge u(x_{\gamma,\alpha},t_{\gamma,\alpha})\}})(x_{\gamma,\alpha}) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)|p| \le 0\,; \tag{4.23}$$

$$b - \left((c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})> v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}})(y_{\gamma,\alpha}) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)|q| \ge 0\,. \tag{4.24}$$

By subtracting (4.24) to (4.23) we get

$$\begin{aligned} \eta + &\left((c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}})(y_{\gamma,\alpha}) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right) |q| \\ - &\left((c_0^\varepsilon \star 1_{\{u(\cdot,t_{\gamma,\alpha})\geq u(x_{\gamma,\alpha},t_{\gamma,\alpha})\}})(x_{\gamma,\alpha}) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right) |p| \leq 0\,. \end{aligned} \tag{4.25}$$

From the fact that $\Phi_{\gamma,\alpha}(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) \geq \Phi_{\gamma,\alpha}(x, x, t_{\gamma,\alpha})$ it follows that

$$\begin{aligned} v(x,t_{\gamma,\alpha}) - v(y_{\gamma,\alpha},t_{\gamma,\alpha}) \;\; &\geq \;\; u(x,t_{\gamma,\alpha}) - u(x_{\gamma,\alpha},t_{\gamma,\alpha}) - 2\alpha|x|^2 \\ &+ \;\; \frac{|x_{\gamma,\alpha}-y_{\gamma,\alpha}|^2}{\gamma^2} + \alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2)\,. \end{aligned}$$

In particular from the above inequality we deduce that

$$\{u(\cdot,t_{\gamma,\alpha}) \geq u(x_{\gamma,\alpha},t_{\gamma,\alpha})\} \cap \{v(\cdot,t_{\gamma,\alpha}) \leq v(y_{\gamma,\alpha},t_{\gamma,\alpha})\} \subset \{|x|^2 \geq R^2_{\alpha,\gamma}\}\,,$$

where $R^2_{\alpha,\gamma} = \dfrac{1}{2\alpha}\left(\dfrac{|x_{\gamma,\alpha}-y_{\gamma,\alpha}|^2}{\gamma^2} + \alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2)\right)$.

Thus

$$\{u(\cdot,t_{\gamma,\alpha}) \geq u(x_{\gamma,\alpha},t_{\gamma,\alpha})\} \subset \{v(\cdot,t_{\gamma,\alpha}) > v(y_{\gamma,\alpha},t_{\gamma,\alpha})\} \cup \{|x|^2 \geq R^2_{\alpha,\gamma}\}\,. \tag{4.26}$$

Given $\gamma \leq \gamma_0$ the following two cases may occur.

Case 1. For all α small and for some $\tilde{C}_\gamma > 0$ we have

$$\frac{|x_{\gamma,\alpha}-y_{\gamma,\alpha}|^2}{\gamma^2} \geq \tilde{C}_\gamma^2\,.$$

In this case we have

$$\{|x - x_{\alpha,\gamma}| \geq R_{\alpha,\gamma}\} \subset \{|x| \geq \tilde{R}_{\alpha,\gamma}\}, \tag{4.27}$$

where $\tilde{R}_{\alpha,\gamma} = -|x_{\alpha,\gamma}| + R_{\alpha,\gamma}$ satisfies the following *lemma* which proof is postponed

Lemma 2.5. *We have the following estimate on $\tilde{R}_{\alpha,\gamma}$*

$$\tilde{R}_{\alpha,\gamma} = R_{\alpha,\gamma} - |x_{\alpha,\gamma}| \geq \frac{\tilde{C}_\gamma^2}{8\sqrt{C_0}\sqrt{\alpha}}.$$

Now let us choose $\delta > 0$ such that $\delta C_\gamma \leq \dfrac{\eta}{3}$, $C_\gamma > 0$ being an upper bound of $|p|, |q|$ depending on γ and independent of α small enough. Since $c_0^\varepsilon \in W^{1,1}(\mathbb{R}^n)$, we have for α small

$$\int_{B^c(0,\tilde{R}_{\alpha,\gamma})} c_0^\varepsilon(x)\,dx \leq \delta\,.$$

and

$$\begin{aligned}
&|(c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}})(x_{\gamma,\alpha}) - (c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}})(y_{\gamma,\alpha})| \\
\leq &||Dc_0^\varepsilon||_1 |x_{\gamma,\alpha} - y_{\gamma,\alpha}|.
\end{aligned}$$

By using the inclusions (4.26) and (4.27) from (4.25) we get

$$\begin{aligned}
0 \;&\geq\; \eta + |q| c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}}(y_{\gamma,\alpha}) - |p| c_0^\varepsilon \star 1_{\{u(\cdot,t_{\gamma,\alpha})\geq u(x_{\gamma,\alpha},t_{\gamma,\alpha})\}}(x_{\gamma,\alpha}) \\
&-\; \frac{1}{2}\int c_0^\varepsilon(x)dx(|q|-|p|) \\
&\geq\; \eta + |q| c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}}(y_{\gamma,\alpha}) - |p| c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}}(x_{\gamma,\alpha}) \\
&-\; |p| c_0^\varepsilon \star 1_{B^c(0,R_{\alpha,\gamma})}(x_{\gamma,\alpha}) - \frac{1}{2}||c_0^\varepsilon||_1(|p-q|) \qquad (4.28) \\
&\geq\; \eta - ||Dc_0^\varepsilon||_1 |x_{\gamma,\alpha} - y_{\gamma,\alpha}|(2\frac{|x_{\gamma,\alpha} - y_{\gamma,\alpha}|}{\gamma^2} + \alpha + \alpha|y_{\gamma,\alpha}|^2) \\
&-\; \frac{3}{2}||c_0||_1\{2\alpha + \alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2)\} - |p|\int_{B^c(0,\tilde{R}_{\alpha,\gamma})} c_0^\varepsilon(x)\,dx \\
&\geq\; \eta - ||Dc_0^\varepsilon||_1 |x_{\gamma,\alpha} - y_{\gamma,\alpha}|(2\frac{|x_{\gamma,\alpha} - y_{\gamma,\alpha}|}{\gamma^2} + \alpha + \alpha|y_{\gamma,\alpha}|^2) \\
&-\; \frac{3}{2}||c_0||_1\{2\alpha + \alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2)\} - \delta C_\gamma\,.
\end{aligned}$$

By taking in (4.28) the $\limsup_{\alpha\to 0}$ and using (4.22) we get a contradiction and we can conclude.

Case 2. There is a subsequence $\alpha_n > 0$ which we still denote by α such that

$$\frac{|x_{\gamma,\alpha} - y_{\gamma,\alpha}|^2}{\gamma^2} \to 0, \quad \text{as } \alpha \to 0\,.$$

In this case we have $\lim_{\alpha\to 0}|p| = 0$ and $\lim_{\alpha\to 0}|q| = 0$. On the other hand, from (4.25) we have the following estimate

$$0 \geq \eta - \frac{1}{2}\|c_0^\varepsilon\|_{L^1}(|p|+|q|). \tag{4.29}$$

By letting in (4.29) $\alpha \to 0$, we get a contradiction and we can conclude.

Proof of *Lemma* 2.5

By assumptions, we have

$$\frac{|x_{\gamma,\alpha} - y_{\gamma,\alpha}|^2}{\gamma^2} \geq \tilde{C}_\gamma^2.$$

We then deduce

$$\begin{aligned}
R_{\gamma,\alpha}^2 - |x_{\gamma,\alpha}|^2 \geq & \frac{\tilde{C}_\gamma^2}{2\alpha} - \frac{1}{2}(|x_{\gamma,\alpha}|^2 - |y_{\gamma,\alpha}|^2) \\
\geq & \frac{\tilde{C}_\gamma^2}{2\alpha} - \frac{1}{2}\left(|x_{\gamma,\alpha} - y_{\gamma,\alpha}|(|x_{\gamma,\alpha}| + |y_{\gamma,\alpha}|)\right) \\
\geq & \frac{\tilde{C}_\gamma^2}{2\alpha} - \frac{\gamma C_0}{\sqrt{\alpha}} \\
\geq & \frac{\tilde{C}_\gamma^2}{4\alpha} \quad \text{if } \alpha \text{ is small enough}
\end{aligned}$$

where we have used (4.17) for the third line. Moreover, using (4.17), we deduce

$$R_{\gamma,\alpha} \leq \sqrt{\frac{C_0}{\alpha}}$$

so

$$\tilde{R}_{\gamma,\alpha} = R_{\gamma,\alpha} - |x_{\gamma,\alpha}| = \frac{R_{\gamma,\alpha}^2 - |x_{\gamma,\alpha}|^2}{R_{\gamma,\alpha} + |x_{\gamma,\alpha}|} \geq \frac{\tilde{C}_\gamma^2}{4\alpha}\,\frac{1}{2\sqrt{\frac{C_0}{\alpha}}} \geq \frac{\tilde{C}_\gamma^2}{8\sqrt{C_0}\sqrt{\alpha}}.$$

This ends the proof of the *lemma.*

Theorem 2.6. (Existence and uniqueness for the ε-problem)
Let $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$ such that

$$|Du_0| < B_0 \quad \textit{in } \mathbb{R}^n \tag{4.30}$$

then there is a unique solution of (4.4).

Proof of Theorem 2.6
The uniqueness comes from the comparison principle and the existence is a straightforward consequence of Perron's method (see Da Lio, Kim, Slepčev [69] Theorem 1.2). Indeed, it suffices to remark that $u^{\pm}(x,t) = u_0(x) \pm \|c_0^\varepsilon\|_1 B_0 t$ are respectively super and subsolution of (4.4).

Proposition 2.7. (Lipschitz estimates in space)
The unique solution of (4.4) is Lipschitz continuous :

$$|Du^\varepsilon(\cdot,t)|_{L^\infty(\mathbb{R}^n)} \leq |Du^\varepsilon(\cdot,0)|_{L^\infty(\mathbb{R}^n)} \tag{4.31}$$

Proof of Proposition 2.7
The estimate (4.31) follows from the fact that the equation is invariant by space translation. Indeed, if we set $v(x,t) = u^\varepsilon(x+h,t) + |Du_0|_{L^\infty(\mathbb{R}^n)}|h|$, then it is easy to check that v is still a supersolution to the problem (4.4). Moreover, $v(x,0) \geq u(x,0)$, so, by comparison principle, $v(x,t) \geq u(x,t)$ for all $t \in [0,\infty)$ *i.e.* $u(x,t) - u(x+h,t) \leq |Du_0|_{L^\infty(\mathbb{R}^n)}|h|$. Using similarly a subsolution, we deduce the result.

3 The limit problem

Definition 3.1. (Viscosity sub/super/solution for mean curvature type motions)
A function $u^0 \in B_{\text{loc}}USC(\mathbb{R}^n \times [0,T])$ is a viscosity subsolution of (4.5)-(4.6)-(4.7) if it satisfies :

(i) $u^0(x,0) \leq u_0(x)$ *in* $\mathbb{R}^n$,

(ii) for every $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ *and for every function* $\Phi \in C^\infty(\mathbb{R}^n \times [0,\infty))$ *such that* $u^0 - \Phi$ *has a maximum at* (x_0,t_0)*, the following holds :*

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) + F_*\left(D\Phi, D^2\Phi\right) \leq 0. \tag{4.32}$$

A function $u^0 \in B_{\text{loc}}LSC(\mathbb{R}^n \times [0,T])$ is a viscosity supersolution of (4.5)-(4.6)-(4.7) if it satisfies :

(i) $u^0(x,0) \geq u_0(x)$ *in* $\mathbb{R}^n$,

(ii) for every $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ *and for every function* $\Phi \in C^\infty(\mathbb{R}^n \times [0,\infty))$ *such that* $u^0 - \Phi$ *has a minimum at* (x_0,t_0)*, the following holds :*

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) + F^*\left(D\Phi, D^2\Phi\right) \geq 0. \tag{4.33}$$

A continuous function u^0 is a viscosity solution of (4.5)-(4.6)-(4.7) if, and only if, it is a sub and a supersolution of (4.5)-(4.6)-(4.7).

This definition comes from the general definition of viscosity solution for discontinuous Hamiltonians first given by Ishii [113] (see also Crandall, Ishii, Lions [61]). We need an equivalent definition which eliminates, at least partially, the difficulty related to the fact that $D\Phi$ may be equal to zero.

Theorem 3.2. (Equivalent definition for mean curvature type motions)
We can replace in Definition 3.1 Condition (4.32) by

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) + F\left(D\Phi, D^2\Phi\right) \leq 0 \text{ if } D\Phi(x_0,t_0) \neq 0$$

or

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) \leq 0 \text{ if } D\Phi(x_0,t_0) = 0 \text{ and } D^2\Phi(x_0,t_0) = 0$$

and Condition (4.33) by

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) + F\left(D\Phi, D^2\Phi\right) \geq 0. \text{ if } D\Phi(x_0,t_0) \neq 0$$

or

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) \leq 0 \text{ if } D\Phi(x_0,t_0) = 0 \text{ and } D^2\Phi(x_0,t_0) = 0$$

and the definition remains equivalent.

The equivalence between these two definitions was first proved by Barles, Georgelin [22] for the isotropic mean curvature motion and their proof adapts here without any difficulty.

It is well known that this problem admits a unique viscosity solution. See for instance Bellettini, Novaga [33] [34], Chen, Giga, Goto [58] and Evans, Spruck [83]. Moreover, we have the following comparison principle :

Theorem 3.3. (Comparison principle for the limit problem)
If $u \in B_{\mathrm{loc}}USC(\mathbb{R}^n \times [0,T])$ is a subsolution of (4.5) and $v \in B_{\mathrm{loc}}LSC(\mathbb{R}^n \times [0,T])$ is a supersolution of (4.5) satisfying $u(x,0) \leq v(x,0)$ $\forall x \in \mathbb{R}^n$, then $u(x,t) \leq v(x,t)$ $\forall (x,t) \in \mathbb{R}^n \times (0,T)$.

In this Theorem, we do not need any assumption on the behaviour of the solution at infinity, since the equation is geometric.

4 Convergence of the velocity for a test function

4.1 Link with other works

In this subsection, we show in an heuristic way the links and the differences between our result and previous strongly related works such as Barles, Georgelin [22], Chambolle, Novaga [56], Evans [79], Ishii [116] and Ishii, Pires, Souganidis [120]. In particular, we explain the term $1/|\ln \varepsilon|$ in our scaling. We make the computation formally for a general kernel K_0 with the parabolic scaling, *i.e.*

$$K_0^\varepsilon(x) = \frac{1}{\varepsilon^{n+1}} K_0\left(\frac{x}{\varepsilon}\right).$$

We assume that K_0 is symmetric, *i.e.*, $K_0(-x) = K_0(x)$ and admits a moment of order two for every section, *i.e.*, for every $p \in \mathbf{S}^{n-1}$

$$\int_{p^\perp} K_0(x)|x|^2 < \infty \tag{4.34}$$

We want to show formally that for every regular function φ, the velocity

$$c^\varepsilon = K_0^\varepsilon \star 1_{\{\varphi \geq 0\}}(0) - \frac{1}{2}\int K_0^\varepsilon$$

converges to anisotropic mean curvature. To simplify the computation, we finally assume that the zero level set of φ is the graph of a function h, *i.e.*, more precisely that $\varphi(x',x_n) = h(x') - x_n$ where $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$ and $D_{x'}h(0) = 0$. We have

$$\begin{aligned}
c^\varepsilon =& \int_{\{x_n \le h(x')\}} K_0^\varepsilon - \int_{\{x_n \le 0\}} K_0^\varepsilon \\
=& \frac{1}{\varepsilon} \int_{\{0 \le x_n \le \frac{h(\varepsilon x')}{\varepsilon}\}} K_0(x)dx \\
\simeq& \int_{x' \in \mathbb{R}^{n-1}} \left(\frac{1}{\varepsilon} \int_0^{\frac{\varepsilon}{2} D^2 h(0)(x',x')} K_0(x', x_n)\, dx_n \right) dx' \\
\simeq& \int_{x' \in \mathbb{R}^{n-1}} \frac{1}{2} K_0(x',0) D^2h(0)(x',x')\, dx' \\
=& \text{trace}\left(A(p)(Id - p \otimes p)(D^2\varphi)\right) \qquad \text{with } |D\varphi(0)| = 1,
\end{aligned}$$

where $p = \dfrac{D\varphi}{|D\varphi|}$ and $A(p) = \int_{x \in p^\perp \simeq \mathbb{R}^{n-1}} \frac{1}{2} K_0(x)\ x \otimes x\ dx$. So, formally, if (4.34) holds, then the velocity c^ε converges to anisotropic mean curvature. Barles, Georgelin [22] and Evans [79] used this result to prove the convergence of the Merriman, Bence, Osher scheme [143]. For the proof, they used the kernel

$$K_0(x) = \frac{1}{(4\pi)^{n/2}} e^{-\frac{x^2}{4}}$$

which satisfied the assumptions. This result was then generalised for a threshold dynamics by Ishii [116] and Ishii, Pires, Souganidis [120] to more general kernels assuming also the symmetry of the kernel and (4.34). A by-product of our work shows that for general kernels, our limit mean curvature motion is of variational type (see Theorem 1.11).

The main difference in our case is that c_0 behaves like $\dfrac{1}{|x|^{n+1}}$ and so (4.34) does not hold. This explain the presence of the small term $\dfrac{1}{|\ln \varepsilon|}$ in our scaling which will compensate the bad decay of our kernel. Indeed to make a renormalization of the integral $\displaystyle\int_{x' \in \mathbb{R}^{n-1}} \frac{1}{2} K_0(x',0) D^2h(0)(x',x')\, dx'$ finite, we have to multiply by a term going to zero faster. We denote by $J(\varepsilon)$ this term (*i.e.*, we use the scaling $c_0^\varepsilon(x) = \frac{J(\varepsilon)}{\varepsilon^{n+1}} c_0\left(\frac{x}{\varepsilon}\right)$). Using the same computation as above, we obtain :

$$\begin{aligned}
c^\varepsilon =& J(\varepsilon) \frac{1}{\varepsilon} \int_{\{0 \le x_n \le \frac{h(\varepsilon x')}{\varepsilon}\} \cap \{|x'| \le \delta/\varepsilon\}} c_0(x) dx + J(\varepsilon)\mathcal{I}_1 \\
\simeq& J(\varepsilon) \int_{\{|x'| \le \delta/\varepsilon\}} \frac{1}{2} c_0(x',0) D^2h(0)(x',x')\, dx' + J(\varepsilon)\mathcal{I}_1
\end{aligned}$$

where

$$\mathcal{I}_1 = \frac{1}{\varepsilon}\int_{\{0\leq x_n \leq \frac{h(\varepsilon x')}{\varepsilon}\}\cap\{|x'|\geq\delta/\varepsilon\}} c_0(x)dx \leq \frac{1}{\varepsilon}\int_{\left(B_{\delta/\varepsilon}(0)\right)^c} c_0(x)dx$$

Using the particular form of c_0 for $|x|\geq 1$, we deduce that

$$\mathcal{I}_1 \leq \frac{1}{\varepsilon}\int_{\delta/\varepsilon}^{\infty} dr \frac{1}{r^2}\int_{\theta\in\mathbf{S}^{n-1}} d\theta\ g(\theta)$$

and so $\mathcal{I}_1$ is finite. This implies that the last term $J(\varepsilon)\mathcal{I}_1$ goes to zero as $\varepsilon\to 0$. We then decompose the first integral in two terms :

$$\begin{aligned}
&J(\varepsilon)\int_{\{|x'|\leq\delta/\varepsilon\}} \frac{1}{2}c_0(x',0)D^2h(0)(x',x')\,dx'\\
=&J(\varepsilon)\int_{|x'|\leq 1} \frac{1}{2}c_0(x',0)D^2h(0)(x',x')\,dx'\\
&+J(\varepsilon)\int_{|x'|\in(1,\delta/\varepsilon)} \frac{1}{2}c_0(x',0)D^2h(0)(x',x')\,dx'.
\end{aligned}$$

Since c_0 is bounded, we remark that the first term goes to zero as ε goes to zero. Then, the only interesting term is the second one. Using again the particular form of c_0 for $|x|\geq 1$, we deduce that

$$\begin{aligned}
&J(\varepsilon)\int_{|x'|\in(1,\delta/\varepsilon)} \frac{1}{2}c_0(x',0)D^2h(0)(x',x')\,dx'\\
=&J(\varepsilon)\int_{\theta\in\mathbf{S}^{n-2}} d\theta\frac{1}{2}D^2h(0)(\theta,\theta)g(\theta)\int_1^{\delta/\varepsilon}\frac{1}{r}dr\\
=&J(\varepsilon)(\ln\frac{\delta}{\varepsilon})\int_{\theta\in\mathbf{S}^{n-2}}\frac{1}{2}g(\theta)D^2h(0)(\theta,\theta)d\theta\\
=&J(\varepsilon)(\ln\frac{\delta}{\varepsilon})\ \mathrm{trace}\left(A(p)D^2\varphi\right).
\end{aligned}$$

So the correct scaling is to take $J(\varepsilon)=|\ln\varepsilon|$ and we finally obtain

$$c^\varepsilon \to \mathrm{trace}\left(A(p)D^2\varphi\right) \quad \text{when } |D\varphi(0)|=1.$$

4.2 Proof of convergence

In this section, we prove rigorously the convergence result for test functions.

Let us define (for $M=D^2\varphi$, $p=D\varphi$)

$$G(M,p)=\frac{-1}{|p|}F(M,p).$$

For a $n \times n$ matrix M we set the norm

$$|M| = \sup_{\xi \in B_1(0)} |M \cdot \xi|. \tag{4.35}$$

We define the modulus of continuity of the function g by

$$\omega_g(r) = \sup_{|\theta'-\theta| \le r, \quad \theta,\theta' \in \mathbf{S}^{n-1}} |g(\theta') - g(\theta)|.$$

Then we have the following fundamental estimate for test function independent on time :

Proposition 4.1. (Error estimate on the velocity for a test function) *Let us assume that $\varphi \in C^2(\mathbb{R}^n)$ and that $D\varphi(x_0) \neq 0$. For $c_0^\varepsilon(\cdot) = \dfrac{1}{\varepsilon^{n+1}|\ln \varepsilon|} \; c_0\left(\dfrac{\cdot}{\varepsilon}\right)$, let us define*

$$c^\varepsilon = (c_0^\varepsilon \star 1_{\{\varphi(\cdot) > \varphi(x_0)\}})(x_0) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon.$$

Let us call $b = |D\varphi(x_0)|$, and for any $a \geq |D^2\varphi|_{L^\infty(B_1(x_0))}$, let us introduce the relative modulus of continuity of $D^2\varphi$ at x_0, defined for $0 < r < 1$ by

$$\omega(r) = \begin{cases} \displaystyle\sup_{x \in B_r(x_0)} \frac{|D^2\varphi(x) - D^2\varphi(x_0)|}{a} & \text{if} \quad a \neq 0 \\ \\ 0 & \text{if} \quad a = 0. \end{cases}$$

We fix $\delta_1 \leq 1$ such that

$$\omega(\delta_1) \leq 1.$$

We define $\delta_0 = \min(1, \dfrac{b}{3a}, \delta_1)$. There exists a constant $C = C(n, \sup_{\mathbb{R}^n} c_0) > 0$ such that for $0 < \varepsilon < \delta$ with $0 < \delta \leq \delta_0/2$, we have

$$|c^\varepsilon - G(D^2\varphi(x_0), D\varphi(x_0))| \leq C \cdot e\,(\varepsilon, \delta, \delta_0)$$

with

$$e\,(\varepsilon, \delta, \delta_0) = \frac{1}{|\ln \varepsilon|}\left(\frac{1}{\delta} + \frac{1}{\delta_0}|\ln \delta|\right) + \frac{1}{\delta_0}\left(\omega_g\left(\frac{\delta}{\delta_0}\right) + \omega(2\delta) + \frac{\delta}{\delta_0}\right).$$

Before to prove proposition 4.1, let us give a corollary.

Corollary 4.2. (Convergence of the velocity for a test function) *Let us assume that $\varphi \in C^2(\mathbb{R}^n \times (0, +\infty))$ and that $D\varphi(x_0, t_0) \neq 0$. If $(x_\varepsilon, t_\varepsilon) \longrightarrow (x_0, t_0)$, then*

$$c^\varepsilon := \left((c_0^\varepsilon \star 1_{\{\varphi(\cdot, t_\varepsilon) > \varphi(x_\varepsilon, t_\varepsilon)\}})(x_\varepsilon, t_\varepsilon) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right) \quad \longrightarrow \quad G(D^2\varphi(x_0, t_0), D\varphi(x_0, t_0))$$

Proof of Corollary 4.2
This is a straightforward consequence of the fact that we can choose the relative modulus of continuity ω uniformly in a neighbourhood of (x_0, t_0) and then estimate $c^\varepsilon - G(D^2\varphi(x_\varepsilon, t_\varepsilon), D\varphi(x_\varepsilon, t_\varepsilon))$ using Proposition 4.1. We conclude choosing a suitable sequence $\delta = \delta(\varepsilon) = \frac{1}{\sqrt{|\ln \varepsilon|}}$.

Proof of Proposition 4.1
Up to change the coordinates, we can assume that $x_0 = 0$, $\varphi(x_0) = 0$, $D\varphi(x_0) = be_n$ with $b > 0$. We denote $x' = (x_1, ..., x_{n-1})$ a point of $\mathbb{R}^{n-1}$ and $x = (x', x_n) \in \mathbb{R}^n$. Then using the implicit function Theorem, we can assume that there exists a neighbourhood

$$Q_\delta = B_\delta^{n-1} \times (-\delta, \delta) \subset \mathbb{R}^n$$

of the origin such that the level set $\{\varphi = 0\}$ can be written

$$\{\varphi = 0\} \cap Q_\delta = \{(x', x_n) \in Q_\delta, \quad x_n = h(x')\}$$

for a suitable function $h \in C^2(B_\delta^{n-1}; (-\delta, \delta))$.

Then we have the following result which will be proved later :

Lemma 4.3. *Let δ_0 as defined in Proposition 4.1. For $0 < \delta \leq \delta_0/2$, we have*

$$\forall x' \in B_\delta^{n-1}, \quad (x', h(x')) \in Q_\delta$$

and

$$\left| \frac{h(x') - \frac{1}{2}D^2h(0) \cdot (x', x')}{|x'|^2} \right| \leq \frac{a}{b}\left(\omega(2\delta) + 8\frac{\delta}{\delta_0}\right).$$

Moreover

$$\frac{\partial^2 h}{\partial x_i \partial x_j}(0) = -\frac{1}{|D\varphi(0)|}\ \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(0), \qquad i, j = 1, ..., n-1$$

and

$$|h(x')| \leq 6\frac{a}{b}|x'|^2 \quad \text{for } x' \in B_\delta^{n-1}.$$

We have

$$\begin{aligned} c^\varepsilon &= (c_0^\varepsilon \star 1_{\{\varphi(\cdot)>0\}})(0) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \\ &= (c_0^\varepsilon \star 1_{\{\varphi(\cdot)>0\}})(0) - (c_0^\varepsilon \star 1_{\{x_n>0\}})(0) \\ &= -(c_0^\varepsilon \star 1_{\{\varphi(\cdot)\leq 0\}\cap\{x_n>0\}})(0) + (c_0^\varepsilon \star 1_{\{\varphi(\cdot)>0\}\cap\{x_n<0\}})(0) \\ &= -\{(I)_\varepsilon + (II)_\varepsilon\} \end{aligned}$$

where

$$(I)_\varepsilon = (c_0^\varepsilon \star 1_{Q_\delta \cap \{\varphi(\cdot)\le 0\}\cap\{x_n>0\}})(0) - (c_0^\varepsilon \star 1_{Q_\delta \cap \{\varphi(\cdot)> 0\}\cap\{x_n<0\}})(0)$$

and

$$(II)_\varepsilon = (c_0^\varepsilon \star 1_{(\mathbb{R}^n\setminus Q_\delta) \cap \{\varphi(\cdot)\le 0\}\cap\{x_n>0\}})(0) - (c_0^\varepsilon \star 1_{(\mathbb{R}^n\setminus Q_\delta) \cap \{\varphi(\cdot)> 0\}\cap\{x_n<0\}})(0).$$

We have for $\delta > \varepsilon$

$$\begin{aligned}
|(II)_\varepsilon| &\le \int_{\mathbb{R}^n\setminus Q_\delta} c_0^\varepsilon \\
&= \frac{1}{\varepsilon|\ln\varepsilon|}\int_{\mathbb{R}^n\setminus Q_{\frac{\delta}{\varepsilon}}} c_0 \\
&\le \frac{C}{\delta|\ln\varepsilon|}.
\end{aligned}$$

Let us now compute the term $(I)_\varepsilon$. We have for $\delta \le \delta_0/2$

$$(I)_\varepsilon = \int_{B_\delta^{n-1}} dx' \int_0^{h(x')} dx_n \, \frac{1}{\varepsilon^{n+1}|\ln\varepsilon|} \; c_0\left(\frac{x}{\varepsilon}\right).$$

Let us define (with $x=(x',x_n)=|x|\theta$)

$$(I)'_\varepsilon = \int_{B_\delta^{n-1}\setminus B_\varepsilon^{n-1}} dx' \int_0^{h(x')} dx_n \, \frac{1}{|\ln\varepsilon|} \; \frac{g(\theta)}{(|x'|^2+|x_n|^2)^{\frac{n+1}{2}}}.$$

Then

$$\begin{aligned}
|(I)_\varepsilon - (I)'_\varepsilon| &\le \int_{B_\varepsilon^{n-1}} dx' \int_0^{|h(x')|} dx_n \, \frac{1}{\varepsilon^{n+1}|\ln\varepsilon|} \; c_0\left(\frac{x}{\varepsilon}\right) \\
&\le \int_{B_\varepsilon^{n-1}} dx' \int_0^{6\frac{a}{b}|x'|^2} dx_n \, \frac{1}{\varepsilon^{n+1}|\ln\varepsilon|} \; \left(\sup_{\mathbb{R}^n} c_0\right) \\
&\le \frac{6a}{b|\ln\varepsilon|} \; \left(\frac{\sup_{\mathbb{R}^n} c_0}{n+1}\right)
\end{aligned}$$

where we have used the fact that $|h(x')| \le 6\frac{a}{b}|x'|^2$ for $|x'|\le\delta\le\delta_0/2$. We now compute $(I)'_\varepsilon$

$$\begin{aligned}
(I)'_\varepsilon &= \int_{B_\delta^{n-1}\setminus B_\varepsilon^{n-1}} dx' \int_0^{\frac{h(x')}{|x'|^2}} d\zeta \, \frac{|x'|^2}{|\ln\varepsilon|} \; \frac{g\left(\frac{\left(x',|x'|^2\zeta\right)}{\sqrt{|x'|^2+(|x'|^2\zeta)^2}}\right)}{\left(|x'|^2+(|x'|^2\zeta)^2\right)^{\frac{n+1}{2}}} \\
&= \int_{B_\delta^{n-1}\setminus B_\varepsilon^{n-1}} \frac{1}{|\ln\varepsilon|}\frac{dx'}{|x'|^{n-1}} \left(\int_0^{\frac{h(x')}{|x'|^2}} d\zeta \;\; \frac{g\left(\frac{\left(\frac{x'}{|x'|},|x'|\zeta\right)}{\sqrt{1+|x'|^2\zeta^2}}\right)}{\left(1+|x'|^2\zeta^2\right)^{\frac{n+1}{2}}} \right).
\end{aligned}$$

Let us define

$$\begin{aligned}
(I)''_\varepsilon &= \int_{B_\delta^{n-1}\setminus B_\varepsilon^{n-1}} \frac{1}{|\ln\varepsilon|}\frac{dx'}{|x'|^{n-1}} \left(\int_0^{\frac{1}{2}D^2h(0)\cdot\left(\frac{x'}{|x'|},\frac{x'}{|x'|}\right)} d\zeta \;\; g\left(\frac{x'}{|x'|}\right)\right) \\
&= \frac{\ln(\delta/\varepsilon)}{|\ln\varepsilon|}\left(\int_{\theta\in\mathbf{S}^{n-2}\subset\{x_n=0\}} d\theta \;\; \frac{1}{2}\, g(\theta)\cdot D^2h(0)\cdot(\theta,\theta)\right).
\end{aligned}$$

We define

$$(I)''_0 = \int_{\theta\in\mathbf{S}^{n-2}\subset\{x_n=0\}} d\theta \;\; \left(\frac{1}{2}\, g(\theta)\cdot D^2h(0)\cdot(\theta,\theta)\right)$$

i.e. we have from Lemma 4.3

$$\begin{aligned}
-(I)''_0 &= \int_{\theta\in\mathbf{S}^{n-2}\subset\{x_n=0\}} d\theta \;\; \left(\frac{1}{2}\, g(\theta)\cdot \frac{1}{|D\varphi(0)|}D^2\varphi(0)\cdot(\theta,\theta)\right) \\
&= \frac{1}{|D\varphi(0)|}\text{trace}\;\left(D^2\varphi(0)\cdot A\left(\frac{D\varphi(0)}{|D\varphi(0)|}\right)\right) \\
&= G(D^2\varphi(0), D\varphi(0))
\end{aligned}$$

where A is defined in (4.7).
Then we have

$$\begin{aligned}
|(I)''_\varepsilon - (I)''_0| &\le \frac{|\ln\delta|}{|\ln\varepsilon|}\left(\int_{\theta\in\mathbf{S}^{n-2}} d\theta\right)\;\left(\sup_{\mathbf{S}^{n-1}} g\right)\frac{1}{2}|D^2h(0)| \\
&\le \frac{|\ln\delta|}{|\ln\varepsilon|}(n-1)|B_1^{n-1}|\;\left(\sup_{\mathbf{S}^{n-1}} g\right)\frac{a}{2b}.
\end{aligned} \tag{4.36}$$

We now want to estimate the difference between $(I)'_\varepsilon$ and $(I)''_\varepsilon$. To this end, we first set $v = (\frac{x'}{|x'|}, |x'|\zeta)$, $\theta = \left(\frac{x'}{|x'|}, 0\right)$. Then using only the fact that $|\theta| = 1$ and the identity $\langle v-\theta, \theta\rangle = 0$ for the scalar product, we get $0 \leq |v| - 1 \leq |v-\theta|$, and $\left|\frac{v}{|v|} - \theta\right| \leq 2|v-\theta|$. We then estimate

$$\begin{aligned}
\left|\frac{g\left(\frac{v}{|v|}\right)}{|v|^{n+1}} - g(\theta)\right| &\leq \left|g\left(\frac{v}{|v|}\right) - g(\theta)\right| + g(\theta)\left(|v|^{n+1} - 1\right) \\
&\leq \omega_g\left(\left|\frac{v}{|v|} - \theta\right|\right) + \left(\sup_{\mathbf{S}^{n-1}} g\right)(n+1)|v|^n\left(|v| - 1\right) \\
&\leq \omega_g(2|v-\theta|) + \left(\sup_{\mathbf{S}^{n-1}} g\right)(n+1)\left(1 + |v-\theta|\right)^n |v-\theta| \\
&\leq \omega_g(2|v-\theta|) + \left(\sup_{\mathbf{S}^{n-1}} g\right)(n+1)2^n|v-\theta|
\end{aligned}$$

where for the last line, we have moreover used the fact that $|v-\theta| \leq 1$ when $|x'| \leq \delta$, $|\zeta| \leq \frac{1}{2}\left|D^2h(0)\cdot(\theta,\theta)\right| \leq \frac{a}{2b}$, and $\delta \leq \delta_0/2$.
Using $|v-\theta| \leq \frac{\delta a}{2b}$, we bound the last term by the quantity

$$e_1 = \omega_g\left(\frac{\delta a}{b}\right) + \left(\sup_{\mathbf{S}^{n-1}} g\right)(n+1)2^{n-1}\frac{\delta a}{b}.$$

Using Lemma 4.3 with

$$e_2 = \frac{a}{b}\left(\omega(2\delta) + 8\frac{\delta}{\delta_0}\right)$$

we then estimate

$$\begin{aligned}
|(I)''_\varepsilon - (I)'_\varepsilon| &\leq \int_{B_\delta^{n-1}\setminus B_\varepsilon^{n-1}} \frac{1}{|\ln\varepsilon|}\frac{dx'}{|x'|^{n-1}}\left\{e_2\cdot\left(\sup_{\mathbf{S}^{n-1}} g\right) + \frac{a}{2b}\cdot e_1\right\} \\
&\leq \frac{\ln(\delta/\varepsilon)}{|\ln\varepsilon|}\left(\int_{\theta\in\mathbf{S}^{n-2}} d\theta\right)\left\{e_2\cdot\left(\sup_{\mathbf{S}^{n-1}} g\right) + \frac{a}{2b}\cdot e_1\right\} \\
&\leq (n-1)|B_1^{n-1}|\left\{e_2\cdot\left(\sup_{\mathbf{S}^{n-1}} g\right) + \frac{a}{2b}\cdot e_1\right\}.
\end{aligned}$$

Finally we get (using $\dfrac{3a}{b} \le \dfrac{1}{\delta_0}$, $\delta \le \dfrac{\delta_0}{2} \le \dfrac{1}{2}$),

$$\begin{aligned}
|c^\varepsilon + (I)_0''| &\le |(II)_\varepsilon| + |(I)_\varepsilon - (I)_\varepsilon'| + |(I)_\varepsilon' - (I)_\varepsilon''| + |(I)_\varepsilon'' - (I)_0''| \\
&\le \frac{C}{|\ln\varepsilon|}\left(\frac{1}{\delta} + \frac{a}{b}|\ln\delta|\right) + C\left(\frac{a}{b}\omega_g\left(\frac{\delta a}{b}\right) + \frac{a}{b}\omega(2\delta) + \frac{a}{b}\frac{\delta}{\delta_0}\right) \\
&\le \frac{C}{|\ln\varepsilon|}\left(\frac{1}{\delta} + \frac{1}{\delta_0}|\ln\delta|\right) + C\frac{1}{\delta_0}\left(\omega_g\left(\frac{\delta}{\delta_0}\right) + \omega(2\delta) + \frac{\delta}{\delta_0}\right)
\end{aligned}$$

where the constant C only depends on the dimension n and c_0. More precisely we have $C = C\left(n,\ \int_{\mathbb{R}^n\setminus B_1} c_0,\ \sup_{\mathbb{R}^n} c_0,\ \sup_{\mathbf{S}^{n-1}} g\right) = C(n,\ \sup_{\mathbb{R}^n} c_0)$. This ends the proof of Proposition 4.1.

Proof of *Lemma* 4.3

Using the notations $\varphi_i = \dfrac{\partial\varphi}{\partial x_i}$, and $\varphi_{ij} = \dfrac{\partial^2\varphi}{\partial x_i \partial x_j}$, and taking the derivatives of the relation $\varphi(x', h(x')) = 0$, we get

$$\begin{cases}
h_i = -\dfrac{\varphi_i}{\varphi_n}, \quad i = 1, ..., n-1 \\[2ex]
h_{ij} = -\dfrac{1}{\varphi_n}\left(\varphi_{ij} + \varphi_{in}h_j + \varphi_{jn}h_i + \varphi_{nn}h_i h_j\right), \quad i, j = 1, ..., n-1.
\end{cases}$$

Now, by definition of a, we have $|D^2\varphi(x)| \le a$ for $x \in B_1$. Therefore for $0 < \delta \le 1$, we get

$$|D\varphi(x) - D\varphi(0)| \le a\delta \quad \text{for} \quad x \in B_\delta.$$

Let us define $\delta_0'' \in (0, +\infty]$ such that $a\delta_0'' = \dfrac{1}{2}|D\varphi(0)|$ and $\delta_0' = \min(1, \delta_0'')$. Then for $b = |D\varphi(0)| = \varphi_n(0)$ and $0 < \delta \le \delta_0'$ we get

$$a\delta_0 \le a\delta_0' \le \frac{b}{2} \le \varphi_n(x) \le |D\varphi(x)| \quad \text{for} \quad x \in B_\delta.$$

Using the elementary estimate : $\forall x \in B_\delta$,

$$\left|\frac{f(x)}{g(x)} - \frac{f(0)}{g(0)}\right| \le \frac{1}{g(0)(\inf_{B_\delta} g)}\left(|f(x) - f(0)|\, g(0) + |f(0)|\, |g(x) - g(0)|\right) \tag{4.37}$$

and using the fact that $\varphi_i(0) = 0$ for $i = 1, ..., n-1$, we get

$$Dh(0) = 0 \quad \text{and} \quad |Dh(x')| \le \frac{\delta}{\delta_0} \quad \text{for} \quad (x', h(x')) \in B_\delta.$$

Still using (4.37), we get for $(x', h(x')) \in B_\delta$ and $0 < \delta \le \delta_0$

$$\begin{aligned} |D^2h(x') - D^2h(0)| &\le \frac{2}{b^2}\left((a\ \omega(\delta)) \cdot b + a \cdot (a\delta)\right) + \frac{2}{b}\left(2a\frac{\delta}{\delta_0} + a\left(\frac{\delta}{\delta_0}\right)^2\right) \\ &\le \frac{2a}{b}\left(\omega(\delta) + 4\frac{\delta}{\delta_0}\right) \end{aligned}$$

where we have used the fact that $\dfrac{a}{b} \le \dfrac{1}{2\delta_0}$.
Using the Taylor formula with $h(0) = 0 = Dh(0)$, we get

$$\left|h(x') - \frac{1}{2}D^2h(0)\cdot(x', x')\right| \le \int_0^1 dt \int_0^t ds\ \left|D^2h(sx') - D^2h(0)\right| \cdot |x'|^2$$

and then for $(x', h(x')) \in B_\delta$

$$\left|\frac{h(x') - \frac{1}{2}D^2h(0)\cdot(x', x')}{|x'|^2}\right| \le \frac{a}{b}\left(\omega(\delta) + 4\frac{\delta}{\delta_0}\right) =: J(\delta).$$

Let us now assume that $0 < 2\delta \le \delta_0$. Then $Q_\delta = B_\delta^{n-1} \times (-\delta, \delta) \subset B_{2\delta}$, and for $x' \in B_\delta^{n-1}$ we have (using $|D^2h(0)| \le a/b$)

$$|h(x')| \le \delta^2\left(\frac{1}{2}\frac{a}{b} + J(2\delta)\right) < \delta$$

while $\omega(2\delta) \le 1$ and $\dfrac{6a\delta}{b} \le 1$. Therefore for $0 < 2\delta \le \delta_0$, we get that $(x', h(x')) \in Q_\delta \subset B_{2\delta}$ if $x' \in B_\delta^{n-1}$, and then

$$\left|\frac{h(x') - \frac{1}{2}D^2h(0)\cdot(x', x')}{|x'|^2}\right| \le \frac{a}{b}\left(\omega(2\delta) + 8\frac{\delta}{\delta_0}\right).$$

We then deduce

$$|h(x')| \le |x'|^2\frac{a}{b}\left(\omega(2\delta) + 8\frac{\delta}{\delta_0} + \frac{1}{2}\right) \le 6\frac{a}{b}|x'|^2,$$

which ends the proof of Lemma 4.3.

Finally, we give the following corollary which will be usefull to get *a priori* estimate at initial time :

Corollary 4.4. (Error estimate for a particular test function)
For $B, \eta > 0$, we consider the function

$$\varphi(x) = B\sqrt{\eta^2 + |x|^2}.$$

Then, there exists a constant $C' = C'(n, \sup_{\mathbb{R}^n} c_0) > 0$ such that for

$$c^\varepsilon(x) = (c_0^\varepsilon \star 1_{\{\varphi(\cdot)>\varphi(x)\}})(x) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon$$

we have pointwise, for $|x_0| \geq 6\sqrt{2}\,\varepsilon$ and $3 \geq \eta \geq 6\sqrt{2}\,\varepsilon$:

$$|c^\varepsilon(x_0)|D\varphi(x_0)| + F(D^2\varphi(x_0), D\varphi(x_0))| \leq C' \cdot \frac{B}{\eta}.$$

Proof of Corollary 4.4

Let us first remark that we do not change the result if we divide φ by B (because F is geometric), so we can assume that $B = 1$.

For all x, we have

$$D\varphi(x) = \frac{x}{\sqrt{\eta^2 + |x|^2}}, \qquad D^2\varphi(x) = \frac{1}{\sqrt{\eta^2 + |x|^2}}\,(Id - p(x) \otimes p(x))$$

where $p(x) = D\varphi(x),\ |p(x)| \leq 1,\ \forall x.$

We have, for all $x,\ x_0$:

$$\begin{aligned}
&D^2\varphi(x) - D^2\varphi(x_0)\\
=&\left(\frac{1}{\sqrt{\eta^2+|x|^2}} - \frac{1}{\sqrt{\eta^2+|x_0|^2}}\right)(Id - p(x)\otimes p(x))\\
&- \frac{1}{\sqrt{\eta^2+|x_0|^2}}\,(p(x)\otimes(p(x)-p(x_0)) + (p(x)-p(x_0))\otimes p(x_0)).
\end{aligned}$$

Moreover, the following holds

$$\begin{aligned}
\left|\frac{1}{\sqrt{\eta^2+|x|^2}} - \frac{1}{\sqrt{\eta^2+|x_0|^2}}\right| \leq& \frac{\left|\sqrt{\eta^2+|x|^2} - \sqrt{\eta^2+|x_0|^2}\right|}{\eta^2}\\
\leq& \frac{||x|^2 - |x_0|^2|}{\eta^2\left(\sqrt{\eta^2+|x|^2} + \sqrt{\eta^2+|x_0|^2}\right)}\\
\leq& \frac{||x| - |x_0||\,(|x| + |x_0|)}{\eta^2(|x| + |x_0|)}\\
\leq& \frac{|x - x_0|}{\eta^2}
\end{aligned}$$

and, using the bound $|D^2\varphi| \leq \frac{1}{\eta}$, we get

$$|p(x) - p(x_0)| = |D\varphi(x) - D\varphi(x_0)| \leq \frac{|x - x_0|}{\eta}.$$

We set $a = \frac{1}{\eta} \geq |D^2\varphi|$. We then get, with the notation of Proposition 4.1 :

$$\frac{|D^2\varphi(x) - D^2\varphi(x_0)|}{a} \leq \frac{3|x - x_0|}{\eta}, \qquad \omega(r) \leq \frac{3r}{\eta}.$$

Then we can apply Proposition 4.1 with $a = \dfrac{1}{\eta}$, $b = |D\varphi(x_0)| > 0$, $\delta_1 = \frac{\eta}{3}$,
$2\delta = \delta_0 = \min\left(\dfrac{b}{3a},\ \delta_1\right) = \dfrac{b}{3a}$ (because $b \leq 1$).
We deduce that there exists a constant $C' = C'(n, \sup_{\mathbb{R}^n} c_0) > 0$ such that for $\delta > \varepsilon > 0$:

$$|c^\varepsilon(x_0)|D\varphi(x_0)| + F(D^2\varphi(x_0), D\varphi(x_0))| \leq C'\left(\frac{1}{\eta} + \frac{1}{\eta|\ln\varepsilon|}\right) \leq \frac{C'}{\eta}.$$

Moreover, the condition $\delta > \varepsilon$ is equivalent to $b > \dfrac{6\varepsilon}{\eta}$. We then deduce conditions on $|x_0|$ and η :

1. If $|x_0| \leq \eta$, then $b \geq \dfrac{|x_0|}{\sqrt{2}\eta}$ and it suffices to take $|x_0| > 6\sqrt{2}\varepsilon$.
2. If $|x_0| \geq \eta$, then $b \geq \dfrac{1}{\sqrt{2}}$ and it suffices to take $\eta > 6\sqrt{2}\varepsilon$.

5 *A priori* estimate at initial time

Proposition 5.1. (Modulus of continuity in time)
There is a constant $C'' = C''(n, \sup_{\mathbb{R}^n} c_0) > 0$ such that for every $x_0 \in \mathbb{R}^n$ and $t > 0$ we have, for $\eta > 6\sqrt{2}\varepsilon$, and $\varepsilon \in (0, 1/2)$

$$|u^\varepsilon(x_0, t) - u_0(x_0)| \leq |Du_0|_{L^\infty(\mathbb{R}^n)} \cdot \left\{\eta + t \cdot \frac{C''}{\eta}\right\}.$$

Remark 5.2. *Since $|Du^\varepsilon(\cdot, t)|_{L^\infty(\mathbb{R}^n)} \leq |Du_0|_{L^\infty(\mathbb{R}^n)}$ (see Proposition 2.7), we also have, for $\varepsilon \in (0, 1/2)$ and $\forall \eta > 6\sqrt{2}\varepsilon$*

$$|u^\varepsilon(x_0, t + s) - u^\varepsilon(x_0, s)| \leq |Du_0|_{L^\infty(\mathbb{R}^n)} \cdot \left\{\eta + t \cdot \frac{C''}{\eta}\right\}.$$

Proof of Proposition 5.1
We consider the following function

$$\varphi(x, t) = B_0\sqrt{\eta^2 + |x|^2} + u_0(x_0) - B_0|x_0| + L \cdot t$$

with $B_0 = |Du_0|_{L^\infty(\mathbb{R}^n)}$ and L that will be precised later. To prove the result, it suffices to show that for $L = C''\frac{B_0}{\eta}$ and C'' large enough, then φ is a supersolution of (4.4). Indeed, by comparison principle (Theorem 2.2), we will then have

$$u^\varepsilon(x_0,t) \le \varphi(x_0,t) \le B_0\left(\eta + t\cdot\frac{C''}{\eta}\right) + u_0(x_0).$$

Let $(x,t) \in \mathbb{R}^n \times (0,\infty)$. To prove that φ is a supersolution of (4.4) at (x,t), since φ is $C^\infty(\mathbb{R}^n \times (0,\infty))$, it suffices to show that φ satisfies the equation pointwise, *i.e.*

$$\varphi_t(x,t) \ge c^\varepsilon |D\varphi(x,t)|.$$

The proof is now decomposed into two cases :

1. $|x| \le 6\sqrt{2}\,\varepsilon$. In this case, we have

$$c^\varepsilon|D\varphi(x,t)| \le \frac{\|c_0\|_{L^1}}{\varepsilon|\ln\varepsilon|}\,\frac{B_0|x|}{\eta} \le \frac{6\sqrt{2}\|c_0\|_{L^1}B_0}{|\ln\varepsilon|\eta}.$$

   So it suffices to take $L \ge \dfrac{6\sqrt{2}\|c_0\|_{L^1}}{\left|\ln\frac{1}{2}\right|}\dfrac{B_0}{\eta}$.

2. $|x| \ge 6\sqrt{2}\,\varepsilon$. In this case we will show that φ is a supersolution of

$$\varphi_t + F(D^2\varphi, D\varphi) \ge L - L_0 \tag{4.38}$$

   for $L_0 = \dfrac{B_0}{\eta}\displaystyle\sup_{q\in\mathbf{S}^{n-1}} \operatorname{trace}\left(A\left(\frac{q}{|q|}\right)\right)$ and then we will use Corollary 4.4.

   We set $M = D^2\varphi$. We can choose a basis such that

$$A\left(\frac{p}{|p|}\right) = \begin{pmatrix} A_{n-1}\left(\frac{p}{|p|}\right) & 0 \\ 0 & 0 \end{pmatrix},$$

   where the last vector of the basis is $\frac{p}{|p|}$, with $p = D\varphi$. We set

$$M = B_0\begin{pmatrix} M_{n-1} & M_n \\ {}^tM_n & M_{nn} \end{pmatrix},$$

   where $M_{n-1} = \dfrac{1}{\sqrt{\eta^2+|x|^2}}Id$, M_n is a vector and $M_{nn} = \dfrac{\eta^2}{(\eta^2+|x|^2)^{\frac{3}{2}}}$. We then deduce that

$$\operatorname{trace}\left(M.A\left(\frac{p}{|p|}\right)\right) = \frac{B_0}{\sqrt{\eta^2+|x|^2}}\operatorname{trace}(A_{n-1}) \le \frac{B_0}{\eta}\operatorname{trace}\left(\mathrm{A}\left(\frac{\mathrm{p}}{|\mathrm{p}|}\right)\right).$$

We then deduce that

$$\begin{aligned}\varphi_t(x,t) + F(D^2\varphi, D\varphi) =& L - \text{trace}\left(MA\left(\frac{p}{|p|}\right)\right)\\ \geq& L - \frac{B_0}{\eta}\sup_{\mathbf{S}^{n-1}} \text{trace}\left(A\left(\frac{p}{|p|}\right)\right)\\ =& L - L_0.\end{aligned}$$

We now prove that φ is a supersolution of (4.4), *i.e.*

$$\varphi_t(x,t) \geq c^\varepsilon |D\varphi(x,t)|,$$

where $c^\varepsilon = (c_0^\varepsilon \star 1_{\{\varphi(\cdot,t)>\varphi(x,t)\}})(x,t) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon$. We have pointwise

$$\begin{aligned}\varphi_t \geq& - F(D^2\varphi, D\varphi) + L - L_0\\ \geq& \, c^\varepsilon|D\varphi| + L - L_0 - F(D^2\varphi, D\varphi) - c^\varepsilon|D\varphi|\\ \geq& \, c^\varepsilon|D\varphi| + L - L_0 - C' \cdot \frac{B_0}{\eta},\end{aligned}$$

where we have used Corollary 4.4. It is sufficient to take

$$L \geq \frac{B_0 C''}{\eta}$$

with

$$C'' = \sup_{q\in\mathbf{S}^{n-1}} \text{trace}\left(A\left(\frac{q}{|q|}\right)\right) + C' + \frac{6\sqrt{2}|c_0|_{L^1}}{\ln\frac{1}{2}}. \tag{4.39}$$

Moreover, trace(A) is bounded by $|g|_{L^\infty}$ which is controlled by $|c_0|_{L^\infty}$ (since $c_0(x) = g(x)$ if $|x| = 1$). So, by Corollary 4.4, $C'' = C''(n, \sup_{\mathbb{R}^n} c_0)$.

Using similarly a subsolution, we deduce the result. This ends the proof of the proposition.

Corollary 5.3. *There exists a constant $C > 0$ such that for any $\varepsilon \leq 1/2$, the solution u^ε of (4.4) satisfies :*

$$|u^\varepsilon(x,t+h) - u^\varepsilon(x,t)| \leq C|Du_0|_{L^\infty(\mathbb{R}^n)}\sqrt{h}, \quad \forall x \in \mathbb{R}^n,\ \forall t \geq 0,\ h \in [0,h_0],$$

with $h_0 = h_0(n, \sup_{\mathbb{R}^n} C_0) > 0$.

Proof of Corollary 5.3

We can optimise the estimate of Remark 5.2 to obtain, if $\eta = \sqrt{tC''} \leq 3$:

$$|u^\varepsilon(x_0,t+s) - u^\varepsilon(x_0,s)| \leq 2|Du_0|_{L^\infty(\mathbb{R}^n)} \cdot \sqrt{C''}\sqrt{t} \quad \text{if } \sqrt{t} > \frac{6\sqrt{2}\varepsilon}{\sqrt{C''}}.$$

Moreover, for all ε, we have :

$$|u^\varepsilon(x_0,t+s)-u^\varepsilon(x_0,s)|\le \frac{t}{\varepsilon|\ln\varepsilon|}|c_0|_{L^1}|Du_0|_{L^\infty(\mathbb{R}^n)}.$$

But, for $\sqrt{t}\le \frac{6\sqrt{2}\varepsilon}{\sqrt{C''}}$ and $\varepsilon\le\frac{1}{2}$, the following holds (using (4.39))

$$\frac{t}{\varepsilon|\ln\varepsilon|}|c_0|_{L^1}|Du_0|_{L^\infty(\mathbb{R}^n)}\le |Du_0|_{L^\infty(\mathbb{R}^n)}\sqrt{t}\frac{6\sqrt{2}}{\sqrt{C''}}\frac{|c_0|_{L^1}}{|\ln\frac{1}{2}|}\le |Du_0|_{L^\infty(\mathbb{R}^n)}\sqrt{t}\sqrt{C''},$$

so, $\forall t\le \dfrac{9}{C''}$, s, we have

$$|u^\varepsilon(x_0,t+s)-u^\varepsilon(x_0,s)|\le 2|Du_0|_{L^\infty(\mathbb{R}^n)}\sqrt{C''}\sqrt{t}.$$

Iterating the estimate on time intervals of length T satisfying $\sqrt{TC''}\le 3$, we get the result. This ends the proof of the corollary.

6 Proof of the convergence Theorem

Proof of Theorem 1.4

We use the half-relaxed limits introduced by Barles, Perthame [26], and defined by :

$$\overline{u}(x,t)=\limsup_{\varepsilon\to 0,\ y\to x,\ s\to t} u^\varepsilon(y,s)$$

and

$$\underline{u}(x,t)=\liminf_{\varepsilon\to 0,\ y\to x,\ s\to t} u^\varepsilon(y,s).$$

We will show that $\overline{u}$ (resp. $\underline{u}$) is a viscosity subsolution (resp. supersolution) of (4.5)-(4.6)-(4.7).

We argue by contradiction. Assume that there exists $\phi\in C^2$ such that $\overline{u}-\phi$ reaches a global strict maximum at (x_0,t_0) and such that

$$\phi_t(x_0,t_0)+F_*(D^2\phi,D\phi)=\theta>0. \tag{4.40}$$

Two cases may occur :

1. $|D\phi(x_0,t_0)|\ne 0$.

   We then deduce that there exists $(x_\varepsilon,t_\varepsilon)\to(x_0,t_0)$ such that $u^\varepsilon-\phi$ reaches a maximum at $(x_\varepsilon,t_\varepsilon)$. Using the fact that u^ε has linear growth, we can assume (by adding a term like $|x-x_0|^4+|t-t_0|^2$ to ϕ if necessary) that this maximum is global. Since u^ε is a solution of (4.4), the following holds :

$$\phi_t(x_\varepsilon,t_\varepsilon)\le\left((c_0^\varepsilon\star 1_{\{u^\varepsilon(\cdot,t_\varepsilon)\ge u^\varepsilon(x_\varepsilon,t_\varepsilon)\}})(x_\varepsilon)-\frac{1}{2}\int_{\mathbb{R}^n}c_0^\varepsilon\right)|D\phi(x_\varepsilon,t_\varepsilon)|.$$

Moreover, $\forall x \neq x_\varepsilon$, we have $u^\varepsilon(x,t_\varepsilon) - \phi(x,t_\varepsilon) < u^\varepsilon(x_\varepsilon,t_\varepsilon) - \phi(x_\varepsilon,t_\varepsilon)$. So $\{u^\varepsilon(\cdot,t_\varepsilon) \geq u^\varepsilon(x_\varepsilon,t_\varepsilon)\} \subset \{\phi(\cdot,t_\varepsilon) > \phi(x_\varepsilon,t_\varepsilon)\} \cup \{x_\varepsilon\}$. We then deduce :

$$\phi_t(x_\varepsilon,t_\varepsilon) \leq \left((c_0^\varepsilon \star 1_{\{\phi(\cdot,t_\varepsilon)>\phi(x_\varepsilon,t_\varepsilon)\}})(x_\varepsilon) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right) |D\phi(x_\varepsilon,t_\varepsilon)|.$$

We can use Corollary 4.2 and pass to the limit in ε. The following holds :

$$\begin{aligned}\phi_t(x_0,t_0) \leq& G(D^2\phi(x_0,t_0), D\phi(x_0,t_0))|D\phi(x_0,t_0)| \\ =& -F(D^2\phi(x_0,t_0), D\phi(x_0,t_0)),\end{aligned}$$

what contradicts (4.40) (since $F(M,p) = F_*(M,p)$ for $p \neq 0$).

2. $|D\phi(x_0,t_0)| = 0$ and $|D^2\phi(x_0,t_0)| = 0$. As in the first case, there exist $(x_\varepsilon,t_\varepsilon) \to (x_0,t_0)$ such that $u^\varepsilon - \phi$ reaches a global maximum at $(x_\varepsilon,t_\varepsilon)$ (up to add a term like $|x-x_0|^4 + |t-t_0|^2$ to ϕ if necessary). We set

$$c^\varepsilon[\phi](x_\varepsilon,t_\varepsilon) = \left((c_0^\varepsilon \star 1_{\{\phi(\cdot,t_\varepsilon)>\phi(x_\varepsilon,t_\varepsilon)\}})(x_\varepsilon) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right).$$

By assumptions, for all $\eta > 0$, there exists $r > 0$ such that

$$|D^2\phi(x,t)| \leq \eta \quad \text{if } (x,t) \in Q_{2r}(x_0,t_0)$$

where $Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r, t_0 + r)$.
Subcase A : $|D\phi(x_\varepsilon,t_\varepsilon)| > 12\varepsilon\eta r$.
We set

$$\mathcal{I}(x,t) = c^\varepsilon[\phi](x,t)|D\phi| + F_*(D^2\phi, D\phi)$$

and

$$\phi^r(x,t) = \frac{1}{r^2}\phi(x_0 + rx, t_0 + rt).$$

Straightforward computations give with $\bar{x}_\varepsilon = \dfrac{x_\varepsilon}{r}$, $\bar{t}_\varepsilon = \dfrac{t_\varepsilon}{r}$

$$\begin{aligned}\mathcal{I}(x_\varepsilon,t_\varepsilon) =& F_*(D^2\phi^r, D\phi^r) + \frac{|\ln \frac{\varepsilon}{r}|}{|\ln \varepsilon|}|D\phi^r|c^{\varepsilon/r}[\phi^r](\bar{x}_\varepsilon,\bar{t}_\varepsilon) \\ =& F_*(D^2\phi^r, D\phi^r) + \left(1 - \frac{|\ln r|}{|\ln \varepsilon|}\right)|D\phi^r|c^{\varepsilon/r}[\phi^r](\bar{x}_\varepsilon,\bar{t}_\varepsilon) \\ =& \left(1 - \frac{|\ln r|}{|\ln \varepsilon|}\right)\mathcal{I}_1 + \mathcal{I}_2\end{aligned}$$

where

$$\mathcal{I}_1 = F_*(D^2\phi^r, D\phi^r) + |D\phi^r|c^{\varepsilon/r}[\phi^r](\bar{x}_\varepsilon,\bar{t}_\varepsilon) \quad \text{and} \quad \mathcal{I}_2 = \frac{|\ln r|}{|\ln \varepsilon|}F_*(D^2\phi^r, D\phi^r).$$

We can then apply Proposition 4.1 to $\mathcal{I}_1$ with

$$a = 2\eta \geq |D^2\phi^r|, \quad b = |D\phi^r(\bar{x}_\varepsilon, \bar{t}_\varepsilon)| \to 0, \quad 2\delta = \delta_0 = \frac{b}{6\eta}$$

and get (with an abuse of notation for a generic constant C)

$$\begin{aligned}|\mathcal{I}_1| \leq & Cb\left\{\frac{1}{\delta_0} + \frac{1}{\delta_0}\frac{|\ln\delta|}{|\ln\varepsilon|} + \frac{1}{\delta_0|\ln\varepsilon|}\right\}\\ \leq & C\left\{\eta + \eta + \frac{\eta}{|\ln\varepsilon|}\right\}\\ \leq & C\eta\end{aligned}$$

for ε small enough to get b small enough. We then deduce that for ε small enough we have

$$|\mathcal{I}(x_\varepsilon, t_\varepsilon)| \leq C\eta$$

and so

$$\begin{aligned}\phi_t(x_\varepsilon, t_\varepsilon) + F_*(D^2\phi, D\phi) = & \phi_t(x_\varepsilon, t_\varepsilon) - c^\varepsilon[\phi](x_\varepsilon, t_\varepsilon) + F_*(D^2\phi, D\phi) + c^\varepsilon[\phi](x_\varepsilon, t_\varepsilon)\\ \leq & |\mathcal{I}(x_\varepsilon, t_\varepsilon)|\\ \leq & C\eta.\end{aligned}$$

Subcase B : $|D\phi(x_\varepsilon, t_\varepsilon)| \leq 12\varepsilon\eta r$.

Then we have

$$c^\varepsilon[\phi](x_\varepsilon, t_\varepsilon)|D\phi| \leq \frac{|c_0|_{L^1}}{\varepsilon|\ln\varepsilon|}|D\phi| \leq \frac{12\eta r}{|\ln\varepsilon|}|c_0|_{L^1}$$

and using $F_*(D^2\phi, D\phi) = 0$ in (x_0, t_0), we also deduce that for ε small enough we have

$$\phi_t(x_\varepsilon, t_\varepsilon) + F_*(D^2\phi, D\phi) \leq C\eta.$$

Sending $\varepsilon \to 0$, we get

$$\phi(x_0, t_0) + F_*(D^2\phi, D\phi) \leq C\eta$$

and so

$$\theta \leq C\eta$$

which is a contradiction for η small enough.

Finally, we have shown that $\overline{u}$ is a subsolution. The proof to show that $\underline{u}$ is a supersolution is exactly the same.

Moreover, by corollary 5.3, we have :

$$|u^\varepsilon(\cdot, t) - u_0(\cdot)| \leq C|t|^{\frac{1}{2}}, \quad \text{for } 0 \leq t \leq 1$$

where C is a constant which depends only on n, $\sup_{\mathbb{R}^n} c_0$ and $|Du_0|_{L^\infty}$. So $\overline{u}(\cdot,0) = \underline{u}(\cdot,0) = u_0(\cdot)$. Since $\overline{u}$ is a subsolution and $\underline{u}$ is a supersolution, we deduce by the comparison principle (Theorem 3.3) that

$$\overline{u}(x,t) \leq \underline{u}(x,t) \quad \forall (x,t)$$

and so $\overline{u} = \underline{u} = u^0$, *i.e.* u^ε converges locally uniformly on compact sets of $\mathbb{R}^n \times [0,\infty)$ to u^0 which is the unique solution of (4.5)-(4.6)-(4.7). This ends the proof of the Theorem.

7 Proof of Theorem 1.7

We now prove Theorem 1.7. We need the following proposition :

Proposition 7.1. (The matrix A is an hessian)
Let $n \geq 2$. Let $g \in C^0(\mathbb{R}^n \backslash \{0\})$ such that $g(\lambda p) = \dfrac{g(p)}{|\lambda|^{n+1}}$. We set

$$A\left(\frac{p}{|p|}\right) = \int_{\theta \in \mathbf{S}^{n-2} = \mathbf{S}^{n-1} \cap \left\{\langle x, \frac{p}{|p|}\rangle = 0\right\}} \left(\frac{1}{2} g(\theta)\theta \otimes \theta\right) \, d\theta$$

with $A(\lambda p) = \dfrac{1}{|\lambda|} A(p)$ for $\lambda \neq 0$. Then, the function $G := -\dfrac{1}{2\pi}\mathcal{F}(L_g)$ (where L_g and the Fourier transform are given in Definition 1.6) is such that $G(\lambda p) = |\lambda| G(p)$ and satisfies

$$A(p) = D^2 G(p).$$

For the proof of this proposition, we will need the following *lemma*

Lemma 7.2. (The Curl of the matrix A)
Under the assumptions of Proposition 7.1, the Curl of A, defined by

$$Curl(A) = (\partial_k A_{ij} - \partial_i A_{kj})_{i,j,k}$$

is zero, and there exists a distribution Φ such that $A(p) = D^2\Phi(p)$. Moreover, $\Phi \in C^0(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \backslash \{0\})$, and Φ is unique if we assume $\Phi(-p) = \Phi(p)$ and $\Phi(0) = 0$. We then have $\Phi(\lambda p) = |\lambda|\Phi(p)$, $\forall \lambda \in \mathbb{R}\backslash\{0\}$, $\forall p \in \mathbb{R}^n$.

Proof of *Lemma* 7.2
In this proof, we denote by $e \cdot f$ the scalar product between e and f.
First, we compute $\partial_k A_{ij}(p)$ for $p \neq 0$ and where $g \in C^1(\mathbb{R}^n \backslash \{0\})$ and ∂_k indicates the derivation in the direction e_k. Because we will compute the Curl of the row vectors of the matrix, it is sufficient to choose an orthonormal basis $(e_1, ..., e_n)$ such that e_1 is parallel to p. Two cases may occur :

1. e_k is parallell to p ($e_k \parallel p$). Then,

$$\partial_k A_{ij}(p) = -\frac{p \cdot e_k}{|p|^2} A_{ij}(p).$$

2. e_k is perpendicular to p ($e_k \perp p$). In this case (see Figure 4.1), we have to consider variations at the first order of the integral defining $A(p)$ for $\theta \in p^\perp \cap \mathbf{S}^{n-1}$ to $\theta \in (p + \varepsilon e_k)^\perp \cap \mathbf{S}^{n-1}$ for ε arbitrarily small. Let us consider a unit vector $\theta \in p^\perp \cap \mathbf{S}^{n-1}$ that we write

$$\theta = (\cos\alpha)e' + (\sin\alpha)e_k$$

with $\sin\alpha = \theta \cdot e_k$ and $e' \perp p$, $e' \perp e_k$. At the first order, this vector becomes (by infinitesimal rotation)

$$(\cos\alpha)e' + (\sin\alpha)(e_k + \varepsilon q) \in (p + \varepsilon e_k)^\perp \cap \mathbf{S}^{n-1},$$

where we define

$$q = \frac{-p}{|p|}, \quad \bar{g}(\theta) = g(\theta)\theta \otimes \theta.$$

Then the following holds

$$\partial_k A_{ij}(p) = \frac{1}{|p|^2} \int_{\mathbf{S}^{n-1} \cap p^\perp} d\theta \, \frac{1}{2} q \cdot \nabla \bar{g}(\theta)(\theta \cdot e_k)(e_i, e_j).$$

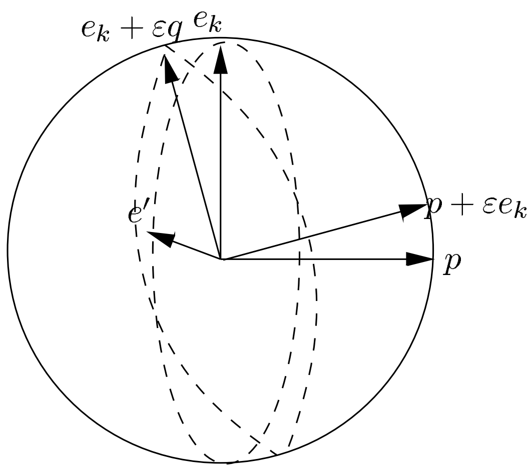

FIG. 4.1 – Computation at the first order of $\partial_k A_{ij}(p)$, case $|p| = 1$.

Moreover,

$$\begin{aligned}
&q \cdot \nabla \bar{g}(\theta)(\theta \cdot e_k)(e_i, e_j))\\
=&(q \cdot \nabla g(\theta))(\theta \cdot e_k)(\theta \cdot e_i)(\theta \cdot e_j) + g(\theta)(\theta \cdot e_k)\left(q \cdot e_i(\theta \cdot e_j) + (\theta \cdot e_i) q \cdot e_j\right)\\
=&q \cdot \nabla g(\theta)(\theta \cdot e_k)(\theta \cdot e_i)(\theta \cdot e_j) + (q \cdot e_i)\bar{g}(\theta)(e_k, e_j) + (q \cdot e_j)\bar{g}(\theta)(e_k, e_i).
\end{aligned}$$

We are now able to compute the Curl of A. To do this we separate in several cases :

1. $e_k,\ e_i,\ e_j \parallel p$. Then, $A_{ij}(p) = A_{kj}(p) = 0$ and so $\partial_k A_{ij} - \partial_i A_{kj} = 0$.
2. $e_k,\ e_i \parallel p,\ e_j \perp p$. In the same way, $\partial_k A_{ij} - \partial_i A_{kj} = 0$
3. $e_k,\ e_j \parallel p,\ e_i \perp p$. Then $\partial_k A_{ij} = 0$ and $\partial_i A_{kj} = 0$ (since $\theta \cdot e_j = \theta \cdot e_k = 0$).
4. $e_k \perp p,\ e_i,\ e_j \parallel p$. It is the same case as 3.
5. $e_k \parallel p,\ e_i,\ e_j \perp p$. Then $\partial_k A_{ij} = -\frac{1}{|p|} A_{ij}$ (if $e_k = \frac{p}{|p|}$)and

$$\begin{aligned}
\partial_i A_{kj} =& \frac{1}{|p|^2} \int_{\mathbf{S}^{n-1} \cap p^\perp} d\theta\ \frac{1}{2} q \cdot e_k \bar{g}(\theta)(e_i, e_j)\\
=& -\frac{1}{|p|^2} \int_{\mathbf{S}^{n-1} \cap p^\perp} d\theta\ \frac{1}{2} \bar{g}(\theta)(e_i, e_j)\\
=& -\frac{1}{|p|} A_{ij}(p).
\end{aligned}$$

   We have used the fact that $q \cdot e_k = \frac{-p}{|p|} \cdot \frac{p}{|p|} = -1$. So $\partial_k A_{ij} - \partial_i A_{kj} = 0$.
6. $e_k,\ e_j \perp p,\ e_i \parallel p$. It is the same case as 4.
7. $e_k,\ e_i \perp p,\ e_j \parallel p$. Then

$$\partial_k A_{ij}(p) = \frac{1}{|p|^2} \int_{S^{n-1} \cap p^\perp} d\theta\ (q \cdot e_j)\bar{g}(\theta)(e_k, e_i) = \partial_i A_{kj}(p).$$

8. $e_k,\ e_i,\ e_j \perp p$. Then

$$\partial_k A_{ij}(p) = \frac{1}{|p|^2} \int_{S^{n-1} \cap p^\perp} d\theta\ \frac{1}{2}(q \cdot \nabla g(\theta))(\theta \cdot e_k)(\theta \cdot e_i)(\theta \cdot e_j) = \partial_i A_{kj}(p).$$

   We have used the fact that $q \cdot e_i = q \cdot e_j = q \cdot e_k = 0$.

We then deduce that $\mathrm{Curl}(A) = 0$ on $\mathbb{R}^n\backslash\{0\}$. We now remark that

$$\begin{aligned}\langle -(\mathrm{CurlA})_{\mathrm{i,j,k}}, \varphi\rangle &= \int_{\mathbb{R}^n} A_{ij}\partial_k\varphi - A_{kj}\partial_i\varphi \\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\backslash B_\varepsilon} A_{ij}\partial_k\varphi - A_{kj}\partial_i\varphi \\ &= \lim_{\varepsilon\to 0}\left(\int_{\mathbb{R}^n\backslash B_\varepsilon} -(\partial_k A_{ij} - \partial_i A_{kj})\varphi + \int_{\partial B_\varepsilon}(A_{ij}n_k - A_{kj}n_i)\varphi\right) \\ &= \lim_{\varepsilon\to 0}\varepsilon^{n-2}\int_{\partial B_1}(A_{ij}(\theta)\theta_k - A_{kj}(\theta)\theta_i)\varphi(\varepsilon\theta)d\theta \\ &= \begin{cases}\varphi(0)\int_{\mathbf{S}^1}(A_{ij}(\theta)\theta_k - A_{kj}(\theta)\theta_i)d\theta & \text{if} \quad n=2 \\ 0 & \text{if} \quad n\neq 2\end{cases}\end{aligned}$$

In particular, we have used the fact that for $n=1$, $A\equiv 0$. Now, using the symmetry of g, we deduce that $A(-\theta) = A(\theta)$ and then by antisymmetry the last integral on $\mathbf{S}^1$ vanishes. Therefore

$$\mathrm{Curl}(A) = 0 \quad \text{on } \mathbb{R}^n.$$

By a passage to the limit, this is still true if $g\in C^0$ (and not only $g\in C^1$).

To deduce that there exists Φ such that $A = D^2\Phi$, we use the following *lemma* :

Lemma 7.3. (Vectors fields with zero Curl are gradients)
Let $f = (f_1, ..., f_n)(x) \in \mathcal{D}'(\mathbb{R}^n)$ be such that $Curl(f) = (\partial_k f_i - \partial_i f_k)_{i,k} = 0$, then there is $h\in\mathcal{D}'(\mathbb{R}^n)$ such that $f_i = \partial_i h$.

For the proof of this *lemma*, we refer to Schwartz [166] Chapter II, Paragraph 6, Theorem VI p59.

We denote by $f_j = (f_{j1}, ..., f_{jn}) = (A_{j1}, ..., A_{jn})$. Using the fact that $Curl(A) = 0$, we deduce that for all $j\in\{1,..,n\}$, $Curl(f_j) = 0$. Then, by *Lemma* 7.3 there are h_j such that $f_j = \nabla h_j$. Using the fact that A is symmetric, we deduce that $\partial_j h_i - \partial_i h_j = 0$. Applying again *Lemma* 7.3, we deduce that there is Φ such that $h = \nabla\Phi$ and so $A = D^2\Phi$. Let us remark that Φ is unique up to a polynomial of degree 1. Let $\Phi^s(p) = \frac{1}{2}(\Phi(p) + \Phi(-p))$. Then $A = D^2\Phi^s$ and then Φ^s is unique up to a constant. Moreover, $D^2\Phi(p)$ behaves like $\frac{1}{|p|}$ for small p and then $D^2\Phi\in L^{n-\varepsilon}$ for every $\varepsilon > 0$. Therefore $\Phi\in W^{2,n-\varepsilon}_{Loc}$ and by Sobolev injections $\Phi\in C^0(\mathbb{R}^n)$. We deduce that there is a unique Φ such that

$$\Phi(-p) = \Phi(p) \quad \text{and} \quad \Phi(0) = 0. \tag{4.41}$$

Finally, we remark that

$$D^2\left(\frac{\Phi(\lambda p)}{|\lambda|}\right) = |\lambda|(D^2\Phi)(\lambda p) = |\lambda|A(\lambda p) = A(p) = D^2\Phi(p)$$

Therefore $\Phi(\lambda p) = |\lambda|\Phi(p)$ if Φ satisfies (4.41).

Proof of Proposition 7.1

We show that $\Phi = -\dfrac{1}{2\pi}\mathcal{F}(L_g)$ (where Φ is defined in *Lemma* 7.2). Let $\varphi \in \mathcal{S}$. The following holds :

$$\begin{aligned}
\langle -D^2_{\xi\xi}\mathcal{F}(L_g)(\xi), \varphi\rangle(\zeta,\zeta) =& \langle \mathcal{F}(-ix \otimes ixL_g(x)), \varphi\rangle(\zeta,\zeta) \\
=& \langle L_g, (x\otimes x)\mathcal{F}(\varphi)\rangle(\zeta,\zeta) \\
=& \langle L_g, (x\cdot\zeta)^2\mathcal{F}(\varphi)(x)\rangle \\
=& \int_{\mathbb{R}^n} dx\, \frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\mathcal{F}(\varphi)(x) \\
=& \langle \mathcal{F}\left(\frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\right), \varphi\rangle.
\end{aligned}$$

We then have the following lemma

Lemma 7.4. *Let $n \geq 2$. Let $g \in C^0(\mathbb{R}^n\backslash\{0\})$ such that $g(\lambda p) = \dfrac{g(p)}{|\lambda|^{n+1}}$. Then, the following holds*

$$\mathcal{F}\left(\frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\right)(\xi) = 2\pi A(\xi)(\zeta,\zeta).$$

We just give here a formal proof. The complete proof is given in Appendix.

By definition of Fourier transform, we have formally for $\xi \neq 0$, with $\theta =$

$\dfrac{x}{|x|}$, $r = |x|$

$$\begin{aligned}
\mathcal{F}\left(\frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\right) &= \int_{\mathbb{R}^n} \frac{g\left(\frac{x}{|x|}\right)(x\cdot\zeta)^2 e^{-i\xi\cdot x}}{|x|^{n+1}}dx \\
&= \int_{\mathbb{R}^n} \frac{g\left(\theta\right)(\theta\cdot\zeta)^2 e^{-i\xi\cdot x}}{|x|^{n-1}}dx \\
&= \int_{\mathbf{S}^{n-1}\times(0,\infty)} g\left(\theta\right)(\theta\cdot\zeta)^2 e^{-i\xi\cdot\theta r} d\theta dr \\
&= \int_{\mathbf{S}^{n-1}} d\theta\ g(\theta)(\theta\cdot\zeta)^2 \int_0^\infty dr\ \left(\frac{e^{i\xi\cdot\theta r}+e^{-i\xi\cdot\theta r}}{2}\right) \\
&= \int_{\mathbf{S}^{n-1}} d\theta g(\theta)(\theta\cdot\zeta)^2 \int_{-\infty}^\infty dr \frac{e^{i\xi\cdot\theta r}}{2} \\
&= \frac{2\pi}{|\xi|}\int_{\mathbf{S}^{n-1}\cap\xi^\perp} d\theta\ \frac{1}{2}g(\theta)(\theta\cdot\zeta)^2 \\
&= 2\pi A(\xi)(\zeta,\zeta),
\end{aligned}$$

where we have used the fact that $\mathcal{F}(1) = 2\pi\delta_0$ in 1D, that formally gives

$$\int_{-\infty}^{+\infty} dr\ e^{i\xi\cdot\theta r} = 2\pi\delta_0(\xi\cdot\theta) = \frac{2\pi}{|\xi|}\delta_0\left(\frac{\xi}{|\xi|}\cdot\theta\right).$$

This achieves the formal proof of Lemma 7.4.

We then get

$$-D^2\mathcal{F}(L_g)(\xi) = 2\pi A(\xi) = 2\pi D^2\Phi.$$

Moreover $\mathcal{F}(L_g)(-\xi) = \mathcal{F}(L_g)(\xi)$ and $\mathcal{F}(L_g)(0) = 0$. Therefore, by *Lemma* 7.2 we deduce that

$$\Phi = -\frac{1}{2\pi}\mathcal{F}(L_g)$$

and $\Phi(\lambda p) = |\lambda|\Phi(p)$. This achieves the proof of the proposition.

We now prove Theorem 1.7 :

Proof of Theorem 1.7

Let us first compute $\mathrm{div}\nabla G\left(\frac{Du}{|Du|}\right)$. We set $p = Du$. The following holds :

$$\begin{aligned}
\mathrm{div}\nabla G\left(\frac{Du}{|Du|}\right) &= \sum_i \frac{\partial}{\partial x_i}\left(\frac{\partial G}{\partial x_i}\left(\frac{p}{|p|}\right)\right)\\
&= \sum_{i,j} \frac{\partial^2 G}{\partial x_i \partial x_j}\left(\frac{p}{|p|}\right)\frac{\partial}{\partial x_i}\left(\frac{D_j u}{|Du|}\right)\\
&= \frac{1}{|p|}\sum_{i,j} \frac{\partial^2 G}{\partial x_i \partial x_j}\left(\frac{p}{|p|}\right)\left(D^2_{ij}u - \frac{D^2_{\cdot i}u\cdot p\otimes p_j}{|p|^2}\right)\\
&= \frac{1}{|p|}\mathrm{trace}\left(D^2G\left(\frac{p}{|p|}\right)\left(I - \frac{p\otimes p}{|p|^2}\right)D^2u\right).
\end{aligned}$$

Moreover, for $\lambda > 0$, we have $G(\lambda p) = \lambda G(p)$. Then by derivation we get

$$p \cdot \nabla G(\lambda p) = G(p).$$

Taking the gradient, we get

$$\nabla G(p) = \nabla G(\lambda p) + p \cdot D^2 G(\lambda p)\lambda$$

which implies for $\lambda = 1$

$$p \cdot D^2 G(P) = 0.$$

This implies that $D^2G\left(\frac{p}{|p|}\right)\left(I - \frac{p\otimes p}{p^2}\right) = D^2G\left(\frac{p}{|p|}\right)$. We then deduce :

$$\mathrm{div}\nabla G\left(\frac{Du}{|Du|}\right) = \frac{1}{|p|}\mathrm{trace}\left(A\left(\frac{p}{|p|}\right)\cdot D^2u\right).$$

This show the first part of the Theorem.

In the two dimensional case, we simply remark that we have

$$g(\theta)\theta\otimes\theta = D^2G(\theta^\perp)$$

which implies the result. This ends the proof of Theorem 1.7

Proof of Theorem 1.11

We can then rewrite $A(p)$ as

$$\begin{aligned}
A(p) &= \int_{x\in p^\perp} \frac{1}{2}K_0(x)\; x\otimes x\; dx\\
&= \int_{\theta\in p^\perp\cap \mathbf{S}^{n-1}} d\theta\; \frac{1}{2}\left(\int_{(0,+\infty)} dr\; r^n K_0(r\theta)\right)\theta\otimes\theta\\
&= \int_{\theta\in p^\perp\cap \mathbf{S}^{n-1}} d\theta\; \frac{1}{2}g(\theta)\;\theta\otimes\theta
\end{aligned}$$

with $g(\theta) = \int_{(0,+\infty)} dr\ r^n K_0(r\theta)$. So, by applying Theorem 1.7, we see that the mean curvature motion defined by (4.5)-(4.6) using the matrix $A(p)$, is of variational type.

Proof of Proposition 1.8

The idea to build a function g which changes its sign, such that

$$\int_{\mathbf{S}^{n-1}\cap p^\perp} d\theta\ \frac{1}{2} g(\theta)\ \theta \otimes \theta \geq 0$$

for all $p \in \mathbb{R}^n \backslash \{0\}$ is simple. First, we consider the set

$$\mathbf{S} = \cup_{i=1}^n (\mathbf{S}^{n-1} \cap \{x_i = 0\})$$

and we remark that any hyperplane Π which contains the origine intersect $\mathbf{S}$ with an angle $\alpha \geq \alpha_0$ with $\alpha_0 > 0$ independant of Π. We then define g on $\mathbf{S}^{n-1}$ as a mollification of $\delta_{\mathbf{S}} - \eta$ for η small enough where $\delta_{\mathbf{S}}$ is a Dirac mass on $\mathbf{S}^{n-1}$ with support the set $\mathbf{S}$.

We now make the rigorous construction. We denote by $(e_i)_{i=1,\dots,n}$ a orthonormal basis of $\mathbb{R}^n$. We use the following *lemma*

Lemma 7.5. *For $\varepsilon \in (0,1]$, there exist $g_i^\varepsilon \in C^\infty(\mathbf{S}^{n-1})$, for $i = 1, ..., n$, such that for all $\Psi^\varepsilon \in C^\infty(\mathbf{S}^{n-1})$, $\forall p_\varepsilon \in \mathbf{S}^{n-1}$, if $p_\varepsilon \to p_0$, $\|\Psi^\varepsilon - \Psi^0\|_{L^\infty(\mathbf{S}^{n-1})} \to 0$*

$$\int_{\mathbf{S}^{n-1}\cap p_\varepsilon^\perp \simeq \mathbf{S}^{n-2}} d\theta\ g_i^\varepsilon(\theta)\Psi^\varepsilon(\theta) \longrightarrow \frac{1}{\sin\widehat{(p_0, e_i)}} \int_{\mathbf{S}^{n-1}\cap p_0^\perp \cap e_i^\perp \simeq \mathbf{S}^{n-3}} d\theta\ \Psi^0(\theta) \quad \text{as } \varepsilon \to 0$$

provided p_0 is not parallel to e_i and where $\widehat{(p_0, e_i)} \in [0, \frac{\pi}{2}]$ denotes the angle between p_0 and e_i. Moreover,

$$g_i^\varepsilon(\theta) = 0 \quad \text{if } |\langle \theta, e_i\rangle| \geq \varepsilon. \tag{4.42}$$

The proof is postponed.

We set

$$g^\varepsilon = \sum_{i=1}^n g_i^\varepsilon - \eta$$

with η a small parameter to be precised. We remark that by (4.42) for ε small enough, g^ε is not nonnegative. We want to show that there exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$, for all $p,\ \xi \in \mathbf{S}^{n-1}$

$$\int_{\mathbf{S}^{n-1}\cap p^\perp \simeq \mathbf{S}^{n-2}} d\theta\ g^\varepsilon(\theta)\ \langle \theta, \xi\rangle^2 \geq 0. \tag{4.43}$$

We will prove (4.43) by contradiction, using the following *lemma* :

Lemma 7.6. *There exists $C_0 > 0$ such that $\forall p \in \mathbf{S}^{n-1}$, $\forall \xi \in \mathbf{S}^{n-1} \cap p^\perp$, $\exists\ i_0 \in \{1, ..., n\}$ such that*

$$\int_{\mathbf{S}^{n-1}\cap p^\perp \cap e_{i_0}^\perp \simeq \mathbf{S}^{n-3}} d\theta \ \langle \theta, \xi \rangle^2 \geq C_0$$

and

$$\widehat{(p, e_{i_0})} \geq C_0$$

where $\widehat{(p, e_{i_0})} \in [0, \frac{\pi}{2}]$ denotes the angle between p and e_{i_0}.

The proof is postponed.

We now prove (4.43) by contradiction assuming that there exists a subsequence $\varepsilon_k \to 0$ such that there exists $p_k \in \mathbf{S}^{n-1}$, $\xi_k \in \mathbf{S}^{n-1} \cap p_k^\perp$ such that

$$\int_{\mathbf{S}^{n-1}\cap p_k^\perp \simeq \mathbf{S}^{n-2}} d\theta \ g^{\varepsilon_k}(\theta) \ \langle \theta, \xi_k \rangle^2 \leq 0.$$

Up to extract a subsequence, we can assume that $p_k \to p_\infty$ and $\xi_k \to \xi_\infty$ with p_∞, $\xi_\infty \in \mathbf{S}^{n-1}$. We then have with the index i_0 given by *Lemma* 7.6 for $p = p_\infty$, $\xi = \xi_\infty$

$$\begin{aligned} 0 \geq \int_{\mathbf{S}^{n-1}\cap p_k^\perp} d\theta \ g^{\varepsilon_k}(\theta) \ \langle \theta, \xi_k \rangle^2 &\geq \int_{\mathbf{S}^{n-1}\cap p_k^\perp} d\theta \ g_{i_0}^{\varepsilon_k}(\theta) \ \langle \theta, \xi_k \rangle^2 - \eta \int_{\mathbf{S}^{n-1}\cap p_k^\perp} d\theta \\ &\geq \int_{\mathbf{S}^{n-1}\cap p_k^\perp} d\theta \ g_{i_0}^{\varepsilon_k}(\theta) \ \langle \theta, \xi_k \rangle^2 - \eta |\mathbf{S}^{n-2}|. \end{aligned}$$

By passing to the limit, using *Lemma* 7.5, we then obtain

$$0 \geq \frac{1}{\sin(\widehat{p_\infty, e_{i_0}})} \int_{\mathbf{S}^{n-1}\cap p_\infty^\perp \cap e_{i_0}^\perp} d\theta \ \langle \theta, \xi_\infty \rangle^2 - \eta |\mathbf{S}^{n-2}| \geq \frac{C_0}{\sin C_0} - \eta |\mathbf{S}^{n-2}|$$

where C_0 is given in *Lemma* 7.6. This is a contradiction for η small enough.

We now prove *Lemma* 7.6

Proof of *Lemma* 7.6
We perform the proof by contradiction. If the result is false, then $\exists\ C_k \to 0$, $\exists\ p_k \in \mathbf{S}^{n-1}$, $\exists\ \xi_k \in \mathbf{S}^{n-1} \cap p_k^\perp$ such that for all $i \in \{1, ..., n\}$

$$0 \leq \widehat{(p_k, e_i)} \leq C_k \tag{4.44}$$

or

$$\int_{\mathbf{S}^{n-1}\cap p_k^\perp \cap e_i^\perp} d\theta \ (\xi_k \cdot \theta)^2 \leq C_k \quad \text{if } \widehat{(p_k, e_i)} \neq 0. \tag{4.45}$$

We distinguish two cases :

Case 1. There exist two indices i such that (4.44) holds. Up to reorganise the indices, we can assume that (4.44) holds for $i = 1, 2$. We deduce by extracting a subsequence and passing to the limit that there exists $p_\infty = \lim p_k$ such that $\widehat{(p_\infty, e_i)} = 0$ for $i = 1, 2$, which is a contradiction.

Case 2. There exists two indices i such that (4.45) holds. Up to reorganise the indices, we can assume that (4.45) holds for $i = 1, 2$. In this case, by passing to the limit, up to extract a subsequence, we obtain

$$\int_{\mathbf{S}^{n-1}\cap p_\infty^\perp \cap e_i^\perp} d\theta \, (\xi_\infty \cdot \theta)^2 = 0 \quad \forall i = 1, 2.$$

We then deduce that $\xi_\infty \in \mathbf{S}^{n-1} \cap p_\infty^\perp$ is parallel to e_i, for $i = 1, 2$ which is a contradiction.

Finally, in dimension $n \geq 3$, we are either in case 1 or case 2, so we obtained a contradiction.

Proof of *Lemma* 7.5

We set $\tilde{g}_i^\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{x\cdot e_i}{\varepsilon}\right)$ where $\rho \in C_c^\infty(\mathbb{R}, \mathbb{R})$ and satisfies :

$$\rho \geq 0, \quad \operatorname{supp}(\rho) \subset [-1, 1], \quad \int_{\mathbb{R}} \rho(x)dx = 1.$$

We then set

$$g_i^\varepsilon(\theta) = \int_0^\infty r^{n-1}\tilde{g}_i^\varepsilon(r\theta)f(r)\, dr$$

with $f \in C_c^\infty((0,\infty), \mathbb{R})$ satisfying $\int_0^\infty f(r)r^{n-2}\, dr = 1$. For all $\Psi^\varepsilon \in C^\infty(\mathbf{S}^{n-1})$ and $p_\varepsilon \in \mathbf{S}^{n-1}$, let us define

$$\mathcal{I}^\varepsilon = \int_{\mathbf{S}^{n-1}\cap p_\varepsilon^\perp \simeq \mathbf{S}^{n-2}} d\theta\; g_i^\varepsilon(\theta)\Psi^\varepsilon(\theta).$$

To simplify the notations, let us set $\Psi = \Psi^\varepsilon$ and $p = p_\varepsilon$. We then have, if p is not parallel to e_i

$$\begin{aligned}\mathcal{I}^\varepsilon &= \int_{\mathbf{S}^{n-1}\cap p^\perp \simeq \mathbf{S}^{n-2}} d\theta\; \Psi(\theta) \int_0^\infty dr\; r^{n-1}\tilde{g}_i^\varepsilon(r\theta)f(r) \\ &= \int_{p^\perp} dx\; \tilde{g}_i^\varepsilon(x)\tilde{\Psi}(x)\end{aligned}$$

where

$$\tilde{\Psi}(x) = f(|x|)\Psi\left(\frac{x}{|x|}\right)|x|.$$

Using the definition of $\tilde{g}_i^\varepsilon$, we then have, by denoting $\alpha_i = \widehat{(p, e_i)}$ the angle between p and e_i and using the change of coordinates $x = (y', y_n)$ with $y' \in p^\perp$ and $y_n \in \mathbb{R}$, that

$$\begin{aligned}
\mathcal{I}^\varepsilon &= \int_{p^\perp} dx\ \frac{1}{\varepsilon}\rho\left(\frac{x \cdot e_i}{\varepsilon}\right)\tilde{\Psi}(x) \\
&= \int_{p^\perp} dy'\ \frac{1}{\sin\alpha_i}\left(\frac{\sin\alpha_i}{\varepsilon}\right)\rho\left(\frac{y'.e_i'}{\left(\frac{\varepsilon}{\sin\alpha_i}\right)\sin\alpha_i}\right)\tilde{\Psi}(y', 0) \\
&= \frac{1}{\sin\alpha_i}\int_{p^\perp} dy'\ \frac{1}{\varepsilon'}\rho\left(\frac{y' \cdot \frac{e_i'}{|e_i'|}}{\varepsilon'}\right)\tilde{\Psi}(y', 0)
\end{aligned}$$

where $\varepsilon' = \dfrac{\varepsilon}{\sin\hat{\theta}_i}$ and e_i' is the orthogonal projection of e_i onto the hyperplane $p^\perp$. In particular, e_i' satisfies $|e_i'| = \sin\alpha_i$. Passing to the limit in ε, with $p_\varepsilon \to p_0$, $\Psi^\varepsilon \to \Psi^0$, $\alpha_i = \alpha_i^\varepsilon = \widehat{(p_\varepsilon, e_i)} \to \alpha_i^0 = \widehat{(p_0, e_i)}$ and $\tilde{\Psi}^\varepsilon = f(|x|)\Psi^\varepsilon\left(\dfrac{x}{|x|}\right)|x| \to \tilde{\Psi}^0 = f(|x|)\Psi^0\left(\dfrac{x}{|x|}\right)|x|$, yields

$$\begin{aligned}
\mathcal{I}^\varepsilon \ &\longrightarrow \frac{1}{\sin\alpha_i^0}\int_{p^\perp \cap e_i'^\perp} dy'\ \tilde{\Psi}^0(y', 0) \\
&= \frac{1}{\sin\alpha_i^0}\int_{p^\perp \cap e_i^\perp} dy'\ \tilde{\Psi}^0(y', 0) \\
&= \frac{1}{\sin\alpha_i^0}\int_{\mathbf{S}^{n-3} \simeq \mathbf{S}^{n-1} \cap p^\perp \cap e_i^\perp} d\theta\ \left(\int_0^\infty dr\ r^{n-3} f(r) r\right)\Psi^0(\theta) \\
&= \frac{1}{\sin\alpha_i^0}\int_{\mathbf{S}^{n-3} \simeq \mathbf{S}^{n-1} \cap p^\perp \cap e_i^\perp} d\theta\ \Psi^0(\theta).
\end{aligned}$$

This ends the proof of the *lemma.*

8 Heuristical convergence and some properties of the energies

8.1 Monotonicity of the energy

We begin this section by showing that the energy associated to (4.4) is nonincreasing in time. We recall that (4.4) is formally associated to the following energy :

$$\mathcal{E}^\varepsilon(u^\varepsilon) = \int_\lambda \overline{\mathcal{E}^\varepsilon}(\lambda) d\lambda \tag{4.46}$$

where

$$\overline{\mathcal{E}^\varepsilon}(\lambda) = \int_{\mathbb{R}^n} -\frac{1}{2}\left(\overline{c_0}^\varepsilon \star \rho_\lambda^\varepsilon\right)\rho_\lambda^\varepsilon$$

with

$$\rho_\lambda^\varepsilon = 1_{\{u^\varepsilon>\lambda\}}, \quad \overline{c_0}^\varepsilon = c_0^\varepsilon - \left(\int_{\mathbb{R}^n} c_0^\varepsilon\right)\delta_0.$$

Formally, we have :

$$\frac{d\overline{\mathcal{E}^\varepsilon}(\lambda)}{dt} = \int_{\mathbb{R}^n} -\left(\overline{c_0}^\varepsilon \star \rho_\lambda^\varepsilon\right)(\rho_\lambda^\varepsilon)_t$$

which is defined only on the support of $|D\rho_\lambda^\varepsilon|$ (since (4.4) formally implies $(\rho_\lambda^\varepsilon)_t = (\overline{c_0}^\varepsilon \star \rho_\lambda^\varepsilon)|D\rho_\lambda^\varepsilon|$). Moreover, $\overline{c_0}^\varepsilon \star \rho_\lambda^\varepsilon = c_0^\varepsilon \star \rho_\lambda^\varepsilon - \left(\int c_0^\varepsilon\right)\delta_0 \star \rho_\lambda^\varepsilon$. If we set η_n a regularisation of the Dirac mass, we then have $\eta_n \star \rho_\lambda^\varepsilon = \frac{1}{2}$ on the support of $|D\rho_\lambda^\varepsilon|$ (see Figure 4.2).

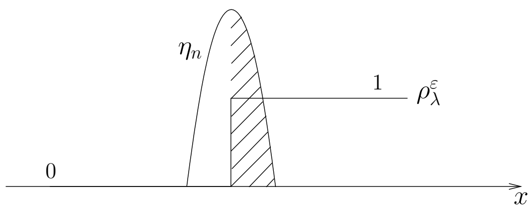

FIG. 4.2 – The convolution of ρ_λ^ε with the Dirac.

So, we can assume that $\overline{c_0}^\varepsilon \star \rho_\lambda^\varepsilon = c_0^\varepsilon \star \rho_\lambda^\varepsilon - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon$ on the support of $|D\rho_\lambda^\varepsilon|$. We then deduce that

$$\frac{d\overline{\mathcal{E}^\varepsilon}(\lambda)}{dt} = \int_{\mathbb{R}^n} -\left(c_0^\varepsilon \star \rho_\lambda^\varepsilon - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)^2 |D\rho_\lambda^\varepsilon|.$$

This implies :

$$\frac{d\mathcal{E}^\varepsilon(u^\varepsilon)}{dt} = \int d\lambda \int_{\mathbb{R}^n} -\left(c_0^\varepsilon \star \rho_\lambda^\varepsilon - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)^2 |D\rho_\lambda^\varepsilon| \leq 0.$$

So the energy is nonincreasing in time.

8.2 Formal convergence of the energy

We set $\mathcal{E}(u^0) = \int G(Du^0)$, the energy associated to the mean curvature motion. We have formally

$$\frac{d}{dt}\mathcal{E}(u^0) = \int \nabla G\left(\frac{Du^0}{|Du^0|}\right)\cdot Du_t^0 = \int -\left(\mathrm{div}\nabla G\left(\frac{Du^0}{|Du^0|}\right)\right)^2 |Du^0|.$$

Moreover, still formally we have

$$\begin{array}{rcl}\frac{d}{dt}\mathcal{E}^\varepsilon(u^\varepsilon) & = & \int d\lambda \int -\left(c_0^\varepsilon \star \rho_\lambda^\varepsilon - \frac{1}{2}\int c_0^\varepsilon\right)^2 |D\rho_\lambda^\varepsilon| \\ & \to & \int d\lambda \int -\left(\text{trace}\left(A\left(\frac{Du^0}{|Du^0|}\right)D^2u^0\right)\right)^2 |D\rho_\lambda^0| \\ & = & \int d\lambda \int -\left(\text{div}\left(\nabla G\left(\frac{Du^0}{|Du^0|}\right)\right)\right)^2 |D\rho_\lambda^0| \\ & = & \int -\left(\text{div}\left(\nabla G\left(\frac{Du^0}{|Du^0|}\right)\right)\right)^2 |Du^0|. \end{array}$$

So, formally,

$$\frac{d}{dt}\mathcal{E}^\varepsilon(u^\varepsilon) \to \frac{d}{dt}\mathcal{E}(u^0).$$

The work of Garroni, Müller [94], suggests that we should have $\int_{x,\lambda}\frac{1}{2}(c_0^\varepsilon \star \rho_\lambda^\varepsilon)\rho_\lambda^\varepsilon \to \int d\lambda \int_{\Gamma_\lambda} G\left(\frac{Du^0}{|Du^0|}\right)$, where Γ_λ is the λ level set of u^0. We deduce that (using formally the coarea formula for BV functions)

$$\begin{array}{rcl}\int_{x,\lambda}\frac{1}{2}(c_0^\varepsilon \star \rho_\lambda^\varepsilon)\rho_\lambda^\varepsilon & \to & \int d\lambda \int_x G\left(\frac{Du^0}{|Du^0|}\right)|D\rho_\lambda^0| \\ & = & \int_x G\left(\frac{Du^0}{|Du^0|}\right)|Du^0| \\ & = & \int G(Du^0) \end{array}$$

and so, formally

$$\mathcal{E}^\varepsilon(u^\varepsilon) \to \mathcal{E}(u^0).$$

9 Appendix : some *lemmata* on Fourier transform

Lemma 9.1. *The distribution L_g associated to g (see definition 1.6) satisfies the following properties :*

$$L_g(\lambda\cdot) = \frac{1}{\lambda^{n+1}}L_g \quad \forall\lambda > 0, \tag{4.47}$$

$$\mathcal{F}(L_g)(\lambda\cdot) = \lambda\mathcal{F}(L_g) \quad \forall\lambda > 0, \tag{4.48}$$

where $\mathcal{F}(L_g)$ is the Fourier transform of L_g defined by

$$\forall\varphi \in \mathcal{S},\ \langle\mathcal{F}(L_g),\varphi\rangle = \langle L_g,\mathcal{F}(\varphi)\rangle.$$

Proof of *Lemma* 9.1
Equation (4.47) results from the definition of L_g which by construction is of homogeneity of degree $-(n+1)$. This can be rigorously shown using the general definition for a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$

$$\forall\varphi \in C_c^\infty(\mathbb{R}^n), \quad \langle u(\lambda\cdot,\varphi\rangle := \frac{1}{\lambda^n}\langle u,\varphi\left(\frac{\cdot}{\lambda}\right)\rangle. \tag{4.49}$$

We now prove (4.48). A straightforward computation for $\varphi \in \mathcal{S}$ gives

$$\mathcal{F}(\varphi)(\lambda\cdot) = \mathcal{F}\left(\frac{1}{\lambda^n}\varphi\left(\frac{\cdot}{\lambda}\right)\right)(\cdot). \tag{4.50}$$

Using the definition (4.49), one can show that (4.50) is still true for element of $\mathcal{S}'$. Hence, we have

$$\begin{aligned}\mathcal{F}(L_g)(\lambda\cdot) =& \mathcal{F}\left(\frac{1}{\lambda^n}L_g\left(\frac{\cdot}{\lambda}\right)\right)(\cdot)\\ =& \mathcal{F}\left(\frac{\lambda^{n+1}}{\lambda^n}L_g(\cdot)\right)(\cdot)\\ =& \lambda\mathcal{F}(L_g(\cdot))(\cdot)\end{aligned}$$

where we have use (4.47). This ends the proof of the *lemma*.

Proof of *Lemma* 7.4

Let $R_0 > r_0 > 0$ and $\varphi \in C^\infty(\mathbb{R}^n)$ with Supp $\varphi \subset B_{R_0}(0)\backslash B_{r_0}(0)$. Let $\Psi_\lambda(y) = \Psi(\lambda y)$ for $y \in \mathbb{R}$ with $\Psi \in C_c^\infty(\mathbb{R})$ such that

$$\text{Supp }\Psi \subset [-1,1], \quad \Psi \equiv 1 \text{ on } [-\frac{1}{2},\frac{1}{2}], \quad 0 \le \Psi \le 1, \quad \Psi(-y) = \Psi(y).$$

Let us consider $f \in C_c^\infty([0,+\infty))$ with Supp $f \subset [r_0, R_0]$ and such that

$$\int_0^\infty f(\bar{r})\bar{r}^n d\bar{r} = 1.$$

Let us assume first that $g \in C^\infty(\mathbf{S}^{n-1})$. Let us compute for $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}\mathcal{I} &= \left\langle \mathcal{F}\left(\frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\right), \varphi\right\rangle\\ &= \left\langle \frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2, \mathcal{F}(\varphi)\right\rangle\\ &= \int_{\mathbb{R}^n} dx\ \frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\left(\int_{\mathbb{R}^n} d\xi\ e^{-i\xi\cdot x}\varphi(\xi)\right)\left(\int_0^\infty f(\bar{r})\bar{r}^n d\bar{r}\right)\end{aligned}$$

Since $\left|\Psi_\lambda\left(\frac{|x|}{\bar{r}}\right)\right| \le 1$ and $\Psi_\lambda\left(\frac{|x|}{\bar{r}}\right) \to 1$ as $\lambda \to 0$, we deduce by Dominated Convergence Theorem that

$$\begin{aligned}\mathcal{I} &= \lim_{\lambda\to 0}\int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+} dx\ d\xi\ d\bar{r}\quad \frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\Psi_\lambda\left(\frac{|x|}{\bar{r}}\right)e^{-i\xi\cdot x}f(\bar{r})\bar{r}^n\varphi(\xi)\\ &= \lim_{\lambda\to 0}\int_{\mathbf{S}^{n-1}\times\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}_+} d\theta\ d\xi\ d\bar{r}\ dr\quad g(\theta)(\theta\cdot\zeta)^2\Psi_\lambda\left(\frac{r}{\bar{r}}\right)e^{-i\xi\cdot\theta r}f(\bar{r})\bar{r}^n\varphi(\xi)\end{aligned}$$

where $\theta = \dfrac{x}{|x|}$, $r = |x|$. We set $r = \bar{r}s$, $\bar{x} = \theta\bar{r}$, $\bar{s} = |\xi|s$ and we get

$$\begin{aligned}
\mathcal{I} &= \lim_{\lambda\to 0} \int_{\mathbf{S}^{n-1}\times\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}_+} d\theta\, d\xi\, d\bar{r}\, \bar{r}ds \quad g(\theta)(\theta\cdot\zeta)^2\Psi_\lambda(s)\, e^{-i\xi\cdot\theta\bar{r}s} f(\bar{r})\bar{r}^n\varphi(\xi)\\
&= \lim_{\lambda\to 0} \int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+} d\bar{x}\, d\xi\, ds \quad f(|\bar{x}|)g\left(\frac{\bar{x}}{|\bar{x}|}\right)(\bar{x}\cdot\zeta)^2\Psi_\lambda(s)e^{-i\xi\cdot\bar{x}s}\varphi(\xi)\\
&= \lim_{\lambda\to 0} \int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+} d\bar{x}\, d\xi\, d\bar{s} \quad f(|\bar{x}|)\, g\left(\frac{\bar{x}}{|\bar{x}|}\right)(\bar{x}\cdot\zeta)^2\Psi_{\frac{\lambda}{|\xi|}}(\bar{s})\, e^{-i\frac{\xi}{|\xi|}\cdot\bar{x}\bar{s}}\ \frac{\varphi(\xi)}{|\xi|}\\
&= \lim_{\lambda\to 0} \int_{\mathbb{R}^n} d\xi\ \frac{\varphi(\xi)}{|\xi|} \int_{\mathbb{R}^n} d\bar{x}\ \Phi(\bar{x}) \int_{R_+} d\bar{s}\ \Psi_{\frac{\lambda}{|\xi|}}(\bar{s})\, e^{-i\frac{\xi}{|\xi|}\cdot\bar{x}\bar{s}}
\end{aligned}$$

where $\Phi(\bar{x}) = f(|\bar{x}|)g\left(\dfrac{\bar{x}}{|\bar{x}|}\right)(\bar{x}\cdot\zeta)^2 \in C_c^\infty(\mathbb{R})$ and $\text{Supp}\ \Phi \subset B_{R_0}(0)\backslash B_{r_0}(0)$. Using the fact that $\Phi(-\bar{x}) = \Phi(\bar{x})$, we deduce

$$\begin{aligned}
\mathcal{I} &= \lim_{\lambda\to 0} \int_{\mathbb{R}^n} d\xi\ \frac{\varphi(\xi)}{|\xi|} \int_{\mathbb{R}^n} d\bar{x}\ \Phi(\bar{x}) \int_{R_+} d\bar{s}\ \Psi_{\frac{\lambda}{|\xi|}}(\bar{s})\ \frac{e^{-i\frac{\xi}{|\xi|}\cdot\bar{x}\bar{s}} + e^{i\frac{\xi}{|\xi|}\cdot\bar{x}\bar{s}}}{2}\\
&= \lim_{\lambda\to 0} \int_{\mathbb{R}^n} d\xi\ \frac{\varphi(\xi)}{2|\xi|} \int_{\mathbb{R}^n} d\bar{x}\ \Phi(\bar{x}) \int_{R} d\bar{s}\ \Psi_{\frac{\lambda}{|\xi|}}(\bar{s})\ e^{-i\frac{\xi}{|\xi|}\cdot\bar{x}\bar{s}}
\end{aligned}$$

We set $\bar{x} = \bar{x}' + \bar{y}e_\xi$ with $\bar{x}' \in e_\xi^\perp$, $\bar{y} \in \mathbb{R}$ and $e_\xi = \dfrac{\xi}{|\xi|}$ and get

$$\begin{aligned}
\mathcal{I} &= \lim_{\lambda\to 0} \int_{\mathbb{R}^n} d\xi\ \frac{\varphi(\xi)}{2|\xi|} \int_{\mathbb{R}^{n-1}} d\bar{x}' \int_{\mathbb{R}} d\bar{y}\ \Phi(\bar{x}',\bar{y}) \int_{\mathbb{R}} d\bar{s}\ \Psi_{\frac{\lambda}{|\xi|}}(\bar{s})\ e^{-i\bar{y}\bar{s}}\\
\mathcal{I} &= \lim_{\lambda\to 0} \int_{\mathbb{R}^n} d\xi\ \frac{\varphi(\xi)}{2|\xi|} \int_{\mathbb{R}^{n-1}} d\bar{x}'\ J_{\frac{\lambda}{|\xi|}}(\bar{x}')
\end{aligned}$$

where

$$\begin{aligned}
J_{\frac{\lambda}{|\xi|}}(\bar{x}') &= \int_R d\bar{y}\ \Phi(\bar{x}',\bar{y})\mathcal{F}\left(\Psi_{\frac{\lambda}{|\xi|}}\right)(\bar{y})\\
&= \left\langle \mathcal{F}\left(\Psi_{\frac{\lambda}{|\xi|}}\right), \Phi(\bar{x}',\cdot)\right\rangle
\end{aligned}$$

We claim the following whose proof is postponed

Lemma 9.2. *We have*

$$\mathcal{F}(\Psi_\mu) \to 2\pi\delta_0$$

in $\mathcal{S}'(\mathbb{R})$ *as* $\mu \to 0$.

Using this result and the fact that $\left|J_{\frac{\lambda}{|\xi|}}(\bar{x}')\right| \leq |\mathcal{F}(\Phi(x',\cdot))|_{L^1(\mathbb{R})}$, we deduce that

$$\begin{aligned}
\mathcal{I} =&2\pi \int_R d\xi\ \frac{\varphi(\xi)}{2|\xi|} \int_{\mathbb{R}^{n-1}} d\bar{x}'\ \Phi(\bar{x}',0)\\
=&2\pi \int_R d\xi\ \frac{\varphi(\xi)}{2|\xi|} \int_{\mathbb{R}^{n-1}} d\bar{x}'\ f(|\bar{x}|)g\left(\frac{\bar{x}}{|\bar{x}|}\right)(\bar{x}\cdot\zeta)^2\\
=&2\pi \int_R d\xi\ \frac{\varphi(\xi)}{2|\xi|} \int_{\mathbf{S}^{n-1}\cap\xi^\perp} d\theta\ g(\theta)(\theta\cdot\zeta)^2 \int_{R_+} d\bar{r}\ f(\bar{r})\bar{r}^n\\
=&2\pi \int_R d\xi\ \frac{\varphi(\xi)}{|\xi|} A\left(\frac{\xi}{|\xi|}\right)(\zeta,\zeta)\\
&2\pi \int_R d\xi\ \varphi(\xi)A(\xi)(\zeta,\zeta)\\
=&2\pi\langle A(\xi)(\zeta,\zeta),\varphi\rangle
\end{aligned}$$

We then have shown that

$$\left\langle \mathcal{F}\left(\frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\right),\varphi\right\rangle = 2\pi\langle A(\xi)(\zeta,\zeta),\varphi\rangle.$$

By a passage to the limit, this is still true if $g\in C^0$ (and not only C^∞) and for all $\varphi\in C_c^\infty(\mathbb{R}^n\backslash\{0\})$. We then deduce that

$$\mathcal{F}\left(\frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\right) - 2\pi A(\xi)(\zeta,\zeta) = T$$

with Supp $T\subset\{0\}$, and then the distribution T is a finite sum of derivatives of Dirac mass : $T=\sum a_\alpha\delta_0^{(\alpha)}$. Using the fact that $\delta_0^{(\alpha)}(\lambda\xi)=\frac{1}{\lambda^{n+|\alpha|}}\delta_0^{(\alpha)}(\xi)$ with $|\alpha|=\alpha_1+...+\alpha_n$, and the homogeneity of degree -1 of $D^2\mathcal{F}(L_g)$, we deduce that for $n\geq 2$, $T=0$ and

$$\mathcal{F}\left(\frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\right) = 2\pi A(\xi)(\zeta,\zeta).$$

This ends the proof of the *lemma*.

Proof of *Lemma* 9.2
Let $\varphi_1\in\mathcal{S}(\mathbb{R})$. The following holds

$$\begin{aligned}
\langle\mathcal{F}(\Psi_\mu)-2\pi\delta_0,\varphi_1\rangle =&\langle\mathcal{F}(\Psi_\mu)-\mathcal{F}(1),\varphi_1\rangle\\
=&\langle\Psi_\mu-1,\mathcal{F}(\varphi_1)\rangle.
\end{aligned}$$

So, it just remains to show that $\Psi_\mu \to 1$ in $\mathcal{S}'(\mathbb{R})$ as $\mu \to 0$. Let $\varphi \in \mathcal{S}(\mathbb{R})$. The following holds

$$\begin{aligned}
\langle \Psi_\mu - 1, \varphi \rangle &= \int_{\mathbb{R}} dx \ (\Psi_\mu(x) - 1)\,\varphi(x) \\
&= \int_{\mathbb{R}} dx \ (\Psi(\mu x) - 1)\varphi(x) \\
&= \int_{|x| \geq \frac{1}{2\mu}} dx \ (\Psi(\mu x) - 1)\varphi(x) \\
&\leq \int_{|x| \geq \frac{1}{2\mu}} dx \ |\varphi(x)| \\
&\leq C\mathcal{N}_2(\varphi) \int_{|x| \geq \frac{1}{2\lambda}} dx \ \frac{1}{1+x^2} \\
&\to 0 \quad \text{as } \mu \to 0
\end{aligned}$$

where we have used the definition of $\mathcal{N}_p(\varphi) = \sup_{|\alpha|,|\beta| \leq p} \left| |x|^\alpha \frac{d^\beta \varphi(x)}{dx^\beta} \right|$ and the fact that

$$(1+x^2)|\varphi(x)| \leq C\mathcal{N}_2(\varphi).$$

This ends the proof of the *lemma*.

Acknowledgements

The authors would like to thanks G. Barles, G. Bellettini, A. Bossavit, A. Chambolle, B. Devincre, C. Imbert, A. Finel, Y. Le Bouar, M. Novaga, F. Murat and P.E. Souganidis for fruitfull discussions in the preparation of this paper. The authors also thank an unknown referee for his suggestions and for correcting a mistake in a previous version of the paper. The two last authors were supported by the contract JC 1025 called "ACI jeunes chercheuses et jeunes chercheurs" of the French Ministry of Research (2003-2005).

Chapitre 5

Une estimation d'erreur pour un nouveau schéma pour le mouvement par courbure moyenne

Ce chapitre est issu de [88].
Dans ce travail, nous proposons un nouveau schéma numérique pour le mouvement par courbure moyenne anisotrope. La solution du schéma n'est pas unique mais pour toutes les solutions numériques, nous donnons une estimation d'erreur entre la solution continue et son approximation numérique. Cette estimation d'erreur n'est pas optimale mais, à notre connaissance, c'est la première estimation pour des équations de type mouvement par courbure moyenne. Le schéma est également utilisable pour calculer la solution de l'équation de la dynamique des dislocations.

An error estimate for a new scheme for mean curvature motion

N. Forcadel

Abstract

In this work, we propose a new numerical scheme for the anisotropic mean curvature equation. The solution of the scheme is not unique, but for all numerical solutions, we provide an error estimate between the continuous solution and the numerical approximation. This error estimate is not optimal, but as far as we know, this is the first one for mean curvature type equation. Our scheme is also applicable to compute the solution to dislocations dynamics equation.

AMS Classification : 65M06, 65M12, 65M15, 49L25, 53C44.

Keywords : Mean curvature motion, error estimate, numerical scheme, dislocations dynamics.

1 Introduction

Mean curvature motion has been largely studied in terms of both theory and computation, in particular due to the large number of applications like front propagation, image processing, fluid dynamics...(see for instance Sethian [163] and Osher, Paragios [153]).

The level set framework has been used in both theoretical and numerical problems. Theoretically, the mean curvature equation has been well understood using the framework of viscosity solutions by Chen, Giga, Goto [58] and Evans, Spruck [83]. However, this equation has serious problems for the question of numerical approximations. Nevertheless, there are several works on this question. First, let us mention the numerical method of Osher, Sethian [154]. This method is very used in practice but, as far as we know, there is no convergence result. Another algorithm is the Merriman, Bence, Osher scheme [143] in which motion by mean curvature is viewed as singular limit of a diffusion equation with threshold. The convergence of this scheme has been proved by Barles, Georgelin [22] and Evans [79] (see also Ishii [116], Ishii, Pires, Souganidis [120], and Chambolle, Novaga [56]). A class of convergent schemes for nonlinear parabolic equations including mean curvature motion have been proposed by Crandall, Lions [65]. Let us mention also the recent work of Oberman [149]. In these last two works, two different scales are used. The first one is the space step Δx and the second one is the size of the stencil ε. As it was pointed out in [65], these two scales are very important to approximate degenerate equations like mean curvature equation.

The goal of our work is to propose a new scheme for mean curvature motion and to prove an error estimate between the continuous solution and its numerical approximation. This error estimate is not optimal, but as far as we know, this is the first one for complete discretized scheme for mean curvature type motion.

The idea is to use a recent work of Da Lio, Monneau and the author [67] concerning the convergence of dislocations dynamics to mean curvature motion. Dislocations are linear defects which move in crystals. Their dynamics can be represented by a non-local first order Hamilton-Jacobi equation (see Alvarez, Hoch, Le Bouar and Monneau [10]). The first goal of this work is to prove an error estimate between the solution of dislocations dynamics and the solution of mean curvature motion. To do this, we will use in a more quantitative way the definition of viscosity solutions for mean curvature motion proposed by Barles, Georgelin [22] by considering a regularization with quartics.

The second goal of this work is to propose a numerical scheme for dislocations dynamics. The main properties of this scheme is that it is implicit (so there is no CFL condition) and uses two different scales. Moreover, this scheme is not monotone and does not admit a unique solution. It is only "almost monotone" (see Lemma 5.2). Nevertheless, the fact that it is implicit and the monotonicity of the velocity, allows us to "freeze" the velocity and to prove a Crandall Lions type [63] error estimate for any solutions (we refer to Alvarez, Carlini, Monneau and Rouy [7, 8] for error estimate for dislocations dynamics in the non monotone case). It is also possible to explicit the scheme. In this case, we are able to prove the convergence of the scheme under a CFL condition, but error estimates in this case are still open. This comes from the fact that there is no consistency error for the scheme (see Proposition 5.6).

Finally, let us mention some works on error estimate for numerical scheme for Hamilton-Jacobi equations. For the first order case, we refer to Souganidis [173]. For the second order case, when the Hamiltonian is convex, we refer to Krylov [128, 129] and Barles, Jakobsen [23, 24]. When the Hamiltonian is uniformly elliptic, we refer to Caffarelli, Souganidis [48].

Let us now explain how this paper is organized. In Section 2, we state the main results of this work. In Section 3, we prove the main result concerning mean curvature type motion. Section 4 is devoted to prove the error estimate between the solution of dislocations dynamics and the one of mean curvature motion. In Section 5, we study the numerical scheme for dislocations dynamics and we prove an error estimate between the continuous solution and its numerical approximation. Some numerical simulations are provided in Section 6.

2 Main Results

2.1 Error estimate for Mean Curvature Motion

In order to propose a numerical scheme for anisotropic mean curvature motion, we will use the work of Da Lio *et al.* [67] that we briefly recall here. Given a function g defined on the unit sphere $\mathbf{S}^{N-1}$of $\mathbb{R}^N$ by

$$g \in \mathrm{Lip}(\mathbf{S}^{N-1}), \quad g(-\theta) = g(\theta) \geq 0, \quad \forall \theta \in \mathbf{S}^{N-1} \tag{5.1}$$

we consider kernels $c_0 \in L^\infty(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N)$ satisfying

$$\begin{cases} c_0(x) = \dfrac{1}{|x|^{N+1}} \; g\left(\dfrac{x}{|x|}\right) & \text{if} \quad |x| \geq 1, \\ \\ c_0(-x) = c_0(x) \geq 0, \quad \forall x \in \mathbb{R}^N \end{cases} \tag{5.2}$$

and we use the following scaling for $0 < \varepsilon < e^{-1}$

$$c_0^\varepsilon(x) = \frac{1}{\varepsilon^{N+1} |\ln \varepsilon|} \; c_0\left(\frac{x}{\varepsilon}\right). \tag{5.3}$$

The term $|\ln \varepsilon|$ comes from the bad decay at infinity of the kernel c_0 (see [67, Section 4.1]).

We then consider the following auxiliary problem

$$\begin{cases} u_t^\varepsilon(x,t) = \left((c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t) \geq u^\varepsilon(x,t)\}})(x) - \dfrac{1}{2} \displaystyle\int_{\mathbb{R}^N} c_0^\varepsilon \right) |Du^\varepsilon(x,t)| & \text{in} \quad \mathbb{R}^N \times (0,T), \\ \\ u^\varepsilon(\cdot, 0) = u_0^\varepsilon & \text{in} \quad \mathbb{R}^N \end{cases} \tag{5.4}$$

where u_t^ε denotes the derivative with respect to the time variable, Du^ε indicates the gradient of u^ε with respect to the space variables, the convolution is done in space only and $1_{\{u^\varepsilon(\cdot,t) \geq u^\varepsilon(x,t)\}}$ is the characteristic function of the set $\{u^\varepsilon(\cdot,t) \geq u^\varepsilon(x,t)\}$ (which is equal to 1 on the set and 0 outside). This equation arises in the theory of dislocations dynamics (see Alvarez, Hoch, Le Bouar and Monneau [10]) and uses the Slepčev formulation [169] to consider the simultaneous evolutions of all the level sets of the function u^ε. We refer to Da Lio *et al.* [67] for the study of this model and in particular to [67, Definition 2.1] for the definition of viscosity solutions for Problem (5.4). In particular, we recall that we take the indicatrice of $\{u^\varepsilon(\cdot,t) \geq u^\varepsilon(x,t)\}$ for sub-solution and the indicatrice of $\{u^\varepsilon(\cdot,t) > u^\varepsilon(x,t)\}$ for super-solution.

The main result of [67] is that, if $u_0^\varepsilon = u_0$, then the unique solution u^ε of (5.4) converges locally uniformly on compact sets to the solution u^0 of the following limit equation

$$\begin{cases} u_t^0 - \text{trace}\left(D^2u^0 \cdot A\left(\dfrac{Du^0}{|Du^0|}\right)\right) = 0 & \text{in} \quad \mathbb{R}^N \times (0,T) \\ u^0(\cdot,0) = u_0 & \text{in} \quad \mathbb{R}^N \end{cases} \tag{5.5}$$

with

$$A\left(\frac{p}{|p|}\right) = \int_{\theta \in \mathbf{S}^{N-2} = \mathbf{S}^{N-1} \cap \left\{\langle x, \frac{p}{|p|}\rangle = 0\right\}} \left(\frac{1}{2} g(\theta)\, \theta \otimes \theta\right) \, d\theta \tag{5.6}$$

and where $M \cdot A$ and $\langle \cdot, \cdot \rangle$ denote respectively the product between the two matrices and the usual scalar product. Our first main result is an error estimate between u^ε and u^0 :

Theorem 2.1. (Error estimate for Mean Curvature Motion)
Let $N \geq 1$ and $T \leq 1$. Assume that $u_0, u_0^\varepsilon \in \mathrm{Lip}(\mathbb{R}^N)$, $g \in \mathrm{Lip}(\mathbf{S}^{N-1})$ and that c_0 given in (5.2) satisfies $c_0 \in W^{1,1}(\mathbb{R}^N)$. Then, there exists a constant K_1 depending on N, $\sup_{\mathbb{R}^N} c_0$, $|Dg|_{L^\infty(\mathbf{S}^{N-1})}$ and $|Du_0|_{L^\infty(\mathbb{R}^N)}$ such that the difference between the solution u^ε of (5.4) and the solution u^0 of (5.5) is given by

$$\sup_{\mathbb{R}^N \times (0,T)} |u^\varepsilon - u^0| \leq K_1 \left(\frac{T}{|\ln \varepsilon|}\right)^{\frac{1}{6}} + \sup_{\mathbb{R}^N} |u_0^\varepsilon - u_0|. \tag{5.7}$$

2.2 Discrete-continuous error estimate for dislocations dynamics

To approximate the solution u^0 of (5.5), we then have to approximate the solution u^ε of (5.4). Up to a change of variable (see Corollary 3.1), it suffices to approximate the solution u of

$$\begin{cases} u_t(x,t) = \left((c_0 \star 1_{\{u(\cdot,t) \geq u(x,t)\}})(x) - \dfrac{1}{2}\displaystyle\int c_0\right) |Du(x,t)| & \text{in} \quad \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = \bar{u}_0 & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{5.8}$$

Given a mesh size $\Delta x_1, ..., \Delta x_N, \Delta t$ and a lattice $Q_T^\Delta = Q^\Delta \times \{0, ..., (\Delta t) N_T\}$ where $Q^\Delta = \{(i_1 \Delta x_1, ..., i_N \Delta x_N),\ I = (i_1, ..., i_N) \in \mathbb{Z}^N\}$ and N_T is the integer part of $T/\Delta t$, we will denote by $(x_1, ..., x_N, t_n) = (x_I, t_n)$ the node of the lattice $(i_1 \Delta x_1, ..., i_N \Delta x_N, n\Delta t)$ and by v_I^n the value of a numerical approximation of the exact solution $u(x_I, t_n)$. We set $\Delta x = \sqrt{\Delta x_1^2 + ... + \Delta x_N^2}$ the space mesh size. We shall assume throughout that $\Delta x + \Delta t \leq 1$.

The discrete solution v is computed iteratively by solving the implicit scheme

$$v_I^0 = \tilde{u}_0(x_I), \qquad \frac{v_I^{n+1} - v_I^n}{\Delta t} = c^{\Delta}[v]_I^{n+1} G(v^{n+1})_I \tag{5.9}$$

where $\tilde{u}_0$ is an approximation of $\bar{u}_0$ and $G(v^{n+1})_I$ is a suitable approximation of the gradient of v^{n+1} taken at point x_I. The non-local velocity is the discrete convolution

$$c^{\Delta}[v]_I^{n+1} = c^{\Delta}[v](x_I, t_{n+1}) = \sum_{J \in \mathbb{Z}^N} \bar{c}^0_{I-J} 1_{\{v_J^{n+1} \geq v_I^{n+1}\}} \Delta x_1 ... \Delta x_N - \frac{1}{2} \sum_{J \in \mathbb{Z}^N} \bar{c}^0_J \Delta x_1 ... \Delta x_N \tag{5.10}$$

with

$$\bar{c}^0_I = \frac{1}{|Q_I|} \int_{Q_I} c_0(x)\, dx \tag{5.11}$$

where Q_I is the square cell centred in x_I

$$Q_I = [x_{i_1} - \Delta x_1/2, x_{i_1} + \Delta x_1/2] \times ... \times [x_{i_N} - \Delta x_N/2, x_{i_N} + \Delta x_N/2]. \tag{5.12}$$

Finally, let us define

$$v_{\#}(y, t_n) = \sum_I v(x_I, t_n) \chi_{Q_I}(y) \tag{5.13}$$

where χ_{Q_I} is the indicator function of Q_I.

The approximation of the gradient is obtained using the Osher, Sethian scheme [154] (we can also use the one proposed by Rouy, Tourin [159]). It is monotone, consistent and depends on the sign of the non-local velocity. Its precised definition is recalled in Section 5.

Since the velocity c^{Δ} is non-local and not continuous, we have to give a sense to the equality in the scheme (5.9). In fact, we will use the analogue of the Slepčev formulation [169] for discrete sub and super-solution (see Definition 5.1) and we will use a discrete version of the Perron's method to construct a discrete solution. The solution of the scheme is not unique but, for any solutions, we have the following Crandall-Lions type [63] error estimate :

Theorem 2.2. (Discrete-continuous error estimate for (5.8))
Let $N \geq 1$ and $T \leq 1$. Assume that $\bar{u}_0, \tilde{u}_0 \in W^{1,\infty}(\mathbb{R}^N)$ and that c_0 given in (5.2) satisfies $c_0 \in W^{1,1}(\mathbb{R}^N)$. Assume that $\Delta x + \Delta t \leq 1$. Then there exists a constant $K_2 > 0$ depending only on N, $|c_0|_{W^{1,1}(\mathbb{R}^N)}$, $|D\bar{u}_0|_{L^\infty(\mathbb{R}^N)}$ and $|D\tilde{u}_0|_{L^\infty(\mathbb{R}^N)}$ such that the error estimate between the continuous solution u of (5.8) and any discrete solution v of the finite difference scheme (5.9) is given by

$$\sup_{\mathbb{R}^N \times \{0, ..., t_{N_T}\}} |u - v_{\#}| \leq K_2 \sqrt{T} \, (\Delta x + \Delta t)^{1/2} + \sup_{\mathbb{R}^N} |\bar{u}_0 - (\tilde{u}_0)_{\#}|$$

provided $\Delta x + \Delta t \leq \dfrac{1}{K_2^2}$.

Remark 2.3. *It is possible to explicit the computation of the gradient, i.e., to replace the term $G(v^{n+1})_I$ by $G(v^n)_I$ in the scheme (5.9) and to consider the solution v of*

$$v_I^0 = \tilde{u}_0(x_I), \qquad \frac{v_I^{n+1} - v_I^n}{\Delta t} = c^\Delta[v]_I^{n+1} G(v^n)_I. \tag{5.14}$$

In this case, as usual, we have to satisfy a CFL condition like for instance

$$\Delta t \leq \frac{\Delta x}{2|c_0|_{L^1(\mathbb{R}^N)}}$$

for the Osher Sethian discretisation of the gradient. Under this additional assumption, Theorem 2.2 remains true with v solution of the scheme (5.14).

2.3 Discrete continuous error estimate for Mean Curvature Motion

Using the above results, we will prove in Section 3 the following theorem :

Theorem 2.4. (Discrete-continuous error estimate for the mean curvature motion)
Let $N \geq 1$ and $T \leq 1$. Let us denote by v^ε a solution of (5.9)-(5.10)-(5.11) with initial condition $\tilde{u}_0^\varepsilon$ (which is an approximation of u_0) and with c_0^ε in the place of c_0. Assume that $u_0, \tilde{u}_0^\varepsilon \in W^{1,\infty}(\mathbb{R}^N)$, $g \in \mathrm{Lip}(\mathbf{S}^{N-1})$ and that c_0 given in (5.2) satisfies $c_0 \in W^{1,1}(\mathbb{R}^N)$. Assume also that $\Delta x + \Delta t \leq 1$. Then there exists a constant $K_3 > 0$ depending only on N, $\sup_{\mathbb{R}^N} c_0$, $|Dg|_{L^\infty(\mathbf{S}^{N-1})}$, $|c_0|_{W^{1,1}(\mathbb{R}^N)}$, $|Du_0|_{L^\infty(\mathbb{R}^N)}$ and $|D\tilde{u}_0^\varepsilon|_{L^\infty(\mathbb{R}^N)}$ such that the error estimate between the continuous solution u^0 of (5.5) and its numerical approximation v^ε is given by

$$\sup_{\mathbb{R}^N \times \{0,..,t_{N_T}\}} |u^0 - v_\#^\varepsilon| \leq K_3 \left(\frac{T}{|\ln \varepsilon|}\right)^{1/6} + \sup_{\mathbb{R}^N} |u_0 - (\tilde{u}_0^\varepsilon)_\#|$$

where $\varepsilon \geq K_3(\Delta x + \sqrt{\Delta t})$.

Remark 2.5. *If $T \geq 1$, since $|Du^0|_{L^\infty(\mathbb{R}^N \times (0,T))} \leq |Du_0|_{L^\infty(\mathbb{R}^N)}$ and "$|Dv^\varepsilon|_{L^\infty(\mathbb{R}^N)}$"$\leq |D\tilde{u}_0^\varepsilon|_{L^\infty(\mathbb{R}^N)}$ (see Proposition 5.4 for the exact setting), we can iterate the process and get a linear estimate in T, i.e*

$$\sup_{\mathbb{R}^N \times \{0,..,t_{N_T}\}} |u^0 - v_\#^\varepsilon| \leq \frac{K_3}{|\ln \varepsilon|^{1/6}} T + \sup_{\mathbb{R}^N} |u_0 - (\tilde{u}_0^\varepsilon)_\#|.$$

Remark 2.6. *We can truncate the kernel c_0 at infinity and consider*

$$\tilde{c}_0^R(x) = \begin{cases} c_0(x) & \text{if } |x| \leq R, \\ 0 & \text{else.} \end{cases}$$

In this case, we make an error of order $\int_{\mathbb{R}^N\setminus B_R(0)} c_0 \le \frac{K}{R}$ *and we can make the computation on a finite stencil, even if* Δx *goes to zero. This is possible, if we choose* ε *of the same order of* Δx*. The condition* $\varepsilon \ge K(\Delta x + \sqrt{\Delta t})$ *in Theorem 2.4 then implies that we need to impose a CFL condition* $\Delta t \le K\Delta x^2$ *(which is classical for second order equation).*

At the opposite, if we do not impose any CFL condition, we can choose Δt *larger than* Δx^2*, but we have to choose* ε *of the same order of* $\sqrt{\Delta t}$ *and so to make the convolution on larger and larger stencils as* Δx *goes to zero.*

As we mention in the introduction, it is also possible to explicit the computation of the velocity in the scheme (5.9) and to consider

$$v_I^0 = \tilde{u}_0(x_I), \qquad \frac{v_I^{n+1} - v_I^n}{\Delta t} = c^\Delta[v]_I^n G(v^n)_I, \tag{5.15}$$

with c_0^ε in the place of c_0 in the definition of c^Δ.

We can prove the convergence of the scheme (see Theorem 2.7) under the CFL condition

$$\Delta t \le \frac{\varepsilon|\ln\varepsilon|}{2|c_0|_{L^1(\mathbb{R}^N)}}\Delta x. \tag{5.16}$$

But, in this case, we are not able to prove an error estimate. The reason is that when we implicit the velocity, it "freezes" it, and then we can use the consistency error on the scheme with given velocity. At the opposite, for the explicit scheme, we have to control the consistency error of the velocity which is not possible as we point out in Proposition 5.6.

We define

$$v^{\varepsilon,\delta}(x,t) = v^\varepsilon(x_I, t_n) \quad \text{if} \quad x \in Q_I,\ t \in [t_n, t_{n+1})$$

where $\delta = (\Delta x, \Delta t)$ and v^ε is the solution of the scheme (5.15)-(5.10)-(5.11) with the kernel c_0^ε in the place of c_0. Then we have the following convergence result :

Theorem 2.7. (Convergence of the explicit scheme)
Assume (5.16)*. Under the assumptions of Theorem 2.4, the function* $v^{\varepsilon,\delta}$ *converges locally uniformly on compact sets as* $\delta \to 0$ *to* u^ε *solution of* (5.4)*.*

Remark 2.8. *In Theorem 2.7, if we take the limit* $\delta \to 0$ *and* $\varepsilon \to 0$ *with* $\delta \ll \varepsilon$*, we will approach the solution of* (5.5)*. The condition* $\delta \ll \varepsilon$ *implies that, in practice, we have to make the convolution in larger and larger stencil as* $\delta \to 0$*.*

Remark 2.9. *In the scheme* (5.15)*, it is possible to implicit the computation of the gradient and thus to withdraw the CFL condition* (5.16) *to get the same result as Theorem 2.7.*

Notation In what follows, we will denote by K a generic constant, which will then satisfy $K + K = K$, $K \cdot K = K$, and so on.

3 Numerical scheme for mean curvature motion

3.1 Proof of Theorem 2.4

Using a rescaling argument, we will prove the following corollary of Theorem 2.2 :

Corollary 3.1. (Discrete-continuous error estimate for (5.4))
Let us denote by v^ε a solution of (5.9)-(5.10)-(5.11) with initial condition $\tilde{u}_0^\varepsilon$ (which is an approximation of u_0^ε) and with c_0^ε in the place of c_0. Under the assumptions of Theorem 2.2, there exists a constant $K > 0$ depending only on $|c_0|_{W^{1,1}(\mathbb{R}^N)}$, $|Du_0^\varepsilon|_{L^\infty(\mathbb{R}^N)}$ and $|D\tilde{u}_0^\varepsilon|_{L^\infty(\mathbb{R}^N)}$ such that the error estimate between the continuous solution u^ε of (5.4) and its numerical approximation v^ε is given by

$$\sup_{\mathbb{R}^N\times\{0,\dots,t_{N_T}\}} |u^\varepsilon - v^\varepsilon_\#| \le K\frac{1}{\varepsilon}\sqrt{\frac{T}{|\ln\varepsilon|}}\sqrt{\varepsilon\Delta x + \frac{\Delta t}{|\ln\varepsilon|}} + \sup_{\mathbb{R}^N}|u_0^\varepsilon - (\tilde{u}_0^\varepsilon)_\#|$$

provided $\varepsilon\Delta x + \dfrac{\Delta t}{|\ln\varepsilon|} \le \dfrac{\varepsilon^2}{K^2}$.

Proof of Corollary 3.1
First we remark that a simple change of variable gives (for u and v solutions respectively of (5.8) and (5.9))

$$u^\varepsilon(y,\tau) = \varepsilon u\left(\frac{y}{\varepsilon}, \frac{\tau}{\varepsilon^2|\ln\varepsilon|}\right), \qquad u_0^\varepsilon(y) = \varepsilon\bar{u}_0\left(\frac{y}{\varepsilon}\right)$$

$$v^\varepsilon(y,\tau) = \varepsilon v\left(\frac{y}{\varepsilon}, \frac{\tau}{\varepsilon^2|\ln\varepsilon|}\right), \qquad \tilde{u}_0^\varepsilon(y) = \varepsilon\tilde{u}_0\left(\frac{y}{\varepsilon}\right).$$

Let us denote by $x = \dfrac{y}{\varepsilon}$, $t = \dfrac{\tau}{\varepsilon^2|\ln\varepsilon|}$, $\Delta x = \dfrac{\Delta y}{\varepsilon}$, $\Delta t = \dfrac{\Delta\tau}{\varepsilon^2|\ln\varepsilon|}$, $T = \dfrac{\Gamma}{\varepsilon^2|\ln\varepsilon|}$, $s_n = n\Delta\tau$ and by N_Γ the integer part of $\Gamma/\Delta\tau$. Then, the following inequality holds :

$$\begin{aligned}
\sup_{R^N\times\{0,\dots,s_{N_\Gamma}\}} |u^\varepsilon - v^\varepsilon_\#| =& \varepsilon \sup_{R^N\times\{0,\dots,t_{N_T}\}} |u - v_\#| \\
\le& K\frac{\sqrt{\Gamma}}{\sqrt{|\ln\varepsilon|}}\sqrt{\frac{\Delta y}{\varepsilon} + \frac{\Delta\tau}{\varepsilon^2|\ln\varepsilon|}} + \varepsilon\sup_{\mathbb{R}^N}|\bar{u}_0 - (\tilde{u}_0)_\#| \\
\le& K\frac{\sqrt{\Gamma}}{\varepsilon\sqrt{|\ln\varepsilon|}}\sqrt{\varepsilon\Delta y + \frac{\Delta\tau}{|\ln\varepsilon|}} + \sup_{\mathbb{R}^N}|u_0^\varepsilon - (\tilde{u}_0^\varepsilon)_\#|
\end{aligned}$$

provided that

$$\frac{\Delta y}{\varepsilon} + \frac{\Delta \tau}{\varepsilon^2 |\ln \varepsilon|} \leq \frac{1}{K^2},$$

i.e.

$$\varepsilon \Delta y + \frac{\Delta \tau}{|\ln \varepsilon|} \leq \frac{\varepsilon^2}{K^2}.$$

This ends the proof of the corollary.

Proof of Theorem 2.4
The proof of Theorem 2.4 is now very easy. Indeed, using Theorem 2.1 and Corollary 3.1, we deduce that

$$\begin{aligned}|u^0 - v^\varepsilon| \leq & |u^0 - u^\varepsilon| + |u^\varepsilon - v^\varepsilon| \\ \leq & K\left(\frac{T}{|\ln \varepsilon|}\right)^{\frac{1}{6}} + \frac{K}{\varepsilon\sqrt{|\ln \varepsilon|}}\sqrt{T}\sqrt{\varepsilon \Delta x + \Delta t} \\ & + \sup_{\mathbb{R}^N} |u_0^\varepsilon - (\tilde{u}_0^\varepsilon)_\#| + \sup_{\mathbb{R}^N} |u_0^\varepsilon - u_0|.\end{aligned}$$

Taking $\varepsilon \geq K(\Delta x + \sqrt{\Delta t})$ implies the prevalence of the first term with respect to the second one and so this implies the desired estimate for the choice of initial condition $u_0^\varepsilon = u_0$.

3.2 Proof of Theorem 2.7

The idea of the proof is borrowed from Barles, Souganidis [28]. Let us set

$$\overline{v}^\varepsilon = \limsup_{\delta \to 0}{}^* \, v^{\varepsilon,\delta}, \quad \underline{v}^\varepsilon = \liminf_{\delta \to 0}{}_* \, v^{\varepsilon,\delta}.$$

We now prove that $\overline{v}^\varepsilon$ and $\underline{v}^\varepsilon$ are respectively sub and super-solution of (5.4). Let $\varphi \in C^\infty(\mathbb{R}^N \times [0,T))$ such that $\overline{v}^\varepsilon - \varphi$ reaches a strict maximum at the point (x_0, t_0) with $t_0 \in (0,T)$. Then, there exists $\delta \to 0$, $(x^\delta, t^\delta) \to (x_0, t_0)$ such that $v^{\varepsilon,\delta} - \varphi$ reaches a maximum at (x^δ, t^δ). We denote by $(x_I^\delta, t_n^\delta) \in Q_T^\Delta$ the node such that $v^{\varepsilon,\delta}(x^\delta, t^\delta) = v^{\varepsilon,\delta}(x_I^\delta, t_n^\delta)$. This implies that, for all (x,t) :

$$v^{\varepsilon,\delta}(x,t) \leq v^{\varepsilon,\delta}(x_I^\delta, t_n^\delta) - \varphi(x^\delta, t^\delta) + \varphi(x,t).$$

Using the monotony of the scheme for given velocity and the fact that v is solution of the scheme, we get

$$\begin{aligned}&\frac{\varphi(x^\delta, t^\delta) - \varphi(x^\delta, t^\delta - \Delta t)}{\Delta t} \\ \leq & c^\varepsilon[v^{\varepsilon,\delta}](x_I^\delta, t^\delta - \Delta t)\left\{\begin{array}{r} G^+(D^+\varphi(x^\delta, t^\delta - \Delta t), D^-\varphi(x^\delta, t^\delta - \Delta t)) \\ \text{if} \quad c[v^{\varepsilon,\delta}](x_I^\delta, t^\delta - \Delta t) \geq 0 \\ G^-(D^+\varphi(x^\delta, t^\delta - \Delta t), D^-\varphi(x^\delta, t^\delta - \Delta t)) \\ \text{if} \quad c[v^{\varepsilon,\delta}](x_I^\delta, t^\delta - \Delta t) \leq 0 \end{array}\right.\end{aligned}$$

with

$$c^\varepsilon[u](x,t) = c_0^\varepsilon \star 1_{\{u(\cdot,t)\geq u(x,t)\}}(x) - \frac{1}{2}\int_{\mathbb{R}^N} c_0^\varepsilon$$

and where we have used Lemma 5.5 to obtain the velocity. Sending $\delta \to 0$, using Slepčev Lemma [169, equation (5)] and the consistency of the scheme for given velocity, we get

$$\varphi_t(x_0,t_0) \leq \left(c_0^\varepsilon \star 1_{\{\overline{v}^\varepsilon(\cdot,t_0)\geq \overline{v}^\varepsilon(x_0,t_0)\}}(x_0) - \frac{1}{2}\int_{\mathbb{R}^N} c_0^\varepsilon \right) |D\varphi(x_0,t_0)|.$$

This proves that $\overline{v}^\varepsilon$ is a sub-solution of (5.4). The proof for $\underline{v}^\varepsilon$ is the same and we skip it. Moreover, using a barrier argument (using the equivalent of Proposition 5.4) we get that $\overline{v}^\varepsilon(\cdot,0) = \underline{v}^\varepsilon(\cdot,0) = u_0^\varepsilon$. Then we have that $v^\varepsilon = \overline{v}^\varepsilon = \underline{v}^\varepsilon$ is solution of (5.4). This ends the proof of the theorem.

Remark 3.2. *In the scheme* (5.15)*, we have made the choice to take the velocity* $c^\Delta[v]_I^n$*. An another possibility is to take*

$$\tilde{c}^\Delta[v]_I^{n+1} = \sum_{J\in\mathbb{Z}^N} \bar{c}^0_{I-J} 1_{\{v_J^{n+1}>v_I^{n+1}\}} \Delta x_1 \ldots \Delta x_N - \frac{1}{2}\sum_{J\in\mathbb{Z}^N} \bar{c}^0_J \Delta x_1\ldots\Delta x_N.$$

This construct two discrete solution v and $\tilde{v}$. So, we can take any discrete function comprised between $\tilde{v}$ and v. In fact, this is equivalent to define a discrete solution as a sub and a super-solution as in Definition 5.1. With this definition, v will be the greater sub-solution and $\tilde{v}$ will be the lower super-solution and Theorem 2.7 will be true for every solutions.

3.3 What happens if we change the kernel ?

A natural question is what happens if we change the kernel and if we take a kernel which decrease more quickly at infinity. For this kind of kernel K_0, the natural scaling is the following one (see [67, Section 4.1])

$$K_0^\varepsilon(x) = \frac{1}{\varepsilon^{n+1}} K_0\left(\frac{x}{\varepsilon}\right).$$

In fact, using the same arguments as the one we use for the proof of Theorem 2.1, we can prove the same kind of error estimate. For example for $K_0(x) = \dfrac{1}{|x|^{n+p}}$ for $|x| \geq 1$, with $p \geq 3$, we get

$$|u^\varepsilon - u^0| \leq K\sqrt{\varepsilon}T^{1/4}.$$

This is the best estimate for ε small that we can obtain for general kernel, as we can see in Step 3, Case 1 of the proof of Theorem 2.1.

The main difference is that the estimate of Corollary 3.1 is replaced by

$$\sup_{\mathbb{R}^N\times\{0,\dots,t_{N_T}\}} |u^\varepsilon - v^\varepsilon_\#| \le K\frac{\sqrt{T}}{\varepsilon}\sqrt{\varepsilon\Delta x + \Delta t} + \sup_{\mathbb{R}^N} |u_0^\varepsilon - (\tilde{u}_0^\varepsilon)_\#| \quad \text{for } \varepsilon\Delta x + \Delta t \le \frac{\varepsilon^2}{K^2}.$$

Finally, we obtain for the choice $u_0^\varepsilon = u_0$

$$\sup_{\mathbb{R}^N\times\{0,\dots,t_{N_T}\}} |u^0 - (v^\varepsilon)_\#| \le K\sqrt{\varepsilon}T^{\frac{1}{4}} + \sup_{\mathbb{R}^N} |u_0 - \tilde{u}_0^\varepsilon|$$

if $\varepsilon \ge K\left(\sqrt{\Delta x} + (\Delta t)^{\frac{1}{3}}\right)$. This implies that $\dfrac{\Delta x}{\varepsilon} \to 0$ as $\Delta x \to 0$. So, when Δx goes to zero, we have to make the convolution on all the space which can be very expensive in practice. This approach can also be used for Gaussian kernel and should give the same estimates. In particular, this could give an error estimate for the equivalent version of the classical Bence, Merriman, Osher [143] algorithm.

4 Proof of Theorem 2.1

Before to prove Theorem 2.1, we need some notation. Let us define

$$F(M,p) = \operatorname{trace}\left(M\cdot A\left(\frac{p}{|p|}\right)\right) \quad \text{and} \quad G(M,p) = \frac{-1}{|p|}F(M,p)$$

where we recall that $A(p)\cdot p = 0\ \forall p \in \mathbf{S}^{N-1}$.

Then we have the following fundamental estimate for balls :

Lemma 4.1. (Error estimate for a ball)
Let $\varphi \in C^2(\mathbb{R}^n)$ with $D\varphi(x_0) \neq 0$, be such that the set $\{\varphi(x) \ge \varphi(x_0)\}$ is a ball of radius R. For $c_0^\varepsilon(\cdot) = \dfrac{1}{\varepsilon^{n+1}|\ln\varepsilon|}\ c_0\left(\dfrac{\cdot}{\varepsilon}\right)$, let us define

$$c^\varepsilon = (c_0^\varepsilon \star 1_{\{\varphi(\cdot)>\varphi(x_0)\}})(x_0) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon.$$

Then, there exists a constant $K = K(N, \sup_{\mathbb{R}^n} c_0, |Dg|_{L^\infty(\mathbf{S}^{N-1})}) > 0$ such that for $0 < \varepsilon < \delta$ with $0 < \delta \le R/2$, we have

$$|c^\varepsilon + G(D^2\varphi(x_0), D\varphi(x_0))| \le K\cdot e\,(\varepsilon,\delta,R)$$

with

$$e\,(\varepsilon,\delta,R) = \frac{1}{|\ln\varepsilon|}\left(\frac{1}{\delta} + \frac{1}{R}|\ln\delta|\right) + \frac{\delta}{R^2}.$$

This is a straightforward consequence of Da Lio *et al.* [67, Proposition 4.1] where their Lemma 4.3 is replaced by a direct estimate for an explicit function $h(x') = R - \sqrt{R^2 - |x'|^2}$ with $x' \in \mathbb{R}^{N-1}$ and $|x'| \leq R$.

Proof of Theorem 2.1
Let u^0 be the solution of the mean curvature motion (5.5). The idea of the proof is to regularise the function u^0 by a kind of sup-convolution but using quartic penalization and then to plug the regularised function $u^{0,\alpha}$ into (5.4). This regularisation allows us to control quantitatively the first and the second derivatives in space of $u^{0,\alpha}$. This is a quantitative version of the definition of Barles and Georgelin [22] for viscosity solutions of mean curvature motion where they used this kind of regularisation to prove that one can take test functions such that $D\varphi \neq 0$ or $D\varphi = 0$ and $D^2\varphi = 0$ in the definition of viscosity solutions. This kind of arguments is also used to obtained the comparison principle for mean curvature type equations.

The proof is decomposed into five steps :
Step 1. Regularisation of u^0
By classical estimates for mean curvature type equations, we have

$$\|Du^0\|_{L^\infty(\mathbb{R}^N \times (0,T))} \leq \|Du_0\|_{L^\infty(\mathbb{R}^N)}. \tag{5.17}$$

We regularise u^0 by considering a spatial sup-convolution by quartics

$$u^{0,\alpha}(x,t) = \sup_{y \in \mathbb{R}^n} \left\{ u^0(x-y,t) - \frac{1}{4\alpha}|y|^4 \right\} \tag{5.18}$$

$$= \sup_{z \in \mathbb{R}^n} \left\{ u^0(z,t) - \frac{1}{4\alpha}|z-x|^4 \right\}. \tag{5.19}$$

Since u^0 is Lipschitz continuous, the supremum is reached.
Step 2. Estimate between u^0 and u^0
For $x \in \mathbb{R}^N$ and $t \in (0,T)$, we denote by $\bar{x} = \bar{x}(x,t)$ a point where the maximum is reached in (5.19). Then

$$u^{0,\alpha}(x,t) = u^0(\bar{x},t) - \frac{1}{4\alpha}|\bar{x}-x|^4 \geq u^0(x,t). \tag{5.20}$$

Therefore

$$\frac{1}{4\alpha}|\bar{x}-x|^4 \leq u^0(\bar{x},t) - u^0(x,t) \leq K|\bar{x}-x| \tag{5.21}$$

and so

$$|\bar{x}-x| \leq K\alpha^{1/3}. \tag{5.22}$$

Moreover, by (5.20), we get

$$0 \leq u^{0,\alpha}(x,t) - u^0(x,t) \leq u^0(\bar{x},t) - u^0(x,t) \leq K|\bar{x}-x|. \tag{5.23}$$

Using (5.22) and (5.23), we then deduce for $x \in \mathbb{R}^N$ and $t \in (0,T)$

$$|u^{0,\alpha}(x,t) - u^0(x,t)| \leq K\alpha^{1/3} \tag{5.24}$$

Before to continue the proof, we need some notation. We now want to compare u^ε and $u^{0,\alpha}$. To do that, we will prove that $u^{0,\alpha} \pm (f(\varepsilon,\alpha)t + K\alpha^{\frac{1}{3}} + \sup_{\mathbb{R}^N} |u_0^\varepsilon - u_0|)$ are respectively super and sub-solutions of Problem (5.4) for

$$f(\varepsilon,\alpha) = K\left(\frac{1}{\sqrt{|\ln\varepsilon|}} + \frac{\ln|\ln\varepsilon|}{\alpha^{1/3}|\ln\varepsilon|} + \frac{1}{\alpha^{2/3}}\left(\frac{1}{\sqrt{|\ln\varepsilon|}} + \varepsilon\right)\right).$$

Let $\psi \in C^\infty(\mathbb{R}^n \times (0,T))$ such that $u^{0,\alpha} - \psi$ reaches a global strict maximum at $(x_0,t_0) \in \mathbb{R}^N \times (0,T)$. Since $u^{0,\alpha}$ is semi-convex in the space variable, the functions $u^{0,\alpha}$ is differentiable with respect to x at the point (x_0,t_0).

Let us denote by

$$\varphi_z(x) = u^0(z,t_0) - \frac{1}{4\alpha}|z-x|^4.$$

Then

$$u^{0,\alpha}(x,t_0) = \sup_{z\in\mathbb{R}^N} \varphi_z(x).$$

We denote by $\bar{x}_0$ a point where the maximum is reached for $x = x_0$ and by $\varphi = \varphi_{\bar{x}_0}$. We then deduce that

$$\psi(x,t_0) \geq u^{0,\alpha}(x,t_0) \geq \varphi(x) \tag{5.25}$$

with equality for $x = x_0$. This implies that

$$D\varphi(x_0) = Du^{0,\alpha}(x_0,t_0) = D\psi(x_0,t_0) \tag{5.26}$$

and

$$D^2\varphi(x_0) \leq D^2\psi(x_0,t_0). \tag{5.27}$$

Moreover, with $h = |\bar{x}_0 - x_0|$, we have (by (5.22))

$$h \leq K\alpha^{\frac{1}{3}}, \tag{5.28}$$

$$|D\varphi(x_0)| = \frac{h^3}{\alpha} \tag{5.29}$$

and the set $\{\varphi(x) \geq \varphi(x_0)\}$ is a ball of radius $R = h$. Let us define the error

$$e(\varphi) = |c^\varepsilon[\varphi](x_0)|D\varphi(x_0)| - F(D^2\varphi(x_0), D\varphi(x_0))|$$

where

$$c^\varepsilon[\varphi](x_0) = \left(c_0^\varepsilon \star 1_{\{\varphi(\cdot)\geq\varphi(x_0)\}}\right)(x_0) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon.$$

Step 3. $e(\varphi) \le f(\varepsilon, \alpha)$

We distinguish three cases :

Case 1. $h \le 2\varepsilon$. In this case, we directly have the estimate (using (5.29))

$$|c^\varepsilon[\varphi](x_0)|D\varphi(x_0)|| \le |c_0^\varepsilon|_{L^1(\mathbb{R}^n)}\frac{h^3}{\alpha} \le \frac{h^2|c_0|_{L^1(\mathbb{R}^n)}}{\varepsilon|\ln\varepsilon|}\frac{h}{\alpha} \le \frac{K}{\alpha^{2/3}}\frac{\varepsilon}{|\ln\varepsilon|}$$

where we have also used (5.28). Using moreover the fact that

$$|F(D^2\varphi(x_0), D\varphi(x_0))| \le \frac{K}{h}|D\varphi(x_0)| \le K\frac{h^2}{\alpha},$$

we then deduce that

$$e(\varphi) \le \frac{K}{\alpha^{2/3}}\varepsilon \le f(\varepsilon,\alpha).$$

Case 2. $2\varepsilon < h \le \dfrac{2}{\sqrt{|\ln\varepsilon|}}$. Using (5.29) and Lemma 4.1, with $\delta = h/2$, we then deduce that

$$\begin{aligned} e(\varphi) \le & \frac{Kh^3}{\alpha}\left(\frac{1}{|\ln\varepsilon|}\left(\frac{1}{h}+\frac{1}{h}|\ln\delta|\right)+\frac{1}{h}\right) \le \frac{Kh}{\alpha^{2/3}} \le \frac{K}{\alpha^{2/3}}\frac{1}{\sqrt{|\ln\varepsilon|}} \\ \le & f(\varepsilon,\alpha). \end{aligned}$$

Case 3. $h > \dfrac{2}{\sqrt{|\ln\varepsilon|}}$. We set $\delta = \dfrac{1}{\sqrt{|\ln\varepsilon|}}$. We then have $\varepsilon < \delta < h/2$. So by Lemma 4.1, we deduce that

$$\begin{aligned} e(\varphi) \le & \frac{Kh^3}{\alpha}\left(\frac{1}{\sqrt{|\ln\varepsilon|}}+\frac{1}{h}\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}+\frac{1}{h^2}\frac{1}{\sqrt{|\ln\varepsilon|}}\right) \\ \le & K\left(\frac{1}{\sqrt{|\ln\varepsilon|}}+\frac{1}{\alpha^{1/3}}\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}+\frac{1}{\alpha^{2/3}}\frac{1}{\sqrt{|\ln\varepsilon|}}\right) \\ \le & f(\varepsilon,\alpha). \end{aligned}$$

Step 4. Estimate between u^0 and u

By construction, we have, since $u^{0,\alpha}$ is a sub-solution of (5.5) (but does not satisfy the initial condition)

$$\begin{aligned} \psi_t(x_0,t_0) \le & F(D^2\psi(x_0,t_0), D\psi(x_0,t_0)) \\ \le & c^\varepsilon[\varphi](x_0)|D\psi(x_0,t_0)| + F(D^2\varphi(x_0), D\varphi(x_0)) - c^\varepsilon[\varphi](x_0)|D\varphi(x_0)| \\ \le & c^\varepsilon[u^{0,\alpha}(\cdot,t_0)](x_0)|D\psi(x_0,t_0)| + f(\varepsilon,\alpha) \end{aligned}$$

where we have used (5.25) for the last line. On the other hand, we have

$$u^{0,\alpha}(\cdot,0) - K\alpha^{1/3} - \sup_{\mathbb{R}^N} |u_0^\varepsilon - u_0| \le u_0^\varepsilon$$

where we have used (5.24) at the limit $t = 0$. We then deduce that $\tilde{u}^{0,\alpha}(x,t) = u^{0,\alpha}(x,t) - f(\varepsilon,\alpha)t - K\alpha^{1/3} - \sup_{\mathbb{R}^N} |u_0^\varepsilon - u_0|$ is sub-solution of (5.4). This implies that $\tilde{u}^{0,\alpha} \le u^\varepsilon$ and so

$$u^{0,\alpha} - u^\varepsilon \le f(\varepsilon,\alpha)t + K\alpha^{1/3} + \sup_{\mathbb{R}^N} |u_0^\varepsilon - u_0|. \tag{5.30}$$

Step 5. Estimate between u^0 and u
Using (5.24) and (5.30), we then get

$$u^0 - u^\varepsilon = u^0 - u^{0,\alpha} + u^{0,\alpha} - u^\varepsilon \le f(\varepsilon,\alpha)t + K\alpha^{1/3} + \sup_{\mathbb{R}^N} |u_0^\varepsilon - u_0|.$$

But

$$\begin{aligned} f(\varepsilon,\alpha)t + K\alpha^{1/3} =& K\left(\frac{1}{\sqrt{|\ln\varepsilon|}} + \frac{\ln|\ln\varepsilon|}{\alpha^{1/3}|\ln\varepsilon|} + \frac{1}{\alpha^{2/3}}\left(\frac{1}{\sqrt{|\ln\varepsilon|}} + \varepsilon\right)\right)t + K\alpha^{1/3} \\ \le& K\left(\frac{T}{\sqrt{|\ln\varepsilon|}}\right)^{\frac{1}{3}} \end{aligned}$$

for $\alpha = \dfrac{Kt}{\sqrt{|\ln\varepsilon|}}$ and $t \le T \le 1$. We then get

$$u^0 - u^\varepsilon \le K\left(\frac{T}{\sqrt{|\ln\varepsilon|}}\right)^{\frac{1}{3}} + \sup_{\mathbb{R}^N} |u_0^\varepsilon - u_0|.$$

A lower bound is proved similarly (using the same result as Lemma 4.1 where the set $\{\varphi(x) \ge \varphi(x_0)\}$ is replaced by $\{\varphi(x) > \varphi(x_0)\}$). This implies (5.7) for $T \le 1$. This ends the proof of the theorem.

5 Numerical scheme for dislocations dynamics

5.1 Definitions and preliminary results

We recall here the notation used in the scheme. The discrete solution v is computed iteratively by solving the implicit scheme

$$v_I^0 = \tilde{u}_0(x_I), \qquad \frac{v_I^{n+1} - v_I^n}{\Delta t} = c^\Delta[v]_I^{n+1} G(v^{n+1})_I \tag{5.31}$$

where the non-local velocity is defined in (5.10). The approximation of the gradient of v^{n+1} at the point x_I is given by

$$G(v^{n+1})_I = \begin{cases} G^+(D^+v_I^{n+1}, D^-v_I^{n+1}) & \text{if} \quad c^\Delta[v]_I^{n+1} \geq 0, \\ G^-(D^+v_I^{n+1}, D^-v_I^{n+1}) & \text{if} \quad c^\Delta[v]_I^{n+1} < 0 \end{cases}$$

where $G^\pm$ is a suitable approximation of the Euclidean norm and $D^\pm v^n(x_I) = (D^\pm_{x_1} v^n(x_I), ..., D^\pm_{x_N} v^n(x_I))$ are the discrete gradients. The terms $D^\pm_{x_i} v^n(x_I)$ are the standard forward and backward first order differences, *i.e.* for a general function $f(x_I)$:

$$D^+_{x_i} f(x_I) = \frac{f(x_{I^{i,+}}) - f(x_I)}{\Delta x_i},$$

$$D^-_{x_i} f(x_I) = \frac{f(x_I) - f(x_{I^{i,-}})}{\Delta x_i},$$

where

$$I^{k,\pm} = (i_1, ..., i_{k-1}, i_k \pm 1, i_{k+1}, ..., i_N). \tag{5.32}$$

The approximations of the Euclidean norm $G^\pm$ are those proposed by Osher and Sethian [154] (we can also use the ones proposed by Rouy, Tourin [159]) :

$$G^+(P,Q) = \left(\sum_{i=1,..,N} \max(P_i, 0)^2 + \sum_{i=1,..,N} \min(Q_i, 0)^2 \right)^{\frac{1}{2}}, \tag{5.33}$$

$$G^-(P,Q) = \left(\sum_{i=1,..,N} \min(P_i, 0)^2 + \sum_{i=1,..,N} \max(Q_i, 0)^2 \right)^{\frac{1}{2}}.$$

We recall that the functions $G^\pm$ are Lipschitz continuous with respect to the discrete gradients, *i.e.*

$$|G^\pm(P_1, P_2) - G^\pm(P_1', P_2')| \leq (|P_1 - P_1'| + |P_2 - P_2'|). \tag{5.34}$$

They are consistent with the Euclidean norm

$$G^\pm(P, P) = |P|, \tag{5.35}$$

and $G^\pm = G^\pm(P_1^+, ..., P_N^+, P_1^-, ..., P_N^-)$ enjoy suitable monotonicity with respect to each variable

$$\frac{\partial G^+}{\partial p_i^+} \geq 0, \quad \frac{\partial G^+}{\partial p_i^-} \leq 0, \quad \frac{\partial G^-}{\partial p_i^+} \leq 0, \quad \frac{\partial G^-}{\partial p_i^-} \geq 0. \tag{5.36}$$

To define discrete solutions, we need the following notation

$$\tilde{c}^\Delta[v]_I^{n+1} = \sum_{J \in \mathbb{Z}^N} \bar{c}^0_{I-J} 1_{\{v_J^{n+1} > v_I^{n+1}\}} \Delta x_1 ... \Delta x_N - \frac{1}{2} \sum_{J \in \mathbb{Z}^N} \bar{c}^0_J \Delta x_1 ... \Delta x_N, \tag{5.37}$$

$$\tilde{G}(v^{n+1})_I = \begin{cases} G^+(D^+v_I^{n+1}, D^-v_I^{n+1}) & \text{if} \quad \tilde{c}^\Delta[v]_I^{n+1} \geq 0, \\ G^-(D^+v_I^{n+1}, D^-v_I^{n+1}) & \text{if} \quad \tilde{c}^\Delta[v]_I^{n+1} < 0 \end{cases}$$

where $\bar{c}_0$ is defined in (5.11). Finally, for simplicity of presentation, let us denote by

$$s[v]_I^{n+1} = \text{Sign}(c^\Delta[v]_I^{n+1}), \quad \tilde{s}[v]_I^{n+1} = \text{Sign}(\tilde{c}^\Delta[v]_I^{n+1}).$$

Definition 5.1. (Numerical sub, super and solution of the scheme)
We say that v is a discrete sub-solution (resp super-solution) of the scheme (5.9) if for all $I \in \mathbb{Z}^N$, $n \in \mathbb{N}$, we have

$$v_I^{n+1} \leq v_I^n + \Delta t \; c^\Delta[v]_I^{n+1} G(v^{n+1})_I$$

$$\left(\text{resp. } v_I^{n+1} \geq v_I^n + \Delta t \; \tilde{c}^\Delta[v]_I^{n+1} \tilde{G}(v^{n+1})_I\right).$$

We say that v is a discrete solution if and only if it is a sub and a super-solution.

Lemma 5.2. (Almost monotonicity of the scheme)
The scheme (5.9) *is almost monotone in the following sense. Let v, w be two discrete functions and assume that there is I such that $w_I = v_I$ and $w_J \geq v_J$ for $J \neq I$. Then*

$$c^\Delta[v]_I G(v)_I \leq c^\Delta[w]_I G(w)_I$$

and

$$\tilde{c}^\Delta[v]_I G(v)_I \leq \tilde{c}^\Delta[w]_I G(w)_I.$$

Proof of Lemma 5.2
First, we remark that $c^\Delta[v]_I \leq c^\Delta[w]_I$. Then, there is three cases :

1. $c^\Delta[v]_I \leq 0 \leq c^\Delta[w]_I$. In this case, the result is trivial.
2. $0 \leq c^\Delta[v]_I \leq c^\Delta[w]_I$. Using the monotonicity of G, we get that $G(v)_I \leq G(w)_I$. This implies the result.
3. $c^\Delta[v]_I \leq c^\Delta[w]_I \leq 0$. Using the monotonicity of G, we get that $G(v)_I \geq G(w)_I$. This implies the result.

The proof for $\tilde{c}^\Delta$ is the same and we skip it. This ends the proof of the lemma.

The existence of a solution for the scheme is not trivial (since it is implicit and non-local). This is the subject of the following proposition :

Proposition 5.3. (Existence of solution for the scheme (5.9))
There exists, at least, one discrete solution v of the scheme (5.9) in the sense of Definition 5.1.

Proof of Proposition 5.3
We assume that there exists a solution v^n at step n and we will construct a solution v^{n+1} at step $n+1$. First, we remark that if $(v^{n+1,i})_i$ is a family indexed by i of discrete sub-solution at step $n+1$, then $v^{n+1} = \max\ v^{n+1,i}$ is still a sub-solution (it suffices to use Lemma 5.2). Moreover, $w^+ = \sup_{Q^\Delta} |v^n|$ and $w^- = -\sup_{Q^\Delta} |v^n|$ are respectively discrete super and sub-solutions to the scheme (5.9). Then, let us define

$$v^{n+1} = \max\{w \text{ subsolution at step } n+1 \text{ s.t } w \le w^+\}.$$

Then v^{n+1} is a discrete sub-solution at step $n+1$. Now let us prove that v^{n+1} is a super-solution. By contradiction, assume that there is I such that

$$\frac{v_I^{n+1} - v_I^n}{\Delta t} < \tilde{c}[v]_I^{n+1} G^{\tilde{s}[v]_I^n}(D^+v_I^{n+1}, D^-v_I^{n+1}).$$

This implies in particular that $v_I^{n+1} < w^+$. Now, let us consider, the solution w_I (in a sense similar to Definition 5.1) of

$$\frac{w_I - v_I^n}{\Delta t} = c^{\Delta}_{v^{n+1}}[w]_I \left\{ \begin{array}{ll} G^+(D^+_{v^{n+1}}w_I, D^-_{v^{n+1}}w_I) & \text{if} \quad c^{\Delta}_{v^{n+1}}[w]_I \ge 0 \\ G^-(D^+_{v^{n+1}}w_I, D^-_{v^{n+1}}w_I) & \text{if} \quad c^{\Delta}_{v^{n+1}}[w]_I < 0 \end{array} \right. ,$$

where

$$c^{\Delta}_{v^{n+1}}[w]_I = \sum_{J\in\mathbb{Z}^N} \bar{c}^0_{I-J} 1_{\{v_J^{n+1} \ge w_I\}} \Delta x_1 ... \Delta x_N - \frac{1}{2} \sum_{J\in\mathbb{Z}^N} \bar{c}^0_J \Delta x_1 ... \Delta x_N$$

and $D^{\pm}_{v^{n+1}} w_I = (D^{\pm}_{x_1,v^{n+1}} w_I, ..., D^{\pm}_{x_N,v^{n+1}} w_I)$ with

$$D^+_{x_i,v^{n+1}} w_I = \frac{v^{n+1}_{I^{i,+}} - w_I}{\Delta x_i},$$

$$D^-_{x_i,v^{n+1}} w_I = \frac{w_I - v^{n+1}_{I^{i,-}}}{\Delta x_i},$$

where $I^{k,\pm}$ is defined in (5.32). The existence of such a solution comes from the fact that the left hand side is non-decreasing in w_I and the right hand side is non-increasing. Then, it is easy to prove (using Lemma 5.2) that $w_I > v_I^{n+1}$ and that w defined by

$$w_J = \left\{ \begin{array}{ll} w_I & \text{if } J = I \\ v_J^{n+1} & \text{otherwise} \end{array} \right.$$

is a discrete sub-solution of (5.9) at step $n+1$. This contradicts the definition of v^{n+1} and ends the proof of the proposition.

Proposition 5.4. (Properties of the discrete solutions)
Assume that $\tilde{u}_0 \in W^{1,\infty}(\mathbb{R}^N)$*. Let us denote by* $L = |D\tilde{u}_0|_{L^\infty(\mathbb{R}^N)}$ *and*

$$\overline{v} = \sup\{v \textit{ subsolution of } (5.9) \textit{ with initial condition } v^0 = \tilde{u}_0, v \leq |\tilde{u}_0|_{L^\infty(\mathbb{R}^N)}\}. \tag{5.38}$$

For $k \in \mathbb{Z}^N \backslash \{0\}$*, we denote by* $k\Delta X = (k_1\Delta x_1, ..., k_N \Delta x_N)$*. Then, the following estimates hold*

$$\frac{\overline{v}(x_I, t_n) - \overline{v}(x_I + k\Delta X, t_n)}{|k\Delta X|} \leq L, \tag{5.39}$$

$$\frac{\overline{v}(x_I, t_n) - \overline{v}(x_I + k\Delta X, t_n)}{|k\Delta X|} \geq -L, \tag{5.40}$$

$$\left| \frac{\overline{v}_I^{n+1} - \overline{v}_I^n}{\Delta t} \right| \leq \frac{\sqrt{N}}{2} |c_0|_{L^1(\mathbb{R}^N)} \, L. \tag{5.41}$$

The same estimates hold for

$$\underline{v} = \inf\{v \textit{ supersolution of } (5.9) \textit{ with initial condition } v^0 = \tilde{u}_0, v \geq -|u_0|_{L^\infty(\mathbb{R}^N)}\}. \tag{5.42}$$

Proof of Proposition 5.4
For $k \in \mathbb{Z}^N \backslash \{0\}$, let us denote by

$$\tilde{u}_{0,k}(x_I) = \tilde{u}_0(x_I + k\Delta X) + L|k\Delta X| \geq \tilde{u}_0(x_I)$$

and by $\overline{v}_k$ the greater sub-solution, with initial condition $\tilde{u}_{0,k}$, *i.e.*

$$\overline{v}_k = \sup\{v_k \text{ subsolution of } (5.9) \text{ with initial condition } v_k^0 = \tilde{u}_{0,k}, v_k \leq |u_0|_{L^\infty(\mathbb{R}^N)}\}.$$

First, since the equation does not see the constants, we remark that $\overline{v} - L|k\Delta X|$ is a sub-solution of (5.9) with initial condition $\overline{v}^0 - L|k\Delta X| \leq \tilde{u}_{0,k}$. So, by definition of $\overline{v}_k$, we have

$$\overline{v} - L|k\Delta X| \leq \overline{v}_k.$$

Moreover, using the fact that the scheme is invariant by translation in space, we deduce that $\overline{v}(x_I + k\Delta X, t_n)$ is the greater sub-solution with initial condition $\overline{v}(x_I + k\Delta X, 0) = \tilde{u}_{0,k}(x_I)$. We then deduce that

$$\overline{v}_k(x_I, t_n) = \overline{v}(x_I + k\Delta X, t_n).$$

This implies that

$$\frac{\overline{v}(x_I, t_n) - \overline{v}(x_I + k\Delta X, t_n)}{|k\Delta X|} \leq L.$$

The proof of (5.40) is similar and we skip it.

To prove (5.41), we use the fact that $\overline{v}$ is a solution and the two previous estimates. This implies

$$-\frac{\sqrt{N}}{2}|c_0|_{L^1(\mathbb{R}^N)}L \leq \frac{\overline{v}_I^{n+1}-\overline{v}_I^n}{\Delta t} \leq \frac{\sqrt{N}}{2}|c_0|_{L^1(\mathbb{R}^N)}L$$

which is the desired result.

Before to prove Theorem 2.2, we need the following lemma :

Lemma 5.5. (Equivalent formulation for the discrete velocity)
The discrete velocity $c^\Delta[v]$ can be rewritten as

$$c^\Delta[v]_I^n = \left(c_0 \star 1_{\{v_\#(.,t_n)\geq v_\#(x_I,t_n)\}}\right)(x_I) - \frac{1}{2}|c_0|_{L^1(\mathbb{R}^N)}$$

where $v_\#$ is defined in (5.13).

Proof of Lemma 5.5
The idea of the proof is borrowed from Alvarez *et al.* [8]. Using the definition of the discrete velocity and $\bar{c}^0$, we get

$$\begin{aligned}
c^\Delta[v]_I^n &= \sum_{J\in\mathbb{Z}^N} \bar{c}^0_{I-J}1_{\{v_J^n\geq v_I^n\}}\Delta x_1 ... \Delta x_N - \frac{1}{2}\sum_{J\in\mathbb{Z}^N}\bar{c}^0_J \Delta x_1 ... \Delta x_N\\
&= \sum_{J\in\mathbb{Z}^N}\int_{Q_{J-I}} c_0(y)\; dy\; 1_{\{v_J^n\geq v_I^n\}} - \frac{1}{2}\sum_{J\in\mathbb{Z}^N}\int_{Q_J} c_0(y)\; dy\\
&= \sum_{J\in\mathbb{Z}^N}\int_{Q_J} c_0(x_I-y)\; 1_{\{v_J^n\geq v_I^n\}}\; dy\; - \frac{1}{2}|c_0|_{L^1(\mathbb{R}^N)}\\
&= \sum_{J\in\mathbb{Z}^N}\int_{Q_J} c_0(x_I-y)\; 1_{\{v_\#(y,t_n)\geq v_\#(x_I,t_n)\}}\; dy\; - \frac{1}{2}|c_0|_{L^1(\mathbb{R}^N)}\\
&= \left(c_0 \star 1_{\{v_\#(\cdot,t_n)\geq v_\#(x_I,t_n)\}}\right)(x_I) - \frac{1}{2}|c_0|_{L^1(\mathbb{R}^N)}.
\end{aligned}$$

This ends the proof of the lemma.

We now prove Theorem 2.2 :
Proof of Theorem 2.2
The proof is an adaptation of the one of Crandall Lions [63], revisited by Alvarez *et al.* [7]. Nevertheless, for the reader's convenience, we give the main steps in order to show the new difficulties due to the non-local term. The main idea of the proof is the same as the one of comparison principles, *i.e.* to consider the maximum of $u - v_\#$, to duplicate the variable and to use the viscosity inequalities to get the result.

The proof splits into four steps.

We first assume that

$$\bar{u}_0(x_I) \geq (\tilde{u}_0(x_I))_\#, \quad \text{for all } I \in \mathbb{Z}^N. \tag{5.43}$$

and we set

$$\mu^0 = \sup_{\mathbb{R}^N}(\bar{u}_0 - (\tilde{u}_0)_\#) \geq 0. \tag{5.44}$$

We denote throughout by K various constant depending only on N, $|c_0|_{W^{1,1}(\mathbb{R}^N)}$, $|D\bar{u}_0|_{L^\infty(\mathbb{R}^N)}$, $|D\tilde{u}_0|_{L^\infty(\mathbb{R}^N)}$ and μ^0.

Step 1 : Estimate on v

We have the following estimate for the discrete solution

$$-Kt_n \leq \bar{u}_0(x) - v_\#(x,t_n) \leq Kt_n + \mu^0. \tag{5.45}$$

To show this, it suffices to use the estimate (5.41) of Proposition 5.4. This implies that, for $x \in Q_I$ (with the notation $\overline{v}$ and $\underline{v}$ defined in Proposition 5.4)

$$\begin{aligned}(\tilde{u}_0)_\#(x) - Kt_n = v_I^0 - Kt_n \leq \underline{v}_I^n \leq & v_I^n = v_\#(x,t_n) \\ \leq & \overline{v}_I^n \leq Kt_n + v_I^0 = Kt_n + (\tilde{u}_0)_\#(x).\end{aligned}$$

This implies the desired estimate.

Before continuing the proof, we need a few notations. We put

$$\mu = \sup_{\mathbb{R}^N \times [0,T]} (u - v_\#).$$

We want to bound from above μ by μ^0 plus a constant. For every $0 < \alpha \leq 1$, $0 < \gamma \leq 1$ and $0 < \eta \leq 1$, we set

$$M_\eta^{\alpha,\gamma} = \sup_{\mathbb{R}^N \times \mathbb{R}^N \times (0,T) \times \{0,\ldots,t_{N_T}\}} \Psi_\eta^{\alpha,\gamma}(x,y,t,t_n)$$

with

$$\Psi_\eta^{\alpha,\gamma}(x,y,t,t_n) = u(x,t) - v_\#(y,t_n) - \frac{|x-y|^2}{2\gamma} - \frac{|t-t_n|^2}{2\gamma} - \eta t - \alpha(|x|^2 + |y|^2).$$

We shall drop the super and subscripts on Ψ when no ambiguity arises as concerning the value of the parameter.

The main difference with the classical Crandall Lions proof, is to consider the function $v_\#$ in the place of v in the separation of variables. This allows us to treat the non-local velocity.

Since u is Lipschitz continuous and $T \leq 1$, we have

$$|u(x,t)| \leq K(1+|x|). \tag{5.46}$$

Moreover by Step 1, we have

$$\begin{aligned}|v(x_i,t_n)| \leq & |v(x_i,t_n) - \bar{u}_0(x_i)| + |\bar{u}_0(x_i)| \\ \leq & Kt_n + K(1+|x_i|) \\ \leq & K(1+|x_i|).\end{aligned}$$

We then deduce that Ψ achieves its maximum at some point that we denote by (x^*, y^*, t^*, t_n^*).

Step 2 : Estimates for the maximum point of Ψ

The maximum point of Ψ enjoys the following estimates

$$\alpha|x^*| + \alpha|y^*| \leq K \tag{5.47}$$

and

$$|x^* - y^*| \leq K\gamma, \quad |t^* - t_n^*| \leq (K+2\eta)\gamma. \tag{5.48}$$

Indeed, by inequality $\Psi(x^*, y^*, t^*, t_n^*) \geq \Psi(0,0,0,0) \geq 0$, we obtain

$$\begin{aligned}\alpha|x^*|^2 + \alpha|y^*|^2 \leq & u(x^*,t^*) - v_{\#}(y^*,t_n^*) \leq K(1+|x^*|+|y^*|) \\ \leq & K + \frac{K^2}{\alpha} + \frac{\alpha}{2}|x^*|^2 + \frac{\alpha}{2}|y^*|^2.\end{aligned}$$

This implies (5.47), since $\alpha \leq 1$.

The first bound of (5.48) follows from the Lipschitz regularity in space of u, from the inequality $\Psi(x^*, y^*, t^*, t_n^*) \geq \Psi(y^*, y^*, t^*, t_n^*)$ and from (5.47). The second bound of (5.48) is obtained in the same way, using the inequality $\Psi(x^*, y^*, t^*, t_n^*) \geq \Psi(x^*, y^*, t_n^*, t_n^*)$.

Step 3 : A better estimate for the maximum point of Ψ

Inequality (5.47) can be strengthened to

$$\alpha|x^*|^2 + \alpha|y^*|^2 \leq K. \tag{5.49}$$

Indeed, using the Lipschitz regularity of u, the inequality $\Psi(x^*, y^*, t^*, t_n^*) \geq \Psi(0,0,0,0)$, Step 1 and equation (5.48), yields

$$\begin{aligned}\alpha|x^*|^2 + \alpha|x_i^*|^2 \leq & u(x^*,t^*) - v_{\#}(y^*,t_n^*) + u_0(y^*) - u_0(y^*) \\ \leq & K(|x^*-y^*| + t^*) + Kt_n^* + \mu^0 \leq K.\end{aligned}$$

Step 4 : Upper bound of μ

We have the bound $\mu \leq K\sqrt{T}\,(\Delta x + \Delta t)^{\frac{1}{2}} + \mu^0$ if $\Delta x + \Delta t \leq \frac{1}{K^2}$.

First, we claim that for η large enough, we have either $t^* = 0$ or $t_n^* = 0$. We argue by contradiction. Then the function $(x,t) \mapsto \Psi(x, y^*, t, t_n^*)$ achieves its maximum at a point of $\mathbb{R}^N \times (0,T)$. Then, using the fact that u is a sub-solution of (5.8), we deduce that

$$\eta + p_t^* \leq c[u](x^*, t^*)|p_x^* + 2\alpha x^*| \tag{5.50}$$

where

$$c[u](x,t) = c_0 \star 1_{\{u(\cdot,t) \geq u(x,t)\}}(x) - \frac{1}{2}\int_{\mathbb{R}^N} c_0$$

and

$$p_t^* = \frac{t^* - t_n^*}{\gamma}, \qquad p_x^* = \frac{x^* - y^*}{\gamma}.$$

Since $t_n^* > 0$, we also have $\Psi(x^*, y^*, t^*, t_n^*) \geq \Psi(x^*, y, t^*, t_n)$ for $t_n \geq t_n^* - \Delta t$. This implies

$$v_\#(y^*, t_n^*) - v_\#(y, t_n) \leq \varphi(y^*, t_n^*) - \varphi(y, t_n) \quad \text{for } t_n \geq t_n^* - \Delta t \tag{5.51}$$

for $\varphi(y, t_n) = -\left(\frac{|x^* - y|^2}{2\gamma} + \frac{|t^* - t_n|^2}{2\gamma} + \alpha|y|^2\right)$. We denote by x_I^* the node such that $y^* \in Q_{I^*}$ (the unit cube centred in x_I^*). We then deduce

$$\begin{aligned}
\frac{\varphi(y^*, t_n^*) - \varphi(y^*, t_n^* - \Delta t)}{\Delta t} &\geq \frac{v_\#(y^*, t_n^*) - v_\#(y^*, t_n^* - \Delta t)}{\Delta t} \\
&= \frac{v(x_I^*, t_n^*) - v(x_I^*, t_n^* - \Delta t)}{\Delta t} \\
&\geq \tilde{c}^\Delta[v](x_I^*, t_n^*) G^{s[v](x_I^*, t_n^*)}(D^+ v(x_I^*, t_n^*), D^- v(x_I^*, t_n^*)) \\
&\geq \tilde{c}^\Delta[v](x_I^*, t_n^*) G^{s[v](x_I^*, t_n^*)}(D^+ \varphi(y^*, t_n^*), D^- \varphi(y^*, t_n^*))
\end{aligned}$$

where we have use the monotonicity of $G^\pm$ and (5.51) with $t_n = t_n^*$ for the last line. We denote by $s^* = s[v](x_I^*, t_n^*)$. Straightforward computations of the discrete derivative of φ yield

$$\begin{aligned}
&p_t^* + \frac{\Delta t}{2\gamma} \\
&\geq \tilde{c}^\Delta[v](x_I^*, t_n^*) G^{s^*}\left(p_x^* - \frac{\Delta x}{2\gamma} - \alpha(2y^* + \Delta x), p_x^* + \frac{\Delta x}{2\gamma} - \alpha(2y^* - \Delta x)\right).
\end{aligned} \tag{5.52}$$

Subtracting the above inequality to (5.50) yields

$$
\begin{aligned}
\eta \leq & \frac{\Delta t}{2\gamma} + c[u](x^*,t^*)|p_x^* + 2\alpha x^*| \\
& - \tilde{c}^\Delta[v](x_I^*,t_n^*)G^{s^*}\left(p_x^* - \frac{\Delta x}{2\gamma} - \alpha(2y^*+\Delta x), p_x^* + \frac{\Delta x}{2\gamma} - \alpha(2y^*-\Delta x)\right) \\
\leq & \frac{\Delta t}{2\gamma} + (c[u](x^*,t^*) - \tilde{c}^\Delta[v](x_I^*,t_n^*))|p_x^*| + \alpha K|x^*| \\
& + \tilde{c}^\Delta[v](x_I^*,t_n^*)\Bigg|G^{s^*}\left(p_x^* - \frac{\Delta x}{2\gamma} - \alpha(2y^*+\Delta x), p_x^* + \frac{\Delta x}{2\gamma} - \alpha(2y^*-\Delta x)\right) \\
& \qquad - G^{s^*}(p_x^*,p_x^*)\Bigg| \\
\leq & \frac{\Delta t}{2\gamma} + (c[u](x^*,t^*) - \tilde{c}^\Delta[v](x_I^*,t_n^*))|p_x^*| + \alpha K|x^*| + K\left(\frac{\Delta x}{\gamma} + 2\alpha|y^*| + 2\alpha\Delta x\right).
\end{aligned}
$$

We now have to bound the term

$$
\begin{aligned}
\mathcal{I} =& (c[u](x^*,t^*) - \tilde{c}^\Delta[v](x_I^*,t_n^*))|p_x^*| \\
=& (c_0 \star 1_{\{u(\cdot,t^*)\geq u(x^*,t^*)\}}(x^*) - c_0 \star 1_{\{v_\#(\cdot,t_n^*)>v_\#(y^*,t_n^*)\}}(x_I^*))|p_x^*|.
\end{aligned}
$$

We have to distinguish two cases :

Case 1. Assume that for $\gamma > 0$ fixed, $\dfrac{|x^*-y^*|^2}{\gamma} \geq C_\gamma^2$ for some constant $C_\gamma > 0$ and for all α small.

By inequality $\Psi(x^*,y^*,t^*,t_n^*) \geq \Psi(x,x,t^*,t_n^*)$, we deduce that

$$
\begin{aligned}
& u(x,t^*) - u(x^*,t^*) \\
\leq & v_\#(x,t_n^*) - v_\#(y^*,t_n^*) - \frac{|x^*-y^*|^2}{2\gamma} - \alpha(|x^*|^2 + |y^*|^2) + 2\alpha|x|^2.
\end{aligned}
$$

We then get the inclusion

$$\{u(\cdot,t^*) \geq u(x^*,t^*)\} \cap \{v_\#(\cdot,t_n^*) \leq v_\#(y^*,t_n^*)\} \subset \{|x|^2 \geq R_{\alpha,\gamma}^2\}$$

where $R_{\alpha,\gamma}^2 = \frac{1}{2\alpha}\left(\frac{|x^*-y^*|^2}{2\gamma} + \alpha(|x^*|^2+|y^*|^2)\right)$. We also have

$$\{|x-x^*| \geq R_{\alpha,\gamma}\} \subset \{|x| \geq \tilde{R}_{\alpha,\gamma}\}$$

where $\tilde{R}_{\alpha,\gamma} = R_{\alpha,\gamma} - |x^*| \to \infty$ as $\alpha \to 0$ (see Da Lio *et al.* [67, Lemma 2.5]). We then obtain

$$\begin{aligned}\mathcal{I} \leq & (c_0 \star 1_{\{v_\#(\cdot,t_n^*)>v_\#(y^*,t_n^*)\}}(x^*) - c_0 \star 1_{\{v_\#(\cdot,t_n^*)>v_\#(y^*,t_n^*)\}}(x_I^*) + c_0 \star 1_{B^c(0,R_{\alpha,\gamma})}(x^*))|p_x^*| \\ \leq & K\left(|Dc_0|_{L^1(\mathbb{R}^N)}|x^* - x_I^*| + \int_{B^c(0,\tilde{R}_{\alpha,\gamma})} c_0(x)dx\right) \\ < & K(\gamma + \Delta x) + o_\alpha(1)\end{aligned}$$

where we have used (5.48) to bound $|p_x^*|$ in the second line.

Case 2. Assume that there exists a subsequence $\alpha_n > 0$ which we still denote by α such that

$$\frac{|x^* - y^*|^2}{\gamma} \to 0 \quad \text{as } \alpha \to 0.$$

We then get that $p_x^* \to 0$ as $\alpha \to 0$. Since the velocities are bounded, we get that $\mathcal{I} = o_\alpha(1)$.

To sum up, we have shown that

$$\mathcal{I} < K(\gamma + \Delta x) + o_\alpha(1).$$

We then get that

$$\eta < K\left(\frac{\Delta x + \Delta t}{\gamma} + \gamma\right) + o_\alpha(1).$$

Putting $\eta^* = K\left(\frac{\Delta x + \Delta t}{\gamma} + \gamma\right) + o_\alpha(1)$, we then conclude that either $t^* = 0$ or $t_n^* = 0$ for $\eta \geq \eta^*$.

Whenever $t^* = 0$, we get for $\eta^* \leq \eta \leq 1$

$$\begin{aligned}M = & \Psi(x^*, y^*, 0, t_n^*) \leq u_0(x^*) - v_\#(y^*, t_n^*) \\ \leq & K|x^* - y^*| + Kt_n^* + \mu^0 \\ \leq & K\gamma + \mu^0\end{aligned}$$

The same result holds whenever $t_n^* = 0$. We then deduce that for $\eta = \eta^*$ and $(x, t_n) \in \mathbb{R}^N \times \{0, ..., t_{N_T}\}$

$$u(x, t_n) - v_\#(x, t_n) - \eta^* T - 2\alpha|x_I|^2 \leq M \leq K\gamma + \mu^0.$$

Sending $\alpha \to 0$, taking the supremum over (x, t_n) and choosing $\gamma = T^{1/2}(\Delta x + \Delta t)^{1/2}$, we get

$$\mu \leq K(\Delta x + \Delta t)^{1/2}\sqrt{T} + \mu^0$$

provided $\Delta x + \Delta t \leq \frac{1}{K^2}$. Using the same arguments of Alvarez *et al.*[7, Theorem 2], we easily deduce the result in the general case. This ends the proof of Theorem 2.2.

We now point out that there is no consistency error for the scheme (5.15). If there is a consistency error, then there will be a consistency error for the velocities, *i.e.* for all $\varphi \in C^1(\mathbb{R}^N \times (0,T))$ with $\|\varphi\|_{C^1(\mathbb{R}^N\times(0,T))} \leq C$ then

$$c^\Delta[\varphi_{|Q_T^\Delta}]_I^n - c[\varphi](x_I, t_n) \leq f(\Delta x, \Delta t) \tag{5.53}$$

with $f(\Delta x, \Delta t) \to 0$ as $(\Delta x, \Delta t) \to 0$.

Proposition 5.6. (No consistency error)
There is no consistency error for the scheme (5.15)*, i.e. equation* (5.53) *does not hold.*

Proof of Proposition 5.6
We have to prove that there exists a constant $C_0 > 0$ such that for all $\Delta x, \Delta t > 0$ there exists a function φ such that

$$c^\Delta[\varphi_{|Q_T^\Delta}]_I^n - c[\varphi](x_I, t_n) \geq C_0 \quad \text{and } |\varphi|_{C^1(\mathbb{R}^N\times(0,T))} \leq C.$$

To see this, it suffices to take a function φ which oscillates as shown in Figure 5.1.

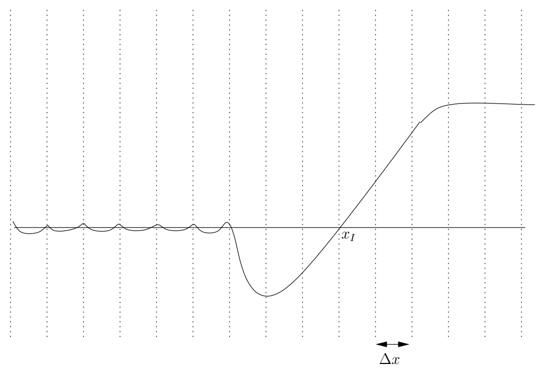

FIG. 5.1 – Graph of the function φ defined in $\mathbb{R}$.

Remark 5.7. *There is no consistency error, but the scheme is consistent. Indeed, if we fix the function φ in the previous proof, then using Slepčev Lemma [169, equation (5)], we get*

$$\lim_{\Delta x \to 0} c^\Delta[\varphi_{|Q_T^\Delta}]_I^n - c[\varphi](x_I, t_n) = 0.$$

We now point out that the two scheme (5.9) and (5.15) are not monotone :

Proposition 5.8. (Non monotonicity of the scheme)
The two scheme (5.9) *and* (5.15) *are not monotone for general $\vec{c}^0$.*

Proof of Proposition 5.8

1. **The implicit scheme** (5.9)
   We give a simple counter-example in the one dimensional case. We take $\tilde{u}_i^n = v_i^n = i\Delta x$. The goal is to construct a sub-solution u^{n+1} and a super-solution v^{n+1} such that
   $$u_i^{n+1} = i\Delta x + c_1$$
   $$v_i^{n+1} = i\Delta x + c_2.$$
   A simple computation gives
   $$c_1 = \Delta t \sum_{j\geq 0} \bar{c}_j^0 \quad \text{and} \quad c_2 = \Delta t \sum_{j>0} \bar{c}_j^0.$$
   We then deduce that $c_1 - c_2 = \Delta t \bar{c}_0^0 > 0$ (if $\bar{c}_0^0 > 0$). This implies that we can construct two different solutions (the supremum of sub-solutions and the infimum of super-solutions). This implies that the scheme (5.9) is not monotone.
2. **The explicit scheme** (5.15)
   We also give a counter-example in the one dimensional case. We take
   $$u_i^n = \begin{cases} -\delta & \text{if } i = 0 \\ 1-\delta & \text{if } i = -1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad v_i^n = \begin{cases} \delta & \text{if } i = 0 \\ 1-\delta & \text{if } i = -1 \\ 0 & \text{otherwise} \end{cases}$$
   We then have $u_i^n \leq v_i^n$ for all $i \in \mathbb{Z}$. We then prove that $u_0^{n+1} > v_0^{n+1}$ for δ small enough. First we remark that $c^\Delta[v^n]_0 \leq 0$ (if $\sum_{i\neq 0,-1} \bar{c}_j^0 \geq \bar{c}_0^0 + \bar{c}_{-1}^0$), so $v_0^{n+1} \leq \delta$. Moreover,
   $$\begin{aligned} u_0^{n+1} =& -\delta + \frac{\Delta t}{\Delta x}\left(\sum_j \bar{c}_j^0 - \frac{1}{2}\sum_j \bar{c}_j^0\right)\sqrt{1+\delta^2} \\ \geq& -\delta + \frac{\Delta t}{\Delta x}\left(\frac{1}{2}|c_0|_{L^1(\mathbb{R})}\right) \\ >& \delta \end{aligned}$$
   if
   $$\delta < \frac{\Delta t}{2\Delta x}\left(\frac{1}{2}|c_0|_{L^1(\mathbb{R})}\right).$$
   This contradicts the monotonicity of the scheme.

This ends the proof of the Proposition.

6 Numerical Simulations

In this section, we provide some numerical simulations. In a first subsection, we explain a method to solve numerically the implicit scheme (5.9)-(5.10)-(5.11). Then we provide a simple simulation concerning a collapsing circle, with the implicit scheme and an another one, to highlight the fattening phenomena, with the explicit scheme.

6.1 How to solve the implicit scheme ?

To solve (5.9)-(5.10)-(5.11), we will use an iterative process. Assume that we have a solution v^n at step n. We now want to compute a solution v^{n+1} at step $n+1$. To do this, for $\tilde{w} \in \mathbb{R}^{Q^\Delta}$, we denote by $w = \Phi(\tilde{w})$ the solution of the following auxiliary scheme :

$$\frac{w_I - v_I^n}{\Delta t} = c_{\tilde{w}}^\Delta[w]_I \left\{ \begin{array}{lll} G^+(D_{\tilde{w}}^+ w_I, D_{\tilde{w}}^- w_I) & \text{if} & c_{\tilde{w}}^\Delta[w]_I \geq 0 \\ G^-(D_{\tilde{w}}^+ w_I, D_{\tilde{w}}^- w_I) & \text{if} & c_{\tilde{w}}^\Delta[w]_I < 0 \end{array} \right. \tag{5.54}$$

where

$$c_{\tilde{w}}^\Delta[w]_I = \sum_{J \in \mathbb{Z}^N} \bar{c}_{I-J}^0 1_{\{\tilde{w}_J \geq w_I\}} \Delta x_1 ... \Delta x_N - \frac{1}{2} \sum_{J \in \mathbb{Z}^N} \bar{c}_J^0 \Delta x_1 ... \Delta x_N$$

and $D_{\tilde{w}}^\pm w_I = (D_{x_1,\tilde{w}}^\pm w_I, ..., D_{x_N,\tilde{w}}^\pm w_I)$ with

$$D_{x_i,\tilde{w}}^+ w_I = \frac{\tilde{w}_{I^{i,+}} - w_I}{\Delta x_i},$$

$$D_{x_i,\tilde{w}}^- w_I = \frac{w_I - \tilde{w}_{I^{i,-}}}{\Delta x_i},$$

where $I^{k,\pm}$ is defined in (5.32). Since the left hand side of (5.54) is non-decreasing in w_I and the right hand side is non-increasing, there exists a unique solution (in a sense similar of Definition 5.1) which can be computed using a dichotomy process.

The important point is that if $\tilde{w}$ is a sub-solution (resp. super-solution) then $w = (w_I)_I$ is still a sub-solution (resp. super-solution) and satisfies $w_I \geq \tilde{w}_I$ (resp. $w_I \leq \tilde{w}_I$) (see the proof of Proposition 5.3).

The idea is then to define $w^0 = v^n - C$ where the constant C is such that w^0 is a sub-solution of (5.9) and then to construct iteratively $w^{k+1} = \Phi(w^k)$. Setting $v^{n+1} = \lim_{k \to \infty} w^k$, we have that v^{n+1} is a solution of (5.9).

6.2 A collapsing circle

In this subsection, we provide a simple test with the implicit scheme concerning the evolution of a circle. The goal of this simple simulation is just to check that the circle will disappear with the good time. We take a circle of radius 1. The parameters are $\Delta x = 0,05$, $\Delta t = 0,01$ and $\varepsilon = 0,3$. Moreover, we take the kernel

$$c_0^\varepsilon = \begin{cases} 1 & \text{if} \quad |x| \leq 0,05 \\ \frac{1}{|\ln \varepsilon||x|^3} & \text{if} \quad 0,05 \leq |x| \leq 2 \\ 0 & \text{if} \quad |x| \geq 2 \end{cases}$$

The initial condition is the distance to the circle. The result is shown in Figure 5.2.

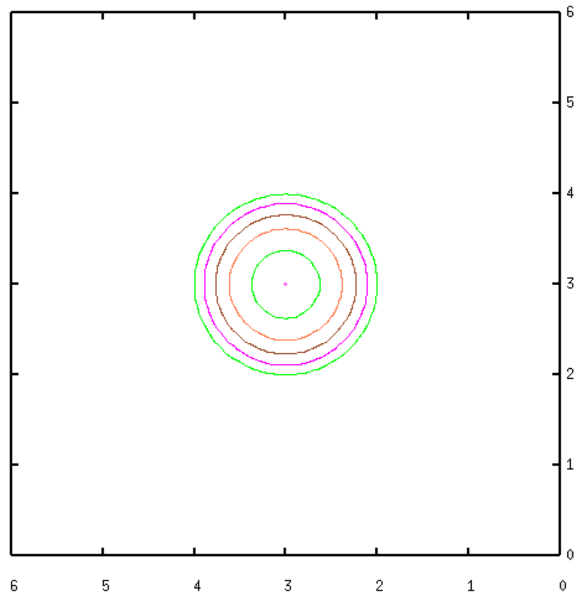

FIG. 5.2 – Evolution of a circle of radius 1 at time $0, 0.1, 0.2, 0.3, 0.4$ and 0.49

Numerically, the disappearing time is comprise between 0.49 and 0.50 which correspond to what we expect theoretically (the real time is 0.50).

6.3 Fattening phenomena

The second test is concerning with the evolution of the 8 to point out the fattening phenomena. This test has been made with the explicit scheme. We take two circles of radius $0,58$ such that they are tangent in one point and we look at the evolution of the level set 0 and $-0,06$. The parameters are $\Delta x_1 = \Delta x_2 = 0,01$, $\Delta t = 0,0001$ and $\varepsilon = 0,1$. Moreover, we take the kernel

$$c_0^\varepsilon(x) = \begin{cases} 0 & \text{if} \quad |x| \leq 0,3 \\ \frac{1}{|\ln \varepsilon||x|^3} & \text{if} \quad 0,05 \leq |x| \leq \frac{10}{3} \\ 0 & \text{if} \quad |x| \geq \frac{10}{3} \end{cases}$$

The results are provided in Figure 5.3.

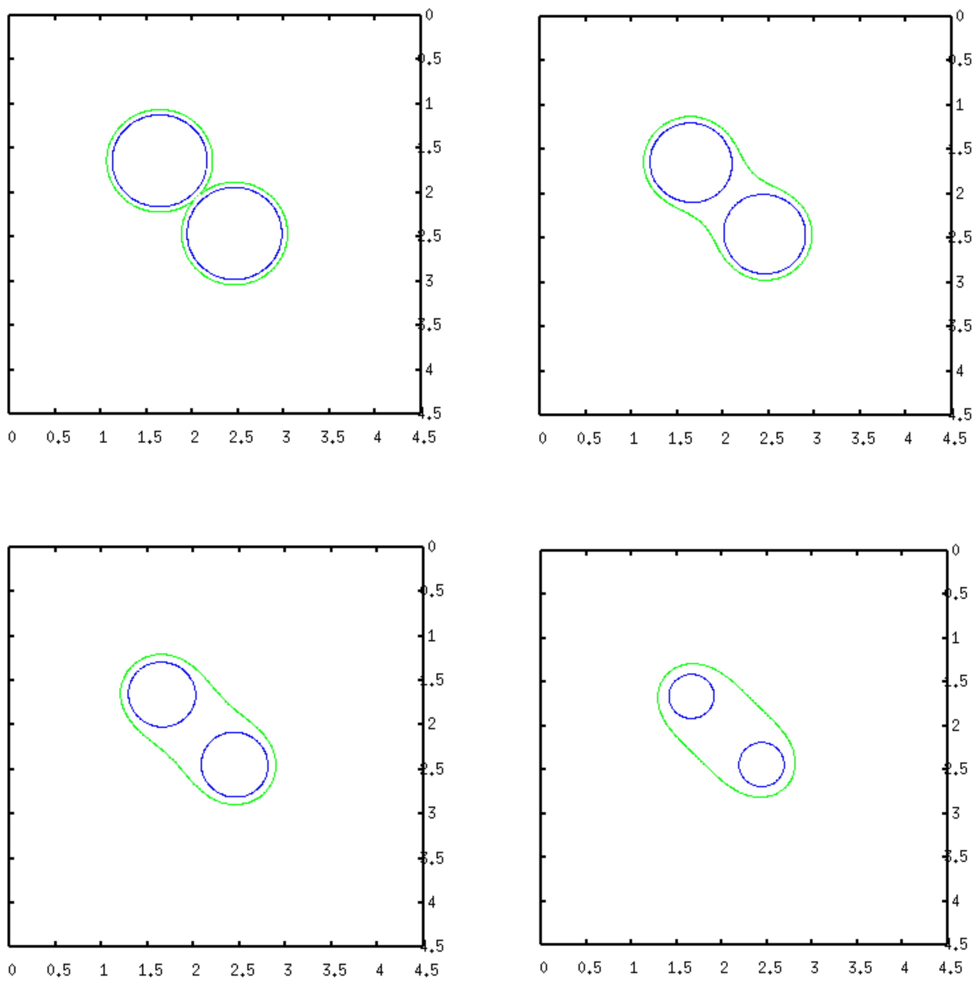

FIG. 5.3 – Evolution of the 8 at time 0, 0.05, 0.1 and 0.15.

Acknowledgements

The author would like to thanks R. Monneau for proposing him the problem and for fruitful discussions in the preparation of this paper and E. Carlini for enlighting discussions about numerical simulations. The author was supported by the contract JC 1025 called "ACI jeunes chercheuses et jeunes chercheurs" of the French Ministry of Research (2003-2006).

Chapitre 6

Convergence d'une méthode Fast Marching généralisée pour une équation eikonale non convexe

Ce chapitre est issu d'un travail en collaboration avec E. Carlini, M. Falcone et R. Monneau [55].
Dans ce travail, nous présentons un nouvel algorithme de type Fast Marching pour une équation eikonale non-convexe modélisant l'évolution d'un front dans sa direction normale. L'algorithme est une extension de la Fast Marching Method (proposée par Sethian) dans le sens où le nouveau schéma peut également prendre en compte des vitesses dépendantes du temps, sans aucune restriction de signe. Nous analysons les propriétés de l'algorithme et nous montrons sa convergence dans la classe des solutions de viscosités discontinues. Nous présentons également quelques simulations numériques pour des fronts se propageant dans $\mathbb{R}^2$.

Convergence of a Generalized Fast Marching Method for a non-convex eikonal equation

E. Carlini , M. Falcone, N. Forcadel, R. Monneau

Abstract

We present a new Fast Marching algorithm for a non-convex eikonal equation modeling front evolutions in the normal direction. The algorithm is an extension of the Fast Marching Method since the new scheme can deal with a *time-dependent* velocity without *any restriction on its sign*. We analyze the properties of the algorithm and we prove its convergence in the class of discontinuous viscosity solutions. Finally, we present some numerical simulations of fronts propagating in $\mathbb{R}^2$.

AMS Classification : 65M06, 65M12, 49L25.

Keywords : Hamilton Jacobi equations, fast marching scheme, convergence, viscosity solutions.

1 Introduction

The goal of this paper is to propose and analyze a numerical scheme to compute the evolution of a front driven by its normal velocity $c(x,t)$ under very general assumptions on c. In particular, we will remove the usual assumption which assigns to c a constant sign during the evolution. This means that the front can oscillate and pass several times over the same points. The initial front is the boundary of an open set Ω_0, which is represented by a characteristic function $1_{\Omega_0} - 1_{\Omega_0^c}$, defined equal to 1 on Ω_0 and -1 on its complementary set. Mathematically, we are interested in the discontinuous viscosity solution $\theta(x,t)$ of the following equation

$$\begin{cases} \theta_t(x,t) = c(x,t)|D\theta(x,t)| & \text{on} \quad \mathbb{R}^N \times (0,T) \\ \theta(\cdot,0) = 1_{\Omega_0} - 1_{\Omega_0^c}. \end{cases} \tag{6.1}$$

Here the support of the discontinuities of the function θ localizes the front we are interested in. This work is motivated by the numerical computation of dislocations dynamics where the velocity of the front can change sign (see Rodney, Le Bouar, Finel [158]).

A very popular method to describe the evolution of a front is the Level Sets method (see the seminal paper by Osher and Sethian [154] as well as the monographies [162, 163], [152]), where the discontinuous solution θ is replaced by a continuous function, and the equation is discretized using finite difference method with a CFL condition of the type $\Delta t\|c\|_\infty \leq \Delta x$ for explicit schemes, where Δx is the space step and Δt is the time step.

Another well-known method is the Fast Marching Method (FMM) (see Sethian [161, 163]), where the unknown of the problem is the time $t(x)$ the front reaches the point x. This method works for non negative (non positive) velocities and provides a very efficient scheme which concentrates the computational effort on a neighborhood of the front. To be more precise, keeping in mind the viewpoint of discontinuous solutions, in the usual FMM we define the Accepted region (A+) as the discretization of the region $\{\theta = 1\}$ and the Narrow Band (NB-) as the discretization of the boundary $\partial\{\theta = 1\}$, which is at the discrete level contained in the region $\{\theta = -1\}$. The algorithm computes the new values only at the nodes belonging to the narrow band and accepts just one of them, the one corresponding to the minimum value (see Kim [125] for a faster implementation). In the case when c cannot change sign we have a monotone (increasing or decreasing) evolution and the front passes just one time on every point of the computational domain. The corresponding arrival time of the front is univalued so that the evolutive problem reduces to a stationary problem (the eikonal equation). Note that in this method, there is no time step, because the time is itself the unknown of the problem so that the original evolutive problem (6.1) reduces to a stationary problem as remarked in [85] and [151].

To set this paper into perspective, let us recall that the FMM was initially developed for (6.1) with time independent velocities $c(x) > 0$ (see Sethian [163] and Tsitsiklis [179] for the method previously developed on graphs). This FMM scheme has been proved to be convergent, using a relation between the FMM solution and the numerical solution to finite difference schemes for the Level Sets formulation, for which it is known that these schemes are convergent (see Cristiani and Falcone [66]). More recently, the method has been extended to more general Hamilton-Jacobi equation by Sethian and Vladimirsky [164, 165] and it has been also adapted to the case of time-dependent non-negative velocities $c(x,t) \geq 0$ by Vladimirsky [180]. However, up to our knowledge, no proof of convergence has been given for the variable sign velocity case.

The goal of this paper is to propose a new Generalized Fast Marching Method (GFMM) which works for general velocities $c(x,t)$ without sign restrictions. This implies that the evolution is not necessarily monotone and that the time of arrival of the front can be multivalued. Then, in our GFMM it is natural to introduce two Accepted regions (A_+) and (A_-), and two Narrow bands (F_+) and (F_-) in order to be able to take into account the changes of sign of the velocity. The typical picture is Fig. 6.1. In some sense we track two fronts : one moving with positive velocity and one moving with negative velocity. A preliminary version of this new scheme has been proposed in [54], however in that first version no proof was given and some small but very important details, which make the scheme work in the general case, were missing.

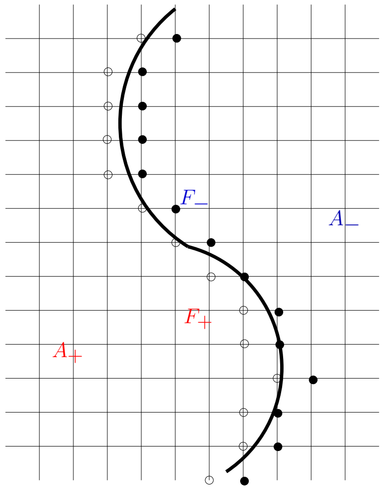

FIG. 6.1 – The narrow bands F_+ and F_-.

Our GFMM has a great potential for several future developments. Let us only mention the application to dislocations dynamics that we will study in a future work.

We introduce in Section 2 the GFMM scheme. Let us observe that there are several subtilities, that do not appear in the usual FMM for $c(x) > 0$. These new features seem necessary to make the scheme work for general $c(x, t)$. Let us list a few of them. First, where the velocity changes sign in space, we need somehow to regularize it to avoid instabilities (in time) of the front. Second, because the time step is somehow the difference $\Delta t_n = t_{n+1} - t_n$ between two computed times, we need our algorithm to ensure that this time step remains bounded from above by a given time step Δt. In fact, this is necessary since the convergence result shows that to improve our approximation of the solution the discretization parameters Δt and Δx must go to 0 and when the velocity is very close to zero, if that bound is not respected, the algorithm may generate a sequence of time steps Δt_n non convergent to zero. Third, we may get computed times $\tilde{t}_{n+1}$ which satisfy $\tilde{t}_{n+1} < t_n$, if for instance the velocity is always equal to zero except at time t_n. In this case, it is necessary to update the time with the value t_n and not $\tilde{t}_n$. Fourth, when the front is close to a given point we have to choose carefully if we update or not the value of the time at this given point. This really depends on the position of the discrete front at time t_n and at time t_{n+1} and on the definition of the new accepted points.

The main result of this paper is Theorem 2.5 which shows the convergence of our GFMM algorithm. When the discontinuous solution is unique, this result states that the numerical solution converges to the discontinuous viscosity solution as $\Delta x, \Delta t$ go to zero. In the case where the discontinuous viscosity solution is not unique, the result only claims that the upper semicontinuous envelope (obtained by a $limsup^*$, see (6.8) for a definition) of the numerical solution is a discontinuous viscosity subsolution and, conversely, that the lower semicontinuous envelope (obtained by the $liminf_*$) is a supersolution.

Another novelty is the proof of convergence of this GFMM algorithm. In fact, we can not use the relation with the usual schemes for the eikonal equation as in the case of non-negative velocities $c(x) > 0$ and we need a *direct proof*. It is interesting to remark that, even in the case of non-negative velocity, our proof is new. However, the idea of our proof is inspired by the paper by Barles and Georgelin [22] on fronts driven by Mean Curvature where they prove convergence for a scheme in the framework of discontinuous viscosity solutions (we also refer to Barles, Souganidis [28] for convergence in the framework of continuous viscosity solutions). Basically, it is sufficient to consider a test function touching the upper semicontinuous envelope of the numerical solution (obtained as $\Delta x, \Delta t$ go to zero) which violates the subsolution property and to derive from this some properties of the discrete solution for non-zero $\Delta x, \Delta t$. This corresponds to consider test functions touching the discrete analogue of the discontinuous function θ in order to get a contradiction with the basic properties of the algorithm.

The paper is organized as follows. In Section 2 we introduce our notation, present our GFMM algorithm and the main result of this paper, *i.e.* the convergence of the algorithm (Theorem 2.5). Several comments and the explanation of the subtleties of the algorithm are discussed in Section 3. Section 4 is devoted to prove comparison principles and symmetry for GFMM. In Section 5, several preliminary results are presented, focusing on properties of discrete times and on the geometry of the level sets of test functions. In Section 6, we use the results of Section 5 to prove the subsolution property of the $limsup^*$ envelope of the numerical solution, while the comparison principle of Section 4 is used to prove this subsolution property at the initial time. The main result of Section 6 is the proof of our main Theorem (Theorem 2.5). Finally, in Section 7 we present some numerical simulations and comment these results in connection with our theoretical results.

2 The GFMM algorithm and the main result

In this section we give details for our GFMM algorithm for unsigned velocity. Let us start introducing our definitions and notation.

Let us consider a lattice $Q \equiv \{x_I = (x_{i_1}, .., x_{i_N}) = (i_1\Delta x, .., i_N\Delta x),\ I = (i_1, .., i_N) \in \mathbb{Z}^N\}$ with space step $\Delta x > 0$. We will also use a time step $\Delta t > 0$.

The following definitions will be useful in the following.

Definition 2.1. *The* **neighborhood of the node** $I \in \mathbb{Z}^N$ *is the set* $V(I) \equiv \{J \in \mathbb{Z}^N : |J - I| \leq 1\}$.

Definition 2.2. *Given the speed* $c_I^n \equiv c(x_I, t_n)$ *we define the function*

$$\widehat{c}_I^n \equiv \begin{cases} 0 & if\ there\ exists\ J \in V(I)\ such\ that\ (c_I^n c_J^n < 0\ and\ |c_I^n| \leq |c_J^n|), \\ c_I^n & otherwise. \end{cases}$$

Definition 2.3. *The* **numerical boundary** ∂E *of a set* $E \subset \mathbb{Z}^N$ *is*

$$\partial E \equiv V(E) \backslash E$$

with

$$V(E) = \left\{ J \in \mathbb{Z}^N, \quad \exists I \in E, \quad J \in V(I) \right\}$$

Definition 2.4. *Given a field* θ_I^n *with values* $+1$ *and* -1, *we define the two phases*

$$\Theta_\pm^n \equiv \{I : \theta_I^n = \pm 1\},$$

and the fronts

$$F_\pm^n \equiv \partial \Theta_\mp^n, \quad F^n \equiv F_+^n \cup F_-^n.$$

In the description of the algorithm we will use the following notations :

$$\pm g \geq 0 \ \text{ for } I \in F_\pm \tag{6.2}$$

means

$$+g \geq 0 \ \text{ for } I \in F_+ \ \text{ and } \ -g \geq 0 \text{ for } I \in F_-. \tag{6.3}$$

Moreover,

$$\min_\pm\{0, g_\pm\} \equiv \min\{0, g_+, g_-\} \text{ and } \max_\pm\{0, g_\pm\} \equiv \max\{0, g_+, g_-\}. \tag{6.4}$$

2.1 The algorithm step-by-step

We describe now our GFMM algorithm for unsigned velocity. As one can see, to track correctly the evolution we need to introduce a discrete function $u_I^n \in \mathbb{R}^+$ defined for $I \in F^n$ to represents the approximated physical time for the front propagation at the nodes $I = (i_1, .., i_N)$ of the fronts at the n-th iteration of the algorithm.

Initialization

1. Set $n = 1$
2. *Initialize the field* θ^0 as
$$\theta_I^0 = \begin{cases} 1 & \text{for } x_I \in \Omega_0 \\ -1 & \text{elsewhere} \end{cases}$$

3. *Initialize the time on* F^0
   $u_I^0 = 0$ for all $I \in F^0$
   Main cycle

4. *Initialize* $\widehat{u}^{n-1}$ *everywhere on the grid*

$$\widehat{u}_{\pm,J}^{n-1} = \begin{cases} u_J^{n-1} & \text{for } J \in F_{\pm}^{n-1} \\ \infty & \text{elsewhere.} \end{cases}$$

5. *Compute* $\tilde{u}^{n-1}$ *on* F^{n-1} *as*
   Let $I \in F_{\pm}^{n-1}$, then
   (a) if $\pm\widehat{c}_I^{n-1} \geq 0$, $\tilde{u}_I^{n-1} = \infty$,
   (b) if $\pm\widehat{c}_I^{n-1} < 0$, we compute $\tilde{u}_I^{n-1}$ as the solution of the following second order equation :

$$\begin{aligned} \sum_{k=1}^{N} \left(\max_{\pm} \left(0, \tilde{u}_I^{n-1} - \widehat{u}_{+,I^{k,\pm}}^{n-1} \right) \right)^2 &= \frac{(\Delta x)^2}{|\widehat{c}_I^{n-1}|^2} \quad \text{if } I \in F_-^{n-1}, \\ \sum_{k=1}^{N} \left(\max_{\pm} \left(0, \tilde{u}_I^{n-1} - \widehat{u}_{-,I^{k,\pm}}^{n-1} \right) \right)^2 &= \frac{(\Delta x)^2}{|\widehat{c}_I^{n-1}|^2} \quad \text{if } I \in F_+^{n-1}, \end{aligned} \tag{6.5}$$

   where

$$I^{k,\pm} = (i_1, .., i_{k-1}, i_k \pm 1, i_{k+1}, .., i_N).$$

6. $\tilde{t}_n = \min \left\{ \tilde{u}_I^{n-1},\ I \in F^{n-1} \right\}$.
7. $\widehat{t}_n = \min\{\tilde{t}_n, t_{n-1} + \Delta t\}$.
8. $t_n = \max(t_{n-1}, \widehat{t}_n)$
9. if $t_n = t_{n-1} + \Delta t$ and $t_n < \tilde{t}_n$ go to 4 with $n := n + 1$.
10. *Initialize the new accepted point*
    $NA_{\pm}^n = \{I \in F_{\pm}^{n-1},\ \tilde{u}_I^{n-1} = \tilde{t}_n\}$, $NA^n = NA_+^n \cup NA_+^n$
11. *Reinitialize* θ^n

$$\theta_I^n = \begin{cases} -1 & \text{for} & I \in NA_+^n \\ 1 & \text{for} & I \in NA_-^n \\ \theta_I^{n-1} & \text{elsewhere} & \end{cases}$$

12. *Reinitialize* u^n *on* F^n
    (a) If $I \in F^n \backslash V(NA^n)$, then $u_I^n = u_I^{n-1}$.
    (b) If $I \in NA^n$ then $u_I^n = t_n$.
    (c) If $I \in (F^{n-1} \cap V(NA^n)) \backslash NA^n$, then $u_I^n = u_I^{n-1}$.

(d) If $I \in V(NA^n)\backslash F^{n-1}$ then $u_I^n = t_n$

13. Set $n := n+1$ and go to 4

Let us describe a few features of this new algorithm :

1. We know, at each time step, the time on the fronts, *i.e.* on both side of the front. This is necessary to allow the changes of the velocity sign in time.
2. In *step 5*, we use the regularized velocity $\hat{c}$ and not c in order to stabilize the front.
3. *step 7* avoids large jumps in time and guarantees that $t_n - t_{n-1} \leq \Delta t$ with Δt small enough.
4. *step 9* allows to increase the time. For example, if at time step n, we have $\widehat{c}_I^{n-1} = 0 \; \forall I \in F^{n-1}$, then there will not be new accepted points and the time will not change and the algorithm will be blocked without steps 7 and 9.
5. *step 8* guarantees that the physical time t_n does not decrease.
6. In *step 12*, for the reinitialization of u_I^n, we change its value only if a point of the neighborhood of the point I has been accepted. Moreover when u_I^n is updated, we use the physical time t_n and not $\tilde{t}_n$ or $\hat{t}_n$.

These choices, which can appear strange with respect to the classical FMM scheme, will be motivated in Section 3, giving also some examples which justify the new definition.

2.2 The main result

The scheme approximates the evolution of the fronts by a double *Narrow band* and the physical time by the sequence $\{t_k, k \in \mathbb{N}\}$, defined at the step 8 in the algorithm. Such sequence is nondecreasing and we can extract a subsequence $\{t_{k_n}, n \in \mathbb{N}\}$ strictly increasing such that

$$t_{k_n} = t_{k_n+1} = \ldots = t_{k_{n+1}-1} < t_{k_{n+1}}.$$

We denote by S_I^n the square cell $S_I^n = [x_I, x_I + \Delta x[\times[t_{k_n}, t_{k_{n+1}}[$ with

$$[x_I, x_I + \Delta x[= \Pi_{\alpha=1}^N [x_{i_\alpha}, x_{i_\alpha} + \Delta x[$$

and by ε the couple

$$\varepsilon = (\Delta x, \Delta t).$$

Let us define the following functions :

$$\theta^\varepsilon(x,t) = \begin{cases} \sup\{\theta_I^m : k_n \leq m \leq k_{n+1} - 1\} & \text{if } (x,t) \in S_I^n \text{ and } c(x_I, t_{k_n}) > 0 \\ \inf\{\theta_I^m : k_n \leq m \leq k_{n+1} - 1\} & \text{if } (x,t) \in S_I^n \text{ and } c(x_I, t_{k_n}) < 0 \\ \theta_I^m, \; \forall m : \; k_n \leq m \leq k_{n+1} - 1 & \text{if } (x,t) \in S_I^n \text{ and } c(x_I, t_{k_n}) = 0. \end{cases} \tag{6.6}$$

This definition is equivalent to the following

$$\theta^\varepsilon(x,t) = \theta_I^{k_{n+1}-1} \text{ if } (x,t) \in S_I^n. \tag{6.7}$$

We define the half-relaxed limits

$$\overline{\theta}^0(x,t) = \limsup_{\varepsilon\to 0, y\to x, s\to t} \theta^\varepsilon(y,s), \quad \underline{\theta}^0(x,t) = \liminf_{\varepsilon\to 0, y\to x, s\to t} \theta^\varepsilon(y,s). \tag{6.8}$$

We make the following assumption

(A) The velocity $c \in W^{1,\infty}(\mathbb{R}^N \times [0,T])$, for some constant $L > 0$ we have $|c(x',t') - c(x,t)| \leq L(|x'-x| + |t'-t|)$, and Ω_0 is a C^2 open set, with bounded boundary $\partial\Omega_0$.

Theorem 2.5. (Convergence Result)
Under assumption (A), $\overline{\theta}^0$ (resp. $\underline{\theta}^0$) is a viscosity sub-solution (resp. super-solution) of (6.1). In particular, if (6.1) satisfies a comparison principle, then $\overline{\theta}^0 = (\underline{\theta}^0)^*$ and $(\overline{\theta}^0)_* = \underline{\theta}^0$ is the unique viscosity solution of (6.1).

Remark 2.6. *When the uniqueness holds, this is up to the upper and lower semi-continuous envelopes.*

Remark 2.7. *Note that when $c > 0$, our GFMM algorithm is a modified FMM algorithm where the time on the narrow band is computed using only the accepted points. In this monotone case the viscosity solution of (6.1) is unique and our result provides a convergence result (see also Test 3 in the last section).*

Remark 2.8. *The Lipschitz-continuity in time of the velocity could be relaxed to continuity, but is assumed here to simplify the presentation of the proofs, which are already quite complicated.*

3 Justifications and examples

In this section we will show that in the variable sign scheme it is necessary to introduce new variables to track correctly the evolution of the front.

3.1 Introduction of the numerical speed $\widehat{c}_I^n$

Let us show by an example in dimension $N = 1$ what would happen choosing c_I^n instead than $\widehat{c}_I^n$.
Consider the speed

$$c(x) = \begin{cases} -\delta & \text{if } x < x_I \\ \delta & \text{if } x \geq x_I \end{cases},$$

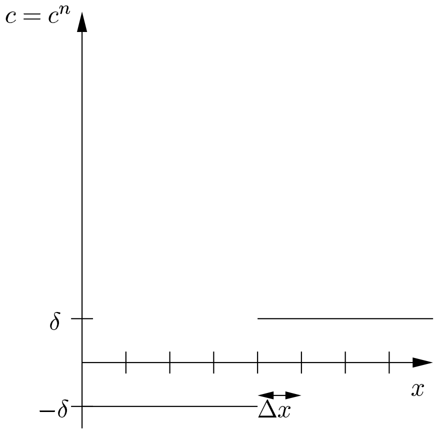

FIG. 6.2 – The velocity c^n.

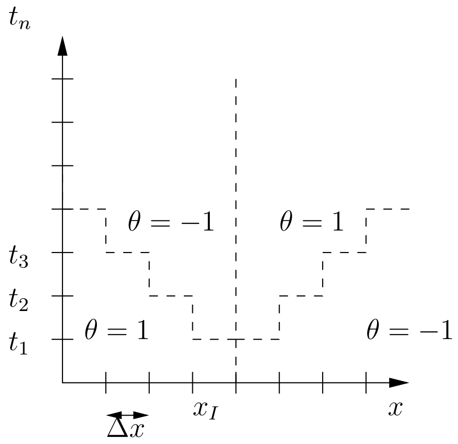

FIG. 6.3 – Evolution with the velocity c^n.

as plotted in Fig.6.2.

Suppose $\frac{\Delta x}{\delta} = \Delta t$, and $\theta_J^0 = 1$ for $J \leq I$, $\theta_J^0 = -1$ for $J > I$. Then the nodes I, $I + 1$ will be accepted at the iteration 1, with $t_1 = \Delta t$ and the front will duplicate, see Fig. 6.3.

Let us now consider the same example but with the numerical speed $\widehat{c}_I^n$, as plotted in Fig.6.4. In this case the nodes on the front have always speed zero, and the front does not move as one would expect.

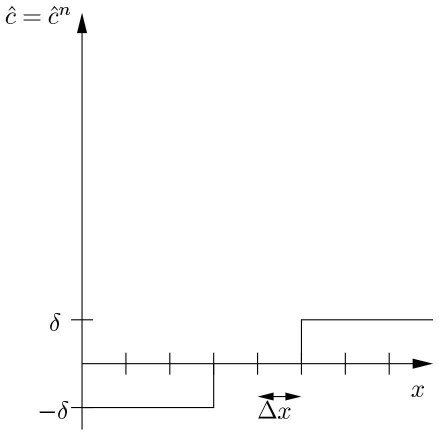

FIG. 6.4 – The velocity $\widehat{c}^n$.

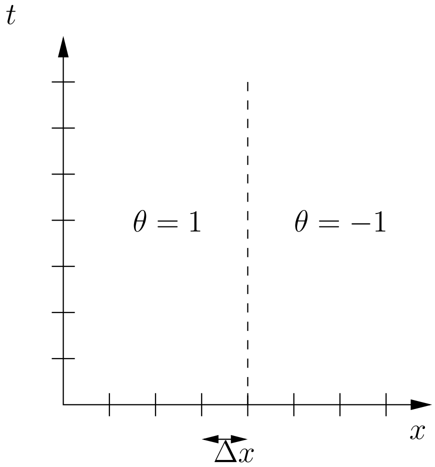

FIG. 6.5 – Evolution with the velocity $\widehat{c}^n$.

3.2 Introduction of the time step

In the case of monotone evolution with speed depending only on the space variables, the FMM algorithm approximates the corresponding stationary equation of (6.1) and then no time step Δt is required. In our more general case, the dependence of the speed in time makes necessary to introduce a time step Δt in the algorithm. We show its need by an example. We provide for simplicity the example in dimension $N = 1$.

Let us take as speed a linear function in time : $c(t) = T - t$, where T is a fixed positive constant. In this case the sequence of the discrete time $\{t_n\}_{n\in\mathbb{N}}$ is given by :

$$t_{n+1} = t_n + \frac{\Delta x}{|c(t_n)|}.$$

Let us define the sequence $\tau_n = T - t_n$. Such a sequence verifies : $\tau_n = f(\tau_{n-1})$ where $f(\tau) = \tau - \frac{\Delta x}{\tau}$. Since f is invertible for any $\tau > 0$, we then define $\tau_n = \Delta x$ and we evaluate τ_k by $\tau_k = f^{-1}(\tau_{k+1})$ for any $k < n$, then $\tau_k < \tau_{k-1}$.

Then we have defined

$$\begin{cases} t_m = T - \tau_m, & m = n, n-1, ..., n_0 \\ n_0 = \min\{m, & t_m \geq 0\}. \end{cases}$$

It results $t_n = T - \Delta x$ and then $t_{n+1} = T + 1 - \Delta x$, i.e. $t_{n+1} - t_n = 1$. We want to avoid this situations, since a big increment in the sequence of the discrete time step can bring a loss of informations and in general the algorithm would not converge to the correct evolution of the fronts. We show such a distribution of discrete time together with the linear speed in Fig.6.6 and we show the wrong evolution of a front in this case in Fig.6.7.

If instead we introduce a threshold Δt as in the GFMM algorithm step 7, then it results $t_{n+1} = t_n + \Delta t$, and we get the correct evolution for Δt small enough (see Fig. 6.8).

3.3 Why we update the front using t instead of $\widehat{t}$

We explain by an example in dimension $N = 1$, why it is correct to assign the value t_n instead of $\widehat{t_n}$ on the front F^n in the Step 12 of the algorithm.

Fixed $p \in \mathbb{N}$, we define $\Delta s = p\Delta t$ and $\delta = \frac{\Delta x}{\Delta t}$ and we consider a piecewise constant speed for $t \geq 0$:

$$c(t) = \begin{cases} \delta & \text{if} \quad t \in [(2k-1)\Delta s, 2k\Delta s) \text{ for some } k \in \mathbb{N}\backslash\{0\}, \\ 0 & \text{otherwise} \end{cases}$$

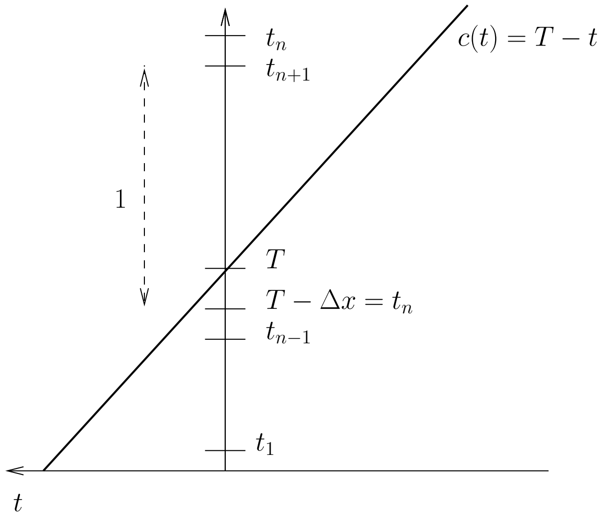

FIG. 6.6 – Jump in the discrete time without threshold Δt in the case of a linear in time velocity

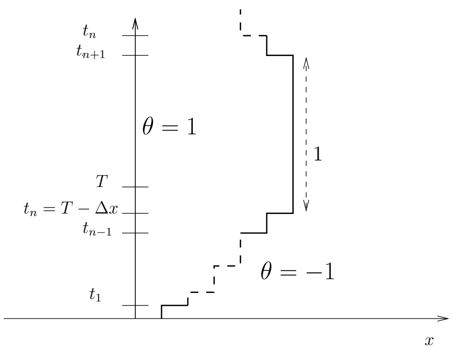

FIG. 6.7 – Front evolution without the threshold Δt

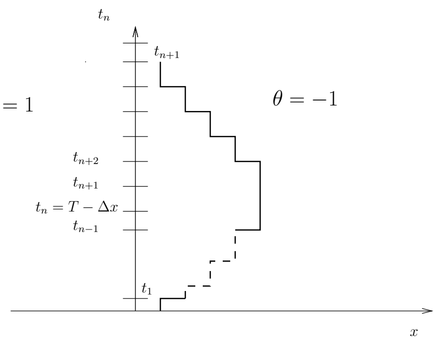

FIG. 6.8 – Front evolution with the threshold Δt

as in Fig.6.9 on the left.

If we update the front using $\widehat{t}$ instead of t, the corresponding evolution will be the one plotted on the right of Fig.6.9. In fact the front will start to move using the velocity given at time $t_p = p\Delta t$ and since $\tilde{t}_{p+1} = 0 + \frac{\Delta x}{c(t_p)} = \frac{\Delta x}{c(t_p)} = \Delta t$, then $\widehat{t}_{p+1} = \min\{\tilde{t}_{p+1}, t_p + \Delta t\} = \min\{\Delta t, \Delta s + \Delta t\} = \Delta t$ and $t_{p+1} = \max(t_p, \widehat{t}_{p+1}) = \max(\Delta s, \Delta t) = \Delta s = t_p$. So the front will propagate on the right at the iteration $p, ..., 2p-1$ but at constant time $t = t_p$ (Fig.6.9).
On the contrary, if we update the front using t, the front will start to move using the velocity given at time $t_p = p\Delta t$ and the iteration $p+1$ will be as before. But since $\tilde{t}_{p+2} = t_{p+1} + \frac{\Delta x}{c(t_p)} = \Delta s + \frac{\Delta x}{c(t_p)} = \Delta s + \Delta t$, then $\widehat{t}_{p+2} = \min\{\tilde{t}_{p+2}, t_{p+1} + \Delta t\} =$

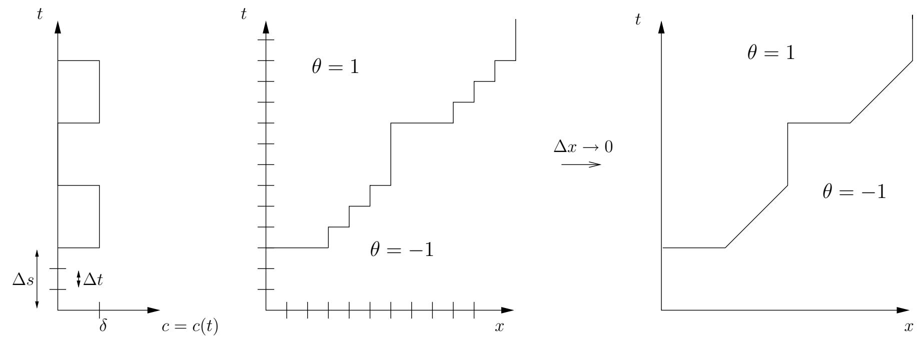

FIG. 6.9 – Wrong evolution when the time is updated with $\hat{t}$.

$\min\{\Delta t+\Delta s, \Delta t+\Delta s\}$ and $t_{p+2} = \max(t_{p+1}, \widehat{t_{p+1}}) = \max(\Delta s+\Delta t, \Delta t) = \Delta s+\Delta t$. So the front will propagate on the right at the iteration $p, ..., 2p-1$ but at linear times $t_m = m\Delta t$ (Fig.6.10).

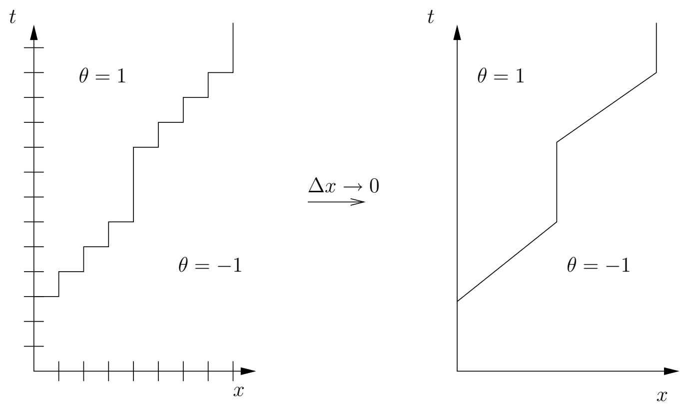

FIG. 6.10 – Correct evolution when the time is updated with t.

4 Comparison principles for the GFMM algorithm

As we said in the introduction, our convergence result will be proved in the framework of discontinuous viscosity solutions. To this end the role of comparison principles is crucial.

In this Section, we first present a property of symmetry of the algorithm, then present some comparison principles in some special cases and finally a counter-example to a general comparison principle.

This Section shows in particular that statements on our GFMM algorithm are highly non-trivial in general.

4.1 Symmetry of the algorithm

The following lemma claims that if we change the sign of the velocity and the sign of the two phases at the initial time, then the GFMM algorithm computes the same front.

Lemma 4.1. (Symmetry of the GFMM algorithm)
We denote by $\overline{\theta}^0[\theta^0, c]$ and $\underline{\theta}^0[\theta^0, c]$ the functions constructed by the GFMM algorithm with initial condition θ^0 and velocity c. Then we have

$$\underline{\theta}^0[\theta^0, c] = -\overline{\theta}^0[-\theta^0, -c].$$

Proof of Lemma 4.1
With the same kind of notation, we remark that

$$\theta_I^n[-\theta^0, -c] = -\theta_I^n[\theta^0, c].$$

We then have, for $x \in [x_I, x_I + \Delta x[,\ t \in [t_{k_n}, t_{k_{n+1}}[$ and $c(x_I, t_{k_n}) > 0$

$$\begin{aligned}\theta^\varepsilon[\theta^0, c](x,t) &= \sup\{\theta_I^k[\theta^0, c],\ k_n \leq k \leq k_{n+1} - 1\}\\ &= -\inf\{-\theta_I^k[\theta^0, c],\ k_n \leq k \leq k_{n+1} - 1\}\\ &= -\inf\{\theta_I^k[-\theta^0, -c],\ k_n \leq k \leq k_{n+1} - 1\}\\ &= -\theta^\varepsilon[-\theta^0, -c].\end{aligned}$$

The result is similar for $c(x_I, t_{k_n}) \leq 0$. Therefore

$$\underline{\theta}^0[\theta^0, c] = -\overline{\theta}^0[-\theta^0, -c].$$

□

4.2 Comparison principles

Proposition 4.2. (Comparison principle for the time)
We denote by u_I^n (resp. v_I^n) the numerical solution at the point (x_I, t_n) of the GFMM algorithm with velocity c_u (resp. c_v). We assume that there exists $T > 0$ such that for all $(x,t) \in \mathbb{R}^N \times [0,T]$

$$\inf_{s \in [t-\Delta t, t],\ s \geq 0} c_v(x,s) \geq \sup_{s \in [t-\Delta t, t],\ s \geq 0} (c_u(x,s))^+$$

where $(f)^+$ is the positive part of f. We assume that

$$\{\theta_u^0 = 1\} \subset \{\theta_v^0 = 1\} \quad \text{and} \quad v^0 = u^0 = 0.$$

We define $\bar{m}$ and $\bar{k}$ such that

$$\begin{cases} t_{\bar{m}} \leq T < t_{\bar{m}+1} \\ s_{\bar{k}} \leq T < s_{\bar{k}+1} \end{cases}$$

where $(t_m)_m$ and $(s_m)_m$ are respectively the sequence of time constructed by the GFMM algorithm with velocity c_u and c_v. We then consider

$$v_I = \begin{cases} v_I^0 & \text{if} \quad \theta_{v,I}^0 = 1 \\ v_I^k & \text{if} \quad I \in NA_v^k \quad \text{for some} \quad k \leq \bar{k}+1 \\ s_{\bar{k}+1} & \text{if} \quad \theta_{v,I}^{\bar{k}} = -1 \end{cases}$$

Then, $\forall l \leq \bar{m}$, $\forall I \in NA_u^l$, we have

$$v_I \leq u_I^l.$$

Remark 4.3. *Here the notation for $\theta_u, \theta_v, NA_u^l$ and further notation in the sequel are obvious and are not explained. Moreover we also remark that the front for v passes at most one time at a given point because $c_v \geq 0$.*

Proof of Proposition 4.2
We argue by contradiction. We denote by $m(u)$ the first index such that there exists $I \in NA_u^{m(u)}$ such that

$$u_I^{m(u)} < v_I \tag{6.9}$$

We define

$$k(v) \quad \text{such that} \quad I \in NA_v^{k(v)}$$

with the convention that $k(v) = \bar{k}+1$ if $\theta_{v,I}^{\bar{k}} = -1$. This implies that

$$t_{m(u)} = u_I^{m(u)} < v_I = s_{k(v)}$$

The proof distinguishes two cases.

1. $\boldsymbol{I \in NA^{(\ )} \subset F^{(\ )\ 1}}$ **.**
   We claim that for all $J \in V(I) \setminus \{I\}$, we have

$$\widehat{u}_{+,J}^{m(u)-1} \geq \widehat{v}_{+,J}^{k(v)-1} \tag{6.10}$$

   Indeed assume that $\widehat{u}_{+,J}^{m(u)-1} < \infty$ (if $\widehat{u}_{+,J}^{m(u)-1} = \infty$, then (6.10) holds), then $J \in F_{+,u}^{m(u)-1}$ and we have

$$t_{m(u)} \geq \widehat{u}_{+,J}^{m(u)-1} \geq v_J.$$

It just remains to show that $v_J = \widehat{v}_{+,J}^{k(v)-1}$. We argue by contradiction. Assume that $\widehat{v}_{+,J}^{k(v)-1} = \infty$, *i.e* $J \in \{\theta_v^{k(v)-1} = -1\}$. Then $v_J \geq s_{k(v)}$. This contradicts the fact that $v_J \leq t_{m(u)} < s_{k(v)}$ and proves (6.10).

We define
$$k^* := \sup\{k, s_k \leq t_{m(u)}\} < k(v).$$
In particular, we have $t_{m(u)} - \Delta t \leq s_{k^*} \leq t_{m(u)}$. Since for all $J \in V(I) \cap F_{+,u}^{m(u)-1}$

$$t_{m(u)} \geq \widehat{u}_{+,J}^{m(u)-1} \geq \widehat{v}_{+,J}^{k(v)-1}$$

we deduce that
$$s_{k^*} \geq \widehat{v}_{+,J}^{k(v)-1}. \tag{6.11}$$
Indeed, $+\infty > \widehat{v}_{+,J}^{k(v)-1} > s_{k^*}$ would imply that there exists $k' > k^*$ such that $t_{m(u)} \geq \widehat{v}_{+,J}^{k(v)-1} = s_{k'}$ which contradicts the definition of k^*.
Then we claim that for all $J \in V(I) \cap F_{+,u}^{m(u)-1}$

$$\widehat{v}_{+,J}^{k(v)-1} = \widehat{v}_{+,J}^{k^*}. \tag{6.12}$$

We now prove the claim (6.12). First, because we have $\widehat{v}_{+,J}^{k(v)-1} < +\infty$, we deduce that $\theta_{v,J}^{k(v)-1} = 1$ and then there exists $k \leq k(v) - 1$ such that if $k \geq 1$, then $J \in NA_v^k$ and $\widehat{v}_{+,J}^{k(v)-1} = v_J^k = s_k$, and if $k = 0$, then $\theta_{v,J}^0 = 1$ and $\widehat{v}_{+,J}^{k(v)-1} = v_J^0 = 0$.
Assume by contradiction that $k > k^*$. Then

$$\widehat{v}_{+,J}^{k(v)-1} = v_J^k = s_k \geq s_{k^*+1} > s_{k^*}$$

Contradiction with (6.11). Therefore $k \leq k^*$. Now we have $\theta_{v,I}^{k(v)} = 1$ and $\theta_{v,I}^m = -1$ for $m \leq k(v) - 1$. Therefore $J \in F_{+,v}^{k^*}$ and

$$\widehat{v}_{+,J}^{k(v)-1} = v_J^k = \widehat{v}_{+,J}^{k^*}$$

which ends the proof of the claim (6.12). We deduce that

$$\widehat{v}_{+,J}^{k(v)-1} = \widehat{v}_{+,J}^{k^*} \leq \widehat{u}_{+,J}^{m(u)-1},$$

where we have used (6.10). We define the following function

$$f_{\widehat{u}^m}^2(t) = \sum_{k=1}^{N} \left(\max_{\pm} \left(0, t - \widehat{u}_{+,I^{k,\pm}}^m\right) \right)^2 .$$

We then have, using the fact that $\tilde{v}_I^{k^*} \geq s_{k^*+1} > s_{k^*}$

$$f_{\widehat{v}^{k^*}}(s_{k^*+1}) \leq f_{\widehat{v}^{k^*}}(\tilde{v}_I^{k^*}) = \left|\frac{\Delta x}{\widehat{c}_{I,v}^{k^*}}\right| \leq \left|\frac{\Delta x}{\widehat{c}_{I,u}^{m(u)-1}}\right| = f_{\widehat{u}^{m(u)-1}}(\tilde{u}_I^{m(u)-1}) \leq f_{\widehat{v}^{k^*}}(\tilde{u}_I^{m(u)-1})$$

We then deduce that

$$s_{k^*+1} \leq \tilde{u}_I^{m(u)-1} \leq u_I^{m(u)} = t_{m(u)}.$$

This is absurd.

$\square$

2. $\boldsymbol{I \in NA_+^{(\ )} \subset F_+^{(\ )\ 1}}$ **.**
   We consider the following subcases
   (a) $I \in \{\theta_v^0 = 1\}$. Then $v_I = v_I^0 = 0 = u_I^0 \leq u_I^{m(u)}$. This is absurd.
   (b) $I \in \{\theta_v^0 = -1\}$. Then $\theta_{u,I}^0 = -1$ and so there exists $n < m(u)$ such that

   $$\theta_{u,I}^{n-1} = -1 \quad \text{and} \quad \theta_{u,I}^n = 1.$$

   This implies that

   $$u_I^n \geq v_I > u_I^{m(u)} \geq u_I^n.$$

   This is absurd. $\square$

Remark 4.4. *If we implicit the computation of the gradient, i.e. the computation of $\tilde{u}$ in step 5, the situation seems better and one could expect to prove a general comparison principle without restriction on the velocity.*

We now rephrase this comparison principle for the functions θ^ε and prove it.

Corollary 4.5. (Comparison principle with a nonnegative velocity)
Under the assumptions of Proposition 4.2, we have for all $(x,t) \in \mathbb{R}^N \times [0,T]$

$$\theta_u^\varepsilon(x,t) \leq \theta_v^\varepsilon(x,t).$$

Proof of Corollary 4.5
By contradiction, assume that there exist x_I and t such that

$$\theta_u^\varepsilon(x_I,t) = 1 \quad \text{and} \quad \theta_v^\varepsilon(x_I,t) = -1. \tag{6.13}$$

We denote by t the first time such that (6.13) holds. We then have, since $c_v \geq 0$,

$$\theta_u^\varepsilon(x_I,s) = -1 \quad \text{if} \quad s < t.$$

We then deduce that there exists $m(u)$ such that $t = t_{m(u)}$, $I \in NA_u^{m(u)}$ and $u_I^{m(u)} = t_{m(u)} = t$. Moreover, since the index I has not been already accepted for v, we have $v_I > t = u_I^{m(u)}$. This is absurd. $\square$

Corollary 4.6. (Comparison principle for a nonpositive velocity)
We denote by u_I^n (resp. v_I^n) the numerical solution at the point (x_I, t_n) of the GFMM algorithm with velocity c_u (resp. c_v). We assume that there exists $T > 0$ such that for all $(x,t) \in \mathbb{R}^N \times [0,T]$

$$\sup_{s\in[t-\Delta t,t],\ s\geq 0} c_u(x,s) \leq \inf_{s\in[t-\Delta t,t],\ s\geq 0} -(c_v(x,s))^-$$

where $(f)^- \geq 0$ is the negative part of f. We assume that

$$\{\theta_v^0 = -1\} \subset \{\theta_u^0 = -1\} \quad \text{and} \quad v^0 = u^0 = 0.$$

Then, for all $(x,t) \in \mathbb{R}^N \times [0,T]$, we have

$$\theta_u^\varepsilon(x,t) \leq \theta_v^\varepsilon(x,t).$$

Proof of Corollary 4.6
This is a straightforward consequence of Corollary 4.5 and the fact that $\theta^\varepsilon[-\theta^0, -c] = -\theta^\varepsilon[\theta^0, c]$ (with the notation of Lemma 4.1). □

4.2.1 Counter-example for the comparison principle in general

We now give a counter-example for a more general comparison principle for which the two velocities can change their signs.

Proposition 4.7. (Counter-example)
Let $N = 1$. We assume that the velocity c_u and c_v are null everywhere except on a node I for which $c_u(x_I, \cdot) \geq c_v(x_I, \cdot)$ are given in Figure 6.11 and 6.12 respectively with $\dfrac{\Delta x}{\delta} = k\Delta t$. We also suppose that

$$\theta_{u,J}^0 = 1 \text{ if and only if } J \leq I \quad \text{and} \quad \theta_{v,J}^0 = 1 \text{ if and only if } J < I.$$

Then

$$\theta_{u,I}^{k+3} = -1 \quad \text{and} \quad \theta_{v,I}^{k+3} = 1.$$

Proof of Proposition 4.7

1. For the GFMM associated to u.
   The node I will be accepted with a time $u_I^k = t_k = k\Delta t$ and we will affect the value $t_k = k\Delta t$ to u_I^k. Then the velocity will change of sign and the node I will be accepted again with a time $2k\Delta t$ (see Figure 6.11).
2. For the GFMM associated to v.
   Since the velocity is nonpositive, nothing will move. The current time t_n will continue to increase and we will have $t_{k+2} = (k+2)\Delta t$ and then the velocity will become positive. The node I will then be accepted with a time $v_I^{k+3} = t_{k+2} = (k+2)\Delta t$ (see Figure 6.12).

We then conclude that the node I will be accepted for v before to be accepted for u and so no comparison principle can hold in general.

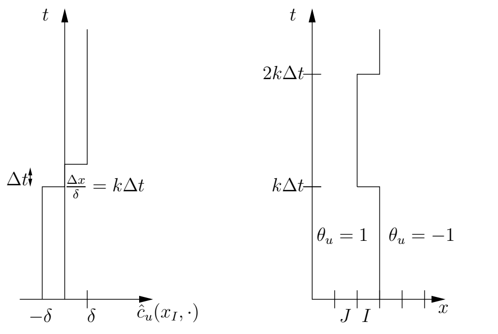

FIG. 6.11 – Velocity and evolution of the front for u.

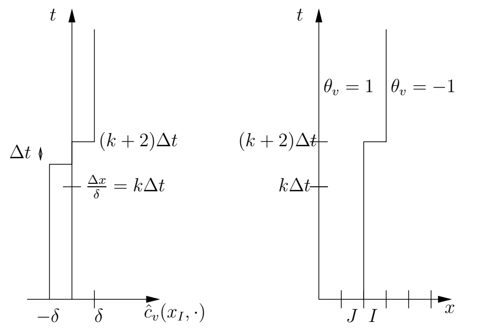

FIG. 6.12 – Velocity and evolution of the front for v.

5 Preliminary results on the discrete time and on the level sets of test functions

The GFMM algorithm described in Section 2 has several properties which fit the physics of the problem we want to solve. We present in this Section several results that will be used in the proof of Proposition 6.1 which is crucial for the proof of our main result of convergence.

In a first subsection, we present some properties of the various times $\hat{u}, t, \tilde{t}$ appearing in our algorithm, and in a second subsection we give some geometrical consequences of the existence of test functions tangent from above to our function θ^ε.

5.1 Preliminary results on the discrete time

Lemma 5.1. (Time character of the $\widehat{u}$)
Assume there exists $\delta > 0$ and $(I,n) \in \mathbb{Z}^N \times \mathbb{N}$ such that $c(x_I,t_n) \geq \delta > 0$, $\theta_I^{n-1} = -1$ and $\theta_I^n = 1$ (resp. $c(x_I,t_n) \leq -\delta < 0$, $\theta_I^{n-1} = 1$ and $\theta_I^n = -1$), then for any $J \in V(I) \cap F_+^{n-1}$ (resp. $J \in V(I) \cap F_-^{n-1}$), we have for $\Delta x \leq \dfrac{\delta^2}{16L}$

$$\widehat{u}_{+,J}^{n-1} = \sup\{t_m \leq t_{n-1},\ \theta_J^{m-1} = -1, \theta_J^p = 1,\ \textit{for}\, m \leq p \leq n-1\} > t_n - \frac{4\Delta x}{\delta}$$

with the convention that $\widehat{u}_{+,J}^{n-1} = 0$ if $\theta_J^p = 1$ for $0 \leq p \leq n-1$

(resp. $\widehat{u}_{-,J}^{n-1} = \sup\{t_m \leq t_{n-1},\ \theta_J^{m-1} = 1, \theta_J^p = -1,\ \textit{for}\, m \leq p \leq n-1\} > t_n - \dfrac{4\Delta x}{\delta}.$

with the convention that $\widehat{u}_{+,J}^{n-1} = 0$ if $\theta_J^p = -1$ for $0 \leq p \leq n-1$).

This lemma claims in fact that the $\widehat{u}^{n-1}_{+,J}$ is defined as the last time at which the front passed through J. Intuitively, this comes from the fact that, since the velocity is locally non-negative and since the front has crossed the node x_I at time t_n, it has crossed the node x_J at a time closed to t_n.

Proof of Lemma 5.1
We only do the proof in the case $c > 0$ (the case $c < 0$ is similar). By assumptions, c is Lipschitz-continuous with constant L, and there exists $\delta_0 \leq \delta/(4L)$ such that for all $(x_J, t_m) \in B_{\delta_0}(x_I) \times [t_n - \delta_0, t_n + \delta_0]$, we have

$$\widehat{c}^m_J \geq \frac{\delta}{2}.$$

This implies that

$$\theta^m_I = -1 \text{ for all } m \text{ such that } \; t_n - \delta_0 \leq t_m \leq t_{n-1}. \tag{6.14}$$

Let $J \in V(I) \cap F^{n-1}_+$. We define

$$m_J = \sup\{m \leq n-1,\ \theta^{m-1}_J = -1, \theta^m_J = 1\}.$$

We claim that for all $J \in V(I) \cap F^{n-1}_+$, we have $t_{m_J} > t_n - \delta_0$ for Δx small enough. Indeed, by contradiction, assume that there exists $J \in V(I) \cap F^{n-1}_+$ such that $t_{m_J} \leq t_n - \delta_0$. Let us define $p \geq 0$ such that

$$t_n = ... = t_{n-p} > t_{n-p-1}.$$

We then have $\widehat{u}^{n-p-1}_{+,J} \leq t_n - \delta_0$ and $\tilde{u}^{n-p-1}_I \geq t_{n-p} = t_n$. Using the fact that

$$\sum_{k=1}^{N} \left(\max_{\pm} \left(0, \tilde{u}^{n-p-1}_I - \widehat{u}^{n-p-1}_{+,I^{k,\pm}} \right) \right)^2 = \left(\frac{\Delta x}{\widehat{c}^{n-p-1}_I} \right)^2$$

we then deduce that

$$\delta_0 = t_{n-p} - (t_n - \delta_0) \leq \tilde{u}^{n-p-1}_I - \widehat{u}^{n-p-1}_J \leq \frac{2\Delta x}{\delta}.$$

This is absurd for the choice $\delta_0 = \dfrac{4\Delta x}{\delta} \leq \dfrac{\delta}{4L}$ which is valid for Δx small enough. Moreover, using (6.14), we deduce that $J \in F^m$ for all $m_J \leq m \leq n-1$. This implies that $\widehat{u}^{n-1}_{+,J} = u^{n-1}_J = u^{m_J}_J = t_{m_J}$. $\square$

The following lemma is concerned with the fact that we can control the decay of the time $\tilde{t}_n$ given by the GFMM algorithm, by the variations in time of the velocity.

Lemma 5.2. (Error estimate between t_n and $\tilde{t}_n$)
Assume that there exists $I \in NA^n$ such that $|\widehat{c}_I^{n-1}| \geq \delta > 0$. Then, the following estimate holds

$$\left(t_n - \tilde{t}_n\right)^+ \leq \frac{2L}{\delta^2}\Delta x \Delta t \quad \textit{if} \quad \Delta t \leq \frac{\delta}{2L}$$

Proof of Lemma 5.2
We only treat the case $c_I^{n-1} \geq \delta > 0$ (the other case is similar). Assume that $\tilde{t}_n < t_n$, then necessarily $t_n = t_{n-1}$. We define $p > 0$ such that

$$t_{n-p-1} < t_{n-p} = ... = t_{n-1} = t_n.$$

In particular, we have

$$t_{n-p} \leq \tilde{t}_{n-p} \leq \tilde{u}_J^{n-p-1} \quad \forall J \in F^{n-p-1}$$

and

$$\tilde{t}_n = \tilde{u}_I^{n-1} \leq \tilde{u}_J^{n-1} \quad \forall J \in F^{n-1}.$$

We claim that $I \in F_-^{n-p-1}$. Indeed, assume that $I \notin F_-^{n-p-1}$. Using the fact that $\theta_I^{n-p-1} = -1$ (since $\widehat{c}_I > 0$), we deduce that for all $J \in V(I) \cap F_+^{n-1}$, we have $\theta_J^{n-p-1} = -1$ and so $\widehat{u}_{+,J}^{n-1} = t_n$, this means that also the node J has been accepted at the physical time t_n . This implies that $\tilde{u}_I^{n-1} > t_n$ and this is absurd.

Moreover, because $t_{n-p} - t_{n-p-1} \leq \Delta t$, we have $\widehat{c}_I^{n-p-1} \geq \frac{\delta}{2}$ for $\Delta t \leq \dfrac{\delta}{2L}$. We then have

$$\sum_{k=1}^{N}\left(\max_{\pm}\left(0, \tilde{u}_I^{n-p-1} - \widehat{u}_{+,I^{k,\pm}}^{n-p-1}\right)\right)^2 = \left(\frac{\Delta x}{\widehat{c}_I^{n-p-1}}\right)^2 \tag{6.15}$$

and

$$\sum_{k=1}^{N}\left(\max_{\pm}\left(0, \tilde{u}_I^{n-1} - \widehat{u}_{+,I^{k,\pm}}^{n-1}\right)\right)^2 = \left(\frac{\Delta x}{\widehat{c}_I^{n-1}}\right)^2. \tag{6.16}$$

Let us compare $\widehat{u}_{+,J}^{n-1}$ and $\widehat{u}_{+,J}^{n-p-1}$ for $J \in V(I) \cap F_+^{n-1}$. If $J \notin F_+^{n-p-1}$, then u_J changes values during the iterations $n - p \leq m \leq n-1$, and for such m we have $\widehat{u}_{+,J}^{n-1} = u_J^m = t_m = t_n$. Since $\tilde{t}_n < t_n$, then this node $J \in V(I)$ does not contribute to the evaluation of (6.16) and

$$\sum_{k=1}^{N}\left(\max_{\pm}\left(0, \tilde{t}_n - \widehat{u}_{+,I^{k,\pm}}^{n-p-1}\right)\right)^2 = \sum_{k=1}^{N}\left(\max_{\pm}\left(0, \tilde{t}_n - \widehat{u}_{+,I^{k,\pm}}^{n-1}\right)\right)^2. \tag{6.17}$$

Let us denote by

$$f_q(v) = \left\{\sum_{k=1}^{N}\left(\max_{\pm}\left(0, v - \widehat{u}_{+,I^{k,\pm}}^{q}\right)\right)^2\right\}^{1/2}.$$

The function f_q verifies for any $q \in \mathbb{N}$ such that $I \in F_-^q$:

$$f_q(\tilde{u}_I^q) = \frac{\Delta x}{|\widehat{c}_I^q|}, \quad f_q'(v) \geq 1.$$

Then

$$\begin{aligned}
t_n - \tilde{t}_n &\leq \tilde{u}_I^{n-p-1} - \tilde{t}_n \leq f_{n-p-1}(\tilde{u}_I^{n-p-1}) - f_{n-p-1}(\tilde{t}_n) \\
&= f_{n-p-1}(\tilde{u}_I^{n-p-1}) - f_{n-1}(\tilde{t}_n) = \Delta x \left(\frac{1}{|\widehat{c}_I^{n-p-1}|} - \frac{1}{|\widehat{c}_I^{n-1}|} \right) \\
&\leq \Delta x \frac{|\widehat{c}_I^{n-p-1} - \widehat{c}_I^{n-p}|}{|\widehat{c}_I^{n-p-1}||\widehat{c}_I^{n-p}|} \\
&\leq \frac{\Delta x |\partial_t c|_{L^\infty} |t_{n-p} - t_{n-p-1}|}{|\widehat{c}_I^{n-p-1}||\widehat{c}_I^{n-p}|} \\
&\leq \frac{2\Delta x |\partial_t c|_{L^\infty} \Delta t}{\delta^2}.
\end{aligned}$$

□

5.2 Preliminary results on the level sets of test functions

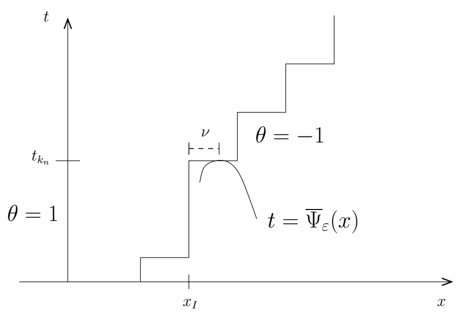

FIG. 6.13 – Test function from below

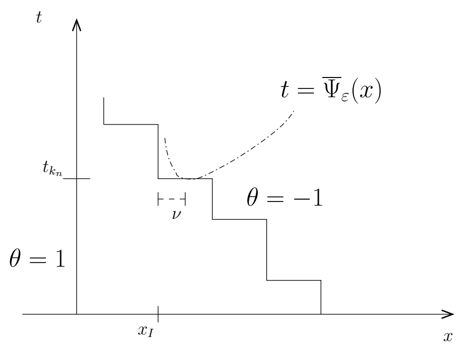

FIG. 6.14 – Test function from above

Lemma 5.3. (Separation of the phases of θ^ε by the level set of a test function)
Let $\varphi \in C^2$ in a neighborhood V of (x_0, t_0) such that $\varphi_t(x_0, t_0) > 0$ (resp. $\varphi_t(x_0, t_0) < 0$). There exist $\delta_0 > 0$, $r > 0$, $\tau > 0$ such that if $\max_{\bar{V}}((\theta^\varepsilon)^* - \varphi)$ is reached at some point $(x_\varepsilon, t_\varepsilon) \in B_{\delta_0}(x_0, t_0) \subset V$ with $(\theta^\varepsilon)^*(x_\varepsilon, t_\varepsilon) = 1$, then there exists $\Psi_\varepsilon \in C^2(B_r(x_0), (t_0 - \tau, t_0 + \tau))$ such that

(i) *For all* $(x_J, t_m) \in Q_{r,\tau}(x_0, t_0) = B_r(x_0) \times (t_0 - \tau, t_0 + \tau)$

$$\theta^\varepsilon(x_J, t_m) = 1 \quad \Longrightarrow \quad t_m \geq \Psi_\varepsilon(x_J) \text{ (\textit{resp.} } t_m \leq \Psi_\varepsilon(x_J)).$$

(ii) *There exists* $(I, n) \in \mathbb{Z}^N \times \mathbb{N}$ *such that*

$$(x_\varepsilon, t_\varepsilon) \in \bar{Q}_I^n = [x_I, x_I + \Delta x] \times [t_{k_n}, t_{k_{n+1}}], \quad (\theta^\varepsilon)^*(x_I, t_{k_n}) = 1, \quad t_{k_n} = \Psi_\varepsilon(x_I)$$

and

$$\theta_I^{\overline{n}} = 1, \quad \theta_I^m = -1 \quad m_0 \leq m \leq \overline{n} - 1$$
$$\big(\textit{resp. } \theta_I^{\overline{n}} = -1, \quad \theta_I^m = 1 \quad m_0 \leq m \leq \overline{n} - 1\big)$$

where

$$\overline{n} = \inf\big\{k, \quad k_n \leq k \leq k_{n+1} - 1, \quad \theta_I^k = 1 \quad (\textit{resp.} \quad \theta_I^k = -1)\big\}$$

and $m_0 = \inf\{m,\ t_m \geq t_0 - \tau\}$.

(iii) *The following Taylor expansion holds*

$$\Psi_\varepsilon(x_J) = \Psi_\varepsilon(x_I) - \frac{D\varphi(x_0, t_0)}{\varphi_t(x_0, t_0)}(x_J - x_I) + (\Delta x)\ O\left(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|\right).$$

(iv) *If* $\varphi_t(x_0, t_0) < 0$, *then for all* $(x_J, t_{k_n}) \in Q_{r,\tau}(x_0, t_0) = B_r(x_0) \times (t_0 - \tau, t_0 + \tau)$

$$\theta^\varepsilon(x_J, t_{k_{n-1}}) = 1 \quad \textit{and} \quad \theta^\varepsilon(x_J, t_{k_n}) = -1 \quad \Longrightarrow \quad t_{k_n} \leq \Psi_\varepsilon(x_J).$$

Proof of Lemma 5.3

We consider the case $\varphi_t(x_0, t_0) > 0$. The other case can be treated in a similar way. We define $\varphi_\varepsilon = \varphi + ((\theta^\varepsilon)^* - \varphi)(x_\varepsilon, t_\varepsilon)$. In particular, we have $(\theta^\varepsilon)^* \leq \varphi_\varepsilon$ and $(\theta^\varepsilon)^*(x_\varepsilon, t_\varepsilon) = \varphi_\varepsilon(x_\varepsilon, t_\varepsilon) = 1$. We start by proving (i) and (ii). The proof is decomposed in several steps.

Step 1. We have $\boldsymbol{t \ = t}$.

Indeed, assume that $t_\varepsilon \in (t_{k_n}, t_{k_{n+1}})$. Using the fact that $(\theta^\varepsilon)^*(x_\varepsilon, t_\varepsilon) = 1$, we deduce that $(\theta^\varepsilon)^*(x_\varepsilon, t) = 1$ for $t_{k_n} \leq t \leq t_{k_{n+1}}$ and so $\varphi_t(x_\varepsilon, t_\varepsilon) = 0$. This is absurd for δ_0 small enough since $\varphi_t(x_0, t_0) > 0$.

Step 2. We have $\boldsymbol{(\theta\ )\ = -1}$ on all $\boldsymbol{Q^{\ \ 1} =]x\ , x\ + \Delta x[\times]t_{\ \ 1}, t\ [}$ such that $\boldsymbol{(x\ , t\ ) \in \bar{Q}^{\ \ 1}}$.

Indeed, since $\varphi_\varepsilon(x_\varepsilon, t_{k_n}) = 1$ and $(\varphi_\varepsilon)_t > 0$, we deduce that $\varphi_\varepsilon(x_\varepsilon, t) < 1$ if $t < t_{k_n}$. Using the fact that $(\theta^\varepsilon)^* - \varphi_\varepsilon$ reaches a maximum in (x_ε, t_{k_n}), yields

$$(\theta^\varepsilon)^*(x_\varepsilon, t) \leq \varphi_\varepsilon(x_\varepsilon, t) < 1 \quad \text{if } t < t_{k_n}$$

and so

$$(\theta^\varepsilon)^*(x_\varepsilon, t) = -1 \quad \text{if } t < t_{k_n}.$$

Using the semi-continuity of $(\theta^\varepsilon)^*$, one deduce that

$$(\theta^\varepsilon)^* = -1 \quad \text{on all } Q_J^{n-1} \text{ such that } (x_\varepsilon, t_{k_n}) \in \bar{Q}_J^{n-1}.$$

Step 3. There exists $\boldsymbol{I \in \mathbb{Z}}$, such that $\boldsymbol{(x\ , t\ ) \in \bar{Q}}$ and $\boldsymbol{(\theta\ )\ = 1}$ on $\boldsymbol{Q}$.
By contradiction, assume that on all cubes Q_J^n such that $(x_\varepsilon, t_{k_n}) \in \bar{Q}_J^n$, we have $(\theta^\varepsilon)^* = -1$. Then, using *Step 2*, we deduce that $(\theta^\varepsilon)^* = -1$ in a neighborhood of (x_ε, t_{k_n}). This is absurd since $(\theta^\varepsilon)^*(x_\varepsilon, t_{k_n}) = 1$.

Before continuing the proof, we need a few notation. We set

$$\overline{n} = \inf\{k, \quad k_n \leq k \leq k_{n+1} - 1,\ \theta_I^k = 1\}.$$

In particular, we have $\theta_I^{\overline{n}} = 1$ and $\theta_I^{\overline{n}-1} = -1$.

Since $(\varphi_\varepsilon)_t(x_\varepsilon, t_{k_n}) > 0$ for ε small enough, by Implicit Function Theorem, there exists a neighborhood V_ε of $(x_\varepsilon, t_\varepsilon)$ and a function $\bar{\Psi}_\varepsilon$ such that

$$\{\varphi_\varepsilon(x,t) < 1\} \Leftrightarrow \{t < \bar{\Psi}_\varepsilon(x)\}$$

in V_ε. Using the fact that $(\theta^\varepsilon)^* \leq \varphi_\varepsilon$ yields

$$\{(\theta^\varepsilon)^* = 1\} \subset \{t \geq \bar{\Psi}_\varepsilon(x)\}. \tag{6.18}$$

Moreover, for δ_0 small enough, *i.e.* for $(x_\varepsilon, t_\varepsilon)$ closed enough to (x_0, t_0), we can assume that $V_\varepsilon \supset Q_{r,\tau}(x_0, t_0) = B_r(x_0) \times (t_0 - \tau, t_0 + \tau)$. We define $\nu = x_\varepsilon - x_I \in [0, \Delta x)^N$ and $\Psi_\varepsilon(x) = \bar{\Psi}_\varepsilon(x + \nu)$. In particular, we have $\Psi_\varepsilon(x_I) = t_{k_n}$.

Step 4. For all $\boldsymbol{(x\ , t\ ) \in Q\ (x_0, t_0) = B\ (x_0) \times (t_0 - \tau, t_0 + \tau)}$

$$\boldsymbol{\theta\ (x\ , t\ ) = 1 \quad \Longrightarrow \quad t\ \geq \Psi\ (x\ ).}$$

To prove this, we consider the collection of nodes

$$\mathcal{C} = \left\{(x_J, t_{k_m}) \in Q_{r,\tau}(x_0, t_0) \cap \{\theta^\varepsilon = 1\}\right\}.$$

By inclusion (6.18), we then have $t_{k_m} \geq \bar{\Psi}_\varepsilon(x_J)$, $\forall (x_J, t_{k_m}) \in \mathcal{C}$. We deduce $\forall (x_J, t_{k_m}) \in \mathcal{C}$

$$\bar{Q}_J^m = [x_J, x_J + \Delta x] \times [t_{k_m}, t_{k_{m+1}}] \subset \{t \geq \bar{\Psi}_\varepsilon(x)\}.$$

This implies that

$$(x_J + (x_\varepsilon - x_I), t_{k_m}) \in \{t \geq \bar{\Psi}_\varepsilon(x)\}$$

and so

$$(x_J, t_{k_m}) \in \{t \geq \Psi_\varepsilon(x)\}$$

which implies i) because any $t_{m'}$ can be written t_{k_m} for a suitable m.

Step 5. We have $\boldsymbol{\theta} = -1$ for $\boldsymbol{m_0 \leq m \leq \overline{n} - 1}$ where $\boldsymbol{m_0 = \inf\{m,\ t \geq t_0 - \tau\}}$.
By contradiction, suppose that there exists $m_0 \leq m \leq \overline{n}-1$ such that $\theta_I^m = 1$. We then define m_1 as

$$m_1 = \sup\{m \leq \overline{n} - 1,\ \theta_I^m = 1\}.$$

In particular, we have $\theta_I^{m_1+1} = -1$ (since $\theta_I^{\overline{n}-1} = -1$). Two cases may occur :

(a) $t_{m_1} = t_{k_n} = t_{\overline{n}}$.
In this case, we have $\widehat{c}_I^{m_1} = \widehat{c}_I^{\overline{n}-1} > 0$ (since $\theta_I^{\overline{n}-1} = -1$ and $\theta_I^{\overline{n}} = 1$). This contradicts the fact that $\theta_I^{m_1} = 1$ and $\theta_I^{m_1+1} = -1$.

(b) $t_{m_1} < t_{k_n} = t_{\overline{n}}$.
In this case, we have $\theta^\varepsilon(x_I, t_{m_1}) = 1$ and $t_{m_1} < t_{k_n} = \Psi_\varepsilon(x_I)$. This contradicts *Step 4*.

We now prove *(iii)*.
By Implicit Functions Theorem, we have $\varphi_\varepsilon(x, \bar{\Psi}_\varepsilon(x)) = 1$. Deriving yields

$$\varphi_t(x, \bar{\Psi}_\varepsilon(x)) D\bar{\Psi}_\varepsilon(x) + D\varphi(x, \bar{\Psi}(x)) = 0.$$

Taking $x = x_\varepsilon$ yields

$$D\Psi_\varepsilon(x_I) = -\frac{D\varphi(x_\varepsilon, \bar{\Psi}_\varepsilon(x_\varepsilon))}{\varphi_t(x_\varepsilon, \bar{\Psi}_\varepsilon(x_\varepsilon))} = -\frac{D\varphi(x_I, t_{k_n})}{\varphi_t(x_I, t_{k_n})} + O(\Delta x)$$

and so

$$D\Psi_\varepsilon(x_I) = -\frac{D\varphi(x_0, t_0)}{\varphi_t(x_0, t_0)} + O(|x_I - x_0| + |t_{k_n} - t_0| + \Delta x).$$

Moreover, by Taylor expansion, we get, if $|\varphi(x_\varepsilon, t_\varepsilon) - 1|$ is small enough, for all $J \in V(I)$

$$\begin{aligned} \Psi_\varepsilon(x_J) =& \Psi_\varepsilon(x_I) + (x_J - x_I) \cdot D\Psi_\varepsilon(x_I) + O(|\Delta x|^2) \\ =& \Psi_\varepsilon(x_I) - \frac{D\varphi(x_0, t_0)}{\varphi_t(x_0, t_0)} \cdot (x_J - x_I) + (\Delta x)\ O\left(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|\right). \end{aligned}$$

where "the O is uniform in ε". This ends the proof of (iii).

It just remains to show that if $\varphi_t(x_0, t_0) < 0$, then for all $(x_J, t_{k_n}) \in Q_{r,\tau}(x_0, t_0) = B_r(x_0) \times (t_0 - \tau, t_0 + \tau)$

$$\theta^\varepsilon(x_J, t_{k_{n-1}}) = 1 \quad \text{and} \quad \theta^\varepsilon(x_J, t_{k_n}) = -1 \quad \Longrightarrow \quad t_{k_n} \leq \Psi_\varepsilon(x_J).$$

In this case, inclusion (6.18) is replaced by

$$\{(\theta^\varepsilon)^* = 1\} \subset \{t \leq \bar{\Psi}_\varepsilon(x)\}.$$

By definition of θ^ε, for all $y \in [x_J, x_J + \Delta x]$, we have $(\theta^\varepsilon)^*(y, t_{k_n}) = 1$. Taking $y = x_J + \nu$, we then deduce that

$$t_{k_n} \leq \bar{\Psi}_\varepsilon(y) = \bar{\Psi}_\varepsilon(x_J + \nu) = \Psi_\varepsilon(x_J).$$

□

Lemma 5.4. (Approximate horizontal level set in the i-direction for negative velocity)
Under the notation and assumptions of Lemma 5.3 with $\varphi_t(x_0, t_0) < 0$, let us suppose that there exists $\delta_0 > 0$ such that $c < -\delta < 0$ on $B_{\delta_0}(x_0, t_0)$.
Let us assume moreover that $(x_I, t_{\overline{n}}) \in B_{\delta_0}(x_0, t_0)$, $\theta_I^{\overline{n}-1} = 1$ and $\theta_I^{\overline{n}} = -1$. If for some fixed $i \in \{1, ..., N\}$ we have

$$\tilde{u}_I^{\overline{n}-1} - \widehat{u}_{-,I^{i,+}}^{\overline{n}-1} < 0 \quad \text{and} \quad \tilde{u}_I^{\overline{n}-1} - \widehat{u}_{-,I^{i,-}}^{\overline{n}-1} < 0$$

then

$$\left| \frac{D\varphi(x_0, t_0)}{\varphi_t(x_0, t_0)} \cdot e_i \right| \leq o(1).$$

Proof
We first prove that if $\tilde{u}_I^{\overline{n}-1} - \widehat{u}_{-,J}^{\overline{n}-1} < 0$ for some $J \in V(I) \setminus \{I\}$, then

$$\Psi_\varepsilon(x_I) - \Psi_\varepsilon(x_J) \leq o(\Delta x).$$

There are two cases : $\widehat{u}_{-,J}^{\overline{n}-1} = \infty$ or $\widehat{u}_{-,J}^{\overline{n}-1} < \infty$.
If $\widehat{u}_{-,J}^{\overline{n}-1} < \infty$ then $J \in F_-^{\overline{n}-1}$. By Lemma 5.1 it results

$$\widehat{u}_{-,J}^{\overline{n}-1} = \sup\{t_m \leq t_{\overline{n}-1},\ \theta_J^{m-1} = 1, \theta_J^p = -1, \quad \text{for} \quad m \leq p \leq \overline{n} - 1\}$$

and by Lemma 5.3 (iv) we have $\widehat{u}_{-,J}^{\overline{n}-1} \leq \Psi_\varepsilon(x_J)$.
We then deduce that

$$0 > \tilde{u}_I^{\overline{n}-1} - \widehat{u}_{-,J}^{\overline{n}-1} \geq \tilde{t}_{\overline{n}} - \Psi_\varepsilon(x_J) = \Psi_\varepsilon(x_I) - \Psi_\varepsilon(x_J) - (t_{\overline{n}} - \tilde{t}_{\overline{n}}).$$

We apply Lemma 5.2 and we obtain

$$\Psi_\varepsilon(x_I) - \Psi_\varepsilon(x_J) \leq o(\Delta x).$$

If $\widehat{u}_{-,J}^{\overline{n}-1} = \infty$ then necessarily $\theta_J^{\overline{n}-1} = 1$, now either $\theta_J^{\overline{n}} = 1$, respectively either $\theta_J^{\overline{n}} = -1$. Then we can apply Lemma 5.3 (i), respectively (iv), and we get $t_{\overline{n}} \leq \Psi_\varepsilon(x_J)$. We deduce then

$$\Psi_\varepsilon(x_I) - \Psi_\varepsilon(x_J) \leq t_{\overline{n}} - t_{\overline{n}} \leq 0.$$

Using Lemma 5.3 (iii) for $J = I^{i,\pm}$, we deduce that

$$\pm \Delta x \frac{D\varphi(x_0, t_0)}{\varphi_t(x_0, t_0)} \cdot e_i \leq o(\Delta x).$$

□

Lemma 5.5. (Decay of θ^ε in the gradient direction of a test function) *Let φ be C^2 in a neighborhood V of (x_0,t_0) and let us suppose there exist $\delta_0>0$ such that $\max_{\overline{V}}((\theta^\varepsilon)^*-\varphi)=(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon)-\varphi(x_\varepsilon,t_\varepsilon)$ with $(x_\varepsilon,t_\varepsilon)\in B_{\delta_0}(x_0,t_0)\subset V$ and $(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon)=1$. Then, there exists a node $(I,n)\in\mathbb{Z}^N\times\mathbb{N}$ such that $\theta_I^{k_{n+1}-1}=1$ with $(x_\varepsilon,t_\varepsilon)\in\partial Q_I^n=\partial(]x_I,x_I+\Delta x[\times]t_{k_n},t_{k_{n+1}}[)$ such that if $\mp e_i\cdot D\varphi(x_0,t_0)>0$ then*

$$\theta^\varepsilon(x,t)=-1\quad\text{in } Q^n_{I^{i,\pm}}=]x_{I^{i,\pm}},x_{I^{i,\pm}}+\Delta x[\times]t_{k_n},t_{k_{n+1}}[.$$

Proof of Lemma 5.5

Since $(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon)=1$, there exists a node $(I,n)\in\mathbb{Z}^N\times\mathbb{N}$ such that $\theta_I^{k_{n+1}-1}=1$ with $(x_\varepsilon,t_\varepsilon)\in\partial Q_I^n=\partial(]x_I,x_I+\Delta x[\times]t_{k_n},t_{k_{n+1}}[)$.

Assume for example that

$$e_i\cdot D\varphi(x_0,t_0)<0$$

and let us suppose by contradiction that $\theta^\varepsilon=1$ in

$$Q^n_{I^{i,+}}=]x_{I^{i,+}},x_{I^{i,+}}+\Delta x[\times]t_{k_n},t_{k_{n+1}}[.$$

We define $\varphi_\varepsilon=\varphi+((\theta^\varepsilon)^*-\varphi)(x_\varepsilon,t_\varepsilon)$. In particular, we have

$$(\theta^\varepsilon)^*\le\varphi_\varepsilon\quad\text{and}\quad(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon)=\varphi_\varepsilon(x_\varepsilon,t_\varepsilon)=1.$$

Since $(\theta^\varepsilon)^*\le\varphi_\varepsilon$, the following inclusion holds

$$\{(\theta^\varepsilon)^*=1\}\subset\{\varphi_\varepsilon\ge 1\}.$$

We define $x_\varepsilon^{i,\lambda}=x_\varepsilon+\lambda e_i$ with $0\le\lambda\le\Delta x$ such that $(\theta^\varepsilon)^*(x_\varepsilon^{i,\lambda},t_\varepsilon)=1$. Then $\varphi_\varepsilon(x_\varepsilon^{i,\lambda},t_\varepsilon)\ge 1$ and

$$\frac{\varphi_\varepsilon(x_\varepsilon^{i,\lambda},t_\varepsilon)-\varphi_\varepsilon(x_\varepsilon,t_\varepsilon)}{\lambda}\ge 0.$$

Taking the limit for $\lambda\to 0$, we obtain

$$e_i\cdot D\varphi(x_\varepsilon,t_\varepsilon)=e_i\cdot D\varphi_\varepsilon(x_\varepsilon,t_\varepsilon)\ge 0.$$

This ends the proof, since it contradicts the assumption. □

Lemma 5.6. (Bound on $|t_\varepsilon-t_{\overline{m}_0}|$ for negative velocity) *Under the notation and assumptions of Lemma 5.5, if we suppose there exists $\delta>0$ and $\delta_0>0$ such that $c(x,t)<-\delta<0$ in $(x,t)\in B_{2\delta_0}(x_0,t_0)\subset V$ then the following estimate holds*

$$|t_\varepsilon-t_{\overline{m}_0}|\le\frac{\Delta x}{\delta}$$

with

$$t_{\overline{m}_0}=\sup\{t_m\le t_{k_n}\ :\ \theta^{m-1}_{I^{i,+}}=1,\ \theta^m_{I^{i,+}}=-1\}\ \textit{if}\ -e_i\cdot D\varphi(x_0,t_0)>0$$

$$(\textit{resp. } t_{\overline{m}_0}=\sup\{t_m\le t_{k_n}\ :\ \theta^{m-1}_{I^{i,-}}=1,\ \theta^m_{I^{i,-}}=-1\}\ \textit{if}\ +e_i\cdot D\varphi(x_0,t_0)>0)$$

where I is defined in Lemma 5.5.

Proof of Lemma 5.6
Let us define

$$\overline{m}_0 = \sup\{\ m \le k_{n+1}-1,\ \theta^{m-1}_{I^{i,\pm}} = 1, \theta^{m}_{I^{i,\pm}} = -1\}.$$

For $\Delta x, \Delta t$ small enough, we can assume that $(x_K, t_m) \in B_{2\delta_0}(x_0, t_0)$ for $K = I, I^{i,\pm}$ and $\overline{m}_0 \le m \le k_{n+1}$. Since $c < 0$ in $B_{2\delta_0}(x_0, t_0)$, $\theta_I^{k_{n+1}-1} = 1$ implies $\theta_I^m = 1$ for all $\overline{m}_0 \le m \le k_{n+1}-1$, and by definition of $\overline{m}_0$, $\theta^m_{I^{i,\pm}} = -1$ for all $\overline{m}_0 \le m \le k_{n+1}-1$. This means that $I^{i,\pm} \in F^m_-$ for all $\overline{m}_0 \le m \le k_{n+1}-1$ and so

$$\widehat{u}^m_{-,I^{i,\pm}} = t_{\overline{m}_0} \text{ for } \overline{m}_0 \le m \le k_{n+1}-1. \tag{6.19}$$

In particular, $\widehat{u}^{k_{n+1}-1}_{-,I^{i,\pm}} = t_{\overline{m}_0}$ and by the definition of the $\widehat{t}_{k_{n+1}}$ it results $\tilde{u}_I^{k_{n+1}-1} \ge \widehat{t}_{k_{n+1}}$ with $\widehat{t}_{k_{n+1}} = t_{k_{n+1}}$, since $t_{k_{n+1}} > t_{k_n}$.
By the equation

$$\sum_{k=1}^{N}\left(\max_{\pm}\left(0, \tilde{u}_I^{k_{n+1}-1} - \widehat{u}^{k_{n+1}-1}_{-,I^{k,\pm}}\right)\right)^2 = \left(\frac{\Delta x}{\widehat{c}_I^{k_{n+1}-1}}\right)^2,$$

we conclude that

$$t_\varepsilon - t_{\overline{m}_0} \le t_{k_{n+1}} - t_{\overline{m}_0} \le \tilde{u}_I^{k_{n+1}-1} - \widehat{u}^{k_{n+1}-1}_{-,I^{i,\pm}} \le \frac{(\Delta x)}{|\widehat{c}_I^{k_{n+1}-1}|} \le \frac{\Delta x}{\delta}.$$

□

6 Proof of Theorem 2.5

This section is dedicated to the proof of the main theorem, which is preceded by two important propositions.

The first proposition will show that the limit function $\overline{\theta}^0$ is a sub-solution in all the domain excepted for the initial time, whereas the second proposition will show that the limit function $\overline{\theta}^0$ is a sub-solution at the initial time. The reason why we need to treat a part the initial condition is that the proof of the first proposition is based on the definition of discontinuous viscosity sub-solution (see Barles [18] and Crandall, Ishii, Lions [61]) consisting in testing the equation by smooth functions, but this definition does not hold at the initial time. Then we treat the initial condition using the technique of barriers.

At the end of this section, we give the main proof using both results.

Proposition 6.1. (Sub-solution property of the limit)
The function $\overline{\theta}^0$ *is a sub-solution of the equation*

$$\theta_t(x,t) = c(x,t)|D\theta(x,t)|$$

on $\mathbb{R}^N \times (0,T)$.

Proof of Proposition 6.1
By contradiction, assume that there are (x_0,t_0) and $\varphi \in C^2$ such that $\overline{\theta}^0 - \varphi$ reaches a strict maximum at (x_0,t_0) with $\overline{\theta}^0(x_0,t_0) = \varphi(x_0,t_0)$ and

$$\varphi_t(x_0,t_0) = \alpha + c(x_0,t_0)|D\varphi(x_0,t_0)| \tag{6.20}$$

with $\alpha > 0$. Since the maximum of $\overline{\theta}^0 - \varphi$ is strict, there exists $(x_\varepsilon,t_\varepsilon) \to (x_0,t_0)$ as $\Delta x \to 0$ such that

$$\max((\theta^\varepsilon)^* - \varphi) = ((\theta^\varepsilon)^* - \varphi)(x_\varepsilon,t_\varepsilon).$$

In particular, we have $(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon) = 1$ for $\Delta x, \Delta t$ small enough. Indeed, by contradiction, suppose that $(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon) = -1$. Using the fact that $(\theta^\varepsilon)^*$ is upper semi-continuous, we obtain $(\theta^\varepsilon)^* = -1$ a neighborhood of $(x_\varepsilon,t_\varepsilon)$. We then deduce that $\varphi_t(x_\varepsilon,t_\varepsilon) = D\varphi(x_\varepsilon,t_\varepsilon) = 0$ and so

$$0 = \varphi_t(x_\varepsilon,t_\varepsilon) - c(x_\varepsilon,t_\varepsilon)|D\varphi(x_\varepsilon,t_\varepsilon)| \to \varphi_t(x_0,t_0) - c(x_0,t_0)|D\varphi(x_0,t_0)| = \alpha$$

This is absurd.
If $|D\varphi(x_0,t_0)| \neq 0$, we note that we can rewrite inequality (6.20) as

$$\varphi_t(x_0,t_0) = \bar{c}|D\varphi(x_0,t_0)| \quad \text{with } \bar{c} > c(x_0,t_0) \tag{6.21}$$

We denote by

$$\vec{n}_0 = \frac{D\varphi(x_0,t_0)}{|D\varphi(x_0,t_0)|}. \tag{6.22}$$

To continue the proof, we have to distinguish several cases :

1. $\boldsymbol{c(x_0,t_0) > 0}$.
   In this case, we have in particular, $\varphi_t(x_0,t_0) > 0$. Then we can apply Lemma 5.3 and we deduce that there exist $\Psi_\varepsilon \in C^2$ and $(I,n) \in \mathbb{Z}^N \times \mathbb{N}$ such that $(x_I,t_{k_n}) \to (x_0,t_0)$ as $\varepsilon = (\Delta x, \Delta t) \to 0$,

   $$(\theta^\varepsilon)^*(x_I,t_{k_n}) = 1, \quad t_{k_n} = \Psi_\varepsilon(x_I)$$

   and

   $$\theta_I^{\overline{n}} = 1, \quad \theta_I^{\overline{n}-1} = -1,$$

where $\bar{n}$ is defined in Lemma 5.3. Using Lemma 5.1 and Lemma 5.3 (i), we deduce also that for all $J \in V(I)\setminus\{I\}$ such that $\theta_J^{\overline{n}-1} = 1$, we have

$$\widehat{u}_{+,J}^{\overline{n}-1} \geq \Psi_\varepsilon(x_J).$$

This implies for all $J \in V(I) \cap F_+^{\overline{n}-1}$, using also the (general) fact that $\tilde{u}_I^{\overline{n}-1} \leq t_{\overline{n}} = t_{k_n}$,

$$\tilde{u}_I^{\overline{n}-1} - \widehat{u}_{+,J}^{\overline{n}-1} \leq t_{\overline{n}} - \widehat{u}_{+,J}^{\overline{n}-1} = \Psi_\varepsilon(x_I) - \widehat{u}_{+,J}^{\overline{n}-1} \leq \Psi_\varepsilon(x_I) - \Psi_\varepsilon(x_J). \tag{6.23}$$

By the GFMM algorithm (Step 5), $\tilde{u}_I^{\overline{n}-1}$ is solution of the equation

$$\left(\frac{\Delta x}{c(x_I, t_{\overline{n}-1})}\right)^2 = \sum_{i=1}^{N}\left(\max_{\pm}\left(0, \tilde{u}_I^{\overline{n}-1} - \widehat{u}_{+,I^{i,\pm}}^{\overline{n}-1}\right)\right)^2$$

If $|D\varphi(x_0,t_0)| \neq 0$, by adding (6.23) for $J = I^{i,\pm}$ on all direction $i \in \mathcal{C} \subset \{1,..,N\}$ such that

$$\tilde{u}_I^{\overline{n}-1} - \widehat{u}_{+,I^{i,+}}^{\overline{n}-1} \geq 0 \;\text{ or }\; \tilde{u}_I^{\overline{n}-1} - \widehat{u}_{+,I^{i,-}}^{\overline{n}-1} \geq 0$$

and by using Lemma 5.3 *(iii)*, we can estimate

$$\begin{aligned}
\left(\frac{\Delta x}{c(x_I, t_{\overline{n}-1})}\right)^2 =& \sum_{i\in\mathcal{C}}\left(\max_{\pm}\left(\tilde{u}_I^{\overline{n}-1} - \widehat{u}_{+,I^{i,\pm}}^{\overline{n}-1}\right)\right)^2 \leq \sum_{i\in\mathcal{C}}\max_{\pm}\left(\Psi_\varepsilon(x_I) - \Psi_\varepsilon(x_{I^{i,\pm}})\right)^2 \\
\leq& \frac{(\Delta x)^2}{\bar{c}^2}\sum_{i\in\mathcal{C}}(\vec{n_0}\cdot e_i)^2 + (\Delta x)^2 O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|) \\
\leq& \frac{(\Delta x)^2}{\bar{c}^2} + (\Delta x)^2 O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|)
\end{aligned}$$

where $\bar{c}$ and $\vec{n_0}$ are defined in (6.21) and (6.22) respectively.
It follows that

$$\frac{1}{c^2(x_I, t_{\overline{n}-1})} - \frac{1}{\bar{c}^2} \leq O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|).$$

Taking the limit $\varepsilon = (\Delta x, \Delta t) \to 0$, we obtain a contradiction.
If $D\varphi(x_0,t_0) = 0$, we get in the same way

$$\frac{1}{c^2(x_I, t_{\overline{n}-1})} \leq O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|).$$

Taking the limit $\varepsilon \to 0$, since we have assumed $c(x_0,t_0) > 0$, we obtain a contradiction.

2. $\boldsymbol{c(x_0,t_0)<0}$.
   In this case, we have no informations on the sign of φ_t, so we have to distinguish several cases :

   (a) $\boldsymbol{\varphi\ (x_0,t_0)<0}$.
   Note that, in this case, $|D\varphi(x_0,t_0)|\neq 0$ and (6.21) holds with $0>\bar{c}>c(x_0,t_0)$.
   Then we can apply Lemma 5.3 and we deduce that there exist $\Psi_\varepsilon\in C^2$ and $(I,n)\in\mathbb{Z}^N\times\mathbb{N}$ such that

$$(\theta^\varepsilon)^*(x_I,t_{k_n})=1,\quad t_{k_n}=\Psi_\varepsilon(x_I)$$

   and

$$\theta_I^{\overline{n}}=-1,\quad \theta_I^{\overline{n}-1}=1,$$

   where $\bar{n}$ is defined in Lemma 5.3. Using Lemma 5.1 and Lemma 5.3 (iv), we deduce also that for all $J\in V(I)\backslash\{I\}$ such that $\theta_J^{\overline{n}-1}=-1$, we have

$$\widehat{u}_{-,J}^{\overline{n}-1}\leq\Psi_\varepsilon(x_J).$$

   This implies that for all $J\in V(I)\cap F_-^{\overline{n}-1}$

$$\begin{aligned}\tilde{u}_I^{\overline{n}-1}-\widehat{u}_{-,J}^{\overline{n}-1}\geq&\tilde{t}_{\overline{n}}-\Psi_\varepsilon(x_J)\\ =&t_{\overline{n}}-\Psi_\varepsilon(x_J)+(\tilde{t}_{\overline{n}}-t_{\overline{n}})=\Psi_\varepsilon(x_I)-\Psi_\varepsilon(x_J)+(\tilde{t}_{\overline{n}}-t_{\overline{n}})\end{aligned}$$

   Since $c(x_0,t_0)\neq 0$, there exists $\delta,\ \delta_0>0$ such that $|c|\geq\delta>0$ on $B_{\delta_0}(x_0,t_0)$ and we can apply Lemma 5.2 to get

$$\tilde{u}_I^{\overline{n}-1}-\widehat{u}_{-,J}^{\overline{n}-1}\geq\Psi_\varepsilon(x_I)-\Psi_\varepsilon(x_J)+o(\Delta x).$$

   Using Lemma 5.3 (iii) yields

$$\tilde{u}_I^{\overline{n}-1}-\widehat{u}_{-,J}^{\overline{n}-1}\geq\frac{1}{\bar{c}}\vec{n_0}\cdot(x_J-x_I)+(\Delta x)\ O(\Delta x+|x_I-x_0|+|t_{k_n}-t_0|)+o(\Delta x).$$

   By adding the previous equation for $J=I^{i,\pm}$ on all direction $i\in\mathcal{C}\subset\{1,..,N\}$ such that

$$\tilde{u}_I^{\overline{n}-1}-\widehat{u}_{-,I^{i,+}}^{\overline{n}-1}\geq 0\ \text{ or }\ \tilde{u}_I^{\overline{n}-1}-\widehat{u}_{-,I^{i,-}}^{\overline{n}-1}\geq 0$$

we obtain, since $|D\varphi(x_0,t_0)| \neq 0$

$$\begin{aligned}\left(\frac{\Delta x}{c(x_I,t_{\overline{n}-1})}\right)^2 =& \sum_{i=1}^{N}\left(\max_{\pm}\left(0,\tilde{u}_I^{\overline{n}-1}-\widehat{u}_{-,I^{i,\pm}}^{\overline{n}-1}\right)\right)^2\\ =& \sum_{i\in\mathcal{C}}\left(\tilde{u}_I^{\overline{n}-1}-\widehat{u}_{-,I^{i,\pm}}^{\overline{n}-1}\right)^2\\ \geq& \frac{(\Delta x)^2}{\bar{c}^2}\sum_{i\in\mathcal{C}}(\vec{n_0}\cdot e_i)^2\\ &+(\Delta x)^2\ O(\Delta x+|x_I-x_0|+|t_{k_n}-t_0|)+o(\Delta x)^2\end{aligned} \tag{6.24}$$

If $i\notin\mathcal{C}$ (*i.e.* $\tilde{u}_I^{\overline{n}-1}-\widehat{u}_{-,I^{i,+}}^{\overline{n}-1}<0$ and $\tilde{u}_I^{\overline{n}-1}-\widehat{u}_{-,I^{i,-}}^{\overline{n}-1}<0$), then by Lemma 5.4, we deduce that

$$\left|\frac{1}{\bar{c}}\Delta x\ \vec{n_0}\cdot e_i\right| = o(\Delta x). \tag{6.25}$$

By combining (6.24) and (6.25), we get

$$\begin{aligned}\left(\frac{\Delta x}{c(x_I,t_{\overline{n}-1})}\right)^2 \geq& \frac{(\Delta x)^2}{\bar{c}^2}\sum_{i=1}^{N}(\vec{n_0}\cdot e_i)^2\\ &+(\Delta x)^2\ O(\Delta x+|x_I-x_0|+|t_{k_n}-t_0|)+o(\Delta x^2)\\ =&\frac{(\Delta x)^2}{\bar{c}^2}+(\Delta x)^2\ O(\Delta x+|x_I-x_0|+|t_{k_n}-t_0|)+o(\Delta x^2)\end{aligned}$$

This implies

$$\frac{1}{c^2(x_I,t_{\overline{n}-1})}-\frac{1}{\bar{c}^2}\geq O(\Delta x+|x_I-x_0|+|t_{k_n}-t_0|)+o(1).$$

Taking the limit $\varepsilon=(\Delta x,\Delta t)\to 0$, we get the contradiction since $|c(x_0,t_0)|>|\bar{c}|$.

(b) $\boldsymbol{\varphi\ (x_0,t_0)>0}$.

Since $c(x_0,t_0)<0$, we have by the algorithm that $\dfrac{\partial(\theta^\varepsilon)^*}{\partial t}\leq 0$. We define $\varphi_\varepsilon=\varphi+((\theta^\varepsilon)^*-\varphi)(x_\varepsilon,t_\varepsilon)$. In particular, we have $(\theta^\varepsilon)^*\leq\varphi_\varepsilon$ and

$$(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon)=\varphi_\varepsilon(x_\varepsilon,t_\varepsilon)=1.$$

We have $t_\varepsilon=t_{k_n}$. Indeed, assume that $t_\varepsilon\in(t_{k_n},t_{k_{n+1}})$. Using the fact that $(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon)=1$, we deduce that $(\theta^\varepsilon)^*(x_\varepsilon,t)=1$ for $t_{k_n}\leq t\leq t_{k_{n+1}}$ and so $\varphi_t(x_\varepsilon,t_\varepsilon)=0$. This is absurd for ε small enough since $\varphi_t(x_0,t_0)>0$. Using the fact that $(\varphi_\varepsilon)_t>0$, we deduce that $(\theta^\varepsilon)^*(x_\varepsilon,t)\leq\varphi_\varepsilon(x_\varepsilon,t)<1$ for $t<t_{k_n}$. This is absurd since $\dfrac{\partial(\theta^\varepsilon)^*}{\partial t}\leq 0$.

(c) $\boldsymbol{\varphi\ (x_0, t_0) = 0}$.
Since the equation (6.20) holds with $\alpha > 0$, we have, in particular, $|D\varphi(x_0, t_0)| \neq 0$. Then, there exists a direction $\pm e_i$ such that $\mp e_i \cdot D\varphi(x_0, t_0) > 0$. Using Lemma 5.5, we deduce that there exists $(I, n) \in \mathbb{Z}^N \times \mathbb{N}$ such that $\theta_I^{k_{n+1}-1} = 1$ and $\theta^\varepsilon = -1$ on $Q^n_{I^{i,\pm}} =]x_{I^{i,\pm}}, x_{I^{i,\pm}} + \Delta x[\times]t_{k_n}, t_{k_{n+1}}[$. We define $t_{\overline{m}_0}$ such that

$$\overline{m}_0 = \sup\{m \ :\ t_m \leq t_{k_n},\ \theta^{m-1}_{I^{i,\pm}} = 1,\ \theta^m_{I^{i,\pm}} = -1\}.$$

In particular, $(\theta^\varepsilon)^*(x, t_{\overline{m}_0}) = 1$ for all $x \in [x_{I^{i,\pm}}, x_{I^{i,\pm}} + \Delta x]$.
We define $\varphi_\varepsilon = \varphi + ((\theta^\varepsilon)^* - \varphi)(x_\varepsilon, t_\varepsilon)$. In particular, we have $(\theta^\varepsilon)^* \leq \varphi_\varepsilon$ and

$$(\theta^\varepsilon)^*(x_\varepsilon, t_\varepsilon) = \varphi_\varepsilon(x_\varepsilon, t_\varepsilon) = 1.$$

Since the following inclusion $\{(\theta^\varepsilon)^* = 1\} \subset \{\varphi_\varepsilon \geq 1\}$ holds, $\varphi_\varepsilon(x, t_{\overline{m}_0}) \geq 1$ for all $x \in [x_{I^{i,\pm}}, x_{I^{i,\pm}} + \Delta x]$.
Let $\nu \in [0, \Delta x]^N$ be such that $x_\varepsilon = x_I + \nu$ and let us define $y \equiv x_{I^{i,\pm}} + \nu$ and $\overline{\varphi}(\cdot, \cdot) \equiv \varphi_\varepsilon(\cdot + \nu, \cdot)$. Then it yields $\overline{\varphi}(x_I, t_\varepsilon) = \varphi_\varepsilon(x_\varepsilon, t_\varepsilon) = 1$, and $\overline{\varphi}(x_{I^{i,\pm}}, t_{\overline{m}_0}) = \varphi_\varepsilon(y, t_{\overline{m}_0}) \geq 1$.

To obtain the contradiction, we consider the expansion of $\overline{\varphi}$ up to the first order

$$\begin{aligned} 0 \leq& \overline{\varphi}(x_{I^{i,\pm}}, t_{\overline{m}_0}) - \overline{\varphi}(x_I, t_\varepsilon) \\ \leq& (x_{I^{i,\pm}} - x_I) \cdot D\overline{\varphi}(x_I, t_\varepsilon) + (t_{\overline{m}_0} - t_\varepsilon)\partial_t\overline{\varphi}(x_I, t_\varepsilon) + O((\Delta x)^2 + |t_\varepsilon - t_{\overline{m}_0}|^2). \end{aligned}$$

Now by Lemma 5.6 and using the fact that $\partial_t\varphi(x_0, t_0) = 0$ we obtain

$$\pm e_i \cdot D\overline{\varphi}(x_I, t_\varepsilon)\Delta x + o(\Delta x) \geq 0,$$

that is absurd, since by assumption $\pm e_i \cdot D\varphi(x_0, t_0) < 0$.

3. $\boldsymbol{c(x_0, t_0) = 0}$.
In this case, we have

$$\varphi_t = \alpha > 0$$

and we can apply Lemma 5.3. Hence, there exists $r,\ \tau > 0$, a function $\Psi_\varepsilon \in C^2(B_r(x_0), (t_0 - \tau, t_0 + \tau))$ and a node $(I, n) \in \mathbb{Z}^N \times \mathbb{N}$ such that

$$(\theta^\varepsilon)^*(x_I, t_{k_n}) = 1, \quad t_{k_n} = \Psi_\varepsilon(x_I)$$

and for all $J \in V(I),\ t_m \in (t_0 - \tau, t_0 + \tau)$, we have

$$\theta^\varepsilon(x_J, t_m) = 1 \implies t_m \geq \Psi_\varepsilon(x_J) \tag{6.26}$$

We define m_0 such that

$$t_{m_0-1} < t_0 - \tau \leq t_{m_0}.$$

For all $J \in (V(I) \setminus \{I\}) \cap \{\theta^{\overline{n}-1} = 1\}$ (with $\bar{n}$ defined in Lemma 5.3), we define

$$m_J = \sup\{k \leq \overline{n},\ \theta_J^{k-1} = -1\}$$

We distinguish two cases :

(a) There exists $J \in (V(I) \setminus \{I\}) \cap \{\theta^{\overline{n}-1} = 1\}$ such that $m_J < m_0$.

Using the fact that $\theta_I^k = -1$ for $m_0 \leq k \leq \overline{n} - 1$ (see Lemma 5.3 *(ii)*), we have that $J \in F_+^k$, $\forall\ m_0 \leq k \leq \overline{n} - 1$ and we deduce that

$$\widehat{u}_{+,J}^{\overline{n}-1} = u_J^{\overline{n}-1} \leq t_{m_0} \quad \text{and} \quad \theta^\varepsilon(x_J, t_{m_0}) = 1.$$

By (6.26), we then have $t_{m_0} \geq \Psi_\varepsilon(x_J)$.
We now assume that $|D\varphi| \neq 0$ (the case $|D\varphi| = 0$ can be treated in a similar way). Using Lemma 5.3, we deduce that

$$t_{m_0} \geq \Psi_\varepsilon(x_J) = t_{k_n} - \frac{1}{\bar{c}}\ \vec{n}_0 \cdot (x_J - x_I) + (\Delta x)\ O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|),$$

and so

$$\begin{aligned} t_{k_n} - t_{m_0} \leq & \frac{1}{\bar{c}}\ \vec{n}_0 \cdot (x_J - x_I) + (\Delta x)\ O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|) \\ \leq & \frac{\Delta x}{\bar{c}} + (\Delta x)\ O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|). \end{aligned}$$

Sending $\Delta x, \Delta t$ to 0, yields

$$t_0 - (t_0 - \tau) = \tau \leq 0.$$

This is absurd.

(b) For all $J \in (V(I) \setminus \{I\}) \cap \{\theta^{\overline{n}-1} = 1\}$, $m_J \geq m_0$.
We then have $\theta^\varepsilon(x_J, t_{m_J}) = 1$ and so by (6.26) we have $\widehat{u}_{+,J}^{\overline{n}-1} = t_{m_J} \geq \Psi(x_J)$.
We now assume that $|D\varphi| \neq 0$ (the case $|D\varphi| = 0$ can be treated in a similar way). Using Lemma 5.3, we deduce that

$$\widehat{u}_{+,J}^{\overline{n}-1} \geq \Psi(x_J) = t_{k_n} - \frac{1}{\bar{c}}\vec{n}_0 \cdot (x_J - x_I) + (\Delta x)\ O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|),$$

and so

$$\tilde{u}_I^{\overline{n}-1} - \widehat{u}_{+,J}^{\overline{n}-1} \leq t_{k_n} - \widehat{u}_{+,J}^{\overline{n}-1} \leq \frac{1}{\bar{c}}\vec{n}_0 \cdot (x_J - x_I) + (\Delta x)\ O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|)$$

By adding for $J = I^{i,\pm}$ on all directions $i \in \mathcal{C} \subset \{1,..,N\}$ such that

$$\tilde{u}_I^{\overline{n}-1} - \widehat{u}_{+,I^i}^{\overline{n}-1} = \max(\tilde{u}_I^{\overline{n}-1} - \widehat{u}_{+,I^{i,+}}^{\overline{n}-1}, \tilde{u}_I^{\overline{n}-1} - \widehat{u}_{+,I^{i,-}}^{\overline{n}-1}) \geq 0,$$

we deduce that

$$\begin{aligned}\left(\frac{\Delta x}{\widehat{c}_I^{\overline{n}-1}}\right)^2 &= \sum_{i\in\mathcal{C}} \left(\tilde{u}_I^{n} - \widehat{u}_{+,I^i}^{n-1}\right)^2 \\ &\leq \left(\frac{\Delta x}{\bar{c}}\right)^2 + (\Delta x)^2\ O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|).\end{aligned}$$

i.e.

$$\frac{1}{|\widehat{c}_I^{\overline{n}-1}|^2} \leq \frac{1}{\bar{c}^2} + O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|)$$

Sending $\Delta x, \Delta t$ to 0, yields a contradiction since $\bar{c} > c(x_0, t_0) = 0$. $\square$

We construct a barrier sub-solution and we prove that $\overline{\theta}^0$ defined by (6.8) satisfies the initial condition of (6.1) :

Proposition 6.2. (Initial condition)
We have the following inequality :

$$\overline{\theta}^0(\cdot, 0) \leq \left(1_{\Omega_0} - 1_{\Omega_0^c}\right)^*. \tag{6.27}$$

Proof of Proposition 6.2
For $\alpha > 0$ which will be precised later, we consider the following function

$$v(x) = \alpha\ \mathrm{dist}(x, \Omega_0). \tag{6.28}$$

and we define, for all $I \in \mathbb{Z}^N$

$$v_I = v(x_I).$$

We then define for $x_I \in \Omega_0^c$ a velocity $\infty > c_{v,I} > 0$ by solving

$$\sum_{k=1}^{N} (\max_{\pm}(0, v_I - \widehat{v}_{I^{k,\pm}}))^2 = \left(\frac{\Delta x}{c_{v,I}}\right)^2,$$

where

$$\widehat{v}_J = \begin{cases} v_J & \text{if} \quad v_J \leq v_I \\ \infty & \text{if} \quad v_J > v_I. \end{cases}$$

This define a GFMM with velocity $c_{v,I}$ and whose solution is v_I. On the one hand, using the fact that $|v_I - v_J| \leq \alpha \Delta x$, yields for $J \in V(I)$

$$c_{v,I} \geq \frac{1}{\alpha \sqrt{N}} \tag{6.29}$$

On the other hand, the C^2 regularity of $\partial\Omega_0$ implies that $c_{v,I}$ is uniformly bounded as $\Delta x \to 0$ in a neighborhood of $\partial\Omega_0$.

Moreover, we can define θ_v^ε in the following way

$$\theta_v^\varepsilon(x,t) = \begin{cases} 1 & \text{if} \quad x \in [x_I, x_I + \Delta x[\text{ and } t \geq v_I \\ -1 & \text{if} \quad x \in [x_I, x_I + \Delta x[\text{ and } t < v_I. \end{cases}$$

We denote by u the solution of the GFMM algorithm with velocity $c(x,t)$. We then have

$$\theta_{u,I}^0 = 1 \Rightarrow x_I \in \Omega_0 \Rightarrow v_I = 0 \Rightarrow \theta_{v,I}^0 = 1.$$

and so

$$\{\theta_u^0 = 1\} \subset \{\theta_v^0 = 1\}.$$

Moreover, using (6.29), we deduce that for α small enough, we have, for all $t \geq 0$

$$c_{v,I} \geq (c(x_I, t))^+.$$

Using the comparison principle Corollary 4.5, we deduce that

$$\theta_v^\varepsilon(x,t) \geq \theta^\varepsilon(x,t).$$

We denote by $v^\varepsilon(x) = \sup\limits_{y \in [x - \Delta x, x]} v(y)$ and $\theta_{v^\varepsilon}(x,t) = 1_{\{v^\varepsilon(x) \geq t\}} - 1_{\{v^\varepsilon(x) < t\}}$. It is easy to check that

$$(\theta_{v^\varepsilon})^*(x,t) \geq (\theta_v^\varepsilon)^*(x,t) \geq (\theta^\varepsilon)^*(x,t).$$

Passing to the limit $\varepsilon \to 0$, we then obtain for $t > 0$

$$1_{\{v(x) \geq t\}} - 1_{\{v(x) < t\}} = \theta_v(x,t) \geq \overline{\theta}^0(x,t)$$

and so

$$\left(1_{\Omega_0} - 1_{\Omega_0^c}\right)^* \geq \overline{\theta}^0(x,0).$$

This implies that $\overline{\theta}^0$ satisfies the initial condition (6.27). $\square$

Proof of Theorem 2.5 The proof of Theorem 2.5 is now quite simple. Indeed, using Theorem 6.1 and Proposition 6.2, we get that $\overline{\theta}^0$ is a viscosity sub-solution of (6.1).

For the super-solution property of $\underline{\theta}^0$, it suffices to use the symmetry of $\overline{\theta}^0$ and $\underline{\theta}^0$ (see Lemma 4.1). Indeed, by contradiction, assume that there are (x_0, t_0) and $\varphi \in C^2$ such that $\underline{\theta}^0 - \varphi$ reaches a strict minimum at (x_0, t_0) with

$$\varphi_t(x_0, t_0) = -\alpha + c(x_0, t_0)|D\varphi(x_0, t_0)|$$

with $\alpha > 0$ and $t_0 > 0$. Let us define $c_1 = -c$, $\varphi_1 = -\varphi$ and $\overline{\theta}_1^0 = \overline{\theta}^0[-\theta^0, -c]$. Then, using Lemma 4.1, we get that $\overline{\theta}_1^0 - \varphi_1$ reaches a strict maximum at (x_0, t_0) with $\overline{\theta}_1^0(x_0, t_0) = \varphi_1(x_0, t_0)$ and

$$(\varphi_1)_t(x_0, t_0) = \alpha + c_1(x_0, t_0)|D\varphi(x_0, t_0)|.$$

This contradicts the sub-solution property of $\overline{\theta}_1^0$. For the initial condition, we use the same arguments of those of Proposition 6.2.

Moreover, if (6.1) satisfies a comparison principle, then $\overline{\theta}^0 \leq (\underline{\theta}^0)^*$ and $(\overline{\theta}^0)_* \leq \underline{\theta}^0$. Since, by definition, $\overline{\theta}^0 \geq \underline{\theta}^0$, we get that $\overline{\theta}^0 = (\underline{\theta}^0)^*$ and $(\overline{\theta}^0)_* = \underline{\theta}^0$ is a solution of (6.1). This exactly means that $\overline{\theta}^0$ and $\underline{\theta}^0$ are solutions, which is then unique (when the comparison principle holds for a special choice of the initial data), up to the upper and the lower semi-continuous envelopes. □

7 Numerical tests

We are going to verify our algorithm by some numerical tests in dimension $N = 2$. First we will give in two cases the representation formula of the solution so that we will be able to obtain numerical errors comparing it with the numerical solution obtained by the GFMM algorithm.

Representation formulas for hyperplanes and spheres propagating with linear speed

We verify that hyperplanes and spheres in $\mathbb{R}^N$, that propagate with a linear speed along the normal direction, keep their shapes during the evolution remaining respectively hyperplanes and spheres.

These manifolds can be characterized by the level set of a polynomial $P(x) : \mathbb{R}^N \to \mathbb{R}$ of degree 1 and 2. We denote by $P(x,t)$ the polynomials with coefficients depending on t.

Each point x s.t. $P(x, t_0) = 0$ verifies the following dynamics :

$$\begin{cases} \dot{y}(t) & = -c(y(t), t)\frac{DP(y(t),t)}{|DP(y(t),t)|}, \\ y(t_0) & = x \end{cases}$$

since they propagate with speed c along the unit normal to the manifold. These trajectories are known as *characteristics*. Then we just need to check that the evolution of each point of the manifold verifies the equation $P(y(t),t)=0$, i.e. deriving with respect to t

$$P_t(y(t),t) - |DP(y(t),t)|c(y(t),t) = 0, \tag{6.30}$$

for any linear speed $c(x,t)=a(t)x+b(t)$ and for any $P(x,t)$ representing hyperplanes or spheres.

Hyperplanes : $P(x,t)=\alpha(t)x+\beta(t)$
It results $P_t(x,t)=\dot\alpha(t)x+\dot\beta(t)$ and $|DP(x(t),t)|=|\alpha(t)|$ then $P(x,t)$ verifies (6.30) with coefficients such that :

$$\begin{cases} \dot\alpha(t) = |\alpha(t)|a(t) \\ \dot\beta(t) = |\alpha(t)|b(t) \end{cases}$$

Spheres : $P(x,t)=R(t)^2-|x-x_0(t)|^2$
It results $P_t(x,t)=2(x-x_0(t))\dot x_0(t)+2R(t)\dot R(t)$ and $|DP(x(t),t)|=2|x-x_0(t)|$ then $P(x,t)$ verifies (6.30) with coefficients such that :

$$\begin{cases} \dot x_0(t) = a(t)R(t) \\ \dot R(t) = x_0(t)a(t)+b(t) \end{cases}$$

Test 1 : a rotating line
We choose as initial data a line $P(x,0)=x_2+1.5x_1$ and then as representing function :

$$\theta(x,0) = \begin{cases} 1 & \text{if } x_2+1.5x_1>0 \\ -1 & \text{otherwise.} \end{cases} \tag{6.31}$$

We choose as velocity $c(x,t)=x_1$. We have proved that a line propagating with linear speed stays a line. Applying the result of the previous section, we obtain that $P(x,t)=\alpha(t)x+\beta(t)$ has coefficients solving the following o.d.e.

$$\begin{cases} \dot\alpha_1(t) = \sqrt{1+\alpha_1(t)^2} \\ \alpha_1(0)=1.5, \end{cases} \qquad \begin{cases} \dot\alpha_2(t) = 0 \\ \alpha_2(0)=1. \end{cases}$$

Solving, we obtain $P(x,t)=\sinh(t+\operatorname{arcsinh}(\alpha_1(0)))x_1+x_2$.
We compute the discrete solution in the numerical domain $D=[-1,1]\times[-1,1]$ and we evaluate the error at final time T=0.5. We use the discrete L^1-norm

$$\|\theta(x_I,T)-\theta_I^m\|_1 = \sum_{\{I:x_I\in D\}} |\theta(x_I,T)-\theta_I^m|\Delta x^2,$$

with m the number of iterations corresponding to reach the final time T. The table 6.1 shows the error for the tests run with $26, 51, 101, 201$ number of nodes for each side of the square domain. The convergence is approximately of order 1.
Fig.6.15 shows the 0-level set of the discrete solution at each time interval 0.1. The line is rotating clockwise and it will reach in infinity time the x_2 axe. The last line is plotted with the exact solution in thicker line.

Δx	L^1-error
0.08	0.102
0.04	0.0576
0.02	0.0304
0.01	0.0160

TAB. 6.1 – Numerical errors for test 1

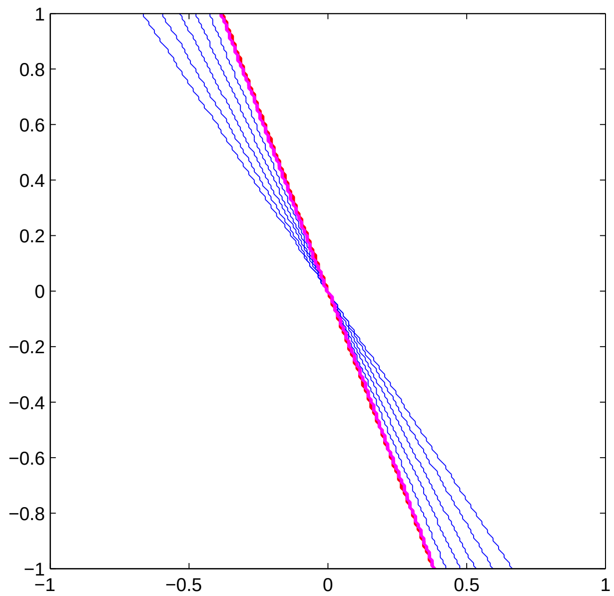

FIG. 6.15 – A rotating line

Test 2 : propagation of a circle
We choose as initial data a circle $P(x,0) = x_1^2 + x_2^2 - 1$ and then as representing function :

$$\theta(x,0) = \begin{cases} 1 & x_1^2 + x_2^2 - 1 < 0 \\ -1 & \text{otherwise.} \end{cases} \tag{6.32}$$

We choose as velocity $c(x,t) = 0.1t - x_1$. We have proved that a circle propagating with linear speed stays a circle. Applying the result of the previous section, we obtain that $P(x,t) = (x_1 - x_{0,1}(t))^2 + (x_2 - x_{0,2}(t))^2 - R(t)^2$ has coefficients solving the following o.d.e.

$$\begin{cases} \dot{x}_{0,1}(t) = -R(t) \\ x_{0,1}(0) = 0 \end{cases} \quad \begin{cases} \dot{x}_{0,2}(t) = 0 \\ x_{0,2}(0) = 0 \end{cases} \quad \begin{cases} \dot{R}(t) = -x_{0,1}(t) + 0.1t \\ R(0) = 1 \end{cases}$$

Solving, we obtain $x_{0,1}(t) = 1/20(2t + 11(\exp(-t) - \exp(t))$ and $R(t) = 1/20(-2 + 11(\exp(t) + \exp(-t)))$.
We compute the discrete solution in the numerical domain $D = [-2,2] \times [-2,2]$ and we evaluate the error at final time T=0.5.
We use the discrete L^1-norm, defined in the previous test.
The table 6.2 shows the error for the tests run with $51, 101, 201, 401$ number of nodes for each side of the square domain. The convergence is approximately of order 1.
Fig.6.16 shows the 0-level set of the discrete solution and at each time interval 0.1, the circle is expanding ad its centre is propagating on the left.

Δx	L^1-error
0.08	0.4992
0.04	0.2784
0.02	0.1288
0.01	0.0582

TAB. 6.2 – Numerical errors for test 2

Test 3 : comparison between the FMM and GFMM algorithm
When the evolution is monotone, i.e. $c(x) > 0$, there exists a link between the evolutive and the stationary equation(see [85] and [151]) :

$$\begin{cases} c(x)|DT(x)| = 1 & x \in \Omega, \\ T(x) = 0 & x \in \partial\Omega. \end{cases}$$

In this case the discrete function u_I^n, computed by the GFMM algorithm, approximates the solution $T(x)$ outside the set Ω.
The two schemes, the FMM and the GFMM, are run in the case the speed is $c(x,t) = 1$ with initial set Ω a circle centred in the origin with radius 0.5.
For this choice of speed, the solution $T(x)$ corresponds at the distance function of the point x from the set Ω.
We compare the two schemes computing the errors in the $||\cdot||_\infty$ discrete norm :

$$|||T(x_I) - u_I||_\infty \equiv \sup_{\{I : x_I \in D\}} |T(x_I) - u_I|.$$

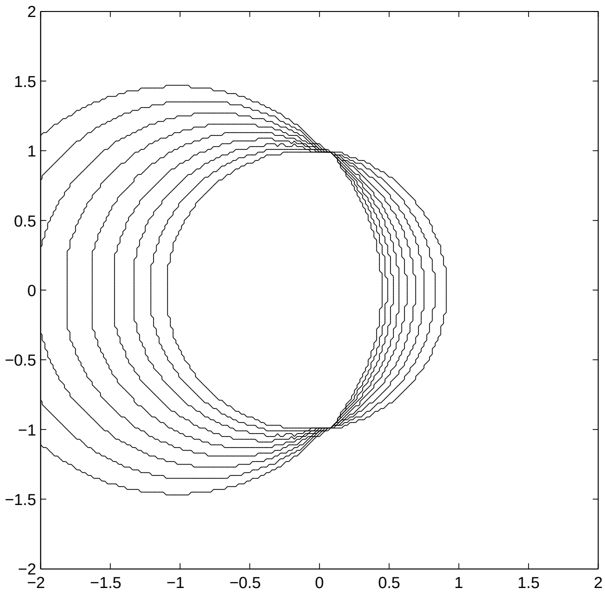

FIG. 6.16 – A propagating circle

Δx	FMM	GFMM
0.08	0.065	0.078
0.04	0.033	0.039
0.02	0.020	0.018

TAB. 6.3 – Numerical errors for test 3

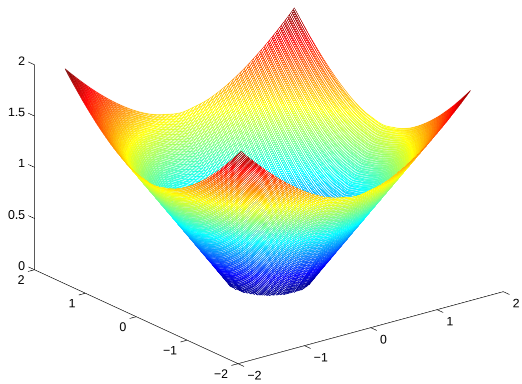

FIG. 6.17 – The discrete time u of a propagating circle with positive constant speed

As one can see, the GFMM scheme produces in this particular case almost the same results of the FMM scheme (as implemented in the HJpack library [107]). The results are slightly different in particular because the time computed in the narrow band in the classical FMM uses not only the accepted points but also the points of the narrow band.

Test 4 : two collapsing circles
We choose as initial data two circles and as velocity $c(x,t) = 1 - t$. The two circles grow as far as the speed is positive. At $t = 1$, when the velocity changes sign, they start to decrease. Fig.6.18 on the left shows the 0-level set of the discrete solution at each time interval 0.2 until $t = 1$ and Fig.6.18 on the right shows the 0-level set of the discrete solution at each time interval 0.2 for the time interval $[1.2, 2.4]$.

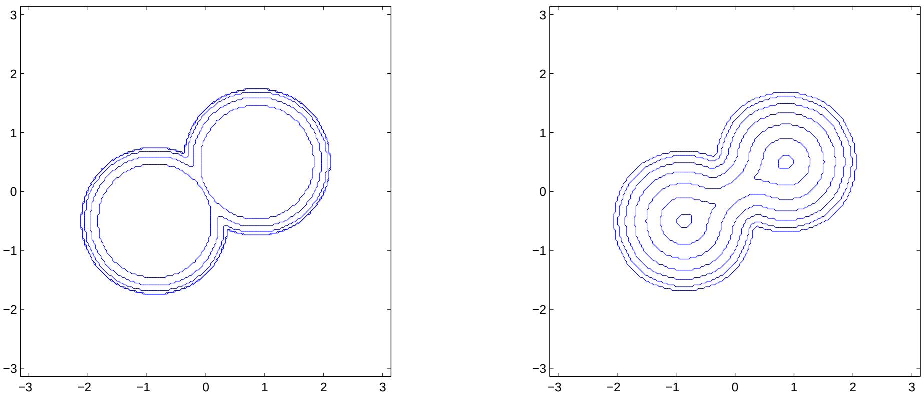

FIG. 6.18 – Two propagating circles

Acknowledgements
This work was supported by the project ACI no. 1025 (2003-2007) from the french ministry of research and by the Italian MIUR 2005 Project "Modellistica numerica per il calcolo scientifico ed applicazioni avanzate". The first author has been supported by Indam (Istituto Nazionale di Alta Matematica). The authors wish to thank Y. Le Bouar and A. Finel for fruitful discussions.

Chapitre 7

Existence de solution pour un modèle décrivant la dynamique de jonctions entre dislocations

Ce chapitre est issu d'un travail en cours avec R. Monneau [90].
On étudie une version dynamique d'un modèle de champ de phase de Koslowski et Ortiz pour des réseaux de dislocations planaires. On considère un champ de vecteurs bi-dimensionnel qui décrit les transitions de phases entre des phases constantes. Chaque transition de phases correspond à une ligne de dislocation et cette description permet la formation de jonctions entre les dislocations. Ce champ de vecteur est supposé satisfaire une équation de Hamilton-Jacobi non locale avec une viscosité non nulle. Pour ce modèle, nous montrons l'existence d'une solution faible globale en temps.

Existence of solutions for a model describing the dynamics of junctions between dislocations

N. Forcadel, R. Monneau

Abstract We study a dynamical version of a multi-phase field model of Koslowski and Ortiz for planar dislocation networks. We consider a two-dimensional vector field which describes phase transitions between constant phases. Each phase transition corresponds to a dislocation line, and the vectorial field description allows the formation of junctions between dislocations. This vector field is assumed to satisfy a non-local vectorial Hamilton-Jacobi equation with non-zero viscosity. For this model, we prove the existence for all time of a weak solution.
AMS Classification : 35K15, 74K30.
Keywords : Dislocation dynamics, non-local equations, junctions, Hamilton-Jacobi.

1 Introduction

1.1 Physical motivation

Dislocations are line defects in crystal, and their motion is at the origin of plastic properties of metals. In these crystals, we can observe self-organised structures, like Frank networks, *i.e.* networks of dislocations related by junctions. See for instance page 190 in Hull, Bacon [109] for such networks in BCC iron, or page 188 for hexagonal networks in FCC crystals. In the present paper, we consider a special case of a network contained in a single slip plane, where the dislocations can move. We are interested in particular in the motion of the junctions between dislocations, which remains a quite open question, both from the modelling point of view, and from the mathematical analysis point of view (see for instance the work of Rodney, Le Bouar, Finel [158]). Let us mention, for the stationary case, the work of Garroni [92]. The goal of the present paper is to propose and to study a model for the dynamics of junctions of dislocations.

The question of junctions has several other physical applications and there is various literature on this subject. Let us mention for instance the problem of crystal growth or grain growth (see Taylor [177, 178] and Bronsard, Reitich [44]). We also refer to Bonnet [40] for problems concerning the minimisation of the Mumford-Shah functional.

1.2 A phase field model for the dynamics of junctions

In a phase field model, the dislocation can be represented as the phase transition of a phase parameter $\rho(x) = \rho_1(x)e^1 + \rho_2(x)e^2 \in \mathbb{R}^2$ defined for $x = x_1e^1 + x_2e^2$ in the plane $\mathbb{R}^2$ with (e^1, e^2) an orthonormal basis. The energy of the dislocations, in the presence of a constant exterior applied stress $\sigma^0 \in \mathbb{R}^2$, is then given (see the model of Koslowski and Ortiz [127]) by

$$\mathcal{E}(\rho) = \int_{\mathbb{R}^2} -\frac{1}{2}(C^0 \star \rho)\cdot\rho - \sigma^0\cdot\rho + W(\rho), \tag{7.1}$$

where the precise meaning of this expression will be given later.
For any phase transition between two states A and B, the difference $B - A$ needs physically to be the Burgers vector of the dislocation, *i.e.* a vector of the lattice $\Lambda = \mathbb{Z}a^1 + \mathbb{Z}a^2$ of the crystal we are considering, with general basis (a^1, a^2). This information is encoded in the potential $W : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ which is is assumed to be minimal on Λ and to have the periodicity of the lattice Λ :

$$W(\rho + a) = W(\rho) \quad \text{for any} \quad a \in \Lambda. \tag{7.2}$$

In this model, junctions of three dislocations of Burgers vectors $b^1, b^2, b^3 \in \Lambda$ with $b^1 + b^2 + b^3 = 0$ is expected, like for instance as the phase transitions between the states $0, b^1, -b^3$ (see Figure 7.1).

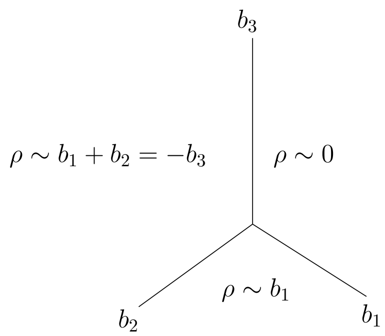

FIG. 7.1 – The junction of three dislocations as phase transitions of ρ.

When the material is submitted to an exterior shear stress, it makes the dislocations move. The dynamics of a given dislocation line is physically given by its normal velocity, which is called the resolved Peach-Koehler force. This force is the sum of the resolved exterior shear stress and the stress created by all the dislocations lines, including the line itself.
In the expression giving the energy (7.1), the kernel $C^0(x)$ is a 2×2 symmetric matrix which takes into account the long range elastic interactions between dislocations

and

$$(C^0 \star \rho)_i = \sum_{j=1,2} C^0_{ij} \star \rho_j, \quad \text{for} \quad i = 1,2$$

where $\star$ denotes the usual convolution. In (7.1) and throughout the paper, we denote by $A \cdot B$ the scalar product between two vectors $A, B \in \mathbb{R}^2$.
The resolved stress $\sigma[\rho]$ created by all the dislocations is then formally given by the opposite of the gradient of the energy $-\mathcal{E}'(\rho)$, and can be expressed as the following non-local quantity

$$\sigma[\rho] = \sigma^0 + C^0 \star \rho - W'_\rho(\rho). \tag{7.3}$$

The phase parameter $\rho(t,x) \in \mathbb{R}^2$ is then assumed to satisfy the following equation

$$\begin{cases} (\rho_k)_t = |\nabla \rho|^{-1} \displaystyle\sum_{i=1,2} \sum_{j=1,2} (\sigma[\rho])_i \nabla_j \rho_i \nabla_j \rho_k + \varepsilon \Delta \rho_k, \text{ for } k = 1,2 & \text{in } (0,T) \times \mathbb{T}^2, \\ \rho(0,x) = \rho^0(x) & \text{on } \mathbb{T}^2, \end{cases} \tag{7.4}$$

where σ is given in (7.3), $\rho_t = \dfrac{\partial \rho}{\partial t}$ and $\nabla_j \rho = \dfrac{\partial \rho}{\partial x_j}$ for $j = 1,2$, and

$$|\nabla \rho|^2 = \sum_{i=1,2} \sum_{j=1,2} |\nabla_j \rho_i|^2.$$

Here the parameter $1 > \varepsilon > 0$ is a small viscosity introduced in the model, in order to regularise the equation, but which has no real physical meaning. Given any time $T > 0$, we will study this equation, not on the the whole plane, but on the particular torus $\mathbb{T}^2 = \mathbb{R}^2 \backslash \mathbb{Z}^2$, in order to simplify the analysis. This kind of periodic conditions is also meaningful physically, if we want to describe periodic networks of dislocations. This means in particular that $\sigma[\rho]$ is given by (7.3) where the convolution is done on the torus.

Finally, let us mention that our model (7.4) has some similarities with the model of Allen, Cahn [2] on the motion of curved anti-phase boundaries in which they consider gradient flow associated with a free-energy functional. This led to the study of scalar Ginzburg-Landau type diffusion equation like

$$u_t = \Delta u - W'(u).$$

1.3 Main result

We make the following assumption on the kernel $C^0 : \mathbb{T}^2 \longrightarrow \mathbb{R}^{2\times 2}_{sym}$

(A) We assume that there exists a constant $m > 0$, such that for any $k \in \mathbb{R}^2$, the Fourier coefficients of the kernel $\widehat{C}^0(k) = \int_{\mathbb{T}^2} dx\, e^{-2i\pi k \cdot x} C^0(x)$ satisfy $\widehat{C}^0(k) = M(k)$, where for any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and any $p = (p_1, p_2) \in \mathbb{R}^2$

$$\left\{\begin{array}{l} M \in C^\infty(\mathbb{R}^2\backslash\{0\}\,;\mathbb{R}^{2\times 2}_{sym}), \quad M(-\xi) = M(\xi), \quad M(\xi) = |\xi| M\left(\dfrac{\xi}{|\xi|}\right) \\ \\ \dfrac{|\xi|\ |p|^2}{m} \geq -\displaystyle\sum_{i=1,2}\sum_{j=1,2} p_i \cdot M_{ij}(\xi) \cdot p_j \geq m|\xi|\ |p|^2 \quad \text{with} \quad |p|^2 = \sum_{i=1,2} |p_i|^2 \end{array}\right. \tag{7.5}$$

We also make the following assumption on the potential $W : \mathbb{T}^2 \longrightarrow \mathbb{R}_+$

(B) We assume that $W \in C^2$ and W satisfies (7.2).

Then we have the following result for the model of dynamics of junctions between dislocations :

Theorem 1.1. (Existence of a solution)
Under assumptions (A) and (B), if $\rho^0 \in \left(H^2(\mathbb{T}^2)\right)^2$, then for any constant applied stress $\sigma^0 \in \mathbb{R}^2$ and for any time $T > 0$, there exists a solution ρ of (7.4) with $\rho \in C^0\left([0,T); \left(L^{\frac{4}{3}}(\mathbb{T}^2)\right)^2\right)$.

The uniqueness of the solution is not known, neither the existence of a solution when $\varepsilon = 0$. Let us mention that equation (7.4) is a non-local system of scalar equations, and can be sketched as the following equation

$$v_t = |\nabla v|^2 + \Delta v \tag{7.6}$$

Indeed, this comes from our assumption (A) that the convolution with the kernel behaves like a first order operator. A lot of work has been done on equations (or systems) like equation (7.6). Let us mention for instance the works of Boccardo, Murat, Puel [35, 36, 37] in which they study general equations including equation (7.6) and prove existence results.

Equation (7.6) is also similar to the Navier-Stokes equations written for the potential A such that the velocity of the fluid is given by $u = \text{curl } A$ (see for instance Leray [136]).

1.4 Organisation of the paper

In Section 2, we give some remarks on the modelling. In Section 3, we study an approximate problem of equation (7.4) where the right hand side is approached by some term at most linear in the solution. The main result is proved in Section 4. In a first subsection, we give some *a priori* estimates for the solution of the approximate problem and then in a second subsection we pass to the limit in the approximate problem.

1.5 Notation

In what follows, we will denote by C a generic constant, which will then satisfy $C + C = C$, $C \cdot C = C$, and so on. We also use the following set :

$$W^{2,1;p}(Q_T) = \left\{ u \in L^p(Q_T);\ u_t \in L^p(Q_T) \text{ and } \frac{\partial^2 u}{\partial x_i, \partial x_j} \in L^p(Q_T) \text{ for } i,j = 1,2 \right\}$$

where $Q_T = (0,T) \times \mathbb{T}^2$.

2 Preliminary remarks on the modelling

2.1 Dynamics of a single dislocation

In this section, we consider a special case where a dislocation of Burgers vector $b^1 \in \Lambda$ is described by the phase transition of a scalar parameter $\bar{\rho}$ such that

$$\rho = \bar{\rho}\, b^1$$

Then the resolved shear stress that makes the dislocation move, is given by

$$\bar{\sigma}[\bar{\rho}] = b^1 \cdot \sigma[\bar{\rho}\, b^1]$$

and the dislocation line (described by the phase parameter $\bar{\rho}$) moves with normal velocity proportional to this shear stress. More precisely, $\bar{\rho}$ solves the following eikonal equation where a term with small viscosity $\varepsilon > 0$ has been added

$$\bar{\rho}_t = |b^1|^{-1}\ \bar{\sigma}[\bar{\rho}]|\nabla\bar{\rho}| + \varepsilon\Delta\bar{\rho}$$

with viscosity $\varepsilon = 0$, where $\bar{\rho}_t = \dfrac{\partial\bar{\rho}}{\partial t}$ and $\nabla\bar{\rho} = \dfrac{\partial\bar{\rho}}{\partial x_1}\ e^1 + \dfrac{\partial\bar{\rho}}{\partial x_1}\ e^2$. In order to simplify the analysis and get an existence result, we only consider the case with additional velocity $\varepsilon > 0$. Then, we easily check that $\rho = \bar{\rho}\ b^1$ satisfies (7.4).

2.2 Explicit expression of $\widehat{C}^0$ for isotropic materials

Given a particular Burgers vector b^1, let us consider

$$\bar{c}^0 = (b^1)^T \cdot C^0 \cdot b^1.$$

In the special case of isotropic linear elasticity with constant coefficients, we recall (see for instance a limit case of the Peierls-Nabarro model in Alvarez *et al.* [10]) that we have for $k = (k_1, k_2)$

$$\widehat{\bar{c}}^0(k) = -\frac{\mu|b^1|^2}{2}\left(\frac{\frac{1}{1-\nu}\,(k \cdot e)^2 + \left(k^\perp \cdot e\right)^2}{|k|}\right) \quad \text{with} \quad e = \frac{b^1}{|b^1|}$$

where $k^\perp = (-k_2, k_1)$ is the vector obtained by a rotation of k of angle $\pi/2$. Here $\mu > 0$ is a Lamé coefficient and $\nu \in (-1, 1/2)$ is the Poisson ratio.
We deduce that

$$\widehat{C}^0(k) = -\frac{\mu}{2|k|}\left(\frac{1}{1-\nu}k \otimes k + k^\perp \otimes k^\perp\right)$$

which satisfies assumption (A).

3 An approximate problem

We first start to approximate the right hand side of equation (7.4) by some term at most linear in the solution. To this end, we introduce a function h^n defined by

$$h^n(r) = h^0(r-n)$$

with

$$h^0(r) = \begin{cases} 1 \quad \text{if} \quad r \leq 0 \\ \\ 1-r \quad \text{if} \quad 0 \leq r \leq 1 \\ \\ 0 \quad \text{if} \quad r \geq 1 \end{cases}$$

We then look at the following approximate problem :

$$\left\{ \begin{array}{lll} \rho_t - \epsilon\Delta\rho = f^n[\rho] & \text{on} & Q_T := (0,T) \times \mathbb{T}^2 \\ \\ \rho(0,\cdot) = \rho^0 & \text{on} & \mathbb{T}^2 \end{array} \right. \tag{7.7}$$

where

$$f^n[\rho] = h^n(|\nabla\rho|)\ |\nabla\rho|^{-1}\ (\nabla\rho)^T \cdot \nabla\rho \cdot \sigma[\rho]$$

and $\sigma[\rho]$ is given in (7.3), and is at most linear in ρ.

The natural idea to find a solution to equation (7.7), is to define the map Φ which associates to any function u, the solution $\rho = \Phi(u)$ of

$$\left\{ \begin{array}{lll} \rho_t - \epsilon\Delta\rho = f^n[u] & \text{on} & Q_T := (0,T) \times \mathbb{T}^2 \\ \\ \rho(0,\cdot) = \rho^0 & \text{on} & \mathbb{T}^2 \end{array} \right. \tag{7.8}$$

and to prove that Φ has a fixed point in a suitable space. This way, we will prove the following result

Theorem 3.1. (Existence of a solution for the approximate problem)
If $\rho^0 \in (H^1(\mathbb{T}^2))^2$, then for any $n \geq 1$ and any $T > 0$, there exists a solution ρ^n of (7.7) with $\rho^n \in L^2((0,T);(H^2(\mathbb{T}^2))^2) \cap C^0([0,T);(L^2(\mathbb{T}^2))^2)$.

In this section, we will make the proof of this theorem. In a first subsection, we will collect some preliminary results, and in a second subsection we will prove that Φ has a fixed point.

3.1 Preliminary results

The following estimate on the stress will be important.

Lemma 3.2. (Estimate on $\sigma[\rho]$)
For any $p \in (1, +\infty)$, there exists a constant C (depending on p, on the constant σ^0, on the potential W and on the constant m defined in assumption (A)), such that for any $\rho \in (W^{1,p}(\mathbb{T}^2))^2$, we have

$$\|\sigma[\rho]\|_{(L^p(\mathbb{T}^2))^2} \leq C\left(1 + \|\nabla\rho\|_{(L^p(\mathbb{T}^2))^{2\times 2}}\right). \tag{7.9}$$

Partial proof of Lemma 3.2
Let us make the proof for $p = 2$. We have with $\sigma = \sigma[\rho]$

$$\begin{aligned}
|\sigma - \sigma^0 + W'(\rho)|^2_{(L^2(\mathbb{T}^2))^2} &= \sum_{k\in\mathbb{Z}^2} |(\widehat{C^0 \star \rho})(k)|^2 \\
&= \sum_{k\in\mathbb{Z}^2} |\widehat{C}^0(k)\cdot\widehat{\rho}(k)|^2 \\
&\leq \frac{1}{m^2}\sum_{k\in\mathbb{Z}^2} |k|^2|\widehat{\rho}(k)|^2 \\
&\leq \frac{1}{(2\pi m)^2}\sum_{k\in\mathbb{Z}^2} |\widehat{\nabla\rho}(k)|^2 \\
&= \frac{1}{(2\pi m)^2}|\nabla\rho|^2_{(L^2(\mathbb{T}^2))^{2\times 2}}
\end{aligned}$$

Therefore

$$|\sigma[\rho]|_{(L^2(\mathbb{T}^2))^2} \leq |\sigma^0| + |W'|_{(L^\infty(\mathbb{T}^2))^2} + \frac{1}{2\pi m}|\nabla\rho|_{(L^2(\mathbb{T}^2))^{2\times 2}}$$

which provides the result in the case $p = 2$.
The proof for the general case $p \in (1, +\infty)$ is given in Appendix. □

We will also need the following result.

Lemma 3.3. (Estimate on $f^n[u]$)
If $u \in (H^1(\mathbb{T}^2))^2$, then $f^n[u] \in (L^2(\mathbb{T}^2))^2$ with the following estimate :

$$\|f^n[u]\|_{(L^2(\mathbb{T}^2))^2} \leq C(n+1)\left(1 + \|\nabla u\|_{L^2(\mathbb{T}^2)^{2\times 2}}\right)$$

where the constant C depends on σ^0, on the potential W and on the constant m defined in assumption (A).

Proof of Lemma 3.3
Since $\operatorname{supp}(h^n) \subset\ [0, n+1]$, the following holds

$$|f^n[u]| \leq (n+1)|\sigma[u]|, \tag{7.10}$$

where we have used the fact that $|B^T \cdot B \cdot p| \leq |B|^2|p|$ for $B \in \mathbb{R}^{2\times 2}$ and $p \in \mathbb{R}^2$. Then

$$\begin{aligned}\|f^n[u]\|_{(L^2(\mathbb{T}^2))^2} \leq & (n+1)\|\sigma[u]\|_{(L^2(\mathbb{T}^2))^2} \\ \leq & C(n+1)\left(1 + \|\nabla u\|_{L^2(\mathbb{T}^2)^{2\times 2}}\right)\end{aligned}$$

where we have used Lemma 3.2 $\square$

We now recall some classical results. We start with the following classical parabolic estimates for the following equation

$$\begin{cases} g_t - \epsilon \Delta g = f & \text{on} \quad Q_T := (0,T) \times \mathbb{T}^2 \\ \\ g(0,\cdot) = g^0 & \text{on} \quad \mathbb{T}^2 \end{cases} \tag{7.11}$$

Proposition 3.4. (Parabolic estimates for the heat equation)
Let $g^0 \in H^1(\mathbb{T}^2)$ and $f \in L^2(Q_T)$. Then there exists a unique solution g to (7.11) with

$$g \in L^2((0,T); H^2(\mathbb{T}^2)) \cap L^\infty((0,T); H^1(\mathbb{T}^2)), \quad g_t \in L^2(Q_T). \tag{7.12}$$

We have the following estimate

$$\begin{aligned}&\sup_{0\leq t\leq T} \|g(t)\|_{H^1(\mathbb{T}^2)} + \|g\|_{L^2((0,T);H^2(\mathbb{T}^2))} + \|g_t\|_{L^2((0,T);L^2(\mathbb{T}^2))} \\ \leq & C_T \left(\|f\|_{L^2(Q_T)} + \|g^0\|_{H^1(\mathbb{T}^2)}\right),\end{aligned} \tag{7.13}$$

where the constant C_T only depends on T and ε.
Moreover we have

$$\sup_{0\leq t\leq T} \int_{\mathbb{T}^2} g^2(t) + 4\varepsilon \int_0^T \int_{\mathbb{T}^2} |\nabla g|^2 \leq 2\int_{\mathbb{T}^2} (g^0)^2 + 4T \int_0^T \int_{\mathbb{T}^2} f^2 \tag{7.14}$$

Proof of Proposition 3.4
For the proof of (7.12)-(7.13), we refer to Evans [81, Theorem 5 page 360].
To prove (7.14), we simply multiply equation (7.11) by g and integrate over $(0,t)$ in time, taking the supremum for $0 \leq t \leq T$. We get

$$\sup_{0\leq t\leq T} \int_{\mathbb{T}^2} \frac{g^2(t)}{2} + \varepsilon \int_0^T \int_{\mathbb{T}^2} |\nabla g|^2 \leq \int_{\mathbb{T}^2} \frac{(g^0)^2}{2} + \int_0^T \int_{\mathbb{T}^2} |g\ f|$$

We now use the fact that

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^2}|g\,f| &\leq \left(\int_0^T\int_{\mathbb{T}^2}g^2\right)^{\frac{1}{2}}\cdot\left(\int_0^T\int_{\mathbb{T}^2}f^2\right)^{\frac{1}{2}}\\ &\leq \left(T\sup_{0\leq t\leq T}\int_{\mathbb{T}^2}g^2(t)\right)^{\frac{1}{2}}\cdot\left(\int_0^T\int_{\mathbb{T}^2}f^2\right)^{\frac{1}{2}}\\ &\leq \frac{1}{4}\sup_{0\leq t\leq T}\int_{\mathbb{T}^2}g^2(t)+T\int_0^T\int_{\mathbb{T}^2}f^2\end{aligned}$$

which implies the result. □

We also recall the

Theorem 3.5. (Schaefer's fixed point theorem)
Let X be a real Banach space. Suppose that

$$\Phi : X \to X$$

is a continuous and compact mapping. Assume further that the set

$$\{u\in X,\quad u=\lambda\Phi(u)\quad \textit{for some}\quad \lambda\in[0,1]\}$$

is bounded. Then Φ has a fixed point.

For the proof of this theorem, we refer to Evans [81, Theorem 4 page 504].

Finally, we will need some compactness argument and a weak continuity property contained in the following two classical results :

Proposition 3.6. (Compactness)
We recall that

$$W^{2,1;2}(Q_T)=\left\{g\in L^2((0,T);H^2(\mathbb{T}^2)),\quad g_t\in L^2(Q_T)\right\}.$$

Then the injection

$$W^{2,1;2}(Q_T)\longrightarrow L^2((0,T);H^1(\mathbb{T}^2))\quad \textit{is compact.}$$

For the proof of this result, we refer to Lions [137, Theorem 5.1 page 58].

Proposition 3.7. (Continuity)
With the notation of Proposition 3.6, let us consider a sequence $(g^m)_m$ such that

$$g^m\rightharpoonup g\quad \textit{weakly in}\quad W^{2,1;2}(Q_T)$$

We assume also that $g^m_{|t=0}=\rho^0$. Then

$$g_{|t=0}=\rho^0.$$

This result is classical but for the reader's convenience we give the proof in Appendix.

3.2 Proof of Theorem 3.1

We are now ready to make the proof of Theorem 3.1. To this end, for any $T > 0$, we set

$$X_T = L^2((0,T); H^1(\mathbb{T}^2)).$$

In all what follows the index n is assumed fixed. We first remark that if $u \in X_T^2$, then $f^n[u] \in (L^2(Q_T))^2$, and then we can consider the solution ρ of

$$\begin{cases} \rho_t - \epsilon \Delta \rho = f^n[u] & \text{on} \quad Q_T := (0,T) \times \mathbb{T}^2 \\ \\ \rho(0,\cdot) = \rho^0 & \text{on} \quad \mathbb{T}^2 \end{cases} \tag{7.15}$$

which satisfies $\rho \in X_T^2$ because of the parabolic estimates Proposition 3.4. Then we set $\Phi(u) = \rho$, and see that Φ maps X_T^2 into X_T^2. We will prove that Φ admits a fixed point using Schaefer's fixed point theorem. We do the proof in four steps.

Step 1 : weak continuity of Φ
Let us consider sequences $(u^m)_m, (\rho^m)_m$ such that

$$\begin{cases} u^m \in X_T^2, \quad \rho^m = \Phi(u^m) \\ \\ u^m \longrightarrow u \quad \text{in} \quad X_T^2 \end{cases}$$

From Lemma 3.3, we deduce that

$$||f^n[u^m]||_{(L^2(Q_T))^2} \leq C(n+1)\left(1 + ||u^m||_{X_T^2}\right) \tag{7.16}$$

From the parabolic estimates (Proposition 3.4), we deduce that ρ^m is bounded in $\left(W^{2,1;2}(Q_T)\right)^2$, *i.e.* there exists a constant $C > 0$ such that

$$||\rho^m||_{(W^{2,1;2}(Q_T))^2} \leq C \tag{7.17}$$

Therefore, up to a subsequence, we have

$$\rho^m \rightharpoonup \rho \quad \text{in} \quad \left(W^{2,1;2}(Q_T)\right)^2$$

and from Proposition 3.7, we deduce that

$$\rho_{|t=0} = \rho^0 \quad \text{on} \quad \mathbb{T}^2$$

We now claim that

$$f^n[u^m] \longrightarrow f^n[u] \quad \text{in} \quad L^1(Q_T) \tag{7.18}$$

Indeed, we can write

$$f^n[u] = g^n(\nabla u) \cdot \sigma[u] \quad \text{with} \quad g^n(\nabla u) := h^n(|\nabla u|)\ |\nabla u|^{-1}\ (\nabla u)^T \cdot \nabla u$$

From the proof of Lemma 3.2, for $p = 2$, we already deduce that

$$\sigma[u^m] \longrightarrow \sigma[u] \quad \text{in} \quad L^2(Q_T) \tag{7.19}$$

From the convergence of u^m to u in X_T^2, we deduce that up to a subsequence we have $\nabla u^m \longrightarrow \nabla u$ a.e. in Q_T. Now from the fact that g^n is continuous and bounded, we deduce in particular that

$$g^n(\nabla u^m) \longrightarrow g^n(\nabla u) \quad \text{in} \quad L^2(Q_T) \tag{7.20}$$

Then the convergence (7.18) follows from (7.19) and (7.20).
Therefore we conclude that ρ solves (7.15). Finally, by uniqueness of the solutions of (7.15), we deduce that the limit ρ does not depend on the choice of the subsequence, and then that the full sequence converges :

$$\rho^m \rightharpoonup \rho \quad \text{weakly in} \quad \left(W^{2,1;2}(Q_T)\right)^2, \quad \text{with} \quad \rho = \Phi(u)$$

Step 2 : compactness of Φ
The compactness (and the usual continuity) of Φ follows from the compactness of the injection $\left(W^{2,1;2}(Q_T)\right)^2 \longrightarrow X_T^2$ (see Proposition 3.6).

Step 3 : a priori bounds on the solutions of $u = \lambda\Phi(u)$ for T small
Let us consider a solution u of

$$u = \lambda\Phi(u) \quad \text{for some} \quad \lambda \in [0,1] \tag{7.21}$$

Then from the parabolic estimates (7.14), we have

$$\begin{aligned} & \sup_{0\le t\le T} \int_{\mathbb{T}^2} |u(t)|^2 + 4\varepsilon \int_0^T \int_{\mathbb{T}^2} |\nabla u|^2 \\ \le \quad & 2\int_{\mathbb{T}^2} |\rho^0|^2 + 4T \int_0^T \int_{\mathbb{T}^2} |\lambda f^n[u]|^2 \\ \le \quad & 2\int_{\mathbb{T}^2} |\rho^0|^2 + 8TC^2(n+1)^2 \left(T + \int_0^T \int_{\mathbb{T}^2} |\nabla u|^2 \right) \end{aligned}$$

where in the third line we have used Lemma 3.3 and the fact that $|\lambda| \le 1$. Therefore for

$$T \le T^* := \left(4C^2(n+1)^2\right)^{-1} \varepsilon \tag{7.22}$$

we have

$$\sup_{0\le t\le T} \int_{\mathbb{T}^2} |u(t)|^2 + 2\varepsilon \int_0^T \int_{\mathbb{T}^2} |\nabla u|^2 \le 2\int_{\mathbb{T}^2} |\rho^0|^2 + 2\varepsilon T$$

which proves that there exists a constant $C > 0$ such that any solution of (7.21) satisfies

$$||u||_{X_T^2} \leq C$$

We can then apply the Schaefer's fixed point Theorem (Theorem 3.5), to deduce that Φ has a fixed point on X_T^2, and therefore there is a solution ρ of (7.7) on the time interval $(0, T)$ if T satisfies (7.22), *i.e.* if T is small enough independently on the initial data ρ^0.

Step 4 : solution for any time
Let us call $\rho(\rho^0, t)$ the function $\rho(t, \cdot)$ obtained at Step 3 as a solution of (7.7) on the time interval $[0, T^*)$ with initial data ρ^0. From the parabolic estimates (Proposition 3.4), we also know that $\rho(t, \cdot) \in (H^1(\mathbb{T}^2))^2$ for any $t \in [0, T^*)$. Then we can define with $\tau = T^*/2$

$$u(0) = \rho^0 \quad \text{and} \quad u(t) = \rho\left(u(k\tau), t\right) \quad \text{if} \quad k\tau \leq t < (k+1)\tau \quad \text{with} \quad k \in \mathbb{N}.$$

Using the fact that $u_t \in L^2_{loc}((0, +\infty); (L^2(\mathbb{T}^2)^2)$, and the fact that the problem is invariant by translation in time, we can easily check that u solves (7.7) for any $T > 0$ and provides the desired solution $\rho^n = u$ of Theorem 3.1.
This ends the proof of Theorem 3.1.

4 A priori estimates and proof of Theorem 1.1

4.1 A priori estimates

We have the following *a priori* estimates :

Lemma 4.1. (*A priori* estimates)
There exists a constant $C > 0$ such that for all $T > 0, n \geq 1$ and $0 < \varepsilon < 1$, any solution ρ^n of (7.7) given by Theorem 3.1, satisfies

$$\|\rho^n\|^2_{L^\infty((0,T);(H^{\frac{1}{2}}(\mathbb{T}^2))^2)} \leq Ce^{\frac{CT}{\varepsilon}}, \tag{7.23}$$

$$\|\rho^n\|^2_{L^2((0,T);(H^{\frac{3}{2}}(\mathbb{T}))^2)} \leq \frac{C}{\varepsilon}e^{\frac{CT}{\varepsilon}}, \tag{7.24}$$

and

$$\left\| h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-\frac{1}{2}}\nabla\rho^n \cdot \sigma[\rho^n] \right\|^2_{(L^2(Q_T))^2} \leq Ce^{\frac{CT}{\varepsilon}}. \tag{7.25}$$

Proof of Lemma 4.1

Step 1 : Preliminaries on the energy

We first recall the expression of the energy for a general $\mathbb{Z}^2$-periodic smooth function $\rho(x) = (\rho_1(x), \rho_2(x))$

$$\mathcal{E}(\rho) = \int_{\mathbb{T}^2} -\frac{1}{2}(C^0 \star \rho)\cdot\rho - \sigma^0\cdot\rho + W(\rho)$$

For future use, we start to evaluate from below the first term in the energy, using Fourier series

$$\begin{aligned}\int_{\mathbb{T}^2} -(C^0\star\rho)\cdot\rho &= \mathrm{Re}\left(\sum_{k\in\mathbb{Z}^2} -(\widehat{C^0\star\rho})(k)\cdot\widehat{\rho}^*(k)\right)\\ &= \mathrm{Re}\left(\sum_{k\in\mathbb{Z}^2} -(\widehat{C}^0(k)\cdot\widehat{\rho}(k))\cdot\widehat{\rho}^*(k)\right)\\ &\geq m\sum_{k\in\mathbb{Z}^2}|k|\ |\widehat{\rho}(k)|^2\end{aligned}$$

where we have used in the first line the fact that ρ and C^0 are real, and in the last line we have used assumption (A). Then we define

$$||\rho||^2_{(\dot{H}^{\frac{1}{2}}(\mathbb{T}^2))^2} := \sum_{k\in\mathbb{Z}^2}|k|\ |\widehat{\rho}(k)|^2.$$

Similarly, we compute

$$\begin{aligned}\int_{\mathbb{T}^2} -(C^0\star(\nabla\rho)^T):\nabla\rho =&(2\pi)^2\mathrm{Re}\left(\sum_{k\in\mathbb{Z}^2} -(\widehat{C}^0(k)\cdot\widehat{\rho}(k)\otimes(ik)):(ik)^*\otimes\widehat{\rho}^*(k)\right)\\ =&(2\pi)^2\mathrm{Re}\left(\sum_{k\in\mathbb{Z}^2} -|k|^2(\widehat{C}^0(k)\cdot\widehat{\rho}(k)):\widehat{\rho}^*(k)\right)\\ \geq&(2\pi)^2 m\sum_{k\in\mathbb{Z}^2}|k|^3\ |\widehat{\rho}(k)|^2\end{aligned}$$

where we have used assumption (A) in the last line. Then we define

$$||\rho||^2_{(\dot{H}^{\frac{3}{2}}(\mathbb{T}^2))^2} := \sum_{k\in\mathbb{Z}^2}|k|^3\ |\widehat{\rho}(k)|^2.$$

Step 2 : Estimate on the time-derivative of the energy
Let us fix $T > 0$. We know that any solution ρ^n given by Theorem 3.1 belongs to the space $W^{2,1;2}(Q_T)$. In particular, using the following general fact (because of assumption (A))

$$\int_{\mathbb{T}^2} -(C^0 \star \rho) \cdot \rho = \mathrm{Re}\left(\sum_{k \in \mathbb{Z}^2} -|k| (\hat{C}^0\left(\frac{k}{|k|}\right) \cdot \widehat{\rho}(k)) \cdot \widehat{\rho}^*(k) \right)$$

we deduce that the energy $\mathcal{E}(\rho^n(t))$ is well-defined for almost every $t \in [0,T)$, and that for almost every time $t \in [0,T)$, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(\rho^n(t)) &= \int_{\mathbb{T}^2} -\sigma[\rho^n] \cdot \rho^n_t \\
&= \int_{\mathbb{T}^2} -h^n(|\nabla \rho^n|) |\nabla \rho^n|^{-1} \; |\nabla \rho^n \cdot \sigma[\rho^n]|^2 - \varepsilon \sigma[\rho^n] \cdot \Delta \rho^n \\
&= \int_{\mathbb{T}^2} -h^n(|\nabla \rho^n|) |\nabla \rho^n|^{-1} \; |\nabla \rho^n \cdot \sigma[\rho^n]|^2 \\
&\quad - \int_{\mathbb{T}^2} \varepsilon \left\{ W''(\rho^n) : ((\nabla \rho^n)^T \cdot \nabla \rho^n) - (C^0 \star (\nabla \rho^n)^T) : \nabla \rho^n \right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(\rho^n(t)) + \int_{\mathbb{T}^2} h^n(|\nabla \rho^n|) |\nabla \rho^n|^{-1} \; |\nabla \rho^n \cdot \sigma[\rho^n]|^2 \\
\leq & C\varepsilon \left\{ \int_{\mathbb{T}^2} |\nabla \rho^n|^2 + C^0 \star (\nabla \rho^n)^T) : \nabla \rho^n \right\}
\end{aligned} \tag{7.26}$$

But now (up to change the constant line to line)

$$\begin{aligned}
\|\nabla \rho^n\|^2_{(L^2(\mathbb{T}^2))^{2\times 2}} &\leq C \sum_{k \in \mathbb{Z}^2} |k|^2 |\widehat{\rho}^n(k)|^2 \\
&\leq C \sum_{k \in \mathbb{Z}^2} |k|^{\frac{3}{2}} |\widehat{\rho}^n(k)| \cdot |k|^{\frac{1}{2}} |\widehat{\rho}^n(k)| \\
&\leq C \left(\sum_{k \in \mathbb{Z}^2} \frac{1}{2\alpha} |k|^3 |\widehat{\rho}^n(k)|^2 + \frac{\alpha}{2} |k| |\widehat{\rho}^n(k)|^2 \right) \\
&\leq C \left(\int_{\mathbb{T}^2} -\frac{1}{\alpha} (C^0 \star (\nabla \rho^n)^T) : \nabla \rho^n + \int_{\mathbb{T}^2} -\alpha (C^0 \star \rho^n) \cdot \rho^n \right),
\end{aligned}$$

where $\widehat{\rho^n}(k)$ are the Fourier coefficients of ρ^n and α is a constant which will be precised later. We then deduce finally that :

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(\rho^n(t)) + \int_{\mathbb{T}^2} h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1}\ |\nabla\rho^n\cdot\sigma[\rho^n]|^2\\
&\le -C\varepsilon\left(1-\frac{1}{\alpha}\right)\int_{\mathbb{T}^2} -(C^0\star(\nabla\rho^n)^T):\nabla\rho^n + C\varepsilon\alpha\int_{\mathbb{T}^2} -(C^0\star\rho^n)\cdot\rho^n \qquad (7.27)\\
&\le -C\varepsilon\|\rho^n(t)\|^2_{(\dot H^{\frac{3}{2}}(\mathbb{T}^2))^2} + C\varepsilon\left(1+\mathcal{E}(\rho^n(t)) + |\sigma^0|\left|\int_{\mathbb{T}^2}\rho^n(t)\right|\right).
\end{aligned}$$

for α chosen large enough, with C a suitable positive constant.

Step 3 : Estimate on the time-derivative of the mean-value of the solution
Integrating in space equation (7.7), we get

$$\frac{d}{dt}\int_{\mathbb{T}^2}\rho^n(t) \ = \int_{\mathbb{T}^2} h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1}\ (\nabla\rho^n)^T\cdot\nabla\rho^n\cdot\sigma[\rho^n]$$

and then

$$\begin{aligned}
\frac{d}{dt}\left|\int_{\mathbb{T}^2}\rho^n(t)\right| \ &\le \int_{\mathbb{T}^2} (h^n(|\nabla\rho^n|)|\nabla\rho^n|)^{\frac{1}{2}}\cdot\Big((h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1})^{\frac{1}{2}}|\nabla\rho^n\cdot\sigma[\rho^n]|\Big)\\
&\le \int_{\mathbb{T}^2}(1+|\sigma^0|)h^n(|\nabla\rho^n|)|\nabla\rho^n|\\
&\qquad +\tfrac{1}{4(1+|\sigma^0|)}h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1}|\nabla\rho^n\cdot\sigma[\rho^n]|^2
\end{aligned} \qquad (7.28)$$

Step 4 : Estimate on the energy
Setting

$$F^n(t) = 1+\mathcal{E}(\rho^n(t)) + (1+|\sigma^0|)\left|\int_{\mathbb{T}^2}\rho^n(t)\right| + (1+|\sigma^0|)^4, \qquad (7.29)$$

we deduce from (7.27) and (7.28) that

$$\begin{aligned}
&\frac{d}{dt}F^n(t) + \frac{3}{4}\int h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1}|\nabla\rho^n\cdot\sigma[\rho^n]|^2\\
\le &-C\varepsilon\|\rho^n(t)\|^2_{(\dot H^{\frac{3}{2}}(\mathbb{T}^2))^2} + C\varepsilon\left(1+\mathcal{E}(\rho^n(t)) + |\sigma^0|\left|\int_{\mathbb{T}^2}\rho^n(t)\right|\right)\\
&+(1+|\sigma^0|)^2\int_{\mathbb{T}^2} h^n(|\nabla\rho^n|)|\nabla\rho^n|.
\end{aligned}$$

Now we remark that

$$\begin{aligned}
(1+|\sigma^0|)^2\int_{\mathbb{T}^2} h^n(|\nabla\rho^n|)|\nabla\rho^n| \le &(1+|\sigma^0|)^2\int_{\mathbb{T}^2}|\nabla\rho^n|\\
\le &\frac{C\varepsilon}{2}\int_{\mathbb{T}^2}|\nabla\rho^n|^2 + \frac{(1+|\sigma^0|)^4}{2C\varepsilon}
\end{aligned}$$

Using the fact that (since the domain is bounded)

$$\int_{\mathbb{T}^2} |\nabla \rho^n|^2 \leq \|\rho^n(t)\|^2_{(\dot{H}^{\frac{3}{2}}(\mathbb{T}^2))^2},$$

we get

$$\begin{aligned}
&\frac{d}{dt}F^n(t) + \frac{3}{4}\int_{\mathbb{T}^2} h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1}|\nabla\rho^n\cdot\sigma[\rho^n]|^2 + \frac{C\varepsilon}{2}\|\rho^n(t)\|^2_{(\dot{H}^{\frac{3}{2}}(\mathbb{T}^2))^2} \\
&\leq C\varepsilon\left(1 + \mathcal{E}(\rho^n(t)) + |\sigma^0|\left|\int_{\mathbb{T}^2}\rho^n(t)\right|\right) + \frac{(1+|\sigma^0|)^4}{2C\varepsilon} \\
&\leq \frac{C}{\varepsilon}F^n(t).
\end{aligned} \tag{7.30}$$

This implies, using Gronwall Lemma,

$$F^n(t) \leq F^n(0)e^{\frac{C}{\varepsilon}t}. \tag{7.31}$$

Step 5 : Estimate on ρ^n
Let us first remark that

$$\mathcal{E}(\rho^n(t)) \geq \int -\frac{1}{2}(C^0 \star \rho^n)\cdot\rho^n - |\sigma^0|\left|\int_{\mathbb{T}^2}\rho^n(t)\right|. \tag{7.32}$$

Using (7.31), (7.32) and the definition of $F^n(t)$, yields

$$\int_{\mathbb{T}^2} -\frac{1}{2}(C^0\star\rho^n)\cdot\rho^n + \left|\int_{\mathbb{T}^2}\rho^n(t)\right| \leq Ce^{\frac{C}{\varepsilon}t}.$$

Using Step 1, we then get

$$\|\rho^n\|^2_{L^\infty((0,T);(\dot{H}^{\frac{1}{2}}(\mathbb{T}^2))^2)} \leq Ce^{\frac{C}{\varepsilon}T} \quad \text{and} \quad \left|\int_{\mathbb{T}^2}\rho^n(t)\right| \leq Ce^{\frac{C}{\varepsilon}T}.$$

This implies (7.23). Taking the integral $\int_0^T$ in (7.30) and using the fact that $\forall t \leq T$, $F^n(t) \geq 0$, we get

$$\|h_n(|\nabla\rho^n|)|\nabla\rho^n|^{-\frac{1}{2}}|\nabla\rho^n\cdot\sigma[\rho^n]|\|^2_{(L^2(Q_T))^2} + \varepsilon C\|\rho^n\|^2_{L^2((0,T);(\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)))} \leq Ce^{\frac{C}{\varepsilon}T}$$

which implies (7.24) and (7.25).

□

4.2 Proof of Theorem 1.1

We are now able to prove Theorem 1.1. In this section, we denote by C a generic constant which can depend on ρ^0, ε and T but which do not depend on n.

Proof of Theorem 1.1
Let $T > 0$. The idea of the proof is to pass to the limit in Equation (7.7). The only difficulty is to prove that the non-linear term $f^n[\rho^n]$ converges in a certain sense to $|\nabla\rho|^{-1}(\nabla\rho)^T \cdot \nabla\rho \cdot \sigma[\rho]$, where ρ is the limit of ρ^n in an appropriate norm. The proof is decomposed into five steps :

Step 1 : a priori bound on $f^n[\rho^n]$
We have the following estimate on $f^n[\rho^n]$:

$$\|f^n[\rho^n]\|_{\left(L^{\frac{4}{3}}(Q_T)\right)^2} \leq C. \tag{7.33}$$

To prove this, let us write

$$f^n[\rho^n] = \left(|\nabla\rho^n|^{-1}(\nabla\rho^n)^T\right) \cdot \left(|\nabla\rho^n|^{\frac{1}{2}}\right) \left(h_n(|\nabla\rho^n|)|\nabla\rho^n|^{-\frac{1}{2}}\nabla\rho^n \cdot \sigma[\rho^n]\right).$$

Using (7.25), we have that the last term is bounded in $(L^2(Q_T))^2$ by C. Moreover, the first term is bounded by 1 in $(L^\infty(Q_T))^{2\times 2}$, then we just have to bound the term $|\nabla\rho^n|^{\frac{1}{2}}$ in $L^4(Q_T)$. Using (7.24), we have

$$\||\nabla\rho^n|^{\frac{1}{2}}\|_{L^4(Q_T)} = \left(\int_{Q_T} |\nabla\rho^n|^2\right)^{\frac{1}{4}} \leq \|\rho^n\|^{\frac{1}{2}}_{L^2((0,T);(H^{\frac{3}{2}}(\mathbb{T}))^2)} \leq C. \tag{7.34}$$

This implies (7.33).

Step 2 : Strong convergence of $\nabla\rho^n$ in $L^2((0,T);(L^{\frac{4}{3}}(\mathbb{T}^2))^{2\times 2})$
Using the parabolic estimates for the heat equation (see [130, ch 4.3 p 80 and ch 4.9 p 341]) and Step 1, we get

$$\|\nabla\rho^n\|_{W^{\frac{1}{2};\frac{4}{3}}((0,T);(L^{\frac{4}{3}}(\mathbb{T}^2))^{2\times 2})} \leq C \tag{7.35}$$

where for a Banach space B

$$W^{\frac{1}{2};p}((0,T);B) = \left\{ g \in L^p((0,T);B),\ \int_0^T \int_0^T \frac{\|g(t)-g(s)\|_B^p}{|t-s|^{\frac{1}{2}p+1}}\, dt\, ds < \infty \right\}.$$

Moreover, using (7.24) we get

$$\|\nabla\rho^n\|_{L^2\left((0,T);\left(H^{\frac{1}{2}}(\mathbb{T}^2)\right)^{2\times 2}\right)} \leq C. \tag{7.36}$$

We then use the following lemma :

Lemma 4.2. (Compactness result)
Let $(g_n)_n$ be a sequence uniformly bounded in

$$L^2\left((0,T);H^{\frac{1}{2}}(\mathbb{T}^2)\right)\cap W^{\frac{1}{2};\frac{4}{3}}\left((0,T);L^{\frac{4}{3}}(\mathbb{T}^2)\right),$$

then, for a subsequence, $g_n \to g$ strongly in $L^2((0,T);L^{\frac{4}{3}}(\mathbb{T}^2))$.

Formally, the proof uses the fact that $H^{\frac{1}{2}} \subset L^{\frac{4}{3}}$ with compact injection in space while the compactness in time comes from (7.35). We refer to Simon [167, Corollary 5, p.86] for a more general result and for the proof of this lemma.

Using (7.35), (7.36) and Lemma 4.2, we then deduce that, for a subsequence, $\nabla\rho^n \to \nabla\rho$ strongly in $L^2\left((0,T);\left(L^{\frac{4}{3}}(\mathbb{T}^2)\right)^2\right)$ and almost everywhere.

Step 3 : Weak convergence of $\sigma[\rho^n]$ in $L^2\left((0,T);(L^4(\mathbb{T}^2))^2\right)$
We have $H^{\frac{1}{2}}(\mathbb{T}^2) \subset L^4(\mathbb{T}^2)$ with continuous injection (see Adams [1, Theorem 7.57 p217]). So $L^2((0,T);H^{\frac{1}{2}}(\mathbb{T}^2)) \subset L^2((0,T);L^4(\mathbb{T}^2))$ with continuous injection. We then deduce from (7.24) that

$$\|\nabla\rho^n\|_{L^2\left((0,T);(L^4(\mathbb{T}^2))^{2\times 2}\right)} \le C. \tag{7.37}$$

Using Lemma 3.2, we then get

$$\|C^0 \star \rho^n\|_{L^2\left((0,T);(L^4(\mathbb{T}^2))^2\right)} \le C. \tag{7.38}$$

Using the fact that the application $W^{2,1,\frac{4}{3}}_{x,t}(Q_T) \mapsto L^{\frac{4}{3}}(Q_T)$ is compact and the converse of Lebesgue Theorem, we deduce that $W'(\rho^n) \to W'(\rho)$ almost everywhere. This implies that $\sigma[\rho^n] \rightharpoonup \sigma[\rho]$ in $L^2((0,T);(L^4(\mathbb{T}^2))^2$.

Step 4 : Passing to the limit
Using Step 2 and Step 3 and the fact that $|\nabla\rho^n|^{-1}\nabla\rho^n$ is bounded by 1, we deduce that

$$f_n[\rho^n] \to |\nabla\rho|^{-1}(\nabla\rho)^T\cdot\nabla\rho\cdot\sigma[\rho] \quad \text{in the distributions sense.}$$

By passing to the limit in (7.7), we obtain

$$\rho_t - \epsilon\Delta\rho = |\nabla\rho|^{-1}(\nabla\rho)^T\cdot\nabla\rho\cdot\sigma[\rho] \text{ in } \mathcal{D}'((0,T)\times\mathbb{T}^2). \tag{7.39}$$

Step 5 : Initial Condition
Using the fact that ρ^n_t are bounded uniformly in $L^{\frac{4}{3}}(Q_T)$ (by parabolic estimates for the heat equation and Step 1), we deduce that (uniformly in n)

$$\|\rho^n(t+h)-\rho^n(t)\|_{(L^{\frac{4}{3}}(\mathbb{T}^2))^2} \le Ch^{\frac{1}{4}}\|\rho^n_t\|_{L^{\frac{4}{3}}((0,T);(L^{\frac{4}{3}}(\mathbb{T}^2))^2)}$$

and then $\rho \in C^0((0,T);(L^{\frac{4}{3}}(\mathbb{T}^2))^2)$ and $\rho_{|t=0} = \rho^0$.

This achieves the proof of Theorem 1.1. □

5 Appendix

Full proof of Lemma 3.2
Here we do the proof for any $p \in (1,+\infty)$. Under assumption (A), there exists a constant $C > 0$ only depending on p, such that the following result holds for any $\tilde{\rho} \in W^{1,p}(\mathbb{R}^2)$

$$|\tilde{C}^0 \star_{\mathbb{R}^2} \tilde{\rho}|_{(L^p(\mathbb{R}^2))^2} \quad \leq \quad \frac{C}{m}|\nabla\tilde{\rho}|_{(L^p(\mathbb{R}^2))^{2\times 2}}$$

where the Fourier transform of $\tilde{C}^0$ satisfies $\widehat{\tilde{C}}^0 = M$ with M as in (7.5).
This result can be found in the scalar case on $\mathbb{R}^n$ in Stein [175], see proposition 5 page 251, or Coifman, Meyer [59], Theorem 9 page 39 and Proposition 2 page 41. See also Calderon-Zygmund inequalities Theorem 2.7.2 in Morrey [145]. Here the convolution by $\tilde{C}^0$ is a multiplier operator in the class S^1 of pseudo-differential operators. We then get the result in the vectorial case, summing the scalar components. See also the book of Garroni, Menaldi [95] for complements on integro-differential operators. The fact that the result holds on the torus $\mathbb{T}^2$ is then classical. We prove it for the convenience of the reader. To this end, we consider a smooth function φ such that

$$\varphi(x) = 1 \quad \text{on} \quad [-1/3, 1/3]^2, \quad \text{and} \quad \text{supp}\, \varphi \subset [-2/3, 2/3]^2, \quad \text{and} \quad 0 \leq \varphi \leq 1$$

such that

$$\sum_{k\in\mathbb{Z}^2} \varphi(x-k) = 1.$$

For any smooth function $\rho : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which is $\mathbb{Z}^2$-periodic, we then set for $K > 0$

$$(S_{2K}\rho)(x) = \sum_{|k|\leq 2K,\ k\in\mathbb{Z}^2} \varphi(x-k)\rho(x).$$

Therefore we get for $K > 0$ large enough

$$\begin{aligned}
& |B_K| \left\{ |\tilde{C}^0 \star_{\mathbb{R}^2} \rho|_{L^p((-1/2,1/2)^2)} + 0(1/K) \right\} \\
\leq \quad & |\tilde{C}^0 \star_{\mathbb{R}^2} \rho|_{L^p(B_K)} \\
\leq \quad & |\tilde{C}^0 \star_{\mathbb{R}^2} (S_{2K}\rho)|_{L^p(\mathbb{R}^2)} + |\tilde{C}^0 \star (\rho - (S_{2K}\rho))|_{L^p(B_K)} \\
\leq \quad & \frac{C}{m}|\nabla(S_{2K}\rho)|_{L^p(\mathbb{R}^2)} + +|\tilde{C}^0 \star (\rho - (S_{2K}\rho))|_{L^p(B_K)} \\
\leq \quad & \frac{C}{m}|B_{2K}| \left\{ |\nabla\rho|_{(L^p(-1/2,1/2))^{2\times 2}} + 0(1/K) \right\} + |\rho|_{(L^\infty(\mathbb{R}^2))^2}|B_K| \int_{|z|\geq K-1} |\tilde{C}^0(z)|.
\end{aligned}$$

Using the fact that $\int_{|z|\geq K-1} |\tilde{C}^0(z)| = O(1/K)$, dividing by $|B_K|$ and taking the limit as $K \longrightarrow +\infty$, we get

$$|\tilde{C}^0 \star_{\mathbb{R}^2} \rho|_{(L^p(\mathbb{T}^2))^2} \quad \leq \quad \frac{C}{m}\frac{|B_2|}{|B_1|}|\nabla\rho|_{(L^p(\mathbb{T}^2))^{2\times 2}}$$

i.e.

$$|C^0 \star_{\mathbb{T}^2} \rho|_{(L^p(\mathbb{T}^2))^2} \quad \leq \quad \frac{4C}{m}|\nabla\rho|_{(L^p(\mathbb{T}^2))^{2\times 2}}$$

with

$$C^0(x) = \sum_{k\in\mathbb{Z}^2} \tilde{C}^0(x-k).$$

We then get the final result by density of smooth functions in $(W^{1,p}(\mathbb{T}^2))^2$. □

Proof of Lemma 3.7
For simplicity of notation, we denote by $g(t)$ the function $x \mapsto g(t,x)$. We have

$$\begin{aligned}\|g^m(t) - \rho^0\|_{(L^2(\mathbb{T}^2))^2} &\leq \int_0^t ds\ \|g_t^m(s)\|_{(L^2(\mathbb{T}^2))^2} \\ &\leq \sqrt{t}\ \|g_t^m\|_{(L^2(Q_T))^2}.\end{aligned}$$

Using the fact that g^m is bounded uniformly in $W^{2,1;2}(Q_T)$ (since $g^m \rightharpoonup g$ in $W^{2,1;2}(Q_T)$), we get

$$\|g^m(t) - \rho^0\|_{(L^2(\mathbb{T}^2))^2} \leq C\sqrt{t}. \tag{7.40}$$

Now let $\varphi \in C_c^\infty([0,+\infty),\mathbb{R})$ be such that $\varphi \geq 0$. Using (7.40), we get that

$$\int_0^t ds\ \|g^m(s) - \rho^0\|^2_{(L^2(\mathbb{T}^2))^2}\varphi(s) \leq C\int_0^t ds\ s\ \varphi(s).$$

Using Fatou's Lemma, we deduce that

$$\int_0^t \left(\|g(s) - \rho^0\|^2_{(L^2(\mathbb{T}^2))^2} - Cs\right)\varphi(s) \leq 0.$$

Using that $\varphi \geq 0$ is arbitrary, we deduce that for almost every t, we have

$$\|g(t) - \rho^0\|^2_{(L^2(\mathbb{T}^2))^2} \leq \sqrt{Ct}.$$

This implies the result. □

Acknowledgement. The authors thank A. El Hajj for fruitful discussion in the preparation of this paper. This work was supported by the ACI JC 1025.

Chapitre 8

Homogénéisation de la dynamique des dislocations et de certains systèmes de particules avec interactions par paires

Ce chapitre est issu d'un travail en collaboration avec C. Imbert et R. Monneau [89].

Ce travail concerne l'homogénéisation d'une équation de Hamilton-Jacobi non locale du premier ordre décrivant la dynamique de plusieurs lignes de dislocations et l'homogénéisation de certains systèmes de particules avec des interactions par paires. Le premier objectif est d'établir une connexion entre la dynamique rescalée d'un nombre croissant de lignes de dislocations et la dynamique de densité de dislocations, passant d'un modèle discret (lignes de dislocation) à un modèle continu (densité de dislocations). Une première réponse à ce problème a été donnée dans un papier de Imbert, Monneau et Rouy mais la définition géométrique des fronts n'était pas complètement satisfaisante. Ce problème est complètement résolu ici. L'équation limite est une équation de diffusion non linéaire mettant en jeu un opérateur de Lévy du premier ordre. Cet opérateur intégral garde en mémoire les interactions à longue distance alors que la non linéarité garde en mémoire les interactions à courte distance. Les techniques que nous utilisons s'avèrent être les bonnes pour obtenir des résultats d'homogénéisation pour la dynamique de particules avec des interactions par paires. Les systèmes d'ODE que nous considérons sont très proches des modèles de Frenkel-Kontorova. On montre que la fonction de répartition rescalée des particules converge vers la solution continue d'une équation de diffusion non linéaire.

Homogenization of the dislocation dynamics and of some particle systems with two-body interactions

N. Forcadel, C. Imbert, R. Monneau

Abstract
This paper is concerned with the homogenization of a non-local first order Hamilton-Jacobi equation describing the dynamics of several dislocation lines and the homogenization of some particle systems with two-body interactions. The first objective is to establish a connection between the rescaled dynamics of a increasing number of dislocation lines and the dislocation dynamics density, passing from a discrete model (dislocation lines) to a continuous one (dislocation density). A first answer to this problem was presented in a paper by Rouy and the two last authors [112] but the geometric definition of the fronts was not completely satisfactory. This problem is completely solved here. The limit equation is a nonlinear diffusion equation involving a first order Lévy operator. This integral operator keeps memory of the long range interactions, while the nonlinearity keeps memory of short ones. The techniques and tools we introduce turn out to be the right ones to get homogenization results for the dynamics of particles in two-body interaction. The systems of ODEs we consider are very close to overdamped Frenkel-Kontorova models. We prove that the rescaled "cumulative distribution function" of the particles converges towards the continuous solution of a nonlinear diffusion equation.
AMS Classification : 35B10, 35B27, 35F20, 45K05, 47G20, 49L25..
Keywords : periodic homogenization, Hamilton-Jacobi equations, moving fronts, two-body interactions, integro-differential operators, Lévy operator, dislocation dynamics, Slepčev formulation, Frenkel-Kontorova model.

1 Introduction

In this paper, we study a non-local Hamilton-Jacobi equation describing the dynamics of dislocation lines in interaction and we apply these results to get homogenization for both the dislocation dynamics and some one dimensional particle systems.

A model for the dynamics of a single dislocation is proposed in Alvarez *et al.* [10] by using the so-called level set approach. We adapt here this model to describe the motion of several dislocation lines moving in two-body interaction. The level set approach permits to describe such geometric motions by considering a function $u : \mathbb{R}^+ \times \mathbb{R}^N$ such that, for any (but fixed) $\alpha \in [0,1)$, the dislocation line Γ_k^α at time t coincides with

$$\{x : u(t,x) = \alpha + k\}$$

for $k = 1, \ldots, N$. After rescaling the problem, we therefore obtain the following equation

$$\left\{ \begin{array}{lll} \partial_t u^\varepsilon = \left(c\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) + M^\varepsilon \left[\frac{u^\varepsilon(t,\cdot)}{\varepsilon}\right](x) \right) |\nabla u^\varepsilon| & \text{in} & (0,+\infty) \times \mathbb{R}^N, \\ u^\varepsilon(0,x) = u_0(x) & \text{on} & \mathbb{R}^N \end{array} \right. \tag{8.1}$$

where M^ε is a 0 order non-local operator defined by

$$M^\varepsilon [U](x) = \int_{\mathbb{R}^N} dz\, J(z) E\left(U(x+\varepsilon z) - U(x)\right) \tag{8.2}$$

where E is a modification of the integer part :

$$E(\alpha) = k + \frac{1}{2} \quad \text{if} \quad k \leq \alpha < k+1. \tag{8.3}$$

The non-local operator M^ε describes the interactions between dislocation lines. Their interactions are thus completely characterized by a kernel J. We assume that $J \in W^{1,1}(\mathbb{R}^N)$ is an *even nonnegative* function with the following behaviour at infinity

$$\exists\ R_0 > 0 \text{ and } \exists\ g \in C^0(\mathbf{S}^{N-1}),\ g \geq 0 \text{ s.t. } J(z) = \frac{1}{|z|^{N+1}}\, g\left(\frac{z}{|z|}\right) \text{ for } |z| \geq R_0. \tag{8.4}$$

Let us mention that such an assumption is natural for dislocations and can be (slightly) generalized. In the special case where J has a bounded support (choose $g = 0$), we also assume

$$\inf_{e \in [0,1)^N} \int_{\mathbb{R}^N} dz\ \min(J(z), J(z+e)) > 0 \qquad \text{if } N \geq 2. \tag{8.5}$$

As far as the forcing term c and the initial datum are concerned, we assume

$$\left\{ \begin{array}{l} c(\tau,y) \text{ is Lipschitz continuous and } \mathbb{Z}^{N+1}\text{-periodic w.r.t. } (\tau,y); \\ u_0 \in W^{2,\infty}(\mathbb{R}^N). \end{array} \right. \tag{8.6}$$

Our first aim is to say what happens to the solution u^ε of (8.1) as $\varepsilon \to 0$.

This paper follows [111] and [112]. The main difference lies in the fact that the model we propose here describes better the geometric motion of the dislocation lines (see Section 3). Let us explain this briefly. In the level set approach for the motion of a single front, the initial front is described as the 0-level set of a function u_0

that is used as the initial datum of a Cauchy problem. The front at time $t > 0$ is then defined as the 0-level set of the solution u of the Cauchy problem. If now one considers two functions u_0 and v_0 with the same 0-level set, the geometric motion is well defined if the corresponding 0-level sets of the solutions of the Cauchy problem coincide. Considering now the motion of a finite number of fronts, the model in [112] did not satisfy such a property. But it does with the new model we consider in this paper (see Theorem 4.7). We will refer to this property as the consistency of the definition of the fronts.

The technical difficulty when trying to solve (8.1) is : how to deal with the integer part E, since it is discontinuous ? Our first try in [112] was to regularize E, make a change of unkown function and perform the homogenization in this framework (see also [111] for similar techniques for local equations). Here, on the contrary, we want to keep the model with the integer part in order to get consistency of the definition of the fronts. We use a notion of viscosity solutions for non-local equations introduced by Slepčev [169]. It consists in considering the simultaneous evolution of all the level sets of the function u^ε (see Definition 4.1). Such a definition ensures the stability of solutions, a key property in the viscosity solution approach.

The first aim of this work is to pass from a discrete and microscopic model involving the evolution of a finite number of dislocation lines to a continuous and macroscopic one describing the evolution of a dislocation density. To do so, we prove a homogenization result, *i.e.* we prove that the limit u^0 of u^ε as $\varepsilon \to 0$ exists and is the (unique) solution of a homogenized (or effective) equation. The function u^0 is understood as a "cumulative distribution function" associated with dislocations and its gradient represents the dislocation density. As in [112], the effective equation is

$$\begin{cases} \partial_t u^0 = \overline{H}^0\left(\mathcal{I}_1[u^0(t,\cdot)], \nabla u^0\right) & \text{in} \quad (0,+\infty)\times\mathbb{R}^N, \\ u^0(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N \end{cases} \tag{8.7}$$

where $\overline{H}^0$ is a continuous function and $\mathcal{I}_1$ is an anisotropic Lévy operator of order 1 associated with the function g appearing in (8.4). It is defined for any function $U \in C_b^2(\mathbb{R}^N)$ for $r > 0$ by

$$\begin{aligned} \mathcal{I}_1[U](x) = \; & \int_{|z|\le r}(U(x+z) - U(x) - \nabla_x U(x)\cdot z)\ \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz \\ & + \int_{|z|\ge r}(U(x+z) - U(x))\ \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz \end{aligned} \tag{8.8}$$

(notice that the latter expression is independent of r since J is even). As explained in [112], this Lévy operator $\mathcal{I}_1$ only keeps the memory of the long range interactions between dislocations, as the effective Hamiltonian $\overline{H}^0$ will keep the memory of the short range interactions (see the proof of Lemma 6.1 below). As usual in periodic homogenization, our aim is two-fold : to determine the so-called *effective Hamiltonian* $\overline{H}^0$ and to prove the convergence of u^ε towards u^0.

The second aim of this work is to apply these techniques to get homogenization results for the following system of ODEs

$$\dot{y}_i = F - V_0'(y_i) - \sum_{j\in\{1,\dots,N_\varepsilon\}\setminus\{i\}} V'(y_i - y_j) \quad \text{for } i = 1, \dots, N_\varepsilon \tag{8.9}$$

where F is a constant given force, V_0 is a 1-periodic potential and V is a potential taking into account two-body interactions. One can think of y_i as the "position" of dislocation straight lines. The key fact for applying the results about the solution of (8.1) is that, under proper assumptions on V_0 and V, the function

$$\rho^\varepsilon(t,x) = \varepsilon\left(-\frac{1}{2} + \sum_{i=1}^{N_\varepsilon} H(x - \varepsilon y_i(t/\varepsilon))\right)$$

(where H is the Heaviside function — see below for a definition) satisfies (8.1) for some c and J well chosen and suitable initial data. Hence, the rescaled "cumulative distribution function" ρ^ε of particles is proved to converge towards the unique solution of the corresponding nonlinear diffusion equation (8.7). See in particular [142] for interesting results concerning homogenization of some gradient systems with wiggly energies.

We would like to conclude this introduction by mentioning that our work is focused on a particular equation with a particular scaling in ε, directly inspired from the dislocation dynamics. It would be interesting to consider several extensions of this model. A first extension could take into account the presence of Franck-Read sources or of mean curvature motion terms. The study of different scalings in ε and different decays at infinity for the kernel J is another interesting question. We also want to point out that getting some error estimates, both for the homogenization process and for the numerical computation of the effective Hamiltonian, would be also very interesting. To finish with, let us mention that we will study in a future work the homogenization of classical Frenkel-Kontorova models which are systems of ODEs.

Organization of the article. The paper is organized as follows. In Section 2, we present our main results. In Section 3, we give a physical derivation of equation (8.1). In Section 4, we recall the definition of viscosity solutions for equations like (8.1) and (8.12), we give a stability result, a comparison principle and existence results. The proof of the ergodicity of the problem (Theorem 2.1) is presented in Section 5. Section 6 is devoted to the proof of the convergence (Theorem 2.5). In Section 7, we establish the qualitative properties of the effective Hamiltonian described in Theorem 2.6. Finally, in Section 8, we apply our approach to the case of systems of particles (Theorem 8.1 and 2.11).

Notation. The open ball of radius r centered at x is classically denoted $B_r(x)$. When x is the origin, $B_r(0)$ is simply denoted B_r and the unit ball B_1 is denoted B. The cylinder $(t-\tau, t+\tau) \times B_r(x)$ is denoted $Q_{\tau,r}(t,x)$. The indicator function of a subset $A \subset C$ is denoted by 1_A : it equals 1 on A and 0 on $C \setminus A$.

The quantity $\lfloor x \rfloor$ denotes the floor integer parts of a real number x. Let $H(x)$ denote the Heaviside function :

$$H(r) = \begin{cases} 1 & \text{if } r \geq 0, \\ 0 & \text{if } r < 0. \end{cases}$$

It is convenient to introduce the unbounded measure on $\mathbb{R}^N$ defined on $\mathbb{R}^N \setminus \{0\}$ by :

$$\mu(dz) = \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz \tag{8.10}$$

and such that $\mu(\{0\}) = 0$. For the reader's convenience, we recall here the five integro-differential operators appearing in this work :

$$\begin{aligned}
M^{\varepsilon}[U](x) &= \int_{\mathbb{R}^N} E(U(x+\varepsilon z) - U(x)) J(z)\, dz \\
M_p^{\alpha}[U](x) &= \int_{\mathbb{R}^N} \{E(U(x+z) - U(x) + p\cdot z + \alpha) - p \cdot z\} J(z)\, dz, \\
M_p[U](x) &= \int_{\mathbb{R}^N} \{E(U(x+z) - U(x) + p\cdot z) - p \cdot z\} J(z)\, dz, \\
M[U](x) &= \int_{\mathbb{R}^N} \{E(U(x+z) - U(x))\} J(z)\, dz, \\
\mathcal{I}_1[U](x) &= \int_{\mathbb{R}^N} (U(x+z) - U(x) - 1_{B_r}(z) \nabla_x U(x) \cdot z)\ \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz.
\end{aligned}$$

To each operator M, we associate $\tilde{M}$ which is defined in the same way but where E is replaced with E_* defined as follows

$$E_*(\alpha) = k + \frac{1}{2} \quad \text{if } \ k < \alpha \leq k+1.$$

See Section 4 for further details.

2 Main results

2.1 General homogenization results

We explained in the introduction that our first aim is to get homogenization results for (8.1). In other words, we want to say what happens to the solution u^{ε} of

(8.1) as $\varepsilon \to 0$. We classically try to prove that u^ε converges to the solution u_0 of an effective equation. In order to both determine the effective equation and prove the convergence, it is also classical to perform a formal expansion, that is to write u^ε as $u_0 + \varepsilon v$. One must next find an equation (E) solved by v. The function v is classically called a corrector and the equation it satisfies is referred to as the cell equation or cell problem. In our case, such a problem is associated with any constant $L \in \mathbb{R}$ and any $p \in \mathbb{R}^N$:

$$\lambda + \partial_\tau v = \Big(c(\tau, y) + L + M_p[v(\tau,\cdot)](y) \Big) |p + \nabla v| \text{ in } \quad (0,+\infty) \times \mathbb{R}^N \tag{8.11}$$

where

$$M_p[U](y) = \int dz\, J(z) \left\{ E\left(U(y+z) - U(y) + p \cdot z \right) - p \cdot z \right\}.$$

The construction of correctors v satisfying (8.11) is one of the important problem we have to solve. It is done by considering the solution w of

$$\begin{cases} \partial_\tau w = \Big(c(\tau, y) + L + M_p[w(\tau,\cdot)](y) \Big) |p + \nabla w| & \text{in} \quad (0,+\infty) \times \mathbb{R}^N, \\ w(0,y) = 0 & \text{on} \quad \mathbb{R}^N. \end{cases} \tag{8.12}$$

and by looking for some $\lambda \in \mathbb{R}$ such that $w - \lambda\tau$ is bounded. Here is the precise result.

Théorème 2.1 (Ergodicity). *Under the assumptions* (8.4)-(8.5)-(8.6)*, for any $L \in \mathbb{R}$ and $p \in \mathbb{R}^N$, there exists a unique $\lambda \in \mathbb{R}$ such that the continuous viscosity solution of* (8.12) *(in the sense of Definition 4.1) satisfies : $\frac{w(\tau,y)}{\tau}$ converges towards λ as $\tau \to +\infty$, locally uniformly in y. The real number λ is denoted by $\overline{H}^0(L,p)$. Moreover, the function $\overline{H}^0$ satisfies*

$$\overline{H}^0 \textit{ is continuous in } (L,p) \textit{ and nondecreasing in } L. \tag{8.13}$$

Remark 2.2. *Condition* (8.5) *is related to the periodicity assumption on the velocity. Indeed, assumption* (8.5) *is crucial in our analysis and we do not know if ergodicity holds or not if this assumption is not fullfilled in dimension $N \geq 2$.*

Remark 2.3. *Condition* (8.13) *ensures the existence of solutions for the homogenized equation* (8.7) *(see Theorem 4.6).*

Remark 2.4. *A superscript* 0 *appears in the effective Hamiltonian. The reason is the same as in [112] : we will have to study the ergodicity of a family of Hamiltonians in order to prove the convergence. With the notation of Section 5, we have $\overline{H}^0(L,p) = \overline{H}(L,p,0)$.*

With correctors in hand, we can now prove the convergence of the sequence u^ε. The second main result of this paper is the following convergence result.

Théorème 2.5 (Convergence)**.** *Under the assumptions* (8.4)-(8.5)-(8.6)*, the bounded continuous viscosity solution* u^ε *of* (8.1) *(in the sense of Definition 4.1) with initial data* $u_0 \in W^{2,\infty}(\mathbb{R}^N)$*, converges as* $\varepsilon \to 0$ *locally uniformly in* (t,x) *towards the unique bounded viscosity solution* u^0 *of* (8.7).

Recall that the first homogenization problem we are trying to solve comes from dislocation theory. We thus would like to be able to get an interpretation of the homogenization result we obtain in terms of dislocation theory. This is the reason why we look for qualitative properties of $\overline{H}^0$. Considering the one dimensional special case and a driving force independent of time, we obtain the following

Théorème 2.6 (Qualitative properties of $\overline{H}^0$)**.** *Under the assumptions* $N=1$*,* $c=c(y)$ *and* $\int_{(0,1)} c = 0$*, the function* $\overline{H}^0(L,p)$ *is continuous and satisfies the following properties :*

1. *If* $c \equiv 0$ *then* $\overline{H}^0(L,p) = L|p|$.
2. **(Bound)** *If* C *denotes* $\|c\|_\infty + \frac{1}{2}\|J\|_{L^1}$*, we have*
$$\left|\frac{\overline{H}^0(L,p)}{|p|} - L\right| \leq C \quad \textit{for} \quad (L,p) \in \mathbb{R}\times\mathbb{R}.$$
3. **(Sign of the Hamiltonian)**
$$\overline{H}^0(L,p)L \geq 0 \quad \textit{for} \quad (L,p) \in \mathbb{R}\times\mathbb{R}.$$
4. **(Monotonicity in** L**)** *The function* $\overline{H}^0(L,p)$ *satisfies for* $C = \|c\|_\infty + (|p| + \frac{1}{2})\|J\|_{L^1}$ *:*
$$\frac{|\overline{H}^0|}{(|L|-C)^+} \geq \frac{\partial \overline{H}^0}{\partial L} \geq \frac{|\overline{H}^0|}{|L|+C}.$$
5. **(Modulus of continuity in** L**)** *There exists a constant* C_1 *only depending on* $\|\nabla c\|_\infty$ *such that*
$$0 \leq \overline{H}^0(L+L',p) - \overline{H}^0(L,p) \leq \frac{C_1|p|}{|\ln L'|} \quad \text{for } 0 < L' \leq \frac{1}{2}.$$
6. **(Antisymmetry in** L**)** *If there exists* $a \in \mathbb{R}$ *such that* $-c(y) = c(y+a)$*, then :*
$$\overline{H}^0(-L,-p) = -\overline{H}^0(L,p).$$

7. **(Symmetry in p)** *If there exists $a \in \mathbb{R}$ such that $c(-y) = c(y+a)$, then :*

$$\overline{H}^0(L,-p) = \overline{H}^0(L,p).$$

8. **(0-plateau property)** *If $c \not\equiv 0$, then there exists $r_0 > 0$ (only depending on $\|c\|_\infty$ and $J_{|\mathbb{R}\setminus[-1,1]}$) such that :*

$$\overline{H}^0(L,p) = 0 \quad \textit{for} \quad (L,p) \in B_{r_0}(0) \subset \mathbb{R}^2.$$

9. **(Non-zero Hamiltonian for large p)** *Let us assume that $c \in W^{2,\infty}(\mathbb{R})$, $J \in W^{1,\infty}(\mathbb{R})$ and :*

$$h_0, h_1 \in L^1(\mathbb{R})$$

where

$$h_0(z) = \sup_{a\in[-1/2,1/2]} |J(z+a)|, \quad h_1(z) = \sup_{a\in[-1/2,1/2]} |J'(z+a)|.$$

Then there exists a constant $C > 0$ (depending on $\|c\|_{W^{2,\infty}}$, $\|h_0\|_{L^1}$, $\|h_1\|_{L^1}$) such that :

$$L\overline{H}^0(L,p) > 0 \quad \textit{for} \quad |L| > C/|p| \quad \textit{and} \quad |p| > C.$$

Remark 2.7. *Notice that assuming $\int c = 0$ is not a restriction at all.*

Remark 2.8. *The qualitative properties 8 and 9 of the homogenized Hamiltonian shows that there is a cooperative collective behaviour. More precisely, increasing the dislocation density allows to move the dislocations that were locked for small densities (and small enough L).*
The 0-plateau property for small p is related to the work of Aubry [15] on the breaking of analyticity of the hull function. In particular, from De La Llave [70], it is possible to see that for any Diophantine number p and for any c small enough (depending on p), with c and V analytic, we have

$$L\overline{H}^0(L,p) > 0 \quad \text{for } L \neq 0.$$

The threshold $\frac{C}{p}$ for large p is related to the well-known pile-up effect for dislocations in front of an obstacle. Indeed, it is know that for an applied stress L_a and a pile-up of p dislocations stuck on the obstacle, the internal stress field created by the the obstacle is $F = pL_a$ (see [106] page 766 for further details). Therefore to make the dislocations to move, we need to apply a stress of the order F/p, which is exactly the result we get.

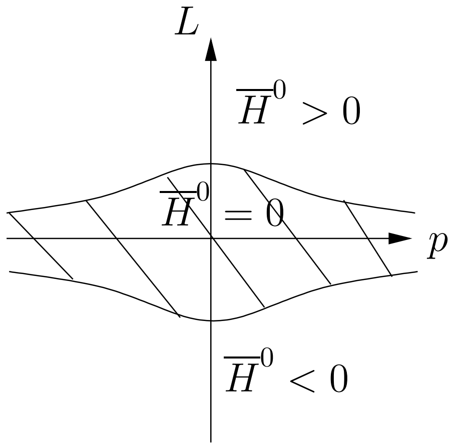

Fig. 8.1 – Schematic representation of the effective Hamiltonian

Remark 2.9. *In the case $c \equiv 0$, self-similar solutions of* (8.7) *were obtained by Head (see for instance [72, 104]).*

The typical profile of $\overline{H}^0$ is represented in Figure 8.1. We also refer to Ghorbel [96] and Ghorbel, Hoch, Monneau [97] for simulations.

Remark 2.10. *The boundary of the set $\{\overline{H}^0(L,p) = 0\}$ is given by two graphs $h^-(p) \leq L \leq h^+(p)$, but it is not known if h^+ and h^- are continuous.*

2.2 Application to the homogenization of particle systems with two-body interactions

As explained in the Introduction, we are able to apply the homogenization results of (8.1) to the system of ODEs (8.9) because under appropriate assumptions the function ρ of (τ, y) defined by

$$\rho(\tau, y) = -\frac{1}{2} + \sum_{i=1}^{N_\varepsilon} H(y - y_i(\tau)) \tag{8.14}$$

(where H is the Heaviside function — see the Introduction for a definition) is a solution of (8.1) with $\varepsilon = 1$ and c independent on time where

$$c(y) = V_0'(y) - F \qquad \text{and} \qquad J = V'' \text{ on } \mathbb{R}\backslash\{0\}. \tag{8.15}$$

See Theorem 8.1 for a precise statement.

Before presenting the results for (8.9), let us make precise the assumptions on F, V_0 and V and make some comments on them. We recall that F is a constant given force and V_0 is a 1-periodic potential. As far as V is concerned, we assume

Assumption (H)

$(H0)$ $V \in W^{1,\infty}_{\mathrm{Loc}}(\mathbb{R})$ and $V'' \in W^{1,1}(\mathbb{R}\backslash\{0\})$,

$(H1)$ V is symmetric, *i.e.* $V(-y) = V(y)$,

$(H2)$ V is nonincreasing and convex on $(0, +\infty)$,

$(H3)$ $V'(y) \to 0$ as $|y| \to +\infty$,

$(H4)$ there exists $R_0 > 0$ and a constant g_0 such that $V''(y)y^2 = g_0$ for $|y| \geq R_0$.

In $(H4)$, V'' will play the role of the function J appearing in (8.2) and condition $(H4)$ is equivalent to (8.4). It is possible to slightly generalize Assumption $(H4)$ by only assuming an asymptotic behaviour of V'' instead of assuming that it coincides with g_0/y^2 outside a fixed ball. The system of ODEs (8.9) has some similarities with

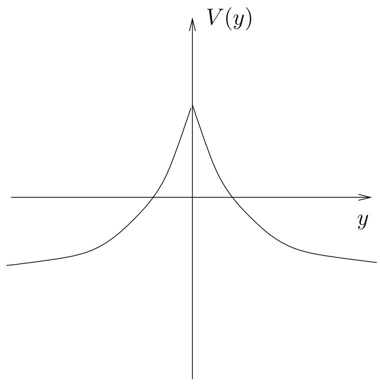

FIG. 8.2 – Typical profile for a potential V satisfying assumptions (H).

the overdamped Frenkel-Kontorova model [126], except that in the classical Frenkel-Kontorova model only interactions between nearest neighboors are considered (see Hu, Qin, Zheng [108]; see also Aubry [15], Aubry, Le Daeron [16] as far as stationnary solutions are concerned). We plan to study the homogenization of the classical Frenkel-Kontorova model in a future work.

Then we have the following homogenization result for our particle system.

Théorème 2.11 (Homogenization of the particle system)**.** *Assume that* V_0 *is* 1*-periodic,* V_0' *is Lipschitz continuous and* V *satisfies* (H)*. Assume that* $y_1(0) < \ldots < y_{N_\varepsilon}(0)$ *are given by the discontinuities of the function* $\rho_0^\varepsilon(x) = \varepsilon E\left(\frac{u_0(x)}{\varepsilon}\right)$ *with* E *defined by* (8.3)*, for some given nondecreasing function* $u_0 \in W^{2,\infty}(\mathbb{R})$*. Define* ρ *as in* (8.14) *and consider*

$$\rho^\varepsilon(t,x) = \varepsilon\rho\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right).$$

Then ρ^ε *converges towards the solution* $u^0(t,x)$ *of* (8.7) *where the operator* $\mathcal{I}_1$ *is defined by* (8.8) *with* $g\left(z/|z|\right) = g_0$ *and* $\overline{H}^0$ *is given in Theorem 2.1 with* c *and* J *defined in* (8.15)*.*

Remark 2.12. *In the case of short range interactions, i.e.,* $g_0 = 0$ *in (H4), the homogenized equation* (8.7) *is a local Hamilton-Jacobi equation and the dislocation density* $\frac{\partial u^0}{\partial x}$ *satisfies formally a hyperbolic equation.*

Remark 2.13. *Theorem 2.11 is still true (with other constants depending possibly on these quantities) with a potential* $V_0(t, x)$ *periodic in* x *and* t *(see Theorem 2.5). Moreover, the regularity of the initial data* u_0 *can be considerably weakened if necessary.*

Since our first goal was to study dislocation dynamics, the following generalization is of special interest.

Théorème 2.14 (The dislocation case)**.** *Theorems 2.11 and 2.6 (excepted point 4. and point 9.) are still true with* $V(x) = -\ln|x|$ *and* c *and* J *defined by* (8.15).

Remark 2.15. *Here the annihilation of particles is not included in* (8.9)*, but our approach with equation* (8.1) *could be developed in this case.*

3 Physical derivation of the model for dislocation dynamics

Dislocations are line defects in crystals. Their typical length is of the order of $10^{-6}m$ and their thickness of the order of $10^{-9}m$. When the material is submitted to shear stress, these lines can move in the crystallographic planes and their complicated dynamics is one of the main explanation of the plastic behaviour of metals.

In the present paper we are interested in describing the effective dynamics for the collective motion of dislocation lines with the same Burgers's vector and all contained in a *single* slip plane $\{x_3 = 0\}$ of coordinates $x = (x_1, x_2)$, and moving in a periodic medium. At the end of this derivation, we will see that the dynamics of dislocations is described by equation (8.1) in dimension $N = 2$.

Several obstacles to the motion of dislocation lines can exist in real life : precipitates, inclusions, other pinned dislocations or other moving dislocations, *etc.* We will describe all these obstacles by a given field

$$c(t, x) \tag{8.16}$$

that we assume to be periodic in space and time. Another natural force exists : this is the Peach-Koehler force acting on a dislocation j. This force is the sum of the interactions with the other dislocations k for $k \neq j$, and of the self-force created by the dislocation j itself.

The level set approach for describing dislocation dynamics at this scale consists in considering a function v such that the dislocation $k \in \mathbb{Z}$ is basically described by the level set $\{v = k\}$. Let us first assume that v is smooth.

As explained in [10], the Peach-Koehler force at the point x created by a dislocation j is well-described by the expression

$$c_0 \star 1_{\{v \geq j\}}$$

where $1_{\{v \geq j\}}$ is the characteristic function of the set $\{v \geq j\}$ which is equal to 1 or 0. In a general setting, the kernel c_0 can change sign. In the special case where the dislocations have the same Burgers vector and move in the same slip plane, a monotone formulation (see Alvarez et al. [10], Da Lio et al. [67]) is physically acceptable. Indeed, the kernel can be chosen as

$$c_0 = J - \delta_0$$

where J is nonnegative and δ_0 denotes the Dirac mass. The negative part of the kernel is somehow concentrated at the origin. Moreover, we assume that J satisfies the symmetry condition : $J(-z) = J(z)$ and $\int_{\mathbb{R}^2} J = 1$ so that we have (at least formally) $\int_{\mathbb{R}^2} c_0 = 0$. The kernel J can be computed from physical quantities (like the elastic coefficients of the crystal, the Burgers vector of the dislocation line, the slip plane of the dislocation, the Peierls-Nabarro parameter, *etc.*). We set formally

$$\left(\delta_0 \star 1_{\{v \geq j\}}\right)(x) := \begin{cases} 1 & \text{if} \quad v(x) > j \\ \frac{1}{2} & \text{if} \quad v(x) = j \\ 0 & \text{if} \quad v(x) < j \end{cases}$$

We remark in particular that the Peach-Koehler force is discontinuous on the dislocation line in this modeling (see Figure 8.3).

Let us now assume that for integers $N_1, N_2 \geq 0$, we have $-N_1 - 1/2 < v < N_2 + 1/2$. Then the Peach-Koehler force at the point x on the dislocation j (*i.e.* $v(x) = j$) created by dislocations for $k = -N_1, ..., N_2$ is given by the sum

$$\left((J - \delta_0) \star \sum_{k=-N_1}^{N_2} 1_{\{v \geq k\}} \right)(x) = \left(J \star E(v - v(x))\right)(x) \tag{8.17}$$

with E defined in (8.3). Defining the normal velocity to dislocation lines as the sum of the periodic field (8.16) and the Peach-Koehler force (8.17), we see that the dislocation line $\{v = j\}$ for integer j, is formally a solution of the following level set (or eikonal) equation :

$$v_t = \left(c + J \star E(v - v(x))\right)|\nabla v| \tag{8.18}$$

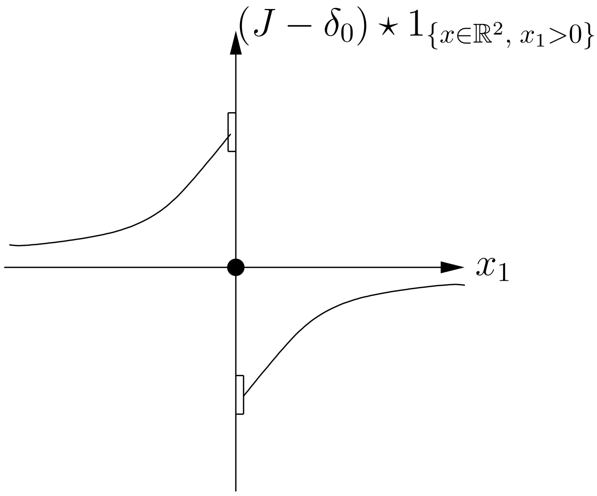

FIG. 8.3 – Typical profile for the Peach-Koeller force created by a dislocation straight line.

which is exactly (8.1) with $\varepsilon = 1$.

We refer in particular to [112] for a mechanical interpretation of the homogenized equation and the references therein for other studies of models with dislocation densities. See in particular [156] for the homogenization of one-dimensional models giving some rate-independent plasticity macroscopic models.

4 Viscosity solutions for non-local equations (8.1) and (8.12)

In this paper, we have to deal with Hamilton-Jacobi equations involving integro-differential operators. For Equations (8.1) and (8.12), we will use a definition of viscosity solutions first introduced by Slepčev [169]. As far as Equation (8.7) is concerned, the reader is referred to [112] for a definition for viscosity solution and for the proof of a comparison principle in the class of bounded functions.

Let us first recall the definition of relaxed lower semi-continuous (lsc for short) and upper semi-continuous (usc for short) limits of a family of functions u^ε which is locally bounded uniformly w.r.t. ε :

$$\limsup{}^* u^\varepsilon(t,x) = \limsup_{\varepsilon\to 0, s\to t, y\to x} u^\varepsilon(s,y) \quad \text{and} \quad \liminf{}_* u^\varepsilon(t,x) = \liminf_{\varepsilon\to 0, s\to t, y\to x} u^\varepsilon(s,y).$$

If the family contains only one element, we recognize the usc envelope and the lsc envelope of a locally bounded function u :

$$u^*(t,x) = \limsup_{s\to t, y\to x} u(s,y) \quad \text{and} \quad u_*(t,x) = \liminf_{s\to t, y\to x} u(s,y).$$

4.1 Definition of viscosity solutions

In this subsection, we will give the definition of viscosity solution for the following problem

$$\begin{cases} \partial_t u = \big(c(t,x) + M_p^\alpha[u(t,\cdot)](x)\big)\,|p+\nabla u| & \text{in} \quad (0,+\infty)\times\mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N \end{cases} \tag{8.19}$$

where M_p^α is defined by

$$M_p^\alpha[U](x) = \int_{\mathbb{R}^N} dz\; J(z)\left\{E(U(x+z) - U(x) + p\cdot z + \alpha) - p\cdot z\right\}.$$

It will be convenient to define the following associated operator

$$\tilde{M}_p^\alpha[U](x) = \int_{\mathbb{R}^N} dz\; J(z)\left\{E_*(U(x+z) - U(x) + p\cdot z + \alpha) - p\cdot z\right\}.$$

where we recall that

$$E_*(\alpha) = k + \frac{1}{2} \quad \text{if} \quad k < \alpha \leq k+1.$$

We now recall the definition of viscosity solutions introduced by Slepčev in [169] :

Definition 4.1 (Viscosity solutions for (8.19))**.** *A upper semi-continuous (resp. lower semi-continuous) function $u : \mathbb{R}^+\times\mathbb{R}^N \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of* (8.19) *if $u(0,x) \leq u_0^*(x)$ in $\mathbb{R}^N$ (resp. $u(0,x) \geq (u_0)_*(x)$) and for any $(t,x) \in (0,\infty)\times\mathbb{R}^N$ and any test function $\phi \in C^2(\mathbb{R}^+\times\mathbb{R}^N)$ such that $u-\phi$ attains a maximum (resp. a minimum) at the point $(t,x)\in(0,+\infty)\times\mathbb{R}^N$, then we have*

$$\partial_t\phi(t,x) \leq \big(c(t,x) + M_p^\alpha[u(t,\cdot)](x)\big)\,|p+\nabla\phi|$$

$$\left(\text{resp.} \quad \partial_t\phi(t,x) \geq \left(c(t,x) + \tilde{M}_p^\alpha[u(t,\cdot)](x)\right)|p+\nabla\phi|\right).$$

A function u is a viscosity solution of (8.19) *if u^* is a viscosity subsolution and u_* is a viscosity supersolution.*

4.2 Stability results for (8.19)

In this subsection, we will prove a general stability result for the non-local term. The following proposition permits to show all the classical stability results for viscosity solutions we need.

Proposition 4.2 (Stability of the solutions of (8.19)). *Let $(u_n)_n$ be a sequence of uniformly bounded usc functions (resp. lsc functions) and let $\overline{u}$ denote* $\limsup^* u^n$ *(resp. $\underline{u} = \liminf^* u^n$). Let $(t_n, x_n, p_n, \alpha_n) \to (t_0, x_0, p, \alpha)$ in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ be such that $u_n(t_n, x_n) \to \overline{u}(t_0, x_0)$ (resp. $u_n(t_n, x_n) \to \underline{u}(t_0, x_0)$). Then*

$$\limsup_{n\to\infty} M_{p_n}^{\alpha_n}[u_n(t_n,\cdot)](x_n) \leq M_p^{\alpha}[\overline{u}(t_0,\cdot)](x_0) \tag{8.20}$$

$$\left(\text{resp. } \liminf_{n\to\infty} \tilde{M}_{p_n}^{\alpha_n}[u_n(t_n,\cdot)](x_n) \geq \tilde{M}_p^{\alpha}[\underline{u}(t_0,\cdot)](x_0)\right).$$

This result is a consequence of the stability of the Slepčev definition and in particular of the following lemma (whose proof is given in [169]) :

Lemma 4.3. *Let $(f_n)_n$ be a sequence of measurable functions on $\mathbb{R}^N$, and consider*

$$\overline{f} = \limsup{}^* f_n$$

and

$$\underline{f} = \liminf{}^* f_n.$$

Let $(a_n)_n$ be a sequence of $\mathbb{R}$ converging to zero. Then

$$\mathcal{L}(\{f_n \geq a_n\} \backslash \{\overline{f} \geq 0\}) \to 0 \quad \text{as} \quad n \to \infty$$

and

$$\mathcal{L}(\{\underline{f} > 0\} \backslash \{f_n > a_n\}) \to 0 \quad \text{as} \quad n \to \infty$$

where $\mathcal{L}(A)$ denotes the Lebesgue measure of measurable set A.

Proof of Proposition 4.2. We just prove the result for $\overline{u}$. Let $\varepsilon > 0$. Using the strong decay at infinity of J and the fact that $|E(r) - r| \leq \frac{1}{2}$, we know that there exists R such that for any $n \in \mathbb{N}$

$$\left| \int_{|z|\geq R} J(z)\{E(u_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n \cdot z + \alpha_n) - p_n \cdot z\} \right| \leq \frac{\varepsilon}{4},$$

$$\left| \int_{|z|\geq R} J(z)\{E(\overline{u}(t_0, x_0 + z) - \overline{u}(t_0, x_0) + p \cdot z + \alpha) - p \cdot z\} \right| \leq \frac{\varepsilon}{4}. \tag{8.21}$$

Moreover, using the uniform bound on the sequence $(u_n)_n$, we deduce that for $|z| \leq R$, there exists $N_0 \in \mathbb{N}$, $N_0 \geq 1$, such that

$$|u_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n \cdot z + \alpha_n| \leq N_0$$

and

$$|\overline{u}(t_0, x_0 + z) - \overline{u}(t_0, x_0) + p \cdot z + \alpha| \leq N_0.$$

We notice that

$$E(\beta) = \sum_{k\geq 1} 1_{\{\beta \geq k\}} - \sum_{k\leq 0} 1_{\{\beta < k\}} + \frac{1}{2}. \tag{8.22}$$

We then get that

$$\begin{aligned}
&\int_{|z|\leq R} J(z)\{E(u_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n \cdot z + \alpha_n) - p_n \cdot z\} \\
&\quad - \int_{|z|\leq R} J(z)\{E(\overline{u}(t_0, x_0 + z) - \overline{u}(t_0, x_0) + p \cdot z + \alpha) - p \cdot z\} \\
\leq &\int_{|z|\leq R} J(z)\Big\{ \sum_{k=1}^{N_0} \big(1_{\{u_n(t_n,x_n+z)-u_n(t_n,x_n)+p_n\cdot z+\alpha_n\geq k\}} - 1_{\{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p\cdot z+\alpha\geq k\}}\big) \\
&\qquad - \sum_{k=-N_0}^{0} \big(1_{\{u_n(t_n,x_n+z)-u_n(t_n,x_n)+p_n\cdot z+\alpha_n< k\}} - 1_{\{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p\cdot z+\alpha< k\}}\big) \Big\}
\end{aligned} \tag{8.23}$$

Since the two sums are finite, we get, using Lemma 4.3, that for n big enough

$$\begin{aligned}
&\sum_{k=1}^{N_0} \int_{|z|\leq R} dz\ J(z)\ \big(1_{\{u_n(t_n,x_n+z)-u_n(t_n,x_n)+p_n\cdot z+\alpha_n\geq k\}} - 1_{\{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p\cdot z+\alpha\geq k\}}\big) \\
&\qquad\qquad \leq \frac{\varepsilon}{4} \\
&\sum_{k=-N_0}^{0} \int_{|z|\leq R} dz\ J(z)\ \big(1_{\{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p\cdot z+\alpha< k\}} - 1_{\{u_n(t_n,x_n+z)-u_n(t_n,x_n)+p_n\cdot z+\alpha_n< k\}}\big) \\
&\qquad\qquad \leq \frac{\varepsilon}{4}.
\end{aligned} \tag{8.24}$$

Using (8.21), (8.23) and (8.24) we deduce that

$$M_p^\alpha[u_n(t_n,\cdot)](x_n) \leq M_p^\alpha[\overline{u}(t_0,\cdot)](x_0) + \varepsilon$$

for n big enough. This implies (8.20). □

4.3 Comparison principles

In this subsection, we will prove a comparison principle for (8.19).

Théorème 4.4 (Comparison Principle)**.** *Let $T > 0$ and assume that $J \in W^{1,1}(\mathbb{R}^N)$. Consider an initial datum $u_0 \in W^{2,\infty}(\mathbb{R}^N)$ and a forcing term $c \in W^{1,\infty}([0,+\infty) \times \mathbb{R}^N)$. Let u be a bounded upper semi-continuous subsolution of (8.19) and v be a bounded lower semi-continuous supersolution. Then $u(t,x) \le v(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}^N$.*

Proof. Suppose by contradiction that $M_0 = \sup_{(0,T)\times\mathbb{R}^N}(u(t,x) - v(t,x)) > 0$. For all $0 < \gamma < 1$, $\eta > 0$, $\beta > 0$, we define

$$\Phi^\eta_{\gamma,\beta}(t,x,y) = u(t,x) - v(t,y) - \eta t + p\cdot(x-y) - e^{K_0 t}\left(\frac{|x-y|^2}{2\gamma} + \beta(|x|^2+|y|^2)\right)$$

where K_0 is a constant which will be chosen latter. We observe that

$$\limsup_{|x|,|y|\to\infty} \Phi^\eta_{\gamma,\beta}(t,x,y) = -\infty$$

so $\Phi^\eta_{\gamma,\beta}$ reaches its maximum at a point $(\bar t, \bar x, \bar y) \in [0,T] \times \mathbb{R}^N \times \mathbb{R}^N$. Moreover, the constant K_0 being fixed, we have $\overline{M}_0 = \sup \Phi^\eta_{\gamma,\beta} \ge \frac{M_0}{2}$ for η and β small enough. Standard arguments show that

$$|\bar x - \bar y|^2 \le C_0\gamma, \quad \beta(|\bar x|^2 + |\bar y|^2) \le C_0 \tag{8.25}$$

with C_0 depending on $\|u\|_\infty$, $\|v\|_\infty$, p and K_0.

We claim that there exists $0 < \gamma < 1$ such that for all β small enough, we have $\bar t > 0$. Indeed, if for all $0 < \gamma < 1$ there exists $\beta > 0$ (small) such that $\bar t = 0$, then the following estimates holds :

$$\begin{aligned}\frac{M_0}{2} \le \overline{M}_0 \le& u(0,\bar x) - v(0,\bar y) + |p||\bar x - \bar y| \\ \le& (\|Du_0\|_\infty + |p|)|\bar x - \bar y|.\end{aligned}$$

Using (8.25), we get a contradiction if γ is small enough and we prove the claim. Using Ishii's Lemma yields that there are $a, b \in \mathbb{R}$ and $\bar p, \bar q_x, \bar q_y \in \mathbb{R}^N$ such that

$$\begin{aligned} a - b = \eta + K_0 e^{K_0 \bar t}\left(\frac{|\bar x - \bar y|^2}{2\gamma} + \beta(|\bar x|^2 + |\bar y|^2)\right), \\ \bar p = e^{K_0\bar t}\frac{\bar x - \bar y}{\gamma}, \quad \bar q_x = 2\beta e^{K_0 \bar t}\bar x, \quad \bar q_y = 2\beta e^{K_0\bar t}\bar y, \\ a - \left(c(\bar t, \bar x) + M^\alpha_p[u(\bar t,\cdot)](\bar x)\right)|\bar p + \bar q_x| \le 0 \end{aligned}$$

and

$$b - \left(c(\bar t, \bar y) + \tilde{M}^\alpha_p[v(\bar t,\cdot)](\bar y)\right)|\bar p - \bar q_y| \ge 0.$$

Subtracting the two last inequalities, we get

$$\begin{aligned}&\eta + K_0 e^{K_0 \bar{t}} \left(\frac{|\bar{x}-\bar{y}|^2}{2\gamma} + \beta(|\bar{x}|^2 + |\bar{y}|^2) \right) \\ &\le \big(c(\bar{t},\bar{x}) + M_p^\alpha[u(\bar{t},\cdot)](\bar{x})\big)\, |\bar{p}+\bar{q}_x| - \Big(c(\bar{t},\bar{y}) + \tilde{M}_p^\alpha[v(\bar{t},\cdot)](\bar{y})\Big)\, |\bar{p}-\bar{q}_y|. \qquad (8.26)\end{aligned}$$

We define

$$\mathcal{A} = \{z : E(u(\bar{t},z) - u(\bar{t},\bar{x}) + p\cdot(z-\bar{x}) + \alpha) \le E_*(v(\bar{t},z) - v(\bar{t},\bar{y}) + p\cdot(z-\bar{y}) + \alpha)\}. \qquad (8.27)$$

The inequality $\Phi^\eta_{\gamma,\beta}(\bar{t},\bar{x},\bar{y}) \ge \Phi^\eta_{\gamma,\beta}(\bar{t},x,x)$ yields

$$\begin{aligned}&u(\bar{t},z) - u(\bar{t},\bar{x}) + p\cdot(z-\bar{x}) + \alpha \\ &\le v(\bar{t},z) - v(\bar{t},\bar{y}) + p\cdot(z-\bar{y}) + \alpha - e^{K_0\bar{t}} \left(\frac{|\bar{x}-\bar{y}|^2}{2\gamma} + \beta(|\bar{x}|^2+|\bar{y}|^2) - 2\beta|z|^2 \right).\end{aligned}$$

This implies that

$$\mathcal{A}^c \subset \{|z| \ge R_{\gamma,\beta}\},$$

where

$$(R_{\gamma,\beta})^2 = \frac{1}{2\beta}\left(\frac{|\bar{x}-\bar{y}|^2}{2\gamma} + \beta(|\bar{x}|^2+|\bar{y}|^2) \right).$$

We now distinguish two cases.

Case 1. There exists a constant $\tilde{C}_\gamma > 0$ such that for any β small enough we have

$$\frac{|\bar{x}-\bar{y}|}{2\gamma} \ge \tilde{C}_\gamma.$$

In this case, we have

$$\{|z-\bar{x}| \ge R_{\gamma,\beta}\} \subset \{|z| \ge \tilde{R}_{\gamma,\beta}\} \qquad (8.28)$$

where $\tilde{R}_{\gamma,\beta} = -|\bar{x}| + R_{\gamma,\beta} \to +\infty$ as $\beta \to 0$ (see Da Lio *et al.* [67, Lemma 2.5]). This implies that

$$\begin{aligned}&M_p^\alpha[u(\bar{t},\cdot)](\bar{x}) \\ &= \int_{\mathbb{R}^N} dz\; J(\bar{x}-z)\,\{E(u(\bar{t},z) - u(\bar{t},\bar{x}) + p\cdot(z-\bar{x}) + \alpha) - p\cdot(z-\bar{x})\} \\ &\le \int_{\mathbb{R}^N} dz\; J(\bar{x}-z)\,\{E_*(v(\bar{t},z) - v(\bar{t},\bar{y}) + p\cdot(z-\bar{y}) + \alpha) - p\cdot(z-\bar{x})\} + o_\beta(1).\end{aligned}$$

Using (8.26) we then get

$$
\begin{aligned}
&\eta + K_0 e^{K_0\bar t}\left(\frac{|\bar x-\bar y|^2}{2\gamma}+\beta(|\bar x|^2+|\bar y|^2)\right)\\
\le&\,(c(\bar t,\bar x)-c(\bar t,\bar y))\,|\bar p+\bar q_x| + c(\bar t,\bar y)\,(|\bar p+\bar q_x|-|\bar p-\bar q_y|)\\
&+M_p^\alpha[u(\bar t,\cdot)](\bar x)\,(|\bar p+\bar q_x|-|\bar p-\bar q_y|)\\
&+\left(M_p^\alpha[u(\bar t,\cdot)](\bar x)-\tilde M_p^\alpha[v(\bar t,\cdot)](\bar y)\right)|\bar p-\bar q_y|\\
\le&\|\nabla c\|_\infty|\bar x-\bar y||\bar p+\bar q_x| + \|c\|_\infty|\bar q_x+\bar q_y| + \left(2\|u\|_\infty+\alpha+\frac12\right)\|J\|_{L^1(\mathbb{R}^N)}|\bar q_x+\bar q_y|\\
&+\left(\int_{\mathbb{R}^N} dz\ J(\bar x-z)\left\{E_*(v(\bar t,z)-v(\bar t,\bar y)+p\cdot(z-\bar y)+\alpha)-p\cdot(z-\bar y)\right\}+\right.\\
&+\int_{\mathbb{R}^N} dz\ J(\bar x-z)(p\cdot(\bar x-\bar y))\\
&\left.-\int_{\mathbb{R}^N} dz\ J(\bar y-z)\left\{E_*(v(\bar t,z)-v(\bar t,\bar y)+p\cdot(z-\bar y)+\alpha)-p\cdot(z-\bar y)\right\}\right)|\bar p-\bar q_y|\\
&+o_\beta(1)|\bar p-\bar q_y|\\
\le& e^{K_0\bar t}\frac{|\bar x-\bar y|^2}{2\gamma}\left(2\|\nabla c\|_\infty+2\|\nabla J\|_{L^1(\mathbb{R}^N)}\left(2\|v\|_\infty+\frac12+\alpha\right)+2\|J\|_{L^1(\mathbb{R}^N)}\right)+o_\beta(1)
\end{aligned}
$$

where we have used the definition of $\bar p$ and that $|\bar q_x|,\ |\bar q_y| = o_\beta(1)$. Taking

$$K_0 = 2\|Dc\|_\infty + 2\|DJ\|_{L^1(\mathbb{R}^N)}(2\|v\|_\infty+\frac12+\alpha)+2\|J\|_{L^1(\mathbb{R}^N)},$$

we get a contradiction for β small enough.

Case 2. there exists a subsequence β_n, such that

$$\frac{|\bar x-\bar y|}{2\gamma}\to 0 \quad \text{as} \quad n\to+\infty.$$

In this case, we have $|\bar p+\bar q_x|\to 0$ and $|\bar p-\bar q_y|\to 0$ as $n\to+\infty$. Sending $n\to+\infty$ in (8.26), we get a contradiction.

This ends the proof of the theorem. □

4.4 Existence results

Théorème 4.5. *Consider $u_0\in W^{2,\infty}(\mathbb{R}^N)$, $c\in W^{1,\infty}([0,+\infty)\times\mathbb{R}^N)$ and $J\in W^{1,1}(\mathbb{R}^N)$. For $\varepsilon>0$, there exists a (unique) bounded continous viscosity solution u^ε of* (8.1). *Moreover, there exists a constant C independent on $\varepsilon>0$ such that,*

$$|u^\varepsilon(t,x)-u_0(x)|\le Ct. \tag{8.29}$$

Proof. As is explained in [112, Proof of Theorem 6] (see also Alvarez, Tourin [11] or Imbert [110, Theorem 3]), to apply the Perron's method for non-local equations, it suffices to prove that there exists a constant $C > 0$ (independent of ε) such that $u_0 \pm Ct$ are respectively a super and a subsolution. The only difficulty is to bound, for every C, the term $\left|M^\varepsilon\left[\frac{u_0(\cdot)+Ct}{\varepsilon}\right]\right|_\infty$ by a constant C_1 independent of C and ε. To do this, it suffices to remark that

$$\begin{aligned}&\left|M^\varepsilon\left[\frac{u_0(\cdot)+Ct}{\varepsilon}\right]\right|_\infty\\ \le&\frac{1}{2}\|J\|_{L^1}+\int_{\mathbb{R}^N}dz\ J(z)\left|\frac{u_0(x+\varepsilon z)}{\varepsilon}-\frac{u_0(x)}{\varepsilon}-\nabla u_0(x)\cdot\frac{z}{\varepsilon}1_B(z)\right|\end{aligned}$$

and to use [112, Proof of Theorem 6] to get a constant $\tilde{C}_1 = \tilde{C}_1(R_0, N, ||u_0||_{W^{2,\infty}})$ such that

$$\int_{\mathbb{R}^N}dz\ J(z)\left|\frac{u_0(x+\varepsilon z)}{\varepsilon}-\frac{u_0(x)}{\varepsilon}-\nabla u_0(x)\cdot\frac{z}{\varepsilon}1_B(z)\right|\le\tilde{C}_1$$

(with R_0 appearing in (8.4)). Then taking $C_1 = \frac{1}{2}\|J\|_{L^1} + \tilde{C}_1$ and $C = (\|c\|_\infty + C_1)\|\nabla u_0\|$, we get that $u_0 \pm Ct$ are respectively a super and a subsolution. This achieves the proof of the theorem. $\square$

We recall the existence and uniqueness result for (8.7).

Théorème 4.6 ([112, Proposition 3])**.** *Assume that $u_0 \in W^{2,\infty}(\mathbb{R}^N)$, $g \ge 0$, $g \in C^0(\mathbf{S}^{N-1})$ and $\overline{H}^0$ is continuous in (L,p) and nondecreasing in L, then the homogenized equation (8.7) has a unique bounded continuous viscosity solution u^0.*

4.5 Consistency of the definition of the geometric motion

As explained in Sections 1 and 3, Eq. (8.1) is the rescaled level set equation corresponding to the motion of N fronts submitted to monotone two-body interactions. The classical level set approach is well adapted for describing the motion of fronts since it can be proved (at least for local equations) that if (8.1) is solved for two initial data u_0 and v_0 that have the same 0-level set, then so have the two corresponding solutions. It turns out that the classical proof of [27] can be adapted to our framework. Before explaining it, let us state precisely the result.

Théorème 4.7. *Consider two bounded uniformly continuous functions u_0, v_0 and two corresponding solutions u and v of (8.1) with $\varepsilon = 1$. Fix any $\alpha \in [0,1)$, and assume that u_0 and v_0 satisfy for any $k \in \mathbb{Z}$*

$$\{u_0 < k+\alpha\} = \{v_0 < k+\alpha\} \qquad \& \qquad \{u_0 > k+\alpha\} = \{v_0 > k+\alpha\}.$$

Then the solutions u and v satisfy

$$\{u(t,\cdot) < k+\alpha\} = \{v(t,\cdot) < k+\alpha\} \qquad \& \qquad \{u(t,\cdot) > k+\alpha\} = \{v(t,\cdot) > k+\alpha\}.$$

The proof of this theorem relies on the invariance of the set of sub/super solutions of a level set equation under the action of monotone semicontinuous functions. Such a result is classical in the level set approach literature.

Proposition 4.8. *Assume that $\theta : \mathbb{R} \to \mathbb{R}$ is nondecreasing and upper semicontinuous (resp. lower semicontinuous). Assume also that*

$$\theta(v) - v \textit{ is } 1 - \textit{periodic in } v. \tag{8.30}$$

Assume that $\varepsilon = 1$ in (8.1). *Consider also a subsolution (resp. supersolution) u of* (8.1). *Then $\theta(u)$ is also a subsolution (resp. supersolution) of* (8.1).

The proof of this proposition is postoned and we now explain how to use it to prove Theorem 4.7.

Proof of Theorem 4.7. We only do the proof for bounded fronts since the general case imply further technicalities we want to avoid, see for instance [121]. This is the reason why we assume that for any $k \in \mathbb{Z}$ (case $\alpha = 0$)

$$\{u_0 = k\} = \{v_0 = k\} \text{ is bounded.}$$

We now follow the lines of the original proof of [27]. Hence, we introduce two nondecreasing functions ϕ and ψ

$$\phi(r) = \begin{cases} \inf\{v_0(y) : u_0(y) \geq r\} & \text{if } r \leq M \\ \phi(M) & \text{if } r > M \end{cases}$$

$$\psi(r) = \begin{cases} \sup\{v_0(y) : u_0(y) < r\} & \text{if } r \geq m \\ \psi(m) & \text{if } r < m \end{cases}$$

where $M = \sup u_0$ and $m = \inf u_0$. It is clear that ϕ is upper semicontinuous and ψ is lower semicontinuous. We now consider increasing extension $\tilde{\phi}$, $\tilde{\psi}$ of ϕ and ψ that satisfy $\tilde{\psi}(k) = k = \tilde{\phi}(k)$ for $k \in \mathbb{Z}$ and define

$$\begin{cases} \bar{\phi}(v) = \inf_{k\in\mathbb{Z}}\{\tilde{\phi}(v+k) - k\} \\ \bar{\psi}(v) = \sup_{k\in\mathbb{Z}}\{\tilde{\psi}(v+k) - k\} \end{cases}$$

(in fact the infimum or supremum are only on finite values of k because u_0 and v_0 are bounded). By noticing that $\bar{\phi}(u_0) \leq v_0 \leq \bar{\psi}(u_0)$ (because $v_0 \leq \Psi(u_0+\varepsilon)$ for any $\varepsilon > 0$) and using Proposition 4.8, we conclude that $\bar{\phi}(u) \leq v \leq \bar{\psi}(u)$. It is now easy to conclude. □

It remains to prove Proposition 4.8.

Proof of Proposition 4.8. We just need to check that the non-local term can be handled in the classical proof. We only treat the case of subsolutions. Consider first $\theta \in C^1(\mathbb{R})$ such that $\theta' > 0$. Consider φ a test function from above satisfying $\theta(u) \leq \varphi$ with equality at (t_0, x_0). Then $u \leq \theta^{-1}(\varphi)$ and

$$\partial_t(\theta^{-1}(\varphi)) \leq c[u]|\nabla_x \theta^{-1}(\varphi)|$$

with

$$c[u] = c(t_0, x_0) + M[u(t_0, \cdot)](x_0).$$

Because from (8.22)

$$E(u(t_0, x_0 + z) - u(t_0, x_0)) \leq E(\theta(u(t_0, x_0 + z)) - \theta(u(t_0, x_0)))$$

we deduce that

$$\partial_t \varphi \leq c[\theta(u)]|\nabla_x \varphi|,$$

i.e. $\theta(u)$ is a subsolution in the sense of Definition 4.1.

In the general case, use the following lemma whose proof is left to the reader.

Lemma 4.9. *For a usc nondecreasing function θ, there exists $\theta^\varepsilon \in C^1$ such that $(\theta^\varepsilon)' > 0$, $\theta^\varepsilon \geq \theta$ and $\limsup^* \theta^\varepsilon = \theta$.*

On one hand, one can prove that such an approximation satisfies $\limsup^* \theta^\varepsilon(u) = \theta(u)$. On the other hand, θ^ε still satisfies (8.30). Hence $\theta^\varepsilon(u)$ is a subsolution of (8.1) by the previous case and we conclude that so is $\theta(u)$ by the stability result. □

5 Ergodicity

As explained in the Introduction, we will need in the proof of convergence to add a parameter $\alpha > 0$ in the cell problem. For the solution w of

$$\begin{cases} \partial_\tau w = \big(c(\tau, y) + L + M_p^\alpha[w(\tau, \cdot)](y)\big)\,|p + \nabla_y w| & \text{in} \quad (0, +\infty) \times \mathbb{R}^N \\ w(0, y) = 0 & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{8.31}$$

we prove a result that is stronger than Theorem 2.1.

Théorème 5.1 (Estimates for the initial value problem)**.** *There exists a unique $\lambda = \lambda(L, p, \alpha)$ such that the (unique) bounded continuous viscosity solution $w \in C([0, +\infty) \times \mathbb{R}^N)$ of* (8.31) *satisfies :*

$$|w(\tau, y) - \lambda\tau| \leq C_3, \tag{8.32}$$

$$|w(\tau, y) - w(\tau, z)| \leq C_1, \quad \textit{for all } y,\ z \in \mathbb{R}^N \tag{8.33}$$

$$|\lambda - (L + \alpha\|J\|_{L^1})|p|| \leq (\|c\|_\infty + \frac{1}{2}\|J\|_{L^1})|p| =: C_2 \tag{8.34}$$

where

$$C_1 = \frac{(2\|c\|_\infty + \|J\|_{L^1})}{c_0}, \quad C_3 = 5C_1 + 2C_2,$$

$$c_0 = \inf_{\delta \in [0,1/2)^N} \int_{\mathbb{R}^N} dz \ \min\left(J(z-\delta), J(z+\delta)\right) > 0. \tag{8.35}$$

In the case $N = 1$, we can choose :

$$C_1 = |p|. \tag{8.36}$$

Theorem 2.1 is a consequence of (8.32). The existence of bounded solutions $v(\tau, y) = w(\tau, y) - \lambda\tau$ of

$$\lambda + \partial_\tau v = \big(c(\tau, y) + L + M_p^\alpha[v(\tau, \cdot)](y)\big)\, |p + \nabla_y v| \quad \text{on } (0, +\infty) \times \mathbb{R}^N \tag{8.37}$$

is a straighforward consequence of the previous theorem :

Corollary 5.2 (Existence of bounded correctors)**.** *There exists a solution v of* (8.37) *that satifies :*

$$|v(\tau, y)| \le C_3,$$

$$|v(\tau, y) - v(\tau, z)| \le C_1.$$

Remark 5.3. *To construct periodic sub and supersolution of* (8.11) *we can also classically consider*

$$\delta v^\delta + v^\delta_\tau = \big(c_1(\tau, y) + L + M_p[v^\delta(\tau, \cdot)](y)\big)\, |p + \nabla_y v^\delta|$$

and take the limit $\delta \to 0$ to obtain exact correctors.

In order to solve the homogenized equation and to prove the convergence theorem, further properties of the number λ given by Theorem 5.1 are needed.

Corollary 5.4 (Properties of the effective Hamiltonian)**.** *The real number λ defines a continuous function $\overline{H} : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ that satisfies :*

$$\overline{H}(L, p, \alpha) \to \pm\infty \quad \textit{as} \quad L \to \pm\infty, \tag{8.38}$$

$$\overline{H}(L, p, \alpha) \to \pm\infty \quad \textit{as} \quad \alpha \to \pm\infty. \tag{8.39}$$

Moreover, $\overline{H}(L, p, \alpha)$ is nonincreasing in L and α.

In a first subsection, we successively prove Theorem 5.1 in dimension $N \ge 2$ and Corollary 5.4. Theorem 5.1 in the case $N = 1$ will be proved in a second subsection.

5.1 Proof of Theorem 5.1 and Corollary 5.4

Proof of Theorem 5.1 in the case $N \geq 2$. We proceed in several steps.

STEP 1 : BARRRIERS AND EXISTENCE OF A SOLUTION. We proceed as in the proof of Theorem 4.5. We remark that $w^{\pm}(\tau, y) = C^{\pm}\tau$ with $C^{\pm} = (L + \alpha\|J\|_{L^1})|p| \pm C_2$ (with C_2 defined in (8.34)) are respectively a super- and a subsolution of (8.31). Hence, there exists a unique bounded continuous viscosity solution of (8.31) that satisfies :

$$|w(\tau, y) - (L + \alpha\|J\|_{L^1})|p|\tau| \leq (\|c\|_\infty + \frac{1}{2}\|J\|_{L^1})|p|\tau. \tag{8.40}$$

Remark that by uniqueness, w is $\mathbb{Z}^N$-periodic with respect to y.

STEP 2 : CONTROL OF THE OSCILLATIONS W.R.T. SPACE, UNIFORMLY IN TIME. We proceed as in [112] by considering the functions $M(\tau)$, $m(\tau)$ and $q(\tau)$ defined by :

$$M(\tau) = \sup_{y \in \mathbb{R}^N} w(\tau, y), \quad m(\tau) = \inf_{y \in \mathbb{R}^N} w(\tau, y) \quad \text{and} \quad q(\tau) = M(\tau) - m(\tau) \geq 0.$$

The supremum and infimum are attained since w is 1-periodic with respect to y. In particular, we can assume that :

$$M(\tau) = w(\tau, Y_\tau) \quad \text{and} \quad m(\tau) = w(\tau, y_\tau) \quad \text{and} \quad Y_\tau - y_\tau \in [0,1)^N.$$

Now m, M and q satisfy in the viscosity sense :

$$\begin{aligned}
\frac{dM}{d\tau}(\tau) \leq& \Big(c(\tau, Y_\tau) + L + M_p^\alpha[w(\tau,\cdot)](Y_\tau)\Big)|p| \\
\leq& (\|c\|_\infty + \frac{1}{2}\|J\|_{L^1} + L + \alpha\|J\|_{L^1})|p| \\
&+ \int dz\, J(z)(w(\tau, Y_\tau + z) - w(\tau, Y_\tau))|p|, \\
\frac{dm}{d\tau}(\tau) \geq& \Big(c(\tau, y_\tau) + L + M_p^\alpha[w(\tau,\cdot)](y_\tau)\Big)|p| \\
\geq& (-\|c\|_\infty - \frac{1}{2}\|J\|_{L^1} + L + \alpha\|J\|_{L^1})|p| \\
&+ \int dz\, J(z)(w(\tau, y_\tau + z) - w(\tau, y_\tau))|p|, \\
\frac{dq}{d\tau}(\tau) \leq& (2\|c\|_\infty + \|J\|_{L^1})|p| + \mathcal{L}(\tau)|p|
\end{aligned}$$

where

$$\mathcal{L}(\tau) = \int dz\, J(z)(w(\tau, Y_\tau + z) - w(\tau, Y_\tau)) - \int dz\, J(z)(w(\tau, y_\tau + z) - w(\tau, y_\tau)).$$

Let us estimate $\mathcal{L}(\tau)$ from above by using the definition of y_τ and Y_τ. To do so, let us introduce $\delta_\tau = \frac{Y_\tau - y_\tau}{2} \in [0, \frac{1}{2})^N$ and $c_\tau = \frac{Y_\tau + y_\tau}{2}$ and write :

$$\begin{aligned}\mathcal{L}(\tau) &= \int dz\, J(z-\delta_\tau)(w(\tau, c_\tau + z) - w(\tau, Y_\tau))\\ &- \int dz\, J(z+\delta_\tau)(w(\tau, c_\tau + z) - w(\tau, y_\tau))\\ &\le \min_{\delta\in[0,\frac{1}{2})^N} \int dz\, \min\{J(z-\delta), J(z+\delta)\}(w(\tau, y_\tau) - w(\tau, Y_\tau)) = -c_0 q(\tau).\end{aligned}$$

We conclude that q satisfies in the viscosity sense :

$$\frac{dq}{d\tau}(\tau) \le (2\|c\|_\infty + \|J\|_{L^1})|p| - c_0|p|q(\tau)$$

with $q(0) = 0$ from which we obtain $q(\tau) \le C_1$ for any $\tau \ge 0$ which can be rewritten under the following form :

$$|w(\tau, y) - w(\tau, z)| \le C_1 \quad \text{for any } \tau \ge 0, \quad y, z \in \mathbb{R}^N. \tag{8.41}$$

Step 3 : control of the oscillations w.r.t. time. We keep following the construction of the correctors of [112] by introducing, in order to estimate oscillations w.r.t. time, the two quantities :

$$\lambda^+(T) = \sup_{\tau\ge 0} \frac{w(\tau+T, 0) - w(\tau, 0)}{T} \quad \text{and} \quad \lambda^-(T) = \inf_{\tau\ge 0} \frac{w(\tau+T, 0) - w(\tau, 0)}{T}$$

and proving that they have a common limit as $T \to +\infty$. In order to do so, we first estimate λ^+ from above. This is a consequence of the comparison principle for (8.31) on the time interval $[\tau, \tau+\tau_0]$ for every $\tau_0 > 0$: since $w(\tau, 0) + C_1 + w^+(t)$ is a supersolution, we get for $t \in [0, \tau_0]$:

$$w(\tau + t, y) \le w(\tau, 0) + C_1 + C^+ t \tag{8.42}$$

where $C^\pm = (L + \alpha\|J\|_{L^1})|p| \pm C_2$. Similarly, we get

$$w(\tau, 0) - C_1 + C^- t \le w(\tau + t, y). \tag{8.43}$$

We then obtain for $\tau_0 = t = T$:

$$(L + \alpha\|J\|_{L^1})|p| - C_2 - \frac{C_1}{T} \le \lambda^-(T) \le \lambda^+(T) \le (L + \alpha\|J\|_{L^1})|p| + C_2 + \frac{C_1}{T}$$

By definition of $\lambda^\pm(T)$, for any $\delta > 0$, there exists $\tau^\pm \ge 0$ such that

$$\left|\lambda^\pm(T) - \frac{w(\tau^\pm + T, 0) - w(\tau^\pm, 0)}{T}\right| \le \delta.$$

Let us consider $\beta \in [0,1)$ such that $\tau^+ - \tau^- - \beta = k$ is an integer. Next, consider $\Delta = w(\tau^+,0) - w(\tau^- + \beta, 0)$. From (8.41), we get :

$$w(\tau^+, y) \leq w(\tau^- + \beta, y) + 2C_1 + \Delta = w(\tau^+ - k, y) + 2C_1 + \Delta.$$

The comparison principle for (8.31) on the time interval $[\tau^+, \tau^+ + T]$ (using the fact that $c(\tau, y)$ is $\mathbb{Z}$-periodic in τ) therefore implies that :

$$\begin{array}{rcl} w(\tau^+ + T, y) & \leq & w(\tau^+ - k + T, y) + 2C_1 + \Delta \\ & = & w(\tau^- + \beta + T, y) + 2C_1 + w(\tau^+, 0) - w(\tau^- + \beta, 0). \end{array}$$

Choosing $y = 0$ in the previous inequality yields :

$$w(\tau^+ + T, 0) - w(\tau^+, 0) \leq w(\tau^- + \beta + T, 0) - w(\tau^- + \beta, 0) + 2C_1$$

and setting $t = \beta \leq 1$ and $\tau = \tau^- + T$ in (8.42) and $\tau = \tau^-$ in (8.43) finally yields :

$$T\lambda^+(T) \leq T\lambda^-(T) + 2\delta + 2(C_1 + C_2) + 2C_1.$$

Since this is true for any $\delta > 0$, we conclude that :

$$|\lambda^+(T) - \lambda^-(T)| \leq \frac{4C_1 + 2C_2}{T}.$$

Now arguing as in [111, 112], we conclude that $\lim_{T\to+\infty} \lambda^\pm(T)$ exist and are equal to λ and :

$$|\lambda^\pm(T) - \lambda| \leq \frac{4C_1 + 2C_2}{T}. \tag{8.44}$$

STEP 4 : CONCLUSION. Estimate (8.40) implies (8.34). From (8.44), we conclude that :

$$\left| \frac{w(\tau + T, 0) - w(\tau, 0)}{T} - \lambda \right| \leq \frac{4C_1 + 2C_2}{T}$$

which implies that :

$$|w(T, 0) - \lambda T| \leq 4C_1 + 2C_2$$

and finally (8.32) derives from this inequality and (8.41). The uniqueness of λ follows from (8.32) for instance. □

Proof of Corollary 5.4. The only point to be proved is the continuity of $\overline{H}$ since (8.34) implies the other properties. Let us consider a sequence (L_n, p_n, α_n) such that $(L_n, p_n, \alpha_n) \to (L_0, p_0, \alpha_0)$ and set $\lambda_n = \lambda(L_n, p_n, \alpha_n)$. We remark that by (8.32), we have for any $\tau > 0$

$$\left| \lambda_n - \frac{w_n(\tau, 0)}{\tau} \right| \leq \frac{C_3}{\tau}$$

for some constant C_3 that we can choose independent on n. Stability of viscosity solutions for (8.31) implies that $w_n \to w_0$ locally uniformly w.r.t. (τ, y). This implies that $\limsup_{n\to+\infty} |\lambda_n - \lambda_0| \leq \frac{2C_3}{\tau}$ for any $\tau > 0$. Hence, we conclude that $\lim_{n\to+\infty} \lambda_n = \lambda_0$.

Finally the monotonicity in L and α of $\overline{H}(L,p,\alpha)$ comes from the comparison priniciple.

□

5.2 Proof of Theorem 5.1 in the case $N = 1$

Before proving Theorem 5.1 in the one dimensional case, we need the following lemma :

Lemma 5.5. *Let w be the solution of* (8.31) *in dimension $N = 1$. Then the function $y \mapsto p(p \cdot y + w(\tau, y))$ is nondecreasing for any $\tau \geq 0$, i.e.,*

$$p(p + w_y(\tau, y)) \geq 0.$$

Proof. We only do the proof in the case $p > 0$, since the case $p < 0$ is similar and the case $p = 0$ is trivial. We want to prove that

$$M_0 = \inf_{\Omega_T} \{w(\tau, x) - w(\tau, y) + p \cdot (x - y)\} \geq 0$$

where $\Omega_T = \{(\tau, x, y),\ 0 \leq \tau \leq T,\ y \leq x\}$. By contradiction, assume that $M_0 \leq -\delta < 0$. For $\eta > 0$, we consider

$$\Phi_\eta(\tau, x, y) = w(\tau, x) - w(\tau, y) + p \cdot (x - y) + \frac{\eta}{T - \tau}$$

and

$$M_\eta = \inf_{\Omega_T} \Phi_\eta(\tau, x, y) \leq -\frac{\delta}{2} \tag{8.45}$$

for η small enough.

By the space periodicity of w, we remark that

$$\begin{cases} \Phi_\eta(\tau, x+1, y+1) = \Phi_\eta(\tau, x, y) \\ \Phi_\eta(\tau, x-1, y) = \Phi_\eta(\tau, x, y) - p \quad \text{if } x - 1 \geq y \end{cases}$$

so the minimum is reached at a point $(\bar{\tau}, \bar{x}, \bar{y})$ with $0 \leq \bar{x} - \bar{y} < 1$ and $\bar{\tau} < T$ (because w is bounded by the barrier functions). Moreover $\bar{x} > \bar{y}$ and $\bar{t} > 0$; indeed, otherwise we can check easily that the minimum would be nonnegative. Using Ishii's Lemma (see Crandall, Ishii, Lions [61, Lemma 8.3]), we then get that there exist $(a, -p) \in \overline{D}^+ w(\bar{\tau}, \bar{y})$ and $(b, -p) \in \overline{D}^- w(\bar{\tau}, \bar{x})$ with

$$a - b = \frac{\eta}{(T - \bar{\tau})^2}$$

such that (see equation (8.31))

$$a \leq 0 \quad \text{and} \quad b \geq 0.$$

Subtracting the two above inequalities yields a contradiction.

□

Proof of Theorem 5.1 in the case $N = 1$. Let us assume that $p > 0$ since the case $p < 0$ is proved similarly and the case $p = 0$ is trivial.

Consider the solution w of (8.31). Lemma 5.5 ensures that $u(\tau, y) = w(\tau, y) + p \cdot y$ is nondecreasing :

$$\nabla_y u \geq 0.$$

To simplify the notation, let us drop the time dependence. We also know that w is 1-periodic in y, so for all $0 \leq y \leq z \leq 1$, we have

$$py + w(y) \leq pz + w(z) \leq p(y+1) + w(y).$$

This implies that

$$|w(y) - w(z)| \leq p.$$

The rest of the proof in the same as in the case $N \geq 2$. □

6 The proof of convergence

This Section is devoted to the proof of Theorem 2.5.

Proof of Theorem 2.5. We consider the upper semicontinuous function

$$\overline{u} = \limsup{}^* u^\varepsilon.$$

By Theorem 4.5, it is bounded for bounded times and $\overline{u}(0, x) = u_0(x)$. As usual, we are going to prove that it is a subsolution of (8.7). Similarly, we can prove that $\underline{u} = \liminf{}_* u^\varepsilon$ is a bounded supersolution of (8.7) such that $\underline{u}(0, x) = u_0(x)$. Theorem 4.6 thus yield the result.

Let us prove that $\overline{u}$ is a subsolution of (8.7). We argue (classically) by contradiction by assuming that there exists a point (t_0, x_0), $t_0 > 0$, and a test function $\phi \in C^2$ such that $\overline{u} - \phi$ attains a global zero strict maximum at (t_0, x_0) and :

$$\partial_t \phi(t_0, x_0) = \overline{H}^0(L_0, p) + \theta = \overline{H}(L_0, p, 0) + \theta$$

with

$$\begin{aligned} L_0 = & \int_{|z| \leq 2} \{\phi(t_0, x_0 + z) - \phi(t_0, x_0) - \nabla \phi(t_0, x_0) \cdot z\} \mu(dz) \\ & + \int_{|z| \geq 2} \{\overline{u}(t_0, x_0 + z) - \overline{u}(t_0, x_0)\} \mu(dz) \end{aligned} \tag{8.46}$$

and $p = \nabla\phi(t_0, x_0)$ and $\theta > 0$. By Corollary 5.4, there exists $\alpha > 0$ and $\beta > 0$ such that :

$$\partial_t\phi(t_0, x_0) = \overline{H}(L_0 + \beta, p, \alpha) + \frac{\theta}{2}. \tag{8.47}$$

In the following, λ denotes $\overline{H}(L_0 + \beta, p, \alpha)$.

We now construct a supersolution ϕ^ε of (8.1) on a small ball centered at (t_0, x_0) by using the perturbed test function method (see [77, 78]). Precisely, we consider :

$$\phi^\varepsilon(t,x) = \begin{cases} \phi(t,x) + \varepsilon\, v\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) - \eta_r & \text{if } (t,x) \in (t_0/2, 2t_0) \times B_1(x_0), \\ u^\varepsilon(t,x) & \text{if not} \end{cases}$$

where η_r is chosen later and the corrector v is a bounded solution of (8.37) associated with $(L, p, \alpha) = (L_0 + \beta, p, \alpha)$ given by Corollary 5.2. We will prove that ϕ^ε is a supersolution of (8.1) on $B_r(t_0, x_0)$ (for r and ε small enough – this is made precise later) and that $\phi^\varepsilon \geq u^\varepsilon$ outside. In particular, r is chosen small enough so that $B_r(t_0, x_0) \subset (t_0/2, 2t_0) \times B_1(x_0)$.

Let us first focus on “boundary conditions”. For ε small enough ($i.e.$ $0 < \varepsilon \leq \varepsilon_0(r) < r$), since $\overline{u} - \phi$ attains a strict maximum at (t_0, x_0), we can ensure that :

$$u^\varepsilon(t,x) \leq \phi(t,x) + \varepsilon v\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) - \eta_r \quad \text{for } (t,x) \in (t_0/3, 3t_0) \times B_3(x_0) \setminus B_r(t_0, x_0) \tag{8.48}$$

for some $\eta_r = o_r(1) > 0$. Hence, we conclude that $\phi^\varepsilon \geq u^\varepsilon$ outside of $B_r(t_0, x_0)$.

We now turn to the equation. Consider a test function ψ such that $\phi^\varepsilon - \psi$ attains a local minimum at $(\bar{t}, \overline{x}) \in B_r(t_0, x_0)$. This implies that $v - \Gamma$ attains a local minimum at $(\overline{\tau}, \overline{y})$ where $\overline{\tau} = \frac{\bar{t}}{\varepsilon}$, $\overline{y} = \frac{\overline{x}}{\varepsilon}$ and

$$\Gamma(\tau, y) = \frac{1}{\varepsilon}(\psi - \phi)(\varepsilon\tau, \varepsilon y).$$

Since v is a viscosity solution of (8.37), we conclude that :

$$\lambda + \partial_\tau\Gamma(\overline{\tau}, \overline{y}) \geq \left(c(\overline{\tau}, \overline{y}) + L + \tilde{M}_p^\alpha[v(\overline{\tau}, \cdot)](\overline{y})\right)|p + \nabla\Gamma(\overline{\tau}, \overline{y})|$$

from which we deduce :

$$\begin{aligned}&\left(\partial_t\phi(t_0, x_0) - \frac{\theta}{2}\right) + \partial_t\psi(\bar{t}, \overline{x}) - \partial_t\phi(\bar{t}, \overline{x}) \\ \geq &\left(c(\overline{\tau}, \overline{y}) + L + \tilde{M}_p^\alpha[v(\overline{\tau}, \cdot)](\overline{y})\right) \left|\nabla\psi(\bar{t}, \overline{x}) - (\nabla\phi(\bar{t}, \overline{x}) - \nabla\phi(t_0, x_0))\right|.\end{aligned}$$

Hence, we get :

$$\partial_t\psi(\bar t,\bar x)\geq\left(c\left(\frac{\bar t}{\varepsilon},\frac{\bar x}{\varepsilon}\right)+L_0+\beta+\tilde M_p^\alpha\left[v\left(\frac{\bar t}{\varepsilon},\cdot\right)\right]\left(\frac{\bar x}{\varepsilon}\right)\right)|\nabla\psi(\bar t,\bar x)+o_r(1)|+\frac{\theta}{2}+o_r(1)$$

where $o_r(1)$ only depends on local bounds of ϕ and its derivatives. Since c is bounded and :

$$|\tilde M_p^\alpha[v(\overline{\tau},\cdot)](\overline{y})|\leq(\text{osc } v(\tau,\cdot)+\alpha\|J\|_{L^1})+\frac{1}{2}\|J\|_{L^1}\leq\left(C_1+\alpha+\frac{1}{2}\right)\|J\|_{L^1},$$

where C_1 is given in Theorem 5.1, we conclude that :

$$\partial_t\psi(\bar t,\bar x)\geq\left(c\left(\frac{\bar t}{\varepsilon},\frac{\bar x}{\varepsilon}\right)+L_0+\beta+\tilde M_p^\alpha\left[v\left(\frac{\bar t}{\varepsilon},\cdot\right)\right]\left(\frac{\bar x}{\varepsilon}\right)\right)|\nabla\psi(\bar t,\bar x)|. \tag{8.49}$$

Now recall that L_0 is defined by (8.46) and use the following lemma :

Lemma 6.1. *There exists $\varepsilon_0>0$ such that for any $\varepsilon\leq\varepsilon_0$, $r\leq r_0$ and $(t,x)\in B_r(t_0,x_0)$:*

$$\tilde M^\varepsilon\left[\frac{\phi^\varepsilon(t,\cdot)}{\varepsilon}\right](x)\leq L_0+\beta+\tilde M_p^\alpha\left[v\left(\frac{t}{\varepsilon},\cdot\right)\right]\left(\frac{x}{\varepsilon}\right). \tag{8.50}$$

The proof of this lemma is postponed. Combining (8.49) and (8.50), we conclude that for any $\varepsilon\leq\varepsilon_0$ and $r\leq r_0$:

$$\partial_t\psi(\bar t,\bar x)\geq\left(c\left(\frac{\bar t}{\varepsilon},\frac{\bar x}{\varepsilon}\right)+\tilde M^\varepsilon\left[\frac{\phi^\varepsilon(t,\cdot)(x)}{\varepsilon}\right]\right)|\nabla\psi(\bar t,\bar x)|.$$

We conclude that ϕ^ε is a supersolution of (8.1) on $B_r(t_0,x_0)$ and $\phi^\varepsilon\geq u^\varepsilon$ outside. Using the comparison principle, this implies $\phi^\varepsilon(t,x)\geq u^\varepsilon(t,x)$. Passing to the supremum limit at the point (t_0,x_0), we obtain : $\phi(t_0,x_0)\geq\overline{u}(t_0,x_0)+\eta_r$ which is a contradiction. The proof of Theorem 2.5 is now complete. □

It remains to prove Lemma 6.1.

Proof of Lemma 6.1. It is convenient to use the notation : $\tau=t/\varepsilon$ and $y=x/\varepsilon$. We simply divide the domain of integration in two parts : short range interaction and long range interaction. Precisely :

$$\begin{aligned}\tilde M^\varepsilon\left[\frac{\phi^\varepsilon(t,\cdot)(x)}{\varepsilon}\right] &= \int dz\,J(z)\,E_*\left(\frac{\phi^\varepsilon(t,x+\varepsilon z)-\phi^\varepsilon(t,x)}{\varepsilon}\right)\\ &= \int_{|z|\leq r_\varepsilon}dz\,\{\ldots\}+\int_{\varepsilon|z|\geq\varepsilon r_\varepsilon}dz\,\{\ldots\}=T_1+T_2\end{aligned}$$

and we choose r_ε such that $r_\varepsilon \to +\infty$ and $\varepsilon r_\varepsilon \to 0$ as $\varepsilon \to 0$. Let us estimate from above each term. For $(t,x) \in B_r(t_0,x_0)$ and $|z| \le r_\varepsilon$, we are sure that $(t, x+\varepsilon z) \in (t_0/2, 2t_0) \times B_1(x_0)$ for ε small enough, and :

$$\begin{aligned}
T_1 &= \int_{|z|\le r_\varepsilon} dz\, J(z)\, E_*\left(\frac{\phi^\varepsilon(t,x+\varepsilon z) - \phi^\varepsilon(t,x)}{\varepsilon}\right)\\
&= \int_{|z|\le r_\varepsilon} dz\, J(z)\, E_*\left(\frac{\phi(t,x+\varepsilon z) - \phi(t,x)}{\varepsilon} + v(\tau,y+z) - v(\tau,y)\right)\\
&= \int_{|z|\le r_\varepsilon} dz\, J(z) \Bigg\{ E_*\Bigg(\nabla\phi(t,x)\cdot z + v(\tau,y+z) - v(\tau,y)\\
&\qquad\qquad + \frac{\phi(t,x+\varepsilon z) - \phi(t,x) - \varepsilon\nabla\phi(t,x)\cdot z}{\varepsilon}\Bigg) - \nabla\phi(t,x)\cdot z\Bigg\}.
\end{aligned}$$

To get the last line of the previous inequality, we used that J is even. Choose next ε small enough and r_ε big enough so that

$$\frac{\phi(t,x+\varepsilon z) - \phi(t,x) - \varepsilon\nabla\phi(t,x)\cdot z}{\varepsilon} \le C\varepsilon(r^\varepsilon)^2 \le \frac{\alpha}{2}$$

$$\int_{|z|\ge r_\varepsilon} dz\; J(z)\left\{E_*\left(v\left(\cdot, \frac{x}{\varepsilon} + z\right) - v\left(\cdot, \frac{x}{\varepsilon}\right) + \frac{\alpha}{2} + z\cdot\nabla\phi(t,x)\right) - z\cdot\nabla\phi(t,x)\right\} \le \frac{\beta}{4}.$$

Hence we obtain

$$T_1 \le \tilde{M}^{\alpha/2}_{\nabla\phi(t,x)}\left[v\left(\frac{t}{\varepsilon},\cdot\right)\right]\left(\frac{x}{\varepsilon}\right) + \frac{\beta}{4}. \tag{8.51}$$

We now claim that :

$$\tilde{M}^{\alpha/2}_{\nabla\phi(t,x)}\left[v\left(\frac{t}{\varepsilon},\cdot\right)\right]\left(\frac{x}{t}\right) \le \tilde{M}^{\alpha}_{\nabla\phi(t_0,x_0)}\left[v\left(\frac{t}{\varepsilon},\cdot\right)\right]\left(\frac{x}{t}\right) + \beta/4 \tag{8.52}$$

for r small enough. To see this, consider $R_\beta > 0$ such that :

$$\int_{|z|\ge R_\beta} dz\, J(z)\{E_*(\nabla\phi(t,x)\cdot z + v(\tau,y+z) - v(\tau,y)) - \nabla\phi(t,x)\cdot z\} \le \beta/8,$$

$$\int_{|z|\ge R_\beta} dz\, J(z)\{E_*(\nabla\phi(t_0,x_0)\cdot z + v(\tau,y+z) - v(\tau,y)) - \nabla\phi(t_0,x_0)\cdot z\} \le \beta/8.$$

Now for $|z| \le R_\beta$ and r small enough :

$$|(\nabla\phi(t,x)\cdot z - \nabla\phi(t_0,x_0))\cdot z| \le \alpha/2$$

and we get (8.52). Combining this inequality with (8.51), we obtain :

$$T_1 \le \tilde{M}_p^\alpha\left[v\left(\frac{t}{\varepsilon},\cdot\right)\right]\left(\frac{x}{t}\right)+\beta/2. \tag{8.53}$$

We now turn to T_2. We can choose $r_\varepsilon \ge R_0$ where R_0 appears in (8.4) so that $J(z)=g(z/|z|)|z|^{-N-1}$ for $|z| \ \ge r_\varepsilon$. Hence

$$\begin{aligned}T_2 &= \int_{\varepsilon|z|\ge\varepsilon r_\varepsilon} dz\, J(z)\, E_*\left(\frac{\phi^\varepsilon(t,x+\varepsilon z)-\phi^\varepsilon(t,x)}{\varepsilon}\right)\\ &= \int_{|q|\ge\varepsilon r_\varepsilon} \mu(dq)\, E_*^\varepsilon(\phi^\varepsilon(t,x+q)-\phi^\varepsilon(t,x))\end{aligned}$$

with μ defined by (8.10) and where $E_*^\varepsilon(\alpha)=\varepsilon E_*\left(\frac{\alpha}{\varepsilon}\right)$. Remark that $|E_*^\varepsilon(\alpha)-\alpha|\le\frac{\varepsilon}{2}$ and use (8.48) to get :

$$\begin{aligned}T_2 \ &\le \ \int_{|q|\ge\varepsilon r_\varepsilon}\mu(dq)\,(\phi^\varepsilon(t,x+q)-\phi^\varepsilon(t,x))+\frac{\varepsilon}{2}\mu(\mathbb{R}^N\setminus B_{\varepsilon r_\varepsilon})\\ &\le \ \int_{\varepsilon r_\varepsilon\le|q|\le 2}\mu(dq)\,\{\phi(t,x+q)-\phi(t,x)+\varepsilon\operatorname{osc} v(\tau,\cdot)\}\\ &\quad +\int_{|q|\ge 2}\mu(dq)\,\{u^\varepsilon(t,x+q)-\phi(t,x)\}+\frac{C}{r_\varepsilon}+C'\eta_r.\end{aligned}$$

Now use that μ is even and get

$$\begin{aligned}T_2 \ &\le \ \int_{\varepsilon r_\varepsilon\le|q|\le 2}\mu(dq)\,\{\phi(t,x+q)-\phi(t,x)-\nabla\phi(t,x)\cdot q\}\\ &\quad +\int_{|q|\ge 2}\mu(dq)\,\{u^\varepsilon(t,x+q)-\phi(t,x)\}+C'\left(\frac{1}{r_\varepsilon}+\varepsilon+\eta_r\right).\end{aligned}$$

Now remark that :

$$\begin{aligned}&\int_{\varepsilon r_\varepsilon\le|q|\le 2}\mu(dq)\,\{\phi(t,x+q)-\phi(t,x)-\nabla\phi(t,x)\cdot q\}\\ \le &\int_{|q|\le 2}\mu(dq)\,\{\phi(t_0,x_0+q)-\phi(t_0,x_0)-\nabla\phi(t_0,x_0)\cdot q\}+C\varepsilon r_\varepsilon+o_r(1)\end{aligned}$$

and keeping in mind that $\phi(t_0,x_0)=\overline{u}(t_0,x_0)$, we also have

$$\int_{2\le|q|}\mu(dq)\,\{u^\varepsilon(t,x+q)-\phi(t,x)\}\le\int_{2\le|q|}\mu(dq)\,\{\overline{u}(t_0,x_0+q)-\overline{u}(t_0,x_0)\}+o_r(1).$$

Indeed, it is equivalent to

$$\limsup_{y\to x_0,\varepsilon\to 0}\int_{2\le|q|}\mu(dq)\left\{u^\varepsilon(t,y+q)-\phi(t,y)\right\}\le\int_{2\le|q|}\mu(dq)\left\{\overline{u}(t_0,x_0+q)-\overline{u}(t_0,x_0)\right\}$$

and such an inequality is a consequence of Fatou's lemma.

Combining all the estimates yields :

$$\begin{aligned}T_2\le&\int_{|q|\le 2}\mu(dq)\left\{\phi(t_0,x_0+q)-\phi(t_0,x_0)-\nabla\phi(t_0,x_0)\cdot q\right\}\\&+\int_{2\le|q|}\mu(dq)\left\{\overline{u}(t_0,x_0+q)-\overline{u}(t_0,x_0)\right\}+C\varepsilon r_\varepsilon+C'\left(\frac{1}{r_\varepsilon}+\varepsilon\right)+o_r(1)+\beta/4\\\le& L_0+\beta/2.\end{aligned}\tag{8.54}$$

Combining (8.53) and (8.54) yields (8.50). □

7 Qualitative properties of the effective Hamiltonian

In this section, we consider the special case of the one-dimensional space and of a driving force independent of time : $c(\tau,y)=c(y)$. Before proving Theorem 2.6 in Subsection 7.3, we establish gradient estimates in Subsection 7.1 and then construct sub/super/correctors independent on time in Subsection 7.2.

7.1 Gradient estimates

We recall that w denotes the solution of the following Cauchy problem

$$\begin{cases}\partial_t w=\Big(c(x)+L+M_p[w(t,\cdot)](x)\Big)|p+\nabla w| & \text{in}\quad (0,+\infty)\times\mathbb{R}^N,\\ w(0,x)=0 & \text{on}\quad \mathbb{R}^N.\end{cases}\tag{8.55}$$

Lemma 7.1 (Lipschitz estimates on the solution)**.** *The solution w of* (8.55) *is Lipschitz continuous w.r.t. x and satisfies :*

$$\|p+\nabla_x w(t,\cdot)\|_\infty\le|p|e^{t\|\nabla c\|_\infty}.\tag{8.56}$$

Proof. The function $u(t,x)=p\cdot x+w(t,x)$ is a solution of

$$\partial_t u=\left(c(x)+L+M[u(t,\cdot)](x)\right)|\nabla u|\quad\text{on }(0,+\infty)\times\mathbb{R}.\tag{8.57}$$

Consider the sup-convolution of u :

$$u^\beta(t,x) = \sup_{y\in\mathbb{R}^N}\left\{u(t,y) - e^{Kt}\frac{|x-y|^2}{2\beta}\right\} = u(t,x_\beta) - e^{Kt}\frac{|x-x_\beta|^2}{2\beta}. \tag{8.58}$$

We claim that u^β is a subsolution of the equation (8.57) for K large enough. Indeed, for any $(\eta,q)\in D^{1,+}u^\beta(t,x)$, it is classical that $\left(\eta + Ke^{Kt}\ \frac{|x-x_\beta|^2}{2\beta}, q\right)\in D^{1,+}u(t,x_\beta)$, $q = -e^{Kt}\frac{x-x_\beta}{\beta}$ and $|x-x_\beta|\ \le C\sqrt{\beta}$ where C depends on $\|w\|_\infty$, so that :

$$\begin{aligned}\eta + Ke^{Kt}\frac{|x-x_\beta|^2}{2\beta} &\le (c(x_\beta) + M_p[w(t,\cdot)](x_\beta))\,|q| \\ &\le \big(c(x) + M_p[w^\beta(t,\cdot)](x)\big)\,|q| + \|\nabla c\|_\infty e^{Kt}\frac{|x-x_\beta|^2}{\beta}\end{aligned}$$

where we used that

$$w(t,x_\beta+z) - w(t,x_\beta) \le w^\beta(t,x+z) - w^\beta(t,x) \quad \text{with} \quad w^\varepsilon(t,x) = u(t,x) - p\cdot x$$

(this comes from (8.58) and the fact that $u(t,x_\beta+z) - e^{Kt}\frac{|x-x_\beta|^2}{2\beta} \le u^\beta(t,x+z)$). Choosing now $K = 2\|\nabla c\|_\infty$ permits to get that u^β is a subsolution. Next, remark that :

$$u^\beta(0,x) \le u(0,x) + \sup_{r>0}\left\{\|\ \nabla u_0\|_\infty r - \frac{r^2}{2\beta}\right\} \le u(0,x) + \beta\frac{\|\nabla u_0\|^2}{2}.$$

Hence, the comparison principle (see Theorem 4.4 applied to $u^\beta(t,x) - p\cdot x$ and $w(t,x)$) implies that

$$u^\beta \le u + \beta\frac{\|\nabla u_0\|^2}{2}.$$

Rewrite this inequality as follows

$$u(t,y) \le u(t,x) + \beta\frac{\|\nabla u_0\|^2}{2} + e^{Kt}\ \frac{|x-y|^2}{2\beta}.$$

Optimizing with respect to β permits to conclude.

□

7.2 Sub- and supercorrectors

We give an alternative characterization of the ergodicity of (8.55) that complements Theorem 2.1 in the special case $c(\tau,y) = c(y)$. We will use it repeatedly in the proof of Theorem 2.6. More precisely, we are interested in the following stationnary equation :

$$\lambda = (c(y) + L + M_p[v](y))\,|p + \nabla_y v| \quad \text{on } \mathbb{R}^N. \tag{8.59}$$

Lemma 7.2. *The function* $\overline{H}^0$ *satisfies :*

$$\begin{aligned}\overline{H}^0(p,L) &= \max\{\lambda : \textit{there exists a 1-periodic subsolution of } (8.59)\}\\ &= \min\{\lambda : \textit{there exists a 1-periodic supersolution of } (8.59)\}.\end{aligned} \quad (8.60)$$

Remark 7.3. *Such a characterization is classical in the context of homogenization of Hamilton-Jacobi equation. See for instance [138] for such a characterization for local Hamilton-Jacobi equations.*

Proof. We first remark that we can construct λ^- and λ^+ such that $v = 0$ is respectively a sub- and a supersolution of (8.59) so that the sets at stake in (8.60) are not empty. Let $\lambda^{\min}$ and $\lambda^{\max}$ denote respectively the maximum and the minimum defined in (8.60). The fact that the infimum and the supremum defining $\lambda^{\min}$ and $\lambda^{\max}$ are attained is a consequence of the stability of viscosity solutions and L^∞ a priori bounds (that are easy to obtain).

Let w be the solution of (8.55) and consider the upper relaxed limit $\overline{v}_\infty$ as n goes to infinity (resp. the lower relaxed limit $\underline{v}_\infty$) of $v_n(\tau, y) = v(\tau + n, y)$ with $v(\tau, y) = w(\tau, y) - \overline{H}^0(p, L)\tau$. Then consider the suppremum in time of $\overline{v}_\infty$ (resp. the infimum in time of $\underline{v}_\infty$). This allows us to construct a subsolution (resp. a supersolution) of (8.59). This implies that $\overline{H}^0(p, L) \leq \lambda^{\min}$ (resp. $\lambda^{\max} \leq \overline{H}^0(p, L)$).

Now let v^- be a periodic subsolution of (8.59) for $\lambda^{\min}$. Since (8.59) does not see the constants, we can assume that $v^- \leq 0$. We have that $v^- + \lambda^{\min}\tau$ is a subsolution of (8.55). By comparison principle, we deduce that

$$w \geq v^- + \lambda^{\min}\tau$$

where w is the solution of (8.55). Dividing by τ and sending $\tau \to \infty$, we get that $\overline{H}^0(p, L) \geq \lambda^{\min}$. The proof that $\lambda^{\max} \geq \overline{H}^0(p, L)$ is similar. This ends the proof of the lemma. $\square$

The second technical lemma we need in the proof of Theorem 2.6 is the construction of sub- and supercorrectors of (8.59) with some monotonicity properties and with precise estimates on their oscillations.

Lemma 7.4. (Existence of sub and supercorrectors) *For any* $p \in \mathbb{R}$ *and* $L \in \mathbb{R}$, *there exists* $\lambda \in \mathbb{R}$, *a subcorrector* $\underline{v}(y)$ *and a supercorrector* $\overline{v}(y)$ *which are* 1-*periodic in* y *and satisfy*

$$\begin{aligned}&\lambda \leq (c + L + M_p[\underline{v}])\,|p + \nabla_y \underline{v}|, \quad \textit{with} \quad p(p + \nabla_y \underline{v}) \geq 0 \quad \textit{on} \quad \mathbb{R},\\ &\lambda \geq \left(c + L + \tilde{M}_p[\overline{v}]\right)|p + \nabla_y \overline{v}|, \quad \textit{with} \quad p(p + \nabla_y \overline{v}) \geq 0 \quad \textit{on} \quad \mathbb{R}\end{aligned}$$

such that

$$\max \underline{v} - \min \underline{v} \leq |p| \quad \textit{and} \quad \max \overline{v} - \min \overline{v} \leq |p|. \quad (8.61)$$

There exists a discontinuous corrector v which satisfies

$$\lambda = (c + L + M_p[v])\,|p + \nabla_y v| \quad on \quad \mathbb{R}, \tag{8.62}$$

$$\max v - \min v \leq 2|p| \quad \text{and} \quad |v|_\infty \leq |p|.$$

The unique solution w of (8.55) *satisfies for all $\tau \geq 0$*

$$|w(\tau, y) - \lambda\tau|_\infty \leq 2|p|. \tag{8.63}$$

Proof. When $p = 0$, we observe that (8.59) is satisfied with $\lambda = 0$ and $v = 0$. Hence (8.61) is clearly satisfied. Let us now assume that $p > 0$ since the case $p < 0$ is similar.

STEP 1. Consider the solution w of (8.55), *i.e.*, the solution w of (8.31) with $\alpha = 0$. The proof of Theorem 5.1 in the case $N = 1$ implies that $\tilde{v}(\tau, y) = w(\tau, y) - \lambda\tau$ satisfies

$$|\tilde{v}(\tau, y) - \tilde{v}(\tau, z)| \leq |p| \quad \text{and}\; p(p + \nabla_y \tilde{v}) \geq 0$$

and so

$$\max_y \tilde{v} - \min_y \tilde{v} \leq |p|.$$

STEP 2. Consider the upper relaxed limit $\overline{v}_\infty$ as n goes to infinity (resp. the lower relaxed limit $\underline{v}_\infty$) of $v_n(\tau, y) = \tilde{v}(\tau + n, y)$ Then consider the supremum in time of $\overline{v}_\infty$ (resp. the infimum in time of $\underline{v}_\infty$). This allows us to build a subsolution $\overline{v}$ (resp. a supersolution $\underline{v}$) of (8.62) that satisfies the expected properties.

STEP 3. Finally, we have $\underline{v} + \operatorname{osc} v \geq \overline{v}$ and $\underline{v} + \operatorname{osc} v$ is still a supersolution of (8.62). Then the Perron's method allows us to build a discontinuous periodic corrector v which satisfy $\operatorname{osc} v \leq 2p$. Moreover, since the equation does not see constants, we have that $v(y) - K$ is solution of (8.62) and so we can assume that $|v|_\infty \leq |p|$ for a good choice of the constant K.

To prove that the solution w of (8.55) satisfies $|w - \lambda\tau|_\infty \leq 2|p|$, it suffices to remark that $\underline{v} + \operatorname{osc} \underline{v} + \lambda\tau$ (resp. $\overline{v} + \operatorname{osc} \overline{v} + \lambda\tau$) is supersolution (resp. subsolution) of (8.55) and to use the comparison principle for equation (8.55). This ends the proof of the lemma. □

7.3 Proof of Theorem 2.6

We now turn to the proof itself.

Proof of Theorem 2.6.
1. When $c \equiv 0$, notice that $\lambda = L|p|$ and $v = 0$ do satisfy (8.59).

2. This is a consequence of (8.34).

3. Using the monotonicity of $\overline{H}^0(L,p)$ in L (see Corollary 5.4) we only need to prove that $\overline{H}^0(0,p) = 0$. Thanks to Lemma 7.2, it is enough to construct a sub- and a supersolution of :

$$0 = c + M_p[v].$$

Let us consider the solution v of :

$$\begin{cases} \partial_\tau v = c(y) + M_{p,\delta}[v(\tau,\cdot)](y) & \text{in } \mathbb{R}^+ \times \mathbb{R} \\ v(0,y) = 0 & \text{in } \mathbb{R} \end{cases}$$

with

$$M_{p,\delta}[v](y) = \int_{\mathbb{R}} dz\ J(z)\left\{E^\delta\left(v(y+z) - v(y) + p\cdot z\right) - p\cdot z\right\}$$

where E^δ is a smooth approximation of E, such that $E^\delta(z) - z$ is 1-periodic, $E^\delta(-z) = -E^\delta(z)$ and E^δ is increasing. From the 1-periodicity of c_1, we deduce the 1-periodicity of v. It is also easy to prove (by adapting the proof of Lemma 7.1) that v is Lipschitz continuous in space and time for all finite time. Let us consider $p = P/Q$ with $P \in \mathbb{Z}$, $Q \in \mathbb{N}\setminus\{0\}$ and dropping for a while the dependence on τ, we set $u(y) = v(y) + p\cdot y$. Assuming temporarily that J decays to zero at infinity sufficiently quickly, recalling that J is even and using the fact that $v(y)$ is 1-periodic in y, we compute :

$$\begin{aligned}
&\int_{(-Q/2,Q/2)} dy \int_{\mathbb{R}} dz\ J(z)\left\{E^\delta\left(v(y+z) - v(y) + p\cdot z\right) - p\cdot z\right\}\\
&= \int_{(-Q/2,Q/2)} dy \sum_{k\in\mathbb{Z}} \int_{((k-1/2)Q,(k+1/2)Q)} dz\ J(z)\left\{E^\delta\left(u(y+z) - u(y)\right)\right\}\\
&= K + \int_{(-Q/2,Q/2)} dy \sum_{k\in\mathbb{Z}} \int_{(-Q/2,Q/2)} dz\ J(z+kQ)\left\{E^\delta\left(u(y+z) - u(y)\right)\right\}
\end{aligned}$$

where

$$\begin{aligned}
K &= \int_{(-Q/2,Q/2)} dy \int_{(-Q/2,Q/2)} dz \sum_{k\in\mathbb{Z}} kQJ(z+kQ)\\
&= P\left(\int_{(-Q/2,Q/2)} dz \sum_{k\in\mathbb{Z}} (z+kQ)J(z+kQ) - \int_{(-Q/2,Q/2)} dz J_Q(z)\right)\\
&= P\int_{\mathbb{R}} d\bar{z}\ \bar{z}J(\bar{z})\\
&= 0.
\end{aligned}$$

Hence, with the notation

$$J_Q(z) = \sum_{k\in\mathbb{Z}} J(z+kQ) = J_Q(-z),$$

we get

$$\begin{aligned}
&\int_{(-Q/2,Q/2)} dy \int_{\mathbb{R}} dz\ J(z)\left\{E^\delta\left(v(y+z)-v(y)+p\cdot z\right)-p\cdot z\right\}\\
&= \int_{(-Q/2,Q/2)} dy \int_{(-Q/2,Q/2)} dz\ J_Q(z)\left\{E^\delta\left(u(y+z)-u(y)\right)\right\}\\
&= \int_{(-Q/2,Q/2)} dz\ J_Q(z)\int_{(z-Q/2,z+Q/2)} dx\ \left\{E^\delta\left(u(x)-u(x-z)\right)\right\}\\
&= \int_{(-Q/2,Q/2)} dz\ J_Q(z)\int_{(-z-Q/2,-z+Q/2)} dx\ \left\{E^\delta\left(u(x)-u(x+z)\right)\right\}\\
&= -\int_{(-Q/2,Q/2)} dz\ J_Q(z)\int_{(-z-Q/2,-z+Q/2)} dx\ \left\{E^\delta\left(u(x+z)-u(x)\right)\right\}\\
&= -\int_{(-Q/2,Q/2)} dz\ J_Q(z)\int_{(-Q/2,Q/2)} dx\ \left\{E^\delta\left(u(x+z)-u(x)\right)\right\}
\end{aligned}$$

(the last line is obtained by using the 1-periodicity of $u(x+z)-u(x)$ in x). Finally comparing the second line and the last one of the previous equality, we deduce that :

$$\int_{(-Q/2,Q/2)} dy \int_{\mathbb{R}} dz\ J(z)\left\{E^\delta\left(v(y+z)-v(y)+p\cdot z\right)-p\cdot z\right\}=0.$$

By taking a limit, we deduce that this is still true without assuming that J has strong decay at infinity.

Using now the fact that the equation is satisfied almost everywhere (because the solution is Lipschitz continuous) we get by integration of the equation and the fact that $\int_{(0,1)} c = 0$:

$$\partial_\tau\left(\int_{(-Q/2,Q/2)} dy\ v(\tau,y)\right)=0.$$

We deduce (by periodicity of v) that

$$\int_{(0,1)} dy\ v(\tau,y)=0 \quad \text{for any} \quad \tau\geq 0$$

for any rational p. We conclude that this is true for any $p\in\mathbb{R}$ (just by taking a limit). On the other hand, we deduce as in the proof of Theorem 5.1 in the case $N=1$ that

$$\max_y v(\tau,y)-\min_y v(\tau,y)\leq |p|.$$

and similarly that

$$|v(\tau, y)| \leq C(p).$$

Considering the semi-relaxed limits of $v(\tau, y)$ with the supremum (resp. the infimum) in time, we build a subsolution $\underline{v}$ (resp. a supersolution $\overline{v}$) of the following equation

$$0 = c + M_{p,\delta}[v].$$

Taking now the limit as δ goes to zero, we build a subsolution $\underline{v}$ (resp. a supersolution $\overline{v}$) of the limit equation

$$0 = c + M_p[v]$$

and the proof of $\overline{H}^0(0, p) = 0$ is complete.

4. The monotonicity of $\overline{H}^0(L, p)$ in L is a straightforward consequence of the comparison principle.

We next consider $L_2 > L_1 > 0$ with $\lambda_i = \overline{H}^0(L_i, p)$ for $i = 1, 2$ and $\lambda_1 > 0$ and $p \neq 0$. The other cases can be treated similarly. From Lemma 7.4, we get a subcorrector $\underline{v}_1$ satisfying

$$0 < \lambda_1 \leq |p + \nabla_y \underline{v}_1| \left(c + L_1 + M_p[\underline{v}_1]\right)$$

such that (8.61) holds true. Therefore

$$M_p[\underline{v}_1] \leq \int_{\mathbb{R}} dz \ J(z)[E(|p| + p \cdot z)) - p \cdot z] \leq \left(|p| + \frac{1}{2}\right) \|J\|_{L^1}$$

and we get :

$$0 \leq c + L_1 + M_p[\underline{v}_1] \quad \text{and} \quad 0 < \delta \leq |p + \nabla_y \underline{v}_1|$$

for some constant $\delta \geq \lambda_1/(L_1 + C_p) > 0$ with $C_p = \|c\|_\infty + \left(|p| + \frac{1}{2}\right) \|J\|_{L^1}$. And then

$$\lambda_1 + \delta(L_2 - L_1) \leq |p + \nabla_y \underline{v}_1| \left(c_1 + L_2 + M_p[\underline{v}_1]\right)$$

which implies by Lemma 7.2 that $\lambda_2 \geq \lambda_1 + \delta(L_2 - L_1)$, *i.e.*

$$\frac{\lambda_2 - \lambda_1}{L_2 - L_1} \geq \frac{\lambda_1}{L_1 + C_p}.$$

More generally, we have for $L_2 L_1 > 0$, $|L_2| > |L_1|$ and for some universal constant $C > 0$

$$\frac{\overline{H}^0(L_2, p) - \overline{H}^0(L_1, p)}{L_2 - L_1} \geq \frac{|\overline{H}^0(L_1, p)|}{|L_1| + C_p}.$$

Similarly, let us consider a supercorrector $\overline{v}_1$ given by Lemma 7.4 which satisfies :

$$\lambda_1 \geq |p + \nabla_y \overline{v}_1| \left(c + L_1 + \tilde{M}_p[\overline{v}_1]\right). \tag{8.64}$$

As soon as $L_1 > C_p$, we get again that

$$c + L_1 + \tilde{M}_p[\overline{v}_1] > 0$$

from which we deduce

$$\frac{\lambda_1}{L_1 - C_p} \geq |p + \nabla_y \overline{v}_1|. \tag{8.65}$$

Next, using (8.64) and (8.65), write :

$$\begin{aligned} \lambda_1 + (L_2 - L_1)\frac{\lambda_1}{L_1 - C_p} &\geq \lambda_1 + (L_2 - L_1)|p + \nabla_y \overline{v}_1| \\ &\geq |p + \nabla_y \overline{v}_1| \left(c_1 + L_2 + M_p[\overline{v}_1]\right) \end{aligned}$$

and deduce from Lemma 7.2 that :

$$\lambda_2 \leq \lambda_1 + (L_2 - L_1)\frac{\lambda_1}{L_1 - C_p}.$$

This implies the result.

5. Let $T > 0$. The function w denotes the solution of (8.12) and w' denotes the solution of

$$\left\{ \begin{array}{lll} \partial_\tau w' = (c(y) + L + L' + M_p[w'(\tau, \cdot)](y))|p + \nabla w'| & \text{in} & (0, +\infty) \times \mathbb{R} \\ w(0, y) = 0 & \text{on} & \mathbb{R} \end{array} \right. \tag{8.66}$$

We set $\lambda = \overline{H}^0(L, p)$ and $\lambda' = \overline{H}^0(L + L', p)$. Since $|p + \nabla_y w(\tau, \cdot)|_\infty \leq |p| e^{T\|\nabla c\|_\infty}$ for $\tau \in [0, T)$ (Lemma 7.1), we get that $w(\tau, y) + \tau L'|p|e^{T|\nabla c|_\infty}$ is a supersolution of (8.66) on $(0, T] \times \mathbb{R}$. Hence, the comparison principle implies

$$w'(\tau, y) \leq w(\tau, y) + L'|p|Te^{T\|\nabla c\|_\infty} \quad \text{on } (0, T) \times \mathbb{R}.$$

Using the fact that $|w(\tau, y) - \lambda\tau|,\ |w'(\tau, y) - \lambda'\tau| \leq 2|p|$ and that $\overline{H}^0(L, p)$ is nondecreasing in L, we get that

$$0 \leq (\lambda' - \lambda)T \leq |p| \left(L'Te^{T|\nabla c|_\infty} + 4\right)$$

and so

$$|\lambda' - \lambda| \leq |p| \left(L'e^{T|\nabla c|_\infty} + \frac{4}{T}\right).$$

Taking $T = \frac{|\ln L'|}{2\|\nabla c\|_\infty}$, we get the result for $0 \leq L' \leq \frac{1}{2}$.

6. The following arguments must be adapted by using sub and supercorrectors in order to use Lemma 7.2. However, we assume for the sake of clarity that there exist correctors for any (L, p). Hence, we have with $\lambda = \overline{H}^0(L, p)$:

$$\lambda = |p + \nabla_y v| \left(c + L + M_p[v]\right).$$

Therefore $\overline{v}(y) = -v(y+a)$ satisfies :

$$-\lambda = |-p + \nabla_y \overline{v}(y)| \left(-c(y+a) - L + \tilde{M}_{-p}[\overline{v}](y)\right)$$

and $-c(y+a) = c(y)$ by assumption. Hence $\overline{H}^0(-L,-p) = -\lambda$.

7. We have with $\lambda = \overline{H}^0(L,p)$:

$$\lambda = |p + \nabla_y v| \left(c + L + M_p[v]\right).$$

Therefore $\check{v}(y) = v(-y+a)$ satisfies (because $J(-z) = J(z)$) :

$$\lambda = |-p + \nabla_y \check{v}(y)| \left(c(-y+a) + L + M_{-p}[\check{v}](y)\right).$$

By assumption, $c(-y+a) = c(y)$, hence : $\overline{H}^0(L,-p) = \overline{H}^0(L,p)$.

8. Analysis for small p. From the construction of sub/super/correctors satisfying $p(p + \nabla_y v) \geq 0$ (see Lemma 7.4), we know that on one period we have for $p > 0$ (the analysis is similar for $p < 0$)

$$0 \leq p \cdot z + v(z+y) - v(y) \leq p \quad \text{for} \quad z \in [0,1].$$

Therefore $\max v(z) - \min v(z) \leq p$ in general. We will now estimate $M_p[v]$ for small p. If $0 < p < 1$ and for $|z| < \lfloor 1/p \rfloor - 1$, using the periodicity of v and the monotonicity of $v(y) + py$, we deduce that $|p \cdot z + v(z+y) - v(y)| \leq p(\lfloor 1/p \rfloor - 1) < 1$ and then

$$\begin{aligned} M_p[v](y) &= \int_{|z| > \lfloor 1/p \rfloor - 1} J(z) \left\{ E\left(v(y+z) - v(y) + p \cdot z\right) - p \cdot z \right\} \\ &= \int_{|z| > \lfloor 1/p \rfloor - 1} J(z) \left\{ E\left(v(y+z) - v(y) + p \cdot z\right) - \left(v(y+z) - v(y) + p \cdot z\right) \right\} \\ &\quad + \int_{|z| > \lfloor 1/p \rfloor - 1} J(z) \left\{ v(y+z) - v(y) \right\} \end{aligned}$$

which implies that

$$|M_p[v]| \leq \left(\int_{|z| > \lfloor 1/p \rfloor - 1} J(z) \right) \left(\frac{1}{2} + \max v - \min v \right) \longrightarrow 0 \quad \text{as} \quad p \to 0.$$

Therefore for L and p small enough we see that $L + c(y) + M_p[v](y)$ changes sign if c changes sign. This gives a contradiction for the existence of sub and supercorrectors with non-zero λ.

9. Analysis for large p. We first consider the periodic solution v (with zero mean value) of

$$0 = c(y) + M_\infty[v](y)$$

with

$$M_\infty[v](y) = \int_{\mathbb{R}} dz\ J(z)(v(y+z) - v(y))$$

which satisfies

$$|v|_\infty \leq C|c|_\infty, \quad |v'|_\infty \leq C|c'|_\infty, \quad |v''|_\infty \leq C|c''|_{L^\infty}.$$

To get this result, it is sufficient to study the periodic solutions of

$$\alpha v^\alpha = c + M_\infty[v^\alpha]$$

that can be obtained by the usual Perron's method. The comparison principle (together with the properties of J) gives the bounds on the oscillations of v^α, independent on α. This implies similar bounds on the derivatives of v^α, which gives enough compactness to pass to the limit as $\alpha \to 0$.

For clarity, we assume that $y = 0$ and $v(y) = 0$. The other cases can be treated similarly. Let us set

$$u(z) = v(z) + p \cdot z$$

and let us estimate for $p > 0$ large

$$M_0 = \int_{\mathbb{R}} dz J(z)\left(E(u(z)) - u(z)\right).$$

Because $u'(z) = v'(z) + p > 0$ for $p > 0$ large enough, we can define u^{-1} by $u^{-1}(u(x)) = u(u^{-1}(x)) = x$. We can then rewrite with $\bar{u} = u(z)$

$$M_0 = \int_{\mathbb{R}} \frac{d\bar{u}}{u'(u^{-1}(\bar{u}))} J(u^{-1}(\bar{u}))\left(E(\bar{u}) - \bar{u}\right).$$

We set

$$G(\bar{u}) = \frac{J(u^{-1}(\bar{u}))}{u'(u^{-1}(\bar{u}))}$$

and write

$$M_0 = \sum_{k\in\mathbb{Z}} \int_{k-1/2}^{k+1/2} d\bar{u}\ G(\bar{u})\left(E(\bar{u}) - \bar{u}\right)$$

and set

$$I_k = [u^{-1}(k-1/2), u^{-1}(k+1/2)].$$

We get for $\bar{u} \in [k-1/2, k+1/2]$

$$G(\bar{u}) = G(k) + g_k(\bar{u})$$

with

$$|g_k(\bar{u})| \leq \sup_{z\in I_k} \left| \frac{J'(z)}{u'(z)} - \frac{J(z)u''(z)}{u'^2(z)} \right| \cdot \sup_{\bar{u}\in[k-1/2,k+1/2]} |(u^{-1})'(\bar{u})| \cdot 1/2.$$

By definition of u, we also have

$$\left|u^{-1}(k \pm 1/2) - \frac{k \pm 1/2}{p}\right| \leq \frac{|v|_\infty}{p}.$$

Therefore for p large enough

$$|g_k(\bar{u})| \leq 1/2 \cdot \left(\frac{h_1(k/p)}{(p - |v'|_\infty)^2} + |v''|_\infty \frac{h_0(k/p)}{(p - |v'|_\infty)^3}\right)$$

and then, since $\int_{k-1/2}^{k+1/2} d\bar{u}(E(\bar{u}) - \bar{u}) = 0$ and $|E(\bar{u}) - \bar{u}| \leq \frac{1}{2}$, we get

$$\begin{aligned} |M_0| &\leq 1/2 \sum_{k\in\mathbb{Z}} \int_{k-1/2}^{k+1/2} du \; |g_k(u)| \\ &\leq 1/4 \left(\frac{p}{(p-|v'|_\infty)^2}\left(\sum_{k\in\mathbb{Z}} \frac{1}{p} h_1(k/p)\right) + \frac{p|v''|_\infty}{(p-|v'|_\infty)^3}\left(\sum_{k\in\mathbb{Z}} \frac{1}{p} h_0(k/p)\right)\right). \end{aligned}$$

This implies that

$$|M_0| \leq C/p \quad \text{for} \quad p \quad \text{large enough}$$

with C depending on $\|c\|_{W^{2,\infty}}$, $\|h_0\|_{L^1}$ and $\|h_1\|_{L^1}$. More generally, we get the same estimate for

$$M(y) = \int_{\mathbb{R}} dz \; J(z) \left(E(v(y+z) - v(y) + p \cdot z) - (v(y+z) - v(y) + p \cdot z)\right).$$

We now compute

$$\begin{aligned} (p + \nabla_y v)(L + c + M_p[v]) &\geq (p + \nabla_y v)(L + c + M_\infty[v] - C/p) \\ &\geq (p - |v'|_\infty)(L - C/p). \end{aligned}$$

This implies that

$$\overline{H}^0(L, p) \geq (p - |v'|_\infty)(L - C/p) > 0$$

for $p > 0$ large enough, if $L > C/p$. The case $L < 0$ can be treated similarly.

The proof of Theorem 2.6 is now complete. □

8 Application : homogenization of a particle system

8.1 The general case

This section is devoted to the proof of Theorem 2.11. It is a consequence of Theorem 2.5 and of the following result :

Théorème 8.1 (Link between the system of ODEs and the PDE)**.** *Assume that V_0 is 1-periodic, that V_0' is Lipschitz continuous and that V satisfies (H). If we have $y_1(0) < ... < y_{N_\varepsilon}(0)$ at the initial time, then the cumulative distribution function ρ defined in* (8.14) *is a discontinuous viscosity solution (in the sense of Definition 4.1) of*

$$\partial_\tau \rho = (c\,(y) + M\,[\rho(\tau,\cdot)]\,(y))\,|\nabla \rho| \quad in \quad (0,+\infty)\times\mathbb{R}, \tag{8.67}$$

with

$$c(y) = V_0'(y) - F \qquad and \qquad J = V'' \text{ on } \mathbb{R}\backslash\{0\}. \tag{8.68}$$

Conversely, if u is a bounded and continuous viscosity solution of (8.67) *satisfying for some time $T>0$ and for all $\tau \in (0,T)$*

$$u(\tau, y) \text{ is increasing in } y,$$

then the points $y_i(\tau)$, defined by $u(\tau, y_i(\tau)) = i - \frac{1}{2}$ for $i\in\mathbb{Z}$, satisfy the system (8.9) *on $(0,T)$.*

Before presenting the proof of this theorem, let us explain how to derive Theorem 2.11.

Proof of Theorem 2.11. By construction, we have

$$(\rho_0^\varepsilon)^*(y) = \rho_0^\varepsilon(y) \le u_0(y) \le (\rho_0^\varepsilon)_*(y) + \varepsilon.$$

Using the fact that ρ^ε is a discontinuous viscosity solution of (8.1) (except for the initial condition) and the comparison principle (see Theorem 4.4), we deduce that (with u^ε the continuous solution of (8.1))

$$\rho^\varepsilon(\tau,y) \le u^\varepsilon(\tau,y) \le (\rho^\varepsilon)_*(\tau,y) + \varepsilon$$

and so

$$u^\varepsilon(\tau,y) - \varepsilon \le \rho^\varepsilon(\tau,y) \le u^\varepsilon(\tau,y). \tag{8.69}$$

Sending $\varepsilon\to 0$, we get that $\rho^\varepsilon \to u^0$ which gives the result. □

We now turn to the proof of Theorem 8.1.

Remark that under Assumption (H), V' is not differentiable at 0 and in order to be sure that the solution of (8.9) exists and is unique, we need to prove that the distance between two particles is bounded from below. Precisely, we prove :

Lemma 8.2 (Lower bound on the distance between particles)**.** *Assume that the potential V_0' is Lipschitz continuous and V satisfies (H). Let y_i, $i = 1,..,N_\varepsilon$ be the solution of* (8.9)*. Then, the distance $d(\tau)$ between two particles is bounded from below by*

$$d(\tau) := \min\{|y_i(\tau) - y_j(\tau)|, i\neq j\} \ge d_0 e^{-\|V_0''\|_\infty \tau},$$

where d_0 is the minimal distance between two particles at initial time : $d_0 = d(0)$.

Proof of Lemma 8.2. Let d_i be the distance between y_i and y_{i+1} : $d_i(\tau) = y_{i+1}(\tau) - y_i(\tau)$. At $\tau = 0$, $d(0) > 0$. Hence consider the first time τ^* for which $d(\tau^*) = 0$ and choose i such that $d(\tau) = d_i(\tau)$ in a neighbourhood of τ^*. Let us denote by $\bar{d}_{j-i} = d_j - d_i \geq 0$. We then have

$$\begin{aligned}
\dot{d} =& -\left(V_0'(y_{i+1}) - V_0'(y_i)\right) - \left(\sum_{j\neq i+1} V'(y_{i+1} - y_j) - \sum_{j\neq i} V'(y_i - y_j)\right) \\
=& -\left(V_0'(y_{i+1}) - V_0'(y_i)\right) \\
& - \left\{\sum_{k=0}^{i-1} V'((1+k)d + \bar{d}_0 + .. + \bar{d}_{-k}) - \sum_{k=0}^{N_\varepsilon - i-2} V'((1+k)d + \bar{d}_1 + .. + \bar{d}_{k+1})\right. \\
& \left. - \sum_{k=0}^{i-2} V'((1+k)d + \bar{d}_{-1} + .. + \bar{d}_{-k-1}) + \sum_{k=0}^{N_\varepsilon - i-1} V'((1+k)d + \bar{d}_0 + .. + \bar{d}_k)\right\}.
\end{aligned}$$

We set

$$a_k^- = (1+k)d + \bar{d}_0 + .. + \bar{d}_{-k} \geq 0, \quad a_k^+ = (1+k)d + \bar{d}_0 + .. + \bar{d}_k.$$

We then get (since $\bar{d}_0 = 0$)

$$\begin{aligned}
\dot{d} =& -\left(V_0'(y_{i+1}) - V_0'(y_i)\right) \\
& - \left\{\sum_{k=0}^{i-2} (V'(a_k^-) - V'(a_k^- + \bar{d}_{-k-1})) + \sum_{k=0}^{N_\varepsilon - i-2} (V'(a_k^+) - V'(a_k^+ + \bar{d}_{k+1}))\right. \\
& \left. + V'(a_{i-1}^-) + V'(a_{N_\varepsilon - i-1}^+)\right\}.
\end{aligned}$$

Since $V' \leq 0$ and V' is nondecreasing, we deduce that

$$\dot{d} \geq -d\|V_0''\|_\infty$$

which gives the result. □

It remains to prove Theorem 8.1.

Proof of Theorem 8.1. Theorem 8.1 is a consequence of the following lemma.

Lemma 8.3 (Link between the velocities)**.** *Assume that V_0' is Lipschitz continuous and V satisfies (H). Assume that $(y_i(\tau))_{i=1,...,N_\varepsilon}$ solves the system of ODEs* (8.9) *with $y_1 < ... < y_{N_\varepsilon}$ at the initial time. Then for all $i = 1, ..., N_\varepsilon$, we have*

$$-\dot{y}_i = c(y_i) + M[u(\tau, \cdot)](y_i) \tag{8.70}$$

where c and J are defined by (8.15) *and $u(\tau, y)$ is a continuous function such that :*

$$\left\{\begin{array}{l} u(\tau,y)=\rho^*(\tau,y) \quad \textit{for} \quad y=y_j(\tau),\ j=1,..,N_\varepsilon \\ 0<u<N_\varepsilon \\ u \textit{ is increasing in } y \end{array}\right. \tag{8.71}$$

where ρ is defined by (8.14).

Proof of Lemma 8.3. To simplify the notation, let us drop the time dependence.

$$\begin{aligned} M[u](y_{i_0}) &= \int dz J(z) E(u(y_{i_0}+z)-u(y_{i_0})) \\ &= -\left(\int dz J(z)\right)(i_0-1)+\int dz J(z)\rho(y_{i_0}+z) \\ &= -(2V'(0-))(i_0-1)+J\star\rho(y_{i_0}). \end{aligned}$$

We now compute $J\star\rho$.

$$\begin{aligned} (J\star\rho)(y_{i_0}) &= -\frac{1}{2}\int_{\mathbb{R}} J(z)dz+\sum_{i=1}^{N_\varepsilon}\int J(z)H(y_{i_0}-z-y_i)dz \\ &= \sum_{i<i_0}\int_{y_i-y_{i_0}}^{+\infty} J(z)dz+\sum_{i>i_0}\int_{y_i-y_{i_0}}^{+\infty} J(z)dz \\ &= (i_0-1)(V'(0-)-V'(0+))-\sum_{i\neq i_0} V'(y_i-y_{i_0}) \\ &= 2(i_0-1)V'(0-)-\sum_{i\neq i_0} V'(y_i-y_{i_0}). \end{aligned}$$

This finally gives

$$M[u](y_{i_0})=-\sum_{i\neq i_0} V'(y_i-y_{i_0}).$$

The proof is now complete. □

The fact that ρ is a discontinuous solution of (8.67) is a straighforward consequence of Lemma 8.3 and of Definition 4.1.

We prove the converse. Use Proposition 4.8 and conclude that $\rho=E_*(u)$ (resp. $\rho^*=E(u)$) is a viscosity supersolution (resp. subsolution) of

$$\partial_\tau\rho=\tilde{c}(\tau,y)\ \partial_y\rho \quad \text{with} \quad \tilde{c}(\tau,y)=c(y)+M[u(\tau,\cdot)](y)=c(y)+\tilde{M}[u(\tau,\cdot)](y)$$

where $\tilde{c}$ is in fact a prescribed velocity for ρ. Using the fact that u is increasing in y, we define $y_i(\tau)=\inf\{y,\ u(\tau,y)\geq i-1/2\}=(u(\tau,\cdot))^{-1}(i-1/2)$ and we consider the functions $\tau \mapsto y_i(\tau)$. They are continuous because u is increasing in y and is continuous in (τ, y).

We next show that y_i are viscosity solutions of (8.9). To do so, consider a test function φ such that $\varphi(\tau) \le y_i(\tau)$ and $\varphi(\tau_0) = y_i(\tau_0)$. Define next $\hat{\varphi}(\tau, y) = i - 1/2 + (y - \varphi(\tau))$. It satisfies :

$$\hat{\varphi}(\tau_0, y_i(\tau_0)) = \rho^*(\tau_0, y_i(\tau_0)) \quad \text{and} \quad \hat{\varphi}(\tau, y) \ge \rho^*(\tau, y) \quad \text{for } y_i(\tau) - 1 < y.$$

This implies, with $\bar{c}_i = \tilde{c}(\cdot, y_i)$, that

$$\varphi_\tau(\tau_0) \ge -\bar{c}_i(\tau_0) = F - V_0'(y_i) - \sum_{j \ne i} V'(y_i - y_j).$$

This proves that y_i are viscosity supersolutions of (8.9). The proof for subsolutions is similar and we skip it. Moreover, since $\bar{c}_i$ is continuous, we deduce that y_i is C^1 and it is therefore a classical solution of

$$\dot{y}_i(\tau) = -\bar{c}_i(\tau) = F - V_0'(y_i) - \sum_{j \ne i} V'(y_i - y_j).$$

This ends the proof of the theorem. $\square$

Remark 8.4. *In the case of particles, we have formally the microscopic energy for $\rho = \rho^{micro}$*

$$E^{micro} = \int_{\mathbb{R}} \frac{1}{2}(V^{micro} \star \rho_y)\rho_y + V_0^{micro}\rho_y$$

with $V^{micro} = V$ and $V_0^{micro} = V_0 - F \cdot y$. After a rescaling, we get at the limit a formal macroscopic energy for $\rho = \rho^{macro}$

$$E^{macro} = \int_{\mathbb{R}} \frac{1}{2}(V^{macro} \star \rho_x)\rho_x + V_0^{macro}\rho_x$$

where $V^{macro} = -g_0 \ln|x|$ only keeps the memory of the dislocation part of V^{micro} (the long range interactions), and $V_0^{macro} = -F \cdot x + \int_{\mathbb{R}/\mathbb{Z}} dy V_0(y)$.

8.2 Extension to the dislocation case $V(x) = -\ln|x|$

This subsection is devoted to the proof of Theorem 2.14.

Proof of Theorem 2.14.
STEP 1 : APPROXIMATION OF V. For $\delta > 0$, let the regularization V_δ of the potential $V(x) = -\ln|x|$ be such that $V_\delta \in C^0(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{0\})$ and defined by

$$V_\delta = \begin{cases} -\ln|x| & \text{if } |x| > \delta \\ \text{linear if } x \in (-\delta, \delta)\backslash\{0\} \end{cases}$$

We can easily check that V_δ satisfies assumptions (H) with $g_0 = 1$. Using the fact that the distance between two particles is bounded from below, we deduce that ρ defined in (8.14) satisfies

$$\partial_t u = (c(x) + L + M[u(t,\cdot)](x))\,|\nabla u| \quad \text{on } (0,T)\times\mathbb{R}. \tag{8.72}$$

with

$$c(y) = V_0'(y) - F \qquad \text{and} \qquad J = V_\delta'' \text{ on } \mathbb{R}\backslash\{0\} \tag{8.73}$$

as soon as $\delta \le \delta(T) := d_0 e^{-|V_0''|_\infty T}$. Indeed, Since Lemma 8.2 implies that for all $t \in [0,T]$, we have $d(t) \ge d_0 e^{-|V_0''|_\infty T}$, we can replace the potential V with V_δ for $\delta \le d_0 e^{-|V_0''|_\infty T}$ and the ODEs system remains equivalent.

In the sequel, we use the following functions

$$J_\delta(z) = \begin{cases} V_\delta''(z) & \text{for } \ z \in \mathbb{R}\backslash\{0\} \\ 0 & \text{for } \ z = 0 \end{cases}$$

$$M_{p,\delta}[U](x) = \int_{\mathbb{R}} dz\ J_\delta(z)\left\{E(U(x+z) - U(x) + p\cdot z) - p\cdot z\right\}$$

and $\overline{H}^0_\delta$ denotes the effective Hamiltonian associated with J_δ.

Step 2 : estimate for the approximate effective Hamiltonian.

Lemma 8.5 (Lipschitz continuous sub/supercorrectors). *Let v denote the solution of*

$$\overline{H}^0_\delta = (c(x) + L + M_{p,\delta}[v])\,|p + \nabla v| \quad \text{in} \quad \mathbb{R}$$

and consider

$$v^\varepsilon(x) = \sup_y \left(v(y) - \frac{|x-y|^2}{2\varepsilon}\right), \quad v_\varepsilon(x) = \inf_y \left(v(y) + \frac{|x-y|^2}{2\varepsilon}\right).$$

There then exists a constant $C(p)$ depending only on $|p|$ such that for $\delta,\ \eta \in (0, C(p)\sqrt{\varepsilon})$, we have

$$\overline{H}^0_\delta(L,p) \le \left(c(x) + L + C(p)\sqrt{\varepsilon}\|\nabla c\|_\infty + M_{p,\eta}[v^\varepsilon]\right)|p + \nabla v^\varepsilon|, \tag{8.74}$$

$$\overline{H}^0_\delta(L,p) \ge \left(c(x) + L - C(p)\sqrt{\varepsilon}\|\nabla c\|_\infty + \tilde{M}_{p,\eta}[v_\varepsilon]\right)|p + \nabla v_\varepsilon|. \tag{8.75}$$

Proof. We just make the proof for v^ε and we assume that

$$v^\varepsilon(x) = v(x_\varepsilon) - \frac{|x - x_\varepsilon|^2}{2\varepsilon}.$$

It is classical that v^ε is Lipschitz continuous and

$$\|\nabla v^\varepsilon\|_\infty \le \frac{C}{\sqrt{\varepsilon}} \tag{8.76}$$

where the constant C depends only on $\|v\|_\infty \le 2|p|$. Hence $C = C(p)$. Moreover, for any $q \in D^{1,+}v^\varepsilon(x)$, it is classical that $q \in D^{1,+}v(x_\varepsilon)$, $q = \frac{x-x_\varepsilon}{\varepsilon}$ and $|x-x_\varepsilon| \le C(p)\sqrt{\varepsilon}$, where the constant C depends on $\|v\|_\infty$. We thus deduce that

$$\begin{aligned}\overline{H}^0_\delta(L,p) &\le (c(x_\varepsilon) + L + M_{p,\delta}[v](x_\varepsilon))\,|p+q| \\ &\le \left(c(x) + L + C(p)\sqrt{\varepsilon}\|\nabla c\|_\infty + M_{p,\delta}[v^\varepsilon](x)\right)|p+q|\end{aligned}$$

where we have used that

$$v(x_\varepsilon + z) - v(x_\varepsilon) \le v^\varepsilon(x+z) - v^\varepsilon(x).$$

Now using (8.76), we can replace $M_{p,\delta}[v^\varepsilon](x)$ with $M_{p,\eta}[v^\varepsilon](x)$ for $\eta,\ \delta \in (0, C(p)\sqrt{\varepsilon})$. Indeed, for $|z| \le \delta$ or $|z| \le \eta$, we have

$$|v^\varepsilon(x+z) - v^\varepsilon(x) - p\cdot z| \le (C(p)\varepsilon^{-1/2} + |p|)\max(\delta,\eta) < 1$$

as soon as $\max(\delta,\eta) \le \tilde{C}(p)\sqrt{\varepsilon}$ and we obtain

$$\begin{aligned}&\int_{|z|\le\max(\delta,\eta)} dz\ J_\delta(z)E_*(v_\varepsilon(x+z) - v_\varepsilon(x) - p\cdot z) \\ =&\int_{|z|\le\max(\delta,\eta)} dz\ J_\eta(z)E_*(v_\varepsilon(x+z) - v_\varepsilon(x) - p\cdot z) = 0.\end{aligned}$$

This implies (8.74). □

We derive from the previous lemma the convergence of $\overline{H}^0_\delta$ towards $\overline{H}^0_0$. We even get an error estimate :

Lemma 8.6 (Limit of $\overline{H}^0_\delta$)**.** *There exists $\overline{H}^0_0(L,p)$ such that*

$$\left|\overline{H}^0_\delta(L,p) - \overline{H}^0_0(L,p)\right| \le \frac{C(p)}{\ln\frac{1}{\delta}} \tag{8.77}$$

for δ small enough.

Proof. We deduce from Lemma 8.5 that $\overline{H}^0_\delta(L,p) \le \overline{H}^0_\eta(L+L',p)$ for $\eta,\ \delta \in (0, C(p)\sqrt{\varepsilon})$ and with $L' = C\sqrt{\varepsilon}\|\nabla c\|_\infty$. Since the constants are independent on L, L', we deduce that

$$\overline{H}^0_\delta(L-L',p) \le \overline{H}^0_\eta(L,p) \le \overline{H}^0_\delta(L+L',p) \quad \text{for all } \eta,\ \delta \in (0, C(p)\sqrt{\varepsilon}).$$

Setting $\overline{H}^0_0(L,p) = \limsup_{\eta\to 0}\overline{H}^0_\eta(L,p)$ and using Theorem 2.6, point 5, we then get (8.77). □

One can now check that $\overline{H}_0^0$ satisfies Theorem 2.6 excepted the point 4 and point 9. We now turn to the proof of convergence itself.

Step 3 : the proof of convergence. Let us consider $T > 0$ and try to prove that ρ^ε converges towards $u0$ on $(0,T) \times \mathbb{R}$. In order to do so, choose for any $\varepsilon > 0$ a $\delta = \delta(\varepsilon)$ such that ρ^ε is a (discontinuous) solution of (8.1) with J replaced with $J_{\delta(\varepsilon)}$. We still have (8.69) but now u^ε is associated with $J_{\delta(\varepsilon)}$ instead of a fixed J. We need to check that the convergence proof of Theorem 2.5 can be adapted. This is possible thanks to (8.77), that permits to pass from the exact effective Hamiltonian $\overline{H}^0$ to the approximate one $\overline{H}^0_{\delta(\varepsilon)}$, and thanks to Lemma 8.5 that permits to prove that Lemma 6.1 is still satisfied by using a regularized supercorrector v_ε. Let us give more details. We first prove that the initial condition is satisfied. Since u_0 is Lipschitz continuous, we get that

$$M_\delta^\varepsilon\left[\frac{u_0(\cdot)+Ct}{\varepsilon}\right] = \int_{|z|>\frac{1}{\|\nabla u_0\|_\infty}} dz\ J_\delta(z) E\left(\frac{u_0(x+\varepsilon z)-u_0(x)}{\varepsilon}\right)$$

is independent of δ if $\delta \le \frac{1}{\|\nabla u_0\|_\infty}$, and so (8.29) remains true with a constant C independent of ε and δ.

We now turn to the equation. The idea is to use Lipschitz continuous sub/super-correctors to control the distance between particles. Using notation of Section 6 we now prove by contradiction that $\overline{u}$ is a subsolution of (8.7). Hence we assume that

$$\partial_t\phi(t_0,x_0) = \overline{H}_0^0(L_0,p)+\theta' = \overline{H}_\delta^0(L_0,p,0)+\theta$$

for δ small enough and where we have used (8.77). Moreover, we can replace (8.47) with

$$\partial_t\phi(t_0,x_0) = \overline{H}_\delta^0(L_0+2\beta,p,\alpha)+\frac{\theta}{2}.$$

We now replace the bounded corrector of Section 6 with a Lipschitz continuous supercorrector. More precisely, we consider the solution v of

$$\lambda = \big(c(y)+L+M^\alpha_{p,\delta}[v](y)\big)\,|p+\nabla_y v|$$

and we set

$$v_\varepsilon(x) = \inf_y\left(v(y)+\frac{|x-y|^2}{2\varepsilon}\right) = v(x_\varepsilon)+\frac{|x-x_\varepsilon|^2}{2\varepsilon}$$

where $L = L_0 + 2\beta$ and $\lambda = \overline{H}_\delta(L_0,p,0)$. By Lemma 8.5, we deduce that v_ε is a supersolution of

$$\lambda \ge \Big(c(y)+L-C\sqrt{\varepsilon}\|\nabla c\|_\infty+\tilde{M}^\alpha_{p,\delta}[v_\varepsilon](x)\Big)\,|p+\nabla v_\varepsilon|$$

for $\delta \leq C(p)\sqrt{\varepsilon}$. We now consider ε such that $C\sqrt{\varepsilon}\|\nabla c\|_\infty \leq \beta$ and we get that v_ε is a supersolution of

$$\lambda \geq \left(c(y) + L_0 + \beta + \tilde{M}^\alpha_{p,\delta}[v_\varepsilon](x)\right)|p + \nabla v_\varepsilon|.$$

Using the fact that v_ε is Lipschitz continuous, we get that $\tilde{M}^\alpha_{p,\delta}[v_\varepsilon]$ is independent of δ for $\delta \leq C(p)\sqrt{\varepsilon}$ and so the rest of the proof is exactly the same as the one of Theorem 2.5 with J replaced with J_δ. $\square$

Aknowledgement. The authors thank A. Finel, Y. Le Bouar and L. Proville for fruitful discussion in the preparation of this paper. This work was supported by the ACI JC 1025 and the ANR MICA (2006-2009).

Chapitre 9

Convergence d'un schéma pour un système couplé non-local modélisant la dynamique de densité de dislocations

Ce chapitre est une version rallongée d'un travail en collaboration avec A. El Hajj [75].
Dans ce papier, nous étudions un système couplé non local qui intervient dans la théorie de la dynamique de densité de dislocations. Dans le cadre des solutions de viscosité, nous prouvons un résultat d'existence et d'unicité en temps long pour la solution du modèle. Nous proposons également un schéma numérique et nous montrons une estimation d'erreur de type Crandall-Lions entre la solution continue et son approximation numérique. A notre connaissance, il s'agit de la première estimation de type Crandall-Lions pour un système d'Hamilton-Jacobi. Nous présentons également quelques simulations numériques.

A convergent scheme for a non-local coupled system modelling dislocations densities dynamics

A. El Hajj, N. Forcadel

Abstract

In this paper, we study a non-local coupled system that arises in the theory of dislocations densities dynamics. Within the framework of viscosity solutions, we prove a long time existence and uniqueness result for the solution of this model. We also propose a convergent numerical scheme and we prove a Crandall-Lions type error estimate between the continuous solution and the numerical one. As far as we know, this is the first error estimate of Crandall-Lions type for Hamilton-Jacobi systems. We also provide some numerical simulations.

AMS Classification : 35Q72, 49L25, 35F25, 35L40, 65M06, 65M12, 65M15, 74H20, 74H25.

Keywords : Hamilton Jacobi equations, viscosity solutions, dislocations densities dynamics, numerical scheme, error estimate, system.

1 Introduction

1.1 Presentation and physical motivations

A dislocation is a crystal defect which corresponds to a discontinuity in the crystalline structure organisation. This concept has been introduced by Polanyi, Taylor and Orowan in 1934 as the main explanation at the microscopic scale of plastic deformation. A dislocation creates around it a perturbation that can be seen as an elastic field. Under an exterior strain, a dislocation moves according to its Burgers vector which caracterize the intensity and the direction of the defect displacement (see Hirth and Lothe [106] for an introduction to dislocations).

Here, we are interested in dislocations densities dynamics. More precisely, we consider edge dislocations, $i.e$ the Burgers vectors and dislocations are in the same plane. These dislocations are moving with the Burgers vectors $\pm\vec{b}$ (see figure 9.1). This model has been introduced by Groma, Balogh as a coupled system, namely a transport problem where the velocity is given by the elasticity equations in the 2-D case (see [103]).

If the 2-D domain is 1-periodic in x_1 and x_2, and if the dislocations densities depend only on the variable $x = x_1 + x_2$ (where (x_1, x_2) are the coordinates of a point in $\mathbb{R}^2$), when $\vec{b} = (1, 0)$, the 2-D model of [103] reduces to the system of coupled

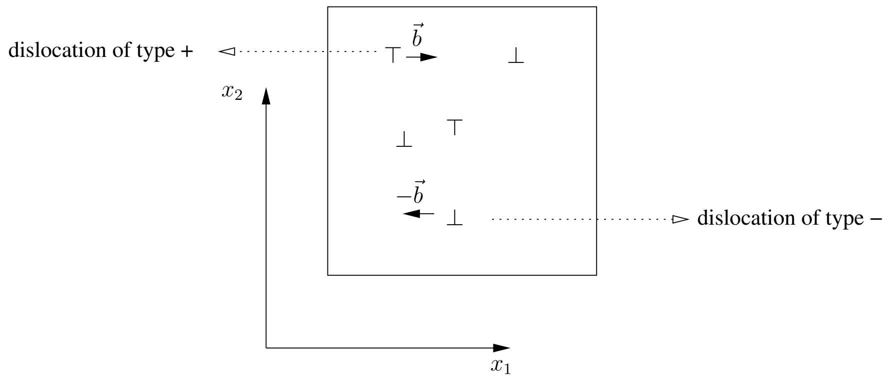

FIG. 9.1 – The cross-section of the dislocations lines.

1-D non local Hamilton-Jacobi equations (see Section 3)

$$\begin{cases} (\rho_+)_t = -\left(\rho_+ - \rho_- + \displaystyle\int_0^1 (\rho_+(x,t) - \rho_-(x,t))\, dx + L(t)\right) |D\rho_+| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad \mathbb{R} \times (0,T) \\ (\rho_-)_t = \left(\rho_+ - \rho_- + \displaystyle\int_0^1 (\rho_+(x,t) - \rho_-(x,t))\, dx + L(t)\right) |D\rho_-| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad \mathbb{R} \times (0,T) \end{cases} \tag{9.1}$$

where ρ_+, ρ_- are the unknown scalars such that $(\rho_+ - \rho_-)$ represents the plastic deformation, their space derivatives $D\rho_\pm := \frac{\partial \rho_\pm}{\partial x}$ are the dislocations densities and $L(t)$ represents the exterior shear stress field. From a physical viewpoint, $D\rho_\pm \geq 0$, however, here we do not make this assumption to remain on a more general framework. The initial conditions for the system (9.1) are defined as follows :

$$\rho_\pm(x,0) = \rho^0_\pm(x) = P^0_\pm(x) + L_0 x \text{ on } \mathbb{R} \tag{9.2}$$

where $P^0_\pm$ are periodic of period 1 and Lipschitz continuous. In particular, $\rho^0_+ - \rho^0_-$ is a 1-periodic function. L_0 is a given constant which is the total densities of type $\pm$, *i.e.* we suppose that initially, we have the same total density of type $+$ and $-$.

1.2 Main Results

The first goal of our paper is to prove the existence and uniqueness for the solution of the non-local system (9.1)-(9.2). A natural framework for our study is the viscosity solution theory. We refer to Barles [18], Bardi, Capuzzo-dolcetta [17] and Crandall, Ishii, Lions [61] for a good introduction to this theory in the scalar case. We also refer to Ishii, Koike [117] and Ishii [115] for the vectorial case and to Engler, Lenhart [76], Ishii, Koike [118], Lenhart [133], Lenhart, Belbas [134], Lenhart, Yamada [135] and Yamada [184] for some applications.

We have the following existence and uniqueness result for the non local system :

Theorem 1.1. (Existence and uniqueness for the non-local problem) *For all $T > 0$, for all $L_0 \in \mathbb{R}$, suppose that $\rho^0_\pm \in \mathrm{Lip}(\mathbb{R})$ satisfy (9.2) and $L \in W^{1,\infty}(\mathbb{R}^+)$. Then, the system (9.1)-(9.2) admits a unique viscosity solution $\rho = (\rho_+, \rho_-)$. Moreover, this solution is uniformly Lipschitz continuous in space and time.*

Remark 1.2. *If at initial time, we have $D\rho^0_\pm \geq 0$, then this remains true for $t \geq 0$, i.e., $D\rho_\pm(x,t) \geq 0$ for all $(x,t) \in \mathbb{R} \times [0,T]$. This allows to treat the physical case where $D\rho_\pm \geq 0$.*

The main difficulty comes from the fact that the comparison principle does not hold because of the non-local term. In order to overcome this problem, we classically use a fixed point method by freezing the non-local term. In a first time, we give an existence and uniqueness result for the local problem (this is a simple adaptation of [117]). Then, we use Lipschitz estimates on the solution to prove the short time existence and uniqueness for the non-local system. In the third step, we obtained the result for all time by iterating the process.

Here, we are interested in the dislocations densities dynamics. Some others models have been proposed to describe the dynamics of dislocations lines. We recall some recent results. A non-local Hamilton-Jacobi equation have been proposed by Alvarez, Hoch, Le Bouar and Monneau [10] [9] for modelling dislocation dynamics. They also proved a short time existence and uniqueness result for this model. We also refer to Alvarez, Cardaliaguet, Monneau [6] and Barles, Ley [25] for a long time result under certain monotony assumptions and to Forcadel [87] for a short time result for dislocations dynamics with a mean curvature term.

The second result is a numerical analysis of the non-local system (9.1). We propose a numerical scheme for our non-local system. Then, we give an error estimate between the continuous solution and the numerical one.

We want to approximate the solution of (9.1)-(9.2). Given a mesh size $\Delta x,\ \Delta t$, we define

$$\Xi = \{i\Delta x,\ i \in \mathbb{Z}\} \quad \Xi_T = \Xi \times \{0, ..., (\Delta t)N_T\}$$

where N_T is the integer part of $T/\Delta t$. We refer generically to the lattice by Δ in the sequel. The discrete running point is (x_i, t_n) with $x_i = i(\Delta x),\ t_n = n(\Delta t)$. We assume that $\Delta x + \Delta t \leq 1$. The approximation of the solution ρ_k at the node (x_i, t_n) is written indifferently as $v_k(x_i, t_n)$ or $v^n_{k,i}$ according to whether we view it as a function defined on the lattice or as a sequence.

Now, we will introduce the numerical scheme. The main difficulty is due to the non-local term, which requires the availability of the solution we are intending to approximate. To solves this problem, we fix the solution $v_i^n = (v_{+,i}^n, v_{-,i}^n)$ at each time step on the interval $[t_n, t_{n+1}]$ and we apply the following monotone scheme,

$$v_i^0 = (v_{+,i}^0, v_{-,i}^0) = \tilde{\rho}^0(x_i) = (\tilde{\rho}_+^0, \tilde{\rho}_-^0), \tag{9.3}$$

where $\tilde{\rho}_\pm^0(x_i)$ is an approximation of $\rho_\pm^0(x_i)$; and $\forall k \in \{+,-\}$

$$v_{k,i}^{n+1} = v_{k,i}^n + \Delta t C_k^\Delta[v](x_i,t_n) \begin{cases} E^+\left(D^+v_{k,i}^n, D^-v_{k,i}^n\right) & \text{if } C_k^\Delta[v](x_i,t_n) \geq 0 \\ E^-\left(D^+v_{k,i}^n, D^-v_{k,i}^n\right) & \text{if not} \end{cases} \tag{9.4}$$

where

$$C_k^\Delta[v](x_i,t_n) = -k\left(v_{+,i}^n - v_{-,i}^n + a^\Delta[v](t_n)\right)$$

and the non-local term $a^\Delta[v](t_n)$ is given by

$$a^\Delta[v](t_n) = \sum_{i=0}^{N_x-1} \Delta x\left(v_+(x_i,t_n) - v_-(x_i,t_n)\right) + L(t_n) \tag{9.5}$$

where N_x is the integer part of $1/\Delta x$. $E^\pm$ are the approximation of the Euclidean norm proposed by Osher and Sethian [154] :

$$E^+(P,Q) = \left(\max(P,0)^2 + \min(Q,0)^2\right)^{\frac{1}{2}}, \tag{9.6}$$

$$E^-(P,Q) = \left(\min(P,0)^2 + \max(Q,0)^2\right)^{\frac{1}{2}}$$

and $D^+v_{k,i}^n$, $D^-v_{k,i}^n$ are the discrete gradient for all $n \in \{0,...,N_T\}$, $i \in \mathbb{Z}$ and $k \in \{+,-\}$:

$$D^+v_{k,i}^n = \frac{v_{k,i+1}^n - v_{k,i}^n}{\Delta x}, \tag{9.7}$$

$$D^-v_{k,i}^n = \frac{v_{k,i}^n - v_{k,i-1}^n}{\Delta x}.$$

Finally, we assume the following uniform CFL condition (see the beginning of Section 5.2 for more details)

$$\Delta t \leq \frac{1}{2L_2}\Delta x \tag{9.8}$$

where

$$L_2 = 2M + 2$$

with $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$.

We then have the following error estimate :

Theorem 1.3. (Discrete-continuous error estimate) *Let $T \geq 0$. Assume that $\Delta x + \Delta t \leq 1$, $L \in W^{1,\infty}(\mathbb{R} \times [0,T))$ and that the CFL condition (9.8) holds. Then there exists a constant $K > 0$ depending only on $\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$, $\|L\|_{W^{1,\infty}(0,T)}$ and $\max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ such that the error estimate between the continuous solution ρ of the system (9.1)-(9.2) and the discrete solution v of the finite difference scheme (9.3)-(9.4) is given by*

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |\rho_k - v_k| \leq K\left((T+\sqrt{T})\,(\Delta x + \Delta t)^{1/2} + \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|\right)$$

provided $K\left((T+\sqrt{T})(\Delta x + \Delta t)^{\frac{1}{2}} + \max_{k\in\{+,-\}} \sup_{\Xi}(\rho_k^0 - v_k^0)\right) \leq 1.$

Remark 1.4. *In the condition* $K\left((T+\sqrt{T})(\Delta x + \Delta t)^{\frac{1}{2}} + \max_{k\in\{+,-\}} \sup_{\Xi}(\rho_k^0 - v_k^0)\right) \leq 1$*, we can replace the right hand side by any positive constant.*

In fact in the proof of this theorem, we mimic the continuous problem by considering the approximate solution of (9.1) as a fixed point of a local system. We are inspired by [8] to prove a Crandall-Lions rate of convergence [63], between the continuous solution of (9.1) and the numerical one. As far as we know, this is the first error estimate of Crandall-Lions type for Hamilton-Jacobi systems. We also refer to Jakobsen, Karlsen [122] and Jakobsen, Karlsen, Risebro [123] where they proved an error estimate for a weakly coupled system of the form

$$(u_i)_t + H_i(t,x,u_i,Du_i) = G_i(t,x,u) \quad \text{in } \mathbb{R}^N \times (0,T) \tag{9.9}$$

for $i = 1,...,M$. Their error estimate is in $O(\Delta t)$ for a semi-discrete splitting algorithm that they propose to approach the solution of (9.9). However, we obtain an error estimate in $O(\sqrt{\Delta t + \Delta x})$ because we also discretize in space.

In the dynamics of dislocations lines case, the model have also been numerically studied by Alvarez, Carlini, Monneau and Rouy [7, 8]. In their paper, they proposed a numerical scheme for the non-local Hamilton-Jacobi equation and they proved a Crandall-Lions type rate of convergence.

Let us now explain how the paper is organised. In Section 2, we fix some notations. We present the formal derivation of the model in Section 3. Then, in Section 4, we study the continuous problem. First in Subsection 4.1, we give an existence and uniqueness result for a local system. Then, in Subsection 4.2, we prove Theorem 1.1 by using a fixed point method. In Section 5, we prove a Crandall-Lions type error estimate for the local problem and then we prove Theorem 1.3 on the non-local problem. Some numerical examples are displayed in Section 6 where we show some tests illustrating our error estimate and then an evolution approximation of dislocation densities.

2 Notation

For simplicity of presentation, we fix some notations :

1. *Order relation* : for $r = (r_1,\ r_2)\,,\ \ s = (s_1,\ s_2) \in \mathbb{R}^2$, we say that $r \leq s$ if $r_k \leq s_k$ for $k \in \{1, 2\}$.
2. *Addition vector-scalar* : for $r = (r_1,\ r_2) \in \mathbb{R}^2,\ \lambda \in \mathbb{R}$, we denote by $r + \lambda$ the vector $(r_1 + \lambda,\ r_2 + \lambda)$.
3. *P-periodic plus L_0-linear function* : we say that ρ is $P-$periodic plus L_0-linear if there exists a vectorial periodic in space function $P^\rho = (P^\rho_+, P^\rho_-)$ of period P and a constant L_0 such that

$$\rho(x,t) = P^\rho(x,t) + L_0 x = \left(P^\rho_+(x,t) + L_0 x,\ P^\rho_-(x,t) + L_0 x\right).$$

3 Modelling

We denote by $\mathbf{X}$ the vector $\mathbf{X} = (x_1, x_2)$. We consider a crystal with periodic deformation, namely the case where the total displacement of the crystal $U = (U_1, U_2) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^2$ can be decomposed in a 1-periodic displacement $u = (u_1, u_2)$ and a linear displacement $A(t)^t\mathbf{X}$ with $A(t)$ a given 2×2 matrix which represents the shear stress

$$A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}.$$

The displacement U is then given by

$$U(\mathbf{X}, t) = u(\mathbf{X}, t) + A(t)^t\mathbf{X}$$

and we define the total strain by

$$\varepsilon(U) = \frac{1}{2}(\nabla U + {}^t\nabla U) = \frac{1}{2}\left(\nabla u + {}^t\nabla u + A(t) + {}^tA(t)\right).$$

where the coefficients of ∇u are $(\nabla u)_{ij} = \dfrac{\partial u_i}{\partial x_j}$, $i, j \in \{1, 2\}$.
This total strain is decomposed in the form

$$\varepsilon(U) = \varepsilon^e(U) + \varepsilon^p,$$

where $\varepsilon^e(U)$ is the elastic deformation and ε^p the plastic deformation which is connected to the densities of dislocations by

$$\varepsilon^p = \varepsilon^0\left(\rho_+ - \rho_-\right), \tag{9.10}$$

where $\rho_\pm$ represent the edge dislocation of type $\pm$, such that $\vec{b}.\nabla\rho_\pm \geq 0$ is the density of dislocation of type $\pm$, $\vec{b} = (b_1, b_2)$ is the Burgers vector and

$$\varepsilon^0 = \frac{1}{2}\left(\vec{b} \otimes \vec{b}^\perp + \vec{b}^\perp \otimes \vec{b}\right) \tag{9.11}$$

where $\vec{b}^\perp$ is a vector orthogonal to $\vec{b}$ and $\left(\vec{b} \otimes \vec{b}^\perp\right)_{ij} = b_i b_j^\perp$.

The stress is then given by

$$\sigma = \Lambda : \varepsilon^e(U), \tag{9.12}$$

i.e. the coefficients of the matrix σ are :

$$\sigma_{ij} = \sum_{k,l\in\{1,2\}} \Lambda_{ijkl}\varepsilon^e_{kl}(U) \quad i,j \in \{1,2\}$$

with $\Lambda = (\Lambda_{ijkl})_{ijkl}$, $i,j,k,l = 1,2$, Λ_{ijkl} are the elastic constant coefficients of the material, satisfying for $m > 0$:

$$\sum_{ijkl=1,2} \Lambda_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq m \sum_{ij=1,2} \varepsilon_{ij}^2 \tag{9.13}$$

for all symmetric matrix $\varepsilon = (\varepsilon_{ij})_{ij}$ *i.e.* such that $\varepsilon_{ij} = \varepsilon_{ji}$.

The functions $\rho_\pm$ and u are then solutions of the coupled system (see Groma, Balogh [103], [102] and Groma [101]) :

$$\left\{\begin{array}{lll} \text{div}\,\sigma & = 0 & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \sigma & = \Lambda : (\varepsilon(U) - \varepsilon^p) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \varepsilon(U) & = \frac{1}{2}\left(\nabla u + {}^t\nabla u + A(t) + {}^tA(t)\right) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \varepsilon^p & = \varepsilon^0\left(\rho_+ - \rho_-\right) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ & & \\ (\rho_\pm)_t & = \pm(\sigma : \varepsilon^0)\vec{b}.\nabla\rho_\pm & \text{in } \mathbb{R}^2 \times (0\ \ T), \end{array}\right. \tag{9.14}$$

i.e. in the coordinates

$$\begin{cases} \sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} & = 0 & \text{in } \mathbb{R}^2 \times (0 \;\; T), \\ \sigma_{ij} & = \sum_{k,l \in \{1,2\}} \Lambda_{ijkl} \left(\varepsilon_{kl}(U) - \varepsilon^p_{kl} \right) & \text{in } \mathbb{R}^2 \times (0 \;\; T), \\ \varepsilon_{ij}(U) & = \frac{1}{2} \Big((\nabla u)_{ij} + (\nabla u)_{ij} + A_{ij}(t) + A_{ji}(t) \Big) & \text{in } \mathbb{R}^2 \times (0 \;\; T), \\ \varepsilon^p_{ij} & = \varepsilon^0_{ij} \left(\rho_+ - \rho_- \right) & \text{in } \mathbb{R}^2 \times (0 \;\; T), \\ (\rho_\pm)_t & = \pm \left(\sum_{i,j \in \{1,2\}} \sigma_{ij} \varepsilon^0_{ij} \right) \vec{b}.\nabla \rho_\pm & \text{in } \mathbb{R}^2 \times (0 \;\; T), \end{cases} \tag{9.15}$$

where the unknowns of the system are $\rho_\pm$ and the displacement $u = (u_1, u_2)$ and with ε^0 defined by (9.11). The sign $\pm$ comes from $\pm$ in $\pm\vec{b}$.

To simplify, we consider the homogeneous case. The coefficients Λ_{ijkl} are such that

$$\sigma = 2\mu\varepsilon^e(U) + \lambda tr(\varepsilon^e(U))I_d, \tag{9.16}$$

where $\mu > 0$ and $\lambda + \mu > 0$ (consequence of (9.13)) are the Lamé coefficients and I_d the identity matrix. Then, the following *lemma* holds :

Lemma 3.1. (Equivalence between 2-D and 1-D models) *If we assume that the Burgers vector is $\vec{b} = (1, 0)$, and that the densities of dislocations and u only depend on one variable $x = x_1 + x_2$ (as shown in Figure 9.2), the 2-D problem (9.14), with Λ defined by (9.16) is equivalent to the 1-D problem*

$$\begin{cases} (\rho_+)_t & = -C_1 \left((\rho_+ - \rho_-) + C_2 \int_0^1 (\rho_+ - \rho_-) + L(t) \right) D\rho_+ & \text{in } \mathbb{R} \times (0 \;\; T) \\ (\rho_-)_t & = C_1 \left((\rho_+ - \rho_-) + C_2 \int_0^1 (\rho_+ - \rho_-) + L(t) \right) D\rho_- & \text{in } \mathbb{R} \times (0 \;\; T) \end{cases} \tag{9.17}$$

where $L(t) = -\frac{(\lambda+2\mu)}{(\lambda+\mu)}(A_{12}(t) + A_{21}(t))$, $C_1 = \frac{(\lambda+\mu)\mu}{\lambda+2\mu}$, and $C_2 = \frac{\mu}{(\lambda+\mu)}$.

Proof of lemma 3.1

We can rewrite the first equation of (9.14) and (9.16) as

$$\operatorname{div}\left(2\mu\varepsilon(U) + \lambda tr(\varepsilon(U))I_d\right) = \operatorname{div}\left(2\mu\varepsilon^p + \lambda tr(\varepsilon^p)I_d\right).$$

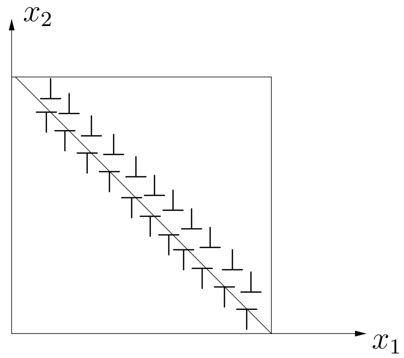

FIG. 9.2 – 1D sub-model for invariance by translation in the direction (-1,1)

This implies by (9.10)

$$\mu\Delta u + (\lambda+\mu)\nabla(\text{div } u) = \mu \begin{pmatrix} \dfrac{\partial(\rho_+ - \rho_-)}{\partial x_2} \\ \\ \dfrac{\partial(\rho_+ - \rho_-)}{\partial x_1} \end{pmatrix}.$$

Using the fact that $x = x_1 + x_2$, yields

$$2\mu \begin{pmatrix} \dfrac{\partial^2 u_1}{\partial x^2} \\ \\ \dfrac{\partial^2 u_2}{\partial x^2} \end{pmatrix} + (\lambda+\mu) \begin{pmatrix} \dfrac{\partial^2(u_1+u_2)}{\partial x^2} \\ \\ \dfrac{\partial^2(u_1+u_2)}{\partial x^2} \end{pmatrix} = \mu \begin{pmatrix} \dfrac{\partial(\rho_+ - \rho_-)}{\partial x} \\ \\ \dfrac{\partial(\rho_+ - \rho_-)}{\partial x} \end{pmatrix}.$$

Now, by adding the two above equations, we obtain

$$\frac{\partial^2(u_1+u_2)}{\partial x^2} = \frac{\mu}{\lambda+2\mu}\left(\frac{\partial(\rho_+ - \rho_-)}{\partial x}\right).$$

Integrating the above equation yields, since u is 1-periodic :

$$\frac{\partial(u_1+u_2)}{\partial x} = \frac{\mu}{\lambda+2\mu}\left((\rho_+ - \rho_-) - \int_0^1 (\rho_+ - \rho_-)\right). \tag{9.18}$$

Using the fact that

$$(\sigma : \varepsilon^0) = \sigma_{12} = 2\mu(\varepsilon^e(U))_{12} = \mu\left(\frac{\partial(u_1+u_2)}{\partial x} + A_{12}(t) + A_{21}(t) - (\rho_+ - \rho_-)\right)$$

and (9.18) yields

$$(\sigma : \varepsilon^0) = -\frac{(\lambda+\mu)\mu}{\lambda+2\mu}\left((\rho_+ - \rho_-) + \frac{\mu}{2(\lambda+\mu)}\int_0^1 (\rho_+ - \rho_-) + L(t)\right)$$

where $L(t) = -\dfrac{(\lambda + 2\mu)}{(\lambda + \mu)}(A_{12}(t) + A_{21}(t))$. We then deduce, if $\vec{b} = (1, 0)$, that the system (9.14) can be rewritten as (9.17). As the constant C_1, C_2 are positive, to simplify the notations, we can put them to 1 in the following without lost of generality on the system (9.1).

4 The continuous problem

To prove the existence and uniqueness result for the non-local problem, we use a fixed point method. In order to do that we freeze the non-local term and we study the following local problem

$$\begin{cases} (\rho_+)_t = -\left(\rho_+ - \rho_- + a(t)\right)|D\rho_+| & \text{on } \mathbb{R} \times (0, T), \\ (\rho_-)_t = \left(\rho_+ - \rho_- + a(t)\right)|D\rho_-| & \text{on } \mathbb{R} \times (0, T), \\ \rho_+(\cdot, 0) = \rho_+^0 & \text{on } \mathbb{R}, \\ \rho_-(\cdot, 0) = \rho_-^0 & \text{on } \mathbb{R}. \end{cases} \tag{9.19}$$

The assumptions are the following :

(H1) $a \in W^{1,\infty}(\mathbb{R}^+)$,

(H2) $\rho^0 = (\rho_+^0, \rho_-^0)$ is 1−periodic plus L_0−linear, *i.e.*, $\rho_\pm^0(x) = P_\pm^0(x) + L_0 x$ where $P_\pm^0$ are periodic of period 1 and L_0 is a constant.

(H3) $P_\pm^0 \in \mathrm{Lip}(\mathbb{R})$.

4.1 The local problem

In this subsection, we will show some existence and uniqueness results for the local Hamilton-Jacobi system (9.19) within the framework of viscosity solution. We denote by $USC(\mathbb{R}\times(0,T))$ (resp. $LSC(\mathbb{R}\times(0,T))$) the set of locally bounded upper (resp. lower) semi-continuous functions. For $k \in \{+, -\}$, we define the following Hamiltonian

$$H_k(t, \rho, p) = k(\rho_+ - \rho_- + a(t))|p|.$$

We recall the definition of viscosity solution for Problem (9.19), proposed by Ishii, Koike [117].

Definition 4.1. (Subsolutions, supersolutions, solutions) *: A function $\rho \in USC(\mathbb{R}\times]0,T[)$ is a viscosity subsolution of (9.19), if*

- $\rho(\cdot, t = 0) \leq \rho_0$
- *for all $k \in \{+, -\}$ and for any test-function $\phi \in C^1(\mathbb{R}\times]0,T[)$ such that $\rho_k - \phi$ reaches a local maximum at a point $(x_0, t_0) \in \mathbb{R}\times]0,T[$, we have*

$$\phi_t(x_0, t_0) + H_k(t_0, \rho(x_0, t_0), D\phi(x_0, t_0)) \leq 0$$

In a similar way, a function $\rho \in LSC(\mathbb{R}\times]0,T[)$ *is a viscosity supersolution of (9.19) if*

- $\rho(\cdot, t=0) \geq \rho_0$
- *for all* $k \in \{+,-\}$ *and for any test-function* $\phi \in C^1(\mathbb{R}\times]0,T[)$ *such that* $\rho_k - \phi$ *reaches a local minimmum at a point* $(x_0,t_0) \in \mathbb{R}\times]0,T[$*, we have*

$$\phi_t(x_0,t_0) + H_k(t_0, \rho(x_0,t_0), D\phi(x_0,t_0)) \geq 0$$

Finally, ρ *is a viscosity solution of (9.19) if and only if* ρ *is sub- and supersolution of (9.19).*

The key point is that our system is quasi monotone in the sense of Ishii, Koike [117, (A.1)], (see *Lemma* 4.2 below) and so we can extend their results to our system in unbounded domain and with unbounded initial condition using the well-known arguments of the scalar case.

Lemma 4.2. (Quasi-monotony of the Hamiltonian)
The Hamiltonian H *is quasi-monotone, i.e., for all vectors* r *and* s *such that*

$$r_j - s_j = \max_{k\in\{+,-\}} (r_k - s_k) \geq 0$$

then

$$H_j^1(t,r,p) - H_j^1(t,s,p) \geq 0. \tag{9.20}$$

Proof of Lemma 4.2 :
Let r and s be two vectors such that $r_j - s_j = \max_{k\in\{+,-\}} (r_k - s_k) \geq 0$. We have

$$\begin{aligned} H_j(t,r,p) - H_j(t,s,p) &= j(r_+ - r_- + a(t))|p| - j(s_+ - s_- + a(t))|p| \\ &= j|p|((r_+ - s_+) - (r_- - s_-)) \\ &= |p| sign((r_+ - s_+) - (r_- - s_-))((r_+ - s_+) - (r_- - s_-)) \\ &= |p||(r_+ - s_+) - (r_- - s_-)| \geq 0. \end{aligned}$$

This ends the proof.

Proposition 4.3. (Comparison principle)
Let $\rho \in USC(\Omega\times]0,T[)$ *and* $v \in LSC(\Omega\times]0,T[)$ *be respectively sub and supersolutions of (9.1). We assume that there exists* $C > 0$ *such that*

$$\rho_0(x) - Ct \leq \rho, v \leq \rho_0(x) + Ct. \tag{9.21}$$

Then if $\rho(\cdot,0) \leq v(\cdot,0)$ in $\mathbb{R}$ *then* $\rho \leq v$ *in* $\mathbb{R}\times]0,T[$.

Proof of Proposition 4.3
Let us denote by $M_{\text{sup}} = \max\limits_k \sup\limits_{\mathbb{R}\times[0,T]} (\rho_k - v_k)$. It is sufficient to prove that this maximum is non positive. Let us suppose by contradiction the positivity of M_{sup}. We duplicate the variables by considering, for all ε, β, η and α positive

$$\psi(x, y, t, s, k) = \rho_k(x, t) - v_k(y, s) - \frac{|x-y|^2}{2\varepsilon} - \frac{|t-s|^2}{2\beta} - \frac{\eta}{T-t} - \alpha(|x|^2 + |y|^2).$$

We note that $\psi(x, y, t, s, k)$ is USC in $(\mathbb{R}\times[0, T[)^2$, because of the term $\frac{|x-y|^2}{2\varepsilon} + \frac{|t-s|^2}{2\beta}$. We can think that the maximum of ψ noted $M(\varepsilon, \beta, \alpha, \eta) = \sup\limits_{x,y,t,s,k} \psi(x, y, t, s, k)$, is similar with M_{sup}.
This idea is justify by the following lemma

Lemma 4.4. *: Let $(\bar{x}, \bar{y}, \bar{t}, \bar{s}, \bar{k})$ be a maximum of ψ. If we define $M' = \lim\limits_{h\to 0} M_h$, where $M_h = \sup\limits_{\substack{|x-y|\le h \\ |t-s|\le h}} (\rho(x, t) - v(y, s))$ Then the following properties hold*

1. $\lim\limits_{\alpha\to 0} \alpha|\bar{x}| = \lim\limits_{\alpha\to 0} \alpha|\bar{y}| = 0$
2. $\limsup\limits_{(\varepsilon,\beta,\alpha,\eta)\to 0} M(\varepsilon, \beta, \alpha, \eta) = M'$
3. $\limsup\limits_{(\varepsilon,\beta,\alpha,\eta)\to 0} u_{\bar{k}}(\bar{x}, \bar{t}) - v_{\bar{k}}(\bar{y}, \bar{s}) = M'$
4. $\frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} \to 0$ *and* $\frac{|\bar{t}-\bar{s}|^2}{2\beta} \to 0$ *when* $(\varepsilon, \beta, \alpha, \eta) \to 0$
5. *$\bar{t}, \bar{s}$ are positive if ε, β, α and η are sufficiently small.*

The proof is postponed.

We take ε, β, α and η small enough such that $\bar{t} > 0$ and $\bar{s} > 0$ (see Lemma 4.4). Using the fact that ρ and v are sublinear, we deduce that

$$\liminf_{|x|,|y|\to\infty} \psi(x, y, t, s) = -\infty$$

and so the function ψ reaches a maximum at a point $(\bar{x}, \bar{y}, \bar{t}, \bar{s}, \bar{k})$. We then deduce that the function

$$(x, t) \to \rho_{\bar{k}}(x, t) - \left[v_{\bar{k}}(\bar{y}, \bar{s}) + \frac{|x-\bar{y}|^2}{2\varepsilon} + \frac{|t-\bar{s}|^2}{2\beta} + \frac{\eta}{T-t} + \alpha(|x|^2 + |\bar{y}|^2) \right]$$

reaches a maximum at $(\bar{x}, \bar{t})$. By using the test-function

$$\phi(x, t) = v_{\bar{k}}(\bar{y}, \bar{s}) + \frac{|x-\bar{y}|^2}{2\varepsilon} + \frac{|t-\bar{s}|^2}{2\beta} + \frac{\eta}{T-t} + \alpha(|x|^2 + |\bar{y}|^2)$$

and the fact that ρ is a subsolution of (9.19), we deduce

$$\frac{(\bar{t}-\bar{s})}{\beta}+\frac{\eta}{(T-\bar{t})^2}+H_{\bar{k}}\left(\bar{t},u(\bar{x},\bar{t}),\frac{(\bar{x}-\bar{y})}{\varepsilon}+2\alpha\bar{x}\right)\leq 0. \tag{9.22}$$

In the same way, we have

$$\frac{(\bar{t}-\bar{s})}{\beta}+H_{\bar{k}}\left(\bar{s},v(\bar{y},\bar{s}),\frac{(\bar{x}-\bar{y})}{\varepsilon}-2\alpha\bar{y}\right)\geq 0. \tag{9.23}$$

By subtracting (9.23) to (9.22), we deduce

$$\frac{\eta}{(T-\bar{t})^2}+H_{\bar{k}}\left(\bar{t},\rho(\bar{x},\bar{t}),\frac{(\bar{x}-\bar{y})}{\varepsilon}+2\alpha\bar{x}\right)-H_{\bar{k}}\left(\bar{s},v(\bar{y},\bar{s}),\frac{(\bar{x}-\bar{y})}{\varepsilon}-2\alpha\bar{y}\right)\leq 0. \tag{9.24}$$

By using Lemma 4.4, we have, up to extraction of a subsequence

$$\lim_{\beta\to 0}\bar{x}=\tilde{x},\quad \lim_{\beta\to 0}\bar{y}=\tilde{y},$$
$$\lim_{\beta\to 0}\bar{t}=\lim_{\beta\to 0}\bar{s}=\tau.$$

Sending β to 0 in (9.24), we deduce

$$\frac{\eta}{(T-\tau)^2}+H_{\tilde{k}}\left(\tau,\rho(\tilde{x},\tau),\frac{(\tilde{x}-\tilde{y})}{\varepsilon}+2\alpha\tilde{x}\right)-H_{\tilde{k}}\left(\tau,v(\tilde{y},\tau),\frac{(\tilde{x}-\tilde{y})}{\varepsilon}-2\alpha\tilde{y}\right)\leq 0. \tag{9.25}$$

with $\tilde{k}=\lim\limits_{\beta\to 0}\bar{k}$.

Moreover, we have

$$\rho_{\tilde{k}}(\tilde{x},\tau)-v_{\tilde{k}}(\tilde{y},\tau)\geq \rho_k(\tilde{x},\tau)-v_k(\tilde{y},\tau)\geq 0.$$

By adding and by subtracting the term $H_{\tilde{k}}\left(\tau,\rho(\tilde{x},\tau),\frac{(\tilde{x}-\tilde{y})}{\varepsilon}-2\alpha\tilde{y}\right)$ in the inequality (9.25) and by using Lemma 4.2, we deduce that

$$\frac{\eta}{(T-\tau)^2}+H_{\tilde{k}}\left(\tau,\rho(\tilde{x},\tau),\frac{(\tilde{x}-\tilde{y})}{\varepsilon}+2\alpha\tilde{x}\right)-H_{\tilde{k}}\left(\tau,\rho(\tilde{x},\tau),\frac{(\tilde{x}-\tilde{y})}{\varepsilon}-2\alpha\tilde{y}\right)\leq 0$$

and so

$$\frac{\eta}{(T-\tau)^2}+\tilde{k}(\rho^+(\tilde{x},\tau)-\rho^-(\tilde{x},\tau)+a(\tau))\left(\left|\frac{(\tilde{x}-\tilde{y})}{\varepsilon}+2\alpha\tilde{x}\right|-\left|\frac{(\tilde{x}-\tilde{y})}{\varepsilon}-2\alpha\tilde{y}\right|\right)\leq 0.$$

Using the fact that $\tilde{k}((\rho^+(\tilde{x},\tau)-\rho^-(\tilde{x},\tau)+a(\tau))$ is bounded (see (9.21)), and sending $\alpha\to 0$, we obtain using Lemma 4.4

$$\frac{\eta}{(T-\tau)^2}\leq 0,$$

this contradiction ends the proof of the proposition.

Proof of lemma 4.4

Thanks to the positivity of $M_{\sup}$, we have

$$M(\varepsilon, \beta, \alpha, \eta) > 0$$

for η and α small enough. We then deduce

$$\begin{aligned}
\alpha(|\bar{x}|^2 + |\bar{y}^2|) &\leq \rho_{\bar{k}}(\bar{x}, \bar{t}) - v_{\bar{k}}(\bar{y}, \bar{s}) - \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} \\
&\leq \rho_{\bar{k},0}(\bar{x}) - \rho_{\bar{k},0}(\bar{y}) + C(\bar{t} + \bar{s}) - \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} \\
&\leq K(1 + |\bar{x} - \bar{y}|) - \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} \\
&\leq K + \tfrac{1}{2}(K^2 + |\bar{x} - \bar{y}|^2) - \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} \\
&\leq K_1
\end{aligned}$$

for $\varepsilon \leq 1$, where we have used (9.21) for the second line. Multiplying the previous inequality by α, yields

$$\lim_{\alpha \to 0} \alpha|\bar{x}| = \lim_{\alpha \to 0} \alpha|\bar{y}| = 0. \tag{9.26}$$

In the same way, we have

$$\frac{|\bar{x} - \bar{y}|^2}{4\varepsilon} + \frac{|\bar{t} - \bar{s}|^2}{2\beta} \leq K_1 \tag{9.27}$$

and so

$$\lim_{\varepsilon \to 0} |\bar{x} - \bar{y}|^2 = \lim_{\beta \to 0} |\bar{t} - \bar{s}|^2 = 0. \tag{9.28}$$

We recall that $M_h = \sup\limits_{\substack{|x-y| \leq h \\ |t-s| \leq h}} (\rho(x,t) - v(y,s))$. Let $(x_n^h, y_n^h, t_n^h, s_n^h)$ be such that

$$\rho(x_n^h, t_n^h) - v(y_n^h, s_n^h) \geq M_h - \frac{1}{n}$$

with $|x_n^h - y_n^h| \leq h$ and $|t_n^h - s_n^h| \leq h$. We then have

$$\begin{aligned}
&M_h - \frac{1}{n} - \frac{h^2}{2\varepsilon} - \frac{h^2}{2\beta} - \alpha\left(|x_n^h|^2 + |y_n^h|^2\right) \\
\leq &\rho(x_n^h, t_n^h) - v(y_n^h, s_n^h) - \frac{|x_n^h - y_n^h|^2}{2\varepsilon} - \frac{|t_n^h - s_n^h|^2}{2\beta} - \alpha\left(|x_n^h|^2 + |y_n^h|^2\right) \\
\leq &M(\varepsilon, \beta, \alpha, \eta) \\
\leq &\rho(\bar{x}, \bar{t}) - v(\bar{y}, \bar{s}).
\end{aligned}$$

We send $\beta \to 0$ and then $\alpha \to 0$ and we obtain

$$\begin{aligned} M_h - \frac{1}{n} - \frac{h^2}{2\varepsilon} - \frac{h^2}{2\beta} &\leq \liminf_{\alpha\to 0} \liminf_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}) \\ &\leq \limsup_{\alpha\to 0} \limsup_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}). \end{aligned}$$

Passing to the limit in h, yields

$$\begin{aligned} M' - \frac{1}{n} &\leq \liminf_{\alpha\to 0} \liminf_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}) \\ &\leq \limsup_{\alpha\to 0} \limsup_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}). \end{aligned}$$

We then take $\lim\limits_{\varepsilon\to 0}$ and get

$$\begin{aligned} M' - \frac{1}{n} &\leq \liminf_{\varepsilon\to 0} \liminf_{\alpha\to 0} \liminf_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}) \\ &\leq \limsup_{\varepsilon\to 0} \limsup_{\alpha\to 0} \limsup_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}) \\ &\leq \limsup_{\varepsilon\to 0} \limsup_{\alpha\to 0} \left(\sup_{\substack{|x-y|\leq K\sqrt{\varepsilon} \\ |t-s|\leq K\sqrt{\beta}}} \rho(x,t) - v(y,s) \right) \\ &\leq \lim_{h\to 0} \sup_{\substack{|x-y|\leq h \\ |t-s|\leq h}} \rho(x,t) - v(y,s) \\ &\leq M'. \end{aligned}$$

We then deduce that

$$\liminf_{\varepsilon\to 0} \liminf_{\alpha\to 0} \liminf_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}) = \limsup_{\varepsilon\to 0} \limsup_{\alpha\to 0} \limsup_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}) = M'.$$

In the same way, we get

$$\liminf_{\varepsilon\to 0} \liminf_{\alpha\to 0} \liminf_{\beta\to 0} M(\varepsilon,\beta,\alpha,\eta) = \limsup_{\varepsilon\to 0} \limsup_{\alpha\to 0} \limsup_{\beta\to 0} M(\varepsilon,\beta,\alpha,\eta) = M'$$

and we deduce that

$$\limsup_{\varepsilon\to 0} \limsup_{\alpha\to 0} \limsup_{\beta\to 0} \left(\frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} + \frac{|\bar{t}-\bar{s}|^2}{2\beta} \right) = 0$$

Finally, let us suppose, for example, that $\bar{t}=0$, then, because of $u_0 \le v_0$, we have

$$\begin{aligned} u_{\bar{k}}(\bar{x},0)-v_{\bar{k}}(\bar{y},\bar{s}) &\le u_{\bar{k}}(\bar{x},0)-v_{\bar{k}}(\bar{x},0)+v_{\bar{k}}(\bar{x},0)-v_{\bar{k}}(\bar{y},\bar{s}) \\ &\le v_{\bar{k}}(\bar{x},0)-v_{\bar{k}}(\bar{y},\bar{s}). \end{aligned}$$

However $M'>0$ and when ε, β, α and η are sufficiently small, we get a contradiction. When $\bar{s}=0$, a similar proof could be given which ends the proof of the Lemma 4.4.

We now prove existence for the problem (9.19). We use the Perron's method for systems proved by Ishii and Koike in [117]. We then have the following theorem

Theorem 4.5. (Existence for the local problem)
Assume (H1)-(H2)-(H3), then there exists a unique viscosity solution ρ of problem (9.19). Moreover, the solution satisfies

$$\rho_{\pm,0}(x)-(M+\|a\|_{L^\infty})B_0 t \le \rho_{\pm}(x,t) \le \rho_{\pm,0}(x)+(M+\|a\|_{L^\infty})B_0 t, \qquad (9.29)$$

where $M=\|P_{+,0}-P_{-,0}\|_{L^\infty}$ and $B_0=\|D\rho_0\|_{L^\infty}$.

Proof of Theorem 4.5
By Perron's method, it suffices to construct viscosity sub and supersolution of (9.19). We claim that $\overline{\rho}=\rho_0+(M+\|a\|_{L^\infty})B_0 t$ and $\underline{\rho}=\rho_0-(M+\|a\|_{L^\infty})B_0 t$ are respectively super and subsolution. Indeed, formally

$$\begin{aligned} -k\left(\overline{\rho}_+-\overline{\rho}_-+a(t)\right)|D\overline{\rho}_k| &\le \left|P_+^0-P_-^0+a(t)\right| |D\rho_{k,0}| \\ &\le \left(\|P_+^0-P_-^0\|_{L^\infty}+\|a\|_{L^\infty}\right)\|D\rho_0\|_{L^\infty} \\ &\le (M+\|a\|_{L^\infty})B_0=(\bar{\rho}_k)_t. \end{aligned}$$

The proof for the subsolution is exactly the same and we skip it. To achieve the proof, it suffices to use the comparison principle to obtain the uniqueness.

Proposition 4.6. (Regularity of the solution)
The solution ρ of (9.19) is Lipschitz continuous in space and time. More precisely, ρ satisfies :

$$\|D\rho_{\pm}\|_{L^\infty} \le B_0 \qquad (9.30)$$

and

$$\|(\rho_{\pm})_t\|_{L^\infty} \le (L_a T+M+\|a\|_{L^\infty})B_0, \qquad (9.31)$$

where L_a is the Lipschitz constant of a, $B_0=\|D\rho_0\|_{L^\infty}$ and $M=\|\rho_{+,0}-\rho_{-,0}\|_{L^\infty}$.

Proof of Proposition 4.6
We set $\rho^h_\pm(x,t) = \rho_\pm(x+h,t)$. Since Problem (9.19) is invariant in space, ρ^h and $\rho^h + \|\rho_{\pm,0}(\cdot) - \rho_{\pm,0}(\cdot + h)\|_{L^\infty}$ are still solutions . Using comparison principle, yields

$$\begin{aligned}|\rho_\pm - \rho^h_\pm| \leq & \|\rho_{\pm,0}(\cdot) - \rho_{\pm,0}(\cdot + h)\|_{L^\infty}\\ \leq & B_0 h.\end{aligned}$$

So, ρ is Lipschitz continuous in space.

For the estimate in time, we consider $v(x,t) = \rho(x,t+h)$. It is easy to check that v is a supersolution of

$$\begin{cases} (v_+)_t = -\left(v_+ - v_- + a(t)\right)|Dv_+| - L_a B_0 h \\ (v_-)_t = \left(v_+ - v_- + a(t)\right)|Dv_-| - L_a B_0 h. \end{cases} \tag{9.32}$$

Indeed, formally

$$\begin{aligned}(v_k)_t(x,t) =& (\rho_k)_t(x,t+h)\\ =& -k\left(\rho_+(x,t+h) - \rho_-(x,t+h) + a(t+h)\right)|D\rho_k(x,t+h)|\\ =& -k\left(v_+(x,t) - v_-(x,t) + a(t)\right)|Dv_k(x,t)| + k(a(t) - a(t+h)||Dv_k|\\ \geq & -k\left(v_+(x,t) - v_-(x,t) + a(t)\right)|Dv_k(x,t)| - L_a h B_0.\end{aligned}$$

Moreover, $\tilde{\rho} = \rho - L_a B_0 h t - (M + \|a\|_{L^\infty}) B_0 h$ is a solution of the same equation and satisfies $v(\cdot,0) \geq \tilde{\rho}(\cdot,0)$ (see Theorem 4.5). So, by comparison principle for (9.32), we deduce that

$$\rho(x,t) - \rho(x,t+h) \leq (L_a t + M + \|a\|_{L^\infty}) B_0 h \leq (L_a T + M + \|a\|_{L^\infty}) B_0 h.$$

Using the same arguments with $\rho(x,t-h)$, we deduce that ρ is Lipshitz continuous in time.

Proposition 4.7. (Caracterization of the solution)
The solution ρ is $(1, L_0)-$periodic plus linear, i.e.

$$\rho = \begin{pmatrix} P^\rho_+ + L_0 x \\ P^\rho_- + L_0 x \end{pmatrix},$$

where the linear part L_0 is the same of the one of ρ_0 and the period of $P^\rho_\pm$ is 1.

Proof of Poposition 4.7
We set $P^\rho_k = \rho_k - L_0 x$. It suffices to show that P^ρ_k are periodic of period 1. The vector function P^ρ satisfies

$$\begin{cases} (P^\rho_+)_t = -\left(P^\rho_+ - P^\rho_- + a(t)\right)|DP^\rho_+ + L_0| \\ (P^\rho_-)_t = \left(P^\rho_+ - P^\rho_- + a(t)\right)|DP^\rho_- + L_0| \\ P^\rho_+(\cdot,0) = P^0_+ \\ P^\rho_-(\cdot,0) = P^0_-. \end{cases}$$

We then set $v(x,t) = P^\rho(x+1,t)$. By the periodicity of $P^0_\pm$, we obtain that v satisfies the same problem as P^ρ and so, by uniqueness, $v = P^\rho$. This achieves the proof of the proposition.

Finally, we proved the following theorem :

Theorem 4.8. (The local problem) *Let $T \geq 0$. Assume (H1)-(H2)-(H3). We set $M = \|P^0_+ - P^0_-\|_{L^\infty(\mathbb{R})}$ and $B_0 = \max_{k\in\{+,-\}} \|D\rho^0_k\|_{L^\infty(\mathbb{R})}$. Then, the following holds :*

(i) **Comparison principle.** *Let $\rho \in USC(\mathbb{R}\times(0,T))$ and $v \in LSC(\mathbb{R}\times(0,T))$ be respectively sub and super-solution of (9.1)-(9.2). We assume that there exists a constant $C > 0$ such that* (9.21) *holds. If* $\rho(\cdot,0) \leq v(\cdot,0)$ in $\mathbb{R}$ *then $\rho \leq v$ in* $\mathbb{R}\times[0,T]$.

(ii) **Existence.** *There exists a unique viscosity solution ρ of problem (9.19) satisfying* (9.29). *Moreover, the solution is $1-$periodic plus L_0-linear.*

(iii) **Regularity.** *The solution ρ of (9.19) is Lipschitz continuous in space and time and satisfies*

$$\max_{k\in\{+,-\}} \|D\rho_k\|_{L^\infty(\mathbb{R}\times(0,T))} \leq B_0,$$

$$\max_{k\in\{+,-\}} \|(\rho_k)_t\|_{L^\infty(\mathbb{R}\times(0,T))} \leq B_0(M + \|a\|_{L^\infty(0,T)}).$$

(iv) **Estimate on the solution.** *The solution ρ satisfies*

$$\|\rho_+ - \rho_-\|_{L^\infty(\mathbb{R}\times(0,T))} \leq \|\rho^0_+ - \rho^0_-\|_{L^\infty(\mathbb{R})}.$$

Proof of Theorem 4.8

The comparison principle is just an extension of the one of Ishii, Koike [117, Th 4.7] for quasi-monotone Hamiltonians. For the existence, it suffices to use Perron's method by remarking that $\rho \pm (M + \|a\|_{L^\infty(0,T)})B_0 t$ are resp. super and sub-solution of (9.19). The fact that ρ is $1-$periodic plus L_0-linear comes from the fact that $\rho(x+1,t) + L_0$ is also solution of (iv).

The Lipschitz estimate in space comes from the fact that Problem (9.19) is invariant by space translation. To obtain the Lipschitz estimate in time, it is sufficient to bound the velocity using (9.29).

We now prove (iv). We set

$$m_+(t) = \sup_{x\in(0,1)} \rho_+(x,t) \quad \text{and} \quad m_-(t) = \inf_{x\in(0,1)} \rho_-(x,t).$$

It is easy to chech that m_+ (resp. m_-) is subsolution (resp. supersolution) of $u_t = 0$ which implies the upper bound of (iv). The lower bound is proved similarly. This ends the proof of the theorem.

4.2 The non-local problem

Before to prove Theorem 1.1, we need the following *lemma* :

Lemma 4.9. (Stability of the solution with respect to the velocity) *Let $T \geq 0$. We consider for $i = 1, 2$ two different equations*

$$\begin{cases} (\rho^i_k)_t = -k\left(\rho^i_+ - \rho^i_- + a_i(t)\right)|D\rho^i_k| & \text{for} \quad k \in \{+,-\} \\ \rho^i_k(\cdot, 0) = \rho^0_k & \text{for} \quad k \in \{+,-\} \end{cases} \tag{9.33}$$

where the coefficients a_i satisfy $(H1)$ and the initial conditions $\rho^0 = (\rho^0_+, \rho^0_-)$ satisfy (H2)-(H3). Then, we have

$$\max_{k\in\{+,-\}} \|\rho^2_k - \rho^1_k\|_{L^\infty(\mathbb{R}\times(0,T))} \leq B_0 T \|a_2 - a_1\|_{L^\infty(0,T)},$$

where ρ^i for $i = 1, 2$ are the solutions of (9.33) given by Theorem 4.8.

Proof of *Lemma* 4.9
We set $K = \|a_2 - a_1\|_{L^\infty(0,T)}$. We remark that ρ^2 is a sub-solution of

$$(\rho_k)_t + k\left(\rho_+ - \rho_- + a_1(t)\right)|D\rho_k| - KB_0 = 0.$$

Moreover $\rho^1 + KB_0 t$ is solution of the same problem. By comparison principle, we then deduce

$$\max_{k\in\{+,-\}} \|\rho^2_k - \rho^1_k\|_{L^\infty(\mathbb{R}\times(0,T))} \leq KB_0 T.$$

This is the estimate we want.

We have the following *lemma* whose proof is trivial :

Lemma 4.10. (Stability of the velocity a) *Let ρ^1, ρ^2 be 1-periodic plus L_0-linear. We set $a[\rho^i](t) = \int_0^1 \rho^i_+(x,t) - \rho^i_-(x,t)dx + L(t)$. Then the following holds*

$$\|a[\rho^2] - a[\rho^1]\|_{L^\infty(0,T)} \leq 2 \max_{k\in\{+,-\}} \|\rho^2_k - \rho^1_k\|_{L^\infty(\mathbb{R}\times(0,T))}.$$

We now prove Theorem 1.1.

Proof of Theorem 1.1
We define the set :

$$U_T = \left\{ \rho = \begin{pmatrix} \rho_+ \\ \rho_- \end{pmatrix} \in \left(L^\infty_{\text{Loc}}\right)^2, s.t. \left| \begin{array}{l} \|\rho_+ - \rho_-\|_{L^\infty} \leq M \\ \rho \text{ is 1-periodic plus } L_0\text{-linear} \\ \displaystyle\max_{k\in\{+,-\}} \|D\rho_k\|_{L^\infty} \leq B_0 \\ \displaystyle\max_{k\in\{+,-\}} \|(\rho_k)_t\|_{L^\infty} \leq B_0(2M + \|L\|_{L^\infty(0,T)}) \end{array} \right. \right\},$$

where L_0 is defined in $(H2)$, $B_0 = \max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ and $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$. For $\rho \in U_T$, we set

$$a[\rho](t) = \int_0^1 \rho_+(x,t) - \rho_-(x,t)dx + L(t). \tag{9.34}$$

We see that for any $\rho \in U_T$, $a[\rho]$ satisfies $(H1)$ with $\|a[\rho]\|_{L^\infty(0,T)} \le M + \|L\|_{L^\infty(0,T)}$.

For $\rho \in U_T$, we then define $v = G(\rho) = (G_+(\rho), G_-(\rho))$ as the unique viscosity solution for $k = 1, 2$ (see Theorem 4.8) of

$$\begin{cases} (v_k)_t = -k\,(v_+ - v_- + a[\rho](t))\,|Dv_k| & \text{on} \quad (0,T)\times\mathbb{R}, \\ v_k(\cdot,0) = \rho_k^0 & \text{on} \quad \mathbb{R}. \end{cases} \tag{9.35}$$

We will show that $G : U_T \to U_T$ is a strict contraction for T small enough. First, we will prove that G is well defined. By Theorem 4.8, we know that v is $1-$ periodic plus L_0-linear. Moreover, we have

$$\max_{k\in\{+,-\}} \|Dv_k\|_{L^\infty(\mathbb{R}\times(0,T))} \le B_0,$$

$$\max_{k\in\{+,-\}} \|(v_\pm)_t\|_{L^\infty(\mathbb{R}\times(0,T))} \le B_0(M + \|a\|_{L^\infty(0,T)}) \le B_0(2M + \|L\|_{L^\infty(0,T)})$$

and

$$\|v_+ - v_-\|_{L^\infty(\mathbb{R}\times(0,T))} \le M$$

and so $v \in U_T$.

It thus remains to show that G is a contraction. For $v^i = G(\rho^i)$, according to *Lemma* 4.9 and *Lemma* 4.10, we have

$$\begin{aligned} \|v^2 - v^1\|_{L^\infty(\mathbb{R}\times(0,T))} = & \sup_{\{k\in\{+,-\}\}} \|v_k^2 - v_k^1\|_{L^\infty} \le B_0T\|a[\rho^2] - a[\rho^1]\|_{L^\infty(0,T)} \\ \le & 2B_0T\|\rho^1 - \rho^2\|_{L^\infty(\mathbb{R}\times(0,T))} \le \frac{1}{2}\|\rho^1 - \rho^2\|_{L^\infty(\mathbb{R}\times(0,T))} \end{aligned}$$

for $T \le T^* = \dfrac{1}{4B_0}$. And so G is a contraction on U_T which is a closed set. So, there exists a unique viscosity solution of (9.1)-(9.2) in U_T on $(0,T^*)$ where $T^* = \dfrac{1}{4B_0}$. By iterating this process, one can construct a solution for all $T > 0$. Indeed, T^* depends only on B_0 which does not change with time.

Proposition 4.11. (Estimate for the non-local solution) *Let $T \ge 0$. The solution ρ of (9.1)-(9.2) satisfies*

$$\|\rho_+ - \rho_-\|_{L^\infty(\mathbb{R}\times(0,T))} \le M$$

where $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$.

The proof is the same of the one of the local case, see Theorem 4.8 (iv).

5 Numerical scheme

5.1 Approximation of the local system

In this subsection, we propose a finite difference scheme for the local system (9.19). Given a discrete velocity a^{Δ}, we consider the discrete solution v that approximates the solution of (9.19), given by the following explicit scheme for all $k \in \{+,-\}$

$$v^0_{k,i} = \tilde{\rho}^0_k(x_i), \tag{9.36}$$

$$v^{n+1}_{k,i} = v^n_{k,i} + \Delta t \left(C_k^{\Delta,\mathrm{Loc}}[v](x_i,t_n)\right) \left\{ \begin{array}{ll} E^+\left(D^+v^n_{k,i}, D^-v^n_{k,i}\right) & \text{if } C_k^{\Delta,\mathrm{Loc}}[v](x_i,t_n) \geq 0 \\ E^-\left(D^+v^n_{k,i}, D^-v^n_{k,i}\right) & \text{if not} \end{array} \right. \tag{9.37}$$

where $\tilde{\rho}^0(x_i)$ are defined in (9.3), $E^{\pm}$ are the approximation of the Euclidean norm proposed by Osher and Sethian [154] defined in (9.6) (we also can use the one proposed by Rouy, Tourin [159]), $D^+v^n_k$, $D^-v^n_k$ are the discrete gradient defined in (9.7) and

$$C_k^{\Delta,\mathrm{Loc}}[w](x_i,t_n) = -k(w_+(x_i,t_n) - w_-(x_i,t_n) + a^{\Delta}(t_n)) \tag{9.38}$$

where a^{Δ} is an approximation of a satisfying

$$a^{\Delta}(t_n) = a(t_n). \tag{9.39}$$

In particular, the functions $E^{\pm}$ are Lipschitz continuous with respect to the discrete gradients, *i.e.*

$$\left|E^{\pm}(P,Q) - E^{\pm}(P',Q')\right| \leq (|P-P'| + |Q-Q'|)\,. \tag{9.40}$$

They are consistent with the Euclidean norm

$$E^{\pm}(P,P) = |P| \tag{9.41}$$

and enjoy suitable monotonicity with respect to each variable

$$\frac{\partial E^+}{\partial P^+} \geq 0, \quad \frac{\partial E^+}{\partial P^-} \leq 0, \quad \frac{\partial E^-}{\partial P^+} \geq 0, \quad \frac{\partial E^-}{\partial P^-} \leq 0. \tag{9.42}$$

Denoting by S^k the operator on the right-hand side of (9.37), we can rewrite the scheme more compactly as

$$v^0_{k,i} = \tilde{\rho}^0_k(x_i), \quad v^{n+1}_k = S^k v^n.$$

Finally, we also assume that the mesh satisfies the following CFL condition (cf Remark 5.2)

$$\Delta t \leq \frac{1}{2L_1}\Delta x \tag{9.43}$$

where

$$L_1 = \|a\|_{L^\infty(0,T)} + M + 2.$$

Theorem 5.1. (Crandall-Lions rate of convergence) *Let $T \leq 1$. Assume that $\Delta x + \Delta t \leq 1$. Assume that $a \in W^{1,\infty}(\mathbb{R} \times [0,T))$ and that the CFL condition (9.43) holds. Then there exists a constant $K > 0$ depending only on $\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$, $\|a\|_{W^{1,\infty}(0,T)}$ and $\max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ such that the error estimate between the continuous solution ρ of the system (9.19) and the discrete solution v of the finite difference scheme (9.36)-(9.37) is given by*

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |\rho_k(x_i,t_n) - v_{k,i}^n| \leq K\sqrt{T}\,(\Delta x + \Delta t)^{1/2} + \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0(x_i) - v_{k,i}^0|$$

provided $K(\Delta x + \Delta t)^{\frac{1}{2}} + \max_{k\in\{+,-\}} \sup_{\Xi} (\rho_k^0(x_i) - v_{k,i}^0) \leq 1.$

Remark 5.2. (Monotony of the scheme) *Under the assumptions of Theorem 5.1, we have*

$$\begin{aligned}|v_{+,i}^n - v_{-,i}^n| \leq & |v_{+,i}^n - \rho_+(x_i,t_n)| + |\rho_+(x_i,t_n) - \rho_-(x_i,t_n)| + |\rho_-(x_i,t_n) - v_{-,i}^n| \\ \leq & 2 + M\end{aligned}$$

where we have used Theorem 4.8 (iv) for the second term. We then deduce that the discrete velocity is uniformly bounded :

$$C_k^{\Delta,\mathrm{Loc}}[v] \leq \|a\|_{L^\infty(0,T)} + M + 2 = L_1.$$

Then, one can show that the scheme is monotone in the following sense : let v and w be two discrete functions such that $v_i^n \leq w_i^n$; then

$$S^k(v^n)(x_i) \leq S^k(w^n)(x_i), \quad \text{for } k \in \{+,-\}.$$

For the proof of Theorem 5.1, we need the following *lemma* :

Lemma 5.3. *If v_i^n is the numerical solution of (9.36)-(9.37), then*

$$-Kt_n - \mu^0 \leq \rho^0(x_i) - v(x_i,t_n) \leq Kt_n + \mu^0 \tag{9.44}$$

where $K = 2(\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})} + \|a\|_{L^\infty(0,T)}) \max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ *and*

$$\mu^0 = \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0(x_i) - v_{k,i}^0| \geq 0. \tag{9.45}$$

Proof of Lemma 5.3

To prove this, we set $w_\pm(x_i,t_n) = \rho_\pm^0(x_i) - Kt_n - \mu^0$ and we show that for K large enough w is a discrete sub-solution. Indeed, we have

$$\begin{aligned}&w_{\pm,i}^{n+1} - (S^\pm w^n)_i \\ =& - K\Delta t - \Delta t C_\pm^{\Delta,\mathrm{Loc}}[\rho^0](x_i,t_n) E^{sgn(C_\pm^{\Delta,\mathrm{Loc}}[\rho^0](x_i,t_n))}(D^+\rho_\pm^0(x_i), D^-\rho_\pm^0(x_i)) \\ =& - \Delta t \left(K \mp (\rho_+^0(x_i) - \rho_-^0(x_i) + a^\Delta(t_n)) E^{sgn(C_\pm^{\Delta,\mathrm{Loc}}[\rho^0](x_i,t_n))}(D^+\rho_\pm^0(x_i), D^-\rho_\pm^0(x_i))\right) \\ \leq& - \Delta t \left(K - 2(\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})} + \|a\|_{L^\infty(0,T)}) \max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}\right)\end{aligned}$$

where $C_k^{\Delta,\mathrm{Loc}}[w](x_i,t_n)$ is defined in (9.38) and $sgn(f)$ is the sign of f.

So, for every $K \geq 2(\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})} + \|a\|_{L^\infty(0,T)}) \max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$, w is a discrete sub-solution. Moreover

$$w_{\pm,i}^0(x_i) = \rho_\pm^0(x_i) - \mu^0 \leq v_\pm^0(x_i).$$

Using the monotony of the scheme, we deduce $w_i^n \leq v_i^n$ and so

$$\rho^0(x_i) - v_i^n \leq Kt_n + \mu_0.$$

The lower bound is proved similarly.

We now give the proof of Theorem 5.1

Proof of Theorem 5.1

The proof is an adaptation for systems of the one of Crandall Lions [63], revisited by Alvarez *et al.* [7]. The proof splits into three steps. We denote throughout by K various constant depending only on $\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$, $\max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ and $\|a\|_{W^{1,\infty}(0,T)}$.

We first assume that

$$\rho^0(x_i) \geq v_i^0 \tag{9.46}$$

and we set

$$\mu^0 = \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0(x_i) - v_{k,i}^0| \geq 0. \tag{9.47}$$

We set a few notations. We put

$$\mu = \max_{k\in\{+,-\}} \sup_{\Xi_T} (\rho_k(x_i,t_n) - v_{k,i}^n).$$

For every $0 < \alpha \leq 1,\ 0 < \varepsilon \leq 1$ and $\sigma > 0$, we set

$$M_\sigma^{\alpha,\varepsilon} = \sup_{\mathbb{R}\times[0,T]\times\Xi_T\times\{+,-\}} \Psi_\sigma^{\alpha,\varepsilon}(x,t,x_i,t_n,k),$$

with

$$\Psi_\sigma^{\alpha,\varepsilon}(x,t,x_i,t_n,k) = \rho_k(x,t) - v_k(x_i,t_n) - \frac{|x-x_i|^2}{2\varepsilon} - \frac{|t-t_n|^2}{2\varepsilon} - \sigma t - \alpha|x|^2 - \alpha|x_i|^2.$$

We shall drop the super and subscripts on Ψ when no ambiguity arises as concerning the value of the parameter.

Since ρ^0 is Lipschitz continuous and $T \leq 1$, we have by (9.29)

$$|\rho_\pm(x,t)| \leq K(1+|x|). \tag{9.48}$$

Moreover by *Lemma* 5.3 we have

$$\begin{aligned}|v_\pm(x_i,t_n)| \leq & |v_\pm(x_i,t_n)-\rho^0_\pm(x_i)|+|\rho^0_\pm(x_i)|\\ \leq & Kt_n+K(1+|x_i|)\\ \leq & K(1+|x_i|).\end{aligned}$$

We then deduce that Ψ achieves its maximum at some point that we denote by $(x^*,t^*,x_i^*,t_n^*,k^*)$.

Step 1 : Estimates for the maximum point of Ψ

The maximum point of Ψ enjoys the following estimates

$$\alpha|x^*|+\alpha|x_i^*|\leq K, \tag{9.49}$$

and

$$|x^*-x_i^*|\leq K\varepsilon, \quad |t^*-t_n^*|\leq (K+2\sigma)\varepsilon. \tag{9.50}$$

Indeed, by inequality $\Psi(x^*,t^*,x_i^*,t_n^*,k^*)\geq \Psi(0,0,0,0,k^*)\geq 0$, we obtain

$$\begin{aligned}\alpha|x^*|^2+\alpha|x_i^*|^2 \leq & \rho_{k^*}(x^*,t^*)-v_{k^*}(x_i^*,t_n^*)\leq K(1+|x^*|+|x_i^*|)\\ \leq & K+\frac{K^2}{\alpha}+\frac{\alpha}{2}|x^*|^2+\frac{\alpha}{2}|x_i^*|^2.\end{aligned}$$

This implies (9.49), since $\alpha\leq 1$.

The first bound of (9.50) follows from the Lipschitz in space regularity of ρ (see Theorem 4.8 (iii)), from the inequality $\Psi(x^*,t^*,x_i^*,t_n^*,k^*)\geq \Psi(x_i^*,t^*,x_i^*,t_n^*,k^*)$ and from (9.49). Indeed, this implies

$$\begin{aligned}\frac{|x^*-x_i^*|^2}{2\varepsilon} \leq & \rho_{k^*}(x^*,t^*)-\rho_{k^*}(x_i^*,t^*)-\alpha|x^*|^2+\alpha|x_i^*|^2\\ \leq & K|x^*-x_i^*|+\alpha|x^*-x_i^*|(|x^*|+|x_i^*|)\leq K|x^*-x_i^*|.\end{aligned}$$

The second bound of (9.50) is obtained in the same way, using the inequality $\Psi(x^*,t^*,x_i^*,t_n^*,k^*)\geq \Psi(x^*,t_n^*,x_i^*,t_n^*,k^*)$ and Theorem 4.8 (iii).

Step 2 : A better estimate for the maximum point of Ψ

Inequality (9.49) can be strengthened to

$$\alpha|x^*|^2+\alpha|x_i^*|^2\leq K. \tag{9.51}$$

Indeed, using the Lipschitz regularity of ρ, the inequality $\Psi(x^*,t^*,x_i^*,t_n^*,k^*)\geq \Psi(0,0,0,0,k^*)$ and equations (9.29), (9.44) and (9.50), yields

$$\begin{aligned}\alpha|x^*|^2+\alpha|x_i^*|^2 \leq & \rho_{k^*}(x^*,t^*)-v_{k^*}(x_i^*,t_n^*)+\rho(x_i^*,0)-\rho(x_i^*,0)\\ \leq & K(|x^*-x_i^*|+t^*)+Kt_n^*+\mu^0\leq K.\end{aligned}$$

Step 3 : Upper bound of μ

We have the bound $\mu \leq K\sqrt{T}\,(\Delta x+\Delta t)^{\frac{1}{2}}+\mu^0$ if $\Delta x+\Delta t \leq \frac{1}{K^2}$.
First, we claim that for σ large enough, we have either $t^*=0$ or $t_n^*=0$. Suppose the contrary. Then the function $(x,t)\mapsto \Psi(x,t,x_i^*,t_n^*,k^*)$ achieves its maximum at a point of $\mathbb{R}\times(0,T]$. Using the fact that ρ is a sub-solution of the continuous problem, we obtain the inequality

$$\sigma + p_t^* \leq -k^*(\rho_+ - \rho_- + a(t^*))|p_x^* + 2\alpha x^*| \tag{9.52}$$

with $p_t^* = \frac{t^*-t_n^*}{\varepsilon}$, $p_x^* = \frac{x^*-x_i^*}{\varepsilon}$.
Since $t_n^*>0$, we also have $\Psi(x^*,t^*,x_i^*,t_n^*,k^*) \geq \Psi(x^*,t^*,x_i,t_n^*-\Delta t,k^*)$. This implies

$$v_{k^*}(\cdot,t_n^*-\Delta t) \geq \varphi(\cdot,t_n^*-\Delta t) + v_{k^*}(x_i^*,t_n^*) - \varphi(x_i^*,t_n^*)$$

for $\varphi(x_i,t_n) = -\frac{|x^*-x_i|^2}{2\varepsilon} - \frac{|t^*-t_n|^2}{2\varepsilon} - \alpha|x_i|^2$. Using the fact that the scheme is monotone and commutes with the addition of constants, yields

$$\begin{aligned}
&v_{k^*}(x_i^*,t_n^*)\\
=&S^{k^*}\left(v_{k^*}(\cdot,t_n^*-\Delta t)\right)(x_i^*)\\
\geq&\varphi(x_i^*,t_n^*-\Delta t) + v_{k^*}(x_i^*,t_n^*) - \varphi(x_i^*,t_n^*)\\
&+\Delta t\left(c_{k^*}^{\Delta,\mathrm{Loc}}[v](x_i^*,t_n^*)\right)E^{l^*}\left(D^+\varphi(x_i^*,t_n^*-\Delta t), D^-\varphi(x_i^*,t_n^*-\Delta t)\right)
\end{aligned}$$

where $l^* = sgn\left(c_{k^*}^{\Delta,\mathrm{Loc}}[v](x_i^*,t_n^*)\right)$. We set

$$c[v] = -c_{k^*}^{\Delta,\mathrm{Loc}}[v](x_i^*,t_n^*), \quad c[\rho] = k^*\left(\rho_+(x^*,t^*) - \rho_-(x^*,t^*) + a(t^*)\right).$$

We then obtain the super-solution inequality :

$$\frac{\varphi(x_i^*,t_n^*) - \varphi(x_i^*,t_n^*-\Delta t)}{\Delta t} \geq -c[v]E^{l^*}(D^+\varphi(x_i^*,t_n^*-\Delta t), D^-\varphi(x_i^*,t_n^*-\Delta t)).$$

Straightforward computations of the discrete derivative of φ yield

$$p_t^* + \frac{\Delta t}{2\varepsilon} \geq -c[v]E^{l^*}\left(p_x^* - \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^*+\Delta x), p_x^* + \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^*-\Delta x)\right).$$

Subtracting the above inequality to (9.52), we deduce

$$\begin{aligned}
\sigma \leq&\frac{\Delta t}{2\varepsilon} - c[\rho]|p_x^*+2\alpha x^*| + c[v]E^{l^*}\left(p_x^* - \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^*+\Delta x), p_x^* + \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^*-\Delta x)\right)\\
\leq&\frac{\Delta t}{2\varepsilon} - \left(c[\rho]-c[v]\right)|p_x^*| + \alpha K|x^*|\\
&+|c[v]|\left|E^{l^*}\left(p_x^* - \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^*+\Delta x), p_x^* + \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^*-\Delta x)\right) - E^{l^*}\left(p_x^*,p_x^*\right)\right|\\
\leq&\frac{\Delta t}{2\varepsilon} - \left(c[\rho]-c[v]\right)|p_x^*| + K\alpha|x^*| + K\frac{\Delta x}{\varepsilon} + 2\alpha K|x_i^*| + 2\alpha K\Delta x
\end{aligned}$$

where we have used, for the second line, the fact that

$$c[\rho] \leq M + 2B_0(M + \|a\|_{L^\infty(0,T)})T \leq K$$

with $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$ and $B_0 = \max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ (see Theorem 4.8). Now, since $\rho_{k^*}(x^*, t^*) - v_{k^*}(x_i^*, t_n^*) = \max_{k\in\{+,-\}} (\rho_k(x^*, t^*) - v_k(x_i^*, t_n^*)) \geq 0$, by *Lemma* 4.2, we obtain

$$\begin{aligned}
-(c[\rho] - c[v])|p_x^*| =& - k^* \left(\rho_+(x^*, t^*) - \rho_-(x^*, t^*) + a(t^*)\right)|p_x^*| \\
&+ k^* \left(v_+(x_i^*, t_n^*) - v_-(x_i^*, t_n^*) + a(t^*)\right)|p_x^*| \\
&+ k^* \left(a^\Delta(t_n^*) - a(t^*)\right)|p_x^*| \\
\leq& \left|a^\Delta(t_n^*) - a(t^*)\right||p_x^*| \leq K|t_n^* - t^*||p_x^*|
\end{aligned}$$

where we have used (9.39). This implies

$$\begin{aligned}
\sigma \leq& \frac{\Delta t}{2\varepsilon} + K|t^* - t_n^*||p_x^*| + K\alpha|x^*| + K\frac{\Delta x}{\varepsilon} + 2\alpha K|x_i^*| + 2\alpha K\Delta x \\
\leq& K\frac{\Delta x + \Delta t}{\varepsilon} + K\alpha^{1/2} + K\varepsilon.
\end{aligned}$$

Putting

$$\sigma^* = \sigma^*(\Delta x + \Delta t, \varepsilon, \alpha) = K\frac{\Delta x + \Delta t}{\varepsilon} + K(\alpha^{1/2} + \varepsilon),$$

we therefore conclude that we must have $t^* = 0$ or $t_n^* = 0$ provided $\sigma \geq \sigma^*$. Whenever $t^* = 0$, we deduce from *Lemma* 5.3 and from (9.50) that

$$\begin{aligned}
M_\sigma^{\alpha,\varepsilon} =& \Psi(x^*, 0, x_i^*, t_n^*, k^*) \leq \rho_{k^*}^0(x^*) - v_{k^*}(x_i^*, t_n^*) \\
\leq& \rho_{k^*}^0(x^*) - \rho_{k^*}^0(x_i^*) + Kt_n^* + \mu^0 \\
\leq& K\left(|x^* - x_i^*| + t_n^*\right) + \mu^0 \leq K(1+\sigma)\varepsilon + \mu^0.
\end{aligned}$$

Similarly, whenever $t_n^* = 0$, we deduce from the Lipschitz regularity of ρ and from (9.50) that

$$\begin{aligned}
M_\sigma^{\alpha,\varepsilon} =& \Psi(x^*, t^*, x_i^*, 0, k^*) \leq \rho_{k^*}(x^*, t^*) - v_{k^*}(x_i^*, 0) \\
\leq& K\left(|x^* - x_i^*| + t^*\right) + \mu^0 \leq K(1+\sigma)\varepsilon + \mu^0.
\end{aligned}$$

To sum up, we have shown that

$$M_\sigma^{\alpha,\varepsilon} \leq K(1+\sigma)\varepsilon + \mu^0 \leq K\varepsilon + \mu^0$$

provided $\sigma^* = K\frac{\Delta x + \Delta t}{\varepsilon} + K(\alpha^{1/2} + \varepsilon) \leq \sigma \leq 1$.We then deduce that, for every (x_i, t_n) and for every k, we have

$$\begin{aligned}
&\rho_k(x_i, t_n) - v_k(x_i, t_n) - \left(K\frac{\Delta x + \Delta t}{\varepsilon} + K(\alpha^{1/2} + \varepsilon)\right)T - 2\alpha|x_i|^2 \leq M_\sigma^{\alpha,\varepsilon} \\
\leq& K\varepsilon + \mu^0.
\end{aligned}$$

Sending $\alpha \to 0$, taking the supremum over (x_i, t_n), the maximum over k and choosing $\varepsilon = T^{1/2} (\Delta x + \Delta t)^{1/2}$, we conclude that

$$\begin{aligned}\max_{k\in\{+,-\}} \sup_{\Xi_T}(\rho_k(x_i,t_n) - v^n_{k,i}) =&\mu \\ \le& K\,(\Delta x + \Delta t)^{1/2}\sqrt{T} + \max_{k\in\{+,-\}} \sup_{\Xi}(\rho^0_k(x_i) - v^0_{k,i}),\end{aligned} \tag{9.53}$$

provided that $\Delta x,\ \Delta t$ are small enough $T \le 1$, $\mu_0 \le 1$ and (9.46) is assumed.

In the general case, we consider $\overline{\rho} = \rho + \mu^1$ with $\mu^1 = \max\limits_{k\in\{+,-\}} \sup\limits_{\Xi}(v^0_{k,i} - \rho^0_k(x_i))$. We remark that $\overline{\rho}$ is solution of (9.19) and satisfies $\overline{\rho}^0(x_i) \ge v^0_i$. Then (9.53) is true with $\overline{\rho}$ in place of ρ, *i.e.*

$$\max_{k\in\{+,-\}} \sup_{\Xi_T}(\rho_k(x_i,t_n)+\mu^1-v^n_{k,i}) \le K\,(\Delta x + \Delta t)^{1/2}\sqrt{T}+ \max_{k\in\{+,-\}} \sup_{\Xi}(\rho^0_k(x_i)+\mu^1-v^0_{k,i}),$$

which still implies (9.53) with $\max\limits_{k\in\{+,-\}} \sup\limits_{\Xi} |\rho^0_k(x_i) - v^0_{k,i}|$.

The lower bound for the error estimate is obtained by exchanging ρ and v. As the proof is similar to the above, we omit it.

5.2 Approximation of the non-local system

To solve numerically the non-local system (9.1)-(9.2), we use the finite difference scheme (9.3)-(9.4)-(9.5). We also assume the CFL condition (9.8). In particular, using Proposition 4.11, we deduce that the CFL condition (9.43) is satisfied uniformly for all a defined by (9.5) because

$$\|a[\rho]\|_{L^\infty(0,T)} \le M + \|L\|_{L^\infty(0,T)}$$

and so $L_1 \le L_2$.

Let $\bar{T} \ge 0$ which will be chosen later. To prove our convergence result, we mimic the continuous case and we rewrite the scheme (9.3)-(9.4)-(9.5) as a fixed point. Before proving Theorem 1.3 we need to introduce some notations and *lemmata*. Defining $X^{1,\Delta}_T = \mathbb{R}^{\{0,\dots,N_T\}}$ and $X^{2,\Delta}_T = (\mathbb{R}^2)^{\mathbb{Z}\times\{0,\dots,N_T\}}$, the set of discrete functions defined on $\{0,...,N_T\}$ and on the mesh Ξ_T respectively, we denote by $G^\Delta : X^{1,\Delta}_{\bar{T}} \to X^{2,\Delta}_{\bar{T}}$ the operator that gives the discrete solution v of the local Problem (9.37) for a given velocity $a^\Delta \in X^{1,\Delta}_{\bar{T}}$, *i.e.*

$$(G^\Delta_+(a^\Delta), G^\Delta_-(a^\Delta)) = G^\Delta(a^\Delta) = v.$$

In particular, the scheme (9.3)-(9.4)-(9.5) can be rewritten as a fixed point of $G^\Delta(a^\Delta[\cdot])$, *i.e.*

$$v = G^\Delta(a^\Delta[v])$$

with $a^\Delta[\cdot]$ defined in (9.5). We set, for all $T \le \bar{T}$:

$$U_T^\Delta = \left\{ w \in X_T^{2,\Delta} : \left| \begin{array}{l} \sup\limits_{\Xi_T} |D_x^+ w_\pm| \le B_0, \\ \sup\limits_{\Xi_T} |D_t^+ w_\pm| \le 2B_0(2M + \|L\|_{L^\infty(0,T)} + 4), \\ \sup\limits_{\Xi_T} |w_+ - w_-| \le M + 2 \end{array} \right. \right\}$$

and

$$V_T^\Delta = \left\{ a^\Delta \in X_T^{1,\Delta} : \left| \sup_{\{0,\dots,N_T\Delta t\}} |a^\Delta| \le M + \|L\|_{L^\infty} + 2 \right. \right\}$$

where $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$. One can easily check that

$$\{(\rho)^\Delta \mid \rho \in U_T\} \subset U_T^\Delta$$

and

$$\left\{(a)^\Delta : \left| \ \|a\|_{L^\infty(0,T)} \le M + \|L\|_{L^\infty(0,T)} \right. \right\} \subset V_T^\Delta$$

where $(f)^\Delta$ is the restriction to Ξ_T of the continuous function f. We have the following *Lemma* :

Lemma 5.4. *Assume that (9.8) holds. Then for all $T \le \bar{T}$, the following inclusion holds*

(i) $a^\Delta[U_T^\Delta] \subset V_T^\Delta$,

(ii) $G^\Delta(V_T^\Delta) \subset U_T^\Delta$.

Proof of *Lemma* 5.4

The proof of (i) is just a simple computation. We prove (ii).

Let $a^\Delta \in V_T^\Delta$ and $v = G^\Delta(a^\Delta)$. We set $w(x_i, t_n) = v(x_{i+1}, t_n) - \Delta x B_0$. Then w is still solution of the discrete scheme (9.37) and satisfies $w^0 \le v^0$. Using the monotony of the scheme yields

$$\frac{v_\pm(x_{i+1}, t_n) - v_\pm(x_i, t_n)}{\Delta x} \le B_0.$$

Using Theorem 4.8, we deduce

$$|v_+ - v_-| \le M + 2. \tag{9.54}$$

For the estimate in time, we have, using (9.54),

$$\begin{aligned} \left| \frac{v_i^{n+1} - v_i^n}{\Delta t} \right| &\le 2B_0 |C_k^{\Delta,\mathrm{Loc}}[v](x_i, t_n)| \\ &\le 2B_0 (M + 2 + \sup_{\{0,\dots,N_T\Delta t\}} |a^\Delta|) \\ &\le 2B_0 (2M + \|L\|_{L^\infty(0,T)} + 4). \end{aligned}$$

So $G^\Delta(V_T^\Delta) \subset U_T^\Delta$. This ends the proof of the *lemma*.

We now have to prove some consistency and stability results for the velocity a^Δ and for the operator G^Δ.

Lemma 5.5. (Consistency for the discrete velocity $a^\Delta[\cdot]$) *There is a constant $K = 2B_0 + M$ such that, for every mesh Δ, for every $0 \le T \le \bar{T}$ and for $\rho \in U_T$, we have*

$$\sup_{\{0,\dots,N_T\Delta t\}} |\,(a[\rho])^\Delta - a^\Delta[(\rho)^\Delta]| \le K\Delta x$$

where $(\rho)^\Delta$ is the restriction to Ξ_T of the continuous function ρ and $a[\cdot]$ is defined in (9.34).

Proof of *Lemma* 5.5
We set $\tilde{\rho}(x,t) = \rho_+(x,t) - \rho_-(x,t)$. The following holds :

$$\begin{aligned}
\left|a[\rho](t_n) - a^\Delta[(\rho)^\Delta](t_n)\right| &= \left|\int_0^1 \tilde{\rho}(x,t_n)dx - \sum_{i=0}^{N_x-1} \Delta x \tilde{\rho}(x_i,t_n)\right| \\
&\le \sum_{i=0}^{N_x-1} \left|\int_{i\Delta x}^{(i+1)\Delta x} \tilde{\rho}(x,t_n)dx - \Delta x \tilde{\rho}(x_i,t_n)\right| \\
&\quad + \int_{N_x\Delta x}^1 \tilde{\rho}(x,t_n)dx \\
&\le \Delta x \sum_{i=0}^{N_x-1} \sup_{[i\Delta x,(i+1)\Delta x]} |\tilde{\rho}(\cdot,t_n) - \tilde{\rho}(x_i,t_n)| + M\Delta x \\
&\le \Delta x(2B_0 + M).
\end{aligned}$$

We have the following *lemma* which proof is just a simple computation

Lemma 5.6. (Stability property of the velocity $a^\Delta[\cdot]$) *For every mesh Δ, for every $0 \le T \le \bar{T}$ and every $v_1, v_2 \in U_T^\Delta$, the following holds*

$$\sup_{\{0,\dots,N_T\Delta t\}} |a^\Delta[v_2] - a^\Delta[v_1]| \le 2 \max_{k\in\{+,-\}} \sup_{\Xi_T} |v_2 - v_1|.$$

Lemma 5.7. (Stability property of the operator G^Δ) *There is a constant $K = 2B_0$ so that, for every mesh Δ satisfying the uniform CFL condition (9.8), for all $0 \le T \le \bar{T}$ and all a_1^Δ, $a_2^\Delta \in V_T^\Delta$*

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |G_k^\Delta(a_2^\Delta) - G_k^\Delta(a_1^\Delta)| \le KT \sup_{\{0,\dots,N_T\Delta t\}} |a_2^\Delta - a_1^\Delta|.$$

Proof of *Lemma* 5.7
We set $v_i = G^\Delta\left(a_i^\Delta\right)$. Using the fact that

$$c_1 E^{sgn(c_1)} - c_2 E^{sgn(c_2)} \leq |c_1 - c_2| \max(E^+, E^-)$$

yields

$$\begin{aligned}
&v_{2,k}^{n+1} - v_{2,k}^n + k\Delta t\left(v_{2,+}^n - v_{2,-}^n + a_1^\Delta(t_n)\right) E^{sgn(v_{2,+}^n - v_{2,-}^n + a_1^\Delta(t_n))}(D^+ v_2^n, D^- v_2^n)\\
\leq &\Delta t|a_2^\Delta(t_n) - a_1^\Delta(t_n)| \max(E^+(D^+ v_2^n, D^- v_2^n), E^-(D^+ v_2^n, D^- v_2^n))\\
\leq &2B_0 \Delta t \sup_{\{0,\dots,N_T\Delta t\}} |a_1^\Delta - a_2^\Delta|.
\end{aligned}$$

Moreover $\tilde{v}_1(x_i, t_n) = v_1(x_i, t_n) + 2B_0 \sup\limits_{\{0,\dots,N_T\Delta t\}} |a_1^\Delta - a_2^\Delta| t_n$ is solution of the same discrete equation. Since the scheme is monotone, one deduces that

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |G_k^\Delta(a_2^\Delta) - G_k^\Delta(a_1^\Delta)| \leq 2B_0 T \sup_{\{0,\dots,N_T\Delta t\}} |a_2^\Delta - a_1^\Delta|.$$

This achieves the proof.

We now prove Theorem 1.3.

Proof of Theorem 1.3
We use the main idea of Alvarez *et al.* [7].
We first assume that $T \geq \bar{T}$ and we set, for every $l \geq 1$:

$$Q_l^\Delta = \Delta x\mathbb{Z} \times \{\Delta t N_l, \dots, \Delta t N_{l+1}\}$$

where N_l is the integer part of $\dfrac{l\bar{T}}{\Delta t}$. As in the continuous case, on each interval $(l\bar{T}, (l+1)\bar{T})$, we can iterate the process (since $\bar{T}$ depends only on B_0 which does not change with time) and construct, using a fix point method (denoting by G and G^Δ), ρ and v respectively solution of (9.1)-(9.2) and (9.3)-(9.4)-(9.5). We then have the inequality

$$\begin{aligned}
\max_{k\in\{+,-\}} \sup_{Q_l^\Delta} |\rho_k - v_k| &\leq \max_{k\in\{+,-\}} \sup_{Q_l^\Delta} |G_{k,l}(a[\rho]) - G_{k,l}^\Delta(a^\Delta[v])|\\
&\leq \max_{k\in\{+,-\}} \sup_{Q_l^\Delta} \left|G_{k,l}(a[\rho]) - G_{k,l}^\Delta\left((a[\rho])^\Delta\right)\right|\\
&\quad + \max_{k\in\{+,-\}} \sup_{Q_l^\Delta} \left|G_{k,l}^\Delta\left((a[\rho])^\Delta\right) - G_{k,l}^\Delta\left(a^\Delta[v]\right)\right|,
\end{aligned}$$

where the function $G_l^\Delta\left((a[\rho])^\Delta\right) = \left(G_{+,l}^\Delta\left((a[\rho])^\Delta\right), G_{-,l}^\Delta\left((a[\rho])^\Delta\right)\right)$ (resp. $G_l(a[\rho])$) is simply the discrete solution of (9.36) (resp. the continuous solution of (9.1)) with the velocity $a[\rho]$ and initial condition v^{N_l} (resp. $\rho(\cdot, N_l\Delta t)$). From Theorem 5.1, we then deduce

$$\begin{aligned}\max_{k\in\{+,-\}} \sup_{Q_l^\Delta} \left|G_{k,l}(a[\rho]) - G_{k,l}^\Delta\left((a[\rho])^\Delta\right)\right| \leq & K\sqrt{\bar{T}\Delta x} + \max_{k\in\{+,-\}} \sup_{\Delta x\mathbb{Z}\times N_l\Delta t} |\rho_k - v_k| \\ \leq & lK\sqrt{\bar{T}\Delta x} + \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|. \quad (9.55)\end{aligned}$$

For the second term, we use *Lemmata* 5.5, 5.6 and 5.7 to obtain

$$\begin{aligned}&\max_{k\in\{+,-\}} \sup_{Q_l^\Delta} \left|G_{k,l}^\Delta\left((a[\rho])^\Delta\right) - G_{k,l}^\Delta\left(a^\Delta[v]\right)\right| \\ \leq & K\bar{T} \sup_{\{N_l\Delta t,\ldots,N_{l+1}\Delta t\}} \left|(a[\rho])^\Delta - a^\Delta[v]\right| \\ \leq & K\bar{T} \sup_{\{N_l\Delta t,\ldots,N_{l+1\Delta t}\}} \left(\left|(a[\rho])^\Delta - a^\Delta\left[(\rho)^\Delta\right]\right| + \left|a^\Delta\left[(\rho)^\Delta\right] - a^\Delta[v]\right|\right) \\ \leq & K\bar{T}\left(\Delta x + \max_{k\in\{+,-\}} \sup_{Q_l^\Delta} |\rho_k - v_k|\right).\end{aligned}$$

This implies, for $\bar{T}\Delta x \leq 1$ and $K\bar{T} < 1$,

$$\max_{k\in\{+,-\}} \sup_{Q_l^\Delta} |\rho_k - v_k| \leq \frac{lK}{1-K\bar{T}}\sqrt{\bar{T}\Delta x} + \left(\max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|\right)\frac{1}{1-K\bar{T}}.$$

We now take $\bar{l} \geq 1$ such that

$$\bar{l}\bar{T} \leq T \leq (\bar{l}+1)\bar{T}.$$

Then the following holds :

$$\begin{aligned}\max_{k\in\{+,-\}} \sup_{\Xi_T} |\rho_k - v_k| \leq & \frac{\bar{l}K}{1-K\bar{T}}\sqrt{\bar{T}\Delta x} + \left(\max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|\right)\frac{1}{1-K\bar{T}} \\ \leq & KT\sqrt{\Delta x} + K\left(\max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|\right), \text{ if } T \geq \bar{T}.\end{aligned}$$

where we have used the fact that $\bar{T}$ depends only on B_0.

Notice that, in the case where $T \leq \bar{T}$, from Theorem 5.1, (9.55) is replaced by

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} \left|G_k(a[\rho]) - G_k^\Delta\left((a[\rho])^\Delta\right)\right| \leq K\sqrt{T\Delta x} + \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|$$

and so we obtain

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |\rho_k - v_k| \leq K\sqrt{T\Delta x} + K\left(\max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|\right), \text{ if } T \leq \bar{T}.$$

This ends the proof of the theorem.

6 Numerical results

In this Section, we present some numerical simulations of the 1-D Groma-Balogh problem (9.1)-(9.2) discretized by the numerical scheme (9.3)-(9.4)-(9.5).

6.1 Numerical error estimate

Here, we show a numerical test in order to confirm our error estimate for local system. Let us fix $L(t) = 0$ even if it is not physicaly relevant, let us choose the following initial conditions : $\rho_+^0(x) = -|x - 1/2| + 1/2$, and $\rho_-^0(x) = -|2x - 1| + 1$ on $[0, 1[$ (and extend it by periodicity on $\mathbb{R}$).

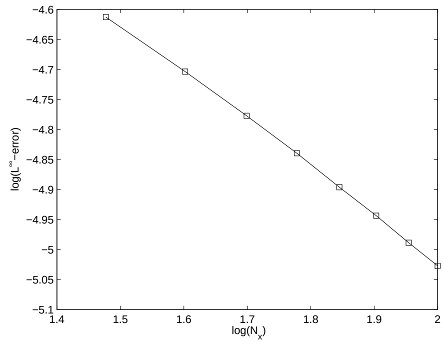

FIG. 9.3 – $\log(L^\infty - error)$ of $|u_{N_T} - u_{N_T-1}|$ versus $\log(N_x)$ at $T = \frac{1}{2}$

Figure (9.3) show the behaviour of the L^∞-error versus the discretization parameter Δx. The regression slope is close to 0.7 and the ideal regression is $\frac{1}{2}$. Hence, the behaviour of this errors confirms that our error seems optimal.

6.2 Dislocation density dynamics

In this paragraph, we are interested by the evolution of dislocations densities for the 1-D Groma-Balogh model (9.1)-(9.2) under the uniformly applied shear stress $L(t) = 3t$.

In this simulations, we choose an example of concentrated dislocations densities, *i.e.* where dislocations densities are initially periodic, and equal to zero on some sub-intervals of $[0,1[$ (see Figure 9.4).

This initial condition means that there exists some regions without dislocations, and others with concentrated dislocations.

Intuitively, dislocations are intended to be uniformly distributed in the whole crystal as shown in (Figure 9.6) where finally a uniform distribution in all the crystal is observed, *i.e.* the density of dislocations becomes a constant.

We remark that when $L(t)$ is non-stationary, our system behaves as a diffusion equation (see [43] for further details). But evidently when $L(t) = 0$, with the same initial condition, the system does not evolve.

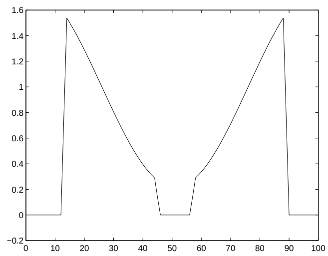

FIG. 9.4 – dislocations density $(D\rho^0_+(.) = D\rho^0_-(.))$

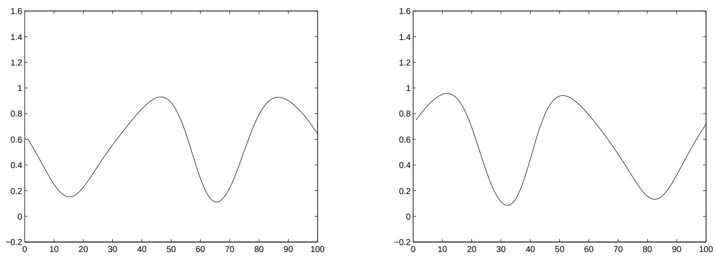

FIG. 9.5 – On the left : density $(D\rho_+(., \frac{1}{2}))$; on the right : dislocations density $(D\rho_-(., \frac{1}{2}))$

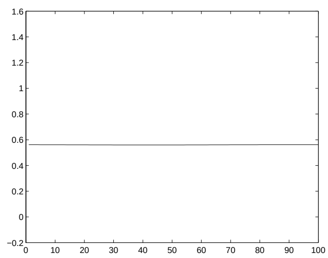

FIG. 9.6 – dislocations density $(D\rho_+(.,3) = D\rho_-(.,3))$

Acknowledgements
The authors would like to thanks O. Alvarez, M. El Rhabi, C. Imbert, and R. Monneau for fruitful discussions in the preparation of this paper. This work was supported by the contract JC 1025 called "ACI jeunes chercheuses et jeunes chercheurs" of the French Ministry of Research (2003-2005).

Conclusion et Perspectives

Les différents travaux de cette thèse ont permis de répondre à plusieurs questions importantes pour la dynamique de lignes de dislocations aussi bien du point de vue de l'analyse mathématique que de l'analyse numérique. Plusieurs travaux ont également ouvert des pistes de recherches que je voudrais détailler ici :

Méthode Fast Marching
La généralisation que nous avons proposée dans [55] de la méthode Fast Marching a de nombreuses applications potentielles. L'objectif pourrait être, pour la propagation de fronts, de proposer une alternative aux méthodes basées sur les level-set qui ne permettent pas une bonne localisation des fronts quand le gradient de la fonction est trop faible. Un exemple, comme nous l'avons indiqué dans l'introduction, est d'utiliser cette méthode pour résoudre numériquement la dynamique des dislocations.

Une autre application est le traitement d'images numériques. Avec C. Gout et C. Le Guyader, nous sommes en train de développer une nouvelle méthode de segmentation utilisant l'algorithme proposé dans [55]. Cette méthode utilise une fonction *edge-detector* qui va localiser les contours de l'image et donc donner une direction de propagation. Cette méthode semble très robuste et nous a déjà permis d'obtenir des résultats encourageants. Nous présentons dans la Figure 9.8 un premier résultat pour la segmentation d'une arthère du coeur.

Dans un groupe de travail ayant lieu au CERMICS, nous essayons également de développer une méthode de type Fast Marching pour des équations de transport linéaire. L'objectif est de réussir à montrer un résultat de convergence pour cette méthode. Cette étude a lieu dans le cadre d'un contrat entre le CEA et l'ENPC avec des applications industrielles.

Il y a également beaucoup d'autres choses à explorer autour de cette méthode, comme la généralisation à des maillages non structurés, à des équations plus générales (autre que l'équation eikonale), ou encore au mouvement par courbure moyenne (en utilisant par exemple les travaux [67] et [88]). En effet, ces deux travaux donnent une façon d'obtenir une approximation de la courbure moyenne de manière localisée.

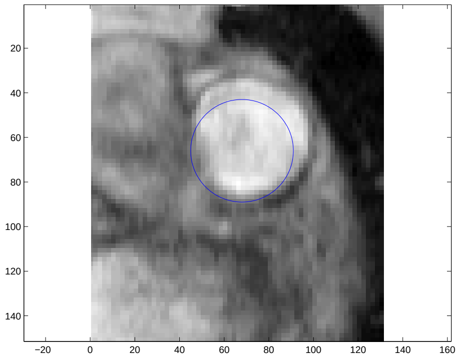

FIG. 9.7 – Condition initiale.

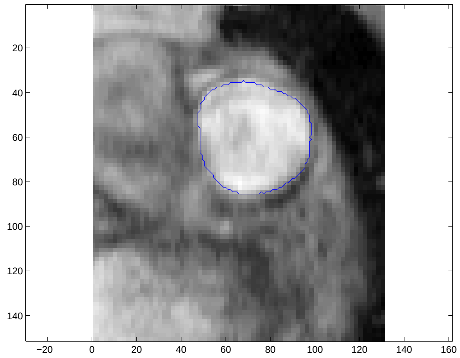

FIG. 9.8 – Résultat final.

J'aimerais utiliser cela pour proposer une méthode de type Fast Marching pour le mouvement par courbure moyenne.

Homogénéisation

Dans la théorie de la dynamique des dislocations, le réseau cristallin va influencer la dynamique des lignes de dislocations. En utilisant des outils d'homogénéisation, j'essaie de comprendre l'influence de ces réseaux sur la dynamique. Cela revient à ajouter à la vitesse normale un potentiel périodique $W(x/\varepsilon)$ (représentant les vallées de Peierls) et de passer à la limite quand ε tend vers zéro. A la limite on s'attend à récupérer une équation non-locale anisotrope où l'anisotropie vient de l'influence du réseau.

Les méthodes utilisées dans [89] permettent d'obtenir des résultats d'homogénéisation pour certains systèmes de particules. En utilisant ces méthodes, on peut espérer obtenir des résultats similaires pour d'autres modèles et en particulier pour des systèmes avec interactions aux plus proches voisins comme le modèle de Frenkel-Kontorova.

Un autre aspect intéressant est l'analyse numérique des problèmes d'homogénéisation et en particulier le calcul de l'Hamiltonien effectif.

Unicité en temps long pour la dynamique des dislocations

Une question importante qui reste ouverte pour la dynamique des dislocations est la question de l'unicité en temps long. Dans [91], nous avons montré l'existence d'une solution faible en temps long pour la dynamique des dislocations avec un terme de courbure moyenne. Un challenge intéressant (et très difficile) est de comprendre la question de l'unicité (ou de la non-unicité) pour cette équation.

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