Contributions to decomposition methods in stochastic optimization

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ÉCOLE DOCTORALE MSTIC
MATHEMATIQUES, SCIENCES ET TECHNOLOGIES
DE L'INFORMATION ET DE LA COMMUNICATION

THÈSE

pour obtenir le titre de

Docteur

de l'Université Paris-Est

Spécialité : Mathématiques Appliquées

Présentée et soutenue par

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Contributions to Decomposition Methods in Stochastic Optimization

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préparée au sein de l'équipe Optimisation et Systèmes du laboratoire CERMICS de l'École des Ponts ParisTech

soutenue le 25 Juin 2014

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Abstract

Stochastic optimal control addresses sequential decision-making under uncertainty. As applications leads to large-size optimization problems, we count on decomposition methods to tackle their mathematical analysis and their numerical resolution. We distinguish two forms of decomposition. In *chained decomposition*, like Dynamic Programming, the original problem is solved by means of successive smaller subproblems, solved one after the other. In *parallel decomposition*, like Progressive Hedging, the original problem is solved by means of parallel smaller subproblems, coordinated and updated by a master algorithm.

In the first part of this manuscript, *Dynamic Programming: Risk and Convexity*, we focus on chained decomposition; we address the well known time decomposition that constitutes Dynamic Programming with two questions. In Chapter 2, we extend the traditional additive in time and risk neutral setting to more general ones for which we establish time-consistency. In Chapter 3, we prove a convergence result for the Stochastic Dual Dynamic Programming Algorithm in the case where (convex) cost functions are no longer polyhedral.

Then, we turn to parallel decomposition, especially decomposition methods obtained by dualizing constraints (spatial or non-anticipative). In the second part of this manuscript, *Duality in Stochastic Optimization*, we first point out that such constraints lead to delicate duality issues (Chapter 4). We establish a duality result in the pairing $\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$ in Chapter 5. Finally, in Chapter 6, we prove the convergence of the Uzawa Algorithm in $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$.

The third part of this manuscript, *Stochastic Spatial Decomposition Methods*, is devoted to the so-called *Dual Approximate Dynamic Programming Algorithm*. In Chapter 7, we prove that a sequence of relaxed optimization problems epiconverges to the original one, where almost sure constraints are replaced by weaker conditional expectation ones and that corresponding σ-fields converge. In Chapter 8, we give theoretical foundations and interpretations to the Dual Approximate Dynamic Programming Algorithm.

Résumé

Le contrôle optimal stochastique (en temps discret) s'intéresse aux problèmes de décisions séquentielles sous incertitude. Les applications conduisent à des problèmes d'optimisation de grande taille. En réduisant leur taille, les méthodes de décomposition permettent le calcul numérique des solutions. Nous distinguons ici deux formes de décomposition. La *décomposition chaînée*, comme la Programmation Dynamique, résout successivement des sous-problèmes de petite taille. La *décomposition parallèle*, comme le Progressive Hedging, consiste à résoudre itérativement et parallèlement les sous-problèmes coordonnés par un algorithme maître.

Dans la première partie de ce manuscrit, *Dynamic Programming : Risk and Convexity*, nous nous intéressons à la décomposition chaînée, en particulier temporelle, connue sous le nom de Programmation Dynamique. Dans le chapitre 2, nous étendons le cas traditionel, risque-neutre, de la somme en temps des coûts à un cadre plus général pour lequel nous établissons des résultats de cohérence temporelle. Dans le chapitre 3, nous étendons le résultat de convergence de l'algorithme SDDP (*Stochastic Dual Dynamic Programming Algorithm*) au cas où les fonctions de coûts (convexes) ne sont plus polyèdrales.

Puis, nous nous tournons vers la décomposition parallèle, en particulier vers les méthodes de décomposition obtenues en dualisant les contraintes (contraintes spatiales presque sûres, ou de non-anticipativité). Dans la seconde partie de ce manuscrit, *Duality in Stochastic Optimization*, nous commençons par souligner que de telles contraintes peuvent soulever des problèmes de dualité délicats (chapitre 4). Nous établissons un résultat de dualité dans les espaces pairés $\left(\mathrm{L}^\infty, \mathrm{L}^1\right)$ au chapitre 5. Finalement, au chapitre 6, nous montrons un résultat de convergence de l'algorithme d'Uzawa dans $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$. qui requiert l'existence d'un multiplicateur optimal.

La troisième partie de ce manuscrit, *Stochastic Spatial Decomposition Methods*, est consacrée à l'algorithme connu sous le nom de DADP (*Dual Approximate Dynamic Programming Algorithm*). Au chapitre 7, nous montrons qu'une suite de problèmes d'optimisation —dans lesquelles une contrainte presque sûre est relaxée en une contrainte en espérance conditionnelle— épi-converge vers le problème original si la suite des tribus converge vers la tribu globale. Finalement, au chapitre 8, nous présentons l'algorithme DADP, des interprétations, des résultats de convergence basés sur la seconde partie du manuscript.

Notations

We lay out the general rules and conventions followed in the manuscript:

- the random variables are written in bold,
- the letter x refers to a state, u refers to a control and w refers to a noise,
- the symbol $\sharp$ refers to optimality,
- the letters j and J refer to the objective function, and the letter Θ to the constraint function.

Here are the main notations:

$[\![a, b]\!]$	set of integers between a and b
$\{u_n\}_{n=n_0}^{n_1}$	sequence $\{u_{n_0}, u_{n_0+1}, \cdots, u_{n_1}\}$ (also written $\{u_n\}_{n_0}^{n_1}$)
$[A_t]_{t=t_1}^{t_2}$	Cartesian product of sets $A_{t_1} \times \cdots \times A_{t_2}$
w.r.t.	with respect to
$\sigma(\boldsymbol{X})$	σ-field generated by the random variable $\boldsymbol{X}$
$\boldsymbol{X} \preceq \mathcal{F}$	the random variable $\boldsymbol{X}$ is measurable w.r.t. the σ-field $\mathcal{F}$
$\boldsymbol{X} \preceq \boldsymbol{Y}$	the random variable $\boldsymbol{X}$ is $\sigma(\boldsymbol{Y})$-measurable
$x_n \to x$	the sequence $(x_n)_{n\in\mathbb{N}}$ (strongly) converges towards x
$x_n \rightharpoonup x$	the sequence $(x_n)_{n\in\mathbb{N}}$ weakly-converges towards x
$(\Omega, \mathcal{F}, \mathbb{P})$	probability space equipped with σ-algebra $\mathcal{F}$ and probability $\mathbb{P}$
$\mathcal{F}(E, F)$	space of functions mapping E into F
$\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}; E)$	space of all $\mathcal{F}$-measurable functions with finite moment of order p taking value in E
$\mathrm{L}^p(\Omega, \mathcal{F}, \mathbb{P}; E)$	Banach space of all equivalence classes of $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ functions, up to almost sure equality
$\mathbb{E}$, $\mathbb{E}_{\mathbb{P}}$	mathematical expectation w.r.t. probability $\mathbb{P}$
$\mathbb{P}$-a.s., a.s.	$\mathbb{P}$-almost surely
$\overline{\lim}$	upper limit
$\underline{\lim}$	lower limit
χ_A	indicator function taking value 0 on A, and $+\infty$ elsewhere
$\mathbb{1}_A$	characteristic function taking value 1 on A, and 0 elsewhere
$\mathrm{dom}\, f$	domain of f, i.e. set of points where f is finite
$f \equiv g$	means that the functions f and g are equal everywhere
$\mathrm{Aff}(A)$	affine hull of the set A
$\lvert A \rvert$	cardinal of the (finite) set A
$\langle y\, , x \rangle_{\mathcal{Y},\mathcal{X}}$	duality pairing of $y \in \mathcal{Y}$ against $x \in \mathcal{X}$
$x \cdot y$	usual Euclidian scalar product of $x \in \mathbb{R}^n$ against $y \in \mathbb{R}^n$
$X^\star$	topological dual of X (i.e. the space of the continuous linear forms on X)
$\mathrm{int}(A)$	interior of set A
$\mathrm{ri}(A)$	relative interior of set A
$\mathcal{P}(A)$	the set of subsets of A
$\overline{\mathbb{R}}$	the set of extended reals $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$
$\bar{\mathbb{R}}$	the set $\mathbb{R} \cup \{+\infty\}$ (used in Chapter 2)

Contents

Remerciements

Toutes les grandes personnes ont d'abord été des enfants, mais peu d'entre elles s'en souviennent.

Le petit prince

À la fin de ces trois années de thèse, je me rends compte que j'ai énormément de monde à remercier – et certains à plus d'un titre–, pour leur aide, conseils et soutien durant cette période. J'espère n'oublier personne, même si cela me semble une tâche impossible.

Mes remerciements vont tout d'abord à mes deux rapporteurs, A. Shapiro et Roger. Wets qui m'ont fait l'honneur de bien vouloir lire et commenter mes travaux. Ils ne cèdent en gratitude qu'à mes directeurs de thèse, P. Carpentier et M. De Lara, qui, aidé de J-P. Chancelier, m'ont accompagnés tout au long de cette thèse. Je leur suis infiniment reconnaissant pour leurs conseils –mathématiques, méthodologiques, linguistiques ou stylistiques– et du temps qu'ils m'ont consacrés. Je remercie également le jury pour leur présence, leur remarques et leur lecture de ce manuscrit, quand bien même les astres ce soient alignés contre la présence physique de certains le jour de la soutenance.

Durant ma thèse j'ai eu l'occasion de travailler avec divers chercheurs de grande qualité. Je tiens particulièrement à remercier : A. Ruszczynński pour son attention et ses conseils avisés lors de mon séjour à RUTGERS ; A. Philpott pour sa disponibilité, sa réactivité et son accueil chaleureux lors de mon séjour à Auckland ; V. Guiguès pour sa présence et ses remarques pertinentes lors de mon passage à Rio. J'ai également collaboré avec le département OSIRIS d'EDF R&D, et je tiens à remercier – dans le désordre et sans prétendre à l'exhaustivité – R. Aïd, L. Andrieu, K. Barty, A. Ben Abbès, S. Charousset, A. Dallagi, A. Lenoir, N. Oudjane tant pour les discussions amicales ou mathématiques que pour leur soutien constant. Merci aussi à ceux qui m'ont donné l'occasion d'enseigner durant la thèse et en particulier J.C. Gilbert pour un cours d'optimisation différentiable d'une qualité incomparable et J.F. Delmas pour un exigeant cours de probabilité et statistique. Je n'oublie pas les élèves qui ont eu la patience de m'écouter et m'ont permis de mieux comprendre en me forçant à (ré)expliquer d'une manière toujours nouvelle. Merci aussi à PGMO et au groupe MODE de la SMAI qui participent à la vie de la communauté de l'optimisation française.

Je tiens à souligner que si j'ai pu réaliser ma thèse dans de très bonnes conditions, matérielles et humaines c'est grâce au CERMICS, laboratoire de l'École des Ponts ParisTech au sein duquel je vais avoir la chance de pouvoir poursuivre mon activité de recherche, et à l'UMA laboratoire de l'ENSTA ParisTech, qui m'ont accueilli. Je remercie également le corps des Ponts des Eaux et des Forêts pour m'avoir offert la possibilité de faire une thèse en son sein et en particulier S. Piperno et les membres de la CFD-IPEF pour leur suivi attentif. Une pensée particulière à S. Cach pour sa disponibilité à nous guider dans les méandres administratifs.

Durant ces années de thèse j'ai navigué en compagnie d'un certain nombre d'autres thésards. En premier lieu Jean-Christophe, avec lequel j'ai beaucoup collaboré. Une part non négligeable de mes remerciements doit être dédié à Laurent, toujours prêt à discuter, sur le coin d'une table avec un café de tel ou tel problème. Sans oubliter les autres doctorants du CERMICS et en particulier ceux de l'équipe optimisation, Pauline, Thomas, Axel, Daniel...

Mais avant d'arriver en thèse il a bien fallu apprendre le B.A.BA et pour cela ma reconnaissance va à un certain nombre d'enseignants : à l'instit qui m'a donné des tables d'additions plus difficiles, au professeur de collège qui s'est battu contre "les recettes de cuisine" au profit d'une compréhension de chaque étape, à celle qui m'a demandé de ne pas perdre mon esprit curieux, ou celle qui nous avait accordé le droit de ne pas faire les exercices si on était capables de les faire aux tableaux et enfin à la directrice qui a permis à ce lycée merveilleux d'exister. Mais surtout à mes profs de prépa -et leurs colleurs- et en particulier M. Cornil qui m'a appris la rigueur. Un grand merci aussi aux professeurs de l'X et du master OJME, qui m'ont permis de trouver une branche des mathématiques qui me plaisait. En particulier F. Bonnans, J. Bolte qui me font l'honneur d'être également dans mon jury, mais aussi S. Sorin, S. Gaubert, P. Combettes, P. Cardialaguet, J. Renault...

Je souhaite également souligner le soutien que la communauté mathématique peut apporter sur internet à de jeunes étudiants au travers des forums (tout spécialement maths-forum et les-mathematiques.net) où des gens parfois très compétent prennent le temps de répondre à diverses questions de mathématiques, plus ou moins pointues. Je leur dois certains exemples ou références de ce manuscrit.

Il y a bien sûr tout un groupe d'amis de longue date qui ont su supporter mon envie de lire le Variational Analysis au pied d'une paroi d'escalade, ou mes tentatives désespérées de vulgarisation après deux mojitos et demi. Ils ont aussi été là quand le moral a pu baisser, avant de remonter, car si une chose est certaine c'est que le moral n'est pas une fonction unimodale du temps. Je cite Cyril, Gégé, Manu, Balthos, Ginette, Marina, Adrien, Xel, Guillaume, Camille, Aurélie, Baloo, Sabrina et demande pardon à tout ceux que j'ai pu oublier. Une place à part aussi pour ceux qui ont pu suivre eux aussi un chemin de thèse, je pense à Cyrille, Antony et Martin en particulier. Un merci aussi sincère que farfelu à Abricot qui a fait son possible pour chauffer mes genoux le plus longtemps possible et Edmond Rostand qui a laissé un beau guide de vie. Et surtout, surtout, un grand merci à Lili qui a veillé sur moi durant les mois de rédaction.

Finalement, recourant à l'expression de la langue de Shakespeare : last but not least, je tiens à remercier profondément ma famille pour tout ce qu'elle a fait. Je remercie mon frère pour les heures passées à construire ensemble des mondes merveilleux. Je remercie mon père, pour m'avoir fait jouer avec les maths en me vendant le chien et rachetant la maison avec des pièces jaunes, ou en me racontant qu'"il existe des nombres qui n'existent pas: on les appellent les nombres imaginaires" bien avant l'heure. Je le remercie aussi pour toutes les fois où il m'a aidé à faire des choix d'autant plus difficiles qu'il n'y en a pas de mauvais. Je remercie ma mère pour toutes ses attentions et sa capacité à m'aider et me soutenir quelles que soit les conditions ou les raisons.

Introduction

Mathematicians are like Frenchmen : whatever you say to them they translate into their own language and forthwith it is something entirely different.

Johann Wolfang von Goethe

Ce premier chapitre introductif est l'occasion de situer et présenter les travaux exposés dans ce manuscrit. Dans un premier temps nous présentons le cadre général de l'optimisation stochastique dynamique en temps discret, et donnons un aperçu des méthodes de décomposition. Nous présentons ensuite les trois parties du manuscrit. La première est consacrée à la programmation dynamique. La seconde à la théorie de la dualité dans le cadre de l'optimisation stochastique qui sera utile pour mettre en oeuvre des méthodes de décomposition telle que la décomposition par les prix. La troisième partie exploite les résultats de la seconde pour construire une méthode de décomposition spatiale en optimisation stochastique.

Optimisation Stochastique en Temps Discret

Cadre Général

L'optimisation, au sens mathématique, a pour but de trouver le minimum d'une fonction objectif sous un ensemble de contraintes. La fonction objectif dénotée $J : \mathbb{U} \to \mathbb{R} \cup \{+\infty\}$ peut être, dans un contexte économique, un coût ; dans un contexte physique une énergie ; ou encore, dans un contexte statistique, l'opposé d'un maximum de vraisemblance. Au cours de cette thèse nous utiliserons le vocabulaire du monde économique. La fonction objectif sera donc un *coût*, dénoté classiquement J, son argument un *contrôle* dénoté classiquement u. Notons que, dans de très nombreux cas, un problème réel comporte des incertitudes. Parfois ces incertitudes peuvent être négligées et un cadre déterministe être suffisant. Dans d'autres cas ces incertitudes peuvent être modélisées par une variable aléatoire $\boldsymbol{W}$, le problème devient alors un problème d'optimisation stochastique.

Nous nous intéressons particulièrement à l'*optimisation stochastique dynamique en temps discret à horizon fini.* Pour cela nous considérons un système dynamique contrôlé défini par un état initial x_0 et une équation d'évolution

$$\boldsymbol{X}_{t+1} = f_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_{t+1}) \ .$$

L'*état physique* du système à l'instant $t+1$ est dénoté $\boldsymbol{X}_{t+1}$ et est déterminé par son état à l'instant t ainsi que par le contrôle $\boldsymbol{U}_t$ choisi à l'instant t. Le terme “temps discret” souligne que la variable de temps t est discrète et non continue (auquel cas le système dynamique serait dirigé par une équation différentielle). Le terme “horizon fini” signifie qu'il existe un instant T à partir duquel le comportement du système ne nous intéresse plus.

Nous considérons à chaque pas de temps un coût instantané $L_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_{t+1})$ qui dépend de l'état actuel du système $\boldsymbol{X}_t$, du contrôle choisi $\boldsymbol{U}_t$ et d'un bruit $\boldsymbol{W}_{t+1}$. Nous considérons également un coût final $K(\boldsymbol{X}_T)$ qui dépend de l'état final du système dynamique. Nous avons donc $T+1$ coûts différents, chacun étant aléatoire. Ces suites de coûts aléatoires sont agrégées pour pouvoir être comparées. Il existe diverses manières de les agréger. La plus courante consiste à minimiser l'espérance de la somme en temps de ces coûts. Une zoologie des approches alternatives sera présentée au chapitre 2. Dans le cas usuel le problème d'optimisation s'écrit

$$\min_{\boldsymbol{X},\boldsymbol{U}} \quad \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_{t+1}) + K(\boldsymbol{X}_T)\Big] \tag{1a}$$

$$s.t. \quad \boldsymbol{X}_0 = x_0 \tag{1b}$$

$$\boldsymbol{X}_{t+1} = f_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_{t+1}) \qquad t = 0, \dots, T-1, \tag{1c}$$

$$\theta_t(\boldsymbol{X}_t, \boldsymbol{U}_t) = 0 \qquad t = 0, \dots, T-1, \tag{1d}$$

$$\boldsymbol{U}_t \preceq \mathcal{F}_t \qquad t = 0, \dots, T-1\ . \tag{1e}$$

La notation $\boldsymbol{U}_t \preceq \mathcal{F}_t$, signifie que $\boldsymbol{U}_t$ est mesurable par rapport à $\mathcal{F}_t$. Cette contrainte (contrainte (1e)) représente l'information disponible à l'instant t pour prendre la décision $\boldsymbol{U}_t$. Habituellement, la tribu $\mathcal{F}_t$ est donnée par

$$\mathcal{F}_t = \sigma(\boldsymbol{W}_1, \dots, \boldsymbol{W}_t)\ . \tag{2}$$

En d'autres termes, le contrôle $\boldsymbol{U}_t$ est pris en connaissant tous les bruits passés. Une famille de décisions $\{\boldsymbol{U}_t\}_{t=1}^{T-1}$ qui vérifie les contraintes de mesurabilité (contrainte (1e)), où $\mathcal{F}_t$ est donné par (2), est dite *non-anticipative*, car elle n'anticipe pas le futur.

Méthodes de Décomposition

Un problème d'optimisation stochastique dynamique est *a priori* difficile à résoudre. En effet, supposons que les bruits soient une suite de variables aléatoires indépendantes prenant 3 valeurs, et que chaque contrôle $\boldsymbol{U}_t$ puisse prendre deux valeurs (typiquement marche ou arrêt), alors le nombre de contrôles non anticipatifs est $2^{(3^{T+1}-1)/2}$, ce qui est rapidement colossal. En effet, la complexité du problème est exponentielle en l'horizon de temps, ainsi qu'en la taille des variables. En particulier, tester toutes les solutions d'un problème d'optimisation dynamique stochastique est numériquement impossible dès que l'on sort des problèmes les plus triviaux.

Pour attaquer les problèmes complexes il existe de nombreuses méthodes, exploitant les propriétés spécifiques des problèmes, ou mettant en place des heuristiques. Parmi elles nous nous intéressons aux méthodes de décomposition. Une approche par décomposition consiste à construire, à partir du problème original, un ensemble de sous-problèmes plus simples à résoudre. Itérativement les sous-problèmes sont résolus, puis ajustés jusqu'à ce que les solutions des sous-problèmes permettent de synthétiser la solution du problème global. Nous présentons en §1.2 une approche unifiée des méthodes de décomposition.

Supposons que chaque coût [1] $L_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_{t+1})$ est en fait une somme de coûts locaux

$$L_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_{t+1}) = \sum_{i=1}^{N} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})\ ,$$

1. Oublions quelques temps le coût final K

où $\boldsymbol{U}_t = \{\boldsymbol{U}_t^i\}_{i=1}^N$ et $\boldsymbol{X}_t = \{\boldsymbol{X}_t^i\}_{i=1}^N$. Supposons qu'il en va de même pour la contrainte (1d). Ainsi le problème (1) devient

$$\begin{aligned}
\min_{\boldsymbol{X},\boldsymbol{U}} \quad & \sum_{\omega\in\Omega}\sum_{i=1}^{N}\sum_{t=0}^{T-1} \quad \mathbb{P}\big(\{\omega\}\big)\; L_t^i\Big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\Big) && && \text{(3a)}\\
s.t. \quad & \boldsymbol{X}_{t+1}^i(\omega) = f_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\big) && \forall t, \quad \forall i, \quad \forall \omega && \text{(3b)}\\
& \sum_{i=1}^{N} \theta_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega)\big) = 0 && \forall t, \quad \forall \omega && \text{(3c)}\\
& \boldsymbol{U}_t^i \preceq \mathcal{F}_t && \forall t, \quad \forall i\ , && \text{(3d)}
\end{aligned}$$

On peut noter que le problème d'optimisation consiste à minimiser une somme en temps (variable t), en unité (variable i) et en aléa (variable ω). Sans les contraintes nous aurions donc $|\Omega| \times T \times N$ problèmes indépendants dont on veut minimiser la somme. Si les problèmes sont indépendants (les variables vivent dans un produit cartésien) alors la somme des minima est le minimum de la somme. En d'autres termes il suffit de minimiser chaque coût $L_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\big)$ par rapport à $\boldsymbol{U}_t^i(\omega)$ pour obtenir la solution du problème global. Malheureusement ces différents coûts ne sont pas indépendants. En d'autres termes, les contrôles $\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega)$ doivent répondre à des contraintes :

- en temps, à cause de l'équation de dynamique du système (Contrainte (3b)) ;
- en espace, à cause de la contrainte couplante du problème (Contrainte (3c)) ;
- en aléa, à cause de la contrainte de mesurabilité des contrôles (Contrainte (3d)).

Nous présenterons plus tard comment les méthodes de dualité permettent de remplacer les contraintes par un mécanisme de prix, et donc de décomposer le problème (3) en une somme de problèmes indépendants.

Nous allons commencer par une autre approche, dite de *décomposition chaînées*, où l'on résout successivement des problèmes de plus petite taille. Cette approche porte le nom de Programmation Dynamique.

Autour de la Programmation Dynamique

La programmation dynamique est une méthode générale de résolution d'un problème d'optimisation multi-étape. Elle s'appuie sur la notion d'état, qui sera discutée en 1.2.4.

Dans un premier temps nous faisons une présentation simple et succincte de cette méthode, puis nous présentons les résultats principaux du chapitre 2 qui étend la Programmation Dynamique à un cadre plus général, finalement nous présentons les résultats principaux du chapitre 3 qui exploite la programmation dynamique pour construire un algorithme efficace de résolution de problème d'optimisation stochastique dynamique.

Programmation Dynamique

Considérons le problème (1), en faisant l'importante hypothèse que la suite de bruits $\{\boldsymbol{W}_t\}_{t=1}^{T-1}$ est une suite de variables aléatoires indépendantes. Dans ce cas (sous des conditions d'existence de solution) on sait (voir [12, 18]) qu'il existe un contrôle optimal $\{\boldsymbol{U}_t^\sharp\}_{t=0}^{T-1}$ qui s'écrit comme fonction de l'état $\boldsymbol{X}_t$, i.e.

$$\boldsymbol{U}_t^\sharp = \pi_t\big(\boldsymbol{X}_t^\sharp\big)\ ,$$

où π_t est une stratégie, c'est à dire une fonction qui va de l'espace $\mathbb{X}_t$ des états à l'instant t dans l'espace des contrôles $\mathbb{U}_t$ à l'instant t. Pour construire cette stratégie nous définissons la fonction valeur de Bellman, obtenue par récurrence arrière :

$$V_T\big(x_T\big) = K(x_T) \qquad \forall x_T \in \mathbb{X}_T \ , \tag{4a}$$

$$V_t\big(x_t\big) = \min_{u \in \mathbb{U}_t} \mathbb{E}\Big[L_t\big(x_t, u, \boldsymbol{W}_{t+1}\big) + V_{t+1} \circ f_t\big(x_t, u, \boldsymbol{W}_{t+1}\big)\Big] \qquad \forall x_t \in \mathbb{X}_t \ . \tag{4b}$$

La fonction valeur $V_t : \mathbb{X}_t \to \overline{\mathbb{R}}$ s'interprète comme le coût minimal du système en partant d'un état $x_t \in \mathbb{X}_t$ à l'instant t. Ainsi, l'équation de récurrence (4) s'interprète en disant que le contrôle optimal à l'instant t est celui qui minimise, en moyenne, la somme du coût instantané $L_t\big(x_t, u, \boldsymbol{W}_{t+1}\big)$ et du coût futur $V_{t+1} \circ f_t\big(x_t, u, \boldsymbol{W}_{t+1}\big)$. Une stratégie optimale $\{\pi_t\}_{t=0}^{T-1}$ est alors donnée par

$$\pi_t(x_t) \in \operatorname*{arg\,min}_{u \in \mathbb{U}_t} \quad \mathbb{E}\Big[L_t\big(x_t, u, \boldsymbol{W}_{t+1}\big) + V_{t+1} \circ f_t\big(x_t, u, \boldsymbol{W}_{t+1}\big)\Big] \ .$$

La programmation dynamique est une méthode de décomposition en temps, puisque l'on résout T problèmes à un pas de temps, au lieu d'un problème à T pas de temps. Ainsi la complexité est linéaire en temps, et non plus exponentielle comme le serait une approche gloutonne. En revanche elle nécessite le recours à une notion d'état, et la complexité est exponentielle en la dimension de l'état. Ainsi la Programmation Dynamique ne sera numériquement efficace que si la dimension de l'état n'est pas trop importante (en pratique un état de dimension 4 ou 5 est à la limite de nos capacités de calcul).

Parfois le problème (1) ne satisfait pas l'hypothèse des bruits indépendants, mais on peut se ramener à une forme avec bruits indépendants si on étend l'état. Par exemple, si les bruits ont une dynamique d'ordre 1, à savoir

$$\boldsymbol{W}_{t+1} = \tilde{f}_t\big(\boldsymbol{W}_t, \widehat{\boldsymbol{W}}_t\big) \ , \tag{5}$$

où $\{\widehat{\boldsymbol{W}}_t\}_{t=0}^{T-1}$ est une suite de variables aléatoires indépendantes. Le nouvel état

$$\widehat{\boldsymbol{X}}_t = \big(\boldsymbol{X}_t, \boldsymbol{W}_t\big) \ , \tag{6}$$

est appelé un *état informationnel*, et suit la dynamique

$$\widehat{\boldsymbol{X}}_{t+1} = \Big(f_t\big(\boldsymbol{X}_t, \boldsymbol{W}_t\big), \tilde{f}_t\big(\boldsymbol{W}_t, \widehat{\boldsymbol{W}}_t\big)\Big) \ . \tag{7}$$

Avec cet état étendu nous pouvons utiliser une approche par programmation dynamique.

Le problème (1) considère la somme sur les aléas (espérance) d'un somme temporelle de coûts. Nous présentons dans la section suivante une extension du cadre d'application de la Programmation Dynamique, et ses liens avec la propriété de consistance temporelle.

Cadre Général et Consistance Temporelle

Un problème d'optimisation dynamique stochastique (en temps discret) est un problème de décision séquentielle sous incertitudes. Cela signifie, en reprenant les notations du problème (1), que nous avons une suite de $T+1$ coûts aléatoires $\boldsymbol{C}_t = L_t\big(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_{t+1}\big)$ à "minimiser". Pour pouvoir minimiser il faut pouvoir comparer des processus stochastiques. Une méthode générique simple, consiste à agréger le processus de coûts en un réel, parfois appelé l'équivalent certain du processus de coûts. La théorie des mesures de risque dynamique s'intéresse aux opérateurs associant à un processus de coûts son équivalent certain. Une manière de faire consiste à aggréger en temps les différents coûts, pour obtenir une variable aléatoire, puis à les aggréger en aléa pour obtenir un réel. Par exemple, dans le problème (1), les coûts sont aggrégés en temps par la somme inter-temporelle, puis en aléa par l'espérance. D'autres aggrégations sont possibles. Nous pouvons considérer un agrégateur temporel global $\Phi : \mathbb{R}^{T+1} \to \mathbb{R}$, et un agrégateur $\mathbb{G}$ sur l'aléa global (qui prend pour argument une variable aléatoire et est à valeur dans $\mathbb{R}$). Le problème (1) s'écrit alors

$$
\begin{aligned}
\min_{\boldsymbol{X},\boldsymbol{U}} \quad & \mathbb{G}\Big[\Phi\Big\{L_0(\boldsymbol{X}_0,\boldsymbol{U}_0,\boldsymbol{W}_1),\dots,L_{T-1}(\boldsymbol{X}_{T-1},\boldsymbol{U}_{T-1},\boldsymbol{W}_T),K(\boldsymbol{X}_T)\Big\}\Big] && \text{(8a)}\\
s.t. \quad & \boldsymbol{X}_0 = x_0 && \text{(8b)}\\
& \boldsymbol{X}_{t+1} = f_t(\boldsymbol{X}_t,\boldsymbol{U}_t,\boldsymbol{W}_{t+1}) \qquad t=0,\dots,T-1, && \text{(8c)}\\
& \theta_t(\boldsymbol{X}_t,\boldsymbol{U}_t) = 0 \qquad t=0,\dots,T-1, && \text{(8d)}\\
& \boldsymbol{U}_t \preceq \mathcal{F}_t \qquad t=0,\dots,T-1. && \text{(8e)}
\end{aligned}
$$

Nous présentons au chapitre 2 des conditions, pour résoudre ce problème par Programmation Dynamique. Présentons rapidement ces conditions et leurs conséquences.

- L'agrégateur temporel global peut s'écrire
$$\Phi\Big\{c_0,\cdots,c_T\Big\} = \Phi_0\Big\{c_0,\Phi_1\Big\{c_1,\cdots\Phi_{T-1}\big\{c_{T-1},c_T\big\}\Big\}\Big\}\ .$$
- L'agrégateur en aléa global peut s'écrire
$$\mathbb{G}\Big[J(w_1,\cdots,w_T)\Big] = \mathbb{G}_1\Big[w_1\mapsto\mathbb{G}_2\Big[\cdots w_T\mapsto\mathbb{G}_T\big[J(\boldsymbol{W}_1,\cdots,\boldsymbol{W}_T)\big]\Big]\Big]\ ,$$
où chaque $\mathbb{G}_t$ est un opérateur prenant pour argument des fonctions de w_t et pour valeur des réels.
- Chaque agrégateur $\mathbb{G}_t$ en aléa sur un pas de temps (resp. temporel Ψ_t) est croissant (resp. croissant en sa seconde variable).
- Les agrégateurs commutent, à savoir
$$\mathbb{G}_{t+1}\Big[\Phi_t\big\{\cdot,\cdot\big\}\Big] = \Phi_t\Big\{\cdot,\mathbb{G}_{t+1}\big[\,\cdot\,\big]\Big\}\ .$$

En effet, sous ces hypothèses, nous montrons que le problème (8) peut se réécrire sous forme imbriquée :

$$
\begin{aligned}
\min_{\boldsymbol{X},\boldsymbol{U}} \quad & \mathbb{G}_1\Bigg[\Phi_0\Bigg\{L_0(\boldsymbol{X}_0,\boldsymbol{U}_0,\boldsymbol{W}_1),\mathbb{G}_2\Bigg[\Phi_1\Bigg\{\cdots\\
& \qquad \mathbb{G}_{T-1}\Big[\Phi_{T-1}\Big\{L_{T-1}(\boldsymbol{X}_{T-1},\boldsymbol{U}_{T-1},\boldsymbol{W}_T),\mathbb{G}_T\big[K(\boldsymbol{X}_T)\big]\Big\}\Big]\Bigg\}\Bigg]\Bigg\}\Bigg] && \text{(9a)}\\
s.t. \quad & \boldsymbol{X}_0 = x_0 && \text{(9b)}\\
& \boldsymbol{X}_{t+1} = f_t(\boldsymbol{X}_t,\boldsymbol{U}_t,\boldsymbol{W}_{t+1}) \qquad t=0,\dots,T-1, && \text{(9c)}\\
& \theta_t(\boldsymbol{X}_t,\boldsymbol{U}_t) = 0 \qquad t=0,\dots,T-1, && \text{(9d)}\\
& \boldsymbol{U}_t \preceq \mathcal{F}_t \qquad t=0,\dots,T-1. && \text{(9e)}
\end{aligned}
$$

On déduit naturellement de cette formulation imbriquée une suite de problèmes d'optimisation, indicés par le temps et l'état initial.

$$
\begin{aligned}
(\mathcal{P}_t)(x) \qquad \min_{\boldsymbol{X},\boldsymbol{U}} \quad & \mathbb{G}_t\Bigg[\Phi_t\Bigg\{L_t(\boldsymbol{X}_t,\boldsymbol{U}_t,\boldsymbol{W}_{t+1}),\mathbb{G}_{t+1}\Bigg[\Phi_{t+1}\Bigg\{\cdots\\
& \qquad \mathbb{G}_{T-1}\Big[\Phi_{T-1}\Big\{L_{T-1}(\boldsymbol{X}_{T-1},\boldsymbol{U}_{T-1},\boldsymbol{W}_T),\mathbb{G}_T\big[K(\boldsymbol{X}_T)\big]\Big\}\Big]\Bigg\}\Bigg]\Bigg\}\Bigg]\\
s.t. \quad & \boldsymbol{X}_t = x\\
& \boldsymbol{X}_{\tau+1} = f_\tau(\boldsymbol{X}_\tau,\boldsymbol{U}_\tau,\boldsymbol{W}_{\tau+1}) \qquad \tau=t,\dots,T-1,\\
& \theta_\tau(\boldsymbol{X}_\tau,\boldsymbol{U}_\tau) = 0 \qquad \tau=t,\dots,T-1,\\
& \boldsymbol{U}_\tau \preceq \mathcal{F}_\tau \qquad \tau=t,\dots,T-1.
\end{aligned}
$$

On définit une fonction valeur $V_t : \mathbb{X}_t \mapsto \mathbb{R}$ qui donne en fonction de l'état initial la valeur du problème $(\mathcal{P})$. Cette fonction est obtenue par une récurrence arrière, à savoir

$$V_T(x_T) = K(x_T) \qquad \forall x_T \in \mathbb{X}_T \ , \tag{10a}$$

$$V_t(x_t) = \min_{u \in \mathbb{U}_t} \mathbb{G}_t\Big[\Phi_t\Big\{L_t(x_t, u, \boldsymbol{W}_{t+1}), V_{t+1} \circ f_t(x_t, u, \boldsymbol{W}_{t+1})\Big\}\Big] \qquad \forall x_t \in \mathbb{X}_t \ . \tag{10b}$$

On déduit des fonctions valeurs une stratégie optimale pour le problème (9) (et donc pour le problème (8)) en selectionnant, à la date t, le controle $u \in \mathbb{U}_t$ réalisant le minimum de (10b) (où x_t désigne l'état courant).

Ainsi nous avons un cadre théorique général pour établir une équation de programmation dynamique (equations du type de (10)). Nous avons au passage établi que la suite de problème $\{(\mathcal{P}_t)\}$ était consistante (en temps). En effet nous avons construit une stratégie optimale pour le problème $\mathcal{P}_0(x_0)$, et montré que cette stratégie était également optimale pour les problèmes $\mathcal{P}_t$, avec $t \geq 1$. Au chapitre 2, nous définissons précisément les conditions évoqués plus haut, et démontrons les résultats annoncés. De plus nous nous attardons sur les liens entre ces différents problèmes d'optimisation et les mesures de risque dynamique. En particulier il existe dans cette littérature une notion de consistance temporelle que nous relions à celle évoquée pour les suites de problèmes d'optimisation.

Stochastic Dual Dynamic Programming

Le chapitre 2 étend le cadre de la Programmation Dynamique, mais ne s'occupe pas des difficultés numériques de mise en oeuvre, en particulier du problème de la *malédiction de la dimension*. L'algorithme SDDP (Stochastic Dual Dynamic Programming), connu depuis 1991, exploite l'équation de programmation dynamique pour construire une approximation polyèdrale des fonctions valeurs V_t. L'avantage numérique principal consistant à se ramener à des problèmes que l'on sait résoudre de manière efficace (typiquement des problèmes linéaires), et ainsi de pouvoir attaquer des problèmes de dimension plus grande que ce que n'autorise une simple programmation dynamique. Présentons en quelques mots cet algorithme.

On considère le problème (1), avec l'hypothèse que les bruits sont indépendants. On note V_t la valeur de Bellman associée au problème, obtenue par l'équation (4). On suppose que les fonctions de coût L_t et K soient convexes, et que les fonctions de dynamique f_t soient affines. Dans ce cas les valeurs de Bellman V_t sont convexes. On suppose que l'on dispose, à l'itération k de l'algorithme, d'approximations des fonctions de Bellman $V_t^{(k)}$ qui vérifient $V_t^{(k)} \leq V_t$. L'algorithme se déroule ensuite en deux temps :

- dans une phase avant on détermine une trajectoire de l'état à partir des approximations des fonctions valeurs,
- dans une phase arrière on améliore les approximations des fonctions valeurs au niveau de cette trajectoire.

On tire au hasard une suite d'aléa $\{w_t^{(k)}\}_{t=1}^T$. On en déduit une trajectoire $\{x_t^{(k)}\}_{t=0}^T$ du système obtenue à partir des approximations de la fonction valeur :

$$\begin{aligned} x_0^{(k)} &= 0 \ , \\ u_t^{(k)} &\in \arg\min \mathbb{E}\Big[L_t(x_t^{(k)}, u, \boldsymbol{W}_{t+1}) + V_{t+1}^{(k)} \circ f_t(x_t^{(k)}, u, \boldsymbol{W}_{t+1})\Big] \ , \\ x_{t+1}^{(k)} &= f_t(x_t^{(k)}, u, w_{t+1}^{(k)}) \ . \end{aligned}$$

Notons que si les approximations de la fonction de Bellman étaient exactes $V_t^{(k)} = V_t$, alors la trajectoire obtenue est la trajectoire optimale du problème.

Maintenant que l'on dispose d'une trajectoire $\{x_t^{(k)}\}_{t=0}^T$, on peut déterminer, pour chaque instant t, une *coupe* de la fonction valeurs V_t. Plus précisément, en résolvant le problème

$$\min_u \mathbb{E}\Big[L_t\big(x_t^{(k)}, u, \boldsymbol{W}_{t+1}\big) + V_{t+1}^{(k)} \circ f_t\big(x_t^{(k)}, u, \boldsymbol{W}_{t+1}\big)\Big] \ ,$$

on obtient, par méthode de dualité et en exploitant la convexité de la fonction V_t, une fonction affine

$$\theta_t^{(k)} + \left\langle \beta_t^{(k)} \, , \cdot - x_t^{(k)} \right\rangle \ ,$$

qui est en dessous de la fonction valeur V_t. On peut donc améliorer l'approximation de la fonction V_t, en définissant

$$V_t^{(k+1)}(\cdot) = \max\left\{V_t^{(k)}(\cdot), \theta_t^{(k)} + \left\langle \beta_t^{(k)} \, , \cdot - x_t^{(k)} \right\rangle\right\} \ .$$

Nous montrons au chapitre 3 que cet algorithme converge dans le sens où les fonctions valeurs approximées $V_t^{(k)}$ convergent vers la fonction V_t aux points visités par une trajectoire optimale du système. Le résultat du chapitre 3 étend les preuves jusqu'à présent dans deux directions :

- jusqu'à maintenant les fonctions de coûts L_t et K étaient supposées linéaires, et nous ne faisons qu'une hypothèse de convexité ;
- nous avons construit une classe d'algorithme assez large incluant les diverses variantes de SDDP rencontrées dans la littérature.

Dualité en Optimisation Stochastique

La théorie de la dualité permet de transformer une contrainte en un coût. Cette approche sera utilisée pour construire une méthode de décomposition spatiale.

Dans un premier temps nous présentons le schéma de décomposition par les prix comme motivation pour la seconde partie du manuscrit. Puis nous évoquons les difficultés à établir des résultats de qualification des contraintes dans un espace L^p, $p < +\infty$, requis par la décomposition par les prix. Nous donnons ensuite des résultats de qualification pour l'espace L^∞. Finalement, nous adaptons l'algorithme d'Uzawa (qui requiert de tels résultats de qualification) à l'espace L^∞ en présentant des résultats de convergence.

Décomposition par les Prix

Nous présentons ici, sur un problème simple, la méthode de décomposition par les prix. Cette méthode peut être intuitivement comprise ainsi. Considérons un problème de production où un décideur dispose de N centrale de production (indicé par i), chacune produisant $\theta_i(u_i)$ pour le contrôle u_i, et devant satisfaire une certaine demande. La demande est incoporée dans l'une des fonctions de production de sorte que la contrainte d'égalité offre-demande s'écrit

$$\sum_{i=1}^N \theta_i(u_i) = 0 \ . \tag{11}$$

Par ailleurs, choisir le contrôle u_i coûte $L_i(u_i)$, et l'objectif du décideur est de minimiser la somme (sur i) des coûts.

La décomposition par les prix consiste à remplacer la contrainte (11) par un système de prix. Pour obtenir un bon prix on suppose qu'un coordinateur propose un prix (par unité produite) à toutes les centrales. Chacune annonce alors la quantité qu'elle produit, et le coordinateur peut ajuster son prix. Plus précisément, à l'itération k, le coordinateur fixe un prix $p^{(k)} = -\lambda^{(k)}$ pour la production des centrales $\theta_i(u_i)$. Chaque centrale maximise alors son profit, à savoir les gains obtenu $p^{(k)}\theta_i(u_i)$ par la production moins le coût local $L_i(u_i)$, et obtient une solution $u_i^{(k)}$. Puis le coordinateur compare la somme des productions avec la demande. Si la demande n'est pas satisfaite le prix est augmenté, si la demande est dépassée par la production, le prix est réduit, et l'on peut passer à l'étape $k+1$ avec le nouveau prix.

Mathématiquement parlant, considérons le problème suivant :

$$\begin{aligned}
&\min_{\{u_i\}_{i=1}^N} && \sum_{i=1}^N L_i(u_i) && && (12a)\\
&s.t. && u_i \in U_i^{\mathrm{ad}}, && \forall i \in [\![1,N]\!] , && (12b)\\
& && \sum_{i=1}^N \theta_i(u_i) = 0 . && && (12c)
\end{aligned}$$

Sous des conditions techniques, ce problème est équivalent à

$$\begin{aligned}
&\min_{\{u_i\}_{i=1}^N} \max_{\lambda\in\mathbb{R}} && \sum_{i=1}^N L_i(u_i) + \lambda\Big(\sum_{i=1}^N \theta_i(u_i)\Big) && && (13a)\\
&s.t. && u_i \in U_i^{\mathrm{ad}}, && \forall i \in [\![1,N]\!] . && (13b)
\end{aligned}$$

Si nous disposons d'une hypothèse de qualification des contraintes, nous pouvons échanger les opérateurs min et max dans le problème (13), pour obtenir

$$\begin{aligned}
&\max_{\lambda\in\mathbb{R}} \min_{\{u_i\}_{i=1}^N} && \sum_{i=1}^N L_i(u_i) + \lambda\theta_i(u_i) && && (14a)\\
&s.t. && u_i \in U_i^{\mathrm{ad}}, && \forall i \in [\![1,N]\!] . && (14b)
\end{aligned}$$

On remarque alors que le problème de minimisation intérieure, i.e. à λ fixé, consiste à minimiser une somme de coûts locaux déterminés par des contrôles indépendants. Ainsi, la somme des minimas est le minimum de la somme et le problème (14) devient

$$\begin{aligned}
&\max_{\lambda\in\mathbb{R}} \sum_{i=1}^N \min_{u_i} && L_i(u_i) + \lambda\theta_i(u_i) && (15a)\\
&\qquad\qquad s.t. && u_i \in U_i^{\mathrm{ad}} . && (15b)
\end{aligned}$$

Pour un multiplicateur $\lambda = \lambda^{(k)}$ donné, nous avons N problèmes de minimisation séparés, qui sont les sous-problèmes de la méthode de décomposition. Ils s'écrivent comme suit.

$$\begin{aligned}
&\min_{u_i} && L_i(u_i) + \lambda^{(k)}\theta_i(u_i) && (16a)\\
&s.t. && u_i \in U_i^{\mathrm{ad}} . && (16b)
\end{aligned}$$

Ces problèmes sont mis à jour en ajustant le prix, par exemple avec

$$\lambda^{(k+1)} = \lambda^{(k)} + \rho \sum_{i=1}^N \theta_i(u_i^{(k)}) , \tag{17}$$

où $\rho > 0$ est un pas donné et $u_i^{(k)}$ une solution optimale du problème (16). Cette formule de mise à jour fait partie de l'algorithme d'Uzawa, rappelé et étendu au chapitre 6.

Problèmes de Qualifications des Contraintes en Optimisation Stochastique

Pour pouvoir remplacer une contrainte par un prix il faut utiliser la théorie de la dualité, brièvement évoquée au chapitre 4. Cette théorie consiste à construire une famille de problèmes perturbés à partir du problème d'origine, ce dernier n'étant plus qu'un cas particulier (le cas où la perturbation est nulle). La fonction qui à une perturbation donnée associe la valeur du problème perturbé est appelée fonction valeur. En utilisant des outils d'analyse convexe on peut alors construire un problème dual du problème original, et les propriétés de régularité (semi-continuité inférieure, sous-différentiabilité) permettent d'établir des liens entre le problème initial et son dual. On note toutefois que le dual dépend des perturbations choisies.

Les contraintes d'un problème d'optimisation seront dites *qualifiées* si elles peuvent être remplacées par un prix, ou, en d'autres termes, si les valeurs du problème primal et dual sont égales et que le problème dual admet une solution optimale. Une condition nécessaire et suffisante, mais abstraite, pour cela est que la fonction valeur soit égale à sa bi-conjuguée de Fenchel. Une condition suffisante courante est rappelée à la proposition 4.10.

Cette technologie mathématique met en lumière l'importance du choix des espaces dans lequel on pose le problème d'optimisation, ainsi que de l'espace de perturbation choisi pour construire le problème dual. Dans le cadre de l'optimisation stochastique, pour utiliser des méthodes de gradient on est tenté de se placer dans un espace de Hilbert, par exemple l'espace L^2 des fonctions de carré intégrables. Nous exposons en §4.2 deux exemples montrant les difficultés d'un tel choix. Dans le premier exemple, nous présentons un problème simple, avec toutes les "bonnes propriétés" que l'on pourrait souhaiter à première vue, dont cependant les contraintes ne sont pas qualifiées dans L^2. Dans le second exemple nous montrons que même lorsque les contraintes sont qualifiées, la condition suffisante de qualification n'est pas vérifiée.

Existence de Multiplicateur dans L^1

Le chapitre 4 montre qu'il est difficile d'avoir des contraintes presque sûres qualifiées dans L^2. Le chapitre 5 établit un résultat de qualification des contraintes presque sûres dans L^∞.

Dans ce chapitre nous montrons que, si la fonction coût $J : \mathrm{L}^\infty \to \mathbb{R} \cup \{+\infty\}$ est finie partout, alors des contraintes affines, presque sûres, d'égalité et les contraintes de non-anticipativité admettent un multiplicateur L^1. En d'autres termes il existe un mécanisme de prix qui peut remplacer cette contrainte. Cependant, l'hypothèse de finitude sur L^∞ interdit la présence de contraintes de bornes presque sûres. Nous trouvons dans la littérature (T. Rockafellar et R. Wets) des résultats de qualification de contraintes d'inégalité sous une hypothèse de *relatively complete recourse*

Nous montrons également comment les hypothèses conduisant à la qualification des contraintes s'appliquent sur un problème d'optimisation dynamique stochastique.

Algorithme d'Uzawa dans $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$

Le chapitre 6 est consacré à l'extension de l'algorithme d'Uzawa (défini dans un espace de Hilbert, par exemple L^2 en optimisation stochastique) à l'espace de Banach non réflexif L^∞. En effet l'algorithme d'Uzawa peut être utilisé comme algorithme de coordination dans une méthode de décomposition par les prix, mais requiert une hypothèse de qualification des contraintes. Or le chapitre 4 a montré que la qualification des contraintes dans L^2 est difficile à vérifier, tandis que le chapitre 5 fournit des hypothèses de qualification dans L^∞.

Il y a deux difficultés à passer de L^2 à L^∞. D'une part il faut donner du sens à l'algorithme d'Uzawa, qui exploite l'identification d'un Hilbert avec son dual topologique dans sa phase de mise à jour. D'autre part, il faut adapter la preuve de convergence qui repose sur des estimations classiques dans un espace Hilbertien.

Deux résultats principaux sont à retenir.

- Avec des hypothèses comparables au cas Hilbertien, plus exigeantes sur la continuité des fonctions mises en jeu, mais se contentant de l'existence d'un multiplicateur L^1, nous montrons la convergence (au sens L^∞) d'une sous-suite de la suite de controle générée par l'algorithme d'Uzawa.
- Sous les mêmes hypothèses, mais avec l'existence d'un multiplicateur L^2, nous renforçons le résultat classique de convergence L^2 en prouvant la convergence au sens L^∞ de la suite des contrôles donnée par l'algorithme d'Uzawa.

Finalement, nous montrons comment l'algorithme conduit naturellement à une méthode de décomposition par les prix pour un problème d'optimisation dynamique stochastique. Cependant le multiplicateur à manipuler est un processus stochastique et non plus un vecteur d'un espace de dimension finie comme c'était le cas dans un cadre déterministe. Ceci a deux défauts majeurs :

- d'une part le multiplicateur vit dans un espace gigantesque, et l'ajuster prendra un grand nombre d'itérations ;
- d'autre part les sous-problèmes obtenus ne sont pas forcément beaucoup plus simples à résoudre que le problème d'origine.

Ces points sont traités dans la troisième partie du manuscrit.

Décomposition Spatiale en Optimisation Stochastique

Nous montrons, au chapitre 6 qu'une méthode de décomposition par les prix directement appliquée à un problème d'optimisation stochastique dynamique fournit des sous-problèmes difficiles à résoudre. Nous proposons donc d'approximer le problème d'origine pour pouvoir appliquer la décomposition par les prix et obtenir des sous-problèmes numériquement solvables.

Épiconvergence de Problèmes relaxés

Le chapitre 7 s'intéresse à la relaxation de contraintes presque sûres en optimisation stochastique. En effet, considérons le problème sous forme abstraite

$$(\mathcal{P}) \qquad \min_{\boldsymbol{U}\in\mathcal{U}^{\mathrm{ad}}\subset\mathcal{U}} \quad J\big(\boldsymbol{U}\big) \qquad s.t. \quad \Theta\big(\boldsymbol{U}\big)=0$$

On peut le relaxer, c'est à dire affaiblir les contraintes, ou encore élargir l'ensemble des contrôles admissibles. La relaxation que l'on considère consiste à remplacer la contrainte presque sûre

$$\Theta\big(\boldsymbol{U}\big)=0\ ,$$

par une contrainte en espérance conditionnelle

$$\mathbb{E}\Big[\Theta\big(\boldsymbol{U}\big)\ \Big|\ \mathcal{B}\Big]=0\ .$$

Pour une tribu $\mathcal{B}=\mathcal{F}_n$ on note $\big(\mathcal{P}_n\big)$ le problème relaxé.

Le résultat principal du chapitre 7 dit que si

- la fonction objectif $J:\mathcal{U}\to\mathbb{R}$ est continue,
- la fonction contrainte $\Theta:\mathcal{U}\to\mathcal{V}$ est continue,
- la suite de tribu $\big\{\mathcal{F}_n\big\}_{n\in\mathbb{N}}$ converge vers la tribu globale du problème $\mathcal{F}$,

alors la suite de problème $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ épiconverge vers le problème original. En pratique cela signifie que chacune des valeurs d'adhérence de la suite $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ des contrôles optimaux associés aux problèmes relaxés $(\mathcal{P}_n)$ est solution du problème d'origine.

Nous montrons aussi des exemples de fonctions J et Θ qui sont continues. En effet ces fonctions allant d'un espace de variables aléatoires dans un autre espace de variable aléatoire l'hypothèse de continuité est a priori assez abstraite. Nous montrons que la topologie de la convergence en probabilité permet de modéliser un certain nombre de contraintes comme une fonction Θ continue.

Algorithm DADP (Dual Approximate Dynamic Programming)

Le chapitre 8 est consacré à l'algorithme DADP (Dual Approximate Dynamic Programming). Cet algorithme peut-être vu comme une méthode de décomposition par les prix sur un certain type de relaxation du problème d'origine, de telle sorte que les sous-problèmes soient solvables numériquement par programmation dynamique.

On considère le problème 1, où les contrôles et l'état s'écrivent comme une collection de contrôles locaux (i.e. $\boldsymbol{U}_t = \{\boldsymbol{U}_t^1, \dots, \boldsymbol{U}_t^N\}$ et $\boldsymbol{X}_t = \{\boldsymbol{X}_t^1, \dots, \boldsymbol{X}_t^N\}$) et les fonctions de coûts et de contraintes presque sûres comme une somme de fonctions locales (i.e. $L_t(\boldsymbol{X}, \boldsymbol{U}, \boldsymbol{W}) = \sum_{i=1}^n L_t^i(\boldsymbol{X}^i, \boldsymbol{U}^i, \boldsymbol{W})$ et $\theta_t(\boldsymbol{X}, \boldsymbol{U}, \boldsymbol{W}) = \sum_{i=1}^n \theta_t^i(\boldsymbol{X}^i, \boldsymbol{U}^i, \boldsymbol{W})$). Dans ce cas l'algorithme DADP consiste à relaxer la contrainte presque sûre 1d par

$$\mathbb{E}\Big[\sum_{i=1}^n \theta_t^i(\boldsymbol{X}^i, \boldsymbol{U}^i, \boldsymbol{W}) \;\Big|\; \boldsymbol{Y}_t\Big] = 0 \;,$$

où $\boldsymbol{Y}_t$ est un processus d'information vérifiant

$$\boldsymbol{Y}_{t+1} = \tilde{f}_t(\boldsymbol{Y}_t, \boldsymbol{W}_t) \;.$$

Sur le problème approximé on peut alors écrire une décomposition par les prix en dualisant la contrainte appoximée. Le gain par rapport à une décomposition par les prix standards tient au fait que l'on peut se contenter de multiplicateur $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \dots, \boldsymbol{\lambda}_{T-1})$ tel que $\boldsymbol{\lambda}_t$ soit mesurable par rapport à $\boldsymbol{Y}_t$. Ainsi, d'une part l'espace des multiplicateurs est plus petit, d'autre part les sous-problèmes peuvent se résoudre par programmation dynamique avec l'état étendu $(\boldsymbol{X}_t^i, \boldsymbol{Y}_t)$ à comparer à l'état $\boldsymbol{X}_t$ pour la résolution directe par programmation dynamique du problème global.

Le chapitre 8 présente, étape par étape, l'algorithme DADP brièvement évoqué ci-dessus. Nous donnons ensuite diverses interprétations de l'algorithme :
- méthode de décomposition par les prix d'un problème approximé,
- méthode d'approximation du multiplicateur pour une décomposition par les prix du problème original,
- approche par règle de décision du problème dual.

Les résultats des chapitres 4 à 7 sont utilisés pour établir des conditions de convergence de l'algorithme. Finalement une application numérique encourageante est présentée.

Conclusion

Le sujet des méthodes de décomposition-coordination en optimisation stochastique reste très largement inexploré. Sans être exhaustif, citons quelques pistes de développement possibles.
- A l'aide du cadre développé au chapitre 2, les liens entre la consistance temporelle des mesures de risque dynamique et des suites de problèmes d'optimisation doivent être précisés.

- La convergence de l'algorithme SDDP, donnée au chapitre 3, s'appuie sur le fait que les aléas prennent des valeurs discrètes. Il y a de nombreuses raisons de penser que la preuve peut être étendue à des variables aléatoires continues, mais cela nécessite de traiter des difficultés inhérentes au cadre infini-dimensionnel.
- Pour étendre les conditions d'existence de multiplicateur (obtenues au chapitre 5) au cas de contraintes d'inégalité nous pensons qu'il faut adapter les résultats de la littérature qui utilisent la notion de *relatively complete recourse*. Ceux-ci permettront d'avoir un résultat de qualification en présence de bornes sur le contrôle.
- Le résultat de convergence que nous avons obtenu au chapitre 6 pour l'algorithme d'Uzawa devrait pouvoir être amélioré pour obtenir la convergence de la suite des contrôles (pour le moment nous avons simplement la convergence d'une sous-suite).
- Nous avons vu au chapitre 7 qu'une suite de relaxation d'un problème d'optimisation, où une contrainte presque-sûre est remplacée par une contrainte en espérance conditionnelle, épiconverge vers le problème original lorsque l'information converge. Cependant l'algorithme DADP ne cherche pas à faire converger l'information vers l'information globale du problème. Ainsi, il faudrait compléter le résultat d'épiconvergence pour obtenir des estimations d'erreurs liées à l'approximation faite lorsque l'on utilise l'algorithme DADP.
- Sur un plan numérique il faut comparer les algorithmes DADP et SDDP (référence actuelle) sur un problème de gestion d'une vallée hydraulique de grande taille. Dans un second temps, l'algorithme SDDP pourrait être intégré à DADP comme outil de résolution des sous-problèmes.
- Finalement notons que nous avons principalement étudié une approche de décomposition par les prix. Il existe, en déterministe, d'autres méthodes de décomposition à étendre au cadre stochastique.

Chapter 1

Preliminaries

If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

John von Neumann

Contents

Before diving into the core of the manuscript, we develop the framing of multistage stochastic optimization problems in §1.1, and we present a unified treatment of decomposition resolution methods in §1.2. Finally, we detail in §1.3 the setting of a dam management problem, that serves as an illustrative thread running throughout the manuscript.

Introduction

We open this chapter with considerations on mathematical optimization and modelling.

What is Mathematical Optimization

In this manuscript, we consider optimization in the sense of minimizing [1] an *objective* function [2] under constraints. This objective function can be a cost in an economic problem, an energy in a physical problem, a likelihood in a statistical problem, etc. The objective

1. Some applications require to maximize a function (in economics, for instance), which is obviously the same problem as minimizing the opposite function.

2. The community of multi-objective optimization considers multiple objectives at the same time — see [66, 106]

function J maps a set $\mathcal{U}$ of *controls*[3] into $\overline{\mathbb{R}}$. Solving an optimization problem generally means finding the set of minimizers of the objective function, or at least one minimizer $u^\sharp$, as well as the value of the minimum. The field of mathematical optimization is concerned with finding:

- conditions of existence (and sometimes of uniqueness) of a local or global minimizer of the objective function on a set of *admissible controls*;
- sufficient conditions of optimality;
- necessary conditions of optimality that reduce the set of controls to explore;
- algorithms yielding a sequence of controls converging to an optimal control;
- speed of convergence of such algorithms;
- bound on the error made when the algorithm is stopped;
- etc.

The general optimization problem we consider is written

$$\min_{u \in \mathcal{U}^{\mathrm{ad}} \subset \mathcal{U}} \quad J(u) \tag{1.1a}$$

$$s.t. \quad \Theta(u) \in -C \ , \tag{1.1b}$$

where J is the *objective function*, $\mathcal{U}^{\mathrm{ad}}$ is a *constraint set* of a vector space $\mathcal{U}$, C is the *constraint cone* of a vector space $\mathcal{V}$, $\Theta : \mathcal{U} \to \mathcal{V}$ is the *constraint function*, and $u \in \mathcal{U}$ is a *control*. A control u is said to be *admissible* if $u \in \mathcal{U}^{\mathrm{ad}}$ and $\Theta(u) \in -C$. A control $u^\sharp$ is said to be *optimal* if we have $J(u^\sharp) \leq J(u)$, for any admissible control u.

Notice that we have distinguished to types of constraints: a *set membership* constraint $\Theta(u) \in -C$, and an abstract constraint $u \in \mathcal{U}^{\mathrm{ad}}$. The set membership constraint is classically represented by several equality and inequality constraints, and, in this manuscript, we will often treat it by duality, whereas the abstract constraint will be kept as such. Of course, there is latitude in choosing to model a constraint as part of $\Theta(u) \in -C$ or as part of $u \in \mathcal{U}^{\mathrm{ad}}$, since $\mathcal{U}^{\mathrm{ad}}$ can accept any kind of constraint.

The Art of Modelling

In practice, a "real-life" optimization problem is not given in mathematical form, but has to be casted and formulated as such. Crafting a model is a trade-off between, on the one hand, realism and complexity, and, on the other hand, mathematical tractability.

In the special case of fashioning a multistage optimization problem, we distinguish three elements to be identified:

- the *control variables* and their timing;
- the *objective function* (or *criterion*) J, that reflects multiple conflicting interests quantified and weighted each against the other, while other objectives will be formulated as constraints;
- the *constraints* that restrict control variables, and incorporate objectives outside the criterion J.

In this manuscript, we shed special light on constraints and, in the perspective of multistage stochastic optimization, we put forward three types of constraints.

Physical constraints. They result from physical laws, e.g. the maximum speed of a vehicle, the maximum volume of a reservoir, the dynamical evolution of stocks, etc.

Information constraints. They state what is the information available when choosing a control. In a stochastic setting, we will mostly represent them by measurability constraints.

3. We use indifferently the terminology *decision* or *control* (though control is generally reserved to trajectories of decisions).

Objectives as constraints. They represent other objectives than the criterion J. In this sense, they are "soft constraints" reflecting preferences of the decision-maker (like risk constraints).

Physical and information constraints have to be satisfied whatever the cost, as they derive from physical laws. They are called "hard constraints" because we cannot "negotiate" with them. By contrast, constraints that can be negotiated with, at the modelling level, are called "soft constraints". For instance, constraints representing objectives, e.g. risk constraints, could be loosened by the decision-maker. Moreover, some physical constraints could be loosened through complex mechanisms not represented in the mathematical problem (we could upgrade our engine to have a higher speed, or extend a reservoir to have more capacity, etc.). For soft constraints, the multipliers (see the duality theory of Chapter 4) give precious informations, as they can be interpreted as the marginal cost of infinitesimally relaxing a constraint.

1.1 Framing Stochastic Optimization Problems

Before tackling resolution methods in §1.2, we focus on how to frame stochastic optimization problems. We start with stochastic static optimization problems in §1.1.1, then move to multistage stochastic optimization problems in §1.1.2.

1.1.1 Framing of a Static Stochastic Optimization Problem

In most problems, uncertainties abound. In stochastic optimization, these uncertainties are modeled by random variables [4] or stochastic processes, together with their joint probability distributions. [5] Selecting possible classes of probabilities, reflecting in particular dependencies between random variables, is a modelling issue. Specifying the parameters of the law is a statistical problem that has also to be dealt with, although it is not a part of the optimization problem itself.

With uncertainties, the cost itself becomes a random variable. As one cannot easily rank two random variables (when is one random cost "better" than another?), one usually averages out and aggregates the random cost to produce a single number. The most used *random aggregator* is the mean, or mathematical expectation. In some cases (financial problems), the expectation is taken with respect to another probability (namely the risk-neutral probability) than the original one, or alternative random aggregators, representing alternative risk preferences, can be used (see §2.2.2 for a presentation of risk measures). In Chapter 2, we will consider a large spectrum of uncertainty aggregators.

The traditional stochastic optimization problem is formulated as

$$\min_{\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}} \subset \mathcal{U}} \quad \mathbb{E}\big[J(\boldsymbol{U}, \boldsymbol{W})\big] \tag{1.2a}$$

$$s.t. \quad \boldsymbol{U} \preceq \mathcal{B} \tag{1.2b}$$

where

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $\mathbb{E}$ is the mathematical expectation;
- $\mathcal{U}$ is the space of all random variables $\boldsymbol{U} : \Omega \to \mathbb{U}$, where $\mathbb{U}$ is a measurable space;
- $\boldsymbol{W} : \Omega \to \mathbb{W}$ is a random variable that represents *exogenous noise*, where $\mathbb{W}$ is a measurable space;

4. We use *random variable* as a generic term that includes random vectors and stochastic processes. Throughout this manuscript, we write random variables in bold. We consistently use the notation $\boldsymbol{W}$ for the noises, i.e. the exogenous random variables.

5. In a connex area known as *robust optimization* (see [13, 15]), uncertainties are modeled as sets of values that the uncertain parameters can take, and optimization is performed with respect to the worst possible case.

- $J : \mathbb{U} \times \mathbb{W} \to \mathbb{R}$ is the objective function, defined on the product set of controls and uncertainties;
- $\mathcal{B} \subset \mathcal{F}$ is a sigma-algebra, and the notation $\boldsymbol{U} \preceq \mathcal{B}$ stands for "$\boldsymbol{U}$ is a random variable measurable with respect to the sigma-algebra $\mathcal{B}$", namely

$$\boldsymbol{U} \preceq \mathcal{B} \quad \iff \quad \sigma(\boldsymbol{U}) \subset \mathcal{B} \,, \tag{1.3}$$

  and captures *measurability* or *information* constraints; intuitively, the sigma-algebra $\mathcal{B}$ represents the information available to the decision-maker when choosing the control $\boldsymbol{U}$;
- $\mathcal{U}^{\mathrm{ad}}$ is a subset of $\mathcal{U}$ that represents all remaining *constraints* like *set membership* constraints (say, inequality or equality constraints), *risk constraints*, etc.

We wish to highlight the specificities of *stochastic* optimization w.r.t. *deterministic* optimization. In this perspective, we focus on the information constraints, and we lay out different ways to represent them mathematically. Instead of the "algebraic formulation" (1.3), we can use an almost-sure equality:

$$\boldsymbol{U} - \mathbb{E}\big[\boldsymbol{U} \bigm| \mathcal{B}\big] = 0, \quad \mathbb{P} - a.s. \tag{1.4}$$

When the sigma-algebra $\mathcal{B}$ is generated by a random variable $\boldsymbol{X} : \Omega \to \mathbb{X}$, that is, when $\mathcal{B} = \sigma(\boldsymbol{X})$, and when $\mathbb{U}$ is a separable complete metric space, a result due to J. Doob (see [35, Chapter 1, p. 18]) states that $\boldsymbol{U} \preceq \boldsymbol{X}$ is equivalent to the existence of a measurable function $\pi : \mathbb{X} \to \mathbb{U}$ such that $\boldsymbol{U} = \pi(\boldsymbol{X})$. Thus, we obtain a "functional formulation" of an information constraint:

$$\boldsymbol{U} \preceq \sigma(\boldsymbol{X}) \quad \iff \quad \exists \pi : \mathbb{X} \to \mathbb{U} \text{ measurable, such that } \boldsymbol{U} = \pi(\boldsymbol{X}) \,. \tag{1.5}$$

We distinguish two notions of *solution*, depending on the sigma-algebra $\mathcal{B}$ in (1.2b).

Open-Loop. An *open-loop solution* is $\boldsymbol{U} \preceq \{\emptyset, \Omega\}$, that is, a constant random variable. Then, the random variable is represented by its unique value.

Closed-Loop. By contrast, a *closed-loop solution* may depend on the uncertainty: $\boldsymbol{U} \preceq \mathcal{B}$, where $\{\emptyset, \Omega\} \subsetneq \mathcal{B} \subset \mathcal{F}$.

1.1.2 Multistage Stochastic Optimization Problem

By contrast with static stochastic problems, a multistage stochastic problem introduces *stages* — labeled with integers $t = 0, \dots, T-1$, with *horizon* $T \geq 2$ — and several measurability constraints instead of only one in (1.2b). The general multistage stochastic optimization problem reads

$$\min_{(\boldsymbol{U}_0,\dots,\boldsymbol{U}_{T-1}) \in \mathcal{U}^{\mathrm{ad}} \subset \mathcal{U}} \quad \mathbb{E}\big[J(\boldsymbol{U}_0, \cdots, \boldsymbol{U}_{T-1}, \boldsymbol{W})\big] \tag{1.6a}$$

$$s.t. \quad \boldsymbol{U}_t \preceq \mathcal{B}_t, \qquad \forall t \in [\![0, T-1]\!] \,, \tag{1.6b}$$

where
- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $\mathbb{E}$ is the mathematical expectation;
- $\mathcal{U}$ is the space of all random variables $(\boldsymbol{U}_0, \cdots, \boldsymbol{U}_{T-1}) : \Omega \to \mathbb{U}_0 \times \cdots \times \mathbb{U}_{T-1}$, where all $\mathbb{U}_t$ are measurable spaces;
- $\boldsymbol{W} : \Omega \to \mathbb{W}$ is a random variable that represents *exogenous noise*, where $\mathbb{W}$ is a measurable space;
- $J : \mathbb{U}_0 \times \cdots \times \mathbb{U}_{T-1} \times \mathbb{W} \to \mathbb{R}$ is the objective function;
- $\mathcal{B}_t \subset \mathcal{F}$ is a sigma-algebra, for $t \in [\![0, T-1]\!]$, and the condition $\boldsymbol{U}_t \preceq \mathcal{B}_t$ captures *measurability* or *information* constraints at stage t;

- $\mathcal{U}^{\mathrm{ad}}$ is a subset of $\mathcal{U}$ that represents all remaining *constraints*, including ones that connect different stages.

Now, we wish to highlight the specificities of multistage *stochastic* optimization w.r.t. multistage *deterministic* optimization in a setting where information flows sequentially. We distinguish two notions of *solution*, depending on the sigma-algebras $\mathcal{B}_0, \dots, \mathcal{B}_{T-1}$.

Open-Loop. An *open-loop solution* is $(\boldsymbol{U}_0, \dots, \boldsymbol{U}_{T-1})$ such that $\boldsymbol{U}_t \preceq \{\emptyset, \Omega\}$ for all $t \in [\![0, T-1]\!]$, that is, $\boldsymbol{U}_t$ is a constant random variable.

Closed-Loop. By contrast, a *closed-loop solution* may depend on the uncertainty when $\{\emptyset, \Omega\} \subsetneq \mathcal{B}_t \subset \mathcal{F}$ for at least one $t \in [\![0, T-1]\!]$.

The case of *information accumulation* — also called *perfect memory* — is grasped with the inclusions

$$\mathcal{B}_0 \subset \cdots \subset \mathcal{B}_{T-1} . \tag{1.7}$$

Until now, we did not require that the exogenous noise $\boldsymbol{W}$ be a sequence $\{\boldsymbol{W}_0, \dots, \boldsymbol{W}_{T-1}\}$. But, when $\boldsymbol{W}$ is a random process, we can capture the property of *non-anticipativity* by

$$\forall t \in [\![0, T-1]\!], \qquad \mathcal{B}_t \subset \sigma(\boldsymbol{W}_0, \dots, \boldsymbol{W}_t) . \tag{1.8}$$

The formalism (1.6) covers the case where $\mathcal{B}_t \subset \mathcal{F}$ does not depend on past controls $\boldsymbol{U}_0, \dots, \boldsymbol{U}_{t-1}$ (like $\mathcal{B}_t = \sigma(\boldsymbol{W}_0, \dots, \boldsymbol{W}_t)$), and the case where $\mathcal{B}_t \subset \mathcal{F}$ indeed depends on past controls $\boldsymbol{U}_0, \dots, \boldsymbol{U}_{t-1}$ (like $\mathcal{B}_t = \sigma(\boldsymbol{U}_0, \dots, \boldsymbol{U}_{t-1})$).

The two most important multistage stochastic optimization theories can be distinguished according to how they handle the information constraints (1.6b):

- in the Stochastic Programming framework, the information is generally encoded in a tree, and the sigma-algebra $\mathcal{B}_t$ corresponds to the set of nodes at stage t;
- in the Stochastic Optimal Control framework, the sigma-algebra $\mathcal{B}_t$ is $\sigma(\boldsymbol{X}_t)$ generated by an information state $\boldsymbol{X}_t$, produced by a controlled dynamics.

Both theories incorporate a non-anticipativity property, as well as information accumulation (under the Markovian setup in Stochastic Optimal Control). We now present Stochastic Programming and Stochastic Optimal Control with a focus on the information constraints (1.6b).

Stochastic Programming (SP)

In Stochastic Programming, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *scenario space*, where *scenarios* stand for sequences of *uncertainties*. The sequential structure of information arrival about uncertainty is represented either by a subset of a product space or by a so-called *scenario tree* (see Figure 1.1).

For the sake of simplicity, in this manuscript we only consider Stochastic Programming for finite scenario spaces. For a set of scenario Ω we suppose given

- a probability $\mathbb{P}$ on Ω;
- control sets $\mathbb{U}_0, \dots, \mathbb{U}_{T-1}$;
- an uncertainty set $\mathbb{W}$ and a mapping $\boldsymbol{W} : \Omega \to \mathbb{W}$ that represents *exogenous noises*;
- an objective function $J : \mathbb{U}_0 \times \cdots \times \mathbb{U}_{T-1} \times \mathbb{W} \to \mathbb{R}$.

Stochastic Programming with Scenario Space In Stochastic Programming, the finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ can be represented as a subset of a product space

$$\Omega \subset \Omega_0 \times \cdots \times \Omega_{T-1} , \tag{1.9a}$$

where the set Ω_t supports the uncertainties at step t, so that a scenario is denoted by

$$\omega = (\omega_0, \dots, \omega_{T-1}) = \{\omega_s\}_{s=0}^{T-1} . \tag{1.9b}$$

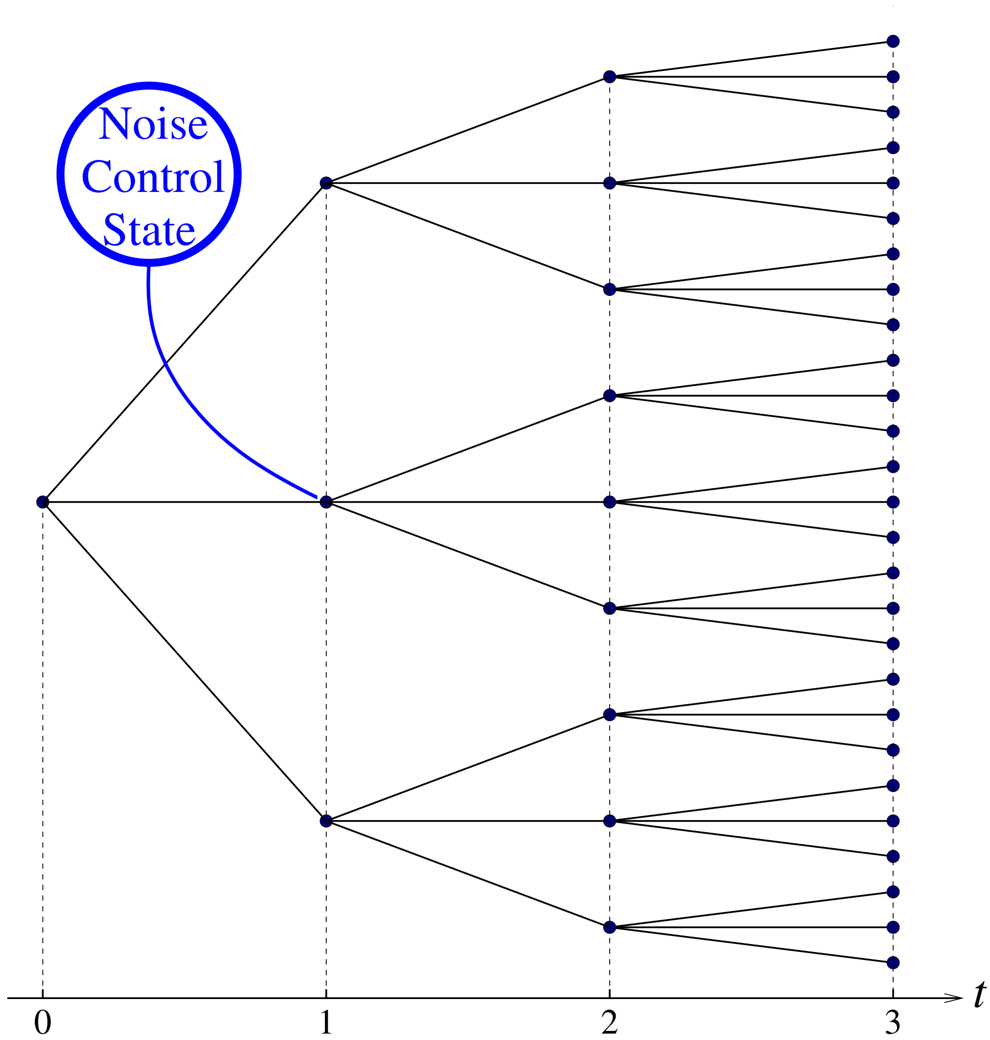


Figure 1.1: A scenario tree

A possible solution is a family of controls $u_t(\omega) \in \mathbb{U}_t$ doubly indexed by step t and uncertainty ω. The non-anticipativity constraint (Constraint (1.6b) where $\mathcal{B}_t = \mathcal{F}_t$) is captured by the requirement that, for all $t \in [\![0, T-1]\!]$,

$$\forall (\omega, \omega') \in \Omega^2, \qquad \{\omega_s\}_{s=0}^t = \{\omega'_s\}_{s=0}^t \quad \Longrightarrow \quad u_t(\omega) = u_t(\omega') \; . \tag{1.10}$$

The general stochastic programming problem reads

$$\begin{aligned} \min_{\{\{u_t(\omega)\}_{\omega\in\Omega}\}_{t=0}^{T-1}} \quad & \sum_{\omega\in\Omega} \mathbb{P}\big(\{\omega\}\big) J\Big(\{u_t(\omega)\}_{t=0}^{T-1}, W(\omega)\Big) \; . \\ s.t. \quad & \text{constraint (1.10)} \end{aligned} \tag{1.11}$$

We develop in Table 1.1 the correspondence between the framing of Stochastic Programming problems with scenario space and the abstract framing of §1.1.2.

Stochastic Programming with Scenario Tree The stochastic programming community often presents problems on a scenario tree. We give a formal definition of a scenario tree (for a finite Ω), and proceed to explain links between the representations.

Definition 1.1. *Consider the sequence $\{\mathcal{N}_t\}_{t=0}^{T-1}$ of partitions of the set Ω, such that $\mathcal{N}_{t+1}$ is a refinement of $\mathcal{N}_t$ (i.e. any element of $\mathcal{N}_{t+1}$ is contained in an element of $\mathcal{N}_t$).*

	Stochastic Programming formulation	Abstract formulation
States of Nature	$\Omega \subset \Omega_0 \times \cdots \times \Omega_{T-1}$ finite set	Ω measurable space
Probability	$\{\mathbb{P}(\{\omega\})\}_{\omega\in\Omega}$	$\mathbb{P}$
Solution	$\{\{u_t(\omega)\}_{\omega\in\Omega}\}_{t=0}^{T-1}$ $\forall\omega \in \Omega, \quad \forall t \in [\![0, T-1]\!],$ $\{\omega_s\}_{s=0}^t = \{\omega'_s\}_{s=0}^t \Rightarrow u_t(\omega) = u_t(\omega')$	$\{\boldsymbol{U}_t\}_{t=0}^{T-1}$ $\forall t \in [\![0, T-1]\!],$ $\boldsymbol{U}_t \preceq \mathcal{B}_t$

Table 1.1: Correspondence between Stochastic Programming with scenario space framing and abstract framing

A scenario forest *is given by* $\mathcal{T} = \big(\{\mathcal{N}_t\}_{t=0}^{T-1}, \mathbb{P}\big)$. *A* scenario tree *is a scenario forest where* $\mathcal{N}_0 = \{\Omega\}$*, and* $\mathcal{N}_{T-1} = \big\{\{\omega\} \mid \omega \in \Omega\big\}$.

Hence, on a scenario tree, a scenario $\omega \in \Omega$ *is associated with a* leaf *of the tree* $\{\omega\} \in \mathcal{N}_{T-1}$*. A* node *of depth* t *of the tree* $\mathcal{T}$*, is an element of* $\mathcal{N}_t$*. A node* n *is said to be an* ascendant *of a node* m *if* $m \subset n$*, we denote by* $a(m)$ *the set of ascendant nodes of* m*. Conversely,* m *is a* descendant *of* n*. For a node* $n \in \mathcal{N}_t$*, we define its set of* children node $r(n)$ *as the nodes* $m \in \mathcal{N}_{t+1}$ *that are descendant of* n*. The* genealogy *of a node is the collection of all its ascendants.*

We also define the functions $n_t : \Omega \to \mathcal{N}_t$ *satisfying* $\omega \in n_t(\omega)$*: its the function mapping the event* ω *with its corresponding node at time* t*.*

Note that, with this construction, from the probability $\mathbb{P}$ on Ω, we have the probability of each nodes $n \in \mathcal{T}$.

From a set of uncertainties $\Omega \subset \Omega_0 \times \cdots \times \Omega_{T-1}$, we can construct a tree in the following way: a node $n_t \in \mathcal{N}_t$ is given by (when non-empty)

$$n_t(\omega) := \big\{\omega' \in \Omega \quad | \quad \forall s \in [\![0, t]\!], \quad \{\omega_s\}_{s=0}^t = \{\omega'_s\}_{s=0}^t\big\} \neq \emptyset ,$$

where $\{\omega_s\}_{s=0}^t$ is a sequence satisfying $\omega_s \in \Omega_s$. Conversely, we easily construct a product set of uncertainties from a tree, and identify the tree with a subset (see Figure 1.4).

A possible solution is a family of controls indexed by the nodes of the tree $\Big\{\big\{u_{n_t}\big\}_{n_t\in\mathcal{N}_t}\Big\}_{t=0}^{T-1}$, where, for any time t, and any node $n_t \in \mathcal{N}_t$, $u_{n_t} \in \mathbb{U}_t$.

In this way, the information constraints (1.10) are automatically captured in the very indexing of a possible solution by the nodes n_t of the tree: at step t, a solution can only depend on past uncertainties $\omega_0, \ldots, \omega_t$.

The general stochastic programming problem reads

$$\min_{\{u_n\}_{n\in\mathcal{T}}} \sum_{\omega\in\Omega} \mathbb{P}\big(\{\omega\}\big) J\Big(\big\{u_n\big\}_{n\in a(\{\omega\})}, W\big(\omega\big)\Big) . \tag{1.12}$$

A usual specific class of problems, additive in time, reads

$$\min_{\big\{\{u_{n_t}\}_{n_t\in\mathcal{N}_t}\big\}_{t=0}} \quad \sum_{t=0}^{T-1} \sum_{n\in\mathcal{N}_t} \sum_{m\in r(n)} \mathbb{P}\big(m\big)\ L_t\Big(X_n, U_m, W_m\Big) \tag{1.13a}$$

$$s.t. \quad X_m = f_t\big(X_n, U_m, W_m\big), \qquad \forall m \in r(n), \quad \forall n \in \mathcal{N}_t, \quad \forall t . \tag{1.13b}$$

In this formulation the variables $\{x_n\}_{n\in\mathcal{T}}$ is called a *physical state*.

We develop in Table 1.2 the correspondence between the framing of Stochastic Programming problems with scenario tree and the abstract framing of §1.1.2.

	Stochastic Programming formulation	Abstract formulation
States of Nature	$\mathcal{T}$ / $\mathcal{N}_{T-1}$ tree (forest) / leaves	Ω measurable space
Information	$\mathcal{N}_t$ nodes at time t	$\mathcal{B}_t$ sigma-algebra
Probability	$\{\mathbb{P}(\{n\})\}_{n\in\mathcal{N}_{T-1}}$	$\mathbb{P}$
Solution	$\big\{\{u_n\}_{n\in\mathcal{N}_t}\big\}_{t=0}^{T-1}$	$\{\boldsymbol{U}_t\}_{t=0}^{T-1}$ $\boldsymbol{U}_t \preceq \mathcal{B}_t$, $\forall t\in[\![0,T-1]\!]$

Table 1.2: Correspondence between Stochastic Programming with scenario tree framing and abstract framing

Stochastic Optimal Control (SOC)

In Stochastic Optimal Control, the information constraints (1.6b) are materialized by means of a so-called *state*. The framing comprises a *Stochastic Dynamic System* (SDS) consisting of

- a sequence $\{\mathbb{X}_t\}_0^T$ of sets of *states*;
- a sequence $\{\mathbb{U}_t\}_0^{T-1}$ of sets of *controls*;
- a sequence $\{\mathbb{W}_t\}_0^{T-1}$ of sets of *uncertainties*,;
- a sequence $\{f_t\}_0^{T-1}$ of *functions*, where $f_t : \mathbb{X}_t \times \mathbb{U}_t \times \mathbb{W}_t \to \mathbb{X}_{t+1}$, play the role of *dynamics* at time t;
- a probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- exogenous noises $\{\boldsymbol{W}_t\}_{t=0}^{T-1}$, where each $\boldsymbol{W}_t$ takes values in $\mathbb{W}_t$;
- an objective function $J : \mathbb{X}_0 \times \cdots \times \mathbb{X}_T \times \mathbb{U}_0 \times \cdots \times \mathbb{U}_{T-1} \times \mathbb{W}_0 \times \cdots \times \mathbb{W}_{T-1} \to \mathbb{R}$.

The sigma-algebras

$$\forall t \in [\![0, T-1]\!], \qquad \mathcal{F}_t = \sigma\big(\boldsymbol{W}_0, \cdots, \boldsymbol{W}_t\big) , \tag{1.14}$$

form the filtration $\mathfrak{F}$ of past noises, and we naturally define $\mathfrak{F}$-adapted processes. For an $\mathfrak{F}$-adapted sequence $\{\boldsymbol{U}_t\}_{t=0}^{T-1}$ of controls — that is, random variables $\boldsymbol{U}_t$ with value in $\mathbb{U}_t$, and such that $\boldsymbol{U}_t \preceq \mathcal{F}_t$ — and an initial state $\overline{x}_0 \in \mathbb{X}_0$, we obtain a sequence $\{\boldsymbol{X}_t\}_{t=0}^{T}$ of states as follows:

$$\forall t \in [\![0, T-1]\!], \qquad \boldsymbol{X}_{t+1} = f_t\big(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t\big) .$$

We observe that, for any time $t \in [\![1, T]\!]$, $\boldsymbol{X}_t$ is measurable w.r.t. $\mathcal{F}_{t-1} \subset \mathcal{F}_t$ by construction.

We denote

$$\boldsymbol{X} = \{\boldsymbol{X}_t\}_{t=0}^{T}, \quad \boldsymbol{U} = \{\boldsymbol{U}_t\}_{t=0}^{T-1}, \quad \boldsymbol{W} = \{\boldsymbol{W}_t\}_{t=0}^{T-1} . \tag{1.15}$$

The general stochastic optimal control problem reads [6]

$$\min_{\boldsymbol{X},\boldsymbol{U}} \quad \mathbb{E}\Big[J(\boldsymbol{X},\boldsymbol{U},\boldsymbol{W})\Big] \tag{1.16a}$$

$$s.t. \quad \boldsymbol{X}_{t+1} = f_t\big(\boldsymbol{X}_t,\boldsymbol{U}_t,\boldsymbol{W}_t\big) \qquad \forall t \in [\![0,T-1]\!] , \tag{1.16b}$$

$$\boldsymbol{U}_t \preceq \mathcal{B}_t \qquad \forall t \in [\![0,T-1]\!] , \tag{1.16c}$$

where $\mathcal{B}_t \subset \mathcal{F}_t$ is a sigma-algebra, for $t \in [\![0,T-1]\!]$, and the conditions $\boldsymbol{U}_t \preceq \mathcal{B}_t$ captures *measurability* or *information* constraints at stage t.

Here again, we wish to highlight the specificities of multistage *stochastic* optimization w.r.t. multistage *deterministic* optimization, in a setting where information flows sequentially. Since $\mathcal{B}_t \subset \mathcal{F}_t$, the condition (1.16c) implies that the control $\boldsymbol{U}_t$ is chosen knowing only the past noises $\boldsymbol{W}_0,\dots,\boldsymbol{W}_t$. This is the so-called *nonanticipativity* constraint: $\boldsymbol{U}_t$ is measurable with respect to $\mathcal{F}_t$.

We distinguish several classes of information structures, depending on $\mathcal{B}_t$ in the condition (1.16c), hence several notions of *solution*.

Open-Loop. An *open-loop solution* is one where the condition (1.16c) reads $\boldsymbol{U}_t \preceq \{\emptyset,\Omega\}$, for all $t \in [\![0,T-1]\!]$. In other words, $\mathcal{B}_t = \{\emptyset,\Omega\}$, for all $t \in [\![0,T-1]\!]$.

Closed-Loop. A solution satisfying the condition (1.16c) is a *closed loop solution* as soon as $\{\emptyset,\Omega\} \subsetneq \mathcal{B}_t \subset \mathcal{F}_t$ for at least one $t \in [\![0,T-1]\!]$. The following subdivisions are helpful in practice.

- In the *Decision-Hazard* setting, $\mathcal{B}_t = \sigma(\boldsymbol{X}_t)$ in (1.16c) so that decisions $\boldsymbol{U}_t \preceq \boldsymbol{X}_t$ are taken before knowing the uncertainty $\boldsymbol{W}_t$ at time t, and only according to the current state $\boldsymbol{X}_t$. By the Doob result (1.5), a solution can be expressed as a *state feedback* $\boldsymbol{U}_t = \pi_t(\boldsymbol{X}_t)$, where $\pi_t : \mathbb{X}_t \to \mathbb{U}_t$.
- In the *Hazard-Decision* setting, $\mathcal{B}_t = \sigma(\boldsymbol{X}_t,\boldsymbol{W}_t)$ in (1.16c) so that decisions $\boldsymbol{U}_t \preceq \sigma(\boldsymbol{X}_t,\boldsymbol{W}_t)$ are taken after knowing the uncertainty at time t, according to the current state $\boldsymbol{X}_t$ and the current uncertainty $\boldsymbol{W}_t$. By the Doob result (1.5), a solution can be expressed as $\boldsymbol{U}_t = \pi_t(\boldsymbol{X}_t,\boldsymbol{W}_t)$, where $\pi_t : \mathbb{X}_t \times \mathbb{W}_t \to \mathbb{U}_t$.
- The largest class of closed loop solutions is of course obtained when $\mathcal{B}_t = \mathcal{F}_t$ for all $t \in [\![0,T-1]\!]$. When the exogenous noises $\{\boldsymbol{W}_t\}_{t=0}^{T-1}$ form a sequence of independent random variables, it can be shown that there is no loss of optimality in reducing the search to the class of Hazard-Decision feedback solutions, namely $\mathcal{B}_t = \sigma(\boldsymbol{X}_t,\boldsymbol{W}_t)$. When the size of the state space $\mathbb{X}_t$ does not increase with t, and neither does $\mathbb{W}_t$, this property has major consequences for numerical applications.
- A smaller class of closed loop solutions is obtained when $\mathcal{B}_t = \mathcal{F}_{t-1}$ for all $t \in [\![0,T-1]\!]$. When the exogenous noises $\{\boldsymbol{W}_t\}_{t=0}^{T-1}$ form a sequence of independent random variables, it can be shown that there is no loss of optimality in reducing the search to the class of state feedback solutions, namely $\mathcal{B}_t = \sigma(\boldsymbol{X}_t)$. When the size of the state space $\mathbb{X}_t$ does not increase with t, this property has major consequences for numerical applications.

This general form (1.16) is not common, and one generally rather considers a time additive expression for the cost function, namely,

$$\min_{\pi=\{\pi_t\}_{t=0}^{T-1}} \quad \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t\big(\boldsymbol{X}_t,\boldsymbol{U}_t,\boldsymbol{W}_t\big) + K\big(\boldsymbol{X}_T\big)\Big] \tag{1.17a}$$

$$s.t. \quad \boldsymbol{X}_{t+1} = f_t\big(\boldsymbol{X}_t,\boldsymbol{U}_t,\boldsymbol{W}_t\big), \qquad \forall t \in [\![0,T-1]\!] , \tag{1.17b}$$

$$\boldsymbol{U}_t = \pi_t(\boldsymbol{X}_t), \quad \pi_t : \mathbb{X}_t \to \mathbb{U}_t, \qquad \forall t \in [\![0,T-1]\!] , \tag{1.17c}$$

6. In Chapter 2, we consider other aggregators in time and uncertainties.

where

- $L_t : \mathbb{X}_t \times \mathbb{U}_t \times \mathbb{W}_t \mapsto \mathbb{R}$ is the *instantaneous cost* at step t, for all $t \in [\![0, T-1]\!]$, and $K : \mathbb{X}_T \to \mathbb{R}$ is the final cost;
- the policies π_t are measurable mappings, for all $t \in [\![0, T-1]\!]$, and capture *information constraints*;

Remark 1.2. *We discuss the notion of state in §1.2.4. The quantity $\boldsymbol{X}_t$ is an* information state *when the condition $\boldsymbol{U}_t \preceq \mathcal{B}_t$ in (1.16c) can be replaced by the condition $\boldsymbol{U}_t \preceq \boldsymbol{X}_t$, where $\boldsymbol{X}_t$ is $\mathcal{F}_t$-measurable.*

In Problem (1.17), the condition (1.17c) suggests that the state $\boldsymbol{X}_t$ is an information state *as the decision are taken in function of it; we say "suggests" because this relies on the implicit assumption that there is no loss of optimality in reducing the search to the class of state feedback solutions, instead of the largest class of adapted controls. In Problem (1.16), $\boldsymbol{X}_t$ is simply the physical state (and might or might not be an information state, depending on additional assumptions).*

As just discussed, the form (1.17) is especially adapted to the case where the exogenous noises $\{\boldsymbol{W}_t\}_{t=0}^{T-1}$ form a sequence of independent random variables. We will come back to that point when we address Dynamic Programming in §1.2.4.

Connection between SP and SOC

The SOC framing includes the SP one, at the expense of introducing a state like in Table 1.3.

	Stochastic Programming formulation	Stochastic Optimal Control formulation
States of Nature	$\Omega \subset \Omega_0 \times \cdots \times \Omega_{T-1}$ finite set	Ω measurable space
Exogenous noise		$\mathbb{W}_t = \Omega_t$, $\boldsymbol{W}_t : \Omega \to \Omega_t$ projection
Probability	$\{\mathbb{P}(\{\omega\})\}_{\omega \in \Omega}$	$\mathbb{P}$
State		$\boldsymbol{X}_t = (\overline{x}_0, \boldsymbol{W}, \boldsymbol{U}_0, \ldots, \boldsymbol{U}_{t-1})$
Information		$\mathcal{F}_t = \sigma(\boldsymbol{W}_0, \cdots, \boldsymbol{W}_t)$
Dynamics		$f_t(x_t, u_t, w_t) = (x_t, u_t)$

Table 1.3: Turning a Stochastic Programming framing into Stochastic Optimal Control framing

Observe that the state $\boldsymbol{X}_t$ at stage t is huge, as it includes all the exogenous noises $\boldsymbol{W}$ and the past controls $\boldsymbol{U}_0, \ldots, \boldsymbol{U}_{t-1}$. Observe also the not common fact that the state $\boldsymbol{X}_t$ at stage t includes *all* the noises $\boldsymbol{W} = \{\boldsymbol{W}_0, \ldots, \boldsymbol{W}_{T-1}\}$, be they past, present or future! As a consequence, the state $\boldsymbol{X}_t$ is *not* $\mathcal{F}_t$-measurable, hence is not observable by the decision-maker at stage t and cannot be the input of any implementable feedback. What is more, the dimension of the state grows with the stages, as reflected in the dynamics that just extends the vector x_t by adding u_t to the right: the state $\boldsymbol{X}_t$ at stage t keeps track of past controls $\boldsymbol{U}_0, \ldots, \boldsymbol{U}_{t-1}$ by accumulating them. This state is called the "maximal state", and it will again be discussed in §1.2.4. This is not an information state as it is not totally observable (see Remark 1.2), whereas we will see that the conditional distribution of the maximal state $\boldsymbol{X}_t$ knowing $\mathcal{F}_t$ is. In practice, depending on the specificities of the model, it may happen that smaller states can be displayed.

1.1.3 Discussion of complexity

We point out in what sense multistage stochastic optimization problems are complex, and then quickly review different approaches to address their numerical resolution.

More precise and involved discussion on the complexity of multistage stochastic optimization problems can be found in [111, 112]. In particular the question of approximating the underlying probability is discussed.

Multistage Stochastic Problems are Complex

To give a feeling of the complexity of a multistage stochastic optimization problem, we assume that controls take their values in a finite set of cardinal n_u. Therefore, there are

$$(n_u)^{T|\Omega|}$$

possible solutions (not all of them are admissible).

To account for non-anticipativity and restrict solutions, we suppose that the sample space Ω is a product of T copies with cardinal n_w, so that $|\Omega| = (n_w)^T$. Hence, the number of possible solutions is

$$(n_u)^{T(n_w)^T} ,$$

and the number of non-anticipative ones is

$$(n_u)^{\sum_{s=0}^{T-1}(n_w)^s} = (n_u)^{\frac{(n_w)^T-1}{n_w-1}} . \tag{1.18}$$

This number is also the number of possible solutions when the set Ω is represented by the leaves of a tree of depth T, each node having n_w children, because then the number of nodes is $\sum_{s=0}^{T-1} n_w^s = \frac{(n_w)^T-1}{n_w-1}$.

Discussing Resolution Methods to Address Complex Optimization Problems

Most "real life" optimization problems are too complex to be numerically solved directly. We briefly list some of the many ways found in the academic literature to tackle complex optimization problems, pointing to well-known references, without aiming at exhaustivity.

Heuristic. We can look for heuristic solution, either by looking for the solutions in a more limited class of solutions (approximate dynamic programming – see [19, 83]– and machine learning –see [54]– are classical approaches), or by cunningly trying to find a good solution through method like simulated annealing (see [60]), or genetic algorithms (see [51]).

Specific problems. We can also make some approximation of the problem itself, and make the most of some mathematical properties of the (approximated problem). For example, one finds very efficient algorithms for linear programming problems (see [33]), quadratic programming, semi-definite programming, conic programming, (see [4, 14, 119]) large classes of mixed integer linear programming (see [68]), etc.

Decomposition. Decomposition approaches (see [12, 30, 98]) consist in partitioning the original optimization problem into several *subproblems* usually coordinated by a *master problem*. We then solve each subproblem independently, and send the relevant part of the solutions to the master problem. The master problem then adjusts the subproblems, that are to be solved again, and so on. The numerical gain is contained in the fact that, if the original problem is of size S, solving N problems of size S/N, even with iterations, might be much faster than solving the original problem.

1.2 Resolution by Decomposition Methods in Multistage Stochastic Optimization

We present, in an unified framework, the main approaches to decompose multistage stochastic optimization problems for numerical resolution.

To fix ideas and simplify the exposition, we present a setting where all variables are parametrized by discrete indexes. For this purpose, suppose given a finite horizon T (so that time $t \in [\![0, T]\!]$), a finite probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, endowed with a filtration $\mathfrak{F} = \{\mathcal{F}_t\}_0^{T-1}$, a finite number N of units (space). We consider the multistage stochastic optimization problem

$$\min_{\boldsymbol{X}, \boldsymbol{U}} \quad \sum_{\omega \in \Omega} \sum_{i=1}^{N} \sum_{t=0}^{T-1} \quad \mathbb{P}\big(\{\omega\}\big)\ L_t^i\Big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\Big) \tag{1.19a}$$

$$s.t. \quad \boldsymbol{X}_{t+1}^i(\omega) = f_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\big) \qquad \forall t, \quad \forall i, \quad \forall \omega \tag{1.19b}$$

$$\sum_{i=1}^{N} \theta_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega)\big) = 0 \qquad \forall t, \quad \forall \omega \tag{1.19c}$$

$$\boldsymbol{U}_t^i \preceq \mathcal{F}_t \qquad \forall t, \quad \forall i\ , \tag{1.19d}$$

where ω is a scenario of uncertainties given by $\omega = \big\{\omega_t\big\}_{t=0}^{T-1}$. The constraint (1.19b) represents the dynamics of each subsystem, the constraint (1.19c) represents the coupling constraint between the subsystems (also called units), and the constraint (1.19d) is the non-anticipativity constraint. Constraints function θ_t^i are assumed to have image in $\mathbb{R}^{n_c}$.

As we have seen in §1.1.2 that the SOC framing includes the SP one, the above setting applies both to SP and SOC problems.

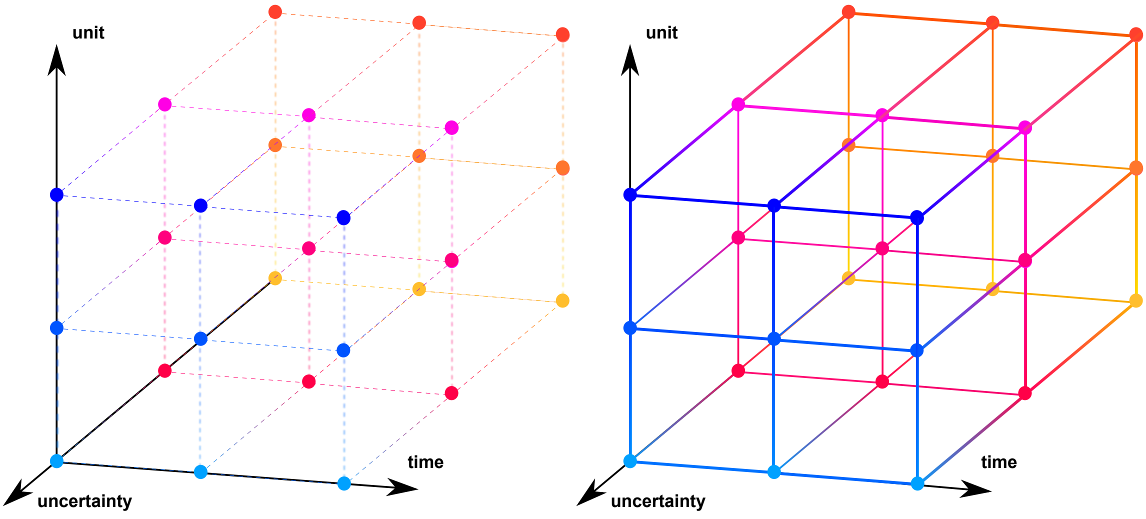
In Problem (1.19), we have local costs — depending on step t, uncertainty ω and unit i — and we minimize their sum over time, uncertainty and space. Without constraints (1.19b)-(1.19d), Problem (1.19) (illustrated in Figure 1.2a) consists in minimizing a sum of independent costs. Hence, the minimum of the sum is the sum of the minimums, and the problem is decomposed. However, the local costs are linked (illustrated in Figure 1.2b)

- in time through the dynamic of the system (e.g. Equation (1.19b));
- in unit through the coupling constraints (e.g. Equation (1.19c));
- and in scenario (uncertainty) through the nonanticipativity constraint (e.g. Equation (1.19d)).

We now lay out different ways to divide the original complex problem into easier to solve subproblems. We propose three angles to decompose the original problem: decomposition in time (step), decomposition in scenario (uncertainty) and decomposition in space (unit), as illustrated in Figure 1.3.

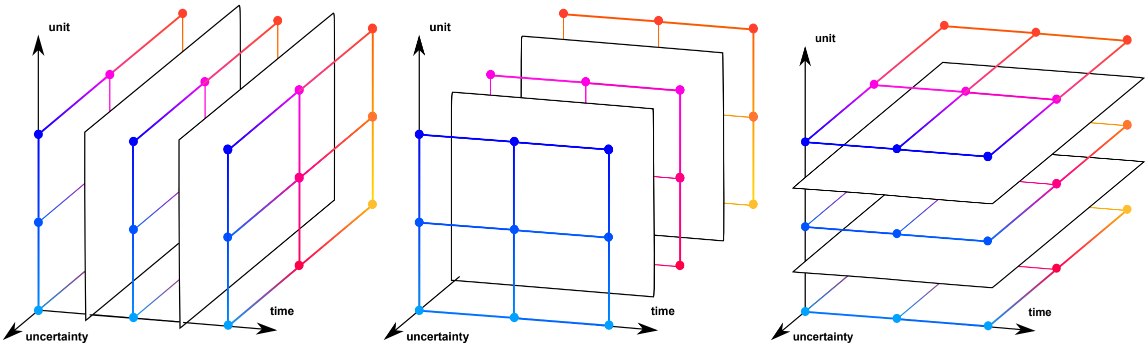
Moreover, we distinguish two types of decomposition.

- In *chained decomposition*, like Dynamic Programming (see [12, 17]), the original problem is solved by means of successive smaller subproblems, solved one after the other (in Dynamic Programming, each subproblem is solved only once). Chained decomposition relies on a specific structure of the coupling constraint, like the flow of time.
- In *parallel decomposition*, like Progressive Hedging (see [98, 115]), the original problem is solved by means of parallel smaller subproblems, coordinated and updated by a master algorithm. These subproblems can be obtained by dualizing the constraint, and have to be solved several times before obtaining an optimal solution to the global problem.



(a) Local costs (b) Local costs linked

Figure 1.2: Representation of the local costs depending on time, uncertainty (scenario) and space (unit) and the links induced by the constraints



(a) Time decomposition (b) Uncertainty decomposition (c) Space decomposition

Figure 1.3: Decomposition according to time, uncertainty (scenario) or space (unit). Each plane carries a problem with coupling in only two dimensions.

1.2.1 Duality and Parallel Decomposition

Before presenting the different decompositions approaches, we now illustrate how the duality theory (recalled in Chapter 4) leads to decomposition schemes. We present here, in a simple setting, the most usual, known as *price decomposition scheme*. For clarity, the units coupling functions θ^i in (1.19c) are assumed, here, to be real valued.

This price decomposition scheme can be intuitively understood as follows. We consider a problem where a team of N units — each of them producing a quantity $\theta_i(u_i)$ function of the local control u_i — has to meet a given demand. Each unit incurs a local cost $L_i(u_i)$, and the problem consists in minimizing the sum of the local costs. The decomposition is obtained by replacing the "production equal demand" equality by a price mechanism. To achieve a proper price, we suppose that a coordinator can impose costs to all units iteratively. At iteration k, the coordinator sets a price $p^{(k)} = -\lambda^{(k)}$ for the output of each unit $\theta_i(u_i)$. Each unit then minimizes the sum of its local production cost $L_i(u_i)$ minus the cash flow produced by the output $p^{(k)}\theta_i(u_i)$, and obtains a solution $u_i^{(k)}$. Then, the coordinator collects the production of all units, makes the sum and compares the result to the demand. If the total production is not enough, he increases the price of the output; if the total production exceeds the demand, he decreases the price.

More precisely, we consider the following problem:

$$\begin{aligned}
\min_{\{u_i\}_{i=1}^N} \quad & \sum_{i=1}^N L_i(u_i) && && (1.20a)\\
s.t. \quad & u_i \in U_i^{\mathrm{ad}}, && \forall i \in [\![1, N]\!] \ , && (1.20b)\\
& \sum_{i=1}^N \theta_i(u_i) = 0 \ , && && (1.20c)
\end{aligned}$$

where the index i can represent unit, time, uncertainties or a mix. Under mild technical conditions, this problem is equivalent to

$$\begin{aligned}
\min_{\{u_i\}_{i=1}^N} \quad \max_{\lambda\in\mathbb{R}} \quad & \sum_{i=1}^N L_i(u_i) + \lambda\Big(\sum_{i=1}^N \theta_i(u_i)\Big) && && (1.21a)\\
s.t. \quad & u_i \in U_i^{\mathrm{ad}}, && \forall i \in [\![1, N]\!] \ . && (1.21b)
\end{aligned}$$

Under a proper *constraint qualification condition*, we can exchange the min operator with the max operator and obtain

$$\begin{aligned}
\max_{\lambda\in\mathbb{R}} \quad \min_{\{u_i\}_{i=1}^N} \quad & \sum_{i=1}^N L_i(u_i) + \lambda\theta_i(u_i) && && (1.22a)\\
s.t. \quad & u_i \in U_i^{\mathrm{ad}}, && \forall i \in [\![1, N]\!] \ . && (1.22b)
\end{aligned}$$

Now, consider the inner minimization problem: the objective function is given as a sum of local costs, each of them determined by local independent controls. Thus, the minimum of the sum is the sum of the minima, and Problem (1.22) can be written as

$$\begin{aligned}
\max_{\lambda\in\mathbb{R}} \quad \sum_{i=1}^N \quad \min_{u_i} \quad & L_i(u_i) + \lambda\theta_i(u_i) && (1.23a)\\
s.t. \quad & u_i \in U_i^{\mathrm{ad}} \ . && (1.23b)
\end{aligned}$$

For a given $\lambda = \lambda^{(k)}$, we now obtain N separate minimization problems, that are the subproblems of the decomposition method:

$$\begin{aligned}
\min_{u_i} \quad & L_i(u_i) + \lambda^{(k)}\theta_i(u_i) && (1.24a)\\
s.t. \quad & u_i \in U_i^{\mathrm{ad}} \ . && (1.24b)
\end{aligned}$$

These subproblems are updated as the multiplier $\lambda^{(k)}$ (or equivalently the price) is updated, like with

$$\lambda^{(k+1)} = \lambda^{(k)} + \rho \sum_{i=1}^N \theta_i(u_i^{(k)}) \ , \tag{1.25}$$

where $\rho > 0$ is a given parameter, and $u_i^{(k)}$ an optimal solution of Problem (1.24). This update formula for the multiplier is part of the equations of the Uzawa algorithm, recalled and extended in Chapter 6.

Remark 1.3. *This price decomposition scheme is the simplest and most well-known of decomposition schemes, but not the only one. In short, the* decomposition by quantity *approach consists in allocating to each subproblem a given quantity of the demand to satisfy, and then update the allocation; the* decomposition by prediction *approach consists in allocating to each subproblem a part of the constraint.*

Notice that, even if the property of having a sum of costs over units seems to be fundamental for decomposition, the Auxiliary Problem Principle (see [30]) allows to extends these decomposition schemes to general (non-additive) costs and constraint functions.

The second part of the manuscript (Chapters 4, 5 and 6) is dedicated to the duality theory in stochastic optimization as a tool for parallel decomposition.

1.2.2 Spatial Decomposition

The spatial decomposition (by prices) relies on the idea of dualizing the coupling constraint (1.19c). It will be developed in §6.3 and in Chapter 8.

We now apply to Problem (1.19) a price decomposition scheme, presented in §1.2.1, by dualizing the spatial constraint (1.19c). Since there are $T\times|\Omega|$ constraints of dimension n_c, the set of multipliers is of dimension $T\times|\Omega|\times n_c$. Problem (1.19), with constraint (1.19c) dualized, reads

$$
\begin{aligned}
\min_{\boldsymbol{X},\boldsymbol{U}} \quad \max_{\boldsymbol{\lambda}} \quad & \sum_{\omega\in\Omega}\mathbb{P}\big(\{\omega\}\big)\sum_{t=0}^{T-1}\bigg(\sum_{i=1}^{N} L_t^i\Big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega),\boldsymbol{W}_t(\omega)\Big) \\
& \qquad\qquad + \boldsymbol{\lambda}_t(\omega)\cdot\sum_{i=1}^{N}\theta_t^i\big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega)\big)\bigg) \\
s.t. \quad & \boldsymbol{X}_{t+1}^i(\omega) = f_t^i\big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega),\boldsymbol{W}_t(\omega)\big) \qquad \forall t,\quad \forall i,\quad \forall\omega\ , \\
& \boldsymbol{U}_t^i \preceq \mathcal{F}_t, \qquad \forall t,\quad \forall i\ .
\end{aligned}
$$

Assuming constraint qualification, this problem is equivalent to

$$
\begin{aligned}
\max_{\boldsymbol{\lambda}} \quad \sum_{i=1}^{N} \quad \min_{\boldsymbol{X}^i,\boldsymbol{U}^i} \quad & \sum_{\omega\in\Omega}\mathbb{P}\big(\{\omega\}\big)\sum_{t=0}^{T-1} \quad L_t^i\Big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega),\boldsymbol{W}_t(\omega)\Big) \\
& \qquad\qquad + \boldsymbol{\lambda}_t(\omega)\cdot\theta_t^i\big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega)\big) \\
s.t. \quad & \boldsymbol{X}_{t+1}^i(\omega) = f_t^i\big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega),\boldsymbol{W}_t(\omega)\big) \qquad \forall t,\quad \forall i,\quad \forall\omega\ , \\
& \boldsymbol{U}_t^i \preceq \mathcal{F}_t, \qquad \forall t,\quad \forall i\ .
\end{aligned}
$$

For a given multiplier $\boldsymbol{\lambda}^{(k)}$, we obtain N parallel inner minimization problems

$$
\begin{aligned}
\min_{\boldsymbol{X}^i,\boldsymbol{U}^i} \quad & \sum_{\omega\in\Omega}\mathbb{P}\big(\{\omega\}\big)\sum_{t=0}^{T-1} \quad L_t^i\Big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega),\boldsymbol{W}_t(\omega)\Big) \\
& \qquad\qquad + \boldsymbol{\lambda}_t^{(k)}(\omega)\cdot\theta_t^i\big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega)\big) \\
s.t. \quad & \boldsymbol{X}_{t+1}^i(\omega) = f_t^i\big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega),\boldsymbol{W}_t(\omega)\big) \qquad \forall t,\quad \forall\omega \\
& \boldsymbol{U}_t^i \preceq \mathcal{F}_t, \qquad \forall t\ .
\end{aligned}
$$

We denote $\boldsymbol{U}_t^{i,(k)}$ and $\boldsymbol{X}_t^{i,(k)}$ an optimal solution. We update the multipliers with

$$
\forall t\in[\![0,T-1]\!],\quad \forall\omega\in\Omega,\quad \boldsymbol{\lambda}_t^{(k+1)}(\omega) = \boldsymbol{\lambda}_t^{(k)}(\omega)+\rho\bigg(\sum_{i=1}^{N}\theta_t^i\Big(\boldsymbol{X}_t^{(k)}(\omega),\boldsymbol{U}_t^{(k)}(\omega)\Big)\bigg)\ , \tag{1.26}
$$

where $\rho>0$ is a given parameter.

Remark 1.4. *As discussed in §1.2.1, this price decomposition has an insightful interpretation. The multiplier $\boldsymbol{\lambda}_t(\omega)$ can be interpreted as the marginal cost of the output at time t along scenario ω. It is worth noting that the prices form a stochastic process $\{\boldsymbol{\lambda}_t\}_{t=0}^{T-1}$, that can be represented as an element of the huge space $\mathbb{R}^{(Tn_c)^{|\Omega|}}$. We show in Remark 6.12 how we can only consider non-anticipative processes. The method presented in Chapter 8 consists precisely in restricting the space of multipliers $\boldsymbol{\lambda}$ over which the maximization is done.*

1.2.3 Scenario Decomposition

The decomposition scenario by scenario consists in dualizing the non-anticipativity constraint, and then solving subproblems for each scenario (using any of the tools available for deterministic problems). The Progressive Hedging (PH) Algorithm stands as the state of the art in this domain, but we also present the Stochastic Pontryaguin approach.

Progressive Hedging (PH)

We consider Problem (1.19) written on a tree $\mathcal{T}$. We then have

$$\min_{\big\{\{u_{n_t}\}_{n_t \in \mathcal{N}_t}\big\}_{t=0}} \quad \sum_{n\in\mathcal{T}} \sum_{m\in r(n)} \sum_{i=1}^{N} \mathbb{P}\big(\{m\}\big)\ L_t^i\Big(\boldsymbol{X}_n^i, \boldsymbol{U}_m^i, \boldsymbol{W}_m\Big) \tag{1.27a}$$

$$s.t. \quad \boldsymbol{X}_m^i = f_t^i\big(\boldsymbol{X}_n^i, \boldsymbol{U}_m^i, \boldsymbol{W}_m\big), \qquad \forall i, \quad \forall m \in r(n), \quad \forall n \in \mathcal{N}_t, \quad \forall t, \tag{1.27b}$$

$$\sum_{i=1}^{N} \theta_t^i\big(\boldsymbol{X}_n^i, \boldsymbol{U}_m^i\big) = 0, \qquad \forall i, \quad \forall m \in r(n), \quad \forall n \in \mathcal{N}_t, \quad \forall t\ . \tag{1.27c}$$

Note that we have one decision u_n per node on the tree; this materializes the information constraint (1.19d), the one that is dualized in the Progressive Hedging algorithm. For this purpose, we introduce new control variables (see Figure 1.4), that is, a sequence $\{u_t\}_{t=0}^{T-1}$ of controls for each scenario ω (associated to a leaf of the tree), as in Problem (1.19). It means that, with a given node $n \in \mathcal{T}$, are associated $|n|$ control variables, that is, one per scenario going through this node. The non-anticipativity constraint (1.19d) is represented by

$$\forall i \in [\![1, n]\!], \quad \forall t \in [\![0, T-1]\!], \quad \forall (\omega, \omega') \in n^2, \qquad U_t^i(\omega) = U_t^i(\omega')\ . \tag{1.28}$$

We introduce $\bar{U}_n$ the mean control on node $n \in \mathcal{N}_t$, defined by

$$\bar{U}_n^i = \frac{\sum_{\omega\in n} U_t^i(\omega)}{|n|}\ . \tag{1.29}$$

We denote by $n_t(\omega)$ the node of depth t in which ω is contained. Hence, Equation (1.28) can be rewritten as

$$\forall t \in [\![0, T-1]\!], \quad \forall \omega \in \Omega, \qquad U_t^i(\omega) = \bar{U}_{n_t(\omega)}^i\ , \tag{1.30}$$

and Problem (1.19) now reads

$$\min_{\boldsymbol{X},\boldsymbol{U}} \quad \sum_{\omega\in\Omega} \mathbb{P}\big(\{\omega\}\big) \sum_{i=1}^{N} \sum_{t=0}^{T-1} L_t^i\Big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\Big) \tag{1.31a}$$

$$s.t. \quad \boldsymbol{X}_{t+1}^i(\omega) = f_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\big), \qquad \forall t, \quad \forall i, \quad \forall \omega \tag{1.31b}$$

$$\sum_{i=1}^{N} \theta_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega)\big) = 0, \qquad \forall t, \quad \forall \omega \tag{1.31c}$$

$$U_t^i(\omega) = \bar{U}_{n_t(\omega)}^i \qquad \forall t, \quad \forall i, \quad \forall \omega\ . \tag{1.31d}$$

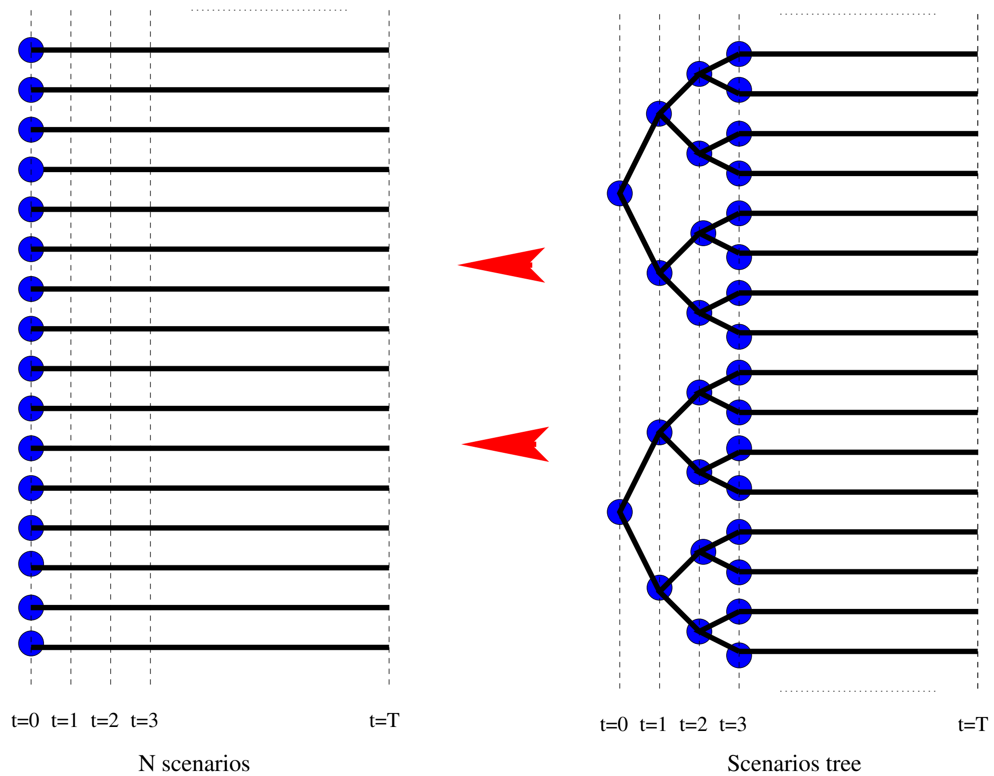


Figure 1.4: From scenario tree to set of scenarios

We dualize Constraint (1.31d), and, under constraint qualification, obtain

$$
\begin{aligned}
\max_{\boldsymbol{\lambda}} \quad \min_{\boldsymbol{X},\boldsymbol{U}} \quad & \sum_{\omega\in\Omega} \mathbb{P}\big(\{\omega\}\big) \sum_{i=1}^{N} \sum_{t=0}^{T-1} L_t^i\Big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\Big) \\
& \qquad + \lambda_t^i(\omega)\big(\boldsymbol{U}_t^i(\omega) - \bar{U}^i_{n_t(\omega)}\big) \\
s.t. \quad & \boldsymbol{X}_{t+1}^i(\omega) = f_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\big), \qquad \forall t, \quad \forall i, \quad \forall \omega \\
& \sum_{i=1}^{N} \theta_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega)\big) = 0, \qquad \forall t, \quad \forall \omega \ ,
\end{aligned}
$$

where Λ is of dimension $|\Omega| \times N \times n_u \times T$. We now fix, for each node $n \in \mathcal{T}$, a mean control $\bar{U}_n^{(k)}$. For each scenario $\omega \in \Omega$, and each stage $t \in [\![0, T-1]\!]$, we fix a multiplier $\lambda_t^{(k)}(\omega)$. The inner minimization of the above problem, for the given multipliers and mean controls, can be done ω per ω, and reads

$$
\min_{\boldsymbol{X}(\omega),\boldsymbol{U}(\omega)} \quad \sum_{i=1}^{N} \sum_{t=0}^{T-1} L_t^i\Big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\Big) \tag{1.32a}
$$

$$
+ \lambda_t^{i,(k)}(\omega)\big(U_t^i(\omega) - \bar{U}^{i,(k)}_{n_t(\omega)}\big) \tag{1.32b}
$$

$$
s.t. \quad \boldsymbol{X}_{t+1}^i(\omega) = f_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\big), \qquad \forall t, \quad \forall i, \quad \forall \omega \tag{1.32c}
$$

$$
\sum_{i=1}^{N} \theta_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega)\big) = 0, \qquad \forall t, \quad \forall \omega \ . \tag{1.32d}
$$

Remark 1.5. *It is interesting to note that the non-anticipativity constraint, written in the form of Equation* (1.30)*, is equivalent to*

$$
\forall i \in [\![1, N]\!], \quad \forall t \in [\![0, T-1]\!], \qquad \boldsymbol{U}_t^i - \mathbb{E}\big[\boldsymbol{U}_t^i \bigm| \mathcal{F}_t\big] = 0 \ . \tag{1.33}
$$

The Progressive Hedging algorithm, schematically presented in Algorithm 1.1, is in fact more elaborated, as it uses an augmented Lagrangian instead of a simple Lagrangian, hence adding a quadratic term in the cost of the subproblems (1.32). We refer the reader to [21, 98] for more details.

Data: Initial multipliers $\big\{\{\lambda_t^{(0)}(\omega)\}_{t=0}^{T-1}\big\}_{\omega\in\Omega}$ and mean control $\big\{\bar{U}_n^{(0)}\big\}_{n\in\mathcal{T}}$;
Result: optimal feedback;
repeat
  forall the *scenario* $\omega \in \Omega$ **do**
    Solves the deterministic minimization problem (1.32) for scenario ω with a measurability penalization, and obtain optimal control $\boldsymbol{U}^{(k+1)}$;
  Update the mean controls

$$\forall t \in [\![0, T-1]\!], \quad \forall n \in \mathcal{N}_t, \qquad \bar{u}_n^{(k+1)} = \frac{\sum_{\omega\in n} \boldsymbol{U}_t^{(k+1)}(\omega)}{|n|} \ ;$$

  Update the measurability penalization with

$$\forall \omega \in \Omega, \quad \forall t \in [\![0, T-1]\!] \qquad \lambda_t^{(k+1)}(\omega) = \lambda_t^{(k)}(\omega) + \rho\big(U_t(\omega)^{(k+1)} - \bar{u}_{n_t(\omega)}^{(k+1)}\big);$$

until $\boldsymbol{U}_t^i - \mathbb{E}\big[\boldsymbol{U}_t^i \bigm| \mathcal{F}_t\big] = 0$;

Algorithm 1.1: General Scheme of Progressive Hedging

Stochastic Pontryaguin

We present an extension to the stochastic framework of Pontryaguin method. More details and numerical experiments can be found in [32].

Ignoring the "spatial" coupling constraint (1.19c), and dualizing [7] the dynamics constraints (1.19b), Problem (1.19) reads

$$\min_{\{\boldsymbol{U}_t \preceq \mathcal{F}_t\}_{t=0}^{T-1}} \left\{ \min_{\boldsymbol{X}} \max_{\boldsymbol{\lambda}\in\Lambda} \sum_{\omega\in\Omega} \sum_{i=1}^{N} \sum_{t=0}^{T-1} \mathbb{P}\big(\{\omega\}\big) \ L_t^i\Big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\Big) \right.$$
$$\left. + \boldsymbol{\lambda}_{t+1}^i(\omega)\Big(f_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\big) - \boldsymbol{X}_{t+1}^i(\omega)\Big) \right\} . \tag{1.34}$$

For a given control process $\boldsymbol{U}^{(k)}$, we consider the inner min-max problem,

$$\begin{aligned} \min_{\boldsymbol{X}} \max_{\boldsymbol{\lambda}} \sum_{\omega\in\Omega} \mathbb{P}\big(\{\omega\}\big) \sum_{i=1}^{N} \sum_{t=0}^{T-1} \quad & L_t^i\Big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^{i,(k)}(\omega), \boldsymbol{W}_t(\omega)\Big) \\ & + \boldsymbol{\lambda}_{t+1}^i(\omega)\Big(f_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^{i,(k)}(\omega), \boldsymbol{W}_t(\omega)\big) - \boldsymbol{X}_{t+1}^i(\omega)\Big) . \end{aligned} \tag{1.35}$$

This problem can be solved ω per ω

$$\begin{aligned} \min_{\boldsymbol{X}(\omega)} \max_{\boldsymbol{\lambda}(\omega)} \sum_{i=1}^{N} \sum_{t=0}^{T-1} \quad & L_t^i\Big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^{i,(k)}(\omega), \boldsymbol{W}_t(\omega)\Big) \\ & + \boldsymbol{\lambda}_{t+1}^i(\omega)\Big(f_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^{i,(k)}(\omega), \boldsymbol{W}_t(\omega)\big) - \boldsymbol{X}_{t+1}^i(\omega)\Big) . \end{aligned} \tag{1.36}$$

7. To be more specific multiplier $\boldsymbol{\lambda}_t^i$ corresponds to the constraint $\boldsymbol{X}_t^i - f_{t-1}^i\big(\boldsymbol{X}_{t-1}^i, \boldsymbol{U}_{t-1}^i, \boldsymbol{W}_{t-1}\big)$. However, we want to have local cost depending on state and control of time t, hence the appearance of multipliers $\boldsymbol{\lambda}_t^i$ and $\boldsymbol{\lambda}_{t+1}^i$ in Problem (1.34).

Assuming that we have the necessary regularity conditions (and since we assumed no bound constraints on $\boldsymbol{U}$ and $\boldsymbol{X}$), we write the first order optimality conditions of this inner min-max problem and deduce the optimal solutions $\boldsymbol{X}^{(k)}$ and $\boldsymbol{\lambda}^{(k)}$ by

$$\boldsymbol{X}_0^{(k)} = x_0 \,, \tag{1.37a}$$

$$\boldsymbol{X}_{t+1}^{(k)} = f_t\big(\boldsymbol{X}_t^{(k)}, \boldsymbol{U}_t^{(k)}, \boldsymbol{W}_t\big) \qquad t \in [\![0, T-1]\!] \,, \tag{1.37b}$$

$$\boldsymbol{\lambda}_T = 0 \,, \tag{1.37c}$$

$$\boldsymbol{\lambda}_t = \nabla_x f_t\big(\boldsymbol{X}_t^{(k)}, \boldsymbol{U}_t^{(k)}, \boldsymbol{W}_t\big)\boldsymbol{\lambda}_{t+1}^{(k)} + \nabla_x L_t\big(\boldsymbol{X}_t^{(k)}, \boldsymbol{U}_t^{(k)}, \boldsymbol{W}_t\big) \qquad t \in [\![1, T-1]\!] \,. \tag{1.37d}$$

These conditions involve a co-state stochastic process $\boldsymbol{\lambda}$ which is *not* $\mathcal{F}$-adapted since the dynamics (1.37c)–(1.37d) propagate backwards and therefore $\boldsymbol{\lambda}_t$ is not $\mathcal{F}_t$-measurable in general.

Given a control trajectory $(\boldsymbol{U}_0^{(k)}, \dots, \boldsymbol{U}_{T-1}^{(k)})$, we can solve these equations by, first integrating Equations (1.37a)-(1.37b) forward to obtain $\{\boldsymbol{X}_t^{(k)}\}_{t=0}^T$, and then integrating Equations (1.37c)-(1.37d) backward to obtain the multiplier process $\{\boldsymbol{\lambda}_t^{(k)}\}_{t=1}^T$. Note that these integrations are performed scenario per scenario, hence in parallel.

Denote by H the function mini-maximized in Problem (1.34), i.e.

$$\begin{aligned} H\big(\boldsymbol{X}, \boldsymbol{U}, \boldsymbol{\lambda}\big) = \sum_{\omega \in \Omega} \mathbb{P}\big(\{\omega\}\big) \sum_{i=1}^{N} \sum_{t=0}^{T-1} \quad & L_t^i\Big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\Big) \\ & + \boldsymbol{\lambda}_{t+1}^i(\omega)\Big(f_t^i\big(\boldsymbol{X}_t^i(\omega), \boldsymbol{U}_t^i(\omega), \boldsymbol{W}_t(\omega)\big) - \boldsymbol{X}_{t+1}^i(\omega)\Big) \,. \end{aligned} \tag{1.38}$$

Define by J the function minimized in Problem (1.34), that is,

$$J(\boldsymbol{U}) = \min_{\boldsymbol{X}} \quad \max_{\boldsymbol{\lambda}} \quad H\big(\boldsymbol{X}, \boldsymbol{U}, \boldsymbol{\lambda}\big) \,. \tag{1.39}$$

The Danskin theorem (also known as the *envelop theorem* in Economics), states that, under proper assumptions, the gradient of the function J at point $\boldsymbol{U}^{(k)}$ is given by

$$\nabla J(\boldsymbol{U}^{(k)}) = \nabla_{\boldsymbol{U}} H(\boldsymbol{X}^{(k)}, \boldsymbol{U}^{(k)}, \boldsymbol{\lambda}^{(k)}) \,. \tag{1.40}$$

Hence, the gradient of J at $\boldsymbol{U}^{(k)}$ is

$$\nabla J(\boldsymbol{U}^{(k)}) = \nabla_u L_t\big(\boldsymbol{X}_t^{(k)}, \boldsymbol{U}_t^{(k)}, \boldsymbol{W}_t\big) + \nabla_u f_t\big(\boldsymbol{X}_t^{(k)}, \boldsymbol{U}_t^{(k)}, \boldsymbol{W}_t\big)\boldsymbol{\lambda}_{t+1}^{(k)} \,. \tag{1.41}$$

As the minimization is done over the $\mathfrak{F}$-adapted controls, a projected gradient step for the minimization of J would be

$$\boldsymbol{U}_t^{(k+1)} = \boldsymbol{U}_t^{(k)} + \rho \mathbb{E}\Big[\nabla_u L_t\big(\boldsymbol{X}_t^{(k)}, \boldsymbol{U}_t^{(k)}, \boldsymbol{W}_t\big) + \nabla_u f_t\big(\boldsymbol{X}_t^{(k)}, \boldsymbol{U}_t^{(k)}, \boldsymbol{W}_t\big)\boldsymbol{\lambda}_{t+1}^{(k)} \;\Big|\; \mathcal{F}_t\Big] \,. \tag{1.42}$$

Equation (1.42) can be used as an update step of the control $\boldsymbol{U}_t^{(k)}$ for this decomposition method.

1.2.4 Time Decomposition

Not all decompositions by duality lead to powerful formulations. For instance, we present a (little used) parallel decomposition approach of time decomposition obtained by dualization of the dynamic constraint.

On the other hand, as there is a natural flow in time, we can write a chained decomposition method, the well-known *Dynamic Programming* approach.

Dualizing the Dynamics Constraints

We apply to Problem (1.19) a price decomposition scheme, presented in §1.2.1, by dualizing the dynamic constraint (1.19b).

Since there are $N \times T \times |\Omega|$ dynamics constraints, the set of multiplier is of dimension $T \times |\Omega| \times N \times n_X$. Dualizing the dynamics constraints (1.19b), Problem (1.19) reads

$$
\begin{aligned}
\min_{\boldsymbol{X},\boldsymbol{U}} \quad \max_{\boldsymbol{\lambda}} \quad & \sum_{\omega\in\Omega}\sum_{i=1}^{N}\sum_{t=0}^{T-1} \quad \mathbb{P}\big(\{\omega\}\big)\; L_t^i\Big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega),\boldsymbol{W}_t(\omega)\Big)\\
& \qquad\qquad + \boldsymbol{\lambda}_t^i(\omega)\boldsymbol{X}_t^i(\omega) - \boldsymbol{\lambda}_{t+1}^i(\omega) f_t^i\big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega),\boldsymbol{W}_t(\omega)\big),\\
s.t. \quad & \sum_{i=1}^{N}\theta_t^i\big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega)\big) = 0, && \forall t, \quad \forall\omega\\
& \boldsymbol{U}_t^i \preceq \mathcal{F}_t && \forall t, \quad \forall i\;.
\end{aligned}
$$

Assuming constraint qualification, and fixing a multiplier $\boldsymbol{\lambda}^{(k)}$, we obtain T separate inner minimization problems

$$
\begin{aligned}
\min_{\boldsymbol{X}_t,\boldsymbol{U}_t} \quad & \sum_{\omega\in\Omega}\sum_{i=1}^{N} \quad \mathbb{P}\big(\{\omega\}\big)\; L_t^i\Big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega),\boldsymbol{W}_t(\omega)\Big)\\
& \qquad\qquad + \boldsymbol{\lambda}_t^{i,(k)}(\omega)\boldsymbol{X}_t^i(\omega) - \boldsymbol{\lambda}_{t+1}^{i,(k)}(\omega) f_t^i\big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega),\boldsymbol{W}_t(\omega)\big),\\
s.t. \quad & \sum_{i=1}^{N}\theta_t^i\big(\boldsymbol{X}_t^i(\omega),\boldsymbol{U}_t^i(\omega)\big) = 0, && \forall\omega\\
& \boldsymbol{U}_t^i \preceq \mathcal{F}_t && \forall i\;.
\end{aligned}
$$

We denote $\boldsymbol{U}_t^{i,(k)}$ and $\boldsymbol{X}_t^{i,(k)}$ an optimal solution. We update the multipliers with

$$
\boldsymbol{\lambda}_{t+1}^{i,(k+1)}(\omega) = \boldsymbol{\lambda}_{t+1}^{i,(k)}(\omega) + \rho\Big(\boldsymbol{X}_{t+1}^{i,(k)}(\omega) - f_t^i\big(\boldsymbol{X}_t^{i,(k)}(\omega),\boldsymbol{U}_t^{i,(k)}(\omega),\boldsymbol{W}_t(\omega)\big)\Big)\;.
$$

This decomposition approach is probably one of the less used decomposition approaches.

Dynamic Programming (DP)

The Dynamic Programming method is a well-known decomposition in time (see [11]). As it is usual, we present the Dynamic Programming in a Decision-Hazard setting. It relies on the assumption that the exogenous noises $\{\boldsymbol{W}_t\}_0^{T-1}$ form a sequence of independent random variables. With this assumption, the original state $\boldsymbol{X} = (\boldsymbol{X}_1,\dots,\boldsymbol{X}_N)$, that follows (1.19b), is a so-called *information state* (see Remark 1.2). This state is the argument of the *value function* V_t: $V_t(x)$ is the best possible future cost starting from time t in state x. The value functions satisfy the *Dynamic Programming Equations*: the V_t are computed backwards, starting from V_T and solving static optimization problems (see Algorithm 1.2). The solutions of these static optimization problems provide an optimal solution as a deterministic function of the current state (state feedback)

$$
\boldsymbol{U}_t^\sharp = \pi_t^\sharp\big(\boldsymbol{X}_t^\sharp\big)\;,
$$

where $\pi_t^\sharp : \mathbb{X}_t \to \mathbb{U}_t$. Observe that the solution, supposed to satisfy the non-anticipativity constraint (1.19d), satisfies what is a stronger constraint, namely $\boldsymbol{U}_t \preceq \boldsymbol{X}_t$. This is an important property of DP: when the exogenous noises $\{\boldsymbol{W}_t\}_{t=0}^{T-1}$ form a sequence of independent random variables, there is no loss of optimality in reducing the search to the class of state feedback solutions, namely $\mathcal{B}_t = \sigma(\boldsymbol{X}_t)$ instead of $\mathcal{B}_t = \mathcal{F}_t$. When the size of the state space $\mathbb{X}_t$ does not increase with t, this property has major consequences for numerical applications: whereas the space of $\mathcal{F}_t$-measurable solutions increases (exponentially) with t, the space of policies $\pi_t : \mathbb{X}_t \to \mathbb{U}_t$ does not.

Data: Problem data (especially initial point x_0 and final cost functions K^i);
Result: Bellman function V_t, Optimal feedback π_t ;
$V_T(x) = 0$;
for $t = T-1$ **to** 0 **do**
 foreach $x_t \in \mathbb{X}_t$ **do**

$$\begin{aligned} V_t(x_t) \quad = \quad & \min_{u=\{u^i\}_1^N} \quad \mathbb{E}\Big[\sum_{i=1}^N \quad L_t^i(x_t^i, u^i, \boldsymbol{W}_t) + V_{t+1}(\boldsymbol{X}_{t+1})\Big] \\ & s.t. \quad \boldsymbol{X}_{t+1}^i = f_t^i(x_t^i, u^i, \boldsymbol{W}_t), \qquad \forall i \\ & \qquad \sum_{i=1}^N \theta_t^i(x_t^i, u^i) = 0 \end{aligned} \tag{1.43}$$

 $\pi_t(x_t)$ is a control u minimizing the above problem ;

Algorithm 1.2: Dynamic Programming Algorithm

The DP chained decomposition is possible because of a causality principle along the time axis (this would not be possible for the uncertainty or for space, except under very specific conditions).

Remark 1.6. *Here, we make the major assumption that the size of the state space $\mathbb{X}_t$ does not increase with t. We suppose that each component of the state takes a finite number n_x of values (hence the state takes at most $(n_x)^N$ values). Solving (1.19) by DP requires to explore*

$$T(n_x)^N n_u \tag{1.44}$$

possible solutions. Comparing with (1.18), we see that DP makes better than brute force whenever

$$\log T + N \log n_x + \log n_u \leq \frac{(n_w)^T - 1}{n_w - 1} \log n_u \ . \tag{1.45}$$

Therefore, the DP algorithm outbeats brute force for a large enough number T of time steps. Indeed, it is linear in time, whereas brute force is exponential in time. However, the complexity of DP is exponential in the number N of subproblems or, in other words, in the dimension of the state: this stands as the curse of dimensionality *(see [12]).*

Discussing DP and the Notion of State

When the exogenous noises $\{\boldsymbol{W}_t\}_{t=0}^{T-1}$ form a sequence of independent random variables, we can write a *Dynamic Programming Equation* (DPE) like (1.43) with state $\boldsymbol{X}$. Now, what happens if this assumption fails? We lay out a theoretical and a practical answer.

The theoretical answer follows [118]. We introduce the "maximal" state (already mentioned in Table 1.3)

$$\widehat{\boldsymbol{X}}_t = (x_0, \boldsymbol{W}, \boldsymbol{U}_0, \ldots, \boldsymbol{U}_{t-1}) \ , \tag{1.46}$$

which satisfies the trivial dynamic equation

$$\widehat{\boldsymbol{X}}_{t+1} = (\widehat{\boldsymbol{X}}_t, \boldsymbol{U}_t) \ . \tag{1.47}$$

Then, there exists a DPE, but with an even larger information state consisting of the conditional distribution of $\widehat{\boldsymbol{X}}_t$ knowing $\mathcal{F}_t$ [118]. Of course, this state is only of theoretical interest.

The practical answer has much to do with the "art of modelling", a compromise between, on the one hand, realism and complexity, and, on the other hand, mathematical tractability. Consider that you want to manage a dam, seen as an electricity storage, over a period of time (see §1.3). The natural physical state is the level of water in the dam, whereas the information state depends on the water inflows (rain, snow melting, etc.). To account for (a weak form of) dependency, we can make the assumption that the inflows are independent random variables, but that their distributions are not stationary, and depend upon time t to reflect seasonal effects. In that case, the physical state is an information state. To account for (a stronger form of) dependency, we can make the assumption that the inflows follow a so-called "order 1 model" (e.g. an AR-1 model)

$$\boldsymbol{W}_{t+1} = \tilde{f}_t\big(\boldsymbol{W}_t, \widehat{\boldsymbol{W}}_t\big) , \tag{1.48}$$

where $\{\widehat{\boldsymbol{W}}_t\}_{t=0}^{T-1}$ is a sequence of independent random variables. Here, an information state is given by

$$\widehat{\boldsymbol{X}}_t = \big(\boldsymbol{X}_t, \boldsymbol{W}_t\big) , \tag{1.49}$$

with the dynamic

$$\widehat{\boldsymbol{X}}_{t+1} = \Big(f_t\big(\boldsymbol{X}_t, \boldsymbol{W}_t\big), \tilde{f}_t\big(\boldsymbol{W}_t, \widehat{\boldsymbol{W}}_t\big)\Big) . \tag{1.50}$$

Of course, more realism pushes for incorporating more delays — $\boldsymbol{W}_{t+1} = \tilde{f}_t\big(\boldsymbol{W}_t, \dots, \boldsymbol{W}_{t-k}, \widehat{\boldsymbol{W}}_t\big)$ — but at the price of increasing the dimension of the information state, now being $\big(\boldsymbol{X}_t, \boldsymbol{W}_t, \dots, \boldsymbol{W}_{t-k}\big)$, hitting the wall of the curse of dimensionality.

If the problem is written on a tree, we can write DPE with the couple physical state x and current node (identified with past noises). This is presented in §3.2.1.

Some approaches mix DP and a state of rather large dimension. For instance, *Stochastic Dual Dynamic Programming Algorithm* (SDDP) makes assumption on the objective function J (convexity) and on the dynamics functions f_t (linearity). With these, the value functions are shown to be convex, so that they can be approximated from below by the class of suprema of finite sets of linear functions. Such a structural property is a mean to partially overcome the curse of dimensionality of DP. In Chapter 3, we will present SDDP as a DP approach where information is encoded in a tree and where value functions are cleverly approximated. Instead of computing the value function for any possible value of the state, the SDDP algorithm iteratively forges approximations of the value function that are improved around the states visited by optimal trajectories.

1.2.5 Summary Table

In Table 1.4, we gather the decompositions listed above. It happens that all the decomposition methods we looked at are parallel, except the Dynamic Programming approach (SDDP being a DP like approach). Indeed, chained decomposition is intimately related to the natural flow of stages. The parallel decompositions that we presented have been deduced from a price decomposition scheme for different constraints. Proving their convergence requires duality results, the main object of the second part of this manuscript (Chapters 4, 5 and 6).

Interestingly, decompositions can be weaved together or mixed, opening the way for a large variety of methods. For instance, we will present and dissect in Chapter 8 the *Dual Approximate Dynamic Programming* method (DADP). With the distinctions we established between decompositions, DADP can be seen as a spatial decomposition, where subproblems can be solved by time decomposition. More precisely, DADP makes it possible to solve subproblems by DP, rendering space and time decompositions compatible. In a different setting, the contributions of Chapter 2 can be seen as conditions of compatibility for time and uncertainty chained decompositions to yield a DPE.

	Decomposition			
	Time		Scenario	Space
	chained	parallel	parallel	parallel
Dynamic Programming	✓			
SDDP	✓			
DADP				✓
Progressive Hedging			✓	
Stochastic Pontryaguin			✓	

Table 1.4: Decomposition Methods

1.3 A Dam Management Example

Here, we detail an example, taken from the energy world, that is used throughout this manuscript as illustration.

Hydroelectricity is the main renewable energy in many countries (16% of global energy, and 13% of France energy). It provides a clean (no greenhouse gases emissions), fast-usable (under 30 seconds) and powerful (20 GW in China) energy that is cheap and substitutable for the thermal one. It is all the more important to ensure its proper use that it comes from a shared limited resource: the reservoirs water. This is the dam hydroelectric production management purpose.

1.3.1 A Single Dam

Let time t vary in $[\![0,T]\!]$. We consider a probability space $\big(\Omega,\mathcal{F},\mathbb{P}\big)$, and a sequence $\{\boldsymbol{W}_t\}_{t=0}^{T-1}$ of random variables with value in $\mathbb{W}$. The random variable $\boldsymbol{W}_t$ represents the random water inflow [8] in the dam at time t. The dam is modeled as a Stochastic Dynamic System, as in §1.1.2, where the physical state $\boldsymbol{X}_t$ is the volume of water available at time t, and the control $\boldsymbol{U}_t$ is the volume of water consumed at time t.

The consumed water at time t induces a cash flow [9] of $-L_t\big(\boldsymbol{X}_t,\boldsymbol{U}_t,\boldsymbol{W}_t\big)$, and the remaining water at the final time t is valued by $-K(\boldsymbol{X}_t)$. We aggregate the random cost with the expectation, and do not take into account any discount factor. Thus, the problem we are interested in is the following

$$\min_{\boldsymbol{X},\boldsymbol{U}} \quad \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t\big(\boldsymbol{X}_t,\boldsymbol{U}_t,\boldsymbol{W}_t\big) + K(\boldsymbol{X}_T)\Big] \tag{1.51a}$$

$$s.t \quad \boldsymbol{X}_{t+1} = f_t\big(\boldsymbol{X}_t,\boldsymbol{U}_t,\boldsymbol{W}_t\big) \qquad \forall t \in [\![0,T-1]\!]\ , \tag{1.51b}$$

$$\boldsymbol{U}_t \preceq \mathcal{F}_t \qquad \forall t \in [\![0,T-1]\!]\ , \tag{1.51c}$$

$$\boldsymbol{U}_t \in U_t^{\mathrm{ad}} \qquad \mathbb{P}-\text{a.s.}, \quad \forall t \in [\![0,T-1]\!]\ , \tag{1.51d}$$

$$\boldsymbol{X}_t \in X_t^{\mathrm{ad}} \qquad \mathbb{P}-\text{a.s.}, \quad \forall t \in [\![0,T-1]\!]\ . \tag{1.51e}$$

8. More information, like the prices of electricity can be contained in the random variable $\boldsymbol{W}_t$
9. As usual the problem being in fact a maximization of cash flow we rewrite it as the minimization of the opposite of those cash-flows.

Constraint (1.51b) is the physical constraint of evolution on the stock of water in the dam. It is given by the physics of the dam, like

$$f_t\big(x, u, w\big) = x - u + w \; .$$

Constraint (1.51c) is the measurability constraint representing what the manager knows when he decides the value of $\boldsymbol{U}_t$. We distinguish two classical cases:

- the *Hazard-Decision* case, where $\mathcal{F}_t = \sigma\big(\boldsymbol{X}_0, \boldsymbol{W}_0, \cdots, \boldsymbol{W}_t\big)$, which means that the manager knows the water input between t and $t+1$ when he decides the consumption in the same period;
- the *Decision - Hazard* case, where $\mathcal{F}_t = \sigma\big(\boldsymbol{X}_0, \boldsymbol{W}_0, \cdots, \boldsymbol{W}_{t-1}\big)$, which means that the manager knows only the past noises and consequently the present volume of water in the dam.

Constraints (1.51d) and (1.51e) are bound constraints on the control and the state, representing the physical limitations of the dam and turbine. Usually we have

$$X_t^{\mathrm{ad}} = [\underline{x}_t, \overline{x}_t] \qquad \text{and} \qquad U_t^{\mathrm{ad}} = [\underline{u}_t, \overline{u}_t] \; .$$

The local cost function L_t represents the (opposite of) the gain obtained by selling the electricity produced by turbining a volume u of water. This gain depends on the market price (included in $\boldsymbol{W}_t$), the water turbined (the control $\boldsymbol{U}_t$) and the level of water in the dam (the state $\boldsymbol{X}_t$): higher level means higher water pressure.

1.3.2 A Chain of Dams

Most times, dams are included in a hydraulic valley, so that dams interact with each other: the water output of one dam is an input for another dam, etc. Hydraulic valley can be quite complex see for example Figure 1.5. but, for the sake of simplicity, we present a cascade of dams as in Figure 1.6. In this setting, the water consumed by dam i is seen as an inflow of dam $i+1$. In particular, we do not consider the cases where one dam receives the outflow of two other dams, neither when the outflow of one dam can go in two different destinations.

Let time t vary in $[\![0, T]\!]$, and dams be labeled with $i \in [\![1, N]\!]$. We consider a probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, and the following real valued random variables:

- $\boldsymbol{X}_t^i$, the *storage level* of dam i at the beginning of period $[t, t+1[$, (*state*)
- $\boldsymbol{U}_t^i$ the *hydro turbine* outflows of dam i during $[t, t+1[$, (*control*)
- $\boldsymbol{Z}_t^i$ the water inflows for dam i from dam $i-1$ during $[t, t+1[$, (*additional control*)
- $\boldsymbol{W}_t^i$, the *external inflows* for dam i during $[t, t+1[$. (*noise*)

The additional control $\boldsymbol{Z}_t^i$ is a useful notation and will be used in §8.1.5 to decompose the system.

The dynamics of the reservoir storage level reads, for the first dam of the chain:

$$\begin{aligned} \boldsymbol{X}_{t+1}^1 &= f_t^1(\boldsymbol{X}_t^1, \boldsymbol{U}_t^1, \boldsymbol{W}_t^1, 0) \; , \\ &= \boldsymbol{X}_t^1 - \boldsymbol{U}_t^1 + \boldsymbol{W}_t^1 \; . \end{aligned}$$

For any other dam $i > 1$, we have

$$\begin{aligned} \boldsymbol{X}_{t+1}^i &= f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t^i, \boldsymbol{Z}_t^i) \; , \\ &= \boldsymbol{X}_t^i - \boldsymbol{U}_t^i + \boldsymbol{W}_t^i + \boldsymbol{Z}_t^i \; , \end{aligned} \tag{1.52}$$

where

$$\boldsymbol{Z}_t^i = \boldsymbol{X}_t^{i-1} - \boldsymbol{U}_t^{i-1} + \boldsymbol{W}_t^{i-1} + \boldsymbol{Z}_t^{i-1} \tag{1.53}$$

is the water inflows in dam i coming from dam $i-1$, it is also the total outflows of dam $i-1$.

Figure 1.5: Chains of dams in France

The bound constraints are

$$\forall t \in [\![0, T-1]\!], \quad \underline{x}_{t+1} \leq \boldsymbol{X}_{t+1} \leq \overline{x}_{t+1} \quad \text{and} \quad \underline{u}_t \leq \boldsymbol{U}_t \leq \overline{u}_t \ . \tag{1.54}$$

Moreover, we assume the *Hazard-Decision* information structure ($\boldsymbol{U}_t^i$ is chosen once $\boldsymbol{W}_t$ is observed), so that

$$\underline{u}^i \leq \boldsymbol{U}_t^i \leq \min\left\{\overline{u}^i, \boldsymbol{X}_t^i + \boldsymbol{W}_t^i + \boldsymbol{Z}_t^i - \underline{x}^i\right\} . \tag{1.55}$$

We consider the multiple step management of a chain of dams, each dam producing electricity that is sold at the same price. Thus, the hydroelectric valley obeys the following valuing mechanism

$$\sum_{i=1}^{N} \sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{Z}_t^i, \boldsymbol{W}_t^i) + K^i(\boldsymbol{X}_T) , \tag{1.56}$$

where K^i is a function valuing the remaining water at time t in the dam i. As this criterion is random, we choose to minimize the expected cost, so that the stochastic optimization problem we address reads

$$\min_{(\boldsymbol{X},\boldsymbol{U},\boldsymbol{Z})} \mathbb{E}\left[\sum_{i=1}^{N}\left(\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{Z}_t^i, \boldsymbol{W}_t^i) + K^i(\boldsymbol{X}_T^i)\right)\right], \tag{1.57a}$$

subject to:

$$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{Z}_t^i, \boldsymbol{W}_t^i), \quad \forall i, \quad \forall t , \tag{1.57b}$$

$$\boldsymbol{Z}_t^{i+1} = g_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{Z}_t^i, \boldsymbol{W}_t^i), \quad \forall i, \quad \forall t , \tag{1.57c}$$

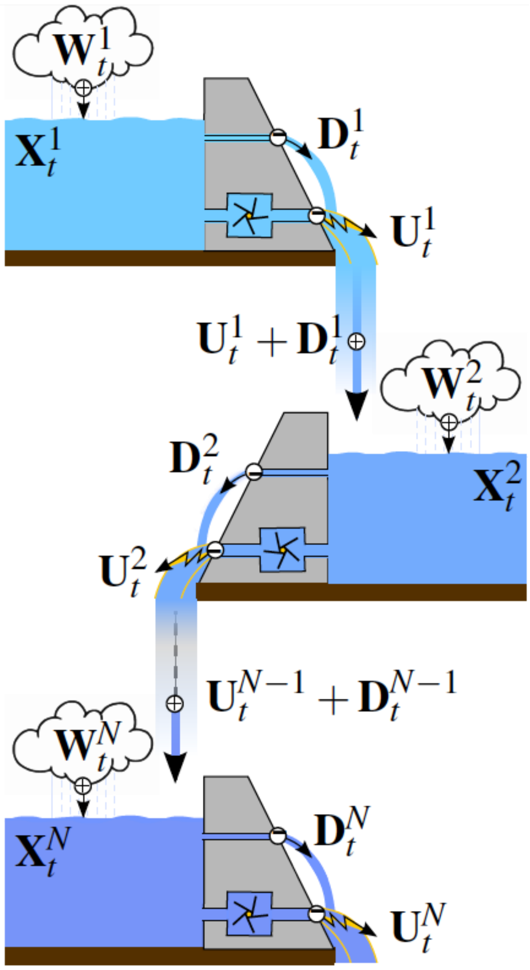


Figure 1.6: A chain of dams scheme

as well as measurability constraints:

$$\boldsymbol{U}_t^i \preceq \mathcal{F}_t, \quad \forall i, \quad \forall t \; . \tag{1.57d}$$

Theoretically, this problem could be solved by methods like Dynamic Programming. However, the dimension of the information state required is the number of dams. Thus, the so-called curse of dimensionality prevents us to apply Dynamic Programming for more than 5 dams.

In Chapter 3, we present an algorithm using approximations of value functions in Dynamic Programming to solve this type of problem for a large number of dams. In Chapter 8, we present a spatial decomposition method.

Conclusion

We conclude this preliminary chapter with a roadmap of the manuscript.

In the first part of this manuscript, *Dynamic Programming: Risk and Convexity*, we focus on chained decomposition, and especially the well-known time decomposition that constitutes Dynamic Programming. In Chapter 2, we extend the traditional additive in time and risk neutral setting to more general ones, for which we establish time-consistency results. We relate the time-consistency property for a sequence of optimization problems with the time-consistency property of a dynamic risk measure. In Chapter 3, we prove a convergence result for the Stochastic Dual Dynamic Programming Algorithm in the case where (convex) cost functions are no longer polyhedral.

Then, we turn to parallel decomposition, especially decomposition methods obtained by dualizing constraints (spatial or non-anticipative). In the second part of this manuscript, *Duality in Stochastic Optimization*, we first point out that such constraints lead to delicate duality issues (Chapter 4). We establish a duality result in the pairing $\left(\mathrm{L}^\infty, \mathrm{L}^1\right)$ in Chapter 5. Finally, in Chapter 6, we prove the convergence of the Uzawa Algorithm in $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$, that requires constraints qualification. This algorithm is used to apply a price decomposition scheme to a multistage stochastic optimization problem.

The third part of this manuscript, *Stochastic Spatial Decomposition Methods*, is devoted to the so-called *Dual Approximate Dynamic Programming Algorithm*. In Chapter 7, we prove that a sequence of relaxed optimization problems epiconverges to the original one, where almost sure constraints are replaced by weaker conditional expectation ones, and that the corresponding sigma-algebras converge. In Chapter 8, we give theoretical foundations and interpretations for the Dual Approximate Dynamic Programming Algorithm.

Part I

Dynamic Programming: Risk and Convexity

Chapter 2

Time-Consistency: from Optimization to Risk Measures

In my next life I want to live my life backwards. You wake up in an old people's home feeling better every day. You get kicked out for being too healthy. You work for 40 years until you're young enough to enjoy your retirement. You party, drink alcohol, and are generally promiscuous. then you become a kid, you play. You have no responsibilities, you become a baby until you are born. And then you spend your last 9 months floating in luxurious spa-like conditions with central heating and room service on tap, larger quarters every day and then Voila! You finish off as an orgasm!

Woody Allen (abbreviated)

Contents

This chapter present a general framework for the chained time decomposition known as Dynamic Programming. Indeed, usually Dynamic Programming is applied to problems considering the expectation of an inter-temporal sum of costs. In [105] a risk-averse dynamic programming theory was developed. Here, we extend the framework by giving general conditions on the aggregator in time (replacing the intertemporal sum) and the aggregator in uncertainties (replacing the expectation) to obtain a Dynamic Programming Equation.

The content of this Chapter has been submitted as an article in a special issue of *European Journal of Operations Research*, dedicated to Time Consistency. It is a common work with M. De Lara.

Introduction

Stochastic optimal control is concerned with sequential decision-making under uncertainty. The theory of dynamic risk measures gives values to stochastic processes (costs) as time goes on and information accumulates. Both theories coin, under the same vocable of *time-consistency* (or *dynamic-consistency*), two different notions. We discuss one after the other.

In stochastic optimal control, we consider a dynamical process that can be influenced by exogenous noises as well as decisions made at every time step. The decision maker wants to optimize a criterion (for instance, minimize a net present value) over a given time horizon. As time goes on and the system evolves, observations are made. Naturally, it is generally more profitable for the decision maker to adapt his decisions to the observations on the system. He is hence looking for policies (strategies, decision rules) rather than simple decisions: a policy is a function that maps every possible history of the observations to corresponding decisions.

The notion of "consistent course of action" (see [73]) is well-known in the field of economics, with the seminal work of [113]: an individual having planned his consumption trajectory is consistent if, reevaluating his plans later on, he does not deviate from the originally chosen plan. This idea of consistency as "sticking to one's plan" may be extended to the uncertain case where plans are replaced by decision rules ("Do thus-and-thus if you find yourself in this portion of state space with this amount of time left", Richard Bellman cited in [41]): [53] addresses "consistency" and "coherent dynamic choice", [61] refers to "temporal consistency".

In this context, we loosely state the property of time-consistency in stochastic optimal control as follows [26]. The decision maker formulates an optimization problem at time t_0 that yields a sequence of optimal decision rules for t_0 and for the following increasing time steps $t_1, \dots, t_N = T$. Then, at the next time step t_1, he formulates a new problem starting at t_1 that yields a new sequence of optimal decision rules from time steps t_1 to T. Suppose the process continues until time T is reached. The sequence of optimization problems is said to be dynamically consistent if the optimal strategies obtained when solving the original problem at time t_0 remain optimal for all subsequent problems. In other words, dynamic consistency means that strategies obtained by solving the problem at the very first stage do not have to be questioned later on.

Now, we turn to dynamic risk measures. At time t_0, you value, by means of a risk measure $\rho_{t_0,T}$, a stochastic process $\big\{\boldsymbol{A}_t\big\}_{t=t_0}^{t_N}$, that represents a stream of costs indexed by the increasing time steps $t_0, t_1, \dots, t_N = T$. Then, at the next time step t_1, you value the tail $\big\{\boldsymbol{A}_t\big\}_{t=t_1}^{t_N}$ of the stochastic process knowing the information obtained and materialized by a σ-field $\mathfrak{F}_{t_1}$. For this, you use a conditional risk measure $\rho_{t_1,T}$ with values in $\mathfrak{F}_{t_1}$-

measurable random variables. Suppose the process continues until time T is reached. The sequence $\{\rho_{t,T}\}_{t=t_0}^{t_N}$ of conditional risk measures is called a dynamic risk measure.

Dynamic or time-consistency has been introduced in the context of risk measures (see [5,28,29,37,84] for definitions and properties of coherent and consistent dynamic risk measures). We loosely state the property of time-consistency for dynamic risk measures as follows. The dynamic risk measure $\{\rho_{t,T}\}_{t=t_0}^{t_N}$ is said to be time-consistent when the following property holds. Suppose that two streams of costs, $\{\underline{\boldsymbol{A}}_t\}_{t=t_0}^{t_N}$ and $\{\overline{\boldsymbol{A}}_t\}_{t=t_0}^{t_N}$, are such that they coincide from time t_i up to time $t_j > t_i$ and that, from that last time t_j, the tail stream $\{\underline{\boldsymbol{A}}_t\}_{t=t_j}^{t_N}$ is valued more than $\{\overline{\boldsymbol{A}}_t\}_{t=t_j}^{t_N}$ ($\rho_{t_j,T}(\{\underline{\boldsymbol{A}}_t\}_{t=t_j}^{t_N}) \geq \rho_{t_j,T}(\{\overline{\boldsymbol{A}}_t\}_{t=t_j}^{t_N})$). Then, the whole stream $\{\underline{\boldsymbol{A}}_t\}_{t=t_i}^{t_N}$ is valued more than $\{\overline{\boldsymbol{A}}_t\}_{t=t_i}^{t_N}$ ($\rho_{t_i,T}(\{\underline{\boldsymbol{A}}_t\}_{t=t_i}^{t_N}) \geq \rho_{t_i,T}(\{\overline{\boldsymbol{A}}_t\}_{t=t_i}^{t_N})$).

We observe that both notions of time-consistency look quite different: the latter is consistency between successive evaluations of a stochastic processes by a dynamic risk measure as information accumulates (a form of monotonicity); the former is consistency between solutions to intertemporal stochastic optimization problems as information accumulates. We now stress the role of information accumulation in both notions of time-consistency, because of its role in how the two notions can be connected. For dynamic risk measures, the flow of information is materialized by a filtration $\{\mathfrak{F}_t\}_{t=t_1}^{t_N}$. In stochastic optimal control, an amount of information more modest than the past of exogenous noises is often sufficient to make an optimal decision. In the seminal work of [12], the minimal information necessary to make optimal decisions is captured in a *state variable* (see [117] for a more formal definition). Moreover, the famous Bellman or *Dynamic Programming Equation (DPE)* provides a theoretical way to find optimal strategies (see [18] for a broad overview on *Dynamic Programming (DP)*).

Interestingly, time-consistency in stochastic optimal control and time-consistency for dynamic risk measures meet in their use of DPEs. On the one hand, in stochastic optimal control, it is well known that the existence of a DPE with state x for a sequence of optimization problems implies time-consistency when solutions are looked after as feedback policies that are functions of the state x. On the other hand, proving time-consistency for a dynamic risk measure appears rather easy when the corresponding conditional risk measures can be expressed by a *nested* formulation that connects successive time steps. In both contexts, such nested formulations are possible only for proper information structures. In stochastic optimal control, a sequence of optimization problems may be consistent for some information structure while inconsistent for a different one (see [26]). For dynamic risk measures, time-consistency appears to be strongly dependent on the underlying information structure (filtration or scenario tree). Moreover, in both contexts, nested formulations and the existence of a DPE are established under various forms of decomposability of operators that display monotonicity and commutation properties.

Our objective is to provide a theoretical framework that offers i) basic ingredients to jointly define dynamic risk measures and corresponding intertemporal stochastic optimization problems ii) common sets of assumptions that lead to time-consistency for both. Our theoretical framework highlights the role of time and risk preferences, materialized in *one-step aggregators*, in time-consistency. Depending on how you move from one-step time and risk preferences to intertemporal time and risk preferences, and depending on their compatibility (commutation), you will or will not observe time-consistency. We also shed light on the relevance of information structure by giving an explicit role to a dynamical system with state $\boldsymbol{X}$.

In §2.1, we present examples of intertemporal optimization problems displaying a DPE, and of dynamic risk measures (time-consistent or not, nested or not). In §2.2, we introduce the basic material to formulate intertemporal optimization problems, in the course of which we define "cousins" of dynamic risk measures, namely *dynamic uncertainty criteria*; we end with definitions of time-consistency, on the one hand, for dynamic risk measures and, in the other hand, for intertemporal stochastic optimization problems. In §2.3, we introduce the notions of time and uncertainty-aggregators, define their composition, and show four ways to craft a dynamic uncertainty criterion from one-step aggregators; then, we provide general sufficient conditions for the existence of a DPE and for time-consistency, both for dynamic risk measures and for intertemporal stochastic optimization problems; we end with applications. In §2.4, we extend constructions and results to Markov aggregators.

2.1 Introductory Examples

The traditional framework for DP consists in minimizing the expectation of the intertemporal sum of costs as in Problem (2.3). As we see it, the intertemporal sum is an aggregation over time, and the mathematical expectation is an aggregation over uncertainties. We claim that other forms of aggregation lead to a DPE with the same state but, before developing this point in §2.3, we lay out in §2.1.1 three settings (more or less familiar) in which a DPE holds. We do the same job for dynamic risk measures in §2.1.2 with time-consistency.

To alleviate notations, for any sequence $\{H_s\}_{s=t_1,\dots,t_2}$ of sets, we denote by $[H_s]_{t_1}^{t_2}$, or by $H_{[t_1:t_2]}$, the Cartesian product

$$H_{[t_1:t_2]} = [H_s]_{t_1}^{t_2} = [H_s]_{s=t_1}^{t_2} = H_{t_1} \times \cdots \times H_{t_2} \ , \tag{2.1a}$$

and a generic element by

$$h_{[t_1:t_2]} = \{h_t\}_{t_1}^{t_2} = \{h_t\}_{t=t_1}^{t_2} = (h_{t_1}, \dots, h_{t_2}) \ . \tag{2.1b}$$

In the same vein, we also use the following notation for any sequence

$$H_{[t_1:t_2]} = \{H_s\}_{t_1}^{t_2} = \{H_s\}_{s=t_1}^{t_2} = \{H_s\}_{s=t_1,\dots,t_2} \ . \tag{2.1c}$$

In this chapter, we denote by $\bar{\mathbb{R}}$ the set $\mathbb{R} \cup \{+\infty\}$.

2.1.1 Examples of DPEs in Intertemporal Optimization

Anticipating on material to be presented in §2.2.1, we consider a dynamical system influenced by exogenous uncertainties and by decisions made at discrete time steps $t = 0$, $t = 1$, …, $t = T - 1$, where T is a positive integer. For any $t \in [\![0, T]\!]$, where $[\![a, b]\!]$ denote the set of integers between a and b, we suppose given a state set $\mathbb{X}_t$, and for any $t \in [\![0, T-1]\!]$ a control set $\mathbb{U}_t$, an uncertainty set $\mathbb{W}_t$ and a mapping f_t that maps $\mathbb{X}_t \times \mathbb{U}_t \times \mathbb{W}_t$ into $\mathbb{X}_{t+1}$. We consider the *control stochastic dynamical system*

$$\forall t \in [\![0, T-1]\!], \qquad X_{t+1} = f_t(X_t, U_t, W_t) \ . \tag{2.2}$$

We call *policy* a sequence $\pi = (\pi_t)_{t \in [\![0,T-1]\!]}$ of mappings where, for all $t \in [\![0, T-1]\!]$, π_t maps $\mathbb{X}_t$ into $\mathbb{U}_t$. We denote by Π the set of all policies. More generally, for all $t \in [\![0, T]\!]$, we call *(tail) policy* a sequence $\pi = (\pi_s)_{s \in [\![t,T-1]\!]}$ and we denote by Π_t the set of all such policies.

Let $\{\boldsymbol{W}_t\}_{t=0}^{T}$ be a sequence of independent random variables (noises). Let $\{J_t\}_0^{T-1}$ be a sequence of cost functions $J_t : \mathbb{X}_t \times \mathbb{U}_t \times \mathbb{W}_t \mapsto \mathbb{R}$, and a final cost function $J_T : \mathbb{X}_T \times \mathbb{W}_T \to \mathbb{R}$.

With classic notations, and assuming all proper measurability and integrability conditions, we consider the dynamic optimization problem

$$\min_{\pi\in\Pi} \quad \mathbb{E}\Big[\sum_{t=0}^{T-1} J_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t) + J_T(\boldsymbol{X}_T, \boldsymbol{W}_T)\Big] , \tag{2.3a}$$

$$\text{s.t.} \quad \boldsymbol{X}_{t+1} = f_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t), \qquad \forall t \in [\![0, T-1]\!] , \tag{2.3b}$$

$$\boldsymbol{U}_t = \pi_t(\boldsymbol{X}_t), \qquad \forall t \in [\![0, T-1]\!] . \tag{2.3c}$$

It is well-known that a DPE with state $\boldsymbol{X}$ can be associated with this problem. The main ingredients for establishing the DPE are the following: the intertemporal criterion is time-separable and additive, the expectation is a composition of expectations over the marginals law (because the random variables $\{\boldsymbol{W}_t\}_{t=0}^T$ are independent), and the sum and the expectation operators are commuting. Our main concern is to extend these properties to other "aggregators" than the intertemporal sum $\sum_{t=0}^{T-1}$ and the mathematical expectation $\mathbb{E}$, and to obtain DPEs with state $\boldsymbol{X}$, thus retrieving time-consistency.

In this example, we aggregate the streams of cost first with respect to time (through the sum over the stages), and then with respect to uncertainties (through the expectation). This formulation is called *TU* for "time then uncertainty". All the examples of this §2.1.1 follow this template.

We do not present proofs of the DPEs exposed here as they fit into the framework developed later in §2.3.

Expected and Worst Case with Additive Costs

We present together two settings in which a DPE holds true. They share the same time-aggregator — time-separable and additive — but with distinct uncertainty-aggregators, namely the mathematical expectation operator and the so-called "fear" operator.

Expectation Operator Consider, for any $t \in [\![0, T]\!]$, a probability $\mathbb{P}_t$ on the uncertainty space $\mathbb{W}_t$ (equipped with a proper σ-algebra), and the product probability $\mathbb{P} = \mathbb{P}_0 \otimes \cdots \otimes \mathbb{P}_T$. In other formulations of stochastic optimization problems, the probabilities $\mathbb{P}_t$ are the image distributions of independent random variables with value in $\mathbb{W}_t$. However, we prefer to ground the problems with probabilities on the uncertainty spaces rather than with random variables, as this approach will more easily easily extend to other contexts without stochasticity.

The so-called *value function* V_t, whose argument is the state x, is the optimal cost-to-go defined by

$$V_t(x) = \min_{\pi\in\Pi_t} \quad \mathbb{E}\Big[\sum_{s=t}^{T-1} J_s(\boldsymbol{X}_s, \boldsymbol{U}_s, \boldsymbol{W}_s) + J_T(\boldsymbol{X}_T, \boldsymbol{W}_T)\Big] , \tag{2.4a}$$

$$\text{s.t.} \quad \boldsymbol{X}_t = x , \tag{2.4b}$$

$$\boldsymbol{X}_{s+1} = f_t(\boldsymbol{X}_s, \boldsymbol{U}_s, \boldsymbol{W}_s), \qquad \forall s \in [\![t, T-1]\!] , \tag{2.4c}$$

$$\boldsymbol{U}_s = \pi_s(\boldsymbol{X}_s) . \tag{2.4d}$$

The DPE associated with problem (2.3) is

$$\begin{cases} V_T(x) &= \mathbb{E}_{\mathbb{P}_T}\Big[J_T(x, \boldsymbol{W}_T)\Big] , \\ V_t(x) &= \min_{u\in\mathbb{U}_t} \mathbb{E}_{\mathbb{P}_t}\Big[J_t(x, u, \boldsymbol{W}_t) + V_{t+1}\circ f_t(x, u, \boldsymbol{W}_t)\Big] , \end{cases} \tag{2.5}$$

for all state $x \in \mathbb{X}_t$ and all time $t \in [\![0, T-1]\!]$.

It is well-known that, if there exists a policy $\pi^\sharp$ (with proper measurability assumptions that we do not discuss here [see [20]]) such that, for each $t \in [\![0, T-1]\!]$, and each $x \in \mathbb{X}_t$, we have

$$\pi_t^\sharp(x) \in \arg\min_{u \in \mathbb{U}_t} \mathbb{E}\Big[J_t(x, u, \boldsymbol{W}_t) + V_{t+1} \circ f_t(x, u, \boldsymbol{W}_t)\Big] , \tag{2.6}$$

then $\pi^\sharp$ is an optimal policy for Problem (2.3).

Time-consistency of the sequence of Problems (2.4), when t runs from 0 to T, is ensured by this very DPE, when solutions are looked after as policies over the state x. We insist that the property of time-consistency may or may not hold depending on the nature of available information at each time step. Here, our assumption is that the state x_t is available for decision-making at each time t. [1]

Remark 2.1. *To go on with information issues, we can notice that the so-called "non-anticipativity constraints", typical of stochastic optimization, are contained in our definition of policies. Indeed, we considered policies are function of the state, which a summary of the* past, *hence cannot anticipate the future. Why can we take the state as a proper summary? If, in Problem (2.3), we had considered policies as functions of past uncertainties (non-anticipativity) and had assumed that the uncertainties are independent, it is well-known that we could have restricted our search to optimal Markovian policies, that is, only functions of the state. This is why, we consider policies only as functions of the state.*

Fear Operator In [16], Pierre Bernhard coined *fear operator* the worst-case operator, widely considered in the field of robust optimization (see [67] and [15]).

We consider the optimization problem

$$\min_{\pi \in \Pi} \quad \sup_{w \in \mathbb{W}_{[0:T]}} \Big[\sum_{t=0}^{T-1} J_t(x_t, u_t, w_t) + J_T(x_T, w_T)\Big], \tag{2.7a}$$

$$\text{s.t.} \quad x_{t+1} = f_t(x_t, u_t, w_t), \tag{2.7b}$$

$$u_t = \pi_t(x_t). \tag{2.7c}$$

Contrarily to previous examples we do not use bold letters for state x, control u and uncertainty w as these variables are not random variables. In [17, Section 1.6], it is shown that the value function

$$V_t(x) = \min_{\pi \in \Pi_t} \quad \sup_{w \in \mathbb{W}_{[t:T]}} \Big[\sum_{s=t}^{T-1} J_s(x_s, u_s, w_s) + J_T(x_T, w_T)\Big] , \tag{2.8a}$$

$$\text{s.t.} \quad x_t = x , \tag{2.8b}$$

$$x_{s+1} = f_s(x_s, u_s, w_s) , \tag{2.8c}$$

$$u_s = \pi_s(x_s) . \tag{2.8d}$$

satisfies the DPE

$$\begin{cases} V_T(x) = \sup_{w_T \in \mathbb{W}_T} J_T(x, w_T) , \\ V_t(x) = \min_{u \in \mathbb{U}_t} \sup_{w_t \in \mathbb{W}_t} \Big[J_t(x, u, w_t) + V_{t+1} \circ f_t(x, u, w_t)\Big] , \end{cases} \tag{2.9}$$

for all state $x \in \mathbb{X}_t$ and all time $t \in [\![0, T-1]\!]$.

1. In the literature on risk measures, information is rather described by filtrations than by variables.

Expectation with Multiplicative Costs

An expected multiplicative cost appears in a financial context if we consider a final payoff $K(\boldsymbol{X}_{T+1})$ depending on the final state of our system, but discounted at rate $r_t(\boldsymbol{X}_t)$. In this case, the problem of maximizing the discounted expected product reads

$$\max_{\pi\in\Pi} \quad \mathbb{E}\Big[\prod_{t=1}^{T-1}\Big(\frac{1}{1+r_t(\boldsymbol{X}_t)}\Big)K(\boldsymbol{X}_T)\Big] .$$

We present another interesting setting where multiplicative cost appears. In control problems, we thrive to find controls such that the state x_t satisfies constraints of the type $x_t \in \mathcal{X}_t \subset \mathbb{X}_t$ for all $t \in [\![0,T]\!]$. In a deterministic setting, the problem has either no solution (there is no policy such that, for all $t \in [\![0,T]\!]$, $x_t \in \mathcal{X}_t$) or has a solution depending on the starting point x_0. However, in a stochastic setting, satisfying the constraint $x_t \in \mathcal{X}_t$, for all time $t \in [\![0,T]\!]$ and $\mathbb{P}-$almost surely, can lead to problems without solution. For example, if we add to a controled dynamic a nondegenerate Gaussian random variable, then the resulting state can be anywhere in the state space, and thus a constraint $\boldsymbol{X}_t \in \mathcal{X}_t \subset \mathbb{X}_t$ where $\mathcal{X}_t$ is, say, a bounded set, cannot be satisfied almost surely.

For such a control problem, we propose alternatively to maximize the probability of satisfying the constraint (see [40], where this is approach is called *stochastic viability*):

$$\max_{\pi\in\Pi} \quad \mathbb{P}\Big(\big\{\forall t\in[\![0,T]\!], \quad \boldsymbol{X}_t \in \mathcal{X}_t\big\}\Big), \tag{2.10a}$$

$$\text{s.t} \quad \boldsymbol{X}_{t+1} = f_t\big(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t\big) , \tag{2.10b}$$

$$\boldsymbol{U}_t = \pi(\boldsymbol{X}_t) . \tag{2.10c}$$

This problem can be written

$$\max_{\pi\in\Pi} \quad \mathbb{E}\Big[\prod_{t=0}^{T}\mathbb{1}_{\{\boldsymbol{X}_t\in\mathcal{X}_t\}}\Big], \tag{2.11a}$$

$$\text{s.t} \quad \boldsymbol{X}_{t+1} = f_t\big(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t\big) , \tag{2.11b}$$

$$\boldsymbol{U}_t = \pi(\boldsymbol{X}_t) . \tag{2.11c}$$

It is shown in [34] that, assuming that noises are independent (i.e the probability $\mathbb{P}$ can be written as a product $\mathbb{P} = \mathbb{P}_0 \otimes \cdots \otimes \mathbb{P}_T$), the associated DPE is

$$\left\{\begin{array}{ll} V_T(x) & = \mathbb{E}\Big[\mathbb{1}_{\{x\in\mathcal{X}_T\}}\Big] , \\ V_t(x) & = \max_{u\in\mathbb{U}_t} \quad \mathbb{E}\Big[\mathbb{1}_{\{x\in\mathcal{X}_t\}}\cdot V_{t+1}\circ f_t(x,u,\boldsymbol{W}_t)\Big] , \end{array}\right. \tag{2.12}$$

for all state $x \in \mathbb{X}_t$ and all time $t \in [\![0,T-1]\!]$.

If there exists a measurable policy $\pi^\sharp$ such that, for all $t \in [\![0,T-1]\!]$ and all $x \in \mathbb{X}_t$,

$$\pi_t^\sharp(x) \in \operatorname*{arg\,max}_{u\in\mathbb{U}_t} \quad \mathbb{E}\Big[\mathbb{1}_{\{x\in\mathcal{X}_t\}}\cdot V_{t+1}\circ f_t(x,u,\boldsymbol{W}_t)\Big] , \tag{2.13}$$

then $\pi^\sharp$ is optimal for Problem (2.10).

2.1.2 Examples of Dynamic Risk Measures

Consider a probability space $\big(\Omega,\mathfrak{F},\mathbb{P}\big)$, and a filtration $\mathfrak{F} = \{\mathfrak{F}_t\}_0^T$. The expression $\{\boldsymbol{A}_s\}_0^T$ denotes an arbitrary, $\mathfrak{F}$-adapted, real-valued, stochastic process.

Anticipating on recalls in §2.2.2, we call *conditional risk measure* a function $\rho_{t,T}$ that maps tail sequences $\{\boldsymbol{A}_s\}_t^T$, where each $\boldsymbol{A}_s$ is $\mathfrak{F}_s$ measurable, into the set of $\mathfrak{F}_t$ measurable random variables. A *dynamic risk measure* is a sequence $\{\rho_{t,T}\}_0^T$ of conditional risk measures.

A dynamic risk measure $\{\rho_{t,T}\}_{t=0}^T$, is said to be *time-consistent* if, for any couples of times $0 \leq \underline{t} < \overline{t} \leq T$, the following property holds true. If two adapted stochastic processes $\{\underline{\boldsymbol{A}}_s\}_0^T$ and $\{\overline{\boldsymbol{A}}_s\}_0^T$ satisfy

$$\underline{\boldsymbol{A}}_s = \overline{\boldsymbol{A}}_s, \qquad \forall s \in [\![\underline{t}, \overline{t}-1]\!] , \tag{2.14a}$$

$$\rho_{\overline{t},T}\big(\{\underline{\boldsymbol{A}}_s\}_{\overline{t}}^T\big) \leq \rho_{\overline{t},T}\big(\{\overline{\boldsymbol{A}}_s\}_{\overline{t}}^T\big) , \tag{2.14b}$$

then we have:

$$\rho_{\underline{t},T}\big(\{\underline{\boldsymbol{A}}_s\}_{\underline{t}}^T\big) \leq \rho_{\underline{t},T}\big(\{\overline{\boldsymbol{A}}_s\}_{\underline{t}}^T\big) . \tag{2.14c}$$

We now lay out examples of dynamic risk measure.

Expectation and Sum

Unconditional Expectation The first classical example, related to the optimization Problem (2.3), consists in the dynamic risk measure $\{\rho_{t,T}\}_{t=0}^T$ given by

$$\forall t \in [\![0, T]\!], \qquad \rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) = \mathbb{E}\Big[\sum_{s=t}^T \boldsymbol{A}_s\Big] . \tag{2.15}$$

We write (2.15) under three forms — denoted by TU, UT, NTU, and discussed later in §2.3.1:

$$\begin{aligned}
\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) &= \mathbb{E}\Big[\sum_{s=t}^T \boldsymbol{A}_s\Big] && (TU)\\
&= \sum_{s=t}^T \mathbb{E}\big[\boldsymbol{A}_s\big] && (UT)\\
&= \mathbb{E}\bigg[\boldsymbol{A}_t + \mathbb{E}\Big[\boldsymbol{A}_{t+1} + \cdots + \mathbb{E}\big[\boldsymbol{A}_{T-1} + \mathbb{E}\big[\boldsymbol{A}_T\big]\big]\cdots\Big]\bigg] && (NTU)
\end{aligned}$$

To illustrate the notion, we show that the dynamic risk measure $\{\rho_{t,T}\}_{t=0}^T$ is time-consistent. Indeed, if two adapted stochastic processes $\underline{\boldsymbol{A}}$ and $\overline{\boldsymbol{B}}$ satisfy (2.14a) and (2.14b), with $\underline{t} = t < \overline{t} \leq T$, we conclude that

$$\begin{aligned}
\rho_{t,T}\big(\{\underline{\boldsymbol{A}}_s\}_t^T\big) =& \mathbb{E}\Big[\sum_{s=t}^{\overline{t}-1} \underline{\boldsymbol{A}}_s + \rho_{\overline{t},T}\big(\{\underline{\boldsymbol{A}}_s\}_{\overline{t}}^T\big)\Big]\\
\leq& \mathbb{E}\Big[\sum_{s=t}^{\overline{t}-1} \overline{\boldsymbol{A}}_s + \rho_{\overline{t},T}\big(\{\overline{\boldsymbol{A}}_s\}_{\overline{t}}^T\big)\Big] = \rho_{t,T}\big(\{\overline{\boldsymbol{A}}_s\}_t^T\big) .
\end{aligned}$$

Conditional Expectation Now, we consider a "conditional variation" of (2.15) by defining

$$\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) = \mathbb{E}\Big[\sum_{s=t}^T \boldsymbol{A}_s \;\Big|\; \mathfrak{F}_t\Big] . \tag{2.16}$$

We write [2] the induced dynamic risk measure $\{\rho_{t,T}\}_{t=0}^T$ under four forms — denoted by TU, UT, NTU, NUT, and discussed later in §2.3.1:

$$\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) = \mathbb{E}^{\mathfrak{F}_t}\Big[\sum_{s=t}^{T}\boldsymbol{A}_s\Big] \tag{TU}$$

$$= \sum_{s=t}^{T}\mathbb{E}^{\mathfrak{F}_t}\big[\boldsymbol{A}_s\big] \tag{UT}$$

$$= \mathbb{E}^{\mathfrak{F}_t}\Big[\boldsymbol{A}_t + \mathbb{E}^{\mathfrak{F}_{t+1}}\Big[\boldsymbol{A}_{t+1} + \cdots + \mathbb{E}^{\mathfrak{F}_{T-1}}\Big[\boldsymbol{A}_{T-1} + \mathbb{E}^{\mathfrak{F}_T}\big[\boldsymbol{A}_T\big]\Big]\cdots\Big]\Big] \tag{NTU}$$

$$= \boldsymbol{A}_t + \mathbb{E}^{\mathfrak{F}_{t+1}}\Big[\boldsymbol{A}_{t+1} + \cdots + \mathbb{E}^{\mathfrak{F}_{T-2}}\big[\boldsymbol{A}_{T-1} + \mathbb{E}^{\mathfrak{F}_{T-1}}\big[\boldsymbol{A}_T\big]\big]\cdots\Big] \tag{NUT}$$

The dynamic risk measure $\{\rho_{t,T}\}_{t=0}^T$ is time-consistent: indeed, if two adapted stochastic processes $\underline{\boldsymbol{A}}$ and $\overline{\boldsymbol{B}}$ satisfy (2.14a) and (2.14b), with $\underline{t} = t < \overline{t} \leq T$, we conclude that

$$\begin{aligned}
\rho_{t,T}\big(\{\underline{\boldsymbol{A}}_s\}_t^T\big) = \;& \mathbb{E}\Big[\sum_{s=t}^{\overline{t}-1}\underline{\boldsymbol{A}}_s + \rho_{\overline{t},T}\big(\{\underline{\boldsymbol{A}}_s\}_{\overline{t}}^T\big)\;\Big|\;\mathfrak{F}_t\Big] \\
\leq \;& \mathbb{E}\Big[\sum_{s=t}^{\overline{t}-1}\overline{\boldsymbol{A}}_s + \rho_{\overline{t},T}\big(\{\overline{\boldsymbol{A}}_s\}_{\overline{t}}^T\big)\;\Big|\;\mathfrak{F}_t\Big] = \rho_{t,T}\big(\{\overline{\boldsymbol{A}}_s\}_t^T\big)\;.
\end{aligned}$$

AV@R and Sum

In the following examples, it is no longer possible to display three or four equivalent expressions for the same conditional risk measure. This is why, we present different dynamic risk measures.

Unconditional AV@R For $0 < \alpha < 1$, we define the *Average-Value-at-Risk* of level α of a random variable $\boldsymbol{X}$ by

$$\mathrm{AV@R}_\alpha\big[\boldsymbol{X}\big] = \inf_{r\in\mathbb{R}}\Big\{r + \frac{\mathbb{E}\big[\boldsymbol{X} - r\big]^+}{\alpha}\Big\}\;. \tag{2.17}$$

Let $\{\alpha_t\}_{t=0}^T$ and $\{\alpha_{t,s}\}_{s,t=0}^T$ be two families in $(0,1)$. We lay out three different dynamic risk measures, given by the following conditional risk measures:

$$\rho_{t,T}\big[\{\boldsymbol{A}_s\}_t^T\big] = \mathrm{AV@R}_{\alpha_t}\Big[\sum_{s=t}^{T}\boldsymbol{A}_s\Big]\;, \tag{TU}$$

$$\rho_{t,T}\big[\{\boldsymbol{A}_s\}_t^T\big] = \sum_{s=t}^{T}\mathrm{AV@R}_{\alpha_{t,s}}\big[\boldsymbol{A}_s\big]\;, \tag{UT}$$

$$\begin{aligned}
\rho_{t,T}^{NTU}\big(\{\boldsymbol{A}_s\}_t^T\big) = \mathrm{AV@R}_{\alpha_{t,t}}\Big[\boldsymbol{A}_t + \mathrm{AV@R}_{\alpha_{t,t+1}}\Big[\boldsymbol{A}_{t+1} + \cdots \\
\mathrm{AV@R}_{\alpha_{t,T}}\big[\boldsymbol{A}_T\big]\cdots\Big]\Big]\;.
\end{aligned} \tag{NTU}$$

The dynamic risk measure $\{\rho_{t,T}^{TU}\}_{t=0}^T$ is not time-consistent, whereas the dynamic risk measure $\{\rho_{t,T}^{UT}\}_{t=0}^T$ and the dynamic risk measure $\{\rho_{t,T}^{NTU}\}_{t=0}^T$ are time consistent, as soon as the levels $\alpha_{t,s}$ do not depend on t.

2. Here, for notational clarity, we denote by $\mathbb{E}^{\mathfrak{F}_t}\big[\,\cdot\,\big]$ the conditional expectation $\mathbb{E}\big[\,\cdot\;\big|\;\mathfrak{F}_t\big]$.

Conditional AV@R For $0 < \alpha < 1$, and a subfield $\mathfrak{G} \subset \mathfrak{F}$ we define the *conditional Average-Value-at-Risk* of level α of a random variable $\boldsymbol{X}$ knowing $\mathfrak{G}$ by

$$\mathrm{AV@R}_{\alpha}^{\mathfrak{G}}\big[\boldsymbol{X}\big] = \inf_{r \ \mathfrak{G}\text{-measurable}} \left\{ r + \frac{\mathbb{E}\big[\big(\boldsymbol{X} - r\big)^{+} \mid \mathfrak{G}\big]}{\alpha} \right\} . \tag{2.19}$$

Let $\{\alpha_t\}_{t=0}^{T}$ and $\{\alpha_{t,s}\}_{s,t=0}^{T}$ be two families in $(0,1)$. We lay out four different dynamic risk measures, given by the following conditional risk measures:

$$\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) = \mathrm{AV@R}_{\alpha_t}^{\mathfrak{F}_t}\left[\sum_{s=t}^{T} \boldsymbol{A}_s\right] , \tag{TU}$$

$$\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) = \sum_{s=t}^{T} \mathrm{AV@R}_{\alpha_{t,s}}^{\mathfrak{F}_t}\left[\boldsymbol{A}_s\right] , \tag{UT}$$

$$\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) = \sum_{s=t}^{T} \mathrm{AV@R}_{\alpha_{t,t}}^{\mathfrak{F}_t}\left[\mathrm{AV@R}_{\alpha_{t,t+1}}^{\mathfrak{F}_{t+1}}\left[\cdots \mathrm{AV@R}_{\alpha_{t,s}}^{\mathfrak{F}_s}\left[\boldsymbol{A}_s\right]\right]\right] , \tag{UT}$$

$$\begin{aligned}\rho_{t,T}\big(\{\boldsymbol{A}_s\}_t^T\big) = \mathrm{AV@R}_{\alpha_{t,t}}^{\mathfrak{F}_t}\Big[\boldsymbol{A}_t +& \\ &\mathrm{AV@R}_{\alpha_{t,t+1}}^{\mathfrak{F}_{t+1}}\Big[\boldsymbol{A}_{t+1} + \cdots \mathrm{AV@R}_{\alpha_{t,T}}^{\mathfrak{F}_T}\left[\boldsymbol{A}_T\right]\cdots\Big]\Big] .\end{aligned} \tag{NTU}$$

Examples of this type are found in papers like [79, 100, 105, 107].

Markovian AV@R Let a policy $\pi \in \Pi$, a time $t \in [\![0,T]\!]$ and a state $x_t \in \mathbb{X}_t$ be fixed. With this and the control stochastic dynamical system (2.2), we define the Markov chain $\{\boldsymbol{X}_s^{x_t}\}_{s=t}^{T}$ produced by (2.3b)–(2.3c) starting from $\boldsymbol{X}_t = x_t$. We also define, for each $s \in [\![t,T]\!]$, the σ-algebra $\mathcal{X}_s^{x_t} = \sigma(\boldsymbol{X}_s^{x_t})$. With this, we define a conditional risk measure by

$$\begin{aligned}\rho_{t,T}^{x_t}\big(\{\boldsymbol{A}_s\}_t^T\big) =& \mathrm{AV@R}_{\alpha_{t,t}}^{\mathcal{X}_t^{x_t}}\Big[\boldsymbol{A}_t + \\ &\mathrm{AV@R}_{\alpha_{t,t+1}}^{\mathcal{X}_{t+1}^{x_t}}\Big[\boldsymbol{A}_{t+1} + \cdots \mathrm{AV@R}_{\alpha_{t,T}}^{\mathcal{X}_T^{x_t}}\left[\boldsymbol{A}_T\right]\cdots\Big]\Big] .\end{aligned} \tag{2.21}$$

Repeating the process, we obtain a family $\Big\{\{\varrho_{t,T}^{x_t}\}_{x_t \in \mathbb{X}_t}\Big\}_{t=0}^{T}$, such that $\{\varrho_{t,T}^{x_t}\}_{t=0}^{T}$ is a dynamic uncertainty criterion, for all sequence $\{x_t\}_{t=0}^{T}$ of states, where $x_t \in \mathbb{X}_t$, for all $t \in [\![0,T]\!]$.

2.2 Time-Consistency: Problem Statement

In §2.2.1, we lay out the basic material to formulate intertemporal optimization problems. In §2.2.2, we define "cousins" of dynamic risk measures, namely *dynamic uncertainty criteria*. In §2.2.3, we provide definitions of time-consistency, on the one hand, for dynamic risk measures and, in the other hand, for intertemporal stochastic optimization problems.

2.2.1 Ingredients for Intertemporal Optimization Problems

In §2.2.1, we recall the formalism of Control Theory, with dynamical system, state, control and costs. Mimicking the definition of adapted processes in Probability Theory, we introduce adapted uncertainty processes. In §2.2.1, we show how to produce an adapted uncertainty process of costs.

Dynamical System, State, Control and Costs

We define a *control T-stage dynamical system*, with $T \geq 2$, as follows. We consider

- a sequence $\{\mathbb{X}_t\}_0^T$ of sets of *states*;
- a sequence $\{\mathbb{U}_t\}_0^{T-1}$ of sets of *controls*;
- a sequence $\{\mathbb{W}_t\}_0^T$ of sets of *uncertainties*, and we define

$$\mathbb{W}_{[0:T]} = [\mathbb{W}_s]_0^T, \quad \text{the set of } \textit{scenarios}, \tag{2.22a}$$
$$\mathbb{W}_{[0:t]} = [\mathbb{W}_s]_0^t, \quad \text{the set of } \textit{head scenarios}, \forall t \in [\![0,T]\!] , \tag{2.22b}$$
$$\mathbb{W}_{[s:t]} = [\mathbb{W}_s]_t^T, \quad \text{the set of } \textit{tail scenarios}, \forall t \in [\![0,T]\!] ; \tag{2.22c}$$

- a sequence $\{f_t\}_0^{T-1}$ of *functions*, where $f_t : \mathbb{X}_t \times \mathbb{U}_t \times \mathbb{W}_t \to \mathbb{X}_{t+1}$, to play the role of *dynamics*;
- a sequence $\{U_t\}_0^{T-1}$ of T *multifunctions* $U_t : \mathbb{X}_t \rightrightarrows \mathbb{U}_t$, to play the role of *constraints*;
- a sequence $\{J_t\}_0^{T-1}$ of *instantaneous cost functions* $J_t : \mathbb{X}_t \times \mathbb{U}_t \times \mathbb{W}_t \mapsto \bar{\mathbb{R}}$, and a final cost function $J_T : \mathbb{X}_T \times \mathbb{W}_T \to \bar{\mathbb{R}}$. [3]

Mimicking the definition of adapted processes in Probability Theory, we introduce the following definition of *adapted uncertainty processes*, where the increasing sequence of head scenarios sets in (2.22b) corresponds to a filtration.

Definition 2.2. *We say that a sequence $A_{[0:T]} = \{A_s\}_0^T$ is an* adapted uncertainty process *if $A_s \in \mathcal{F}(\mathbb{W}_{[0:s]}; \bar{\mathbb{R}})$ (that is, $A_s : \mathbb{W}_{[0:s]} \to \bar{\mathbb{R}}$), for all $s \in [\![0,T]\!]$. In other words, $[\mathcal{F}(\mathbb{W}_{[0:s]}; \bar{\mathbb{R}})]_{s=0}^T$ is the set of adapted uncertainty processes.*

A *policy* $\pi = (\pi_t)_{t \in [\![0,T-1]\!]}$ is a sequence of functions $\pi_t : \mathbb{X}_t \to \mathbb{U}_t$, and we denote by Π the set of all policies. More generally, for all $t \in [\![0,T]\!]$, we call *(tail) policy* a sequence $\pi = (\pi_s)_{s \in [\![t,T-1]\!]}$ and we denote by Π_t the set of all such policies.

We restrict our search of optimal solutions to so-called *admissible* policies belonging to a subset $\Pi^{\mathrm{ad}} \subset \Pi$. An admissible policy $\pi \in \Pi^{\mathrm{ad}}$ always satisfies:

$$\forall t \in [\![0,T-1]\!], \quad \forall x \in \mathbb{X}_t, \qquad \pi_t(x) \in U_t(x) .$$

We can express in Π^{ad} other types of constraints, such as measurability or integrability ones when in a stochastic setting. Naturally, we set $\Pi_t^{ad} = \Pi_t \cap \Pi^{ad}$.

Definition 2.3. *For any time $t \in [\![0,T]\!]$, state $x \in \mathbb{X}_t$ and policy $\pi \in \Pi$, the* flow *$\{X_{t,s}^{x,\pi}\}_{s=t}^T$ is defined by the forward induction:*

$$\forall w \in \mathbb{W}_{[0:T]}, \quad \begin{cases} X_{t,t}^{x,\pi}(w) &= x , \\ X_{t,s+1}^{x,\pi}(w) &= f_s\big(X_{t,s}^{x,\pi}(w), \pi_s(X_{t,s}^{x,\pi}(w)), w_s\big), \quad \forall s \in [\![t,T]\!] . \end{cases} \tag{2.23}$$

The expression $X_{t,s}^{x,\pi}(w)$ is the state $x_s \in \mathbb{X}_s$ reached at time $s \in [\![0,T]\!]$, when starting at time $t \in [\![0,s]\!]$ from state $x \in \mathbb{X}_t$ and following the dynamics (2.2) with the policy $\pi \in \Pi$ along the scenario $w \in \mathbb{W}_{[0:T]}$.

Remark 2.4. *For $0 \leq t \leq s \leq T$, the flow $X_{t,s}^{x,\pi}$ is a function that maps the set $\mathbb{W}_{[0:T]}$ of scenarios into the state space $\mathbb{X}_s$:*

$$X_{t,s}^{x,\pi} : \mathbb{W}_{[0:T]} \to \mathbb{X}_s . \tag{2.24}$$

By (2.23),

3. For notational consistency with the J_t for $t = [\![0,T-1]\!]$, we will often write $J_T(x,u,w)$ to mean $J_T(x,w)$.

- *when* $t > 0$, *the expression* $X^{x,\pi}_{t,s}(w)$ *depends only on the inner part* $w_{[t:s-1]}$ *of the scenario* $w = w_{[0:T]}$, *hence depends neither on the head* $w_{[0:t-1]}$, *nor on the tail* $w_{[s:T]}$,
- *when* $t = 0$, *the expression* $X^{x,\pi}_{0,s}(w)$ *in* (2.23) *depends only on the head* $w_{[0:s-1]}$ *of the scenario* $w = w_{[0:T]}$, *hence does not depend on the tail* $w_{[s:T]}$.

This is why we often consider that the flow $X^{x,\pi}_{t,s}$ *is a function that maps the set* $\mathbb{W}_{[t:s-1]}$ *of scenarios into the state space* $\mathbb{X}_s$*:*

$$\forall s \in [\![1, T]\!], \quad \forall t \in [\![0, s-1]\!], \qquad X^{x,\pi}_{t,s} : \mathbb{W}_{[t:s-1]} \to \mathbb{X}_s \ . \tag{2.25}$$

A *state trajectory* is a realization of the flow $\{X^{x,\pi}_{0,s}(w)\}_{s=0}^{T}$ for a given scenario $w \in \mathbb{W}_{[0:T]}$. The *flow property*

$$\forall t, s, s', \quad t < s' < s, \quad \forall x \in \mathbb{X}_t, \qquad X^{x,\pi}_{t,s} \equiv X^{X^{x,\pi}_{t,s'},\pi}_{s',s} \tag{2.26}$$

expresses the fact that we can stop anywhere along a state trajectory and start again.

Producing Streams of Costs

Definition 2.5. *For a given policy* $\pi \in \Pi$*, and for all times* $t \in [\![0, T]\!]$ *and* $s \in [\![t, T]\!]$*, we define the uncertain costs evaluated along the state trajectories by:*

$$J^{x,\pi}_{t,s} : \quad w \in \mathbb{W}_{[0:T]} \longmapsto J_s\Big(X^{x,\pi}_{t,s}(w), \pi\big(X^{x,\pi}_{t,s}(w)\big), w_s\Big) \ . \tag{2.27}$$

Remark 2.6. *By Remark 2.4,*

- *when* $t > 0$, *the expression* $J^{x,\pi}_{t,s}(w)$ *in* (2.27) *depends only on the inner part* $w_{[t:s]}$ *of the scenario* $w = w_{[0:T]}$, *hence depends neither on the head* $w_{[0:t-1]}$, *nor on the tail* $w_{[s+1:T]}$,
- *when* $t = 0$, *the expression* $J^{x,\pi}_{0,s}(w)$ *in* (2.27) *depends only on the head* $w_{[0:s]}$ *of the scenario* $w = w_{[0:T]}$, *hence does not depend on the tail* $w_{[s+1:T]}$.

This is why we often consider that $J^{x,\pi}_{t,s}$ *is a function that maps the set* $\mathbb{W}_{[t:s]}$ *of scenarios into* $\bar{\mathbb{R}}$*:*

$$\forall s \in [\![0, T]\!], \quad \forall t \in [\![0, s]\!], \qquad J^{x,\pi}_{t,s} : \quad \mathbb{W}_{[t:s]} \to \bar{\mathbb{R}} \ . \tag{2.28}$$

As a consequence, the stream $\{J^{x,\pi}_{0,s}\}_{s=0}^{T}$ *of costs is an adapted uncertainty process.*

By (2.27) and (2.23), we have, for all $t \in [\![0, T]\!]$ and $s \in [\![t+1, T]\!]$,

$$\forall w_{[t:T]} \in \mathbb{W}_{[t:T]}, \quad \left\{ \begin{aligned} J^{x,\pi}_{t,t}(w_t) \quad &= J_t\big(x, \pi_t(x), w_t\big) \ , \\ J^{x,\pi}_{t,s}(w_t, \{w_r\}_{t+1}^{T}) \quad &= J^{f_t(x,\pi_t(x),w_t),\pi}_{t+1,s}(\{w_r\}_{t+1}^{T}) \ . \end{aligned} \right. \tag{2.29}$$

2.2.2 Dynamic Uncertainty Criteria and Dynamic Risk Measures

Now, we stand with a stream $\{J^{x,\pi}_{0,s}\}_{s=0}^{T}$ of costs, which is an adapted uncertainty process by Remark 2.4. To craft a criterion to optimize, we need to aggregate such a stream into a scalar. For this purpose, we define *dynamic uncertainty criterion* in §2.2.2, and relate them to dynamic risk measures in §2.2.2.

Dynamic Uncertainty Criterion

Inspired by the definitions of risk measures and dynamic risk measures in Mathematical Finance, and motivated by intertemporal optimization, we introduce the following definitions of *dynamic uncertainty criterion*, and *Markov dynamic uncertainty criterion*. Examples have been given in §2.1.2.

Definition 2.7. *A* dynamic uncertainty criterion *is a sequence* $\{\varrho_{t,T}\}_{t=0}^{T}$*, such that, for all* $t \in [\![0,T]\!]$*,*

- $\varrho_{t,T}$ *is a mapping*

$$\varrho_{t,T} : \big[\mathcal{F}(\mathbb{W}_{[0:s]};\bar{\mathbb{R}})\big]_{s=t}^{T} \to \mathcal{F}(\mathbb{W}_{[0:t]};\bar{\mathbb{R}}) , \tag{2.30a}$$

- *the restriction of* $\varrho_{t,T}$ *to the domain*[4] $\big[\mathcal{F}(\mathbb{W}_{[t:s]};\bar{\mathbb{R}})\big]_{s=t}^{T}$ *yields constant functions, that is,*

$$\varrho_{t,T} : \big[\mathcal{F}(\mathbb{W}_{[t:s]};\bar{\mathbb{R}})\big]_{s=t}^{T} \to \bar{\mathbb{R}} , \tag{2.30b}$$

A Markov dynamic uncertainty criterion *is a family* $\Big\{\big\{\varrho_{t,T}^{x_t}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T}$*, such that* $\big\{\varrho_{t,T}^{x_t}\big\}_{t=0}^{T}$ *is a dynamic uncertainty criterion, for all sequence* $\big\{x_t\big\}_{t=0}^{T}$ *of states, where* $x_t \in \mathbb{X}_t$*, for all* $t \in [\![0,T]\!]$*.*

We relate dynamic uncertainty criteria and optimization problems as follows.

Definition 2.8. *Given a* Markov dynamic uncertainty criterion $\Big\{\big\{\varrho_{t,T}^{x_t}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T}$*, we define a* Markov optimization problem *as the following sequence of families of optimization problems, indexed by* $t \in [\![0,T]\!]$*, and* $x \in \mathbb{X}_t$*:*

$$(\mathfrak{P}_t)(x) \qquad \min_{\pi\in\Pi^{\mathrm{ad}}} \varrho_{t,T}^{x}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big) . \tag{2.31}$$

Each Problem (2.31) is indeed well defined by (2.30b), because $\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T} \in \big[\mathcal{F}(\mathbb{W}_{[t:s]};\bar{\mathbb{R}})\big]_{s=t}^{T}$ by (2.28).

Dynamic Risk Measures in a Nutshell

We establish a parallel between uncertainty criteria and risk measures. For this purpose, when needed, we implicitely suppose that each uncertainty set $\mathbb{W}_t$ is endowed with a σ-algebra $\mathcal{W}_t$, so that the set $\mathbb{W}_{[0:T]}$ of scenarios is naturally equipped with the filtration

$$\forall t \in [\![0,T]\!], \qquad \mathfrak{F}_t = \mathcal{W}_0 \otimes \cdots \otimes \mathcal{W}_t \otimes \{\emptyset, \mathbb{W}_{t+1}\} \otimes \cdots \otimes \{\emptyset, \mathbb{W}_T\} . \tag{2.32}$$

Then, we make the correspondence between (see also the correspondence Table 2.1)

- the measurable space $(\mathbb{W}_{[0:T]}, \mathfrak{F}_T)$ and the measurable space $(\Omega, \mathfrak{F})$ in §2.2.2,
- the set $\mathcal{F}\big(\mathbb{W}_{[0:t]};\bar{\mathbb{R}}\big)$ of functions and a set $\mathcal{L}_t$ of random variables that are $\mathfrak{F}_t$-measurable in §2.2.2,
- the set $\big[\mathcal{F}(\mathbb{W}_{[s:T]};\bar{\mathbb{R}})\big]_{s=t}^{T}$ and a set $\mathcal{L}_{t,T}$ of adapted processes, as in (2.35) in §2.2.2.

Notice that, when the σ-algebra $\mathcal{W}_t$ is the complete σ-algebra made of all subsets of $\mathbb{W}_t$, $\mathcal{F}\big(\mathbb{W}_{[0:t]};\bar{\mathbb{R}}\big)$ is exactly the space of random variables that are $\mathfrak{F}_t$-measurable.

We follow the seminal work [6], as well as [103, 104], for recalls about risk measures.

4. Where $\mathcal{F}(\mathbb{W}_{[t:s]};\bar{\mathbb{R}})$ is naturally identified as a subset of $\mathcal{F}(\mathbb{W}_{[0:s]};\bar{\mathbb{R}})$.

Static Risk Measures Let $(\Omega, \mathfrak{F})$ be a measurable space. Let $\mathcal{L}$ be a vector space of measurable functions taking values in $\mathbb{R}$ (for example, $\mathcal{L} = \mathrm{L}^p(\Omega, \mathfrak{F}, \mathbb{P}; \mathbb{R})$). We endow the space $\mathcal{L}$ with the following partial order:

$$\forall \boldsymbol{X}, \boldsymbol{Y} \in \mathcal{L}, \qquad \boldsymbol{X} \leq \boldsymbol{Y} \iff \forall \omega \in \Omega, \quad \boldsymbol{X}(\omega) \leq \boldsymbol{Y}(\omega) .$$

Definition 2.9. *A* risk measure *(with domain $\mathcal{L}$) is a mapping $\rho : \mathcal{L} \to \mathbb{R}$.*

A convex risk measure *is a mapping $\rho : \mathcal{L} \to \mathbb{R}$ displaying the following properties:*
- Convexity: $\forall \boldsymbol{X}, \boldsymbol{Y} \in \mathcal{L}, \quad \forall t \in [0,1], \qquad \rho\big(t\boldsymbol{X} + (1-t)\boldsymbol{Y}\big) \leq t\rho\big(\boldsymbol{X}\big) + (1-t)\rho\big(\boldsymbol{Y}\big)$,
- Monotonicity: *if $\boldsymbol{Y} \geq \boldsymbol{X}$, then $\rho\big(\boldsymbol{Y}\big) \geq \rho\big(\boldsymbol{X}\big)$* ,
- Translation equivariance: $\forall c \in \mathbb{R}, \quad \forall \boldsymbol{X} \in \mathcal{L}, \qquad \rho(c + \boldsymbol{X}) = c + \rho(\boldsymbol{X})$.

A coherent risk measure *is a convex risk measure $\rho : \mathcal{L} \to \mathbb{R}$ with the following additional property:*
- Positive homogeneity: $\forall t \geq 0, \quad \forall \boldsymbol{X} \in \mathcal{L}, \qquad \rho(t\boldsymbol{X}) = t\rho(\boldsymbol{X})$.

Let $\mathcal{P}$ be a set of probabilities on $(\Omega, \mathfrak{F})$ and let Υ be a function mapping the space of probabilities on $(\Omega, \mathfrak{F})$ onto $\mathbb{R}$. The functional defined by

$$\rho(\boldsymbol{X}) = \sup_{\mathbb{P} \in \mathcal{P}} \big\{ \mathbb{E}_{\mathbb{P}}\big[\boldsymbol{X}\big] - \Upsilon(\mathbb{P}) \big\} \tag{2.33}$$

is a convex risk measure on a proper domain $\mathcal{L}$ (for instance, the bounded functions over Ω). The expression

$$\rho(\boldsymbol{X}) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\big[\boldsymbol{X}\big] \tag{2.34}$$

defines a coherent risk measure.

Under proper technical assumptions, it can be shown that any convex or coherent risk measure can be represented by the above expressions.

Conditional Risk Mappings We present the *conditional risk mappings* as defined in [103], extending the work of [85].

Let $(\Omega, \mathfrak{F})$ be a measurable space, $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \mathfrak{F}$ be two σ-algebras, and $\mathcal{L}_1 \subset \mathcal{L}_2$ be two vector spaces of functions $\Omega \to \mathbb{R}$ that are measurable with respect to $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively.

Definition 2.10. *A* conditional risk mapping *is a mapping $\rho : \mathcal{L}_2 \to \mathcal{L}_1$.*

A convex conditional risk mapping $\rho : \mathcal{L}_2 \to \mathcal{L}_1$ *has the following properties:*
- Convexity: $\forall \boldsymbol{X}, \boldsymbol{Y} \in \mathcal{L}_2, \quad \forall t \in [0,1], \qquad \rho\big(t\boldsymbol{X} + (1-t)\boldsymbol{Y}\big) \leq t\rho\big(\boldsymbol{X}\big) + (1-t)\rho\big(\boldsymbol{Y}\big)$,
- Monotonicity: *if $\boldsymbol{Y} \geq \boldsymbol{X}$, then $\rho\big(\boldsymbol{Y}\big) \geq \rho\big(\boldsymbol{X}\big)$* ,
- Translation equivariance: $\forall c \in \mathcal{L}_1, \quad \forall \boldsymbol{X} \in \mathcal{L}_2, \qquad \rho(c + \boldsymbol{X}) = c + \rho(\boldsymbol{X})$.

Conditional and Dynamic Risk Measures We follow [105, Section 3]. Let $(\Omega, \mathfrak{F})$ be a measurable space, with a filtration $\mathfrak{F}_1 \subset \cdots \subset \mathfrak{F}_T \subset \mathfrak{F}$, and $\mathcal{L}_1 \subset \cdots \subset \mathcal{L}_T$ be vector spaces of functions $\Omega \to \mathbb{R}$ that are measurable with respect to $\mathfrak{F}_1, \ldots, \mathfrak{F}_T$, respectively. We set

$$\forall t \in [\![0, T]\!], \qquad \mathcal{L}_{t,T} = \mathcal{L}_t \times \cdots \times \mathcal{L}_T . \tag{2.35}$$

An element $\{\boldsymbol{A}_s\}_0^T$ of $\mathcal{L}_{t,T}$ is an *adapted process* since every $\boldsymbol{A}_s \in \mathcal{L}_s$ is $\mathfrak{F}_s$-measurable. Conditional and dynamic risk measures have adapted processes as arguments, to the difference of risk measures that take random variables as arguments.

Definition 2.11. *Let $t \in [\![0, T]\!]$. A* one-step conditional risk mapping *is a conditional risk mapping $\rho_t : \mathcal{L}_{t+1} \to \mathcal{L}_t$. A* conditional risk measure *is a mapping $\rho_{t,T} : \mathcal{L}_{t,T} \mapsto \mathcal{L}_t$.*

A dynamic risk measure *is a sequence $\big\{\rho_{t,T}\big\}_{t=0}^T$ of conditional risk measures.*

Dynamic uncertainty criteria $\{\varrho_{t,T}\}_{t=0}^{T}$, as introduced in Definition 2.7 correspond to *dynamic risk measures.*

Remark 2.12. *A conditional risk measure* $\rho_{t,T} : \mathcal{L}_{t,T} \mapsto \mathcal{L}_t$ *is said to be* monotonous [5] *if, for all* $\{\underline{\boldsymbol{A}}_s\}_{s=t}^{T}$ *and* $\{\overline{\boldsymbol{A}}_s\}_{s=t}^{T}$ *in* $\mathcal{L}_{t,T}$*, we have*

$$\forall s \in [\![t,T]\!], \qquad \underline{\boldsymbol{A}}_s \leq \overline{\boldsymbol{A}}_s \Longrightarrow \rho_{t,T}\big(\{\underline{\boldsymbol{A}}_s\}_{s=t}^{T}\big) \leq \rho_{t,T}\big(\{\overline{\boldsymbol{A}}_s\}_{s=t}^{T}\big) . \tag{2.36}$$

Markov Risk Measures In [105], *Markov risk measures* are defined with respect to a given controlled Markov process. We adapt this definition to the setting developed in the Introduction, and we consider the control stochastic dynamical system (2.3b)

$$\boldsymbol{X}_{t+1} = f_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t) ,$$

where $\{\boldsymbol{W}_t\}_0^T$ is a sequence of independent random variables. Then, for all policy π, when $\boldsymbol{U}_t = \pi_t(\boldsymbol{X}_t)$ we obtain a Markov process $\{\boldsymbol{X}_t\}_{t\in[\![0,T]\!]}$, where $\boldsymbol{X}_t = X_{0,t}^{x_0,\pi}(\{\boldsymbol{W}_s\}_0^{t-1})$ is given by the flow (2.23).

Let $\{\mathfrak{F}_t\}_{t=0}^{T}$ be the filtration defined by $\mathfrak{F}_t = \sigma(\{\boldsymbol{W}_s\}_0^t)$. For any $t \in [\![0,T]\!]$, let $\mathcal{V}_t$ be a set of functions mapping $\mathbb{X}_t$ into $\mathbb{R}$ such that we have $v\big(\boldsymbol{X}_{0,t}^{x_0,\pi}\big) \in \mathcal{L}_t$, for all policy $\pi \in \Pi^{\mathrm{ad}}$.

Definition 2.13. *A one-step conditional risk measure* $\rho_{t-1} : \mathcal{L}_t \to \mathcal{L}_{t-1}$ *is a* Markov risk measure *with respect to the control stochastic dynamical system* (2.3b) *if there exists a function* $\Psi_t : \mathcal{V}_{t+1} \times \mathbb{X}_t \times \mathbb{U}_t \to \mathbb{R}$*, such that, for any policy* $\pi \in \Pi^{\mathrm{ad}}$*, and any function* $v \in \mathcal{V}_{t+1}$*, we have*

$$\begin{aligned} &\rho_{t-1}\Big(\{\boldsymbol{W}_s\}_0^t \mapsto v\Big(\boldsymbol{X}_{0,t+1}^{x_0,\pi}(\{\boldsymbol{W}_s\}_0^t)\Big)\Big) \\ =&\Psi_t\Big(v, \boldsymbol{X}_{0,t}^{x_0,\pi}(\{\boldsymbol{W}_s\}_0^{t-1}), \pi_t\Big(\boldsymbol{X}_{0,t}^{x_0,\pi}(\{\boldsymbol{W}_s\}_0^{t-1})\Big)\Big) . \end{aligned} \tag{2.37}$$

A Markov risk measure is said to be coherent (resp. convex) if, for any state $x \in \mathbb{X}_t$*, any control* $u \in \mathbb{U}_t$*, the function*

$$v \mapsto \Psi_t\big(v, x, u\big) , \tag{2.38}$$

is a coherent (resp convex) risk measure on $\mathcal{V}_{t+1}$ *(equipped with a proper* σ*-algebra).*

Dynamic Markov uncertainty criteria $\{\varrho_{t,T}\}_{t=0}^{T}$, as introduced in Definition 2.7 correspond to *Markov risk measures.*

Correspondence Table

Time-Consistency for Dynamic Risk Measures The literature on risk measures has introduced a notion of *time-consistency for dynamic risk measures*, that we recall here (see [7, 28, 85]).

Definition 2.14. *A dynamic risk measure* $\{\rho_{t,T}\}_{t=0}^{T}$*, where* $\rho_{t,T} : \mathcal{L}_{t,T} \mapsto \mathcal{L}_t$*, is said to be* time-consistent *if, for any couples of times* $0 \leq \underline{t} < \overline{t} \leq T$*, the following property holds true. If two adapted stochastic processes* $\{\underline{\boldsymbol{A}}_s\}_0^T$ *and* $\{\overline{\boldsymbol{A}}_s\}_0^T$ *in* $\mathcal{L}_{0,T}$ *satisfy*

$$\underline{\boldsymbol{A}}_s = \overline{\boldsymbol{A}}_s, \qquad \forall s \in [\![\underline{t}, \overline{t}-1]\!] , \tag{2.39a}$$

$$\rho_{\overline{t},T}\big(\{\underline{\boldsymbol{A}}_s\}_{\overline{t}}^{T}\big) \leq \rho_{\overline{t},T}\big(\{\overline{\boldsymbol{A}}_s\}_{\overline{t}}^{T}\big) , \tag{2.39b}$$

5. In [105, Section 3], a conditional risk measure is necessarily monotonous, by definition.

Risk Measures		Uncertainty Criteria	
measurable space	$(\Omega, \mathfrak{F})$	$(\mathbb{W}_{[0:T]}, \mathfrak{F}_T)$	measurable space
$\mathfrak{F}_t$-measurable		$\mathcal{F}\big(\mathbb{W}_{[0:t]}; \mathbb{R}\big)$	
adapted processes	$\mathcal{L}_{0,T}$	$\big[\mathcal{F}(\mathbb{W}_{[0:s]}; \bar{\mathbb{R}})\big]_{s=0}^{T}$	adapted uncertainty processes
dynamic risk measure	$\{\rho_{t,T}\}_{t=0}^{T}$	$\{\varrho_{t,T}\}_{t=0}^{T}$	dynamic uncertainty criteria
Markov dynamic risk measure	$\Big\{\big\{\rho_{t,T}^{x_t}\big\}_{x_t \in \mathbb{X}_t}\Big\}_{t=0}^{T}$	$\Big\{\big\{\varrho_{t,T}^{x_t}\big\}_{x_t \in \mathbb{X}_t}\Big\}_{t=0}^{T}$	Markov dynamic uncertainty criterion

Table 2.1: Correspondence Table

then we have:

$$\rho_{\underline{t},T}\big(\{\underline{\boldsymbol{A}}_s\}_{\underline{t}}^{T}\big) \leq \rho_{\underline{t},T}\big(\{\overline{\boldsymbol{A}}_s\}_{\underline{t}}^{T}\big) . \tag{2.39c}$$

Remark 2.15. *In [105], the equality (2.39a) is replaced by the inequality*

$$\forall s \in [\![\underline{t}, \overline{t}]\!], \qquad \underline{\boldsymbol{A}}_s \leq \overline{\boldsymbol{A}}_s . \tag{2.39d}$$

Depending whether we choose (2.39a) or (2.39d) as assumption to define a time-consistent dynamic risk measure, we have to adapt or not an assumption in Theorem 2.31 (see Remark 2.32).

2.2.3 Definitions of Time-Consistency

With the formalism of §2.2.2, we give a definition of time-consistency for Markov optimization problems in §2.2.3, and for Markov dynamic uncertainty criteria in §2.2.3.

Time-Consistency for Markov Optimization Problems

With the formalism of §2.2.2, we here give a definition of time-consistency for Markov optimization problems. We refer the reader to Definition 2.8 for the terminology.

Consider the Markov optimization problem $\Big\{\big\{(\mathfrak{P}_t)(x)\big\}_{x \in \mathbb{X}_t}\Big\}_{t=0}^{T}$ defined in (2.31). For the clarity of exposition, suppose for a moment that any optimization Problem $(\mathfrak{P}_t)(x)$ has a unique solution, that we denote $\pi^{t,x} = \{{\pi_s}^{t,x}\}_{s=t}^{T-1} \in \Pi_t^{\mathrm{ad}}$. Consider $0 \leq \underline{t} < \overline{t} \leq T$. Suppose that, starting from the state $\underline{x}$ at time $\underline{t}$, the flow (2.23) drives you to

$$\overline{x} = X_{\underline{t},\overline{t}}^{\underline{x},\pi}(w), \quad \pi = \pi^{\underline{t},\underline{x}} \tag{2.40}$$

at time $\overline{t}$, along the scenario $w \in \mathbb{W}_{[0:T]}$ and adopting the optimal policy $\pi^{\underline{t},\underline{x}} \in \Pi_{\underline{t}}^{\mathrm{ad}}$. Arrived at $\overline{x}$, you solve $(\mathfrak{P}_{\overline{t}})(\overline{x})$ and get the optimal policy $\pi^{\overline{t},\overline{x}} = \{{\pi_s}^{\overline{t},\overline{x}}\}_{s=\overline{t}}^{T-1} \in \Pi_{\overline{t}}^{\mathrm{ad}}$. Time-consistency holds true when

$$\forall s \geq \overline{t}, \qquad {\pi_s}^{\overline{t},\overline{x}} = {\pi_s}^{\underline{t},\underline{x}} , \tag{2.41}$$

that is, when the "new" optimal policy, obtained by solving $(\mathfrak{P}_{\overline{t}})(\overline{x})$, coincides, after time $\overline{t}$, with the "old" optimal policy, obtained by solving $(\mathfrak{P}_{\underline{t}})(\underline{x})$. In other words, you "stick to your plans" (here, a plan is a policy) and do not reconsider your policy whenever you stop along an optimal path and optimize ahead from this stop point.

To account for non-uniqueness of optimal policies, we propose the following formal definition.

Definition 2.16. *For any policy $\pi \in \Pi$, suppose given a Markov dynamic uncertainty criterion $\Big\{\big\{\varrho_{t,T}^{x_t,\pi}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T}$. We say that the Markov optimization problem*

$$(\mathfrak{P}_t)(x) \qquad \min_{\pi\in\Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{x,\pi}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big), \quad \forall t \in [\![0,T]\!], \quad \forall x \in \mathbb{X}_t\ . \tag{2.42}$$

is time-consistent *if, for any couple of times $\underline{t} \leq \overline{t}$ in $[\![0,T]\!]$ and any state $\underline{x} \in \mathbb{X}_{\underline{t}}$, the following property holds: there exists a policy $\pi^\sharp = \{\pi_s^\sharp\}_{s=\underline{t}}^{T-1} \in \Pi_{\underline{t}}^{\mathrm{ad}}$ such that*

- *$\{\pi_s^\sharp\}_{s=\underline{t}}^{T-1}$ is optimal for Problem $\mathfrak{P}_{\underline{t}}(\underline{x})$;*
- *the tail policy $\{\pi_s^\sharp\}_{s=\overline{t}}^{T-1}$ is optimal for Problem $\mathfrak{P}_{\overline{t}}(\overline{x})$, where $\overline{x} \in \mathbb{X}_{\overline{t}}$ is any state achieved by the flow $X_{\underline{t},\overline{t}}^{\underline{x},\pi^\sharp}$ in (2.23).*

We stress that the above definition of time-consistency of a sequence of families of optimization problems is contingent on the state x and on the dynamics $\big\{f_t\big\}_0^{T-1}$ by the flow (2.23). In particular, we assume that, at each time step, the control is taken only in function of the state: this defines the class of solutions as policies that are feedbacks of the state x.

Time-Consistency for Markov Dynamic Uncertainty Criteria

We provide a definition of time-consistency for Markov dynamic uncertainty criteria, inspired by the definitions of time-consistency for, on the one hand, dynamic risk measures (recalled in §2.2.2) and, on the other hand, Markov optimization problems. We refer the reader to Definition 2.7 for the terminology.

Definition 2.17. *The Markov dynamic uncertainty criterion $\{\{\varrho_{t,T}^{x_t}\}_{x_t\in\mathbb{X}_t}\}_{t=0}^{T}$ is said to be* time-consistent *if, for any couple of times $0 \leq \underline{t} < \overline{t} \leq T$, the following property holds true.*

If two adapted uncertainty processes $\{\underline{A}_s\}_0^T$ and $\{\overline{A}_s\}_0^T$, satisfy

$$\underline{A}_s = \overline{A}_s, \qquad \forall s \in [\![\underline{t},\overline{t}]\!]\ , \tag{2.43a}$$

$$\rho_{\overline{t},T}^{\overline{x}}\big(\{\underline{A}_s\}_{\overline{t}}^T\big) \leq \rho_{\overline{t},T}^{\overline{x}}\big(\{\overline{A}_s\}_{\overline{t}}^T\big), \qquad \forall \overline{x} \in \mathbb{X}_{\overline{t}}\ , \tag{2.43b}$$

then we have:

$$\rho_{\underline{t},T}^{\underline{x}}\big(\{\underline{A}_s\}_{\underline{t}}^T\big) \leq \rho_{\underline{t},T}^{\underline{x}}\big(\{\overline{A}_s\}_{\underline{t}}^T\big), \qquad \forall \underline{x} \in \mathbb{X}_{\underline{t}}\ . \tag{2.43c}$$

This Definition 2.17 of time-consistency is quite different from Definition 2.16. Indeed, if the latter looks after consistency between solutions to intertemporal optimization problems, the former is a monotonicity property. Several authors establish connections between these two definitions [23, 56, 78, 105] for case specific problems. In the following §2.3, we provide what we think is one of the most systematic connections between time-consistency for Markov dynamic uncertainty criteria and time-consistency for intertemporal optimization problems.

2.3 Proving Joint Time-Consistency

In §2.3.1, we introduce the notions of time and uncertainty-aggregators, define their composition, and outline the general four ways to craft a dynamic uncertainty criterion from one-step aggregators. In §2.3.2, we present two ways to craft a nested dynamic uncertainty criterion; for each of them, we provide sufficient monotonicity assumptions on one-step aggregators that ensure time-consistency and the existence of a DPE. In §2.3.3, we introduce two commutation properties, that will be the key ingredients for time-consistency and for the existence of a DPE in non-nested cases. In §2.3.4, we present two ways to craft a non-nested dynamic uncertainty criterion; for each of them, we provide sufficient monotonicity and commutation assumptions on one-step aggregators that ensure time-consistency and the existence of a DPE.

2.3.1 Aggregators and their Composition

We introduce the notions of time and uncertainty-aggregators, define their composition, and outline the general four ways to craft a dynamic uncertainty criterion from one-step aggregators.

One-Step Time-Aggregators and their Composition

Time preferences are reflected in how streams of costs — elements of $\bar{\mathbb{R}}^{T+1}$, like $\{J_{0,t}^{x,\pi}(w)\}_{t=0}^{T}$ introduced in Definition 2.5 — are aggregated with respect to time thanks to a function $\Phi : \bar{\mathbb{R}}^{T+1} \to \bar{\mathbb{R}}$, called *multiple-step time-aggregator*. Commonly, multiple-step time-aggregators are built progressively backward. In §2.1.1, the multiple-step time-aggregator is the time-separable and additive $\Phi\big\{c_s\big\}_{s=0}^{T} = \sum_{s=0}^{T} c_s$, obtained as the initial value of the backward induction $\sum_{s=t}^{T} c_s = (\sum_{s=t+1}^{T} c_s) + c_t$; the time-separable and multiplicative aggregator $\Phi\big\{c_s\big\}_{s=0}^{T} = \prod_{s=0}^{T} c_s$ is the initial value of the backward induction $\prod_{s=t}^{T} c_s = (\prod_{s=t+1}^{T} c_s)c_t$. A multiple-step time-aggregator aggregates the $T+1$ costs $\{J_{0,t}^{x,\pi}(w)\}_{t=0}^{T}$, whereas a one-step time-aggregator aggregates two costs, the current one and the "cost-to-go" (as in [117]).

Definition 2.18. *A* multiple-step time-aggregator *is a function mapping $\bar{\mathbb{R}}^k$ into $\bar{\mathbb{R}}$, where $k \geq 2$. When $k = 2$, we call* one-step time-aggregator *a function mapping $\bar{\mathbb{R}}^2$ into $\bar{\mathbb{R}}$.*

A one-step time-aggregator is said to be non-decreasing *if it is non-decreasing in its second variable.*

We define the composition of time-aggregators as follows.

Definition 2.19. *Let $\Phi^1 : \bar{\mathbb{R}}^2 \to \bar{\mathbb{R}}$ be a one-step time-aggregator and $\Phi^k : \bar{\mathbb{R}}^k \to \bar{\mathbb{R}}$ be a multiple-step time-aggregator. We define $\Phi^1 \odot \Phi^k : \bar{\mathbb{R}}^{k+1} \to \bar{\mathbb{R}}$ by*

$$\Big(\Phi^1 \odot \Phi^k\Big)\big\{c_1, c_2, \dots, c_{k+1}\big\} = \Phi^1\Big\{c_1, \Phi^k\big\{c_2, \dots, c_{k+1}\big\}\Big\} . \tag{2.44}$$

Quite naturaly, we define the composition of sequences of one-step time-aggregators as follows.

Definition 2.20. *Consider a sequence $\big\{\Phi_t\big\}_{t=0}^{T-1}$ of one-step time-aggregators $\Phi_t : \bar{\mathbb{R}} \times \bar{\mathbb{R}} \to \bar{\mathbb{R}}$, for $t \in [\![0, T-1]\!]$. For all $t \in [\![0, T-1]\!]$, we define the composition $\bigodot_{s=t}^{T-1} \Phi_s$ as the multiple-step time-aggregator from $\bar{\mathbb{R}}^{T+1-t}$ towards $\bar{\mathbb{R}}$, inductively given by*

$$\bigodot_{t=T-1}^{T-1} \Phi_t = \Phi_{T-1} \ \textit{and} \ \Big(\bigodot_{s=t}^{T-1} \Phi_s\Big) = \Phi_t \odot \Big(\bigodot_{s=t+1}^{T-1} \Phi_s\Big) . \tag{2.45a}$$

That is, for all sequence $c_{[t:T]}$ where $c_s \in \bar{\mathbb{R}}$, we have:

$$\Big(\bigodot_{s=t}^{T-1} \Phi_s\Big)\big(c_{[t:T]}\big) = \Phi_t\Big\{c_t, \Big(\bigodot_{s=t+1}^{T-1} \Phi_s\Big)\big(c_{[t+1:T]}\big)\Big\} . \tag{2.45b}$$

Example 2.21. *Consider the sequence $\{\Phi_t\}_{t=0}^{T-1}$ of one-step time-aggregators given by*

$$\Phi_t\{c_t, c_{t+1}\} = \alpha_t(c_t) + \beta_t(c_t)c_{t+1}, \quad \forall t \in [\![0, T-1]\!] , \tag{2.46}$$

where $(\alpha_t)_{t\in[\![0,T-1]\!]}$ and $(\beta_t)_{t\in[\![0,T-1]\!]}$ are sequences of functions, each mapping $\bar{\mathbb{R}}$ into $\mathbb{R}$. We have

$$\Big(\bigodot_{s=t}^{T-1} \Phi_s\Big)\{c_s\}_t^T = \sum_{s=t}^{T} \Big(\alpha_s(c_s) \prod_{r=t}^{s-1} \beta_r(c_r)\Big), \quad \forall t \in [\![0, T-1]\!] , \tag{2.47}$$

with the convention that $\alpha_T(c_T) = c_T$.

Example 2.22. *Consider the one-step aggregators*

$$\Phi\{c_1, c_2\} = c_1 + c_2, \quad \Psi\{c_1, c_2\} = c_1 c_2 .$$

The first one Φ corresponds to the sum, as in (2.3); the second one Ψ corresponds to the product, as in (2.11). As an illustration, we form four compositions (multiple-step time-aggregators):

$$\begin{aligned}
\Phi \odot \Phi\{c_1, c_2, c_3\} &= \Phi\big\{c_1, \Phi\{c_2, c_3\}\big\} = c_1 + c_2 + c_3 ,\\
\Psi \odot \Psi\{c_1, c_2, c_3\} &= \Psi\big\{c_1, \Psi\{c_2, c_3\}\big\} = c_1 c_2 c_3 ,\\
\Phi \odot \Psi\{c_1, c_2, c_3\} &= \Phi\big\{c_1, \Psi\{c_2, c_3\}\big\} = c_1 + c_2 c_3 ,\\
\Psi \odot \Phi\{c_1, c_2, c_3\} &= \Psi\big\{c_1, \Phi\{c_2, c_3\}\big\} = c_1(c_2 + c_3).
\end{aligned}$$

We extend the composition $\Big(\bigodot_{s=t}^{T-1} \Phi_s\Big) : \bar{\mathbb{R}}^{T+1-t} \to \bar{\mathbb{R}}$ into a mapping (2.48) as follows.

Definition 2.23. *Consider a sequence $\{\Phi_t\}_{t=0}^{T-1}$ of one-step time-aggregators, for $t \in [\![0, T-1]\!]$. For $t \in [\![0, T-1]\!]$, we define the composition*[6] *$\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle$ as a mapping*

$$\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle : \Big(\mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}})\Big)^{T-t+1} \to \mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}}) \tag{2.48}$$

by, for any $\{A\}_t^T \in \Big(\mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}})\Big)^{T-t+1}$,

$$\bigg(\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big(\{A\}_t^T\Big)\bigg)(w) = \Big(\bigodot_{s=t}^{T-1} \Phi_s\Big)\big(\{A_t(w)\}_t^T\big), \quad \forall w \in \mathbb{W}_{[0:T]} . \tag{2.49}$$

In other words, we simply plug the values $\{A_t(w)\}_t^T$ into $\Big(\bigodot_{s=t}^{T-1} \Phi_s\Big)$.

One-Step Uncertainty-Aggregators and their Composition

As with time, risk or uncertainty preferences are materialized by a function $\mathbb{G} : \mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}}) \to \bar{\mathbb{R}}$, called *multiple-step uncertainty-aggregator*. A multiple-step aggregator is usually defined on a subset $\mathbb{F}$ of $\mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}})$ (for example the measurable and integrable functions), and then extended to $\mathcal{F}(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}})$ by setting $\mathbb{G}[A] = +\infty$ for any function $A \notin \mathbb{F}$. Indeed, as we are interested in minimizing $\mathbb{G}$, being not defined or equal to $+\infty$ amount to the same result.

In the first part of §2.1.1, the multiple-step uncertainty-aggregator is the extended expectation with respect to the probability $\mathbb{P}$; still denoted by $\mathbb{E}_{\mathbb{P}}$, it is defined as the usual expectation if the operand is measurable and integrable, and as $+\infty$ otherwise. In the second part of §2.1.1, the multiple-step uncertainty-aggregator is the fear operator, namely the supremum $\sup_{w\in\mathbb{W}_{[0:T]}}$ over scenarios in $\mathbb{W}_{[0:T]}$.

6. We will consistently use the symbol $\langle \quad \rangle$ to denote a mapping with image a set of functions.

Definition 2.24. *Let $t \in [\![0,T]\!]$ and $s \in [\![t,T]\!]$. A* $[t\!:\!s]$-multiple-step uncertainty-aggregator *is a mapping*[7] $\mathbb{G}^{[t:s]}$ *from* $\mathcal{F}(\mathbb{W}_{[t:s]};\bar{\mathbb{R}})$ *into* $\bar{\mathbb{R}}$. *When $t = s$, we call* $\mathbb{G}^{[t:t]}$ *a* t-one-step uncertainty-aggregator.

A $[t\!:\!s]$-multiple-step uncertainty-aggregator is said to be non-decreasing *if, for any functions*[8] $\underline{D}_t$ *and* $\overline{D}_t$ *in* $\mathcal{F}(\mathbb{W}_{[t:s]};\bar{\mathbb{R}})$, *we have*

$$\big(\forall w_{[t:s]} \in \mathbb{W}_{[t:s]}, \qquad \underline{D}_t\big(w_{[t:s]}\big) \leq \overline{D}_t\big(w_{[t:s]}\big)\big) \quad \Longrightarrow \quad \mathbb{G}^{[t:s]}\big[\underline{D}_t\big] \leq \mathbb{G}^{[t:s]}\big[\overline{D}_t\big] \;.$$

Definition 2.25. *Let $t \in [\![1,T]\!]$ and $s \in [\![t,T]\!]$. To a $[t\!:\!s]$-multiple-step uncertainty-aggregator $\mathbb{G}^{[t:s]}$, we attach a mapping*[9]

$$\left\langle \mathbb{G}^{[t:s]} \right\rangle : \mathcal{F}(\mathbb{W}_{[0:s]};\bar{\mathbb{R}}) \to \mathcal{F}(\mathbb{W}_{[0:t-1]};\bar{\mathbb{R}}) \;, \tag{2.50a}$$

obtained by freezing the first variables as follows. For any $A : \mathbb{W}_{[0:s]} \to \bar{\mathbb{R}}$, and any $w_{[0:s]} \in \mathbb{W}_{[0:s]}$, we set

$$\Big(\left\langle \mathbb{G}^{[t:s]} \right\rangle [A]\Big)\big(w_{[0:t-1]}\big) = \mathbb{G}^{[t:s]}\Big[w_{[t:s]} \mapsto A\big(w_{[0:t-1]}, w_{[t:s]}\big)\Big] \;. \tag{2.50b}$$

Multiple-step uncertainty-aggregators are commonly built progressively backward: in §2.1.1, the expectation operator $\mathbb{E}_{\mathbb{P}_0 \otimes \cdots \otimes \mathbb{P}_T}$ is the initial value of the induction $\mathbb{E}_{\mathbb{P}_t \otimes \cdots \otimes \mathbb{P}_T} = \mathbb{E}_{\mathbb{P}_t}\mathbb{E}_{\mathbb{P}_{t+1} \otimes \cdots \otimes \mathbb{P}_T}$; the fear operator $\sup_{w \in \mathbb{W}_{[0:T]}}$ is the initial value of the induction $\sup_{w \in \mathbb{W}_{[t:T]}} = \sup_{w_t \in \mathbb{W}_t} \sup_{w \in \mathbb{W}_{[t+1:T]}}$.

We define the composition of uncertainty-aggregators as follows.

Definition 2.26. *Let $t \in [\![0,T]\!]$ and $s \in [\![t+1,T]\!]$. Let $\mathbb{G}^{[t:t]} : \mathcal{F}(\mathbb{W}_t;\bar{\mathbb{R}}) \to \bar{\mathbb{R}}$ be a t-one-step uncertainty-aggregator, and $\mathbb{G}^{[t+1:s]} : \mathcal{F}(\mathbb{W}_{[t+1:s]};\bar{\mathbb{R}}) \to \bar{\mathbb{R}}$ be a $[t+1\!:\!s]$-multiple-step uncertainty-aggregator. We define the $[t\!:\!s]$-multiple-step uncertainty-aggregator $\mathbb{G}^{[t:t]} \boxdot \mathbb{G}^{[t:s]}$ by*

$$\Big(\mathbb{G}^{[t:t]} \boxdot \mathbb{G}^{[t:s]}\Big)\big[A_t\big] = \mathbb{G}^{[t:t]}\Big[w_t \mapsto \mathbb{G}^{[t+1:s]}\big[w_{[t+1:s]} \mapsto A_t\big(w_t, w_{[t+1:s]}\big)\big]\Big] \;, \tag{2.51}$$

for all function $A_t \in \mathcal{F}\big(\mathbb{W}_{[t:s]};\bar{\mathbb{R}}\big)$.

Quite naturaly, we define the composition of sequences of one-step uncertainty-aggregators as follows.

Definition 2.27. *We say that a sequence $\big\{\mathbb{G}_t\big\}_{t=0}^{T}$ of one-step uncertainty-aggregators is a* chained sequence *if $\mathbb{G}_t$ is a t-one-step uncertainty-aggregator, for all $t \in [\![0,T]\!]$.*

Consider a chained sequence $\big\{\mathbb{G}_t\big\}_{t=0}^{T}$ of one-step uncertainty-aggregators. For $t \in [\![0,T]\!]$, we define the composition $\underset{s=t}{\overset{T}{\boxdot}} \mathbb{G}_s$ as the $[t\!:\!T]$-multiple-step uncertainty-aggregator

$$\underset{s=t}{\overset{T}{\boxdot}} \mathbb{G}_s : \mathcal{F}\big(\mathbb{W}_{[t:T]};\bar{\mathbb{R}}\big) \to \bar{\mathbb{R}} \;, \tag{2.52}$$

inductively given by

$$\underset{s=T}{\overset{T}{\boxdot}} \mathbb{G}_s = \mathbb{G}_T \text{ and } \Big(\underset{s=t}{\overset{T}{\boxdot}} \mathbb{G}_s\Big) = \mathbb{G}_t \boxdot \Big(\underset{s=t+1}{\overset{T}{\boxdot}} \mathbb{G}_s\Big) \;. \tag{2.53a}$$

That is, for all function $B_t \in \mathcal{F}\big(\mathbb{W}_{[t:T]};\bar{\mathbb{R}}\big)$, we have:

$$\Big(\underset{s=t}{\overset{T}{\boxdot}} \mathbb{G}_s\Big)\big[B_t\big] = \mathbb{G}_t\Big[w_t \mapsto \Big(\underset{s=t+1}{\overset{T}{\boxdot}} \mathbb{G}_s\Big)\big[w_{[t+1:T]} \mapsto B_t\big(w_t, w_{[t+1:T]}\big)\big]\Big] \;. \tag{2.53b}$$

---

7. The superscript notation indicates that the domain of the mapping $\mathbb{G}^{[t:s]}$ is $\mathcal{F}(\mathbb{W}_{[t:s]};\bar{\mathbb{R}})$ (not to be confused with $\mathbb{G}_{[t:s]} = \big\{\mathbb{G}_r\big\}_{r=t}^{s}$).

8. We will consistently use the symbol D to denote a function in $\mathcal{F}\big(\mathbb{W}_{[t:s]};\bar{\mathbb{R}}\big)$, that is, $D : \mathbb{W}_{[t:s]} \to \bar{\mathbb{R}}$.

9. See Footnote 6 about the notation $\langle \quad \rangle$.

Crafting Dynamic Uncertainty Criteria from Aggregators

We outline four ways to craft a dynamic uncertainty criterion from aggregators. Let $A_{[0:T]} = \{A_s\}_{s=0}^{T}$ denote an arbitrary adapted uncertainty process (that is, $A_s : \mathbb{W}_{[0:s]} \to \bar{\mathbb{R}}$, as in Definition 2.2).

Non Nested Dynamic Uncertainty Criteria The two following ways to craft a dynamic uncertainty criterion $\{\varrho_{t,T}\}_{t=0}^{T}$ display a natural economic interpretation in term of preferences over streams of uncertain costs like $A_{[0:T]}$. They mix time and uncertainty preferences, either first with respect to uncertainty then with respect to time (UT) or first with respect to time, then with respect to uncertainty (TU). However, they are not directly amenable to a DPE.

TU, or time, then uncertainty. Let $t \in [\![0, T]\!]$ be fixed.

- First, we aggregate $A_{[t:T]}$ with respect to time by means of a multiple-step time-aggregator Φ^t from $\bar{\mathbb{R}}^{T-t+1}$ towards $\bar{\mathbb{R}}$, and we obtain $\Phi^t\big(A_{[t:T]}\big)$.
- Second, we aggregate $\Phi^t\big(A_{[t:T]}\big)$ with respect to uncertainty by means of a multiple-step uncertainty-aggregator $\mathbb{G}^{[t:T]}$, and we obtain

$$\varrho_{t,T}\big(A_{[t:T]}\big) = \Big\langle \mathbb{G}^{[t:T]} \Big\rangle \Big[\Phi^t\big(A_{[t:T]}\big) \Big] . \tag{2.54}$$

All the examples in §2.1.1 belong to this TU class, and some in §2.1.2.

UT, or uncertainty, then time.

- First, we aggregate $A_{[t:T]}$ with respect to uncertainty by means of a sequence $\big[{\mathbb{G}_s}^{[t:s]}\big]_{s=t}^{T}$ of multiple-step time-aggregators ${\mathbb{G}_t}^{[t:s]} : \mathcal{F}(\mathbb{W}_{[t:s]}; \bar{\mathbb{R}}) \to \bar{\mathbb{R}}$, and we obtain a sequence $\Big\{ \Big\langle {\mathbb{G}_s}^{[t:s]} \Big\rangle \big[A_s\big] \Big\}_{s=t}^{T}$.
- Second, we aggregate $\Big\{ \Big\langle {\mathbb{G}_s}^{[t:s]} \Big\rangle \big[A_s\big] \Big\}_{s=t}^{T}$ by means of a multiple-step time-aggregator Φ^t from $\bar{\mathbb{R}}^{T-t+1}$ towards $\bar{\mathbb{R}}$, and we obtain

$$\varrho_{t,T}\big(A_{[t:T]}\big) = \Phi^t\Big(\Big\{ \Big\langle {\mathbb{G}_s}^{[t:t]} \Big\rangle \big[A_s\big] \Big\}_{s=t}^{T}\Big) . \tag{2.55}$$

Some examples in §2.1.2 belong to this UT class.

Nested Dynamic Uncertainty Criteria The two following ways to craft a dynamic uncertainty criterion $\{\varrho_{t,T}\}_{t=0}^{T}$ do not display a natural economic interpretation in term of preferences [65], but they are directly amenable to a DPE. Indeed, they are produced by a backward induction, nesting uncertainty and time. Consider

- on the one hand, a sequence $\big\{\Phi_t\big\}_{t=0}^{T-1}$ of one-step time-aggregators,
- on the other hand, a chained sequence $\big\{\mathbb{G}_t\big\}_{t=0}^{T}$ of one-step uncertainty-aggregators.

NTU, or nesting time, then uncertainty, then time, etc. We define a dynamic uncertainty criterion by the following backward induction:

$$\varrho_{T,T}\big(A_T\big) = \langle \mathbb{G}_T \rangle \big[A_T\big] , \tag{2.56a}$$

$$\varrho_{t,T}\Big(\big\{A_s\big\}_{s=t}^{T}\Big) = \langle \mathbb{G}_t \rangle \Big[\Phi_t \Big\{ A_t, \varrho_{t+1,T}\Big(\big\{A_s\big\}_{s=t+1}^{T}\Big) \Big\} \Big], \quad \forall t \in [\![0, T-1]\!] . \tag{2.56b}$$

By the Definition 2.25 of $\langle \mathbb{G}_t \rangle$, we have, by construction, produced a dynamic uncertainty criterion $\{\varrho_{t,T}\}_{t=0}^{T}$ (see Definition 2.7). Indeed, recalling that $A_s : \mathbb{W}_{[0:s]} \to \bar{\mathbb{R}}$), for $s \in [\![0,T]\!]$, we write

$$\underbrace{\varrho_{T,T}\big(A_T\big)}^{\mathcal{F}\left(\mathbb{W}_{[0:T-1]};\bar{\mathbb{R}}\right)} = \langle \mathbb{G}_T \rangle \Big[\overbrace{A_T}^{\mathcal{F}\left(\mathbb{W}_{[0:T]};\bar{\mathbb{R}}\right)} \Big] ,$$

$$\underbrace{\varrho_{t,T}\Big(\big\{A_s\big\}_{s=t}^{T}\Big)}_{\mathcal{F}\left(\mathbb{W}_{[0:t-1]};\bar{\mathbb{R}}\right)} = \langle \mathbb{G}_t \rangle \Bigg[\Phi_t \Big\{ \underbrace{A_t}_{\mathcal{F}\left(\mathbb{W}_{[0:t]};\bar{\mathbb{R}}\right)} , \underbrace{\varrho_{t+1,T}\Big(\overbrace{\big\{A_s\big\}_{s=t+1}^{T}}^{\left[\mathcal{F}\left(\mathbb{W}_{[0:s]};\bar{\mathbb{R}}\right)\right]_{s=t+1}^{T}} \Big)}_{\mathcal{F}\left(\mathbb{W}_{[0:t]};\bar{\mathbb{R}}\right)} \Big\} \Bigg] ,$$

$$\forall t \in [\![0, T-1]\!] .$$

NUT, or nesting uncertainty, then time, then uncertainty, etc. We define a dynamic uncertainty criterion by the following backward induction:

$$\varrho_{T,T}\big(A_T\big) = \langle \mathbb{G}_T \rangle \big[A_T\big] , \tag{2.57a}$$

$$\varrho_{t,T}\Big(\big\{A_s\big\}_{s=t}^{T}\Big) = \Phi_t \Big\{ \langle \mathbb{G}_t \rangle \big[A_t\big], \langle \mathbb{G}_t \rangle \Big[\varrho_{t+1,T}\Big(\big\{A_s\big\}_{s=t+1}^{T}\Big)\Big] \Big\}, \tag{2.57b}$$

$$\forall t \in [\![0, T-1]\!] .$$

Some examples in §2.1.2 belong to this nested class, made of NTU and NUT.

2.3.2 Time-Consistency for Nested Dynamic Uncertainty Criteria

Consider

- on the one hand, a sequence $\big\{\Phi_t\big\}_{t=0}^{T-1}$ of one-step time-aggregators,
- on the other hand, a chained sequence $\big\{\mathbb{G}_t\big\}_{t=0}^{T}$ of one-step uncertainty-aggregators.

With these ingredients, we present two ways to craft a nested dynamic uncertainty criterion $\{\varrho_{t,T}\}_{t=0}^{T}$, as introduced in Definition 2.7. For each of them, we establish time-consistency.

NTU Dynamic Uncertainty Criterion

With a slight abuse of notation, we define the sequence $\big\{(\mathfrak{P}_t^{\mathrm{NTU}})(x)\big\}_{t=0}^{T}$ of optimization problems parameterized by the state $x \in \mathbb{X}_t$ as the nesting

$$
\begin{aligned}
(\mathfrak{P}_t^{\mathrm{NTU}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \quad & \mathbb{G}_t \Bigg[\Phi_t \Big\{ J_t\big(x_t, u_t, w_t\big), \\
& \quad \mathbb{G}_{t+1} \Bigg[\Phi_{t+1} \Big\{ J_{t+1}\big(x_{t+1}, u_{t+1}, w_{t+1}\big), \cdots \\
& \qquad \mathbb{G}_{T-1} \Big[\Phi_{T-1} \Big\{ J_{T-1}\big(x_{T-1}, u_{T-1}, w_{T-1}\big), \\
& \qquad\quad \mathbb{G}_T \big[J_T\big(x_T, w_T\big)\big] \Big\} \Big] \cdots \Big\} \Bigg] \Big\} \Bigg] ,
\end{aligned} \tag{2.58a}
$$

$$s.t. \quad x_t = x , \tag{2.58b}$$

$$x_{s+1} = f_s\big(x_s, u_s, w_s\big) , \tag{2.58c}$$

$$u_s = \pi_s\big(x_s\big) , \tag{2.58d}$$

$$u_s \in U_s\big(x_s\big) , \tag{2.58e}$$

where constraints are satisfied for all $s \in [\![t, T-1]\!]$.

Definition 2.28. *We construct inductively a* NTU-dynamic uncertainty criterion $\big\{\varrho_{t,T}^{NTU}\big\}_{t=0}^{T}$ *by, for any adapted uncertainty process* $\big\{A_s\big\}_{s=0}^{T}$*,*

$$\varrho_T^{NTU}\big(A_T\big) = \langle\mathbb{G}_T\rangle\,\big[A_T\big]\ , \tag{2.59a}$$

$$\varrho_{t,T}^{NTU}\Big(\big\{A_s\big\}_{s=t}^{T}\Big) = \langle\mathbb{G}_t\rangle\left[\Phi_t\bigg\{A_t,\varrho_{t+1,T}^{NTU}\Big(\big\{A_s\big\}_{s=t+1}^{T}\Big)\bigg\}\right],\quad \forall t\in[\![0,T-1]\!]\ . \tag{2.59b}$$

We define the Markov optimization problem (2.58) formally by

$$(\mathfrak{P}_t^{\mathrm{NTU}})(x)\qquad \min_{\pi\in\Pi_t^{\mathrm{ad}}}\varrho_{t,T}^{\mathrm{NTU}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big),\quad \forall t\in[\![0,T]\!],\quad \forall x\in\mathbb{X}_t\ , \tag{2.60}$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (2.27).

Definition 2.29. *We define the* value functions *inductively by the DPE*

$$V_T^{NTU}(x) = \mathbb{G}_T\Big[J_T(x,\cdot)\Big],\quad \forall x\in\mathbb{X}_T\ , \tag{2.61a}$$

$$V_t^{NTU}(x) = \inf_{u\in U_t(x)}\mathbb{G}_t\left[\Phi_t\bigg\{J_t(x,u,\cdot),V_{t+1}^{NTU}\circ f_t(x,u,\cdot)\bigg\}\right],\qquad \forall t\in[\![0,T-1]\!],\quad \forall x\in\mathbb{X}_t\ . \tag{2.61b}$$

The following Proposition 2.30 expresses sufficient conditions under which any Problem $(\mathfrak{P}_t^{\mathrm{NTU}})(x)$, for any time $t\in[\![0,T-1]\!]$ and any state $x\in\mathbb{X}_t$, can be solved by means of the value functions $\{V_t^{\mathrm{NTU}}\}_{t=0}^{T}$ in Definition 2.29.

Proposition 2.30. *Assume that*

- *for all* $t\in[\![0,T-1]\!]$*,* Φ_t *is non-decreasing,*
- *for all* $t\in[\![0,T]\!]$*,* $\mathbb{G}_t$ *is non-decreasing.*

Assume that there exists[10] *an admissible policy* $\pi^\sharp\in\Pi^{\mathrm{ad}}$ *such that*

$$\pi_t^\sharp(x)\in\arg\min_{u\in U_t(x)}\mathbb{G}_t\left[\Phi_t\bigg\{J_t(x,u,\cdot),V_{t+1}^{NTU}\circ f_t(x,u,\cdot)\bigg\}\right],\qquad \forall t\in[\![0,T-1]\!],\quad \forall x\in\mathbb{X}_t\ . \tag{2.62}$$

Then, $\pi^\sharp$ *is an optimal policy for any Problem* $(\mathfrak{P}_t^{NTU})(x)$*, for all* $t\in[\![0,T]\!]$ *and for all* $x\in\mathbb{X}_t$*, and*

$$V_t^{NTU}(x) = \min_{\pi\in\Pi_t^{\mathrm{ad}}}\varrho_{t,T}^{NTU}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big),\qquad \forall t\in[\![0,T]\!],\quad \forall x\in\mathbb{X}_t\ . \tag{2.63}$$

Proof. In the proof, we drop the superscript in the value function V_t^{NTU}, that we simply denote by V_t. Let $\pi\in\Pi^{\mathrm{ad}}$ be a policy. For any $t\in[\![0,T]\!]$, we define $V_t^\pi(x)$ as the intertemporal cost from time t to time T when following policy π starting from state x:

$$V_t^\pi(x) = \varrho_{t,T}^{\mathrm{NTU}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big),\qquad \forall t\in[\![0,T]\!],\quad \forall x\in\mathbb{X}_t\ . \tag{2.64}$$

10. It may be difficult to prove the existence of a measurable selection among the solutions of (2.62). Since it is not our intent to consider such issues, we make the assumption that an admissible policy $\pi^\sharp\in\Pi^{\mathrm{ad}}$ exists, where the definition of the set Π^{ad} is supposed to include all proper measurability conditions.

This expression is well defined because $J^{x,\pi}_{t,s} : \mathbb{W}_{[t:s]} \to \bar{\mathbb{R}}$, for $s \in [\![t, T]\!]$ by (2.28).

First, we show that the functions $\{V^{\pi}_t\}_{t=0}^{T}$ satisfy a backward equation "à la Bellman":

$$V^{\pi}_t(x) = \mathbb{G}_t\Big[\Phi_t\Big\{J_t(x, \pi_t(x), \cdot), V^{\pi}_{t+1} \circ f_t(x, \pi_t(x), \cdot)\Big\}\Big], \quad \forall t \in [\![0, T-1]\!], \quad \forall x \in \mathbb{X}_t \ . \tag{2.65}$$

Indeed, we have,

$$\begin{aligned}
V^{\pi}_T(x) &= \varrho^{\mathrm{NTU}}_{T,T}\Big(J^{x,\pi}_{T,T}\Big) && \text{by the definition (2.64) of } V^{\pi}_T(x), \\
&= \varrho^{\mathrm{NTU}}_{T,T}\Big(J_T(x, \cdot)\Big) && \text{by (2.27) that defines } J^{x,\pi}_{T,T}, \\
&= \langle \mathbb{G}_T \rangle \big[J_T(x, \cdot)\big] && \text{by the definition (2.59a) of } \varrho^{\mathrm{NTU}}_T, \\
&= \mathbb{G}_T\big[J_T(x, \cdot)\big] && \text{by Definition 2.25 of } \langle \mathbb{G}_T \rangle.
\end{aligned}$$

We also have, for $t \in [\![0, T-1]\!]$,

$$\begin{aligned}
V^{\pi}_t(x) &= \varrho^{\mathrm{NTU}}_{t,T}\Big(\big\{J^{x,\pi}_{t,s}\big\}_{s=t}^{T}\Big) \\
&\quad \text{by the definition (2.64) of } V^{\pi}_t(x), \\
&= \langle \mathbb{G}_t \rangle \Big[\Phi_t\Big\{J^{x,\pi}_{t,t}, \varrho^{\mathrm{NTU}}_{t+1,T}\Big(\big\{J^{x,\pi}_{t,s}\big\}_{s=t+1}^{T}\Big)\Big\}\Big] \\
&\quad \text{by the definition (2.59b) of } \varrho^{\mathrm{NTU}}_{t+1,T}, \\
&= \langle \mathbb{G}_t \rangle \Big[\Phi_t\Big\{J^{x,\pi}_{t,t}, \varrho^{\mathrm{NTU}}_{t+1,T}\Big(\big\{J^{f_t(x,\pi_t(x),\cdot),\pi}_{t+1,s}\big\}_{s=t+1}^{T}\Big)\Big\}\Big] \\
&\quad \text{by the flow property (2.29)}, \\
&= \langle \mathbb{G}_t \rangle \Big[\Phi_t\Big\{J^{x,\pi}_{t,t}, V^{\pi}_{t+1} \circ f_t(x, \pi_t(x), \cdot)\Big\}\Big] \\
&\quad \text{by the definition (2.64) of } V^{\pi}_t(x), \\
&= \langle \mathbb{G}_t \rangle \Big[\Phi_t\Big\{J_t(x, \pi_t(x), \cdot), V^{\pi}_{t+1} \circ f_t(x, \pi_t(x), \cdot)\Big\}\Big] \\
&\quad \text{by the flow property (2.29)}, \\
&= \mathbb{G}_t\Big[\Phi_t\Big\{J_t(x, \pi_t(x), \cdot), V^{\pi}_{t+1} \circ f_t(x, \pi_t(x), \cdot)\Big\}\Big] \\
&\quad \text{by Definition 2.25 of } \langle \mathbb{G}_t \rangle.
\end{aligned}$$

Second, we show that $V_t(x)$, as defined in (2.61) is lower than the value of the optimization problem $\mathfrak{P}^{\mathrm{NTU}}_t(x)$ in (2.58). For this purpose, we denote by (H_t) the following assertion

$$(H_t): \qquad \forall x \in \mathbb{X}_t, \quad \forall \pi \in \Pi^{\mathrm{ad}}, \qquad V_t(x) \leq V^{\pi}_t(x) \ .$$

By definition of $V^{\pi}_T(x)$ in (2.64) and of $V_T(x)$ in (2.61a), assertion (H_T) is true.

Now, assume that (H_{t+1}) holds true. Let x be an element of $\mathbb{X}_t$. Then, by definition of $V_t(x)$ in (2.61b), we obtain

$$V_t(x) \leq \inf_{\pi \in \Pi^{\mathrm{ad}}} \mathbb{G}_t\Big[\Phi_t\Big\{J_t\big(x, \pi_t(x), \cdot\big), V_{t+1} \circ f_t\big(x, \pi_t(x), \cdot\big)\Big\}\Big] \ , \tag{2.66}$$

since, for all $\pi \in \Pi^{\mathrm{ad}}$ we have $\pi_t(x) \in U_t(x)$. By (H_{t+1}) we have, for any $\pi \in \Pi^{\mathrm{ad}}$,

$$V_{t+1} \circ f_t\big(x, \pi_t(x), \cdot\big) \leq V^{\pi}_{t+1} \circ f_t\big(x, \pi_t(x), \cdot\big) \ .$$

From monotonicity of Φ_t and monotonicity of $\mathbb{G}_t$, we deduce:

$$\begin{aligned} &\mathbb{G}_t\Big[\Phi_t\Big\{J_t\big(x,\pi_t(x),\cdot\big), V_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big\}\Big] \\ \leq \quad &\mathbb{G}_t\Big[\Phi_t\Big\{J_t\big(x,\pi_t(x),\cdot\big), V^{\pi}_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big\}\Big] . \end{aligned} \tag{2.67}$$

We obtain:

$$\begin{aligned} V_t(x) &\leq \inf_{\pi\in\Pi^{\mathrm{ad}}} \mathbb{G}_t\Big[\Phi_t\Big\{J_t\big(x,\pi_t(x),\cdot\big), V_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big\}\Big] && \text{by (2.66)},\\ &\leq \inf_{\pi\in\Pi^{\mathrm{ad}}} \mathbb{G}_t\Big[\Phi_t\Big\{J_t\big(x,\pi_t(x),\cdot\big), V^{\pi}_{t}\circ f_{t+1}\big(x,\pi_t(x),\cdot\big)\Big\}\Big] && \text{by (2.67)},\\ &= \inf_{\pi\in\Pi^{\mathrm{ad}}} V^{\pi}_t(x) \text{ by the definition (2.64) of } V^{\pi}_t(x). \end{aligned}$$

Hence, assertion (H_t) holds true.

Third, we show that the lower bound $V_t(x)$ for the value of the optimization problem $\mathfrak{P}^{\mathrm{NTU}}_t(x)$ is achieved for the policy $\pi^\sharp$ in (2.62). For this purpose, we consider the following assertion

$$(H'_t): \qquad \forall x\in\mathbb{X}_t, \qquad V^{\pi^\sharp}_t(x) = V_t(x) .$$

By definition of $V^{\pi^\sharp}_T(x)$ in (2.64) and of $V_T(x)$ in (2.61a), (H'_T) holds true. For $t\in[\![0,T-1]\!]$, assume that (H'_{t+1}) holds true. Let x be in $\mathbb{X}_t$. We have

$$\begin{aligned} V_t(x) &= \mathbb{G}_t\Big[\Phi_t\Big\{J_t\big(x,\pi^\sharp_t(x),\cdot\big), V_{t+1}\circ f_t\big(x,\pi^\sharp_t(x),\cdot\big)\Big\}\Big] && \text{by definition of } \pi^\sharp \text{ in (2.62)},\\ &= \mathbb{G}_t\Big[\Phi_t\Big\{J_t\big(x,\pi^\sharp_t(x),\cdot\big), V^{\pi^\sharp}_{t+1}\circ f_t\big(x,\pi^\sharp_t(x),\cdot\big)\Big\}\Big] && \text{by } (H'_{t+1})\\ &= V^{\pi^\sharp}_t(x) && \text{by (2.64)}. \end{aligned}$$

Hence (H'_t) holds true, and the proof is complete by induction. □

The following Theorem 2.31 is our main result on time-consistency in the NTU case.

Theorem 2.31. *Assume that*

- *for all $t\in[\![0,T-1]\!]$, Φ_t is non-decreasing,*
- *for all $t\in[\![0,T]\!]$, $\mathbb{G}_t$ is non-decreasing.*

Then

1. *the NTU-dynamic uncertainty criterion $\big\{\varrho^{NTU}_{t,T}\big\}_{t=0}^{T}$ defined by (2.59) is time-consistent;*
2. *the Markov optimization problem $\Big\{\big\{(\mathfrak{P}^{NTU}_t)(x)\big\}_{x\in\mathbb{X}_t}\Big\}_{t=0}^{T}$ defined in (2.58) is time-consistent, as soon as there exists an admissible policy $\pi^\sharp\in\Pi^{\mathrm{ad}}$ such that (2.62) holds true.*

Proof. In the proof, we drop the superscripts in V^{NTU}_t, $(\mathfrak{P}^{\mathrm{NTU}}_t)(x)$ and $\varrho^{\mathrm{NTU}}_{t,T}$.

The second assertion is a straightforward consequence of the property that $\pi^\sharp$ is an optimal policy [11] for *all* Problems $(\mathfrak{P}_t)(x)$. Hence, the Markov optimization problem (2.58) is time-consistent.

11. In all rigor, we should say that, for all $t\in[\![0,T-1]\!]$, the *tail* policy $\{\pi^\sharp_s\}_{s=t}^{T-1}$ is an optimal policy for Problem $(\mathfrak{P}_t)(x)$, for any $x\in\mathbb{X}_t$.

We now prove the first assertion.

Let $\underline{t} < \bar{t}$ be both in $[\![0, T]\!]$. Consider two adapted uncertainty processes $\{\underline{A}_s\}_0^T$ and $\{\overline{A}_s\}_0^T$, where $\underline{A}_s : \mathbb{W}_{[0:T]} \to \bar{\mathbb{R}}$ and $\overline{A}_s : \mathbb{W}_{[0:T]} \to \bar{\mathbb{R}}$, satisfying (2.39a) and (2.39b), that is,

$$\underline{A}_s = \overline{A}_s, \qquad \forall s \in [\![\underline{t}, \bar{t}]\!] , \tag{2.68a}$$

$$\varrho_{\bar{t},T}\big(\{\underline{A}_s\}_{\bar{t}}^T\big) \leq \varrho_{\bar{t},T}\big(\{\overline{A}_s\}_{\bar{t}}^T\big) , \tag{2.68b}$$

We show by backward induction that, for all $t \in [\![\underline{t}, \bar{t}]\!]$, the following statement (H_t) holds true:

$$(H_t) \qquad \varrho_{t,T}\big(\{\underline{A}_s\}_t^T\big) \leq \varrho_{t,T}\big(\{\overline{A}_s\}_t^T\big) . \tag{2.69}$$

First, we observe that $(H_{\bar{t}})$ holds true by assumption (2.68b). Second, let us assume that, for $t > \underline{t}$, the assertion (H_t) holds true. Then, by (H_t), and as $\underline{A}_{t-1} = \overline{A}_{t-1}$ by (2.68a), monotonicity [12] of Φ_{t-1} yields

$$\Phi_{t-1}\Big\{\underline{A}_{t-1}, \varrho_{t,T}\big(\{\underline{A}_s\}_t^T\big)\Big\} \leq \Phi_{t-1}\Big\{\overline{A}_{t-1}, \varrho_{t,T}\big(\{\overline{A}_s\}_t^T\big)\Big\} .$$

Monotonicity of $\mathbb{G}_{t-1}$ then gives

$$\langle \mathbb{G}_{t-1} \rangle \Big[\Phi_{t-1}\Big\{\underline{A}_{t-1}, \varrho_{t,T}\big(\{\underline{A}_s\}_t^T\big)\Big\}\Big] \leq \langle \mathbb{G}_{t-1} \rangle \Big[\Phi_{t-1}\Big\{\overline{A}_{t-1}, \varrho_{t,T}\big(\{\overline{A}_s\}_t^T\big)\Big\}\Big] .$$

By definition of $\varrho_{t-1,T}$ in (2.59), we obtain (H_{t-1}). This ends the proof by induction. □

Remark 2.32. *As indicated in Remark 2.15, if we choose the inequality*

$$\forall s \in [\![\underline{t}, \bar{t}]\!], \qquad \underline{A}_s \leq \overline{A}_s , \tag{2.70}$$

as assumption to define a time-consistent dynamic uncertainty criterion (rather than the equality (2.43a)*), we have to make, in Theorem 2.31, the assumption "for all* $t \in [\![0, T-1]\!]$*,"*

- *"the two-variables function* $(c_t, c_{t+1}) \mapsto \Phi_t(c_t, c_{t+1})$ *is non-decreasing",*
- *instead of "for all* c_t*, the single variable function* $c_{t+1} \mapsto \Phi_t(c_t, c_{t+1})$ *is non-decreasing".*

NUT Dynamic Uncertainty Criterion

With a slight abuse of notation, we define the sequence $\big\{(\mathfrak{P}_t^{\mathrm{NUT}})(x)\big\}_{t=0}^T$ of optimization problems parameterized by the state $x \in \mathbb{X}_t$ as the nesting

$$\begin{aligned}
(\mathfrak{P}_t^{\mathrm{NUT}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \quad & \Phi_t\bigg\{\mathbb{G}_t\Big[J_t\big(x_t, u_t, w_t\big)\Big], \mathbb{G}_t\bigg[\\
& \Phi_{t+1}\Big\{\mathbb{G}_{t+1}\Big[J_{t+1}\big(x_{t+1}, u_{t+1}, w_{t+1}\big)\Big], \cdots \\
& \Phi_{T-1}\Big\{\mathbb{G}_{T-1}\Big[J_{T-1}\big(x_{T-1}, u_{T-1}, w_{T-1}\big)\Big], \\
& \mathbb{G}_T\Big[J_T\big(x_T, w_T\big)\Big]\Big\} \cdots \Big\}\bigg]\bigg\} ,
\end{aligned} \tag{2.71a}$$

$$s.t. \quad x_t = x , \tag{2.71b}$$

$$x_{s+1} = f_s\big(x_s, u_s, w_s\big) , \tag{2.71c}$$

$$u_s = \pi_s(x_s) , \tag{2.71d}$$

$$u_s \in U_s(x_s) , \tag{2.71e}$$

where constraints are satisfied for all $s \in [\![t, T-1]\!]$.

12. Recall that, by Definition 2.18, Φ_{t-1} is non-decreasing in its second argument. Remark 2.32 below will enlighten this comment.

Definition 2.33. *We construct inductively a* NUT-dynamic uncertainty criterion $\big\{\varrho_{t,T}^{NUT}\big\}_{t=0}^{T}$ *by, for any adapted uncertainty process* $\big\{A_s\big\}_{s=0}^{T}$,

$$\varrho_T^{NUT}\big(A_T\big) = \langle \mathbb{G}_T \rangle \big[A_T\big] \, , \tag{2.72a}$$

$$\varrho_{t,T}^{NUT}\Big(\big\{A_s\big\}_{s=t}^{T}\Big) = \Phi_t\Big\{ \langle \mathbb{G}_t \rangle \big[A_t\big], \langle \mathbb{G}_t \rangle \Big[\varrho_{t+1,T}^{NUT}\Big(\big\{A_s\big\}_{s=t+1}^{T}\Big)\Big]\Big\}, \quad \forall t \in [\![0, T-1]\!] \, . \tag{2.72b}$$

We define the Markov optimization problem (2.71) formally by

$$(\mathfrak{P}_t^{\mathrm{NUT}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{\mathrm{NUT}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big), \quad \forall t \in [\![0, T]\!], \quad \forall x \in \mathbb{X}_t \, , \tag{2.73}$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (2.27).

Definition 2.34. *We define the* value functions *inductively by the DPE*

$$V_T^{NUT}(x) = \mathbb{G}_T\Big[J_T(x,\cdot)\Big], \quad \forall x \in \mathbb{X}_T \, , \tag{2.74a}$$

$$V_t^{NUT}(x) = \inf_{u \in U_t(x)} \Phi_t\Big\{\mathbb{G}_t\Big[J_t(x,u,\cdot)\Big], \mathbb{G}_t\Big[V_{t+1}^{NUT} \circ f_t(x,u,\cdot)\Big]\Big\} \, , \quad \forall t \in [\![0, T-1]\!], \quad \forall x \in \mathbb{X}_t \, . \tag{2.74b}$$

The following Proposition 2.35 expresses sufficient conditions under which any Problem $(\mathfrak{P}_t^{\mathrm{NUT}})(x)$, for any time $t \in [\![0, T-1]\!]$ and any state $x \in \mathbb{X}_t$, can be solved by means of the value functions $\{V_t^{\mathrm{NUT}}\}_{t=0}^{T}$ in Definition 2.34.

Proposition 2.35. *Assume that*

- *for all* $t \in [\![0, T-1]\!]$, Φ_t *is non-decreasing,*
- *for all* $t \in [\![0, T]\!]$, $\mathbb{G}_t$ *is non-decreasing.*

Assume that there exists[13] *an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that*

$$\pi_t^\sharp(x) \in \arg\min_{u \in U_t(x)} \Phi_t\Big\{\mathbb{G}_t\Big[J_t(x,u,\cdot)\Big], \mathbb{G}_t\Big[V_{t+1}^{NUT} \circ f_t(x,u,\cdot)\Big]\Big\}, \quad \forall t \in [\![0, T-1]\!], \quad \forall x \in \mathbb{X}_t \, . \tag{2.75}$$

Then, $\pi^\sharp$ *is an optimal policy for any Problem* $(\mathfrak{P}_t^{NUT})(x)$, *for all* $t \in [\![0, T]\!]$ *and for all* $x \in \mathbb{X}_t$, *and*

$$V_t^{NUT}(x) = \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{NUT}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big), \quad \forall t \in [\![0, T]\!], \quad \forall x \in \mathbb{X}_t \, . \tag{2.76}$$

Proof. In the proof, we drop the superscript in the value function V_t^{NUT}, that we simply denote by V_t. Let $\pi \in \Pi^{\mathrm{ad}}$ be a policy. For any $t \in [\![0, T]\!]$, we define $V_t^\pi(x)$ as the intertemporal cost from time t to time T when following policy π starting from state x:

$$V_t^\pi(x) = \varrho_{t,T}^{\mathrm{NUT}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big), \quad \forall t \in [\![0, T]\!], \quad \forall x \in \mathbb{X}_t \, . \tag{2.77}$$

This expression is well defined because $J_{t,s}^{x,\pi} : \mathbb{W}_{[t:s]} \to \bar{\mathbb{R}}$, for $s \in [\![t, T]\!]$ by (2.28).

13. See Footnote 10.

First, we show that the functions $\{V_t^\pi\}_{t=0}^T$ satisfy a backward equation "à la Bellman":

$$V_t^\pi(x) = \Phi_t\Big\{\mathbb{G}_t\Big[J_t(x,\pi_t(x),\cdot)\Big], \mathbb{G}_t\Big[V_{t+1}^\pi \circ f_t(x,\pi_t(x),\cdot)\Big]\Big\}, \quad \forall t \in [\![0,T-1]\!], \quad \forall x \in \mathbb{X}_t \ . \tag{2.78}$$

Indeed, we have,

$$\begin{aligned}
V_T^\pi(x) &= \varrho_{T,T}^{\mathrm{NUT}}\Big(J_{T,T}^{x,\pi}\Big) && \text{by the definition (2.77) of } V_T^\pi(x),\\
&= \varrho_{T,T}^{\mathrm{NUT}}\Big(J_T(x,\cdot)\Big) && \text{by (2.27) that defines } J_{T,T}^{x,\pi},\\
&= \langle \mathbb{G}_T\rangle\big[J_T(x,\cdot)\big] && \text{by the definition (2.72a) of } \varrho_T^{\mathrm{NTU}},\\
&= \mathbb{G}_T\big[J_T(x,\cdot)\big] && \text{by Definition 2.25 of } \langle \mathbb{G}_T\rangle.
\end{aligned}$$

We also have, for $t \in [\![0,T-1]\!]$,

$$\begin{aligned}
V_t^\pi(x) &= \varrho_{t,T}^{\mathrm{NUT}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^T\Big)\\
&\qquad\qquad \text{by the definition (2.77) of } V_t^\pi(x),\\
&= \Phi_t\Big\{\langle \mathbb{G}_t\rangle\Big[J_{t,t}^{x,\pi}\Big], \langle \mathbb{G}_t\rangle\Big[\varrho_{t+1,T}^{\mathrm{NUT}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t+1}^T\Big)\Big]\Big\}\\
&\qquad\qquad \text{by the definition (2.72b) of } \varrho_{t+1,T}^{\mathrm{NUT}},\\
&= \Phi_t\Big\{\langle \mathbb{G}_t\rangle\Big[J_{t,t}^{x,\pi}\Big], \langle \mathbb{G}_t\rangle\Big[\varrho_{t+1,T}^{\mathrm{NUT}}\Big(\big\{J_{t+1,s}^{f_t(x,\pi_t(x),\cdot),\pi}\big\}_{s=t+1}^T\Big)\Big]\Big\}\\
&\qquad\qquad \text{by the flow property (2.29)}\\
&= \Phi_t\Big\{\langle \mathbb{G}_t\rangle\Big[J_{t,t}^{x,\pi}\Big], \langle \mathbb{G}_t\rangle\Big[V_{t+1}^\pi \circ f_t(x,\pi_t(x),\cdot)\Big]\Big\}\\
&\qquad\qquad \text{by the definition (2.77) of } V_t^\pi(x),\\
&= \Phi_t\Big\{\langle \mathbb{G}_t\rangle\Big[J_t(x,\pi_t(x),\cdot)\Big], \langle \mathbb{G}_t\rangle\Big[V_{t+1}^\pi \circ f_t(x,\pi_t(x),\cdot)\Big]\Big\}\\
&\qquad\qquad \text{by the flow property (2.29)}\\
&= \Phi_t\Big\{\mathbb{G}_t\Big[J_t(x,\pi_t(x),\cdot)\Big], \mathbb{G}_t\Big[V_{t+1}^\pi \circ f_t(x,\pi_t(x),\cdot)\Big]\Big\}\\
&\qquad\qquad \text{by Definition 2.25 of } \langle \mathbb{G}_t\rangle.
\end{aligned}$$

Second, we show that $V_t(x)$, as defined in (2.74) is lower than the value of the optimization problem $\mathfrak{P}_t^{\mathrm{NUT}}(x)$ in (2.71). For this purpose, we denote by (H_t) the following assertion

$$(H_t): \qquad \forall x \in \mathbb{X}_t, \quad \forall \pi \in \Pi^{\mathrm{ad}}, \qquad V_t(x) \leq V_t^\pi(x) \ .$$

By definition of $V_T^\pi(x)$ in (2.77) and of $V_T(x)$ in (2.74a), assertion (H_T) is true.

Now, assume that (H_{t+1}) holds true. Let x be an element of $\mathbb{X}_t$. Then, by definition of $V_t(x)$ in (2.74b), we obtain

$$V_t(x) \leq \inf_{\pi \in \Pi^{\mathrm{ad}}} \Phi_t\Big\{\mathbb{G}_t\Big[J_t\big(x,\pi_t(x),\cdot\big)\Big], \mathbb{G}_t\Big[V_{t+1} \circ f_t\big(x,\pi_t(x),\cdot\big)\Big]\Big\} \ , \tag{2.79}$$

since, for all $\pi \in \Pi^{\mathrm{ad}}$ we have $\pi_t(x) \in U_t(x)$. By (H_{t+1}) we have, for any $\pi \in \Pi^{\mathrm{ad}}$,

$$V_{t+1} \circ f_t\big(x,\pi_t(x),\cdot\big) \leq V_{t+1}^\pi \circ f_t\big(x,\pi_t(x),\cdot\big) \ .$$

From monotonicity of Φ_t and monotonicity of $\mathbb{G}_t$, we deduce:

$$\begin{aligned} & \Phi_t\Big\{\mathbb{G}_t\Big[J_t\big(x,\pi_t(x),\cdot\big)\Big],\mathbb{G}_t\Big[V_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big]\Big\} \\ \leq\quad & \Phi_t\Big\{\mathbb{G}_t\Big[J_t\big(x,\pi_t(x),\cdot\big)\Big],\mathbb{G}_t\Big[V^{\pi}_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big]\Big\} . \end{aligned} \tag{2.80}$$

We obtain:

$$\begin{aligned} V_t(x) &\leq \inf_{\pi\in\Pi^{\mathrm{ad}}} \Phi_t\Big\{\mathbb{G}_t\Big[J_t\big(x,\pi_t(x),\cdot\big)\Big],\mathbb{G}_t\Big[V_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big]\Big\} \text{ by (2.79)}, \\ &\leq \inf_{\pi\in\Pi^{\mathrm{ad}}} \Phi_t\Big\{\mathbb{G}_t\Big[J_t\big(x,\pi_t(x),\cdot\big)\Big],\mathbb{G}_t\Big[V^{\pi}_{t+1}\circ f_{t+1}\big(x,\pi_t(x),\cdot\big)\Big]\Big\} \text{ by (2.80)}, \\ &= \inf_{\pi\in\Pi^{\mathrm{ad}}} V^{\pi}_t(x) \text{ by the definition (2.77) of } V^{\pi}_t(x). \end{aligned}$$

Hence (H_t) holds true.

Third, we show that the lower bound $V_t(x)$ for the value of the optimization problem $\mathfrak{P}^{\mathrm{NUT}}_t(x)$ is achieved for the policy $\pi^\sharp$ in (2.75). For this purpose, we consider the following assertion

$$(H'_t): \qquad \forall x\in\mathbb{X}_t, \qquad V^{\pi^\sharp}_t(x) = V_t(x) \; .$$

By definition of $V^{\pi^\sharp}_T(x)$ in (2.77) and of $V_T(x)$ in (2.74a), (H'_T) holds true. For $t\in[\![0,T-1]\!]$, assume that (H'_{t+1}) holds true. Let x be in $\mathbb{X}_t$. We have

$$\begin{aligned} V_t(x) &= \Phi_t\Big\{\mathbb{G}_t\Big[J_t\big(x,\pi^\sharp_t(x),\cdot\big)\Big],\mathbb{G}_t\Big[V_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big]\Big\} && \text{by definition of } \pi^\sharp \text{ in (2.75)}, \\ &= \Phi_t\Big\{\mathbb{G}_t\Big[J_t\big(x,\pi^\sharp_t(x),\cdot\big)\Big],\mathbb{G}_t\Big[V^{\pi^\sharp}_{t+1}\circ f_t\big(x,\pi_t(x),\cdot\big)\Big]\Big\} && \text{by } (H'_{t+1}) \\ &= V^{\pi^\sharp}_t(x) && \text{by (2.77)}. \end{aligned}$$

Hence (H'_t) holds true, and the proof is complete by induction. □

The following Theorem 2.36 is our main result on time-consistency in the NUT case.

Theorem 2.36. *Assume that*

- *for all $t\in[\![0,T-1]\!]$, Φ_t is non-decreasing,*
- *for all $t\in[\![0,T]\!]$, $\mathbb{G}_t$ is non-decreasing.*

Then

1. *the NUT-dynamic uncertainty criterion $\big\{\varrho^{NUT}_{t,T}\big\}_{t=0}^{T}$ defined by (2.72) is time-consistent;*
2. *the Markov optimization problem $\big\{\big\{(\mathfrak{P}^{NUT}_t)(x)\big\}_{x\in\mathbb{X}_t}\big\}_{t=0}^{T}$ defined in (2.71) is time-consistent, as soon as there exists an admissible policy $\pi^\sharp\in\Pi^{\mathrm{ad}}$ such that (2.75) holds true.*

Proof. In the proof, we drop the superscripts in V^{NUT}_t, $(\mathfrak{P}^{\mathrm{NUT}}_t)(x)$ and $\varrho^{\mathrm{NUT}}_{t,T}$.

The second assertion is a straightforward consequence of the property that $\pi^\sharp$ is an optimal policy [14] for *all* Problems $(\mathfrak{P}_t)(x)$. Hence, the Markov optimization problem (2.71) is time-consistent.

14. See Footnote 11.

We now prove the first assertion. We suppose given a policy $\pi \in \Pi$, and a sequence $\{x_s\}_0^T$ of states, where $x_s \in \mathbb{X}_s$.

Let $\underline{t} < \bar{t}$ be both in $[\![0, T]\!]$. Consider two adapted uncertainty processes $\{\underline{A}_s\}_0^T$ and $\{\overline{A}_s\}_0^T$, where $\underline{A}_s : \mathbb{W}_{[0:T]} \to \bar{\mathbb{R}}$ and $\overline{A}_s : \mathbb{W}_{[0:T]} \to \bar{\mathbb{R}}$, satisfying (2.39a) and (2.39b), that is,

$$\begin{aligned} \underline{A}_s &= \overline{A}_s, && \forall s \in [\![\underline{t}, \bar{t}]\!] , && (2.81a) \\ \varrho_{\bar{t},T}\big(\{\underline{A}_s\}_{\bar{t}}^T\big) &\le \varrho_{\bar{t},T}\big(\{\overline{A}_s\}_{\bar{t}}^T\big) , && && (2.81b) \end{aligned}$$

We show by backward induction that, for all $t \in [\![\underline{t}, \bar{t}]\!]$, the following statement (H_t) holds true:

$$(H_t) \qquad \varrho_{t,T}\big(\{\underline{A}_s\}_t^T\big) \le \varrho_{t,T}\big(\{\overline{A}_s\}_t^T\big) . \tag{2.82}$$

First, we observe that $(H_{\bar{t}})$ holds true by assumption (2.81b). Second, let us assume that, for $t > \underline{t}$, the assertion (H_t) holds true. Then, by (H_t), monotonicity of $\mathbb{G}_{t-1}$ gives

$$\langle \mathbb{G}_{t-1} \rangle \Big[\varrho_{t,T}\big(\{\underline{A}_s\}_t^T\big) \Big] \le \langle \mathbb{G}_{t-1} \rangle \Big[\varrho_{t,T}\big(\{\overline{A}_s\}_t^T\big) \Big] .$$

As $\underline{A}_{t-1} = \overline{A}_{t-1}$ by (2.81a), monotonicity [15] of Φ_{t-1} yields

$$\Phi_{t-1}\Big\{ \underline{A}_{t-1}, \langle \mathbb{G}_{t-1} \rangle \Big[\varrho_{t,T}\big(\{\underline{A}_s\}_t^T\big) \Big] \Big\} \le \Phi_{t-1}\Big\{ \overline{A}_{t-1}, \langle \mathbb{G}_{t-1} \rangle \Big[\varrho_{t,T}\big(\{\overline{A}_s\}_t^T\big) \Big] \Big\} .$$

By definition of $\varrho_{t-1,T}$ in (2.72), we obtain (H_{t-1}). This ends the proof by induction. □

2.3.3 Commutation of Aggregators

We introduce two notions of commutation between time and uncertainty aggregators.

TU-Commutation of Aggregators

The following notion of TU-commutation between time and uncertainty aggregators stands as one of the key ingredients for a DPE.

Definition 2.37. *Let $t \in [\![0, T]\!]$ and $s \in [\![t+1, T]\!]$. A $[t\!:\!s]$-multiple-step uncertainty-aggregator $\mathbb{G}^{[t:s]}$ is said to* TU-commute *with a one-step time-aggregator Φ if*

$$\mathbb{G}^{[t:s]}\Big[w_{[t:s]} \mapsto \Phi\big\{ c, D_t\big(w_{[t:s]}\big) \big\} \Big] = \Phi\Big\{ c, \mathbb{G}^{[t:s]}\Big[w_{[t:s]} \mapsto D_t\big(w_{[t:s]}\big) \Big] \Big\} , \tag{2.83}$$

for any function $D_t \in \mathcal{F}(\mathbb{W}_{[t:s]}; \bar{\mathbb{R}})$ and any extended scalar $c \in \bar{\mathbb{R}}$.

In particular, a one-step time-aggregator Φ TU-commutes with a one-step uncertainty-aggregator $\mathbb{G}^{[t:t]}$ if

$$\mathbb{G}^{[t:t]}\Big[\Phi\big\{ c, C_t \big\} \Big] = \Phi\Big\{ c, \mathbb{G}^{[t:t]}\big[C_t \big] \Big\} , \tag{2.84}$$

for any function [16] $C_t \in \mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}})$ and any extended scalar $c \in \bar{\mathbb{R}}$.

Example 2.38. *If $(\mathbb{W}_t, \mathcal{F}_t, \mathbb{P}_t)$ is a probability space and if*

$$\Phi\big\{ c, c_t \big\} = \alpha(c) + \beta(c) c_t , \tag{2.85}$$

where $\alpha : \bar{\mathbb{R}} \to \mathbb{R}$ and $\beta : \bar{\mathbb{R}} \to \mathbb{R}_+$, then the extended[17] expectation $\mathbb{G}^{[t:t]} = \mathbb{E}_{\mathbb{P}_t}$ TU-commutes with Φ.

15. See Footnote 12.
16. We will consistently use the symbol C_t to denote a function in $\mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}})$, that is, $C_t : \mathbb{W}_t \to \bar{\mathbb{R}}$.
17. We set $\beta \ge 0$, so that, when $C_t \in \mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}})$ is not integrable with respect to $\mathbb{P}_t$, the equality (2.83) still holds true.

Proposition 2.39. *Consider a sequence $\{\Phi_t\}_{t=0}^{T-1}$ of one-step time-aggregators and a chained sequence $\{\mathbb{G}_t\}_{t=0}^{T}$ of one-step uncertainty-aggregators. Suppose that, for any $0 \leq t < s \leq T$, $\mathbb{G}_s$ TU-commutes with Φ_t.*

Then, $\left\langle \overset{T}{\underset{s=t}{\boxdot}} \mathbb{G}_s \right\rangle$ TU-commutes with Φ_r, for any $0 \leq r < t \leq T$, that is,

$$\left\langle \overset{T}{\underset{s=t}{\boxdot}} \mathbb{G}_s \right\rangle \Big[\Phi_r \big\{ c_r, A \big\} \Big] = \Phi_r \Big\{ c, \left\langle \overset{T}{\underset{s=t}{\boxdot}} \mathbb{G}_s \right\rangle \big[A \big] \Big\}, \qquad \forall\, 0 \leq r < t \leq T \,, \tag{2.86}$$

for any extended scalar $c \in \bar{\mathbb{R}}$ and any function $A \in \mathcal{F}\big(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}}\big)$.

Proof. We prove by induction that

$$\Big(\overset{T}{\underset{s=t}{\boxdot}} \mathbb{G}_s \Big) \Big[\Phi_r \big\{ c, D_t \big\} \Big] = \Phi_r \Big\{ c, \Big(\overset{T}{\underset{s=t}{\boxdot}} \mathbb{G}_s \Big) \big[D_t \big] \Big\}, \qquad \forall\, 0 \leq r < t \leq T \,, \tag{2.87}$$

for any extended scalar $c \in \bar{\mathbb{R}}$ and any function $D_t \in \mathcal{F}\big(\mathbb{W}_{[t:T]}; \bar{\mathbb{R}}\big)$. For $t \in [\![1, T]\!]$, let (H_t) be the following assertion

$$\begin{aligned}(H_t): \quad & \forall r \in [\![0, t-1]\!], \quad \forall c \in \bar{\mathbb{R}}, \quad \forall D_t \in \mathcal{F}\big(\mathbb{W}_{[t:T]}; \bar{\mathbb{R}}\big) \,, \\ & \Big(\overset{T}{\underset{s=t}{\boxdot}} \mathbb{G}_s \Big) \Big[\Phi_r \big\{ c, D_t \big\} \Big] = \Phi_r \Big\{ c, \Big(\overset{T}{\underset{s=t}{\boxdot}} \mathbb{G}_s \Big) \big[D_t \big] \Big\} \,. \end{aligned} \tag{2.88}$$

The assertion (H_T) is

$$\begin{aligned}(H_T): \quad & \forall r \in [\![0, T-1]\!], \quad \forall c \in \bar{\mathbb{R}}, \quad \forall D_T \in \mathcal{F}\big(\mathbb{W}_T; \bar{\mathbb{R}}\big), \\ & \mathbb{G}_T \Big[\Phi_r \big\{ c, D_T \big\} \Big] = \Phi_r \big\{ c, \mathbb{G}_T \big[D_T \big] \big\} \,. \end{aligned}$$

Thus, the assertion (H_T) is true, since it coincides the property that, for any $0 \leq r < T$, $\mathbb{G}_T$ TU-commutes with Φ_r (apply (2.83) where $t = T$, $\Phi = \Phi_r$).

Now, suppose that (H_{t+1}) holds true. Let $r < t$, $c \in \bar{\mathbb{R}}$ and $D_t \in \mathcal{F}\big(\mathbb{W}_{[t:T]}; \bar{\mathbb{R}}\big)$. We have

$$\begin{aligned}
& \Big(\overset{T}{\underset{s=t}{\boxdot}} \mathbb{G}_s \Big) \Big[\Phi_r \big\{ c, D_t \big\} \Big] \,, \\
& \quad = \quad \mathbb{G}_t \bigg[w_t \mapsto \Big(\overset{T}{\underset{t+1}{\boxdot}} \mathbb{G}_s \Big) \Big[w_{[t+1:T]} \mapsto \Phi_r \Big\{ c, D_t\big(w_t, w_{[t+1:T]}\big) \Big\} \Big] \bigg] \,, \\
& \qquad \text{by the definition (2.53) of composition,} \\
& \quad = \quad \mathbb{G}_t \bigg[w_t \mapsto \Phi_r \Big\{ c, \Big(\overset{T}{\underset{s=t+1}{\boxdot}} \mathbb{G}_s \Big) \Big[w_{[t+1:T]} \mapsto D_t\big(w_t, w_{[t+1:T]}\big) \Big] \Big\} \bigg] \\
& \qquad \text{by } (H_{t+1}) \text{ since } r < t < t+1, \\
& \qquad \text{and where, for all } w_t,\ D_{t+1} : w_{[t+1:T]} \mapsto D_t\big(w_t, w_{[t+1:T]}\big) \in \mathcal{F}\big(\mathbb{W}_{[t:T]}; \bar{\mathbb{R}}\big) \,, \\
& \quad = \quad \Phi_r \bigg\{ c, \mathbb{G}_t \Big[w_t \mapsto \Big(\overset{T}{\underset{s=t+1}{\boxdot}} \mathbb{G}_s \Big) \Big[w_{[t+1:T]} \mapsto D_t\big(w_t, w_{[t+1:T]}\big) \Big] \Big] \bigg\} \,, \\
& \qquad \text{by commutation property (2.83) of } \mathbb{G}_t \text{ with } \Phi = \Phi_r, \text{ since } 0 \leq r < t \leq T, \\
& \qquad \text{and where } C_t : w_t \mapsto \Big(\overset{T}{\underset{s=t+1}{\boxdot}} \mathbb{G}_s \Big) \big[w_{[t+1:T]} \mapsto D_t\big(w_t, w_{[t+1:T]}\big) \big] \in \mathcal{F}\big(\mathbb{W}_t; \bar{\mathbb{R}}\big) \,, \\
& \quad = \quad \Phi_r \Big\{ c, \Big(\overset{T}{\underset{s=t}{\boxdot}} \mathbb{G}_s \Big) \big[D_t \big] \Big\} \text{ by the definition (2.53) of composition.}
\end{aligned}$$

This ends the induction, hence the proof of (2.87). Then, (2.86) easily follows by the extensions of Definitions 2.23 and 2.25. □

UT-Commutation of Aggregators

The following notion of UT-commutation between time and uncertainty aggregators stands as one of the key ingredients for a DPE. In practice, it is much more restrictive than TU-commutation.

Definition 2.40. *Let $t \in [\![0,T]\!]$. A multiple-step time-aggregator $\Phi : \bar{\mathbb{R}}^{k+1} \to \bar{\mathbb{R}}$ is said to* UT-commute *with a one-step uncertainty-aggregator $\mathbb{G}^{[t:t]}$ if*

$$\left\langle \mathbb{G}^{[t:t]} \right\rangle \left[\Phi\Big(\big\{ A_s \big\}_{s=0}^{k} \Big) \right] = \Phi\Big(\Big\{ \left\langle \mathbb{G}^{[t:t]} \right\rangle \big[A_s \big] \Big\}_{s=0}^{k} \Big) , \tag{2.89}$$

for any adapted uncertainty process $\big\{ A_s \big\}_{s=0}^{k}$.

In particular, a one-step time-aggregator Φ UT-commutes with a one-step uncertainty-aggregator $\mathbb{G}^{[t:t]}$ if

$$\mathbb{G}^{[t:t]} \Big[\Phi\big\{ B_t, C_t \big\} \Big] = \Phi\Big\{ \mathbb{G}^{[t:t]} \big[B_t \big], \mathbb{G}^{[t:t]} \big[C_t \big] \Big\} , \tag{2.90}$$

for any functions B_t, C_t in $\mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}})$. Comparing (2.90) with (2.84), we observe that UT-commutation requires a property bearing on the first argument of the one-step time-aggregator Φ, whereas TU-commutation does not. In practical applications, UT-commutation is much more restrictive than TU-commutation.

Example 2.41. *If $(\mathbb{W}_t, \mathcal{F}_t, \mathbb{P}_t)$ is a probability space, then the extended expectation $\mathbb{G}^{[t:t]} = \mathbb{E}_{\mathbb{P}_t}$ UT-commutes with Φ, given by $\Phi\big\{ c, c_t \big\} = \alpha(c) + \beta(c) c_t$ in (2.85), only in the case where α is linear and β is a constant. Comparing with Example 2.38, UT-commutation appears much more restrictive than TU-commutation.*

Proposition 2.42. *Consider a sequence $\big\{ \Phi_t \big\}_{t=0}^{T-1}$ of one-step time-aggregators and a chained sequence $\big\{ \mathbb{G}_t \big\}_{t=0}^{T}$ of one-step uncertainty-aggregators. Suppose that, for any $0 \leq t < s \leq T$, Φ_s TU-commutes with $\mathbb{G}_t$.*

Then, $\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle$ TU-commutes with $\mathbb{G}_r$, for any $r \in [\![0, t-1]\!]$, that is, for any $\big\{ A_s \big\}_{s=t}^{T}$, where $A_s \in \mathcal{F}\big(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}}\big)$,

$$\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \Big\{ \mathbb{G}_r \big[A_s \big] \Big\}_{s=t}^{T} \Big\} = \mathbb{G}_r \Big[\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{ A_s \big\}_{s=t}^{T} \Big\} \Big], \quad \forall 0 \leq r < t \leq T . \tag{2.91}$$

Proof. We prove by induction that

$$\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \Big\{ \mathbb{G}_r \big[C_s \big] \Big\}_{s=t}^{T} \Big\} = \mathbb{G}_r \Big[\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{ C_s \big\}_{s=t}^{T} \Big\} \Big], \quad \forall 0 \leq r < t \leq T , \tag{2.92}$$

for any $\big\{ C_s \big\}_{s=t}^{T}$, where $C_s \in \mathcal{F}\big(\mathbb{W}_r; \bar{\mathbb{R}}\big)$.

For $t \in [\![0, T-1]\!]$, let (H_t) be the following assertion

$$\begin{aligned} (H_t): \quad & \forall r \in [\![0, t-1]\!], \quad \forall s \in [\![t, T]\!], \quad \forall C_s \in \mathcal{F}\big(\mathbb{W}_r; \bar{\mathbb{R}}\big) , \\ & \Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \Big\{ \mathbb{G}_r \big[C_s \big] \Big\}_{s=t}^{T} \Big\} = \mathbb{G}_r \Big[\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{ C_s \big\}_{s=t}^{T} \Big\} \Big] . \end{aligned} \tag{2.93}$$

The assertion (H_{T-1}) is

$$\begin{aligned} (H_{T-1}): \quad & \forall r \in [\![0, T-2]\!], \quad \forall C_T \in \mathcal{F}\big(\mathbb{W}_r; \bar{\mathbb{R}}\big), \quad \forall C_{T-1} \in \mathcal{F}\big(\mathbb{W}_r; \bar{\mathbb{R}}\big) , \\ & \langle \Phi_{T-1} \rangle \Big\{ \mathbb{G}_r \big[C_{T-1} \big], \mathbb{G}_r \big[C_T \big] \Big\} = \mathbb{G}_r \Big[\langle \Phi_{T-1} \rangle \big\{ C_{T-1}, C_T \big\} \Big] . \end{aligned} \tag{2.94}$$

Thus, the assertion (H_{T-1}) is true, since it coincides the property that, for any $0 \leq r < T$, Φ_{T-1} TU-commutes with $\mathbb{G}_r$ (apply (2.89) where $t = T$, $\Phi = \Phi_{T-1}$, $A_s = C_s$).

Now, suppose that (H_{t+1}) holds true. With $r < t$, and $C_s \in \mathcal{F}(\mathbb{W}_r; \bar{\mathbb{R}})$, for all $s \in [\![t, T]\!]$, we have

$$
\begin{aligned}
\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{ \mathbb{G}_r\big[C_s\big] \big\}_{s=t}^{T} \Big\} &= \Phi_t \Big\{ \mathbb{G}_r\big[C_t\big], \Big\langle \bigodot_{s=t+1}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{ \mathbb{G}_r\big[C_s\big] \big\}_{s=t+1}^{T} \Big\} \Big\} \\
&\quad \text{by the definition (2.45) of composition,} \\
&= \Phi_t \Big\{ \mathbb{G}_r\big[C_t\big], \mathbb{G}_r \Big[\Big\langle \bigodot_{s=t+1}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{ C_s \big\}_{s=t+1}^{T} \Big\} \Big] \Big\} \\
&\quad \text{by } (H_{t+1}) \text{ since } r < t < t+1 \\
&= \mathbb{G}_r \Big[\Phi_t \Big\{ C_t, \Big\langle \bigodot_{s=t+1}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{ C_s \big\}_{s=t+1}^{T} \Big\} \Big\} \Big] \\
&\quad \text{by commutation property (2.89) of } \mathbb{G}_r \text{ with } \Phi = \Phi_t \\
&\quad \text{since } 0 \leq r < t \leq T, \\
&= \mathbb{G}_r \Big[\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle \Big\{ \big\{ C_s \big\}_{s=t}^{T} \Big\} \Big] \\
&\quad \text{by the definition (2.45) of composition.}
\end{aligned}
$$

This ends the induction, hence the proof of (2.92). The property that $\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle$ TU-commutes with $\mathbb{G}_r$, for any $r \in [\![0, t-1]\!]$, easily follows by the extensions of Definitions 2.23 and 2.25. $\square$

2.3.4 Time-Consistency for Non Nested Dynamic Uncertainty Criteria

Consider

- on the one hand, a sequence $\{\Phi_t\}_{t=0}^{T-1}$ of one-step time-aggregators,
- on the other hand, a chained sequence $\{\mathbb{G}_t\}_{t=0}^{T}$ of one-step uncertainty-aggregators.

With these ingredients, and with the compositions $\Big(\boxdot_{s=t}^{T} \mathbb{G}_s \Big)$ and $\Big\langle \boxdot_{s=t}^{T} \mathbb{G}_s \Big\rangle$ introduced in Definitions 2.27 and 2.25, and $\Big\langle \bigodot_{s=t}^{T-1} \Phi_s \Big\rangle$ in Definition 2.23, we present two ways to craft a non-nested dynamic uncertainty criterion $\{\varrho_{t,T}\}_{t=0}^{T}$, as introduced in Definition 2.7. For each of them, we provide a DPE under the assumption that time and uncertainty aggregators commute.

TU Dynamic Uncertainty Criterion

With a slight abuse of notation, we define the sequence $\big\{(\mathfrak{P}_t^{\mathrm{TU}})(x)\big\}_{t=0}^{T}$ of optimization problems parameterized by the state $x \in \mathbb{X}_t$ as

$$
\begin{aligned}
(\mathfrak{P}_t^{\mathrm{TU}})(x) \qquad & \min_{\pi\in\Pi_t^{\mathrm{ad}}} \; \mathbb{G}_t\Bigg[\mathbb{G}_{t+1}\Bigg[\cdots\mathbb{G}_T\Bigg[\\
&\qquad \Phi_t\Big\{J_t\big(x_t,u_t,w_t\big), \\
&\qquad\quad \Phi_{t+1}\Big\{J_{t+1}\big(x_{t+1},u_{t+1},w_{t+1}\big),\cdots && (2.95a)\\
&\qquad\qquad \Phi_{T-1}\Big\{J_{T-1}\big(x_{T-1},u_{T-1},w_{T-1}\big),J_T\big(x_T,w_T\big)\Big\} \\
&\qquad\qquad\qquad \cdots\Big\}\Big\}\Bigg]\cdots\Bigg]\Bigg]\;, \\
s.t.\quad & x_t = x\;, && (2.95b)\\
& x_{s+1} = f_s\big(x_s,u_s,w_s\big)\;, && (2.95c)\\
& u_s = \pi_s(x_s)\;, && (2.95d)\\
& u_s \in U_s(x_s)\;, && (2.95e)
\end{aligned}
$$

where constraints are satisfied for all $s \in [\![t, T-1]\!]$.

We define the Markov optimization problem (2.95) formally by

$$
(\mathfrak{P}_t^{\mathrm{TU}})(x) \qquad \min_{\pi\in\Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{\mathrm{TU}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big), \quad \forall t\in[\![0,T]\!], \quad \forall x\in\mathbb{X}_t\;, \tag{2.96}
$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (2.27), and where $\varrho_{t,T}^{\mathrm{TU}}$ is defined as follows.

When we compose

$$
\big[\mathcal{F}(\mathbb{W}_{[0:s]};\bar{\mathbb{R}})\big]_{s=t}^{T} \xrightarrow{\left\langle \bigodot_{s=t}^{T-1}\Phi_s\right\rangle} \mathcal{F}(\mathbb{W}_{[0:T]};\bar{\mathbb{R}}) \xrightarrow{\left\langle \boxdot_{s=t}^{T}\mathbb{G}_s\right\rangle} \mathcal{F}(\mathbb{W}_{[0:t-1]};\bar{\mathbb{R}}), \tag{2.97}
$$

we obtain the following Definition.

Definition 2.43. *We define the dynamic uncertainty criterion* $\{\varrho_{t,T}^{TU}\}_{t=0}^{T}$ *by*[18]

$$
\varrho_{t,T}^{TU} = \Big\langle \boxdot_{s=t}^{T}\mathbb{G}_s\Big\rangle \circ \Big\langle \bigodot_{s=t}^{T-1}\Phi_s\Big\rangle, \quad \forall t\in[\![0,T-1]\!]\;. \tag{2.98}
$$

When we plug the stream $\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}$ of costs, introduced in Definition 2.5, into the operator above, this two-stage process displays a natural economic interpretation in term of preferences: we mix time and uncertainty preferences, first with respect to time, then with respect to uncertainty.

- We aggregate streams $\big\{J_{t,s}^{x,\pi}(w)\big\}_{s=t}^{T}$ of costs, *first with respect to time*, thanks to the function $\Big(\bigodot_{s=t}^{T-1}\Phi_s\Big) : \bar{\mathbb{R}}^{T+1} \to \bar{\mathbb{R}}$. However, the result $\Big(\bigodot_{s=t}^{T-1}\Phi_s\Big)\Big(\big\{J_{t,s}^{x,\pi}(w)\big\}_{s=t}^{T}\Big)$ still depends upon the scenario w.

18. With the convention that $\Big(\bigodot_{r=T}^{T-1}\Phi_r\Big)$ is the identity mapping.

- Then, we aggregate uncertain intertemporal costs $w \mapsto \left(\bigodot_{s=t}^{T-1} \Phi_s\right)\left(\left\{J_{t,s}^{x,\pi}(w)\right\}_{s=t}^{T}\right)$ — elements of the set $\mathcal{F}(\mathbb{W}_{[t:T]};\bar{\mathbb{R}})$ of functions — *second with respect to uncertainty*, thanks to the multiple-step uncertainty-aggregator $\boxdot_{s=t}^{T} \mathbb{G}_s : \mathcal{F}(\mathbb{W}_{[t:T]};\bar{\mathbb{R}}) \to \bar{\mathbb{R}}$.

The following Theorem 2.44 is our main result on time-consistency in the TU case.

Theorem 2.44. *Assume that*

- *for any* $0 \leq s < t \leq T$, $\mathbb{G}_t$ *TU-commutes with* Φ_s,
- *for all* $t \in [\![0, T-1]\!]$, Φ_t *is non-decreasing,*
- *for all* $t \in [\![0, T]\!]$, $\mathbb{G}_t$ *is non-decreasing.*

Then

1. *the TU-dynamic uncertainty criterion* $\left\{\varrho_{t,T}^{TU}\right\}_{t=0}^{T}$ *defined by* (2.98) *is time-consistent;*

2. *the Markov optimization problem* $\left\{\left\{(\mathfrak{P}_t^{TU})(x)\right\}_{x \in \mathbb{X}_t}\right\}_{t=0}^{T}$ *defined in* (2.95) *is time-consistent, as soon as there exists an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that* (2.62) *holds true, where the value functions are the* $\left\{V_t^{NTU}\right\}_{t=0}^{T}$ *in Definition 2.29.*

Proof. Since, for any $0 \leq s < t \leq T$, $\mathbb{G}_t$ TU-commutes with Φ_s, the TU- dynamic uncertainty criterion $\{\varrho_{t,T}^{\mathrm{TU}}\}_{t=0}^{T}$, given by Definition 2.43, coincides with $\{\varrho_{t,T}^{\mathrm{NTU}}\}_{t=0}^{T}$, given by Definition 2.28. Indeed, we prove that $\{\varrho_{t,T}^{\mathrm{TU}}\}_{t=0}^{T}$ satisfies the backward induction (2.59).

With the convention [19] that $\left(\bigodot_{r=T}^{T-1} \Phi_r\right)$ is the identity mapping, we have $\varrho_T^{\mathrm{TU}} = \langle \mathbb{G}_T \rangle$, that is, (2.59a). For any $\left\{A_s\right\}_t^T \in \left[\mathcal{F}(\mathbb{W}_{[0:s]};\bar{\mathbb{R}})\right]_{s=t}^{T}$, we have:

$$
\begin{aligned}
\varrho_t^{\mathrm{TU}}\left(\left\{A_s\right\}_{s=t}^{T}\right) &= \left\langle \boxdot_{r=t}^{s} \mathbb{G}_r \right\rangle \left[\left\langle \bigodot_{r=t}^{T-1} \Phi_r \right\rangle \left\{ \left\{A_s\right\}_{s=t}^{T} \right\} \right] \text{ by (2.98),} \\
&= \mathbb{G}_t \left[\left\langle \boxdot_{r=t+1}^{s} \mathbb{G}_r \right\rangle \left[\left\langle \bigodot_{r=t}^{T-1} \Phi_r \right\rangle \left\{ \left\{A_s\right\}_{s=t}^{T} \right\} \right] \right] \text{ by (2.53),} \\
&= \mathbb{G}_t \left[\left\langle \boxdot_{r=t+1}^{s} \mathbb{G}_r \right\rangle \Phi_t \left\{ A_t, \left(\bigodot_{r=t+1}^{T-1} \Phi_r \right) \left\{A_s\right\}_{s=t+1}^{T} \right\} \right] \text{ by (2.45),} \\
&= \mathbb{G}_t \left[\Phi_t \left\{ A_t, \left\langle \boxdot_{r=t+1}^{s} \mathbb{G}_r \right\rangle \left[\left(\bigodot_{r=t+1}^{T-1} \Phi_r \right) \left\{A_s\right\}_{s=t+1}^{T} \right] \right\} \right] \\
&\quad \text{by commutation property (2.91),} \\
&= \mathbb{G}_t \left[\Phi_t \left(A_t, \varrho_{t+1}^{\mathrm{TU}} \left(\left\{A_s\right\}_{s=t+1}^{T} \right) \right) \right] \text{ by (2.98).}
\end{aligned}
$$

□

19. See Footnote 18

UT Dynamic Uncertainty Criterion

With a slight abuse of notation, we define the sequence $\big\{(\mathfrak{P}_t^{\mathrm{UT}})(x)\big\}_{t=0}^{T}$ of optimization problems parameterized by the state $x \in \mathbb{X}_t$ as

$$
\begin{aligned}
(\mathfrak{P}_t^{\mathrm{UT}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \quad & \Phi_t \Bigg\{ \mathbb{G}_t \Big[J_t\big(x_t, u_t, w_t\big) \Big], \\
& \quad \Phi_{t+1} \Bigg\{ \mathbb{G}_t \mathbb{G}_{t+1} \Big[J_{t+1}\big(x_{t+1}, u_{t+1}, w_{t+1}\big), \cdots \\
& \qquad \Phi_{T-1} \Big\{ \mathbb{G}_t \cdots \mathbb{G}_{T-1} \Big[J_{T-1}\big(x_{T-1}, u_{T-1}, w_{T-1}\big) \Big], \\
& \qquad\qquad \mathbb{G}_t \cdots \mathbb{G}_T \big[J_T\big(x_T, w_T\big) \big] \Big\} \Big] \cdots \Bigg\} \Bigg\} , && (2.99a) \\
s.t. \quad & x_t = x \;, && (2.99b) \\
& x_{s+1} = f_s\big(x_s, u_s, w_s\big) \;, && (2.99c) \\
& u_s = \pi_s\big(x_s\big) \;, && (2.99d) \\
& u_s \in U_s\big(x_s\big) \;, && (2.99e)
\end{aligned}
$$

where constraints are satisfied for all $s \in [\![t, T-1]\!]$.

We define the Markov optimization problem (2.99) formally by

$$
(\mathfrak{P}_t^{\mathrm{UT}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{\mathrm{UT}}\Big(\big\{ J_{t,s}^{x,\pi} \big\}_{s=t}^{T} \Big), \quad \forall t \in [\![0, T]\!], \quad \forall x \in \mathbb{X}_t \;, \tag{2.100}
$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (2.27), and where $\varrho_{t,T}^{\mathrm{UT}}$ is defined as follows. We define the mapping

$$
\Big\{ \Big(\mathop{\boxdot}_{r=t}^{s} \mathbb{G}_r \Big) \Big\}_{s=t}^{T} : \big[\mathcal{F}(\mathbb{W}_{[t:s]}; \bar{\mathbb{R}}) \big]_{s=t}^{T} \to \bar{\mathbb{R}}^{T+1} \;, \tag{2.101}
$$

for any $\big\{D_r\big\}_{r=t}^{T} \in \big[\mathcal{F}(\mathbb{W}_{[t:s]}; \bar{\mathbb{R}})\big]_{s=t}^{T}$, componentwise by

$$
\Big(\mathop{\boxdot}_{r=t}^{s} \mathbb{G}_r \Big) \Big[\big\{ D_s \big\}_{s=t}^{T} \Big] = \Big\{ \Big(\mathop{\boxdot}_{r=t}^{s} \mathbb{G}_r \Big) \big[D_s \big] \Big\}_{s=t}^{T} . \tag{2.102}
$$

In the same way, we define the mapping (see Definition 2.25):

$$
\Big\{ \Big\langle \mathop{\boxdot}_{r=t}^{s} \mathbb{G}_r \Big\rangle \Big\}_{s=t}^{T} : \big[\mathcal{F}(\mathbb{W}_{[0:s]}; \bar{\mathbb{R}}) \big]_{s=t}^{T} \to \Big(\mathcal{F}(\mathbb{W}_{[0:t]}; \bar{\mathbb{R}}) \Big)^{T+1} . \tag{2.103}
$$

Definition 2.45. *We define the dynamic uncertainty criterion* $\{\varrho_{t,T}^{UT}\}_{t=0}^{T}$ *by*

$$
\varrho_{t,T}^{UT} = \Big\langle \mathop{\odot}_{s=t}^{T-1} \Phi_s \Big\rangle \circ \Big\{ \Big\langle \mathop{\boxdot}_{r=t}^{s} \mathbb{G}_r \Big\rangle \Big\}_{s=t}^{T}, \quad \forall t \in [\![0, T-1]\!] \;. \tag{2.104}
$$

The expression $\varrho_{t,T}^{\mathrm{UT}}$ is the output of the composition [20]

$$
\big[\mathcal{F}(\mathbb{W}_{[0:s]}; \bar{\mathbb{R}}) \big]_{s=t}^{T} \xrightarrow{\Big\{ \Big\langle \mathop{\boxdot}_{r=t}^{s} \mathbb{G}_r \Big\rangle \Big\}_{s=t}^{T}} \Big(\mathcal{F}(\mathbb{W}_{[0:t]}; \bar{\mathbb{R}}) \Big)^{T+1} \xrightarrow{\Big(\mathop{\odot}_{s=t}^{T-1} \Phi_s \Big)} \mathcal{F}(\mathbb{W}_{[0:t]}; \bar{\mathbb{R}}) \;.
$$

When we plug the stream $\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}$ of costs, introduced in Definition 2.5, into the operator above, this two-stage process displays a natural economic interpretation in term of preferences: we mix time and uncertainty preferences, first with respect to uncertainty, then with respect to time.

20. With the convention that $\mathcal{F}(\mathbb{W}_{[0:-1]}; \bar{\mathbb{R}}) = \bar{\mathbb{R}}$, we have $\varrho_0^{\mathrm{UT}} : \big[\mathcal{F}(\mathbb{W}_{[0:s]}; \bar{\mathbb{R}})\big]_{s=t}^{T} \to \bar{\mathbb{R}}$.

- We aggregate the stream $\{J^{x,\pi}_{t,s}\}_{s=t}^{T}$ of uncertain costs, *first with respect to uncertainty*, producing

$$\Big\{\mathop{\boxdot}_{r=t}^{s}\mathbb{G}_r\big[J^{x,\pi}_{t,s}\big]\Big\}_{s=t}^{T}=\Big\{\mathbb{G}_t\Big[J^{x,\pi}_{t,t}\Big],\dots,\Big(\mathop{\boxdot}_{r=t}^{T}\mathbb{G}_r\Big)\big[J^{x,\pi}_{t,T}\big]\Big\}\,, \tag{2.105}$$

thanks to the multiple-step uncertainty-aggregators $\mathop{\boxdot}_{r=t}^{s}\mathbb{G}_r : \mathcal{F}(\mathbb{W}_{[t:s]};\bar{\mathbb{R}})\to\bar{\mathbb{R}}$, for $s\in[\![t,T]\!]$. However, the resulting quantity $\Big\{\Big(\mathop{\boxdot}_{r=t}^{s}\mathbb{G}_r\Big)\big[J^{x,\pi}_{t,s}\big]\Big\}_{s=t}^{T}$ still depends upon time s.
- Then, we aggregate the time sequence $\Big\{\Big(\mathop{\boxdot}_{r=t}^{s}\mathbb{G}_r\Big)\big[J^{x,\pi}_{t,s}\big]\Big\}_{s=t}^{T}$ of costs, *second with respect to time*, thanks to $\Big(\mathop{\odot}_{r=t}^{T-1}\Phi_r\Big):\bar{\mathbb{R}}^{T+1}\to\bar{\mathbb{R}}$.

The following Theorem 2.46 is our main result on time-consistency in the UT case.

Theorem 2.46. *Assume that*
- *for any $0\le s<t\le T$, $\mathbb{G}_t$ UT-commutes with Φ_s,*
- *for all $t\in[\![0,T-1]\!]$, Φ_t is non-decreasing,*
- *for all $t\in[\![0,T]\!]$, $\mathbb{G}_t$ is non-decreasing.*

Then
1. *the UT-dynamic uncertainty criterion $\{\varrho^{UT}_{t,T}\}_{t=0}^{T}$ defined by (2.104) is time-consistent;*
2. *the Markov optimization problem $\big\{\{(\mathfrak{P}^{UT}_t)(x)\}_{x\in\mathbb{X}_t}\big\}_{t=0}^{T}$ defined in (2.99) is time-consistent, as soon as there exists an admissible policy $\pi^\sharp\in\Pi^{\mathrm{ad}}$ such that (2.75) holds true, where the value functions are the $\{V^{NUT}_t\}_{t=0}^{T}$ in Definition 2.34.*

Proof. Since, for any $0\le s<t\le T$, $\mathbb{G}_t$ UT-commutes with Φ_s, the UT- dynamic uncertainty criterion $\{\varrho^{\mathrm{UT}}_{t,T}\}_{t=0}^{T}$, given by Definition 2.45, coincides with $\{\varrho^{\mathrm{NUT}}_{t,T}\}_{t=0}^{T}$, given by Definition 2.33.

Indeed, we prove that $\{\varrho^{\mathrm{UT}}_{t,T}\}_{t=0}^{T}$ satisfies the backward induction (2.72).

With the convention [21] that $\Big(\mathop{\odot}_{r=T}^{T-1}\Phi_r\Big)$ is the identity mapping, we have $\varrho^{\mathrm{UT}}_{T}=\langle\mathbb{G}_T\rangle$, that is, (2.72a). For any $\{A_s\}_t^T\in\big[\mathcal{F}(\mathbb{W}_{[0:s]};\bar{\mathbb{R}})\big]_{s=t}^{T}$, we have:

$$\begin{aligned}
\varrho^{\mathrm{N}}_t\Big(\{A_s\}_t^T\Big) &= \Big(\mathop{\odot}_{r=t}^{T-1}\Phi_r\Big)\Big\{\Big\langle\mathop{\boxdot}_{r=t}^{s}\mathbb{G}_r\Big\rangle\big[A_s\big]\Big\}_{s=t}^{T} \text{ by (2.104)},\\
&=\Phi_t\Bigg\{\mathbb{G}_t\big[A_t\big],\Big(\mathop{\odot}_{r=t+1}^{T-1}\Phi_r\Big)\Big\{\Big\langle\mathop{\boxdot}_{r=t}^{s}\mathbb{G}_r\Big\rangle\big[A_s\big]\Big\}_{s=t+1}^{T}\Bigg\} \text{ by (2.45)},\\
&=\Phi_t\Bigg\{\mathbb{G}_t\big[A_t\big],\Big(\mathop{\odot}_{r=t+1}^{T-1}\Phi_r\Big)\Big\{\mathbb{G}_t\Big[\Big\langle\mathop{\boxdot}_{r=t+1}^{s}\mathbb{G}_r\Big\rangle\big[A_s\big]\Big]\Big\}_{s=t+1}^{T}\Bigg\} \text{ by (2.53)},\\
&=\Phi_t\Bigg\{\mathbb{G}_t\big[A_t\big],\Big(\mathop{\odot}_{r=t+1}^{T-1}\Phi_r\Big)\mathbb{G}_t\Bigg[\Big\{\Big\langle\mathop{\boxdot}_{r=t+1}^{s}\mathbb{G}_r\Big\rangle\big[A_s\big]\Big\}_{s=t+1}^{T}\Bigg]\Bigg\} \text{ by (2.102)},\\
&=\Phi_t\Bigg\{\mathbb{G}_t\big[A_t\big],\mathbb{G}_t\Bigg[\Big(\mathop{\odot}_{r=t+1}^{T-1}\Phi_r\Big)\Big\{\Big\langle\mathop{\boxdot}_{r=t+1}^{s}\mathbb{G}_r\Big\rangle\big[A_s\big]\Big\}_{s=t+1}^{T}\Bigg]\Bigg\}\\
&\qquad\qquad\qquad\qquad\text{by commutation property (2.91)},\\
&=\Phi_t\Bigg\{\mathbb{G}_t\big[A_t\big],\mathbb{G}_t\Big[\varrho^{\mathrm{N}}_t\Big(\{A_s\}_{s=t+1}^{T}\Big)\Big]\Bigg\} \text{ by (2.104)},
\end{aligned}$$

21. See Footnote 18

This ends the proof. □

2.3.5 Applications

Now, we present applications of Theorem 2.44, that is, the TU case. Indeed, Theorems 2.31 and 2.36 in the nested cases NTU and NUT are less interesting because they cover cases where time-consistency is commonplace since it only depends on monotonocity assumptions. Regarding Theorem 2.46, it is not powerful because UT-commutation appears much more restrictive than TU-commutation: in practice, Theorem 2.46 only applies to linear one-step time-aggregators $\Phi\{c,d\} = \alpha c + \beta d$ (see Example 2.41), that obviously commute with expectations.

Coherent Risk Measures

We introduce a class of TU-dynamic uncertainty criteria, that are related to coherent risk measures (see Definition 2.9), and we show that they display time-consistency. We thus extend, to more general one-step time-aggregators, results known for the sum (see e.g. [105, 109]).

We denote by $\mathcal{P}(\mathbb{W}_t)$ the set of probabilities over $(\mathbb{W}_t, \mathcal{W}_t)$. Let $\mathcal{P}_0 \subset \mathcal{P}(\mathbb{W}_0)$, ..., $\mathcal{P}_T \subset \mathcal{P}(\mathbb{W}_T)$. If A and B are sets of probabilities, then $A \otimes B$ is defined as

$$A \otimes B = \{\mathbb{P}_A \otimes \mathbb{P}_B | \mathbb{P}_A \in A, \quad \mathbb{P}_B \in B\} . \tag{2.106}$$

Let $(\alpha_t)_{t\in[\![0,T-1]\!]}$ and $(\beta_t)_{t\in[\![0,T-1]\!]}$ be sequences of functions, each mapping $\bar{\mathbb{R}}$ into $\mathbb{R}$, with the additional property that $\beta_t \geq 0$, for all $t \in [\![0, T-1]\!]$. We set, for all $t \in [\![0,T]\!]$,

$$\varrho^{\mathrm{co}}_{t,T}\big(\{A_s\}_{s=t}^T\big) = \sup_{\mathbb{P}_t \in \mathcal{P}_t} \mathbb{E}_{\mathbb{P}_t}\Big[\cdots \sup_{\mathbb{P}_T \in \mathcal{P}_T} \mathbb{E}_{\mathbb{P}_T}\Big[\sum_{s=t}^{T}\Big(\alpha_s(A_s)\prod_{r=t}^{s-1}\beta_r(A_r)\Big)\Big]\cdots\Big], \tag{2.107}$$

for any adapted uncertain process $\{A_t\}_0^T$, with the convention that $\alpha_T(c_T) = c_T$.

Proposition 2.47. *Time-consistency holds true for*

- *the dynamic uncertainty criterion* $\{\varrho^{co}_{t,T}\}_{t=0}^T$ *given by* (2.107),
- *the Markov optimization problem*

$$\min_{\pi \in \Pi^{\mathrm{ad}}} \varrho^{co}_{t,T}\big(\{J^{x,\pi}_{t,s}\}_{s=t}^T\big), \quad \forall t \in [\![0,T]\!], \quad \forall x \in \mathbb{X}_t , \tag{2.108}$$

where $J^{x,\pi}_{t,s}(w)$ *is defined by* (2.27), *as soon as there exists an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that, for all* $t \in [\![0, T-1]\!]$, *for all* $x \in \mathbb{X}_t$,

$$\pi^\sharp_t(x) \in \arg\min_{u \in U_t(x)} \sup_{\mathbb{P}_t \in \mathcal{P}_t} \Big\{\mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(J_t(x,u,\cdot)\big) + \beta_t\big(J_t(x,u,\cdot)\big) V_{t+1} \circ f_t(x,u,\cdot)\Big]\Big\} ,$$

where the value functions are given by the following DPE

$$V_T(x) = \sup_{\mathbb{P}_T \in \mathcal{P}_T} \mathbb{E}_{\mathbb{P}_T}\big[J_T(x,\cdot)\big] , \tag{2.109a}$$

$$V_t(x) = \min_{u \in U_t(x)} \sup_{\mathbb{P}_t \in \mathcal{P}_t} \Big\{\mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(J_t(x,u,\cdot)\big) + \beta_t\big(J_t(x,u,\cdot)\big) V_{t+1} \circ f_t(x,u,\cdot)\Big]\Big\} . \tag{2.109b}$$

Proof. The setting is that of Theorem 2.44 and Proposition 2.30, where

- the one-step time-aggregators are defined by

$$\forall t \in [\![0,T-1]\!], \quad \forall (c_t, c_{t+1}) \in \bar{\mathbb{R}}^2, \qquad \Phi_t\{c_t, c_{t+1}\} = \alpha_t(c_t) + \beta_t(c_t) c_{t+1} , \tag{2.110a}$$

- the one-step uncertainty-aggregators are defined by

$$\forall t \in [\![0, T-1]\!], \quad \forall C_t \in \mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}}), \qquad \mathbb{G}_t\big[C_t\big] = \sup_{\mathbb{P}_t \in \mathcal{P}_t} \mathbb{E}_{\mathbb{P}_t}\big[C_t\big] . \tag{2.110b}$$

The DPE (2.109) is the DPE (2.61), which holds true as soon as the assumptions of Theorem 2.44 hold true.

First, we prove that, for any $0 \leq t < s \leq T$, $\mathbb{G}_s$ TU-commutes with Φ_t. Indeed, letting c_t be an extended real number in $\bar{\mathbb{R}}$ and C_s a function in $\mathcal{F}(\mathbb{W}_s; \bar{\mathbb{R}})$, we have [22]

$$\begin{aligned}
\mathbb{G}_s\big[\Phi_t\{c_t, C_s\}\big] &= \sup_{\mathbb{P}_s \in \mathcal{P}_s} \Big\{\mathbb{E}_{\mathbb{P}_s}\big[\alpha(c_t) + \beta(c_t) C_s\big]\Big\} && \text{by (2.110b) and (2.110a)}, \\
&= \alpha_t(c_t) + \beta_t(c_t) \sup_{\mathbb{P}_s \in \mathcal{P}_s} \Big\{\mathbb{E}_{\mathbb{P}_s}[C_s]\Big\} && \text{as } \beta_t \geq 0 \, , \\
&= \alpha_t(c_t) + \beta_t(c_t) \mathbb{G}_s[C_s] && \text{by (2.110b)}, \\
&= \Phi_t\{c_t, \mathbb{G}_s[C_s]\} && \text{by (2.110a)}.
\end{aligned}$$

Second, we observe that $\mathbb{G}_t$ is non-decreasing (see Definition 2.24), and that $c_{t+1} \in \bar{\mathbb{R}} \mapsto \Phi_t\big\{c_t, c_{t+1}\big\} = \alpha_t(c_t) + \beta_t(c_t) c_{t+1}$ is non-decreasing, for any $c_t \in \bar{\mathbb{R}}$.

This ends the proof. □

The one-step uncertainty-aggregators $\mathbb{G}_t$ in (2.110b) correspond to a coherent risk measure, by Definition 2.9 and the comments that follow it.

Our result differs from [105, Theorem 2] in two ways. On the one hand, in [105], arguments are given to show that there exists an optimal Markovian policy among the set of adapted policies (that is, having a policy taking as argument the whole past uncertainties would not give a better cost than a policy taking as argument the current value of the state). We do not tackle this issue since we directly deal with policies as functions of the state. Where we suppose that there exists an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that (2.62) holds true, [105] gives conditions ensuring this property. On the other hand, where [105] restricts to the sum to aggregate instantaneous costs, we consider more general one-step time-aggregators Φ_t. For instance, our results applies to the product of costs.

Convex Risk Measures

We introduce a class of TU-dynamic uncertainty criteria, that are related to convex risk measures (see Definition 2.9), and we show that they display time-consistency. We consider the same setting as for coherent risk measures, with the restriction that $\beta_t \equiv 1$ and an additional data $(\Upsilon_t)_{t \in [\![0,T]\!]}$.

Let $\mathcal{P}_0 \subset \mathcal{P}(\mathbb{W}_0)$, ..., $\mathcal{P}_T \subset \mathcal{P}(\mathbb{W}_T)$, and $(\Upsilon_t)_{t \in [\![0,T]\!]}$ be sequence of functions, each mapping $\mathcal{P}(\mathbb{W}_t)$ into $\bar{\mathbb{R}}$. Let $(\alpha_t)_{t \in [\![0,T]\!]}$ be sequence of functions, each mapping $\bar{\mathbb{R}}$ into $\mathbb{R}$. We set, for all $t \in [\![0, T]\!]$,

$$\varrho^{\mathrm{cx}}_{t,T}\big(\{A_s\}_t^T\big) = \sup_{\mathbb{P}_t \in \mathcal{P}_t} \mathbb{E}_{\mathbb{P}_t}\bigg[\cdots \sup_{\mathbb{P}_T \in \mathcal{P}_T} \mathbb{E}_{\mathbb{P}_T}\bigg[\sum_{s=t}^{T} \Big(\alpha_s(A_s) - \Upsilon_s(\mathbb{P}_s)\Big)\bigg] \cdots\bigg], \tag{2.111}$$

for any adapted uncertain process $\big\{A_t\big\}_0^T$, with the convention that $\alpha_T(c_T) = c_T$.

Proposition 2.48. *Time-consistency holds true for*

- *the dynamic uncertainty criterion* $\{\varrho^{cx}_{t,T}\}_{t=0}^T$ *given by* (2.111),

22. This result can also be obtained by use of Proposition 2.52 with $I = \mathcal{P}_s$.

- *the Markov optimization problem*

$$\min_{\pi\in\Pi^{\mathrm{ad}}} \varrho^{cx}_{t,T}\big(\{J^{x,\pi}_{t,s}\}_{s=t}^{T}\big), \quad \forall t \in [\![0,T]\!], \quad \forall x \in \mathbb{X}_t \;, \tag{2.112}$$

where $J^{x,\pi}_{t,s}(w)$ *is defined by* (2.27)*, as soon as there exists an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that, for all* $t \in [\![0,T-1]\!]$*, for all* $x \in \mathbb{X}_t$*,*

$$\pi^\sharp_t(x) \in \arg\min_{u\in U_t(x)} \sup_{\mathbb{P}_t\in\mathcal{P}_t} \Big\{\mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(J_t(x,u,\cdot)\big) + V_{t+1}\circ f_t(x,u,\cdot)\Big] - \Upsilon_t(\mathbb{P}_t)\Big\} \;,$$

where the value functions are given by the following DPE

$$V_T(x) = \sup_{\mathbb{P}_T\in\mathcal{P}_T} \mathbb{E}_{\mathbb{P}_T}\big[J_T(x,\cdot)\big] - \Upsilon_T(\mathbb{P}_T) \;, \tag{2.113a}$$

$$V_t(x) = \min_{u\in U_t(x)} \sup_{\mathbb{P}_t\in\mathcal{P}_t} \Big\{\mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(J_t(x,u,\cdot)\big) + V_{t+1}\circ f_t(x,u,\cdot)\Big] - \Upsilon_t(\mathbb{P}_t)\Big\} \;. \tag{2.113b}$$

Proof. The setting is that of Theorem 2.44 and Proposition 2.30, where

- the one-step time-aggregators are defined by

$$\Phi_t\big\{c_t, c_{t+1}\big\} = \alpha_t(c_t) + c_{t+1}, \quad \forall t \in [\![0,T-1]\!], \quad \forall \big(c_t, c_{t+1}\big) \in \bar{\mathbb{R}}^2 \;, \tag{2.114a}$$

- the one-step uncertainty-aggregators are defined by

$$\mathbb{G}_t\big[C_t\big] = \sup_{\mathbb{P}_t\in\mathcal{P}_t} \mathbb{E}_{\mathbb{P}_t}\big[C_t\big] - \Upsilon_t(\mathbb{P}_t), \quad \forall t \in [\![0,T-1]\!], \quad \forall C_t \in \mathcal{F}(\mathbb{W}_t;\bar{\mathbb{R}}) \;. \tag{2.114b}$$

The DPE (2.113) is the DPE (2.61), which holds true as soon as the assumptions of Theorem 2.44 hold true.

First, we prove that, for any $t \in [\![0,T-1]\!]$ and $s \in [\![t+1,T]\!]$, $\mathbb{G}_s$ TU-commutes with Φ_t. Indeed, letting c_t be an extended real number in $\bar{\mathbb{R}}$ and C_s a function in $\mathcal{F}(\mathbb{W}_s;\bar{\mathbb{R}})$, we have [23]

$$\begin{aligned}
\mathbb{G}_s\big[\Phi_t\{c_t, C_s\}\big] &= \sup_{\mathbb{P}_s\in\mathcal{P}_s} \Big\{\mathbb{E}_{\mathbb{P}_s}\big[\alpha(c_t) + C_s\big] - \Upsilon_s(\mathbb{P}_s)\Big\} \text{ by (2.114b) and (2.114a)} \\
&= \alpha_t(c_t) + \sup_{\mathbb{P}_s\in\mathcal{P}_s} \Big\{\mathbb{E}_{\mathbb{P}_s}[C_s] - \Upsilon_s(\mathbb{P}_s)\Big\} \\
&= \alpha_t(c_t) + \mathbb{G}_s[C_s] \text{ by (2.114b)} \\
&= \Phi_t\{c_t, \mathbb{G}_s[C_s]\} \text{ by (2.114a).}
\end{aligned}$$

Second, we observe that $\mathbb{G}_t$ is non-decreasing (see Definition 2.24), and that $c_{t+1} \in \bar{\mathbb{R}} \mapsto \Phi_t\big\{c_t, c_{t+1}\big\} = \alpha_t(c_t) + c_{t+1}$ is non-decreasing, for any $c_t \in \bar{\mathbb{R}}$.

This ends the proof. □

The one-step uncertainty-aggregators $\mathbb{G}_t$ in (2.114b) correspond to a convex risk measure, by Definition 2.9 and the comments that follow it.

Worst-Case Risk Measures (Fear Operator)

A special case of coherent risk measures consists of the worst case scenario operators, also called "fear operators" and introduced in §2.1. For this subclass of coherent risk measures, we show that time-consistency holds for a larger class of time-aggregators than the ones above.

23. This result can also be obtained by use of Proposition 2.52 with $I = \mathcal{P}_s$.

For any $t \in [\![0, T-1]\!]$, let $\widetilde{\mathbb{W}}_t$ be a non empty subset of $\mathbb{W}_t$, and let $\Phi_t : \bar{\mathbb{R}}^2 \to \bar{\mathbb{R}}$ be a function which is continuous and non-decreasing in its second variable. We set, for all $t \in [\![0, T]\!]$,

$$\begin{aligned}\varrho^{\mathrm{wc}}_{t,T}(\{A_s\}_t^T) = \sup_{\{w_s\}_t^T \in \widetilde{\mathbb{W}}_t \times \cdots \times \widetilde{\mathbb{W}}_T} \Phi_t \Big\{ A_t(\{w_s\}_t^T), \Phi_{t+1}\Big\{ \cdots, \\ \Phi_{T-1}\Big\{ A_{T-1}(w_{T-1}, w_T), A_T(w_T) \Big\}\Big\}\Big\},\end{aligned} \tag{2.115}$$

for any adapted uncertain process $\{A_t\}_0^T$.

Proposition 2.49. *Time-consistency holds true for*

- *the dynamic uncertainty criterion $\{\varrho^{wc}_{t,T}\}_{t=0}^T$ given by (2.115),*
- *the Markov optimization problem*

$$\min_{\pi \in \Pi^{\mathrm{ad}}} \varrho^{wc}_{t,T}(\{J^{x,\pi}_{t,s}\}_{s=t}^T) , \tag{2.116}$$

where $J^{x,\pi}_{t,s}(w)$ is defined by (2.27), as soon as there exists an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that, for all $t \in [\![0, T-1]\!]$, for all $x \in \mathbb{X}_t$,

$$\pi^\sharp_t(x) \in \arg\min_{u \in U_t(x)} \sup_{w_t \in \widetilde{\mathbb{W}}_t} \Phi_t\Big\{ J_t(x,u,w_t), V_{t+1} \circ f_t(x,u,w_t) \Big\} ,$$

where the value functions are given by the following DPE

$$V_T(x) = \sup_{w_T \in \widetilde{\mathbb{W}}_T} J_T(x, w_T) , \tag{2.117a}$$

$$V_t(x) = \min_{u \in U_t(x)} \sup_{w_t \in \widetilde{\mathbb{W}}_t} \Phi_t\Big\{ J_t(x,u,w_t), V_{t+1} \circ f_t(x,u,w_t) \Big\} . \tag{2.117b}$$

Proof. The setting is that of Theorem 2.44 and Proposition 2.30, where the one-step uncertainty-aggregators are defined by

$$\mathbb{G}_t\big[C_t\big] = \sup_{w_t \in \widetilde{\mathbb{W}}_t} C_t(w_t), \quad \forall t \in [\![0, T-1]\!], \quad \forall C_t \in \mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}}) . \tag{2.118}$$

The DPE (2.117) is the DPE (2.61), which holds true as soon as the assumptions of Theorem 2.44 hold true.

First, we prove that, for any $t \in [\![0, T-1]\!]$ and $s \in [\![t+1, T]\!]$, $\mathbb{G}_s$ TU-commutes with Φ_t. Indeed, letting c_t be an extended real number in $\bar{\mathbb{R}}$ and C_s a function in $\mathcal{F}(\mathbb{W}_s; \bar{\mathbb{R}})$, we have [24]

$$\begin{aligned}\mathbb{G}_s\big[\Phi_t\{c_t, C_s\}\big] &= \sup_{w_s \in \widetilde{\mathbb{W}}_s} \Big[\Phi_t\big\{c_t, C_s(w_s)\big\}\Big] && \text{by (2.118)}, \\ &= \Phi_t\Big\{c_t, \sup_{w \in \widetilde{\mathbb{W}}_s} \big[C_s(w_s)\big]\Big\} && \text{by continuity of } \Phi_t\{c_t, \cdot\} , \\ &= \Phi_t\big\{c_t, \mathbb{G}_s[C_s]\big\} && \text{by (2.118)}.\end{aligned}$$

Second, we observe that $\mathbb{G}_t$ is non-decreasing (see Definition 2.24), and that $c_{t+1} \mapsto \Phi_t(c_t, c_{t+1})$ is non-decreasing for any $c_t \in \bar{\mathbb{R}}$, by assumption.

This ends the proof. □

Note that $\varrho^{\mathrm{wc}}_{t,T}$ is simply the fear operator on the Cartesian product $\widetilde{\mathbb{W}}_t \times \cdots \times \widetilde{\mathbb{W}}_T$. An example of monotonous one-step time-aggregator is $\Phi_t\big\{c_t, c_{t+1}\big\} = \max\big\{c_t, c_{t+1}\big\}$, used in the so-called Rawls or maximin criterion [34].

24. This result can also be obtained by use of Proposition 2.52 with $I = \widetilde{\mathbb{W}}_s$.

2.3.6 Complements on TU-Commuting Aggregators

Here, we present how we can construct new TU-commuting aggregators from known TU-commuting aggregators. We do not consider UT-commutation, since we have seen that it appears much more restrictive than TU-commutation (see Example 2.41).

For this purpose, we consider a fixed non empty set I and a mapping Γ from $\bar{\mathbb{R}}^I$ to $\bar{\mathbb{R}}$.

Time-Aggregators

Let $(\Phi^i)_{i\in I}$ be a family of one-step time-aggregators. Thanks to the mapping $\Gamma : \bar{\mathbb{R}}^I \to \bar{\mathbb{R}}$, we define the one-step time-aggregator $\Gamma\big[(\Phi^i)_{i\in I}\big]$ by

$$\Gamma\big[(\Phi^i)_{i\in I}\big]\big\{c, d\big\} = \Gamma\Big(\big\{\Phi^i\{c, d\}\big\}_{i\in I}\Big) , \tag{2.119}$$

for all $c \in \bar{\mathbb{R}}$ and $d \in \bar{\mathbb{R}}$.

Proposition 2.50. *Let $t \in [\![0, T]\!]$ and $\mathbb{G}_t$ be a t-one-step uncertainty-aggregator. Suppose that*

- *$\mathbb{G}_t$ TU-commutes with ψ^i, for all $i \in I$,*
- *for all $i \in I$ and for all $C_t^i \in \mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}})$,*

$$\mathbb{G}_t\Big[\Gamma\big(\{C_t^i\}_{i\in I}\big)\Big] = \Gamma\bigg(\Big\{\mathbb{G}_t\big[C_t^i\big]\Big\}_{i\in I}\bigg) . \tag{2.120}$$

Then $\mathbb{G}_t$ TU-commutes with $\Gamma\big[(\Phi^i)_{i\in I}\big]$.

Proof. We set $\Phi = \Gamma\big[(\Phi^i)_{i\in I}\big]$. For $c \in \bar{\mathbb{R}}$ and $C_t \in \mathcal{F}(\mathbb{W}_t, \bar{\mathbb{R}})$, we have

$$\begin{aligned}
\mathbb{G}_t\Big[\Phi\big\{c, C_t\big\}\Big] &= \mathbb{G}_t\Big[\Gamma\Big(\big\{\Phi^i\{c, C_t\}\big\}_{i\in I}\Big)\Big] && \text{by definition of } \Phi \text{ in (2.119)},\\
&= \Gamma\bigg(\Big\{\mathbb{G}_t\big[\Phi^i\{c, C_t\}\big]\Big\}_{i\in I}\bigg) && \text{by (2.120) with } C_t^i = \Phi^i\{c, C_t\} ,\\
&= \Gamma\bigg(\Big\{\Phi^i\big\{c, \mathbb{G}_t[C_t]\big\}_{i\in I}\Big\}\bigg) && \text{by TU-commutation (2.83)},\\
&= \Phi\big\{c, \mathbb{G}_t[C_t]\big\} && \text{by definition of } \Phi \text{ in (2.119)}.
\end{aligned}$$

By Definition 2.37, this ends the proof. □

Uncertainty-Aggregators

Let $t \in [\![0, T]\!]$ and $\{\mathbb{G}_t{}^i\}_{i\in I}$ be a family of t-one-step uncertainty-aggregators. Thanks to the mapping $\Gamma : \bar{\mathbb{R}}^I \to \bar{\mathbb{R}}$, we define the t-one-step uncertainty-aggregator $\Gamma\Big[\{\mathbb{G}_t{}^i\}_{i\in I}\Big]$ by

$$\forall C_t \in \mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}}), \qquad \Gamma\Big[\{\mathbb{G}_t{}^i\}_{i\in I}\Big]\big[C_t\big] = \Gamma\Big(\big\{\mathbb{G}_t{}^i\big[C_t\big]\big\}_{i\in I}\Big) . \tag{2.121}$$

We do not give the proof of the next Proposition 2.51, as it follows the same line as that of Proposition 2.50.

Proposition 2.51. *Let Φ be a one-step time-aggregator. Suppose that*

- *Φ TU-commutes with $\mathbb{G}_t{}^i$, for all $i \in I$,*
- *for all $c \in \bar{\mathbb{R}}$, for all $i \in I$ and for all $c^i \in \bar{\mathbb{R}}$,*

$$\Phi\Big(c, \Gamma\big(\{c^i\}_{i\in I}\big)\Big) = \Gamma\bigg(\Big\{\Phi\Big(c, \{c^i\}\Big)_{i\in I}\Big\}\bigg) . \tag{2.122}$$

Then Φ TU-commutes with $\Gamma\Big[\{\mathbb{G}_t{}^i\}_{i\in I}\Big]$.

As a corollary, we obtain the following practical result.

Proposition 2.52. *Let Φ be a one-step time-aggregator. Suppose that*

- *$\mathbb{G}_t{}^i$ TU-commutes with Φ, for all $i \in I$,*
- *for all $c \in \bar{\mathbb{R}}$, $\Phi\big\{c,\cdot\big\}$ is continuous and non-decreasing.* [25]

Then, the t-one-step uncertainty-aggregator $\sup_{i\in I}\mathbb{G}_t{}^i$ TU-commutes with Φ, and so does $\inf_{i\in I}\mathbb{G}_t{}^i$, provided $\inf_{i\in I}\mathbb{G}_t{}^i$ never takes the value $-\infty$.

Proof. We are going to show that (2.119) holds true, and then the proof is a straightforward application of Proposition 2.51. We set $\bar{\mathbb{G}}_t = \theta\sup_{i\in I}\mathbb{G}_t{}^i + (1-\theta)\inf_{i\in I}\mathbb{G}_t{}^i$, with $\theta \in [0,1]$ (only at the end, do we take $\theta \in \{0,1\}$). For any $(c, C_t) \in \bar{\mathbb{R}} \times \mathcal{F}(\mathbb{W}_t, \bar{\mathbb{R}})$, we have

$$
\begin{aligned}
\bar{\mathbb{G}}_t\Big[\Phi\big\{c, C_t\big\}\Big] &= (\theta\sup_{i\in I} + (1-\theta)\inf_{i\in I})\mathbb{G}_t{}^i\Big[\Phi\big\{c, C_t\big\}\Big] \text{ by definition of } \bar{\mathbb{G}}_t\ ,\\
&= (\theta\sup_{i\in I} + (1-\theta)\inf_{i\in I})\Phi\Big\{c, \mathbb{G}_t{}^i\big[C_t\big]\Big\} \text{ by TU-commutation (2.83)},\\
&= \theta\sup_{i\in I}\Phi\Big\{c, \mathbb{G}_t{}^i\big[C_t\big]\Big\} + (1-\theta)\inf_{i\in I}\Phi\Big\{c, \mathbb{G}_t{}^i\big[C_t\big]\Big\}\ ,\\
&= \theta\Phi\Big\{c, \sup_{i\in I}\mathbb{G}_t{}^i\big[C_t\big]\Big\} + (1-\theta)\Phi\Big\{c, \inf_{i\in I}\mathbb{G}_t{}^i\big[C_t\big]\Big\}\ ,\\
&\quad \text{by continuity and monotonicity of } \Phi\big\{c,\cdot\big\}\ ,\\
&= \Phi\Big\{c, (\theta\sup_{i\in I} + (1-\theta)\inf_{i\in I})\mathbb{G}_t{}^i\big[C_t\big]\Big\} \text{ when } \theta \in \{0,1\}\ .
\end{aligned}
$$

The rest of the proof is a straightforward application of Proposition 2.51. □

The following Proposition 2.53 is an easy extension of Proposition 2.52.

Proposition 2.53. *Suppose that the assumptions of Proposition 2.52 hold true. Let $\underline{I}_j \subset I$, $j \in \underline{J}$ and $\overline{I}_j \subset I$, $j \in \overline{J}$ be finite families of non empty subsets of I.*

- *If Φ is affine in its second variable, that is, if*

$$\Phi\big\{c, d\big\} = \alpha(c) + \beta(c)d\ , \tag{2.123}$$

and if $(\{\underline{\theta}_j\}_{j\in\underline{J}}, \{\overline{\theta}_j\}_{j\in\overline{J}})$ are non-negative scalars that sum to one, the convex combination

$$\sum_{j\in\underline{J}}\underline{\theta}_j\inf_{i\in\underline{I}_j}\mathbb{G}_t{}^i + \sum_{j\in\overline{J}}\overline{\theta}_j\sup_{i\in\overline{I}_j}\mathbb{G}_t{}^i \tag{2.124}$$

of infimum or supremum of subfamilies of $\{\mathbb{G}_t{}^i\}_{i\in I}$ TU-commutes with Φ, provided $\inf_{i\in\underline{I}_j}\mathbb{G}_t{}^i$ never takes the value $-\infty$.

- *If Φ is linear in its second variable, that is, if*

$$\Phi\big\{c, d\big\} = \beta(c)d\ , \tag{2.125}$$

and if $(\{\underline{\theta}_j\}_{j\in\underline{J}}, \{\overline{\theta}_j\}_{j\in\overline{J}})$ are non-negative scalars, the combination

$$\sum_{j\in\underline{J}}\underline{\theta}_j\inf_{i\in\underline{I}_j}\mathbb{G}_t{}^i + \sum_{j\in\overline{J}}\overline{\theta}_j\sup_{i\in\overline{I}_j}\mathbb{G}_t{}^i \tag{2.126}$$

of infimum or supremum of subfamilies of $\{\mathbb{G}_t{}^i\}_{i\in I}$ TU-commutes with Φ, provided $\inf_{i\in\underline{I}_j}\mathbb{G}_t{}^i$ never takes the value $-\infty$.

25. Instead of the continuity of $\Phi\{c,\cdot\}$, we can assume that, for all $C_t \in \mathcal{F}(\mathbb{W}_t, \bar{\mathbb{R}})$, $\sup_{i\in I}\mathbb{G}_t{}^i[C_t]$ is achieved (always true for I finite).

2.4 Extension to Markov Aggregators

Here, we extend the results of §2.3 to the case where we allow one-step time and uncertainty aggregators of depend on the state. The difficulty of this extension is mainly one of notations. We do not give the proofs because they follow the sketch of those in §2.3.2 and in §2.3.4. We will reap the benefits of this extension in §2.4.6, where we present applications.

2.4.1 Markov Time-Aggregators and their Composition

We allow one-step time-aggregators to depend on the state as follows (Definition 2.54 differs from Definition 2.18 only through the indexation by the state).

Definition 2.54. *Let $t \in [\![0,T]\!]$. A* one-step Markov time-aggregator *is a family $\big\{\Phi_t^{x_t}\big\}_{x_t\in\mathbb{X}_t}$ of one-step time-aggregators $\Phi_t^{x_t} : \bar{\mathbb{R}}^2 \to \bar{\mathbb{R}}$ indexed by the state $x_t \in \mathbb{X}_t$.*

Now, we introduce the composition of one-step Markov time-aggregators.

Definition 2.55. *Let $\Big\{\big\{\Phi_t^{x_t}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T-1}$ be a sequence of one-step Markov time-aggregators. Let $t \in [\![0,T-1]\!]$. Given a policy $\pi \in \Pi$ and $x_t \in \mathbb{X}_t$, we define the composition $\Big\langle \overset{x_t,\pi}{\underset{t\leq s\leq T-1}{\bigodot}} \Phi_s \Big\rangle : \big[\mathcal{F}(\mathbb{W}_{[0:T]};\bar{\mathbb{R}})\big]_t^T \to \mathcal{F}(\mathbb{W}_{[0:T]};\bar{\mathbb{R}})$ by*

$$\bigg(\Big\langle \overset{x_t,\pi}{\underset{t\leq s\leq T-1}{\bigodot}} \Phi_s \Big\rangle \Big\{\{A_s\}_t^T\Big\}\bigg)(w) = \Big(\overset{x_t,\pi}{\underset{t\leq s\leq T-1}{\bigodot}} \Phi_s^{X_{t,s}^{x_t,\pi}(w)}\Big)\Big\{\{A_s(w)\}_t^T\Big\}\;, \tag{2.127}$$

for all scenario $w \in \mathbb{W}_{[0:T]}$, for any sequence $\{A_s\}_{s=t}^T \in \Big(\mathcal{F}(\mathbb{W}_{[0:T]};\bar{\mathbb{R}})\Big)^{T-t+1}$, that is, where $A_s \in \mathcal{F}\big(\mathbb{W}_{[0:T]};\bar{\mathbb{R}}\big)$.

Notice that the extension, to one-step Markov time-aggregators, of the composition involves the dynamical system (2.2) and a policy (whereas, in Definition 2.23, the composition is independent of the policy).

Remark 2.56. *Observe that we have defined $\Big\langle \overset{x_t,\pi}{\underset{t\leq s\leq T-1}{\bigodot}} \Phi_s \Big\rangle$, defined over functions, but not $\Big(\overset{x_t,\pi}{\underset{t\leq s\leq T-1}{\bigodot}} \Phi_s\Big)$, defined over extended reals. Observe also that the image by $\Big\langle \overset{x_t,\pi}{\underset{t\leq s\leq T-1}{\bigodot}} \Phi_s \Big\rangle$ of any sequence $c_{[t:T]}$ of extended reals is not an extended real, but is a function:*

$$\bigg(\Big\langle \overset{x_t,\pi}{\underset{t\leq s\leq T-1}{\bigodot}} \Phi_s \Big\rangle \big\{c_{[t:T]}\big\}\bigg)(w) = \Big(\overset{x_t,\pi}{\underset{t\leq s\leq T-1}{\bigodot}} \Phi_s^{X_{t,s}^{x_t,\pi}(w)}\Big)\big\{c_{[t:T]}\big\}\;. \tag{2.128}$$

2.4.2 Markov Uncertainty-Aggregators and their Composition

We allow one-step uncertainty-aggregators to depend on the state as follows (Definition 2.57 differs from Definition 2.24 only through the indexation by the state).

Definition 2.57. *Let $t \in [\![0,T-1]\!]$. A* t-one-step Markov uncertainty-aggregator *is a family $\big\{\mathbb{G}_t^{x_t}\big\}_{x_t\in\mathbb{X}_t}$ of t-one-step uncertainty-aggregators indexed by the state $x_t \in \mathbb{X}_t$.*

We say that a sequence $\big\{\{\mathbb{G}_t^{x_t}\}_{x_t\in\mathbb{X}_t}\big\}_{t=0}^T$ of one-step Markov uncertainty-aggregators is a chained sequence *if $\mathbb{G}_t^{x_t}$ is a t-one-step uncertainty-aggregator, for all $t \in [\![0,T]\!]$.*

The extension, to one-step Markov uncertainty-aggregators, of the composition involves the dynamical system (2.2) and a policy (whereas, in Definition 2.27, the composition is independent of the policy). The formal definition is as follows.

Definition 2.58. *Consider a chained sequence* $\Big\{\big\{\mathbb{G}_t^{x_t}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T}$ *of one-step Markov uncertainty-aggregators.*

For a policy $\pi \in \Pi$*, for* $t \in [\![0,T]\!]$ *and for a state* $x_t \in \mathbb{X}_t$*, we define the composition* $\underset{t\le s\le T}{\overset{x_t,\pi}{\boxdot}} \mathbb{G}_s$ *as a functional mapping* $\mathcal{F}\Big(\mathbb{W}_{[t:T]};\bar{\mathbb{R}}\Big)$ *into* $\bar{\mathbb{R}}$*, inductively given by*

$$\underset{T\le s\le T}{\overset{x_T,\pi}{\boxdot}} \mathbb{G}_s = \mathbb{G}_T^{x_T} \ , \tag{2.129a}$$

and then backward by, for any function $D_t \in \mathcal{F}\Big(\mathbb{W}_{[t:T]};\bar{\mathbb{R}}\Big)$*,*

$$\Big(\underset{t\le s\le T}{\overset{x_t,\pi}{\boxdot}} \mathbb{G}_s\Big)\big[D_t\big] = \mathbb{G}_t^{x_t}\Bigg[w_t \mapsto \\ \Bigg(\underset{t+1\le s\le T}{\overset{f_t(x_t,\pi_t(x_t),w_t),\pi}{\boxdot}} \mathbb{G}_s\Bigg)\Big[w_{[t+1:T]} \mapsto D_t\Big(w_t, w_{[t+1:T]}\Big)\Big]\Bigg] \ . \tag{2.129b}$$

2.4.3 Time-Consistency for Nested Dynamic Uncertainty Criteria

Consider

- on the one hand, a sequence $\Big\{\big\{\Phi_t^{x_t}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T-1}$ of one-step Markov time-aggregators,
- on the other hand, a chained sequence $\Big\{\big\{\mathbb{G}_t^{x_t}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T}$ of one-step Markov uncertainty-aggregators.

With these ingredients, we present two ways to design a Markov dynamic uncertainty criterion as introduced in Definition 2.7.

NTU Dynamic Markov Uncertainty Criterion

Definition 2.59. *Let a policy* $\pi \in \Pi$ *be given. We construct inductively a* NTU-Markov dynamic uncertainty criterion $\Big\{\big\{\varrho_{t,T}^{x_t,\pi,NTU}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T}$ *by*

$$\varrho_T^{x_T,\pi,NTU}\big(A_T\big) = \big\langle \mathbb{G}_T^{x_T} \big\rangle \big[A_T\big] \ , \tag{2.130a}$$

$$\varrho_{t,T}^{x_t,\pi,NTU}\Big(\big\{A_s\big\}_{s=t}^{T}\Big) = \big\langle \mathbb{G}_t^{x_t} \big\rangle \Bigg[\Phi_t^{x_t}\Big\{A_t, \varrho_{t+1,T}^{f_t(x_t,\pi_t(x_t),\cdot),\pi,NTU}\Big(\big\{A_s\big\}_{s=t+1}^{T}\Big)\Big\}\Bigg], \\ \forall t \in [\![0, T-1]\!] \ , \tag{2.130b}$$

for any sequence $\{x_s\}_0^T$ *of states, where* $x_s \in \mathbb{X}_s$*.*

We define the Markov optimization problem

$$(\mathfrak{P}_t^{\mathrm{MNTU}})(x) \qquad \min_{\pi\in\Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{x_t,\pi,\mathrm{NTU}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big), \quad \forall t \in [\![0,T]\!], \quad \forall x \in \mathbb{X}_t \ , \tag{2.131}$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (2.27).

Definition 2.60. *We define the* value functions *inductively by the DPE*

$$V_T^{MNTU}(x) = \mathbb{G}_T^x\Big[J_T(x,\cdot)\Big], \quad \forall x \in \mathbb{X}_T \ , \tag{2.132a}$$

$$V_t^{MNTU}(x) = \inf_{u \in U_t(x)} \mathbb{G}_t^x\left[\Phi_t^x\Big\{J_t(x,u,\cdot), V_{t+1}^{MNTU} \circ f_t(x,u,\cdot)\Big\}\right] , \quad \forall t \in [\![0, T-1]\!], \quad \forall x \in \mathbb{X}_t \ . \tag{2.132b}$$

The following Proposition 2.61 expresses sufficient conditions under which any Problem $(\mathfrak{P}_t^{\mathrm{MNTU}})(x)$, for all $t \in [\![0,T]\!]$ and for all $x \in \mathbb{X}_t$, can be solved by means of the value functions in Definition 2.60.

Proposition 2.61. *Assume that*

- *for all $t \in [\![0, T-1]\!]$, for all $x_t \in \mathbb{X}_t$, $\Phi_t^{x_t}$ is non-decreasing,*
- *for all $t \in [\![0, T]\!]$, for all $x_t \in \mathbb{X}_t$, $\mathbb{G}_t^{x_t}$ is non-decreasing.*

Assume that there exists[26] *an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that*

$$\pi_t^\sharp(x) \in \operatorname*{arg\,min}_{u \in U_t(x)} \mathbb{G}_t^x\left[\Phi_t^x\Big\{J_t(x,u,\cdot), V_{t+1}^{MNTU} \circ f_t(x,u,\cdot)\Big\}\right], \quad \forall t \in [\![0, T-1]\!], \quad \forall x \in \mathbb{X}_t \ . \tag{2.133}$$

Then, $\pi^\sharp$ is an optimal policy for any Problem $(\mathfrak{P}_t^{MNTU})(x)$, for all $t \in [\![0,T]\!]$ and for all $x \in \mathbb{X}_t$, and

$$V_t^{MNTU}(x) = \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{x,\pi,NTU}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^T\Big), \quad \forall t \in [\![0,T]\!], \quad \forall x \in \mathbb{X}_t \ . \tag{2.134}$$

The following Theorem 2.62 is our main result on time-consistency in the NTU Markov case.

Theorem 2.62. *Assume that*

- *for all $t \in [\![0, T-1]\!]$, for all $x_t \in \mathbb{X}_t$, $\Phi_t^{x_t}$ is non-decreasing,*
- *for all $t \in [\![0, T]\!]$, for all $x_t \in \mathbb{X}_t$, $\mathbb{G}_t^{x_t}$ is non-decreasing.*

Then

1. *for all policy $\pi \in \Pi$, the NTU-Markov dynamic uncertainty criterion $\Big\{\big\{\varrho_{t,T}^{x_t,\pi,NTU}\big\}_{x_t \in \mathbb{X}_t}\Big\}_{t=0}^T$ defined by (2.130) is time-consistent;*
2. *the Markov optimization problem $\big\{\big\{(\mathfrak{P}_t^{MNTU})(x)\big\}_{x \in \mathbb{X}_t}\big\}_{t=0}^T$ defined in (2.131) is time-consistent, as soon as there exists an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that (2.133) holds true.*

26. See Footnote 10.

NUT Dynamic Markov Uncertainty Criterion

Definition 2.63. *Let a policy $\pi \in \Pi$ be given. We construct inductively a* NUT-Markov dynamic uncertainty criterion $\Big\{ \big\{ \varrho_{t,T}^{x_t,\pi,NUT} \big\}_{x_t \in \mathbb{X}_t} \Big\}_{t=0}^{T}$ *by*

$$\varrho_T^{x_T,\pi,NUT}(A_T) = \langle \mathbb{G}_T^{x_T} \rangle \big[A_T \big] \; , \tag{2.135a}$$

$$\varrho_{t,T}^{x_t,\pi,NUT}\big(\{A_s\}_{s=t}^{T}\big) = \Phi_t^{x_t} \Big\{ \langle \mathbb{G}_t^{x_t} \rangle \Big[A_t \Big], \langle \mathbb{G}_t^{x_t} \rangle \Big[\varrho_{t+1,T}^{f_t(x_t,\pi_t(x_t),\cdot),\pi,NUT}\Big(\big\{ A_s \big\}_{s=t+1}^{T} \Big) \Big] \Big\}, \quad \forall t \in [\![0, T-1]\!] \; , \tag{2.135b}$$

for any sequence $\{x_s\}_{s=0}^{T}$ of states, where $x_s \in \mathbb{X}_s$.

We define the Markov optimization problem

$$(\mathfrak{P}_t^{\mathrm{MNUT}})(x) \qquad \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{x_t,\pi,\mathrm{NUT}}\Big(\big\{ J_{t,s}^{x,\pi} \big\}_{s=t}^{T} \Big), \quad \forall t \in [\![0, T]\!], \quad \forall x \in \mathbb{X}_t \; , \tag{2.136}$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (2.27).

Definition 2.64. *We define the* value functions *inductively by the DPE*

$$V_T^{MNUT}(x) = \mathbb{G}_T^{x}\Big[J_T(x, \cdot) \Big], \quad \forall x \in \mathbb{X}_T \; , \tag{2.137a}$$

$$V_t^{MNUT}(x) = \inf_{u \in U_t(x)} \Phi_t^{x} \Big\{ \mathbb{G}_t^{x}\big[J_t(x,u,\cdot) \big], \mathbb{G}_t^{x}\big[V_{t+1}^{MNUT} \circ f_t(x,u,\cdot) \big] \Big\} \; , \quad \forall t \in [\![0, T-1]\!], \quad \forall x \in \mathbb{X}_t \; . \tag{2.137b}$$

The following Proposition 2.65 expresses sufficient conditions under which any Problem $(\mathfrak{P}_t^{\mathrm{MNUT}})(x)$, for all $t \in [\![0, T]\!]$ and for all $x \in \mathbb{X}_t$, can be solved by means of the value functions in Definition 2.64.

Proposition 2.65. *Assume that*

- *for all $t \in [\![0, T-1]\!]$, for all $x_t \in \mathbb{X}_t$, $\Phi_t^{x_t}$ is non-decreasing,*
- *for all $t \in [\![0, T]\!]$, for all $x_t \in \mathbb{X}_t$, $\mathbb{G}_t^{x_t}$ is non-decreasing.*

Assume that there exists[27] *an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that*

$$\pi_t^\sharp(x) \in \arg\min_{u \in U_t(x)} \Phi_t^{x} \Big\{ \mathbb{G}_t^{x}\big[J_t(x,u,\cdot) \big], \mathbb{G}_t^{x}\big[V_{t+1}^{MNUT} \circ f_t(x,u,\cdot) \big] \Big\}, \quad \forall t \in [\![0, T-1]\!], \quad \forall x \in \mathbb{X}_t \; . \tag{2.138}$$

Then, $\pi^\sharp$ is an optimal policy for any Problem $(\mathfrak{P}_t^{MNUT})(x)$, for all $t \in [\![0, T]\!]$ and for all $x \in \mathbb{X}_t$, and

$$V_t^{MNUT}(x) = \min_{\pi \in \Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{x,\pi,NUT}\Big(\big\{ J_{t,s}^{x,\pi} \big\}_{s=t}^{T} \Big), \quad \forall t \in [\![0, T]\!], \quad \forall x \in \mathbb{X}_t \; . \tag{2.139}$$

The following Theorem 2.66 is our main result on time-consistency in the NUT Markov case.

27. See Footnote 10.

Theorem 2.66. *Assume that*

- *for all* $t \in [\![0, T-1]\!]$, *for all* $x_t \in \mathbb{X}_t$, $\Phi_t^{x_t}$ *is non-decreasing,*
- *for all* $t \in [\![0, T]\!]$, *for all* $x_t \in \mathbb{X}_t$, $\mathbb{G}_t^{x_t}$ *is non-decreasing.*

Then

1. *for all policy* $\pi \in \Pi$, *the NUT-Markov dynamic uncertainty criterion* $\Big\{\big\{\varrho_{t,T}^{x_t,\pi,NUT}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T}$ *defined by* (2.135) *is time-consistent;*
2. *the Markov optimization problem* $\big\{\big\{(\mathfrak{P}_t^{MNUT})(x)\big\}_{x\in\mathbb{X}_t}\big\}_{t=0}^{T}$ *defined in* (2.136) *is time-consistent, as soon as there exists an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that* (2.138) *holds true.*

2.4.4 Commutation of Markov Aggregators

We extend the results on commutation obtained in §2.3.3 to Markov time and uncertainty aggregators. We do not give the proofs.

Consider a sequence $\Big\{\big\{\Phi_t^{x_t}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T-1}$ of one-step Markov time-aggregators and a sequence $\Big\{\big\{\mathbb{G}_t^{x_t}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T}$ of one-step Markov uncertainty-aggregators.

TU-Commutation of Markov Aggregators

The following Proposition 2.67 extends Proposition 2.39 to one-step Markov aggregators.

Proposition 2.67. *Suppose that, for any* $0 \leq t < s \leq T$, *for any states* $x_t \in \mathbb{X}_t$ *and* $x_s \in \mathbb{X}_s$, $\mathbb{G}_s^{x_s}$ *TU-commutes with* $\Phi_t^{x_t}$.

Then, for any policy $\pi \in \Pi$, *any* $0 \leq r < t \leq T$, *any states* $x_t \in \mathbb{X}_t$ *and* $x_r \in \mathbb{X}_r$, $\Big\langle \underset{t\leq s\leq T}{\overset{x_t,\pi}{\boxdot}} \mathbb{G}_s \Big\rangle$ *and* $\Big\langle \Phi_r^{x_r} \Big\rangle$ *TU-commute, that is,*

$$\Big\langle \underset{t\leq s\leq T}{\overset{x_t,\pi}{\boxdot}} \mathbb{G}_s \Big\rangle \Big[\Big\langle \Phi_r^{x_r} \Big\rangle \big\{c, A\big\}\Big] = \Big\langle \Phi_r^{x_r} \Big\rangle \Big\{c, \Big\langle \underset{t\leq s\leq T}{\overset{x_t,\pi}{\boxdot}} \mathbb{G}_s \Big\rangle \big[A\big]\Big\}, \tag{2.140}$$

for any extended scalar $c \in \bar{\mathbb{R}}$ *and any function* $A \in \mathcal{F}\big(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}}\big)$.

UT-Commutation of Markov Aggregators

The following Proposition 2.68 extends Proposition 2.42 to one-step Markov aggregators.

Proposition 2.68. *Suppose that, for any* $0 \leq t < s \leq T$, *for any states* $x_t \in \mathbb{X}_t$ *and* $x_s \in \mathbb{X}_s$, $\Phi_s^{x_s}$ *TU-commutes with* $\mathbb{G}_t^{x_t}$.

Then, for any policy $\pi \in \Pi$, *for any* $0 \leq r < t \leq T$, *any states* $x_r \in \mathbb{X}_r$ *and* $x_t \in \mathbb{X}_t$, $\Big\langle \underset{t\leq s\leq T-1}{\overset{x_t,\pi}{\odot}} \Phi_s \Big\rangle$ *TU-commutes with* $\langle \mathbb{G}_r^{x_r} \rangle$, *that is,*

$$\Big\langle \underset{t\leq s\leq T-1}{\overset{x_t,\pi}{\odot}} \Phi_s \Big\rangle \Big\{\Big\{ \langle \mathbb{G}_r^{x_r} \rangle \big[A_s\big]\Big\}_t^T\Big\} = \langle \mathbb{G}_r^{x_r} \rangle \Big[\Big\langle \underset{t\leq s\leq T-1}{\overset{x_t,\pi}{\odot}} \Phi_s \Big\rangle \Big\{\big\{A_s\big\}_t^T\Big\}\Big], \tag{2.141}$$

for any $\big\{A_s\big\}_{s=t}^T$, *where* $A_s \in \mathcal{F}\big(\mathbb{W}_{[0:T]}; \bar{\mathbb{R}}\big)$.

2.4.5 Time-Consistency for Non Nested Dynamic Uncertainty Criteria

TU Dynamic Markov Uncertainty Criterion

Definition 2.69. *Let a policy $\pi \in \Pi$ be given. We define the* TU-Markov dynamic uncertainty criterion $\Big\{\big\{\varrho_{t,T}^{x_t,\pi,TU}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T}$ *by*[28]

$$\varrho_{t,T}^{x_t,\pi,TU} = \Big\langle \underset{t\le s\le T}{\overset{x_t,\pi}{\boxdot}} \mathbb{G}_s \Big\rangle \circ \Big\langle \underset{t\le s\le T-1}{\overset{x_t,\pi}{\odot}} \Phi_s \Big\rangle , \quad \forall t \in [\![0,T]\!], \quad \forall x_t \in \mathbb{X}_t \; . \tag{2.142}$$

We define the Markov optimization problem

$$(\mathfrak{P}_t^{\mathrm{MTU}})(x) \qquad \min_{\pi\in\Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{\mathrm{MTU}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big), \quad \forall t \in [\![0,T]\!], \quad \forall x \in \mathbb{X}_t \; , \tag{2.143}$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (2.27).

The following Theorem 2.70 is our main result on time-consistency in the TU Markov case.

Theorem 2.70. *Assume that*

- *for any $0 \le s < t \le T$, for any states $x_t \in \mathbb{X}_t$ and $x_s \in \mathbb{X}_s$, $\mathbb{G}_t^{x_t}$ TU-commutes with $\Phi_s^{x_s}$,*
- *for all $t \in [\![0,T-1]\!]$, for all $x_t \in \mathbb{X}_t$, $\Phi_t^{x_t}$ is non-decreasing,*
- *for all $t \in [\![0,T]\!]$, for all $x_t \in \mathbb{X}_t$, $\mathbb{G}_t^{x_t}$ is non-decreasing.*

Then

1. *the TU-Markov dynamic uncertainty criterion $\big\{\varrho_{t,T}^{x_t,\pi,TU}\big\}_{t=0}^{T}$ defined by (2.142) is time-consistent;*
2. *the Markov optimization problem $\big\{\big\{(\mathfrak{P}_t^{x_t,\pi,MTU})(x)\big\}_{x\in\mathbb{X}_t}\big\}_{t=0}^{T}$ defined in (2.143) is time-consistent, as soon as there exists an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that (2.133) holds true, where the value functions are the $\big\{V_t^{NTU}\big\}_{t=0}^{T}$ in Definition 2.60.*

UT Dynamic Markov Uncertainty Criterion

For UT-Markov dynamic uncertainty criteria, we have to restrict the definition to the case where the sequence $\Big\{\big\{\Phi_t^{x_t}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T-1}$ of one-step Markov time-aggregators is a sequence $\big\{\Phi_t\big\}_{t=0}^{T-1}$ of one-step time-aggregators (see Remark 2.56).

Definition 2.71. *Let a policy $\pi \in \Pi$ be given. We define the* UT-Markov dynamic uncertainty criterion $\Big\{\big\{\varrho_{t,T}^{x_t,\pi,UT}\big\}_{x_t\in\mathbb{X}_t}\Big\}_{t=0}^{T}$ *by*[29]

$$\varrho_{t,T}^{x_t,\pi,UT} = \Big\langle \overset{T-1}{\underset{s=t}{\odot}} \Phi_s \Big\rangle \circ \Big\langle \underset{t\le s\le T}{\overset{x_t,\pi}{\boxdot}} \mathbb{G}_s \Big\rangle, \quad \forall t \in [\![0,T]\!], \quad \forall x_t \in \mathbb{X}_t \; . \tag{2.144}$$

We define the Markov optimization problem

$$(\mathfrak{P}_t^{\mathrm{MUT}})(x) \qquad \min_{\pi\in\Pi_t^{\mathrm{ad}}} \varrho_{t,T}^{\mathrm{MUT}}\Big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\Big), \quad \forall t \in [\![0,T]\!], \quad \forall x \in \mathbb{X}_t \; , \tag{2.145}$$

where the functions $J_{t,s}^{x,\pi}$ are defined by (2.27).

The following Theorem 2.72 is our main result on time-consistency in the UT Markov case.

28. See Footnote 18
29. See Footnote 18

Theorem 2.72. *Assume that*

- *for any* $0 \leq s < t \leq T$, *for any states* $x_t \in \mathbb{X}_t$, $\mathbb{G}_t^{x_t}$ *UT-commutes with* Φ_s,
- *for all* $t \in [\![0, T-1]\!]$, Φ_t *is non-decreasing,*
- *for all* $t \in [\![0, T]\!]$, *for all* $x_t \in \mathbb{X}_t$, $\mathbb{G}_t^{x_t}$ *is non-decreasing.*

Then

1. *the UT-Markov dynamic uncertainty criterion* $\big\{\varrho_{t,T}^{x_t,\pi,UT}\big\}_{t=0}^{T}$ *defined by* (2.144) *is time-consistent;*
2. *the Markov optimization problem* $\big\{\big\{(\mathfrak{P}_t^{x_t,\pi,MUT})(x)\big\}_{x\in\mathbb{X}_t}\big\}_{t=0}^{T}$ *defined in* (2.145) *is time-consistent, as soon as there exists an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that* (2.138) *holds true, where the value functions are the* $\big\{V_t^{NUT}\big\}_{t=0}^{T}$ *in Definition 2.64. (where* Φ_t *does not depend on* x_t*).*

2.4.6 Applications

Now, we present applications of Theorem 2.70, that is, the TU Markov case (see the discussion introducing §2.3.5).

Coherent Markov Risk Measures

We introduce a class of TU Markov dynamic uncertainty criteria, that are related to coherent risk measures (see Definition 2.9), and we show that they display time-consistency.

For all $t \in [\![0, T]\!]$ and all $x_t \in \mathbb{X}_t$, let be given $\mathcal{P}_t(x_t) \subset \mathcal{P}(\mathbb{W}_t)$. Let $(\alpha_t)_{t\in[\![0,T-1]\!]}$ and $(\beta_t)_{t\in[\![0,T-1]\!]}$ be sequences of functions, each mapping $\mathbb{X}_t \times \bar{\mathbb{R}}$ into $\mathbb{R}$, with the additional property that $\beta_t \geq 0$, for all $t \in [\![0, T-1]\!]$. Notice that, to the difference with the setting in §2.3.5, α_t and β_t can be functions of the state x.

For a policy $\pi \in \Pi$, for $t \in [\![0, T]\!]$ and for a state $x_t \in \mathbb{X}_t$, we set

$$
\begin{aligned}
\varrho_{t,T}^{x_t,\pi,\mathrm{co}}\big(\{A_s\}_{s=t}^{T}\big) = \sup_{\mathbb{P}_t \in \mathcal{P}_t(x_t)} \mathbb{E}_{\mathbb{P}_t}\Big[\cdots \sup_{\mathbb{P}_T \in \mathcal{P}_T(X_{t,T}^{x_t,\pi})} \mathbb{E}_{\mathbb{P}_T}\Big[\\
\sum_{s=t}^{T} \Big(\alpha_s\big(X_{t,s}^{x_t,\pi}, A_s\big) \prod_{r=t}^{s-1} \beta_r\big(X_{t,r}^{x_t,\pi}, A_r\big)\Big)\Big] \cdots \Big],
\end{aligned}
\tag{2.146}
$$

for any adapted uncertain process $\big\{A_t\big\}_0^T$, with the convention that $\alpha_T(x_T, c_T) = c_T$.

Proposition 2.73. *Time-consistency holds true for*

- *the Markov dynamic uncertainty criterion* $\{\{\varrho_{t,T}^{x_t,\pi,co}\}_{x_t\in\mathbb{X}_t}\}_{t=0}^{T}$ *given by* (2.146),
- *the Markov optimization problem*

$$
\min_{\pi \in \Pi^{\mathrm{ad}}} \varrho_{t,T}^{x,\pi,co}\big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^{T}\big), \quad \forall t \in [\![0, T]\!], \quad \forall x \in \mathbb{X}_t \ , \tag{2.147}
$$

where $J_{t,s}^{x,\pi}(w)$ *is defined by* (2.27), *as soon as there exists an admissible policy* $\pi^\sharp \in \Pi^{\mathrm{ad}}$ *such that, for all* $t \in [\![0, T-1]\!]$, *for all* $x \in \mathbb{X}_t$,

$$
\begin{aligned}
\pi_t^\sharp(x) \in \mathop{\arg\min}_{u \in U_t(x)} \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \Big\{ \mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(x, J_t(x,u,w_t)\big) \\
+ \beta_t\big(x, J_t(x,u,w_t)\big) V_{t+1} \circ f_t(x,u,w_t) \Big] \Big\} ,
\end{aligned}
$$

where the value functions are given by the following DPE

$$V_T(x) = \sup_{\mathbb{P}_T \in \mathcal{P}_T(x)} \mathbb{E}_{\mathbb{P}_T}\big[J_T(x,\cdot)\big] , \tag{2.148a}$$

$$\begin{aligned} V_t(x) = \min_{u \in U_t(x)} \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \Big\{ \mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(x, J_t(x,u,\cdot)\big) \\ + \beta_t\big(x, J_t(x,u,\cdot)\big) V_{t+1} \circ f_t(x,u,\cdot) \Big] \Big\} . \end{aligned} \tag{2.148b}$$

With the one-step Markov uncertainty-aggregator

$$\mathbb{G}_t^x\big[\,\cdot\,\big] = \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \mathbb{E}_{\mathbb{P}_t}\big[\,\cdot\,\big] , \tag{2.149}$$

the expression $\Big\langle \mathbb{G}_t^{\boldsymbol{X}_{0,t-1}} \Big\rangle$ (see Definition 2.25) defines a coherent Markov risk measure (Definition 2.13). The associated function Ψ_t in (2.37) is given by

$$\Psi_t\big(v,x,u\big) = \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \mathbb{E}_{\mathbb{P}_t}\Big[v \circ f_t\big(x,u,\cdot\big) \Big] . \tag{2.150}$$

We see by (2.34) that, for any state $x \in \mathbb{X}_t$, and any control $u \in \mathbb{U}_t$, the function $v \mapsto \Psi_t\big(v,x,u\big)$, is a coherent risk measure (see Definition 2.13).

Convex Markov Risk Measures

We introduce a class of TU-dynamic uncertainty criteria, that are related to convex risk measures (see Definition 2.9), and we show that they display time-consistency. We consider the same setting as for coherent risk measures, with the restriction that $\beta_t \equiv 1$ and an additional data $(\Upsilon_t)_{t \in [\![0,T]\!]}$.

For all $t \in [\![0,T]\!]$ and all $x_t \in \mathbb{X}_t$, let be given $\mathcal{P}_t(x_t) \subset \mathcal{P}(\mathbb{W}_t)$. Let $(\Upsilon_t)_{t \in [\![0,T]\!]}$ be a sequence of functions Υ_t mapping $\mathbb{X}_t \times \mathcal{P}(\mathbb{W}_t)$ into $\bar{\mathbb{R}}$. Let $(\alpha_t)_{t \in [\![0,T]\!]}$ be a sequence of functions α_t mapping $\mathbb{X}_t \times \bar{\mathbb{R}}$ into $\mathbb{R}$. Notice that, to the difference with the setting in §2.3.5, α_t and Υ_t can be functions of the state x.

For a policy $\pi \in \Pi$, a time $t \in [\![0,T]\!]$ and a state $x_t \in \mathbb{X}_t$, we set

$$\begin{aligned} \varrho_{t,T}^{x_t,\pi,\mathrm{cx}}\big(\{A_s\}_{s=t}^T\big) = \sup_{\mathbb{P}_t \in \mathcal{P}_t(x_t)} \mathbb{E}_{\mathbb{P}_t}\Bigg[\cdots \sup_{\mathbb{P}_T \in \mathcal{P}_T(x_T)} \mathbb{E}_{\mathbb{P}_T}\Bigg[\\ \sum_{s=t}^T \Big(\alpha_s\big(x_s, A_s\big) - \Upsilon_s\big(x_s, \mathbb{P}_s\big) \Big) \Bigg] \cdots \Bigg], \end{aligned} \tag{2.151}$$

for any adapted uncertain process $\big\{A_t\big\}_0^T$, with the convention that $\alpha_T(c_T) = c_T$.

Proposition 2.74. *Time-consistency holds true for*

- *the dynamic uncertainty criterion* $\{\{\varrho_{t,T}^{x_t,\pi,cx}\}_{x_t \in \mathbb{X}_t}\}_{t=0}^T$ *given by* (2.151),
- *the Markov optimization problem*

$$\min_{\pi \in \Pi^{\mathrm{ad}}} \varrho_{t,T}^{x,\pi,cx}\big(\big\{J_{t,s}^{x,\pi}\big\}_{s=t}^T\big), \quad \forall t \in [\![0,T]\!], \quad \forall x \in \mathbb{X}_t , \tag{2.152}$$

where $J_{t,s}^{x,\pi}(w)$ is defined by (2.27)*, as soon as there exists an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that, for all $t \in [\![0,T-1]\!]$, for all $x \in \mathbb{X}_t$,*

$$\pi_t^\sharp(x) \in \mathop{\arg\min}_{u \in U_t(x)} \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \Big\{ \mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(x, J_t(x,u,\cdot)\big) + V_{t+1} \circ f_t(x,u,\cdot) \Big] - \Upsilon_t(x,\mathbb{P}_t) \Big\} ,$$

where the value functions are given by the following DPE

$$V_T(x) = \sup_{\mathbb{P}_T \in \mathcal{P}_T(x)} \Big\{ \mathbb{E}_{\mathbb{P}_T}\Big[\alpha_T\big(x, J_T(x,\cdot)\big) \Big] - \Upsilon_T(x, \mathbb{P}_T) \Big\} \,, \tag{2.153a}$$

$$V_t(x) = \min_{u \in U_t(x)} \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \Big\{ \mathbb{E}_{\mathbb{P}_t}\Big[\alpha_t\big(x, J_t(x,u,\cdot)\big) + V_{t+1} \circ f_t(x,u,\cdot) \Big] - \Upsilon_t(x, \mathbb{P}_t) \Big\} \,. \tag{2.153b}$$

With the one-step Markov uncertainty-aggregator

$$\mathbb{G}_t^x\big[\,\cdot\,\big] = \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \Big\{ \mathbb{E}_{\mathbb{P}_t}\big[\,\cdot\,\big] - \Upsilon_t(x, \mathbb{P}_t) \Big\} \,, \tag{2.154}$$

the expression $\Big\langle \mathbb{G}_t^{\boldsymbol{X}_{0,t-1}} \Big\rangle$ (see Definition 2.25) defines a convex Markov risk measure (Definition 2.13). The associated function Ψ_t in (2.37) is given by

$$\Psi_t\big(v,x,u\big) = \sup_{\mathbb{P}_t \in \mathcal{P}_t(x)} \Big\{ \mathbb{E}_{\mathbb{P}_t}\Big[v \circ f_t\big(x,u,\boldsymbol{W}_t\big) \Big] - \Upsilon_t(x, \mathbb{P}_t) \Big\} \,. \tag{2.155}$$

We see by (2.34) that, for any state $x \in \mathbb{X}_t$, and any control $u \in \mathbb{U}_t$, the function $v \mapsto \Psi_t\big(v,x,u\big)$, is a convex risk measure (see Definition 2.13).

2.5 Discussion

We discuss how our assumptions and results in §2.3 relate to other results in the literature on time-consistency for dynamic risk measures

First, we examine the connections between time-consistency for Markov dynamic uncertainty criteria and the existence of a DPE. When we analyze the literature on time-consistency for risk measures with our tools (aggregators), we observe that

- most, if not all results, are obtained for the specific case of linear one-step time-aggregators $\Phi_t\big\{c_t, c_{t+1}\big\} = c_t + c_{t+1}$,
- a key ingredient to obtain time-consistency is an equation like (2.156a), which corresponds to the commutation of one-step uncertainty-aggregators with the sum (that is, with the linear one-step time-aggregators actually used).

Therefore, Theorems 2.31, 2.36, 2.44, 2.46 in §2.3 provide an umbrella for most of the results establishing time-consistency for dynamic risk measures, and yields extensions to more general time-aggregators than the sum. In [23], time-consistency for dynamic risk measures is not defined by a monotonicity property like in [105] but in line with the existence of a DPE. In [56], the time-consistency property is comparable to Definition 2.16, though being restricted to the multiplicative time-aggregator.

We discuss to some extent [105] where time-consistency for dynamic risk measures plus an additional assumption like (2.156a) lead to the existence of a DPE, within the original framework of *Markov risk measures* sketched above. Here is the statement of Theorem 1 in [105], with the notations of §2.2.2.

Theorem 2.75 ([105]). *Suppose that a dynamic risk measure $\big\{\rho_{t,T}\big\}_{t=0}^{T}$ satisfies, for all $t \in [\![0,T]\!]$, and all $\boldsymbol{A}_t \in \mathcal{L}_t$ the conditions*

$$\rho_{t,T}\Big(\big\{\boldsymbol{A}_s\big\}_{s=t}^{T}\Big) \;=\; \boldsymbol{A}_t + \rho_{t,T}\Big(\big\{0, \boldsymbol{A}_{t+1}, \cdots, \boldsymbol{A}_T\big\}\Big) \,, \tag{2.156a}$$

$$\rho_{t,T}\Big(\big\{0\big\}_{s=t}^{T}\Big) \;=\; 0 \,. \tag{2.156b}$$

Then ρ is time-consistent iff, for all $0 \leq s \leq t \leq T$ and all $\{\boldsymbol{A}_s\}_0^T \in \mathcal{L}_{0,T}$, the following identity is true:

$$\rho_{s,T}\Big(\big\{\boldsymbol{A}_r\big\}_{r=s}^T\Big) = \rho_{s,t}\Big(\big\{\boldsymbol{A}_r\big\}_{r=s}^t, \rho_{t,T}\big(\big\{\boldsymbol{A}_r\big\}_{r=t}^T\big)\Big) . \tag{2.157}$$

In [105, Section 5], the finite horizon problem corresponds to Problem (2.95), starting at $t = 0$, where the one-step uncertainty aggregator $\mathbb{G}_t$ in (2.95) corresponds to the one-step conditional risk measure ρ_t, the one-step time-aggregator Φ_t in (2.95) corresponds to the sum, and the cost J_t in (2.95) is denoted c_t in [105]. Commutation of the one-step time-aggregators Φ_t and the one-step uncertainty-aggregators $\mathbb{G}_s$ is ensured through the equivariance translation property (2.156a) of a coherent measure of risk. Monotonicity of the uncertainty aggregator $\mathbb{G}_s$ corresponds to the monotonicity property of a coherent risk measure, and monotonicity of the time aggregator is obvious. Thus, Theorem 2.44 leads to the same DPE as [105, Theorem 2].

Let us now focus on the differences between [105] and our results. In [105], arguments are given to show that there exists an optimal Markovian policy among the set of adapted policies (that is, having a policy taking as argument the whole past uncertainties would not give a better cost than a policy taking as argument the current value of the state). We do not tackle this issue since we directly deal with policies as functions of the state. Where we suppose that there exists an admissible policy $\pi^\sharp \in \Pi^{\mathrm{ad}}$ such that (2.62) holds true, [105] gives conditions ensuring this property. Finally, where [105] restricts to the sum to aggregate instantaneous costs, we consider more general one-step time-aggregators Φ_t. Moreover where we give a sufficient condition for a Markovian policy to be optimal, [105] gives a set of assumptions such that this sufficient condition is also necessary (typically assumption ensuring that minimums are attained).

Second, we discuss the possibility to modify a Markov optimization problem or a dynamic risk measure, in order to make it time-consistent (if it were not originally). When sequences of optimization problems are not time-consistent with the original “state”, they can be made time-consistent by extending the state. In [26], this is done for a sequence of optimization problem under a chance constraint. In [107, Example 1], the sum of AV@R of costs is considered (given by the dynamic risk measure defined in 2.1.2 and labeled (TU)). This formulation is not time consistent. However, exploiting the formulation (2.19) of AV@R, we suggest to extend the state and add the variables $\{r_s\}_0^T$ so that, after transformation, we obtain a problem with expectation as uncertainty aggregator, and sum as time aggregator, thus yielding time-consistency. In [78], it is shown how a large class of possibly time-inconsistent dynamic risk measures, called spectral risk measures and constructed as a convex combination of AV@R, can be made time-consistent by what we interpret as an extension of the state.

Chapter 3

Stochastic Dual Dynamic Programming Algorithm

It is really true what philosophy tells us, that life must be understood backwards. But with this, one forgets the second proposition, that it must be lived forwards.

Søren Kierkegaard

Contents

In Chapter 2 we presented a general framework for Dynamic Programming without any numerical considerations. Here, we are interested with approaches that circumvent the curse of dimensionality. Indeed, we study algorithms that exploits analytical properties of the value functions (mainly convexity) to construct approximations of those value functions.

By contrast with the rest of the manuscript, the formalism of this chapter is the formalism of the Stochastic Programming community, where the uncertainties are encoded on a tree. We have seen in §1.1.2 that this framework can be translated to the stochastic optimal control framework.

The contents of this chapter has been accepted (up to minor modifications) by the *Mathematics of Operations Research* journal, under the name "On the convergence of decomposition methods for multistage stochastic convex programs". It is a common work with A. Philpott and P. Girardeau. The abstract is the following.

This chapter prove the almost-sure convergence of a class of sampling-based nested decomposition algorithms for multistage stochastic convex programs in which the stage costs are general convex functions of the decisions, and uncertainty is modelled by a scenario tree. As special cases, our results imply the almost-sure convergence of SDDP, CUPPS and DOASA when applied to problems with general convex cost functions.

Introduction

Multistage stochastic programs with recourse are well known in the stochastic programming community, and are becoming more common in applications. We are motivated in this paper by applications in which the stage costs are nonlinear convex functions of the decisions. Production functions are often modelled as nonlinear concave functions of allocated resources. For example Finardi and da Silva [47] use this approach to model hydro electricity production as a concave function of water flow. Smooth nonlinear value functions also arise when one maximizes profit with linear demand functions (see e.g. [81]) giving a concave quadratic objective or when coherent risk measures are defined by continuous distributions in multistage problems [108].

Having general convex stage costs does not preclude the use of cutting plane algorithms for attacking these problems. Kelley's cutting plane method [57] was originally devised for general convex objective functions, and can be shown to converge to an optimal solution (see e.g. Ruszczynski [102, Theorem 7.7]), although on some instances this convergence might be very slow [69]. Our goal in this paper is to extend the convergence result of [102] to the setting of multi-stage stage stochastic convex programming.

The most well-known application of cutting planes in multi-stage stochastic programming is the stochastic dual dynamic programming (SDDP) algorithm of Pereira and Pinto [77]. This algorithm constructs feasible dynamic programming policies using an outer approximation of a (convex) future cost function that is computed using Benders cuts. The policies defined by these cuts can be evaluated using simulation, and their performance measured against a lower bound on their expected cost. This provides a convergence criterion that may be applied to terminate the algorithm when the estimated cost of the candidate policy is close enough to its lower bound. The SDDP algorithm has led to a number of related methods [27,38,39,55,80] that are based on the same essential idea, but seek to improve the method by exploiting the structure of particular applications. We call these methods DOASA for Dynamic Outer-Approximation Sampling Algorithms but they are now generically named SDDP methods.

SDDP methods are known to converge almost surely on a finite scenario tree when the stage problems are linear programs. The first formal proof of such a result was published by Chen and Powell [27] who derived this for their CUPPS algorithm. This proof was extended by Linowsky and Philpott [64] to cover other SDDP algorithms. The convergence proofs in [27] and [64] make use of an unstated assumption regarding the independence of sampled random variables and convergent subsequences of algorithm iterates. This assumption was identified by Philpott and Guan [80], who gave a simpler proof of almost sure convergence of SDDP methods based on the finite convergence of the nested decomposition algorithm (see [38]). This does not require the unstated assumption, but exploits the fact that the collection of subproblems to be solved has a finite number of dual extreme points. This begs the question of whether SDDP methods will converge almost surely for general convex stage problems, where the value functions may admit an infinite number of subgradients.

In this paper we propose a different approach from the one in [27] and [64] and show how a proof of convergence for sampling-based nested decomposition algorithms on finite scenario trees can be established for models with convex subproblems (which may not have polyhedral value functions). Our result is proved for a general class of methods including all the variations discussed in the literature ([27, 38, 39, 55, 77, 80]). The proof establishes convergence with probability 1 as long as the sampling in the forward pass is independent of previous realizations. Our proof relies heavily on the independence assumption and makes use of the Strong Law of Large Numbers. In contrast to [80] we have not shown that convergence is guaranteed in all procedures for constructing a forward pass that visit every node of the scenario tree an infinite number of times.

The result we prove works in the space of state variables expressed as random variables adapted to the filtration defined by the scenario tree. Because this tree has a finite number of nodes, this space is compact, and so we may extract convergent subsequences for any infinite sequence of states. Unlike the arguments in [27] and [64], these subsequences are not explicitly constructed, and so we can escape the need to assume properties of them that we wish to be inherited from independent sampling. More precisely Lemma 3.12 gives us the required independence.

Although the value functions we construct admit an infinite number of subgradients, our results do require an assumption that serves to bound the norms of the subgradients used. This assumption is an extension of relatively complete recourse that ensures that some infeasible candidate solutions to any stage problem can be forced to be feasible by a suitable control. Since we are working in the realm of nonlinear programming, some constraint qualification of this form will be needed to ensure that we can extract subgradients. In practice, SDDP models use penalties on constraint violations to ensure feasibility, which implicitly bounds the subgradients of the Bellman functions. Our recourse assumptions are arguably weaker, since we do not have a result that shows that they enable an equivalent formulation with an exact penalization of infeasibility.

The paper is laid out as follows. We first consider a deterministic multistage problem, in which the proof is easily understandable. This is then extended in §3.2 to a stochastic problem formulated in a scenario tree. We close with some remarks about the convergence of sampling algorithms.

3.1 Deterministic Case

Our convergence proofs are based around showing that a sequence of outer approximations formed by cutting planes converges to the true Bellman function in the neighborhood of the optimal state trajectories. We begin by providing a proof that Kelley's cutting plane method [57] converges when applied to the optimization problem:

$$W^* := \min_{u \in \mathcal{U}} W(u),$$

where $\mathcal{U}$ is a nonempty convex subset of $\mathbb{R}^m$, and W is a convex finite function on $\mathbb{R}^m$. The result we prove is not directly used in the more complex results that follow, but the main ideas on which the proofs rely are the same. We believe the reader will find it convenient to already have the scheme of the proof in mind when studying the more important results later on.

Kelley's method generates a sequence of iterates $\left(u^j\right)_{j\in\mathbb{N}}$ by solving, at each iteration, a piecewise linear model of the original problem. The model is then enhanced by adding a cutting plane based on the value $W\left(u^j\right)$ and subgradient g^j of W at u^j. The model at iteration k is denoted by

$$W^k(u) := \max_{1\le j\le k} \left\{ W\left(u^j\right) + \left\langle g^j, u-u^j \right\rangle \right\},$$

and $\theta^k := \min_{u\in\mathcal{U}} W^k(u) = W^k\left(u^{k+1}\right)$. We have the following result.

Lemma 3.1. *If W is convex with uniformly bounded subgradients on $\mathcal{U}$ and $\mathcal{U}$ is compact then*

$$\lim_{k\to+\infty} W\left(u^k\right) = W^*.$$

Proof. This proof is taken from Ruszczynski [102, Theorem 7.7] (see also [101]). Let K_ε be the set of indexes k such that $W^* + \varepsilon < W\left(u^k\right) < +\infty$. The proof consists in showing that K_ε is finite.

Suppose $k_1, k_2 \in K_\varepsilon$ and k_1 is strictly smaller than k_2. We have that $W\left(u^{k_1}\right) > W^* + \varepsilon$ and that $W^* \geq \theta^{k_1}$. Since a new cut is generated at u^{k_1}, we have

$$W\left(u^{k_1}\right) + \left\langle g^{k_1}, u - u^{k_1}\right\rangle \leq W^{k_1}(u) \leq W^{k_2-1}(u), \quad \forall u \in \mathcal{U},$$

where g^{k_1} is an element of $\partial W\left(u^{k_1}\right)$. In particular, choosing $u = u^{k_2}$ gives

$$W\left(u^{k_1}\right) + \left\langle g^{k_1}, u^{k_2} - u^{k_1}\right\rangle \leq W^{k_1}\left(u^{k_2}\right) \leq W^{k_2-1}\left(u^{k_2}\right) = \theta^{k_2-1} \leq W^*.$$

But $\varepsilon < W\left(u^{k_2}\right) - W^*$, so

$$\varepsilon < W\left(u^{k_2}\right) - W\left(u^{k_1}\right) - \left\langle g^{k_1}, u^{k_2} - u^{k_1}\right\rangle,$$

and as $g^{k_2} \in \partial W(u^{k_2})$, the subgradient inequality for $u = u^{k_1}$ yields

$$W\left(u^{k_2}\right) - W\left(u^{k_1}\right) \leq \left\langle g^{k_2}, u^{k_2} - u^{k_1}\right\rangle.$$

Therefore, since W has uniformly bounded subgradients, there exists $\kappa > 0$ such that

$$\varepsilon < 2\kappa \left\|u^{k_2} - u^{k_1}\right\|, \qquad \forall k_1, k_2 \in K_\varepsilon, k_1 \neq k_2.$$

Because $\mathcal{U}$ is compact, K_ε has to be finite. Otherwise there would exist a convergent subsequence of $\left\{u^k\right\}_{k \in K_\varepsilon}$ and this last inequality could not hold for sufficiently large indexes within K_ε. This proves the lemma. □

Note that Lemma 3.1 does not imply that the sequence of iterates $\left(u^k\right)_{k \in \mathbb{N}}$ converges [1]. For instance, if the minimum of W is attained on a "flat" part (if W is not strictly convex), then the sequence of iterates may not converge. However, the lemma shows that the sequence of W values at these iterates will converge.

3.1.1 Multistage Setting

We now consider the multistage case. Let T be a positive integer. We first consider the following deterministic optimal control problem.

$$\min_{x,u} \quad \sum_{t=0}^{T-1} C_t\left(x_t, u_t\right) + V_T\left(x_T\right) \tag{3.1a}$$

$$\text{s.t.} \quad x_{t+1} = f_t\left(x_t, u_t\right), \qquad \forall t \in [\![0, T-1]\!], \tag{3.1b}$$

$$x_0 \text{ is given}, \tag{3.1c}$$

$$x_t \in \mathcal{X}_t, \qquad \forall t \in [\![0, T]\!], \tag{3.1d}$$

$$u_t \in \mathcal{U}_t(x_t), \qquad \forall t \in [\![0, T-1]\!]. \tag{3.1e}$$

In what follows we let $\mathrm{Aff}(\mathcal{X})$ denote the affine hull of $\mathcal{X}$, and define

$$B_t(\delta) = \{y \in \mathrm{Aff}(\mathcal{X}_t) \quad | \quad \| y \| < \delta\}.$$

We make the following assumptions (H_1):

1. for $t = 0, \ldots, T$, $\emptyset \subset \mathcal{X}_t \subset \mathbb{R}^n$,

1. even though because $\mathcal{U}$ is compact, there exists a convergent subsequence.

2. for $t = 0, \ldots, T-1$, multifunctions $\mathcal{U}_t : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ are assumed to be convex [2] and nonempty convex compact valued,
3. the final cost function V_T and functions C_t, $t = 0, \ldots, T$ are assumed to be convex lower semicontinuous proper functions,
4. for $t = 0, \ldots, T-1$, functions f_t are affine,
5. the final cost function V_T is finite-valued and Lipschitz-continuous on $\mathcal{X}_T$,
6. for $t = 0, \ldots, T-1$, there exists $\delta_t > 0$, defining $\mathcal{X}'_t := \mathcal{X}_t + B_t(\delta_t)$, such that :
   (a) $\forall x \in \mathcal{X}'_t, \quad \forall u \in \mathcal{U}_t(x), \qquad C_t(x,u) < \infty$,
   (b) for every $x \in \mathcal{X}'_t$,
   $$f_t(x, \mathcal{U}_t(x)) \cap \mathcal{X}_{t+1} \neq \emptyset.$$

Assumptions $(H_1(1) - (5))$ are made to guarantee that problem (3.1) is a convex optimization problem. Since this problem is in general nonlinear, it also requires a constraint qualification to ensure the existence of subgradients. This is the purpose of Assumption $(H_1(6))$. This assumption means that we can always move from $\mathcal{X}_t$ a distance of $\frac{\delta_t}{2}$ in any direction and stay in $\mathcal{X}'_t$, which is a form of recourse assumption that we call *extended relatively complete recourse* (ERCR). We note that this is less stringent than imposing complete recourse, which would require $\mathcal{X}'_t = \mathbb{R}^n$. Finally we note that we never need to evaluate $C_t(x,u)$ with $x \in \mathcal{X}'_t \backslash \mathcal{X}_t$, so we may only assume that there exists a convex function, finite on $\mathcal{X}'_t$, that coincides with C_t on $\mathcal{X}_t$. Of course not all convex cost functions satisfy such a property e.g. $x \mapsto x\log(x)$ cannot be extended below $x = 0$ while maintaining convexity.

We are now in a position to describe an algorithm for the deterministic control problem (3.1). The Dynamic Programming (DP) equation associated with (3.1) is as follows. For all $t \in [\![0, T-1]\!]$, let

$$V_t(x_t) = \begin{cases} \min_{u_t \in \mathcal{U}_t(x_t)} & \underbrace{\{C_t(x_t,u_t) + V_{t+1}(f_t(x_t,u_t))\}}_{:=W_t(x_t,u_t)}, & x_t \in \mathcal{X}_t \\ +\infty, & & \text{otherwise.} \end{cases} \tag{3.2}$$

Here the quantity $W_t(x_t, u_t)$ is the future optimal cost starting at time t from state x and choosing decision u_t, so that $V_t(x) = \min_{u \in \mathcal{U}_t(x_t)} W_t(x,u)$.

The cutting plane method works as follows. At iteration 0, define functions V_t^0, $t \in [\![0, T-1]\!]$, to be identically equal to $-\infty$. At time T, since we know exactly the end value function, we impose $V_T^k = V_T$ for all iterations $k \in \mathbb{N}$. At each iteration k, the process is the following.

Starting with $x_0^k = x_0$, at any time stage t, solve

$$\theta_t^k = \min_{\substack{u_t \in \mathbb{R}^m \\ x \in \mathrm{Aff}(\mathcal{X}_t)}} \quad C_t(x,u_t) + V_{t+1}^{k-1} \circ f_t(x,u_t), \tag{3.3a}$$
$$\text{s.t.} \quad x = x_t^k \qquad [\beta_t^k] \tag{3.3b}$$
$$f_t(x,u_t) \in \mathcal{X}_{t+1} \tag{3.3c}$$
$$u_t \in \mathcal{U}_t(x) \tag{3.3d}$$

Here $\beta_t^k \in \mathrm{Aff}(\mathcal{X}_t)$ is a vector of Lagrange multipliers for the constraint $x = x_t^k$. We denote a minimizer of (3.3) by u_t^k. Its existence is guaranteed by ERCR. Note that constraint

2. Recall that a multifunction $\mathcal{U}$ on convex set $\mathcal{X}$ is called *convex* if $(1-\lambda)\mathcal{U}(x) + \lambda\mathcal{U}(y) \subseteq \mathcal{U}((1-\lambda)x + \lambda y)$ for every $x, y \in \mathcal{X}$ and $\lambda \in (0,1)$.

(3.3c) can be seen as an induced constraint on u_t. Thus we can define the multifunctions $\tilde{\mathcal{U}}_t : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ by, for all $x \in \mathbb{R}^n$,

$$\tilde{\mathcal{U}}_t(x) := \{u \in \mathcal{U}_t(x) \quad | \quad f_t(x, u_t) \in \mathcal{X}_{t+1}\}. \tag{3.4}$$

We can easily check that $\tilde{\mathcal{U}}_t$ is convex (by linearity of f_t and convexity of $\mathcal{U}_t$) and convex compact valued (as the intersection of a compact convex set and a closed convex set). Moreover ERCR guarantee that $\tilde{\mathcal{U}}_t(x) \neq \emptyset$ for any $x \in \mathcal{X}_t$. Thus (3.3) can be written as

$$\theta_t^k = \min_{\substack{u_t \in \tilde{\mathcal{U}}_t(x) \\ x \in \mathrm{Aff}\left(\mathcal{X}_t\right)}} \quad C_t\left(x, u_t\right) + V_{t+1}^{k-1} \circ f_t\left(x, u_t\right), \tag{3.5a}$$

$$\text{s.t.} \quad x = x_t^k. \qquad [\beta_t^k] \tag{3.5b}$$

Now define, for any $x \in \mathbb{R}^n$:

$$V_t^k\left(x\right) := \max\left\{V_t^{k-1}\left(x\right), \theta_t^k + \left\langle \beta_t^k, x - x_t^k \right\rangle\right\}, \tag{3.6}$$

and move on to the next time stage $t+1$ by defining $x_{t+1}^k = f_t\left(x_t^k, u_t^k\right)$.

Remark 3.2. *The assumption that β_t^k is in Aff$(\mathcal{X}_t)$ is made for technical reasons, and loses little generality. Indeed if $\beta_t^k \in \mathbb{R}^n$ is an optimal Lagrange multiplier, then so is its projection on Aff$(\mathcal{X}_t)$. In practice we would expect Aff$(\mathcal{X}_t)$ to be the same dimension for every t. If this dimension happened to be d strictly less than n, then we might change the formulation (by a transformation of variables) so that Aff$(\mathcal{X}_t) = \mathbb{R}^d$.*

Remark 3.3. *Observe that our algorithm uses V_{t+1}^{k-1} when solving the two-stage problem (3.3) at stage t, although most implementations of SDDP and related algorithms proceed backwards and are thus able to use the freshly updated V_{t+1}^k (although see e.g. [27] for a similar approach to the one proposed here). In the stochastic case we present a general framework that encompasses backward passes.*

Note that only the last future cost function V_T is known exactly at any iteration. All the other ones are lower approximations consisting of the maximum of k affine functions at iteration k. We naturally have the same lower approximation for function W_t. Thus we define for any (x, u) in $\mathbb{R}^{n+m}$

$$W_t^k\left(x, u\right) := C_t\left(x, u\right) + V_{t+1}^k \circ f_t\left(x, u\right), \tag{3.7}$$

and recall

$$W_t\left(x, u\right) := C_t\left(x, u\right) + V_{t+1} \circ f_t\left(x, u\right). \tag{3.8}$$

Using this notation we have

$$\theta_t^k = \min_{u \in \tilde{\mathcal{U}}_t(x_t^k)} W_t^{k-1}\left(x_t^k, u\right) = W_t^{k-1}\left(x_t^k, u_t^k\right) \tag{3.9}$$

Since by (3.6)

$$V_t^k\left(x_t^k\right) = \max_{k' \leq k}\left\{\theta_t^{k'} + \left\langle \beta_t^{k'}, x_t^k - x_t^{k'} \right\rangle\right\}$$

it follows that

$$V_t^k\left(x_t^k\right) \geq W_t^{k-1}(x_t^k, u_t^k). \tag{3.10}$$

Figure 3.1 gives a view of the relations between all these values at a given iteration.

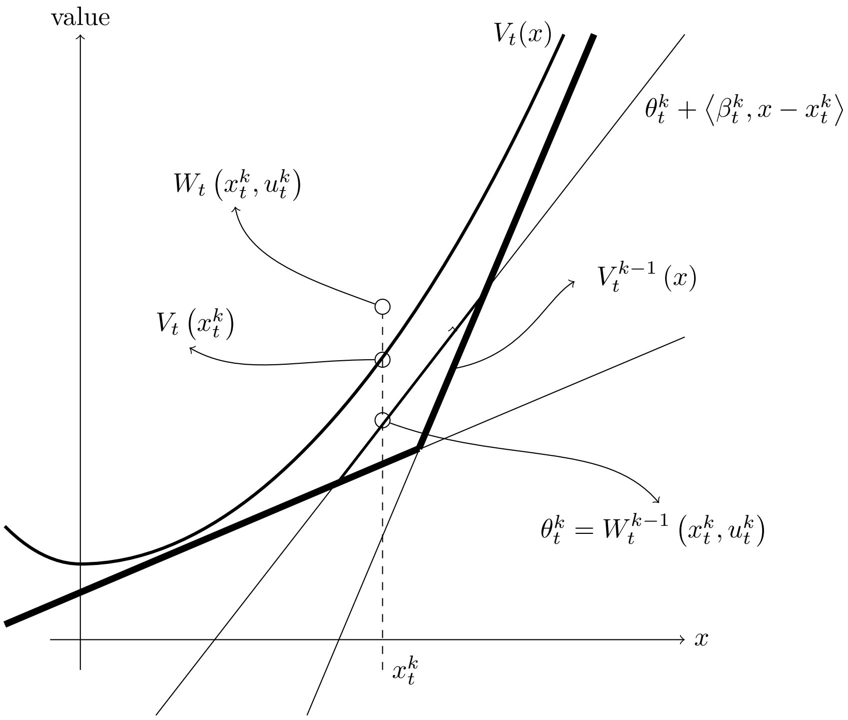


Figure 3.1: Relation between the values at a given iteration

3.1.2 Proof of Convergence in the Deterministic Case.

We begin by showing some regularity and monotonicity results for the value functions and their approximations.

Under assumptions (H_1), we define for $t \in [\![0, T-1]\!]$, and for all $x \in \mathbb{R}^n$, the extended value function

$$\tilde{V}_t(x) = \inf_{u \in \mathcal{U}_t(x)} \left\{ C_t(x,u) + V_{t+1} \circ f_t(x,u) \right\}. \tag{3.11}$$

Note that the infimum could be taken on $\tilde{\mathcal{U}}_t(x) \subseteq \mathcal{U}_t(x)$ as $V_{t+1} = \infty$ when $f_t(x,u) \notin \mathcal{X}_{t+1}$. It is convenient to extend the definition to $t = T$ by defining $\tilde{V}_T = V_T$. We also observe that $\tilde{V}_t \leq V_t$ as these are identical on the domain of V_t.

Lemma 3.4. *For $t \in [\![0, T-1]\!]$,*

(i) the value function V_t is convex and Lipschitz continuous on $\mathcal{X}_t$;

(ii) $V_t^k \leq \tilde{V}_t \leq V_t$, and β_t^k is defined;

(iii) the sequences $(\beta_t^k)_{k \in \mathbb{N}}$ are bounded.

Proof. (i) We first show the convexity and Lipschitz continuity of V_t on $\mathcal{X}_t$. We proceed by induction backward in time. By assumption V_T is convex and Lipschitz continuous on $\mathcal{X}_T$. Assume the result is true for V_{t+1}. The function $\tilde{V}_t(x)$ is convex by lemma 3.9. Now by ERCR, for any $x \in \mathcal{X}_t'$, $\tilde{\mathcal{U}}_t(x) \neq \emptyset$. This implies that, for $x \in \mathcal{X}_t'$, for $u \in \tilde{\mathcal{U}}_t(x)$,

$$\tilde{V}_t(x) \leq C_t(x,u) + V_{t+1} \circ f_t(x,u) < +\infty.$$

By $(H_1(3))$ and the induction hypothesis, for any $x \in \mathcal{X}_t'$,

$$u \mapsto C_t(x,u) + V_{t+1} \circ f_t(x,u)$$

is lower semi-continuous, and so the compactness of $\mathcal{U}_t(x)$ ensures that the infimum in the definition of $\tilde{V}_t(x)$ is attained, and therefore $\tilde{V}_t(x) > -\infty$. $\tilde{V}_t$ is Lipschitz continuous on

$\mathcal{X}_t$ as $\mathcal{X}_t$ is a compact subset of the relative interior of its domain. Finally remarking that $V_t(x) = \tilde{V}_t(x)$ if $x \in \mathcal{X}_t$ gives the conclusion.

(ii) As observed above the inequality $\tilde{V}_t \leq V_t$ is immediate as the two functions are identical on the domain of V_t.

To show $V_t^k \leq \tilde{V}_t$ let us proceed by induction forward in k. Assume that for all $t \in [\![0, T-1]\!]$, β_t^{k-1} is defined and $V_t^{k-1} \leq \tilde{V}_t$. Note that

$$-\infty = V_t^0 \leq \tilde{V}_t,$$

so this is true for $k = 1$ (β_t^0 is never used). We now define, for all $t \in [\![0, T-1]\!]$ and all $x \in \mathbb{R}^n$,

$$\hat{V}_t^k(x) = \min_{u \in \tilde{\mathcal{U}}_t(x)} \left\{ C_t(x,u) + V_{t+1}^{k-1} \circ f_t(x,u) \right\}.$$

By hypothesis on $\tilde{\mathcal{U}}_t$, $\hat{V}_t^k$ is convex and finite on $\mathcal{X}_t'$ which strictly contains $\mathcal{X}_t$. Thus $\hat{V}_t^k$ restricted to $\mathrm{Aff}(\mathcal{X}_t)$ is subdifferentiable at any point of $\mathcal{X}_t$. Moreover by definition of β_t^k in (3.3)

$$\beta_t^k \in \partial\Big(\hat{V}_t^k|_{\mathrm{Aff}(\mathcal{X}_t)}\Big). \tag{3.12}$$

Thus β_t^k is defined. By the induction hypothesis and inequality $\tilde{V}_{t+1} \leq V_{t+1}$ we have that

$$V_{t+1}^{k-1} \circ f_t \leq V_{t+1} \circ f_t.$$

Thus the definitions of $\hat{V}_t^k$ and $\tilde{V}_t$ yield

$$\hat{V}_t^k \leq \tilde{V}_t. \tag{3.13}$$

we have by (3.12) that

$$\theta_t^k + \left\langle \beta_t^k, x - x_t^k \right\rangle \leq \hat{V}_t^k(x) \leq \tilde{V}_t(x) \tag{3.14}$$

by (3.13). The definition of V_t^k in (3.6) gives

$$V_t^k(x) = \max\left\{ V_t^{k-1}(x), \theta_t^k + \left\langle \beta_t^k, x - x_t^k \right\rangle \right\}$$

which shows $V_t^k(x) \leq \tilde{V}_t(x)$ by (3.14) and the induction hypothesis. Thus (ii) follows for all k by induction.

(iii) Finally we show the boundedness of $(\beta_t^k)_{k \in \mathbb{N}}$. By definition of β_t^k we have for all $y \in \mathbb{R}^n$,

$$V^k(y) \geq V^k(x_t^k) + \left\langle \beta_t^k, y - x_t^k \right\rangle. \tag{3.15}$$

Recall that $\mathcal{X}_t' = \mathcal{X}_t + B_t(\delta_t)$, so substituting $y = x_t^k + \frac{\delta_t \beta_t^k}{2\|\beta_t^k\|}$ in (3.15) whenever $\beta_t^k \neq 0$ yields

$$\left\|\beta_t^k\right\| \leq \frac{2}{\delta_t}\left[V_t^k\left(x_t^k + \frac{\delta_t}{2}\frac{\beta_t^k}{\|\beta_t^k\|}\right) - V_t^k\left(x_t^k\right)\right].$$

We define the compact subset $\mathcal{X}_t''$ of $\operatorname{dom} \tilde{V}_t$ as $\mathcal{X}_t'' := \mathcal{X}_t + B_t\left(\frac{\delta_t}{2}\right)$. Now as $x_t^k \in \mathcal{X}_t$ we have that $x_t^k + \frac{\delta_t}{2}\frac{\beta_t^k}{\|\beta_t^k\|} \in \mathcal{X}_t''$. Consequently, by (ii),

$$V_t^k\left(x_t^k + \frac{\delta_t}{2}\frac{\beta_t^k}{\|\beta_t^k\|}\right) \leq \max_{x \in \mathcal{X}_t''} \tilde{V}_t(x) < +\infty.$$

Moreover by construction the sequence of functions $(V_t^k)_{k\in\mathbb{N}}$ is increasing, thus

$$V_t^k(x_t^k) \geq V_t^1(x_t^k) \geq \min_{x\in\mathcal{X}_t} V_t^1(x) > -\infty.$$

Thus we have that, for all $k \in \mathbb{N}^*$ and $t \in [\![0, T-1]\!]$,

$$\left\|\beta_t^k\right\| \leq \frac{2}{\delta_t}\left[\max_{x\in\mathcal{X}_t''} \tilde{V}_t(x) - \min_{x\in\mathcal{X}_t} V_t^1(x)\right]. \tag{3.16}$$

This completes the proof. □

Corollary 3.5. *Under assumptions* (H_1)*, the functions* V_t^k*,* $t \in [\![0, T-1]\!]$*, are* $\alpha-$*Lipschtiz for some constant* α *for all* $k \in \mathbb{N}^*$*.*

Proof. By (3.6) and (3.16) the subgradients of V_t^k are bounded by

$$\alpha = \max_{t\in[\![0,T-1]\!]} \frac{2}{\delta_t}\left[\max_{x\in\mathcal{X}_t''} \tilde{V}_t - \min_{x\in\mathcal{X}_t} V_t^1(x)\right].$$

□

We now prove that both the upper and lower estimates of V_t converge to the exact value function under assumptions (H_1).

Theorem 3.6. *Consider the sequence of decisions* $\left(u^k\right)_{k\in\mathbb{N}}$ *generated by (3.3) and (3.6), where each* u^k *is itself a sequence of decisions in time* $u^k = u_0^k, \dots, u_{T-1}^k$*, and consider the corresponding sequence of state values* $\left(x^k\right)_{k\in\mathbb{N}}$*. Under assumptions* (H_1)*, for any* $t \in [\![0, T-1]\!]$ *we have that:*

$$\lim_{k\to+\infty} W_t\left(x_t^k, u_t^k\right) - V_t\left(x_t^k\right) = 0 \textit{ and } \lim_{k\to+\infty} V_t\left(x_t^k\right) - V_t^k\left(x_t^k\right) = 0.$$

Proof. The demonstration proceeds by induction backwards in time. At time $t+1$, the induction hypothesis is the second statement of the theorem. That is,

$$\lim_{k\to+\infty} V_{t+1}\left(x_{t+1}^k\right) - V_{t+1}^k\left(x_{t+1}^k\right) = 0.$$

In other words the cuts for the future cost function tend to be exact at x_{t+1}^k as k tends to ∞. The induction hypothesis is clearly true at the last time stage T for which we defined the approximate value function V_T^k to be equal to the (known) end value function V_T.

We have to show the induction hypothesis, namely

$$\lim_{k\to+\infty} V_t\left(x_t^k\right) - V_t^k\left(x_t^k\right) = 0$$

for time t. Recall (3.10) gives

$$V_t^k\left(x_t^k\right) \geq W_t^{k-1}\left(x_t^k, u_t^k\right) = C_t\left(x_t^k, u_t^k\right) + V_{t+1}^{k-1}\left(x_{t+1}^k\right).$$

Using the definition (3.8) of W_t, we can replace $C_t\left(x_t^k, u_t^k\right)$ to get

$$V_t^k\left(x_t^k\right) \geq W_t\left(x_t^k, u_t^k\right) + \left(V_{t+1}^{k-1}\left(x_{t+1}^k\right) - V_{t+1}\left(x_{t+1}^k\right)\right).$$

Subtracting $V_t\left(x_t^k\right)$ we obtain

$$V_t^k\left(x_t^k\right) - V_t\left(x_t^k\right) \geq W_t\left(x_t^k, u_t^k\right) - V_t\left(x_t^k\right) + \left(V_{t+1}^{k-1}\left(x_{t+1}^k\right) - V_{t+1}\left(x_{t+1}^k\right)\right).$$

Now as V_t^k is a lower approximation of V_t we have

$$V_t^k\left(x_t^k\right) - V_t\left(x_t^k\right) \leq 0,$$

and by Dynamic Programming equation (3.2)

$$W_t\left(x_t^k, u_t^k\right) - V_t\left(x_t^k\right) \geq 0.$$

Moreover the induction hypothesis at time $t+1$ gives

$$V_{t+1}^k\left(x_{t+1}^k\right) - V_{t+1}\left(x_{t+1}^k\right) \xrightarrow{k\to\infty} 0,$$

which by virtue of Lemma 3.10 (with V_{t+1} replacing f) implies[3]

$$\lim_{k\to+\infty} V_{t+1}\left(x_{t+1}^k\right) - V_{t+1}^{k-1}\left(x_{t+1}^k\right) = 0$$

so

$$V_t^k\left(x_t^k\right) - V_t\left(x_t^k\right) \xrightarrow{k\to\infty} 0,$$

and

$$W_t\left(x_t^k, u_t^k\right) - V_t\left(x_t^k\right) \xrightarrow{k\to\infty} 0,$$

which gives the result. □

Theorem 3.6 indicates that the lower approximation at each iteration tends to be exact on the sequence of state trajectories generated throughout the algorithm. This does not mean that the future cost function will be approximated well everywhere in the state space. It only means that the approximation gets better and better in the neighborhood of an optimal state trajectory.

3.2 Stochastic Case with a Finite Distribution

3.2.1 Stochastic Multistage Problem Formulation.

We now consider that the cost function and dynamics at each time t are influenced by a random outcome that has a discrete and finite distribution. We write the problem on the complete tree induced by this distribution. The set of all nodes is denoted by $\mathcal{N}$ and $\{0\}$ is the root node. We denote nodes by m and n. (We trust that the context will dispel any confusion from the use of m and n as dimensions of variables u and x.) A node n here represents a time interval and a state of the world (which has probability Φ_n) that pertains over this time interval. We say that a node n is an ascendant of m if it is on the path from the root node to node m (including m). We will denote $a(m)$ the set of all ascendants of m, and the depth of node n is one less than the number of its ascendants. For simplicity we identify this with a time index t, although the results hold true for scenario trees for which this is not the case. For every node $m \in \mathcal{N}\backslash\{0\}$, $p(m)$ represents its parent, and $r(m)$ its set of children nodes. Finally $\mathcal{L}$ is the set of leaf nodes of the tree (i.e. those that have degree 1).

3. Corollary 3.5 ensures the $\alpha-$Lipschitz assumption on V_{t+1}^k, and the other assumptions are obviously verified.

This gives the following stochastic program:

$$\begin{aligned}
\min_{x,u} \quad & \sum_{n\in\mathcal{N}\backslash\{\mathcal{L}\}} \sum_{m\in r(n)} \Phi_m C_m\left(x_n, u_m\right) + \sum_{m\in\mathcal{L}} \Phi_m V_m\left(x_m\right) && \text{(3.17a)}\\
\text{s.t.} \quad & x_m = f_m\left(x_{p(m)}, u_m\right), \qquad \forall m \in \mathcal{N}\backslash\{0\}, && \text{(3.17b)}\\
& x_0 \text{ is given}, && \text{(3.17c)}\\
& x_m \in \mathcal{X}_m, \qquad \forall m \in \mathcal{N}, && \text{(3.17d)}\\
& u_m \in \mathcal{U}_m(x_{p(m)}), \qquad \forall m \in \mathcal{N}\backslash\{0\}. && \text{(3.17e)}
\end{aligned}$$

The reader should note that randomness (that appears in the cost and in the dynamics) is realized before the decision is taken in this model. Hence the control affecting the stock [4] x_n is actually indexed by m, a child node of n. Put differently, the control adapts to randomness: there are as many controls as there are children nodes of n. Observe that we also now admit the possibility that $\mathcal{X}_t$ and $\mathcal{U}_t(x)$ might vary with scenario-tree node, so we denote them by $\mathcal{X}_m$ and $\mathcal{U}_m(x_{p(m)})$.

We make the following assumptions (H_2):

1. for all $n \in \mathcal{N}$, $\mathcal{X}_n$ is nonempty convex compact;
2. for all $m \in \mathcal{N}\backslash\{0\}$, the multifunction $\mathcal{U}_m$ is nonempty convex and convex compact valued;
3. all functions C_n, $n \in \mathcal{N}\backslash\mathcal{L}$, V_m, $m \in \mathcal{L}$, are convex lower semicontinuous proper functions;
4. for all $m \in \mathcal{N}\backslash\{0\}$, the functions f_m are affine;
5. for all $m \in \mathcal{L}$, V_m is Lipschitz-continuous on $\mathcal{X}_m$;
6. There exists $\delta > 0$ such that for all nodes $n \in \mathcal{N}\backslash\mathcal{L}$,
   (a) $\forall x \in \mathcal{X}_n + B(\delta), \quad \forall m \in r(n), \qquad f_m(x, \mathcal{U}_m(x)) \cap \mathcal{X}_m \neq \emptyset$,
   (b) $\forall x \in \mathcal{X}_n + B(\delta), \quad \forall u \in \mathcal{U}_m(x), \qquad C_n(x, u) < \infty$.

The convex functions V_m define the future cost of having x_m remaining in stock at the end of the stage represented by leaf node $m \in \mathcal{L}$. Given an optimal control, we can define (applying the Dynamic Programming principle to Problem (3.17)) a future cost function V_n recursively for the other nodes $n \in \mathcal{N}\backslash\mathcal{L}$ by

$$V_n(x_n) = \sum_{m\in r(n)} \frac{\Phi_m}{\Phi_n} \min_{u_m\in\mathcal{U}_m(x_n)} \underbrace{\left\{C_m\left(x_n, u_m\right) + V_m\left(f_m\left(x_n, u_m\right)\right)\right\}}_{W_m(x_n, u_m)}. \tag{3.18}$$

In general the future cost function at each node can be different from those at other nodes at the same stage. In the special case where the stochastic process defined by the scenario tree is stage-wise independent, the future cost function is identical at every node at stage t. Some form of stage-wise independence is typically assumed in applications as it enables cuts to be shared across nodes at the same stage, however we do not require this for our proof.

The algorithm that we consider is an extension of the deterministic algorithm of the previous section applied, at each iteration, to a set of nodes chosen randomly in the tree at which we update estimates of the future cost function. We assume that all other nodes have null updates, in the sense that they just inherit the future cost function from the previous iteration.

4. We do not make any stage-wise independence assumptions on the random variables that affect the system. Hence there is no reason why variable x_n should be called a state variable and we prefer calling it a stock.

We now describe the algorithm formally. We start the process with $\hat{\theta}_n^0 = -\infty$, $\hat{\beta}_n^0 = 0$, for each $n \in \mathcal{N}$, and impose $V_n^k = V_n$ for all nodes $n \in \mathcal{L}$ and all $k \in \mathbb{N}$. We then carry out a sequence of simulations and updates of the future cost functions as follows.

Simulation Starting at the root node, generate stocks and decisions for all possible successors (in other words, visit the whole tree forward) by solving (3.18) with V^{k-1} instead of V. Denote the obtained stock variables by $(x_n^k)_{n\in\mathcal{N}}$ and the control variables by $(u_n^k)_{n\in\mathcal{N}\setminus\{0\}}$. Also, for each node $n \in \mathcal{N}$, impose $\theta_n^k = V_n^{k-1}(x_n^k)$ and $\beta_n^k \in \partial V_n^{k-1}(x_n^k)$.

Update Select non-leaf nodes $n_1, n_2, \ldots, n_I$ in the tree. For each i, $x_{n_i}^k$ is a random variable which is equal to one of the x_n^k. For each selected node n_i, and for every child node m of node n_i, solve:

$$\begin{aligned}
\hat{\theta}_m^k = \min_{\substack{u_m\in\mathbb{R}^m \\ x\in\mathrm{Aff}\left(\mathcal{X}_{n_i}\right)}} \quad & C_m\left(x, u_m\right) + V_m^{k-1} \circ f_m\left(x, u_m\right), && (3.19a)\\
\text{s.t.} \quad & x = x_{n_i}^k \qquad [\hat{\beta}_m^k] && (3.19b)\\
& u_m \in \mathcal{U}_m(x) && (3.19c)\\
& f_m(x, u_m) \in \mathcal{X}_m && (3.19d)
\end{aligned}$$

As before $\hat{\beta}_m^k$ is a Lagrange multiplier at optimality. We also define the multifunctions

$$\tilde{\mathcal{U}}_m : x \mapsto \{u \in \mathcal{U}_m(x) \quad | \quad f_m(x, u_m) \in \mathcal{X}_m\}.$$

For each selected node n_i, replace the values $\theta_{n_i}^k$ and $\beta_{n_i}^k$ obtained during the simulation with

$$\theta_{n_i}^k = \sum_{m\in r(n_i)} \frac{\Phi_m}{\Phi_{n_i}} \hat{\theta}_m^k$$

and

$$\beta_{n_i}^k = \sum_{m\in r(n_i)} \frac{\Phi_m}{\Phi_{n_i}} \hat{\beta}_m.$$

Finally, we update all future cost functions. For every node n, and any $x \in \mathcal{X}_t$,

$$V_n^k\left(x\right) := \max\left\{V_n^{k-1}\left(x\right), \theta_n^k + \left\langle \beta_n^k, x - x_n^k \right\rangle\right\} = \max_{k'\le k}\left\{\theta_n^k + \left\langle \beta_n^k, x - x_n^k \right\rangle\right\}. \tag{3.20}$$

We will make use of the following definitions, where $m \in r(n)$:

$$W_m\left(x_n, u_m\right) := C_m\left(x_n, u_m\right) + V_m\left(f_m\left(x_n, u_m\right)\right) \tag{3.21}$$

$$W_m^k\left(x_n, u_m\right) := C_m\left(x_n, u_m\right) + V_m^k\left(f_m\left(x_n, u_m\right)\right) \tag{3.22}$$

In the case where node $n \in \mathcal{N}$ is selected at iteration k, in other words $n = n_i$, these definitions then give

$$\hat{\theta}_m^k = \min_{u\in\tilde{\mathcal{U}}_m(x_n^k)} W_m^{k-1}\left(x_n^k, u\right) = W_m^{k-1}\left(x_n^k, u_m^k\right).$$

This leads to

$$V_n^k\left(x_n^k\right) \ge \sum_{m\in r(n)} \frac{\Phi_m}{\Phi_n} W_m^{k-1}\left(x_n^k, u_m^k\right). \tag{3.23}$$

Note that we actually only update future cost functions on the selected nodes. Since the cuts we add at all other nodes are binding on the current model (by construction in the simulation), there is no point in storing them. Therefore, in practice, one does not need to sample the whole scenario tree but just enough to attain all selected nodes. In our proof, we need to look at what happens even on the nodes that are not selected.

The way we select nodes at which to compute cuts varies with the particular algorithm implementation. For example DOASA uses a single forward pass to select nodes, and then computes cuts in a backward pass. We represent these selections of nodes using a *selection random variable* $y^k = (y^k_n)_{n\in\mathcal{N}}$ that is equal to 1 if node n is selected at iteration k and 0 otherwise. This gives a *selection stochastic process* $(y^k)_{k\in\mathbb{N}}$, taking values in $\{0,1\}^{|\mathcal{N}\backslash\mathcal{L}|}$, that describes a set of nodes in the tree at which we will compute new cuts in iteration k. We let $(\mathcal{F}_k)_{k\in\mathbb{N}}$ denote the filtration generated by $(y^k)_{k\in\mathbb{N}}$.

To encompass algorithms such as DOASA and SDDP the selection stochastic process can be viewed as consisting of infinitely many finite subsequences, each consisting of $\tau > 0$ selections (consisting for example of a sequence of selections of nodes in a backward pass). This cannot be done arbitrarily, and the way that $(y^k)_{k\in\mathbb{N}}$ is constructed must satisfy some independence conditions from one iteration to the next.

Definition 1. *Let τ be a positive integer. The process $(y^k)_{k\in\mathbb{N}}$ is called a τ-*admissible selection process *if*

(i) $\forall m \in \mathcal{N}\backslash\mathcal{L}, \quad \forall k \in \mathbb{N}, \quad \forall \kappa \in \{0,\dots,\tau-1\},$

$$y_m^{k\tau+\kappa} = 1 \quad\Longrightarrow\quad \forall n \in a(m), \quad y_n^{k\tau} = y_n^{k\tau+1} = \cdots = y_n^{k\tau+\kappa-1} = 0;$$

and the process defined by

$$\tilde{y}_n^k := \max\{y_n^{k\tau}, y_n^{k\tau+1}, y_n^{k\tau+2}, \dots, y_n^{k\tau+\tau-1}\} \tag{3.24}$$

satisfies

(ii) for all $m \in \mathcal{N}\backslash\mathcal{L}$, $(\tilde{y}^k_m)_{k\in\mathbb{N}}$ is i.i.d. and for all $k \in \mathbb{N}$, and all $m \in \mathcal{N}\backslash\mathcal{L}$, $\tilde{y}^k_m$ is independent of $\mathcal{F}_{k\tau-1}$;

(iii) $\forall n \in \mathcal{N}\backslash\mathcal{L}, \qquad \mathbb{P}(\tilde{y}^k_n = 1) > 0.$

Property (i) guarantees that when $\tau > 1$, the updating of cutting planes is done backwards between steps $k\tau$ and $(k+1)\tau$. This means that if the linear approximation of the value function V_n is updated at step $k\tau + \kappa$ then neither it or any approximation at any ascendant node has been updated since step $k\tau - 1$. This implies, as shown in lemma 3.11, that $x^{k\tau+\kappa}$ has not changed since the step $k\tau$, i.e., if $y_n^{k\tau+\kappa} = 1$ then $x^{k\tau+\kappa} = x^{k\tau}$. We explain in section 3.3. how the selection processes of CUPPS (with $\tau = 1$) and SDDP(with $\tau = T-1$) are τ-admissible.

Property (ii) provides the independence of the selections that we will use to prove convergence and property (iii) guarantees that all nodes are selected with positive probability. Without any independence assumption it would be easy to create a case in which the future cost function at a given node is updated only when the current stock variable on this node is in a given region, for instance. In such a case the future cost function could not gather any information about the other parts of the space that the stock variable might visit. In other words, this independence assumption ensures that the values that are optimal can be attained an infinite number of times. We remark that there is no independence assumption over the nodes n for $(y^k_n)_{n\in\mathcal{N}\backslash\mathcal{L}}$ at k fixed. Thus the selection process could be forced to select whole branches of the tree for example, as it would for the CUPPS algorithm. More generally, we have independence when for fixed τ, $(y^{k\tau})_{k\in\mathbb{N}}$ is i.i.d and the next $\tau - 1$ selection values are determined deterministically from $y^{k\tau}$, more precisely if for all $\kappa \in \{0, \dots, \tau-1\}$, there is a deterministic function ϕ_κ such that $y^{k\tau+\kappa} = \phi_\kappa(y^{k\tau})$. On the other hand we have independence when the selection subsequence $(y^{k\tau}, y^{k\tau+1}, \dots, y^{k\tau+\tau-1})_{k\in\mathbb{N}}$ is i.i.d.

In Section 3.3 we shows that usual algorithm can be represented with a $\tau-$admissible selection process.

3.2.2 Proof of Convergence in the Stochastic Case.

For every $n \in \mathcal{N}\backslash\mathcal{L}$ we can define under assumptions (H_2) the extended value function

$$\tilde{V}_n(x) = \sum_{m\in r(n)} \frac{\Phi_m}{\Phi_n} \inf_{u\in\tilde{\mathcal{U}}_m(x)} \left\{C_m(x,u) + V_m \circ f_m(x,u)\right\},$$

and we note that $\tilde{V}_n$ is finite on $\mathcal{X}_n'$. We now state a lemma analogous to lemma 3.4.

Lemma 3.7. *For every $n \in \mathcal{N}$,*

(i) the value function V_n is convex and Lipschitz-continuous on $\mathcal{X}_t$;

(ii) $V_n^k \leq \tilde{V}_n \leq V_n$, and β_n^k is defined;

(iii) the sequences $(\beta_n^k)_{k\in\mathbb{N}}$ are bounded, thus there is α_n such that V_n^k is α_n-Lipschitz.

Proof. We give only a sketch of the proof as it follows exactly the proof of its deterministic counterpart lemma 3.4.

(i) By induction backward on the tree $\tilde{V}_n$, is convex and finite valued on $\mathcal{X}_n'$ as the positive sum of convex finite valued functions, and thus Lipschitz continuous on $\mathcal{X}_n$ leading to the result as $\tilde{V}_n = V_n$ on $\mathcal{X}_n$.

(ii) Assume that for all $n \in \mathcal{N}\backslash\mathcal{L}$ we have $V_n^{k-1} \leq \tilde{V}_n$. We define, for a node $n \in \mathcal{N}\backslash\mathcal{L}$ $x \in \mathbb{R}^n$,

$$\hat{V}_n^k(x) = \sum_{m\in r(n)} \frac{\Phi_m}{\Phi_n} \min_{u\in\tilde{\mathcal{U}}_n(x)} C_m(x,u) + V_m^{k-1} \circ f_m(x,u).$$

By hypothesis on $\tilde{\mathcal{U}}_m$, $\hat{V}_n^k$ is convex and finite on $\mathcal{X}_t'$ thus its restriction on $\mathrm{Aff}(\mathcal{X}_t)$ is subdifferentiable on $\mathcal{X}_t$. By definition $\hat{\beta}_m^k \in \partial\hat{V}_n^k(x_m^k)$, and thus $\hat{\beta}_m^k$ is defined. By the induction hypothesis and inequality $\tilde{V}_m \leq V_m$ we have that

$$\forall m \in r(n), \qquad \hat{V}_m^{k-1} \circ f_m \leq V_m \circ f_m.$$

Thus definitions of $\hat{V}_n^k$ and $\tilde{V}_n$ yield $\hat{V}_n^k \leq \tilde{V}_n$. By definition of β_n^k and construction of V_n^k we have that $V_n^k \leq \tilde{V}_n$. Induction leads to inequality (ii).

(iii) Finally we show the boundedness of $(\beta_n^k)_{k\in\mathbb{N}}$. As β_n^k is an element of $\partial V^k(x_n^k)$, we have

$$V^k(y) \geq V^k(x_n^k) + \left\langle \beta_n^k, y - x_n^k \right\rangle. \tag{3.25}$$

so substituting, if $\beta_n^k \neq 0$, $y = x_n^k + \frac{\delta\beta_n^k}{2\|\beta_n^k\|}$ in (3.25) yields

$$\left\|\beta_n^k\right\| \leq \frac{2}{\delta}\left[V_n^k\left(x_n^k + \frac{\delta}{2}\frac{\beta_n^k}{\|\beta_n^k\|}\right) - V_n^k\left(x_n^k\right)\right].$$

Thus we have that, for all $k \in \mathbb{N}$ and $n \in \mathcal{N}$,

$$\left\|\beta_n^k\right\| \leq \frac{2}{\delta}\left[\max_{x\in\mathcal{X}_n+B(\delta/2)} \tilde{V}_n(x) - \min_{x\in\mathcal{X}_n} V_n^1(x)\right]. \tag{3.26}$$

Which ends the proof.

□

Theorem 3.8. *Consider the sequence of decisions* $\left(u^k\right)_{k\in\mathbb{N}}$ *generated by the above described procedure under assumptions* (H_2)*, where each* u^k *is itself a set of decisions on the complete tree, and consider the corresponding sequence of state values* $\left(x^k\right)_{k\in\mathbb{N}}$*. Assume that the selection process is* τ*-admissible for some integer* $\tau > 0$*.*
Then we have that, $\mathbb{P}$*-almost surely:*

$$\lim_{k\to+\infty} \sum_{m\in r(n)} \frac{\Phi_m}{\Phi_n} W_m\left(x_n^{k\tau}, u_m^{k\tau}\right) - V_n\left(x_n^{k\tau}\right) = 0.$$

and

$$\lim_{k\to+\infty} V_n\left(x_n^{k\tau}\right) - V_n^{k\tau}\left(x_n^{k\tau}\right) = 0.$$

Proof. Because the selection process for nodes in the update step is stochastic, decision variables as well as approximate future cost functions are stochastic throughout the course of the algorithm. Thus, during the whole proof, all equalities or inequalities are taken $\mathbb{P}$-almost surely.

The demonstration follows the same approach as the proof of Theorem 3.6. Let T be the maximum depth of the tree. We proceed by backward induction on nodes of fixed depth. The induction hypothesis is

$$\lim_{k\to+\infty} V_m\left(x_m^{k\tau}\right) - V_m^{k\tau}\left(x_m^{k\tau}\right) = 0$$

for each node m of depth $t+1$. Since for every leaf of the tree those two quantities are equal, by definition, the induction hypothesis is true for every node $n \in \mathcal{L}$.

We start by proving the result for iterations $k\tau$ such that n is selected in the next $\tau - 1$ steps, i.e. such that $\tilde{y}_n^k = 1$. Define $\kappa_k \in \{0, \dots, \tau-1\}$ such that $y^{k\tau+\kappa_k} = 1$. Recall that by lemma 3.11 we have $x_n^{k\tau+\kappa_k} = x_n^{k\tau}$.

We have by (3.23)

$$\begin{aligned}
V_n^{k\tau+\kappa_k}\left(x_n^{k\tau}\right) &= V_n^{k\tau+\kappa_k}\left(x_n^{k\tau+\kappa_k}\right)\\
&\geq \sum_{m\in r(n)} \frac{\Phi_m}{\Phi_n} \min_{u_m\in\tilde{\mathcal{U}}_m(x_n^{k\tau})} \left\{W_m^{k\tau+\kappa_k-1}\left(x_n^{k\tau}, u_m\right)\right\}\\
&\geq \sum_{m\in r(n)} \frac{\Phi_m}{\Phi_n} \min_{u_m\in\tilde{\mathcal{U}}_m(x_n^{k\tau})} \left\{W_m^{k\tau-1}\left(x_n^{k\tau}, u_m\right)\right\}\\
&= \sum_{m\in r(n)} \frac{\Phi_m}{\Phi_n} W_m^{k\tau-1}\left(x_n^{k\tau}, u_m^{k\tau}\right)
\end{aligned}$$

which implies

$$\begin{aligned}
V_n^{k\tau+\kappa_k}\left(x_n^{k\tau}\right) &\geq \sum_{m\in r(n)} \frac{\Phi_m}{\Phi_n}\left[C_m\left(x_n^{k\tau}, u_m^{k\tau}\right) + V_m^{k\tau-1}\left(x_m^{k\tau}\right)\right],\\
&= \sum_{m\in r(n)} \frac{\Phi_m}{\Phi_n}\left[W_m\left(x_n^{k\tau}, u_m^{k\tau}\right) + \left(V_m^{k\tau-1}\left(x_m^{k\tau}\right) - V_m\left(x_m^{k\tau}\right)\right)\right].
\end{aligned}$$

Thus,

$$V_n^{k\tau+\kappa_k}\left(x_n^{k\tau}\right) - V_n\left(x_n^{k\tau}\right) \geq \sum_{m\in r(n)} \left(\frac{\Phi_m}{\Phi_n} W_m\left(x_n^{k\tau}, u_m^{k\tau}\right)\right) - V_n\left(x_n^{k\tau}\right) + \sum_{m\in r(n)} \frac{\Phi_m}{\Phi_n}\left(V_m^{k\tau-1}\left(x_m^{k\tau}\right) - V_m\left(x_m^{k\tau}\right)\right).$$

Note that, as $V_n^{k\tau+\kappa_k}$ is a lower approximation of V_n we know that

$$V_n^{k\tau+\kappa_k}\left(x_n^{k\tau}\right) - V_n\left(x_n^{k\tau}\right) \leq 0,$$

and, by Dynamic Programming Equation (3.18), that

$$\sum_{m\in r(n)} \left(\frac{\Phi_m}{\Phi_n} W_m\left(x_n^{k\tau}, u_m^{k\tau}\right)\right) - V_n\left(x_n^{k\tau}\right) \geq 0.$$

The induction hypothesis

$$\lim_{k\to+\infty} V_m\left(x_m^{k\tau}\right) - V_m^{k\tau}\left(x_m^{k\tau}\right) = 0$$

and Lemma 3.10 (with V_m replacing f)[5] implies

$$\lim_{k\to+\infty} V_m\left(x_m^{k\tau}\right) - V_m^{k\tau-1}\left(x_m^{k\tau}\right) = 0.$$

Thus

$$V_n\left(x_n^{k\tau}\right) - V_n^{k\tau+\kappa_k}\left(x_n^{k\tau}\right) \xrightarrow[\tilde{y}_n^k=1]{k\to\infty} 0,$$

and

$$\sum_{m\in r(n)} \left(\frac{\Phi_m}{\Phi_n} W_m\left(x_n^{k\tau}, u_m^{k\tau}\right)\right) - V_n\left(x_n^{k\tau}\right) \xrightarrow[\tilde{y}_n^k=1]{k\to\infty} 0.$$

Thus lemma 3.10 applied with $\kappa = \tau$ gives

$$V_n\left(x_n^{k\tau}\right) - V_n^{k\tau+\kappa_k-\tau}\left(x_n^{k\tau}\right) \xrightarrow[\tilde{y}_n^k=1]{k\to\infty} 0,$$

and by monotonicity we have $V_n^{k\tau+\kappa_k-\tau} \leq V_n^{k\tau} \leq V_n$, which finally yields

$$V_n\left(x_n^{k\tau}\right) - V_n^{k\tau}\left(x_n^{k\tau}\right) \xrightarrow[\tilde{y}_n^k=1]{k\to\infty} 0. \tag{3.27}$$

Now we prove that the values also converge for the iterations k such that $\tilde{y}_n^k = 0$, i.e. the iterations for which node n is not selected between step $k\tau$ and step $(k+1)\tau - 1$. By contradiction, suppose the values do not converge. Then by lemma 3.10 we have that $V_n(x_n^{k\tau}) - V_n^{k\tau-1}(x_n^{k\tau})$ does not converge to 0. It follows that there is some $\varepsilon > 0$ such that $\mathcal{K}_\varepsilon$ is infinite where

$$\mathcal{K}_\epsilon := \{k \in \mathbb{N} \quad | \quad V_n\left(x_n^{k\tau}\right) - V_n^{k\tau-1}\left(x_n^{k\tau}\right) \geq \varepsilon\}. \tag{3.28}$$

Let z^j denote the j-th element of the set $\{y_n^{k\tau} | k \in \mathcal{K}_\varepsilon\}$. Note that the random variables $V^{k\tau-1}$ and $x_n^{k\tau}$ are measurable with respect to $\mathcal{F}_{k\tau-1} := \sigma\big((y^{k'})_{k'<k\tau}\big)$, and thus so is $1_{k\in\mathcal{K}_\varepsilon}$

5. Lemma 3.7 (iii) provides a Lipschitz condition on V_m^k.

from which $\tilde{y}_n^k$ is independent. Moreover the $\sigma-$algebra generated by the past realizations of $\tilde{y}_n^k$ is included in $\mathcal{F}_{k\tau-1}$. This implies by lemma 3.12 that random variables $(z^j)_{j\in\mathbb{N}}$ are i.i.d. and share the same probability law as $\tilde{y}_n^0$.

According to the Strong Law of Large Numbers [52, page 294] applied to the random sequence $(z^j)_{j\in\mathbb{N}}$, we should then have

$$\frac{1}{N}\sum_{j=1}^{N} z^j \xrightarrow{N\to+\infty} \mathbb{E}[z^1] = \mathbb{E}[\tilde{y}_n^0] = \mathbb{P}\big(\tilde{y}_n^0 = 1\big) > 0.$$

However, $\mathcal{K}_\varepsilon \cap \{\tilde{y}_n^k = 1\}$ is finite because of (3.27) thus we know that there is only a finite number of indexes j such that $z^j = 1$, the rest being equal to 0. So

$$\frac{1}{N}\sum_{j=1}^{N} z^j \xrightarrow{N\to+\infty} 0,$$

which is a contradiction. This shows that

$$V_n\left(x_n^{k\tau}\right) - V_n^{k\tau-1}\left(x_n^{k\tau}\right) \xrightarrow[\tilde{y}_n^k=0]{k\to\infty} 0.$$

and monotonicity shows that,

$$V_n\left(x_n^{k\tau}\right) - V_n^{k\tau}\left(x_n^{k\tau}\right) \xrightarrow[\tilde{y}_n^k=0]{k\to\infty} 0.$$

which completes the induction. $\square$

3.2.3 Application to Known Algorithms.

In order to illustrate on our result we will apply it to two well known algorithms. For simplicity we will assume that the tree represents a T-step stochastic decision problem in which every leaf of the tree is of depth T.

We first define the CUPPS algorithm [27] in this setting. Here at each major iteration we choose a $T-1$-step scenario and compute the optimal trajectory while at the same time updating the value function for each node of the branch. In our setting, this uses a 1-admissible selection process $(y^k)_{k\in\mathbb{N}}$ defined by an i.i.d. sequence of random variables, with y^0 selecting a single branch of the tree. Theorem 3.8 shows that for every node n the upper and lower bound converges, that is

$$\lim_{k\to+\infty}\sum_{m\in r(n)}\left(\frac{\Phi_m}{\Phi_n}W_m\left(x_n^k, u_m^k\right)\right) - V_n\left(x_n^k\right) = 0$$

and

$$\lim_{k\to+\infty} V_n\left(x_n^k\right) - V_n^k\left(x_n^k\right) = 0.$$

We now place the SDDP algorithm [77] and DOASA algorithm [80] in our framework. There are two phases in each major iteration of the SDDP algorithm, namely a forward pass, and a backward pass of $T-1$ steps. Given a current polyhedral outer approximation of the Bellman function $(V_n^{\tilde{k}-1})_{n\in\mathcal{N}\setminus\mathcal{L}}$, a major iteration $\tilde{k}$ of the SDDP algorithm consists in:

- selecting uniformly a number N of scenarios ($N = 1$ for DOASA);
- simulating the optimal strategy for the problem, that is solving problem (3.19) to determine a trajectory (for each scenario) $(x_{n_t}^{\tilde{k}})_{t\in\{0,\dots,T-1\}}$ where $(n_t)_{t\in\{0,\dots,T-1\}}$ defines one of the selected scenarios;

- For $t = T-1$ down to $t = 0$
  for each scenario solving problem (3.19) with $V_m^{\tilde{k}}$ instead of $V_m^{\tilde{k}-1}$,
  and defining

$$V_{n_t}^{\tilde{k}}(x) = \max\{V_{n_t}^{\tilde{k}-1}(x), \theta_{n_t}^{\tilde{k}} + \langle \beta_{n_t}^{\tilde{k}}, x - x_{n_t}^{\tilde{k}} \rangle\}.$$

SDDP fits into our framework as follows. Given N, we define the $T-1$-admissible selection process, $(y^{(T-1)k})_{k\in\mathbb{N}}$ by an i.i.d. sequence of random variables with y^0 selecting uniformly a set of N pre-leaves (i.e. nodes whose children are leaves) of the tree. Then for $\kappa \in \{1, \dots, T-2\}$, $k \in \mathbb{N}$, $n \in \mathcal{N}\backslash\mathcal{L}$, we define

$$y_n^{k(T-1)+\kappa} := \begin{cases} 1 & \text{if there exist } m \in r(n) \text{ such that } y_m^{k(T-1)+\kappa-1} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This algorithm is the same as SDDP with N randomly sampled forward passes per stage, but without the cut sharing feature used when random variables are stage-wise independent. Since for every node n of the tree (excepting the leaves) there is a κ such that $\mathbb{P}(y_n^{k(T-1)+\kappa} = 1) > 0$, theorem 3.8 guarantees the convergence of the lower bound for every node. This remains true when cuts are shared since the proof of almost-sure convergence is unaffected by the addition of extra valid cutting planes at any point during the course of the algorithm. The proof of theorem 3.8 gives

$$V_n\left(x_n^{k\tau}\right) - V_n^{k\tau}\left(x_n^{k\tau}\right) \xrightarrow[\tilde{y}_n^k=0]{k\to\infty} 0,$$

and with shared cuts we obtain an improved value function $\breve{V}_n^{k\tau}$ satisfying

$$V_n^{k\tau}\left(x_n^{k\tau}\right) \leq \breve{V}_n^{k\tau}\left(x_n^{k\tau}\right) \leq V_n\left(x_n^{k\tau}\right)$$

that must satisfy

$$V_n\left(x_n^{k\tau}\right) - \breve{V}_n^{k\tau}\left(x_n^{k\tau}\right) \xrightarrow[\tilde{y}_n^k=0]{k\to\infty} 0.$$

3.3 Discussion

The convergence result we have proved assumes that we compute new cuts at scenario-tree nodes that are selected independently from the history of the algorithm. This enables us to use the Strong Law of Large Numbers in the proof. Previous results for multistage stochastic linear programming [80] require a selection process that visits each node in the tree infinitely often, which is a weaker condition than independence, since it follows by the Borel-Cantelli Lemma [52, page 288]. An example would be the deterministic round-robin selection mentioned in [80]. We do not have a proof of convergence for such a process in the nonlinear case. It is important to observe that the polyhedral form of V_t that was exploited in the proof of [80] is absent in our problem, and this difference could prove to be critical.

The convergence result is proved for a general scenario tree. In SDDP algorithms, the random variables are usually assumed to be stage-wise independent (or made so by adding state variables). This means that the future cost functions $V_m(x)$ are the same at each node m at depth t. This allows cutting planes in the approximations to be shared across these nodes. As we have shown above, the convergence result we have shown here applies to this situation as a special case. It is worth noting that the class of algorithms covered by our result is larger than the examples presented in the literature. For example an algorithm where we select randomly a node on the whole tree, and then update backwards from there is proven to converge. One could also think of combining SDDP and CUPPS algorithms.

In the case where one would want to add cuts at different nodes in the tree in the update step of our procedure, the solving of the subproblems can be done in parallel. This is the case in CUPPS, where a whole branch of the tree is selected at each iteration. It also allows us to consider different selection strategies, where nodes at a given iteration could be selected throughout the tree depending on some criteria defined by the user. In the first few iterations, this could highly increase efficiency of the approximation and, because the solving of the subproblems can be parallelized, would not be very time-consuming. One should bear in mind however that, at some point, the algorithm has to come back to an appropriate selection procedure, i.e. one that satisfies the independence assumption, in order to ensure convergence of the algorithm.

Appendix: Technical lemmas

Lemma 3.9. *If $J : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ is convex, $\mathcal{U} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is convex then $\phi(x) := \min_{u \in \mathcal{U}(x)} J(u)$ is convex. Moreover if J is lower-semicontinuous, and $\mathcal{U}$ compact non-empty valued, then the infimum exists and is attained.*

Proof. We define

$$I(u, x) := \begin{cases} 0 & \text{if } u \in \mathcal{U}(x) \\ +\infty & \text{otherwise} \end{cases}$$

Then $\phi(x) = \min_{u \in \mathbb{R}^m} J(u) + I(u, x)$. Fix $u_1 \in \mathcal{U}(x_1)$ and $u_2 \in \mathcal{U}(x_2)$, then for every $\lambda \in [0, 1]$ $\lambda u_1 + (1 - \lambda)u_2 \in \mathcal{U}(\lambda x_1 + (1 - \lambda)x_2)$ by convexity of $\mathcal{U}$. This shows that $I(u, x)$ is convex, whereby ϕ is convex as the marginal function of a jointly convex function. The second part of the lemma follows immediately from the compactness $\mathcal{U}$ and lower-semicontinuity of J. □

Lemma 3.10. *Suppose f is convex and $\mathcal{X}$ is compact, and suppose for any integer κ, the sequence of α-Lipschitz convex functions $f^k, k \in \mathbb{N}$ satisfies*

$$f^{k-\kappa}(x) \leq f^k(x) \leq f(x), \textit{ for all } x \in \mathcal{X}.$$

Then for any infinite sequence $x^k \in \mathcal{X}$

$$\lim_{k \to +\infty} f\left(x^k\right) - f^k\left(x^k\right) = 0 \iff \lim_{k \to +\infty} f\left(x^k\right) - f^{k-\kappa}\left(x^k\right) = 0.$$

Proof. If $\lim_{k \to +\infty} f\left(x^k\right) - f^{k-\kappa}\left(x^k\right) = 0$ then pointwise monotonicity of f^k shows that $\lim_{k \to +\infty} f\left(x^k\right) - f^k\left(x^k\right) = 0$. For the converse, suppose that the result is not true. Then there is some subsequence $\left(f^{k(l)}\right)_{l \in \mathbb{N}}$ and $x^{k(l)} \in \mathcal{X}$ with

$$\lim_{k \to +\infty} f\left(x^{k(l)}\right) - f^{k(l)}\left(x^{k(l)}\right) = 0 \tag{3.29}$$

and $\varepsilon > 0, L \in \mathbb{N}$ with

$$f\left(x^{k(l)}\right) - f^{k(l)-\kappa}\left(x^{k(l)}\right) > \varepsilon$$

for every $l > L$. Since $\mathcal{X}$ is compact, we may assume (by taking a further subsequence) that $\left(x^{k(l)}\right)_{l \in \mathbb{N}}$ converges to $x^* \in \mathcal{X}$. For sufficiently large l, the Lipschitz continuity of $f^{k(l)}$ and $f^{k(l)-\kappa}$ gives

$$\begin{aligned} \left| f^{k(l)}(x^*) - f^{k(l)}\left(x^{k(l)}\right) \right| &\leq \alpha \| x^{k(l)} - x^* \| < \frac{\varepsilon}{4}, \\ \left| f^{k(l)-\kappa}\left(x^{k(l)}\right) - f^{k(l)-\kappa}(x^*) \right| &\leq \alpha \| x^{k(l)} - x^* \| < \frac{\varepsilon}{4}, \end{aligned}$$

and (3.29) implies that for sufficiently large l

$$f\left(x^{k(l)}\right) - f^{k(l)}\left(x^{k(l)}\right) < \frac{\varepsilon}{4}.$$

It follows that

$$\begin{aligned}
f^{k(l)}\left(x^*\right) - f^{k(l)-\kappa}\left(x^*\right) &= f^{k(l)}\left(x^*\right) - f^{k(l)}\left(x^{k(l)}\right) \\
&\quad + f^{k(l)}\left(x^{k(l)}\right) - f\left(x^{k(l)}\right) \\
&\quad + f\left(x^{k(l)}\right) - f^{k(l)-\kappa}\left(x^{k(l)}\right) \\
&\quad + f^{k(l)-\kappa}\left(x^{k(l)}\right) - f^{k(l)-\kappa}\left(x^*\right) \\
&> \frac{\varepsilon}{4},
\end{aligned}$$

since $f\left(x^{k(l)}\right) - f^{k(l)-\kappa}\left(x^{k(l)}\right)$ is greater than ε, and the other three terms each have an absolute value smaller than $\varepsilon/4$. Consequently $f^{k(l)}\left(x^*\right) > f^{k(l)-\kappa}\left(x^*\right) + \frac{\varepsilon}{4}$, for infinitely many l which contradicts the fact that $f^k\left(x^*\right)$ is bounded above by $f(x^*)$. □

Lemma 3.11. *If $(y^k)_{k\in\mathbb{N}}$ is a τ-admissible selection process then for all $k \in \mathbb{N}$, $\kappa \in \{0, \dots, \tau-1\}$, and all $n \in \mathcal{N}\backslash\mathcal{L}$ we have*

$$y_n^{k\tau+\kappa} = 1 \quad \Longrightarrow \quad \left\{ \begin{array}{lll} x_n^{k\tau+\kappa} & = x_n^{k\tau}, & \\ V_n^{k\tau+\kappa-1} & = V_n^{k\tau-1} & \text{if } k \geq 1. \end{array} \right.$$

Proof. Let n, k and κ be such that $y_n^{k\tau+\kappa} = 1$. Let $a(n) := (n_0, n_1, \dots, n_t)$ be the sequence of ascendants of $n_t := n$, i.e. n_0 is the root node, and for all $t' < t$, $n_{t'} = p(n_{t'+1})$. Define the hypothesis $H(t, \kappa)$:

(a) $x_{n_t}^{k\tau+\kappa} = x_{n_t}^{k\tau}$,

(b) $V_{n_t}^{k\tau+\kappa-1} = V_{n_t}^{k\tau-1}$, if $t \geq 1$.

Let $\kappa' < \kappa$ and assume that for κ' and all $t' \leq t$, $H(t', \kappa')$ holds true. This is satisfied for $\kappa' = 0$. Let $t' < t$ and assume $H(t', \kappa'+1)$ is true. Since x_0 is fixed, this is satisfied for $t' = 0$. By definition of $u_{n_{t+1}}^{k\tau+\kappa'+1}$ we have

$$u_{n_{t'+1}}^{k\tau+\kappa'+1} \in \operatorname*{arg\,min}_{u\in\tilde{\mathcal{U}}\left(x_{n_{t'}}^{k\tau+\kappa'+1}\right)} \left\{ C_{n_{t'+1}}\big(x_{n_{t'}}^{k\tau+\kappa'+1}, u\big) + V_{n_{t'+1}}^{k\tau+\kappa'} \circ f_{n_{t'+1}}\big(x_{n_{t'}}^{k\tau+\kappa'+1}, u\big) \right\}$$

thus by $H(t', \kappa'+1)$ (a) we have

$$u_{n_{t'+1}}^{k\tau+\kappa'+1} \in \operatorname*{arg\,min}_{u\in\tilde{\mathcal{U}}\left(x_{n_{t'}}^{k\tau}\right)} \left\{ C_{n_{t+1}}\big(x_{n_{t'}}^{k\tau}, u\big) + V_{n_{t'+1}}^{k\tau+\kappa'} \circ f_{n_{t+1}}\big(x_{n_{t'}}^{k\tau}, u\big) \right\}.$$

Now as $n_{t'+1}$ is an ascendant of n and $\kappa' < \kappa$ by property (i) of definition 1, we have that the representation of $V_{n_{t'+1}}$ is not updated at iteration κ', i.e.

$$V_{n_{t'+1}}^{k\tau+\kappa'} = V_{n_{t'+1}}^{k\tau+\kappa'-1}.$$

And thus $H(t'+1, \kappa')$ (b) gives $H(t'+1, \kappa'+1)$ (b), i.e.

$$V_{n_{t'+1}}^{k\tau+\kappa'} = V_{n_{t'+1}}^{k\tau-1},$$

therefore

$$u_{n_{t'+1}}^{k\tau+\kappa'+1} \in \underset{u\in\tilde{\mathcal{U}}\left(x_{n_{t'}}^{k\tau}\right)}{\arg\min} \left\{C_{n_{t+1}}\left(x_{n_{t'}}^{k\tau}, u\right) + V_{n_{t'+1}}^{k\tau-1} \circ f_{n_{t+1}}\left(x_{n_{t'}}^{k\tau}, u\right)\right\},$$

and consequently [6]

$$u_{n_{t'+1}}^{k\tau+\kappa'+1} = u_{n_{t'+1}}^{k\tau},$$

which gives by definition $H(t'+1, \kappa'+1)$ (a). Induction on t' gives $H(t', \kappa'+1)$ for all $t' \leq t$, and induction on κ' establishes $H(t, \kappa)$ for all $\kappa \in [\![0, \tau]\!]$. $\square$

Lemma 3.12. *Let $(w^k)_{k\in\mathbb{N}}$ be a stochastic process with value in $\{0,1\}$ adapted to a filtration $(\mathcal{F}_k)_{k\in\mathbb{N}}$, such that the number of terms that are non-zero is almost surely infinite. Let $(y^k)_{k\in\mathbb{N}}$ be a sequence of i.i.d discrete random variables. Define the filtration $\mathcal{B}_k := \sigma\big(\mathcal{F}_k \cup \sigma(y^1, \dots, y^{k-1})\big)$ and assume that for all $k \in \mathbb{N}$, y^k is independent of $\mathcal{B}_k$. Let $k(j)$ denote the j^{th} integer such that $w^k = 1$, i.e. $k(0) = 0$ and for all $j > 0$,*

$$k(j) := \min\{l > k(j-1) | w^l = 1\}.$$

Finally we define for all $j > 0$, the j^{th} value of (y^k) such that $w^k = 1$, i.e.

$$z^j := y^{k(j)}.$$

Then $(z^k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. random variables equal in law to y^0.

Proof. Let Y denote the support of y^0. We start with z^1. For $i \in Y$,

$$\begin{aligned}
\mathbb{P}(z^1 = i) &= \sum_{l=1}^{\infty} \mathbb{P}\big(\{\forall l' < l, \quad w^{l'} = 0\} \cap \{w^l = 1\} \cap \{y^l = i\}\big) && \text{by } \{0,1\}\text{definition} \\
&= \sum_{l=1}^{\infty} \mathbb{P}\big(\{y^l = i\}\big)\mathbb{P}\big(\{\forall l' < l, \quad w^{l'} = 0\} \cap \{w^l = 1\}\big) && \text{by independence} \\
&= \mathbb{P}\big(\{y^0 = i\}\big) \sum_{l=1}^{\infty} \mathbb{P}\big(\{\forall l' < l, \quad w^{l'} = 0\} \cap \{w^l = 1\}\big) && \text{as } (y^l) \text{ is i.i.d.} \\
&= \mathbb{P}\big(\{y^0 = i\}\big)
\end{aligned}$$

as the sequence $(w^k)_{k\in\mathbb{N}}$ must contain a 1 almost surely. Thus z^1 is equal in law to y^0.

Now suppose that $\mathbf{z} = (z^1, \dots, z^m)$ is a sequence of i.i.d. random variables. Let $k_1, \cdots, k_m$ be m ordered integers, and fix $\mathbf{b} \in \{0,1\}^n$ and $i \in Y$. We have

$$\begin{aligned}
&\mathbb{P}\Big(\{\mathbf{z} = \mathbf{b}\} \cap \{z^{m+1} = i\} \cap \{k(1) = k_1, \dots, k(m) = k_m\}\Big) \\
&\quad = \sum_{\nu=0}^{\infty} \mathbb{P}\Big(\{\mathbf{z} = \mathbf{b}\} \cap \{k(1) = k_1, \dots, k(m) = k_m\} \cap \{y^\nu = i\} \cap \{\nu = k(m+1)\}\Big) \\
&\quad = \sum_{\nu=0}^{\infty} \mathbb{P}(y^\nu = i)\mathbb{P}\Big(\{\mathbf{z} = \mathbf{b}\} \cap \{k(1) = k_1, \dots, k(m) = k_m\} \cap \{\nu = k(m+1)\}\Big) \\
&\quad = \mathbb{P}\big(y^0 = i\big)\mathbb{P}\Big(\{\mathbf{z} = \mathbf{b}\} \cap \{k(1) = k_1, \dots, k(m) = k_m\}\Big).
\end{aligned}$$

For the last equality we have used the fact that (y^k) is i.i.d. and the fact that $k(m+1)$ is almost surely finite and thus $\big(\{\nu = k(m+1)\}\big)_{\nu\in\mathbb{N}}$ is a partition of the set of events.

6. This requires that the choice of optimal control among the set of minimizers is deterministic (say that with minimum norm).

Summing over the possible realizations of $k(1), \ldots, k(m)$, we obtain

$$\mathbb{P}\big(\{\mathbf{z} = \mathbf{b}\} \cap \{z^{m+1} = i\}\big) = \mathbb{P}\big(\mathbf{z} = \mathbf{b}\big)\mathbb{P}(y_0 = i).$$

Now summing over the possible realizations of $\mathbf{b}$ shows that z^{m+1} is equal in law to y^0. Thus

$$\begin{aligned} \mathbb{P}\big(\{\mathbf{z} = \mathbf{b}\} \cap \{z^{m+1} = i\}\big) &= \mathbb{P}\big(\{\mathbf{z} = \mathbf{b}\} \cap \{y^0 = i\}\big) \\ &= \mathbb{P}\big(\mathbf{z} = \mathbf{b}\big)\mathbb{P}(y^0 = i) \\ &= \mathbb{P}\big(\mathbf{z} = \mathbf{b}\big)\mathbb{P}(z^{m+1} = i) \end{aligned}$$

which shows that z^{m+1} is independent of $\mathbf{z}$ and equal in law to y^0. Induction over m completes the proof. $\square$

Part II

Duality in Stochastic Optimization

Chapter 4

Constraint Qualification in Stochastic Optimization

Learn from yesterday, live for today, hope for tomorrow. The important thing is to not stop questioning.

Albert Einstein

Contents

With this Chapter 4, we open the part of the manuscript devoted to constraints in stochastic optimization, and we lay out ways to tackle constraints through duality methods.

We first recall basic materials in the abstract theory of duality, and then discuss, through simple examples, the adequation of the usual sufficient conditions of constraint qualification to stochastic optimization problems under almost sure constraints.

Introduction

In the stochastic optimization Problem,

$$\begin{aligned} \min_{\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}} \in \mathcal{U}} \quad & \mathbb{E}\big[J(\boldsymbol{U})\big] \\ s.t. \quad & \Theta(\boldsymbol{U}) \in -C \end{aligned}$$

an admissible control has to satisfy the following constraint

$$\Theta(\boldsymbol{U}) \in -C \qquad \mathbb{P}\text{-a.s.}$$

If the probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ is not finite, the above constraint can be seen as an infinite number of constraints. In most cases, the Karush-Kuhn-Tucker conditions of optimality for a constrained problem are given for a finite number of constraints. Dealing with an infinite number of constraints raises functional analysis questions over which functional spaces, endowed with which topology, are chosen for the controls and the multipliers.

In a few words, the abstract duality point of view consists in embedding an optimization problem $(\mathcal{P}_0)$ into a family of optimization problems $(\mathcal{P}_p)$ indexed by a perturbation $p \in \mathcal{Y}$. We denote by $\varphi(p)$ the value of the perturbed problem $(\mathcal{P}_p)$. The dual problem $(\mathcal{D}_0)$ consists in computing the value of the biconjugate $\varphi^{\star\star}(0)$.

We recall that properties and links between the primal problem $(\mathcal{P}_0)$ and its dual $(\mathcal{D}_0)$ are strongly related to the regularity of the value function φ at $p = 0$. More precisely an optimal solution $\lambda^\sharp$ of the dual $(\mathcal{D}_0)$ is an element of the subdifferential of φ at $p = 0$. This is the well-known marginal interpretation of the multiplier: $\lambda^\sharp$ is the marginal value of a perturbation p of the problem (e.g. a modification of the constraints of the problem).

In §4.1, we present basics in the theory of abstract duality, detailing the links between regularity of φ and existence of optimal multipliers (that is solutions to the dual problem). We also expose the special case of the Lagrangian duality and a sufficient condition of qualification. In §4.2, we work out two examples underlying the difficulties of using the duality theory in a stochastic optimization framework. Indeed simple almost sure constraint are shown to be non-qualified or qualified but not satisfying the generic sufficient condition of qualification.

4.1 Abstract Duality Theory

We recall here the abstract theory of duality that can be found in [24, 45, 89].

4.1.1 Introducing the Framework

A family of perturbed optimization problem

We consider paired spaces[1] $(\mathcal{U}, \mathcal{U}^\star)$, for example a Banach space and its topological dual (see §A.1.4 for more informations). The space $\mathcal{U}$ is called the space of *controls*. In order to study the following optimization problem:

$$(\mathcal{P}_0) \qquad \inf_{u \in \mathcal{U}} \quad \mathfrak{J}(u) \ , \tag{4.1}$$

where $\mathfrak{J} : \mathcal{U} \to \overline{\mathbb{R}}$, we introduce a space $\mathcal{Y}$ of *perturbations* paired with $\mathcal{Y}^\star$. Elements of $\mathcal{Y}$ are denoted p for "perturbation", and elements of its paired space $\mathcal{Y}^\star$ are denoted λ and called *multipliers*. We introduce a *perturbed cost function* $G : \mathcal{U} \times \mathcal{Y} \to \overline{\mathbb{R}}$ that satisfies the following equation.

$$G(\cdot, 0) \equiv \mathfrak{J}(\cdot) \ . \tag{4.2}$$

We consider the family $\big\{(\mathcal{P}_p)\big\}_{p \in \mathcal{Y}}$ of perturbed optimization problem induced by G:

$$(\mathcal{P}_p) \qquad \inf_{u \in \mathcal{U}} \quad G(u, p) \ , \tag{4.3}$$

and denote $\varphi(p)$ its value, i.e.

$$\varphi(p) \ := \ \inf_{u \in \mathcal{U}} \ G(u, p) \ . \tag{4.4}$$

By (4.2), we know that $\varphi(0)$ is the value of the original optimization problem (4.1).

1. All topological assumptions are done with respect to the topologies compatible with the paired spaces.

Introducing the Lagrangian

We introduce the *Lagrangian* function associated with the family of perturbed problems $\{(\mathcal{P}_p)\}_{p\in\mathcal{Y}}$.

Definition 4.1. *The* Lagrangian $L : \mathcal{U} \times \mathcal{Y}^\star \to \overline{\mathbb{R}}$ *associated to* $\{(\mathcal{P}_p)\}_{p\in\mathcal{Y}}$ *is given by*

$$L(u,\lambda) := \inf_{p\in\mathcal{Y}} \{G(u,p) + \langle \lambda , p\rangle\} . \tag{4.5}$$

Fact 4.2. *If for any control $u \in \mathcal{U}$ the function $G_u : p \mapsto G(u,p)$ is convex and l.s.c.(for the topology attached to the pairing $(\mathcal{Y},\mathcal{Y}^\star)$), then the primal problem $(\mathcal{P}_0)$ (defined in (4.1)) can be written*

$$(\mathcal{P}_0) \qquad \inf_{u\in\mathcal{U}} \ \sup_{\lambda\in\mathcal{Y}^\star} \qquad L(u,\lambda) . \tag{4.6}$$

Proof. By definition of the Fenchel conjugate of G_u (see Definition A.37) we have

$$G_u^\star(\lambda) = \sup_{p\in\mathcal{Y}} \Big\{\langle \lambda , p\rangle - G_u(p)\Big\} .$$

Thus by Definition 4.1 we have

$$\forall u \in \mathcal{U}, \quad \forall \lambda \in \mathcal{Y}^\star, \qquad L\big(u,\lambda\big) = -G_u^\star\big(-\lambda\big) .$$

Consequently the biconjugate of G_u reads

$$G_u^{\star\star}(p) = \sup_{\lambda\in\mathcal{Y}^\star} \Big\{\langle \lambda , p\rangle + L\big(u,-\lambda\big)\Big\} .$$

Changing λ into $-\lambda$ and taking $p = 0$ in the previous expression we obtain

$$G_u^{\star\star}(0) = \sup_{\lambda\in\mathcal{Y}^\star} L\big(u,\lambda\big) .$$

As G_u is assumed to be convex and l.s.c, we have by Theorem A.38 that $G_u^{\star\star} = G_u$. Then Equation (4.2) yields

$$\mathfrak{J}(u) = \sup_{\lambda\in\mathcal{Y}^\star} L\big(u,\lambda\big) ,$$

and minimization over $u \in \mathcal{U}$ yields (4.6) □

Introducing the dual problem

With Equation (4.6) in mind, we define the dual problem $(\mathcal{D}_0)$ of problem $(\mathcal{P}_0)$ as

$$(\mathcal{D}_0) \qquad \sup_{\lambda\in\mathcal{Y}^\star} \ \inf_{u\in\mathcal{U}} \qquad L(u,\lambda) . \tag{4.7}$$

Fact 4.3. *The dual problem $(\mathcal{D}_0)$ has value $\varphi^{\star\star}(0)$, where function φ is given by* (4.4).

Proof. For any multiplier $\lambda \in \mathcal{Y}^\star$, we have, by Definition 4.1

$$\begin{aligned}
\inf_{u\in\mathcal{U}} \ L(u,\lambda) &= \inf_{u\in\mathcal{U}} \ \inf_{p\in\mathcal{Y}} \ \{\langle \lambda , p\rangle + G(u,p)\} \\
&= \inf_{p\in\mathcal{Y}} \ \{\langle \lambda , p\rangle + \varphi(p)\} && \text{by (4.4)} \\
&= -\varphi^\star(-\lambda) . && \text{by Definition } A.37
\end{aligned}$$

Then, we deduce that the value of $(\mathcal{D}_0)$ is given by

$$\sup_{\lambda\in\mathcal{Y}^\star} \ \inf_{u\in\mathcal{U}} \ L(u,\lambda) = \sup_{\lambda\in\mathcal{Y}^\star} \ -\varphi^\star(-\lambda) = \varphi^{\star\star}(0) ,$$

which end the proof. □

Note that Fact 4.3 allows to introduce the dual problem directly as the problem of computing $\varphi^{\star\star}(0)$.

Relations between the primal and dual problems

Fact 4.4. *The* weak duality relation *states that the value of the primal problem* $(\mathcal{P}_0)$ *is higher than the value of the dual problem* $(\mathcal{D}_0)$*. We call* duality gap *the (non-negative) difference between the value of the primal and dual problems.*

Proof. Indeed by Theorem A.38 we know that $\varphi^{\star\star} \leq \varphi$. Definition of $\varphi(0)$ and Fact 4.3 ends the proof. □

Furthermore we give in the next proposition some links between the regularity of the value function φ at 0 (given by (4.4)) and the relation between the primal problem $(\mathcal{P}_0)$ (in (4.1)) and the dual problem $(\mathcal{D}_0)$ (in (4.7)). Those results, and more, can be found in [24, 89].

Proposition 4.5. *If the value function* φ *is convex (which is the case if the perturbed cost* G *is jointly convex in* (u,p)*), and finite*[2] *at* 0 *we have:*

- $\varphi(0) = \inf(\mathcal{P}_0)$
- $\varphi^{\star\star}(0) = \sup(\mathcal{D}_0)$ *and* $\arg\max(\mathcal{D}_0) = \partial\varphi^{\star\star}(0)$ *(that can be empty);*
- φ *is l.s.c.at* 0 *iff there is no duality gap, i.e* $\inf(\mathcal{P}_0) = \sup(\mathcal{D}_0)$*;*
- φ *is subdifferentiable at* 0 *if there is no duality gap and there is a solution to the dual problem i.e* $\inf(\mathcal{P}_0) = \max(\mathcal{D}_0)$ *and* $\arg\max \mathcal{D}_0 \neq \emptyset$*.*

Definition 4.6. *Problem* $\left(\mathcal{P}_p\right)$ *is said to be* calm *if* $\varphi(p) < \infty$ *and* $\partial\varphi(p) \neq \emptyset$*.*

4.1.2 A Specific Type of Perturbation

We now show how the classical theory of dualization is inscribed in this abstract duality theory. The main point is to formulate Problem $(\mathcal{P}_0)$ as a problem under constraints, and to perturb it by perturbing additively the constraint.

Constructing the Lagrangian Duality

Recall Problem (1.1),

$$(\mathcal{P}_0) \qquad \inf_{u\in\mathcal{U}^{\mathrm{ad}}} \quad J(u) \qquad s.t. \quad \Theta(u) \in -C$$

where $J : \mathcal{U} \to \overline{\mathbb{R}}$ is a proper l.s.c.convex function, $\mathcal{U}^{\mathrm{ad}}$ a non empty closed convex set, $C \subset \mathcal{Y}$ a closed convex cone and $\Theta : \mathcal{U} \to \mathcal{Y}$ a continuous $C-$convex function (see Definition A.48). Note that the link with §4.1.1 is given by

$$\mathfrak{J} = J + \chi_{\mathcal{U}^{\mathrm{ad}}} + \chi_{\Theta(\cdot)\in -C}$$

Let us define the following perturbed cost function

$$G(u,p) = J(u) + \chi_{\mathcal{U}^{\mathrm{ad}}}(u) + \chi_{\left\{(u,p)\in\mathcal{U}\times\mathcal{Y} \mid \Theta(u)-p\in -C\right\}}(u,p) , \tag{4.8}$$

and we have, as required,

$$G(\cdot,0) = \mathfrak{J}(\cdot) .$$

Then, Problem (1.1) can be embedded in the following family of perturbed problems

$$(\mathcal{P}_p) \qquad \inf_{u\in\mathcal{U}^{\mathrm{ad}}} J(u) + \chi_{\{\Theta(u)-p\in -C\}}(u,p) . \tag{4.9}$$

2. The convexity and finiteness assumptions are sufficient but not always necessary.

which is equivalent to

$$\inf_{u \in \mathcal{U}^{\mathrm{ad}}} \quad J(u)$$
$$s.t. \quad \Theta(u) - p \in -C$$

Equivalently, through Proposition A.39, we can write problem $(\mathcal{P}_p)$

$$(\mathcal{P}_p) \qquad \inf_{u \in \mathcal{U}^{\mathrm{ad}}} \quad \sup_{\lambda \in C^\star} \qquad J(u) + \big\langle \lambda, \Theta(u) - p \big\rangle \; .$$

The Lagrangian introduced in §4.1.1, associated to the family of problems $\{\mathcal{P}_p\}_{p \in \mathcal{Y}}$, reads

$$L(u, \lambda) = J(u) + \big\langle \lambda, \Theta(u) \big\rangle + \chi_{\mathcal{U}^{\mathrm{ad}}} - \chi_{C^\star}(\lambda) \; . \tag{4.10}$$

Thus Problem $(\mathcal{P}_0)$ reads

$$(\mathcal{P}_0) \qquad \inf_{u \in \mathcal{U}^{\mathrm{ad}}} \quad \sup_{\lambda \in C^\star} \qquad L(u, \lambda) \; , \tag{4.11}$$

and Problem $(\mathcal{D}_0)$ reads

$$(\mathcal{D}_0) \qquad \sup_{\lambda \in C^\star} \quad \inf_{u \in \mathcal{U}^{\mathrm{ad}}} \qquad L(u, \lambda) \; . \tag{4.12}$$

Conditions of Qualification

We give now conditions under which problems $(\mathcal{P}_0)$ (in (4.11)) and $(\mathcal{D}_0)$ (in (4.12)) are equivalent in the sense that the set of solutions of Problem (4.11) is the same as the set of solution of Problem (4.12), and their values are equals.

Definition 4.7. *Recall that Problem $(\mathcal{P}_0)$ admits at least one solution, and is convex. Then the constraint*

$$\Theta(u) \in -C \tag{4.13}$$

is said to be qualified *if the problem $\big(\mathcal{P}_0\big)$ is calm, that is if one of the two following equivalent statements holds.*

i) $\partial\varphi(0) \neq \emptyset$, where φ is defined as in (4.4).

ii) There is no duality gap and the dual problem $(\mathcal{D}_0)$ has an optimal solution.

If Problem $(\mathcal{P}_0)$ admits an optimal solution, these assertions are equivalent to

iii) The Lagrangian L, defined in (4.10), admits a saddle point on $\mathcal{U}^{\mathrm{ad}} \times C^\star$, i.e. there exists $(u^\sharp, \lambda^\sharp) \in \mathcal{U}^{\mathrm{ad}} \times C^\star$ ($C^\star$ is defined in SA.4) such that

$$\forall u \in U^{\mathrm{ad}}, \quad \forall \lambda \in C^\star, \qquad L(u^\sharp, \lambda) \leq L(u^\sharp, \lambda^\sharp) \leq L(u, \lambda^\sharp) \; .$$

Note that it is quite difficult to check these conditions. Thus, we need sufficient conditions of qualification. We are going to reformulate classical conditions of qualification in our framework.

We begin by a lemma on the regularity of the perturbed cost function G.

Lemma 4.8. *The function G defined in Equation (4.8) is jointly convex and l.s.c.*

Proof. As J is convex by assumption, in order to show the joint convexity of G it is enough to show that the set

$$\big\{(u, p) \in \mathcal{U}^{\mathrm{ad}} \times \mathcal{Y} \mid \Theta(u) - p \in -C\big\} \tag{4.14}$$

is convex.

For this purpose, consider two pairs (u_1, p_1) and (u_2, p_2) such that,

$$\forall i \in \{1,2\}, \qquad \Theta(u_i) - p_i \ \in -C \ ,$$

and $t \in [0,1]$. As $\mathcal{U}^{\mathrm{ad}}$ is convex, and $\mathcal{Y}$ is a vector space, $\mathcal{U}^{\mathrm{ad}} \times \mathcal{Y}$ is convex and we have

$$t(u_1, p_1) + (1-t)(u_2, p_2) \in \mathcal{U}^{\mathrm{ad}} \times \mathcal{Y} \ .$$

Moreover convexity of C gives

$$t\Theta(u_1) + (1-t)\Theta(u_2) - \big(tp_1 + (1-t)p_2\big) \ \in -C \ .$$

Now, by C-convexity of Θ, we have

$$\Theta\big(tu_1 + (1-t)u_2\big) - \big(t\Theta(u_1) + (1-t)\Theta(u_2)\big) \in -C \ .$$

Moreover as C is a closed convex cone, we have $C + C = C$ (see Lemma A.47), thus,

$$\Theta\big(tu_1 + (1-t)u_2\big) - \big(tp_1 + (1-t)p_2\big) \ \in -C \ .$$

and we have shown the convexity of the set (4.14) and thus the convexity of G.

Continuity of Θ, closedness of C and closedness of $\mathcal{U}^{\mathrm{ad}}$ give the closedness of the set

$$\big\{(u,p) \in \mathcal{U} \times \mathcal{Y} \mid u \in \mathcal{U}^{\mathrm{ad}}, \quad \Theta(u) - p \ \in -C\big\} \ ,$$

hence the lower semicontinuity of the function

$$\chi_{\big\{(u,p)\in\mathcal{U}\times\mathcal{Y} \mid u\in\mathcal{U}^{\mathrm{ad}}, \quad \Theta(u)-p \,\in -C\big\}} \ .$$

Finally, lower semicontinuity of function J gives the lower semicontinuity of function G. $\square$

As G defined in (4.8) is jointly convex, the value function φ defined in (4.4) is also convex (see Proposition A.45). Consequently a sufficient condition for the constraint (4.13) to be qualified is for φ to be continuous at 0. Indeed continuity of a convex function implies its subdifferentiability (see [9, Proposition 2.36]). Moreover recall that:

- a convex function, defined on a topological linear space, is continuous at a point in the interior of its domain if and only if it is locally bounded above at this point (see [9, Proposition 2.14]);
- a proper l.s.c.convex function, defined on a Banach space, is continuous on the interior of its domain (see [9, Proposition 2.16]).

However, there is no general reason for φ to be l.s.c.. Nonetheless we have the following proposition (see [24, Proposition 2.153])

Proposition 4.9. *Assume that $\mathcal{U}$ and $\mathcal{Y}$ are Banach spaces, that the perturbed cost function G is proper, convex and l.s.c., and $\varphi(0) < +\infty$ (where the value function φ is given by (4.4)). Then $0 \in \mathrm{ri}\big(\mathrm{dom}(\varphi)\big)$ implies that $\partial\varphi(0) \neq \emptyset$, hence*

We now give the usual constraint qualification condition.

Proposition 4.10. *Assume that $\mathcal{U}$ and $\mathcal{Y}$ are Banach spaces, and that the perturbed cost function G is proper, convex and l.s.c.. Then, under the following assumption*

$$(CQC) \qquad 0 \in \mathrm{ri}\Big(\Theta\big(\mathcal{U}^{\mathrm{ad}} \cap \mathrm{dom}(J)\big) + C\Big) \ , \tag{4.15}$$

Constraint (4.13) is qualified.

Proof. $(\mathcal{P}_p)$ defined in (4.9) is feasible iff
$$\exists u \in \mathcal{U}^{\mathrm{ad}} \cap \mathrm{dom}(J), \qquad \Theta(u) - p \in -C ,$$
which can be written
$$p \in \Theta\big(\mathcal{U}^{\mathrm{ad}} \cap \mathrm{dom}(J)\big) + C ,$$
thus,
$$\mathrm{dom}(\varphi) = \Theta\big(\mathcal{U}^{\mathrm{ad}} \cap \mathrm{dom}(J)\big) + C .$$
Proposition 4.9 ends the proof. □

Proposition 4.10 is sometimes stated without $\mathcal{U}^{\mathrm{ad}}$ or $\mathrm{dom}(J)$. Indeed the cost function can always be replaced by $J + \chi_{\mathcal{U}^{\mathrm{ad}}}$ and in this case the minimization in u is done on the whole space $\mathcal{U}$. In other words we could easily choose $\mathcal{U}^{\mathrm{ad}}$ to be a subset of $\mathrm{dom}(J)$.

Remark 4.11. *The condition* (CQC) *(in (4.15)) is equivalent to*
$$0 \in \mathrm{ri}\Big(\mathrm{dom}(\varphi)\Big) . \tag{4.16}$$
This sufficient condition is quite strong (as it will be illustrated in the next section). Indeed in most cases a convex function is subdifferentiable also on the border of its domain. For example if $f : E \to \mathbb{R}$ is a finite convex function, and C a closed convex set, then the l.s.c.function $f + \chi_C$ is subdifferentiable at any point of its domain (i.e C).

An example of function that is not subdifferentiable on the border of its domain would be
$$\varphi(x) = \begin{cases} +\infty & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x\log(x) & \text{if } x > 0 \end{cases} .$$
At $x = 0$ this function admits a tangent (toward the interior of the domain) with infinite slope, thus, is not subdifferentiable. If the function admitted a finite sloped tangent it would be subdifferentiable.

Almost Sure Constraint in $L^p\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$ Display Empty Interior

We claim that the sufficient condition of qualification (4.15) is scarcely satisfied in a stochastic optimization setting if we choose $\mathcal{Y} = \mathrm{L}^p\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ with $p < \infty$. By contrast, if $\mathcal{Y} = \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, this condition is more often satisfied.

Proposition 4.12. *Consider a probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, where $\mathcal{F}$ is not finite modulo $\mathbb{P}$.*[3] *Consider $p \in [1, \infty)$, and a set $U^{\mathrm{ad}} \subsetneq \mathbb{R}^n$ that is not an affine subspace of $\mathbb{R}^n$. Then, the set*
$$\mathcal{U}^{\mathrm{ad}} = \Big\{ \boldsymbol{U} \in \mathrm{L}^p\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big) \quad \big| \quad \boldsymbol{U} \in U^{\mathrm{ad}} \quad \mathbb{P}\text{-a.s.} \Big\} ,$$
has an empty relative interior in L^p.

Proof. Consider $\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}}$, $p \in [1, +\infty)$ and $x \in \mathrm{Aff}(U^{\mathrm{ad}}) \backslash U^{\mathrm{ad}}$. We are going to exhibit a sequence $\big\{\boldsymbol{U}_n\big\}_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$, we have $\boldsymbol{U}_n \notin U^{\mathrm{ad}}$ and $\boldsymbol{U}_n \to_{\mathrm{L}^p} \boldsymbol{U}$.

Since $\mathcal{F}$ is not finite modulo $\mathbb{P}$, we can consider a sequence of $\mathcal{F}$-measurable events $\{A_n\}_{n \in \mathbb{N}}$ with $\mathbb{P}(A_n) > 0$ and such that $\lim_n \mathbb{P}(A_n) = 0$. Then, we define
$$\boldsymbol{U}_n = \begin{cases} x & \text{on } A_n, \\ \boldsymbol{U} & \text{elsewhere.} \end{cases}$$
We have $\big\|\boldsymbol{U}_n - \boldsymbol{U}\big\|_p = \big\|\big(\boldsymbol{U} - x\big)\mathbb{1}_{A_n}\big\|$. Thus, dominated convergence theorem ensures that $\boldsymbol{U}_n \to_{L^p} \boldsymbol{U}$. However, by construction, for any n, we have that $\boldsymbol{U}_n \notin \mathcal{U}^{\mathrm{ad}}$. □

3. See Definition 5.1.

The space L^∞, endowed with the norm topology, is better suited for almost sure constraint as shown in the next proposition.

Proposition 4.13. *Consider a set* $U^{\mathrm{ad}} \subset \mathbb{R}^n$ *such that* $\mathrm{int}(U^{\mathrm{ad}}) \neq \emptyset$. *Then the set*

$$\mathcal{U}^{\mathrm{ad}} = \Big\{ \boldsymbol{U} \in \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big) \quad | \quad \boldsymbol{U} \in U^{\mathrm{ad}} \quad \mathbb{P}\text{-}a.s. \Big\} ,$$

has a non-empty interior.

Proof. Consider $u \in \mathrm{int}(U^{\mathrm{ad}})$, and $\varepsilon > 0$ such that $B_{\mathbb{R}^n}(u, \varepsilon) \subset U^{\mathrm{ad}}$. Then the (constant) random variable $\boldsymbol{U} \equiv u$ is such that, for all random variable $\boldsymbol{V} \in B_{\mathrm{L}^\infty}(\boldsymbol{U}, \varepsilon)$, i.e. such that $\|\boldsymbol{U} - \boldsymbol{V}\|_{\mathrm{L}^\infty} < \epsilon$, we have $\boldsymbol{V} \in B_{\mathbb{R}^n}(u, \varepsilon) \subset U^{\mathrm{ad}}$ $\mathbb{P}$-a.s. Thus $\boldsymbol{V} \in \mathcal{U}^{\mathrm{ad}}$. □

The following practical corollary is a direct application of Proposition 4.13 and Proposition 4.10.

Corollary 4.14. *Consider a closed convex set* $U^{\mathrm{ad}} \subset \mathbb{R}^n$. *Consider the affine constraint function* $\Theta : \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big) \to \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^p\big)$ *such that there is a matrix* $A \in M_{p,n}(\mathbb{R})$ *and a vector* $b \in \mathbb{R}^p$ *with*

$$\forall \boldsymbol{U} \in \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big), \qquad \Theta(\boldsymbol{U}) = A\boldsymbol{U} + b, \quad \mathbb{P}\text{-}a.s.$$

Assume that J *is convex, proper, and continuous on* $\mathcal{U}$. *If* $0 \in \mathrm{ri}\,\big(AU^{\mathrm{ad}} + b\big)$ *then the constraint*

$$\Theta(\boldsymbol{U}) = 0 ,$$

in the following problem,

$$\begin{aligned} \min_{\boldsymbol{U} \in \mathrm{L}^\infty} \quad & J(\boldsymbol{U}) \\ s.t. \quad & \Theta(\boldsymbol{U}) = 0 \\ & \boldsymbol{U} \in U^{\mathrm{ad}} \qquad\qquad\qquad \mathbb{P}\text{-}a.s. \end{aligned}$$

is qualified.

4.2 Working Out Two Examples on Constraint Qualification

In this section, we develop two examples that reveal delicate issues related to duality in stochastic optimization. In a first example we show that, even on a seemingly innocuous problem (inspired by R. Wets) there might not exist a dual multiplier in L^2. In a second example we show that a multiplier might exist even if the sufficient qualification condition (CQC) (4.15) is not satisfied.

4.2.1 An Example with a Non-Qualified Constraint

We elaborate on an example from R. Wets [4]. Where R. Wets focused on a discretization of the probability space approach to show that when refining the discretization the multiplier would converges toward a singular measure. On the other hand we, cast the problem in a strongly convex setting and derive directly the conditions of qualification.

Let $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ be a probability space. Let $\boldsymbol{\xi}$ be a random variable uniform on $[1, 2]$, $\alpha > 0$ a positive real number. We consider the optimization problem

4. CEA-EDF-INRIA 2013 summer school.

$$\inf_{x,\boldsymbol{Y}} \quad \frac{x^2}{2} + \mathbb{E}\frac{(\boldsymbol{Y}+\alpha)^2}{2} \tag{4.17a}$$

$$x \geq a \tag{4.17b}$$

$$(x - \boldsymbol{Y}) \geq \boldsymbol{\xi} \tag{4.17c}$$

$$\boldsymbol{Y} \geq 0 \tag{4.17d}$$

where x is a deterministic real variable and $\boldsymbol{Y}$ is a random variable. For technical issues we assume that $2 - \alpha < a$.

We can easily find the optimal solution. Noting that $\alpha \geq 0$, a careful look on the constraints shows that, $\boldsymbol{Y}$ being positive, x has to be greater than $\boldsymbol{\xi}$ almost surely, thus, x is greater than $\operatorname{essupp}(\boldsymbol{\xi}) = 2$. Consequently, from $\alpha \geq 0$, we see that

$$\begin{cases} x^\sharp & = \max\{2, a\} \\ \boldsymbol{Y}^\sharp & \equiv 0 \end{cases}$$

is an optimal solution of Problem (4.17) and yields a value of

$$\frac{\max\{a, 2\}^2}{2} + \frac{\alpha^2}{2} \ .$$

Now using the notations of abstract duality (§4.1), we set the set of perturbation

$$\mathcal{Y} = \mathbb{R} \times \mathrm{L}^2\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}\big) \times \mathrm{L}^2\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}\big).$$

Consider the family of perturbed problem, in L^2 spaces,

$$\begin{aligned} \inf_{x,\boldsymbol{Y}} \quad & \underbrace{\frac{x^2}{2} + \mathbb{E}\Big[\frac{(\boldsymbol{Y}+\alpha)^2}{2}\Big]}_{:=J(x,\boldsymbol{Y})} \\ & x \geq a + p_1 \\ & (x - \boldsymbol{Y}) \geq \xi + \boldsymbol{P}_2 \quad \mathbb{P}\text{-a.s.} \\ & \boldsymbol{Y} \geq 0 + \boldsymbol{P}_3 \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{4.18}$$

with $\boldsymbol{P} = (p_1, \boldsymbol{P}_2, \boldsymbol{P}_3) \in \mathcal{Y}$, and denote by $\varphi\big(\boldsymbol{P}\big)$ its value. Note that the perturbation $\boldsymbol{P}$ has a deterministic part (p_1) and a stochastic part ($\boldsymbol{P}_2$ and $\boldsymbol{P}_3$).

Problem (4.18) is cast in the general framework of §4.1.2 with the constraint function

$$\Theta(x, \boldsymbol{Y}) = \Big(a - x, \boldsymbol{\xi} - x + \boldsymbol{Y}, -\boldsymbol{Y}\Big) \ ,$$

and the cone of constraints

$$C = \big\{(v_1, \boldsymbol{V}_2, \boldsymbol{V}_3) \in \mathcal{Y} \quad | \quad v_1 \geq 0, \quad \boldsymbol{V}_2 \geq 0 \quad \mathbb{P}\text{-a.s.}, \quad \boldsymbol{V}_3 \geq 0 \quad \mathbb{P}\text{-a.s.}\big\} \ .$$

Lower semicontinuity of the value function φ at 0

As J is convex and Θ is C-convex, we obtain by Lemma 4.8 that φ is convex. Moreover, the value function φ related to Problem (4.18) can be made explicit, and its regularity at 0 studied (in order to find properties of the dual and primal problems).

Fact 4.15. *Then, in an* L^2*-neighborhood of* 0*, the value function* φ *related to Problem* (4.18) *is given by*

$$\begin{aligned}\varphi(\boldsymbol{P}) = & \Big(\max \Big\{ a+p_1, \operatorname{essupp}\big(\boldsymbol{\xi} + \boldsymbol{P}_2 + \max\{-\alpha, \boldsymbol{P}_3\}\big), 0 \Big\}\Big)^2 \\ & + \mathbb{E}\left[\frac{\big((\boldsymbol{P}_3+\alpha)^+\big)^2}{2} \right] + \chi_{\{\operatorname{essupp}(\boldsymbol{P}_3+\boldsymbol{P}_2+\boldsymbol{\xi})<\infty\}} \; .\end{aligned} \tag{4.19}$$

with optimal solution

$$\begin{cases} x^\sharp = & \max\big\{a+p_1, \operatorname{essupp}\big(\max\{-\alpha, \boldsymbol{P}_3\} + \xi + \boldsymbol{P}_2\big), 0\big\} \\ \boldsymbol{Y}^\sharp = & \max\{\boldsymbol{P}_3, -\alpha\} \end{cases} \tag{4.20}$$

Proof. Note that Problem (4.18) admits a solution if

$$\operatorname{essupp}\left(\boldsymbol{P}_3 + \boldsymbol{P}_2 + \boldsymbol{\xi}\right) < \infty \; . \tag{4.21}$$

Indeed, in this case, the solution $(x^\sharp, \boldsymbol{Y}^\sharp)$ defined in (4.20) is admissible. On the other hand if (4.21) does not hold, then the two last constraints of Problem (4.18) cannot be satisfied almost surely with a finite x.

Now, if conditions (4.21) hold true, then, for a given admissible $\boldsymbol{Y}$, the solution of

$$\begin{aligned} \min_x \quad & \frac{x^2}{2} + \mathbb{E}\Big[\frac{(\boldsymbol{Y}+\alpha)^2}{2}\Big] \\ & x \geq a + p_1 \\ & x \geq \boldsymbol{Y} + \xi + \boldsymbol{P}_2 \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

is given by

$$x^\sharp = \max\left\{a+p_1, \operatorname{essupp}\left(\boldsymbol{Y} + \boldsymbol{\xi} + \boldsymbol{P}_2\right), 0\right\} \; , \tag{4.22}$$

with value

$$\frac{\Big(\max\left\{a+p_1, \operatorname{essupp}\left(\boldsymbol{Y} + \boldsymbol{\xi} + \boldsymbol{P}_2\right), 0\right\}\Big)^2}{2} + \frac{\mathbb{E}\big[(\boldsymbol{Y}+\alpha)^2\big]}{2} \; .$$

Thus, we consider the minimization in $\boldsymbol{Y}$ of

$$\begin{aligned} \min_{\boldsymbol{Y}} \quad & \frac{\Big(\max\left\{a+p_1, \operatorname{essupp}\left(\boldsymbol{Y} + \boldsymbol{\xi} + \boldsymbol{P}_2\right), 0\right\}\Big)^2}{2} + \frac{\mathbb{E}\big[(\boldsymbol{Y}+\alpha)^2\big]}{2} \\ s.t. \quad & \boldsymbol{P}_3 \leq \boldsymbol{Y} \end{aligned}$$

The first term of the sum is non decreasing with respect to $\boldsymbol{Y}$. Moreover, for $\|\boldsymbol{P}\|_{\mathrm{L}^2}$ small enough, as $2-\alpha < a$, $a+p_1 > 2-\alpha$ and $\operatorname{essupp}\left(\boldsymbol{Y}+\boldsymbol{\xi}+\boldsymbol{P}_2\right) < \operatorname{essupp}\boldsymbol{Y} + 2 - \alpha$. Hence, if $\boldsymbol{Y} \leq 0$, $\left\{a+p_1, \operatorname{essupp}\left(\boldsymbol{Y}+\boldsymbol{\xi}+\boldsymbol{P}_2\right), 0\right\} = a + p_1$. As $\alpha \geq 0$ the second term is also non decreasing with respect to $\boldsymbol{Y}$ from $-\alpha$. Thus,

$$\boldsymbol{Y}^\sharp = \max\left\{\boldsymbol{P}_3, -\alpha\right\} \; ,$$

is optimal for any $x \in \mathbb{R}$.

Now from (4.22) we obtain

$$\boldsymbol{X}^\sharp = \max\left\{a+p_1, \operatorname{essupp}\left(\boldsymbol{Y}^\sharp+\xi+\boldsymbol{P}_2\right), 0\right\} = \max\left\{a+p_1, \operatorname{essupp}\left(\max \boldsymbol{P}_3, -\alpha+\xi+\boldsymbol{P}_2\right), 0\right\} .$$

Evaluating the cost function J at $(\boldsymbol{X}^\sharp, \boldsymbol{Y}^\sharp)$ coupled with the condition of admissibility (4.21) yields the expression of φ. □

Fact 4.16. *The value function φ is l.s.c.at* 0.

Proof. We now show that φ is l.s.c.(for the strong and weak L^2 topologies) at 0.

- As the mapping $x \mapsto (x+\alpha)^+$ is a contraction, if $\big(\boldsymbol{P}_n\big)_{n\in\mathbb{N}}$ converges in L^2 towards $\boldsymbol{P}$, then $\big((\boldsymbol{P}_n+\alpha)^+\big)_{n\in\mathbb{N}}$ converges in L^2 towards $(\boldsymbol{P}+\alpha)^+$. Thus the function
$$h_1 : \boldsymbol{P} \mapsto \mathbb{E}\frac{((\boldsymbol{P}_3+\alpha)^+)^2}{2}$$
is continuous.
- The mapping
$$\boldsymbol{P} \mapsto \xi + \boldsymbol{P}_2 - \max\{\boldsymbol{P}_3, -\alpha\}$$
is continuous. So, by Lemma A.56, the mapping
$$\boldsymbol{P} \mapsto \operatorname{essupp}\big(\boldsymbol{\xi} + \boldsymbol{P}_2 + \max\big\{-\alpha, \boldsymbol{P}_3\big\}\big)$$
is l.s.c, and as the mapping $\boldsymbol{P} \mapsto a + p_1$ is continuous we have (Lemma A.54) that the mapping
$$h_2 : \boldsymbol{P} \mapsto \max\Big\{a + p_1, \operatorname{essupp}\Big(\xi + \boldsymbol{P}_2 + \max\big\{-\alpha, \boldsymbol{P}_3\big\}\Big), 0\Big\}$$
is l.s.c.As the function h_2 is non-negative, we have $h_2^2 = (h_2^+)^2$. Moreover the mapping $x \mapsto (x^+)^2$ is non-decreasing, thus we obtain by Lemma A.53 the lower semicontinuity of h.
- By Lemma A.56 we know that the function $\boldsymbol{P} \mapsto \operatorname{essupp}\big(\boldsymbol{\xi} + \boldsymbol{P}_2 + \boldsymbol{P}_3\big)$ is l.s.c., thus its level sets are closed and the set
$$D = \big\{\boldsymbol{P} \in \mathrm{L}^2 \mid \operatorname{essupp}(\boldsymbol{P}_3 + \boldsymbol{P}_2 + \boldsymbol{\xi}) < M\big\} ,$$
is closed.

Finally, h_1 being continuous, and h_2 l.s.c.$h_1 + h_2$ is l.s.c., and by Lemma A.55, the function
$$\varphi = h_1 + h_2 + \chi_D$$
is l.s.c.. ☐

From the lower semicontinuity of the value function at the origin, Problem (4.17) and its dual in the sense of §4.1.2 have the same value. By Proposition 4.5, there is no duality gap.

Non-Subdifferentiability of φ at 0

Fact 4.17. *If $a < 2$, the value function φ (defined by (4.4) and given by (4.19)) is not subdifferentiable at* 0.

Proof. The proof is by contradiction. Suppose that there exists $\boldsymbol{\lambda}^\sharp \in \partial\varphi(0) \subset \mathrm{L}^2$. Then we have, for all $\boldsymbol{P} \in \mathrm{L}^2$,
$$\varphi(\boldsymbol{P}) - \varphi(0) \geq \langle \boldsymbol{\lambda}^\sharp, \boldsymbol{P} \rangle . \tag{4.23}$$
We now display a family of perturbations that implies that the L^2 norm of $\boldsymbol{\lambda}^\sharp$ is not finite, hence a contradiction.

Consider the perturbation $\varepsilon^2 \boldsymbol{P}_\varepsilon$, where
$$\boldsymbol{P}_\varepsilon = \Big(0, \underbrace{-1/\varepsilon \mathbb{1}_{\boldsymbol{\xi}\in[2-\varepsilon,2]}}_{\boldsymbol{P}_{2,\varepsilon}}, 0\Big) .$$

As we have

$$\boldsymbol{\xi} + \varepsilon^2 \boldsymbol{P}_{2,\varepsilon} = \boldsymbol{\xi} \mathbb{1}_{\{\boldsymbol{\xi}\in[0,2-\varepsilon)\}} + (\boldsymbol{\xi} - \varepsilon)\mathbb{1}_{\{\boldsymbol{\xi}\in[2-\varepsilon,2]\}} ,$$

we obtain

$$\operatorname{essupp}\big(\boldsymbol{\xi} + \varepsilon^2 \boldsymbol{P}_{2,\varepsilon}\big) = 2 - \varepsilon .$$

Moreover, for $0 < \varepsilon \leq 2 - a$, ($\varepsilon$ exists as $a < 2$),

$$\Big(\max\Big\{a + \underbrace{p_{1,\varepsilon}}_{=0}, \operatorname{essupp}\big(\boldsymbol{\xi} + \varepsilon^2 \boldsymbol{P}_{2,\varepsilon}\big), 0\Big\}\Big)^2 = \big(2 - \varepsilon\big)^2 ,$$

which in turn yields

$$\varphi(\varepsilon^2 \boldsymbol{P}_\varepsilon) - \varphi(0) = (2-\varepsilon)^2 - 2^2 = -2\varepsilon + \varepsilon^2 .$$

Consequently, from the subgradient inequality (4.23) we obtain

$$\frac{\varphi(\varepsilon^2 \boldsymbol{P}_\varepsilon) - \varphi(0)}{\varepsilon^2} = -\frac{2}{\varepsilon} + 1 \geq \langle \boldsymbol{\lambda}^\sharp, \boldsymbol{P}_\varepsilon \rangle .$$

Consequently, for $\varepsilon < 1/2$, we have $\langle \boldsymbol{\lambda}^\sharp, \boldsymbol{P}_\varepsilon \rangle < 0$, and thus,

$$\big| -\frac{2}{\varepsilon} + 1 \big| \leq |\langle \boldsymbol{\lambda}^\sharp, \boldsymbol{P}_\varepsilon \rangle| .$$

However the Cauchy-Schwartz inequality yields

$$\big| -\frac{2}{\varepsilon} + 1 \big| \leq |\langle \boldsymbol{\lambda}^\sharp, \boldsymbol{P}_\varepsilon \rangle| \leq ||\boldsymbol{\lambda}^\sharp||_2 \cdot ||\boldsymbol{P}_\varepsilon||_2 = ||\boldsymbol{\lambda}^\sharp||_2 .$$

Taking the limits $\varepsilon \to 0$ leads to a contradiction. Therefore $\boldsymbol{\lambda}^\sharp$ does not exist, which means that $\partial\varphi(0) = \emptyset$: φ is not subdifferentiable at 0. □

From this fact we conclude that the dual problem (defined by L^2 perturbations) has no solution for $a < 2$.

Working out the Dual Problem

We now write the dual problem (for L^2 perturbations) of Problem (4.17), and derive a maximizing sequence that does not converge in L^2, but converges toward an element of $\big(\mathrm{L}^\infty\big)^\star$.

Following §4.1.2 the dual of Problem (4.17) is given by

$$\sup_{\boldsymbol{\lambda}\geq 0} \quad \inf_{x,\boldsymbol{Y}} \qquad \frac{x^2}{2} + \mathbb{E}\Big[\frac{(\boldsymbol{Y}+\alpha)^2}{2}\Big] + \lambda_1(a - x) + \mathbb{E}\big[\boldsymbol{\lambda}_2(\xi - x + \boldsymbol{Y}) - \boldsymbol{\lambda}_3 \boldsymbol{Y}\big] , \tag{4.24}$$

where $\boldsymbol{\lambda} = \big(\lambda_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3\big)$ is an element of $\mathbb{R} \times \mathrm{L}^2\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}\big) \times \mathrm{L}^2\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}\big)$.

Fact 4.18. *The optimal value of the dual problem is given by*

$$\frac{\alpha^2}{2} + \max\{a, 2\}^2/2$$

which is equal to $\varphi(0)$ (by (4.19)). Thus, as already obtained in Fact 4.16, there is no duality gap.

If $a < 2$, then we can construct a maximizing sequence of Problem (4.24) that does not converges in L^2. If $a > 2$, then we have an optimal solution to Problem (4.24) that lies in L^2, thus φ is subdifferentiable at 0.

Proof. For a given multiplier $\boldsymbol{\lambda} = (\lambda_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)$, the minimization part of Problem (4.24) can be written as

$$\underbrace{\min_x \left\{ \frac{x^2}{2} - (\lambda_1 + \mathbb{E}[\boldsymbol{\lambda}_2])x + a\lambda_1 + \mathbb{E}\big[\boldsymbol{\lambda}_2 \boldsymbol{\xi}\big] \right\}}_{(A)} + \underbrace{\min_{\boldsymbol{Y}} \left\{ \mathbb{E}\Big[\frac{(\boldsymbol{Y} + \alpha)^2}{2} + (\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_3)\boldsymbol{Y} \Big] \right\}}_{(B)} .$$

Part (A) is easily minimized as it is a second order polynom in x, with value

$$-\frac{(\lambda_1 + \mathbb{E}[\boldsymbol{\lambda}_2])^2}{2} + a\lambda_1 + \mathbb{E}\big[\boldsymbol{\lambda}_2 \boldsymbol{\xi}\big] .$$

Part (B) is also easily solved as we can interchange the inf and the expectation (because $\boldsymbol{Y}$ is measurable with respect to $\boldsymbol{\xi}$, see [96, Theorem 14.60]). Thus the minimum is attained for

$$\boldsymbol{Y}^\sharp = -\alpha + \boldsymbol{\lambda}_3 - \boldsymbol{\lambda}_2 ,$$

with value

$$\mathbb{E}\Big[\frac{(\boldsymbol{\lambda}_3 - \boldsymbol{\lambda}_2)^2}{2} + (\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_3)(-\alpha + \boldsymbol{\lambda}_3 - \boldsymbol{\lambda}_2) \Big] .$$

Thus Problem (4.24) now becomes

$$\sup_{\lambda_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3 \geq 0} \left\{ -\frac{\lambda_1^2}{2} + \big(a - \mathbb{E}[\boldsymbol{\lambda}_2]\big)\lambda_1 - \frac{\Big(\mathbb{E}[\boldsymbol{\lambda}_2]\Big)^2}{2} + \mathbb{E}\big[\boldsymbol{\lambda}_2 \boldsymbol{\xi}\big] + \mathbb{E}\Big[-\frac{(\boldsymbol{\lambda}_3 - \boldsymbol{\lambda}_2)^2}{2} + \alpha(\boldsymbol{\lambda}_3 - \boldsymbol{\lambda}_2) \Big] \right\} .$$

For given $\boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_3$, the maximization in λ_1 is quadratic (in $\mathbb{R}$). The unconstrained optimum being $\lambda_1 = a - \mathbb{E}[\boldsymbol{\lambda}_2]$, thus the optimum is $\lambda_1^\sharp = (a - \mathbb{E}[\boldsymbol{\lambda}_2])^+$. Maximization in $\boldsymbol{\lambda}_3$ can be done under the expectation and thus, the optimum is achieved for ${\boldsymbol{\lambda}_3}^\sharp = (\boldsymbol{\lambda}_2 + \alpha)^+$. As α and $\boldsymbol{\lambda}_2$ are non-negative we have ${\boldsymbol{\lambda}_3}^\sharp = \boldsymbol{\lambda}_2 + \alpha$. Hence the remaining maximization problem in $\boldsymbol{\lambda}_2$ reads

$$\sup_{\boldsymbol{\lambda}_2 \geq 0} \left\{ \frac{((a - \mathbb{E}[\boldsymbol{\lambda}_2])^+)^2}{2} - \frac{(\mathbb{E}[\boldsymbol{\lambda}_2])^2}{2} + \mathbb{E}\big[\boldsymbol{\lambda}_2 \boldsymbol{\xi}\big] + \frac{\alpha^2}{2} \right\} . \tag{4.25}$$

First we solve Problem (4.25) over the set of $\boldsymbol{\lambda}_2$ such that $\mathbb{E}\big[\boldsymbol{\lambda}_2\big] \geq a$. In this case, we have to solve

$$\frac{\alpha^2}{2} + \sup_{\substack{\boldsymbol{\lambda}_2 \geq 0 \\ \mathbb{E}[\boldsymbol{\lambda}_2] \geq a}} \left\{ -\frac{(\mathbb{E}[\boldsymbol{\lambda}_2])^2}{2} + \mathbb{E}\big[\boldsymbol{\lambda}_2 \boldsymbol{\xi}\big] \right\} .$$

This problem can be written

$$\frac{\alpha^2}{2} + \sup_{M \geq a} \sup_{\substack{\boldsymbol{\lambda}_2 \geq 0 \\ \mathbb{E}[\boldsymbol{\lambda}_2] = M}} \left\{ -\frac{M^2}{2} + \mathbb{E}\big[\boldsymbol{\lambda}_2 \boldsymbol{\xi}\big] \right\} ,$$

and the supremum in $\boldsymbol{\lambda}_2$ is obtained by concentrating the mass on the highest value of $\boldsymbol{\xi}$. A maximizing sequence is given by $Mk\mathbb{1}_{\{(2-1/k) \leq \boldsymbol{\xi} \leq 2\}} \in \mathrm{L}^2$, which converges (up to the canonical injection) in $\big(\mathrm{L}^\infty\big)^\star$ towards $\boldsymbol{\lambda}_2^\sharp = M\delta_{\{\xi = 2\}}$. Moreover $\sup_{M \geq a} \big\{2M - M^2/2\big\}$ is attained at $M^\sharp = \max\{2, a\}$ and has the following value

$$2\max\{a, 2\} - \frac{\max\{a, 2\}^2}{2} .$$

Now consider the set of multipliers such that $\mathbb{E}[\boldsymbol{\lambda}_2] \leq a$. Then, Problem (4.25) reads

$$\frac{\alpha^2}{2} + \sup_{\substack{\boldsymbol{\lambda}_2 \geq 0 \\ \mathbb{E}[\boldsymbol{\lambda}_2] \leq a}} \left\{ \frac{a^2}{2} - a\mathbb{E}[\boldsymbol{\lambda}_2] + \mathbb{E}\big[\boldsymbol{\lambda}_2 \boldsymbol{\xi}\big] \right\} .$$

Thus, we need to solve

$$\sup_{0 \leq M \leq a} \sup_{\substack{\boldsymbol{\lambda}_2 \geq 0 \\ \mathbb{E}[\boldsymbol{\lambda}_2] = M}} \frac{\alpha^2}{2} + \left\{ \frac{a^2}{2} - aM + \mathbb{E}\big[\boldsymbol{\lambda}_2 \boldsymbol{\xi}\big] \right\}$$

and, as previously, we concentrate the mass of $\boldsymbol{\lambda}_2$ over the highest values of $\boldsymbol{\xi}$, leading to

$$\frac{\alpha^2}{2} + \sup_{0 \leq M \leq a} \left\{ \frac{a^2}{2} + (2-a)M \right\} .$$

which is maximized for $M = a$ if $a < 2$, and maximized for $M = 0$ if $a \geq 2$. Note that in this case the optimal multiplier is no longer a singular measure.

Collecting results we consider separately the case where $a < 2$ and where $a \geq 2$.

- Assume that $a < 2$. Then, maximization over the set of multipliers such that $\mathbb{E}[\boldsymbol{\lambda}_2] \geq a$, yields a value of $\alpha^2/2+2$ whereas maximization over the set of multipliers such that $\mathbb{E}[\boldsymbol{\lambda}_2] < a$ yields a value of $\alpha^2/2+2a-a^2/2$ which is smaller. Consequently the optimal value of Problem (4.25) is
$$\alpha^2/2 + 2 ,$$
and a maximizing sequence is
$$\boldsymbol{\lambda}_2^{(k)} = 2k\mathbb{1}_{\{(2-1/k) \leq \boldsymbol{\xi} \leq 2\}} .$$
- Assume that $a \geq 2$. Then, maximization over the set of multipliers such that $\mathbb{E}[\boldsymbol{\lambda}_2] \geq a$, yields a value of $\alpha^2/2 + 2a - a^2/2$ whereas maximization over the set of multipliers such that $\mathbb{E}[\boldsymbol{\lambda}_2] < a$ yields a value of $\alpha^2/2 + a^2/2$ which is bigger. Consequently the optimal value of Problem (4.25) is
$$\alpha^2/2 + a^2/2 ,$$
and the supremum is attained in
$$\boldsymbol{\lambda}_2^\sharp = 0 .$$

This ends the proof. □

We have thus seen on this example that:

- The value function φ is l.s.c. at 0 (in the L^2 topology), and thus there is no duality gap. This is checked through explicit computation of the dual problem.
- If $a < 2$ the function φ is not subdifferentiable at 0, thus the constraints are not qualified. We can however construct an optimal solution in $\big(\mathrm{L}^\infty\big)^\star$.
- If $a \geq 2$, there exists an optimal multiplier in L^2, and thus the constraints are qualified.

Remark 4.19. *Note that if $a < 2$ then the constraints on the random variable $\boldsymbol{Y}$ imply constraints on the variable x. Indeed, according to Constraint (4.17c) we have $x \geq \boldsymbol{Y} + \boldsymbol{\xi}$, and by Constraint (4.17d) we obtain $x \geq \boldsymbol{\xi}$ ($\mathbb{P}$-a.s.), and as x is deterministic this is equivalent to $x \geq \operatorname{essupp}(\boldsymbol{\xi}) = 2$. This last constraint is stronger than Constraint (4.17b). This is an* induced constraints.

On the other hand, when $a \geq 2$, we are in the so-called relatively complete recourse *case as for every $x \geq a$, there is an admissible $\boldsymbol{Y}$. In other words there is no induced constraints. Hence, results by R.T.Rockafellar and R. Wets (see [93]) imply the existence of a L^1 multiplier in this case.*

4.2.2 Second Example: Sufficient Condition is not Necessary

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider the following minimization problem,

$$\inf_{\boldsymbol{U} \leq 1} \quad \frac{1}{2}\mathbb{E}\big[\boldsymbol{U}^2\big] \tag{4.26a}$$

$$s.t. \quad \boldsymbol{U} = 0 \quad \mathbb{P}\text{-a.s.} \tag{4.26b}$$

where the solutions are looked after in the space $\mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$.

The unique admissible solution is $\boldsymbol{U}^\sharp = 0$, and the optimal value is 0.

Fact 4.20. *In Problem (4.26) the $\mathbb{P}$-almost sure constraint $\boldsymbol{U} = 0$ is qualified (for the Banach L^2) but does not satisfy the sufficient constraint qualification (CQC) given in (4.15). However it satisfies (CQC) for the Banach L^∞.*

Proof. We embed Problem (4.26) in the following family of problems indexed by a L^2 perturbation $\boldsymbol{P}$,

$$\inf_{\boldsymbol{U} \leq 1} \quad \frac{1}{2}\mathbb{E}\big[\boldsymbol{U}^2\big]$$

$$s.t. \quad \boldsymbol{U} = \boldsymbol{P} \quad \mathbb{P}\text{-a.s.}$$

with value $\varphi(\boldsymbol{P})$. We easily obtain that

$$\varphi(\boldsymbol{P}) = \frac{\|\boldsymbol{P}\|_2^2}{2} + \chi_{\{\boldsymbol{P} \leq 1\}} \; .$$

As for all $\boldsymbol{P} \in \mathrm{L}^2$ we have $\varphi(\boldsymbol{P}) \geq \varphi(0)$, it comes by definition that φ is l.s.c.at 0 and that $0 \in \partial\varphi(0)$.

Moreover the dual problem is given by

$$\sup_{\boldsymbol{\lambda} \in \mathrm{L}^2} \quad \inf_{\boldsymbol{U} \leq 1} \quad \mathbb{E}\Big[\frac{\boldsymbol{U}^2}{2} + \boldsymbol{\lambda}\boldsymbol{U}\Big] \quad = \quad \sup_{\boldsymbol{\lambda} \in \mathrm{L}^2} -\mathbb{E}\Big[\frac{\boldsymbol{\lambda}^2}{2}\Big] \quad = \quad 0 \; .$$

Consequently there is no duality gap, and an optimal multiplier is $\boldsymbol{\lambda}^\sharp = 0$.

However, in the framework of §4.1.2, we have chosen

$$\mathcal{U}^{\mathrm{ad}} = \{\boldsymbol{U} \in \mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}) \quad | \quad \boldsymbol{U} \leq 1 \quad \mathbb{P}\text{-a.s.}\}$$

and $\Theta = \mathrm{Id}$, $C = \{0\}$. Thus $\Theta(\mathcal{U}^{\mathrm{ad}} \cap \mathrm{dom}(J)) + C = \mathcal{U}^{\mathrm{ad}}$ which is of empty interior (by Proposition 4.12). Consequently this example does not satisfy the sufficient qualification condition (CQC) (see (4.15)).

If we consider $\mathcal{U} = \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ we have

$$\mathcal{U}^{\mathrm{ad}} = \{\boldsymbol{U} \in \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}) \quad | \quad \boldsymbol{U} \leq 1 \quad \mathbb{P}\text{-a.s.}\}$$

and

$$\Theta(\mathcal{U}^{\mathrm{ad}} \cap \mathrm{dom}(J)) + C = \mathcal{U}^{\mathrm{ad}} \; .$$

Finally we have that, through Proposition 4.13,

$$0 \in \mathrm{int}\,\mathcal{U}^{\mathrm{ad}} \; ,$$

which ends the proof. □

4.3 Discussion

We have presented the classical abstract framework for duality in convex optimization, and applied it to almost sure constraints in stochastic optimization.

Working with L^p spaces, with $p < +\infty$, we have shown on simple, seemingly innocuous examples, that:

- the constraint might not be qualified,
- even when the constraint is qualified, the usual sufficient condition of qualification may fail.

We conclude with the observation that L^p spaces, with $p < +\infty$, might not be the proper setting to treat almost sure constraint by duality.

By contrast, the L^∞ topology might be better suited for almost sure constraint. Unfortunately, the topological dual of L^∞ is well-known to be a rich space, difficult to handle. In the next chapter, we provide conditions that lead to constraint qualification in stochastic optimization problems, using the $\left(\mathrm{L}^\infty, \mathrm{L}^1\right)$ duality.

Chapter 5

Constraint Qualification in $\left(\mathrm{L}^\infty, \mathrm{L}^1\right)$

Mathematics consists in proving the most obvious things in the least obvious way.

G. Pólya

Obvious is the most dangerous word in mathematics.

E. Bell

Contents

In Chapter 4, we recalled the abstract theory of duality, with a focus on constraint qualification. We underlined that constraint qualification of almost sure constraints in stochastic optimization raises specific issues. In particular, we have shown on generic examples that the topologies on L^p, for $p < +\infty$, may often fail to yield qualified almost sure constraints. We have also seen that in L^∞, paired with its dual, the sufficient condition of qualification applies. In this Chapter 5, we provide conditions under which almost sure constraints are qualified in the duality $\left(\mathrm{L}^\infty, \mathrm{L}^1\right)$.

In §5.1, we present several topologies on $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$, each topology inducing a different duality pairing, hence different results about constraint qualification. In §5.2, we provide our main result, mainly extending the work of R. Wets in [116]. Finally, in §5.3, we showcase an application to a multistage stochastic optimization problem.

5.1 Topologies on $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$

In this section, we recall results of functional analysis. More precisely we present some topologies on the set $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$ of essentially bounded functions [1].

Moreover, we have restrained ourselves to random variables taking values in $\mathbb{R}^d$. Indeed a control process of a T-stage problem, each stage having finite dimensional controls and state, is a stochastic process taking values in a finite dimensional space. If we were interested in problem with continuous controls we could consider random variables taking values into $[0, T]$ (e.g. $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{[0,T]}\big)$). Some results are available for more general image spaces.

Almost all the following results can be found, in a more general setting, in functional analysis books. For an easy but insightful introduction (for the Lebesgue measure) see [25], for general results in infinite dimension see [3, 22]. A brief selection of general results is given in §A.1.

5.1.1 The space $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$

Let $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ be a probability space. The set of essentially bounded $\mathcal{F}$-measurable functions $\mathcal{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$ can be equipped with an equivalence relation $\sim$ stating that $\boldsymbol{X} \sim \boldsymbol{Y}$ iff $\boldsymbol{X} = \boldsymbol{Y}$ $\mathbb{P} - a.s.$. The set of equivalence classes is denoted by

$$\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big) = \mathcal{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big) / \sim .$$

For notational simplicity, we will sometimes write L^∞ instead of $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$.

The set $\mathrm{L}^0\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$ is the space of (equivalence class of) $\mathcal{F}$-measurable functions taking values in $\mathbb{R}^d$. The set $\mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$ is the subspace of $\mathrm{L}^0\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$ of integrable functions, i.e. such that

$$\forall \boldsymbol{X} \in \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big), \qquad \mathbb{E}\big[\|\boldsymbol{X}\|_{\mathbb{R}^d}\big] := \int_\Omega \|\boldsymbol{X}(\omega)\|_{\mathbb{R}^d} d\mathbb{P}(\omega) < +\infty .$$

The usual norm of L^1 is, for every $\boldsymbol{X} \in \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$, $\big\|\boldsymbol{X}\big\|_{\mathrm{L}^1} = \mathbb{E}\big[\,\big\|\boldsymbol{X}\big\|_{\mathbb{R}^d}\,\big]$.

Definition 5.1. *A σ-algebra $\mathcal{F}$ is said to be not finite modulo $\mathbb{P}$ if one of following equivalent assertions holds true:*

i)

$$\inf\Big\{\mathbb{P}\big(A\big) \quad \big| \quad A \in \mathcal{F}, \quad \mathbb{P}\big(A\big) > 0\Big\} = 0 , \tag{5.1}$$

ii) The number of $\mathcal{F}$-measurable events $A \in \Omega$ of positive probability is not finite,

iii) there exists a $\mathcal{F}$-measurable, real valued, random variable $\boldsymbol{X}$ such that,

$$\forall n \in \mathbb{N}, \qquad \mathbb{P}\big(\boldsymbol{X} = n\big) > 0 .$$

Proof. Consider the following equivalence relation on the σ-algebra $\mathcal{F}$

$$A \sim B \quad \Longleftrightarrow \quad \mathbb{P}\big(A \Delta B\big) = 0 ,$$

where

$$A \Delta B := \big(A \setminus B\big) \cup \big(B \setminus A\big) .$$

We work in the class of equivalence $\mathcal{F}/\sim$.

1. Those topologies can be defined on any Banach space (or even for some cases on topological spaces) and most results recalled here remain true.

i) $\Rightarrow$ ii) Let $\{A_n\}_{n\in\mathbb{N}}$ be a minimizing sequence of (5.1), such that

$$\forall n \in \mathbb{N}, \qquad 0 < \mathbb{P}(A_n) \leq \frac{1}{2^n} \; .$$

Define, for all $n \in \mathbb{N}$, the sets

$$B_n := \cup_{k\geq n} A_k \; ,$$

and

$$C_n := A_n \backslash B_{n+1} \; .$$

Thus $\{B_n\}_{n\in\mathbb{N}}$ and $\{C_n\}_{n\in\mathbb{N}}$ are also minimizing sequences of (5.1), indeed $\mathbb{P}(B_n) \leq 1/2^{n-1}$, and $\mathbb{P}(C_n) \leq 1/2^n$.
Moreover there is a subsequence $\{C_{n_k}\}_{k\in\mathbb{N}}$ such that each term is of positive probability. Otherwise, we would have $N \in \mathbb{N}$ such that

$$\forall n \geq N, \qquad \mathbb{P}(C_n) = 0 \; ,$$

hence,

$$\forall n > N, \qquad A_N \subset B_n \; .$$

As $\mathbb{P}(A_N) > 0$ it contradicts the fact that $\{B_n\}_{n\in\mathbb{N}}$ is a minimizing subsequence.
Thus $\{C_{n_k}\}_{k\in\mathbb{N}}$ is a non-finite sequence of $\mathcal{F}$-measurable, disjoint, events of positive probability.

i) $\Rightarrow$ iii) We choose $\boldsymbol{X} = \sum_{k=1}^{\infty} k \mathbb{1}_{C_{n_k}}$.

ii) $\Rightarrow$ i) If the infimum in (5.1) is $\varepsilon > 0$ the number of disjoint event of positive probability is finite (at most $1/\varepsilon$). Thus the number of events in $\mathcal{F}/\sim$ of positive probability is finite.

iii) $\Rightarrow$ ii) Each events $\Big(\{\boldsymbol{X} = n\}\Big)_{n\in\mathbb{N}}$ is of positive probability.

□

Remark 5.2. *If the σ-algebra $\mathcal{F}$ is finite modulo $\mathbb{P}$ then $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ is a finite dimensional space. As it is an Hilbert space, the weak and weak* topologies (and hence the Mackey topology), presented hereafter, are equivalent.*

For any $\boldsymbol{X} \in \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ we denote by $\|\boldsymbol{X}\|_\infty$ the essential supremum of $\boldsymbol{X}$, i.e.

$$\|\boldsymbol{X}\|_\infty := \inf\Big\{ M \in \mathbb{R} \cup \{+\infty\} \quad \Big| \quad \mathbb{P}\big(\|\boldsymbol{X}\|_{\mathbb{R}^d} \geq M\big) = 0 \Big\} \; . \tag{5.2}$$

The topology $\tau_{\|\|}$ induced by $\|\cdot\|_{\mathrm{L}^\infty}$ is called the *norm topology*. The convergence of a sequence $\{\boldsymbol{X}_n\}_{n\in\mathbb{N}}$ of random variables toward $\boldsymbol{X}$ in the norm topology is simply denoted by $\boldsymbol{X}_n \to \boldsymbol{X}$.

Moreover, we define by $(\mathrm{L}^\infty)^\star$ the set of continuous (for the norm topology) linear forms on L^∞. The natural norm on $(\mathrm{L}^\infty)^\star$ is defined by, for any $v \in (\mathrm{L}^\infty)^\star$,

$$\|v\|_{(\mathrm{L}^\infty)^\star} = \sup\big\{ |v(\boldsymbol{X})| \quad \big| \quad \boldsymbol{X} \in \mathrm{L}^\infty, \quad \|\boldsymbol{X}\|_\infty \leq 1 \big\} \; .$$

We have the following results.

Fact 5.3. *We gather here some useful results on L^∞ and its topological dual $\Big(\mathrm{L}^\infty\Big)^\star$. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*

- *If $\mathcal{F}$ is not finite modulo $\mathbb{P}$, $(\mathrm{L}^\infty, \tau_{\|\|})$ is a non reflexive Banach space.*
- *The Banach space $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ is separable iff the σ-algebra $\mathcal{F}$ is finite modulo $\mathbb{P}$.*

- *The Banach space* L^∞ *is dense (for the norm topology) in* L^p*, for any* $p \in [1, \infty]$ *(recall that a probability measure is finite and see [3, Theorem 13.8] or §B.2).*
- *The topological dual of the Banach space* L^∞*, denoted by* $\big(\mathrm{L}^\infty\big)^\star$*, is isometrically isomorphic to the set* $ba\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ *of all finitely additive, finite, signed measures defined on* Ω*, which are absolutely continuous with respect to* $\mathbb{P}$*, equipped with the total variation norm (see [42, Theorem IV.8.16]).*
- *There is a canonical injection* i *of* L^1 *into* $\big(\mathrm{L}^\infty\big)^\star$*, where for all* $\boldsymbol{Y} \in \mathrm{L}^1$*,*

$$i(\boldsymbol{Y}) : \boldsymbol{X} \mapsto \mathbb{E}\big[\boldsymbol{Y} \cdot \boldsymbol{X}\big] \; .$$

We give a short proof of the second point, when $d = 1$,

Proof. If $\mathcal{F}$ is not finite, there exists a random variable $\boldsymbol{X}$ such that for every $n \in \mathbb{N}$, $\mathbb{P}(\boldsymbol{X} = n) > 0$. For any sequence $u \in \big\{0, 1\big\}^{\mathbb{N}}$, where $u = (u_n)_{n \in \mathbb{N}}$, we define the bounded random variable

$$\boldsymbol{X}_u = \sum_{n \in \mathbb{N}} u_n \mathbb{1}_{\big\{\boldsymbol{X} = n\big\}} \; .$$

Note that for every sequence u and v in $\big\{0, 1\big\}^{\mathbb{N}}$ such that $u \neq v$, we have

$$\big\|\boldsymbol{X}_u - \boldsymbol{X}_v\big\|_\infty = \sup_{n \in \mathbb{N}} |u_n - v_n| = 1 \; .$$

Thus we have an enumerable number of points in L^∞ equally distant to each other, thus no countable sequence in L^∞ can be dense, and thus L^∞ is not separable.

On the other hand if $\mathcal{F}$ is finite it is generated by a finite partition (say of cardinal N), and $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$ is isomorphic to $\mathbb{R}^{N \cdot n}$. □

5.1.2 Weak and Weak* Topology of L^∞

Weak Topology $\sigma\big(\mathrm{L}^\infty, \big(\mathrm{L}^\infty\big)^\star\big)$

Recall that $\big(\mathrm{L}^\infty\big)^\star$ is the set of continuous (for the norm topology) linear forms on L^∞. The *weak topology* $\sigma\big(\mathrm{L}^\infty, \big(\mathrm{L}^\infty\big)^\star\big)$ is, by definition, the coarsest topology such that every element in $\big(\mathrm{L}^\infty\big)^\star$ is still continuous.

Fact 5.4. *The weak topology is separated. We denote by* $\boldsymbol{X}_n \rightharpoonup \boldsymbol{X}$ *the fact that a sequence* $\big(\boldsymbol{X}_n\big)_{n \in \mathbb{N}}$ *weakly converges toward* $\boldsymbol{X}$*. We have*

$$\boldsymbol{X}_n \rightharpoonup \boldsymbol{X} \quad \iff \quad \forall\, \boldsymbol{Y} \in \big(\mathrm{L}^\infty\big)^\star, \qquad \langle \boldsymbol{Y}, \boldsymbol{X}_n \rangle \to \langle \boldsymbol{Y}, \boldsymbol{X} \rangle \; . \tag{5.3}$$

Fact 5.5. *The weak topology is coarser than the strong topology:*

$$\sigma\big(\mathrm{L}^\infty, \big(\mathrm{L}^\infty\big)^\star\big) \subset \tau_{\|\|} \; .$$

We have the following additional properties linking the weak and strong topologies.

- *If* $\boldsymbol{X}_n \to \boldsymbol{X}$*, then* $\boldsymbol{X}_n \rightharpoonup \boldsymbol{X}$*.*
- *If* $\boldsymbol{X}_n \rightharpoonup \boldsymbol{X}$*, then* $\big\|\boldsymbol{X}_n\big\|_\infty$ *is bounded and* $\big\|\boldsymbol{X}\big\|_\infty \leq \underline{\lim}_n \big\|\boldsymbol{X}_n\big\|_\infty$*.*
- *If* $\boldsymbol{X}_n \rightharpoonup \boldsymbol{X}$ *and* $\boldsymbol{Y}_n \xrightarrow{(\mathrm{L}^\infty)^\star} \boldsymbol{Y}$*, then* $\langle \boldsymbol{Y}_n, \boldsymbol{X}_n \rangle \to \langle \boldsymbol{Y}, \boldsymbol{X} \rangle$*.*
- *A convex set is closed in the norm topology iff it is closed in the weak-topology.*
- *Consequently a convex function is l.s.c for the norm topology iff it is l.s.c for the weak topology.*

Note that this fact is not restrained to L^∞

Fact 5.6 (Eberlein–Smulian)**.** *A set is weakly compact iff it is weakly sequentially compact (see [3, 6.35]).*

Weak* Topology $\sigma(\mathrm{L}^\infty, \mathrm{L}^1)$

We go on by coarsening the topology [2] $\sigma(\mathrm{L}^\infty, (\mathrm{L}^\infty)^\star)$.

We define by $\sigma(\mathrm{L}^\infty, \mathrm{L}^1)$ the coarsest topology on L^∞ such that every L^1-*linear form* on $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ is continuous, i.e. such that for every $\boldsymbol{Y} \in \mathrm{L}^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, the linear form $\boldsymbol{X} \mapsto \mathbb{E}[\boldsymbol{X} \cdot \boldsymbol{Y}]$ is continuous. As $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ is the topological dual (for the norm topology of L^1) of $\mathrm{L}^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, $\sigma(\mathrm{L}^\infty, \mathrm{L}^1)$ is the so-called *weak** topology of L^∞.

We denote by $\boldsymbol{X}_n \overset{*}{\rightharpoonup} \boldsymbol{X}$ the fact that sequence $(\boldsymbol{X}_n)_{n\in\mathbb{N}}$ weakly* converges toward $\boldsymbol{X}$, and we have

$$\boldsymbol{X}_n \overset{*}{\rightharpoonup} \boldsymbol{X} \quad \iff \quad \forall\, \boldsymbol{Y} \in \mathrm{L}^1, \qquad \mathbb{E}[\boldsymbol{Y} \cdot \boldsymbol{X}_n] \to \mathbb{E}[\boldsymbol{Y} \cdot \boldsymbol{X}] \;. \tag{5.4}$$

The main interest of the weak* topology is given by the Banach-Alaoglu-Bourbaki Theorem (see [3, Theorem 5.105]), recalled in the following fact.

Fact 5.7. *The norm-closed unit ball is weak* compact, hence any bounded and weak*-closed set is weak*-compact.*

Notice that, if $\mathcal{F}$ is not finite modulo $\mathbb{P}$, the unit ball is not compact in the weak topology. Indeed, if it were the case Kakutani's theorem would imply that L^∞ is reflexive.

Fact 5.8. *We have the following inclusion of topologies:*

$$\sigma(\mathrm{L}^\infty, \mathrm{L}^1) \subset \sigma(\mathrm{L}^\infty, (\mathrm{L}^\infty)^\star) \subset \tau_{\|\|} \;.$$

We have the following additional properties on the weak* topology, where $\{\boldsymbol{X}_n\}_{n\in\mathbb{N}}$ is a sequence of L^∞, and $\{\boldsymbol{Y}_n\}_{n\in\mathbb{N}}$ is a sequence of L^1.

- *The weak* topology is separated.*
- *If $\boldsymbol{X}_n \rightharpoonup \boldsymbol{X}$, then $\boldsymbol{X}_n \overset{*}{\rightharpoonup} \boldsymbol{X}$.*
- *If $\boldsymbol{X}_n \overset{*}{\rightharpoonup} \boldsymbol{X}$, then $\|\boldsymbol{X}_n\|_\infty$ is bounded and $\|\boldsymbol{X}\|_\infty \leq \underline{\lim}\, \|\boldsymbol{X}_n\|_\infty$.*
- *If $\boldsymbol{X}_n \overset{*}{\rightharpoonup} \boldsymbol{X}$ and $i(\boldsymbol{Y}_n) \xrightarrow{(\mathrm{L}^\infty)^\star} i(\boldsymbol{Y})$, then $\langle \boldsymbol{Y}_n, \boldsymbol{X}_n \rangle \to \langle \boldsymbol{Y}, \boldsymbol{X} \rangle$.*

5.1.3 Mackey topology $\tau(\mathrm{L}^\infty, \mathrm{L}^1)$

The weak* topology $\sigma(\mathrm{L}^\infty, \mathrm{L}^1)$ is defined as the coarsest topology such that the L^1 linear forms are continuous. The Mackey topology $\tau(\mathrm{L}^\infty, \mathrm{L}^1)$ is defined as the finest topology such that the only continuous linear forms are the L^1 linear form.

Thus the Mackey topology is finer than the weak* topology. Hence, it is easier for a functional $J : \mathrm{L}^\infty \to \overline{\mathbb{R}}$ to be Mackey continuous, than to be weak* continuous.

Fact 5.9. *We have the following inclusion of topologies:*

$$\sigma(\mathrm{L}^\infty, \mathrm{L}^1) \subset \tau(\mathrm{L}^\infty, \mathrm{L}^1) \subset \sigma(\mathrm{L}^\infty, (\mathrm{L}^\infty)^\star) \subset \tau_{\|\|} \;.$$

We have the following additional properties.

- $\boldsymbol{X}_n \rightharpoonup \boldsymbol{X} \quad \Longrightarrow \quad \boldsymbol{X}_n \xrightarrow{\tau(\mathrm{L}^\infty, \mathrm{L}^1)} \boldsymbol{X} \quad \Longrightarrow \quad \boldsymbol{X}_n \overset{*}{\rightharpoonup} \boldsymbol{X}.$
- *The Mackey topology $\tau(\mathrm{L}^\infty, \mathrm{L}^1)$ is separated.*
- *A convex set is closed in $\tau(\mathrm{L}^\infty, \mathrm{L}^1)$ iff it is closed in $\sigma(\mathrm{L}^\infty, \mathrm{L}^1)$.*
- $\boldsymbol{X}_n \xrightarrow{\tau(\mathrm{L}^\infty, \mathrm{L}^1)} \boldsymbol{X} \quad \Longrightarrow \quad \forall \boldsymbol{Y} \in \mathrm{L}^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}), \qquad \mathbb{E}\big[\boldsymbol{Y}\, \|\boldsymbol{X}_n - \boldsymbol{X}\|_{\mathbb{R}^d}\big] \to 0 \;.$

2. As noted by H.Brezis, one could be surprised that we thrive to obtain coarser topologies. The reason is that a coarser topology implies more compact sets, which are useful for existence results.

There is a practical characterization of the convergence of a sequence in the Mackey topology by M.Nowak [71, Theorem 2.3]

Proposition 5.10. *The sequence $\{\boldsymbol{X}_n\}_{n\in\mathbb{N}}$ $\tau(\mathrm{L}^\infty, \mathrm{L}^1)$-converges toward $\boldsymbol{X}$ iff*

$$\begin{cases} \exists p \in [1, +\infty), & \boldsymbol{X}_n \xrightarrow{\mathrm{L}^p} \boldsymbol{X} \\ \sup_n \left\|\boldsymbol{X}_n\right\|_\infty < +\infty \end{cases}$$

Remark 5.11. *We have presented different topologies on L^∞ as they induce different pairings. For example we can consider:*

- *the natural pairing $\left(\mathrm{L}^\infty, (\mathrm{L}^\infty)^\star\right)$ of a Banach space with its topological dual, where L^∞ is either endowed with the strong or weak topology;*
- *the pairing $\left(\mathrm{L}^\infty, \mathrm{L}^1\right)$ that coincide with the previous one (up to canonical injection), where L^∞ is either endowed with the weak* or Mackey topology.*

In the following section we present a duality result using the pairing $\left(\mathrm{L}^\infty, \mathrm{L}^1\right)$, where L^∞ is endowed with the Mackey topology $\tau(\mathrm{L}^\infty, \mathrm{L}^1)$.

5.2 A Duality Result Through Mackey-Continuity

In [116] R.Wets exhibited conditions such that the non-anticipativity constraints are qualified in the pairing $(\mathrm{L}^\infty, \mathrm{L}^1)$. Here we extend the results to more general affine constraint.

In §5.2.1 we present the optimization problem. In §5.2.2 we gives some results of weak* closedness of an affine subspace of L^∞. Those results are used in §5.2.3 which follow closely the proof given in [116]. In §5.2.4 we discuss one important continuity assumption made in the proof.

5.2.1 Problem Statement

Let $\mathcal{U}$ be $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, and $\mathcal{U}^{\mathrm{ad}}$ be an affine subspace of $\mathcal{U}$.

We consider a cost function $j : \mathbb{R}^d \times \Omega \to \mathbb{R} \cup \{+\infty\}$, assumed to be a convex normal integrand (see [96] for definitions and properties), with the following assumption, known as *strict feasibility condition*,

$$\exists\, \varepsilon > 0, \quad \exists\, \boldsymbol{U}_0 \in \mathcal{U}^{\mathrm{ad}}, \quad \forall u \in \mathbb{R}^d, \quad \|u\|_{\mathbb{R}^d} \leq \varepsilon \quad \Longrightarrow \quad j(\boldsymbol{U}_0 + u, \cdot) < +\infty \quad \mathbb{P}\text{-a.s.} \tag{5.5}$$

This strict feasibility condition is essential for the results. We define the objective function $J : \mathcal{U} \to \mathbb{R}$ by

$$J : \boldsymbol{U} \mapsto \mathbb{E}\big[j(\boldsymbol{U})\big] := \int_\Omega j\big(\boldsymbol{U}(\omega), \omega\big) d\mathbb{P}(\omega) \; . \tag{5.6}$$

Finally, we consider the problem

$$\min_{\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}} \subset \mathcal{U}} \mathbb{E}\big[j(\boldsymbol{U})\big] \; . \tag{5.7}$$

We consider the pairing $\langle \boldsymbol{Y}\,, \boldsymbol{X} \rangle$, where $\boldsymbol{Y} \in \mathrm{L}^1$, $\boldsymbol{X} \in \mathrm{L}^\infty$ given by

$$\langle \boldsymbol{Y}\,, \boldsymbol{X} \rangle := \mathbb{E}\big[\boldsymbol{Y} \cdot \boldsymbol{X}\big] \; .$$

5.2.2 Weak* Closedness of Affine Subspaces of $\mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d\big)$

We show conditions for affine subspaces of $\mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d\big)$ to be weak* closed, and give some examples. This closedness assumption is required to prove the duality result in §5.2.3. Simultaneously we obtain weak* continuity [3] results for linear operators useful in Chapter 6.

Proposition 5.12. *Consider a linear operator*

$$\Theta : \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n\big) \to \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m\big) \ ,$$

and a vector $b \in \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m\big)$. *Assume that there exist a linear operator*

$$\Theta^\dagger : \mathrm{L}^1\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m\big) \to \mathrm{L}^1\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n\big) \ ,$$

such that:

$$\forall \boldsymbol{X} \in \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n\big), \quad \forall \boldsymbol{Y} \in \mathrm{L}^1\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m\big), \qquad \Big\langle \boldsymbol{Y}\ ,\Theta\big(\boldsymbol{X}\big)\Big\rangle = \Big\langle \Theta^\dagger\big(\boldsymbol{Y}\big)\ ,\boldsymbol{X}\Big\rangle \ . \tag{5.8}$$

Then the linear operator Θ *is weak** *continuous and the affine set*

$$\mathcal{U}^{\mathrm{ad}} = \Big\{ \boldsymbol{X} \in \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n\big) \quad \big| \quad \Theta\big(\boldsymbol{X}\big) = \boldsymbol{B} \Big\} \ , \tag{5.9}$$

is weak* *closed.*

Proof. Consider a net $\big(\boldsymbol{X}_i\big)_{i\in\mathcal{I}}$ in $\mathcal{U}^{\mathrm{ad}} \subset \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n\big)$ converging weakly* towards $\boldsymbol{X}$, and a random variable $\boldsymbol{Y} \in \mathrm{L}^1\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m\big)$.

We have, for any $i \in \mathcal{I}$, by definition of $\Theta^\dagger$,

$$\mathbb{E}\big[\langle \boldsymbol{Y}, \Theta\big(\boldsymbol{X}_i\big)\rangle\big] = \mathbb{E}\big[\langle \Theta^\dagger\big(\boldsymbol{Y}\big), \boldsymbol{X}_i\rangle\big] \ .$$

As $\Theta^\dagger\big(\boldsymbol{Y}\big) \in \mathrm{L}^1\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n\big)$, the linear form

$$\boldsymbol{X} \mapsto \mathbb{E}\big[\Theta^\dagger\big(\boldsymbol{Y}\big)\cdot\boldsymbol{X}\big] \ ,$$

is weak* continuous (by definition of the weak* topology). Hence,

$$\lim_{\boldsymbol{X}_i\to\boldsymbol{X}} \mathbb{E}\big[\Theta^\dagger\big(\boldsymbol{Y}\big)\cdot\boldsymbol{X}_i\big] = \mathbb{E}\big[\Theta^\dagger\big(\boldsymbol{Y}\big)\cdot\boldsymbol{X}\big] = \mathbb{E}\big[\boldsymbol{Y}\cdot\Theta\big(\boldsymbol{X}\big)\big] \ .$$

In other words the net $\Big\{\Theta\big(\boldsymbol{X}_i\big)\Big\}_{i\in\mathcal{I}}$ converges weakly* toward $\Theta\big(\boldsymbol{X}\big)$. Hence, the function

$$\Theta : \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n\big) \to \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m\big)$$

is continuous if both spaces are endowed with their weak* topology.

As $\{\boldsymbol{B}\}$ is a weak*-closed set, we have that $\mathcal{U}^{\mathrm{ad}} = \Theta^{-1}\big(\{\boldsymbol{B}\}\big)$ is weak*-closed.

□

Corollary 5.13. *Consider a matrix* $A \in \mathcal{M}_{m,n}(\mathbb{R})$, *and a random variable* $\boldsymbol{B} \in \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m\big)$. *Then the linear operator* $\Theta : \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n\big) \to \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m\big)$, *defined by*

$$\forall \boldsymbol{X} \in \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n\big), \qquad \Theta\big(\boldsymbol{X}\big) = A\boldsymbol{X} \ , \tag{5.10}$$

is weak* *continuous, hence the affine space*

$$\mathcal{U}^{a.s.} := \big\{ \boldsymbol{U} \in \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n\big) \quad \big| \quad A\boldsymbol{U} = \boldsymbol{B} \quad \mathbb{P}-a.s. \big\} \ , \tag{5.11}$$

is weakly* *closed.*

3. By weak* continuity we means the continuity of the function from L^∞ to L^∞ both endowed with the weak* topology.

Proof. The operator $\Theta^\dagger : \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m\big) \to \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$, defined by

$$\forall \boldsymbol{Y} \in \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m\big), \qquad \Theta^\dagger\big(\boldsymbol{Y}\big) = A^T \boldsymbol{Y} \ ,$$

is linear and such that

$$\forall \boldsymbol{X} \in \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big), \quad \forall \boldsymbol{Y} \in \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m\big), \qquad \big\langle \boldsymbol{Y} \,, \Theta\big(\boldsymbol{X}\big) \big\rangle = \big\langle \Theta^\dagger\big(\boldsymbol{Y}\big) \,, \boldsymbol{X} \big\rangle \ .$$

Thus, Θ is weak* continuous by Proposition 5.12, and $\mathcal{U}^{\text{a.s.}}$ in (5.11) is weak* closed. $\square$

Corollary 5.14. *Consider a filtration $\mathfrak{F} = \{\mathcal{F}_0\}_1^{T-1}$ on $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$. Then, for $t \in [\![0, T-1]\!]$, the linear operator $\Theta_t : \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big) \to \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$, defined by*

$$\forall \boldsymbol{X} \in \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big), \qquad \Theta_t\big(\boldsymbol{X}\big) = \mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}_t\big] - \boldsymbol{X} \ ,$$

is weak* continuous.

Hence, the linear space

$$\mathcal{N} := \Big\{ \boldsymbol{U} \in \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{dT}\big) \quad \Big| \quad \forall t \in [\![0, T-1]\!], \quad \mathbb{E}\big[\boldsymbol{U}_t \bigm| \mathcal{F}_t\big] = \boldsymbol{U}_t \Big\} \ , \tag{5.12}$$

is weakly* closed.

Proof. We construct the right operator to apply Proposition 5.12.

For $t \in [\![1, n]\!]$, linear operator $\Theta_t^\dagger : \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big) \to \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$, defined by

$$\forall \boldsymbol{Y} \in \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big), \qquad \Theta_t^\dagger\big(\boldsymbol{Y}\big) = \mathbb{E}\big[\boldsymbol{Y} \bigm| \mathcal{F}_t\big] - \boldsymbol{Y} \ ,$$

coincide with Θ on $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$ and, is such that

$$\forall \boldsymbol{X} \in \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big), \quad \forall \boldsymbol{Y} \in \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big), \qquad \big\langle \boldsymbol{Y} \,, \Theta_t\big(\boldsymbol{X}\big) \big\rangle = \big\langle \Theta_t^\dagger\big(\boldsymbol{Y}\big) \,, \boldsymbol{X} \big\rangle \ .$$

Indeed,

$$\begin{aligned}
\mathbb{E}\Big[\boldsymbol{Y} \cdot \Theta_t\big(\boldsymbol{X}\big)\Big] &= \mathbb{E}\Big[\boldsymbol{Y} \cdot \mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}_t\big]\Big] - \mathbb{E}\big[\boldsymbol{Y} \cdot \boldsymbol{X}\big] \\
&= \mathbb{E}\Big[\mathbb{E}\big[\boldsymbol{Y} \bigm| \mathcal{F}_t\big] \cdot \boldsymbol{X}\Big] - \mathbb{E}\big[\boldsymbol{Y} \cdot \boldsymbol{X}\big] && \text{by Lemma B.3} \\
&= \mathbb{E}\Big[\Theta_t^\dagger\big(\boldsymbol{Y}\big) \cdot \boldsymbol{X}\Big] \ .
\end{aligned}$$

Hence, Proposition 5.12 gives the weak* continuity of Θ_t.

We have

$$\mathcal{N} = \bigcap_{t=1}^{T-1} \Theta_t^{-1}\Big(\{0\}\Big) \ ,$$

thus, $\mathcal{N}$ is weak* closed. $\square$

5.2.3 A duality theorem

In this section we show the following first order optimality conditions.

Theorem 5.15. *Assume that j is a convex normal integrand, that $\mathcal{U}^{\text{ad}}$ is a weak* closed affine subspace of $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$ and that J given by (5.6) is continuous in the Mackey topology $\tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$ at some point $\boldsymbol{U}_0 \in \mathcal{U}^{\text{ad}} \cap \operatorname{dom}(J)$. Then the control $\boldsymbol{U}^\sharp \in \mathcal{U}^{\text{ad}}$ is an optimal solution to*

$$\inf_{\boldsymbol{U} \in \mathcal{U}^{\text{ad}}} \mathbb{E}\big[j\big(\boldsymbol{U}\big)\big]$$

if and only if there exist $\boldsymbol{\lambda}^\sharp \in \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$ such that

- $\boldsymbol{U}^\sharp$ *minimizes on* $\mathcal{U}$ *the following Lagrangian*[4]

$$L(\boldsymbol{U}, \boldsymbol{\lambda}^\sharp) = \mathbb{E}\Big[j(\boldsymbol{U}) + \boldsymbol{U} \cdot \boldsymbol{\lambda}^\sharp\Big] ,$$

- *and* $\boldsymbol{\lambda}^\sharp \in \Big(\mathcal{U}^{\mathrm{ad}}\Big)^\perp$.

In order to show this Theorem, we need a few preliminary results. We follow the work of R.Wets in [116] for non-anticipativity constraints.

Consider the problem

$$\inf_{\boldsymbol{U} \in \mathcal{U}} J(\boldsymbol{U}) + \chi_{\mathcal{U}^{\mathrm{ad}}}(\boldsymbol{U}) . \tag{5.13}$$

In Lemma 5.16 we show that J is weak*-l.s.c, and hence $\tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$-l.s.c. However Theorem 5.17 requires a stronger assumption: the continuity of J at a point $\boldsymbol{U}_0$. This assumption is discussed in §5.2.4.

Lemma 5.16. *If j is a normal convex integrand satisfying (5.5), then the Fenchel conjugate (defined in Definition A.37) of J in the pairing $\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$, is given by*

$$J^\star(\boldsymbol{\lambda}) = \mathbb{E}\big[j^\star(\boldsymbol{\lambda})\big] = \int_\Omega j^\star(\boldsymbol{\lambda}(\omega), \omega) d\mathbb{P}(\omega) ,$$

and

$$j^\star(\lambda, \omega) = \sup_{u \in \mathbb{R}^d} \big\{ u \cdot \lambda - j(u, \omega) \big\} .$$

Moreover, we have

$$J^{\star\star} = J .$$

Thus, J is weak*-l.s.c.

Proof. It is a direct application of [88, Theorem 3]. □

Theorem 5.17. *Assume that $\mathcal{U}^{\mathrm{ad}}$ is an affine space. Assume that j is a convex normal integrand, and that J given by (5.6) is continuous in the Mackey topology $\tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$ at some point $\boldsymbol{U}_0 \in \mathcal{U}^{\mathrm{ad}} \cap \mathrm{dom}(J)$. Then, we have*

$$\inf_{\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}}} J(\boldsymbol{U}) = \max_{\boldsymbol{\lambda} \in \big(\mathcal{U}^{\mathrm{ad}}\big)^\perp} -J^\star(\boldsymbol{\lambda}) .$$

Proof. Notice that as the set $\mathcal{U}^{\mathrm{ad}}$ is weak* closed convex, the function $\chi_{\mathcal{U}^{\mathrm{ad}}}$ is also convex and weak* l.s.c., and hence $\tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$-l.s.c.

By using an extension of Fenchel's duality theorem as given in [87, Theorem 1] we have

$$\inf_{\boldsymbol{U} \in \mathcal{U}} \Big\{ J(\boldsymbol{U}) + \chi_{\mathcal{U}^{\mathrm{ad}}}(\boldsymbol{U}) \Big\} = \max_{\boldsymbol{\lambda} \in \mathrm{L}^1} \Big\{ -J^\star(\boldsymbol{\lambda}) - \chi^\star_{\mathcal{U}^{\mathrm{ad}}}(\boldsymbol{\lambda}) \Big\} . \tag{5.14}$$

Indeed both functions are convex, and J is continuous in the Mackey topology $\tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$ at $\boldsymbol{U}_0 \in \mathcal{U}^{\mathrm{ad}}$, where $\chi_{\mathcal{U}^{\mathrm{ad}}}$ is finite.

Moreover,

$$\chi^\star_{\mathcal{U}^{\mathrm{ad}}}(\boldsymbol{\lambda}) = \max_{\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}}} \langle \boldsymbol{\lambda}, \boldsymbol{U} \rangle = \chi_{\big(\mathcal{U}^{\mathrm{ad}}\big)^\perp}(\boldsymbol{\lambda}) . \tag{5.15}$$

We conclude by combining (5.14) and (5.15). □

A by product of this proof is given, as $\mathcal{U}^{\mathrm{ad}}$ is an affine space, in Equation (5.15).

4. Recall that $\mathcal{U}^{\mathrm{ad}}$ is an affine space, hence we do not need to specify the point at which the dual cone, given in Definition A.40, is evaluated.

Corollary 5.18. *Suppose that the assumptions of Theorem 5.17 hold true. Then, a control $\boldsymbol{U}^\sharp$ minimizes J if and only if there exists $\boldsymbol{\lambda}^\sharp \in \big(\mathcal{U}^{\mathrm{ad}}\big)^\perp$ such that*

$$-\boldsymbol{\lambda}^\sharp \in \partial J\big(\boldsymbol{U}^\sharp\big) .$$

Moreover, those $\boldsymbol{\lambda}^\sharp$ are the points where $J^\star$ achieves its minimum over $\big(\mathcal{U}^{\mathrm{ad}}\big)^\perp$.

Proof. Throughout the proof we consider the Mackey topology $\tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$ on $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$. Thus, the topological dual of L^∞ is L^1, and any subgradients are elements of L^1.

Consider a control $\boldsymbol{U}^\sharp \in \mathcal{U}^{\mathrm{ad}}$. Note that $\boldsymbol{U}^\sharp$ minimizes J on $\mathcal{U}^{\mathrm{ad}}$ iff $0 \in \partial\big(J + \chi_{\mathcal{U}^{\mathrm{ad}}}\big)\big(\boldsymbol{U}^\sharp\big)$. By [88, Theorem 3], this is equivalent to $0 \in \partial\big(J\big)\big(\boldsymbol{U}^\sharp\big) + \partial\big(\chi_{\mathcal{U}^{\mathrm{ad}}}\big)\big(\boldsymbol{U}^\sharp\big)$, and thus to the existence of $\boldsymbol{\lambda}^\sharp \in \mathrm{L}^1$ such that $\boldsymbol{\lambda}^\sharp \in \partial\big(\chi_{\mathcal{U}^{\mathrm{ad}}}\big)\big(\boldsymbol{U}^\sharp\big)$ and $-\boldsymbol{\lambda}^\sharp \in \partial\big(J\big)\big(\boldsymbol{U}^\sharp\big)$. Finally, we have

$$\partial\big(\chi_{\mathcal{U}^{\mathrm{ad}}}\big)\big(\boldsymbol{U}^\sharp\big) = \big(\mathcal{U}^{\mathrm{ad}}\big)^\perp .$$

Indeed $\boldsymbol{\lambda} \in \partial\big(\chi_{\mathcal{U}^{\mathrm{ad}}}\big)\big(\boldsymbol{U}^\sharp\big)$ iff

$$\forall \boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}}, \qquad \big\langle \boldsymbol{\lambda}, \boldsymbol{U} - \boldsymbol{U}^\sharp \big\rangle \leq 0 ,$$

and, as $\mathcal{U}^{\mathrm{ad}}$ is a vector space, it is equivalent to

$$\forall \boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}}, \qquad \big\langle \boldsymbol{\lambda}, \boldsymbol{U} - \boldsymbol{U}^\sharp \big\rangle = 0 .$$

Thus the existence of $\boldsymbol{U}^\sharp$ minimizing J over $\mathcal{U}^{\mathrm{ad}}$, implies that $\boldsymbol{\lambda}^\sharp \in \big(\mathcal{U}^{\mathrm{ad}}\big)^\perp$ and $-\boldsymbol{\lambda}^\sharp \in \partial J\big(\boldsymbol{U}^\sharp\big)$.

On the other hand assume that there is such a $\boldsymbol{\lambda}^\sharp$.

As $-\boldsymbol{\lambda}^\sharp \in \partial J\big(\boldsymbol{U}^\sharp\big)$, we have

$$\forall \boldsymbol{U} \in \mathcal{U}, \qquad J\big(\boldsymbol{U}\big) \geq J\big(\boldsymbol{U}^\sharp\big) + \big\langle -\boldsymbol{\lambda}^\sharp, \boldsymbol{U} - \boldsymbol{U}^\sharp \big\rangle ,$$

which can be written as

$$J\big(\boldsymbol{U}^\sharp\big) \leq \big\langle -\boldsymbol{\lambda}^\sharp, \boldsymbol{U}^\sharp \big\rangle - \underbrace{\sup_{\boldsymbol{U} \in \mathcal{U}} \Big\{ \big\langle -\boldsymbol{\lambda}^\sharp, \boldsymbol{U} \big\rangle - J\big(\boldsymbol{U}\big) \Big\}}_{J^\star\left(-\boldsymbol{\lambda}^\sharp\right)} ,$$

and leads to (the other inequality being always satisfied)

$$J\big(\boldsymbol{U}^\sharp\big) + J^\star\big(-\boldsymbol{\lambda}^\sharp\big) = -\mathbb{E}\big[\boldsymbol{\lambda}^\sharp \cdot \boldsymbol{U}^\sharp\big] .$$

Similarly, as $\boldsymbol{\lambda}^\sharp \in \partial\chi_{\mathcal{U}^{\mathrm{ad}}}\big(\boldsymbol{U}^\sharp\big)$ we have

$$\chi_{\mathcal{U}^{\mathrm{ad}}}\big(\boldsymbol{U}^\sharp\big) + \chi^\star_{\mathcal{U}^{\mathrm{ad}}}\big(\boldsymbol{\lambda}^\sharp\big) = \mathbb{E}\big[\boldsymbol{\lambda}^\sharp \cdot \boldsymbol{U}^\sharp\big] ,$$

and, as $\chi^\star_{\mathcal{U}^{\mathrm{ad}}} = \chi_{\left(\mathcal{U}^{\mathrm{ad}}\right)^\perp}$, (see Equation (5.15)) we obtain

$$\chi_{\mathcal{U}^{\mathrm{ad}}}\big(\boldsymbol{U}^\sharp\big) + \chi_{\left(\mathcal{U}^{\mathrm{ad}}\right)^\perp}\big(\boldsymbol{\lambda}^\sharp\big) = \mathbb{E}\big[\boldsymbol{\lambda}^\sharp \cdot \boldsymbol{U}^\sharp\big] .$$

Thus,

$$\chi_{\mathcal{U}^{\mathrm{ad}}}\big(\boldsymbol{U}^\sharp\big) + \chi_{\left(\mathcal{U}^{\mathrm{ad}}\right)^\perp}\big(\boldsymbol{\lambda}^\sharp\big) = -J\big(\boldsymbol{U}^\sharp\big) - J^\star\big(-\boldsymbol{\lambda}^\sharp\big) ,$$

or, equivalently,

$$\chi_{\mathcal{U}^{\mathrm{ad}}}\big(\boldsymbol{U}^\sharp\big) + J\big(\boldsymbol{U}^\sharp\big) = -J^\star\big(-\boldsymbol{\lambda}^\sharp\big) - \chi_{\left(\mathcal{U}^{\mathrm{ad}}\right)^\perp}\big(\boldsymbol{\lambda}^\sharp\big) = -J^\star\big(-\boldsymbol{\lambda}^\sharp\big) - \chi_{\left(\mathcal{U}^{\mathrm{ad}}\right)^\perp}\big(-\boldsymbol{\lambda}^\sharp\big) ,$$

as $\big(\mathcal{U}^{\mathrm{ad}}\big)^\perp$ is a vector space. Hence, Theorem 5.17 achieves the proof. □

As we said at the beginning we now end the section by the proof of Theorem 5.15.

Proof. By Corollary 5.18, the control $\boldsymbol{U}^\sharp$ is a minimizer of J over $\mathcal{U}^{\mathrm{ad}}$ iff there is $-\boldsymbol{\lambda}^\sharp \in \partial J\big(\boldsymbol{U}^\sharp\big) \cap \big(\mathcal{U}^{\mathrm{ad}}\big)^\perp$. Moreover $\boldsymbol{U}^\sharp$ minimizes the Lagrangian $L(\boldsymbol{U}, \boldsymbol{\lambda}^\sharp)$ on $\mathcal{U}$ iff

$$0 \in \partial\Big(J + \langle \boldsymbol{\lambda}^\sharp, \cdot\rangle\Big)\big(\boldsymbol{U}^\sharp\big) \ ,$$

and, by the continuity assumption and [88, Theorem 3], this condition can be written as

$$0 \in \partial\big(J\big)\big(\boldsymbol{U}^\sharp\big) + \partial_{\boldsymbol{U}}\Big(\langle \boldsymbol{\lambda}^\sharp, \cdot\rangle\Big)\big(\boldsymbol{U}^\sharp\big) \ .$$

As the subdifferential of $\langle \boldsymbol{\lambda}^\sharp, \cdot\rangle$ is $\big\{\boldsymbol{\lambda}^\sharp\big\}$, this is equivalent to $-\boldsymbol{\lambda}^\sharp \in \partial J\big(\boldsymbol{U}^\sharp\big)$. □

5.2.4 Discussing the Local $\tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$-Continuity of J

It is worthwhile to elaborate on the Mackey continuity assumption of J at point $\boldsymbol{U}_0$ in Theorem 5.17. Indeed Lemma 5.16 show that J is weak* l.s.c everywhere, which is equivalent to be Mackey l.s.c everywhere as J is convex. However assuming that J is Mackey upper-semicontinuous at point $\boldsymbol{U}_0$ is a weaker assumption than assuming weak* upper-semicontinuity at point $\boldsymbol{U}_0$.

First we show that if J is finite then J is Mackey continuous. Then, we give conditions on j for J to be finite. Finally we show that, unfortunately, if the optimization problem include almost sure bounds, then the function J cannot be Mackey continuous at a point $\boldsymbol{U}_0$.

Conditions for Mackey Continuity

We show Mackey continuity if J (defined in (5.6)) is finite. First we need a definition and a lemma.

Definition 5.19. *We say that $J : \mathcal{U} \to \mathbb{R}$ has the* Lebesgue property *if for any sequence $\big\{\boldsymbol{U}_n\big\}_{n\in\mathbb{N}}$ such that*
- $\sup_{n\in\mathbb{N}} \big\|\boldsymbol{U}_n\big\|_\infty < +\infty$,
- $\boldsymbol{U}_n \xrightarrow{a.s.} \boldsymbol{U}$,

we have $J(\boldsymbol{U}_n) \to J\big(\boldsymbol{U}\big)$.

Lemma 5.20. *Suppose that j is a convex integrand and that J (defined in (5.6)) is finite everywhere on $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$. Then, J has the Lebesgue property.*

Proof. Consider a sequence $\big(\boldsymbol{U}_n\big)_{n\in\mathbb{N}}$ converging almost surely toward $\boldsymbol{U}$, and such that

$$\sup_{n\in\mathbb{N}} \big\|\boldsymbol{U}_n\big\|_\infty \leq M < +\infty \ .$$

As, for almost all ω, $u \mapsto j(u,\omega)$ is convex and finite, it is also continuous. As j is measurable in ω, it is a Caratheodory integrand, and thus a normal integrand.

By a measurable selection argument [96, Theorem 14.37], there exists $\boldsymbol{V} \in \mathrm{L}^0$ satisfying $\big\|\boldsymbol{V}\big\|_\infty \leq M$ and

$$|j(\boldsymbol{V})| = \max_{\|u\|_{\mathbb{R}^d} \leq M} |j(u,\omega)| < \infty \ ,$$

almost surely. In particular we have, for all $n \in \mathbb{N}$, $|j(\boldsymbol{U}_n)| \leq |j(\boldsymbol{V})|$.

Moreover, by continuity in u of j we have, for almost all ω,

$$j(\boldsymbol{U}_n(\omega), \omega) \to_n j(\boldsymbol{U}(\omega), \omega) \ .$$

Now as $J(\boldsymbol{V}) < +\infty$, Lebesgue dominated convergence theorem ensure that $J(\boldsymbol{U}_n) \to J\big(\boldsymbol{U}\big)$. □

Proposition 5.21. *Assume that j is a convex integrand and that J is finite everywhere on $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$. Then, J is $\tau(\mathrm{L}^\infty, \mathrm{L}^1)$-sequentially continuous.*

Proof. Recall that, for a sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space, the sequence $(x_n)_{n\in\mathbb{N}}$ converges toward x if from any subsequence we can extract a further subsequence converging toward x.

Assume that $(\boldsymbol{U}_n)_{n\in\mathbb{N}}$ $\tau(\mathrm{L}^\infty, \mathrm{L}^1)$-converges toward $\boldsymbol{U}$. Then, by Property 5.10, we have that there is a $p \geq 1$ such that $(\boldsymbol{U}_n)_{n\in\mathbb{N}}$ converges in L^p toward $\boldsymbol{U}$, and, in particular, that $(\boldsymbol{U}_n)_{n\in\mathbb{N}}$ converges in probability.

Consider the sequence $\big(J(\boldsymbol{U}_n)\big)_{n\in\mathbb{N}}$. For any subsequence $\Big(J\big(\boldsymbol{U}_{\phi(n)}\big)\Big)_{n\in\mathbb{N}}$, we are going to construct a sub-subsequence converging towards $J(\boldsymbol{U})$.

As $(\boldsymbol{U}_{\phi(n)})_{n\in\mathbb{N}}$ converges in probability (as a subsequence of a sequence converging in probability) toward $\boldsymbol{U}$, we have a further subsequence $(\boldsymbol{U}_{\psi(n)})_{n\in\mathbb{N}}$ converging almost surely towards $\boldsymbol{U}$. Moreover Property 5.10 ensures that $\sup_{n\in\mathbb{N}} \|\boldsymbol{U}_n\|_\infty < +\infty$. Thus, Lemma 5.20 guarantees convergence of $\big(J(\boldsymbol{U}_{\psi(n)})\big)_{n\in\mathbb{N}}$ toward $J(\boldsymbol{U})$, hence the convergence of $\big(J(\boldsymbol{U}_n)\big)_{n\in\mathbb{N}}$ toward $J(\boldsymbol{U})$. □

Corollary 5.22. *Assume that j is a convex integrand and that J is finite everywhere on $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$. Then, J is $\tau(\mathrm{L}^\infty, \mathrm{L}^1)$-continuous*[5].

Proof. In the proof of [88, Theorem 3], it is shown that, under strict feasibility assumption (satisfied by finiteness of J), we have

$$\exists \boldsymbol{Y}_0 \in \mathrm{L}^1 \text{ such that } j^\star(\boldsymbol{Y}_0, \cdot)^+ \in \mathrm{L}^1 .$$

Using Lemma 5.20, the result is a direct application of [72, Theorem 3.4]. □

Condition on j that Ensures Finiteness of J

As the finiteness of J is an assumption on the integral cost, we gives some set of assumptions on the integrand j that implies that J is finite everywhere on $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$.

Proposition 5.23. *If there exists a $\boldsymbol{U}_0 \in \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ such that $\mathbb{E}\big[|j(\boldsymbol{U}_0, \cdot)|\big] < \infty$ and if the family of functions $\big\{x \mapsto j(x, \omega) \mid \omega \in \Omega\big\}$ is $\mathbb{P}$-almost surely equi-Lipschitz*[6] *on any bounded set, then J is finite.*

Proof. Let $\boldsymbol{U}_0 \in \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ be such that $\mathbb{E}\big[|j(\boldsymbol{U}_0, \cdot)|\big] < +\infty$. Consider $\boldsymbol{U} \in \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Let κ be an almost sure Lipschitz constant of $x \mapsto j(x, \omega)$ on the ball of center 0 and radius $\max\big\{ \|\boldsymbol{U}\|_\infty, \|\boldsymbol{U}_0\|_\infty \big\}$. Then we have, almost surely,

$$|j(\boldsymbol{U}, \omega)| \leq |j(\boldsymbol{U}_0, \omega)| + \kappa |\boldsymbol{U} - \boldsymbol{U}_0| \leq |j(\boldsymbol{U}_0, \omega)| + \kappa\big(\|\boldsymbol{U}\|_\infty + \|\boldsymbol{U}_0\|_\infty \big) ,$$

therefore $J(\boldsymbol{U}) < +\infty$. □

5. This result is stronger than Proposition 5.21. However it relies on a result found in a pre-print, with an involved proof that I have not been able to grasp, whereas the proof of Proposition 5.21 is a personal contribution.

6. In fact we only require that the Lipschitz coefficient is integrable. Moreover we can replace Lipschitz continuity assumption by Hölder continuity assumptions.

Mackey Discontinuity Caused by Almost Sure Bounds

We show that almost sure constraints represented in the objective function J implies that at any point of its domain J is not Mackey-continuous.

Proposition 5.24. *Consider a convex normal integrand $j : \mathbb{R}^d \times \Omega \to \overline{\mathbb{R}}$, Consider a set $U^{\mathrm{ad}} \subsetneq \mathbb{R}^d$ and define the set of random variable*

$$\mathcal{U}^{a.s.} := \Big\{ \boldsymbol{U} \in \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big) \quad \big| \quad \boldsymbol{U} \in U^{\mathrm{ad}} \quad \mathbb{P} - a.s. \Big\} .$$

Then, at any point $\boldsymbol{U}_0 \in \mathrm{dom}\,\big(J\big) \cap \mathcal{U}^{a.s.}$, where J is given by (5.6), the function

$$\widetilde{J} : \boldsymbol{U} \mapsto J(\boldsymbol{U}) + \chi_{\boldsymbol{U} \in \mathcal{U}^{a.s.}} ,$$

is not Mackey continuous.

Proof. Consider a point $x \in \mathbb{R}^d \setminus U^{\mathrm{ad}}$, and a random variable $\boldsymbol{U}_0 \in \mathrm{dom}\,\big(J\big) \cap \mathcal{U}^{\mathrm{a.s.}}$. Let $\boldsymbol{X}$ be random variable uniform on $[0, 1]$. Define the sequence of random variables

$$\boldsymbol{U}_n := \boldsymbol{U}_0 + \big(x - \boldsymbol{U}_0\big) \mathbb{1}_{X \le \frac{1}{n}} .$$

We have

$$\big\|\boldsymbol{U}_n\big\|_\infty \le \big\|\boldsymbol{U}_0 \mathbb{1}_{X \le \frac{1}{n}}\big\|_\infty + \big\|x\big\|_{\mathbb{R}^d} \le \big\|\boldsymbol{U}_0\big\|_\infty + \big\|x\big\|_{\mathbb{R}^d} .$$

Moreover,

$$\big\|\boldsymbol{U}_n - \boldsymbol{U}_0\big\|_{\mathrm{L}^1} = \big\|\big(x - \boldsymbol{U}_0\big)\mathbb{1}_{X \le \frac{1}{n}}\big\|_{\mathrm{L}^1} \le \frac{\big\|x - \boldsymbol{U}_0\big\|_{\mathrm{L}^\infty}}{n} \le \frac{\big\|x\big\|_{\mathrm{L}^\infty} + \big\|\boldsymbol{U}_0\big\|_{\mathrm{L}^\infty}}{n} .$$

Hence, Proposition 5.10 ensure that $\boldsymbol{U}_n \xrightarrow{\tau(\mathrm{L}^\infty, \mathrm{L}^1)} \boldsymbol{U}$. However, as, for any $n \in \mathbb{N}$, $\boldsymbol{U}_n \notin U^{\mathrm{ad}}$ when $\boldsymbol{X} \le \frac{1}{n}$, we have that $\boldsymbol{U}_n \notin \mathcal{U}^{\mathrm{a.s.}}$, hence $\widetilde{J}\big(\boldsymbol{U}_n\big) = +\infty$. And, by assumption $\widetilde{J}\big(\boldsymbol{U}_0\big) < \infty$, thus $\widetilde{J}\big(\boldsymbol{U}_n\big) \nrightarrow \widetilde{J}\big(\boldsymbol{U}_0\big)$. Therefore, $\widetilde{J}$ is not Mackey continuous at $\boldsymbol{U}_0$. □

To sum up, we are able to dualize some affine constraints if there is no non-dualized constraints. In [116] the only type of constraint considered is the so-called non-anticipativity constraints (we show in the following section that they fall in the class of affine constraint that can be dualized). We add to those constraints some affine almost sure constraints. However, we are not able to show the existence of optimal multiplier in presence of almost sure bounds on the control.

5.3 Application to a Multistage Problem

In this section, we present a multistage problem with affine almost sure constraint and show the existence of a multiplier in L^1.

We consider a sequence of noises $\big\{\boldsymbol{W}_t\big\}_{t=0}^{T-1}$, with $\boldsymbol{W}_t \in \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{n_w}\big)$, for any $t \in [\![0, T-1]\!]$. We denote by $\mathcal{F}_t$ the σ-algebra generated by the past noises:

$$\mathcal{F}_t := \sigma\big(\boldsymbol{W}_0, \cdots, \boldsymbol{W}_t\big) ,$$

and by $\mathfrak{F}$ the induced filtration $\mathfrak{F} = \big\{\mathcal{F}_t\big\}_{t=0}^{T-1}$.

given by

$$\forall t \in [\![0, T-1]\!], \qquad \boldsymbol{X}_{t+1} = f_t\big(\boldsymbol{X}_t, \boldsymbol{D}_t, \boldsymbol{W}_t\big) , \tag{5.16}$$

where the control process $\{\boldsymbol{D}_t\}_{t=0}^{T-1}$ is a stochastic process adapted to $\mathfrak{F}$, and for each time $t \in [\![0, T-1]\!]$, $\boldsymbol{D}_t \in \mathrm{L}^\infty\big(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n_d}\big)$.

For each time $t \in [\![0, T-1]\!]$, we consider a local cost $L_t : \mathbb{R}^{n_x+n_d+n_w} \to \mathbb{R}$, and a final cost $K : \mathbb{R}^{n_x} \to \mathbb{R}$. We also consider linear constraint functions $\theta_t : \mathbb{R}^{n_x+n_d} \to \mathbb{R}^{n_c}$, and a sequence of $\mathfrak{F}$-adapted stochastic process $\big\{\boldsymbol{B}_t\big\}_{t\in[\![0,T-1]\!]}$.

Finally the problem reads,

$$
\begin{aligned}
\min_{\boldsymbol{X},\boldsymbol{D}} \quad & \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t\big(\boldsymbol{X}_t, \boldsymbol{D}_t, \boldsymbol{W}_t\big) + K\big(\boldsymbol{X}_T\big)\Big] && \text{(5.17a)}\\
s.t. \quad & \boldsymbol{X}_0 = x_0 && \text{(5.17b)}\\
& \boldsymbol{X}_{t+1} = f_t\big(\boldsymbol{X}_t, \boldsymbol{D}_t, \boldsymbol{W}_t\big), && \text{(5.17c)}\\
& \boldsymbol{D}_t \preceq \mathcal{F}_t, && \text{(5.17d)}\\
& \theta_t\big(\boldsymbol{X}_t, \boldsymbol{D}_t\big) = \boldsymbol{B}_t \qquad \mathbb{P}-a.s. && \text{(5.17e)}
\end{aligned}
$$

Lemma 5.25. *Assume that,*

- *the random noises $\boldsymbol{W}_t$ are essentially bounded;*
- *the local cost functions L_t are finite and convex in (x_t, d_t), continuous in w_t;*
- *the evolution functions f_t are affine in (x_t, d_t), continuous in w_t;*
- *the constraint functions θ_t are affine.*

Then Problem (5.17) can be written

$$
\min_{\boldsymbol{U}\in\mathcal{U}^{\mathrm{ad}}} \qquad J\big(\boldsymbol{U}\big) ,
$$

where

$$
J(\boldsymbol{U}) = \mathbb{E}\big[j(\boldsymbol{U})\big] ,
$$

with j a convex normal integrand. Moreover J is finite on L^∞ and hence is a $\tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$-continuous function, and $\mathcal{U}^{\mathrm{ad}}$ is a $\tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$-closed affine space.

Proof. We first rewrite Problem 5.17 in the framework of §5.2, and then shows the required continuity and closedness properties.

1. We reformulate Problem (5.17). We define the control $\boldsymbol{U} = \big\{\boldsymbol{D}_s\big\}_{s=0}^{T-1} \in \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{Tn_d}\big)$. Then, x_0 being given and constant, we define recursively the functions

$$
\begin{aligned}
x_t : \qquad & \mathbb{R}^{t(n_d+n_w)} \longrightarrow \mathbb{R}^{n_x}\\
& \big\{D_\tau, W_\tau\big\}_{\tau=0}^{t-1} \longmapsto f_{t-1}\Big(x_{t-1}\Big(\big\{D_\tau, W_\tau\big\}_{\tau=0}^{t-2}\Big), D_{t-1}, W_{t-1}\Big)
\end{aligned}
$$

The functions x_t give the value of $\boldsymbol{X}_t$ in function of the past decisions $\{\boldsymbol{D}_s\}_{s=0}^{t-1}$, and noises $\{\boldsymbol{W}_s\}_{s=0}^{t-1}$, and are affine in $\boldsymbol{U}$.

We define (up to $\mathbb{P}$-almost sure equality), the cost

$$
j(\boldsymbol{U}, \cdot) := \sum_{t=0}^{T-1} L_t\Big(x_t\Big(\big\{\boldsymbol{D}_\tau, \boldsymbol{W}_\tau\big\}_{\tau=0}^{t-1}\Big), \boldsymbol{D}_t, \boldsymbol{W}_t\Big) + K\Big(x_T\Big(\big\{\boldsymbol{D}_\tau, \boldsymbol{W}_\tau\big\}_{\tau=0}^{T-1}\Big)\Big) . \qquad (5.18)
$$

Then $J(\boldsymbol{U}) = \mathbb{E}[j(\boldsymbol{U})]$ is the objective function of Problem (5.17), taking into account the initial state constraint (5.17b) and the dynamic constraint (5.17c).

The control $\boldsymbol{U}$ satisfies constraint (5.17d) and is said to be non-anticipative if it is an element of the space $\mathcal{N} \subset \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{Tn_d}\big)$, where

$$
\mathcal{N} := \Big\{\big\{\boldsymbol{D}_s\big\}_{s=0}^{T} \in \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{Tn_d}\big) \quad \Big| \quad \forall s \in [\![0, T-1]\!], \quad \mathbb{E}\big[\boldsymbol{D}_s \bigm| \mathcal{F}_s\big] = \boldsymbol{D}_s\Big\} . \qquad (5.19)
$$

The control $\boldsymbol{U}$ satisfies constraint (5.17e) if it is an element of the subspace $\mathcal{U}^{\text{a.s.}}$ given by

$$\mathcal{U}^{\text{a.s.}} := \Big\{ \{\boldsymbol{D}_s\}_{s=0}^{T-1} \quad \big| \quad \forall s \in [\![0, T-1]\!], \quad \theta_t\Big(x_t\Big(\{\boldsymbol{D}_\tau, \boldsymbol{W}_\tau\}_{\tau=0}^{t-1}\Big), \boldsymbol{D}_t \Big) = \boldsymbol{B}_t \Big\} . \tag{5.20}$$

With these notations, Problem (5.17) can be written

$$\min_{\boldsymbol{U} \in \mathcal{N} \cap \mathcal{U}^{\text{a.s.}}} J\big(\boldsymbol{U}\big) .$$

2. We show the $\tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$-continuity of J.
   As, for any $t \in [\![0, T-1]\!]$, the function L_t is convex, and the function x_t is affine we obtain the convexity in $\boldsymbol{U}$ of function j. Measurability and continuity of j are obvious.
   Moreover, for all $t \in [\![0, T-1]\!]$, the decision variable $\boldsymbol{D}_t$ is bounded as an element of L^∞, and the random noise $\boldsymbol{W}$ is bounded by assumption. Thus the state process $\boldsymbol{X}$ given by (5.16) is also bounded. Furthermore there are constants $\alpha \geq 0$ and $\beta \geq 0$ such that $\|\boldsymbol{X}\|_\infty \leq \alpha + \beta \|\boldsymbol{U}\|_\infty$. Consequently j is a Caratheodory function, and J (as defined in (5.6)) is finite on L^∞.
   Thus, by Corollary 5.22, the function J is $\tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$-continuous.
3. We show the $\tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$-closedness of $\mathcal{U}^{\text{ad}}$.
   Corollary 5.13 and 5.14 ensure that $\mathcal{N}$ and $\mathcal{U}^{\text{a.s.}}$ are weak*-closed affine space, hence $\mathcal{U}^{\text{ad}} = \mathcal{N} \cap \mathcal{U}^{\text{a.s.}}$ is a weak*-closed affine space, thus a $\tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$-closed affine space.

The proof is complete. □

Lemma 5.25, cast the dynamic problem into the static setting of §5.2, and thus ensure the existence of a multiplier for the non-anticipativity constraint coupled with the almost-sure affine constraint. We now discuss, how the multiplier can be decomposed into one for the almost sure constraint, and one for the non-anticipativity constraint.

Proposition 5.26. *We denote by $\mathcal{N}$ the set of non-anticipative controls defined in Equation (5.19), and by $\mathcal{U}^{a.s.}$ the set of controls satisfying (5.17e) given in Equation (5.20).*

If, for all $\boldsymbol{U} \in \mathcal{U}^{a.s.}$, the $\mathfrak{F}$-adapted part of $\boldsymbol{U}$ is also in $\mathcal{U}^{a.s.}$, i.e.

$$\Big\{ \mathbb{E}\big[\boldsymbol{U}_t \mid \mathcal{F}_t\big] \Big\}_{t=0}^{T-1} \in \mathcal{U}^{a.s.} , \tag{5.21}$$

then

$$(\mathcal{U}^{a.s.} \cap \mathcal{N})^\perp = \big(\mathcal{U}^{a.s.}\big)^\perp + \mathcal{N}^\perp . \tag{5.22}$$

Proof. Consider the linear operator $\Theta : \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{Td}\big) \to \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{Td}\big)$, that gives for each stochastic process its $\mathfrak{F}$-adapted part, i.e.

$$\Theta\big(\boldsymbol{U}\big) = \Big\{ \mathbb{E}\big[\boldsymbol{U}_t \mid \mathcal{F}_t\big] \Big\}_{t=0}^{T-1} .$$

Θ is a linear operator, admitting an adjoint, with $\Theta\big(\mathcal{U}\big) = \mathcal{N}$, and $\Theta_{|\mathcal{N}} = \mathrm{Id}$. Moreover, by assumption $\Theta\big(\mathcal{U}^{\text{a.s.}}\big) \subset \mathcal{U}^{\text{a.s.}}$. Hence, Theorem A.43, states that, for any $\boldsymbol{U} \in \mathcal{N} \cap \mathcal{U}^{\text{a.s.}}$,

$$(\mathcal{U}^{\text{a.s.}} \cap \mathcal{N})^\perp_{\boldsymbol{U}} = \big(\mathcal{U}^{\text{a.s.}}\big)^\perp_{\boldsymbol{U}} + \mathcal{N}_0^\perp .$$

Finally, noting that $\mathcal{U}^{\text{a.s.}}$ and $\mathcal{N}$ are affine spaces gives the result. □

Corollary 5.27. *Under assumptions of Lemma 5.25, Problem (5.17) admits a* L^1 *multiplier for the non anticipativity constraint (5.17d) coupled with the almost sure constraints (5.17e).*

Moreover, if the constraint functions θ_t in Problem (5.17) does not depends on $\boldsymbol{X}_t$, then the multiplier can be decomposed into one multiplier for the almost sure constraints, and one for the non-anticipativity constraints.

Proof. From Lemma 5.25 we have the assumptions required to apply Theorem 5.15. Moreover, if the constraint functions θ_t in Problem (5.17) does not depends on $\boldsymbol{X}_t$, then assumption (5.21) is satisfied, and Proposition 5.26 gives the result. □

Conclusion

In this chapter, we have shown that, if the cost function J is finite on L^∞, then almost sure affine equality constraints and non-anticipativity constraints admit a L^1-multiplier. Notice that, when we assume that the cost function J is finite on L^∞, we exclude the possibility of having almost sure constraints that are not dualized.

If we want to incorporate bound constraints on control variables in the optimization problem, we should turn to a series of works by T. Rockafellar and R. Wets. In a first series [86, 91, 93, 97], they work out the theory of duality on a two-stage stochastic optimization problem. In [97], they show a result of non-duality gap. In [91] the Kuhn-Tucker conditions are detailed, whereas in [86] the existence of a multiplier in $\left(\mathrm{L}^\infty\right)^\star$ is shown. Finally, in [93] they introduce a condition, slightly weaker than the well-known assumption of relatively complete recourse, that ensures the existence of a multiplier in L^1. In [92, 94, 95], they adapt these results to a multistage optimization problem.

It appears that, in these papers, two types of assumptions are of the upmost importance: (essential) relatively complete recourse; strict feasibility assumption. We comment one after the other.

- Relatively complete recourse ensures that there is no induced constraint, that is, that the constraints at later stages do not imply constraints at earlier stages. From a multistage application point of view, bound constraints on the state would still be difficult to treat; but bound constraints on the control would be available.
- The strict feasibility assumption is mainly used to show the existence of a multiplier in $\left(\mathrm{L}^\infty\right)^\star$. This assumption forbids the direct use of the results of T. Rockafellar and R. Wets to problems with equality constraints. However, if we look at the proof of [93, Theorem 3], the strict feasibility assumption is used to ensure the existence of a multiplier for the first stage problem (with a linear cost). Hence, the existence of a multiplier in $\left(\mathrm{L}^\infty\right)^\star$ and relatively complete recourse-like assumptions might be enough to show the existence of a multiplier in L^1. Work remains to be done on this subject.

Chapter 6

Uzawa Algorithm in $\mathrm{L}^{\infty}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$

One should always generalize.

Carl Jacobi

Contents

We remind the reader that this second part of the manuscript deals with the treatment of constraints through duality, in stochastic optimization. This Chapter 6 is devoted to the extension of the Uzawa algorithm, as formulated in an Hilbert space (e.g. $\mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P})$), to the Banach space $\mathrm{L}^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$. The issue is the following. The convergence of the Uzawa algorithm relies upon a key assumption of constraint qualification. But, we have seen in Chapter 4 that almost sure constraints generally fail to be qualified for the L^p duality, when $p < +\infty$. In Chapter 5 we derived conditions to obtain an optimal multiplier in the $(\mathrm{L}^{\infty}, \mathrm{L}^1)$ duality. This chapter is devoted to the extension of the Uzawa algorithm, as formulated in an Hilbert space (e.g. $\mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P})$), to the Banach space $\mathrm{L}^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$.

The chapter is organized as follows. In §6.1, we recall optimization results (inequalities and first order optimality conditions), that are well-known in Hilbert spaces, and that remain valid in Banach spaces; we also recall the proof of convergence of Uzawa algorithm in the usual Hilbert spaces case. In §6.2, this proof is used as a canvas for the proof of convergence of Uzawa algorithm in the non reflexive Banach space $\mathrm{L}^{\infty}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$. Finally, in §6.3 we present an application to a multistage example.

Introduction

To address a constraint in an optimization problem, we can dualize it, thus making it disappears as a constraint and appears as a cost. Consequently, the "min" operator is replaced by a "min max" operator. Numerical algorithms address such problems; the Uzawa algorithm is one of them.

For an objective function $J : \mathcal{U} \to \overline{\mathbb{R}}$, a constraint function $\Theta : \mathcal{U} \to \mathcal{V}$, constraint set $\mathcal{U}^{\mathrm{ad}} \subset \mathcal{U}$ and a constraint cone $C \subset \mathcal{V}$, we consider the following problem

$$\min_{u \in \mathcal{U}^{\mathrm{ad}}} \quad J(u) \; , \tag{6.1a}$$

$$s.t. \quad \Theta(u) \in -C \; , \tag{6.1b}$$

where $\mathcal{U}$ (resp. $\mathcal{V}$) is a topological space paired with $\mathcal{U}^\star$ (resp. $\mathcal{V}^\star$). We associate with this problem the Lagrangian $L : \mathcal{U} \times \mathcal{V}^\star \to \overline{\mathbb{R}}$, introduced in Chapter 4, given by

$$L(u, \lambda) := J(u) + \big\langle \lambda \,, \Theta(u) \big\rangle_{\mathcal{V}^\star, \mathcal{V}} \; . \tag{6.2}$$

Thus, Problem (6.1) reads

$$\min_{u \in \mathcal{U}^{\mathrm{ad}}} \quad \max_{\lambda \in C^\star} \qquad J(u) + \big\langle \lambda \,, \Theta(u) \big\rangle_{\mathcal{V}^\star, \mathcal{V}} \; , \tag{6.3}$$

where $C^\star \subset \mathcal{V}^\star$ is the dual cone given by

$$C^\star = \big\{ \lambda \in \mathcal{V}^\star \quad | \quad \forall x \in C, \quad \big\langle \lambda \,, x \big\rangle_{\mathcal{V}^\star, \mathcal{V}} \geq 0 \big\} \; .$$

The dual problem of Problem (6.3) reads

$$\max_{\lambda \in C^\star} \quad \min_{u \in \mathcal{U}^{\mathrm{ad}}} \qquad J(u) + \big\langle \lambda \,, \Theta(u) \big\rangle_{\mathcal{V}^\star, \mathcal{V}} \; , \tag{6.4}$$

and the inner minimization problem for a given multiplier λ is

$$\min_{u \in \mathcal{U}^{\mathrm{ad}}} \qquad J(u) + \big\langle \lambda \,, \Theta(u) \big\rangle_{\mathcal{V}^\star, \mathcal{V}} \; . \tag{6.5}$$

An iteration of the Uzawa algorithm consists in fixing the multiplier λ of the constraint, then solving the inner minimization problem (6.5), and finally updating the multiplier. The update step can be seen, under the right assumptions, as a gradient step over the multiplier. It is described in Algorithm 6.1, where $\mathrm{proj}_A(z)$ is the projection of z on the convex set A.

Data: Initial multiplier $\lambda^{(0)}$, step $\rho > 0$;
Result: Optimal solution $\boldsymbol{U}^\sharp$ and multiplier $\lambda^\sharp$;
repeat

$$u^{(k+1)} \in \operatorname*{arg\,min}_{u \in \mathcal{U}^{\mathrm{ad}}} \quad \Big\{ J(u) + \big\langle \lambda^{(k)} \,, \Theta(u) \big\rangle \Big\} \; , \tag{6.6a}$$

$$\lambda^{(k+1)} = \mathrm{proj}_{C^\star} \Big(\lambda^{(k)} + \rho \Theta\big(u^{(k+1)}\big) \Big) \; . \tag{6.6b}$$

until $\Theta(u^{(k)}) \in -C$;

Algorithm 6.1: Uzawa Algorithm

6.1 Optimization Results and Classical Uzawa Algorithm

In §6.1.1 we show that some inequalities and first order optimality conditions usually presented in an Hilbert setting remain true in a Banach setting. In §6.1.2 we recall the Uzawa algorithm in an Hilbert setting and its proof that is used as a canvas for the proof given in §6.2.

6.1.1 Optimization in a Banach Space

In this synthetic section we underline some relevant differences between Hilbert and Banach spaces, and go on to give some inequalities and optimality conditions that are used in §6.1.2 and §6.2.

Lemma 6.1. *Let $\mathcal{U}$ be a Banach space and $J : \mathcal{U} \to \overline{\mathbb{R}}$ a convex and Gâteaux differentiable function J. We have*

$$\big\langle J'(u) \,, v - u \big\rangle \leq J(v) - J(u) \;.$$

Moreover, if J is strongly convex[1] *of modulus a, we have*

$$a \left\| u - v \right\|^2 \leq \big\langle J'(u) - J'(v) \,, u - v \big\rangle \;.$$

Proof. The usual proof in a Hilbert space remains valid in a Banach space. □

Proposition 6.2. *Let $\mathcal{U}$ be a Banach space. We consider the following problem:*

$$\min_{u \in \mathcal{U}^{\mathrm{ad}}} \quad J(u) + J^{\Sigma}(u) \;. \tag{6.7}$$

We make the following assumptions:

1. *$\mathcal{U}^{\mathrm{ad}}$ is a non empty, closed convex subset of $\mathcal{U}$,*
2. *the function $J : \mathcal{U} \to \overline{\mathbb{R}}$ is convex and Gâteaux-differentiable,*
3. *the function $J^{\Sigma} : \mathcal{U} \to \overline{\mathbb{R}}$ is convex.*

Then, the point $\bar{u} \in \mathcal{U}^{\mathrm{ad}}$ is a solution of Problem (6.7) if and only if

$$\forall u \in \mathcal{U}^{\mathrm{ad}}, \quad \big\langle J'\big(\bar{u}\big) \,, u - \bar{u} \big\rangle + J^{\Sigma}(u) - J^{\Sigma}\big(\bar{u}\big) \geq 0 \;. \tag{6.8}$$

Proof. Assume that $\bar{u}$ is an optimal solution of Problem (6.7). $\mathcal{U}^{\mathrm{ad}}$ being convex, we have

$$\forall t \in (0,1], \quad \forall u \in \mathcal{U}^{\mathrm{ad}}, \qquad J\big(\bar{u} + t(u - \bar{u})\big) + J^{\Sigma}\big(\bar{u} + t(u - \bar{u})\big) \geq J\big(\bar{u}\big) + J^{\Sigma}\big(\bar{u}\big) \;,$$

so that, for any $t \in (0,1]$,

$$\frac{J\big(\bar{u} + t(u - \bar{u})\big) - J\big(\bar{u}\big)}{t} + \frac{J^{\Sigma}\big(\bar{u} + t(u - \bar{u})\big) - J^{\Sigma}\big(\bar{u}\big)}{t} \geq 0 \;.$$

By convexity of J^{Σ} we have

$$\forall t \in (0,1], \qquad \frac{J^{\Sigma}\big(\bar{u} + t(u - \bar{u})\big) - J^{\Sigma}\big(\bar{u}\big)}{t} \leq J^{\Sigma}(u) - J^{\Sigma}\big(\bar{u}\big) \;,$$

and by Gâteaux-differentiability of J we have

$$\lim_{t \to 0^+} \frac{J\big(\bar{u} + t(u - \bar{u})\big) - J\big(\bar{u}\big)}{t} = \big\langle J'\big(\bar{u}\big) \,, u - \bar{u} \big\rangle \;,$$

hence the variational inequality (6.8) holds true.

Now, suppose that (6.8) is satisfied. Then, by convexity of J, we have that

$$\forall u \in \mathcal{U}^{\mathrm{ad}}, \quad \big\langle J'\big(\bar{u}\big) \,, u - \bar{u} \big\rangle \leq J(u) - J\big(\bar{u}\big) \;,$$

thus, the optimality of $\bar{u}$. □

We apply Inequality (6.8) to Problem (6.5), where J is the objective cost, and J^{Σ} is the dual term, and obtain

$$\forall u \in \mathcal{U}^{\mathrm{ad}}, \quad \big\langle J'\big(\bar{u}\big) \,, u - \bar{u} \big\rangle + \big\langle \lambda \,, \Theta(u) - \Theta\big(\bar{u}\big) \big\rangle \geq 0 \;. \tag{6.9}$$

1. See [70, Section 2.1.3] for equivalent definitions of strongly convex functions

6.1.2 Recall of the Convergence Proof of Uzawa Algorithm in a Hilbert Space

Following [45, Ch.VII], we recall the proof of the Uzawa algorithm. This proof will be used as a canvas for the proof an extension of the Uzawa algorithm in the L^∞ case developed in §6.2.

From now on the differential form $J'(u) \in \mathcal{U}^\star$ is associated with the gradient $\nabla J(u) \in \mathcal{U}$.

We make the following assumptions about Problem (6.1).

Hypothesis 1.

1. *The function $J : \mathcal{U} \to \overline{\mathbb{R}}$ is strongly convex of modulus a, and Gâteaux-differentiable.*
2. *The function $\Theta : \mathcal{U} \to \mathcal{V}$ is C-convex (see Definition A.48), and κ-Lipschitz.*
3. *$\mathcal{U}^{\mathrm{ad}}$ is a non empty, closed convex subset of the Hilbert space $\mathcal{U}$.*
4. *C is a non empty, closed convex cone of the Hilbert space $\mathcal{V}$.*
5. *The Lagrangian L (defined in (6.2)) admits a saddle-point $(u^\sharp, \lambda^\sharp)$ on $\mathcal{U}^{\mathrm{ad}} \times C^\star$, that is,*
$$\forall u \in \mathcal{U}^{\mathrm{ad}}, \quad \forall \lambda \in C^\star, \qquad L\big(u^\sharp, \lambda\big) \leq L\big(u^\sharp, \lambda^\sharp\big) \leq L\big(u, \lambda^\sharp\big) \,. \tag{6.10}$$
6. *The step ρ is small enough $(0 < \rho < 2a/\kappa^2)$.*

Let us comment these assumptions.

(a) In general, we do not require condition 5, but obtain it from other assumptions, e.g. through qualification conditions.

(b) The strong convexity of J ensures the uniqueness of $u^\sharp$, first component of the saddle point, in (6.10).

(c) We do not assume that J is l.s.c., as this property is implied by convexity and differentiability:
$$J(v) \geq J(u) + \big\langle \nabla J(u) \,, v - u \big\rangle \;\Rightarrow\; \liminf_{v \to u} J(v) \geq J(u) \,.$$

(d) The right-hand side inequality of (6.10) can be written
$$u^\sharp \in \arg\min_{u \in \mathcal{U}^{\mathrm{ad}}} L\big(u, \lambda^\sharp\big) \,,$$
with the following optimality condition (see (6.9)),
$$\forall u \in \mathcal{U}^{\mathrm{ad}}, \qquad \big\langle \nabla J\big(u^\sharp\big) \,, u - u^\sharp \big\rangle + \big\langle \lambda^\sharp \,, \Theta(u) - \Theta\big(u^\sharp\big)\big\rangle \geq 0 \,.$$

(e) The left-hand side inequality of (6.10) can be written
$$\forall \lambda \in C^\star, \qquad \big\langle \lambda - \lambda^\sharp \,, \Theta\big(u^\sharp\big)\big\rangle \leq 0 \,,$$
which is equivalent to, as $C^\star$ is convex,
$$\lambda^\sharp = \mathrm{proj}_{C^\star}\Big(\lambda^\sharp + \rho\Theta\big(u^\sharp\big)\Big) \,,$$
for any $\rho > 0$.

Theorem 6.3. *Under Hypothesis 1, the Uzawa Algorithm 6.1 is such that the sequence $\{u^{(k)}\}_{k \in \mathbb{N}}$ converges toward $u^\sharp$ in norm.*

Proof. Let $(u^\sharp, \lambda^\sharp)$ be a saddle point of the Lagrangian L given by (6.2). We denote $r^{(k)} = \lambda^{(k)} - \lambda^\sharp$.

1. We have that, as the projection on a convex set is a non-expansive function,

$$\begin{aligned}\big\|r^{(k+1)}\big\|^2 &= \big\|\mathrm{proj}_{C^\star}\Big(\lambda^{(k)} + \rho\Theta\big(u^{(k+1)}\big)\Big) - \mathrm{proj}_{C^\star}\Big(\lambda^\sharp + \rho\Theta\big(u^\sharp\big)\Big)\big\|^2 ,\\ &\leq \big\|r^{(k)} + \rho\big(\Theta\big(u^{(k+1)}\big) - \Theta\big(u^\sharp\big)\big)\big\|^2 .\end{aligned}$$

Developing this expression, and exploiting the κ-Lipschitz continuity of Θ, we obtain

$$\big\|r^{(k+1)}\big\|^2 \leq \big\|r^{(k)}\big\|^2 + 2\rho\big\langle r^{(k)} , \Theta\big(u^{(k+1)}\big) - \Theta\big(u^\sharp\big)\big\rangle + \rho^2\kappa^2\big\|u^{(k+1)} - u^\sharp\big\|^2 . \quad (6.11)$$

2. We apply the optimality condition (6.9), on the one hand for $\lambda = \lambda^{(k)}$ with $u = u^\sharp$ and $\bar{u} = u^{(k+1)}$ and, on the other hand, for $\lambda = \lambda^\sharp$ with $u = u^{(k+1)}$ and $\bar{u} = u^\sharp$. This gives

$$\begin{aligned}&\big\langle \nabla J\big(u^{(k+1)}\big) , u^\sharp - u^{(k+1)}\big\rangle + \big\langle \lambda^{(k)} , \Theta\big(u^\sharp\big) - \Theta\big(u^{(k+1)}\big)\big\rangle \geq 0 ,\\ &\big\langle \nabla J\big(u^\sharp\big) , u^{(k+1)} - u^\sharp\big\rangle \quad\;\; + \big\langle \lambda^\sharp , \Theta\big(u^{(k+1)}\big) - \Theta\big(u^\sharp\big)\big\rangle \quad \geq 0 .\end{aligned}$$

Summing both conditions and using the strong convexity of J, we obtain

$$\begin{aligned}\big\langle \lambda^{(k)} - \lambda^\sharp , \Theta\big(u^{(k+1)}\big) - \Theta\big(u^\sharp\big)\big\rangle &\leq -\big\langle \nabla J\big(u^{(k+1)}\big) - \nabla J\big(u^\sharp\big) , u^{(k+1)} - u^\sharp\big\rangle ,\\ &\leq -a\big\|u^{(k+1)} - u^\sharp\big\|^2 .\end{aligned}$$

3. Using the last inequality Equation (6.11) yields

$$\big\|r^{(k+1)}\big\|^2 \leq \big\|r^{(k)}\big\|^2 - \big(2a\rho - \rho^2\kappa^2\big)\big\|u^{(k+1)} - u^\sharp\big\|^2 .$$

The assumption $0 < \rho < 2a/\kappa^2$ on the step ρ ensures that $2a\rho - \rho^2\kappa^2 > 0$, the sequence $\big\{r^{(k)}\big\}_{k\in\mathbb{N}}$ is decreasing and non-negative, thus convergent. Consequently, the sequence $\big\{\|u^{(k)} - u^\sharp\|\big\}_{k\in\mathbb{N}}$ converges toward 0. □

Notice that this proof relies on i) estimations deduced from optimality conditions that hold in Banach space ii) existence of a saddle-point (which can be obtained with other assumptions) iii) developing a square norm (in (6.11)). This last point might fail in a Banach space.

6.2 Uzawa Algorithm in $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$ Spaces

In Problem (6.1), we considered the case where spaces $\mathcal{U}$ and $\mathcal{V}$ were Hilbert spaces. Now, in the sequel of this chapter, we assume that $\mathcal{U}$ and $\mathcal{V}$ are the following L^∞ spaces:

$$\mathcal{U} = \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n), \quad \mathcal{V} = \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^p) . \quad (6.12)$$

We assume that $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ is a probability space, where the σ-algebra $\mathcal{F}$ is not finite (modulo $\mathbb{P}$, see Definition 5.1). Indeed, when $\mathcal{F}$ is finite, the space $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$ is a finite dimensional vector space, hence a Hilbert space; thus, the convergence result of §6.1.2 holds true.

Moreover, from now on, we assume that we have only equality constraints:

- the cone of constraints in Problem (6.1) is $C = \big\{0\big\}$;
- C-convexity of the constraint function Θ implies that Θ is an affine function;
- $\mathrm{proj}_{C^\star}$ is the identity function.

Thus, Problem (6.1) reads

$$\min_{u\in\mathcal{U}^{\mathrm{ad}}\subset\mathrm{L}^\infty} \quad J(u)\ , \tag{6.13a}$$
$$s.t. \quad \Theta(u)=0\ , \tag{6.13b}$$

and the algorithm defined in 6.1 reads

$$\boldsymbol{U}^{(k+1)} = \mathop{\arg\min}_{\boldsymbol{U}\in\mathcal{U}^{\mathrm{ad}}} \Big\{ J(\boldsymbol{U}) + \big\langle \boldsymbol{\lambda}^{(k)}\,,\Theta(\boldsymbol{U})\big\rangle \Big\}\ , \tag{6.14a}$$
$$\boldsymbol{\lambda}^{(k+1)} = \boldsymbol{\lambda}^{(k)} + \rho\Theta(\boldsymbol{U}^{(k+1)})\ , \tag{6.14b}$$

where we choose to take $\boldsymbol{\lambda}^{(0)}$ in $\mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^p\big)$.

We underline the differences between Hilbert spaces and Banach spaces in §6.2.1, and then explain in §6.2.2 why the Uzawa update (6.14b) is well defined. We give a result of convergence of a subsequence the algorithm in §6.2.3, although using strong assumptions and discuss why we do not obtain the convergence of the whole sequence in §6.2.4.

6.2.1 Discussing Differences Between Hilbert and Banach Spaces

The spaces $\mathcal{U} = \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n\big)$ and $\mathcal{V} = \mathrm{L}^\infty\big(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^p\big)$ given in (6.12) are non-reflexive, non-separable, Banach spaces. Hence they do not have the properties displayed by Hilbert spaces, and useful for optimization.

Perks of an Hilbert Space

In an Hilbert space $\mathcal{H}$ we know that

i) the weak and weak* topologies are identical,

ii) the space $\mathcal{H}$ and its topological dual can be identified.

Point i) allows to formulate existence of minimizer results. Indeed, the weak*-closed bounded subsets of $\mathcal{H}$ are weak* compact, (Banach-Alaoglu Theorem A.24). Hence, weakly closed bounded subsets are weakly compact. A convex set is closed iff it is weakly closed, and a convex function is l.s.c. iff it is weakly l.s.c.. Thus, a convex (strongly) l.s.c. function $f:\mathcal{H}\to\overline{\mathbb{R}}$, coercive on the closed convex subset $\mathcal{U}^{\mathrm{ad}}\subset\mathcal{H}$, admits a minimum on $\mathcal{U}^{\mathrm{ad}}$. Indeed, coercivity implies that we can consider a bounded subset of $\mathcal{U}^{\mathrm{ad}}$; its closed convex hull is weakly compact and, as f is weakly l.s.c., Bolzano Weierstrass theorem ensures the existence of a minimum.

Point ii) allows to write gradient-like algorithms. Indeed, it allows to represent the differential of a (differentiable) function $f:\mathcal{H}\to\overline{\mathbb{R}}$ as the inner product with a vector $g\in\mathcal{H}$ called gradient. Wit this, we can propose gradient-like minimization algorithms as follows: at any iteration k, we have a point $u^{(k)}\in\mathcal{H}$, and the gradient $g^{(k)}=\nabla f\big(u^{(k)}\big)\in\mathcal{H}$; the new point $u^{(k+1)}$ is a linear combination of the former point $u^{(k)}$ and of the gradient $g^{(k)}$, e.g. (6.14b).

Difficulties Appearing in a Banach Space

In a reflexive Banach space E, $i)$ still holds true, and thus the existence of a minimizer remains easy to show. However $ii)$ does not hold any longer. Indeed, the differential of a differentiable function $f:E\to\overline{\mathbb{R}}$ at point $x\in E$ can be represented through a duality product $df(x):h\mapsto\langle g\,,x\rangle$, but g belongs to the topological dual of E, which cannot be identified to E (if E is not an Hilbert space). Thus, a gradient algorithm where $u^{(k+1)}$ is a linear combination of $u^{(k)}\in E$ and $g^{(k)}\in E'$ does not have any sense.

In a non-reflexive Banach space E, neither $i)$ nor $ii)$ hold true. However, if E is the topological dual of a Banach space, then the Banach-Alaoglu theorem (Theorem A.24) holds, and weakly* closed bounded subset, of E are weak* compact. In this case, weak* lower semicontinuity of a function f, coupled with its coercivity, leads to the existence of a minimizer of f (this point is developed in Theorem 6.4).

Here, we warn the reader that we are not sure of the existence of strongly convex functions in a non-reflexive Banach space. As an illustration of the difficulty, it is shown in [24, Remark 3.67] that if f is twice differentiable and strongly convex on a space $\mathcal{H}$, then $\mathcal{H}$ is Hilbertizable.

6.2.2 Making Sense of Uzawa Algorithm in $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ for Equality Constraint

We have seen that a gradient-like formula, for instance the Uzawa update step (6.14b), does not make sense in a generic Banach space. However, we will now show that it is well defined in $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$.

Specificities of $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$

The Banach space $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ is non-reflexive, non-separable because the σ-algebra $\mathcal{F}$ is not finite (Proposition 5.3).

However, as $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ is the topological dual of the Banach space $\mathrm{L}^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$, the Banach-Alaoglu theorem holds, paving the way for a proof of existence of a minimizer (see below). Moreover, $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ can be identified with a subset of its topological dual $\Big(\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)\Big)^\star$. Thus, the update step (6.14b) make sense: it is a linear combination of elements of $\Big(\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)\Big)^\star$. Consequently, $\{\boldsymbol{\lambda}^{(k)}\}_{k\in\mathbb{N}}$ is a sequence of elements of $(\mathrm{L}^\infty)^\star$. Nevertheless, if $\boldsymbol{\lambda}^{(0)}$ is represented by an element of L^∞, then $\{\boldsymbol{\lambda}^{(k)}\}_{k\in\mathbb{N}}$ is represented by a sequence of elements of L^∞. As we make the assumption that $\boldsymbol{\lambda}^{(0)}$ can be represented by an element of L^∞, we consider from now on that $\{\boldsymbol{\lambda}^{(k)}\}_{k\in\mathbb{N}}$ is a sequence of elements of L^∞.

Existence of Solutions

The following theorem shows that there exists a solution to Problem (6.13), and that the minimization problem in the primal step (6.14a) has also a solution.

Theorem 6.4. *Assume that:*

1. *the constraint set $\mathcal{U}^{\mathrm{ad}}$ is weakly* closed,*
2. *the constraint affine function $\Theta : \mathcal{U} \to \mathcal{V}$ is weakly* continuous,*
3. *the objective function $J : \mathcal{U} \to \overline{\mathbb{R}}$ is weak* l.s.c. and coercive on $\mathcal{U}^{\mathrm{ad}}$,*
4. *there exists an admissible control, i.e.*

$$\mathrm{dom}(J) \cap \mathcal{U}^{\mathrm{ad}} \cap \Theta^{-1}\big(\{0\}\big) \neq \emptyset \ .$$

Then, Problem (6.13) admits at least one solution.

Moreover, for any $\boldsymbol{\lambda} \in \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^p)$, the following argmin is not empty:

$$\operatorname*{arg\,min}_{\boldsymbol{U}\in\mathcal{U}^{\mathrm{ad}}} \Big\{ J(\boldsymbol{U}) + \big\langle \boldsymbol{\lambda} \, , \Theta(\boldsymbol{U}) \big\rangle \Big\} \neq \emptyset \ .$$

Finally, if J is strictly convex, then the above argmin is reduced to a single point.

Note that coercivity of J is ensured either when J is strongly convex, or when $\mathcal{U}^{\mathrm{ad}}$ is bounded. We can also replace the coercivity of J by a slightly weaker assumption: we assume that J has a non empty, bounded, level set.

Proof. By weak* continuity of the constraint function Θ, and by weak* closedness of the set $\{0\}$, we have the weak* closedness of the set

$$\Theta^{-1}\big(\{0\}\big) = \big\{\boldsymbol{U} \in \mathcal{U} \quad | \quad \Theta\big(\boldsymbol{U}\big) = 0\big\} .$$

By weak* closedness of the set $\mathcal{U}^{\mathrm{ad}}$ we have the weak* closedness of the set $\Theta^{-1}\big(\{0\}\big) \cap \mathcal{U}^{\mathrm{ad}}$. As J is weak* l.s.c., we then have the weak* lower semicontinuity of the function

$$\tilde{J} : \boldsymbol{U} \mapsto J\big(\boldsymbol{U}\big) + \chi_{\left\{\Theta^{-1}\left(\{0\}\right) \cap \mathcal{U}^{\mathrm{ad}}\right\}}\big(\boldsymbol{U}\big) .$$

By coercivity of J on $\mathcal{U}^{\mathrm{ad}}$ (see Definition A.51), we have the coercivity of $\tilde{J}$. Thus, there exist $\varepsilon > 0$ and $r > 0$ such that

$$\forall u \in \mathcal{U}^{\mathrm{ad}}, \qquad \|u\| \geq r \Longrightarrow J\big(\boldsymbol{U}\big) \geq \inf_{\boldsymbol{V} \in \mathcal{U}} \tilde{J}\big(\boldsymbol{V}\big) + \varepsilon .$$

We obtain

$$\inf_{\boldsymbol{U} \in \mathcal{U}} \tilde{J}\big(\boldsymbol{U}\big) = \inf_{\|\boldsymbol{U}\| \leq r} \tilde{J}\big(\boldsymbol{U}\big) .$$

Moreover, Banach-Alaoglu theorem (Theorem A.24) ensures that the set

$$\big\{\boldsymbol{U} \in \mathcal{U} \quad | \quad \|\boldsymbol{U}\| \leq r\big\}$$

is weak* compact. Thus, weak* lower semicontinuity of $\tilde{J}$ ensures the existence of a minimum of $\tilde{J}$, which is finite, hence the existence of a solution to Problem (6.13).

Furthermore, continuity of the function Θ implies continuity of

$$\boldsymbol{U} \mapsto \big\langle \boldsymbol{\lambda} \,, \Theta\big(\boldsymbol{U}\big)\big\rangle ,$$

and thus weak* lower semicontinuity of

$$\boldsymbol{U} \mapsto J\big(\boldsymbol{U}\big) + \big\langle \boldsymbol{\lambda} \,, \Theta\big(\boldsymbol{U}\big)\big\rangle .$$

With the same ideas as those developed earlier, we obtain the existence of a minimum.

Strict convexity of J implies strict convexity of

$$\boldsymbol{U} \longmapsto J\big(\boldsymbol{U}\big) + \big\langle \boldsymbol{\lambda} \,, \Theta\big(\boldsymbol{U}\big)\big\rangle ,$$

and thus the announced uniqueness of its minimum. □

6.2.3 Convergence Results

We have thus shown that, under assumptions of Theorem 6.4, the Uzawa algorithm (6.14) is well defined, and that the sequence of controls $\big\{\boldsymbol{U}^{(k)}\big\}_{k \in \mathbb{N}}$ (resp. of multipliers $\big\{\boldsymbol{\lambda}^{(k)}\big\}_{k \in \mathbb{N}}$) are elements of $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}\big)$.

We now present a convergence result for algorithm (6.14).

Theorem 6.5. *Assume that*

1. $J : \mathcal{U} \to \overline{\mathbb{R}}$ *is a proper, weak* l.s.c., Gâteaux-differentiable, strongly*[2] *a-convex function,*

2. The existence of a strongly convex function on L^∞ is not clear.

2. *$\Theta : \mathcal{U} \to \mathcal{V}$ is affine, weak* continuous and κ-Lipschitz for the norm L^∞,*
3. *there exists an admissible control, i.e.*

$$\mathrm{dom}(J) \cap \mathcal{U}^{\mathrm{ad}} \cap \Theta^{-1}\big(\{0\}\big) \neq \emptyset ,$$

4. *$\mathcal{U}^{\mathrm{ad}}$ is weak* closed,*
5. *there exists an optimal multiplier (denoted $\boldsymbol{\lambda}^\sharp$) to the constraint $\Theta\big(\boldsymbol{U}\big) = 0$ in $\mathrm{L}^1(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^p)$,*
6. *the step ρ is such that $\rho < \frac{2a}{\kappa}$.*

Then, there exists a subsequence $\big(\boldsymbol{U}^{(n_k)}\big)_{k\in\mathbb{N}}$ of the sequence given by Algorithm (6.14) converging in L^∞ toward the optimal control $\boldsymbol{U}^\sharp$ of Problem (6.13).

Proof. By §6.2.2 and Theorem 6.4, the sequences $\{\boldsymbol{U}^{(k)}\}_{k\in\mathbb{N}}$ and $\{\boldsymbol{\lambda}^{(k)}\}_{k\in\mathbb{N}}$ given by (6.14) are well defined.

We first provide upper bounds and fix notations before giving the convergence result.

Upper bounds. We exploit the fact that an optimal multiplier $\boldsymbol{\lambda}^\sharp$ is in $\mathrm{L}^1(\mathbb{R}^p)$, and that $\mathrm{L}^\infty(\mathbb{R}^p)$ is dense in this space. By density of L^∞ in L^1, we have

$$\forall \varepsilon > 0, \quad \exists \boldsymbol{\lambda}_\varepsilon \in \mathrm{L}^\infty, \qquad \|\boldsymbol{\lambda}^\sharp - \boldsymbol{\lambda}_\varepsilon\|_{\mathrm{L}^1} \leq \varepsilon , \tag{6.15}$$

from which we deduce

$$\forall k \in \mathbb{N}, \qquad \|\boldsymbol{\lambda}^{(k)} - \boldsymbol{\lambda}^\sharp\|_{\mathrm{L}^1} \leq \|\boldsymbol{\lambda}^{(k)} - \boldsymbol{\lambda}_\varepsilon\|_{\mathrm{L}^1} + \varepsilon .$$

For all $\boldsymbol{\lambda} \in \mathrm{L}^\infty(\mathbb{R}^p) \subset \mathrm{L}^2(\mathbb{R}^p) \subset \mathrm{L}^1(\mathbb{R}^p)$, we have (Jensen's inequality)

$$\|\boldsymbol{\lambda}\|^2_{\mathrm{L}^1} \leq \|\boldsymbol{\lambda}\|^2_{\mathrm{L}^2} \leq \|\boldsymbol{\lambda}\|^2_{\mathrm{L}^\infty} .$$

As $(\boldsymbol{\lambda}^{(k+1)} - \boldsymbol{\lambda}_\varepsilon) \in \mathrm{L}^\infty(\mathbb{R}^p) \subset \mathrm{L}^2(\mathbb{R}^p)$, by (6.14b) we have

$$\|\boldsymbol{\lambda}^{(k+1)} - \boldsymbol{\lambda}_\varepsilon\|^2_{\mathrm{L}^2} \;=\; \|\boldsymbol{\lambda}^{(k)} - \boldsymbol{\lambda}_\varepsilon\|^2_{\mathrm{L}^2} + 2\rho\big\langle \boldsymbol{\lambda}^{(k)} - \boldsymbol{\lambda}_\varepsilon \,, \Theta(\boldsymbol{U}^{(k+1)})\big\rangle + \rho^2\big\|\Theta(\boldsymbol{U}^{(k+1)})\big\|^2_{\mathrm{L}^2} .$$

- As Θ is κ-Lipschitz and $\Theta(\boldsymbol{U}^\sharp) = 0$, we obtain

$$\big\|\Theta(\boldsymbol{U}^{(k+1)})\big\|^2_{\mathrm{L}^2} \leq \big\|\Theta(\boldsymbol{U}^{(k+1)}) - \Theta(\boldsymbol{U}^\sharp)\big\|^2_{\mathrm{L}^\infty} \leq \kappa^2\|\boldsymbol{U}^{(k+1)} - \boldsymbol{U}^\sharp\|^2_{\mathrm{L}^\infty} .$$

- From optimality conditions and strong convexity of J (see point 2 in the proof of Theorem 6.3), and using $\Theta\big(\boldsymbol{U}^\sharp\big) = 0$, we obtain

$$\big\langle \boldsymbol{\lambda}^{(k)} - \boldsymbol{\lambda}_\varepsilon \,, \Theta(\boldsymbol{U}^{(k+1)})\big\rangle \leq -a\|\boldsymbol{U}^{(k+1)} - \boldsymbol{U}^\sharp\|^2_{\mathrm{L}^\infty} + \big\langle \boldsymbol{\lambda}^\sharp - \boldsymbol{\lambda}_\varepsilon \,, \Theta(\boldsymbol{U}^{(k+1)})\big\rangle .$$

  Moreover, we have, by κ-Lipschitz continuity of Θ, and by (6.15)

$$\big\langle \boldsymbol{\lambda}^\sharp - \boldsymbol{\lambda}_\varepsilon \,, \Theta(\boldsymbol{U}^{(k+1)})\big\rangle \leq \kappa\varepsilon\|\boldsymbol{U}^{(k+1)} - \boldsymbol{U}^\sharp\|_{\mathrm{L}^\infty} .$$

Finally, we get

$$\begin{aligned}\|\boldsymbol{\lambda}^{(k+1)} - \boldsymbol{\lambda}_\varepsilon\|^2_{\mathrm{L}^2} \;\leq\;& \|\boldsymbol{\lambda}^{(k)} - \boldsymbol{\lambda}_\varepsilon\|^2_{\mathrm{L}^2} - (2a\rho - \rho^2\kappa^2)\|\boldsymbol{U}^{(k+1)} - \boldsymbol{U}^\sharp\|^2_{\mathrm{L}^\infty} \\ &+ 2\rho\kappa\varepsilon\|\boldsymbol{U}^{(k+1)} - \boldsymbol{U}^\sharp\|_{\mathrm{L}^\infty} .\end{aligned} \tag{6.16}$$

$$\begin{aligned} \alpha &:= 2a\rho - \rho^2\kappa^2 > 0 & \beta &:= \rho\kappa/\alpha > 0 \\ q^\varepsilon_k &:= \|\boldsymbol{\lambda}^{(k)} - \boldsymbol{\lambda}_\varepsilon\|^2_{\mathrm{L}^2} \geq 0 & v_k &:= \|\boldsymbol{U}^{(k+1)} - \boldsymbol{U}^\sharp\|_{\mathrm{L}^\infty} \geq 0 \end{aligned} \tag{6.17}$$

With these notations, inequality (6.16) becomes

$$q^\varepsilon_{k+1} \leq q^\varepsilon_k - \alpha v_k^2 + 2\alpha\beta\varepsilon v_k . \tag{6.18}$$

Convergence of a subsequence. Inequality (6.18) can be written as

$$q^\varepsilon_{k+1} \le q^\varepsilon_k - \alpha(v_k - \beta\varepsilon)^2 + \alpha\beta^2\varepsilon^2 \ .$$

We show that $|v_k - \beta\varepsilon| \le \sqrt{\frac{1+\alpha\beta^2}{\alpha}}\varepsilon$ holds true for an infinite number of k.

Indeed, if it were not the case, there would exists $K \in \mathbb{N}$, such that for all $k \ge K$, the inequality

$$(v_k - \beta\varepsilon)^2 > (1 + \alpha\beta^2)\varepsilon^2/\alpha \ ,$$

would hold true. Thus, we would have

$$q^\varepsilon_{k+1} \le q^\varepsilon_k - \alpha(1 + \alpha\beta^2)\varepsilon^2/\alpha + \alpha\beta^2\varepsilon^2 = q^\varepsilon_k - \varepsilon^2 \ ,$$

leading to $q^\varepsilon_k \to_k -\infty$, which is not possible as $q^\varepsilon_k \ge 0$.

Consequently, there is a subsequence $\{v_{s_\varepsilon(k)}\}_{k\in\mathbb{N}}$ that remains in a ball of center zero and radius of order ε. Thus, we can construct a subsequence $\{v_{s(k)}\}_{k\in\mathbb{N}}$ converging toward 0. Now, recalling that, by definition, $v_k = \|\boldsymbol{U}^{(k+1)} - \boldsymbol{U}^\sharp\|_{\mathrm{L}^\infty}$, we obtain the convergence of $\big(\boldsymbol{U}_{s(k)}\big)_{k\in\mathbb{N}}$ toward $\boldsymbol{U}^\sharp$ in L^∞. □

6.2.4 Difficulty to Obtain the Convergence of the Whole Sequence

The result of convergence obtained in Theorem 6.5 is not fully satisfactory, because we made quite strong assumptions (Lipschitz continuity of Θ, strong convexity of J, etc.) but only obtained the convergence of a subsequence toward an optimal solution. We now point out a difficulty if we want to improve this result.

Proposition 6.6. *Assume that* $\sup_{\varepsilon>0} q^\varepsilon_0 < \infty$. *Then the sequence* $\{\boldsymbol{U}_k\}_{k\in\mathbb{N}}$, *given by* (6.14), *converges toward* $\boldsymbol{U}^\sharp$ *in the space* $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{U})$.

Proof. Inequality (6.18) can be written

$$q^\varepsilon_{k+1} \le q^\varepsilon_k - \alpha v_k(v_k - 2\beta\varepsilon) \ . \tag{6.19}$$

Summing up these inequalities from index 0 up to index k leads, as $q^\varepsilon_{k+1} \ge 0$ by definition, to

$$0 \le q^\varepsilon_{k+1} \le q^\varepsilon_0 - \alpha \sum_{l=0}^{k} v_l(v_l - 2\beta\varepsilon) \ .$$

Since $\sup_{\varepsilon>0} q^\varepsilon_0 < \infty$, we deduce that

$$\exists M > 0, \quad \forall \varepsilon > 0, \quad \forall k > 0, \qquad \sum_{l=0}^{k} v_l(v_l - 2\beta\varepsilon) \le M \ . \tag{6.20}$$

Letting ε going to 0, we find that

$$\forall k > 0, \qquad \sum_{l=0}^{k} v_l^2 \le M \ .$$

The series of general term v_k^2 is converging, and thus the sequence $\{v_k\}_{k\in\mathbb{N}}$ converges toward zero. Thus, the sequence $\{\boldsymbol{U}_k\}_{k\in\mathbb{N}}$ converges toward $\boldsymbol{U}^\sharp$ in the space $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{U})$. □

Proposition 6.6 requires an assumption difficult to check. However if there exist an optimal multiplier $\boldsymbol{\lambda}^\sharp$ in L^2 we can take, for all $\varepsilon > 0$, $\boldsymbol{\lambda}^\varepsilon = \boldsymbol{\lambda}^\sharp$, hence, $\varepsilon \mapsto q^\varepsilon_0$ is constant.

Corollary 6.7. *Assume that*

1. $J : \mathcal{U} \to \overline{\mathbb{R}}$ *is a proper, weak* l.s.c., Gâteaux-differentiable, a-convex function,*
2. $\Theta : \mathcal{U} \to \mathcal{V}$ *is affine, weak* continuous and κ-Lipschitz for the norm* L^∞,
3. *there exists an admissible control, i.e.*
$$\mathrm{dom}(J) \cap \mathcal{U}^{\mathrm{ad}} \cap \Theta^{-1}\big(\{0\}\big) \neq \emptyset \ ,$$
4. $\mathcal{U}^{\mathrm{ad}}$ *is weak* closed,*
5. *there exists an optimal multiplier (denoted $\boldsymbol{\lambda}^\sharp$) to the constraint $\Theta\big(\boldsymbol{U}\big) = 0$ in* $\mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^p)$,
6. *the step ρ is such that $\rho < \frac{2a}{\kappa}$.*

Then, the sequence $\big(\boldsymbol{U}^{(k)}\big)_{k\in\mathbb{N}}$ by Algorithm (6.14) converges in L^∞ toward the optimal control $\boldsymbol{U}^\sharp$ of Problem (6.13).

Proof. For all $\varepsilon > 0$, we set $\boldsymbol{\lambda}^\varepsilon = \boldsymbol{\lambda}^\sharp$, hence, $\varepsilon \mapsto q_0^\varepsilon$ is constant, and Proposition 6.6 achieve the proof. □

Note that we obtain a convergence result stronger than the one obtained by Theorem 6.3 if the problem was set in L^2. Indeed, the convergence of the sequence $\big(\boldsymbol{U}^{(n)}\big)_{n\in\mathbb{N}}$ is given in L^∞ instead of L^2.

Remark 6.8. *The assumption $\sup_{\varepsilon>0} q_0^\varepsilon < \infty$ in Proposition 6.6 is quite strong. Without this assumption, Assertion (6.20) does not hold true, and we have only*

$$\forall \varepsilon > 0, \quad \exists M_\varepsilon, \quad \forall k \in \mathbb{N}, \qquad \sum_{l=0}^{k} v_k(v_k - \varepsilon) \leq M_\varepsilon \ . \tag{6.21}$$

The question is: is it enough to show the convergence of $\{v_l\}_{l\in\mathbb{N}}$ toward 0. The answer, negative, is given by Fact 6.9.

Fact 6.9. *There exists a sequence $\big\{u_n\big\}_{n\in\mathbb{N}}$ of non-negative reals such that (6.21) holds true, but that does not converges toward 0.*

Proof. Consider the sequence defined as

$$u_n = \begin{cases} 1/k & \text{if } n \in [\![n_k + 1, n_k + k^2]\!] \\ 1 & \text{if } n = n_k \end{cases} \tag{6.22}$$

where $(n_k)_{k\in\mathbb{N}}$ is defined by

$$\begin{cases} n_0 & = 1 \ , \\ n_{k+1} & = n_k + k^2 + 1, \qquad \forall k \in \mathbb{N} \ . \end{cases} \tag{6.23}$$

In other words, the sequence $\{u_k\}_{k\in\mathbb{N}}$ takes the value 1, then 1/2 four times, then 1, then 1/3 nine times, and so on. In particular, the sequence does not converge toward 0.

We now show that this sequence satisfies (6.21). For a given $\varepsilon > 0$, fix $k_0 \geq 2/\varepsilon$, and $N \in \mathbb{N}$. We have

$$\sum_{n=1}^{N} u_n(u_n - \varepsilon) = \sum_{n=1}^{n_{k_0}} u_n(u_n - \varepsilon) + \sum_{n=n_{k_0}+1}^{N} u_n(u_n - \varepsilon) \ .$$

Let $M_\varepsilon = \sum_{k=1}^{n_{k_0}} u_n(u_n - \varepsilon)$. We show that $\sum_{k=n_{k_0}+1}^{N} u_n(u_n - \varepsilon) \leq 0$.

Indeed for $k \geq k_0$, and $l \in [\![1, k^2]\!]$, we have

$$u_{n_k+l}(u_{n_k+l} - \varepsilon) = \frac{1}{k}\Big(\frac{1}{k} - \varepsilon\Big) \leq -\frac{\varepsilon}{2k} \, .$$

This inequality leads to, for $k \geq k_0$,

$$\sum_{n=n_k}^{n_{k+1}} u_n(u_n - \varepsilon) \leq 1 - \frac{k\varepsilon}{2} \leq 0 \, .$$

Summing up, if we denote by K the largest integer such that $n_K \leq N$, we have

$$\sum_{n=1}^{N} u_n(u_n - \varepsilon) = \underbrace{\sum_{k=1}^{n_{k_0}} u_n(u_n - \varepsilon)}_{M_\varepsilon} + \sum_{k=k_0}^{K} \underbrace{\sum_{l=1}^{k^2+1} u_{n_k+l}(u_{n_k+l} - \varepsilon)}_{\leq 0} + \sum_{n=n_K+1}^{N} \underbrace{u_n(u_n - \varepsilon)}_{\leq 0} \leq M_\varepsilon \, .$$

This ends the proof. □

6.3 Application to a Multistage Problem

We consider a multistage problem, comparable to the one presented in §5.3, but with some more constraint on the control. We suppose that the noise takes a finite number of values, so that the space L^∞ is finite dimensional [3].

We consider a sequence $\{\boldsymbol{W}_t\}_{t=0}^{T-1}$ of noises, with $\boldsymbol{W}_t \in \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{n_w})$, for any $t \in [\![0, T-1]\!]$. We denote by $\mathcal{F}_t$ the σ-algebra generated by the past noises

$$\mathcal{F}_t = \sigma\big(\boldsymbol{W}_0, \cdots, \boldsymbol{W}_t\big) \, ,$$

and by $\mathfrak{F}$ the induced filtration $\mathfrak{F} = \{\mathcal{F}_t\}_{t=0}^{T-1}$.

We consider the dynamical system

$$\forall t \in [\![0, T-1]\!], \qquad \boldsymbol{X}_{t+1} = f_t\big(\boldsymbol{X}_t, \boldsymbol{D}_t, \boldsymbol{W}_t\big) \, , \tag{6.24}$$

where the control process $\{\boldsymbol{D}_t\}_{t=0}^{T-1}$ is a stochastic process adapted to $\mathfrak{F}$, and for each time $t \in [\![0, T-1]\!]$, $\boldsymbol{D}_t \in \mathrm{L}^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n_d})$. The evolutions functions $f_t : \mathbb{R}^{n_x+n_d+n_w} \to \mathbb{R}^{n_x}$ are assumed to be affine in (x, d) and continuous in w.

For each time $t \in [\![0, T-1]\!]$, we consider a convex (jointly in (x, d)) cost $L_t : \mathbb{R}^{n_x+n_d+n_w} \to \mathbb{R}$, and continuous in w, and a convex final cost $K : \mathbb{R}^{n_X} \to \mathbb{R}$. We also consider linear constraint functions $\theta_t : \mathbb{R}^{n_x+n_d} \to \mathbb{R}^{n_c}$, and a $\mathfrak{F}$-adapted sequence of random variables $\{\boldsymbol{B}_t\}_{t=0}^{T-1}$ (they are stochastic target of the constraint function).

Finally, the problem reads,

$$\min_{\boldsymbol{X}, \boldsymbol{D}} \quad \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t\big(\boldsymbol{X}_t, \boldsymbol{D}_t, \boldsymbol{W}_t\big) + K(\boldsymbol{X}_T)\Big] \tag{6.25a}$$

$$s.t. \quad \boldsymbol{X}_0 = x_0 \tag{6.25b}$$

$$\boldsymbol{X}_{t+1} = f_t\big(\boldsymbol{X}_t, \boldsymbol{D}_t, \boldsymbol{W}_t\big), \tag{6.25c}$$

$$\boldsymbol{D}_t \preceq \mathcal{F}_t, \tag{6.25d}$$

$$\boldsymbol{D}_t \in \mathcal{D}_t^{\mathrm{ad}}, \tag{6.25e}$$

$$\boldsymbol{X}_t \in \mathcal{X}_t^{\mathrm{ad}}, \tag{6.25f}$$

$$\theta_t(\boldsymbol{X}_t, \boldsymbol{D}_t) = \boldsymbol{B}_t \qquad \mathbb{P} - a.s. \tag{6.25g}$$

3. This assumption is required to obtain the strong convexity of the global cost. In a finite dimensional setting, most topological consideration are equivalent. However, we choose to still dinstinguish them as we suppose finitess of the alea only to obtain the strong convexity of the cost global cost.

Remark 6.10. *Problem (6.25) differ from Problem (5.17) only through Constraints (6.25e) and (6.25f). Hence, with those constraint we do not have results of existence of an optimal multiplier. However, the existence of an optimal multiplier is an assumption of Proposition 6.11, which allow the existence of Constraints (6.25e) and (6.25f). Moreover, we believe that results in the literature (see [93–95]) could be adapted to show the existence of* L^1*-optimal multiplier even with constraint on the control (Constraint (6.25e)).*

Then, the algorithm given by (6.14) , reads [4]

$$\Big(\boldsymbol{X}^{(k+1)}, \boldsymbol{D}^{(k+1)}\Big) \in \quad \operatorname*{arg\,min}_{\boldsymbol{D},\boldsymbol{X}} \Big\{ \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t\big(\boldsymbol{X}_t, \boldsymbol{D}_t, \boldsymbol{W}_t\big) + \boldsymbol{\lambda}_t^{(k)} \cdot \theta_t\big(\boldsymbol{X}_t, \boldsymbol{D}_t\big) \Big] \Big\} , \tag{6.26a}$$

$$\boldsymbol{\lambda}_t^{(k+1)} = \quad \boldsymbol{\lambda}_t^{(k)} + \rho_t \Big(\theta_t\big(\boldsymbol{X}_t^{(k+1)}, \boldsymbol{D}^{(k+1)}\big) - \boldsymbol{B}_t \Big) , \tag{6.26b}$$

where $\big(\boldsymbol{X}, \boldsymbol{D}\big)$ satisfies constraints (6.25b)-(6.25f), that is, all constraints except the almost sure constraint dualized, i.e. constraint (6.25g).

Proposition 6.11. *Assume that,*

1. *the cost functions L_t are Gâteaux-differentiable (in (x, u)), strongly-convex (in (x, u)) functions and continuous in w;*
2. *the constraint functions $\theta_t : \mathbb{R}^{n_x+n_d} \to \mathbb{R}^{n_c}$ are affine;*
3. *the evolution functions $f_t : \mathbb{R}^{n_x+n_d+n_w} \to \mathbb{R}^{n_x}$ are affine (in (x, u, w));*
4. *the constraint sets $\mathcal{X}_t^{\mathrm{ad}}$ and $\mathcal{U}_t^{\mathrm{ad}}$ are weak* closed, convex;*
5. *there exists an admissible control, i.e. a process $\big(\boldsymbol{X}, \boldsymbol{D}\big)$ satisfying all constraints of Problem (6.25);*
6. *there exists an optimal multiplier process (denoted $\boldsymbol{\lambda}^\sharp$) to the constraint (6.25g); in $\mathrm{L}^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{Tn_c})$ (this is, satisfied if there is neither constraint (6.25e) nor constraint (6.25f)).*

Then, there exists a subsequence $\big(\boldsymbol{D}^{(n_k)}\big)_{k\in\mathbb{N}}$ of the sequence given by Algorithm (6.26) converging in L^∞ toward the optimal control of Problem (6.25).

Proof. We apply the results of §6.2.3 to Problem (6.25). We define the cost function J and constraint function Θ relative to Problem (6.25), and show the required assumptions.

First, we need to cast Problem (6.25) into the framework of Problem (6.13). We define the control $\boldsymbol{U} = \big\{\boldsymbol{D}_s\big\}_{s=0}^{T-1} \in \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{Tn_d}\big) = \mathcal{U}$.

Then, x_0 being given and constant, we define recursively the functions

$$\begin{aligned} x_t : \quad & \mathbb{R}^{T(n_d+n_w)} \longrightarrow \mathbb{R}^{n_x} \\ & \big(U, W\big) \longmapsto f_{t-1}\Big(x_{t-1}\Big(\big\{D_\tau, W_\tau\big\}_{\tau=0}^{t-2}\Big), D_{t-1}, W_{t-1} \Big) \end{aligned}$$

that maps the sequence of controls and noises toward the state. Note that the functions x_t are affine. Hence, the output of the dynamical system (6.24) can be represented by

$$\boldsymbol{X} = A\boldsymbol{U} + B\boldsymbol{W} + C ,$$

were A and B, and C are deterministic matrices.

Now we define the cost function

$$L\big(\boldsymbol{X}, \boldsymbol{U}, \boldsymbol{W}\big) = \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t\big(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t\big) \Big] ,$$

4. We use the notational convention $L_T(x, d, w) = K(x)$.

and the objective function

$$J(\boldsymbol{U}) = \mathbb{E}\Big[L\big(A\boldsymbol{U} + B\boldsymbol{W} + C, \boldsymbol{U}, \boldsymbol{W}\big)\Big] .$$

If each L_t is strongly-convex, then L is also strongly-convex in (x, u), hence J is strongly convex (for the L^2-norm, equivalent to the L^∞-norm by finiteness of the noise). By assumptions on functions L_t and f_t, the objective function J is proper and Gâteaux-differentiable. Note that J is finite on L^∞, consequently Lemma 5.16 implies that J is weak*-l.s.c..

We define the constraint function

$$\Theta = \Big(\Theta_0, \dots, \Theta_T\Big) : \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{Tn_d}\big) \to \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{Tn_c}\big) ,$$

where $\Theta_t : \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{Tn_d}\big) \to \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{n_c}\big)$ is given by

$$\Theta_t(\boldsymbol{U}) = \theta_t\Big(P_t^X\big(A\boldsymbol{U} + B\boldsymbol{W} + C\big), \boldsymbol{D}_t\Big) - \boldsymbol{B}_t = \theta_t\big(P_t^X A\boldsymbol{U}, \boldsymbol{D}_t\big) - \tilde{\boldsymbol{B}}_t , \tag{6.27}$$

where $\tilde{\boldsymbol{B}}_t$ is a $\mathcal{F}_t$-measurable random variable, and P_t^X is the projector such that $P_t^X \boldsymbol{X} = \boldsymbol{X}_t$. In particular Θ is affine. Note that the functions x_t and θ_t are affine on a finite dimensional space, and hence Lipschitz. Consequently, functions Θ_t (and thus Θ) are L^∞-Lipschitz. Moreover, Corollary 5.13 gives the weak* continuity of the constraint function Θ.

We now construct the set $\mathcal{U}^{\mathrm{ad}}$ of admissible controls. Let $\mathcal{X}^{\mathrm{ad}}$ be the Cartesian product $\mathcal{X}_0^{\mathrm{ad}} \times \cdots \times \mathcal{X}_T^{\mathrm{ad}}$, and $\mathcal{D}^{\mathrm{ad}}$ be the Cartesian product $\mathcal{D}_0^{\mathrm{ad}} \times \cdots \times \mathcal{D}_{T-1}^{\mathrm{ad}}$. The linear mappings $\boldsymbol{U} \mapsto A\boldsymbol{U}$, $\boldsymbol{U} \mapsto P_t^U \boldsymbol{U}$ (where $P_t^U \boldsymbol{U} = \boldsymbol{D}_t$) and the constraint functions Θ_t are weak dual continuous (see Corollary 5.13).

Note that the linear mappings $\boldsymbol{X} \mapsto P_t^X\big(A\boldsymbol{U} + B\boldsymbol{W} + C\big)$ are weakly* continuous (see Corollary 5.13). Hence, for $t \in [\![1, T]\!]$, the set

$$\Big\{\boldsymbol{U} \in \mathcal{U} \quad \big| \quad P_t^X\big(A\boldsymbol{U} + B\boldsymbol{W} + C\big) \in \mathcal{X}_t\Big\} ,$$

is weak* closed convex as the the inverse image of a weak* closed convex set by a weak* continuous affine function. Consequently, the set

$$\mathcal{U}_{\mathcal{X}}^{\mathrm{ad}} = \Big\{\boldsymbol{U} \in \mathcal{U} \quad \big| \quad P_t^X\big(A\boldsymbol{U} + B\boldsymbol{W} + C\big) \in \mathcal{X}_t, \quad \forall t \in [\![1, T]\!]\Big\} ,$$

is weak* closed convex as an intersection of such sets.

We denote $\mathcal{N}_d$ the set of essentially bounded, $\mathfrak{F}$-adapted processes with value in $\mathbb{R}^{n_d}$. It is the set $\mathcal{N}^{n_d}$, where $\mathcal{N}$ is defined in (5.12). By Corollary 5.14, the set $\mathcal{N}_d$ is weak* closed convex. In a nutshell, a control $\boldsymbol{U}$ satisfies:

- constraint (6.25e) if it is an element of $\mathcal{D}^{\mathrm{ad}}$;
- constraint (6.25f) if it is an element of $\mathcal{U}_{\mathcal{X}}^{\mathrm{ad}}$;
- constraint (6.25d) if it is an element of $\mathcal{N}_d$.

Hence, the constraint set $\mathcal{U}^{\mathrm{ad}}$ given by

$$\mathcal{U}^{\mathrm{ad}} = \mathcal{D}^{\mathrm{ad}} \cap \mathcal{U}_{\mathcal{X}}^{\mathrm{ad}} \cap \mathcal{N}_d ,$$

is a weak* closed convex set.

Finally, by Corollary 5.27, if $\mathcal{U}^{\mathrm{ad}} = \mathcal{N}_d$, we have optimal multipliers in L^1 for constraints (6.25g).

With those notations, Problem (6.25) reads

$$\begin{aligned} \min_{\boldsymbol{U} \in \mathcal{U}^{\mathrm{ad}}} \quad & J(\boldsymbol{U}) . \\ s.t. \quad & \Theta(\boldsymbol{U}) = 0 \end{aligned}$$

Moreover Algorithm (6.14) correspond to Algorithm (6.26).

Hence, for ρ small enough, all the assumptions in Theorem 6.5 are satisfied; this ends the proof. $\square$

Remark 6.12. *Note that, by Lemma B.3, it is easy to see that, if $\boldsymbol{\lambda}^\sharp$ is an optimal multiplier for constraints (6.25g), then so is its $\mathfrak{F}$-adapted part, that is, the process $\boldsymbol{\mu}^\sharp$ where*

$$\forall t \in [\![0, T]\!], \quad \boldsymbol{\mu}_t^\sharp = \mathbb{E}\big[\boldsymbol{\lambda}_t^\sharp \mid \mathcal{F}_t\big] .$$

Interestingly, the multiplier in $\boldsymbol{\lambda}^{(k)}$ in Algorithm (6.26) is an essentially bounded, $\mathfrak{F}$-adapted stochastic process.

However, we cannot write a Dynamic Programming equation for Problem (6.26a) with the state $\boldsymbol{X}$. Indeed, the multiplier $\boldsymbol{\lambda}^{(k)}$ should be seen as a correlated, $\mathfrak{F}$-adapted noise. Hence, the natural state is the past noises $\{\boldsymbol{W}_s\}_{s=0}^t$, and Dynamic Programming methods are numerically untractable to solve Problem (6.26a).

In Chapter 8, we will present a method where the multiplier is approximated by its conditional expectation with respect to a given information process $\boldsymbol{Y}$, following a dynamic $\boldsymbol{Y}_{t+1} = \tilde{f}_t\big(\boldsymbol{Y}_t, \boldsymbol{W}_t\big)$. This allows to use Dynamic Programming with an extended state $\big(\boldsymbol{X}_t, \boldsymbol{Y}_t\big)$ to solve the minimization part (equation (6.26a)) of Uzawa algorithm.

Conclusion

We have provided conditions ensuring convergence of a subsequence of $\big\{u^{(k)}\big\}_{k\in\mathbb{N}}$, for the Uzawa algorithm in $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}\big)$. Our key assumption is the existence of a saddle point for the Lagrangian in the $\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$ pairing. Work remains to be done on the subject. Indeed, the strong convexity assumption on the objective function usually ensures the convergence of the whole primal sequence $\big\{u^{(k)}\big\}_{k\in\mathbb{N}}$ toward the optimal value. With the bounds that we have derived, we were only able to obtain the convergence of a subsequence of $\big\{u^{(k)}\big\}_{k\in\mathbb{N}}$. Tighter bounds might give better convergence results, and alternative schemes of proof should be investigated.

Moreover, we have made an abstract weak* continuity (or lower-semicontinuity) assumption; we should study its potential of applicability.

Finally, we have restricted ourselves to the case of equality constraints; more generic constraint require a careful look at the projection step in the Uzawa algorithm.

In §6.3, we have applied the Uzawa algorithm to a multistage process. However, we have seen that the minimization part of the Uzawa algorithm is not straightforward in this case. In the final part of this manuscript, we will develop and adapt this idea, in order to apply the Uzawa algorithm for the spatial decomposition of stochastic optimization problems.

Part III

Stochastic Spatial Decomposition Methods

Chapter 7

Epiconvergence of Relaxed Stochastic Problems

Truth is much too complicated to allow anything but approximations.

John von Neumann

Contents

At the end of Chapter 6 we saw that a price decomposition scheme over the coupling spatial constraints does not leads to tractable subproblems. In Chapter 8, we propose a tractable price decomposition scheme over an approximation of the original problem.

In this Chapter, we study the approximation required. Roughly, this approximation relax an almost sure constraint into a conditional expectation constraint. Were the conditioning is done with respect to a σ-algebra $\mathcal{F}_n$. We study the convergence of a sequence of approximated problem when the σ-algebra converges.

Introduction

Stochastic optimization problems often consist in minimizing a cost over a set of random variables. If the set of events is infinite, the minimization is done over an infinite dimensional space. Consequently there is a need for approximation. We are interested in the approximation of almost sure constraints, say $\theta(\boldsymbol{U}) = 0$ almost surely (a.s.), by a conditional expectation constraint like $\mathbb{E}\big[\theta(\boldsymbol{U}) \bigm| \mathcal{F}_n\big] \geq 0$ a.s.

Consider the following problem,

$$\min_{\boldsymbol{U}\in\mathcal{U}} \quad J(\boldsymbol{U}) \ , \tag{7.1a}$$

$$s.t. \quad \theta(\boldsymbol{U}) = 0 \quad \text{a.s.} \ , \tag{7.1b}$$

where the set of controls $\mathcal{U}$ is a set of random variables over a probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$. If Ω is not finite, $\mathcal{U}$ may be of infinite dimension. Moreover the constraint (7.1b) is a functional constraint that can roughly be seen as an infinity of constraints. For tractability purposes we consider approximations of this problem. In order to give theoretical results for the approximations of Problem (7.1) the right notion of convergence is epi-convergence. Indeed, under some additional technical conditions, the epi-convergence ensures the convergence of both the optimal value and the optimal solutions.

One way of approximating Problem (7.1) consists in approximating the probability $\mathbb{P}$. Roughly speaking the Sample Average Approximation procedure consist in simulating a set of scenarios under the real probability $\mathbb{P}$. Then we solve Problem (7.1) under the empirical probability on the set of simulated scenarios. In this literature (see [43], [59]) the authors are interested in problems where the controls are deterministic. However other epi-convergence results have been shown for more general spaces of controls, including spaces of random variables or random processes (see [120] and references therein, as well as [74], [76], [75]). More generally, the idea of discretizing or quantizing the set Ω, for example by use of finite scenario trees has been largely studied in the field of Stochastic Programming (see [110] for a thorough presentation).

Instead of approximating the probability space we propose a way to approximate constraints, especially almost sure constraints. The main idea is to replace a constraint by its conditional expectation with respect to (w.r.t.) a σ-algebra $\mathcal{B}$. This is in some sense an aggregation of constraints. This approximation appears when considering duality schemes for dynamic stochastic optimization problem.

More precisely, we relax the almost sure constraint (7.1b) by replacing it by its conditional expectation, i.e.

$$\mathbb{E}\big[\theta(\boldsymbol{U}) \bigm| \mathcal{B}\big] = 0\ . \tag{7.2}$$

If λ is an integrable optimal multiplier for Constraint (7.1b), then $\lambda_{\mathcal{B}} = \mathbb{E}\big[\lambda \bigm| \mathcal{B}\big]$ is an optimal multiplier for Constraint (7.2). This leads to look for $\mathcal{B}$-measurable multiplier, which may authorize decomposition-coordination methods where the sub-problems are easily solvable. This is presented in Chapter 8.

The chapter is organized as follows. §7.1 presents the general form of the problem considered and its approximation. §7.2 shows, after a few recalls on convergence notions of random variables, functions and σ-algebras, conditions on the sequence of approximate problems guaranteeing its convergence toward the initial problem. The main assumptions are the Kudo's convergence of σ-algebra, and the continuity - as operators - of the constraint function Θ and objective function J. Finally §7.3 gives some examples of continuous objective and constraint functions that represent usual stochastic optimization problems. Finally §7.4 presents a decomposition-coordination algorithm using this type of relaxation and developed in the Chapter 8.

7.1 Problem Statement

We consider a probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ and a topological spaces of controls $\mathcal{U}$. Let $\mathcal{V}$ be the spaces of random variables with value in a Banach $\mathbb{V}$ with finite moment of order $p \in [1, \infty)$, denoted $\mathcal{V} = \mathrm{L}^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{V})$.

We consider now a stochastic optimization problem

$$\min_{\boldsymbol{U} \in \mathcal{U}} \quad J(\boldsymbol{U})\ , \tag{7.3a}$$

$$s.t. \quad \Theta(\boldsymbol{U}) \in -C\ , \tag{7.3b}$$

with J mapping $\mathcal{U}$ into $\mathbb{R} \cup \{+\infty\}$, and Θ mapping $\mathcal{U}$ into $\mathcal{V}$. We assume that $C \subset \mathcal{V}$ is a closed convex cone of $\mathcal{V}$, and that $\mathbb{V}$ is a separable Banach space with separable dual (the fact that C is a cone is not essential for our results).

To give an example of cost operator, assume that $\mathcal{U} \subset L^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{U}\big)$, where $\mathbb{U}$ is a Banach. The usual choice for the criterion is the expected cost $J(\boldsymbol{U}) := \mathbb{E}\big[j(\boldsymbol{U})\big]$, for a suitable cost function $j : \mathbb{U} \to \mathbb{R}$. Other choices could be risk measures (see [6] for example) like Conditional-Value-at-Risk (see [90] for a definition), worst-case or robust approaches. The constraint operator Θ cover various cases, for example

- almost sure constraint: $\Theta\big(\boldsymbol{U}\big)(\omega) := \theta\big(\boldsymbol{U}(\omega)\big)$, where θ maps $\mathbb{U}$ into $\mathbb{V}$ and $\theta\big(\boldsymbol{U}\big) \in -C$ is realized almost surely;
- measurability constraint: $\Theta\big(\boldsymbol{U}\big) := \mathbb{E}\big[\boldsymbol{U} \bigm| \mathcal{B}\big] - \boldsymbol{U}$, with $C = \{0\}$, expresses that $\boldsymbol{U}$ is measurable with respect to the σ-algebra $\mathcal{B}$, that is, $\mathbb{E}\big[\boldsymbol{U} \bigm| \mathcal{B}\big] = \boldsymbol{U}$;
- risk constraint: $\Theta\big(\boldsymbol{U}\big) := \rho(\boldsymbol{U}) - a$, where ρ is a conditional risk measure, and C is the cone of positive random variables.

We introduce a stability assumption of the set C that will be made throughout this paper.

Definition 7.1. *We consider a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ of sub-fields of $\mathcal{F}$. The closed convex cone C is said to be* stable w.r.t. $\big(\mathcal{F}_n\big)_{n\in\mathbb{N}}$, *if for all $n \in \mathbb{N}$ we have*

$$\forall \boldsymbol{V} \in C, \qquad \mathbb{E}[\boldsymbol{V} \mid \mathcal{F}_n] \in C \; .$$

A first widely used example would be $C = \{0\}$, or more generally any deterministic closed convex cone, another example would be the set of almost surely positive random variables.

We now consider the following relaxation of Problem (7.3)

$$\min_{\boldsymbol{U}\in\mathcal{U}} \quad J(\boldsymbol{U}) \; , \tag{7.4a}$$

$$s.t. \quad \mathbb{E}\big[\Theta(\boldsymbol{U}) \bigm| \mathcal{F}_n\big] \in -C \; , \tag{7.4b}$$

where C is assumed to be stable w.r.t the sequence $\big(\mathcal{F}_n\big)_{n\in\mathbb{N}}$.

We denote the set of admissible controls of Problem (7.3)

$$\mathcal{U}^{ad} := \big\{\boldsymbol{U} \in \mathcal{U} \quad \big| \quad \Theta(\boldsymbol{U}) \in -C\big\} \; , \tag{7.5}$$

and the corresponding set of admissible controls of Problem (7.4)

$$\mathcal{U}^{ad}_n := \big\{\boldsymbol{U} \in \mathcal{U} \quad \big| \quad \mathbb{E}\big[\Theta(\boldsymbol{U}) \bigm| \mathcal{F}_n\big] \in -C\big\} \; . \tag{7.6}$$

Problems (7.3) and (7.4) can also be written [1] as

$$\min_{\boldsymbol{U}\in\mathcal{U}} \quad \underbrace{J(\boldsymbol{U}) + \chi_{\mathcal{U}^{ad}}(\boldsymbol{U})}_{:=\tilde{J}(\boldsymbol{U})} \; , \tag{7.7}$$

and

$$\min_{\boldsymbol{U}\in\mathcal{U}} \quad \underbrace{J(\boldsymbol{U}) + \chi_{\mathcal{U}^{ad}_n}(\boldsymbol{U})}_{:=\tilde{J}_n(\boldsymbol{U})} \; . \tag{7.8}$$

Note that we have $\mathcal{F}_n \subset \mathcal{F}$, and that C is stable w.r.t $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$, thus $\mathcal{U}^{ad} \subset \mathcal{U}^{ad}_n$: Problem (7.4) is a relaxation of the original Problem (7.3) as it has the same objective function but a wider set of admissible controls.

1. We use the notation χ_A for the characteristic function of A, that is the function such that $\chi_A(x) = 0$ if $x \in A$, and $\chi_A(x) = +\infty$ elsewhere.

Replacing an almost sure constraint by a conditional expectation constraint is similar to an aggregation of constraints. For example consider a finite set $\Omega = \{\omega_i\}_{i \in [\![1,N]\!]}$ [2], with a probability $\mathbb{P}$ such that, for all $i \in [\![1,N]\!]$, we have $\mathbb{P}(\omega_i) = p_i > 0$. Consider a partition $\mathcal{B} = \{B_l\}_{l \in [\![1,|\mathcal{B}|]\!]}$ of Ω, and the σ-algebra $\mathcal{F}_{\mathcal{B}}$ generated by the partition $\mathcal{B}$. Assume that $C = \{0\}$, then the relaxation presented consists in replacing the constraint

$$\theta(\boldsymbol{U}) = 0 \qquad \mathbb{P}\text{-a.s.}$$

which is equivalent to N constraints

$$\forall i \in [\![1,N]\!], \qquad \theta(\boldsymbol{U}(\omega_i)) = 0 \; ,$$

by the collection of $|\mathcal{B}| \leq N$ (where $|\mathcal{B}|$ is the number of sets in the partition $\mathcal{B}$) constraints

$$\forall l \in [\![1,|\mathcal{B}|]\!], \qquad \sum_{i \in B_l} p_i \theta(\boldsymbol{U}(\omega_i)) = 0 \; .$$

7.2 Epiconvergence Result

In this section we show the epiconvergence of the sequence of approximated cost functions $\{\tilde{J}_n\}_{n \in \mathbb{N}}$ (defined in (7.8)) towards $\tilde{J}$ (defined in (7.7)). First, we recall some results on convergence of random variables, epiconvergence of functions and convergence of σ-algebras. Moreover a technical result is required.

7.2.1 Preliminaries

Assume that $p \in [1,+\infty)$ and denote $q \in (1,+\infty]$ such that $1/q + 1/p = 1$. Recall that $\mathbb{V}$ is a separable Banach space with separable dual $\mathbb{V}^*$.

Convergence of random variables

A sequence $(\boldsymbol{X}_n)_{n \in \mathbb{N}}$ of $\mathrm{L}^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{V})$ is said to converges strongly toward $\boldsymbol{X} \in \mathrm{L}^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{V})$, and denoted $\boldsymbol{X}_n \to_{\mathrm{L}^p} \boldsymbol{X}$ if

$$\lim_{n\to\infty} \mathbb{E}\big[\big\| \boldsymbol{X}_n - \boldsymbol{X} \big\|_{\mathbb{V}}^p \big] = 0 \; .$$

A sequence $(\boldsymbol{X}_n)_{n \in \mathbb{N}}$ of $\mathrm{L}^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{V})$ is said to weakly converges toward $\boldsymbol{X} \in \mathrm{L}^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{V})$, and denoted $\boldsymbol{X}_n \rightharpoonup_{\mathrm{L}^p} \boldsymbol{X}$ if

$$\forall \boldsymbol{X}' \in \mathrm{L}^q(\Omega,\mathcal{F},\mathbb{P};\mathbb{V}^*), \qquad \lim_{n\to\infty} \mathbb{E}\big[\langle \boldsymbol{X}_n - \boldsymbol{X}, \boldsymbol{X}' \rangle_{\mathbb{V},\mathbb{V}^*} \big] = 0 \; .$$

For more details we refer the reader to [99].

Epiconvergence of functions

We first recall the definition of the Painlevé-Kuratowski convergence of sets. Let E be a topological space and consider a sequence $\{A_n\}_{n \in \mathbb{N}}$ of subsets of E. Then the inner limit of $\{A_n\}_{n \in \mathbb{N}}$ is the set of accumulation points of any sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in A_n$, i.e,

$$\underline{\lim}_n A_n = \{ x \in E \quad | \quad \forall n \in \mathbb{N}, \quad x_n \in A_n, \quad \lim_{k\to\infty} x_n = x \} \; , \tag{7.9}$$

2. We denote by $[\![a,b]\!]$ the set of all integers between a and b.

and the outer limit of $\{A_n\}_{n\in\mathbb{N}}$ is the set of accumulation points of any sub-sequence $(x_{n_k})_{k\in\mathbb{N}}$ of a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in A_n$, i.e,

$$\overline{\lim}_n A_n = \{x \in E \quad | \quad \exists \{n_k\}_{k\in\mathbb{N}}, \quad \forall k \in \mathbb{N}, \quad x_{n_k} \in A_{n_k}, \quad \lim_{k\to\infty} x_{n_k} = x\} \, . \tag{7.10}$$

We say that $\{A_n\}_{n\in\mathbb{N}}$ converges toward A in the Painlevé-Kuratowski sense if

$$A = \overline{\lim}_n A_n = \underline{\lim}_n A_n \, .$$

A sequence $\{J_n\}_{n\in\mathbb{N}}$ of functions taking value into $\mathbb{R} \cup \{+\infty\}$ is said to epi-converge toward a function J if the sequence of epigraphs of J_n converges toward the epigraph of J, in the Painlevé-Kuratowski sense. For more details and properties of epi-convergence, see Rockafellar-Wets [96] in finite dimension, and Attouch [8] for infinite dimension.

Convergences of σ-algebras

Let $\mathcal{F}$ be a σ-algebra and $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ a sequence of sub-fields of $\mathcal{F}$. It is said that the sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ Kudo-converges toward the σ-algebra $\mathcal{F}_\infty$, and denoted $\mathcal{F}_n \to \mathcal{F}_\infty$, if for each set $F \in \mathcal{F}$, $\Big(\mathbb{E}\big[1_F \bigm| \mathcal{F}_n\big]\Big)_{n\in\mathbb{N}}$ converges in probability toward $\mathbb{E}\big[1_F \bigm| \mathcal{F}_\infty\big]$.

In [62], Kudo shows that $\mathcal{F}_n \to \mathcal{F}_\infty$ if and only if for each integrable random variable $\boldsymbol{X}$, $\mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}_n\big]$ converges in L^1 toward $\mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}_\infty\big]$. In [82], Piccinini extends this result to the convergence in L^p in the strong or weak sense with the following lemma.

Lemma 7.2. *Let $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ be a probability space and $\big(\mathcal{F}_n\big)_{n\in\mathbb{N}}$ be a sequence of sub-σ-algebras of $\mathcal{F}$. The following statements are equivalent:*

1. *$\mathcal{F}_n \to \mathcal{F}_\infty$,*
2. *$\forall \boldsymbol{X} \in \mathrm{L}^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{V})$, $\quad \mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}_n\big] \to_{\mathrm{L}^p} \mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}_\infty\big]$,*
3. *$\forall \boldsymbol{X} \in \mathrm{L}^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{V})$, $\quad \mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}_n\big] \rightharpoonup_{\mathrm{L}^p} \mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}_\infty\big]$.*

We have the following useful proposition where both the random variable and the σ-algebra are parametrized by n.

Proposition 7.3. *Assume that $\mathcal{F}_n \to \mathcal{F}_\infty$, and $\boldsymbol{X}_n \to_{\mathrm{L}^p} \boldsymbol{X}$ (resp. $\boldsymbol{X}_n \rightharpoonup_{\mathrm{L}^p} \boldsymbol{X}$) then $\mathbb{E}\big[\boldsymbol{X}_n \bigm| \mathcal{F}_n\big] \to_{\mathrm{L}^p} \mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}_\infty\big]$ (resp. $\mathbb{E}\big[\boldsymbol{X}_n \bigm| \mathcal{F}_n\big] \rightharpoonup_{\mathrm{L}^p} \mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}_\infty\big]$).*

Proof. The weak-limit case is detailed in [82]. We show the strong convergence case. If $\boldsymbol{X}_n \to_{\mathrm{L}^p} \boldsymbol{X}$, then

$$\begin{aligned} ||\mathbb{E}\big[\boldsymbol{X}_n \bigm| \mathcal{F}_n\big] - \mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}\big]||_{\mathrm{L}^p} \leq & \, ||\mathbb{E}\big[\boldsymbol{X}_n \bigm| \mathcal{F}_n\big] - \mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}_n\big]||_{\mathrm{L}^p} \\ & + ||\mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}_n\big] - \mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}\big]||_{\mathrm{L}^p} \end{aligned}$$

As the conditional expectation is a contraction operator, we have

$$||\mathbb{E}\big[\boldsymbol{X}_n \bigm| \mathcal{F}_n\big] - \mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}_n\big]||_{\mathrm{L}^p} \leq ||\boldsymbol{X}_n - \boldsymbol{X}||_{\mathrm{L}^p} \to 0 \, .$$

Moreover we have

$$||\mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}_n\big] - \mathbb{E}\big[\boldsymbol{X} \bigm| \mathcal{F}\big]||_{\mathrm{L}^p} \to 0$$

by Lemma 7.2, hence the result. □

We finish by a few properties on the Kudo-convergence of σ-algebras (for more details we refer to [62] and [31]):

1. the topology associated with the Kudo-convergence is metrizable;
2. the set of σ-fields generated by the partitions of Ω is dense in the set of all σ-algebras;
3. if a sequence of random variables $(\boldsymbol{X}_n)_{n\in\mathbb{N}}$ converges in probability toward $\boldsymbol{X}$ and for all $n \in \mathbb{N}$ we have $\sigma(\boldsymbol{X}_n) \subset \sigma(\boldsymbol{X})$, then we have the Kudo-convergence of $\big(\sigma(\boldsymbol{X}_n)\big)_{n\in\mathbb{N}}$ toward $\sigma(\boldsymbol{X})$.

7.2.2 Main result

Recall that $\mathcal{U}$ is endowed with a topology τ, and that $\mathcal{V} = \mathrm{L}^p\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{V}\big)$, with $p \in [1, \infty)$.

Theorem 7.4. *Let $\mathcal{V}$ be endowed with the strong or weak topology. Assume that C is stable w.r.t $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. If the two mappings Θ and J are continuous, and if $(\mathcal{F}_n)_{n\in\mathbb{N}}$ Kudo-converges toward $\mathcal{F}$, then $\{\tilde{J}_n\}_{n\in\mathbb{N}}$ (defined in (7.7)) epi-converges toward $\tilde{J}$ (defined in (7.8)).*

Note that $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is not assumed to be a filtration, and that $\mathcal{F}_n$ is not assumed to be finite.

Proof. To prove the epi-convergence of $\{\tilde{J}_n\}_{n\in\mathbb{N}}$ toward $\tilde{J}$ it is sufficient to show that $\mathcal{U}_n^{ad}$ (defined in (7.6)) converges toward $\mathcal{U}^{ad}$ (defined in (7.5)) in the Painlevé-Kuratowski sense. Indeed it implies the epiconvergence of $(\chi_{\mathcal{U}_n^{ad}})_{n\in\mathbb{N}}$ toward $\chi_{\mathcal{U}^{ad}}$, and adding a continuous function preserves the epi-convergence (Attouch [8, Th 2.15]).

By stability of C w.r.t. $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ we have that, for all $n \in \mathbb{N}$, $\mathcal{U}^{ad} \subset \mathcal{U}_n^{ad}$ and thus $\mathcal{U}^{ad} \subset \liminf_n \mathcal{U}_n^{ad}$ (for any $x \in \mathcal{U}^{ad}$ take the constant sequence equal to x).

We now show that $\mathcal{U}^{ad} \supset \limsup_n \mathcal{U}_n^{ad}$. Let $\boldsymbol{U}$ be an element of $\limsup_n \mathcal{U}_n^{ad}$. By Definition (7.10), there is a sequence $\big\{\boldsymbol{U}_{n_k}\big\}_{k\in\mathbb{N}}$ that τ-converges to $\boldsymbol{U}$, such that for all $k \in \mathbb{N}$, $\mathbb{E}\big(\Theta(\boldsymbol{U}_{n_k})|\mathcal{F}_{n_k}\big) \in -C$. As Θ is continuous, we have $\Theta(\boldsymbol{U}_{n_k}) \to \Theta(\boldsymbol{U})$ strongly (resp. weakly) in L^p. Moreover we have that $\mathcal{F}_{n_k} \to \mathcal{F}$, and consequently by Lemma 7.3,

$$\mathbb{E}\big[\Theta(\boldsymbol{U}_{n_k}) \bigm| \mathcal{F}_{n_k}\big] \to_{L^p} \mathbb{E}\big(\Theta(\boldsymbol{U})|\mathcal{F}\big) = \Theta(U) \ .$$

Thus $\Theta(\boldsymbol{U})$ is the limit of a sequence in $-C$. By closedness of C (weak and strong as C is convex [3]), we have that $\Theta(\boldsymbol{U}) \in -C$ and thus $\boldsymbol{U} \in \mathcal{U}^{ad}$. □

The practical consequences for the convergence of the approximation (7.4) toward the original Problem 7.3 is given in the following Corollary.

Corollary 7.5. *Assume that $\mathcal{F}_n \to \mathcal{F}$, and that J and Θ are continuous. Then the sequence of Problems (7.4) approximates Problem (7.3) in the following sense. If $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ is a sequence of control such that for all $n \in \mathbb{N}$,*

$$\tilde{J}_n(\boldsymbol{U}_n) < \inf_{\boldsymbol{U}\in\mathcal{U}} \tilde{J}_n(\boldsymbol{U}) + \varepsilon_n, \ \text{where} \ \lim_n \varepsilon_n = 0 \ ,$$

then, for every converging sub-sequence $(\boldsymbol{U}_{n_k})_{k\in\mathbb{N}}$, we have

$$\tilde{J}\big(\lim_k \boldsymbol{U}_{n_k}\big) = \min_{\boldsymbol{U}\in\mathcal{U}} \tilde{J}(\boldsymbol{U}) = \lim_k \tilde{J}_{n_k}\big(\boldsymbol{U}_{n_k}\big) \ .$$

Moreover if $\big\{\mathcal{F}_n\big\}_{n\in\mathbb{N}}$ is a filtration, then the convergences are monotonous in the sense that the optimal value is non-decreasing in n.

Proof. The convergence result is a direct application of Attouch [8, Th. 1.10, p. 27]. Monotonicity is given by the fact that, if $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is a filtration, then for $n > m$ then $\mathcal{U}_n^{ad} \subset \mathcal{U}_m^{ad}$. □

3. if C is non-convex we need it to be sequentially closed.

7.2.3 Dynamic Problem

We extend Problem (7.3) into the following dynamic problem

$$\begin{aligned} \min_{\boldsymbol{U}\in\mathcal{U}} \quad & J(\boldsymbol{U}) \ , \\ s.t. \quad & \Theta_t(\boldsymbol{U}_t) \in -C_t \quad \forall t \in [\![1, T]\!] \ , \\ & \boldsymbol{U}_t \preceq \mathcal{F}_t \ , \end{aligned} \tag{7.11}$$

where $\boldsymbol{U}_t \preceq \mathcal{F}_t$ stands for "$\boldsymbol{U}_t$ is $\mathcal{F}_t$-measurable". Here $\boldsymbol{U}$ is a stochastic process of control $(\boldsymbol{U}_t)_{t\in[\![1,T]\!]}$ defined on $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ with value in $\mathbb{U}$. We have T constraints operators Θ_t taking values in $\mathrm{L}^p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{V}_t)$, where $(\mathcal{F}_t)_{t\in[\![1,T]\!]}$ is a sequence of σ-algebra. Note that $(\mathcal{F}_t)_{t\in[\![1,T]\!]}$ is not necessarily a filtration. Then, for each $t \in [\![1, T]\!]$ we define a sequence of approximating σ-algebra $(\mathcal{F}_{n,t})_{n\in\mathbb{N}}$. For all $t \in [\![1, T]\!]$, C_t is a closed convex cone stable w.r.t $\big(\mathcal{F}_{n,t}\big)_{n\in\mathbb{N}}$.

Finally we consider the sequence of approximated problem

$$\begin{aligned} \min \quad & J(\boldsymbol{U}) \ , \\ s.t. \quad & \mathbb{E}\big[\Theta_t(\boldsymbol{U}_t) \bigm| \mathcal{F}_{n,t}\big] \in -C_t \quad \forall t \in [\![1, T]\!] \ . \end{aligned} \tag{7.12}$$

Furthermore we denote

$$\mathcal{U}_t^{ad} := \big\{\boldsymbol{U}_t \in \mathcal{U}_t \quad \big| \quad \Theta_t(\boldsymbol{U}_t) \in -C_t\big\} \ ,$$

and

$$\mathcal{U}_{n,t}^{ad} := \big\{\boldsymbol{U}_t \in \mathcal{U}_t \quad \big| \quad \mathbb{E}\big[\Theta(\boldsymbol{U}_t) \bigm| \mathcal{F}_{n,t}\big] \in -C_t\big\} \ .$$

We define the set of admissible controls for the original problem

$$\mathcal{U}^{ad} = \mathcal{U}_0^{ad} \times \cdots \times \mathcal{U}_T^{ad} \ ,$$

and accordingly for the relaxed problem

$$\mathcal{U}_n^{ad} = \mathcal{U}_{n,0}^{ad} \times \cdots \times \mathcal{U}_{n,T}^{ad} \ .$$

In order to show the convergence of the approximation proposed here, we consider the functions

$$\tilde{J}(\boldsymbol{U}) = J\big(\boldsymbol{U}\big) + \chi_{\mathcal{U}^{ad}}(\boldsymbol{U}) \ ,$$

and

$$\tilde{J}_n(\boldsymbol{U}) = J\big(\boldsymbol{U}\big) + \chi_{\mathcal{U}_n^{ad}}(\boldsymbol{U}) \ ,$$

and show the epi-convergence of $\tilde{J}_n$ to $\tilde{J}$. The interaction between the different time-step are integrated in the objective function J.

Theorem 7.6. *If Θ and J are continuous, and if for all $t \in [\![1, T]\!]$, $(\mathcal{F}_{t,n})_{n\in\mathbb{N}}$ Kudo-converges to $\mathcal{F}_t$, then $\big(\tilde{J}_n\big)_{n\in\mathbb{N}}$ epi-converges to $\tilde{J}$.*

Proof. The proof is deduced from the one of Theorem 7.4. By following the same steps we obtain the Painlevé-Kuratowski convergence of $\mathcal{U}_{n,t}^{ad}$ to $\mathcal{U}_t^{ad}$, and thus the convergence of their Cartesian products. □

7.3 Examples of Continuous Operators

The continuity of J and Θ as operators required in Theorem 7.4 is an abstract assumption. This section presents conditions for some classical constraint and objective functions to be representable by continuous operators. Before presenting those results we show a technical lemma that allows us to prove convergence for the topology of convergence in probability by considering sequences of random variables converging almost surely.

7.3.1 A technical Lemma

Lemma 7.7. *Let $\Theta : E \to F$, where $(E, \tau_{\mathbb{P}})$ is a space of random variables endowed with the topology of convergence in probability, and (F, τ) is a topological space. Assume that Θ is such that if $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ converges almost surely toward $\boldsymbol{U}$, then $\Theta(\boldsymbol{U}_n) \to_\tau \Theta(\boldsymbol{U})$. Then Θ is a continuous operator from $(E, \tau_{\mathbb{P}})$ into (F, τ).*

Proof. We recall first a well known property (see for example [44, Th 2.3.3]). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a topological space. If from any sub-sequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ we can extract a sub-sub-sequence $\{x_{\sigma(n_k)}\}_{k\in\mathbb{N}}$ converging to x^*, then $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Indeed suppose that $\{x_n\}_{n\in\mathbb{N}}$ does not converges toward x^*. Then there exist an open set $\mathcal{O}$ containing x^* and a sub-sequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, $x_{n_k} \notin \mathcal{O}$, and no sub-sub-sequence can converges to x^*, hence a contradiction.

Let $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ be a sequence converging in probability to $\boldsymbol{U}$. We consider the sequence $\{\Theta(\boldsymbol{U}_n)\}_{n\in\mathbb{N}}$ in F. We choose a sub-sequence $\{\Theta(\boldsymbol{U}_{n_k})\}_{k\in\mathbb{N}}$. By assumption $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ converges in probability toward $\boldsymbol{U}$, thus we have $\boldsymbol{U}_{n_k} \to_{\mathbb{P}} \boldsymbol{U}$. Consequently there exist a sub-sub-sequence $\boldsymbol{U}_{\sigma(n_k)}$ converging almost surely to $\boldsymbol{U}$, and consequently $\Theta(\boldsymbol{U}_{\sigma(n_k)}) \to \Theta(\boldsymbol{U})$. Therefore Θ is sequentially continuous, and as the topology of convergence in probability is metrizable, Θ is continuous. □

Remark 7.8. *This Lemma does not imply the equivalence between convergence almost sure and convergence in probability as you cannot endow $\mathcal{U}$ with the "topology of almost sure convergence" as almost sure convergence is not generally induced by a topology.*

However note that $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ converges in probability toward $\boldsymbol{U}$ iff from any sub-sequence of $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ we can extract a further sub-sequence converging almost surely to $\boldsymbol{U}$ (see [44, Th 2.3.2]).

7.3.2 Objective function

Let $\mathcal{U}$ be a space of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, with value in a Banach $\mathbb{U}$.

The most classical objective function is given as $J(\boldsymbol{U}) := \mathbb{E}\big[j(\boldsymbol{U})\big]$, where $j : \mathbb{U} \to \mathbb{R}$ is a measurable, bounded cost function. This objective function expresses a risk-neutral attitude; indeed a random cost with high variance or a deterministic cost with the same expectation are considered equivalent. Recently in order to capture risk-averse attitudes, coherent risk measures (as defined in [6]), or more generally convex risk measures (as defined in [48]), have been prominent in the literature.

Following [104], we call *convex risk measure* an operator $\rho : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ verifying

- Convexity: for all $\lambda \in [0, 1]$ and all $X, Y \in \mathcal{X}$, we have

$$\rho\big(\lambda X + (1-\lambda)Y\big) \leq \lambda\rho\big(X\big) + (1-\lambda)\rho\big(Y\big)\,;$$

- Monotonicity: for all $X, Y \in \mathcal{X}$ such that $X \leq Y$ we have $\rho(X) \leq \rho(Y)$;
- Translation equivariance: for all constant $c \in \mathbb{R}$ and all $X \in \mathcal{X}$, we have $\rho(X + c) = \rho(X) + c$,

where $\mathcal{X}$ is a linear space of measurable functions. We focus on the case where $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$, and assume that $\rho(0) = 0$.

Proposition 7.9. *Let $\mathcal{U}$ be a set of random variables endowed with the topology of convergence in probability, and $J\big(\boldsymbol{U}\big) := \rho\big(j(\boldsymbol{U})\big)$, where $j : \mathbb{U} \to \mathbb{R}$ is continuous and bounded, and ρ a lower semicontinuous convex risk measure. Then, $J : \mathcal{U} \to \mathbb{R}$ is continuous.*

Proof. Note that as j is bounded, $j(\boldsymbol{U}) \in \mathcal{X}$ for any $\boldsymbol{U} \in \mathcal{U}$. Then we know that ([104]) there is a convex set of probabilities $\mathcal{P}$ such that

$$\rho(\boldsymbol{X}) = \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}\big(\boldsymbol{X}\big) - g(\mathbb{Q}) \ ,$$

where g is convex and weak*-lowersemicontinuous on the space of finite signed measures on $(\Omega, \mathcal{F})$. Moreover any probability in $\mathcal{P}$ is absolutely continuous w.r.t $\mathbb{P}$.

Consider a sequence $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{U}$ converging in probability toward $\boldsymbol{U} \in \mathcal{U}$. Let M be a majorant of j, we have $\rho\big(j(\boldsymbol{U})\big) \leq \rho(M) = M < +\infty$. By definition of ρ, for all $\varepsilon > 0$ there is a probability $\mathbb{P}_\varepsilon \in \mathcal{P}$ such that

$$\mathbb{E}_{\mathbb{P}_\varepsilon}\big(j(\boldsymbol{U})\big) - g(\mathbb{P}_\varepsilon) \geq \rho\big(j(\boldsymbol{U})\big) - \varepsilon \ .$$

As $\mathbb{P}_\varepsilon$ is absolutely continuous w.r.t $\mathbb{P}$, the convergence in probability under $\mathbb{P}$ of $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ implies the convergence of probability under $\mathbb{P}_\varepsilon$ and in turn the convergence in law under $\mathbb{P}_\varepsilon$. By definition of convergence in law we have that

$$\lim_n \mathbb{E}_{\mathbb{P}_\varepsilon}\big(j(\boldsymbol{U}_n)\big) - g(\mathbb{P}_\varepsilon) = \mathbb{E}_{\mathbb{P}_\varepsilon}\big(j(\boldsymbol{U})\big) - g(\mathbb{P}_\varepsilon) \ .$$

Let η be a positive real, and set $\varepsilon = \eta/2$, and $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|\mathbb{E}_{\mathbb{P}_\varepsilon}\big(j(\boldsymbol{U}_n)\big) - \mathbb{E}_{\mathbb{P}_\varepsilon}\big(j(\boldsymbol{U})\big)| \leq \frac{\eta}{2} \ . \tag{7.13}$$

Then, recalling that

$$\rho\Big(j\big(\boldsymbol{U}\big)\Big) \geq \mathbb{E}_{\mathbb{P}_{\frac{\eta}{2}}}\big(j(\boldsymbol{U})\big) - g(\mathbb{P}_{\frac{\eta}{2}}) \geq \rho\big(j(\boldsymbol{U})\big) - \frac{\eta}{2} \ , \tag{7.14}$$

we have that for all $n \geq N$,

$$\begin{aligned}
\rho\big(j(\boldsymbol{U}_n)\big) &= \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\Big(j(\boldsymbol{U}_n)\Big) - g(\mathbb{Q}) \\
&\geq \mathbb{E}_{\mathbb{P}_{\frac{\eta}{2}}}\Big(j(\boldsymbol{U}_n)\Big) - g\big(\mathbb{P}_{\frac{\eta}{2}}\big) \\
&\geq \mathbb{E}_{\mathbb{P}_{\frac{\eta}{2}}}\Big(j(\boldsymbol{U})\Big) - g\big(\mathbb{P}_{\frac{\eta}{2}}\big) - \frac{\eta}{2} && \text{by (7.13)}, \\
&\geq \rho\Big(j(\boldsymbol{U})\Big) - \eta && \text{by (7.14)},
\end{aligned}$$

and thus

$$\rho\Big(j\big(\boldsymbol{U}\big)\Big) + \frac{\eta}{2} \geq \rho\Big(j\big(\boldsymbol{U}_n\big)\Big) \geq \rho\Big(j\big(\boldsymbol{U}\big)\Big) - \eta \ .$$

Thus $\lim_n \rho\big(j(\boldsymbol{U}_n)\big) = \rho\big(j(\boldsymbol{U})\big)$. Hence the continuity of J. $\square$

The assumptions of this Proposition can be relaxed in different ways.

In a first place, if the convex risk measure ρ is simply the expectation then we can simply endow $\mathcal{U}$ with the topology of convergence in law. In this case the continuity assumption on j can also be relaxed. Indeed if $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ converges in law toward $\boldsymbol{U}$, and if the set K of points where j is continuous is such that $\mathbb{P}(\boldsymbol{U} \in K) = 1$, then $\mathbb{E}\big[j(\boldsymbol{U}_n)\big]$ converges toward $\mathbb{E}\big[j(\boldsymbol{U})\big]$.

Otherwise assume that $\mathcal{U}$ is a set of random variables endowed with the topology of convergence in probability, and that j continuous. Moreover if we can ensure that $j(\boldsymbol{U})$ is dominated by some integrable (for all probability of $\mathcal{P}$) random variable, then $J : \mathcal{U} \to \mathbb{R}$ is continuous. Indeed we consider a sequence $\big(\boldsymbol{U}_n\big)_{n\in\mathbb{N}}$ almost surely converging to $\boldsymbol{U}$. We modify the proof of Proposition 7.9 by using a dominated convergence theorem to show that $\lim_n \mathbb{E}_{\mathbb{P}_\varepsilon}\big(j(\boldsymbol{U}_n)\big) = \mathbb{E}_{\mathbb{P}_\varepsilon}\big(j(\boldsymbol{U})\big)$. Lemma 7.7 concludes the proof.

7.3.3 Constraint operator

We present some usual constraints and how they can be represented by an operator Θ that is continuous and take values into $\mathcal{V}$.

Almost sure constraint

From Lemma 7.7, we obtain a first important example of continuous constraints.

Proposition 7.10. *Suppose that $\mathcal{U}$ is the set of random variables on $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, with value in $\mathbb{U}$, endowed with the topology of convergence in probability. Assume that $\theta : \mathbb{U} \to \mathbb{V}$ is continuous and bounded. Then the operator $\Theta\big(\boldsymbol{U}\big)(\omega) := \theta\big(\boldsymbol{U}(\omega)\big)$ maps $\mathcal{U}$ into $\mathcal{V}$ and is continuous.*

Proof. The function θ being continuous, is also Borel measurable.Thus for all $\boldsymbol{U} \in \mathcal{U}$, for all Borel set $V \subset \mathbb{V}$, we have

$$\big(\Theta(\boldsymbol{U})\big)^{-1}(V) = \{\omega \in \Omega \quad | \quad \boldsymbol{U}(\omega) \in \theta^{-1}(V)\} \in \mathcal{F} ,$$

thus $\Theta(\boldsymbol{U})$ is $\mathcal{F}$-measurable. Boundedness of θ insure the existence of moment of all order of $\Theta(\boldsymbol{U})$. Thus Θ is well defined.

Suppose that $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ converges to $\boldsymbol{U}$ almost surely. Then by boundedness of θ, we have that $\Big(\big\|\theta\big(\boldsymbol{U}_n\big) - \theta\big(\boldsymbol{U}\big)\big\|_{\mathbb{V}}^p\Big)_{n\in\mathbb{N}}$ is bounded, and thus by dominated convergence theorem we have that

$$\lim_{n\to\infty} \theta\big(\boldsymbol{U}_n\big) = \theta\big(\boldsymbol{U}\big) \quad \text{in } \mathrm{L}^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{V}) ,$$

which is exactly

$$\lim_{n\to\infty} \Theta\big(\boldsymbol{U}_n\big) = \Theta\big(\boldsymbol{U}\big) .$$

Consequently by Lemma 7.7 we have the continuity of Θ. □

We note that boundedness of θ is only necessary in order to use the dominated convergence theorem. Thus an alternative set of assumptions is given in the following proposition.

Proposition 7.11. *Let $\mathcal{B}$ be a sub-field of $\mathcal{F}$. If $\mathcal{U} = \mathrm{L}^{p'}\big(\Omega, \mathcal{B}, \mathbb{P}\big)$, with the topology of convergence in probability, and if θ is γ-Hölder, with $\gamma \leq p'/p$ then $\Theta\big(\boldsymbol{U}\big)(\omega) := \theta\big(\boldsymbol{U}(\omega)\big)$ is well defined and continuous as an operator mapping $\mathcal{U}$ into $\mathcal{V}$.*

Proof. By definition a function θ mapping $\mathbb{U}$ into $\mathbb{V}$ is γ-Hölder if there exist a constant $C > 0$ such that for all u, u' in $\mathbb{U}$ we have

$$\left\|\theta(u) - \theta(u')\right\|_{\mathbb{V}} \leq C \left\|u - u'\right\|_{\mathbb{U}}^{\gamma} ,$$

in particular the 1-Hölder continuity is the Lipschitz continuity.

Following the previous proof we just have to check that the sequence $\Big(\left\|\theta(\boldsymbol{U}_n) - \theta(\boldsymbol{U})\right\|_{\mathbb{V}}^{p} \Big)_{n\in\mathbb{N}}$ is dominated by some integrable variable. Indeed, the Hölder assumption implies

$$\left\|\theta(\boldsymbol{U}_{n_k}) - \theta(\boldsymbol{U})\right\|_{\mathbb{V}}^{p} \leq C^p \left\|\boldsymbol{U}_{n_k} - \boldsymbol{U}\right\|_{\mathbb{U}}^{p\gamma} .$$

And as $p\gamma \leq p'$, and $\boldsymbol{U}_n$ and $\boldsymbol{U}$ are elements of $\mathrm{L}^{p'}\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, $\left\|\boldsymbol{U}_{n_k} - \boldsymbol{U}\right\|_{\mathbb{U}}^{p\gamma}$ is integrable. □

Measurability constraint

When considering a dynamic stochastic optimization problem, measurability constraints are used to represent the nonanticipativity constraints. They can be expressed by stating that a random variable and its conditional expectation are equal.

Proposition 7.12. *We set $\mathcal{U} = \mathrm{L}^{p'}\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{V}\big)$, with $p' \geq p$. Assume that*

- *either $\mathcal{U}$ is equipped with the strong topology, and $\mathcal{V}$ is equipped with the strong or weak topology,*
- *or $\mathcal{U}$ and $\mathcal{V}$ are equipped with the weak topology.*

If $\mathcal{B}$ is a sub-field of $\mathcal{F}$, then $\Theta\big(\boldsymbol{U}\big) := \mathbb{E}\big[\boldsymbol{U} \bigm| \mathcal{B}\big] - \boldsymbol{U}$, is well defined and continuous.

Proof. In a first place note that as $p' \geq p$, $\mathcal{U} \subset \mathcal{V}$; and if $\boldsymbol{V} \in \mathcal{V}$ then $\mathbb{E}\big[\boldsymbol{V} \bigm| \mathcal{B}\big] \in \mathcal{V}$ as the conditional expectation is a contraction. Thus for all $\boldsymbol{U} \in \mathcal{U}$, we have $\Theta(\boldsymbol{U}) \in \mathcal{V}$.

Consider a sequence $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ of $\mathcal{U}$ strongly converging in $L^{p'}$ toward $\boldsymbol{U} \in \mathcal{U}$. We have

$$\begin{aligned} ||\Theta\big(\boldsymbol{U}_n\big) - \Theta\big(\boldsymbol{U}\big)||_p &\leq ||\boldsymbol{U}_n - \boldsymbol{U}||_p + ||\mathbb{E}\big[\boldsymbol{U}_n - \boldsymbol{U} \bigm| \mathcal{B}\big]||_p \\ &\leq 2||\boldsymbol{U}_n - \boldsymbol{U}||_p \\ &\leq 2||\boldsymbol{U}_n - \boldsymbol{U}||_{p'} \to 0 . \end{aligned}$$

Thus the strong continuity of Θ is proven.

Now consider $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ converging weakly in $L^{p'}$ toward $\boldsymbol{U} \in \mathcal{U}$. We have, for all $\boldsymbol{Y} \in L^q$,

$$\begin{aligned} \mathbb{E}\Big[\mathbb{E}\big[\boldsymbol{U}_n \bigm| \mathcal{B}\big] \cdot Y\Big] =& \quad \mathbb{E}\Big[\boldsymbol{U}_n \mathbb{E}\big[Y \bigm| \mathcal{B}\big]\Big] , \\ \underset{n}{\longrightarrow}& \quad \mathbb{E}\Big[\boldsymbol{U} \mathbb{E}\big[Y \bigm| \mathcal{B}\big]\Big] , \\ =& \quad \mathbb{E}\Big[\mathbb{E}\big[\boldsymbol{U} \bigm| \mathcal{B}\big] Y\Big] . \end{aligned}$$

Thus we have the weak convergence of the conditional expectation and therefore of Θ. Finally as the strong convergence imply the weak convergence we have the continuity from $\mathcal{U}$-strong into $\mathcal{V}$-weak. □

Until now the topology of convergence in probability has been largely used. If we endow $\mathcal{U}$ with the topology of convergence in probability in the previous proposition we will obtain continuity of Θ on a subset of $\mathcal{U}$. Indeed if a set of random variables $\mathcal{U}^{ad}$ such that there exist a random variable in $\mathrm{L}^{p'}\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ dominating every random variable in $\mathcal{U}^{ad}$, then a sequence converging almost surely will converge for the $L^{p'}$ norm and we can follow the previous proof to show the continuity of Θ on $\mathcal{U}^{ad}$.

Risk constraints

Risk attitude can be expressed through the criterion or through constraints. We have seen that a risk measure can be chosen as objective function, we now show that conditional risk measure can be used to define some constraints.

Let ρ be a conditional risk mapping as defined in [103], and more precisely ρ maps $\mathcal{U}$ into $\mathcal{V}$ where $\mathcal{U} = L^p\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{U}\big)$ and $\mathcal{V} = L^p\big(\Omega, \mathcal{B}, \mathbb{P}; \mathbb{V}\big)$, with $\mathcal{B} \subset \mathcal{F}$, and verifies the following properties

- Convexity: for all $\lambda \in \mathcal{U}$, $\lambda \in [0,1]$ and all $X, Y \in \mathcal{V}$, we have

$$\rho\big(\lambda X + (1-\lambda)Y\big) \leq \lambda\rho\big(X\big) + (1-\lambda)\rho\big(Y\big) \, ;$$

- Monotonicity: for all $X, Y \in \mathcal{V}$ such that $X \leq Y$ we have $\rho(X) \leq \rho(Y)$;
- Translation equivariance: for all $c \in \mathcal{V}$ and all $X \in \mathcal{U}$, we have $\rho(X+c) = \rho(X) + c$.

Proposition 7.13. *Let $\mathcal{U}$ be endowed with the topology of convergence in probability, and $\mathcal{V}$ endowed with the strong topology. If ρ is a conditional risk mapping, θ is a continuous bounded cost function mapping $\mathbb{U}$ into $\mathbb{R}$, and $a \in \mathcal{V}$, then $\Theta\big(\boldsymbol{U}\big) := \rho\Big(\theta\big(\boldsymbol{U}\big)\Big) - a$ is continuous.*

Proof. Consider a sequence of random variables $\big\{\boldsymbol{U}_n\big\}_{n\in\mathbb{N}}$ converging in probability toward $\boldsymbol{U}_\infty$. Let $\pi : L^p(\Omega, \mathcal{B}, \mathbb{P}; \mathbb{U}) \to \mathrm{L}^p(\Omega, \mathcal{B}, \mathbb{P}; \mathbb{U})$ be a selector of $\mathcal{V} = L^p(\Omega, \mathcal{B}, \mathbb{P}; \mathbb{U})$, i.e. for any $\boldsymbol{U} \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{U})$, $\pi(\boldsymbol{U}) \in \boldsymbol{U}$. For any $\omega \in \Omega$, any $\boldsymbol{U} \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{U})$ we define

$$\rho_\omega(\boldsymbol{U}) := \pi(\rho\big(\boldsymbol{U}\big))(\omega) \, .$$

Note that for $\mathbb{P}$-almost all $\omega \in \Omega$, the function $\Theta_\omega(\boldsymbol{U}) := \rho_\omega\big(\theta(\boldsymbol{U})\big)$, satisfies the conditions of Proposition 7.9. Thus for $\mathbb{P}$-almost all $\omega \in \Omega$, $\big(\Theta_\omega(\boldsymbol{U}_n)\big)_{n\in\mathbb{N}}$ converges toward $\Theta_\omega(\boldsymbol{U}_\infty)$. Thus we have shown that $\big(\Theta(\boldsymbol{U}_n)\big)_{n\in\mathbb{N}}$ converges almost surely toward $\Theta\big(\boldsymbol{U}_\infty\big)$. By boundedness of θ and monotonicity of ρ we obtain the boundedness of $\big(\Theta(\boldsymbol{U}_n)\big)_{n\in\mathbb{N}}$. Thus almost sure convergence and dominated convergence theorem ensure that $\big(\Theta(\boldsymbol{U}_n)\big)_{n\in\mathbb{N}}$ converges in L^p toward $\Theta\big(\boldsymbol{U}_\infty\big)$, hence the continuity of Θ. □

Another widely used risk measure, even if it has some serious drawbacks, is the Value-at-Risk. If $\boldsymbol{X}$ is a real random variable its value at risk of level α can be defined as $VaR_\alpha(\boldsymbol{X}) := \inf\{F_{\boldsymbol{X}}^{-1}(\alpha)\}$ where $F_{\boldsymbol{X}}(x) := \mathbb{P}(\boldsymbol{X} \leq x)$.

Proposition 7.14. *If $\theta : \mathbb{U} \to \mathbb{R}$ is continuous, and if $\mathcal{U}$ is such that every $\boldsymbol{U} \in \mathcal{U}$ have a continuous distribution function, then $\Theta(\boldsymbol{U}) := VaR_\alpha\Big(\theta\big(\boldsymbol{U}\big)\Big)$ is continuous if we have endowed $\mathcal{U}$ with the topology of convergence in law, and a fortiori for the topology of convergence in probability.*

Proof. By definition of convergence in law, if $\boldsymbol{U}_n \to \boldsymbol{U}$ in law, then $\big(\theta\big(\boldsymbol{U}_n\big)\big)_{n\in\mathbb{N}}$ converges in law toward $\theta\big(\boldsymbol{U}\big)$ and we have, for all $x \in \mathbb{R}$, $F_{\theta(\boldsymbol{U}_n)}(x) \to F_{\theta(\boldsymbol{U})}(x)$. Thus $\big(\Theta(\boldsymbol{U}_n)\big)_{n\in\mathbb{N}}$ converges toward $\Theta(\boldsymbol{U})$, and as $\Theta(\boldsymbol{U})$ is real-valued, Θ is continuous. □

Note that in Proposition 7.14 the constraint function take deterministic values. Thus considering the conditional expectation of this constraint yields exactly the same constraint. However consider a constraint $\Theta_1 : \mathcal{U} \to \mathbb{R}$ of this form, and another constraint $\Theta_2 : \mathcal{U} \to \mathcal{V}$. Then if Θ_1 and Θ_2 are continuous, then so is the constraint $\Theta = (\Theta_1, \Theta_2) \to \mathbb{R} \times \mathcal{V}$. Thus we can apply Theorem 7.4 on the coupled constraint.

7.4 Application to a Multistage Problem

In this section we say a few words about how the approximation of an almost sure constraint by a conditional expectation – as presented in section 7.2 – can be used.

We are interested in an electricity production problem with N power stations coupled by an equality constraint. At time step t, each power station i have an internal state $\boldsymbol{X}_t^i$, and is affected by a random exogenous noise $\boldsymbol{W}_t^i$. For each power station, and each time step t, we have a control $\boldsymbol{Q}_t^i \in \mathcal{Q}_{t,i}^{ad}$ that must be measurable with respect to $\mathcal{F}_t$ where $\mathcal{F}_t$ is the σ-algebra generated by all past noises: $\mathcal{F}_t = \sigma\big(\boldsymbol{W}_s^i\big)_{1\leq i\leq n, 0\leq s\leq t}$. Moreover there is a coupling constraint expressing that the total production must be equal to the demand. This constraint is represented as $\sum_{i=1}^N \theta_t^i(\boldsymbol{Q}_t^i) = 0$, where θ_t^i is a continuous bounded function from $\mathbb{Q}_{t,i}^{ad}$ into $\mathbb{V}$, for all $i \in [\![1, n]\!]$. The cost to be minimized is a sum over time and power stations of all current local cost $L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{Q}_t^i, \boldsymbol{W}_t^i\big)$.

Finally the problem reads

$$\min_{\boldsymbol{X},\boldsymbol{Q}} \quad \mathbb{E}\Big[\sum_{i=1}^N \sum_{t=0}^T L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{Q}_t^i, \boldsymbol{W}_t^i\big)\Big] \tag{7.15a}$$

$$s.t. \quad \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{Q}_t^i, \boldsymbol{W}_t^i) \qquad \forall t, \quad \forall i, \tag{7.15b}$$

$$\boldsymbol{X}_0^i = x_0^i \qquad \forall i, \tag{7.15c}$$

$$\boldsymbol{Q}_t^i \in \mathcal{Q}_{t,i}^{ad} \qquad \forall t, \quad \forall i, \tag{7.15d}$$

$$\boldsymbol{Q}_t^i \preceq \mathcal{F}_t \qquad \forall t, \quad \forall i, \tag{7.15e}$$

$$\sum_{i=1}^N \theta_t^i(\boldsymbol{Q}_t^i) = 0 \qquad \forall t, \quad \forall i. \tag{7.15f}$$

For the sake of brevity, we denote by $\mathcal{A}$ the set of random processes $(\boldsymbol{X}, \boldsymbol{Q})$ verifying constraints (7.15b), (7.15c) and (7.15d).

Let assume that all random variables are in L^2 spaces and dualize the coupling constraint (7.15f). We do not study here the relation between the primal and the following dual problem (see [95] and [94] for an alternative formulation involving duality between L^1 and its dual).

$$\begin{aligned} \max_{\boldsymbol{\lambda}\in L^2} \quad \min_{(\boldsymbol{X},\boldsymbol{Q})\in\mathcal{A}} \quad & \mathbb{E}\Big[\sum_{i=1}^N \sum_{t=0}^T L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{Q}_t^i, \boldsymbol{W}_t^i\big) + \boldsymbol{\lambda}_t \theta_t^i(\boldsymbol{Q}_t^i)\Big] \\ s.t. \quad & \boldsymbol{Q}_t^i \preceq \mathcal{F}_t \quad \forall t, \quad \forall i. \end{aligned} \tag{7.16}$$

We solve this problem using a gradient-like algorithm on $\boldsymbol{\lambda}$. Thus for a fixed $\boldsymbol{\lambda}^{(k)}$ we have to solve N problems of smaller size than Problem (7.16).

$$\begin{aligned} \min_{(\boldsymbol{X},\boldsymbol{U})\in\mathcal{A}} \quad & \mathbb{E}\Big[\sum_{t=0}^T L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{Q}_t^i, \boldsymbol{W}_t^i\big) + \boldsymbol{\lambda}_t^{(k)} \theta_t^i(\boldsymbol{Q}_t^i)\Big] \\ s.t. \quad & \boldsymbol{Q}_t^i \preceq \mathcal{F}_t \quad \forall t, \quad \forall i. \end{aligned} \tag{7.17}$$

Note that the process $\boldsymbol{\lambda}^{(k)}$ has no given dynamics but can be chosen to be adapted to the filtration $(\mathcal{F}_t)_{t=1,..,T}$. Consequently solving Problem (7.17) by Dynamic Programming is possible but numerically difficult as we need to keep all the past realizations of the noises in the state. In fact the so-called curse of dimensionality prevent us to solve numerically this problem.

Nevertheless it has recently been proposed in [10] to replace $\boldsymbol{\lambda}_t$ by $\mathbb{E}\big[\boldsymbol{\lambda}_t \bigm| \boldsymbol{Y}_t\big]$, where $\boldsymbol{Y}_t$ is a random variable measurable with respect to $(\boldsymbol{Y}_{t-1}, \boldsymbol{W}_t)$ instead of $\boldsymbol{\lambda}_t$. This is similar to a decision rule approach for the dual as we are restraining the control to a certain class, the $\boldsymbol{Y}_t$-measurable $\boldsymbol{\lambda}$ in our case. Thus Problem (7.17) can be solved by Dynamic Programming with the augmented state $(\boldsymbol{X}_t^i, \boldsymbol{Y}_t)$. It has also been shown that, under some non-trivial conditions, using $\mathbb{E}\big[\boldsymbol{\lambda}_t \bigm| \boldsymbol{Y}_t\big]$ instead of $\boldsymbol{\lambda}_t$ is equivalent to solving

$$\min_{(\boldsymbol{X},\boldsymbol{Q})\in\mathcal{A}} \quad \mathbb{E}\Big[\sum_{i=1}^{N}\sum_{t=0}^{T} L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{Q}_t^i, \boldsymbol{W}_t^i\big)\Big] \tag{7.18a}$$

$$s.t. \quad \boldsymbol{Q}_t^i \preceq \mathcal{F}_t \qquad \forall t, \quad \forall i, \tag{7.18b}$$

$$\mathbb{E}\Big[\sum_{i=1}^{N} \theta_t^i(\boldsymbol{Q}_t^i) \Bigm| Y_t\Big] = 0 \qquad \forall t, \quad \forall i. \tag{7.18c}$$

Problem (7.18) is a relaxation of Problem (7.15) where the almost sure constraint (7.15f) is replaced by the constraint (7.18c). Now consider a sequence of information processes $(\boldsymbol{Y}^{(n)})_{n\in\mathbb{N}}$ each generating a σ-algebra $\mathcal{F}_n$, and their associated relaxation $(\mathcal{P}_n)$ (as specified in Problem 7.18) of Problem (7.15) (denoted $(\mathcal{P})$). Those problems correspond to Problems (7.11) and (7.12) with

$$J(\boldsymbol{U}) = \mathbb{E}\Big[\sum_{i=1}^{N}\sum_{t=0}^{T} L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{Q}_t^i, \boldsymbol{W}_t^i\big)\Big],$$

where $\boldsymbol{U} = (\boldsymbol{Q}^{(i)})_{i\in[\![1,N]\!]}$ and $\boldsymbol{X}_t^i$ follow the dynamic equation (7.15b). We also have

$$\Theta_t(\boldsymbol{U}_t) = \sum_{i=1}^{N} \theta_t^i(\boldsymbol{Q}_t^i)$$

and $C_t = \{0\}$.

Assume that for all $t \in [\![1,T]\!]$, and all $i \in [\![1,N]\!]$ the cost functions L_t^i and constraint function θ_t^i are continuous, and that $\mathcal{Q}_{t,i}^{ad}$ is a compact subset of an Euclidean space. Moreover we assume that the noise variables $\boldsymbol{W}_t^i$ are essentially bounded. Finally we endow the space of control processes with the topology of convergence in probability. Then by induction we have that the state processes and the control processes are essentially bounded, thus so is the cost $L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t^i\big)$. Thus the cost function can be effectively replaced by bounded functions. Consequently Proposition 7.9 insures that J is continuous if $\mathcal{U}$ is equipped with the topology of convergence in probability. Similarly Proposition 7.10 insures that Θ is continuous.

Thus Theorem 7.6 implies that our sequence of approximated problems $(\mathcal{P}_n)$ converges toward the initial problem $(\mathcal{P})$. More precisely assume that $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ is a sequence of ε_n-optimal solution of $\mathcal{P}_n$, i.e. $\boldsymbol{U}_n$ verifying constraint (7.18c) and $J(\boldsymbol{U}_n) < \inf_{\boldsymbol{U}\in\mathcal{U}_n^{ad}} J(\boldsymbol{U}) + \varepsilon_n$, with $(\varepsilon_n)_{n\in\mathbb{N}}$ a sequence of positive real number converging to 0. Then we can extract a subsequence $(\boldsymbol{U}_{n_k})_{k\in\mathbb{N}}$ converging almost surely to an optimal solution of $(\mathcal{P})$, and the limit of the approximated value of $(\mathcal{P}_n)$ converges to the value of $(\mathcal{P})$.

Conclusion

In this Chapter we have considered a sequence of optimization problem $\big(\mathcal{P}_n\big)$ where each problem is a relaxation of an optimization problem $\big(\mathcal{P}\big)$. This relaxation is given by replacing an almost sure constraint by a conditional expectation constraint with respect to a σ-algebra $\mathcal{F}_n$. We have shown that, if the cost and constraint functions are continuous and if the sequence of σ-algebras $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ converges toward the global σ-algebra, then the sequence of optimization problems $\big(\mathcal{P}_n\big)$ converges toward the original problem $\big(\mathcal{P}\big)$.

In the next chapter, we apply this relaxation to a multistage optimization problem in order to obtain a tractable price decomposition scheme.

Chapter 8

Dual Approximate Dynamic Programming Algorithm

If you can't solve a problem, then there is an easier problem you can solve: find it.

Georg Pólya

Contents

In this final chapter, we present a spatial decomposition algorithm that solves an approximation of a multistage stochastic optimization problem. We illustrate the approach on the hydraulic valley example of §1.3.2. This algorithm, called DADP for *Dual Approximate Dynamic Programming*, was first described in [10], following preliminary works in [114] and [50]; numerical studies can be found in [1, 50].

In §8.1, we provide a bird eye view of the method. In §8.2, we describe each step of the DADP algorithm. In §8.3, we use the results of Chapters 4 to 7 to provide theoretical foundations to the algorithm. In §8.4, we display numerical results for the 3-dam valley example (see §1.3.2). Finally, in §8.5, we discuss the design of the information process that appears in the approximation step.

As this Chapter is pretty heavy on notations and indexes, the quantifier $\forall$ will often be omitted. In addition, many variables are indexed by the dynamic subsystem i, and the time t; most of the time, when we omit an index, it means the collection, e.g. $\boldsymbol{X} = \{\boldsymbol{X}_t^i\}_{t\in[\![0,T]\!],i\in[\![1,N]\!]}$, $\boldsymbol{X}^i = \{\boldsymbol{X}_t^i\}_{t=0}^T$, $\boldsymbol{X}_t = \{\boldsymbol{X}_t^i\}_{i=1}^N$. We also assume that spaces $\mathbb{X} = \big[\mathbb{X}_t\big]_1^T$, $\mathbb{U} = \big[\mathbb{U}_t\big]_1^T$ and $\mathbb{W} = \big[\mathbb{W}_t\big]_1^T$ are subsets of finite dimensional vector spaces. More precisely, we denote by n_X the dimension of $\mathrm{Aff}(\mathbb{X}_t)$, n_U the dimension of $\mathrm{Aff}(\mathbb{U}_t)$, and n_W the dimension of $\mathrm{Aff}(\mathbb{W}_t)$; it is for notational sobriety only that these dimensions are assumed to be the same for every time t. The integer n_C denotes the dimension of the image space of the constraint functions θ_t^i.

8.1 Overview of the DADP Method

We consider N stochastic dynamic systems coupled by almost sure equality constraints. The global cost to be minimized is the expectation of a sum over the N systems of the sum over time of local costs. The problem considered is detailed in §8.1.1. Our objective here is to obtain feedbacks (strategies), for a large scale stochastic dynamical problem.

The price decomposition scheme consists in dualizing the coupling constraints, fixing a multiplier, and obtaining N uncoupled subproblems. From the solution of each subproblem we update the multiplier before iterating. However, we show in §8.1.2 that this price decomposition scheme leads to subproblems which are too difficult to solve by Dynamic Programming (dimension of the state too important). Thus, we propose an approximation method called Dual Approximate Dynamic Programming (DADP) and based on the main following ideas [1]:

- relaxing the almost sure coupling equality constraints into conditional expectation constraint,
- using a price decomposition scheme to obtain subproblems,
- solving the subproblems through methods like Dynamic Programming.

The approximation idea behind the Dual Approximate Dynamic Programming (DADP) algorithm is presented in §8.1.3. A presentation of the scheme of DADP method is given in §8.1.4 (a more detailed presentation is done in §8.2). Its application on the hydraulic valley example is presented in §8.1.5.

8.1.1 Presentation of the Spatially Coupled Problem

We are interested in a production problem involving N units. Each unit i has an internal state $\boldsymbol{X}_t^i$ at time step t, and is affected by a random exogenous noise $\boldsymbol{W}^i$. The global exogenous noise $\{\boldsymbol{W}_t\}_0^{T-1}$ is assumed to be time-independent. Time dependence could be represented by extending the state, and incorporating information of the noise in it. On the other hand, for a given time t, the sequence $\{\boldsymbol{W}_t^i\}_{i=1}^N$ is not assumed to be independent (between units). Moreover we assume a Hazard-Decision setting, that is, that the control taken at time t is chosen once the uncertainty $\boldsymbol{W}_t$ is known.

1. Different interpretations of the DADP algorithm are given in 8.3.1.

For each unit $i \in [\![1, N]\!]$, and each time step $t \in [\![0, T-1]\!]$, we have to make a decision $\boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{\mathrm{ad}}$ that must be measurable with respect to $\mathcal{F}_t$, where $\mathcal{F}_t$ is the σ-algebra generated by all past noises:

$$\mathcal{F}_t = \sigma\Big(\big(\boldsymbol{W}_s^i\big)_{1 \leq i \leq N, 0 \leq s \leq t} \Big) .$$

We denote $\mathfrak{F}$ the filtration $\{\mathcal{F}_t\}_0^T$.

We consider an almost sure coupling constraint represented as

$$\sum_{i=1}^{N} \theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) = 0 \qquad \mathbb{P} - a.s. \tag{8.1}$$

For example, each θ_t^i can represent the production of unit i at time t and a constraint on the global production at time t is represented through Equation (8.1). Moreover, as in the dam example (see §1.3.2), if some controls are shared by two dynamical systems, then it is formulated by defining one control for each dynamical system, and stating their equality in (8.1).

Finally, the cost to be minimized is the expectation of a sum over time and over unit of all current local costs $L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big)$.

The overall problem can be formulated

$$\min_{\boldsymbol{X}, \boldsymbol{U}} \quad \mathbb{E}\Big[\sum_{i=1}^{N} \sum_{t=0}^{T} L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) \Big] \tag{8.2a}$$

$$\boldsymbol{X}_0^i = x_0^i \tag{8.2b}$$

$$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t) \tag{8.2c}$$

$$\boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{\mathrm{ad}} \tag{8.2d}$$

$$\boldsymbol{U}_t^i \preceq \mathcal{F}_t \tag{8.2e}$$

$$\sum_{i=1}^{N} \theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) = 0 \tag{8.2f}$$

where constraint (8.2c)-(8.2f) are to be understood for all time $t \in [\![0, T]\!]$ (constraint (8.2c) for $t \in [\![0, T-1]\!]$ only) and constraints (8.2b)-(8.2e) for all unit $i \in [\![1, N]\!]$.

Note that, if it were not for constraint (8.2f), Problem (8.2) would lead to a sum of independent subproblems, that could be optimized independently.

8.1.2 First Idea: Price Decomposition Scheme

In §6.3, we presented how Uzawa algorithm can be applied to a multistage problem. However, in Chapter 6 we did not specify how to solve the minimization problem for a given multiplier. Here, we use the Uzawa algorithm as the master problem in a price decomposition approach to Problem (8.2), and show its limits.

Let us assume that all random variables used in Problem (8.2) are in L^∞, and that the problem has a L^1 optimal multiplier for the *coupling constraint* (8.2f). There are three reasons for choosing the space L^∞. First, assuming that the states and control are essentially bounded is a reasonable modernization for most industrial problems. Second, there exists - see Chapter 5 - condition for existence of multiplier in the $\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$ pairing, whereas the examples of Chapter 4 show that it is more delicate in L^p with $p < \infty$. Third, a convergence in L^∞ has an easier interpretation than a convergence in L^2.

We dualize (see Chapter 4) the coupling constraints (8.2f) (in the $\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$ pairing) to obtain

$$\begin{aligned} \min_{\boldsymbol{X},\boldsymbol{U}} \quad \max_{\boldsymbol{\lambda} \in \mathrm{L}^1} \quad & \mathbb{E}\Big[\sum_{i=1}^{N}\sum_{t=0}^{T} L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) + \boldsymbol{\lambda}_t \cdot \theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big)\Big] \\ & \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t) \\ & \boldsymbol{X}_0^i = x_0^i \\ & \boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{\mathrm{ad}} \\ & \boldsymbol{U}_t^i \preceq \mathcal{F}_t \ . \end{aligned} \tag{8.3}$$

Note that the multiplier $\boldsymbol{\lambda}$ is a stochastic process $\boldsymbol{\lambda} = \big\{\boldsymbol{\lambda}_t\big\}_{t=0}^{T}$.

We now consider the dual problem (see Chapter 4)

$$\begin{aligned} \max_{\boldsymbol{\lambda} \in \mathrm{L}^1} \quad \min_{\boldsymbol{X},\boldsymbol{U}} \quad & \mathbb{E}\Big[\sum_{i=1}^{N}\sum_{t=0}^{T} L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) + \boldsymbol{\lambda}_t \cdot \theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big)\Big] \\ & \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t) \\ & \boldsymbol{X}_0^i = x_0^i \\ & \boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{\mathrm{ad}} \\ & \boldsymbol{U}_t^i \preceq \mathcal{F}_t \ . \end{aligned} \tag{8.4}$$

Fact 8.1. *If there exists an optimal multiplier process $\boldsymbol{\lambda} = \{\boldsymbol{\lambda}_s\}_0^{T-1}$ such that $\boldsymbol{\lambda} \in \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{n_C}\big)$, then there exists an optimal multiplier process that is $\mathfrak{F}$-adapted.*

Proof. Indeed, for $\boldsymbol{\lambda}_t$ in $\mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, the conditional expectation w.r.t the σ-algebra $\mathcal{F}_t$ is defined, and we have,

$$\mathbb{E}\Big[\boldsymbol{\lambda}_t \cdot \mathbb{E}\big[\theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) \bigm| \mathcal{F}_t\big]\Big] = \mathbb{E}\Big[\mathbb{E}\big[\boldsymbol{\lambda}_t \bigm| \mathcal{F}_t\big] \cdot \mathbb{E}\big[\theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) \bigm| \mathcal{F}_t\big]\Big] \ .$$

Hence, we replace $\boldsymbol{\lambda}_t$ by the $\mathcal{F}_t$-measurable $\mathbb{E}\big[\boldsymbol{\lambda}_t \bigm| \mathcal{F}_t\big]$ that yields the same value for Problem (8.4). □

From now on we will consider that the multiplier process $\boldsymbol{\lambda}$ is $\mathfrak{F}$-adapted.

We can solve the maximization part of the dual problem using a gradient-like algorithm on $\boldsymbol{\lambda}$. Thus, for a fixed multiplier process $\boldsymbol{\lambda}^{(k)}$, we have to solve N independent problems of smaller size

$$\begin{aligned} \min_{\boldsymbol{X}^i,\boldsymbol{U}^i} \quad & \mathbb{E}\Big[\sum_{t=0}^{T} L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) + \boldsymbol{\lambda}_t^{(k)} \cdot \theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big)\Big] \\ & \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t) \\ & \boldsymbol{X}_0^i = x_0^i \\ & \boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{\mathrm{ad}} \\ & \boldsymbol{U}_t^i \preceq \mathcal{F}_t \ . \end{aligned} \tag{8.5}$$

Problem (8.2) is a multistage problem with a physical state $\boldsymbol{X}_t = \{\boldsymbol{X}_t^i\}_{i=1}^N$, affected by a time independent noise process $\{\boldsymbol{W}_t\}_0^T$. Hence, the state $\boldsymbol{X}_t$ is an information state in the sense of Dynamic Programming (see §1.2.4) and Problem (8.2) can be solved through Dynamic Programming with a state of dimension $N \times dim(\mathbb{X}_t^i)$.

If it were not for the term $\boldsymbol{\lambda}_t^{(k)} \cdot \theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big)$ in the objective function of Problem (8.5), we would have a problem with a physical state $\boldsymbol{X}_t^i$ affected by the time independent noise process $\{\boldsymbol{W}_t\}_0^T$. Hence, the Dynamic Programming would be far faster: the dimension of the state is divided by N.

Unfortunately, with the term $\boldsymbol{\lambda}_t^{(k)} \cdot \theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big)$ Problem (8.5) is a problem with a physical state $\boldsymbol{X}_t^i$, and two random noises : $\boldsymbol{W}_t$ and $\boldsymbol{\lambda}_t^{(k)}$. The noise $\{\boldsymbol{W}_t\}_0^T$ process is time-independent, but the noise process $\{\boldsymbol{\lambda}_t^{(k)}\}_{t=0}^{T-1}$ is not time independent. All we know is that it is a $\mathfrak{F}$-adapted information process. Hence, *a priori*, Problem (8.5) can be solved by Dynamic Programming, by using $\{\boldsymbol{W}_s\}_{s=0}^{t-1}$ as the information state at time t. However, this state is not necessarily smaller than the state of the global problem (Problem (8.2)).

If we could show that the multiplier process $\boldsymbol{\lambda}_t^{(k)}$ had a dynamic, say

$$\boldsymbol{\lambda}_t^{(k)} = h_t\big(\boldsymbol{\lambda}_{t-1}^{(k)}, \cdots, \boldsymbol{\lambda}_{t-s}^{(k)}, \boldsymbol{W}_t\big) ,$$

then Problem (8.5) could be solved with the information state $\{\boldsymbol{X}_t^i, \boldsymbol{\lambda}_{t-1}^{(k)}, \cdots, \boldsymbol{\lambda}_{t-s}^{(k)}\}$. On a very specific example it has been shown in [114] that the multiplier process has a dynamic. In the following section, we construct an approximation of Problem (8.2) such that its multiplier process is a function of a stochastic process $\boldsymbol{Y}$ with a dynamic. Our goal is to solve Problem (8.5) by Dynamic Programming with the extended information state $\big(\boldsymbol{X}_t^i, \boldsymbol{Y}_t\big)$.

8.1.3 Second Idea: Constraint Relaxation

We have seen in the previous section that, if we apply a price decomposition scheme to Problem (8.2) the subproblems (8.5) cannot be solved numerically by the Dynamic Programming approach because of the curse of dimensionality. Thus, we approximate Problem (8.2) by relaxing the almost sure constraints, in order to obtain subproblems with a smaller dimension state, and thus numerically solvable by Dynamic Programming.

For this purpose, we consider a stochastic process $\big\{\boldsymbol{Y}_t\big\}_{t=0}^{T-1}$ (uncontrolled), called an *information process*, that follows a dynamic

$$\forall t \in [\![0, T-1]\!], \qquad \boldsymbol{Y}_{t+1} = \tilde{f}_t(\boldsymbol{Y}_{t-1}, \boldsymbol{W}_t) , \tag{8.6}$$

where $\tilde{f}_t$ are known deterministic functions. The choice of the information process is arbitrary, but determines the quality of the method. It will be discussed in §8.4 and §8.5.

For simplicity, we present the algorithm with only one information process. However, it can be extended to multiple information processes, affected to different constraints. This will be done in the dam valley example on which we illustrate the method.

We replace, in Problem (8.2), constraint (8.2f) by its conditional expectation w.r.t the information process (see constraint (8.7f)):

$$\min_{\boldsymbol{X},\boldsymbol{U}} \quad \mathbb{E}\Big[\sum_{i=1}^{N}\sum_{t=0}^{T} L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big)\Big] \tag{8.7a}$$

$$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t) \tag{8.7b}$$

$$\boldsymbol{X}_0^i = x_0^i \tag{8.7c}$$

$$\boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{\mathrm{ad}} \tag{8.7d}$$

$$\boldsymbol{U}_t^i \preceq \mathcal{F}_t \tag{8.7e}$$

$$\mathbb{E}\Big[\sum_{i=1}^{N} \theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) \;\Big|\; \boldsymbol{Y}_t\Big] = 0 . \tag{8.7f}$$

This type of relaxation was studied in Chapter 7.

In the $\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$ pairing, we define the Lagrangian function

$$L(\boldsymbol{U}, \boldsymbol{\mu}) = \mathbb{E}\Big[\sum_{i=1}^{N}\sum_{t=0}^{T} L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) + \boldsymbol{\mu}_t \cdot \mathbb{E}\big[\theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) \bigm| \boldsymbol{Y}_t\big]\Big] , \tag{8.8}$$

where the state process $\boldsymbol{X}$ follows the dynamic equation (8.7b).

Thus, we obtain the following dual problem

$$\begin{aligned} \max_{\boldsymbol{\mu}\in \mathrm{L}^1} \quad \min_{\boldsymbol{X},\boldsymbol{U}} \quad & \mathbb{E}\Big[\sum_{i=1}^{N}\sum_{t=0}^{T} L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) + \boldsymbol{\mu}_t \cdot \mathbb{E}\big[\theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) \bigm| \boldsymbol{Y}_t\big]\Big] \\ & \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t) \\ & \boldsymbol{X}_0^i = x_0^i \\ & \boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{\mathrm{ad}} \\ & \boldsymbol{U}_t^i \preceq \mathcal{F}_t \ . \end{aligned} \tag{8.9}$$

Lemma 8.2. *Assume that there exists an optimal process $\boldsymbol{\mu} = \big\{\boldsymbol{\mu}_t\big\}_{t=0}^T$ for the maximization part of Problem (8.9), with $\mu_t \in \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^p\big)$. Then, the process $\boldsymbol{\mu}^\sharp$ defined by*

$$\boldsymbol{\mu}_t^\sharp = \mathbb{E}\big[\boldsymbol{\mu}_t \bigm| \boldsymbol{Y}_t\big] ,$$

is also an optimal solution to Problem (8.9).

Proof. Indeed if $\boldsymbol{\mu}_t \in \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, then we have,

$$\mathbb{E}\Big[\boldsymbol{\mu}_t \cdot \mathbb{E}\big[\theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) \bigm| \boldsymbol{Y}_t\big]\Big] = \mathbb{E}\Big[\mathbb{E}\big[\boldsymbol{\mu}_t \bigm| \boldsymbol{Y}_t\big] \cdot \mathbb{E}\big[\theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) \bigm| \boldsymbol{Y}_t\big]\Big] .$$

Using this equality in (8.8)

$$L(\boldsymbol{U}, \boldsymbol{\mu}) = L(\boldsymbol{U}, \boldsymbol{\mu}^\sharp) ,$$

hence the result. □

Thus, we can restrict ourselves to multiplier processes $\boldsymbol{\mu}$, such that for all time $t \in [\![0, T]\!]$, $\boldsymbol{\mu}_t$ is measurable w.r.t $\boldsymbol{Y}_t$.

Consequently, using once more Lemma B.3, we can write Problem (8.9) as

$$\begin{aligned} \max_{\boldsymbol{\mu}_t \preceq \boldsymbol{Y}_t} \sum_{i=1}^{N} \min_{\boldsymbol{X}^i,\boldsymbol{U}^i} \quad & \mathbb{E}\Big[\sum_{t=0}^{T} L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) + \boldsymbol{\mu}_t \cdot \theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big)\Big] \\ & \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t) \\ & \boldsymbol{X}_0^i = x_0^i \\ & \boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{\mathrm{ad}} \\ & \boldsymbol{U}_t^i \preceq \mathcal{F}_t \ . \end{aligned} \tag{8.10}$$

Problem (8.10) is equivalent to Problem (8.9), but is simpler:

- the multiplier process $\boldsymbol{\mu}$ of Problem (8.10) lives in a smaller linear space,
- the dual cost in the objective function of Problem (8.10) no longer requires to compute a conditional expectation.

Note that, for a given multiplier $\boldsymbol{\mu}_t^{(k)}$, we have to solve the N following separate inner minimization subproblems.

$$
\begin{aligned}
\min_{\boldsymbol{X}^i,\boldsymbol{U}^i} \quad & \mathbb{E}\Big[\sum_{t=0}^{T} L_t^i\big(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_t\big) + \boldsymbol{\mu}_t^{(k)}\cdot\theta_t^i\big(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_t\big)\Big] \\
& \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_t) \\
& \boldsymbol{X}_0^i = x_0^i \\
& \boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{\mathrm{ad}} \\
& \boldsymbol{U}_t^i \preceq \mathcal{F}_t \; .
\end{aligned} \tag{8.11}
$$

Each inner minimization problem can be solved by Dynamic Programming with the extended state $\big(\boldsymbol{X}_t^i,\boldsymbol{Y}_t\big)$. Indeed, fix a multiplier $\boldsymbol{\mu}_t^{(k)}$ measurable w.r.t $\boldsymbol{Y}_t$, and represented by a measurable function $\mu_t^{(k)}$ such that $\mu_t^{(k)}(\boldsymbol{Y}_t)=\boldsymbol{\mu}_t^{(k)}$. Recalling that the noises $\{\boldsymbol{W}_t\}_0^{T-1}$ are assumed to be time-independent, we can write the following Dynamic Programming equation for the inner minimization problem.

$$
\begin{aligned}
V_t^i\big(x_t^i,y_t\big) = \quad \min_{\boldsymbol{U}_t^i\preceq\boldsymbol{W}_t} \quad & \mathbb{E}\Big[L_t^i\big(x_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_t\big) + \mu_t(y_t)\cdot\theta_t^i\big(x_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_t\big) + V_{t+1}^i\big(\boldsymbol{X}_{t+1}^i,\boldsymbol{Y}_{t+1}\big)\Big] \\
& \boldsymbol{X}_{t+1}^i = f_t^i(x_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_t) \\
& \boldsymbol{Y}_{t+1}^i = \tilde{f}_t^i\big(y_t,\boldsymbol{W}_t\big) \\
& \boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{\mathrm{ad}} \; .
\end{aligned}
$$

Thus, we can solve the inner minimization problem for a given multiplier, by applying Dynamic Programming to the N separate problems.

Remark 8.3. *For notational simplicity, we relaxed the almost sure constraint* (8.2f) *in its conditional expectation with respect to one information process (see Equation* (8.7f)*). However, exactly the same approach can be done with several constraints. More precisely, we consider,*

$$
\forall j \in [\![1,J]\!], \qquad \sum_{i=1}^{n} \Theta_t^{i,j}\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_t = 0, \qquad \mathbb{P}-a.s. \; , \tag{8.12}
$$

and their relaxed counterpart

$$
\forall j \in [\![1,J]\!], \qquad \mathbb{E}\Big[\sum_{i=1}^{n} \Theta_t^{i,j}\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_t \;\Big|\; \boldsymbol{Y}_t^j\Big] = 0 \; , \tag{8.13}
$$

where $\{\boldsymbol{Y}_t^j\}_{t\in[\![0,T-1]\!]}$ *is an information process. There is no difficulty in extending the results to this type of relaxation. This is done in the dam example in §8.1.5 and thereafter.*

8.1.4 General Scheme

We now describe more precisely the DADP algorithm in Algorithm 8.1 given an information process $\{\boldsymbol{Y}_t\}_{t=0}^{T-1}$ satisfying (8.6).

Iteration k of Algorithm 8.1 starts with a multiplier process $\boldsymbol{\mu}_t^{(k)}$. The N inner minimization problems (8.11) are solved, for example, by Dynamic Programming.

From these resolutions, we obtain a *slack process* $\boldsymbol{\Delta}_t^{(k)}$ defined by

$$
\boldsymbol{\Delta}_t^{(k)} := \sum_{i=1}^{N} \theta_t^i\big(\boldsymbol{X}_t^{i,(k)},\boldsymbol{U}_t^{i,(k)},\boldsymbol{W}_t\big) \; , \tag{8.14}
$$

where $\{\boldsymbol{X}_t^{i,(k)}, \boldsymbol{U}_t^{i,(k)}\}_{t=0}^T$ is the solution process of Problem (8.11).

Then, we update the multiplier process by a gradient like step

$$\forall t \in [\![0,T]\!], \qquad \boldsymbol{\mu}_t^{(k+1)} := \boldsymbol{\mu}_t^{(k)} + \rho \mathbb{E}\big[\boldsymbol{\Delta}_t^{(k)} \bigm| \boldsymbol{Y}_t\big] , \tag{8.15}$$

for a given $\rho > 0$. As $\boldsymbol{\mu}_t^{(k)}$ is measurable w.r.t $\boldsymbol{Y}_t$, and $\boldsymbol{Y}_t$ is a finite dimensional random variable, a result from Doob (extended in [36, Chapter 1, p.18]) allows us to represent $\boldsymbol{\mu}_t^{(k)}$ as a function $\mu_t^{(k)}$ of $\boldsymbol{Y}_t$. Moreover, if $\boldsymbol{Y}_t$ takes a finite number of values, $\mu_t^{(k)}$ can be represented by a finite dimensional vector. The update (8.15) of the multiplier process $\{\boldsymbol{\mu}_t^{(k)}\}_{t\in[\![0,T]\!]}$ ends iteration k of the DADP algorithm.

Data: Information process evolution functions $\tilde{f}_t$ and starting point y_0 initial multipliers $\boldsymbol{\mu}_t^{(0)}$;
Result: optimal multipliers $\mu_t^\sharp$, admissible feedback ;
repeat
  forall the $i \in [\![1,N]\!]$ **do**
    Solve Problem (8.11) ;
  forall the $t \in [\![0,T-1]\!]$ **do**
    Estimate $\mathbb{E}\big[\boldsymbol{\Delta}_t^k \bigm| \boldsymbol{Y}_t\big]$;
    Update the multiplier : $\boldsymbol{\mu}_t^{(k)}$ (8.15);
until $\mathbb{E}\big[\boldsymbol{\Delta}_t^k \bigm| \boldsymbol{Y}_t\big] \simeq 0$;
Compute admissible feedbacks ;

Algorithm 8.1: General Scheme of DADP

We sum up the information structure and notations:

- $\boldsymbol{W}_t$ is the noise happening at the beginning of the time period $[t, t+1[$,
- $\mathcal{F}_t = \sigma\Big(\{\boldsymbol{W}_\tau\}_{\tau=0}^t\Big)$ is the σ-algebra of all information contained in the noises realized before time $t+1$,
- $\boldsymbol{U}_t = \{\boldsymbol{U}_t^i\}_{i=1}^N$ is the control applied at the end of the time period $[t, t+1[$, measurable w.r.t $\mathcal{F}_t$,
- $\boldsymbol{X}_t = \{\boldsymbol{X}_t^i\}_{i=1}^N$ is the state of the system at the beginning of $[t, t+1[$, is measurable w.r.t $\mathcal{F}_{t-1}$ (note that this time the index is $t-1$),
- $\boldsymbol{Y}_t$ is the information variable measurable w.r.t $\mathcal{F}_t$,
- $\boldsymbol{\lambda}_t$ is the multiplier of the almost sure constraint (8.2f), measurable w.r.t $\mathcal{F}_t$,
- $\boldsymbol{\mu}_t$ is the multiplier of the conditional constraint (8.7f) measurable w.r.t $\sigma\big(\boldsymbol{Y}_t\big) \subset \mathcal{F}_t$, and we have $\boldsymbol{\mu}_t = \mu_t\big(\boldsymbol{Y}_t\big)$, where μ_t is a deterministic function.

8.1.5 An Hydraulic Valley Example

We illustrate the DADP algorithm on the example of a chain of N dams presented in §1.3.2, more thoroughly developed in [1].

First of all, to recover the framework of Problem (8.2), with dynamical system coupled through constraints, we need to duplicate the outflow of dam $i-1$. It means that we consider $\boldsymbol{Z}_t^i$ as a control variable pertaining to the dam i, and submitted to the constraint

$$\forall i \in [\![2,N]\!], \qquad \boldsymbol{Z}_t^i = g_t^{i-1}\big(\underbrace{\boldsymbol{X}_t^{i-1}, \boldsymbol{U}_t^{i-1}, \boldsymbol{Z}_t^{i-1}}_{:=\boldsymbol{H}_t^{i-1}}, \boldsymbol{W}_t^{i-1}\big) , \quad \text{and} \quad \boldsymbol{Z}_t^1 \equiv 0 . \tag{8.16}$$

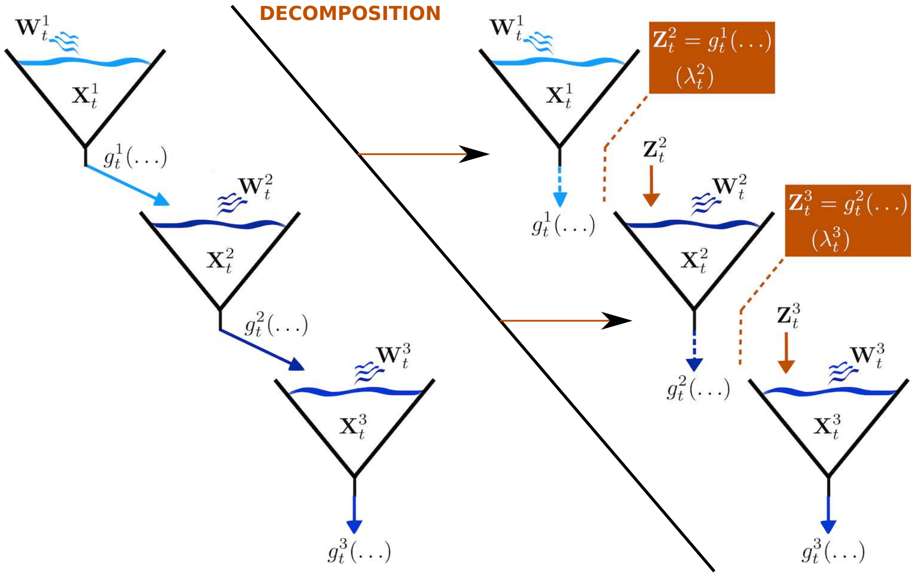


Figure 8.1: From a coupled formulation to a decomposed formulation of the dam problem

Intuitively the control $\boldsymbol{Z}_t^i$ is the water "bought" by the dam i to the dam $i-1$. The price of this exchange is the multiplier $\boldsymbol{\lambda}_t^i$ to constraint (8.16). Note that, if the price is not the actual optimal multiplier, the physical constraint stating that the outflow of dam i goes into dam $i+1$ will not be satisfied. Schematically, this can be seen on Figure 8.1. Note also that there are $N-1$ constraint, hence $N-1$ multiplier processes.

In order to fit the framework of Problem (8.2), the coupling constraint is given by

$$\theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{Z}_t^i, \boldsymbol{W}_t^i\big) = \Big(0, \cdots, 0, \underbrace{-\boldsymbol{Z}_t^i}_{i^{\text{th}}\text{position}}, g_t^i\big(\boldsymbol{H}_t^i, \boldsymbol{W}_t^i\big), 0, \cdots, 0\Big) ,$$

so that constraint (8.2f) coincides with constraint (8.16).

We now explicit the relaxation (8.7). In the rest of the presentation, for notational simplicity, we consider only one information process $\big\{\boldsymbol{Y}_t\big\}_0^T$. However, we can choose one information process per coupling constraint. Hence, for any $i \in [\![2, N]\!]$, we consider an information process $\big\{\boldsymbol{Y}_t^i\big\}_{t=0}^{T-1}$, and we relax constraint (8.16) into

$$\forall i \in [\![2, N]\!], \qquad \mathbb{E}\Big[g_t^{i-1}\big(\boldsymbol{H}_t^{i-1}, \boldsymbol{W}_t^{i-1}\big) - \boldsymbol{Z}_t^i \;\Big|\; \boldsymbol{Y}_t^i\Big] = 0 , \quad \text{and} \quad \boldsymbol{Z}_t^1 \equiv 0 . \tag{8.17}$$

We assume that the information processes $\{\boldsymbol{Y}_t^i\}_{i=1}^N$ satisfy (8.6); more precisely that there are known deterministic functions $\tilde{f}_t^i$ such that

$$\forall i \in [\![1, N]\!], \quad \forall t \in [\![0, T-1]\!], \qquad \boldsymbol{Y}_{t+1}^i = \tilde{f}_t^i\big(\boldsymbol{Y}_t^i, \boldsymbol{W}_t\big) .$$

Thus, the relaxed optimization problem (8.7) reads now

$$\begin{aligned}
\max_{\boldsymbol{\mu}} \quad \min_{\boldsymbol{H}} \quad & \mathbb{E}\Big[\sum_{i=1}^N \sum_{t=0}^T L_t^i\big(\boldsymbol{H}_t^i, \boldsymbol{W}_t\big)\Big] \\
& \boldsymbol{X}_0^i = x_0^i \\
& \boldsymbol{X}_{t+1}^i = f_t^i\big(\boldsymbol{H}_t^i, \boldsymbol{W}_t^i\big) \\
& \boldsymbol{Y}_{t+1}^i = \tilde{f}_t^i\big(\boldsymbol{Y}_t^i, \boldsymbol{W}_t^i\big) \\
& \big(\boldsymbol{U}_t^i, \boldsymbol{Z}_t^i\big) \in \mathcal{U}_{t,i}^{\text{ad}} \\
& \big(\boldsymbol{U}_t^i, \boldsymbol{Z}_t^i\big) \preceq \mathcal{F}_t \\
& \mathbb{E}\big[g_t^{i-1}\big(\boldsymbol{H}_t^{i-1}, \boldsymbol{W}_t^{i-1}\big) - \boldsymbol{Z}_t^i \mid \boldsymbol{Y}_t^i\big] = 0 .
\end{aligned} \tag{8.18}$$

For notational consistency, we introduce a fictitious $\boldsymbol{\mu}_t^{N+1} \equiv 0$, and the dual problem (8.10) reads

$$
\begin{aligned}
\max_{\boldsymbol{\mu}_t^i \preceq \boldsymbol{Y}_t^i} \quad \sum_{i=1}^{N} \quad \min_{\boldsymbol{H}^i} \quad & \mathbb{E}\Big[\sum_{t=0}^{T} L_t^i\big(\boldsymbol{H}_t^i, \boldsymbol{W}_t\big) + \boldsymbol{\mu}_t^{i+1} \cdot g_t^i\big(\boldsymbol{H}_t^i, \boldsymbol{W}_t^i\big) - \boldsymbol{\mu}_t^i \cdot \boldsymbol{Z}_t^i\Big] \\
& \boldsymbol{X}_0^i = x_0^i \\
& \boldsymbol{X}_{t+1}^i = f_t^i\big(\boldsymbol{H}_t^i, \boldsymbol{W}_t^i\big) \\
& \boldsymbol{Y}_{t+1}^i = \tilde{f}_t^i\big(\boldsymbol{Y}_t^i, \boldsymbol{W}_t^i\big) \\
& \big(\boldsymbol{U}_t^i, \boldsymbol{Z}_t^i\big) \in \mathcal{U}_{t,i}^{\mathrm{ad}} \\
& \big(\boldsymbol{U}_t^i, \boldsymbol{Z}_t^i\big) \preceq \mathcal{F}_t \ .
\end{aligned}
\tag{8.19}
$$

8.2 DADP Algorithm Step by Step

Here, we present each step of the DADP algorithm and illustrate it with the dam valley example.

8.2.1 Initialization

The Uzawa algorithm (see Chapter 6) is close to a gradient algorithm for the dual problem. Consequently, we need a good starting point for this gradient algorithm. In some cases, if the random variables were deterministic, the problem could be efficiently solved exactly, yielding the exact Bellman function. From this Bellman function we can determine (see below for an example on the dam valley) the optimal multiplier. Thus, a good idea for an initial μ^0 would be the (deterministic) optimal multiplier for the problem on the mean scenario. More precisely we consider Problem (8.2) where each $\boldsymbol{W}_t$ is replaced by $\mathbb{E}\big[\boldsymbol{W}_t\big]$. This new problem is deterministic and can be solved by specific methods.

Example 8.4. *Let $\{w_t\}_{t=0}^{T-1}$ be a scenario of noise. We consider the following deterministic optimization problem, close to the one presented in §8.1.5*

$$
\min_{(x,u,z)} \quad \sum_{t=0}^{T-1} \sum_{i=1}^{N} L_t^i(x_t^i, u_t^i, w_t^i, z_t^i) + \sum_{i=1}^{N} K_i(x_T^i) \tag{8.20a}
$$

$$
x_{t+1}^i - f_t^i(x_t^i, u_t^i, w_t^i, z_t^i) = 0 \tag{8.20b}
$$

$$
z_t^{i+1} - g_t^i(x_t^i, u_t^i, w_t^i, z_t^i) = 0 \ . \tag{8.20c}
$$

We denote by α_{t+1}^i the multiplier of the dynamic equation (8.20b) and by β_t^{i+1} the multiplier of (8.20c). We dualize (8.20b) and (8.20c) and write the optimality equation on z_t^i (recall that for all $t \in [\![0, T-1]\!]$, we have set $z_{1,t} \equiv 0$).

$$
\forall i \in [\![1, N-1]\!], \quad \forall t \in [\![0, T-1]\!], \qquad \frac{\partial L_t^i}{\partial z} - \alpha_{t+1}^i \cdot \frac{\partial f_t^i}{\partial z} - \beta_t^{i+1} \cdot \frac{\partial g_t^i}{\partial z} + \beta_t^i = 0 \ . \tag{8.21}
$$

We obtain, for all $t \in [\![0, T-1]\!]$, the following backward (in i) equations to determine β_t (recall that $\beta_t^{N+1} \equiv 0$):

$$
\begin{cases}
\beta_t^N = -\frac{\partial L_t^N}{\partial z} + \alpha_{t+1}^N \cdot \frac{\partial f_t^N}{\partial z} \\
\beta_t^i = -\frac{\partial L_t^i}{\partial z} + \alpha_{t+1}^i \cdot \frac{\partial f_t^i}{\partial z} + \beta_t^{i+1} \cdot \frac{\partial g_t^i}{\partial z} & \forall i \in [\![1, N-1]\!]
\end{cases}
\tag{8.22}
$$

In order to obtain the multiplier α we write the Dynamic Programming equation for Problem (8.20).

$$
\begin{cases}
V_T(x_T) & = \sum_{i=1}^{N} K_i(x_T^i) \\
V_t(x_t) & = \min_{(u,z)} \sum_{i=1}^{N} L_t^i(x_t^i, u_t^i, w_t^i, z_t^i) \\
& \qquad + V_{t+1}\Big(f_t^1(x_t^1, u_t^1, w_t^1, z_t^1), \cdots, f_t^N(x_t^N, u_t^N, w_t^N, z_t^N)\Big)
\end{cases}, \tag{8.23}
$$

where the minimum is taken under constraint (8.20c).
After dualization, first order optimality conditions in z_t^i of Problem (8.23) are given by

$$
\frac{\partial L_t^i}{\partial z} + \frac{\partial f_t^i}{\partial z}.\frac{\partial V_{t+1}}{\partial x^i} - \frac{\partial g_t^i}{\partial z}.\beta_t^{i+1} + \alpha_t^i = 0 \;. \tag{8.24}
$$

By identification of (8.24) with (8.21), we deduce the expression of α_t^i:

$$
\alpha_t^i = -\frac{\partial V_t}{\partial x^i}(x_t) \;. \tag{8.25}
$$

Thus, equation (8.22) can be written

$$
\begin{cases}
\beta_t^N = -\frac{\partial L_t^N}{\partial z} + \frac{\partial f_t^N}{\partial z} \cdot \alpha_{t+1}^N & \\
\beta_t^i = -\frac{\partial L_t^i}{\partial z} - \frac{\partial V_{t+1}}{\partial x^i} \cdot \frac{\partial f_t^i}{\partial z} + \beta_t^{i+1} \cdot \frac{\partial g_t^i}{\partial z} & \forall i \in [\![1, N-1]\!]
\end{cases}. \tag{8.26}
$$

Hence, we obtain a starting multiplier $\boldsymbol{\mu}^{(0)}$ by setting $\boldsymbol{\mu}_t^{(0)} \equiv \beta_t$ given by (8.26).

We have seen on the deterministic hydraulic valley example that, if we know the Bellman function of a problem, we can obtain the optimal multiplier $\{\boldsymbol{\mu}_t\}_{t=0}^{T-1}$. Note that the same computation can be done in a non-deterministic setting, and having the exact Bellman functions would give the exact multiplier process as well.

8.2.2 Solving the Inner Problem

At each step of the DADP algorithm (Algorithm 8.1), we have to solve the N inner minimization problems (8.11). For the DADP algorithm, the only output needed to update the multipliers, is the stochastic process $\boldsymbol{\Delta}^{(k)}$ defined in (8.14). Consequently, the inner problems can be solved by any methods available (e.g Stochastic Dual Approximate Dynamic Programming - see Chapter 3).

Without further assumptions, the Dynamic Programming method is available. At iteration k, we initialize the Bellman function $V_T^{i,(k)} \equiv K^i$, and proceed recursively backward in time to construct $V_t^{i,(k)}$. For every possible value of $\boldsymbol{X}_t^i$ denoted x_t^i and value of $\boldsymbol{Y}_t$ denoted y_t, we solve

$$
\begin{aligned}
\min_{\pi_t^i} \quad & \mathbb{E}\Big[L_t^i(x_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t) + \mu_t^{(k)}(\boldsymbol{Y}_t) \cdot \theta_t^i(x_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t) + V_{t+1}^{i,(k)}\Big(\boldsymbol{X}_{t+1}^i, \boldsymbol{Y}_{t+1}\Big)\Big] \\
& \boldsymbol{X}_{t+1}^i = f_t^i(x_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t) \\
& \boldsymbol{Y}_{t+1} = \tilde{f}_t(y_t, \boldsymbol{W}_t) \\
& \boldsymbol{U}_t^i = \pi_t^i(x_t^i, y_t, \boldsymbol{W}_t) \\
& \boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{\mathrm{ad}} \;.
\end{aligned} \tag{8.27}
$$

where the minimization is done on the policies π_t^i that are bounded functions mapping $\mathbb{X}_t^i \times \mathbb{Y}_t \times \mathbb{W}_t$ into $\mathbb{U}_t$. The optimal value of Problem (8.27) is the local Bellman value $V_t^{i,(k)}(x_t^i, y_t)$.

Once all optimal policies $\pi_t^{i,(k)}$ are obtained, we can easily simulate the optimal control and state processes $\big(\boldsymbol{U}_t^{i,(k)}, \boldsymbol{X}_t^{i,(k)}\big)_{t\in[\![0,T]\!]}$, and thus compute the slack process $\Delta_t^{(k)}$ and proceed to the update step.

Obviously, if the support of $\boldsymbol{X}_t^i$ or $\mathbf{Y}_t$ is large (*a fortiori* infinite), the exact computation of $V_t^{i,(k)}$ in (8.27) is not numerically tractable. One solution consists in computing some values of $V_t^{i,(k)}$ and interpolating afterward, as is usual in Dynamic Programming. However, one has to keep in mind that we are solving a relaxed version of Problem (8.2). Moreover this computation has for main objective to update the multiplier process. Thus, we do not need to put too much numerical effort in obtaining a precise approximation of $V_t^{i,(k)}$.

Example 8.5. *On the dam valley example exposed in §8.1.5, the subproblem associated with dam i is given by*

$$
\begin{aligned}
\min_{\boldsymbol{H}^i} \quad & \mathbb{E}\Big[\sum_{t=0}^{T} L_t^i\big(\boldsymbol{H}_t^i, \boldsymbol{W}_t\big) + \mu_t^{i+1,(k)}\big(\mathbf{Y}_t^{i+1}\big)\cdot g_t^i\big(\boldsymbol{H}_t^i, \boldsymbol{W}_t^i\big) - \mu_t^{i,(k)}\big(\mathbf{Y}_t^i\big)\cdot \boldsymbol{Z}_t^i\Big] \\
& \mathbf{Y}_{t+1}^i = \tilde{f}_t^i\big(\mathbf{Y}_t^i, \boldsymbol{W}_t\big) \\
& \mathbf{Y}_{t+1}^{i+1} = \tilde{f}_t^{i+1}\big(\mathbf{Y}_t^{i+1}, \boldsymbol{W}_t\big) \\
& \boldsymbol{X}_{t+1}^i = f_t^i\big(\boldsymbol{H}_t^i, \boldsymbol{W}_t^i\big) \\
& \boldsymbol{X}_0^i = x_0^i \\
& \big(\boldsymbol{U}_t^i, \boldsymbol{Z}_t^i\big) \in \mathcal{U}_{t,i}^{\mathrm{ad}} \\
& \big(\boldsymbol{U}_t^i, \boldsymbol{Z}_t^i\big) \preceq \mathcal{F}_t \ .
\end{aligned}
\tag{8.28}
$$

Problem (8.28) can be solved thanks to the following Dynamic Programming equation[2]

$$
\begin{aligned}
V_t^i\big(x, y^i, y^{i+1}\big) = \mathbb{E}\Big[\min_{(u,z)\in U_{i,t}^{\mathrm{ad}}} \Big\{ & L_t^i\big(x,u,z,\boldsymbol{W}_t\big) + \mu_t^{i+1,(k)}\big(y^{i+1}\big)\cdot g_t^i\big(x,u,z,\boldsymbol{W}_t^i\big) \\
& - \mu_t^{i,(k)} \circ \tilde{f}_{t-1}^i\big(y^i, \boldsymbol{W}_t\big)\cdot z \\
& + V_{t+1}\Big(f_t\big(x,u,z,\boldsymbol{W}_t^i\big), \tilde{f}_t^i\big(y^i,\boldsymbol{W}_t\big), \tilde{f}_t^{i+1}\big(y^{i+1},\boldsymbol{W}_t\big)\Big)\Big\}\Big] .
\end{aligned}
\tag{8.29}
$$

where we compute u and z for each possible value of $\boldsymbol{W}_t$.

Note that we require a state of dimension 3:

- *the physical state x corresponding to the amount of water in dam i;*
- *the information state y^i corresponding to the information process relative to the equality constraint between the outflow of water $g_t^{i-1}(\boldsymbol{H}_t^{i-1}, \boldsymbol{W}_t^i)$ from dam $i-1$ and the inflow of water $\boldsymbol{Z}_t^i$ into dam i;*
- *the information state y^{i+1} corresponding to the information process relative to the equality constraint between the outflow of water $g_t^i(\boldsymbol{H}_t^i, \boldsymbol{W}_t^i)$ from dam i and the inflow of water $\boldsymbol{Z}_t^{i+1}$ into dam $i+1$.*

Note also that the information processes $\boldsymbol{Y}^i$ and $\boldsymbol{Y}^{i+1}$ are not indexed by k as they are uncontrolled processes. In §8.5, we discuss on how the information processes can evolve along the iterations of the algorithm.

---

2. Being in a Hazard-Decision setting, we are able to exchange the minimization and expectation operators.

Solving Problems (8.29) gives an optimal state $\boldsymbol{X}^{i,(k)}$ and controls $\boldsymbol{U}^{i,(k)}, \boldsymbol{Z}^{i,(k)}$. They can be written with feedback law $\eta^{i,(k)}$ and $\gamma^{i,(k)}$ such that

$$\boldsymbol{U}_t^{i,(k)} = \eta^{i,(k)}\big(\boldsymbol{X}_t^{i,(k)}, \boldsymbol{Y}_t^i, \boldsymbol{Y}_t^{i+1}, \boldsymbol{W}_t\big)$$
$$\boldsymbol{Z}_t^{i,(k)} = \gamma^{i,(k)}\big(\boldsymbol{X}_t^{i,(k)}, \boldsymbol{Y}_t^i, \boldsymbol{Y}_t^{i+1}, \boldsymbol{W}_t\big)$$

8.2.3 Multiplier Process Update

We look at the multiplier $\boldsymbol{\mu}_t^{(k)}$ from a functional point of view, that is, we consider a sequence of functions $\mu_t^{(k)}$ such that

$$\forall k \in \mathbb{N}, \quad \forall t \in [\![0, T]\!], \qquad \boldsymbol{\mu}_t^{(k)} = \mu_t^{(k)}\big(\boldsymbol{Y}_t\big) . \tag{8.30}$$

After solving the N subproblems (8.11), we assume that we are able to simulate the slack process $\boldsymbol{\Delta}^{(k)}$ defined in (8.14). We proceed to apply the update step (8.15)

$$\forall t \in [\![0, T]\!], \qquad \boldsymbol{\mu}_t^{(k+1)} := \boldsymbol{\mu}_t^{(k)} + \rho \cdot \mathbb{E}\big[\boldsymbol{\Delta}_t^{(k)} \bigm| \boldsymbol{Y}_t\big] .$$

Remark 8.6. *Note that the update step (8.15) is a gradient step for the dual problem (8.9). If the constraint were $\sum_{i=1}^N \theta_i\big(\boldsymbol{X}_i^t, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) \in -C$ (for example inequality constraints) instead of $\sum_{i=1}^N \theta_i\big(\boldsymbol{X}_i^t, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) = 0$ a projection step would be required.*

Moreover, as the gradient algorithm is known to converge slowly, we might be tempted to use a more advanced update step, for example a quasi-Newton step. Indeed, under smoothness assumptions, the conditional expectation $\mathbb{E}\big[\boldsymbol{\Delta}_t^{(k)} \bigm| \boldsymbol{Y}_t\big]$ gives the gradient of the function maximized in Problem (8.9). This yields good numerical results and we know that, if we find a multiplier process such that the constraint is satisfied, then this multiplier is optimal and the primal solution is the optimal solution (see §8.3.6 for more details).

In order to estimate $\mathbb{E}\big[\boldsymbol{\Delta}_t^{(k)} \bigm| \boldsymbol{Y}_t\big]$ at least two different approaches are possible: Monte-Carlo method, and Fokker-Planck method.

Monte-Carlo We draw a large number S of scenarios of the couple $\big(\boldsymbol{\Delta}_t^{(k)}, \boldsymbol{Y}_t\big)$ denoted by $\big\{\Delta_t^{(k),s}, Y_t^s\big\}_{s=1}^S$, and use classical regression tools to estimate $\boldsymbol{\Delta}_t^{(k)}$ as a function of $\boldsymbol{Y}_t$. For example, if $\boldsymbol{Y}_t$ takes a finite number of values $\big(y_t^l\big)_{l=1}^L$ we can estimate the conditional expectation by Monte-Carlo methods : $\mathbb{E}[\boldsymbol{\Delta}_t^{(k)} \mid \boldsymbol{Y}_t = y_t^{\{l\}}]$ is the empirical mean of $\boldsymbol{\Delta}_t^{(k)}$ over the set of realizations such that $\boldsymbol{Y}_t = y_t^{\{l\}}$.

Fokker-Planck Assume that we have the probability law of $\big(\boldsymbol{X}_t^{i,(k)}, \boldsymbol{Y}_t\big)$. Then, using the optimal feedbacks obtained when solving the subproblems (8.27), we get:

- the exact probability law of $\big(\boldsymbol{X}_{t+1}^{i,(k)}, \boldsymbol{Y}_{t+1}\big)$;
- the probability law of the couple $\big(\boldsymbol{\Delta}_t^{(k)}, \boldsymbol{Y}_t\big)$.

We deduce, by integration of the law of the couple, the law of the conditional expectation $\mathbb{E}[\boldsymbol{\Delta}_t^{(k)} \mid \boldsymbol{Y}_t]$. Hence, we can compute forward the exact law of $\mathbb{E}\big[\boldsymbol{\Delta}_t^{(k)} \bigm| \boldsymbol{Y}_t\big]$.

To compare these methods we give estimations on their complexity. In the following, indexes (k) denote the iteration of the algorithm, $\{l\}$ the values in the support of some random variables, and $[s]$ the realization of a random variable over a scenario.

Remark 8.7. *Assume that every random variable takes a finite number of values and in particular that we have* $\operatorname{supp}(\boldsymbol{Y}_t) = \{y_t^{\{l\}}\}_{l=1}^L$.

Monte-Carlo *By Monte-Carlo method, we do not obtain the exact values of* $\mathbb{E}[\boldsymbol{\Delta}_t^{(k)} \mid \boldsymbol{Y}_t = y_t^{\{l\}}]$. *For a given possible values* $y_t^{\{l\}}$ *of the information process* $\boldsymbol{Y}_t$, *the precision of the estimation of* $\mathbb{E}[\boldsymbol{\Delta}_t^{(k)} \mid \boldsymbol{Y}_t = y_t^{\{l\}}]$ *depends on the number* n_l *of simulated scenarios such that* $y_t^s = y_t^{\{l\}}$. *Indeed, by the Central Limit Theorem, the asymptotic error on* $\mathbb{E}[\boldsymbol{\Delta}_t^{(k)} \mid \boldsymbol{Y}_t = y_t^{\{l\}}]$ *decreases in* $O(1/\sqrt{n_l})$. *If the information process* $\boldsymbol{Y}_t$ *is uniformly distributed on its possible values, then this number is roughly* $n_l \simeq S/L$. *Furthermore, we have* $\min_{l \in [\![1,L]\!]} n_l \leq S/L$. *Moreover, in practice the information process law can be concentrated on some possible values of* $\boldsymbol{Y}_t$, *and there are some realizations* l *such that* n_l *is far smaller than* S/L. *In this case, we cannot trust Monte-Carlo estimate and we may need to use some interpolation methods to estimate* $\mathbb{E}[\boldsymbol{\Delta}_t^{(k)} \mid \boldsymbol{Y}_t = y_t^{\{l\}}]$.

Fokker-Planck *On the other hand, Fokker Plank gives exact results. We determine the complexity of this computation. Assume that we know the law of* $\big(\boldsymbol{X}_t^{i,(k)}, \boldsymbol{Y}_t\big)$, *with finite support of* N_t *elements, and the law of* $\boldsymbol{W}_t$ *(of finite support of* $|\operatorname{supp}\big(\boldsymbol{W}_t\big)|$ *elements). Then, we can compute, using the optimal feedbacks obtained when solving the subproblems, the exact law of* $\big(\boldsymbol{X}_{t+1}^{i,(k)}, \boldsymbol{Y}_{t+1}\big)$ *in* $N_t \times |\operatorname{supp}\big(\boldsymbol{W}_t\big)|$ *operations. Then, computing the exact conditional expectation* $\mathbb{E}\big[\boldsymbol{\Delta}_{t+1}^{(k)} \bigm| \boldsymbol{Y}_{t+1}\big]$ *is straightforward and requires* $|\operatorname{supp}\big(\boldsymbol{Y}_{t+1}\big)|$ *operations. Without any assumptions on the spaces state* $\mathbb{X}_t$ *and* $\mathbb{Y}_t$, *we can only show that* N_t *is growing exponentially with time, more precisely we have* $N_t \leq \prod_{s=0}^{t} |\operatorname{supp}\big(\boldsymbol{W}_s\big)|$, *and this exact computation seems numerically impracticable. However, if we can show that* N_t *is reasonably small, for example if the state space* $\mathbb{X}_t^i$, *and the information state space* $\mathbb{Y}_t$ *are discrete, then computing the law of* $\big(\boldsymbol{X}_{t+1}^{i,(k)}, \boldsymbol{Y}_t\big)$ *(and simultaneously the law of* $\Big(\boldsymbol{\Delta}_t^{(k)}, \boldsymbol{Y}_t\Big)$*) from the law of* $\big(\boldsymbol{X}_t^{i,(k)}, \boldsymbol{Y}_t\big)$ *is of complexity* $O\big(|\mathbb{X}_t^i| \times |\mathbb{Y}_t| \times |\mathbb{W}_t|\big)$, *and deriving the law of* $\mathbb{E}\big[\boldsymbol{\Delta}_{t+1}^{(k)} \bigm| \boldsymbol{Y}_{t+1}\big]$ *is of complexity* $O\big(|\operatorname{supp}\big(\boldsymbol{Y}_{t+1}\big)|\big)$.

Example 8.8. ***Monte-Carlo Approach.*** *In Example 8.5 we obtained feedback laws* $\eta^{i,(k)}$ *and* $\gamma^{i,(k)}$. *Using these feedbacks, and a realization of the noises* $\big(w_t^{[s]}\big)_{t \in [\![0,T]\!]}$ $(s \in [\![1,S]\!])$, *we obtain a sample of* $\boldsymbol{X}^{i,(k)}$, $\boldsymbol{U}^{i,(k)}$, *and* $\boldsymbol{Z}^{i,(k)}$, *with the following equation (and the corresponding starting point)* [3]

$$
\begin{aligned}
u_t^{i,(k),[s]} &= \eta_t^{i,(k)}\Big(x_t^{i,(k),[s]}, y_t^{i+1,[s]}, y_t^{i,[s]}, w_t^{[s]}\Big) \\
z_t^{i,(k),[s]} &= \gamma_t^{i,(k)}\Big(x_t^{i,(k),[s]}, y_t^{i+1,[s]}, y_t^{i,[s]}, w_t^{[s]}\Big) \\
x_{t+1}^{i,(k),[s]} &= f_t\Big(x_t^{i,(k),[s]}, u_t^{i,(k),[s]}, z_t^{i,(k),[s]}, w_t^{i,[s]}\Big) \\
y_{t+1}^{i,[s]} &= \tilde{f}_t\Big(y_t^{i,[s]}, w_t^{[s]}\Big) \\
y_{t+1}^{i+1,[s]} &= \tilde{f}_t\Big(y_t^{i+1,[s]}, w_t^{[s]}\Big)
\end{aligned}
$$

Thus, for S *realizations of* $\boldsymbol{W}$, *we obtain* S *trajectories of* $\boldsymbol{\Delta}^{(k)}$ *given by*

$$
\Delta_t^{i,(k),[s]} := g_t^{i-1}\Big(x_t^{i-1,(k),[s]}, u_t^{i-1,(k),[s]}, z_t^{i-1,(k),[s]}, w_t^{i,[s]}\Big) - z_t^{i,(k),[s]} \ .
$$

We fit a function $\delta_t^{i,(k)}$, *that takes the possible values of* $\boldsymbol{Y}_t^{(i)}$, *and give an estimation of the value of* $\Delta_t^{i,(k)}$. *Then, we update the functional multiplier* $\mu_t^{i,(k)}$ *by*

$$
\mu_t^{i,(k+1)}(\cdot) := \mu_t^{i,(k)}(\cdot) + \rho\delta_t^{i,(k)}(\cdot) \ . \tag{8.31}
$$

3. If the same realization of the noises $\big(w_t^{[s]}\big)_{t \in [\![0,T]\!]}$ are used throughout the algorithm, then the trajectories of the information processes $\boldsymbol{Y}_t^i$ can be computed once at the beginning of the algorithm.

To be more specific, assume that the information process $\mathbf{Y}_t^i$ takes only discrete values denoted by $\{y_t^{i,\{l\}}\}_{l=1}^S$. Then $\delta_t^{i,(k)}$ can be defined as

$$\forall l \in [\![1, L]\!], \qquad \delta_t^{i,(k)}\big(y_t^{i,\{l\}}\big) = \frac{\sum_{s=1}^{S} \Delta_t^{i,(k),[s]} \cdot \mathbb{1}_{\left\{y_t^{i,[s]}=y_t^{i,\{l\}}\right\}}}{\sum_{s=1}^{S} \mathbb{1}_{\left\{y_t^{i,[s]}=y_t^{i,\{l\}}\right\}}} .$$

This is a Monte Carlo estimation of a conditional expectation. However we cannot choose the number of simulations satisfying $y_t^{i,[s]} = y_t^{i,\{l\}}$. More precisely an approximation of this number is $L\mathbb{P}\big(\mathbf{Y}_t^i = y_t^{i,\{l\}}\big)$, and can even be equal to 0. When this number is small the Monte-Carlo estimation cannot be trusted, and some interpolation may be a better approach.

Fokker-Planck approach. *Assume, furthermore, that the local state space $\mathbb{X}_t$, and the information state space $\mathbb{Y}_t$ are discrete. We compute forward in time the law of*

$$\big(\boldsymbol{X}_t^{i-1,(k)}, \boldsymbol{X}_t^{i,(k)}, \mathbf{Y}_t^{i-1}, \mathbf{Y}_t^{i}, \mathbf{Y}_t^{i+1}\big) ,$$

and deduce the exact value of $\delta^{i,(k)}$. The easiest way to understand this computation is to see the pseudo-code in Algorithm 8.2, where $P_t(x^{i-1}, x^i, y^{i-1}, y^i, y^{i+1})$ stands for

$$\mathbb{P}\big(\boldsymbol{X}_t^{i-1,(k)} = x^{i-1}, \boldsymbol{X}_t^{i,(k)} = x^{i}, \mathbf{Y}_t^{i-1} = y^{i-1}, \mathbf{Y}_t^{i} = y^{i}, \mathbf{Y}_t^{i+1} = y^{i+1}\big) ,$$

$\pi^i(y)$ represents $\mathbb{P}\big(\mathbf{Y}_t^i = y\big)$, and $\delta^i(y)$ represents $\delta_t^{i,(k)}$ such that $\mathbb{E}\big[\Delta_t^{i,(k)} \mid \mathbf{Y}_t^i\big] = \delta_t^{i,(k)}\big(\mathbf{Y}_t^i\big)$. The function $\delta_t^{i,(k)}$ is directly used to update the functional multipliers:

$$\mu_t^{i,(k+1)}(\cdot) = \mu_t^{i,(k)}(\cdot) + \rho\delta_t^{i,(k)}(\cdot) .$$

8.2.4 Back to Admissibility

We have stressed that the DADP algorithm solves an approximation of Problem (8.2). More precisely, the coupling constraint (8.2f) is approximated by relaxation (the set of admissible policies is extended). Thus, even once the algorithm has converged, the optimal policy found is not supposed to be admissible with respect to the original almost sure constraint. Hence, any practical implementation must incorporate a way of recovering admissible strategies.

We propose a natural way of obtaining an admissible strategy for problem (8.2). Assume that the method used to solve the inner problem (8.27) yields a local Bellman value function $V_t^i : \mathbb{X}_t^i \times \mathbb{Y}_t \to \mathbb{R}$ that takes as arguments the local state and the information state. Roughly speaking, we then approximate the Bellman function of the global problem as the sum $\hat{V}_t(x, y) := \sum_{i=1}^N V_t^i(x_i, y)$ of the local Bellman functions. An admissible policy π^{ad} is obtained by solving one step of the dynamic programming equation with this approximate value function. More precisely, we have

$$\begin{aligned}
\pi_t^{\mathrm{ad}}(x, y) \in \arg\min_{\pi} \quad & \mathbb{E}\Big[\sum_{i=1}^{n} L_t^i\big(x_i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) + V_{t+1}^i\Big(f_t^i\big(x_i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big)\Big), \tilde{f}_t\big(y, \boldsymbol{W}_t\big)\Big] \\
& \boldsymbol{U}_t = \pi_t(x, y, \boldsymbol{W}_t) \\
& \boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{\mathrm{ad}} \\
& \sum_{i=1}^{N} \theta_i\big(x_i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) = 0
\end{aligned} \tag{8.32}$$

Data: Joint law of $\big(\boldsymbol{X}_t^{i-1,(k)}, \boldsymbol{X}_t^{i,(k)}, \boldsymbol{Y}_t^{i-1}, \boldsymbol{Y}_t^{i}, \boldsymbol{Y}_t^{i+1}\big)$;
Result: Joint law of $\big(\boldsymbol{X}_{t+1}^{i-1,(k)}, \boldsymbol{X}_{t+1}^{i,(k)}, \boldsymbol{Y}_{t+1}^{i-1}, \boldsymbol{Y}_{t+1}^{i}, \boldsymbol{Y}_{t+1}^{i+1}\big)$, exact function $\delta_t^{i,(k)}$;

```
// Initialization
```
$P_{t+1}\big(x^{i-1}, x^i, y^{i-1}, y^i, y^{i+1}\big) = 0$;
`// Joint law of` $\big(\boldsymbol{X}_t^{i-1,(k)}, \boldsymbol{X}_t^{i,(k)}, \boldsymbol{Y}_t^{i-1}, \boldsymbol{Y}_t^{i}, \boldsymbol{Y}_t^{i+1}\big)$
$\delta^i(y^i) = 0, \quad \forall y^i \in \mathbb{Y}_t^i$; `// function` $\delta_t^{i,(k)}$
$\pi^i(y^i) = 0, \quad \forall y^i \in \mathbb{Y}_t^i$; `//` $\mathbb{P}(\boldsymbol{Y}_t^i = y)$
`// Computation of joint laws`
forall the $\big(x^{i-1}, x^i, y^{i-1}, y^i, y^{i+1}, w\big) \in \mathbb{X}_t^{i-1} \times \mathbb{X}_t^{i} \times \mathbb{Y}_t^{i-1} \times \mathbb{Y}_t^{i} \times \mathbb{Y}_t^{i+1} \times \mathbb{W}_t$ **do**

> $p = P_t(x^i, y^i, y^{i+1}) \cdot \mathbb{P}\big(\boldsymbol{W}_t = w\big)$;
> `// probability of` $\big(x^{i-1}, x^i, y^{i-1}, y^i, y^{i+1}, w\big)$
> $u^{i-1} = \eta_t^{i-1,(k)}\big(x^{i-1}, y^{i-1}, y^i, w\big)$;
> $z^{i-1} = \gamma_t^{i-1,(k)}\big(x^{i-1}, y^{i-1}, y^i, w\big)$;
> $u^{i} = \eta_t^{i,(k)}(x^{i}, y^{i}, y^{i+1}, w)$;
> $z^{i} = \gamma_t^{i,(k)}(x^{i}, y^{i}, y^{i+1}, w)$;
> $\hat{x}^{i-1} = f_t^i\Big(x^{i-1}, u^{i-1}, z^{i-1}, w^{i-1}\Big)$; `// value of` $\boldsymbol{X}_{t+1}^{i-1,(k)}$
> $\hat{x}^{i} = f_t^i\Big(x^{i}, u^{i}, z^{i}, w^{i}\Big)$; `// value of` $\boldsymbol{X}_{t+1}^{i,(k)}$
> $\hat{y}^{i-1} = \tilde{f}_t^{i-1}\big(y^{i-1}, w\big)$; `// value of` $\boldsymbol{Y}_{t+1}^{i-1}$
> $\hat{y}^{i} = \tilde{f}_t^{i}\big(y^{i}, w\big)$; `// value of` $\boldsymbol{Y}_{t+1}^{i}$
> $\hat{y}^{i+1} = \tilde{f}_t^{i+1}\big(y^{i+1}, w\big)$; `// value of` $\boldsymbol{Y}_{t+1}^{i}$
> $d^i = g_t^{i-1}\big(x^{i-1}, u^{i-1}, z^{i-1}, w^{i-1}\big) - z^i$; `// slackness`
> $\delta^i(y^i) = \delta^i(y^i) + p \cdot d^i$; `//` $\mathbb{E}\big[\boldsymbol{\Delta}_t^{i,(k)} \cdot \mathbb{1}_{\boldsymbol{Y}_t^i = y_i}\big]$
> $\pi^i(y^i) = \pi^i(y^i) + p$; `//` $\mathbb{P}\big(\boldsymbol{Y}_t^i = y^i\big)$
> $P_{t+1}\big(\hat{x}^{i-1}, \hat{x}^i, \hat{y}^{i-1}, \hat{y}^i, \hat{y}^{i+1}\big) = P_{t+1}\big(\hat{x}^{i-1}, \hat{x}^i, \hat{y}^{i-1}, \hat{y}^i, \hat{y}^{i+1}\big) + p$;

`// Renormalization of` $\delta_t^{i,(k)}$
forall the $y^i \in \mathbb{Y}_t^i$ **do**

> $\delta^i(y^i) = \delta^i(y^i)/\pi^i(y^i)$; `//` $\mathbb{E}\big[\boldsymbol{\Delta}_t^{i,(k)} \cdot \mathbb{1}_{\boldsymbol{Y}_t^i = y^i}\big]/\mathbb{P}\big(\boldsymbol{X}_{t+1}^{(k)} = x, \boldsymbol{Y}_{t+1}^i = y^i\big)$

Algorithm 8.2: Exact update of multiplier μ_t^i for the hydraulic valley example

Notice that the admissible policy π^{ad} obtained depends on the physical state x_t, the information state y_t and the noise $\boldsymbol{W}_t$. It means that, to implement and simulate the strategy π^{ad}, we have to compute the information process $\boldsymbol{Y}_t$. This increases the global state of the system, leading to numerical difficulties. However specific cases, presented in §8.4 and §8.5, are of interest:

- if the information process $\boldsymbol{Y}_t$ is constant, it does not increase the state of the global problem;
- if the information process $\boldsymbol{Y}_t$ is a part of the noise $\boldsymbol{W}_t$, it does not increase the state of the global problem (we are in a hazard-decision setting);
- if the information process $\boldsymbol{Y}_t$ is aimed at mimicking a function of the state $h(\boldsymbol{X}_t)$ then it can be replaced in Problem (8.32) by $h(\boldsymbol{X}_t)$ (more information in §8.5).

This heuristic requires to solve the problem on the whole system, but only for one time step, at the state where the computation of the policy is required. This corresponds to an iteration of policy iteration algorithm, which is known to be efficient (for infinite horizon problem). This heuristic gave some good numerical results (see §8.4), but have few theoretical properties. In particular, we show in §8.4 that a control process obtained when solving Problem (8.7) with $\boldsymbol{Y} = 0$ gives, after this heuristic is applied, better results than a control process obtained with a more precise approximation.

Example 8.9. *Continuing the examples 8.5 and 8.8, we assume that we obtained, after a large number of iterations, controls that are (almost) optimal for Problem (8.18). These controls do not satisfy the almost sure constraint (8.16).*

An heuristic to obtain an admissible solution satisfying the almost sure constraint is given by finding, for each[4] *collection of states* $x_t = \{x_t^i\}_{i=1}^N$*, each collection of information state* $y_t = \{y_t^i\}_{i=1}^{N-1}$*, and each noise* w_t*, an optimal solution*[5] $\big(u_t^{\mathrm{ad}}(x,y,w), z_t^{\mathrm{ad}}(x,y,w)\big)$ *to the following problem*

$$\begin{aligned} \min_{u,z} \quad & \sum_{i=1}^{n} L_t^i(x^i,u^i,z^i,w^i) + V_{t+1}^i\Big(f_t^i(x^i,u^i,z^i,w^i), \tilde{f}_t^i(y^i,w), \tilde{f}_t^{i+1}(y^{i+1},w)\Big) \\ & (u^i,z^i) \in \mathcal{U}_{t,i}^{\mathrm{ad}} \\ & z^i = g^{i-1}(x^{i-1},u^{i-1},z^{i-1},w^{i-1}) \ . \end{aligned} \tag{8.33}$$

Notice that, if we want to implement and simulate the admissible strategy $(u^{\mathrm{ad}}, z^{\mathrm{ad}})$ *obtained, we need to have computed, at each step, the current information process* $\boldsymbol{Y}_t$*. If the information state is a part of the noise (say* $\boldsymbol{W}^{i-1}$*), it is readily available. If the information process is aimed at mimicking some part of the state (say* $\boldsymbol{X}^{i-1}$*), we can simply replace* y^i *by* x^{i-1}*, and* $\tilde{f}^i$ *by* f^i *in the resolution of Problem (8.33).*

8.3 Theoretical Analysis of the DADP Method

We present now how the results obtained in Chapters 4-7 apply to the DADP algorithm.

In the overview of the DADP method, we have given one possible interpretation of the algorithm. Other interpretations are available and presented in §8.3.1. Then, we go on to give convergence results. First, in §8.3.2, we explain how the relaxation of constraint (8.2f) into constraint (8.7f) can be understood, and show an epi-convergence of the approximation result. Then, in §8.3.3, we state the conditions for the existence of a saddle point, and thus, in §8.3.4, the result of convergence of the coordination algorithm for a given information process. In §8.3.6, we recall a complimentary result about the convergence of a dual algorithm. Finally, in §8.3.7, we comment on the bounds obtained through the DADP algorithm.

8.3.1 Interpretations of the DADP method

We present three way of looking at the DADP method. First, as the resolution of a relaxation of the primal problem. Then, as an approximation of the multiplier $\boldsymbol{\lambda}$. Finally, as a decision rule approach (i.e. an addition of constraints) for the dual problem.

DADP as Constraint Relaxation.

DADP can be seen as solving the dual of the relaxation (8.7). Hence, the primal solution obtained is not admissible for the original problem (8.2). This approach was presented in §8.1.3.

4. Obviously, in practice, we do not compute the whole strategy, but only evaluate it at the points required for the simulation.

5. We do not dwell on the fact that a measurable selection theorem should be used to turn $u_t^\sharp$ and $z_t^\sharp$ into a strategy.

DADP as Approximation of $\boldsymbol{\lambda}$.

Another way of looking at the DADP consists in considering the subproblem (8.5) As stated in §8.1.2, this problem cannot be numerically solved by Dynamic Programming, due to the curse of dimensionality. Hence, in the DADP algorithm, we approximate the multiplier $\boldsymbol{\lambda}_t^{(k)}$ by its conditional expectation with respect to the information process $\boldsymbol{Y}_t$.

Note that keeping track of the multiplier process $\boldsymbol{\lambda}_t^{(k)}$ is of no use, as only its conditional expectation with respect to $\boldsymbol{Y}_t$ is updated along the DADP algorithm.

Moreover, assume for the sake of clarity that there is a unique multiplier $\boldsymbol{\lambda}^\sharp \in \mathrm{L}^2$ solution of Problem (8.4), and a unique solution $\boldsymbol{\mu}^\sharp$ of Problem (8.9). Then, we have $\boldsymbol{\mu}_t^\sharp = \mathbb{E}\big[\boldsymbol{\lambda}_t^\sharp \mid \boldsymbol{Y}_t\big]$. In other words, the solution $\boldsymbol{\mu}_t^\sharp$ is the projection (in the L^2 sense) of the optimal multipliers $\boldsymbol{\lambda}_t^\sharp$ on the linear subset of the $\sigma(\boldsymbol{Y}_t)$-measurable random variable.

This is the historic way of presenting and understanding the DADP algorithm (see [10, 50]).

DADP as Decision-Rule Approach to the Dual Problem.

Finally, consider the dual problem (8.4). Solving this problem is quite difficult as $\boldsymbol{\lambda}$ lives in the set of ($\mathfrak{F}$-adapted) integrable stochastic processes. A common approach in the literature is the so-called decision rule approach (see [49, 63] and reference therein). The idea behind this approach consists in reducing the space in which the control is looked for. For example, we can consider feedbacks that are linear in the state only. DADP is a decision-rule approach to problem (8.4) where the optimal multipliers are searched in the space of integrable stochastic processes such that $\boldsymbol{\lambda}_t$ is measurable with respect to $\boldsymbol{Y}_t$. In other words, it means that we have added some constraints to the maximization problem (8.4).

8.3.2 Consistence of the Approximation

In §8.1.1, we showed that, if we apply a spatial decomposition scheme on the original problem (8.2), then the dual multiplier $\boldsymbol{\lambda}$ is a $\mathfrak{F}$-adapted process. Thus solving the inner subproblem (8.11) requires an information state of large dimension.

Accordingly, we considered an approximation of the original problem. This approximation is studied in Chapter 7; more precisely, Problem (8.7) falls in the framework presented in §7.2.3. We show, in the following proposition, that, if we refine this approximation we obtain a converging sequence of optimization problems.

Proposition 8.10. *Consider Problem (8.2). Assume that, for all $i \in [\![1, N]\!]$ and $t \in [\![0, T-1]\!]$,*

- *the cost functions L_t^i, dynamic functions f_t^i and constraint functions θ_t^i are continuous;*
- *the noise variables $\boldsymbol{W}_t$ are essentially bounded;*
- *the constraint sets $\mathcal{U}_{i,t}^{\mathrm{ad}}$ are bounded.*

Consider a sequence of information process $\big\{\boldsymbol{Y}^{(n)}\big\}_{n\in\mathbb{N}}$ and a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of non-negative reals converging toward 0. Denote by $\boldsymbol{U}^{(n)}$ an ε_n-optimal solution to the relaxation (8.7) of Problem (8.2) corresponding to the information process $\boldsymbol{Y}^{(n)}$. Assume that the sequence $\big\{\sigma\big(\boldsymbol{Y}_t^{(n)}\big)\big\}_{n\in\mathbb{N}}$ of σ-algebra Kudo-converges[6] toward $\sigma\big(\boldsymbol{Y}_t^{(\infty)}\big)$.

Then, every cluster point of $\big\{\boldsymbol{U}^{(n)}\big\}_{n\in\mathbb{N}}$, for the topology of the convergence in probability, is an optimal solution of the relaxation (8.7) of Problem (8.4) corresponding to the information process $\boldsymbol{Y}^{(\infty)}$.

6. See §7.2.1 for recalls and pointers for this notion.

Remark 8.11. *The boundedness assumption of the constraint sets $\mathcal{U}_{i,t}^{\mathrm{ad}}$ can be replaced by a coercivity assumption of the local costs L_t^i. In §8.3.3, and thus in §8.3.4, the boundedness of $\mathcal{U}_{i,t}^{\mathrm{ad}}$ is troublesome to ensure the existence of a L^1-multiplier. However, in §8.3.4 the assumed strong convexity of L_t^i ensures the coercivity of L_t^i.*

Proof. We use the results of Chapter 7. Denote by $\boldsymbol{U}'_t = \{\boldsymbol{U}_\tau\}_{\tau=0}^t$ the sequence of controls, and the objective function by

$$J(\boldsymbol{U}'_T) = \mathbb{E}\Big[\sum_{i=1}^N \sum_{t=0}^T L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t^i\big)\Big],$$

where $\{\boldsymbol{X}_t\}_{t=0}^T$ follows the dynamic equations

$$\forall t \in [\![0, T-1]\!], \quad \forall i \in [\![1, N]\!], \qquad \boldsymbol{X}_{t+1}^i = f_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) .$$

Denote the constraint functions by

$$\Theta_t\big(\boldsymbol{U}'_t\big) = \sum_{i=1}^N \theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big)$$

and by $C_t = \{0\}$ the cone of constraints which is stable with respect to any sequence of σ-algebra. We endow the set of controls with the topology of convergence in probability.

By boundedness of $\mathcal{U}_{t,i}^{\mathrm{ad}}$, the control variables $\boldsymbol{U}_t^i$ are essentially bounded. We show by induction that the state process $\boldsymbol{X}_t$ is essentially bounded (true for $t = 0$). If it is true at time t, then, by continuity of the dynamic functions f_t^i, and essential-boundedness of $\boldsymbol{U}_t$, we have that $\boldsymbol{X}_{t+1}$ is essentially bounded. Then, by induction and continuity of L_t^i we have that $L_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t^i\big)$ is essentially bounded. Consequently, the cost function can be replaced by a bounded function. Therefore, Proposition 7.9 ensures that J is continuous. Similarly, Proposition 7.10 ensures that Θ is continuous.

Thus, Theorem 7.6 implies that the sequence of approximated problems converges toward the initial problem if $\sigma\big(\boldsymbol{Y}^{(n)}\big)$ Kudo-converges toward $\mathcal{F}$. More precisely, we can extract a subsequence $(\boldsymbol{U}^{(n_k)})_{k\in\mathbb{N}}$ converging almost surely to an optimal solution of Problem (8.2). More generally, any cluster point (for the convergence in probability topology) of $\big(\boldsymbol{U}^{(n)}\big)_{n\in\mathbb{N}}$ is an optimal solution of Problem (8.2).

□

Corollary 8.12. *Under the assumption of Proposition 8.10, and assuming that, for all time $t \in [\![0, T]\!]$, $\sigma\big(\boldsymbol{Y}_t^{(\infty)}\big) = \mathcal{F}_t$, then any cluster point of $\boldsymbol{U}^{(n)}$ is an optimal solution of Problem (8.2).*

8.3.3 Existence of Saddle Point

In Chapter 5, we gave conditions ensuring the existence of a multiplier in the space L^1 for almost sure affine constraints. Those conditions rely on the property that the objective cost J is finite everywhere on the space L^∞.

A first interesting fact is the following.

Fact 8.13. *If Problem (8.2) admits an optimal multiplier $\boldsymbol{\lambda}^\sharp$ in L^1 for constraint (8.2f), then Problem (8.7) admits an optimal multiplier in L^1 for constraint (8.7f), namely*

$$\boldsymbol{\mu}_t^\sharp = \mathbb{E}\big[\boldsymbol{\lambda}_t^\sharp \bigm| \boldsymbol{Y}_t\big] .$$

Proof. This is a direct application of Lemma 8.2. □

The following Proposition is a direct application of the results obtained in §5.3.

Proposition 8.14. *In Problem (8.2), assume that*
- *the random noises* $\boldsymbol{W}_t$ *are essentially bounded;*
- *the local cost functions* L_t^i *are finite and convex in* (x_i, u_i)*, continuous in* w*;*
- *the dynamic functions* f_t^i *are affine in* (x_i, u_i)*, continuous in* w*;*
- *the constraint sets* $\mathcal{U}_{t,i}^{\mathrm{ad}}$ *are* $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{U}_{i,t}\big)$*;*
- *the constraint functions* θ_t^i *are affine.*

Then, the coupling constraint (8.2f) admits a multiplier in L^1*, hence the relaxed coupling constraint (8.7f) admits a multiplier in* L^1*.*

Proof. It is easy to recover the framework of §5.3. Indeed,
- the state $\boldsymbol{X}_t$ (of §5.3), is the collection $\{\boldsymbol{X}_t^i\}_{i=1}^N$;
- the control $\boldsymbol{D}_t$ is the collection $\{\boldsymbol{U}_t^i\}_{i=1}^N$;
- the dynamic functions f_t are given, coordinate by coordinate, by the dynamic functions f_t^i;
- the cost functions L_t are given by $L_t = \sum_{i=1}^N L_t^i$;
- the measurability constraints are equivalent.

The only noticeable difference of formulation is in the almost sure constraint. As each θ_t^i is an affine function, we can write

$$\sum_t^i \theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big) = A_t \boldsymbol{X}_t + B_t \boldsymbol{U}_t + C_t \boldsymbol{W}_t \; .$$

Hence, the coupling constraint (8.2f) reads

$$A_t \boldsymbol{X}_t + B_t \boldsymbol{U}_t = \boldsymbol{B}_t \; .$$

Corollary 5.27 ends the proof. □

Note that this result does not allow bound constraints on $\boldsymbol{U}_t^i$. Other results about a multiplier in L^1 rely on a relatively complete recourse assumption. In practice, for example on a dam management problem, effective bounds on the control or the state are enforced through penalization, which ensure the finiteness of the cost (and hence the relatively complete recourse assumption). Consequently, it is our belief that we can give better results of existence of an L^1 multiplier, using relatively complete recourse assumption, but that might not extend dramatically the field of applications.

8.3.4 Convergence of Uzawa Algorithm

We have seen that the DADP algorithm is an Uzawa algorithm in L^∞ applied to Problem (8.3). Uzawa algorithm (see Algorithm 6.1) in Banach spaces was presented and studied in Chapter 6. Hence, we have the following convergence proposition.

Proposition 8.15. *Assume that,*
- *the noises* $\boldsymbol{W}_t$ *have a finite support*[7]*;*
- *the local cost* L_t^i *are Gâteaux-differentiable functions, strongly convex (in* (x, u)*) and continuous (in* w*);*
- *the coupling functions* θ_t^i *are affine;*
- *the evolution functions* f_t *are affine (in* (x, u, w)*);*
- *the admissible set* $\mathcal{U}_{i,t}^{\mathrm{ad}}$ *is a weak** *closed, non-empty convex set;*
- *there exists an admissible control, i.e. a process* $\big(\boldsymbol{X}, \boldsymbol{D}\big)$ *satisfying all constraints of Problem (8.7);*

7. The finite support assumption is only required to obtain the strong convexity of the global cost.

- *Constraint* (8.7f) *admits an optimal multiplier in* L^1*.*

Consider the sequence of controls $\{\boldsymbol{U}^{(n)}\}_{n\in\mathbb{N}}$ *generated by Algorithm 8.1. Then, for a step* $\rho > 0$ *small enough, there exists a subsequence* $\{\boldsymbol{U}^{(n_k)}\}_{k\in\mathbb{N}}$ *converging in* L^∞ *toward the optimal control of the relaxed problem* (8.7)*.*

Moreover, if there exists an optimal multiplier[8] *in* L^2 *the whole sequence* $\{\boldsymbol{U}^{(n)}\}_{n\in\mathbb{N}}$ *converges in* L^∞ *toward the optimal control of the relaxed problem* (8.7)*.*

Proof. We apply the results of §6.3 with a slight modification of the constraint function Θ in the proof of Proposition 6.11. Indeed, in (6.27), we replace the function Θ_t by

$$\widehat{\Theta}_t := \mathbb{E}\big[\Theta_t(\cdot) \bigm| \boldsymbol{Y}_t\big] ,$$

which is still Lipschitz for the L^∞-norm. Considering the linear operator

$$\widehat{\Theta}_t^\dagger := \mathbb{E}\big[\Theta_t^\dagger(\cdot) \bigm| \boldsymbol{Y}_t\big] ,$$

we can apply Proposition 5.12 to show the weak* continuity of $\widehat{\Theta}_t$.

If there exists an optimal multiplier in L^2, then Corollary 6.7 instead of Proposition 6.11 concludes the proof. □

8.3.5 Consequences of Finitely Supported Noises

We now assume that the set of uncertainties $\mathbb{W}$ is finite, and point out the consequences for the previous theoretical results. As $\mathbb{W}$ is finite we can represent Problem 8.2 with a finite set of scenarios Ω. Hence, the sets $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, in which live the control $\boldsymbol{U}$ and state $\boldsymbol{X}$, are finite dimensional vector spaces.

In this setting, the results of §8.3.2 are obvious: Kudo-convergence of a sequence of finitely generated σ-algebras implies that the sequence is eventually stationary and equal to the limit. Hence, the sequence of relaxed optimization problems is eventually equal to the limit problem.

With finite Ω, there is no difference between a L^1, L^2 or L^∞ multiplier. In particular affine equality constraints (with a finite objective function) are qualified and admit an optimal multiplier without requiring the other assumptions of §8.3.3. More generally Mangasarian-Fromovitz condition can be used to obtain the existence of multiplier.

The space of controls being a Hilbert space, the usual Uzawa convergence result (recalled in §6.1.2) holds true. In particular the admissible set is only required to be closed convex, and the existence of a multiplier yields the convergence of the whole sequence of controls generated by DADP.

8.3.6 Validity of Solution After Convergence

We have seen that it is difficult to guarantee the convergence of the Uzawa algorithm in the DADP method. However, if we numerically obtain a multiplier process $\boldsymbol{\mu}^{(\infty)}$ such that the corresponding primal solution $\boldsymbol{U}^{(\infty)}$, i.e. the solution of the minimization part of Problem (8.9) for $\boldsymbol{\mu} = \boldsymbol{\mu}^{(\infty)}$, satisfies the coupling constraint (8.7f), then the primal solution $\boldsymbol{U}^{(\infty)}$ is a solution of Problem (8.7). This is a direct consequence of the following proposition from Everett [46].

Proposition 8.16. *Consider an objective function* $J : \mathcal{U} \to \overline{\mathbb{R}}$*, and a constraint function* $\Theta : \mathcal{U} \to \mathcal{V}$*, where* $\mathcal{V}$ *is a topological space paired with* $\mathcal{Y}$*. Consider a closed convex constraint cone* $C \subset \mathcal{V}$*.*

8. The finiteness assumption of the noises implies that an L^1 multiplier is also an L^2 multiplier. We keep the difference only so that the finiteness assumption is used to obtain the strong convexity of the integral cost and nothing else. The complete implication of finitely supported noises is given in §8.3.5.

Suppose there exist $\lambda^\sharp \in C^\star$, and $u^\sharp \in \mathcal{U}$ such that

$$J(u^\sharp) = \min_{u \in \mathcal{U}} \Big\{ J(u) + \langle \lambda^\sharp, \Theta(u) \rangle \Big\} ,$$

and

$$\Theta(u^\sharp) \in -C .$$

Then, we have

$$u^\sharp \in \arg\min_{u \in \mathcal{U}} \Big\{ J(u) \quad | \quad \Theta(u) \in -C \Big\} .$$

Note that there is no assumption required on the set $\mathcal{U}$, or on the objective and constraint functions.

Proof. We have

$$\begin{aligned}
\min_{u \in \mathcal{U}} \Big\{ J(u) \mid \Theta(u) \in -C \Big\} &= \min_{u \in \mathcal{U}} \max_{\lambda \in C^\star} \quad J(u) + \langle \lambda, \Theta(u) \rangle , \\
&\geq \max_{\lambda \in C^\star} \min_{u \in \mathcal{U}} \quad J(u) + \langle \lambda, \Theta(u) \rangle , \\
&\geq \min_{u \in \mathcal{U}} \quad J(u) + \langle \lambda^\sharp, \Theta(u) \rangle , \\
&= J(u^\sharp) + \langle \lambda^\sharp, \Theta(u^\sharp) \rangle , \\
&\geq J(u^\sharp) ,
\end{aligned}$$

where the last inequality is obtained by definition of $C^\star$. Moreover, as $\Theta(u^\sharp) \in -C$, we have,

$$J(u^\sharp) \quad \geq \quad \min_{u \in \mathcal{U}} \quad \Big\{ J(u) \quad | \quad \Theta(u) \in -C \Big\} .$$

This ends the proof. □

Remember that the update step (8.15) is a gradient step for the maximization of the dual problem (8.9), the gradient being obtained by $\mathbb{E}\big[\boldsymbol{\Delta}_t^{(k)} \mid \boldsymbol{Y}_t\big]$. However, it is known that the gradient algorithm converges slowly. Hence, we are tempted to use more efficient algorithm like conjugate gradient or quasi-Newton algorithms. The practical consequence of Proposition 8.16 is that we can numerically use these more efficient algorithms, take note of their convergence, check that the primal solution $\boldsymbol{U}^{(\infty)}$ obtained satisfies constraint (8.7f), and thus guarantee that $\boldsymbol{U}^{(\infty)}$ is indeed an optimal solution of Problem (8.7).

8.3.7 Upper and Lower Bounds

We have seen in §8.1 that the DADP algorithm solves an approximation of the original Problem (8.2). However, it also provides exact lower and upper bounds of the original problem.

Lower Bound. Consider the DADP algorithm at a specific iteration $k \in \mathbb{N}$. For a given multiplier process $\boldsymbol{\mu}^{(k)}$, we solve the N minimization problems (8.11) and compute the sum of the N obtained values. We claim that this sum is a lower bound of the original problem (8.2). Indeed,

- it is the value for $\boldsymbol{\mu} = \boldsymbol{\mu}^{(k)}$ of the function maximized in Problem (8.10), hence, it is lower than the value of Problem (8.10);
- by weak duality inequality (see Fact 4.4), the value of Problem (8.10) is lower than the value of Problem (8.7);
- Problem (8.7) is a relaxation of original minimization problem, thus the value of Problem (8.7) is lower than the value of Problem (8.2).

Upper Bound. Conversely, thanks to a heuristic, e.g. the one presented in §8.2.4, we have an admissible solution for Problem (8.2), and thus an upper bound of the value of Problem (8.2).

Note that we automatically compute the lower bound, whereas the upper bound is only obtained through an heuristic, which can be computationally demanding. Moreover, the gap between the upper and lower bound does not reduce to 0, unless a solution of Problem (8.2) satisfies the relaxed constraint (8.7f).

8.4 Numerical Results for the Hydraulic Valley Example

We present numerical results of a three dam valley problem. This problem is a toy-problem, small enough to be exactly solved by Dynamic Programming. Thus, we have computed the exact solution and can compare the result of the DADP algorithm, that is, the upper and lower bounds obtained, with the optimal solution. Moreover, we have been able to compute the optimal multiplier $\boldsymbol{\lambda}$, for the almost sure coupling constraint, and make some statistical inferences.

A more thorough presentation with more complete results are to be found in [1]. Other numerical results, on a problem with only one coupling constraint, can be found in [50].

8.4.1 Problem Specification

We present here the characteristics of the numerical problem treated.

We consider the hydraulic valley problem presented in §1.3.2, on which the DADP algorithm was described in §8.1.5, and developed in each example of §8.2. We consider a problem with $N = 3$ dams, and a time horizon $T = 11$, inspired by real life problem.

The set $\mathbb{X}_t = \mathbb{X}_t^1 \times \mathbb{X}_t^2 \times \mathbb{X}_t^3$ of state is discretized by step of 2hm^3 of water: $\mathbb{X}_t = \{0, 2, \dots, 60\}^3$. Analogously the control states $\mathbb{U}_t$ is discretized by step of 2hm^3, and we have $\mathbb{U}_t^i = \{0, 2, \dots, 40\}$.

The random variables $\{\boldsymbol{W}_t^1, \boldsymbol{W}_t^2, \boldsymbol{W}_t^3\}$ are discrete with values in $\{0, 2, \dots, 40\}$. They are time independent, but coupled spatially. Hence, we are able to compute exactly the expectations in the Dynamic Programming resolution of Problem (8.11), whereas the update of the multiplier and the upper bound given by the admissible strategy are evaluated by Monte-Carlo over $100,000$ scenarios.

Hence, all variables (except the cost), are assumed to be discrete. This does not fit with the assumptions (convexity of the sets) made earlier, but avoids the question of interpolating.

We have seen that the proof of convergence of DADP (see §8.3.4) requires that the cost functions are strongly convex. Hence we choose for the local costs

$$L_t^i(x, u, z, w) = -p_t u + \varepsilon_u u^2 + \varepsilon_z z^2 \ ,$$

and for the final cost

$$K^i(x) = \varepsilon_x (x - x_T^i)^2 \ ,$$

where x_T^i is the target level of water, and $\{p_t\}_0^T$ is a sequence of deterministic prices modelling the electricity market.

The evolution functions are given by

$$f_t^i(x, u, z, w) = x - u + z + w \ .$$

The controls $\boldsymbol{U}_t^i$ and $\boldsymbol{Z}_t^i$, and the states $\boldsymbol{X}_t^i$ have almost sure bounds.

8.4.2 Information Processes

We study three different information processes.

Constant information process $\boldsymbol{Y}_t \equiv 0$. If the information process $\boldsymbol{Y}$ is constant, then the almost sure constraint, given, for $i \in \{1, 2\}$ and $t \in [\![0, 10]\!]$, by

$$g_t^{i-1}\big(\boldsymbol{H}_t^{i-1}, \boldsymbol{W}_t^{i-1}\big) - \boldsymbol{Z}_t^i = 0 \,, \tag{8.34}$$

is replaced by the following constraint in expectation

$$\mathbb{E}\Big[g_t^{i-1}\big(\boldsymbol{H}_t^{i-1}, \boldsymbol{W}_t^{i-1}\big) - \boldsymbol{Z}_t^i\Big] = 0 \,. \tag{8.35}$$

Hence, the multiplier $\boldsymbol{\mu}$ of the corresponding relaxed problem (8.7) is almost surely constant. More precisely, the sequence $\boldsymbol{\mu}^{(k)}$ is given by a sequence of vectors in $\mathbb{R}^{11}$. The main interest of this choice is the numerical efficiency:

- the state required to solve Problem (8.11), by Dynamic Programming, is simply $\boldsymbol{X}_t^i$;
- the maximization over $\boldsymbol{\mu}$ is a maximization over $\mathbb{R}^{11}$, a reasonably small vector space.

Random inflow of the dam upper stream dam $\boldsymbol{Y}_t^i = \boldsymbol{W}_t^{i-1}$. As $\boldsymbol{Y}_t^i$ is related to the constraint linking the dam $i-1$ with the dam i, we chose the information process to be the inflow of dam $i-1$. Thus, the almost sure constraint given by Equation (8.34) is replaced by

$$\mathbb{E}\Big[g_t^{i-1}\big(\boldsymbol{H}_t^{i-1}, \boldsymbol{W}_t^{i-1}\big) - \boldsymbol{Z}_t^i \;\Big|\; \boldsymbol{W}_t^{i-1}\Big] = 0 \,. \tag{8.36}$$

Constraint (8.36) is closer to the almost sure constraint (8.34) than the expectation constraint (8.35). However, we can still solve Problem (8.11) by a Dynamic Programming approach with state $\boldsymbol{X}_t^i$.

Phantom state of the first dam. As, on this (small) problem, we are able to compute the optimal solution of the coupled problem (8.2), we are also able to compute the exact optimal multipliers $\big(\boldsymbol{\lambda}^{1,\sharp}, \boldsymbol{\lambda}^{2,\sharp}\big)$ (see Example 8.4) . Thus we have studied statistically the correlation between each optimal multiplier $\boldsymbol{\lambda}^{i,\sharp}$ and the trajectory of the optimal states $\boldsymbol{X}^{j,\sharp}$. It appears that each multiplier is mostly correlated with the optimal state of the first dam.

Consequently, we want to construct an information process $\boldsymbol{Y}_t^1 = \boldsymbol{Y}_t^2 = \boldsymbol{Y}_t$ that mimics the process $\boldsymbol{X}_t^1$. We call this information process a *phantom state*. As $\boldsymbol{X}_t^1$ evolves along the iteration of the DADP algorithm we choose to study an information process $\boldsymbol{Y}_t$ that mimics the optimal state $\boldsymbol{X}_t^{1,\sharp}$. For numerical simplicity we have chosen to construct an AR1-like process

$$\boldsymbol{Y}_{t+1} = \alpha_t \boldsymbol{Y}_t + \beta_t \boldsymbol{W}_t^1 + \gamma_t \,, \tag{8.37}$$

the coefficients being obtained by regression on the optimal control. Note that, on a real problem, we do not have the optimal trajectory of the state at our disposal for such regressions. In §8.5, we develop ideas, and point out difficulties when using a phantom state as information process.

8.4.3 Results Obtained

The numerical results are summed up in Table 8.1. They were obtained on a personal laptop with a quad core Intel Core i7 à 2.2 GHz, implemented in C. A quasi-Newton algorithm is used to update the multipliers.

	DADP - $\mathbb{E}$	DADP - $\boldsymbol{W}^{i-1}$	DADP - dyn.	DP
Number of iterations	165	170	25	1
Time (min)	2	3	67	41
Lower Bound	-1.386×10^6	-1.379×10^6	-1.373×10^6	
Final Value	-1.335×10^6	-1.321×10^6	-1.344×10^6	-1.366×10^6
Relative Loss	-2.3%	-3.3%	-1.6%	ref.

Table 8.1: Numerical results on the 3-dam problem

Let us comment these results. For the sake of clarity, the index 0 denotes the results related to the optimal solution to the original problem obtained by Dynamic Programming (i.e. $\sigma(\boldsymbol{Y}_t^0) = \mathcal{F}_t$); the index 1 denotes the results related to the constant information process $\boldsymbol{Y}^1 = 0$; the index 2 denotes the results related to the noise information process $\boldsymbol{Y}^{2,i} = \boldsymbol{W}^{i-1}$; the index 3 denotes the results related to the information process mimicking $\boldsymbol{X}^{1,\sharp}$ given in (8.37). We denote $\underline{V}^i$ (resp $\overline{V}^i$) the lower (resp. upper) bound on the value of the 3-dam problem given by the information process $\boldsymbol{Y}^i$. Note that the upper-bound is also the value of the admissible, and hence implementable, solution obtained. The admissible strategy obtained is denoted $\boldsymbol{U}_i^{\mathrm{ad}}$, whereas the solution of the relaxed problem is denoted $\boldsymbol{U}_i^\sharp$.

- First of all, the constant information process seems, on this example, to be an efficient information process. Indeed, the subproblems (8.28) are quite easy to solve (no increase in dimension) and, the multiplier $\boldsymbol{\mu}$ being deterministic, the sequence $\boldsymbol{\mu}^{(k)}$ simply lives in $\mathbb{R}^{11}$. For both these reasons, the algorithm converges quite quickly (5 times faster than the Dynamic Programming approach on this application). The lower bound $\underline{V}^1$, is 2% under the optimal value, and the upper bound $\overline{V}^1$ (which is also the value of $\boldsymbol{U}_1^{\mathrm{ad}}$) is 2.3% above the optimal value.
- The "back-to-admissibility" heuristic that gives, from a solution to the approximated Problem (8.18), an admissible solution to the 3-dam problem, is not monotonous with respect to information content. Indeed, the information process $\boldsymbol{Y}^i = \boldsymbol{W}^{i-1}$ yields a finer σ-algebra than the constant information process, hence the solution $\boldsymbol{U}_2^\sharp$ is "closer" to the set of admissible solutions than $\boldsymbol{U}_1^\sharp$. In particular, the upper bound $\overline{V}^2$ is tighter than the upper bound $\overline{V}^1$. However, we see in Table 8.1 that the admissible strategy $\boldsymbol{U}_2^{\mathrm{ad}}$ obtained for the noise as information is less efficient than the admissible strategy $\boldsymbol{U}_1^{\mathrm{ad}}$.
- For the three information processes, the DADP algorithm converges reasonably well in a short time, even if the multipliers do not stabilize completely due to the discretization of the different spaces. Obviously the gain in computation time will be more and more visible with problems with states of higher dimension - e.g. for the dam valley problem with $N > 3$. Indeed, solving the dam valley problem has a complexity exponential in the number of dam, whereas one iteration of the DADP algorithm is linear in the number of dams; even if the number of gradient step required increase the DADP algorithm will be relatively faster than the Dynamic Programming approach for an increasing number of dam.

In a nutshell, it seems that, on the hydraulic valley problem, choosing a constant information process is a good compromise. In order to obtain better strategies we need to find a well designed information process $\boldsymbol{Y}_t$. Here, it was possible to find one from the knowledge of the optimal solution of the global problem. In the next section, we discuss the design of information model.

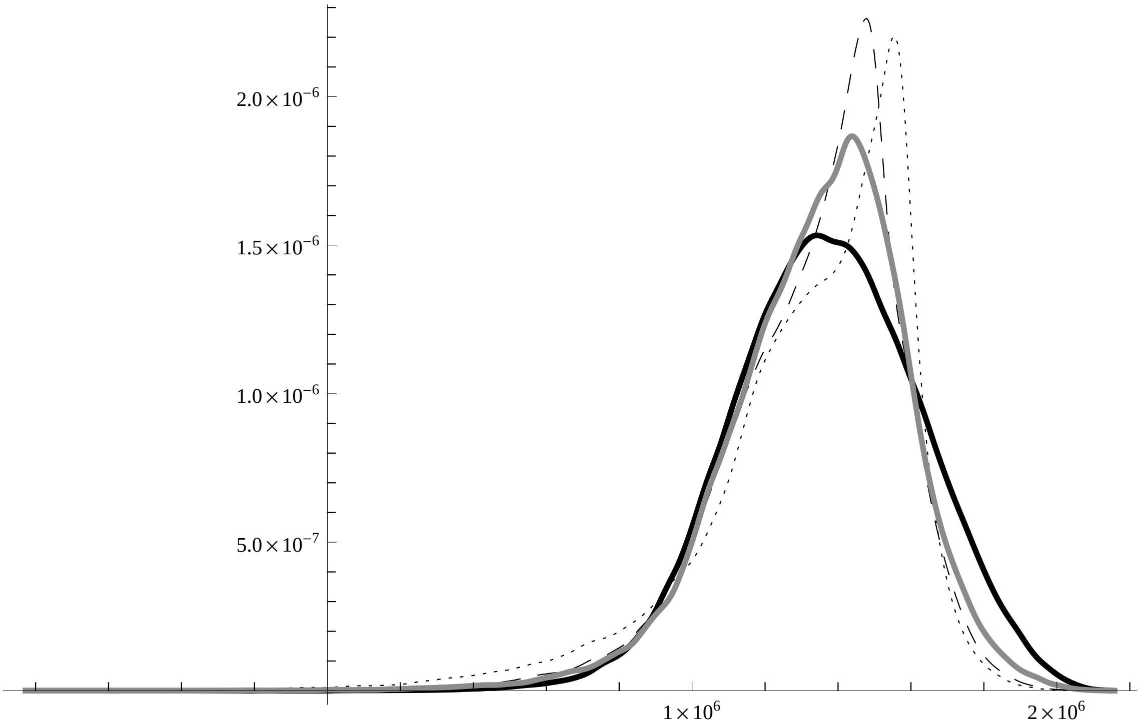


Figure 8.2: Density of gains obtained with Dynamic Programming (bold black), DADP with constant information (dotted line), DADP with $\mathbf{Y}^2$ (fine dotted line) and DADP with $\mathbf{Y}^3$ (bold Gray).

8.5 Discussion on the Information Model

In this section, we give guiding lines on how to construct an information process $\boldsymbol{Y}$. We point out some difficulties related to them. For better understanding, the suggestion of information processes are presented on the dam example. In particular we explore the difficulties in choosing an information process that evolves along the iterations.

8.5.1 A Short List of Information Processes

One of the first options, when implementing the DADP algorithm, is to determine the information process $\boldsymbol{Y}$ that will be used to construct the relaxation (8.7). We present general ideas that were implemented on the dam valley example (see §8.4).

A constant information process

Choosing $\boldsymbol{Y} \equiv 0$ is a way of replacing the almost sure constraint (8.2f) by the expected constraint

$$\mathbb{E}\Big[\sum_{i=1}^{N} \theta_t^i\big(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_t\big)\Big] = 0 \ .$$

This is the simplest information process; it can be implemented very simply, on any problem regardless of its specific properties. The state required for solving the subproblem (8.11) is only the local state $\boldsymbol{X}^i$, and the multiplier μ is almost surely constant.

On the hydraulic valley, but also on other example in [50], this simple information process gives good numerical results.

Part of the noise

The general idea is to use a part of the noise $\boldsymbol{W}$ that is not naturally included in the subproblem. The main interest is that this method does not increase the state of the subproblem (8.11) (hence the resolution by Dynamic Programming is not hindered by this choice of process). Moreover the σ-algebra generated by $\boldsymbol{W}^i$ is richer than the σ-algebra generated by a constant information variable, hence leading to a tighter upper bound.

In the dam valley example, the subproblem i is only concerned with the inflows $\boldsymbol{W}^i$ of dam i. However the outflows of dam $i-1$ seem to be strongly correlated with the inflows of dam $i-1$. Thus, we choose the information variable $\boldsymbol{Y}_t^i = \boldsymbol{W}_t^{i-1}$, and obtain multipliers $\mu_t^{i,(k)}$ depending on the value of $\boldsymbol{W}_t^{i-1}$.

Phantom State

A natural idea for an information process is to use a part of the state $\boldsymbol{X}$ as an information process. However, we stated that an information process has to be an uncontrolled process. Consequently we try to construct a "phantom state", that is, an uncontrolled process that mimics a part of the state.

Recall that, in the dam valley example, $\{\boldsymbol{Y}_t^i\}_{t=0}^{T-1}$ is the information process related to the constraint

$$\boldsymbol{Z}_t^i - g_t^{i-1}\big(\boldsymbol{X}_t^{i-1}, \boldsymbol{U}_t^{i-1}, \boldsymbol{W}_t\big) .$$

Intuitively, this constraint is quite well explained by the value of $\boldsymbol{X}_{t+1}^{i-1}$. We are thus trying to find a "short-memory" stochastic process $\boldsymbol{Y}_t^i$ that mimics $\boldsymbol{X}_{t+1}^{i-1,(k)}$. Here are two ways to construct such a process.

Statistical regression. At iteration $(k-1)$, we have computed the process $\boldsymbol{X}^{i-1,(k-1)}$. It is reasonable to assume that $\boldsymbol{X}^{i-1,(k-1)}$ is close to $\boldsymbol{X}^{i-1,(k)}$. Thus, we can use any statistical regression tools to find a Markov chain mimicking $\boldsymbol{X}^{i-1,(k-1)}$.

Given control. If we have a feedback (η, γ) close to the one used for the dam $i-1$, we can derive a phantom state from it. An idea would be to use the last feedback $(\eta^{(k)}, \gamma^{(k)})$; however, this would lead to a second-guessing effect presented in §8.5.2. Moreover we show in §8.5.2 that the sequence of σ-algebra thus generated might not converge.

Another problem with this choice of information process is that it would lead to an information process evolving with each iteration. Unfortunately there is no theoretical background or interpretation of what the algorithm is solving. Consequently, we suggest to modify the algorithm as presented in Algorithm 8.3, where the variable n corresponds to the step of update of the information process, and k to step of the DADP process for a given information variable $\boldsymbol{Y}^{(n)}$.

8.5.2 Difficulties of the Phantom State

Even if using a phantom state as information process seems a natural idea, it leads to several difficulties. We first show in §8.5.2 that the phantom state (and in particular the σ-algebra generated by the phantom state) might not converges. Then, in §8.5.2, we present the so-called *mirror effect*, that extends the size of the state needed to solve subproblems (8.11). Finally, in §8.5.2, we show an heuristic way of reducing the dimension of the state required to solve subproblems (8.11).

Non Convergence of Phantom State on a Simple Example

We show, on a simple example, that letting the information evolve along the iterations of the algorithm yields some difficulties.

Data: Information process evolution functions $\tilde{f}_t^0$ and starting point y_0;
Multipliers $\boldsymbol{\mu}_t^{(0)}$
Result: Optimal multipliers $\mu_t^\sharp$, admissible feedback;
repeat
  repeat
    forall the $i \in [\![1, N]\!]$ **do**
      Solve Problem (8.11);
    forall the $t \in [\![0, T-1]\!]$ **do**
      Estimate $\mathbb{E}\big[\boldsymbol{\Delta}_t^k \bigm| \boldsymbol{Y}_t\big]$;
      Update the multiplier $\mu^{n,k} \rightsquigarrow \mu^{n,k+1}$;
  until $\mathbb{E}\big[\boldsymbol{\Delta}_t^k \bigm| \boldsymbol{Y}_t\big] \simeq 0$;
  Compute admissible feedbacks;
  Evaluate admissible feedbacks;
  Update the information process $\boldsymbol{Y}^{(n)} \rightsquigarrow \boldsymbol{Y}^{(n+1)}$;
until $\boldsymbol{Y}_t^{(n+1)} \simeq \boldsymbol{Y}_t^{(n)}$;

Algorithm 8.3: General scheme of DADP with evolving information process

We propose to study the following scheme.

- We consider a stochastic problem $(\mathcal{P})$, with an almost sure coupling constraint $\theta(\boldsymbol{U}) = 0$.
- We construct a sequence of approximations of $(\mathcal{P})$ as follows.
  - We consider a first approximated problem $(\mathcal{P}_0)$, where the almost constraint is replaced by an expectation constraint and derive an optimal control $\boldsymbol{U}^{(0)}$.
  - Now, assuming that we have determined an optimal control $\boldsymbol{U}^{(k-1)}$, we consider the problem $(\mathcal{P}_k)$, relaxation of problem $(\mathcal{P})$ where the almost constraint is replaced by $\theta(\boldsymbol{U}^{(k-1)})$, and determine its optimal control $\boldsymbol{U}^{(k)}$.

On an example we show that this scheme might not converge.

Example 8.17. *Consider the following problem* $(\mathcal{P})$

$$\begin{aligned} \min_{\boldsymbol{U} \in \mathrm{L}^2} \quad & \big(\boldsymbol{U} - \boldsymbol{G}\big)^2 \\ s.t. \quad & \boldsymbol{U} = 0 \qquad \mathbb{P} - a.s. \; , \end{aligned}$$

where $\boldsymbol{G}$ *is a standard normal random variable. We assume that the* σ*-algebra* $\mathcal{F}$ *is given by* $\mathcal{F} = \sigma(\boldsymbol{G})$*. The optimal solution of problem* $(\mathcal{P})$ *is the only admissible solution,* $\boldsymbol{U}^\sharp = 0$ *with optimal value* 1*. We relax the almost sure constraint into* $\mathbb{E}\big[\boldsymbol{U}\big] = 0$*. The approximated problem* $(\mathcal{P}_0)$ *reads*

$$\begin{aligned} \min_{\boldsymbol{U} \in \mathrm{L}^2} \quad & \big(\boldsymbol{U} - \boldsymbol{G}\big)^2 \\ s.t. \quad & \mathbb{E}\big[\boldsymbol{U}\big] = 0 \; , \end{aligned}$$

with optimal control $\boldsymbol{U}^{(0)} = \boldsymbol{G}$*, and optimal value* 0*. Then we approximate the constraint by* $\mathbb{E}\big[\boldsymbol{U} \bigm| \boldsymbol{U}^0\big] = 0$*, to obtain problem* $(\mathcal{P}_1)$

$$\begin{aligned} \min_{\boldsymbol{U} \in \mathrm{L}^2} \quad & \big(\boldsymbol{U} - \boldsymbol{G}\big)^2 \\ s.t. \quad & \mathbb{E}\big[\boldsymbol{U} \bigm| \boldsymbol{U}^{(0)}\big] = 0 \; , \end{aligned}$$

with optimal control $\boldsymbol{U}^{(1)} = \boldsymbol{U}^\sharp$*. Thus we have:*

- *for any even number $2k$, the optimal control of $(\mathcal{P}_{2k})$ is $\boldsymbol{U}^{(2k)} \equiv 0$, generating the trivial σ-algebra $\{\emptyset, \Omega\}$;*
- *and for any odd number $2k+1$, the optimal control of $(\mathcal{P}_{2k+1})$ is $\boldsymbol{U}^{(2k+1)} = \boldsymbol{G}$ generating the whole σ-algebra $\mathcal{F}$.*

This shows that using a phantom state to construct an information process $\boldsymbol{Y}^{(n)}$ might not lead to converging σ-algebras $\sigma\big(\boldsymbol{Y}^{(n)}\big)$. Hence, the results of §8.3.2 are not available.

Second Guessing Effect

In this section, we present the so-called "second guessing effect", or how the dimension of the state of the system required to solve the subproblems (8.11) grows with iterations when we update the information model.

We consider the following problem $(\mathcal{P})$

$$\begin{aligned}
\min_{\boldsymbol{X},\boldsymbol{U}} \quad & \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t)\Big] \\
s.t. \quad & \boldsymbol{X}_{t+1} = f\big(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t\big) \\
& \theta_t(\boldsymbol{X}_t) = 0
\end{aligned}$$

We consider the sequence of approximated problems $\big(\mathcal{P}_k\big)$ given as follows:

$\underline{\big(\mathcal{P}_0\big)}$:

$$\begin{aligned}
\min_{\boldsymbol{X}_t,\boldsymbol{U}_t} \quad & \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t)\Big] \\
s.t. \quad & \boldsymbol{X}_{t+1} = \quad f\big(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t\big) \\
& \mathbb{E}\big[\theta_t(\boldsymbol{X}_t)\big] = 0
\end{aligned}$$

which yield an optimal control $\boldsymbol{U}_t^\sharp = \eta^{(0)}(\boldsymbol{X}_t)$. Then, we define an information variable $\boldsymbol{Y}^{(0)}$ which is the "phantom state" following the optimal control of problem $\big(\mathcal{P}_0\big)$. More precisely, we have

$$\begin{cases}
\boldsymbol{Y}_0^{(0)} = x_0 \\
\boldsymbol{Y}_{t+1}^{(0)} = \underbrace{f\big(\boldsymbol{Y}_t^{(0)}, \eta^{(0)}(\boldsymbol{Y}_t^{(0)})\big)}_{:=f^{(0)}\left(\boldsymbol{Y}_t^{(0)}\right)}
\end{cases}$$

and we proceed to define a new problem where the constraint in expectation is refined by a conditional expectation constraint.

$\underline{\big(\mathcal{P}_1\big)}$:

$$\begin{aligned}
\min_{\boldsymbol{X},\boldsymbol{U}} \quad & \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t)\Big] \\
s.t. \quad & \boldsymbol{X}_{t+1} = \quad f\big(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t\big) \\
& \boldsymbol{Y}_{t+1}^{(0)} = f^{(0)}\big(\boldsymbol{Y}_t^{(0)}\big) \\
& \mathbb{E}\big[\theta_t(\boldsymbol{X}_t) \bigm| \boldsymbol{Y}_t^{(0)}\big] = 0
\end{aligned}$$

yields an optimal control $\boldsymbol{U}_t^\sharp = \eta^{(1)}\big(\boldsymbol{X}_t, \boldsymbol{Y}_t^{(0)}\big)$. Then we define an information variable $\boldsymbol{Y}^{(1)}$ as the "phantom state" following the optimal control of problem $\big(\mathcal{P}_1\big)$, i.e.

$$\begin{cases} \boldsymbol{Y}_0^{(1)} = x_0 \\ \boldsymbol{Y}_{t+1}^{(1)} = \underbrace{f\big(\boldsymbol{Y}_t^{(1)}, \eta^{(1)}(\boldsymbol{Y}_t^{(1)}, \boldsymbol{Y}_t^{(0)})\big)}_{:=f^{(1)}\big(\boldsymbol{Y}_t^{(1)}, \boldsymbol{Y}_t^{(0)}\big)} \end{cases}$$

which yield an optimal control $\boldsymbol{U}_t^\sharp = \eta^{(2)}(\boldsymbol{X}_t, \boldsymbol{Y}_t^{(1)}, \boldsymbol{Y}_t^{(0)})$.

More generally, assuming that we have defined an information process $\boldsymbol{Y}^{(k)}$, we can construct the next problem

$\underline{\big(\mathcal{P}_k\big)}$:

$$\begin{aligned} \min_{\boldsymbol{X},\boldsymbol{U}} \quad & \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t)\Big] \\ s.t. \quad & \boldsymbol{X}_{t+1} = f\big(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_t\big) \\ & \boldsymbol{Y}_{t+1}^{(0)} = f^{(0)}\big(\boldsymbol{Y}_t^{(0)}\big) \\ & \boldsymbol{Y}_{t+1}^{(1)} = f^{(1)}\big(\boldsymbol{Y}_t^{(1)}, \boldsymbol{Y}_t^{(0)}\big) \\ & \cdots \\ & \boldsymbol{Y}_{t+1}^{(k)} = f^{(k)}\big(\boldsymbol{Y}_t^{(k)}, \boldsymbol{Y}_t^{(k-1)}, \cdots, \boldsymbol{Y}_t^{(0)}\big) \\ & \mathbb{E}\big[\theta_t(\boldsymbol{X}_t) \bigm| \boldsymbol{Y}_t^{(k)}\big] = 0 \end{aligned}$$

which yields an optimal control $\boldsymbol{U}_t^\sharp = \eta^{(k+1)}(\boldsymbol{X}_t, \boldsymbol{Y}_t^{(k)}, \cdots, \boldsymbol{Y}_t^{(0)})$, and we define

$$\begin{cases} \boldsymbol{Y}_0^{(k+1)} = x_0 \\ \boldsymbol{Y}_{t+1}^{(k+1)} = \underbrace{f\big(\boldsymbol{Y}_t^{(k+1)}, \eta^{(1)}(\boldsymbol{Y}_t^{(k+1)}, \boldsymbol{Y}_t^{(k)}, \cdots, \boldsymbol{Y}_t^{(0)})\big)}_{:=f^{(k+1)}\big(\boldsymbol{Y}_t^{(k+1)}, \boldsymbol{Y}_t^{(k)}, \cdots, \boldsymbol{Y}_t^{(0)}\big)} \end{cases} . \tag{8.38}$$

Thus, we see that if we want to use the "last optimal state" as an information variable in the DADP algorithm, the state size required to solve the subproblems increases for each update of the information process.

Moreover, if we apply this idea to the hydraulic valley problem, the size of the information state increases both with the past information state as presented here, and with the information state of the upstreams dams.

Replacing a Phantom State by an Actual State

We could consider a stochastic information process $\boldsymbol{Y}^i$ mimicking $\boldsymbol{X}^{i-1}$. Then, solving Problem (8.28) by Dynamic programming approach requires to use the state $\big(\boldsymbol{X}_t^i, \boldsymbol{Y}_t^i, \boldsymbol{Y}_t^{i+1}\big)$. However $\boldsymbol{Y}_t^{i+1}$ should mimic $\boldsymbol{X}_t^i$. Thus replacing $\boldsymbol{Y}_t^{i+1}$ by $\boldsymbol{X}_t^i$ in Problem (8.28) seems natural, and reduces the size of the state needed for the Dynamic Programming.

However, it amounts to replace the duality cost

$$\mathbb{E}\Big[\big\langle \boldsymbol{\lambda}_t^i \,, \boldsymbol{Z}_t^i - g_t^{i-1}(\boldsymbol{H}_t^{i-1}, \boldsymbol{W}_t^{i-1})\big\rangle \Bigm| \boldsymbol{Y}_t^i\Big] , \tag{8.39}$$

by

$$\mathbb{E}\Big[\big\langle \boldsymbol{\lambda}_t^i \,, \boldsymbol{Z}_t^i\big\rangle \Bigm| \boldsymbol{X}_t^i\Big] - \mathbb{E}\Big[\big\langle \boldsymbol{\lambda}_t^i \,, g_t^{i-1}(\boldsymbol{H}_t^{i-1}, \boldsymbol{W}_t^{i-1})\big\rangle \Bigm| \boldsymbol{Y}_t^i\Big] . \tag{8.40}$$

Hence, we do not exactly know what problem is solved with this approximation.

Conclusion

When we consider a multistage stochastic optimization problem (like Problem (8.2)), a price decomposition scheme, as described in §8.1.2, is untractable for DP treatment. Indeed, there is no reason that the couple consisting of the original state $\boldsymbol{X}$ and of the multiplier $\boldsymbol{\lambda}$ be Markovian; the multiplier $\boldsymbol{\lambda}$ is an unknown adapted stochastic process. However, if we restrict ourselves to a specific class of multipliers — like those measurable w.r.t. a given "information" process $\boldsymbol{Y}$, corresponding to an approximation of the multiplier $\boldsymbol{\lambda}$ — we are solving a relaxation of the original problem (Problem (8.7)). On this relaxed problem, we have been able to apply a price decomposition scheme, where each subproblem is solved through Dynamic Programming with an extended state $\big(\boldsymbol{X}_t^i, \boldsymbol{Y}_t\big)$. Theoretical results of consistence and convergence of the method were derived from the results of Chapters 4-7. This method has been tested on a problem small enough to be compared with the optimal solution. Numerical tests on a real scale problem remain to be done, and benchmarked against the SDDP algorithm.

Conclusion

Imagination is more important than knowledge.

Einstein

This manuscript is the result of three years of scientific work at École des Ponts-ParisTech, within the Optimization and Systems group — whose senior members are Pierre Carpentier, Jean-Philippe Chancelier and Michel De Lara — and with the participation of Jean-Christophe Alais, as a fellow PhD student. This work stands in the continuity of a series done by the group, oftimes in collaboration with EDF R&D.

Contributions of this Manuscript

This manuscript is a contribution to the domain of decomposition methods in discrete-time stochastic optimal control.

- In Chapter 1, we have presented a global view of decompositions methods in multi-stage stochastic optimization problems.
- In Chapter 2, we have extended the setting of dynamic programming to allow for more general aggregation in time than the intertemporal sum, and for more general aggregation in uncertainty than the expectation. We have focused on the concept of time consistency of optimization problems, and on the links with the time consistency concept in the risk measure literature. The content of Chapter 2 has been submitted to the *European Journal of Operations Research*.
- In Chapter 3, we have extended the results of convergence of the SDDP algorithm. Indeed, till now, the proofs of convergence were relying on the piecewise linearity of the cost functions (in addition to the convexity), whereas our proof only relies on convexity. The content of Chapter 3 has been accepted for publication in *Mathematics of Operations Research* (up to minor modifications).
- In Chapter 4, we have detailed two examples showing that the existence of an optimal multiplier for an almost sure constraint is a delicate issue.
- In Chapter 5, we have extended a result of existence of an L^1-multiplier for almost sure constraints, and have applied it for a multistage problem.
- In Chapter 6, we have provided the Uzawa algorithm in the non-reflexive Banach space $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$, and have shown a result of convergence of a subsequence.
- In Chapter 7, we have shown an epiconvergence result: a sequence of relaxed optimization problems — where almost sure constraints are replaced by weaker conditional expectation ones — epiconverges to the original one when the corresponding σ-fields converge. The content of Chapter 7 has been submitted to *Mathematics of Operations Research*.
- Finally, in Chapter 8, we have presented the so-called DADP algorithm, given different interpretations and provided conditions of convergence.

Perspectives

C'est tout ? Ah non, c'est un peu court jeune homme !
On aurait pû dire, oh Dieu, bien des choses en somme.
En variant le ton, par exemple, tenez:

Cyrano de Bergerac

A substantial body of work stands ahead of us.

- Using the framework of Chapter 2, connections between the time consistency of dynamic risk measures and sequences of optimization problems remains to be further scrutinized.
- The convergence of SDDP still relies on the property that the uncertainties take discrete values. There are reasons to think that the proof could be extended to continuous random variables, but difficulties related to infinite dimensional object have to be overcome.
- To extend our conditions for the existence of multiplier results to inequality constraints we think that existing results relying on relatively complete recourse (hence allowing at least bounds on the control) could be adapted to our setting.
- The proof of convergence of the Uzawa algorithm in L^∞ should be modified in order to prove the convergence of the whole sequence.
- We have seen that the sequence of approximated problem, where the almost sure constraint is replaced by its conditional expectation, converges toward the original problem when information converges. However, DADP works with a fixed information. Hence, we should complete these result with bound errors for a given information process, to make a link with the DADP algorithm.
- The DADP algorithm should be compared with the SDDP algorithm on a large-scale hydraulic valley. In a second phase SDDP could be integrated in the DADP algorithm as an alternative mean of solving the subproblems.

Appendix A

Analysis

"Begin at the beginning", the King said, very gravely,
"and go on till you come to the end: then stop."

Alice in Wonderland

Contents

We give some technical recalls and results.

A.1 Topology and Duality

A.1.1 General Definitions on Topologies

We begin with a survey of the definitions and vocabulary related to topological spaces. In this section E is a $\mathbb{R}$-vector space E.

Definition A.1 (Topology)**.** *A* topology $\tau \subset \mathcal{P}\big(E\big)$ *is a set of subsets of* E *satisfying:*

- $\emptyset$ *and* E *are element of* τ*;*
- *finite intersection of elements of* τ *are element of* τ*;*
- *any union of elements of* τ *is an element of* τ*.*

The elements of τ *are called* τ-open sets*. If* A *is a subset of* E *such that its complement*[1] *is open, it is said to be a* closed set*.*

The couple (E, τ) *is called a* topological space*. We also say that the space* E *is* endowed with*, or* equipped with *the topology* τ*.*

If P *is a set of subset of* E*, then the* topology generated by P*, denoted by* $\tau(P)$*, is the intersection of all topology on* E *containing* P*. We call* P *a base of* $\tau(P)$*.*

Definition A.2 (Convergence of sequence)**.** *Consider a sequence* $\{x_n\}_{n\in\mathbb{N}}$ *of elements of a topological space* $\big(E, \tau\big)$*. The sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is said to be* τ-converging toward $x \in E$

1. The complement of A (in E) is set of elements of E that are not in A.

*if for all open set $O \in \tau$ containing x, there exists $N \in \mathbb{N}$, such that for all $n \geq N$, $x_n \in O$. x is called the limit of $\{x_n\}_{n\in\mathbb{N}}$. A τ-*convergent sequence *$\{x_n\}_{n\in\mathbb{N}}$, is such that there exists $x \in E$, such that $\{x_n\}_{n\in\mathbb{N}}$ is τ-converging toward x.*

A set $A \subset B \subset E$ is said to be dense in B if for all point $x \in B$, there exists a sequence of points in A converging toward x.

Fact A.3. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a topological space $\big(E,\tau\big)$. If from any sub-sequence $\big\{x_{n_k}\big\}_{k\in\mathbb{N}}$ we can extract a sub-sub-sequence $\big\{x_{\sigma(n_k)}\big\}_{k\in\mathbb{N}}$ converging to $x^* \in E$, then $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*.*

Proof. Indeed, suppose that $\{x_n\}_{n\in\mathbb{N}}$ does not converges toward x^*. Then there exist an open set $\mathcal{O}$ containing x^* and a sub-sequence $\big\{x_{n_k}\big\}_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, $x_{n_k} \notin \mathcal{O}$, and no sub-sub-sequence can converges to x^*, hence a contradiction. $\square$

Definition A.4 (Metric spaces)**.** *A* distance *on E is a function $d : E^2 \to \mathbb{R}^+$ satisfying:*
- $\forall x \in E, \qquad d(x,x) = 0;$
- $\forall (x,y) \in E^2, \qquad d(x,y) = d(y,x);$
- $\forall (x,y,z) \in E^3, \qquad d(x,y) \leq d(x,z) + d(z,y);$
- $\forall (x,y) \in E^2, \qquad d(x,y) = 0 \Longrightarrow x = y.$

We call open ball of radius r and center x *(relative to the distance d) the set*

$$B(x,r) = \big\{x' \in E \quad | \quad d(x,x') < r\big\} \, .$$

The topology generated by the distance d, *denoted τ_d, is the topology generated by the set of open balls relative to the distance d. The topological set (E,τ_d) is said to be a* metric space. *A topological set (E,τ) is said to be* metrizable *if there exists a distance d on E such that $\tau = \tau_d$.*

Definition A.5 (Separable and separated topologies)**.** *A topological space $\big(E,\tau\big)$ is said to be* separated *if for any distinct points x and x', there exist two τ-open set $\mathcal{O}$ and $\mathcal{O}'$ such that:*

$$x \in \mathcal{O}, \quad x' \in \mathcal{O}', \quad \mathcal{O} \cap \mathcal{O}' = \emptyset \, .$$

A separated topological space is also called a Hausdorff space.

A separable space *is a topological space that contains a countable dense subset.*

Fact A.6. *A metrizable space is a Hausdorff space.*

Definition A.7 (Compactness)**.** *Consider a subset A of E. An* open cover of A *is an arbitrary collection of open sets $\{b_i\}_{i\in I}$ such that*

$$A \subset \bigcup_{i\in I} B_i \, .$$

A subset A of E is said to be compact *if for all open cover of A there exists a finite number of indexes $\{i_k\}_{k=1}^n$ such that*

$$A \subset \bigcup_{k=1}^{n} B_{i_k} \, .$$

Fact A.8. *A metrizable compact space is separable.*

Definition A.9 (Normed spaces)**.** *A* pseudonorm *on E is a function $n : E \mapsto \mathbb{R}^+$ such that:*
- $\forall \lambda \in \mathbb{R}, \qquad n(\lambda x) = |\lambda| n(x);$
- $\forall (x,y) \in E^2, \qquad n(x+y) \leq n(x) + n(y).$

A norm *on E is a pseudonorm such that $n(x) = 0$ implies that $x = 0$. We often denote a norm by $\|\cdot\|$, meaning that $\|x\| = n(x)$. The function $d : E^2 \to \mathbb{R}^+$, given by $d(x, y) = \|x - y\|$ is a distance. The topology generated by a norm is the topology generated by its corresponding distance. A* normed space *is a topological space, associated with a given norm, and endowed with the topology of the norm.*

In this manuscript the space $\mathbb{R}^d$ is always endowed with the norm[2] topology.

Definition A.10 (Continuity)**.** *Consider two topological spaces $\big(E, \tau\big)$ and $\big(F, \tau'\big)$. Let f be a function mapping E into F.*

- *The* image *by f of $A \subset E$ is the set*

$$f(A) = \big\{ f(x) \in F \quad | \quad x \in A \big\} \ ,$$

and the preimage *by f of $B \subset E$ is the set*

$$f^{-1}\big(B\big) := \big\{ x \in E \quad | \quad f(x) \in A \big\} \ .$$

- *f is said to be* continuous *if*

$$\forall \mathcal{O} \in \tau', \qquad f^{-1}\big(\mathcal{O}\big) \in \tau \ .$$

- *Assume that $F = \overline{\mathbb{R}}$. The* epigraph *of f is the set*

$$\mathrm{epi}(f) = \big\{ (x, y) \in E \times \overline{\mathbb{R}} \quad | \quad y \geq f(x) \big\} \ .$$

The function f is said to be lower semicontinuous (l.s.c.) if its epigraph is closed; and upper semicontinuous if $-f$ is l.s.c.. A function that is both lower and upper semicontinuous is continuous.

Definition A.11. *Consider an arbitrary family of functions $\{f_i\}_{i \in I}$, each mapping E into a topological space $\big(F, \tau'\big)$. The topology generated by the family $\{f_i\}_{i \in I}$, is the coarsest*[3] *topology such that each function f_i is continuous.*

A locally convex topology *is a topology generated by a family of pseudonorm.*

Compactness and lower semicontinuity are very useful to show the existence of solution to an optimization problem, as recalled in the following theorem.

Theorem A.12. *Consider a l.s.c. function $f : E \to \overline{\mathbb{R}}$, and a compact set $A \subset E$. Then we have,*

$$\exists x^\sharp \in A, \quad \forall x \in A, \qquad f(x^\sharp) \leq f(x) \ .$$

In other words a l.s.c. function admits a minimum on a compact set.

Definition A.13 (Topology Comparison)**.** *Consider two topologies τ_1 and τ_2 on E. If $\tau_1 \subset \tau_2$, τ_1 is said to be* coarser *than τ_2, and τ_2 is said to be* finer *than τ_1.*

Fact A.14. *A coarser topology on E implies more compact set, but less continuous and lower semicontinuous functions. A coarser topology on F implies more continuous functions.*

If $\tau_1 \subset \tau_2$, then convergence in τ_2 implies the convergence in τ_1.

2. Any norm, e.g. the euclidian norm, yields the same topology.
3. The discrete topology $\tau = \mathcal{P}(E)$ is such that all functions are continuous.

Definition A.15 (Completeness). *Consider a metric space $\big(E, d\big)$, a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of E. The* diameter *of the sequence $\{x_n\}_{n\in\mathbb{N}}$, is*

$$\operatorname{diam}\Big(\{x_n\}_{n\in\mathbb{N}}\Big) := \sup_{n,m} \quad \{d(x_n, x_m)\} \ .$$

A Cauchy sequence *is a sequence $\{x_n\}_{n\in\mathbb{N}}$, such that*

$$\lim_{k\to\infty} \operatorname{diam}\Big(\{x_{n+k}\}_{n\in\mathbb{N}}\Big) = 0 \ .$$

The metric space $\big(E, d\big)$ is said to be complete *if all its Cauchy sequences admits a limit (in E). A* Banach space *is a complete normed space.*

Definition A.16 (Inner Product). *An* inner product *on E is a function $\langle \cdot \,, \cdot\rangle : E^2 \to \mathbb{R}$ satisfying:*

- $\forall (x, y) \in E^2, \qquad \langle x \,, y\rangle = \langle y \,, x\rangle$;
- $\forall (x, y) \in E^2, \quad \forall \lambda \in \mathbb{R}, \qquad \langle \lambda x \,, y\rangle = \lambda\langle x \,, y\rangle$;
- $\forall (x, y, z) \in E^3, \qquad \langle x + z \,, y\rangle = \langle x \,, y\rangle + \langle z \,, y\rangle$;
- $\forall x \in E, \qquad \langle x \,, x\rangle \geq 0$;
- $\forall x \in E, \qquad \langle x \,, x\rangle = 0 \quad \Longrightarrow \quad x = 0.$

The function $n : E \mapsto \mathbb{R}$, given by $n(x) = \sqrt{\langle x \,, x\rangle}$ is the norm associated with the inner product.

If E is complete for the topology induced by the norm associated with the inner product, it is called an Hilbert space. *In particular, an Hilbert space is a Banach space.*

A.1.2 Topologies on Banach Spaces

Recall that a linear function mapping E into $\mathbb{R}$ is called a *linear form*.

Definition A.17 (Topological dual). *Consider a topological space $\big(E, \tau\big)$. The* topological dual *of E is the set of τ-continuous linear form on E. It is denoted $E^\star$. If $\big(E, \|\|_E\big)$ is a Banach space, we define the dual norm as follows*

$$\forall y \in E^\star, \qquad \|y\|_{E^\star} := \sup\big\{y(x) \quad | \quad \|x\|_E \leq 1\big\} \ .$$

Moreover, $\big(E^\star, \|\|_{E^\star}\big)$ is a Banach space.

Definition A.18 (Reflexive spaces). *Consider a Banach space E, and its topological $E^\star$. The* bidual *of E, denoted by $E^{\star\star}$, is the topological dual of the Banach space $E^\star$.*

We define the evaluation map *$\mathcal{E} : E \to E^{\star\star}$, by*

$$\mathcal{E} : x \mapsto \Big\{E^\star \ni x^\star \mapsto x^\star\big(x\big)\Big\} \ .$$

This is the canonical injection of E into its bidual $E^{\star\star}$.

If $\mathcal{E}$ is surjective, it is an isometric isomorphism of E into $E^{\star\star}$, and we say that E is reflexive, *and identify E with its bidual.*

An Hilbert space is reflexive.

For any Banach space we construct a coarser topology than the norm topology that has the same continuous linear functionals.

Definition A.19 (Weak topology). *Let $\big(E, \tau_{\|\|}\big)$ be a Banach space. The* weak topology *$\sigma\big(E, E^\star\big)$ is the coarsest topology such that the $\tau_{\|\|}$-continuous linear form, are continuous.*

Fact A.20. *Here are some properties of the* weak topology.

- *The weak topology is separated.*

- *A convex set is closed in the norm topology iff it is closed in the weak topology.*
- *Consequently a convex function is l.s.c for the norm topology iff it is l.s.c for the weak topology.*
- *A sequence strongly converging toward x also weakly converges toward x.*

We now consider the topologies on the dual of a Banach space.

Definition A.21 (Weak* topology)**.** *Let E be a Banach space and $E^\star$ its topological dual.*

The weak* topology $\sigma\big(E^\star, E^{\star\star}\big)$ *is the coarsest topology such that the $\tau_{\|\|}$-continuous linear forms are continuous.*

The weak* topology $\sigma\big(E^\star, E\big)$ *is the coarsest topology such that the linear forms on $E^\star$ that are elements of $\mathcal{E}(E)$, where $\mathcal{E}$ is the valuation map defined in Definition A.18, are continuous.*

The Mackey topology $\tau\big(E^\star, E\big)$, *is the finest topology such that the only $\tau\big(E^\star, E\big)$-continuous linear form on $E^\star$ are the elements of $\mathcal{E}(E)$.*

Fact A.22. *The above topologies display the following properties.*

- *We have the following inclusion of topologies*

$$\sigma\big(E^\star, E\big) \subset \tau\big(E^\star, E\big) \subset \sigma\big(E^\star, E^{\star\star}\big) \subset \tau_{\|\|} \,.$$

- *A convex set is $\tau\big(E^\star, E\big)$-closed iff it is $\sigma\big(E^\star, E\big)$-closed.*
- *A convex function $f : E^\star \mapsto \overline{\mathbb{R}}$ is $\tau\big(E^\star, E\big)$-l.s.c. iff it is $\sigma\big(E^\star, E\big)$-l.s.c..*

Fact A.23. *For a reflexive Banach space, the weak, weak* and Mackey topologies coincide.*

The main interest of introducing the weak* topology is given by the two following theorems. Banach Alaoglu theorem shows weak* compactness of large class of sets, and Kakutani theorem shows that it would not be the same with weak compactness in non-reflexive Banach spaces.

Theorem A.24 (Banach-Alaoglu)**.** *Let E be Banach space. Consider its topological dual $E^\star$ endowed with the weak* topology $\sigma\big(E^\star, E\big)$. Then the unit ball*

$$\big\{x' \in E^\star \quad | \quad \|x'\|_{E^\star} \leq 1\big\}$$

is weak* compact.

Proof. See [24, Theorem 2.27]. □

Theorem A.25 (Kakutani)**.** *Let E be Banach space. The unit ball*

$$\big\{x \in E \quad | \quad \|x\|_{E} \leq 1\big\}$$

is weakly compact iff E is reflexive.

We conclude these comparison with a last fact on the unit ball.

Fact A.26. *Let E be Banach space. The unit ball*

$$\big\{x \in E \quad | \quad \|x\|_{E} \leq 1\big\}$$

is norm-compact iff E is finite dimensional.

On the other hand the weak topology present a few difficulties.

Fact A.27. *The following assertion are equivalent.*

i) *The vector space E is finite dimensional.*

ii) *The weak and norm topology coincide.*

iii) *The weak topology is metrizable.*

In any infinite dimensional space we have that:

- *the weak interior of every closed or open ball is empty;*
- *the closed unit sphere is weakly dense in the closed unit ball.*

A.1.3 Sequential Properties

In metric spaces we are used to sequential definition of topological properties. However, in non-metrizable spaces (such as some Banach spaces endowed with weak or weak$\star$ topologies), those definitions differ. We obtain comparable characterization by using a generalization of the sequences: the nets.

In this section we consider two topological spaces $\big(E,\tau\big)$, and $\big(F,\tau'\big)$.

Definition A.28 (net)**.** *Consider a set D. A* direction $\prec$ *on D, is a binary relation satisfying:*

- *$\alpha \prec \beta$ and $\beta \prec \gamma$ imply $\alpha \prec \gamma$;*
- *$\alpha \prec \alpha$;*
- *for all α and β of D, there exists $\gamma \in D$, such that $\alpha \prec \gamma$ and $\beta \prec \gamma$.*

A directed set*, is a set endowed with a partial order.*

A net *in a set X, is a function $x : D \to X$ where D is a directed set. It is denoted $\{x_\alpha\}_{\alpha\in D}$.*

In a topological space $\big(E,\tau\big)$, a net $\{x_\alpha\}_{\alpha\in D}$ is said to converge toward *$x \in E$ if, for any open set $\mathcal{O}$ containing x, there exists an index $\alpha_0 \in D$, such that for all index $\alpha_0 \prec \alpha$, we have $x_\alpha \in \mathcal{O}$.*

Fact A.29. *We have the following characterization of topological properties.*

- *A set $C \subset E$ is closed iff every converging net in C has its limit in C.*
- *A function $f : E \to F$ is continuous iff $x_\alpha \to_\tau x$ implies $f\big(x_\alpha\big) \to_{\tau'} f(x)$.*
- *A function $f : E \to \overline{\mathbb{R}}$ is l.s.c. iff $x_\alpha \to_\tau x$ implies $\underline{\lim}_\alpha f\big(x_\alpha\big) \le f(x)$.*
- *A set $C \subset E$ is compact iff every net in C has a convergent subnet.*

If we replace nets by sequences in the above fact we have *sequential properties*.

Definition A.30 (Sequential properties)**.** *We have the following definition of topological properties.*

- *A set $C \subset E$ is sequentially closed iff every converging sequence in C has its limit in C.*
- *A function $f : E \to F$ is sequentially continuous iff $x_n \to_\tau x$ implies $f\big(x_n\big) \to_{\tau'} f(x)$.*
- *A function $f : E \to \overline{\mathbb{R}}$ is sequentially l.s.c. iff $x_\alpha \to_\tau x$ implies $\underline{\lim}_n f\big(x_n\big) \le f(x)$.*
- *A set $C \subset E$ is sequentially compact iff every sequence in C has a convergent subsequence.*

Fact A.31. *In a first countable space, (e.g. a metric space) sequential properties are equivalent to their topological counterparts.*

Fact A.32. *We have the following implications.*

- *A closed set is sequentially closed.*
- *A continuous function $f : E \to F$, where F is first countable (e.g. metric), is sequentially continuous.*
- *A compact set is sequentially compact.*
- *A l.s.c. function, is sequentially l.s.c.*

We have the following property analogous to Theorem A.12.

Proposition A.33. *A sequentially l.s.c. function admit a minimum on a sequentially compact set.*

Finally we have the Eberlein-Smulian theorem.

Theorem A.34 (Eberlein-Smulian)**.** *In the weak topology of a normed space compactness and sequential compactness coincide.*

Proof. See [2, Thm 10.15]. □

A.1.4 Duality

This section is mostly taken from the book *Conjugate duality and optimization* by R.T.Rockafellar [89]. A fundamental point to the study of duality is the fact that (continuous) linear functions on a given linear space can be seen as elements of a linear space. In the finite dimensional case the (continuous) linear functions on $\mathbb{R}^n$ can be identified to vectors of $\mathbb{R}^n$ through the inner product. This is still the case (for continuous linear functions) through the Riesz representation theorem in Hilbert spaces. To extend this representation we present the notion of paired space.

Definition A.35 (Paired spaces)**.** *Consider two $\mathbb{R}$-vector spaces E and F. A* pairing *of E and F is a bi-linear form $\langle \cdot\,,\cdot \rangle$ on $E \times F$, i.e, for each $x \in E$,*

$$\langle x\,,\cdot \rangle : y \mapsto \langle x\,,y \rangle\;,$$

is a linear function, and for each $y \in Y$,

$$\langle \cdot\,,y \rangle : x \mapsto \langle x\,,y \rangle\;,$$

is also a linear function.

A topology on X is said to be compatible with the pairing *if it is locally convex and such that all linear function $\langle \cdot\,,y \rangle$ is continuous, and such that all continuous linear functions can be represented in such a way. The definition for compatible topologies on Y is symmetric.*

We say that E and F are paired space *if they are two topological vector spaces, endowed with topologies compatible with a pairing.*

Various compatible topologies can be systematically generated and we refer to the functional analysis books for more precision (e.g. [58]).

Definition A.36 (Adjoint Operator)**.** *Consider two paired spaces $\big(X, X'\big)$ and $\big(Y, Y'\big)$, and a linear operator $L : X \to Y$. A linear operator $L^\dagger : Y' \to X'$ is said to be the* adjoint operator of L *if it satisfies*

$$\forall x \in X, \quad \forall y' \in Y', \qquad \langle y'\,,Lx \rangle_{\left(Y',Y\right)} = \langle L^\dagger y'\,,x \rangle_{X',X}\;.$$

Definition A.37 (Fenchel Conjugate)**.** *Consider a function $f : X \to \bar{\mathbb{R}}$. Its* Fenchel conjugate *(for the given pairing) is the function $f^\star : Y \to \bar{\mathbb{R}}$ such that*

$$f^\star(y) = \sup_{x \in X} \Big\{ \langle x\,,y \rangle - f(x) \Big\}\;. \tag{A.1}$$

The biconjugate *is $f^{\star\star} : X \to \bar{\mathbb{R}}$, is given by*

$$f^{\star\star}(x) = \sup_{y \in Y} \Big\{ \langle x\,,y \rangle - f^\star(y) \Big\}\;. \tag{A.2}$$

Theorem A.38. *For any function $f : X \to \bar{\mathbb{R}}$ its conjugate $f^\star$ is a closed convex function on Y, and $f^{\star\star}$ is the lower semicontinuous convex hull of f. In particular if f is a proper convex lower semicontinuous function then $f \equiv f^{\star\star}$.*

We give a useful illustration of this definition. Consider a set $C \subset X$ and its indicator function χ_C. Then $\chi_C^\star$ is the *support function* of C:

$$\chi_C^\star(y) = \sup_{x \in C} \langle x\,,y \rangle\;. \tag{A.3}$$

If C is a cone, then $\chi_C^\star$ is the indicator of its *dual cone* $C^\star$, where

$$C^\star := \left\{ y \in Y \mid \forall x \in C, \quad \langle x \,, y \rangle \geq 0 \right\} . \tag{A.4}$$

In particular if C is a linear subspace of X we have

$$C^\star = \left\{ y \in Y \mid \forall x \in C, \quad \langle x \,, y \rangle = 0 \right\} . \tag{A.5}$$

Proposition A.39. *For a closed convex cone $C \subset X$, we have*

$$\forall x \in X, \qquad \chi_{-C}(x) = \sup_{y \in C^\star} \left\langle y \,, x \right\rangle .$$

Proof. As $-C$ is a closed convex set, χ_{-C} is a l.s.c. convex function, hence

$$\chi_{-C} = \chi_{-C}^{\star\star} .$$

Now by definition

$$\chi_{-C}^\star(y) = \sup_{x \in X} \left\{ \left\langle y \,, x \right\rangle - \chi_{-C}(x) \right\} = \sup_{x \in -C} \left\langle y \,, x \right\rangle = \chi_{C^\star}(y) .$$

And

$$\chi^{\star\star}{-C}(x) = \left(\chi_{C^\star} \right)^\star (x) = \sup_{y \in C^\star} \left\langle y \,, x \right\rangle .$$

Hence the result. □

More generally, we have the following definition of the cone dual of any given set.

Definition A.40 (Dual cone)**.** *For any subset $X \subset E$ and $x \in E$ we define the set $X_x^\perp \subset \mathcal{X}^\star$ as follows :*

$$X_x^\perp := \left\{ y \in \mathcal{X}^\star \quad \middle| \quad \forall z \in \mathcal{X}, \quad \left\langle y \,, z - x \right\rangle \leq 0 \right\} . \tag{A.6}$$

Note that if X is an affine subset of E, then $X_x^\perp$ does not depends on the point x, and the notation can be omitted.

We now give a –possibly new– result over the dual of the intersection of two sets. Consider a topological space $\left(E, \tau\right)$.

Lemma A.41. *Let $L : E \to E$ be a linear operator, admitting an adjoint $L^\dagger$, such that $L(E) = N$ and $L_{|_N} = \mathrm{Id}$ then we have :*

$$N_x^\perp = N_0^\perp = \left\{ y - L^\dagger(y) \quad \middle| \quad y \in E^\star \right\}$$

Proof. Since N is a linear space it is immediate to see that $N_x^\perp$ does not depend on x and that the inequality can be replaced by an equality in the definition of $N_x^\perp$. We thus have

$$N_x^\perp = N_0^\perp = \left\{ y \in E^\star \quad \middle| \quad \forall z \in E, \quad \left\langle y \,, z - x \right\rangle = 0 \right\} .$$

First, fix $y \in E^\star$. For all $z \in N$ we have

$$\left\langle y - L^\dagger(y) \,, z \right\rangle = \left\langle y \,, z - L(z) \right\rangle = 0 \,,$$

where the last equality is deduced from $L_{|_N} = \mathrm{Id}$. We thus have

$$\left\{ y - L^\dagger(y) \quad \middle| \quad y \in E^\star \right\} \subset N_x^\perp .$$

Then, fix $y \in N_0^\perp$. For all $z \in E$ we have $\left\langle y \,, L(z) \right\rangle = 0$ since $L(E) = N$ and thus for all $z \in E$ we have $\left\langle L^\dagger(y) \,, z \right\rangle = 0$. Thus, for all $z \in N$, we obtain that $\left\langle y \,, z \right\rangle = 0$ and $\left\langle L^\dagger(y) \,, z \right\rangle = 0$ which gathered give $\left\langle y - L^\dagger(y) \,, z \right\rangle = 0$. We thus have

$$N_x^\perp \subset \left\{ y - L^\dagger(y) \quad \middle| \quad y \in E^\star \right\} ,$$

which achieves the proof. □

Lemma A.42. *Under assumptions of Lemma A.41 and assuming moreover that $x \in U \cap N$ and that $L(U) \subset U$ then, for all $y \in (U \cap N)_x^\perp$, we have that $L(y) \in U_x^\perp$.*

Proof. Fix $y \in (U \cap N)_x^\perp$. For all $z \in U$ we have that $L^\dagger(z) \in U \cap N$ using the facts that $L(U) \subset U$ and $L(E) = N$. We thus have $\langle y\,, L(z) - x\rangle \le 0$ for all $z \in U$. Now, since $x \in U \cap N$ and $L_{|_N} = \mathrm{Id}$ we have $L(x) = x$ and using the linearity of L we obtain that $\langle L^\dagger(z)\,, z - x\rangle \le 0$ for all $y \in U$ which ends the proof. □

We now prove the main result.

Theorem A.43. *Under assumptions of Lemma A.41 and assuming moreover that $x \in U \cap N$ and that $L(U) \subset U$ we have that*

$$(U \cap N)_x^\perp = U_x^\perp + N_0^\perp\,.$$

Proof. First, proving that $U_x^\perp + N_x^\perp \subset (U \cap N)_x^\perp$ is immediate and since $N_x^\perp = N_0^\perp$ we obtain that $U_x^\perp + N_0^\perp \subset (U \cap N)_x^\perp$.

Let $y \in (U \cap N)_x^\perp$, then we can write y as $y = (y - L^\dagger(y)) + L^\dagger(y)$. Using Lemma A.42 we obtain that $L^\dagger(y) \in U_x^\perp$ and Using Lemma A.41 we obtain that $y - L^\dagger(y) \in N_x^\perp$. We thus have $y \in U_x^\perp + N_x^\perp$. □

A.2 Convexity and Lower Semicontinuity

A.2.1 Convexity

Let E and U be vector spaces.

Definition A.44 (Convexity)**.** *A set $C \subset E$ is said to be convex iff*

$$\forall x \in C, \quad \forall y \in C, \quad \forall t \in [0,1], \qquad tx + (1-t)y \in C\,. \tag{A.7}$$

A function $f : E \to \overline{\mathbb{R}}$ is said to be convex if its epigraph is a convex set, or equivalently if

$$\forall x \in E, \quad \forall y \in E, \quad \forall t \in [0,1], \qquad f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y)\,. \tag{A.8}$$

A function $f : E \to \overline{\mathbb{R}}$ is said to be strictly convex iff

$$\forall x \in E, \quad \forall y \in E, \quad \forall t \in (0,1), \qquad f\big(tx + (1-t)y\big) < tf(x) + (1-t)f(y)\,. \tag{A.9}$$

When a function G has two arguments (x, u) and its convex in the couple (x, u) we say that G is *jointly convex.*

Proposition A.45. *Let $G : E \times U \to \overline{\mathbb{R}}$ be a jointly convex function. Then the function defined by*

$$\varphi(x) = \inf_{u \in E} G(x, u)\,,$$

is convex.

Proof. Consider a pair of point $(x_1, x_2) \in E^2$, and a pair $(u_1, u_2) \in U^2$. We have

$$\varphi\big(tx_1 + (1-t)x_2\big) \le G\big(tx_1 + (1-t)x_2, tu_1 + (1-t)u_2\big) \le tG(x_1, u_1) + (1-t)G(x_2, u_2)\,.$$

taking the infimum in $u_1 \in U$ and $u_2 \in U$ gives the result. □

Definition A.46 (Cone)**.** *A set $C \subset E$ is said to be a* cone *if*

$$\forall t \in \mathbb{R}^+, \quad \forall x \in C, \qquad tx \in C \; . \tag{A.10}$$

A salient cone *is a cone such that there exist a nonzero vector $x \in C$ such that $-x \in C$. A salient convex cone defines a partial order on E with*

$$\forall (x, y) \in C^2, \qquad x \preccurlyeq_C y \quad \iff \quad y - x \in C \; . \tag{A.11}$$

Lemma A.47. *Let $C \subset E$ be a closed convex cone, then*

$$C + C = C \; .$$

Proof. Consider a pair of point (c_1, c_2) of C. By convexity of C, we have $c = \frac{c_1+c_2}{2} \in C$. And as C is a cone, $2c \in C$. Moreover as C is a closed cone we have $0 \in C$, thus $C \subset C + C$. □

Definition A.48. *For a convex cone $C \subset E$, a function $\Theta : U \to E$ is said to be C-convex if*

$$\forall (u, v) \in C^2, \qquad \Theta\big(tu + (1-t)v\big) \preccurlyeq_C t\Theta(u) + (1-t)\Theta(v) \; . \tag{A.12}$$

Proposition A.49. *For a convex cone $C \subset E$, if $\Theta : U \to E$ is a C-convex function then*

$$\mathcal{U}^{ad} = \big\{ u \in U \quad | \quad \Theta(u) \in -C \big\}$$

is convex. Moreover if E and U are endowed with topologies such that Θ is sequentially continuous and C is sequentially closed, then so is $\mathcal{U}^{ad}$.

Note that in metric spaces, and a fortiori in Banach spaces sequential continuity and continuity are equivalent.

Proof. Consider u and v elements of $\mathcal{U}^{ad}$, and $t \in [0, 1]$. As C is convex, $-C$ is also convex, thus

$$t\Theta(u) + (1-t)\Theta(v) \in -C \; .$$

By C-convexity of Θ we have

$$t\Theta(u) + (1-t)\Theta(v) - \Theta\big(tu + (1-t)v\big) \in C \; ,$$

and as C is a convex cone we have

$$C + C \subset C \; ,$$

which leads to

$$-\Theta\big(tu + (1-t)v\big) \in C \; ,$$

and thus $tu + (1-t)v \in \mathcal{U}^{ad}$.

The sequential continuity result is obvious. □

Definition A.50 (Strong convexity)**.** *If E is a normed space, a function is said to be* strongly convex *of modulus $\alpha > 0$, or* α-convex *if*

$$\forall t \in [0, 1], \quad \forall (x, y) \in E^2, \qquad f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) - \frac{\alpha}{2} t(1-t)\|x - y\|_E \; .$$

Definition A.51 (Coercivity)**.** *If E is a normed space, a function $f : E \mapsto \overline{\mathbb{R}}$ is said to be* coercive *on $A \subset E$ if*

$$\forall M \in \mathbb{R}, \quad \exists r > 0, \quad \forall x \in A, \quad \big\|x\big\| \geq r \quad \Longrightarrow \quad f(x) \geq M \; .$$

In particular a strongly convex function is coercive on E.

A.2.2 Results on Semicontinuity

We recall some results on semicontinuous functions that were needed in the manuscript.

Lemma A.52. *Let X and Y be two metric spaces.*[4] *If $f : Y \to \overline{\mathbb{R}}$ is l.s.c and $g : X \to Y$ is continuous, then $f \circ g$ is l.s.c.*

Proof. We consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x. We define $y_n = g(x_n)$. By continuity of g, we have that $\liminf_{x_n\to x} y_n = \liminf_{x_n\to x} g(x_n) = g(x)$. Thus $\liminf_{x_n\to x} f \circ g(x_n) = \liminf_{y_n\to y} f(y_n) \geq f(g(x))$. □

Lemma A.53. *Let X be a metric space. If $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is non-decreasing and $g : X \to \overline{\mathbb{R}}$ is l.s.c, then $f \circ g$ is l.s.c.*

Proof. We consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x. We have that $\liminf_{x_n\to x} g(x_n) \geq g(x)$ and, by monotonicity of f, we have $\liminf_{x_n\to x} f(g(x_n)) \geq f(g(x))$. □

Lemma A.54. *Let X be a metric space. If $f : X \to \overline{\mathbb{R}}$ and $g : X \to \overline{\mathbb{R}}$ are l.s.c functions then so are $\max\{f, g\}$ and $\min\{f, g\}$.*

Proof. Recall that a function is l.s.c iff its epigraph is closed. As $\mathrm{epi}(\max\{f, g\}) = \mathrm{epi}(f) \cap \mathrm{epi}(g)$ and $\mathrm{epi}(\min\{f, g\}) = \mathrm{epi}(f) \cup \mathrm{epi}(g)$, and as $\mathrm{epi}(f)$ and $\mathrm{epi}(g)$ are closed, so are $\mathrm{epi}(\min\{f, g\})$ and $\mathrm{epi}(\max\{f, g\})$. □

Lemma A.55. *Let X be a separated topological space, A a closed convex subset of X, $J : X \to \overline{\mathbb{R}}$ a lower semicontinuous convex function. The function $J + \chi_A$ is convex and l.s.c..*

Proof. By assumption $\mathrm{epi}(J)$ is a closed convex subset of $E \times \mathbb{R}$. As A is a closed convex subset of E, $A \times \mathbb{R}^+$ is also a closed convex subset of $E \times \mathbb{R}$. Thus,

$$\mathrm{epi}\left(J + \chi_A\right) = \mathrm{epi}(J) \cap A \times \mathbb{R}^+$$

is a closed convex set, hence the convexity and lower semicontinuity of $J + \chi_A$. □

Lemma A.56. *Then the mapping $\boldsymbol{X} \mapsto \mathrm{essupp}(\boldsymbol{X})$ is lower semicontinuous in the strong topology of $\mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P})$.*

Proof. To prove semicontinuity it is enough to show that the level sets of $\boldsymbol{X} \mapsto \mathrm{essupp}(\boldsymbol{X})$ are closed. Let's $(\boldsymbol{X}_n)$ be a sequence converging in L^2 to $\boldsymbol{X}$ such that $\mathrm{essupp}(\boldsymbol{X}_n) \leq M$. We can extract a subsequence $(\boldsymbol{X}_{n_k})$ converging almost surely to $\boldsymbol{X}$. Let's call $\Omega' \subset \Omega$ a set of probability 1 verifying $\forall \omega \in \Omega', \lim_{k\to\infty} \boldsymbol{X}_{n_k}(\omega) = \boldsymbol{X}(\omega)$, consequently on this set $\boldsymbol{X} \leq M$ and thus $\left\{\boldsymbol{X} \in \mathrm{L}^2 \mid \mathrm{essupp}(\boldsymbol{X}) \leq M\right\}$ is (sequentially) closed (for the strong or weak topology as the set is convex). □

4. We need the equivalence between sequential and classical properties.

Appendix B

Probability

Always pass on what you have learned.

Yoda

Contents

B.1 Basic Definitions

Definition B.1 (σ-algebra). *A* σ-algebra $\mathcal{F}$ *on a set* Ω*, is a set of subset of* Ω *satisfying:*

- $\emptyset$ *and* Ω *are element of* $\mathcal{F}$*;*
- *countable unions of elements of* $\mathcal{F}$ *are element of* $\mathcal{F}$*;*
- *complement of an element of* $\mathcal{F}$ *is in* $\mathcal{F}$*.*

A measurable space *is a couple* $(\Omega, \mathcal{F})$ *where* $\mathcal{F}$ *is a* σ*-algebra on* Ω*.*

A filtration $\mathfrak{F} = \{\mathcal{F}_t\}_{t=0}^{T-1}$ *is an increasing sequence of* σ*-algebra.*

A function f *mapping a space endowed with a* σ*-algebra into another is said to be* measurable *if any preimage of the* σ*-algebra (of the image set) is in the* σ*-algebra (of the origin set).*

This definition is close to the definition of a topology (see Definition A.1), and a measurable function comparable to a measurable function.

Definition B.2 (Probability). *A* probability *on a measurable space* $(\Omega, \mathcal{F})$ *is a function* $\mathbb{P}$ *mapping* $\mathcal{F}$ *into* $[0, 1]$*, and satisfying:*

- $\mathbb{P}(\Omega) = 1$*;*
- *For any sequence* $\{A_n\}_{n\in\mathbb{N}}$ *of disjunct elements of* $\mathcal{F}$*, we have*

$$\mathbb{P}\big(\bigcup_{n\in\mathbb{N}} A_n\big) = \sum_{n\in\mathbb{N}} \mathbb{P}(A_n) \ .$$

A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ *where* $\mathbb{P}$ *is a probability on* $(\Omega, \mathcal{F})$*.*

The expectation w.r.t. probability $\mathbb{P}$ is given by

$$\mathbb{E}\big[\boldsymbol{X}\big] = \int_\Omega X(\omega) d\mathbb{P}(\omega).$$

The conditional expectation of $\boldsymbol{X}$ w.r.t of sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$, is defined on $\mathrm{L}^2\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ as the $\mathcal{G}$-measurable random variable $\mathbb{E}\big[\boldsymbol{X} \mid \mathcal{G}\big]$ satisfying

$$\forall Y \in \mathrm{L}^2\big(\Omega, \mathcal{F}, \mathbb{P}\big), \qquad \mathbb{E}\Big[\boldsymbol{X}\boldsymbol{Y}\Big] = \mathbb{E}\Big[\mathbb{E}\big[\boldsymbol{X} \mid \mathcal{G}\big]\boldsymbol{Y}\Big] .$$

It is then extended on $\mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}\big)$.

Lemma B.3. *For every $\boldsymbol{\lambda} \in \mathrm{L}^1\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$, every $\boldsymbol{X} \in \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$, and every σ-algebra $\mathcal{B} \subset \mathcal{F}$,*

$$\mathbb{E}\Big[\boldsymbol{\lambda} \cdot \mathbb{E}\big[\boldsymbol{X} \mid \mathcal{B}\big]\Big] = \mathbb{E}\Big[\mathbb{E}\big[\boldsymbol{\lambda} \mid \mathcal{B}\big] \cdot \mathbb{E}\big[\boldsymbol{X} \mid \mathcal{B}\big]\Big] = \mathbb{E}\Big[\mathbb{E}\big[\boldsymbol{\lambda} \mid \mathcal{B}\big] \cdot \boldsymbol{X}\Big] .$$

Proof. Immediate application of the well known equality $\mathbb{E}\Big[\,\cdot\,\Big] = \mathbb{E}\Big[\mathbb{E}\big[\,\cdot \mid \boldsymbol{Y}\big]\Big]$. □

B.2 Recalls on $\mathrm{L}^p\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$-spaces

The subject of $\mathrm{L}^p\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$-spaces is widely studied and lots of results can be found in the literature (e.g. [3, Chap. 13]). However note that we are interested in the case where $\mathbb{P}$ is a probability measure, whereas most authors in the functional analysis literature either works with the Lebesgue measure, or a general measure. A probability measure is more general than the Lebesgue measure, but is of finite total mass.

Proposition B.4. *Consider $1 \le p, q \le +\infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, with the convention $\dfrac{1}{\infty} = 0$. We have, for any $\boldsymbol{X} \in \mathrm{L}^p\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$, and any $\boldsymbol{Y} \in \mathrm{L}^q\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$, the* Hölder inequality

$$\|\boldsymbol{X} \cdot \boldsymbol{Y}\|_1 \le \|\boldsymbol{X}\|_p \cdot \|\boldsymbol{Y}\|_q . \tag{B.1}$$

Proposition B.5. *We have, for all $1 \le r \le s \le \infty$*

$$\mathrm{L}^s\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big) \subset \mathrm{L}^r\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big) , \tag{B.2}$$

and for all $\boldsymbol{X} \in \mathrm{L}^s\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big)$,

$$\|\boldsymbol{X}\|_r \le \|\boldsymbol{X}\|_s . \tag{B.3}$$

Proof. See [3, Corollary 13.3]. Note that (B.1) holds for any measure, (B.2) for any finite measure, and (B.3) for probability measures. □

Definition B.6. *Let $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ be a probability space. A function $\varphi : \Omega \to \mathbb{R}^n$ is called a* step-function *if there is a sequence of $\mathcal{F}$-measurable sets $\big(A_i\big)_{i \in [\![1,N]\!]}$, and a sequence of vectors $\big(a_i\big)_{i \in [\![1,N]\!]}$ such that*

$$\varphi = \sum_{i=1}^{N} a_i \mathbb{1}_{A_i} .$$

The set of all step-function is a vector space whose interest is given in the following proposition.

Proposition B.7. *For any $p \in [1, +\infty)$ the space of all step-functions is norm dense in $\mathrm{L}^p\big(\Omega, \mathcal{F}, \mathbb{P}\big)$. Thus for any $1 \le r \le s \le +\infty$, $\mathrm{L}^r\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ is norm dense in $\mathrm{L}^s\big(\Omega, \mathcal{F}, \mathbb{P}\big)$.*

Proof. See [3, Theorem 13.4]. □

Note that with some more properties on Ω, especially if $\Omega \subset \mathbb{R}^m$ equipped with the Lebesgue measure, the space of smooth functions with compact support is norm dense in any L^p for $p < +\infty$.

B.3 Convergences

We quickly review defininitions of convergence in probability. A sequence of random variable $\{\boldsymbol{X}_n\}_{n\in\mathbb{N}}$ is said to converges toward $\boldsymbol{X}$

- almost surely if $\mathbb{P}\big(\boldsymbol{X}_n \to \boldsymbol{X}\big) = 1$;
- in probability if $\forall\varepsilon > 0$, $\mathbb{P}\big(|\boldsymbol{X}_n - \boldsymbol{X}| > \varepsilon\big) \to 0$;
- in L^p if $||\boldsymbol{X}_n - \boldsymbol{X}||_{\mathrm{L}^p} \to 0$;
- in law if, for all continuous and bounded function $f : \mathbb{X} \to \mathbb{R}$, $\mathbb{E}\big[f(\boldsymbol{X}_n)\big] \to \mathbb{E}\big[f(\boldsymbol{X})\big]$.

We sum up in figure B.1 the following relation between this convergences.

- For $+\infty \geq p > r \geq 1$, convergence in L^p implies convergence in L^r.
- Convergence in L^∞ implies almost sure convergence.
- Convergence in L^p, or almost sure imply convergence in probability which in turn imply convergence in law.
- Dominated convergence theorem can be used together with almost sure convergence to ensure convergence in L^1.
- Convergence in probability implies the almost sure convergence of a subsequence.

Finally, we point out a few topologic properties of this convergences.

- Convergence in L^p is the convergence relative to a norm.
- Convergence in Probability is metrizable.
- Convergence in Law is deduced from a topology (weak* topology).
- Almost sure convergence can not generally be induced by a topology. Indeed, if almost sure convergence were induced by a topology, then Lemma 7.7, would imply that convergence in probability imply almost sure convergence. However note that $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ converges in probability toward $\boldsymbol{U}$ iff from any sub-sequence of $\{\boldsymbol{U}_n\}_{n\in\mathbb{N}}$ we can extract a further sub-sequence converging almost surely to $\boldsymbol{U}$ (see [44, Th 2.3.2]).

$$
\begin{tikzcd}
 & \mathrm{L}^\infty \arrow[ddl, Rightarrow] \arrow[dr, Rightarrow] & \\
 & & \mathrm{L}^p \arrow[d, Rightarrow] \\
a.s. \arrow[rr, "\text{Dominated convergence}"] \arrow[dr, Rightarrow] & & \mathrm{L}^r \arrow[dl, Rightarrow] \\
 & \mathbb{P} \arrow[ul, bend left] \arrow[d, Rightarrow] & \\
 & \mathcal{L} &
\end{tikzcd}
$$

Figure B.1: Summing up the relation between probabilistic convergences.

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