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Thèse pour l'obtention du titre de

DOCTEUR DE L'ÉCOLE POLYTECHNIQUE

Spécialité : mathématiques appliquées

Gwenael MERCIER

Évolutions de courbes et surfaces pour le traitement d'images

Soutenue le 15 septembre 2015 devant le jury composé de

François Alouges	Président du jury
Pierre Bousquet	Rapporteur
Pierre Cardaliaguet	Rapporteur
Antonin Chambolle	Directeur de thèse
Olivier Ley	Examinateur
Otmar Scherzer	Examinateur

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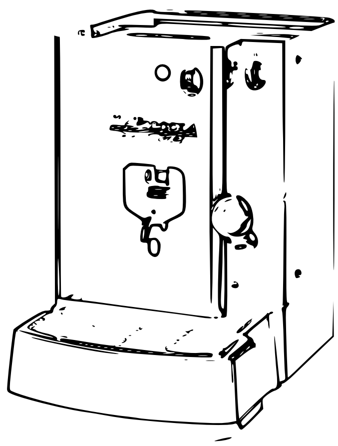

Introduction

Ce manuscrit est divisé en deux parties assez indépendantes. Aussi, nous avons choisi de respecter cette division dans l'introduction. On commencera par un bref résumé des travaux présentés dans les quatre chapitres de ce manuscrit. Nous introduirons ensuite les notions de géométrie différentielle utilisées dans ce manuscrit puis nous examinerons différents points de vue sur l'objet courbure moyenne.

Les sections 4 et 5 présenteront ensuite les résultats de ce manuscrit, en respectant la structure de celui-ci. La première exposera le contexte et les résultats du premier chapitre et la seconde introduira et résumera les chapitres 2 à 4.

1 Résumé

L'objectif de cette thèse a été d'étudier différents problèmes apparaissant naturellement en traitement d'images et mettant en jeu des hypersurfaces de l'espace euclidien à n dimensions. Débruiter une image consiste essentiellement à en lisser les lignes. Ce lissage peut apparaître soit comme le résultat d'une minimisation d'une fonctionnelle, soit comme l'application d'un flot régularisant sur les lignes de l'image. Dans ce manuscrit, nous étudions deux exemples de ces deux approches.

— Dans le chapitre 1, on lisse par minimisation et on s'intéresse à la régularité de la solution. Plus précisément, on travaille sur des généralisations de la minimisation proposée par Rudin, Osher et Fatemi qui pénalise la variation totale. On cherche à montrer que sous différentes hypothèses sur le domaine, les conditions d'attaches aux données ainsi que le choix de la variation totale (isotrope, anisotrope,...), la continuité de l'image observée se transmet forcément au minimiseur, ce qui montre que le débruitage par minimisation ne vas pas faire apparaître de discontinuité non observée.

— Dans le capitre 2, on étudie le flot par courbure moyenne (éventuellement anisotrope), qui est connu pour avoir un effet régularisant [AGLM93]. On y ajoute des obstacles. L'approche choisie est celle des lignes de niveau : la surface est l'image réciproque de 0 par une fonction qu'on fait évoluer. On démontre existence et unicité d'une fonction solution (de viscosité) de l'équation du mouvement par ligne de niveau et on étudie son asymptotique en temps en la comparant à un mouvement minimisant discret.

— Dans le chapitre 3 (travail en collaboration avec M. Novaga), on précise le résultat du chapitre 2 en étudiant le même problème mais sous forme géométrique (ce qui est nettement plus précis que l'approche ligne de niveau). On suit l'approche de

Ecker et Huisken pour montrer qu'il existe une unique solution au mouvement par courbure moyenne avec obstacles en temps court.
— Enfin, dans le dernier chapitre (travail en collaboration avec M. Novaga et P. Pozzi), on fait un premier pas vers l'étude géométrique du mouvement anisotrope (on pourra en particulier traiter les anisotropies cristallines). Uniquement restreints à la dimension deux, on montre, en l'approchant par un mouvement lisse, l'existence d'un mouvement par courbure anisotrope d'une courbe immergée pour un temps petit.

2 Une brève introduction à la géométrie des hypersurfaces

Dans cette partie, nous tâchons de rappeler des notions élémentaires de géométrie différentielle qui nous permettront de définir correctement la courbure moyenne d'une hypersurface, ainsi que la plupart des notions qu'on utilisera dans ce manuscrit. Ces notions sont classiques et cette partie peut-être sautée. Le lecteur souhaitant avoir un peu plus de détails peut, dans un premier temps, se référer à [dC92] qui me semble constituer une excellente introduction. Pour une approche plus avancée et plus exhaustive, on pourra lire [Wil93] qui traite des variétés abstraites en toute généralité (ce que nous ne ferons pas ici).

2.1 Définition et plan tangent

Dans toute la suite, on considèrera uniquement des hypersurfaces (sous-variété de dimension $n-1$) de $\mathbb{R}^n$. Dans toute cette introduction, on travaillera avec des objets $\mathcal{C}^\infty$. On rappelle qu'une sous-variété de dimension $n-1$ est un sous-ensemble de $\mathbb{R}^n$ qui satisfait (définitions équivalentes)
— Pour tout $a \in M$, il existe un voisinage V de a dans $\mathbb{R}^n$ et un difféomorphisme $\varphi : V \to \mathbb{R}^n$ qui envoie $V \cap M$ sur un ouvert de $\mathbb{R}^{n-1}$.
— Pour tout $a \in M$, il existe un voisinage V de a dans $\mathbb{R}^n$ et un paramétrage $F : \Omega \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$ telle que $F(\Omega) = V \cap M$.
— Pour tout $a \in M$, il existe un voisinage V de a dans $\mathbb{R}^n$ et une submersion $f : V \to \mathbb{R}$ (application $\mathcal{C}^1$ de gradient ne s'annulant par sur V) telle que $M \cap V = f^{-1}(0)$.

On rappelle que M est munie en chaque point p d'un plan tangent (un hyperplan de $\mathbb{R}^n$) T_pM. Un vecteur de T_pM est appelé vecteur tangent à M en p. L'ensemble des couples (p, x) tel que $x \in T_pM$ est noté TM et est appelé fibré tangent à M. En effet, si $\pi : TM \to M$ est la première projection canonique, alors pour tout p, la fibre $\pi^{-1}(p)$ est exactement l'espace tangent à M en p T_pM. Ainsi défini, TM est une sous variété de $\mathbb{R}^{2n}$ de dimension $2(n-1)$. En effet, en utilisant la définition 3 de sous-variété, il existe un voisinage V de $p \in M$ et f telle que $M \cap V = f^{(-1)}(0)$. Alors, on remarque que

$$TM \cap (V \times \mathbb{R}^n) = \{(p, t) \in M \times \mathbb{R}^n, \mid f(p) = 0,\ Df_p(t) = 0\},$$

et que l'application $(p, t) \mapsto (f(p), Df_p(t))$ a une différentielle surjective, c'est-à-dire exactement que TM est une sous variété de $\mathbb{R}^{2n}$ de dimension $2n-2$.

On appelle également champ de vecteur toute application lisse $X : M \to TM$ qui à p, fait correspondre un couple (p, t) [1]. Avec les mains, ceci signifie qu'on fait correspondre à

1. Même si la terminologie ne sera pas utilisée ici, on dit que X est une section du fibré tangent TM.

tout point de M, un vecteur tangent à ce point, et que cette correspondance est lisse. On note ΞM l'ensemble des champs de vecteurs sur M.

En tout point p de M, on dispose d'une base naturelle de T_pM en étudiant les dérivées du paramétrage

$$e_i = \partial_i F.$$

Comme DF est injective, les e_i forment bien une base de T_pM.

On appelle également vecteur normal à M tout vecteur orthogonal à T_pM. Lorsque l'on parlera de vecteur normal dans la suite, il sera considéré unitaire et, lorsque la sous variété est fermée, dirigé vers l'extérieur.

2.2 Métrique sur une hypersurface

On souhaite maintenant pouvoir mesurer des distances sur M, distances dont on souhaite qu'elles proviennent de la métrique de l'espace ambiant. On dispose donc naturellement d'un produit scalaire sur l'espace tangent (comme sous-espace de $\mathbb{R}^n$). On peut ainsi définir la longueur d'une courbe $\gamma : [0,1] \to M$ par

$$L(\gamma) = \int_0^1 |\gamma'(t)| dt.$$

Cette formule a parfaitement un sens car pour tout $t \in [0,1]$, γ' est un vecteur tangent à M, il a donc une norme. On peut donc définir la distance entre deux points a et b de M via

$$d(a,b) = \inf_{\substack{\gamma(0)=a \\ \gamma(1)=b}} L(\gamma).$$

Ainsi, la notion de distance ne dépend que du produit scalaire sur T_pM. Pour calculer, on peut donc exprimer le produit scalaire sur T_pM dans la base des (e_i). On note cette matrice

$$g_{ij}(p) = (e_i\,,\, e_j) = (\partial_i F\,,\, \partial_j F)$$

et on l'appelle métrique sur M en p. Remarquons alors que cette matrice est lisse par rapport à p.

On introduit aussi la matrice inverse de $[g_{ij}]$ qu'on note, comme il est d'usage, $[g^{ij}]$.

2.3 Dérivation sur une hypersurface et courbure

Pour définir la courbure, il est nécessaire de pouvoir dériver sur M. Expliquons brièvement comment. Tout d'abord, si f est une fonction sur M, comme un voisinage V de p dans M peut être envoyé de façon difféomorphe par ψ sur un ouvert de $\mathbb{R}^{n-1}$ tel que $\psi(0) = p$, on peut définir l'application linéaire $D_pf : T_pM \to \mathbb{R}$ par

$$D_pf \cdot h = D_0(f \circ \psi) \cdot (D_0\psi^{-1} \cdot h).$$

On vérifie alors aisément que cette définition ne dépend en fait pas de ψ. Pour dériver un champ de vecteurs tangents, la situation est moins simple. Prenons par exemple un cercle dans $\mathbb{R}^3$. Si l'on veut appliquer la formule utilisée pour les applications pour dériver le vecteur tangent au cercle, on obtient un vecteur orthogonal au cercle. Ceci traduit que du point de vue du cercle, le vecteur ne bouge pas. La notion qui apparaît ici est la notion de dérivée covariante, qui correspond à la reprojection de la dérivée classique (dans l'espace ambiant $\mathbb{R}^n$) sur l'espace tangent T_pM. Donnons quelques définitions abstraites

Définition 0.1. *Soit M une sous-variété de $\mathbb{R}^n$. On appelle connexion sur M une application*

$$\nabla : \Xi M \times \Xi M \to \Xi M, \quad (X, Y) \to \nabla_X Y.$$

qui vérifie, pour toutes fonctions f et g sur M,

- $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
- $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$
- $\nabla_X(fY) = f\nabla_X Y + X(f)Y$, *où $X(f)$ désigne la dérivée de f dans la direction X.* *Cette connexion est dite symétrique si*

$$\forall X, Y \in \Xi M, \quad \nabla_X Y - \nabla_Y X = [X, Y], \tag{1}$$

où $[X, Y]$ désigne le crochet de Lie de (X, Y), c'est-à-dire le vecteur défini comme agissant sur une fonction f via

$$[X, Y]f = X(Y(f)) - Y(X(f)).$$

La connexion que l'on va utiliser dans la suite est définie dans le

Théorème 0.1. *Il existe une unique connexion symétrique compatible avec la métrique, c'est-à-dire telle que pour tous champs de vecteurs X, Y, Z, on a*

$$X(\langle Y\,,\, Z\rangle) = \langle \nabla_X Y\,,\, Z\rangle + \langle Y\,,\, \nabla_X Z\rangle\,. \tag{2}$$

Cette connexion est appelée connexion de Levi-Civita[2].

Il est à noter que pour tout tenseur Λ, on a

$$\begin{aligned}\nabla_X \Lambda(X^1, \cdots, X^k) &= \partial_X(\Lambda(X^1, \cdots, X^k)) \\ &\quad - \Lambda(\nabla_X X^1, X_2, \cdots, X^k) - \cdots - \Lambda(X^1, \cdots, X^{k-1}, \nabla_X X^k).\end{aligned} \tag{3}$$

Cette définition permet de définir une géodésique.

Définition 0.2. *Une courbe $\gamma : [0, 1] \to M$, de champ de vecteur tangent $\partial_t = \frac{d\gamma}{dt}$ est appelée géodésique si*

$$\nabla_{\partial t} \partial_t = 0.$$

On a maintenant tous les éléments pour définir le tenseur de courbure.

Définition 0.3. *Le tenseur de courbure R associe à tout couple de champs de vecteurs (X, Y), une application de $\Xi M \to \Xi M$ définie par*

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z.$$

2. Tullio Levi-Civita (1873 – 1941), mathématicien italien.

La courbure indique donc un défaut de commutativité des dérivées covariantes. Si M est un hyperplan de $\mathbb{R}^n$, ce tenseur est clairement nul.

On va maintenant définir la seconde forme fondamentale, qui est d'une autre nature que les objets définis jusque-là. En effet, elle n'est plus intrinsèque à M. Soient X et Y deux champs de vecteurs sur M. On définit alors l'application bilinéaire symétrique (ces deux propriétés viennent directement des axiomes définissant une connexion symétrique, et du fait que $[X, Y] = [X_M, Y_M]$ comme agissant sur les fonctions de M) $A : T_pM \times T_pM \to (T_pM)^\perp$ par

$$A(X, Y) = \partial_{\overline{X}}\overline{Y} - \nabla_X Y \tag{4}$$

où $\overline{\alpha}$ désigne une extension de α à un voisinage de M dans $\mathbb{R}^n$ et $\partial_{\overline{X}}$ désigne la dérivée selon $\overline{X}$ dans $\mathbb{R}^n$ (on aura pu aussi noter $\nabla^{\mathbb{R}^n}$). On peut noter que la définition de A ne dépend pas de l'extension $\overline{X}$ (ni, par symétrie, de celle de Y) choisie. En effet, si $\overline{X} = \sum_{i=1}^{n-1} X_i\partial_i + \overline{X}_n\partial_n$ avec ∂_i tangents à M pour $i \leqslant n-1$, on a

$$\nabla_{\overline{X}}\alpha = X_i\nabla_{\partial_i}\alpha.$$

Or, sur M, $\overline{X}_n = 0$, car $\overline{X} = X$, et ce pour toute les extensions de X possibles. De fait, seule la composante tangente de X compte ici.

2.4 Calculs en coordonnées

Dans toute la suite, la plupart de ces notions seront utilisées uniquement en coordonnées. On dispose donc d'une famille de champs de vecteurs coordonnées (dans le cas d'une paramétrisation F de M telle que $F(x_0) = p$), on peut par exemple choisir la famille des $\partial_i(p) = \partial_i F(x_0)$. Par hypothèse sur F, en tout point q autour de p, les $\partial_i(q)$ forment bien une base de l'hyperplan tangent T_qM. Complétée du vecteur ν normal à M, on dispose d'une base de $\mathbb{R}^n$.

Commençons par exprimer la connexion de Levi-Civita dans cette base. Dans la suite on notera ∇_i pour ∇_{∂_i}. Tous vecteurs X et Y de T_pM s'écrivent

$$X = X^i\partial_i \quad \text{et} \quad Y = Y^j\partial_j.$$

Alors

$$\nabla_X Y = \nabla_{X^i\partial_i}(Y^j\partial_j) = X^i\nabla_i(Y^j\partial_j) = X^i\partial_i(Y^j)\partial_j + X^iY^j\nabla_i(\partial_j) = X(Y^j)\partial_j + X^iY^j\Gamma_{ij}^k\partial_k.$$

Les Γ_{ij}^k sont appelés *symboles de Christoffel* et sont définis [3] par

$$\nabla_i\partial_j = \Gamma_{ij}^k\partial_k.$$

3. On peut en fait les calculer explicitement en écrivant, d'après la compatibilité avec la métrique (2)

$$\partial_i \langle\partial_k\,,\,\partial_j\rangle = \langle\partial_k\,,\,\nabla_i\partial_j\rangle + \langle\nabla_i\partial_k\,,\,\partial_j\rangle\,,$$

$$\partial_j \langle\partial_i\,,\,\partial_k\rangle = \langle\partial_i\,,\,\nabla_j\partial_k\rangle + \langle\nabla_j\partial_i\,,\,\partial_k\rangle$$

et

$$\partial_k \langle\partial_j\,,\,\partial_i\rangle = \langle\partial_j\,,\,\nabla_k\partial_i\rangle + \langle\nabla_k\partial_j\,,\,\partial_i\rangle\,.$$

Sommant les deux premières relations et privant la somme de la troisième, on obtient

$$\partial_i \langle\partial_k\,,\,\partial_j\rangle + \partial_j \langle\partial_i\,,\,\partial_k\rangle - \partial_k \langle\partial_j\,,\,\partial_i\rangle = \langle\partial_k\,,\,\nabla_i\partial_j\rangle + \langle\nabla_i\partial_k\,,\,\partial_j\rangle + \langle\partial_i\,,\,\nabla_j\partial_k\rangle + \langle\nabla_j\partial_i\,,\,\partial_k\rangle - \langle\partial_j\,,\,\nabla_k\partial_i\rangle - \langle\nabla_k\partial_j\,,\,\partial_i\rangle\,.$$

La seconde forme fondamentale peut également se décomposer en coordonnées. En effet, pour $X = X^i\partial_i$ et $Y = Y^j\partial_j$, on a

$$A(X,Y) = X^iY^jA(\partial_i,\partial_j) =: h_{ij}X^iY^j.$$

Opérateurs différentiels sur M. On peut définir, comme dans l'espace euclidien, des opérateurs différentiels usuels sur la sous-variété M (ces opérateur ne dépendent en fait que de la structure de variété sur M).

Définition 0.4. *Soit $f : M \to \mathbb{R}$. On appelle gradient de f en $p \in M$ le vecteur $\nabla f \in T_pM$ défini comme le représentant de $D_pf \in \mathcal{L}(T_pM,\mathbb{R})$ par le théorème de Riesz, c'est-à-dire qu'il vérifie : pour tout $t \in T_pM$, on a*

$$\langle t\,,\, \nabla f\rangle = D_pf \cdot t.$$

En coordonnées, il s'écrit donc

$$(\nabla f)^i = g^{ij}\partial_j f.$$

On définit également, pour un champ de vecteurs X sur M, la divergence de X comme trace de l'application $Y \mapsto \nabla_Y X$ dans T_pM. Ainsi définie, elle s'écrit

$$\mathrm{div}_M X = g^{ij}\partial_j X^i.$$

Enfin, on définit le laplacien comme $\Delta f = \mathrm{div}(\nabla f)$ qui s'exprime en coordonnées par

$$\Delta f = g^{ij}\partial_{ij} f.$$

Pour un tenseur T, c'est

$$\Delta T = g^{mn}\nabla_m\nabla_n T.$$

Coordonnées normales géodésiques. Ces coordonnées, extrêmement pratiques pour faire les calculs, peuvent être définis en tout point $p \in M$ de la façon suivante. Soit (e_i) une base orthonormée de T_pM. On peut montrer l'existence d'une unique géodésique $\gamma : [0,1] \to M$ passant par p et ayant une vitesse $v \in T_pM$ en p, et ce pour v suffisamment petite. On appelle alors $\exp_p(v)$ le point $\gamma(1)$. Ainsi définie l'exponentielle est lisse et est un difféomorphisme d'un voisinage de 0 dans T_pM sur un voisinage de p dans M. On définit alors des coordonnées en définissant l'application

$$(x_1,\cdots,x_{n-1}) \mapsto \exp_p\left(\sum x_ie_i\right),$$

D'après la symétrie de la connexion (1) et le fait que comme ∂_i sont des champs de vecteurs coordonnées, on a $[\partial_i,\partial_j] = 0$, ce qui impose $\nabla_i\partial_j = \nabla_j\partial_i$. On peut donc écrire

$$\begin{aligned}
&\partial_i \langle\partial_k\,,\,\partial_j\rangle + \partial_j\langle\partial_i\,,\,\partial_k\rangle - \partial_k\langle\partial_j\,,\,\partial_i\rangle\\
&\quad = \langle\partial_k\,,\,\nabla_i\partial_j\rangle + \langle\nabla_i\partial_k\,,\,\partial_j\rangle + \langle\partial_i\,,\,\nabla_j\partial_k\rangle + \langle\nabla_i\partial_j - [\partial_i,\partial_j]\,,\,\partial_k\rangle - \langle\partial_j\,,\,\nabla_i\partial_k - [\partial_i,\partial_k]\rangle - \langle\nabla_j\partial_k - [\partial_j,\partial_k]\,,\,\partial_i\rangle\\
&\quad = 2\,\langle\partial_k\,,\,\nabla_i\partial_j\rangle\,.
\end{aligned}$$

Ainsi, on a

$$\Gamma_{ij}^k = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right).$$

sur un voisinage de 0 dans $\mathbb{R}^{n-1}$. Comme la différentielle de $\exp_p$ en 0 est l'identité, on a nécessairement, pour ces coordonnées

$$g_{ij}(p) = \delta_{ij}.$$

De plus, les symboles de Christoffel s'annulent en p. En effet, les géodésiques coordonnées définies par les équations $x_i = a^i s$, où s désigne la longueur d'arc, vérifient (en tant que géodésiques) l'équation différentielle $\nabla_{\partial_s}\partial_s = 0$, c'est-à-dire, en coordonnées, on a $\partial_s = a_i \partial_i$, donc

$$\nabla_s \partial_s = a^i \nabla_i (a^j \partial_j) = a^i \underbrace{\partial_{ii}\gamma^i}_{=0} + \Gamma^k_{ij} a^i a^j = 0.$$

Ainsi, les Γ^k_{ij} sont bien tous nuls.

Remarque. Ces relations bien particulières sont valables uniquement au point p (et absolument pas dans un voisinage).

3 La courbure moyenne, un objet à plusieurs facettes

On présente dans cette partie l'objet courbure moyenne et on donne plusieurs points de vue sur cet objet. Tous les résultats de cette partie sont classiques et la partie toute entière peut être laissée de côté par les spécialistes de ces questions.

3.1 De la courbure linéique à la courbure moyenne

Dans *La courbure des surfaces*, publié en 1767, Euler fait la remarque suivante. « Pour connaître la courbure des lignes courbes, la détermination du rayon osculateur en fournit la plus juste mesure, en nous présentant pour chaque point de la courbure un cercle dont la courbure est précisément la même. Mais quand on demande la courbure d'une surface, la question est fort équivoque et point du tout susceptible d'une réponse absolue, comme dans le cas précédent. Il n'y a que les surfaces sphériques dont on puisse mesurer la courbure, attendu que la courbure d'une sphère est la même qe celle de ses grands cercles et que son rayon en peut être regardé comme la juste mesure. Mais, pour les autres surfaces, on n'en saurait même comparer la courbure avec celle d'une sphère, comme on peut toujours comparer la courbure d'une ligne courbe avec celle d'un cercle ; la raison en est évidente, puisque dans chaque point d'une surface, il peut y avoir une infinité de courbes différentes. »

Nourrie de cette équivoque, c'est Sophie Germain qui, semble-t-il, est la première à parler de courbure moyenne. Alors que les courbures principales au point P (courbures maximale et minimale d'une courbe sur S et passant par P, qu'on note ici $1/f$ et $1/g$) étaient connues d'Euler, celui-ci donnait une formule pour calculer le rayon de courbure en P de la section de S et d'un plan faisant un angle ϕ avec celui de plus grande courbure

$$r = \frac{2fg}{f + g - (f - g)\cos(2\phi)}.$$

De fait, deux plans perpenticulaires qui définissent, en coupant S, deux courbes de courbure r et r' en P vont nécessairement vérifier

$$\frac{1}{r} + \frac{1}{r'} = \frac{1}{f} + \frac{1}{g}.$$

Cette remarque légitime l'introduction d'une quantité égale à la moitié de cette constante : la courbure moyenne. Il me paraît intéressant de remarquer qu'ainsi définie, la courbure moyenne est une quantité purement géomérique.

3.2 Au sens de la géométrie différentielle

Puisqu'on travaille ici en codimension 1, l'application A définie par (4) est en fait, à isomorphisme près (qui correspond à un choix de normale) une forme bilinéaire symétrique. On appelle *courbure moyenne de* M *dans* $\mathbb{R}^n$ la quantité $\frac{1}{n}\operatorname{Tr} A$, la trace de A étant entendue dans n'importe quelle base *orthonormée* de T_pM [4]. Ses valeurs propres sont appelées courbures principales et son déterminant, courbure de Gauss.

La courbure moyenne, comme plus haut, peut se calculer dans la base des (∂_i). Attention néanmoins, comme cette base n'est pas orthonormée, elle se calcule grâce à l'inverse de la matrice [5] $[g_{ij}]$, qu'on a convenu de noter $[g^{ij}]$,

$$H = g^{ij} h_{ij}. \tag{5}$$

Remark. Soit $\mathbf{x}$ le point courant sur M. Alors $\Delta_M \mathbf{x} = -H(\mathbf{x})\nu$, courbure moyenne de M au point $\mathbf{x}$. En effet, rappelons

$$\mathbf{x} = F(x_1, \cdot, x_{n-1}),$$

on a

$$g^{ij}\nabla_j\nabla_i \mathbf{x} = g^{ij}\partial_j\partial_i F = -g^{ij}h_{ij}\nu = -H\nu.$$

Ligne de niveau d'une fonction. Comme vu plus haut, une hypersurface peut être vue localement comme la ligne de niveau $\{u = 0\}$ d'une fonction $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ dont le gradient est non nul, c'est-à-dire $M \cap \Omega = u^{-1}(0)$. La normale à cette surface est alors

$$\nu = \frac{\nabla u}{|\nabla u|}.$$

4. On retrouve ainsi la propriété, présentée plus haut, de conservation de la somme des inverses des rayons de courbure découverte par Sophie Germain.

5. En effet, rappelons que si α est une forme bilinéaire symétrique, elle est diagonalisable (on note D la matrice diagonale qui la représente) dans base orthonormée (e_i) de $\mathbb{R}^n$. Si son expression dans une autre base (f_j) de $\mathbb{R}^n$ est la matrice C, alors D et C sont liées par

$$D = PCP^T$$

où P est la matrice de passage de la base (f_j) à la base (e_i). Comme (e_i) est orthonormée, il est plus simple d'exprimer P^{-1}, matrice de passage de (e_i) à (f_j), qui a comme coefficients on a

$$P^{ij} = (f_j\,,\, e_i)\,.$$

De fait, on a

$$\operatorname{Tr}(B) = \operatorname{Tr}(D) = \operatorname{Tr}(P^T PC) = \operatorname{Tr}((P^{-1}(P^{-1})^T)^{-1}C),$$

ce qui, en observant

$$(P^{-1}(P^{-1})^T)_{ij} = P^{ik}P^{jk} = (e_k\,,\, f_i)\,(e_k\,,\, f_j) = (f_i\,,\, f_j)\,,$$

conduit à $\operatorname{Tr}(B) = \operatorname{Tr}(G^{-1}C)$ où $G_{ij} = (f_i\,,\, f_j)$, matrice de Gram de la base (f_j).
Dans le cas qui nous intéresse, la base (f_j) est celle des (∂_j) et la matrice de Gram correspondante est justement celle de la métrique, ce qui prouve la formule (5).

La courbure s'écrit alors

$$H = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right). \tag{6}$$

Cette définition dépend uniquement de $\{u = 0\}$ (on peut par exemple prendre pour u la distance à M).

3.3 La courbure moyenne comme première variation de l'aire

Au delà d'une quantité géométrique, la courbure moyenne peut-être vue comme quantité variationnelle. Plus précisément, si M est une hypersurface de $\mathbb{R}^n$ (qu'on va supposer compacte, pour simplifier) et si φ est une fonction sur M, on peut perturber M selon φ dans la direction normale en posant, si F est une paramétrisation de M,

$$F_t(x) = F(x) + t\varphi(F(x))\nu(F(x)).$$

Alors, l'aire de la surface M_t s'obtient via la métrique modifiée, elle-même obtenue comme ci-dessus à l'aide des nouveaux vecteurs tangents

$$g_{ij}^t = \langle \partial_i F_t \, , \, \partial_j F_t \rangle = \langle \partial_i F + t(\nabla\varphi \cdot \partial_i F \nu + \varphi \partial_i \nu) \, , \, \partial_j F + t(\nabla\varphi \cdot \partial_j F \nu + \varphi \partial_j \nu) \rangle = g^{ij} - 2t\varphi h_{ij} + o(t).$$

Ainsi, la forme volume devient

$$\sqrt{\det(g_{ij}^t)} = \sqrt{\det(g_{ij})} - t\varphi H + o(t).$$

Ce nouveau point de vue sur la courbure moyenne permet plus facilement de la calculer dans différentes situations.

Cas d'un graphe. Si M est donnée comme graphe d'une fonction $u : \Omega \to \mathbb{R}^{n-1} \to \mathbb{R}$ (supposons Ω borné), alors son aire est obtenue par

$$\mathcal{A}(M) = \int_\Omega \sqrt{1 + |\nabla u|^2}.$$

Une simple dérivation fournit alors

$$H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right).$$

La notion de surface minimale. Si l'on se donne une courbe de Jordan dans $\mathbb{R}^3$, un problème qui remonte à 1760 et qui porte le nom de Joseph Plateau consiste à trouver la surface d'aire minimale ayant pour bord ladite courbe. Ces surfaces minimisant l'aire ont fait et font encore l'objet de nombreux travaux en géométrie différentielle et calcul des variations. La notion de surface minimale est néanmoins ambiguë, car elle réfère alternativement à un minimiseur de l'aire ou à un point critique (c'est-à-dire une surface de courbure moyenne zéro). Bien que des travaux aient tenté d'étendre des propriétés des surfaces minimisantes à des surfaces de courbure moyenne zéro (on peut en particulier citer les travaux d'Ilmanen et collaborateurs, qui, dans leurs preuves, ont étudié les liens étroits entre ces deux types de surfaces, voir [Ilm96, ISZ98]), les deux notions sont loin de coïncider, comme le montre l'exemple suivant.

Le caténoïde. Considérons le caténoïde

$$X(u, v) = (\cosh u \cos v, \cosh u \sin v, u).$$

On peut vérifier aisément qu'il est à courbure moyenne nulle. Néanmoins, considérant deux cercles correspondant à des sections horizontales du caténoïde $\{u = c\}$, ledit caténoïde ne minimise pas forcément l'aire parmi les surfaces qui ont ces deux cercles comme bords. Les deux disques contenus dans les cercles peuvent avoir une aire plus faible (si c est assez grand, $c = 1$ par exemple).

3.4 L'aire d'une surface : une mesure comme une autre

Les ensembles à périmètre fini

Comment généraliser les sous variétés de sorte à pouvoir encore définir une notion de courbure moyenne ? Alors qu'il semble impossible d'étendre l'approche différentielle de courbure à des hypersurfaces qui n'ont plus rien de lisse, il est néanmoins possible d'étendre la notion d'aire, et par là-même, la notion de courbure moyenne comme première variation de l'aire. Inspiré par la récente introduction des fonctions à variations bornées par Tonelli et Cesari, Caccioppoli a proposé de considérer une surface comme le support d'une mesure. On dira par exemple qu'un ensemble $E \subset \mathbb{R}^n$ est de périmètre fini si la dérivée de sa fonction indicatrice au sens des distributions est une mesure. Son périmètre est alors la masse de cette mesure.

On introduira plus précisément ces objets et leur propriétés dans le chapitre 1, mais mentionnons les avantages de cette définition :

- Elle coïncide avec le périmètre classique pour des hypersurfaces lisses,
- Le périmètre satisfait une hypothèse de semi continuité (les suites minimisantes vont converger),
- Les fonctions à variations bornées forment un espace de Banach et s'approchent bien par des fonctions lisses.

Surfaces à courbures prescrites pour les ensembles de périmètre fini. Étant données les définitions précédentes, on peut définir une surface de courbure variationnelle prescrite g comme minimisant par perturbation compacte

$$\mathrm{Per}(E) + \int_E g.$$

En effet, si E est lisse et minimise cette quantité, alors la première variation donne directement $H(x) = -g(x)$.

Généralisation aux varifolds

Nous souhaitons dans cette partie présenter la généralisation des surfaces proposée par Almgren dans [Alm66]. Elle est dans l'esprit de la partie précédente mais demeure plus générale. On pourra aussi consulter [Sim83] pour une introduction complète à cette théorie.

Même si on n'utilisera pas cette théorie dans la suite, il nous semble difficile de ne pas en dire un mot car c'est l'approche historique pour définir un mouvement par courbure moyenne pour des surfaces non lisses (voir plus loin).

Définition 0.5. *On appelle varifold de dimenion $n-1$ toute mesure de Radon sur l'ensemble $G_{n-1} = \mathbb{R}^n \times G(n-1)$ où $G(n-1)$ est la grassmannienne des hyperplans de $\mathbb{R}^n$. L'espace de ces varifolds est noté $V_{n-1}(\mathbb{R}^n)$ et est muni de la topologie faible*

$$V_n \rightharpoonup V \Leftrightarrow \forall \psi \in \mathcal{C}^0_c(G_{n-1}), \quad V_n(\psi) \to V(\psi).$$

À l'image des ensembles, V est dite rectifiable s'il existe une collection dénombrable E_i de graphes $\mathcal{C}^1$ dont on note $T_i(x)$ les plans tangents, et une collection de réels positifs c_i tels que

$$V = \sum_{i=0}^{\infty} c_i v(E_i)$$

où pour tout $B \in G_{n-1}$, $v(E_i) \cdot B = H^{n-1}\left\{x \in E_i \,|\, (x, T_i(x)) \in B\right\}$. On dit que V est un varifold entier (ou à multiplicité entière) si les c_i sont entiers.

On peut définir l'aire d'un varifold V dans $A \subset \mathbb{R}^n$ comme variation totale de V sur $A \times G(n-1)$.

$$\|V\| \cdot A = V(\{(x, S) \in G_{n-1} \,|\, x \in A\}).$$

On peut remarquer que la définition de rectifiablilité est équivalente à l'existence d'un ensemble rectifiable M et d'une fonction positive localement intégrable $\theta : M \to \mathbb{R}$ tels que

$$\|V\| \cdot A = \int_{A \cap M} \theta d\mathcal{H}^{n-1}.$$

Dans toute la suite, on suppose que V est rectifiable.

On souhaite maintenant perturber V à l'aide d'une fonction lisse $h_t(x) : (-\varepsilon, \varepsilon) \times \mathbb{R}^n \to \mathbb{R}^n$, telle que

$$\{x \in \mathbb{R}^n, \; \exists t \in (-\varepsilon, \varepsilon), h_t(x) \neq x\}$$

est un compact de $\mathbb{R}^n$. On note

$$V_t \cdot A = (h_t)_{\#} V \cdot A := \int_{A \cap h_t(M)} \tilde{\theta} d\mathcal{H}^{n-1}$$

où $\tilde{\theta}(y) = \int_{f^{-1}(y) \cap M} \theta(x) d\mathcal{H}^0(x)$. Si J_t est le jacobien de h_t, on peut écrire

$$V_t \cdot A = \int_{M \cap A} J_t \theta d\mathcal{H}^{n-1}.$$

Il est donc raisonnable de définir la première variation de l'aire en dérivant selon t cette définition :

$$\left.\frac{\partial}{\partial t}\right|_{t=0} V_t \cdot \mathbb{R}^n = \int_M \partial_t J_t \theta d\mathcal{H}^{n-1}.$$

Il s'agit donc de différentier le jacobien

$$J_t := \sqrt{\det(Dh_t^T Dh_t)}.$$

On note $X = \partial_t h_t|_{t=0}$. On a $Dh_t = Dh_0 + DX + o(t)$, ce qui conduit à

$$\begin{aligned}\det(Dh_t^T Dh_t) &= \det(Dh_0^T Dh_0)(1 + 2t\,\mathrm{Tr}((Dh_0^T Dh_0)^{-1} DX^T Dh_0)) + o(t)\\ &= 1 + 2t\,\mathrm{Tr}(DX^T)) + o(t) = 1 + 2t\,\mathrm{div}_M X + o(t).\end{aligned}$$

En composant avec la racine, on obtient

$$J_t = J_0 + t\,\mathrm{div}_M X + o(t) = 1 + t\,\mathrm{div}_M X + o(t).$$

De fait,

$$\left.\frac{\partial}{\partial t}\right|_{t=0} V_t \cdot \mathbb{R}^n = \int_M \mathrm{div}_M X\theta d\mathcal{H}^{n-1}.$$

La courbure moyenne peut donc être définie comme un opérateur sur les champs de vecteurs à support compact sur M via $X \mapsto \int_M \mathrm{div}_M X$. Par dualité, ceci peut se récrire comme un opérateur sur les fonctions lisses $g : \mathbb{R}^n \to \mathbb{R}$ (noté δV dans [Bra78])

$$\delta V(g) = \int_M Dg(x)\theta d\mathcal{H}^{n-1}(x).$$

Étant donnée une fonction lisse $\phi : \mathbb{R}^n \to \mathbb{R}$, on peut même définir la variation selon ϕ par

$$\delta(V,\phi)(g) = \left.\frac{\partial}{\partial t} V_t \cdot \phi\right|_{t=0} = \int_M Dg\phi\theta d\mathcal{H}^{n-1} + \int_M gD\phi\theta d\mathcal{H}^{n-1}.$$

Pour définir le mouvement par courbure de Brakke [Bra78] qu'on présentera brièvement dans la partie suivante, il reste à définir un vecteur courbure moyenne (qui correspond à $-H\nu$ dans le cas régulier). Pour ce faire, on introduit la variation totale de δV définie pour tout ouvert G par

$$\|\delta V\| \cdot G = \sup\{g \in \mathcal{C}_0^1(G),\ |g| \leqslant 1\}.$$

Grâce à l'hypothèse de rectifiabilité, $\|\delta V\|$ est absolument continue par rapport à V et il existe deux fonctions η et ξ telles que

$$\delta V = \xi_V \|\delta V\| \quad \text{et} \quad \|\delta V\| = \eta_V \|V\|.$$

On peut donc définir

$$\mathbf{h}(V,x) = -\eta_V(x)\xi(x). \tag{7}$$

La formule de première variation se récrit, avec cette fois-ci g à valeurs dans $\mathbb{R}^n$

$$\delta V(g) = -\int_M g(x) \cdot \mathbf{h}(V,x)\theta(x) d\mathcal{H}^{n-1}(x),$$

ou, avec une fonction $\phi : \mathbb{R}^n \to \mathbb{R}^n$,

$$\delta(V,\phi)(g) = \int_M \mathbf{h}(V,x) \cdot g(x)\phi(x)\theta(x) d\mathcal{H}^{n-1}(x) + \int_M D_M\phi(x) \cdot g(x)\theta(x) d\mathcal{H}^{n-1}(x). \tag{8}$$

4 Un problème stationnaire variationnel

Toute cette partie introduit le chapitre 1.

4.1 Introduction au débruitage d'images

On s'intéresse ici aux problèmes de débruitage par minimisation. Étant donnée une image bruitée $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, on cherche à la débruiter en résolvant

$$\min_u \xi(u, g)$$

où ξ comporte typiquement un terme de régularisation (pénalisant les gradients) et un terme d'attache aux données g.

4.2 Un débruitage naïf par moyenne locale

Soit g une image bruitée par un bruit gaussien, c'est-à-dire que

$$g = g_0 + n$$

où g_0 est l'image idéale (celle qu'on veut observer) et n est un bruit dont les oscillations sont supposées bien plus rapides que celles de l'image. Ainsi, en 1960, Gabor propose de débruiter en convolant l'image avec un noyau gaussien. Cela revient à résoudre l'équation de la chaleur sur un temps court avec condition initiale g. Les résultats sont néanmoins mitigés. L'un des reproches qu'on peut faire à ce modèle est de diffuser dans toutes les directions : en particulier, dans les directions de fort gradient. C'est exactement dire que ce procédé lisse les bords.

Ce reproche majeur peut être contourné assez facilement, car il suffit de diffuser dans toutes les directions sauf dans celle du gradient pour éviter cet écueil. On remplace alors l'équation $u_t = \Delta u$ par, en notant $n = \frac{\nabla u}{|\nabla u|}$,

$$u_t = \Delta u - \partial_{nn} u = \Delta u - \big(n\,,\, D^2 u \cdot n\big) = \mathrm{Tr}\left[\left(I - \frac{\nabla u \otimes \nabla u}{|\nabla u|^2}\right) D^2 u\right] = |\nabla u| \;\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

Cette équation est exactement celle du mouvement par courbure moyenne qu'on présentera dans la dernière partie de cette introduction.

4.3 Le modèle de Rudin Osher et Fatemi

Que l'on consière le débruitage par moyenne locale comme équation d'évolution ou comme minimisation de

$$\int |\nabla u|^{\,2} + \int \frac{(u-g)^2}{2},$$

le principal problème est le non respect des bords, car cette fonctionnelle, dont le cadre naturel de minimisation est H^1, explose en présence des discontinuités de dimension $n-1$ que sont typiquement les bords des objets de l'image. Pour pallier ce problème, Rudin Osher et Fatemi ont proposé dans [ROF92, RO94] de minimiser cette fonctionnelle dans l'espace des fonctions à variations bornées. Ils proposent donc de minimiser

$$TV(u) + \int \frac{(u-g)^2}{2}. \tag{9}$$

Cette formulation peut prendre en compte des bords, et, à ce titre, produit des résultats bien meilleurs en pratique.

Alors que ce modèle a été introduit il y a plus de vingt ans [ROF92, RO94], seuls des travaux récents se sont efforcés à montrer qu'une telle minimisation ne peut pas faire apparaître d'irrégularités dans u qui ne seraient pas déjà présentes dans g. On sait en particulier ([CCN07, Jal12]) que les discontinuités de dimension $n-1$ (sauts, voir détails dans le premier chapitre de ce manuscrit) de u viennent nécessairement de celles de g. Il ne nous est pas connu de résultats sur les discontinuités générales. Nous avons donc souhaité poursuivre ce travail en s'intéressant à la transmission de la continuité de g à u par cette minimisation (un peu généralisée) : on souhaite montrer que cette minimisation ne peut pas créer de discontinuité qui ne serait pas présente dans g.

4.4 Quelques résultats liés à ces problèmes

On présente dans cette partie quelques résultats qui ne s'inscrivent pas tout à fait dans les hypothèses présentées plus haut (en particulier dans l'hypothèse de croissance à l'infini), mais qui reposent sur des techniques similaires à celles qu'on utilisera.

La croissance sur-linéaire. Dans [Cla05], Francis Clarke s'intéresse à la minimisation de

$$\int_\Omega F(\nabla u) \quad u \in W^{1,1}(\Omega) \quad \text{et } u = \phi \text{ sur } \partial\Omega. \tag{10}$$

Il suppose que F est strictement convexe et à croissance d'exposant $p > 1$ ($F(z) \geqslant \sigma|z|^p + \mu$). En outre, il demande à la donnée au bord ϕ de satisfaire, en tout point de la frontière γ, la *condition de pente* suivante : il suppose l'existence d'une fonction affine f de pente inférieure à K, vérifiant $f(\gamma) = \phi(\gamma)$ et, pour tout autre points γ' de $\partial\Omega$,

$$\phi(\gamma) + f(\gamma') \leqslant \phi(\gamma').$$

On a alors le

Théorème 0.2. *Le minimiseur de* (10) *est localement lipschitzien sur* Ω *et semi-continu inférieurement sur* $\overline{\Omega}$. *En outre,* u *est continue sur* $\overline{\Omega}$ *dans les trois cas suivant*

- $\partial\Omega$ *est un polyèdre,*
- $\partial\Omega$ *est* $\mathcal{C}^{1,1}$ *et* $p > \frac{N+1}{2}$,
- Ω *est uniformément convexe.*

Alors que le dernier point est essentiellement un travail de Miranda [Mir65] qu'on présentera (mais étendu à un minimiseur $W^{1,1}$), Clarke montre que l'hypothèse d'uniforme convexité pour Ω n'est pas nécessaire, si l'on demande une coercivité plus grande pour F. Il étend de plus le travail de Miranda à un polyèdre et affaiblit la condition de pente précédente en une condition de pente par dessous (seule l'existence de f^- est demandée).

On notera aussi un travail récent de Pierre Bousquet et Lorenzo Brasco [BB15] qui s'intéresse aux minimiseurs $W^{1,1}$ de

$$\int_\Omega F(\nabla u) + fu$$

avec condition de Dirichlet. Les auteurs prouvent une régularité Lipschitz des minimiseurs sous des hypothèses plus faibles que la stricte convexité pour F. Le schéma de preuve est le même que dans [Mir65] mais demande de nombreuses adaptations car le second membre empêche l'invariance par translation des minimiseurs, tout comme le caractère minimisant des fonctions affines. Il est donc nécessaire de trouver des fonctions barrières plus compliquées et adapter le schéma de Miranda.

Minimiseurs et champs de vecteurs. Dans un article récent [BS15], Beck et Schmidt considèrent les relations entre minimiseurs de

$$\int_\Omega f(x, \nabla u) dx \tag{11}$$

dans l'ensemble des fonctions à variations bornées prenant une donnée $g \in W^{1,1}(\mathbb{R}^n)$ au bord et les maximiseurs du problème dual

$$R_g[\tau] := \int_\Omega (\tau \cdot \nabla g - f^*(x, \tau))\, dx \tag{12}$$

parmi les champs de vecteurs L^∞ à divergence nulle, où f^* est la conjuguée de f en la variable z

$$f^*(x, z^*) = \sup_{z \in \mathbb{R}^n} (z^* \cdot z - f(x, z))\, .$$

Ils supposent que $f(x, z)$ est strictement convexe en z (hypothèse qui n'est pas satisfaite dans le problème qu'on étudie) et qu'elle vérifie une croissance linéaire en z (hypothèse satisfaite dans notre cadre). Alors, si (12) a une solution continue $\sigma(x)$ telle que $\sigma(x) \in \overline{\mathrm{Im}(\partial_z f(x, \cdot)}$ (intérieur de $\overline{\mathrm{Im}(\partial_z f(x, \cdot)})$, les minimiseurs de (11) sont Lipschitz presque partout.

Remark. Il est facile, comme expliqué dans [BS15], de se convaincre formellement de la relation entre les deux problèmes, en remarquant que

$$f(x, z) = f^{**}(x, z) = \sup_{z^*} z^* \cdot z - f^*(x, z^*)$$

ce qui implique formellement que

$$\inf_w \int f(x, \nabla w) = \sup_\tau \left[\inf_w \int (\tau \cdot \nabla w - f^*(x, \tau))\, dx \right].$$

Maintenant, comme τ est à divergence nulle, le membre de droite correspond à la maximisation de (12). En échangeant inf et sup, on a même l'égalité. Néanmoins, le résultat de [BS15] nécessite des hypothèses très faibles de régularité : la fonction f est par exemple seulement mesurable en x.

4.5 Les lignes de niveau minimisent aussi

En utilisant la formule de la coaire, on peut montrer que les minimiseurs de (9) ont des lignes de niveau $E_s = \{u > s\}$ qui minimisent le problème géométrique

$$\mathrm{Per}(E) + \int_E (s - g)$$

où le périmètre s'entend au sens des ensembles de Caccioppoli. Il s'agit de la formulation variationnelle de la propriété « E est à courbure prescrite $g-s$ ». Cette dualité fonctionnelle et géomérique sera utilisée sans cesse dans tout le chapitre 1.

Étudier la continuité d'un minimiseur de (9) revient donc à montrer que les ensembles de niveau ∂E_s et ∂E_t ne peuvent pas se toucher, ce qui est exactement montrer des principes de comparaison (dans l'esprit de [Sim87]) pour des surfaces à courbures prescrites.

Cette vision géométrique des minimiseurs de (9) a déjà été utilisée par plusieurs auteurs (voir [CCN11, Jal12],...), pour montrer des résultats sur les sauts du minimiseur u par rapport aux sauts de g (on décrira ces résultats dans le chapitre 1), tout comme des résultats de continuité uniforme de u en basse dimension.

4.6 Contributions de la thèse

Dans ce manuscrit (chapitre 1), nous avons cherché à étudier les propriétés de continuité des minimiseurs de

$$\int_\Omega F(\nabla u) \tag{13}$$

parmi les fonctions à variations bornées, pour différents cadres :

- Concernant l'attache au données, on a utilisé soit des conditions de Dirichlet au bord du domaine d'étude, soit un terme $\|u-g\|^2$ ajouté à la variation totale.
- La fonction $F(\nabla u)$ pourra être de trois types. Soit une variation totale classique $F(\nabla u) = |\nabla u|$, soit une anisotropie (norme quelconque dans $\mathbb{R}^n$), lisse ou cristalline : $F(\nabla u) = \varphi(\nabla u)$, soit encore une composée d'une fonction convexe f quelconque et d'une anisotropie : $F(\nabla u) = f(\varphi(\nabla u))$.
- Enfin, le domaine Ω pourra lui aussi varier. La première partie du chapitre sera limitée à un domaine borné (qui pourra être convexe ou convexe en moyenne), et la seconde traitera des domaines non bornés.

On commencera ce premier chapitre, avant toute introduction des outils sur les fonctions à variations bornées, par détailler un des premiers résultats de ce type, obtenu par Miranda en 1965 [Mir65], qui traite des domaines uniformément convexes avec conditions au bord de Dirichlet. Même si le résultat en temps que tel n'est pas si intéressant pour notre objectif (il est supposé F strictement convexe), il met en évidence un principe qui restera le nôtre dès qu'on étudiera (13) en domaine borné. Le schéma de preuve est le suivant.

1. On montre un principe de comparaison : si u et v sont deux minimiseurs de $\int F(\nabla u)$ et que $u \leqslant v$ sur $\partial\Omega$, alors cette propriété est transmise à Ω tout entier.
2. Ce principe de comparaison permet, grâce à l'invariance par translation de $F(\nabla u)$, de transmettre la régularité au bord à l'intérieur du domaine
3. La régularité au bord est obtenue grâce à l'uniforme convexité de celui-ci et au principe de comparaison, en remarquant que les fonctions affines sont des minimiseurs de $F(\nabla u)$.

Alors qu'on voit ici que la stricte convexité de F est uniquement utilisée pour obtenir le principe de comparaison, on peut donc utiliser ce schéma de preuve dans notre cadre, sous réserve de pouvoir montrer ce principe de comparaison.

Dans un travail récent [JMN13], Jerrard, Moradifam et Nachman montrent justement un principe de comparaison pour les minimiseurs (sous conditions de Dirichlet) de

$$\int_\Omega \varphi(\nabla u),$$

sous réserve que Ω vérifie une condition de bord plus faible que la convexité (une sorte de stricte convexité en moyenne par rapport à l'anisotropie φ). On peut donc déduire directement la continuité du minimiseur avec le schéma de preuve précédent. C'est l'objet de la section 7. Néanmoins, alors que la preuve de [JMN13] utilise des concepts assez profonds sur les courants minimaux (car ils travaillent avec une anisotropie qui varie en espace), nous avons exploité l'invariance par translation pour donner une preuve complète et plus simple du

Théorème 0.3. *Soit φ une anisotropie lisse et Ω un ouvert borné qui vérifie une condition de barrière relative à φ (voir Définition 1.7) et g une fonction continue sur $\partial\Omega$. Alors, le minimiseur de $\int_\Omega \varphi(\nabla u)$ sous condition $u = g$ sur $\partial\Omega$ est continu.*

Un contrôle du module de continuité. Motivé par un résultat de Caselles, Chambolle et Novaga [CCN11] sur la préservation du module de continuité de g par les minimiseurs de

$$\int_\Omega |\nabla u| + \frac{(u-g)^2}{2},$$

pour Ω convexe borné dans $\mathbb{R}^n$ avec $n \leqslant 8$ (pour assurer la régularité des lignes de niveau de u) avec condition de Neumann au bord, nous avons étudié les minimiseurs de

$$\int_\Omega f(|\nabla u|) + \frac{(u-g)^2}{2}, \tag{14}$$

toujours avec condition de Neumann au bord.

Après avoir approché F par un opérateur lisse et uniformément elliptique et constaté qu'encore une fois, la régularité des minimiseurs était contrôlée par leur régularité au voisinage de $\partial\Omega$ et que cette dernière pouvait être déterminée grâce à la convexité du bord, on a pu montrer le

Théorème 0.4. *Soit Ω un convexe borné de $\mathbb{R}^n$ (sans restriction sur n) et g uniformément continue de module ω. Alors il existe un unique minimiseur de* (14) *dans BV, avec conditions de Neumann au bord. Ce minimiseur est continu, de module ω.*

Un résultat en domaine non borné. Dans cette partie, on se restreindra au modèle de Rudin-Osher et Fatemi (isotrope), avec g continue, mais on ne suppose plus Ω borné. On y montre le

Théorème 0.5. *Soit u un minimiseur (par perturbations compactes) de*

$$\int_\Omega |\nabla u| + \frac{(u-g)^2}{2}$$

avec Ω un ouvert de $\mathbb{R}^n$ quelconque, et g une fonction continue. Alors u est continue.

Comme il n'y a plus nécessairement de bord, le schéma de preuve précédent est inopérant. On va donc montrer un principe de comparaison strict entre les lignes de niveau de u, qui sont des minimiseurs du problème de périmètre vu plus haut

$$E_s := \{u > s\} = \underset{E \Delta E_s \Subset \Omega}{\arg\min} \operatorname{Per}(E) + \int_E s - g. \tag{15}$$

Dans le cadre des surfaces minimales (minimiseurs du périmètre uniquement, sans second membre), ce résultat est bien connu. Il a d'abord été montré par [Mos77] dans un article très court (4 pages) en italien, puis indépendamment, dix ans plus tard, par [Sim87]. Alors que ces deux articles montrent ce principe de comparaison strict pour des minimiseurs, Illmanen a montré dans [Ilm96] qu'il est encore vrai pour des points critiques (c'est-à-dire des hypersurfaces de courbure moyenne zéro).

Pour montrer notre résultat de continuité, nous avons donc souhaité étendre le résultat de Simon aux minimiseurs de (15). Nous ne sommes parvenu qu'à l'étendre aux second membres constants en espace. Néanmoins, c'est suffisant. En effet, si E_s et E_t sont deux lignes de niveau de u (supposons par exemple que $t < s$), $s - g$ et $t - g$ sont à distance supérieure à ε sur une boule suffisamment petite. Sur cette même boule, il va donc exister une constante telle que $t - g \leqslant a \leqslant s - g$.

On montre alors qu'entre E_s et E_t, sur la petite boule construite plus haut, on peut intercaler deux surfaces minimisant

$$\operatorname{Per}(E) - \int_E a$$

avec différentes conditions au bord. Puisqu'elles ne peuvent pas être confondues, le résultat de Simon, étendu aux surfaces à courbure constante, montre qu'elles ne peuvent pas se toucher. Donc E_s et E_t ne le peuvent pas davantage.

5 Variations sur le mouvement par courbure moyenne

5.1 Introduction

L'objectif de toute cette section est de considérer, étant donnée une hypersurface $M_0 \subset \mathbb{R}^n$, une famille d'hypersurfaces M_t telle que pour tout $x \in M_t$, la vitesse de x est normale à M_t , et de module H, courbure moyenne de M_t en x (dans la suite, la normale ν est, dans le cas d'une surface fermée, dirigée vers l'extérieur et H est choisie de sorte qu'une surface enfermant un convexe soit à courbure positive). Plus précisément, on doit donc avoir

$$\frac{\partial x}{\partial t} = -H\nu(x).$$

Cette équation peut ensuite être formulée avec n'importe quelle description de la courbure présentée dans la partie 3.

Exemple. Une sphère $n-1$-dimensionnelle de rayon R se réduit de façon homothétique sous un mouvement par courbure moyenne. Plus précisément, à chaque instant $t \in [0, T)$ pour $T = \frac{R^2}{2}$, la surface obtenue est une sphère de rayon $R_t = R - \sqrt{2nt}$. En effet, la courbure moyenne d'une sphère est exactement l'inverse de son rayon. Supposant alors que M_t reste une sphère de rayon R_t, on doit avoir

$$\frac{\partial x}{\partial t} = -\frac{\partial R_t}{\partial t}\nu = -\frac{1}{R_t}\nu.$$

Ainsi, $R_t = \sqrt{R^2 - 2t}$, tant que la sphère existe, c'est-à-dire pour tout $t \in [0, T)$.

Notre objectif a été d'ajouter des obstacles contraignant la surface mobile à bouger à l'intérieur d'une zone délimitée par deux ouverts fixes Ω^+ et Ω^-, c'est-à-dire

$$\Omega^- \subset M_t \subset \Omega^+.$$

Plusieurs difficultés apparaissent lors de l'étude du mouvement par courbure, certains dus aux obstacles, d'autres pas. L'objectif principal est de pouvoir définir un temps d'existence du mouvement, temps qui ne dépend que des paramètres initiaux du problème. On vient d'en exhiber un dans le cas d'une sphère. Que peut-on dire en toute généralité ? Ensuite, la notion de courbure elle-même n'est pas définie pour une surface qui serait uniquement Lipschitz : comment, alors, peut-on tout de même définir un mouvement ?

Dans la suite de cette partie, on tâchera de présenter (dans un ordre historique) des réponses à ces questions. On présentera les différentes approches sans obstacles, on présentera brièvement leurs avantages et inconvénients et on verra, pour certaines d'entre-elles, comment on peut ajouter la contrainte $\Omega^- \subset M_t \subset \Omega^+$ à leur formulation.

5.2 1978 : Un problème de varifolds

S'appuyant sur la définition généralisée aux varifolds de la courbure moyenne, Brakke définit en 1978 dans [Bra78] le mouvement d'un varifold V_t par courbure moyenne :

$$\overline{D}\|V_t\|(\phi) \leqslant \delta(V_t, \phi)(\mathbf{h}(V_t, \cdot)).$$

Où les notations sont définies dans la section précédente (en particulier (7), (8)) et où $\overline{D}$ est la dérivée supérieure

$$\overline{D}f(t) = \limsup_{s \to t} \frac{f(s) - f(t)}{s - t}.$$

Brakke prouve un théorème d'existence d'un tel mouvement. Pour ce faire, il utilise une approximation de la courbure moyenne h_ε et utilise une discrétisation en temps sous la forme

$$x_{n+1} = x_n + \Delta t h_\varepsilon^n(x_n).$$

Néanmoins, à chaque étape, des corrections sont nécessaires pour prendre en compte les petites irrégularités qui sont ignorées par l'approximation h_ε.

Il ne nous est connu aucune généralisation de ces travaux.

5.3 L'approche géométrique

On s'intéresse ici à l'évolution d'hypersurfaces de $\mathbb{R}^n$ par courbure moyenne, c'est-à-dire qu'on souhaite construire une famille de paramétrisations

$$F_t : M \to \mathbb{R}^n$$

où M est une sous variété de $\mathbb{R}^n$ fixée (on pourrait évidemment travailler avec des ouverts de $\mathbb{R}^{n-1}$, mais on n'aurait alors uniquement une paramétrisation locale, ce qui est un peu moins pratique...), de sorte que

$$\partial_t F_t(x) = -H(F_t(x))\nu(F_t(x)). \tag{16}$$

Remarquons alors que cette équation peut être écrite

$$\partial_t F_t = \Delta_t F_t$$

où Δ_t est le laplacien sur M_t. Cette écriture suggère que cette équation puisse être parabolique. Elle l'est en effet, ce qui permet d'affirmer qu'il existe un temps d'existence et une solution $\mathcal{C}^\infty$ à cette équation.

C'est l'approche de Hamilton dans son étude du flot de Ricci [Ham82] qui a motivé cette approche classique du flot par courbure. En effet, tous les outils utilisés dans [Hui84] (mais aussi dans le chapitre 2 de ce manuscrit) sont présents dans cette étude de Hamilton.

Le théorème principal de [Hui84] s'énonce comme suit

Théorème 0.6. *Soit M_0 une hypersurface lisse et uniformément convexe. Alors, il existe une solution lisse à* (16) *sur un intervalle de temps $[0, T[$. De plus, M_t converge vers un point quand $t \to T$. De plus, on peut normaliser ce flot en volume pour obtenir une existence en tout temps qui converge vers une sphère.*

Le fait qu'une solution existe en temps court n'est pas ici un problème, puisque l'équation (16) est parabolique. Le point intéressant est l'analyse de la situation au temps maximal d'existence. Huisken y prouve que, comme le fait le tenseur de courbure dans [Ham82], la seconde forme fondamentale explose nécessairement. Pour ce faire, il va contrôler toutes les quantités géométriques par l'action du flot en utilisant le principe du maximum, qui stipule que si une fonction f vérifie, sur une famille de surfaces compactes M_t, l'inéquation différentielle

$$(\partial_t - \Delta_M)f^2 \leqslant 0,$$

alors la quantité $\sup_{M_t} f^2$ est décroissante au cours du temps.

Cinq ans plus tard, Ecker et Huisken étudient dans [EH89] l'évolution des graphes posés dans tout $\mathbb{R}^{n-1}$ par courbure moyenne, c'est-à-dire de l'équation

$$\frac{du}{dt} = \sqrt{1 + |\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right). \tag{17}$$

Ils prouvent le théorème suivant

Théorème 0.7. *Soit u_0 une fonction lipschitzienne sur $\mathbb{R}^{n-1}$. Alors il existe une solution à* (17) *pour tout t.*

Encore une fois, la démonstration de ce théorème repose sur des estimations sur les principales quantités géométriques (cette fois-ci, la seconde forme fondamentale, à cause de la contrainte graphe, ne peut pas exploser). À noter que la surface n'étant pas compacte, il est nécessaire d'améliorer le très basique principe du maximum de la section précédente. Ecker et Huisken utilisent donc la formule de monotonie qui porte le nom du premier auteur, et dont la preuve se trouve dans [Hui90].

En plus du théorème cité plus haut, Ecker et Huisken s'intéressent au comportement asymptotique du mouvement, montrant la convergence, après normalisation qui empêche la surface de s'enfuire vers l'infini, vers une surface minimale.

Enfin, en 1991, dans [EH91a], les deux mêmes auteurs parviennent à démontrer un résultat d'existence du mouvement sans hypothèse restrictive sur la surface initiale.

Théorème 0.8. *Soit M_0 une surface localement lipschitzienne. Alors on peut faire évoluer M_0 selon* (16) *pour un temps court qui ne dépend que de M_0.*

La clef de ce théorème est d'être parvenu à estimer la seconde forme fondamentale pour une évolution générale, et de montrer qu'elle ne pouvait pas exploser sur un temps contrôlé uniquement à l'aide de la surface initiale.

5.4 Le point de vue ligne de niveau

L'idée de cette partie est de considérer que la surface qu'on souhaite faire évoluer par courbure moyenne est en fait la ligne de niveau zéro d'une fonction u. Grâce à (6), la surface évoluant M_t est la ligne de niveau zéro de la fonction $u(\cdot,t)$ si et seulement si

$$u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right). \tag{18}$$

C'est cette équation qu'on étudie en oubliant (au moins temporairement) l'aspect géométrique.

Comme cette équation n'est pas sous forme divergence, les définitions classiques de solutions faibles ne pouvaient pas être appliquées. Aussi, Evans et Spruck, dans [ES91], et Chen, Giga et Goto, dans [CGG91] ont proposé d'utiliser la toute récente théorie des solutions de viscosité (introduites par Crandall, Evans, Ishii, Jensen, Lions,... dans [CL83, CEL84, Jen88], voir aussi [CIL92] pour le second ordre). Une fonction u est solution de viscosité de (18) si toutes fonction lisse φ qui touche u par dessus (resp. par dessous) vérifie

$$\varphi_t \leqslant |\nabla \varphi| \operatorname{div}\left(\frac{\nabla \varphi}{|\nabla \varphi|}\right), \tag{19}$$

resp.

$$\varphi_t \geqslant |\nabla \varphi| \operatorname{div}\left(\frac{\nabla \varphi}{|\nabla \varphi|}\right), \tag{20}$$

avec $\varphi_t \leqslant 0$ ou $\varphi_t \geqslant 0$ respectivement lorsque $\nabla\varphi = 0$. Une fonction qui ne vérifiera que la première condition sera appelée sous-solution, tandis qu'une fonction vérifiant uniquement la deuxième prendra le nom de sur-solution. Ces deux articles montrent un théorème d'existence et d'unicité pour (18).

Théorème 0.9. *Soit u_0 continue sur $\Omega \cap \mathbb{R}^n$. Alors, il existe une unique solution continue de viscosité u à* (18).

Alors que les conclusions du théorème sont les mêmes pour les deux articles, les hypothèses diffèrent légèrement car les preuves ne sont pas tout à fait les mêmes. Les deux articles reposent sur le schéma classique de preuves pour les solutions de viscosité : on montre d'abord un principe de comparaison, c'est-à-dire que si u est une sous-solution et v une sur-solution et que $u \leqslant v$ au temps initial, cette inégalité reste vraie pour tout temps. Dans les deux papiers, l'idée de preuve est la même, il s'agit de raisonner par l'absurde en s'intéressant au point de maximum d'une fonction de type

$$\Phi(x, y, t, s) = u(x, t) - v(y, t) - \alpha|x - y|^2 - \varepsilon(|x|^2 + |y|^2)$$

et de montrer que la partie droite de cette fonction fournit naturellement une fonction test en un point de maximum de Φ. Comme u et v sont sous/sur-solutions de viscosité, ces fonctions test vérifient (19)/(20), ce qui conduit à une contradiction.

Pour la partie existence, les deux articles diffèrent un peu. Alors que [CGG91] utilise la classique méthode de Perron, qui dit essentiellement que puisqu'on dispose du principe de comparaison, la solution, si elle existe, ne peut qu'être égale à

$$u_{sol}(x, t) = \sup\{u(x, t) \,|\, u \text{ sous-solution } \},$$

l'article de Evans et Spruck choisit d'approcher l'équation (18) par une famille d'EDP (on retrouve (18) si $\varepsilon = 0$)

$$u_t^\varepsilon = \left(\delta_{ij} - \frac{u_i^\varepsilon u_j^\varepsilon}{|Du^\varepsilon| + \varepsilon^2}\right) u_{ij}. \tag{21}$$

Cette nouvelle équation est uniformément elliptique et la théorie parabolique classique s'applique, fournissant une unique solution lisse u^ε sur tout $\mathbb{R}^+$ qui prend la condition initiale. Evans et Spruck montrent ensuite que la famille u^ε converge vers une solution u de (18).

En plus de l'intérêt technique d'une telle solution approchée, [ES91] fournit aussi une interprétation géométrique de cette approximation qu'il me semble intéressant de reproduire ici, car elle donne naissance à plusieurs travaux sur le flot par courbure.
Soit u^ε une solution lisse de (21). On se plonge alors dans $\mathbb{R}^{n+1}$ et on pose $y = (x, x_{n+1})$ et

$$v^\varepsilon(y, t) = u^\varepsilon(x, t) - \varepsilon x_{n+1}.$$

La fonction v^ε satisfait alors l'équation

$$v_t^\varepsilon = |\nabla v^\varepsilon| \operatorname{div}\left(\frac{\nabla v^\varepsilon}{|\nabla v^\varepsilon|}\right),$$

c'est-à-dire que les lignes de niveau $\{v^\varepsilon = s\}$ (on s'intéresse uniquement à $\Gamma_t^\varepsilon = \{v^\varepsilon(\cdot, t) = 0\}$) évoluent par courbure moyenne. Mais Γ_t^ε est un graphe (il s'écrit $x_{n+1} = \frac{1}{\varepsilon} u^\varepsilon(x, t)$), et on a vu dans la partie précédente que d'après [EH89], son évolution est un graphe et qu'elle existe pour tout temps.

Alors que dans [ES91], cette justification se limite à une heuristique, des travaux récents (voir en particulier [SS14]) se basent sur cette approche pour construire un flot par courbure qui se prolonge au delà des singularités. De plus, cette approche peut aussi être utilisée pour étudier le flot contraint par des obstacles, comme nous le ferons dans ce manuscrit (voir [RS14]).

Cette approche définit un flot géométrique. Alors que le problème géométrique se borne à faire évoluer une hypersurface par courbure, l'approche ligne de niveau fournit en fait une évolution de la famille complète des lignes de niveau $\{u = t\}$. Pour que cette approche soit intéressante du point de vue géométrique, il est nécessaire de montrer que l'évolution de la ligne de niveau $\{u = 0\}$ ne dépend que de la condition initiale $\{u(\cdot, 0) = 0\}$. Autrement dit, n'importe quelle fonction ψ qui vérifie $\{\psi(\cdot, 0) = 0\} = \{u(\cdot, 0) = 0\}$ fournit la même évolution $\{\psi = 0\} = \{u = 0\}$.

Comportement en temps long. S'il est impossible de parler de comportement en temps long dans le cadre défini plus haut, il nous semble important de signaler qu'en ajoutant des conditions de Dirichlet dans un domaine borné, la notion de solution stationnaire de (18) a pleinement un sens et on peut s'intéresser au comportement en temps long des solutions de (18) satisfaisant $u = g$ sur $\partial\Omega$, avec $g \in \mathcal{C}^2(\overline{\Omega})$. Ilmanen, Sternberg et Ziemer ont montré dans [ISZ98] qu'il y avait effectivement convergence de la solution de viscosité de (18) vers une solution stationnaire. De plus, cette solution stationnaire coïncide jusqu'à la dimension $n - 8$ avec une hypersurface minimale stable.

Discussion sur l'unicité. Le théorème cité plus haut fournit un résultat d'unicité à l'équation (18). Il est intéressant de s'interroger sur les liens entre cette unicité et l'unicité d'un mouvement géométrique par courbure. Notons par exemple que puisque la distance à n'importe quel ensemble fermé est une fonction 1-lipschitzienne, et que sa ligne de niveau zéro est exactement l'ensemble en question, le théorème précédent permet de faire évoluer tout ensemble fermé selon (18), y compris par exemple un ensemble de Cantor. Il est donc clair qu'une solution de (18) n'a pas nécessairement de signification géométrique. Le phénomène qui se produit typiquement lorsque la solution de (18) perd son sens géométrique est le développement d'un intérieur non vide pour les lignes de niveau $\{u = \alpha\}$. On pourra par exemple consulter [BNP98] pour des exemples de tels intérieurs.

5.5 Un schéma discret en temps

Dans une autre approche pour définir un flot par courbure moyenne lorsque la surface initiale n'est pas lisse, Almgren, Taylor et Wang ont proposé dans [ATW93] un schéma discret en temps, en résolvant à chaque étape

$$E_{n+1} = \arg\min_E \operatorname{Per}(E) + \int_{E\Delta E_n} d_{E_n}. \tag{22}$$

Cette formulation peut-être justifiée de la façon suivante. Supposons avoir une famille M_t de surfaces compactes lisses évoluant de façon régulière. Soit $p \in M_{t+dt}$ La distance entre p et M_t se calcule, si dt est suffisamment petit de sorte que p reste dans le voisinage de M_t où d_{M_t} est lisse,

$$d(p, M_t) = |q - p|$$

avec $q \in M_t$ vérifiant $\nu_{M_t}(q) = \frac{p-q}{|p-q|}$. Si $v(q)$ est la vitesse normale de M_t en q, on a alors

$$d(p, M_t) = v(q)dt + o(dt) = v(p)dt + o(dt).$$

De fait, en paramétrant la différence symétrique $M_t \Delta M_{t+dt}$ par $(q,s) = q + s\nu(q)$, on peut écrire

$$\int_{M_t \Delta M_{t+dt}} \frac{d(p, M_t)}{dt} = \int_{M_t} v(q) + o(1).$$

Par ailleurs, on a montré que la courbure moyenne de M_t au point q s'exprime comme dérivée de l'aire :

$$\int_{M_t} H(q) = \left.\frac{d}{ds}\right|_{s=0} \mathrm{Per}(M_t + s\nu).$$

Puisqu'on souhaite avoir $v(q) = -H(q)$, il est raisonnable de minimiser (pour rendre égales les dérivées d'ordre 1)

$$\mathrm{Per}(E) + \int_E d(M_t, \cdot).$$

En étudiant la convergence de ce schéma, Almgren, Taylor et Wang montrent le

Théorème 0.10. *Soit M_0 une surface de classe $\mathcal{C}^{3+\alpha}$, avec $\alpha < 1$. Alors, il existe une famille M_t de surfaces $\mathcal{C}^{2+\alpha}$, solutions fortes du mouvement par courbure moyenne géométrique. En outre, cette famille est unique.*

Remark. Ce théorème est en fait plus général que la version citée ci-dessus, puisqu'il autorise une géométrie anisotrope (à anisotropie lisse).

Un premier travail avec obstacles. Cette formulation du mouvement permet très simplement d'ajouter des obstacles (supposons par exemple qu'on impose $O \subset E_n$, il suffit de résoudre le même schéma en forçant, à chaque étape, $O \subset E_n$. C'est ce qu'ont proposé, motivés par de récentes applications en biologie (voir [ABH$^+$08] et [ABH$^+$10]), Almeida, Chambolle et Novaga dans [ACN12].

Ils montrent le

Théorème 0.11. *Le schéma* (22) *converge vers une notion faible de mouvement par courbure moyenne avec obstacle. En outre, en dimension 2, la solution du mouvement avec obstacles est en fait $\mathcal{C}^{1,1}$.*

C'est motivé par ce résutlat que nous avons tâché, dans une partie de ce manuscrit, de voir comment on pouvait, grâce aux approches présentées plus haut, traiter le plus généralement et le plus précisément possible le mouvement par courbure avec obstacle.

5.6 Contributions de la thèse.

Un résultat sur les lignes de niveau avec obstacles

Le premier travail en ce sens est présenté dans le chapitre 2. Il consiste à adapter le cadre de la partie 5.4 au problème avec obstacle. Cela consiste à ajouter à l'équation (18) la contrainte $u^- \leqslant u \leqslant u^+$.

Le premier enjeu de ce travail a été de définir correctement la notion de solution de viscosité avec obstacles. Nous avons choisi la formulation usuelle (voir par exemple [Yam87]) qui consiste à remplacer le hamiltonien $u_t - F(Du, D^2u)$ par

$$\max\left\{ \min\left\{u_t - F(Du, D^2u),\ u^-\right\},\ u^+\right\}.$$

Néanmoins, utiliser ce hamiltonien induit la résolution d'une équation qui n'est pas sous la forme $u_t - G(x, u, Du, D^2u)$, donc les résultats classiques (voir [CIL92]) ne s'appliquent pas directement. Aussi, nous avons choisi de définir plus explicitement ces solutions : par exemple, une sous-solution

Définition 0.6. *Une fonction $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ est appelée sous-solution de viscosité avec obstacles pour la condition initiale g si*
- *u est semi-continue supérieurement,*
- *pour tout $x, t \in \mathbb{R}^n \times [0, T]$, $u^-(x, t) \leqslant u(x, t) \leqslant u^+(x, t)$,*
- *pour tout $x \in \mathbb{R}^n$, $u(x, 0) \leqslant g(x)$,*
- *si φ est une fonction $\mathcal{C}^2$ de x, t, si $(\hat{x}, \hat{t}) \in \mathbb{R}^n \times (0, T]$ est un maximum de $u - \varphi$ et si $u(\hat{x}, \hat{t}) > u^-(\hat{x}, \hat{t})$, alors*

$$\varphi_t + \underline{F}(D\varphi, D^2\varphi) \leqslant 0.$$

C'est avec cette définition que nous travaillerons dans tout le chapitre. On prouve tout d'abord le

Théorème 0.12. *Soit $g : \mathbb{R}^n \times \mathbb{R}$ continue bornée. Alors, il existe une unique solution de viscosité avec condition initiale g à l'équation* (18) *avec contrainte $u^- \leqslant u \leqslant u^+$. De plus, cette solution conserve le module de continuité commun à $u^\pm$ et g.*

Remark. On montre aussi qu'on peut ajouter un terme forçant Lipschitz au mouvement, c'est-à-dire considérer l'équation

$$u_t + F(Du, D^2u) + k(x)|Du| = 0.$$

On a alors le même résultat, sauf que le module de continuité de la solution dégénère exponentiellement.

Pour démontrer ce théorème, nous avons adapté les preuves existant sans obstacles. En utilisant à la fois les articles [ES91] ou [GGIS91] et le guide [CIL92], nous avons essayé de donner une preuve la plus concise possible de ce théorème. La démonstration s'appuie, comme dans les papiers cités plus haut, sur un principe de comparaison prouvé grâce au lemme d'Ishii parabolique et sur la méthode de Perron pour l'existence.

Comme plus haut, nous montrons que la solution u de cette équation définit bien un mouvement géométrique, c'est-à-dire que $\{u(\cdot, t) = 0\}$ dépend uniquement de $\{g = 0\}$ et des obstacles $\{u^\pm = 0\}$. Néanmoins, la présence d'obstacle peut amener une solution qui serait lisse dans le cas sans obstacle, à développer un intérieur non vide. On donne l'exemple de trois points dans $\mathbb{R}^2$.

Finalement, on compare l'évolution visqueuse à celle du mouvement avec obstacle de [ACN12] et [Spa11]. On montre, en utilisant [Tho12], le

Théorème 0.13. *Soit u_h la fonction obtenue en faisant évoluer séparément les lignes de niveau $\{u_h > t\}$ par le schéma* (22) *avec obstacles $\{u^- > t\}$ et $\{u^+ > t\}$. Alors, la famille u_h est équi-continue en temps et en espace, et elle converge vers la solution de viscosité du théorème 0.12.*

Au delà de la comparaison entre les deux approches du mouvement par courbure moyenne avec obstacles, ce résultat est intéressant car il donne à l'équation visqueuse une dimension variationnelle. En particulier, il permet de montrer que si une surface vérifie une hypothèse de *mean-convex hull*, c'est-à-dire minimise le périmètre par variations exétieures, cette propriété est conservée au cours de l'évolution et l'évolution est monotone. Cette propriété permet un passage à la limite dans (18) avec obstacle qui fournit une solution u_∞ de

$$|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0$$

avec obstacles $u^- \leqslant u \leqslant u^+$. On peut, grâce à [ISZ98], énoncer la

Proposition 0.1. *Si $\{u_\infty = s\}$ est d'intérieur vide, alors, en dehors des zones de contact avec $u^\pm$, elle coïncide avec une surface minimale lisse excepté sur un ensemble de dimension au plus $n-8$.*

Un temps d'existence géométrique (avec M. Novaga)

Dans le chapitre 3 de ce manuscrit, nous avons tâché d'adapter le mouvement géométrique introduit par Ecker et Huisken à un cadre avec obstacles. L'idée principale de ce travail a été d'approcher les obstacles par un terme forçant régulier (cela revient si l'on veut à ramollir les obstacles), d'étudier le mouvement ainsi obtenu, puis de passer à la limite dans l'approximation. Cette idée d'approximation a été utilisée en particulier dans [CN13a]. On considère donc d'abord une solution de

$$\frac{\partial F}{\partial t} = -(H + k_\varepsilon)\nu \tag{23}$$

avec k_ε une fonction lisse convergeant vers $M(1 - 1_{\Omega^-} - 1_{\Omega^+})$ avec M une constante fixée choisie suffisamment grande.

La théorie parabolique fournit une solution lisse de (23) jusqu'à un temps T_ε. Pour passer à la limite, il faut au moins montrer que les temps T_ε sont uniformément bornés inférieurement par un temps T strictement positif. On peut alors montrer le

Théorème 0.14. *Soit U un ouvert de $\mathbb{R}^n$ (représentant les obstacles) et $M_0 \subset U$ une surface. On suppose que M_0 et ∂U sont uniformément de classe $C^{1,1}$, ainsi que* $\operatorname{dist}(M_0, \partial U) > 0$. *Alors, il existe $T > 0$ et une solution unique M_t de*

$$v = H \qquad \textit{sur } M_t \cap U, \tag{24}$$

pour tout $t \in [0, T)$, telle que M_t reste de classe $C^{1,1}$.

La preuve de se résultat repose sur le contrôle des quantités géométriques, comme dans [EH91a]. Néanmoins, elle se fait en deux temps, et deux arguments distincts sont utilisés.

On montre d'abord qu'il existe un voisinage de la surface initiale M_0 et un temps T_1, qui dépendent tout deux uniquement de la surface initiale, tels que la surface solution de (23) ne peut se promener que dans ce voisinage pendant le temps T_1. Pour prouver cela, on n'utilise absolument pas la régularité de la surface initiale, mais un principe de comparaison géométrique, qu'on peut par exemple déduire du principe de comparaison pour les solutions de viscosité du mouvement level-sets : si $N_1 \subset N_2$ sont deux surfaces initiales, alors l'inclusion demeure le long du flot. En outre, puisqu'on dispose d'une borne sur la norme $\mathcal{C}^{1,1}$ de M_0, on peut placer des boules de rayon contrôlé de chaque côté de M_0, et ceci en tout point de la surface initiale. Comme on connaît parfaitement l'évolution des boules, on peut borner l'évolution de M_0. Ceci permet aussi de donner une borne Hölder uniforme en ε sur l'évolution temporelle des surfaces.

Maintenant qu'on a uniformément réduit le terrain de jeu des évolutions approchées, on peut borner leur norme Lipschitz uniformément en ε. On s'intéresse pour ça, étant donnée une direction ω arbitraire, à la quantité $(\omega\,,\,\nu)$ qu'on souhaite borner inférieurement. On y parvient (en fait, pour une version localisée) grâce au principe du maximum.

À ce stade, les solutions approchées sont localement des graphes dans des directions qui ne dépendent pas de ε. Reste alors à montrer qu'elles existent toutes sur un intervalle de temps contrôlé. Pour ce faire, il faut analyser ce qui se passe au temps final d'existence. Grâce au voisinage mis en évidence plus haut, les surfaces approchées ne peuvent disparaître, et puisqu'elles sont localement des graphes de norme Lipschitz conrôlée, on ne peut pas perdre l'aspect plongé. Pour un mouvement sans obstacle, on a vu plus haut qu'alors, la seconde forme fondamentale doit nécessairement exploser. On montre que c'est aussi ce qui se passe avec obstacles, et, comme dans [EH91a], on montre que ça ne peut pas se produire.

On peut ensuite passer simplement à la limite par le théorème d'Ascoli et obtenir une solution de viscosité (la définition de telles solution est évidemment locale mais la surface existe globalement).

Un cas particulier : les graphes « entiers ». Comme dans le cas sans obstacles, on peut montrer que pour les graphes définis sur $\mathbb{R}^{n-1}$ tout entier, le mouvement existe pour tout temps, et que l'évolution est lipchitzienne en temps et en espace (alors qu'on avait uniquement Hölder en temps dans le cadre d'une évolution générale). Ces bornes se montrent grâce à la formule de monotonie d'Ecker et Huisken, adaptée à la présence d'un terme forçant.

Un comportement en temps long dans le cas périodique. On a aussi pu montrer, mais dans un cadre périodique uniquement (car l'argument principal est la décroissance de l'aire de la surface), que pour $t \to \infty$, le mouvement pour les graphes converge le long d'une suite vers un graphe minimal, c'est-à-dire une solution de

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

5.7 Vers un mouvement anisotrope et cristallin (avec M. Novaga et P. Pozzi)

Alors que le mouvement par courbure anisotrope a été étudié dans [And01] pour une anisotropie lisse, on s'est intéressé à des anisotropies cristallines. Le caractère non lisse introduit une difficulté importante, puisqu'on ne dispose plus d'une équation parabolique standard, donc l'existence en temps court n'est plus assurée.

Dans toute cette partie, on travaillera uniquement avec $n = 2$. Soit γ une norme sur $\mathbb{R}^2$. On sait que les boules euclidiennes, pour le mouvement par courbure moyenne isotrope, se réduisent de façon homothétique par le flot. On cherche donc à définir une variante anisotrope du flot par courbure où les Wulff shapes (c'est-à-dire les ensembles $\{\gamma \leqslant R\}$) se réduisent de manière auto-similaire. Gage et Li ont montré dans [GL94] qu'il existe effectivement de telles évolutions dans le cas où φ est lisse. L'équation s'écrit, si $u_0 : [0,1] \to \mathbb{R}^2$

$$\frac{u_t}{\gamma(\nu)} = (\gamma''(\nu)\tau \cdot \tau)\kappa\nu.$$

Nous nous sommes penchés sur le cas où ni la courbe initiale u_0 ni la norme φ n'est lisse, et avons prouvé le

Théorème 0.15. *Soit φ une norme de $\mathbb{R}^2$ et $u_0 : S^1 \to \mathbb{R}^2$ une courbe lipschitzienne qui vérifie une condition de φ-courbure bornée. Alors, il existe $T > 0$ qui ne dépend que de φ et de u_0,un champ de vecteur n sur $u(S^1)$ tel que pour presque tout $t \in [0,T)$ et $s \in [0, L(t)]$,*

$$\tilde{u}_t^{\perp} = \varphi(\nu)\,\mathrm{div}\, n\; \nu$$

où ν est la normale euclidienne à $u([0, L(t)])$ (définie presque partout) et n est un champ de vecteurs lipschitzien (Cahn Hoffman) qui vérifie $\varphi(n) = 1$ et dont la divergence représente, dans le cas régulier, la φ-courbure.

L'idée de la preuve est simple, puisqu'elle a consisté à approcher la norme γ par des normes lisses et strictement elliptiques γ_ε, tout comme u_0 par u_0^ε, faire évoluer la courbe approchée avec les normes approchées (on a alors affaire à une équation parabolique standard) et passer à la limite. On contrôle le temps d'existence comme précédemment, en montrant que la φ-courbure doit nécessairement exploser lorsque le mouvement cesse d'exister, ce qui ne se produit pas pour un temps contrôlé.

Chapter 1

On regularity of minimizers for TV

1 Introduction

In this chapter, we study the regularity of minimizers of generalized total variations. More precisely, let Ω be a subset of $\mathbb{R}^n$ and g be a function which is defined on some subset $\tilde{\Omega}$ of $\overline{\Omega}$. We want to analyze the regularity of minimizers of

$$\min_{u \in BV} \int_\Omega F(\nabla u) \tag{1.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is a convex function with linear growth ($\frac{1}{\mu}|x| \leqslant F(x) \leqslant \mu|x|$) and with two possible links between u and g:

1. Either a Dirichlet condition $u = g$ on $\partial\Omega$,
2. or a L^2-distance between u and g

$$d(u,g) = \int_\Omega \frac{(u-g)^2}{2},$$

which is the distance introduced by Rudin, Osher and Fatemi in [ROF92] in their well known denoising model.

In what follows, we will be interested in three types of F:

1. $F(\nabla u) = |\nabla u|$, that is the usual total variation,
2. $F(\nabla u) = \varphi(\nabla u)$, where φ is a norm in $\mathbb{R}^n$, which can be non Euclidian,
3. $F(\nabla u) = f(\varphi(\nabla u))$, where $f : \mathbb{R} \to \mathbb{R}$ is a convex function and φ is a norm in $\mathbb{R}^n$.

The current framework will be recalled in every section.

All along this chapter, our goal is to relate the regularity of u with the regularity of g. More precisely, we want to show that the minimizing procedure preserves continuity. One can even show (see Section 4) that under strong assumptions on the domain, we can control the modulus of continuity of u by the modulus of g.

Let us present now the structure of this chapter.

— In a first section, we recall a work by Miranda [Mir65] which is a prototype of what happens when dealing with functionals like (1.1). We assume that F is strictly convex and introduce the problem in the simple class of Lipschitz functions and in a bounded domain and we show how to solve it. It gives the opportunity to introduce in a very easy way the typical behavior of minimizers of such problems and the typical regularity proof, which shows that the key point is to be able to control the minimizers on the boundary.

— In Section 3, we introduce BV-functions and sets with finite perimeter. In particular, we give the link between minimizers of (1.1) and geometric minimizers of

$$\mathrm{Per}_\varphi(E;\Omega) - \int_{E\cap\Omega} g, \tag{1.2}$$

which is the variational formulation of "E has a prescribed curvature g." We also give some density properties of these geometrical minimizers. We also recall the known regularity results on u which deal with its jump set (hypersurfaces of discontinuity).

— In Section 4, we apply Miranda's scheme of Section 2 directly to study minimizers of

$$\int_\Omega f(|\nabla u|) + \frac{(u-g)^2}{2}$$

with Neumann boundary conditions in a convex domain, and we show that the control on the boundary can be obtained using these boundary conditions. We can obtain a bound on the modulus of continuity of u using the modulus of g, extending a result by Caselles, Chambolle and Novaga [CCN11] to higher dimension.

In the sections which follow, we use level-sets $E_s = \{u > s\}$ and their minimizing property to get regularity results for u. Indeed, showing that u is continuous is equivalent to show that $\partial E_s \cap \partial E_t = \emptyset$ as soon as $s \neq t$.

— In Section 5, we recall the usual Hopf maximum principle for smooth geometric minimizers of (1.2). As it is well known that such minimizers have no reason to be smooth (think of Simons' cone [Sim68]), we have to extend this kind of result to a non smooth framework.

— We first show in Section 6 that one can easily extend this result assuming that only one of the two minimizers is smooth.

— In Section 7, we investigate the problem

$$\min \int_\Omega \varphi(\nabla u)$$

in bounded domains with continuous Dirichlet boundary conditions. We could use the scheme of Miranda, but since the functional is no longer strictly convex, we have to find another way to get a comparison principle for minimizers ($u \leqslant v$ on $\partial\Omega$ implies $u \leqslant v$ in the whole Ω): Jerrard proposed a geometric proof of this principle in [JMN13], with an strict φ-mean convexity assumption on the domain Ω. Since [JMN13] deals with a space dependent φ, he can obtain continuity of the minimizer only in dimension $\leqslant 3$. Taking advantage of the translation invariance, we prove continuity for u in all dimensions, using simpler arguments than in [JMN13]. Nonetheless proof is totally geometric (it deals with level-sets) and remains in the spirit of [JMN13].

— Finally, in Section 8, we come back to the usual Rudin-Osher-Fatemi model (no anisotropy). We show that some results can exist in an unbounded domain but that the situation is more difficult, because we cannot use the boundary as a step towards continuity. As a result, we show that a srong maximum principle for minimal surfaces [Sim87] can be extended to variational constant mean curvature hypersurfaces, and see that it is enough to claim that two different level-sets of a minimizer cannot touch. That is exactly proving that the minimizer u is continuous.

2 A primary work by Miranda

In this section, we would like to focus on one of the first papers on minimizing

$$\int_\Omega F(\nabla u), \quad u = g \text{ on } \partial\Omega. \tag{1.3}$$

It has been published in Italian by Miranda [Mir65]. The assumptions are the following.

— The function F is $\mathcal{C}^2$ and strictly convex,

— the domain Ω is open and bounded,

— we assume that the boundary data g satisfies the so called K-bounded slope condition (BSC): for every $\hat{y} \in \partial\Omega$, there exist two affine functions $f^\pm$, vanishing at $\hat{y}$, such that for every other $y' \in \partial\Omega$, we have

$$f^-(y') + g(\hat{y}) \leqslant g(y') \leqslant f^+(y') + g(\hat{y}). \tag{1.4}$$

The main statement of [Mir65] is

Theorem 1.1. *There exists a unique minimizer of* (1.24) *in the class of Lipschitz functions.*

There is no work on BV (or even in $W^{1,1}$) functions in this paper: every function is at least continuous. Nonetheless, the techniques used to prove this theorem are very fundamental in this whole section. Let us give a few words about the proof.

First, since F is strictly convex, there is at most one (Lipschitz) minimizer to (1.24). And we have the

Proposition 1.1. *Let u and v two minimizers of* (1.24) *with boundary data g and h. Then, if $g \leqslant h$, $u \leqslant v$.*

To show the existence, we minimize (1.3) in the classe of p-Lipschitz functions, providing some function u_p. To make u_p converge, we need to show that they actually all share a Lipschitz constant. This is a regularity result which will be fundamental in what follows.

Thanks to the (BSC) and Proposition 1.1, we control the behavior of a minimizer on the boundary. Indeed, since $f^\pm$ are affine, they are natural minimizers of $\int F(\nabla u)$. Proposition above applied to u and $f^\pm$ shows that the property

$$\forall \gamma \in \partial\Omega, \quad f^-(\gamma) \leqslant g(\gamma) \leqslant f^+(\gamma)$$

can be extended to the whole $\overline{\Omega}$:

$$\forall y \in \overline{\Omega}, \quad f^-(y) + g(\hat{y}) \leqslant u(y) \leqslant f^+(y) + g(\hat{y}).$$

Since the slope of $f^{\pm}$ is bounded by K, we deduce, since $\hat{y}$ can be any point in $\partial\Omega$

$$\forall (y,\hat{y}) \in \overline{\Omega} \times \partial\Omega, \quad |u(y) - u(\hat{y})| \leqslant K|y - \hat{y}|. \tag{1.5}$$

The most important result is the following proposition, which shows that to control the reguarity of a minimizer, it is enough to control it on the boundary.

Proposition 1.2. *Let u be a minimizer of* (1.24) *which satisfies* (1.5). *Then, it is K-Lipschitz.*

Proof. We use that the translational invariance of the integral. If y' and y are two points in Ω, we define $v(x) = u(x + (y' - y))$. Thanks to the comparison principle, $\max u - v$ is reached on the boundary of $\overline{\Omega \cap (\Omega + y' - y)}$, in some $\hat{x}$. As a result,

$$u(x + (y' - y)) - u(x) \;\leqslant u(\hat{x} + y' - y) - u(\hat{x}).$$

But either $\hat{x}$ or $\hat{x} + y' - y$ belongs to $\partial\Omega$. As a result, (1.5) implies

$$u(\hat{x} + y' - y) - u(\hat{x}) \leqslant K|y - y'|.$$

Hence we obtain

$$u(x + (y' - y)) - u(x) \;\leqslant K|y - y'|,$$

which proves that u is K-Lipschitz. □

Remark. Even if this proposition is stated in the framework of Lipschitz functions, it totally applies when u and v are only BV (with boundary data considered as a trace). We will use this translation strategy several times in what follows.

Finally, let us make a remark on the bounded slope condition:

Remark. Let us assume that Ω is uniformly convex and g is $\mathcal{C}^2$. Then, g satisfies the BSC. Pierre Bousquet proved in [Bou07] that if g is only continuous, Theorem 1.1 still holds (in the class of continuous functions instead of Lipschitz ones). The idea is to approximate g by $\mathcal{C}^2$ functions g_i and control the Lipschitz norms of the approximate minimizers. In addition, Bousquet deals with functions in $W^{1,1}$. See also [BB15] for a generalization where affine functions are no longer minimizers.

3 An introduction on BV functions

3.1 Functions of bounded variation

In this section, we give some definitions and classical results (most of them without proof) on BV functions. To learn more details, one can read [Giu84], [AFP00] and, with a more image processing oriented point of view, [CCC+10]. In what follows, Ω will be an open subset of $\mathbb{R}^n$.

Definition 1.1. *Let $u : \Omega \to \mathbb{R}$ such that $u \in L^1(\Omega)$. We say that u has bounded variation and note $u \in BV(\Omega)$ if its distributionnal derivative Du is a Radon measure. Then, we call $TV(u, \Omega)$ the norm of this derivative, as a Radon measure:*

$$TV(u;\Omega) = \sup_{\phi \in \mathcal{C}^1_c(\Omega),\ \|\phi\|_{\mathcal{C}^1} \leqslant 1} (Du\,,\,\phi)\,.$$

In an equivalent way, one can ask that

$$\sup \left\{ \int_\Omega u \operatorname{div} \phi \,\middle|\, \phi \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^n),\ \|\phi\|_{L^\infty} \leqslant 1. \right\}$$

is finite, and prove that this quantity is exactly $TV(u;\Omega)$.

Remark. Since constant functions have a total variation zero, TV is only a seminorm. We usually introduce the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1} + TV(u).$$

Endowed with this norm, the space BV is a Banach space.

Using the last definition of TV, one can easily state a crucial property of total variation.

Proposition 1.3 (Lower semicontinuity). *Let* $u_n \rightharpoonup u$ *in* L^1. *Then,*

$$TV(u) \leqslant \liminf_{n\to\infty} TV(u_n).$$

In addition, the functions in BV can be approximated by smooth functions, thanks to the

Theorem 1.2. *Let* $u \in BV(\Omega)$. *Then, there exists a sequence* $u_n \in \mathcal{C}^\infty \cap W^{1,1}(\Omega)$ *such that*

$$u_n \to u \in L^1$$

and

$$TV(u_n) \to TV(u).$$

It has to be noticed that this density of $W^{1,1}$ in BV enables to extend the Sobolev inequalities:

Proposition 1.4 (Sobolev inequalities). *Let* Ω *be Lipschitz and bounded and* $u \in BV(\Omega)$. *Then, there exists* C_1 *(depending only on* Ω *and the dimension) such that*

$$\left\| u - \fint u \right\|_{L^{\frac{n}{n-1}}} \leqslant C_1 \cdot TV(u). \tag{1.6}$$

If u *has a compact support in* Ω, *then there exists* C_2 *(depending only on* Ω *and the dimension) such that*

$$\|u\|_{L^{\frac{n}{n-1}}} \leqslant C_2 \cdot TV(u). \tag{1.7}$$

3.2 Sets with finite perimeter

Definition 1.2. *Let E be a mesurable set in $\mathbb{R}^n$. We say that it has finite perimeter in Ω if its characteristic function 1_E has bounded variation in Ω. We note*

$$\mathrm{Per}(E,\Omega) := TV_\Omega(1_E).$$

If $\Omega = \mathbb{R}^n$, we note $\mathrm{Per}(E)$. *Such a set is also called a Caccioppoli set.*

Let us state a useful proposition for sets with finite perimeter (the proof uses Theorem 1.2 and Proposition 1.3).

Proposition 1.5. *Let A and B two Caccioppoli sets. Then,*

$$\mathrm{Per}(A\cap B;\Omega) + \mathrm{Per}(A\cup B;\Omega) \leqslant \mathrm{Per}(A;\Omega) + \mathrm{Per}(B;\Omega).$$

Remark. Whereas the $n-1$ dimensional Hausdorff of ∂E measure highly depends on $n-1$ changes on E, $\mathrm{Per}(E)$ does not. For instance, if $E = [0,1]^2 \cup [1,2] \times \{1\} \subset \mathbb{R}^2$, then E differs from the unit square by a set of Lebesgue measure zero, but

$$\mathcal{H}^{n-1}(\partial E) = 5$$

whereas $\mathrm{Per}(E) = TV(1_E) = TV(1_{[0,1]^2}) = 4$.

This remark yields the following question: since $\mathrm{Per}(E)$ does not change by adding or deleting a $n-1$ dimensional subet to E, is there a canonical representative of E? What can we say about ∂E? This leads to the following definition

Definition 1.3. *For every Caccioppoli set $E \subset \Omega$, we note $E^{(1)}$ the set of points with density* 1 *and $E^{(0)}$ the set of points with density zero. More precisely,*

$$E^{(1)} := \left\{ x \in \Omega \,\middle|\, \lim_{r\to 0} \frac{|B_r(x)\cap E|}{|B_r|} = 1 \right\},$$

$$E^{(0)} := \left\{ x \in \Omega \,\middle|\, \lim_{r\to 0} \frac{|B_r(x)\cap E|}{|B_r|} = 0 \right\}.$$

These sets are invariant to negligible modifications of E.

Proposition 1.6. *The difference between E and $E^{(1)}$ has a Lebesgue measure zero, as well as the difference between E^c and $E^{(0)}$.*

Proof. This result comes from the usual Lebesgue differentiation theorem: with $f = 1_E$, it reads

$$\forall x \in \mathbb{R}^n \text{ a.e.}, \lim_{\rho\to 0} \quad \frac{1}{|B_\rho(x)|}\int_{B_\rho(x)} 1_E = 1_E(x).$$

□

Proposition 1.7. *Let $n \geqslant 2$ and E be a Caccioppoli set in $\mathbb{R}^n$. Then, either E or $\mathbb{R}^n \setminus E$ has a finite Lebesgue measure and there exists a dimensional constant C such that*

$$\min\{|E|,\ |\mathbb{R}^n \setminus E|\} \leqslant C \operatorname{Per}(E)^{\frac{n}{n-1}}.$$

Proof. We just use (1.6) on the ball $B_\rho(x)$ for 1_E and $1_{\mathbb{R}^n\setminus E}$ to get

$$\int_{B_\rho(x)} \left| 1_E - ⨍_{B_\rho(x)} 1_E \right|^{\frac{n}{n-1}} \leqslant C \left(\int_{B_\rho(x)} |D1_E| \right)^{\frac{n}{n-1}},$$

which can be rewritten as

$$|E\cap B_\rho(x)| \left(\frac{|(\mathbb{R}^n \setminus E) \cap B_\rho(x)|}{|B_\rho(x)|} \right)^{\frac{n}{n-1}} + |(\mathbb{R}^n\setminus E)\cap B_\rho(x)| \left(\frac{|B_\rho(x) \cap E|}{|B_\rho(x)|} \right)^{\frac{n}{n-1}} \leqslant C \left(\int_{B_\rho(x)} |D1_E| \right)^{\frac{n}{n-1}}$$

which implies

$$\min\{|E \cap B_\rho(x)|,\ |(\mathbb{R}^n \setminus E) \cap B_\rho(x)|\}\cdot \underbrace{\left(\left(\frac{|(\mathbb{R}^n \setminus E) \cap B_\rho(x)|}{|B_\rho(x)|} \right)^{\frac{n}{n-1}} + \left(\frac{|E \cap B_\rho(x)|}{|B_\rho(x)|} \right)^{\frac{n}{n-1}} \right)}_{\geqslant 1} \leqslant C \left(\int_{B_\rho(x)} |D1_E| \right)^{\frac{n}{n-1}}$$

Hence,

$$\min\{|E \cap B_\rho(x)|,\ |(\mathbb{R}^n \setminus E) \cap B_\rho(x)|\}^{\frac{n-1}{n}} \leqslant C \operatorname{Per}(E, B_\rho(x)). \tag{1.8}$$

Letting $\rho \to \infty$, we get the expected inequality. □

Definition 1.4 (Reduced boundary)**.** *A point $x \in \Omega$ belongs to the reduced boundary of E (we note $x \in \partial^* E$) if*

i) For every $\rho > 0$, $\int_{B_\rho(x)} |D1_E| > 0$.

ii) The quantity

$$\nu_\rho(x) = \frac{\int_{B_\rho(x)} D1_E}{\int_{B_\rho(x)} |D1_E|}$$

has a limit $\nu(x)$ with $|\nu(x)| = 1$.

The following proposition shows, in some sense, that the reduced boundary is the smooth part of the boundary, and that all the perimeter is somehow contained in the reduced boundary.

Proposition 1.8. *Let E has finite perimeter. Then,*

$$\operatorname{Per}(E; \Omega) = \int_{\partial^* E\cap\Omega} d\mathcal{H}^{n-1}(x). \tag{1.9}$$

We conclude this section giving a crucial theorem, which is called coarea formula and enables linking the minimizing properties of u and the minimizing properties of its level sets $E_t = \{u > t\}$.

Theorem 1.3. *[Giu84, Th. 1.23]*

$$\int_\Omega |Du| = \int_{-\infty}^{+\infty} \mathrm{Per}(E_t; \Omega) dt. \tag{1.10}$$

3.3 Rudin-Osher-Fatemi denoising procedure

In 1992, Rudin, Osher and Fatemi proposed in [ROF92] a denoising procedure based on total variation. More precisely, if $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a noisy picture, they suggest to regularize it solving

$$u = \underset{v \in BV(\Omega)}{\arg\min} \int_\Omega |Du| + \frac{1}{\lambda} \int_\Omega \frac{(u-g)^2}{2}. \tag{1.11}$$

In what follows, we are interested in anisotropic generalizations of this problem. More precisely, let φ be a smooth, symmetric ($\varphi(-x) = \varphi(x)$) anisotropy (a norm in $\mathbb{R}^n$) such that φ^2 is uniformly convex, we deal with

$$u = \underset{v \in BV(\Omega)}{\arg\min} \int_\Omega \varphi(Du) + \frac{1}{\lambda} \int_\Omega \frac{(u-g)^2}{2}. \tag{1.12}$$

In this equation, the term

$$\int_\Omega \varphi(Du)$$

has to be understood as

$$\int_\Omega \varphi\left(\frac{Du}{|Du|}\right) d(Du)(x) = \sup\left\{ \int_\Omega u \cdot \operatorname{div} \xi \;\middle|\; \varphi^\circ(\xi) \leqslant 1 \right\}$$

where Du is the derivative of the BV-function u (it is therefore a Radon measure), and $\frac{Du}{|Du|}$ is the function provided by Radon-Nikodym theorem when writing that Du is absolutely continuous relatively to $|Du|$. Since the functional $u \mapsto \int_\Omega \varphi(Du) + \frac{1}{\lambda} \int_\Omega \frac{(u-g)^2}{2}$ is strictly convex and semi continuous (thanks to the semi continuity of the total variation), it has a unique minimizer in $BV(\Omega)$.

In all the following, we are searching for the links which may exist between the regularity of g and the regularity of u. Let us recall the results on the jump set of u.

3.4 On the level-sets of minimizers

In this subsection, we give a few results which link the minimizing property of u in (1.12) and the minimizing property of each level-set of u

$$E_t := \{u > t\}. \tag{1.13}$$

To this aim, let us introduce some anisotropic variants of the quantities presented in Section 3.

Definition 1.5. *Anisotropic perimeter Let E has finite perimeter. We can define an anisotropic φ-perimeter by*

$$\mathrm{Per}_\varphi(E,\Omega) := \int_{\partial^* E\cap\Omega} \varphi(\nu_E) d\mathcal{H}^{n-1}. \tag{1.14}$$

Note that if $\varphi = Id$, then, thanks to (1.9), we obtain the usual perimeter. It is easy to show that Per_φ satisfies the same properties as the isotropic perimeter (with the same proofs which basically use the semi continuity of the total variation with respect to the L^1 convergence). For instance,

$$\mathrm{Per}_\varphi(E\cap F) + \mathrm{Per}_\varphi(E\cup F) \leqslant \mathrm{Per}_\varphi(E) + \mathrm{Per}_\varphi(F) \tag{1.15}$$

and the key-tool in what follows, the so called anisotropic coarea formula

Theorem 1.4. *Anisotropic coarea formula Let $u \in BV(\Omega)$. Then,*

$$\int_\Omega \varphi(\nabla u) = \int_{-\infty}^{+\infty} \mathrm{Per}_\varphi(E_t)dt.$$

See for instance [JMN13] of [Jal12] for details on these anisotropic quantities.

Finally, we state the precise consequence of this coarea formula.

Proposition 1.9. *Let $u \in BV(\Omega)$. Then, u minimizes* (1.12) *with $\lambda = 1$ if and only if for every $t \in \mathbb{R}$, the level sets E_t of u minimize*

$$E_t = \arg\min_F \mathrm{Per}_\varphi(F) + \int_\Omega t - g. \tag{1.16}$$

3.5 Jump-set

Let us state here the first regularity results on u which come from regularity of g. They deal with jump set.

Let $u \in BV(\Omega)$. Then, its differential Du is a Radon measure: it can be splitted in two parts

$$Du = Du^a + Du^c$$

where Du^a is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$ and writes

$$Du^a = \nabla u^a \mathcal{L}^n$$

and Du^c is singular. The singular part can also be separated in two quantities $Du^s = Du^c + Du^j$ where Du^c is the Cantor part and Du^j the jump part. One can prove (see [AFP00, Chapter 3.]) that $Du^j = D^s u|_{J_u}$ where J_u is defined by

Definition 1.6. *Let $u \in BV(\Omega)$ and $x_0 \in \Omega$. We say that x_0 is a jump point of u if there exists two real numbers $u^-(x_0) \neq u^+(x_0)$, an orientation $\nu(x)$ such that*

$$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_{B_\rho^+(x_0)} |u^+(x_0) - u(x)|dx = \lim_{\rho\to 0} \frac{1}{\rho^n} \int_{B_\rho^-(x_0)} |u^-(x_0) - u(x)| = 0$$

where

$$B_\rho^\pm(x_0) = \{x \in B_\rho(x_0) \mid \pm (x - x_0\,,\, \nu(x_0)) \geqslant 0\}.$$

We call jump-set *the set of all jump points. We denote it by J_u.*

Roughly speaking, the jump set corresponds to hypersurfaces of discontinuity of u. It is therefore consistent to be interested first in this set, when studying regularity of u. Let us give the results which links the jump set of u and the jump set of g. The first one has been obtained in [CCN07].

Theorem 1.5 (Caselles, Chambolle, Novaga, '07)**.** *Let $g \in BV(\Omega) \cap L^\infty(\Omega)$ where $\Omega \subset \mathbb{R}^n$, and let u minimize the isotropic* (1.11)*. Then*

$$J_u \subset J_g$$

up to a $\mathcal{H}^{n-1}$-negligible set.

In 2012, in his PhD thesis, Khaled Jalalzai extended this result to more general first and second terms in (1.11). To be more understandable, we only state a restriction of his theorem

Theorem 1.6 (Jalalzai, '12)**.** *Let $g \in BV(\Omega) \cap L^\infty(\Omega)$ and u minimize* (1.12)*. Then,*

$$J_u \subset J_g,$$

up to a $\mathcal{H}^{n-1}$-negligible set.

It has to be noticed that this result allows the two terms to have a space dependency.

Finally, we mention a recent paper by Valkonen [Val15], which extends this results to much more general regularizations.

3.6 Density estimates on the minimizers of the geometric problem

In this subsection, we give useful results on the minimizers of the anisotropic perimeter. The main density estimate comes from [GMT93], with slight changes due to the anisotropic framework.

A word on the anisotropies. In this subsection, we will use an anisotropy φ. It just consists in a norm in $\mathbb{R}^n$. We assume that it is smooth and that φ^2 is strongly convex ($D^2\varphi^2 \geqslant \lambda I$ with $\lambda > 0$). As a result, there exist two constants A and B such that

$$\forall |x| = 1, \quad A|x| \leqslant \varphi(x) \leqslant B|x|.$$

Proposition 1.10. *Let E minimize* (1.16) *in B_1 and assume that $0 \in \partial E$. Then, there exists r_0 and a constant $q > 0$ which both depend on the dimension, $\|g - t\|_\infty$ and A and B such that for every $r \leqslant r_0$,*

$$1 - q \geqslant \frac{|B_r \cap E|}{|B_r|} \geqslant q \tag{1.17}$$

Proof. Let $\Lambda = \|g - t\|_{L^\infty(B_1)}$. Applying Sobolev inequality to $f = 1_{E \cap B_r}$ gives

$$\|1_{E\cap B_r}\|_{L^{n/n-1}} \leqslant C \int_B |D1_{E\cap B_r}| \leqslant \frac{C}{A} \operatorname{Per}_\varphi(E \cap B_r; B_1).$$

Note that using trace theorem,

$$\mathrm{Per}_\varphi(E\cap B_r;B_1)=\mathrm{Per}_\varphi(E;B_r)+\int_{\partial B_r}1_E\varphi(\nu)\leqslant \mathrm{Per}_\varphi(E;B_r)+B\int_{\partial B_r}1_E.$$

Since E is minimizing in B_1 and comparing with $E\setminus B_r$, we have

$$\mathrm{Per}_\varphi(E;B_r)+\int_{E\cap B_r}t-g\leqslant \mathrm{Per}_\varphi(B_r;E)\leqslant B\int_{\partial B_r}1_E. \tag{1.18}$$

Putting the two last equations together, we obtain

$$\mathrm{Per}_\varphi(E\cap B_r;B_1)\leqslant 2B\int_{\partial B_r}1_E+|E\cap B_r|\Lambda.$$

For almost every r, we have

$$\int_{\partial B_r}1_E=\frac{d}{dr}|E\cap B_r|$$

which implies

$$\|1_{E\cap B_r}\|_{L^{n/n-1}}=|E\cap B_r|^{\frac{n-1}{n}}\leqslant \frac{2BC}{A}\frac{d}{dr}|E\cap B_r|+\frac{2C\Lambda|E\cap B_r|}{A}\quad a.e.$$

Then, we would like to have

$$\frac{C\Lambda|E\cap B_r|}{A}\leqslant\frac12|E\cap B_r|^{\frac{n-1}{n}}.$$

It is enough to force $|B_r|^{1/n}\leqslant\frac{A}{4C\Lambda}$, that is

$$r\leqslant r_0:=\frac{A}{4\omega_n^{1/n}C\Lambda}.$$

Finally, we have

$$|E\cap B_r|^{\frac{n-1}{n}}\leqslant\frac{4BC}{A}\frac{d}{dr}|E\cap B_r|.$$

Denoting by $\eta(r)$ the quantity $|E\cap B_r|$, we have

$$\left(\eta^{1/n}\right)'\geqslant\frac{A}{4BC}.$$

Integrating between $\frac{r}{2}$ and $r\leqslant r_0$, we obtain

$$0\leqslant|B_{r/2}\cap E|^{1/n}\leqslant|B_r\cap E|^{1/n}-\frac{A}{4BC}r, \tag{1.19}$$

which gives the expected inequality. Using symmetry, we have the other inequality.

□

Corollary 1.1. *With the same assumptions, there exists $q>0$ such that for all $r\leqslant r_0$ and all balls B_r centered at $x_r\in\partial E$,*

$$\mathrm{Per}(E\cap B_r)\geqslant qr^{n-1}.$$

This comes directly from Proposition 1.7.

Let us state another corollary which will be useful in what follows. This corollary is often mentionned as *clean-ball property.*

Corollary 1.2. *Let E be minimizing in B with $0 \in \partial E$. Then, there exists $q > 0$ (depending only on A, B, $\|t-g\|_\infty$ and the dimension) such that for all $r \leqslant r_0$ there exists a ball $B_{qr} \subset E \cap B_r$ of radius qr. In addition, there exists another ball B'_{qr} with the same radius, such that $B'_{qr} \subset \mathbb{R}^n \setminus E \cap B_r$.*

Proof. For a fixed $\delta > 0$, thanks to Vitali covering lemma, one can cover $\overline{B}_{r_0/2}$ with a finite number of balls $(B_{5\delta}(x_i))_{i \in I}$ such that $B_\delta(x_i) \cap B_\delta(x_j) = \emptyset$ for $i \neq j$. Then, let $J \subset I$ defined by

$$i \in J \Leftrightarrow B_\delta(x_i) \cap E \neq \emptyset.$$

Let us imagine, for a moment, that

$$\forall i \in J,\ B_{\delta/4}(x_i) \cap \partial E \neq \emptyset.$$

Then, let $y_i \in B_{\delta/4}(x_i) \cap \partial E$. We have $B_{\delta/4}(y_i) \subset B_\delta(x_i)$. As a result,

$$\forall i \neq j, \quad B_{\delta/4}(y_i) \cap B_{\delta/4}(y_j) = \emptyset.$$

On the other hand, thanks to Corollary 1.1, we have

$$\forall i \in J, \quad \mathrm{Per}(E; B_{\delta/4}(y_i)) \geqslant C\delta^{n-1}$$

which implies

$$\mathrm{Per}(E; B_{r_0/2}) \geqslant C N_\delta \delta^{n-1} \tag{1.20}$$

where N_δ is the cardinal of J. Since the $B_\delta(x_i)$ are disjoints, we have $N_\delta |B_\delta| \leqslant |B_{r_0/2}|$. On the other hand, by Proposition 1.10, we have $N_\delta |B_\delta| \geqslant |E \cap B_{r_0/2}| > C(r_0)^n$ which implies

$$N_\delta \sim \delta^{-n}.$$

As a result, if δ is too small, we have a contradiction with (1.20) and there exists a ball $B_{\delta/4}$ which intersects E and such that $\partial E \cap B_{\delta/4} = \emptyset$. That implies $B_{\delta/4} \subset E$.

We build B' by symmetry. □

Finally, these density estimates give some information on the points of density one.

Proposition 1.11. *Let E be a minimizer of* (1.16)*. Then, the sets $E^{(1)}$ of points with density 1 in E and $E^{(0)}$ of points with density* 0 *in E (see Definition 1.3) are both open subsets of $\mathbb{R}^n$.*

Proof. We will show the result for $E^{(0)}$. We only have to note that Equation (1.19) holds as long as $|B_r \cap E|$ does not vanish (there is no need for 0 to belong to ∂E). As a result, if $|B_r \cap E|$ is too small, $|B_{r/2} \cap E|$ must be zero.

Let $x_0 \in E^{(0)}$. Then, for every ε, we have a radius r_1 such that if $r \leqslant r_1$, then $|E \cap B_{r_1}(x_0)| \leqslant \varepsilon \omega_n r_1^n$. That implies, if ε is sufficiently small, that $|E \cap B_{r_1/2}(x_0)| = 0$. As a result, every point x in $B_{r_1/4}(x_0)$ satisfies

$$|E \cap B_{r_1/4}(x)| = 0.$$

□

3.7 A word on minimizing graphs

Let E minimize (1.16) (we assume for simplicity that $t = 0$) on $Q_1 = \{|x_i| \leqslant 1\}$ and assume that E is a smooth graph over some $n^\perp$ (for simplicity we assume that $n = e_n$), that is

$$E = \{(x', x_n) \in \mathbb{R}^n \mid z \leqslant u(x')\}$$

with $|u| \leqslant 1$. In this whole part, we will use the following notations.

— Every $x \in \mathbb{R}^n$ will be written $x = (x', x_n)$.

— The differential operators on $\mathbb{R}^{n-1} \simeq e_n^\perp$ are denoted with a $'$.

— The normal vector to E at $x = (x', x_n)$ is $\nu(x') = \frac{(\nabla' u, 1)}{\sqrt{1+|\nabla' u|^2}}$.

Lemma 1.1. *The function u satisfies the equation*

$$-\operatorname{div}(\nabla\varphi(\nabla' u, 1)) = g(x', u(x')).$$

Proof. We use the first-variation approach. The area measure is

$$\operatorname{Per}_\varphi(E, Q_1) = \int_{\partial E \cap Q_1} \varphi(\nu) = \int_{Q_1'} \sqrt{1+|\nabla' u|^2}\varphi\left(\frac{(\nabla' u, 1)}{\sqrt{1+|\nabla' u|^2}}\right) = \int_{Q_1'} \varphi(\nabla' u, 1)$$

using the homogeneity of φ. In addition, we have

$$\int_{E \cap B_1} g = \int_{Q_1'} \int_{w=-1}^{u(x')} g(x', w) dw dx'$$

Since u is minimizing, we have, for all v and s,

$$\int_{Q_1'} \varphi(\nabla'(u+sv), 1) + \int_{-1}^{u+sv(x')} g(x', w) \geqslant \int_{Q_1'} \varphi(\nabla'(u), 1) + \int_{-1}^{u(x'))} g(x', w).$$

Then, note that

$$\int_{Q_1'} \varphi(\nabla'(u+sv), 1) = \int_{Q_1'} \varphi(\nabla'(u), 1) + s\int_{Q_1'} \nabla'\varphi(\nabla' u, 1) \cdot \nabla' v + o(s)$$

and

$$\int_{Q_1'} \int_{-1}^{u+sv(x')} g(x', w) = \int_{Q_1'} \int_{-1}^{u(x')} g(x', w) + s\int_{Q_1'} g(x', u(x')) + o(s).$$

As a result, we must have

$$\int_{Q_1'} \nabla'\varphi(\nabla' u, 1) \cdot \nabla' v - v(x') g(x', u(x')) dx' = 0,$$

which can be rewritten using Green formula, and assuming that v has a compact support on Q_1', as

$$\int_{Q_1'} v\left(-\operatorname{div}'(\nabla'\varphi(\nabla' u, 1)) - g(x', u(x'))\right) = 0$$

which finally yields

$$-\operatorname{div}'(\nabla'\varphi(\nabla' u, 1)) = g(x', u(x')).$$

□

Now, we are ready to give the main regularity results of this chapter. Let us begin by a theorem really in the spirit of Section 2.

4 On convex domains with Neumann boundary conditions

In this section, we take full advantage of the remark by Miranda, that everything happens on the boundary. We prove that it is also true when studying (1.12) with Neumann conditions. The assumption of convexity of Ω prevents bad things to happen on the boundary, as we will see. In the first two subsections, we work only with smooth objects. We will see in the last section that the general results could be derived from the smooth ones.
Note that the second subsection contains no result (since we cannot regularize the problem in a appropriate way to proceed similarly as in the first subsection), but consists in a remark on what can be said on smooth minimizers of the anisotropic ROF-functional.

4.1 The isotropic case

In this section, we prove the

Theorem 1.7. *Let Ω be a convex bounded domain and $f : \mathbb{R} \to \mathbb{R}$ be convex and satisfies*

$$f(0) = 0, \quad f(+\infty) = +\infty.$$

Let u be the minimizer of

$$\int_\Omega f(|\nabla u|) + \frac{(u-g)^2}{2}$$

with Neumann conditions on the boundary and assume that g is continuous with modulus ω. Then, u is continuous with modulus ω.

Note that Theorem 1.7 is already known in low dimension ($n \leqslant 8$) and isotropic framework, thanks to [CCN11].
The strategy is to work on the approximate problem

$$\min_u \int_\Omega f_\varepsilon(|\nabla u|) + \frac{(u-g)^2}{2}$$

with $f_\varepsilon \to f$ locally uniformly and $f_\varepsilon \geqslant 0$, smooth and satisfies

$$\varepsilon \leqslant f_\varepsilon'' \leqslant \frac{1}{\varepsilon}$$

as well as $f_\varepsilon'(0) = 0$. Letting $G(x) = f_\varepsilon(|x|)$, G is smooth, uniformly elliptic (we have $\frac{1}{C} I \leqslant D^2 G \leqslant CI$). We can show using the classical Nirenberg translation method, that $u \in H^2$. It is in addition $\mathcal{C}^\infty$ up to the boundary thanks to De Giorgi Nash Moser theorem.

The solution u of the approximate problem is $\mathcal{C}^\infty$ up to the boundary so is K_ε-Lipschitz and satisfies the Euler Lagrange equations

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|} f_\varepsilon'(|\nabla u|)\right) + (u-g) = 0$$

and

$$(\nabla u\,,\,\nu) = 0 \quad \text{ on } \partial\Omega.$$

Lemma 1.2. *Let*

$$L = \sup_{x \neq y \in \overline{\Omega}} \frac{u(x) - u(y)}{\omega(x-y)}.$$

Then, either $L \leqslant 1$ or L is reached on the boundary.

Proof. First, note that this supremum is a maximum, because since $\alpha < 1$, $\frac{u(x)-u(y)}{\omega(x-y)} \to 0$ as soon as $|x-y| \to 0$.

Let us now assume (to get a contradiction), that $L > 1$ and that the maximum is not reached on the boundary. That is, we assume that there exists $\delta > 0$ such that

$$\sup_{\substack{x \in \partial\Omega \\ y \neq x \in \overline{\Omega}}} \frac{u(x) - u(y)}{\omega(x-y)} \leqslant L - 100\delta.$$

We can choose δ such that $L - 100\delta > 1$. Thus, if $|z| \leqslant \eta$ sufficiently small, we can write

$$\forall x \in \partial\Omega,\ |z| \leqslant \eta, \quad u(x-z) - u(x) \leqslant (L-\delta)\omega(z).$$

Let $v = u(\cdot - z) - (L-\delta)\omega(z)$. We have just said that $v \leqslant u$ on $\partial\Omega_z$ where $\Omega_z = (\Omega + z) \cap \Omega$. Using the very definition of u, one can write (on $\Omega \setminus (\Omega + z)$, we will impose $u \vee v = u$ and on $(\Omega + z) \setminus \Omega$, $u \wedge v = v$)

$$\int_\Omega f(|\nabla u|) + \frac{(u-g)^2}{2} \leqslant \int_\Omega f(|\nabla u \vee v|) + \frac{(u \vee v - g)^2}{2}$$

and

$$\int_{(\Omega+z)} f(|\nabla v|) + \frac{(v(x) - (g(x-z) - (L-\delta)\omega(z)))^2}{2}$$
$$\leqslant \int_{(\Omega+z)} f(|\nabla u \wedge v|) + \frac{(u \wedge v(x) - (g(x-z) - (L-\delta)\omega(z)))^2}{2}.$$

We sum this two inequalities and notice that, as measures,

$$f(|\nabla u|) + f(|\nabla v|) \geqslant f(|\nabla u \vee v|) + f(|\nabla u \ \wedge v|),$$

that yields

$$0 \leqslant \int_{\Omega_z} (u \vee v - g)^2 - (u-g)^2 + (u \wedge v(x) - (g(x-z) - (L-\delta)\omega(z)))^2 - (v(x) - (g(x-z) - (L-\delta)\omega(z)))^2$$

which means

$$0 \leqslant \int_{\Omega_z} -2gu \vee v + 2ug - 2(g(x-z) - (L-\delta)\omega(z))u \wedge v + 2(g(x-z) - (L-\delta)\omega(z))v,$$

which is equivalent to

$$0 \leqslant \int_{\Omega_z} (u \vee v - u)\left(-g + g(x-z) - (L-\delta)\omega(z)\right).$$

Since $z \neq 0$, one has $\omega(z) > 0$. In addition, $L - \delta > 1$ so

$$-g + g(x - z) - (L - \delta)\omega(z) < 0$$

(g is ω-uniformly continuous). Finally, $(u \vee v - u) \geqslant 0$ and since the integral is nonnegative, we must have $u \vee v - u = 0$ on the whole Ω_z, which implies $u \geqslant v$ on Ω_z, that is

$$u(x + z) - u(x) \leqslant (L - \delta)\omega(z),$$

which is a contradiction with the definition of L. $\square$

It remains to show that this maximum can actually not be reached on the boundary. For almost every level-set of u, using Sard's theorem, ∇u does not vanish. So, the condition $\nabla u \cdot \nu = 0$ on $\partial\Omega$ provides a non degenerate orthogonality between $\partial\Omega$ and the level sets of u. Let $\hat{x}, \hat{y} \in \overline{\Omega}$ such that

$$L = \frac{u(\hat{x}) - u(\hat{y})}{\omega(x - y)},$$

and assume that $\hat{x} \in \partial\Omega$. That is exactly saying that the distance between $A := \{u = u(\hat{x})\}$ and $B := \{u = u(\hat{y})\}$ is reached on $A \cap \partial\Omega$. In particular, the sphere with center $\hat{y}$ and radius $|\hat{x} - \hat{y}|$ contains $\hat{x}$ and does not cut A. That must imply $(\hat{x} - \hat{y}\, ,\, \nu(\hat{x})) \leqslant 0$, which is impossible because of the uniform convexity of $\partial\Omega$.

Let us deal with the case where $\nabla u(\hat{x}) = 0$. (what we have just done still holds if $\nabla u(\hat{y}) = 0$). In a neighborhood of $\hat{x}$, we have $|u(\hat{x}) - u(x)| = o(\hat{x} - x)$ which provides, recalling the definition of L,

$$|u(x) - u(\hat{y})| \geqslant L\omega(\hat{x} - \hat{y}) + o(|x - \hat{x}|).$$

On the other hand, choosing ω differentiable on $(0, +\infty)$ and increasing ($\omega' \geqslant \lambda > 0$), we obtain

$$\omega(x - \hat{y}) = \omega(\hat{x} - \hat{y}) + \frac{(x - \hat{y}\, ,\, x - \hat{x})}{|x - \hat{y}|}\omega'(|\hat{x} - \hat{y}|) + o(\hat{x} - x).$$

Since $|u(x) - u(\hat{y})| \leqslant L\omega(x - \hat{y})$, this yields

$$\omega(\hat{x} - \hat{y}) + o(x - \hat{x}) \geqslant \omega(\hat{x} - \hat{y}) + \frac{(x - \hat{y}\, ,\, x - \hat{x})}{|x - \hat{y}|}\omega'(|\hat{x} - \hat{y}|) + o(\hat{x} - x)$$

so

$$0 \geqslant \frac{(x - \hat{y}\, ,\, x - \hat{x})}{|x - \hat{y}|}\omega'(|\hat{x} - \hat{y}|),$$

for every x in a neighborhood of $\hat{x}$ in Ω. That is not possible.

Finally, this proves that all the approximate solutions share the same modulus of continuity ω. Passing to the limit shows that the exact solution also has ω as a modulus of continuity.

4.2 A remark on the anisotropic case

We now consider an anisotropic version of what we have just proved, trying to minimize

$$\int_\Omega f(\varphi(\nabla u)) + \frac{(u-g)^2}{2}.$$

with φ any anisotropy (norm in $\mathbb{R}^n$).
The previous scheme of proof, which would consist in approximating this problem by

$$\int_\Omega f_\varepsilon\left(\varphi_\varepsilon(\nabla u)\right) + \frac{(u-g)^2}{2},$$

where f_ε is as in the isotropic case and φ_ε is a 1-homogeneous $\mathcal{C}^\infty$ elliptic approximation of φ, does not necessarly provide a minimizer which is regular up to the boundary. Indeed, the operator $-\operatorname{div}(f'(\varphi(\nabla u)) \cdot \nabla\varphi(\nabla u))$ is singular when $\nabla u = 0$.

Nonetheless, we can notice that if a minimizer of the approximate problem is C^1, it is uniformly continuous with a control on the modulus. Indeed, it would satisfy the Euler-Lagrange equation on the boundary

$$\nabla\varphi(\nabla u) \cdot \nu = 0.$$

Assuming that

$$\forall x, y \in \Omega, \quad |g(x) - g(y)| \leqslant \omega(\varphi^\circ(x-y)),$$

where

$$\varphi^\circ(\xi) = \sup\left\{\langle x\,,\,\xi\rangle \mid \varphi(x) \leqslant 1\right\},$$

one can notice that Lemma 1.2 still holds, with the same proof, but with $\omega(x-y)$ replaced by $\omega(\varphi^\circ(x-y))$.

Now let us show as before that the maximum of

$$\frac{u(x)-u(y)}{\omega(\varphi^\circ(x-y))}$$

can actually not be reached on the boundary. We proceed by contradiction and can consider $\hat{x}$ and $\hat{y}$ which maximize $\varphi^\circ(x-y)$ for $x \in \{u = u(\hat{x})\}$ and $y \in \{u = u(\hat{y})\}$ and assume that $\hat{x} \in \partial\Omega$. Then,

$$\nabla u(\hat{x}) \in \operatorname{Vect}(\nabla\varphi^\circ(\hat{x}-\hat{y}), \nu).$$

To see that, let h be both orthogonal to $\nabla\varphi^\circ(\hat{x}-\hat{y})$ and to ν. Assume that $\nabla u(\hat{x}) \cdot h < 0$ (if not, one consider $-h$). Let w be the projection of h on the boundary $\partial\Omega$. Since $(h\,,\,\nu) = 0$, we have

$$u(\hat{x}+w) = u(\hat{x}) + \nabla u \cdot h + o(h).$$

As a result, for h sufficiently small,

$$u(\hat{x}+w) \leqslant u(\hat{x}) + \frac{\nabla u(\hat{x}) \cdot h}{2} < u(\hat{x}).$$

In addition, $\hat{x}$ is a minimizer on $\{u \leqslant u(\hat{x})\}$ so $\varphi^\circ(\hat{x}+w-\hat{y}) \geqslant \varphi^\circ(\hat{x}-\hat{y})$.

Moreover, since $\nabla\varphi^\circ(\hat{x}-\hat{y})\cdot h = 0$, one can write

$$\varphi^\circ(\hat{x}+w-\hat{y}) = \varphi^\circ(\hat{x}-\hat{y}) + o(h).$$

As a result, if v denotes the projection of w on $\varphi^\circ = \varphi^\circ(\hat{x}-\hat{y})$ (this projection remains in Ω in a neighborhood of $\hat{x}$), we both have $\varphi^\circ(\hat{x}+v-\hat{y}) = \varphi^\circ(\hat{x}-\hat{y})$ and $u(\hat{x}+v) \leqslant u(\hat{x}) + \frac{\nabla u(\hat{x})\cdot h}{2}$, which contradicts the definition of $\hat{x}$.

We proved that

$$\nabla u(\hat{x}) = \alpha\nabla\varphi^\circ(\hat{x}-\hat{y}) + \beta\nu.$$

One can notice that $\alpha, \beta \leqslant 0$ ($\alpha > 0$ would imply that we can decrease u from $u(\hat{x})$ by going towards $\hat{y}$ whereas $\beta > 0$ would imply that going inside Ω from $\hat{x}$ would necessarily decrease u, and both cannot happen).

Finally, one can write (we replace α and β by $-\alpha$ and $-\beta$ taking advantage of the 0-homogeneity of $\nabla\varphi$)

$$\begin{aligned} 0 = \nabla\varphi(\nabla u(\hat{x}))\cdot\nu &= \nabla\varphi(-\alpha\nabla\varphi^\circ(\hat{x}-\hat{y}) - \beta\nu)\cdot\nu \\ &= \frac{\hat{x}-\hat{y}}{\varphi^\circ(\hat{x}-\hat{y})}\cdot\nu(\hat{x}) + \int_0^{-\beta} D^2\varphi(\nabla\varphi^\circ(\hat{x}-\hat{y}) + s\nu)\nu\cdot\nu\, ds \end{aligned}$$

Since φ is convex, one can conclude

$$0 \geqslant \frac{\hat{x}-\hat{y}}{\varphi^\circ(\hat{x}-\hat{y})}\cdot\nu(\hat{x}),$$

where the right hand side is positive using the strict convexity of Ω.

4.3 Non smooth limit

In the two previous subsections, we assumed that all the objects are smooth and that Ω is uniformly convex. What happens if Ω is not uniformly convex and if φ is not smooth?

If Ω is not uniformly convex, we can approximate it by uniformly convex domains Ω_n. For each Ω_n, we have a minimizer u_n which has a controlled modulus of continuity (which does not depend on n). As a result, passing to the limit, we get the same modulus for the minimizer in Ω.

If the anisotropy is not smooth (for example crystalline), we can choose a sequence $\varphi_n \to \varphi$ with $\varphi_n \geqslant \varphi$ and smooth. Then, recalling that $\varphi_n \geqslant \varphi \Leftrightarrow \varphi_n^\circ \leqslant \varphi^\circ$, g is continuous with φ_n-modulus ω for every n, and the minimizers u_n of (1.12) with φ_n instead of φ therefore satisfy

$$\forall x, y \in \Omega, \quad |u(x) - u(y)| \leqslant \omega(\varphi_n^\circ(x-y)) \leqslant \omega(\varphi^\circ(x-y)).$$

As a result, passing to the limit in n, the control on the modulus of continuity still holds with crystalline anisotropies if it does with smooth ones.

5 The usual smooth Hopf maximum principle

In the rest of this chapter, every regularity result will be obtained taking advantage of the minimizing property of level-sets. That means we are involved in studying minimizers of (1.16). In this section, we recall the strong Hopf maximum principle, which is well known to hold for two smooth minimal surfaces (see for example [CM11]).

Theorem 1.8. *Let $f \leqslant g - \varepsilon$ both continuous in Ω and $F \subset G$ two subsets of $\mathbb{R}^n$ with smooth boundaries which minimize respectively (in the family of finite perimeter sets, and with respect to compact perturbations in Ω)*

$$\mathrm{Per}_\varphi(F, \Omega) + \int_{F \cap \Omega} f$$

and

$$\mathrm{Per}_\varphi(G, \Omega) + \int_{G \cap \Omega} g.$$

with an anisotropy φ which is smooth and uniformly elliptic. Then, either $F = G$ or $\partial F \cap \partial G = \emptyset$.

Remark. The assumptions with ε are made to compensate the variations of f and g. If f and g are constant or even Lipschitz, we can take $\varepsilon = 0$.

Proof. The proof is standard. Let us assume that there is some $x \in \partial F \cap \partial G$. Since these two sets are smooth, F and G are graphs of functions u and v over the same hyperplan in a small ball B_r. Since f and g are continuous, we choose r sufficiently small such that

$$\forall x, y \in B_r, \quad f(x) \leqslant g(y).$$

One can assume that $u \leqslant v$ and $x = (x_0, u(x_0)) = (x_0, v(x_0))$. Denoting by $H((\nabla')^2 u, \nabla' u)$ the quantity

$$H((\nabla')^2 u, \nabla' u) = -\,\mathrm{div}' \left(\nabla' \varphi(\nabla' u)\right),$$

The graphs satisfy the equation

$$H((\nabla')^2 u, \nabla' u) = f(x', u(x')) \quad \text{and} \quad H((\nabla')^2 v, \nabla' v) = g(x', v(x')).$$

As a result, thanks to the ε (or if f and g are constant), we have

$$0 \leqslant H((\nabla')^2 v, \nabla' v) - H((\nabla')^2 u, \nabla' u)$$

in a neighborhood of x_0. We can show that since $|\nabla u|$ and $|\nabla v|$ are bounded, H a uniformly elliptic operator. So, let us write (we use the mean value theorem)

$$\begin{aligned} 0 \leqslant \mathrm{div}'(\nabla' \varphi(\nabla' u)) - \mathrm{div}'(\nabla' \varphi(\nabla' v)) &= \partial_t \left[\mathrm{div}' \left(\nabla' \varphi((1-t)\nabla' u + t \nabla' v)\right)\right]\Big|_{t=t_0} \\ &= \mathrm{div}' \left[\partial_t \left(\nabla' \varphi((1-t)\nabla' u + t\nabla' v)\right)\right]\Big|_{t=t_0} \\ &= \mathrm{div}' \left[(\nabla')^2 \varphi((1-t)\nabla' u + t\nabla' v) \cdot \nabla'(v-u)\right]\Big|_{t=t_0}. \end{aligned}$$

which is an elliptic linear divergence type equation $\mathrm{div}(A(x)\nabla w) \leqslant 0$ for $w = v - u$. This directly implies that $v \leqslant u$, so that $u = v$. Finally, $F = G$. □

Remark. If $\varepsilon = 0$ and f is Lipschitz, then we get

$$0 \leqslant \operatorname{div}' \left[(\nabla')^2\varphi((1-t)\nabla' u + t\nabla' v) \cdot \nabla'(v-u)\right] + (v-u)\partial_n f(x', tu(x') + (1-t)v(x'))\big|_{t=t_0}$$

which has the form of

$$-\operatorname{div}(A\nabla w) + w \cdot c(x) = 0,$$

which satisfies also a strong maximum principle (with the decomposition $c(x) = c^+(x) - c^-(x)$).

6 A comparison result with a smooth set

We make the same assumptions as in Section 5 (in particular, φ is smooth), but we only assume that G is smooth.

Theorem 1.9. *Let F and G minimize in Ω*

$$\operatorname{Per}_\varphi(F) + \int_F f$$

and

$$\operatorname{Per}_\varphi(G) + \int_G g$$

with $f \leqslant g - \varepsilon$. We assume that $F \subset G$ and ∂G is a $\mathcal{C}^1$ hypersurface. Then, either $F = G$ or $\partial F \cap \partial G = \emptyset$.

Remark. — As above, when f and g are constant, we do not need ε to be positive (it can be zero).

— This theorem is already known when $f = g = 0$ in a more general version in [SW89] (in particular, the anisotropy can depend on the space variable, and the sets are only stationary, whereas they are minimizing in our framework). Nonetheless, we present a simpler proof of this result, in the spirit of [CC93] (see also [CRS10]).

We replace ∂F by $\operatorname{supp}(D1_F)$ in order to work with a closed set. Let us assume that there exists $x_0 \in \partial F \cap \partial G$. We want to prove that it implies $F = G$. Since ∂G is $\mathcal{C}^1$, ∂G is the graph of some $\mathcal{C}^1$ function $\tilde{v}$ over $\tilde{n}^\perp$, with $\tilde{n}$ the outer normal to G at x_0 (we may assume $x_0 = (0, \tilde{v}(0))$, $\tilde{v}(0) = 0$ and v defined on $B'_{\tilde{\rho}}$.).

In what follows, for every $\tilde{z} \in \mathbb{R}^n$, we will denote by $\tilde{z}'$ the $n-1$ first component of $\tilde{z}$: $\tilde{z} = (\tilde{z}', \tilde{z}_n)$.

Thanks to Corollary 1.2, for every r sufficiently small, there exists a ball $B := B_{qr}(x_r)$ of center x_r and radius qr with $B \subset F \cap B_r(x_0)$. Since $\{\tilde{z}_n = 0\}$ is tangent to G, x_r must have a negative n-th component for r small enough. Let r_0 satisfies this requirement and let $n = x_0 - x_{r_0}$. Then, since $(n\,,\tilde{n}) > 0$, ∂G is also a graph over $n^\perp$ of some function v defined on $B'_\rho(0)$. Once again, we assume $v(0) = 0$ and denote (z', z_n) the components of every $z \in \mathbb{R}^n$. Then, we define

$$\forall |z'| \leqslant \rho, \quad u(z') := \sup\{z_n \in \mathbb{R} \,|\, (z', z_n) \in F\}.$$

Note that since $F \subset G$ and by definition of u, we must have $v \geqslant u$ on B'_ρ.

Moreover, v is a smooth graph over $n^{\perp}$ on B_ρ, so it satisfies (in the strong sense, so also in the viscosity sense)

$$-\operatorname{div}'\big(\nabla'\varphi(\nabla' v,-1)\big)=g.$$

Proposition 1.12. *The function u is upper semicontinuous and is a viscosity subsolution of*

$$-\operatorname{div}'\big(\nabla'\varphi(\nabla' u,-1)\big)=f. \tag{1.21}$$

Proof. Let us first prove that u is upper semicontinous. Let $x_n\to x\in B_\rho$. Then, we have a sequence $(x_k,u(x_k))\in F$, which is bounded above. As a result, there exists a subsequence (still denoted by $(x_k,u(x_k))$) which converges (possibly $u(x_k)\to-\infty$). We want to show that $u(x)\geqslant\limsup_k u(x_k)$. If $u(x_k)\to-\infty$, nothing has to be done. If not, then $(x_k,u(x_k))$ is a converging sequence of F which is closed. So, $(x,\lim u(x_k))$ in F and $u(x)\geqslant\limsup_k u(x_k)$.

Now, let us show that it is a subsolution of (1.21). Assume by contradiction that it does not hold. Then, there exists a smooth function ψ and some $x_1\in B'_\rho$ such that $u-\psi$ has a maximum at x_1 and

$$-\operatorname{div}'\big(\nabla'\varphi'(\nabla'\psi,-1)\big)>f.$$

On can assume that $x_1=0$ and $u(x_1)=\psi(x_1)$ and that the maximum is strict. Let Γ be the graph of ψ. We want to generalize the result by Caffarelli and Cordoba [CC93]. To this aim, we work with the φ-relative distance

$$d_\varphi(x,y)=\varphi^\circ(x-y)\quad\text{where}\quad\varphi^\circ(x)=\sup_{\varphi(\nu)\leqslant 1}\,(x\,,\,\nu)$$

and

$$d_\varphi^\Gamma(x)=\inf\{d_\varphi(x,y)\ \mid\ y\in\Gamma\}.$$

Then, we defined the signed φ-relative distance to Γ by setting

$$d(x',x_n)=d_\varphi^\Gamma(x',x_n)1_{\{x_n\leqslant\psi(x)\}}-d_\varphi^\Gamma(x',x_n)1_{\{x_n\geqslant\psi(x',x_n)\}}.$$

Since Γ is smooth, then there exists a tubular neighborhood of Γ where d is smooth.

Lemma 1.3. *We have*

$$-\operatorname{div}'(\nabla'\varphi(\nabla'\psi,-1))(0)=-\operatorname{div}(\nabla\varphi(\nabla d))(0,0). \tag{1.22}$$

Proof. Let us first notice that $d(x',\psi(x'))=0$, so that $\nabla' d+\partial_n d\nabla'\psi=0$. Hence, since $\nabla\varphi$ is 0-homogeneous and even, we get

$$\frac{\partial\varphi}{\partial x_i}(\nabla\psi,-1)=\frac{\partial\varphi}{\partial x_i}\left(-\frac{\nabla' d}{\frac{\partial d}{\partial x_n}},-1\right)=\frac{\partial\varphi}{\partial x_i}\left(\nabla' d,\frac{\partial d}{\partial x_n}\right)=\frac{\partial\varphi}{\partial x_i}(\nabla d(x',\psi(x')).$$

As a matter of fact,

$$\begin{aligned}\operatorname{div}'(\nabla'\varphi(\nabla\psi,-1)) = \operatorname{div}'(\nabla'\varphi(\nabla d(x',\psi(x')) &= \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}\left(\frac{\partial\varphi(\nabla d(x',\psi(x')))}{\partial x_j}\right)\\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{n} \frac{\partial^2\varphi}{\partial x_i\partial x_j}\frac{\partial}{\partial x_i}\left(\frac{\partial d(x',\psi(x'))}{\partial x_j}\right)\\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{n} \frac{\partial^2\varphi}{\partial x_i\partial x_j}\left(\frac{\partial^2 d}{\partial x_i\partial x_j} + \frac{\partial^2 d}{\partial x_j\partial x_n}\frac{\partial\psi}{\partial x_i}\right).\end{aligned}$$

As a result,

$$\begin{aligned}\operatorname{div}(\nabla'\varphi(\nabla d) &= \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial\varphi(\nabla d)}{\partial x_i}\right) = \sum_{i,j=1}^{n}\frac{\partial^2\varphi(\nabla d)}{\partial x_i\partial x_j}\frac{\partial^2 d}{\partial x_i\partial x_j}\\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{n}\frac{\partial^2\varphi(\nabla d)}{\partial x_i\partial x_j}\frac{\partial^2 d}{\partial x_i\partial x_j} + \sum_{j=1}^{n}\frac{\partial^2\varphi}{\partial x_n\partial x_j}\frac{\partial^2 d}{\partial x_n\partial x_j}\\ &= \operatorname{div}'(\nabla'\varphi(\nabla\psi,-1)) + \sum_{j=1}^{n}\frac{\partial^2\varphi}{\partial x_n\partial x_j}\frac{\partial^2 d}{\partial x_n\partial x_j} - \sum_{i=1}^{n-1}\sum_{j=1}^{n}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\frac{\partial^2 d}{\partial x_n\partial x_j}\frac{\partial\psi}{\partial x_i}\\ &= \operatorname{div}'(\nabla'\varphi(\nabla\psi,-1)) + \sum_{j=1}^{n}\frac{\partial^2 d}{\partial x_n\partial x_j}\left(\sum_{i=1}^{n-1}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\ \frac{\partial\psi}{\partial x_i} - \frac{\partial^2\varphi}{\partial x_n\partial x_j}\right)\\ &= \operatorname{div}'(\nabla'\varphi(\nabla\psi,-1)) - \sum_{j=1}^{n}\frac{\partial^2 d}{\partial x_n\partial x_j}\left(\sum_{i=1}^{n-1}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\frac{\frac{\partial d}{\partial x_i}}{\frac{\partial d}{\partial x_n}} + \frac{\partial^2\varphi}{\partial x_n\partial x_j}\right)\\ &= \operatorname{div}'(\nabla'\varphi(\nabla\psi,-1)) - \sum_{j=1}^{n}\frac{\partial^2 d}{\partial x_n\partial x_j}\frac{1}{\frac{\partial d}{\partial x_n}}\sum_{i=1}^{n}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\frac{\partial d}{\partial x_i}.\end{aligned}$$

Let us show that the last term of the last inequality vanishes. Indeed, one has $\varphi^\circ(\nabla\varphi(\nabla d)) = 1$, whose derivative provides

$$\forall i \leqslant n, \quad \sum_{j=1}^{n}\frac{\partial\varphi^\circ(\nabla\varphi(\nabla d))}{\partial x_j}\cdot\frac{\partial}{\partial i}\left(\frac{\partial\varphi(\nabla d)}{\partial x_j}\right) = 0. \tag{1.23}$$

In addition, thanks to the equality (which holds for any anisotropy) $(\varphi^\circ\nabla\varphi^\circ)(\varphi(\xi)\nabla\varphi(\xi)) = \xi$, one obtains $\nabla\varphi^\circ(\nabla\varphi(\xi)) = \frac{\xi}{\varphi(\xi)}$. Then, (1.23) can be rewritten

$$\forall i, \quad \sum_{j,k=1}^{n}\frac{1}{\varphi(\nabla d)}\frac{\partial d}{\partial x_j}\frac{\partial^2 d}{\partial x_i\partial x_k}\frac{\partial^2\varphi}{\partial x_j\partial x_k} = 0,$$

which implies for $i = n$ (and some changes of indices)

$$\forall i, \quad \sum_{j=1}^{n}\frac{\partial^2 d}{\partial x_n\partial x_j}\left(\sum_{i=1}^{n}\frac{\partial d}{\partial x_i}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right) = 0,$$

what was expected. $\square$

Let δ be small and fixed. Let $\tilde{\Omega}$ be the epigraph of ψ (we have $\partial\tilde{\Omega} = \Gamma$). We are interested in $(\tilde{\Omega} - \delta e_n) \cap B'_\rho$. Then, if δ is small enough,

— $F \setminus (\tilde{\Omega} - \delta e_n) \cap B'_\rho$ is a compact perturbation of F in B_ρ (the minimum is strict).

— $(\tilde{\Omega} - \delta e_n) \cap F$ has a nonempty interior in F (clean ball property).

— If $\tilde{d} = d(\cdot + \delta e_n)$, we have $-\operatorname{div}(\nabla\varphi(\nabla\tilde{d})) \geqslant f + \eta$ ($\eta > 0$) in $(\tilde{\Omega} - \delta e_n) \cap F$ (continuity of d and (1.22)).

Let $\Omega = (\tilde{\Omega} - \delta e_n) \cap F$,. If F were smooth, we would have

$$\int_\Omega -\operatorname{div}(\nabla\varphi(\nabla\tilde{d})) = -\int_{(\Gamma - \delta e_n)\cap F} (\nabla\varphi(\nabla\tilde{d})) \cdot n_\Omega d\sigma - \int_{\partial F \cap \Omega} (\nabla\varphi(\nabla\tilde{d})) \cdot n_\Omega d\sigma$$

which yields, using $-\operatorname{div}(\nabla\varphi(\nabla\tilde{d})) \geqslant f + \eta$ and noting that on ∂F, $n_\Omega = n_F$, we obtain

$$-\int_{\Gamma\cap(F+\delta e_n)} (\nabla\varphi(\nabla d)) \cdot n_\Gamma d\sigma + \int_{\partial F\cap\Omega} (\nabla\varphi(\nabla\tilde{d})) \cdot n_F d\sigma \geqslant \int_\Omega f + \eta.$$

Recall that F is minimizing, we can also write (comparing F to the compact perturbation $F \setminus \Omega$)

$$\int_\Omega f \leqslant \int_{\Gamma\cap(F+\delta e_n)} \varphi(\nu) d\sigma - \int_{\partial F\cap\Omega} \varphi(\nu) d\sigma.$$

Substracting the second inequality to the first one, we obtain

$$\int_\Omega \eta \leqslant -\int_{\Gamma\cap(F+\delta e_n)} \varphi(\nu) d\sigma + \int_{\partial F\cap\Omega} \varphi(\nu) d\sigma$$
$$- \int_{\Gamma\cap(F+\delta e_n)} (\nabla\varphi(\nabla d)) \cdot n_\Gamma d\sigma + \int_{\partial F\cap\Omega} (\nabla\varphi(\nabla\tilde{d})) \cdot n_F d\sigma.$$

Now, note that on Γ, we have $\nabla d = \frac{\nu}{\varphi(\nu)}$. On the other hand, $\nabla\varphi(\nu) \cdot \nu = \varphi(\nu)$ (because of the homogeneity of φ), which implies $\nabla\varphi(\nabla d) = \varphi(\nu)$ on Γ. We can then compute

$$\int_{\Gamma\cap(F+\delta e_n)} (\nabla\varphi(\nabla d)) \cdot n_\Gamma d\sigma = -\int_{\Gamma\cap(F+\delta e_n)} \varphi(\nu) d\sigma.$$

In addition, since $\varphi^\circ(\nabla\varphi(\nabla d)) = 1$, we also have $\nabla\varphi(\nabla d) \cdot \nu \leqslant \varphi(\nu)$. That implies

$$\left| \int_{\partial F\cap\Omega} (\nabla\varphi(\nabla\tilde{d})) \cdot n_F \right| \leqslant \int_{\partial F\cap\Omega} \varphi(n_F) d\sigma.$$

These two relations yield

$$\int_\Omega \eta \leqslant 0$$

which is not possible.

If F is not smooth, we select a sequence of $F_n \to F$ with F_n smooth and $1_{F_n} \to 1_F$ in BV and we reproduce this construction on F_n and pass to the limit (note that η does not depend on n). □

At this stage, u is a viscosity subsolution of

$$-\operatorname{div}'\left(\nabla'\varphi(\nabla' u,-1)\right) = f(x',u(x'))$$

whereas v is a viscosity supersolution of

$$-\operatorname{div}'\left(\nabla'\varphi(\nabla' v,-1)\right) = g(x',v(x')) \geqslant f(x',u(x')).$$

So, v is also a supersolution of (1.21). We also know that $v \geqslant u$. We would like to prove that $v > u$, because that would ensure that $\partial F \cap \partial G = \emptyset$. So, we need a strict comparison principle for viscosity solutions. This is found in [GO05]. Let us check that the assumptions are fullfiled. This article deals with an equation written as (see [GO05, Remark 3.6] for the right hand side)

$$F(Du, D^2u) = h$$

with F satisfying

1. The function $F : \mathbb{R}^n \times \mathcal{S}_n \to \mathbb{R}$ is continuous,
2. There exists a coercive function w such that for all p, X, Y,

$$F(p,X) - F(p,Y) \geqslant w(p, X-Y),$$

3. For every $M, K > 0$ and $|q|, |\tilde{q}| \leqslant K$, $\|X\| \leqslant M$, one has

$$|F(q,X) - F(\tilde{q},X)| \ \leqslant L_{M,K}|q-\tilde{q}|.$$

Here, we have

$$F(p,X) = \sum_{i,k=1}^{n-1} \frac{\partial^2\varphi}{\partial x_i \partial x_k}(p,-1)X_{ik} = \operatorname{Tr}(D'^2\varphi(p,-1)X).$$

It is clearly continous.

— If $p, q \in \mathbb{R}^n$ such that $|p|, |q| \leqslant M$, if $X \in \mathcal{S}^n$ satisfies $|X| \leqslant K$, one obtains

$$|F(p,X) - F(q,X)| = \left|\sum \left(\frac{\partial^2\varphi}{\partial x_i \partial x_k}(p,-1) - \frac{\partial^2\varphi}{\partial x_i \partial x_k}(q,-1)\right) X_{ik}\right| \leqslant L_{M,K}|p-q|.$$

— Let $p \in \mathbb{R}^n$ with $|p| \leqslant M$ and $X, Y \in \mathcal{S}_n$ such that $X \leqslant Y$.
The assumption on φ imply that $p \mapsto \varphi(p,-1)$ is uniformly convex with constant $\lambda(M)$ on every $\{|p| \leqslant M\}$ (see the proposition below) As a result, one has

$$\lambda(M)\operatorname{Tr}(Y-X) \leqslant F(p,X) - F(p,Y) = \sum_{i=1}^{n-1} \frac{\partial^2\varphi}{\partial x_i^2}(p,-1)\lambda_i \leqslant \Lambda \operatorname{Tr}(Y-X)$$

with Λ is the maximum of the spectral radius of $D^2\varphi^2(q)$ for $q = 1$.

Hence, [GO05, Th. 3.1] applies and gives the following alternative: either $u = v$ on B_ρ or $u < v$. That is exactly Theorem 1.9.

Finally, note that in the framework of (1.12), we have $f < g - \varepsilon$ so F and G cannot coincide.

During the proof, we showed the

Proposition 1.13. *The function $\tilde{\varphi} : p \mapsto \varphi(p,-1)$ is uniformly convex on $\{|p| \leqslant M\}$, with a constant $\lambda(M)$.*

Proof. First, recall a few properties of the anisotropy φ. By assumption, the sets $\{\varphi \leqslant t\}$ (Wulff shape of radius t) are homothetic convex subsets which contain a neighborhood of zero. In addition, $D^2\varphi^2 \geqslant \alpha I$. Noticing that

$$D^2\varphi = \frac{1}{\varphi} D^2\varphi^2 - \frac{1}{\varphi} \nabla\varphi \otimes \nabla\varphi,$$

we see that $D^2\varphi$ is positive definite on $T(p,-1)$, the tangent plane to the Wulff shape $\{\varphi = \varphi(p,-1)\}$ at $(p,-1)$, with eigenvalues bigger than $\frac{\alpha}{\varphi(p,-1)}$.
Finally,

Since φ is smooth around $(p,-1)$, to prove the proposition, we only have to control the eigenvalues of $D^2\tilde{\varphi}(p) = D^2\varphi(p,-1)\big|_{\{(e,0)\}}$. Let us write $e = e^T + e^0$ the decomposition of e with respect to $\nabla\varphi(p,-1)^{\perp}$ and $\text{span}(p,-1)$ (note that this projection is not orthogonal). Then,

$$D^2\varphi(p,-1)\cdot(e,e) = \underbrace{D^2\varphi(p,-1)\cdot(e^T,e^T)}_{\geqslant \alpha |e^T|^2} + \underbrace{2D^2\varphi(p,-1)\cdot(e^0,e^T) + D^2\varphi(p,-1)\cdot(e^0,e^0)}_{=0 \text{ since } D^2\varphi(p,-1)\cdot e^0 = 0}.$$

To conclude, we need to show that there exists a constant $\gamma(M)$ such that $|(e,0)^T| \geqslant \gamma(M)|(e,0)|$ as soon as $|p| \leqslant M$. Since there is an angle between $(e,0)$ and $(p,-1)$ which remains far from 0 on $\{|p| \leqslant M\}$, this is equivalent to show that the norm of the projection is controlled, or to show that the angle between $(p,-1)$ and $\nabla\varphi(p,-1)$ remains far from $\frac{\pi}{2}$. This is true using that the Wulff shape is a convex which contains a neighborhood of zero.

Finally, $D^2\varphi$ is uniformly convex with constant $\frac{\alpha\gamma(M)}{\beta(M)}$ where $\beta(M) = \min_{|p|\leqslant M} \varphi(p,-1)$. ☐

7 A result on mean convex domains with Dirichlet conditions

In this section, we link the minimizer u to the image g using Dirichlet conditions on the boundary of the domain. To give the assumptions on Ω, we need the

Definition 1.7. *Let φ be a norm in $\mathbb{R}^n$. We say that Ω satisfies the barrier condition if for every $x_0 \in \partial\Omega$ and $\varepsilon > 0$ sufficiently small, if V minimizes* Per_φ *in*

$$\{W \subset \Omega \mid W \setminus B_\varepsilon(x_0) = \Omega \setminus B_\varepsilon(x_0)\},$$

then

$$\partial V^{(1)} \cap \partial\Omega \cap B_\varepsilon(x_0) = \emptyset.$$

Remark. The barrier condition means that $\partial\Omega$ is not a local minimizer of the perimeter (there is always a inside perturbation of Ω which provides a set with strictly smaller perimeter). Note that if φ is the Euclidean norm and Ω is smooth, this property is the strict mean-convexity of Ω. (positive mean curvature).

Theorem 1.10. *Let φ be a norm in $\mathbb{R}^n$ which is $\mathcal{C}^2$ in $\mathbb{R}^n \setminus B_0(\varepsilon)$ and such that φ^2 is strongly convex. Let also Ω be a bounded Lipschitz open subset which satisfies the barrier condition. Moreover, let g be continuous on $\partial\Omega$. Then, there is a unique minimizer u of*

$$u = \operatorname*{arg\,min}_{\substack{v \in BV \\ v=g \text{ on } \partial\Omega}} \int_\Omega \varphi(\nabla u) \tag{1.24}$$

where the equality $v = g$ on $\partial\Omega$ means, as in [JMN13], that

$$\forall x \in \partial\Omega,\ \lim_{r\to 0} \operatorname*{ess\,sup}_{\substack{y\in\Omega \\ |x-y|\leqslant r}} |v(y) - g(x)| = 0. \tag{1.25}$$

In addition, this minimizer is continuous.

Remark. Since φ is not strictly convex as in Section 2 (because of the homogeneity), we have to find another way to obtain something similar to Proposition 1.1. This is done in [JMN13], which provides such proposition in the case we deal with. Proceeding as in Section 2, we could directly complete the proof (note that due to the space dependency, Jerrard et al. can obtain continuity of the minimizer only in dimension $\leqslant 3$, using the smoothness of the level-sets of u). Nonetheless, since we can take advantage of the translation invariance of the minimizers (which does not exists in [JMN13] because of the space dependency), we give a much simpler proof of the continuity of u. In particular, we will use no deep results neither on topological dimension nor on connected components of regular points of a minimal surface.

For simplicity, we assume that g is defined and continuous on the whole $\mathbb{R}^n$.

We first recall the proof of the existence part of the theorem (it is already done in [JMN13]). Let u be a minimizer of (1.24) in the class

$$\mathcal{A}_f := \{v \in BV(\mathbb{R}^n) \mid v = g \text{ on } \Omega^c\}.$$

It exists by standard techniques of calculus of variation.

We recall that thanks to the coarea formula (used similarly as in Proposition 1.9), the level sets $E_t^{(1)}$ minimize

$$E_t^{(1)} = \operatorname*{arg\,min}_{E\setminus\Omega = F_t\setminus\Omega} \operatorname{Per}_\varphi(E), \tag{1.26}$$

with $F_t := \{g > t\}$, where the exponent (1), as before, means that we consider the subset of points with density one:

$$F^{(1)} := \left\{ x \in \Omega \,\middle|\, \lim_{r\to 0} \frac{|B_r(x) \cap F|}{|B_r|} = 1 \right\}.$$

We recall that thanks to Proposition 1.11 $E_t^{(1)}$ are open subsets.

To show that (1.25) is in fact satisfied by u, we prove the following lemma (which is simply a restatement of [JMN13, Th. 1.1] which we give for convenience).

Lemma 1.4. *Let $\hat{x} \in \partial\Omega$ and let t and ε such that $g(\hat{x}) \leqslant t - \varepsilon$. Then, there exists $\rho > 0$ such that*

$$E_t^{(1)} \cap B_\rho(\hat{x}) = \emptyset.$$

The same result holds for $g(\hat{x}) \geqslant t + \varepsilon$.

Proof. Let us proceed by contradiction: we assume that there exists a sequence $x_n \in E_t^{(1)}$ such that $x_n \to \hat{x}$. Since g is continuous and $u = g \in \Omega^c$, then $x \in E := E_{t-\varepsilon/2}^{(1)}$ and $u < t - \varepsilon/2$ on $B_\rho(\hat{x}) \setminus \Omega$.

Since E cannot intersect $B_\rho(\hat{x}) \setminus \Omega$, we must have

$$B_\rho(\hat{x}) \cap \partial E \subset \overline{\Omega}.$$

But this cannot happen (see [JMN13, Lemma 3.3]). If V minimizes Per_φ in

$$\{W \subset \Omega \,|\, W \setminus B_\rho(\hat{x}) = \Omega \setminus B_\rho(\hat{x})\},$$

then the set $V \cup (E \cap B_\rho(\hat{x}))$ is also a minimizer whose boundary contains $\hat{x}$, contradicting Definition 1.7. □

Now, let u be a minimizer of (1.24). We prove that it is continuous. We will show that its level sets E_t and E_s, for $s < t$, satisfy $E_t^{(1)} \Subset E_s^{(1)}$.

We begin by noting that these two sets cannot touch near $\partial\Omega$.

Lemma 1.5. *Let $s < t$. There exists $\delta > 0$ and $\varepsilon > 0$ such that for every $x \in \Omega \cap \partial E_s^{(1)}$ with $d(x, \partial\Omega) \leqslant \delta$ and $y \in E_t^{(1)} \cap \Omega$, then $d(x,y) \geqslant \varepsilon$.*

This is straightforward using Lemma 1.4, with $\varepsilon = \frac{t-s}{2}$. The compactness of $\partial\Omega$ provides the expected δ.

Before proving Theorem 1.10, we state a very standard but useful

Lemma 1.6. *Let E and $\tilde{E}$ be two minimizers of* (1.26) *with F_t replaced respectively by F and $\tilde{F}$ and assume that $(E \cup \tilde{E}) \setminus E$ is a compact subset of Ω. Then, $E \cup \tilde{E}$ and $E \cap \tilde{E}$ are minimizers of* (1.26) *with F_t replaced respectively with F and $\tilde{F}$.*

Proof. The proof is also very standard but we give it for completeness. We notice that $(E \cup \tilde{E}) \setminus \Omega = E \setminus \Omega = F \setminus \Omega$ so $E \cup \tilde{E}$ is an admissible perturbation for E. One therefore can write

$$\mathrm{Per}_\varphi(E \cup \tilde{E}) \geqslant \mathrm{Per}_\varphi E.$$

Similarly $E \cap \tilde{E}$ is an admissible perturbation for $\tilde{E}$ and we can write,

$$\mathrm{Per}_\varphi(E \cap \tilde{E}) \geqslant \mathrm{Per}_\varphi \tilde{E}.$$

By summing the two inequalities and recalling (1.15), we must have equality in the inequalities. That is the claim. □

Proof of Theorem 1.10. We proceed by contradiction. Let us assume that there exists $x_0 \in \partial E_s^{(1)} \cap \partial E_t^{(1)}$ and let $r_0 = \frac{min(\delta,\varepsilon)}{10}$, where δ and ε are the constants provided by Lemma 1.5. Thanks to this lemma, $d(x_0, \partial\Omega) \geqslant \delta$.

Recalling that $\partial E_s^{(1)}$ and $\partial E_t^{(1)}$ are regular up to a compact set of dimension at most $n-3$ we can choose $\alpha \in \partial E_s^{(1)}$ and $\beta \in \partial E_t^{(1)}$ two regular points such that

$$|\alpha - x_0| \leqslant r_0 \quad \text{and} \quad |\beta - x_0| \leqslant r_0.$$

If $\nu = \alpha - \beta$, note that $|\nu| \leqslant \frac{1}{2}\min(\delta,\varepsilon)$ thus $E_s^{(1)}$ and $E_t^{(1)} + \nu$ do not touch near the boundary $\partial\Omega$.

The regular set $\mathrm{reg}(\partial E_s^{(1)} \cap B_{r_0/2}(x_0))$ **is a set of pieces of parallel hyperplanes.** The point α is regular means that one can find a direction n such that both $\partial E_s^{(1)}$ and $\partial \widetilde{E_t^{(1)}} := \partial E_t^{(1)} + \tilde{\nu}$ are (smooth) graphs around α. Since $\widetilde{E_t^{(1)}} \cap E_s^{(1)}$ and $\widetilde{E_t^{(1)}} \cup E_s^{(1)}$ are also minimizers (thanks to Lemma 1.6) and are both graphs around α, we have two functions $w_1 \leqslant w_2$ such that $w_1(\alpha') = w_2(\alpha')$ and which satisfy the zero φ-mean curvature equation for graphs

$$\mathrm{div}' \left(\nabla' \varphi(\nabla' w_i, -1)\right) = 0.$$

By comparison principle for graphs ([GO05], the one used in Section 6), they must coincide locally.

Notice that this coincidence is true for every pair $\alpha, \beta \in B_{r_0/2}(x_0)$ with $\alpha \in \partial E_s^{(1)}$ and $\beta \in \partial E_t^{(1)} + \nu$. Leaving β and moving only α, this proves that every regular point α of $\partial E_s^{(1)} \cap B_{r_0/2}(x_0)$ has a neighborhood (in $\partial E_s^{(1)}$) which coincides with a neighborhood of β in $\partial E_t^{(1)} + \nu$. As a result, every regular point of $\partial E_s^{(1)} \cap B_{r_0/2}(x_0)$ has the same normal (let us call it ω). Since in addition, the set of regular points is an open subset of $\partial E_s^{(1)}$, the connected components of $\mathrm{reg}(\partial E_s^{(1)})$ are affine hyperplanes parallel to $\omega^\perp$, oriented either by ω or by $-\omega$.

Of course, $\mathrm{reg}((\partial E_t^{(1)}) \cap B_{r_0/2}(x_0))$ satisfies the same property.

These pieces of hyperplans which cross $B_{r_0/4}(x_0)$ **fill** $B_{r_0/4}(x_0)$**.** Indeed, Let $x \in \mathrm{reg}\, \partial E_s^{(1)} \cap B_{r_0/4}(x_0)$. Then, there is a ball $\hat{B}$ (of radius $\hat{r}$) around x such that $\partial E_s^{(1)} \cap \hat{B}$ is exactly a diameter of $\hat{B}$. Let us assume that the normal of $\partial E_s^{(1)}$ is ω in $\hat{B}$. Then, let us consider the cylinder $\hat{C}$ generated by $\hat{B}$ and a vector $e \perp \omega$ in the ball $B_{r_0/2}(x_0)$. One can write, for every R such that $z + Re \in B_{r_0/4}(x_0)$

$$\int_{\substack{z \in e^\perp \\ |z| \leqslant \hat{r}}} \int_{\tau=0}^{Re} |D(\chi_{E_s^{(1)}}(z + \tau e))| = \int_{\hat{C} \cap \partial E_s^{(1)}} |\nu_{E_s^{(1)}} \cdot e| dH^{n-1} = 0$$

because $e \perp \omega$. Then, for almost every $z \in e^\perp$ with $|z| \leqslant \hat{r}$, we have $\tau \mapsto \chi_{E_s^{(1)}}(z + \tau e)$ is constant. That means that if $z + \tau e$ belongs to $E_s^{(1)}$ for some τ, that is true for every τ (and similarly for $\notin E_s^{(1)}$). Finally, the piece of hyperplane of $\mathrm{reg}\, E_s^{(1)}$ which is a diameter of $\hat{B}$ exists in the whole cylinder $\hat{C}$, and since e is arbitrary in $\omega^\perp$, in the whole ball $B_{r_0/4}(x_0)$ (we have to stay sufficiently far from $\partial B_{r_0/2}(x_0)$ in order to keep the whole cylinder inside $B_{r_0/2}(x_0)$).

The point x_0 **is in fact regular** Thanks to previous paragraphs, $\mathrm{reg}\, E_s^{(1)} \cap B_{r_0/4}(x_0)$ is a (finite, for measurability reasons) set of hyperplanes.

In addition, since $x_0 \in \partial E_s^{(1)} \cap (\partial E_t^{(1)} + \nu)$, we have a sequence of points in $\mathrm{reg}\, E_s^{(1)}$ (which therefore belong to hyperplanes) which converge to x_0. Using the finiteness of the set of hyperplanes, x_0 must be in one of them. So, x_0 is in fact a regular point of $E_s^{(1)}$ (the same holds for $E_t^{(1)} + \nu$), and $E_s^{(1)}$ and $E_t^{(1)} + \nu$ coicinde around x_0. That is exactly saying that $\partial E_s^{(1)} \cap (\partial E_t^{(1)} + \nu)$ is open in $\partial E_s^{(1)}$. It is closed by definition. To reach a contradiction, we now need to show that every connected component of $\partial E_s^{(1)}$ has to reach the boundary $\partial\Omega$. $\square$

Proposition 1.14. *There is no connected component of $\partial E_s^{(1)}$ which is compact in Ω.*

Proof. Let us proceed by contradiction and call Γ a compact connected component of $\partial E_s^{(1)}$. We denote by δ the distance between Γ and $\partial\Omega$. One can find a continuous function $f : \partial E_s^{(1)} \to \{0,1\}$ which is 0 on Γ and 1 on $E_s^{(1)} \backslash \{\mathrm{dist}(x,\Gamma) < \delta/2\}$. Since $E_s^{(1)}$ is compact, f is uniformly continuous. Let call ω its modulus of continuity and extend f to the whole Ω by

$$f(x) = \sup_{y \in E_s^{(1)}} f(y) + \omega(x-y).$$

In addition, we may assume that $f \geqslant 1$ on $\partial\Omega$ (eventually replacing f by $\max(f, 1 - \mathrm{dist}(x,\partial\Omega)/\delta)$). Note that $f(x) = \alpha \in (0,1)$ implies that x remains far from $\partial E_s^{(1)}$.

Now, let us introduce C as the connected component of the open subset $\{f < \frac{1}{4}\}$ which contains Γ and set

$$a := \min_{x \in \partial C} u^-(x) \quad \text{and} \quad b := \max_{x \in \partial C} u^+(x).$$

If $a > s$, then we define v such that $v = u$ everywhere but in $C \cap \left\{u \leqslant \frac{a+s}{2}\right\}$ where we set $v = \frac{a+s}{2}$.
Then, we notice that v differs from u only in a neighborhood of Γ and

$$\int_C \varphi\left(\nabla\left(u \vee \frac{a+s}{2}\right)\right) + \int_C \varphi\left(\nabla\left(u \wedge \frac{a+s}{2}\right)\right) \leqslant \int_C \varphi(\nabla u) + \underbrace{\int_C \nabla\left(\frac{a+s}{2}\right)}_{=0}.$$

Then, v is also a minimizer with

$$(\partial\{v > s\}) \cap C = \emptyset,$$

which implies

$$\mathrm{Per}(\{v > s\}) < \mathrm{Per}(\{u > s\}),$$

which cannot happen.

Similarly, if $b \leqslant s$, then we introduce $v = u \wedge s$ in C, $v = u$ in C^c and we also reach a contradiction.

Finally, we cannot have either $\partial C \subset \{u > s\}$ or $\partial C \subset \{u \leqslant s\}$. But on the other hand, we have $\partial C \subset \{f = \frac{1}{4}\}$ which means that ∂C cannot be too close to $\partial E_s^{(1)}$: this is a contradiction.

$\square$

Remark. All the proof above can be reproduce with $E_t^{(1)} = \{u > t\}$ and $E_s^{(1)} = \{v > s\}$, if u and v are two minimizers: that shows $u = v$ a.e.

8 Local continuity

In this section, we get back to the isotropic case (1.11). We want to prove the

Theorem 1.11. *Let $g : \Omega \to \mathbb{R}$ be continuous and bounded and let u be a minimizer of*

$$\int_\Omega |Du| + \int_\Omega \frac{(u-g)^2}{2}. \tag{1.27}$$

Then, u is continuous.

Note that this theorem is local and therefore extends [CCN11, Th. 2] (but for continuous functions only).

We will use the level sets. More precisely, let $E_s \subset E_t$ two level sets of u (with $s > t$). We know that they minimize respectively (with respect to compact perturbations in Ω)

$$\mathrm{Per}(E, \Omega) + \int_{E\cap\Omega} s - g$$

and

$$\mathrm{Per}(E, \Omega) + \int_{E\cap\Omega} t - g.$$

The strategy is the following. We know that two minimal surfaces satisfy a strict comparison principle [Sim87], and we can extend this proof to constant mean curvature surfaces. As a result, we first show that we can create two different constant mean curvature which stands between E_s and E_t. Then, we show that these surfaces do not touch. So, neither can E_s and E_t. As before, we replace E_s and E_t by the set of points of density one.

8.1 Back to constant mean curvature

We assume (and we hope that we can get a contradiction) that there is $x_0 \in \partial E_s \cap \partial E_t$. Note first that since E_s and E_t have mean curvature which are different, they cannot coincide on a neighborhood of x_0. By continuity of g, we can find $\rho > 0$ such that on $B_\rho(x_0)$, we have $g(x_0) - \alpha < g(x) < g(x_0) + \alpha$ with $\alpha := \frac{s-t}{100}$. So, let $a = s - g(x_0) - \alpha$. Then,

$$\forall x, y \in B_\rho(x_0), \quad t - g(x) \leqslant a \leqslant s - g(y).$$

Now, we introduce E with finite perimeter in Ω as the minimizer of

$$E = \underset{G \Delta E_s \subset B_\rho(x_0)}{\arg\min} \ \mathrm{Per}(G, \Omega) + a|G \cap B_\rho(x_0)|$$

and similarly, F with finite perimeter in Ω and minimizing

$$F = \underset{G \Delta E_t \subset B_\rho(x_0)}{\arg\min} \ \mathrm{Per}(G, \Omega) + a|G \cap B_\rho(x_0)|.$$

Note that E and F have variational constant mean curvature a.

Using the standard (weak) comparison principle, we have $E_s \subset E \subset F \subset E_t$. In addition, since E_s and E_t cannot coincide, E and F cannot either. On the other hand, we must have $x_0 \in \partial E \cap \partial F$.

To show that ∂E_s and ∂E_t cannot touch, it is enough to prove that $\partial E \cap \partial F = \emptyset$. That is to prove the

Theorem 1.12. *Let $a \in \mathbb{R}$ and $E \subset F$ such that E and F both minimize (with respect to compact perturbations) in an open subset O,*

$$\mathrm{Per}(E; O) + a|E \cap O|. \tag{1.28}$$

Then, either $E = F$ or $\partial E \cap \partial F = \emptyset$.

In what follows, we take $O = \Omega$ (we can reduce the latter since we only want a local result).

8.2 Properties of minimizers

Before proving Theorem 1.12, we first recall results on minimizers of (1.28) that will be crucial in the proof. We begin by the usual monotonicity formula (see [Mas75])

Proposition 1.15 (Monotonicity formula)**.** *Let E be a minimizer of* (1.28)*. Then, for every $s < r$ and every $x \in \partial E$, we have*

$$\frac{\mathrm{Per}(E, B_r(x))}{r^{n-1}} - \frac{\mathrm{Per}(E, B_s(x))}{s^{n-1}} \geqslant -(n-1)\omega_n|a|(r-s).$$

Remark. That formula explains why we restricted ourself to the isotropic case. We do not know if this monotonicity holds in the anisotropic framework.

Corollary 1.3. *For all $x \in \partial E$ and* $\mathrm{dist}(x, \partial\Omega) > r > 0$ *we have*

$$r^{1-n} \int_{B_r(x)} |D1_E| \geqslant \omega_{n-1} - (n-1)\omega_n|a|r. \tag{1.29}$$

Lemma 1.7. *[MP75, Th. 2] Let (E_λ) be a family of minimizers of* (1.28) *with a_λ ($\in \mathbb{R}$) instead of a, and let E minimize* (1.28)*. We assume that $E_\lambda \to E$ in L^1_{loc} and that $a_\lambda \to a$. Then, for every bounded set D (with Lipschitz boundary) such that*

$$\int_{\partial D} |D1_E| = 0,$$

we have

$$\lim_\lambda \int_{\overline{D}} |D1_{E_\lambda}| = \int_{\overline{D}} |D1_E|.$$

Proof. We take $B \supset D$ with Lipschitz boundary and which satisfies

$$\int_{\partial B} |D1_E| = 0 \tag{1.30}$$

and

$$\lim \int_{\partial B} |1_{E_\lambda} - 1_E| d\mathcal{H}^{n-1} = 0. \tag{1.31}$$

We just compare E_λ to $M_\lambda := (E \cap B) \cup (E_\lambda \cap (\Omega \setminus B))$. We can write

$$\mathrm{Per}(E_\lambda;\Omega) + a_\lambda |E_\lambda \cap \Omega| \leqslant \mathrm{Per}((E \cap B) \cup (E_\lambda \cap (\Omega \setminus B);\Omega) + a_\lambda |(E \cap B) \cup (E_\lambda \cap (\Omega \setminus B))|.$$

Now, just note that E_λ and $(E \cap B) \cup (E_\lambda \cap (\Omega \setminus B))$ coincide outside B and that (using Proposition 1.5 and trace theorem [AFP00, Th. 3.87])

$$\mathrm{Per}((E \cap B) \cup (E_\lambda \cap (\Omega \setminus B));\Omega) = \mathrm{Per}(E;B) + \mathrm{Per}(E_\lambda;\Omega \setminus B) + \int_{\partial B} |1_E - 1_{E_\lambda}| d\mathcal{H}^{n-1}.$$

Putting together these two facts, we obtain

$$\int_B |D1_{E_\lambda}| + \int_{B \cap E_\lambda} a_\lambda \leqslant \int_B |D1_E| + \int_{\partial B} |1_{E_\lambda} - 1_E| d\mathcal{H}^{n-1} + \int_{B \cap E} a_\lambda$$

which yields, taking the limsup of each member and using the L^1-convergence and (1.31),

$$\limsup_{\lambda \to \infty} \int_B |D1_{E_\lambda}| \leqslant \int_B |D1_E|.$$

Now, notice that

$$\begin{aligned} \limsup \int_{\overline{D}} |D1_{E_\lambda}| &\leqslant \int_{\overline{B}} |D1_E| - \liminf \int_{B \setminus \overline{D}} |D1_{E_\lambda}| \\ &\leqslant \int_{\overline{B}} |D1_E| - \int_{B \setminus \overline{D}} |D1_E| \leqslant \int_{\overline{D}} |D1_E|. \end{aligned}$$

On the other hand, the semi-continuity of the total variation (but for the open set $\mathring{B}$) gives the reverse inequality and provides (since $\int_{\partial B} |D1_E| = 0$),

$$\lim_{\lambda \to \infty} \int_B |D1_{E_\lambda}| = \int_B |D1_E|.$$

□

The following theorem, usually called *improvement of flatness*, is the key result in the regularity proof. It can be found in [Mas75].

Theorem 1.13 (De Giorgi)**.** *Let E minimize (1.28) and $\alpha \in (0,1)$. Then, there exits a constant $\sigma(n, \alpha, |a|)$ such that for all $\eta \leqslant \sigma$ and $r \leqslant \eta^2$, if E satisfies*

$$\int_{\overline{B}_r(x)} |D1_E| - \left| \int_{\overline{B}_r(x)} D1_E \right| \leqslant \eta r^{n-1},$$

then, we have

$$\int_{\overline{B}_{\alpha r}(x)} |D1_E| - \left| \int_{\overline{B}_{\alpha r}(x)} D1_E \right| \leqslant \alpha^{1/2} \eta (\alpha r)^{n-1}.$$

8.3 blowups

In this subsection, we analize the convergence of blowups to a minimal cone. In particular, we prove the

Theorem 1.14. *Let E minimize* (1.28)*. Then the sets*

$$E_\lambda := \{x_0 + \frac{1}{\lambda}(x - x_0) \,|\, x \in E\}$$

converge, in Hausdorff sense and up to a subsequence $\lambda_n \to 0$, to some minimizing cone C. In addition, for all K compact of $\operatorname{reg} C$*, there exists a neighborhood V of K such that $E_\lambda \cap V$ converges to $C \cap V$ in $\mathcal{C}^2(V)$.*

We first prove the Hausdorff convergence.

Proposition 1.16. *Let $x_0 \in \partial E$. The sets E_λ converge to a minimizing cone C in Hausdorff distance.*

First, we prove the convergence of E_λ to C in L^1_{loc}. This proof is classical (see for example [Giu84]). Let $r > 0$, then $\operatorname{Per}(E_\lambda, r) = \lambda^{n-1} \operatorname{Per}(E, r/\lambda)$.

Since E_λ is minimizer of

$$\operatorname{Per}(F, r) + \int_{B_r \cap F} \lambda^n a,$$

we have, comparing to a ball B_r,

$$P(E_\lambda, r) + \lambda^n \int_{E_\lambda \cap B_r} a \leqslant P(B_r, r) + \lambda^n \int_{B_r} a \leqslant n\omega_n r^{n-1} + \lambda^n |a| \omega_n r^n \tag{1.32}$$

which shows that $P(E_\lambda, r)$ is bounded above.

The usual compactness ([Giu84, Th. 1.19]) produces a subsequence $\lambda_n \to 0$ and a Caccioppoli set C such that

$$E_{\lambda_n} \to C \quad \text{in } L^1_{loc}.$$

To see that C is minimal, consider a perturbation M of C in a compact subset K (we assume without loss of generality that K satisfies (1.30) and (1.31)). Then, comparing E_λ to $M_\lambda := (M \cap K) \cup (E_\lambda \cap (\Omega \setminus K))$, one ca write (as in the proof of Lemma 1.7)

$$\begin{aligned} \int_K |D1_{E_\lambda}| + \int_{E_\lambda \cap K} a_\lambda & \\ &\leqslant \int_K |D1_{M_\lambda}| + \int_{K \cap M} a_\lambda \\ &\leqslant \int_{\mathring{K}} |D1_{M_\lambda}| + \int_{\partial K} |1_{E_\lambda} - 1_E| + \int_{\mathring{K} \cap M} a_\lambda. \end{aligned}$$

Passing to the limit as in the proof of Lemma 1.7, we get

$$\operatorname{Per}(E; \mathring{K}) \leqslant \operatorname{Per}(M; \mathring{K}).$$

To see that C must be a cone, we notice that it satisfies equality in the monotonicity formula (Proposition 1.15). See [Giu84] for details.

Let us now show a local Hausdorff convergence. If it were false, there would exist $x_n \in E_{\lambda_n}$ and $\delta > 0$ (we choose δ sufficiently small such that Proposition 1.10 applies) such that $\operatorname{dist}(x_n, C) \geqslant \delta$. As a result,

$$E_{\lambda_n} \cap B_{\frac{1}{2}\cdot\delta}(x_n) \cap C = \emptyset.$$

Using Proposition 1.10, we know that $|E_{\lambda_n} \cap B_{\frac{1}{2}\cdot\delta}(x_n)| \geqslant \frac{C\delta^{n-1}}{8n}$, which shows that

$$|E_{\lambda_n} \Delta C| \geqslant \frac{C^{\mathrm{st}}\delta^{n-1}}{8n}.$$

That contradicts the L^1 convergence.

Now, let us inverstigate the regularity of a minimizing set which is close to $\operatorname{reg} C$.

Proposition 1.17. *Let K be a compact subset of $\operatorname{reg} C$. Then, for every $x_0 \in K$, there exists a neighborhood W of x_0, whose size depends only on $\Lambda = |a|$, the dimension and K, such that $E_\lambda \cap W$ is a $\mathcal{C}^2$ surface.*

Proof. This is proven in [MP75, Th. 3]. Since the whole proof uses several papers ([Mas74, MP75, Mas75]) and does not provide information on the uniformity of the convergence, we reproduce it here. By compactness, it is enough to show that for every x_0, there exists a neighborhood W of x_0 such that every $x_\lambda \in \partial E_\lambda \cap W$ belongs to $\operatorname{reg} E_\lambda$.

We recall that the *reduced boundary* of any E, denoted by $\partial^* E$ is the set of the points $x \in \partial E$ such that

$$\lim_{\rho \to 0^+} \frac{\int_{B_\rho(x)} D1_E}{\int_{B_\rho(x)} |D1_E|} = \nu(x) \quad \text{and} \quad |\nu(x)| = 1. \tag{1.33}$$

Since $x_0 \in \partial^* C$ and using (1.32) for C (with $a = 0$), we conclude that

$$\lim_{r \to 0} r^{1-n} \left(\int_{B_r(x_0)} |D1_C| - \left| \int_{B_r(x_0)} D1_C \right| \right) = 0.$$

Choosing r such that $B_r \subset K$ and

$$\int_{\partial B_r} |D1_C| = 0,$$

Lemma 1.7 shows that

$$\lim_{\lambda \to \infty} \int_{\overline{B_r}} |D1_{E_\lambda}| = \int_{\overline{B_r}} |D1_C|.$$

In addition, using the relation (trace theorem)

$$\int_{B_r(x)} D1_{E_\lambda} = \int_{\partial B_r(x)} 1_{E_\lambda}(y) \frac{y - x}{|y - x|} dH^{n-1}(y),$$

we obtain, for almost every r,

$$\lim_{\lambda \to \infty} \left| \int_{\overline{B_r}(x_0)} D1_{E_\lambda} \right| = \left| \int_{\overline{B_r}(x_0)} D1_C \right|.$$

As a result, for every $\varepsilon > 0$, one can choose r_0 and λ_0 such that for all $r \leqslant r_0$ and $\lambda \leqslant \lambda_0$,

$$r^{1-n}\left(\int_{\overline{B}_r(x_0)} |D1_{E_\lambda}| - \left|\int_{\overline{B}_r(x_0)} D1_{E_\lambda}\right|\right) \leqslant \varepsilon.$$

In particular, with $\varepsilon = \frac{\sigma}{2^{n-1}}$, where σ is the constant in Theorem 1.13 (corresponding to some $\alpha < 1$ that is considered fixed in what follows), we fix $\hat{r} \leqslant r_0$. If $r_\lambda = |x_0 - x_\lambda|$ for any sequence $x_\lambda \to x_0$, and choosing λ_0 such that for $\lambda \leqslant \lambda_0$, $r_\lambda \leqslant \frac{\hat{r}}{2}$, we have,

$$\left(\int_{\overline{B}_{\hat{r}}(x_0)} |D1_{E_\lambda}| - \left|\int_{\overline{B}_{\hat{r}}(x_0)} D1_{E_\lambda}\right|\right) \leqslant \sigma(\hat{r} - r_\lambda)^{n-1}.$$

Then, we recall that $B_{\hat{r}-r_\lambda}(x_\lambda) \subset B_{\hat{r}}(x_0)$ and notice that the integral on the left is monotone with respect to the inclusion (because for every $A \subset B_{\hat{r}}(x_0)$,

$$\int_{\overline{B}_{\hat{r}}(x_0)} |D1_{E_\lambda}| - \left|\int_{\overline{B}_{\hat{r}}(x_0)} D1_{E_\lambda}\right| \geqslant 0 \quad),$$

$$\int_{B_{\hat{r}-r_\lambda}(x_\lambda)} |D1_{E_\lambda}| - \left|\int_{\overline{B}_{\hat{r}-r_\lambda}(x_\lambda)} D1_{E_\lambda}\right| \leqslant \sigma(\hat{r} - r_\lambda)^{n-1}.$$

Let us show now that the last inequality implies that $x_\lambda \in \partial^* E_\lambda$, that means there exists

$$\nu(x) := \lim_{r\to 0} \frac{\int_{B_r(x_\lambda)} D1_{E_\lambda}}{\int_{B_r(x_\lambda)} |D1_{E_\lambda}|}, \quad \text{with } |\nu(x)| = 1. \tag{1.34}$$

This is [Mas75, Lemma 2.2]. We introduce the notation

$$\nu_r = \frac{\int_{B_r(x_\lambda)} D1_{E_\lambda}}{\int_{B_r(x_\lambda)} |D1_{E_\lambda}|}.$$

Lemma 1.8. *For every $s < r < r_0$, we have*

$$|\nu_r - \nu_s| \leqslant 2\left(\frac{\int_{B_r(x_\lambda)} |D1_{E_\lambda}| - \left|\int_{B_r(x_\lambda)} D1_{E_\lambda}\right|}{\int_{B_s(x_\lambda)} |D1_{E_\lambda}|}\right)^{1/2}. \tag{1.35}$$

Proof. Let u and v be smaller than 1, we have $|u - v|^2 \leqslant 2 - 2uv$. As a result,

$$|\nu_r - \nu_s| \leqslant \sqrt{2}\left(1 - \frac{\int_{B_r(x_\lambda)} D1_{E_\lambda}}{\int_{B_r(x_\lambda)} |D1_{E_\lambda}|} \cdot \frac{\int_{B_s(x_\lambda)} D1_{E_\lambda}}{\int_{B_s(x_\lambda)} |D1_{E_\lambda}|}\right)^{1/2},$$

which implies

$$\begin{aligned}
|\nu_r - \nu_s| &\leqslant \left(\frac{2}{\int_{B_s(x_\lambda)} |D1_{E_\lambda}|}\left(\int_{B_s(x_\lambda)} |D1_{E_\lambda}| - \frac{\int_{B_r(x_\lambda)} D1_{E_\lambda} \cdot \int_{B_s(x_\lambda)} D1_{E_\lambda}}{\int_{B_r(x_\lambda)} |D1_{E_\lambda}|}\right)\right)^{1/2} \\
&\leqslant \left(\frac{2}{\int_{B_s(x_\lambda)} |D1_{E_\lambda}|}\left(\int_{B_r(x_\lambda)} |D1_{E_\lambda}| - \frac{\int_{B_r(x_\lambda)} D1_{E_\lambda} \cdot \int_{B_r(x_\lambda)} D1_{E_\lambda}}{\int_{B_r(x_\lambda)} |D1_{E_\lambda}|}\right)\right)^{1/2}.
\end{aligned}$$

The last inequality is obtained using that for all $|\eta| \leqslant 1$ and $s \leqslant r$,

$$\int_{B_s} |D1_{E_\lambda}| - \eta \cdot \int_{B_s} D1_{E_\lambda} \leqslant \int_{B_r} |D1_{E_\lambda}| - \eta \cdot \int_{B_r} D1_{E_\lambda}.$$

Indeed, for every $A \subset \mathbb{R}^n$ bounded, we have

$$\int_A |D1_{E_\lambda}| - \eta \cdot \int_{B_s} D1_{E_\lambda} \geqslant 0.$$

Finally, we get

$$|\nu_r - \nu_s| \leqslant 2 \left(\frac{\int_{B_r(x_\lambda)} |D1_{E_\lambda}|}{\int_{B_s(x_\lambda)} |D1_{E_\lambda}|} \cdot \frac{\int_{B_r(x_\lambda)} |D1_{E_\lambda}| - \left| \int_{B_r(x_\lambda)} D1_{E_\lambda} \right|}{\int_{B_r(x_\lambda)} |D1_{E_\lambda}|} \right)^{1/2}$$

which yields (1.35). □

We will prove that $(\nu_{\alpha^k r})_{k\in\mathbb{N}}$ is a Cauchy sequence. Using (1.35), we have

$$|\nu_{\alpha^{k+m} r} - \nu_{\alpha^k r}| \leqslant 2 \sum_{j=0}^{m-1} \left| \frac{\int_{B_{\alpha^{k+j} r}(x_\lambda)} |D1_{E_\lambda}| - \int_{B_{\alpha^{k+j} r}(x_\lambda)} D1_{E_\lambda}}{\int_{B_{r\alpha^{k+j+1}}(x_\lambda)} |D1_{E_\lambda}|} \right|^{1/2}.$$

Thanks to Corollary 1.3, for $r < \frac{\omega_{n-1}}{2(n-1)\omega_n |a|}$, we have

$$\int_{B_{\alpha^i r}(x_\lambda)} |D1_{E_\lambda}| \geqslant \frac{\omega_{n-1}}{2} r^{n-1} \alpha^{i(n-1)}. \tag{1.36}$$

Now, Theorem 1.13 implies that for $r \leqslant \hat{r} - r_\lambda$

$$\int_{\overline{B}_{\alpha^i r}(x)} |D1_{E_\lambda}| - \left| \int_{\overline{B}_{\alpha^i r}(x)} D1_{E_\lambda} \right| \leqslant \alpha^{i/2} \sigma (\alpha^i r)^{n-1}. \tag{1.37}$$

As a result,

$$\begin{aligned} |\nu_{\alpha^{k+m} r} - \nu_{\alpha^k r}| &\leqslant 2 \sum_{j=0}^{m-1} \left(\frac{\sigma \alpha^{(k+j)(n-\frac{1}{2})}}{\omega_{n-1}/2 \cdot \alpha^{(k+j+1)(n-1)}} \right)^{1/2} \\ &\leqslant 2 \left(\frac{2\sigma}{\omega_{n-1}} \right)^{\frac{1}{2}} \alpha^{k/4} \; \frac{1-\alpha^{m/4}}{1-\alpha^{k/4}} \\ &\leqslant 2 \left(\frac{2\sigma}{\omega_{n-1}} \right)^{\frac{1}{2}} \alpha^{k/4} \quad \frac{1}{1-\alpha^{1/4}}, \end{aligned}$$

which shows that $(\nu_{\alpha^k r})_{k\in\mathbb{N}}$ is a Cauchy sequence. Let $\nu(x)$ denote its limit.

Since every $|\nu_{\alpha^i r}(x)| = 1$, we have

$$|\nu(x)| = 1.$$

Then, let us prove that in fact,

$$\lim_{t\to 0} \nu_t(x) = \nu(x).$$

For every t sufficiently small, there exists $i \in \mathbb{N}$ such that

$$r\alpha^{i+1} \leqslant t \leqslant r\alpha^i.$$

Then,

$$|\nu_t(x) - \nu(x)| \leqslant |\nu_t(x) - \nu_{r\alpha^i}(x)| + |\nu_{r\alpha^i}(x) - \nu(x)|.$$

Using equation (1.35), one can write

$$\begin{aligned}
|\nu_t(x) - \nu_{r\alpha^i}(x)| &\leqslant 2\left(\frac{\int_{B_{r\alpha^i}(x_\lambda)} |D1_{E_\lambda}| - \int_{B_{r\alpha^i}(x_\lambda)} D1_{E_\lambda}}{\int_{B_t(x_\lambda)} |D1_{E_\lambda}|}\right)^{1/2} \\
&\leqslant 2\left(\frac{\int_{B_{r\alpha^i}(x_\lambda)} |D1_{E_\lambda}| - \int_{B_{r\alpha^i}(x_\lambda)} D1_{E_\lambda}}{\int_{B_{r\alpha^{i+1}}(x_\lambda)} |D1_{E_\lambda}|}\right)^{1/2} \\
&\leqslant 2\left(\frac{2\sigma}{\omega_{n-1}}\right)^{\frac{1}{2}} \alpha^{i/4}
\end{aligned}$$

using Equations (1.36) and (1.37). This is exactly saying that $x_\lambda \in \partial^* E_\lambda$, and so in $\operatorname{reg} E_\lambda$ (see [Giu84], Th. 4.11). □

Remark. Note that the size of V depends only on the choice of r_0 and ε, that means on the constant σ is Lemma 1.13 (so of the dimension and $|a|$) and of the convergence rate in Lemma 1.7.

We can now conclude the proof of Theorem 1.14. It is enough to notice that since the E_λ have a constant mean curvature, then $\operatorname{reg} E_\lambda$ is in fact analytic, as well as $\operatorname{reg} C$. So, the local Hausdorff convergence of $E_\lambda \to C$ directly provides the $\mathcal{C}^2$ convergence of $E_\lambda \cap V$ to $C \cap V$.

8.4 We can assume that E and F have the same tangent cone

We are now ready to prove the strict comparison principle for constant mean curvature surfaces E and F (Theorem 1.12). We proceed by contradiction and assume that there exists $x_0 \in \partial E \cap \partial F$. We prove that we can assume that E and F have the same tangent cone at x_0. To do so, we use the dimension reducing argument by Bombieri and Giusti [BG72]. Let $C_1 \subset C_2$ the tangent cones to E and F at x_0. Then, there must exist $y \neq 0$ in $\partial C_1 \cap \partial C_2$. Indeed, if not, we could consider a ball $B_r(x_0)$ and $C_1 \cap B_r(x_0)$ and $C_2 \cap B_r(x_0)$ would not touch near $\partial B_r(x_0)$ and would be both minimizing in $B_r(x_0)$ and contain x_0. We could then apply the proof of Theorem 1.10 with E_s and E_t replaced by C_1 and C_2 (which do not touch near the boundary of $B_r(x_0)$, which would provide a contradiction.
We then blow up C_1 and C_2 at y and get two tangent cones C_1^1 and C_2^1 which both contain the line $l = \mathbb{R}y$. Hence $D_1 = C_1^1 \cap (y + l^\perp)$ and $D_2 = C_2^1 \cap (y + l^\perp)$ are two $n-1$-dimensional minimizing cones which are either equal or distinct. If they are distinct, we can reproduce

the scheme for D_1 and D_2, obtaining two $(n-2)$-dimensional minimizing cones C_1^2 and C_2^2. Since there is no singular minimizing cone with dimension smaller than 7, this iteration stops and gives two equal minimizing cones $C_1^k = C_2^k$.

As a result, if we prove Theorem 1.12 with $C_1 = C_2$, we can apply it to C_1^{k-1} and C_2^{k-1} which have, by definition, the same tangent cone at some point. This gives $C_1^{k-1} = C_2^{k-1}$. By (finite) induction, we will obtain $E = F$.

In what follows, we suppose that E and F have the same tangent cone C at x_0. In addition, for simplicity, we take $x_0 = 0$.

8.5 Proof

Note that in what follows, to have the same notations as in [Sim87], we use $T_1 = \partial E$ and $T_2 = \partial F$. We also assume that $x_0 = 0$. The proof is the same as in [Sim87]. Nonetheless, the different blowups have no zero mean curvature anymore and we have to check that their convergence is still $\mathcal{C}^2$ near regular points of the limit. We begin by seeing that [Sim87, Lemma 1] still holds with minimizers of (1.28).

Lemma 1.9. *Let E minimize* (1.28)*, $x_0 = 0 \in \partial E$ and ν denote the unit normal to E. We define Ω_θ the set of points $x \in \operatorname{reg} T_1$ which satisfy*

i) $d(x, \operatorname{sing} E) > \theta|x|$,

ii)

$$\sup \left\{ \frac{|\nu(x) - \nu(y)|}{|x-y|} \,\middle|\, y \in \operatorname{reg} E,\ 0 < |y - x| < \theta|x| \right\} < \frac{1}{\theta|x|}.$$

Then, there exist $\rho_0(x_0, E) > 0$ and $\theta_0(x_0; E) > 0$ such that

$$\forall\, 0 < \rho \leqslant \rho_0,\ \forall\, 0 < \theta \leqslant \theta_0, \quad \Omega_\theta \cap \partial B_\rho(x_0) \neq \emptyset.$$

Proof. The proof is exactly the same as in [Sim87]. We reproduce it here and give some extra details. We proceed by contradiction. If the conclusion of the lemma were false, we could find two sequences $\rho_j \to 0$, $\theta_j \to 0$ such that

$$\left\{ x \in \operatorname{reg} E \,\middle|\, |x| = \rho_j,\ \operatorname{dist}(x, \operatorname{sing} E) > \rho_j\theta_j,\ \sup_{\substack{y \in \operatorname{reg} E \\ |x-y| \leqslant \rho_j\theta_j}} \left[\frac{|\nu(x) - \nu(y)|}{|x-y|} \right] < \frac{1}{\rho_j\theta_j} \right\} = \emptyset. \tag{1.38}$$

Let $E_j = \rho_j^{-1}E$. Thanks to Theorem 1.14, there exists a cone C, a subsequence (which we still denote by j) such that $E_j \to C$ in the Hausdorff sense, and $\mathcal{C}^2$ sense on the neighborhoods of points in $\operatorname{reg} C$. If $y \in \operatorname{reg} C \cap \partial B_1$ (such a point exists because $\mathcal{H}^{n-7}(\operatorname{sing} C) = 0$), there exists $\theta > 0$ and a sequence $y_j \to y$ with $y_j \in B_\theta(y) \cap \partial B_1 \cap \operatorname{reg} E_j$ (we can take y_j on the sphere again), and such that $B_\theta(y) \cap\ E_j \subset \operatorname{reg} E_j$ (thanks to Theorem 1.14). In addition, by the $\mathcal{C}^2$ convergence (and eventually reducing θ again), one can have

$$\forall x, z \in B_\theta(y) \cap \operatorname{reg} E_j, \quad \frac{|\nu(x) - \nu(z)|}{|x-z|} \leqslant \frac{1}{\theta}.$$

Going back to E, we have

$$\forall x, z \in B_{\rho_j\theta}(\rho_j y) \cap \operatorname{reg} E, \qquad \frac{|\nu(x) - \nu(z)|}{|x - z|} \leqslant \frac{1}{\rho_j\theta}. \tag{1.39}$$

Finally, notice that $\rho_j y_j \in \partial B_{\rho_j} \cap \operatorname{reg} E$. In addition, $\operatorname{dist}(\rho_j y_j, \operatorname{sing} E) = \rho_j \operatorname{dist}(y_j, \operatorname{sing} E_j) \geqslant \rho_j\theta$ and, using (1.39) with $z = \rho_j y_j \in B_{\rho_j\theta}(\rho_j y)$, this contradicts (1.38) for j large enough. $\square$

Let ρ_0, θ_0 and $\Omega_\theta \subset \operatorname{reg} T_1$ as in Lemma 1.9 and define, for all $x \in T_1$, $h(x) = \operatorname{dist}(x, \operatorname{spt} T_2)$. Since T_1 and T_2 have the same tangent cones at x_0, one has, for every $\theta \leqslant \theta_0$,

$$\lim_{r \to 0} r^{-1} \sup_{|x|=r, x \in \Omega_\theta} h(x) = 0. \tag{1.40}$$

Indeed, we have in fact

$$\frac{1}{r} \sup_{|x|=r, x\in T_1} d(x, C) = \sup_{|y|=1, y \in r^{-1}T_1} d(y, C) \to 0$$

because of Hausdorff convergence of $r^{-1}T_1$ to C. As the same holds for $x \in T_2$, that gives

$$\frac{1}{r} \sup_{|x|=r, x\in T_2} d(x, C) = \sup_{|y|=1, y \in r^{-1}T_2} d(y, C) \to 0$$

which implies (1.40).
We select $\rho_j \to 0$ such that for all $\rho \leqslant \rho_j$,

$$\rho_j^{-1} \sup_{\substack{x \in \Omega_{\theta_0} \\ |x| = \rho_j}} h(x) \geqslant \frac{1}{2}\rho^{-1} \sup_{\substack{x \in \Omega_{\theta_0} \\ |x| = \rho}} h(x)$$

we have in particular for $\theta < 1$,

$$\sup_{\substack{x \in \Omega_{\theta_0} \\ |x| = \theta\rho_j}} h(x) \leqslant 2\theta \sup_{\substack{x \in \Omega_{\theta_0} \\ |x| = \rho_j}} h(x). \tag{1.41}$$

Let $\rho_j \to 0$ and $T_l^{(j)} = \rho_j^{-1} T_l$. We want to show that $T_l^{(j)}$ are normal graphs over points of $\operatorname{reg} C$.

Lemma 1.10. *For every $l \in 1, 2$, there exist a sequence of $\mathcal{C}^2$ functions $h_l^{(j)}$ which is defined in a connected domain U_j such that for some $\theta_j \to 0$,*

$$\left\{ x \in \operatorname{reg} C \,\middle|\, \operatorname{dist}(x, \operatorname{sing} C) > \theta_j |x|,\ \theta_j < |x| < \theta_j^{-1} \right\} \subset U_j \tag{1.42}$$

and such that

$$\lim_{j \to +\infty} |h_l^{(j)}|^*_{\mathcal{C}^2} = 0, \quad \textit{with} \quad |f|^*_{\mathcal{C}^2} := \sup \frac{|f(x)|}{|x|} + |\nabla f(x)| + |x||\nabla^2 f(x)|. \tag{1.43}$$

and that for every $\theta \in (0,1)$ and every $j \geqslant j(\theta)$, we also have, for $l \in \{1,2\}$,

$$\left\{x \in \operatorname{reg}(\rho_j^{-1}T_l) \,\middle|\, \operatorname{dist}(x, \operatorname{sing} C) > \theta|x|,\ \theta < |x| < \theta^{-1}\right\} \subset G_l^{(j)} \subset \operatorname{reg}(\rho_j^{-1}T_l) \qquad (1.44)$$

where $G_l^{(j)}$ is the graph of $h_l^{(j)}$ (more precisely, $G_l^{(j)} = H_l^{(j)}(U_j)$ where $H_l^{(j)}(x) = x + h_l^{(j)}\nu(x)$ and $\nu(x)$ is the normal of $\operatorname{reg}(C)$ *at x). We also ask that*

$$\rho_j^{-1}(\Omega_{2\theta}) \cap \left\{x \,\middle|\, \theta < |x| < \theta^{-1}\right\} \subset \left\{x \in \operatorname{reg}(\rho_j^{-1}T_1) \,\middle|\, \operatorname{dist}(x, \operatorname{sing} C) > \theta|x|\right\}. \qquad (1.45)$$

Proof. Let $T_l^{(j)} := \rho_j^{-1}T_l$. We construct θ_j as follows. Let θ_1 be any real in $(0,1)$ and for $l \in \{1,2\}$, we consider

$$K_1 := \{x \in \operatorname{reg} C \mid \operatorname{dist}(x, \operatorname{sing} C) \geqslant \theta_1|x|,\ \theta_1 \leqslant |x| \leqslant \theta_1^{-1}\ \}.$$

It is a compact subset of $\operatorname{reg} C$. Thanks to Theorem 1.14, there exists h_1 such that if $y \in T_l^{(j)}$ satisfies $|y-x| \ < h_1$ for some $x \in K_1$, then $y \in \operatorname{reg} T_l^{(j)}$.

Using the Hausdorff convergence of $T_l^{(j)}$ to C on the compact set

$$L_1 = \{x \in \mathbb{R}^n \mid, \operatorname{dist}(x, \operatorname{sing} C) \geqslant \theta_1|x|,\ \theta_1 \leqslant |x| \leqslant \theta_1^{-1}\ \},$$

there exists j_2 such that for every $j \geqslant j_2$ and $y \in L_1 \cap T_l^{(j)}$, there exists $x \in K_1$ with $|x-y| \ \leqslant h_0/2$. That implies that

$$L_1 \cap T_l^{(j)} \subset \operatorname{reg} T_l^{(j)}.$$

We can increase j_2 again such that $L_1 \cap T_l^{(j)}$ is in fact a graph of $h_l^{(j)}$ over K_1 with

$$\|h_l^{(j)}\|_{\mathcal{C}^2} \leqslant \frac{1}{j}.$$

This is possible since the L^∞ convergence of the $h_l^{(j)}$ is provided by the Hausdorff convergence of $T_l^{(j)}$ to C and the $\mathcal{C}^2$ is obtained using the analyticity of $\operatorname{reg} T_l^{(j)}$ as well as $\operatorname{reg} C$. We let $\theta_{j_2} = \frac{\theta_1}{2}$ and for every $j \in [1, j_2-1]$, $\theta_j = \theta_1$. To define j_3, we use the same scheme with θ_{j_2} in place of θ_1: that enables to define θ_k for $k \leqslant j_3$. Then, $\theta_j \to 0$.

We proved (1.43) and (1.44) by construction.

We now prove (1.45). If it does not hold, then there exists θ and $j_k \to \infty$ such that there exists

$$x_k \in \rho_{j_k}^{-1}(\Omega_{2\theta}) \cap \left\{x \,\middle|\, \theta < |x| < \theta^{-1}\right\}$$

and

$$\operatorname{dist}(x_k, \operatorname{sing} C) \leqslant \theta|x_k|.$$

The last equation means that there is $z_k \in \operatorname{sing} C$ such that $|x_k - z_k| \leqslant \theta|x_k|$. One can assume that $z_k \to \overline{z} \in \operatorname{sing} C$ using the local compactness of $\operatorname{sing} C$. Finally, $|x_k - \overline{z}| \leqslant \theta|x_k| + \varepsilon_k$ with $\varepsilon_k \to 0$.

The point $\overline{z}$ is singular, which implies in particular that C cannot be a graph around it. As a result, we have a unit vector ν and two sequences $z^i, \tilde{z}^i \in \operatorname{reg} C$ which converge to $\overline{z}$ and whose normals $\nu(z^i)$ and $\nu(\tilde{z}^i)$ converge respectively to ν and $-\nu$. Since T^j converge $\mathcal{C}^2$ to C in the neighborhood of $\operatorname{reg} C$, there exist (using a diagonal argument) $\alpha_k, \tilde{\alpha}_k \in \operatorname{reg} T_1^{(j_k)}$ such that

$$|\overline{z} - \alpha_k| \leqslant \frac{\theta^2}{10}$$

and the normals $\nu(\alpha_k)$ and $\nu(\tilde{\alpha}_k)$ to $\operatorname{reg} T_1^{j_k}$ satisfy, for k large enough,

$$|\nu(\alpha_k) - \nu(\tilde{\alpha}_k)| \geqslant \frac{3}{2}. \tag{1.46}$$

On the other hand, since $x_k \in \rho_{j_k}^{-1}\Omega_{2\theta}$, we have

$$\sup\left\{ \frac{|\nu(x_k) - \nu(y)|}{|x_k - y|} \,\middle|\, y \in \operatorname{reg} T_1^{j_k},\, 0 < |y - x_k| < 2\theta|x_k| \right\} < \frac{1}{2\theta|x_k|}.$$

Noting that we can choose $y = \alpha_k$ and $y = \tilde{\alpha}_k$ in the last identity, it provides

$$\frac{|\nu(x_k) - \nu(\alpha_k)|}{|x_k - \alpha_k|} \leqslant \frac{1}{2\theta|x_k|}$$

which implies, since $|x_k - \alpha_k| \leqslant 11\frac{\theta|x_k|}{10}$,

$$|\nu(x_k) - \nu(\alpha_k)| \leqslant \frac{11}{20}.$$

The same holds for $\tilde{\alpha}_k$. Summing, we get

$$|\nu(\alpha_k) - \nu(\tilde{\alpha}_k)| \leqslant \frac{11}{10}.$$

This contradicts (1.46), proving (1.45). □

Using this lemma, we have maps

$$p_j : \rho_j^{-1}\Omega_{2\theta} \cap \left\{x \,\middle|\, \theta < |x| < \theta^{-1}\right\} \to U_j$$

with

$$H_1^{(j)}(p_j(x)) = p_j(x) + h_1^{(j)}(p_j(x))\nu(p_j(x)) = x$$

and $\forall \left\{x \,\middle|\, \theta < |x| < \theta^{-1}\right\}$ and j sufficiently large,

$$\frac{1}{2}u_j(p_j(x)) \leqslant \rho_j^{-1} \operatorname{dist}(\rho_j x, T_2) \leqslant 2u_j(p_j(x)) \tag{1.47}$$

where $u_j = h_1^{(j)} - h_2^{(j)}$. The last inequality is provided by the convergence of $\nu_l^{(j)}(x_j)$ to $\nu(x)$ for $x_j \to x$ (and obvious notation). We notice that since $\operatorname{reg} T_1 \cap \operatorname{reg} T_2 = \emptyset$ (Section 5), one can assume that $u_j > 0$. Equation (1.41), after dilation with a factor $\rho_j^{(-1)}$, gives

$$\sup_{\substack{x \in \rho_j^{-1}\Omega_{\theta_0} \\ |x| = \theta}} \rho_j^{-1} h(\rho_j x) \leqslant 2\theta \sup_{\substack{x \in \rho_j^{-1}\Omega_{\theta_0} \\ |x| = 1}} \rho_j^{-1} h(\rho_j x)$$

Using then (1.47), we obtain

$$\sup_{\substack{x\in\rho_j^{-1}\Omega_{\theta_0}\\ |x|=\theta}} u_j(p_j(x)) \leqslant 4\theta \sup_{\substack{x\in\rho_j^{-1}\Omega_{\theta_0}\\ |x|=1}} u_j(p_j(x)). \tag{1.48}$$

Since $\operatorname{reg} T_1$ and $\operatorname{reg} T_2$ are two constant mean curvature submanifolds, we can prove the

Lemma 1.11. *The difference* $u_j := h_1^{(j)} - h_2^{(j)}$ *satisfies an equation of the form*

$$\Delta_C u_j + |A_C|^2 u_j = \operatorname{div}(a_j\cdot\nabla u_j) + b_j\cdot\nabla u_j + c_j u_j \tag{1.49}$$

where Δ_C *is the Laplace-Beltrami operator on* C*,* A_C *the second fundamental form of* C *and* a_j, b_j, c_j *three functions converging uniformly to zero on compact subsets of* $\operatorname{reg} C$*.*

Proof. Let f be a function on $\operatorname{reg} C$ and consider M the normal graph of f over $\operatorname{reg} C$ (we note only C in the rest of the proof). A parametrization of M is (locally) $F:\Omega\to\mathbb{R}^n$ with

$$F(x) = C(x) + f(C(x))\nu(C(x))$$

where $C(x)$ is a local parametrization of C. More precisely, the metric on C is written

$$g_{ij} = (\partial_i C\,,\,\partial_j C)\,.$$

As a result, a tangent vector is written (the $h_{\alpha,\beta}$ are the coefficients of A_C)

$$\tau_i = \partial_i F = \partial_i(C+f\nu) = \partial_i C + \partial_i f\nu + f\partial_i\nu = \partial_i C + \partial_i f\nu + fh_{il}g^{lm}\partial_m C.$$

Thus the metric on M is

$$\begin{aligned}
\tilde g_{ij} &= \Big(\partial_i C + \partial_i f\nu + fh_{il}g^{lm}\partial_m C\,,\,\partial_j C + \partial_j f\nu + fh_{js}g^{st}\partial_t C\Big) &(1.50)\\
&= g_{ij} + \partial_i f\partial_j f + fh_{il}g^{lm}g_{mj} + fh_{js}g^{st}g_{it} + f^2h_{il}g^{lm}h_{js}g^{st}g_{mt} &(1.51)\\
&= g_{ij} + f\left(h_{il}g^{lj} + h_{jl}g^{li}\right) + \partial_i f\partial_j f + f^2g^{lm}h_{il}h_{jm}. &(1.52)
\end{aligned}$$

Note that this metric does not contain any derivatives of order two for f. Using normal coordinates on C, it can be rewritten as

$$\tilde g_{ij} = \delta_{ij}(1+2f\lambda_i + f^2\lambda_i^2) + \partial_i f\partial_j f.$$

The normal to M can be computed in the basis $(\partial_i C,\nu)$ as

$$\tilde\nu = \alpha\nu + \sum_i \beta_i\partial_i C.$$

The coefficients α and β_i satisfy

$$\begin{aligned}
0 = (\tilde\nu\,,\,\partial_i F) &= \Big(\tilde\nu\,,\,\partial_i C + \partial_i f\nu + fh_{il}g^{lm}\partial_m C\Big)\\
&= \beta_j g_{ij} + \alpha\partial_i f + fh_{il}g^{lm}\beta_j g_{jm}\\
&= \beta_j g_{ij} + \alpha\partial_i f + fh_{il}\beta_l.
\end{aligned} \tag{1.53}$$

and

$$(\tilde{\nu}\,,\,\tilde{\nu}) = \alpha^2 + \beta_i\beta_j g_{ij} = 1. \tag{1.54}$$

So, the coefficients α and β depends only on order zero and one derivatives of f.

One also have $\tilde{h}_{ij} = -\left(F_{ij}\,,\,\tilde{\nu}\right)$. Let us compute

$$\begin{aligned}\partial_j(F_i) &= \partial_j\left(\partial_i C + \partial_i f \nu + f h_{il} g^{lm}\partial_m C\right)\\ &= \partial_{ij}C + \partial_{ij} f\nu + \partial_i f\partial_j\nu + \partial_j f h_{il}g^{lm}\partial_m C + f\partial_j(h_{jl})g^{lm}\partial_m C + f h_{il}\partial_j(g^{lm})\partial_m C + f h_{il}g^{lm}\partial_{jm}C.\end{aligned}$$

Hence

$$\begin{aligned}(F_{ij}\,,\,\tilde{\nu}) &= (\partial_{ij}C\,,\,\alpha\nu + \beta_k\partial_k C) + \alpha\partial_{ij}f + \partial_i f\left(\partial_j\nu\,,\,\beta_k\partial_k C\right) + \partial_j f h_{il}g^{lm}\beta_k\left(\partial_k C\,,\,\partial_m C\right)\\ &\quad + f\partial_j(h_{il})g^{lm}\beta_k\left(\partial_k C\,,\,\partial_m C\right) + f h_{il}\partial_j(g^{lm})\beta_k\left(\partial_k C\,,\,\partial_m C\right) + f h_{il}g^{lm}\left(\partial_{jm}C\,,\,\alpha n + \beta_k C_k\right).\end{aligned}$$

In normal coordinates on C (the second fundamental form is written $h_{ij} = \lambda_i\delta_{ij}$), that can be rewritten as

$$\tilde{h}_{ij} = \alpha h_{ij} - \alpha\partial_{ij}f - \partial_i f\lambda_j\beta_j - \partial_j f\lambda_i\beta_i - f\partial_j(h_{il})\beta_l + \alpha f h_{ij}.$$

To compute the mean curvature, we need the inverse of the metric. We compute using normal coordinates in C.

$$\begin{aligned}\tilde{g}^{ij} &= \delta_i j\left(1 - 2f\lambda_i - f^2\lambda_i^2\right) - \partial_i f\partial_j f + 4f^2\delta_{ij}\lambda_i^2 + o(f^2)\\ &= \delta_{ij}(1 - 2f\lambda_i + 3f^2\lambda_i^2) - \partial_i f\partial_j f + o(f^2).\end{aligned}$$

Note that no term in this metric (even in $o(f^2)$) involves second derivative of f. We have to estimate α and β_i. In normal coordinates, we have, using (1.53) and (1.54),

$$\beta_i + \alpha\partial_i f + f\lambda_i\beta_i = 0,$$

which yields

$$\beta_i = -\frac{\alpha\partial_i f}{1 + f\lambda_i} = -\alpha\partial_i f(1 - f\lambda_i + o(f)) = -\alpha\partial_i f + \alpha f\partial_i f\lambda_i + o(f^2).$$

On the other hand, $\alpha^2 + \sum\beta_i^2 = 1$, which means

$$\alpha^2 + \sum_i(-\alpha\partial_i f + f\partial_i f\lambda_i + o(f^2))^2 = 1,$$

or

$$\alpha^2\left(1 + \sum_i(\partial_i f)^2(1 + f\lambda_i o(f^2))^2\right) = \alpha^2(1 + |\nabla f|^2) + o(f^2) = 1.$$

Finally,

$$\alpha = \sqrt{\frac{1}{1 + |\nabla f|^2}} = 1 - \frac{1}{2}|\nabla f|^2 + o(f^2)$$

and

$$\beta_i = -\partial_i f + f\partial_i f \lambda_i + o(f^2)$$

where there is no second derivative of f in $o(f^2)$.

The mean curvature can now be computed using normal coordinates on C (once again, no second derivative in $o(f^2)$).

$$\begin{aligned}
\tilde{H} = \tilde{g}^{ij}\tilde{h}_{ij} &= (\delta_{ij}(1 - 2f\lambda_i + 3f^2\lambda_i^2) - \partial_i f \partial_j f)(\alpha h_{ij} - \alpha\partial_{ij} f - \partial_i f \lambda_j \beta_j - \partial_j f \lambda_i \beta_i - f\partial_j(h_{il})\beta_l + \alpha f h_{ij}) \\
&= \alpha\lambda_i(1 - 2f\lambda_i - 3f^2\lambda_i^2) + \alpha(-\partial_{ii} f + 2f\lambda_i\partial_{ii} f) - 2\partial_i f \lambda_i \beta_i - f\partial_i(h_{il})\beta_l \\
&\quad + \alpha f\lambda_i - 2f^2\alpha\lambda_i^2 - \alpha\lambda_i\partial_i f^2 + o(f^2) \\
&= \lambda_i(1 - 2f\lambda_i - 3f^2\lambda_i^2) - \frac{1}{2}|\nabla f|^2\lambda_i - \partial_{ii} f + 2f\lambda_i\partial_{ii} f \\
&\quad + 2\lambda_i(\partial_i f)^2 + f\partial_i(h_{il})\partial_l f + f\lambda_i - 2f^2\lambda_i^2 - \lambda_i(\partial_i f)^2 + o(f^2) \\
&= \underbrace{(1+f)H - \frac{1}{2}|\nabla f|^2 H}_{=0} - \Delta f - 2f|A|^2 + \operatorname{div}(a\nabla f) + b\cdot\nabla f + cf
\end{aligned}$$

with

$$a_{ij} = 2fh_{ij} \quad \text{and} \quad b_i = -\lambda_i\partial_i f - f\partial_k(h_{ki}) + o(f) \quad \text{and} \quad c = -2f|A|^2 + o(f).$$

So, $h_l^{(j)}$ both satisfy (we denote by H_j the (constant) mean curvature of $T_l^{(j)}$)

$$H_j + \Delta h_l^{(j)} + 2h_l^{(j)}|A|^2 = \operatorname{div}(a\nabla f) + b\cdot\nabla f + cf.$$

Substracting the two equations (and denoting by B the quantity $2A$) and noting that since the two terms $o(h_1^{(j)})$ and $o(h_2^{(j)})$ are smooth and obtained by the same procedure, one has $o(h_1^{(j)}) - o(h_2^{(j)}) = o(u_j)$, we get

$$\begin{aligned}
\Delta u_j + 2u_j|A|^2 &= \operatorname{div}\left(h_1^{(j)}B\nabla(h_1^{(j)}) - h_2^{(j)}B\nabla(h_2^{(j)})\right) \\
&\quad - \sum_i \lambda_i\left[(\partial_i h_1^{(j)})^2 - (\partial_i h_2^{(j)})^2\right] - 2u_j(h_1^{(j)} + h_2^{(j)})|A|^2 + o(u_j) \\
&= \operatorname{div}\left((h_1^{(j)} + h_2^{(j)})B\nabla u_j\right) + \operatorname{div}\left(h_1^{(j)}B\nabla h_2^{(j)} - h_2^{(j)}B\nabla h_1^{(j)}\right) \\
&\quad - \sum_i \lambda_i\partial_i(h_1^{(j)} - h_2^{(j)})\partial_i(h_1^{(j)} + h_2^{(j)}) - 2u_j(h_1^{(j)} + h_2^{(j)})|A|^2 + o(u_j).
\end{aligned}$$

Then, we write

$$\operatorname{div}\left(h_1^{(j)}B\nabla h_2^{(j)} - h_2^{(j)}B\nabla h_1^{(j)}\right) = h_1^{(j)}(\operatorname{div}(B\nabla u_j)) + (h_1^{(j)} - h_2^{(j)})\operatorname{div}(B\nabla h_1^j)$$

and

$$h_1^{(j)}(\operatorname{div}(B\nabla u_j)) = \operatorname{div}(h_1^{(j)}B\nabla u_j) - \nabla h_1^{(j)}\cdot B\nabla u_j$$

to get

$$\begin{aligned}\Delta u_j + 2u_j|A|^2 = \operatorname{div}\left((2h_1^{(j)} + h_2^{(j)})B\nabla u_j\right) \\ + \left(B\nabla h_1^{(j)} - A\nabla(h_1^{(j)} + h_2^{(j)})\right)\cdot\nabla u_j \\ + \left(-2u_j(h_1^{(j)} + h_2^{(j)})|A|^2 + \operatorname{div}(B\nabla h_1^{(j)})\right)u_j + o(u_j).\end{aligned}$$

Then, it remains to see that with

$$a_j := (2h_1^{(j)} + h_2^{(j)})B$$

$$b_j := B\nabla h_1^{(j)} - A\nabla(h_1^{(j)} + h_2^{(j)})$$

and

$$c_j := -2u_j(h_1^{(j)} + h_2^{(j)})|A|^2 + \operatorname{div}(B\nabla h_1^{(j)}) + \varepsilon_j$$

where $o(u_j) = \varepsilon_j u_j$, we have $a_j, b_j, c_j \to 0$ on compact subsets of $\operatorname{reg} C$ and satisfy (1.49). □

The rest of the proof is similar to [Sim87]. Nonetheless, we reproduce it for convenience (and give extra details).

Since $u_j > 0$, one can use Harnack inequality in (1.49) on a compact $K \subset \operatorname{reg} C$. It yields

$$\sup_K u_j \leqslant c_K \inf_K u_j. \tag{1.55}$$

Then, Schauder theory ([GT01, Th. 8.32]) implies that for j large enough,

$$|u_j|_{\mathcal{C}^{1,\alpha}(K)} \leqslant c_K \inf_K u_j.$$

Now, let us fix $y_0 \in \operatorname{reg} C$. Then, the sequence $\alpha_j := (u_j(y_0)^{-1})u_j$ converges, up to a subsequence, in $\mathcal{C}^1_{loc}(\operatorname{reg} C)$ to some function u. Since $\alpha_j(y_0) = 1$ for all j, $|\alpha_j|_{\mathcal{C}^{1,\alpha}(K)}$ is bounded away from zero, and so is $\inf_K u_j$. As a result, $u > 0$ on $\operatorname{reg} C$ (and $u(y_0) = 1$). On the other hand, u is a solution of

$$\Delta_C u + |A_C|^2 u = 0.$$

In particular, $\Delta_C u \leqslant 0$ on $\operatorname{reg} C$.

The last part of the proof consists in applying Bombieri and Giusti Harnack inequality [BG72, Th. 6] for functions on a minimal cone to u on $\operatorname{reg} C$.

Lemma 1.12. *There exists a sequence* $\varphi_j \in \mathcal{C}^\infty_c(\operatorname{reg} C)$ *such that*

– *For every* $x \in \Omega$, $0 \leqslant \varphi_j(x) \leqslant 1$
– *For every* $x \in \operatorname{reg} C$ *such that* $\frac{1}{j} \leqslant |x| \leqslant j$ *and* $\operatorname{dist}(x, \operatorname{sing} C) > \frac{1}{j}$, *we have* $\varphi_j(x) = 1$,
– *For a fixed* $R > 0$, *one has*

$$\int_{\operatorname{reg} C \cap B_R(0)} |\nabla\varphi_j|^2 \to 0. \tag{1.56}$$

Proof. First, note that $\mathcal{H}^{n-2}(\operatorname{sing} C) = 0$, so, for all $\varepsilon > 0$, we can cover $\operatorname{sing} C$ by N_j balls $B_i := B_{\rho_i}(x_i)$, of radius ρ_i such that

$$\rho_i \leqslant \frac{1}{2j} \quad \text{and} \quad \sum_i \rho_i^{n-2} \leqslant \varepsilon.$$

We take $\varepsilon = \frac{1}{j}$ in what follows.
For every i, we introduce a smooth function ψ_i such that $\psi_i(x) = 1$ on B_i and $\psi_i = 0$ on $\Omega \setminus B_{2\rho_i}(x_i)$. Then,

$$\int_\Omega |\nabla \psi_i|^2 \leqslant \left(\frac{\rho_i}{2}\right)^{-2} (4\rho_i^n - \rho_i^n) \leqslant \rho_i^{n-2}.$$

We introduce $\psi^j := 1 - \max(\psi_i)$. Then, as soon as $\operatorname{dist}(x, \operatorname{sing} C) > \frac{1}{j}$, $\operatorname{dist}(x, B_i) > \frac{1}{2j} > \rho_j$ so $\psi_i(x) = 0$ and then $\psi^j(x) = 1$.
Let us define the sets $A_0 = \emptyset$ and

$$\forall 1 \leqslant i \leqslant N_j, \quad A_i := \{\psi^j = 1 - \psi_i\} \setminus \bigcup_{k<i} A_k.$$

One can compute

$$\int_\Omega |\nabla \psi_j|^2 = \sum_i \int_{A_i} |\nabla \psi_i|^2 \leqslant \sum_i \rho_i^{n-2} \leqslant \frac{1}{j}.$$

Finally, we set

$$\varphi_j = \chi_j \circ \psi^j$$

where χ_j is a cut off function such that $\chi_j = 1$ on $B_j(0)$ and 0 on $\Omega \setminus B_{j+1}$. This way, $|\nabla \chi_j| \leqslant 1$. As a result,

$$\forall x \in \Omega, \quad |\nabla \varphi_j(x)| \leqslant |\nabla \psi^j(x)|$$

and φ_j fulfills the requirement of the lemma. □

Now, let $Q > 0$ and $u_Q = \min(u, Q)$. Since $\Delta_C u \leqslant 0$ on $\operatorname{reg} C$, one has, for every $\zeta \geqslant 0$ Lipschitz compactly supported on $\operatorname{reg} C$,

$$\int_{\operatorname{reg} C} \nabla u_Q \cdot \nabla \zeta \geqslant 0.$$

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. With $\zeta = \varphi_j^2 \psi^2 u_Q^{-1}$, we have

$$\int_{\operatorname{reg} C} \nabla u_Q \cdot \left(2u_Q^{-1} \psi \nabla \varphi_j + 2u_Q^{-1} \varphi_j^2 \nabla \psi^2 - \frac{1}{u_Q} \varphi_j^2 \psi^2 \nabla u_Q\right) \geqslant 0.$$

Using the regularity of u, (1.56) and $\varphi_j \to 1$ uniformly on compact sets of $\operatorname{reg} C$, we get that for every $R > 0$,

$$\int_{B_R(0) \cap \operatorname{reg} C} |\nabla u_Q|^2 < +\infty.$$

On the other hand, with $\zeta = \psi \varphi_j$ and assuming $\psi \geqslant 0$, and letting $j \to \infty$ we obtain

$$\int_{\operatorname{reg} C} \nabla u_Q \cdot \nabla \psi \geqslant 0.$$

Thanks to the two last inequalities, one can now apply [BG72, Th. 6] with $p = 1$, which tells that

$$\int_{\operatorname{reg} C\cap B_2(0)} u_Q \leqslant c \inf_{\operatorname{reg} C\cap B_2(0)} u_Q.$$

With $Q \to \infty$, we obtain

$$\int_{\operatorname{reg} C\cap B_2(0)} u \leqslant c \inf_{\operatorname{reg} C\cap B_2(0)} u > 0.$$

Coming back to the functions u_j, on every (non empty) compact $L \subset \operatorname{reg} C \cap B_2(0)$, we have

$$\inf_L u \geqslant \inf_{\operatorname{reg} C\cap B_2(0)} u := \delta > 0.$$

As $\inf_L u_j \to \inf_L u$, one has, for j larger than some j_1,

$$\inf_L u_j \geqslant \frac{\delta}{2}.$$

On the other hand, $u_j(y_0) \to u(y_0) = 1$. So, there exists j_2 such that $\forall j \geqslant j_2$, $u_j(y_0) \geqslant \frac{1}{2}$. Thus, there exists $j_3 = \max(j_1, j_2)$ such that for all $j \geqslant j_3$,

$$\inf_L u_j \geqslant \frac{\delta}{4} u_j(y_0).$$

Remembering (1.55), one deduce that for every $K \subset \operatorname{reg} C \cap B_2(0)$ compact (non empty), one has, for j sufficiently large (depending on K and L),

$$\inf_L u_j \geqslant c_K \sup_K u_j. \tag{1.57}$$

Taking $K = p_j(\rho_j^{-1}\Omega_{\theta_0} \cap \partial B_1)$ and $L = p_j(\rho_j^{-1}\Omega_{\theta_0} \cap \partial B_\theta)$, we see that for small θ, (1.57) and (1.48) cannot happen together. This is a contradiction.

Chapter 2

Mean curvature flow with obstacles: a viscosity approach

1 Introduction

In this article, we introduce the level set formulation of a generalized motion by mean curvature with obstacles. It is well known (see for example [ES91]) that if $u : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with a nonzero gradient at x_0, the mean curvature of the level set $\{u = u(x_0)\}$ is given by $\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$. As a result, making this set evolve by mean curvature yields the following equation for u:

$$u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right). \tag{2.1}$$

This type of motion was first studied in 1991 using viscosity solutions by Evans and Spruck ([ES91]) and Chen, Giga and Goto ([CGG91]).

Motivated by recent works from Almeida, Chambolle and Novaga ([ACN12]) and Spadaro ([Spa11]) about a discrete scheme for the mean curvature flow with obstacles, we are interested in the viscosity solutions of a mean curvature motion constrained by obstacles.

We first present the viscosity framework we use and prove a uniqueness and existence result for bounded uniformly continuous initial data and obstacles, and Lipschitz forcing term. Then, we give a more precise result on the regularity of the solution. We also show that our level-set approach really defines a geometric flow: the α-level set of the solution depends only on the α-level set of the initial data and the obstacles. Nonetheless, as expected, there is no uniqueness: level sets of the solution can develop non empty interiors.

Finally, in Section 4, we compare the approach followed by [Spa11] and [ACN12] to ours, in order to get long time properties of our solution.

2 Notation

In what follows, we consider the equation (slightly more general than (2.1), but the latter has to be kept in mind), for $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$

$$\forall t \geqslant 0, x \in \mathbb{R}^n, \quad u_t + F(Du, D^2u) + k|Du| = 0, \tag{2.2}$$

where $k : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ is a forcing term and $F : \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ satisfies

i) F is continuous in space and time when $p \neq 0$,

ii) F is geometric : $\forall \lambda > 0, \sigma \in \mathbb{R},\ F(\lambda p, \lambda X + \sigma p \otimes p) = \lambda F(p, X)$,

iii) For X and Y symetric matrices with $X \leqslant Y$, $F(p, X) \leqslant F(p, Y)$.

In the following, ∇u, Du and D^2u denote space derivatives only.

We also introduce $\overline{F}$ and $\underline{F}$ which are respectively the upper semicontinuous and lower semicontinuous envelopes of F [1].
To play the role of obstacles, we consider u^- and $u^+ : \mathbb{R}^n \to \mathbb{R}$, with $u^- \leqslant u^+$. The function u will be forced to stay between u^- and u^+. Geometrically, the constraint reads $\{u^+ > s\} \subset \{u > s\} \subset \{u^- > s\}$.

To adapt the classical theory of viscosity solutions (we will use the same scheme of proof as in [CIL92]), the key point is to define correctly sub and super solutions.

Definition 2.1. *A function $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ is said to be a (viscosity) subsolution on $[0, T]$ of the motion equation with obstacles u^+, u^- and initial condition g if*

- *u is upper semicontinous (usc),*
- *for all $x, t \in \mathbb{R}^n \times [0, T]$, $u^-(x, t) \leqslant u(x, t) \leqslant u^+(x, t)$,*
- *for all $x \in \mathbb{R}^n$, $u(x, 0) \leqslant g(x)$,*
- *if φ is a $\mathcal{C}^2$ function of x, t, if $(\hat{x}, \hat{t}) \in \mathbb{R}^n \times (0, T]$ is a maximum of $u - \varphi$ and if $u(\hat{x}, \hat{t}) > u^-(\hat{x}, \hat{t})$, then*

$$\varphi_t + \underline{F}(D\varphi, D^2\varphi) + k|D\varphi| \leqslant 0.$$

Similarly, u is said to be a (viscosity) supersolution of the motion equation with obstacles u^+, u^- and initial condition g if

- *u is lower semicontinous (lsc),*
- *for all $x, t \in \mathbb{R}^n \times [0, T]$, $u^-(x, t) \leqslant u(x, t) \leqslant u^+(x, t)$,*
- *for all $x \in \mathbb{R}^n$, $u(x, 0) \geqslant g(x)$,*
- *if φ is a $\mathcal{C}^2$ function of x, t, if $(\hat{x}, \hat{t}) \in \mathbb{R}^n \times (0, T]$ is a minimum of $u - \varphi$ and if $u(\hat{x}, \hat{t}) < u^+(\hat{x}, \hat{t})$, then*

$$\varphi_t + \overline{F}(D\varphi, D^2\varphi) + k|D\varphi| \geqslant 0.$$

1. This quantity is useful to make the following results apply for the mean curvature motion, where

$$F(p, X) = -\operatorname{Tr}\left(\left(I - \frac{p \otimes p}{|p|^2}\right) X\right).$$

Finally, u is said to be a (viscosity) solution of the motion equation with obstacles u^+, u^- *if* u *is both a super and a sub solution.*
To simplify, we write

$$u_t + F(Du, D^2u) + k|Du| = 0 \quad \text{on } \{u^- \leqslant u \leqslant u^+\}. \tag{2.3}$$

A supersolution (resp subsolution) of the motion equation with obstacles u^+, u^- *will be called a supersolution (resp. subsolution) of* (2.3).

There is another equivalent definition of such solutions, which can be useful (see [CIL92]).

Definition 2.2. *Let* $f : \mathbb{R}^n \times (0,T] \to \mathbb{R}$. *We said that* $(a,p,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n(\mathbb{R})$ *is a superjet for* f *in* (x_0,t_0) *and we denote* $(a,p,X) \in \mathcal{J}^{2,+}f(x_0,t_0)$ *if, when* $x,t \to x_0,t_0$ *in* $\mathbb{R}^n \times (0,T]$,

$$f(x,t) \leqslant f(x_0,t_0) + a(t-t_0) + (p\,,\,x-x_0) + \frac{1}{2}\,(X(x-x_0)\,,\,x-x_0) + o(|t-t_0| + |x-x_0|^2).$$

We likewise say that $(a,p,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n(\mathbb{R})$ *is a subjet for* f *in* (x_0,t_0) *and we denote* $(a,p,X) \in \mathcal{J}^{2,-}f(x_0,t_0)$ *if, for every* $x,t \to x_0,t_0$,

$$f(x,t) \geqslant f(x_0,t_0) + a(t-t_0) + (p\,,\,x-x_0) + \frac{1}{2}\,(X(x-x_0)\,,\,x-x_0) + o(|t-t_0| + |x-x_0|^2).$$

Then, u *is a subsolution of* (2.3) *if it satisfies the three first assumptions of the previous definition and if*

$$\forall x,t \in \mathbb{R}^n \times (0,T], \forall (a,p,X) \in \mathcal{J}^{2,+}u(x,t), \quad u(x) > u^-(x) \Rightarrow a + \underline{F}(p,X) + k|p| \leqslant 0.$$

Of course, u *is a supersolution of* (2.3) *if the three assumptions of the first definition are satisfied and if*

$$\forall x,t \in \mathbb{R}^n \times (0,T], \forall (a,p,X) \in \mathcal{J}^{2,-}u(x,t), \quad u(x) < u^+(x) \Rightarrow a + \overline{F}(p,X) + k|p| \geqslant 0.$$

We also use the following notations.

Definition 2.3. *For* $f : \mathbb{R}^n \to R$, *we denote by* f^* *the upper semicontinuous envelope of* f. *More precisely*

$$f^*(x) = \limsup_{y \to x} f(y).$$

We define in a similar way the lower semicontinuous envelope of f.

$$f_*(x) = \liminf_{y \to x} f(y).$$

Note that f^* *(resp.* f_**) is the smaller (resp. larger) semicontinuous function* g *such that* $g \geqslant f$ *(resp.* $g \leqslant f$*).*

3 Existence and uniqueness

The aim of this section is to show the

Theorem 2.1. *We assume that u^- and u^+ are uniformly continuous and bounded and that k is Lipschitz and bounded. Then, if $g : \mathbb{R}^n \to \mathbb{R}$ is uniformly continuous and $u^-(x,0) \leqslant g(x) \leqslant u^+(x,0)$,* (2.3) *has a unique solution, which is uniformly continuous.*

The proof structure is classical when dealing with viscosity solutions. A comparison principle will show uniqueness, and makes us be able to show existence.

3.1 Uniqueness

We begin by proving a comparison principle, adapted from [CIL92], Theorem 8.2. Throughout the proof, ω will denote a modulus of continuity for u^-, u^+ and g and L a Lipschitz bound on k.

Proposition 2.1 (Comparison principle)**.** *We assume that u is a subsolution and v a supersolution of* (2.3) *on $(0,T)$, and that $u(x,0) \leqslant v(x,0)$. Then, $u \leqslant v$ in $\mathbb{R}^n \times (0,T)$.*

Proof. We proceed by contradiction. Since $\tilde{u} = (u - \frac{\eta}{T-t}) \vee u^-$ is still a subsolution, but with

$$\partial_t \tilde{u} + F(D\tilde{u}, D^2\tilde{u}) + k|D\tilde{u}| \leqslant -c < 0,$$

it is enough to prove the comparison principle with $\tilde{u}$ and then pass to the limit. Suppose that there exists $\overline{x}, \overline{t}$ such that $u(\overline{x},\overline{t}) - v(\overline{x},\overline{t}) \geqslant 2\delta > 0$. One defines

$$\Phi(x,y,t) = u(x,t) - v(y,t) - \frac{\alpha}{4}|x-y|^4 - \frac{\varepsilon}{2}(|x|^2 + |y|^2).$$

If ε is sufficiently small, $\Phi(\overline{x},\overline{x},\overline{t}) \geqslant \delta$. Hence, $M := \max\limits_{x,y,t} \Phi(x,y,t) \geqslant \delta$ (the penalization at infinity reduces searching for the maximum to a compact set). Let $\hat{x}, \hat{y}, \hat{t}$ be a maximum point. Since u and v are bounded, there is C depending only on $\|u\|_\infty$ and $\|v\|_\infty$ such that

$$|\hat{x} - \hat{y}| \leqslant \frac{C}{\alpha^{1/4}}.$$

First, let's show by contradiction that $u(\hat{x},\hat{t}) > u^-(\hat{x},\hat{t})$ and $v(\hat{y},\hat{t}) < u^+(\hat{y},\hat{t})$. Suppose for example that $u(\hat{x},\hat{t}) = u^-(\hat{x},\hat{t})$. Then

$$\begin{aligned} 0 < \delta \leqslant u^-(\hat{x},\hat{t}) - v(\hat{y},\hat{t}) &\leqslant u^-(\hat{y},\hat{t}) + \omega(|\hat{x}-\hat{y}|) - v(\hat{y},\hat{t}) \\ &\leqslant \omega(|\hat{x}-\hat{y}|) + 0 \leqslant \omega(C\alpha^{-1/4}). \end{aligned}$$

Hence, if α is sufficiently large (independently of ε), $\omega(C\alpha^{-1/4}) \leqslant \delta/3$. Contradiction (this shows moreover that $\hat{t} < T$). Similarly, $v(\hat{y},\hat{t}) < u^+(\hat{y},\hat{t})$.

In what follows, α is fixed sufficiently big to satisfy these conclusions.

As

$$M + \frac{\alpha}{4}|x-y|^4 + \frac{\varepsilon}{2}(|x|^2 + |y|^2) \geqslant u(x,t) - v(y,t) \tag{2.4}$$

with equality in $\hat{x}, \hat{y}, \hat{t}$, we are able to apply Ishii's lemma [CIL92] which provides (a, b, X, Y) such that $(a, \underbrace{\alpha|\hat{x}-\hat{y}|^2(\hat{x}-\hat{y})}_{:=\hat{p}} - \varepsilon\hat{x}, X - \varepsilon I) \in \overline{\mathcal{J}}^{2,+}u(\hat{x}, \hat{t})$ and $(b, \alpha|\hat{x}-\hat{y}|^2(\hat{x}-\hat{y}) + \varepsilon\hat{y}, Y + \varepsilon I) \in \overline{\mathcal{J}}^{2,-}v(\hat{y}, \hat{t})$. It provides moreover $a - b = 0$ and

$$-C|\hat{x}-\hat{y}|^2\alpha \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leqslant \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leqslant \alpha C|\hat{x}-\hat{y}|^2 \begin{bmatrix} I & -I \\ -I & I \end{bmatrix},$$

which shows in particular that $X \leqslant Y$ and $\|X\|, \|Y\| \leqslant C_1\alpha|\hat{x}-\hat{y}|^2$.

Since u and v are respectively subsolution and supersolution near $(\hat{x}, \hat{t})$ and $(\hat{y}, \hat{t})$, one has

$$c \leqslant \underbrace{a-b}_{=0} + \overline{F}(\hat{p} - \varepsilon\hat{y}, Y - \varepsilon I) - \underline{F}(\hat{p} + \varepsilon\hat{x}, X + \varepsilon I) + k(\hat{y}, \hat{t})|\hat{p} + \varepsilon\hat{y}| - k(\hat{x}, \hat{t})|\hat{p} - \varepsilon x|.$$

One can write

$$k(\hat{y}, \hat{t})|\hat{p} + \varepsilon\hat{y}| - k(\hat{x}, \hat{t})|\hat{p} - \varepsilon x| \leqslant (k(\hat{y}, \hat{t}) - k(\hat{x}, \hat{t}))|\hat{p} + \varepsilon\hat{y}| + 2k(\hat{x}, t)(|\varepsilon\hat{y}| + |\varepsilon\hat{x}|),$$

which gives (if $N = \|k\|_\infty$)

$$c \leqslant -\underline{F}(\hat{p} + \varepsilon\hat{x}, X + \varepsilon I) + \overline{F}(\hat{p} - \varepsilon y, Y - \varepsilon I) + L(|\hat{x}-\hat{y}|)|\hat{p} + \varepsilon\hat{y}| + N(|\varepsilon\hat{x}| + |\varepsilon\hat{y}|).$$

Then, we want to let ε goes to 0.
Since $M \geqslant \delta > 0$, we have

$$\delta + \frac{1}{4}\alpha|x-y|^4 + \frac{\varepsilon}{2}(|x|^2 + |y|^2) \leqslant u(\hat{x}) - v(\hat{y}) \leqslant 2N,$$

which implies that $\varepsilon|\hat{x}|^2$ is bounded, hence $\varepsilon\hat{x} \to 0$ (same for $\varepsilon\hat{y}$), whereas for $i \in \{2, 3, 4\}$, $\alpha|\hat{x}-\hat{y}|^i$ is bounded (so is $\hat{p}$, X and Y). Indeed, α is fixed here. Hence one can assume that $\hat{p} \to p$, $X \to X_0$, $\alpha|\hat{x}-\hat{y}|^4 \to \mu_\alpha$.
We now use a short lemma, which is proved in [For08] (we cite the preprint of a paper which is published, but whose published version does not contain the result we are interested in), Lemma 2.8, and whose proof is reproduced here for convenience.

Lemma 2.1 (Forcadel, [For08])**.** *One has*

$$\lim_{\alpha\to\infty} \lim_{\varepsilon\to 0} \alpha|\hat{x}-\hat{y}|^4 = 0.$$

Proof. Let $M_h = \sup\limits_{\substack{|x-y|\leqslant h \\ t\in[0,T)}} u(x,t) - v(y,t)$ and (x_h^n, y_h^n, t_h^n) such that $u(x_h^n, t_h^n) - v(y_h^n, t_h^n) \geqslant M_h - \frac{1}{n}$ and $|x_h^n - y_h^n| \leqslant h$. Then,

$$M_h - \frac{1}{n} - \alpha h^4 - \frac{\varepsilon}{2}\left(|x_h^n|^2 + |y_h^n|^2\right) \leqslant M \leqslant u(\hat{x}, \hat{t}) - v(\hat{y}, \hat{t}).$$

As x_h^n and y_h^n do not depend on ε, one can let it go to zero (considering the liminf of the right term) to get

$$M_h - \frac{1}{n} - \alpha h^4 \leqslant \liminf_{\varepsilon\to 0} u(\hat{x}, \hat{t}) - v(\hat{y}, \hat{t}).$$

Let $h \to 0$ (We denote by M' the decreasing limit of M_h). One obtains

$$M' - \frac{1}{n} \leqslant \liminf_{\varepsilon \to 0} (u(\hat{x}, \hat{t}) - v(\hat{y}, \hat{t})).$$

Let α go to infinity :

$$\begin{aligned} M' - \frac{1}{n} &\leqslant \liminf_{\alpha \to \infty} \liminf_{\varepsilon \to 0} (u(\hat{x}, \hat{t}) - v(\hat{y}, \hat{t})) \\ &\leqslant \limsup_{\alpha \to \infty} \left(\sup_{\substack{|x-y| \leqslant C\alpha^{-1/4} \\ t \in [0,T)}} (u(x,t) - v(y,t)) \right) \\ &\leqslant \limsup_{h \to 0} \sup_{|x-y| \leqslant h} (u(x,t) - v(y,t)) = M' \end{aligned}$$

hence

$$\lim_{\alpha \to \infty} \lim_{\varepsilon \to 0} u(\hat{x}, \hat{t}) - v(\hat{y}, \hat{t}) = M'.$$

We prove similarly that $\lim\limits_{\alpha \to \infty} \lim\limits_{\varepsilon \to 0} M = M'$. As a matter of fact,

$$\lim_{\alpha \to \infty} \lim_{\varepsilon \to 0} \left(\alpha |\hat{x} - \hat{y}|^4 + \frac{\varepsilon}{2} (|\hat{x}|^2 + |\hat{y}|^2) \right) = 0,$$

which proves the lemma. □

One can now choose α such that $\lim\limits_{\varepsilon \to 0} \alpha |\hat{x} - \hat{y}|^4 \to \mu_\alpha$ with $\mu_\alpha \leqslant c/2L$ and pass to the liminf in $\varepsilon \to 0$. One gets (using $X \leqslant Y$),

$$\frac{c}{2} \leqslant \liminf \left(\underline{F}(\hat{p}, X) - \overline{F}(\hat{p}, X) \right).$$

To conclude, we distinguish two cases :
- if $p \neq 0$, then $\overline{F}(p, X_0) = \underline{F}(p, X_0)$ and we get the contradiction.
- if $p = 0$, we have $\alpha |\hat{x} - \hat{y}|^2 (\hat{x} - \hat{y}) \underset{\varepsilon \to 0}{\longrightarrow} 0$, so $X_0 = 0$ and $\overline{F}(p, X_0) = \underline{F}(p, X_0) = 0$ and we get the contradiction too.

□

3.2 Existence

Let us state a straightforward but useful proposition.

Proposition 2.2. *Let u be a subsolution of the mean curvature motion without obstacles which satisfies $u \leqslant u^+$. Then, $u_{ob} := u \vee u^-$ is a subsolution of* (2.3) *with obstacles (the same happens for v supersolution and $v_{ob} = v \wedge u^+$).*

Lemma 2.2. *Let $\mathcal{F}$ be a subsolution-of-*(2.3) *family and define $U := \sup\{u(x), u \in \mathcal{F}\}$. Then, U^* is a subsolution of* (2.3).

To prove this lemma, we need the

Proposition 2.3. *Let v be a upper semicontinuous function, $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $(a,p,X) \in J^{2,+}v(x,t)$. Assume there exists a sequence (u_n) of usc functions which satisfy*

i) There exists (x_n, t_n) such that $(x_n, t_n, u_n(x_n, t_n)) \to (x, t, v(x,t))$

ii) $(z_n, s_n) \to (z, s)$ in $\mathbb{R}^n \times \mathbb{R}$ implies $\limsup u_n(z_n, s_n) \leqslant v(z,s)$.

Then, there exists $(\hat{x}_n, \hat{t}_n) \in \mathbb{R}^n \times \mathbb{R}$, $(a_n, p_n, X_n) \in \mathcal{J}^{2,+}u_n(\hat{x}_n, \hat{t}_n)$ such that

$$(\hat{x}_n, \hat{t}_n, u_n(\hat{x}_n, \hat{t}_n), a_n, p_n, X_n) \to (x, t, v(x,t), a, p, X).$$

The proof of the proposition and the lemma can be found in [CIL92], Lemma 4.2 (with obvious changes due to the parabolic situation and obstacles).

We begin by dealing with the case $k = 0$.

Construction of barriers in the non forcing case

In what follows, it will be crucial to show the existence of a subsolution w^- which satisfies $(w^-)^*(x,0) = g(x)$ and a supersolution w^+ such that $(w^+)_*(x,0) = g(x)$. Thanks to the maximum principle, they will force the solution to get the initial data.

Let us construct w^-. Without a forcing term, we note that

$$h(x,t) = -(|x-\xi|^2 + 2nt)$$

is a classical subsolution of (2.3) but with neither initial conditions nor obstacles (it represents the evolution of a circle). We define

$$\theta_\xi(r) = \inf\{g(y) \mid |y-\xi|^2 + r \leqslant 0\}$$

The function θ_ξ is bounded, non decreasing, continuous and satisfies $\theta_0(0) = g(0)$ and $\theta_\xi(-|x-\xi|^2 - 2nt) \leqslant g(x)$. As the equation is geometric, $\theta_\xi(-|x-\xi|^2 - 2nt)$ is also a classical subsolution. Let us then define

$$\phi(x,t) = \left(\sup_\xi \theta_\xi(-|x-\xi|^2 - 2nt)\right)^*.$$

Since $\theta_\xi(-|x-\xi|^2 - 2nt) \leqslant g(x)$ and g is continuous, it is also true for $\phi(x,t)$. In addition, we can check that

$$\phi(x,t) \geqslant \theta_x(-|x-x|^2 - 2nt) = \theta_x(-2nt) \geqslant g(x) - \omega(\sqrt{2nt}). \tag{2.5}$$

where ω is the modulus of continuity of g. Hence, $\phi(x,0) = g(x)$. Thanks to Lemma 2.2, ϕ is a subsolution with $\phi(x,0) \leqslant g(x)$. We conclude this proof defining

$$w^-(x,t) = (\phi(x,t) - \omega(t)) \vee u^-(x,t).$$

It is clear that w^- is a subsolution with obstacles. Indeed, by definition, $w^- \geqslant u^-$. Moreover, $\phi(x,t) - \omega(t) \leqslant g(x) - \omega(t) \leqslant u^+(x,0) - \omega(t) \leqslant u^+(x,t)$. Proposition 2.2 concludes the proof.

The other barrier w^+ is obtained similarly.

Perron's method

We use the usual Perron's method to show the existence of a solution on $[0,T)$ for any T. To do that, we consider

$$W(x,t) = \sup\{w(x,t),\ w \text{ (usc) subsolution on } [0,T)\}.$$

W is well defined because u^- is a subsolution (so the set is not empty). Moreover, since every w is smaller than u^+, so is W.

Applying Lemma 2.2, we show that W^* is a subsolution of (2.3). But, by definition of W, we must have $W^* = W$. Indeed, $W \leqslant w^+$ thanks to the comparison principle. Since w^+ is continuous at $t=0$, $W^* \leqslant (w^+)^* = w^+$ which shows that $W^*(x,0) \leqslant g(x)$ and W^* is a subsolution of (2.3) with initial conditions. By definition of W, $W^* = W$. Moreover, thanks to w^-, one has $W(x,0) = g(x)$.

Before finishing the proof of existence, let's notice a usefull property of the no-forcing-term case.

Remark. If $k(x,t) = 0$, then W is ω-uniformly continuous in space. In time, W is uniformly continuous with modulus $\tilde\omega : r \mapsto \max(\omega(r), \omega(\sqrt{2nr}))$. Indeed, the proof is contained in the following lemma.

Lemma 2.3. Let $u(x,t)$ be a subsolution of (2.3) *with no forcing term (and g, u^-, u^+ ω-uniformly continuous in space and time). Then,*

$$u_{z,\delta}(x,t) = (u(x+z, t+\delta) - \omega(|z|) - \tilde\omega(|\delta|)) \vee u^-(x,t)$$

is also a subsolution.

Proof. To begin, we notice that $u(x+z,t+\delta) - \omega(|z|) - \tilde\omega(|\delta|) \leqslant u^+(x,t)$.
Now, let φ be a smooth function with $\forall x, t,\ u_{z,\delta}(x,t) \leqslant \varphi(x,t)$ with equality at $(\overline{x}, \overline{t})$. Then, either $u_{z,\delta}(\overline{x},\overline{t}) = u^-(\overline{x},\overline{t})$, and nothing has to be done, or $u_{z,\delta}(\overline{x},\overline{t}) > u^-(\overline{x},\overline{t})$. In the second alternative, we have

$$u(\overline{x}+z,\overline{t}+\delta) - \omega(|z|) - \tilde\omega(\delta) > u^-(\overline{x},\overline{t}) = u^-(\overline{x}+z,\overline{t}+\delta) + (u^-(\overline{x},\overline{t}) - u^-(\overline{x}+z,\overline{t}+\delta))$$

hence

$$u(\overline{x}+z,\overline{t}+\delta) > u^-(\overline{x}+z,\overline{t}+\delta) + \underbrace{(u^-(\overline{x},\overline{t}) - u^-(\overline{x}+z,\overline{t}+\delta) + \omega(|z|) + \tilde\omega(|\delta|))}_{\geqslant 0} \geqslant u^-(\overline{x}+z,\overline{t}+\delta).$$

As u is a subsolution at $(\overline{x}+z,\overline{t}+\delta)$ and $u(x+z,t+\delta) \leqslant \varphi(x,t) + \omega(|z|) + \tilde\omega(|\delta|)$ with equality at $(\overline{x}+z,\overline{t}+\delta)$, one can write, with $y = x+z, s = t+\delta$,

$$u(y,t) \leqslant \varphi(y-z, t-\delta) + \omega(|z|) + \tilde\omega(|\delta|) =: \phi(y,t),$$

equality at $(\overline{y},\overline{t})$, and deduce that $\phi_t + \underline{F}(D\phi(\overline{y},\overline{t}), D^2\phi(\overline{y},\overline{t})) \leqslant 0$. Since $D\phi(\overline{y},\overline{t}) = D\varphi(\overline{x},\overline{t})$ (so are the second derivatives), we get

$$\varphi_t + \underline{F}(D\varphi(\overline{x},\overline{t}), D^2\varphi(\overline{x},\overline{t})) \leqslant 0,$$

what was expected.

Concerning the initial conditions, we have (we use (2.5))

$$u(x+z,0+\delta)-\omega(|z|)-\tilde{\omega}(\delta)\leqslant w^{+}(x+z,\delta)-\omega(|z|)-\tilde{\omega}(\delta)\leqslant g(x+z)-\omega(|z|)\leqslant g(x).$$

□

Applying this lemma to W shows $(x,t)\mapsto W(x+z,t+\delta)-\omega(|z|)-\tilde{\omega}(|\delta|)\vee u^{-}(x+z,t)$ is a subsolution. By definition of W, one can write

$$W(x,t)\geqslant(W(x+z,t+\delta)-\omega(|z|)-\tilde{\omega}(\delta))\vee u^{-}(x+z,t)\geqslant W(x+z,t+\delta)-\omega(|z|)-\tilde{\omega}(\delta)$$

which shows exactly that W is uniformly continuous.

We now want to show that W is in fact a supersolution of (2.3). We need the following lemma which is adapted from [CIL92], Lemma 4.4.

Lemma 2.4. *Let u be a subsolution of (2.3). If u_* fails to be a solution of $u_t+\overline{F}(Du,D^2u)+k|Du|\geqslant 0$ in some point $(\hat{x},\hat{t})$ (there exists $(a,p,X)\in\mathcal{J}^{2,-}u_*(\hat{x},\hat{t})$ such that $a+\overline{F}(p,X)+k|p|<0$), then for all sufficiently small κ, there exists a solution u_κ of $u_t+\underline{F}(Du,D^2u)+k|Du|\leqslant 0$ satisfying $u_\kappa(x,t)\geqslant u(x,t)$, $\sup_{\mathbb{R}^n}(u_\kappa-u)>0$, $u_\kappa(x,t)\leqslant u^{+}(x,t)$ and such that u and u_κ coincide for all $|x-\hat{x}|,|t-\hat{t}|\geqslant\kappa$.*

Proof. We can suppose that u_* fails to be a supersolution in $(0,1)$ (this implies in particular $u_*(0,1)<u^{+}(0,1)$). We get $(a,p,X)\in\mathcal{J}^{2,-}u_*(0,1)$ such that $a+\overline{F}(p,X)+k(0,1)|p|<0$. We introduce

$$u_{\delta,\gamma}(x)=u_*(0,1)+\delta+(p\,,\,x)+a(t-1)+\frac{1}{2}\,(Xx\,,\,x)-\gamma(|x|^2+t-1).$$

By upper semicontinuity of $\overline{F}$, $u_{\delta,\gamma}$ is a classical subsolution of $u_t+\overline{F}(Du,D^2u)+k|Du|\leqslant 0$ in $B_r(0,1)$ for γ,δ,r sufficiently small.
Since

$$u(x,t)\geqslant u_*(x,t)\geqslant u_*(0,1)+a(t-1)+(p\,,\,x)+\frac{1}{2}\,(Xx\,,\,x)+o(|x|^2)+o(|t-1|),$$

choosing $\delta=\gamma\frac{r^2+r}{8}$, we get $u(x,t)>u_{\delta,\gamma}(x,t)$ for $\frac{r}{2}\leqslant|x|,|t-1|\leqslant r$ and r sufficiently small. Moreover, we can reduce r again to have $u_{\delta,\gamma}\leqslant u^{+}$ on B_r (Choosing r sufficiently small, one has δ sufficiently small and $u_{\delta,\gamma}(0,1)-u_*(0,1)=\delta<u^{+}(0,1)-u_*(0,1)$. By continuity, one can find a smaller r such that $u_{\delta,\gamma}(x,t)<u^{+}(x,t)$ for all $\frac{r}{2}\leqslant|x|,|t-1|\leqslant r$.).
Thanks to Lemma 2.2, the function

$$\tilde{u}(x,t)=\begin{cases}\max(u(x,t),u_{\delta,\gamma}(x,t)) \text{ if } |x,t-1|<r\\ u(x) \text{ otherwise}\end{cases}$$

is a subsolution of (2.3) (with initial conditions if r is small enough). □

Now, we saw that W is a subsolution of (2.3) (in particular, $W\leqslant u^{+}$). If it is not a supersolution at a point $\hat{x},\hat{t}$, Lemma 2.4 provides $W_\kappa\geqslant W$ subsolutions of (2.3) (with initial condition, even if we have to reduce r again, to make t stay far from zero), which is a contradiction with the definition of W.
Finally, W is the expected solution of (2.3).

With forcing term

1. We assume at this point only that u^-, u^+ and g are K-Lipschitz. Then, thanks to Remark 3.2, there exists a K-Lipschitz solution ψ of the non forcing term equation. Let's set $w^-(x,t) = (\psi(x,t) + NKt) \vee u^-(x,t)$. It satisfies, as soon as $w^- > u^-$,
$$u_t - NK + F(Du, D^2u) = 0, \qquad u(x,0) = g(x).$$
As a consequence, w^- is a continuous subsolution of (2.3) (with forcing term) satisfying $w^-(x,0) = g(x)$. It is a barrier as in 3.2. We build w^+ in a similar way and apply Perron's method to see that W is a solution.
2. Here, u^+, u^- and g are only ω-uniformly continuous. For all $K > 0$, let $g_K = \min_y g(y) + K|x-y|$, $u_K^+ = \max_y u^+(y) - K|x-y|$ and $u_K^- = \min_y g(y) + K|x-y|$. These three new function are K-Lipschitz and converge uniformly to g, u^+ and u^- when $K \to \infty$. Moreover, as g, u^+, u^- are ω-uniformly continuous, so are they.
Thanks to the previous point, for every K, there exists a solution u_K of (2.3) with obstacles u_K^+, u_K^- and with initial data g_K, which is (thanks to the following proposition 2.4, which is admitted for a little time) uniformly continuous with same moduli on $[0,T]$ for every T. One can define, thanks to Ascoli's theorem
$$u(x,t) = \lim_n u_{K_n}(x,t).$$
The function u is continuous. We have to check that it is the solution of the motion with obstacles $u^\pm$.
It is clear that $u^- \leqslant u \leqslant u^+$. Let φ be a smooth function and $(\hat{x}, \hat{t})$ a maximum point of $u - \varphi$ such that $u(\hat{x}, \hat{t}) - u^-(\hat{x}, \hat{t}) =: \eta > 0$. One can assume that the maximum is strict. We then choose ε such that
$$\forall (x,t) \in B_\varepsilon(\hat{x}, \hat{t}), \quad u(x,t) - u^-(x,t) \geqslant \frac{3\eta}{4}.$$
Let
$$\delta := \min_{B_\varepsilon^c} |u - \varphi|.$$
It is positive (if necessary, one can reduce ε again). We choose n_0 such that
$$\forall n \geqslant n_0, \ \|u - u_{K_n}\|_{L^\infty(B_\varepsilon)} \leqslant \max\left(\frac{\eta}{4}, \frac{\delta}{2}\right).$$
Then, for every $n \geqslant n_0$, $u_{K_n} - \varphi$ has a maximum (x_n, t_n) on B_ε reached out of u_K^-. It is easy to show that $(x_n, t_n) \to (\hat{x}, \hat{t})$. Since u_K is a viscosity subsolution, one can write, at (x_n, t_n),
$$\varphi_t + \underline{F}(D\varphi, D^2\varphi) + k|D\varphi| \leqslant 0.$$
By smoothness of φ and semicontinuity of $\underline{F}$, we get the same inequality at $(\hat{x}, \hat{t})$.
We prove that u is a supersolution using the same arguments.

Let's conclude this section by an estimation of the solution's regularity, inspired from [For08].

Proposition 2.4. *Let u be the unique solution of* (2.3)*. Then u is uniformly continuous in space. moreover, one as*

$$\forall x, y, t, \quad |u(x,t) - u(y,t)| \leqslant \omega(e^{Lt}|x-y|).$$

Proof. First, it is well known that one can choose ω to be continuous and nondecreasing. Since u and v are bounded by N, $\omega \wedge 2N$ is a modulus too. In the following, we use this new modulus, sill denoted by ω.

Then, let ρ_n be a $\mathcal{C}^\infty$ nondecreasing function on $[0, \infty[$ such that $0 \leqslant \rho_n - \omega$, for all $r > n+1$, $\rho_n(r) = 2N+1$, and for all $r \in [0,n], \rho_n(r) - \omega(r) \leqslant \frac{1}{n}$. Then, let us define

$$\omega_n(r) = \rho_n + \frac{r}{n^2}.$$

It's clear that $\omega_n(r) \underset{n \to \infty}{\longrightarrow} \omega(r)$. Moreover, for a fixed n, $\omega_n'(r)$ is bounded and stays far from zero. In what follows, we work with ω_n.

We will proceed as in Proposition 2.1. Let $\phi(x,y,t) = \omega_n(e^{Lt}|x-y|)$. We will show by contradiction that $u(x,t) - u(y,t) \leqslant \phi(x,y,t)$. Assume that

$$M := \sup_{(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times [0,T)} u(x,t) - u(y,t) - \phi(x,y,t) > 0.$$

As before, we introduce

$$\tilde{M} = \sup_{x,y,t \leqslant T} u(x,t) - u(y,t) - \phi(x,y,t) - \frac{\alpha}{2}(|x|^2 + |y|^2) - \frac{\gamma}{T-t}.$$

For sufficiently small γ, α, $\tilde{M}$ remains positive and is attained (at $\overline{x}, \overline{y}, \overline{t} < T$). As g is ω-uniformly continuous, $\overline{t} > 0$. Moreover, it is clear that $\overline{x} \neq \overline{y}$.

By assumption,

$$u^-(\overline{x}, \overline{t}) \leqslant u^-(\overline{y}, \overline{t}) + \omega(|\overline{x} - \overline{y}|) \leqslant u^-(\overline{y}, \overline{t}) + \omega_n(|\overline{x} - \overline{y}|) \leqslant u(\overline{y}, \overline{t}) + \phi(\overline{x}, \overline{y}, \overline{t})$$

so

$$\tilde{M} > u^-(\overline{x}, \overline{t}) - u(\overline{y}, \overline{t}) - \phi(\overline{x}, \overline{y}, \overline{t}),$$

which forces $u(\overline{x}, \overline{t}) > u^-(\overline{x}, \overline{t})$. Similarly, $u(\overline{y}, \overline{t}) < u^+(\overline{y}, \overline{t})$.

Applying Ishii's lemma ([CIL92], Th. 8.3) to $\tilde{u}(x,t) = u(x,t) - \frac{\alpha}{2}|x|^2$ and $\tilde{v}(y,t) = u(y,t) + \frac{\alpha}{2}|y|^2$ where

$$\overline{p} = D_x\phi = \frac{\overline{x} - \overline{y}}{|\overline{x} - \overline{y}|} e^{L\overline{t}} \omega_n'(e^{L\overline{t}}|\overline{x} - \overline{y}|) = -D_y\phi \neq 0,$$

$$Z = D_x^2\phi = \frac{e^{L\overline{t}}}{|\overline{x} - \overline{y}|} \omega_n'(e^{L\overline{t}}|\overline{x} - \overline{y}|) I + \frac{(\overline{x} - \overline{y}) \otimes (\overline{x} - \overline{y})}{|\overline{x} - \overline{y}|^3} e^{Lt} \omega_n'(e^{L\overline{t}}|\overline{x} - \overline{y}|) + \frac{(\overline{x} - \overline{y}) \otimes (\overline{x} - \overline{y})}{|\overline{x} - \overline{y}|^2} e^{2L\overline{t}} \omega_n''(e^{L\overline{t}}|\overline{x} - \overline{y}|).$$

and

$$A = D^2\phi = \begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix},$$

we get the following. For all β such that $\beta A < I$, there exists $\tau_1, \tau_2 \in \mathbb{R}$, $X, Y \in \mathcal{S}_n$ such that

$$\begin{gathered} \tau_1 - \tau_2 = \frac{\gamma}{(T-t)^2} + Le^{L\bar{t}}|\overline{x} - \overline{y}|\omega_n'(e^{Lt}|\overline{x} - \overline{y}|), \\ (\tau_1, \overline{p} + \alpha\overline{x}, X + \alpha I) \in \overline{\mathcal{J}}^{2,+}u(\overline{x}, \overline{t}), \\ (\tau_2, \overline{p} - \alpha\overline{y}, Y - \alpha I) \in \overline{\mathcal{J}}^{2,-}u(\overline{y}, \overline{t}), \\ \frac{-1}{\beta}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leqslant \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leqslant (I - \beta A)^{-1}A. \end{gathered}$$

As u is a subsolution and a supersolution, one has

$$\begin{aligned} \tau_1 + k(\overline{x}, \overline{t})|\overline{p} + \alpha\overline{x}| + \underline{F}(\overline{p} + \alpha\overline{x}, X + \alpha I) \leqslant 0, \qquad (2.6) \\ \tau_2 - k(\overline{y}, \overline{t})|\overline{p} - \alpha\overline{y}| + \overline{F}(\overline{p} - \alpha\overline{y}, Y - \alpha I) \geqslant 0. \end{aligned}$$

□

$X \leqslant Y$ in the last equation gives

$$-\tau_2 + k(\overline{y}, \overline{t})|\overline{p} - \alpha\overline{y}| - \overline{F}(\overline{p} - \alpha\overline{y}, X - \alpha I) \leqslant 0. \qquad (2.7)$$

Adding (2.7) to (2.6) yields

$$\begin{aligned} \frac{\gamma}{(T-\overline{t})^2} + Le^{L\bar{t}}|\overline{x} - \overline{y}|\omega_n'(e^{L\bar{t}}|\overline{x} - \overline{y}|) - k(\overline{x}, \overline{t})|\overline{p} + \alpha\overline{x}| + k(\overline{y}, \overline{t})|\overline{p} - \alpha\overline{y}| \\ + \underline{F}(\overline{p} + \alpha\overline{x}, X + \alpha I) - \overline{F}(\overline{p} - \alpha\overline{y}, X - \alpha I) \leqslant 0. \qquad (2.8) \end{aligned}$$

Notice that

$$\begin{aligned} Le^{L\bar{t}}|\overline{x} - \overline{y}|\omega_n'(e^{L\bar{t}}|\overline{x} - \overline{y}|) - k(\overline{x}, \overline{t})|\overline{p}| + k(\overline{y}, \overline{t})|\overline{p}| \\ \geqslant Le^{L\bar{t}}|\overline{x} - \overline{y}|\omega_n'(e^{L\bar{t}}|\overline{x} - \overline{y}|) - L|\overline{x} - \overline{y}|e^{L\bar{t}}\omega_n'(e^{L\bar{t}}|\overline{x} - \overline{y}|) \geqslant 0. \qquad (2.9) \end{aligned}$$

Then, (2.8) becomes

$$\frac{\gamma}{(T-\overline{t})^2} + (|\overline{p}| - |\overline{p} + \alpha\overline{x}|)\, k(\overline{x}, \overline{t}) - (|\overline{p}| - |\overline{p} - \alpha\overline{y}|)\, k(\overline{y}, \overline{t}) + \underline{F}(\overline{p} + \alpha\overline{x}, X + \alpha I) - \overline{F}(\overline{p} - \alpha\overline{y}, X - \alpha I) \leqslant 0.$$

Let α go to zero. $\overline{p}$ and X are bounded : one assumes they converge and still denotes by $\overline{p}, X$ their limit. As $|\overline{p}| \geqslant \frac{1}{n^2}$ (ρ_n is nondecrasing), $\underline{F}(\overline{p}, H) = \overline{F}(\overline{p}, H)$ for all $H \in \mathcal{S}_n$. Moreover, $\alpha\overline{x}, \alpha\overline{y} \to 0$ and k is bounded, hence

$$\frac{\gamma}{(T-\overline{t})^2} \leqslant 0,$$

which is a contradiction. So

$$u(x,t) - u(y,t) \leqslant \omega_n(e^{Lt}|x-y|).$$

It remains to let n go to $+\infty$ to conclude.

3.3 Independence of level sets

In all this subsection, a solution u of the motion with initial data u_0 and obstacles u^- and u^+ will be denoted by $u = [u_0, u^-, u^+]$. The corresponding equation will be denoted by (u_0, u^-, u^+).
To agree with the geometric motion, we have to check that the zero level-set of the solution depends only on the zero level sets of the initial condition u_0 and of the obstacles u^+ and u^-. We begin by proving a weak invariance.

Proposition 2.5. *Let u be the solution of* (2.2) *with obstacles u^+ and u^-, and let ϕ be a continuous nondecreasing functions $[-N, N] \to \mathbb{R}$ such that $\{\phi = 0\} = \{0\}$. Then, the solutions*

$$[u_0 \wedge \phi(u^+)|_{t=0}, u^-, \phi(u^+) \vee u^-],$$

$$(u_0 \vee \phi(u^-)|_{t=0}, \phi(u^-) \vee u^+, u^+]$$

$$\textit{and} \quad [(\phi(u_0) \wedge u^+|_{t=0}) \vee u^-|_{t=0}, u^-, u^+]$$

have the same zero level set as u.

Proof. We will prove that

$$u_\phi = [u_0 \wedge \phi(u^+)|_{t=0}, u^-, \phi(u^+) \vee u^-]$$

has the same zero set as u. All the other equalities can be prove with a similar strategy.

We begin the proof assuming $\phi(x) \geqslant x$. Then, $u_\phi = [u_0, u^-, \phi(u^+)]$.
First, let us notice that the classical invariance proves immediately that $\phi(u)$ is the solution $[\phi(u_0), \phi(u^-), \phi(u^+)]$. In addition, thanks to Proposition 2.2, $u_\phi \wedge u^+$ is a supersolution of (u_0, u^-, u^+). That shows $u_\phi \geqslant u$ thanks to the comparison principle. Similarly, $u_\phi \vee \phi(u^-)$ is a subsolution of the motion with $(u_0, \phi(u^-), \phi(u^+))$, and a subsolution of $(\phi(u_0), \phi(u^-), \phi(u^+))$ since $\phi(u_0) \geqslant u_0$. By comparison, that yields to $u_\phi \leqslant \phi(u)$.

Finally, $u \leqslant u_\phi \leqslant \phi(u)$ and $\{\phi(u) = 0\} = \{u = 0\}$. This shows $\{u = 0\} = \{u_\phi = 0\}$, what was expected.

Assume now that $\phi(x) \leqslant x$. Then, $u \wedge (\phi(u^+) \vee u^-)$ is a supersolution of

$$(u_0, u^-, \phi(u^+) \vee u^-).$$

Since $\phi(u_0) \leqslant u_0$, it is also a supersolution of

$$(u_0 \wedge \phi(u^+)|_{t=0}, u^-, \phi(u^+) \vee u^-)$$

which proves that $u_\phi \leqslant u$. On the other hand, $\phi(u) \vee u^-$ is a subsolution of

$$(\phi(u_0), u^-, \phi(u^+) \vee u^-),$$

so a subsolution of

$$(u_0 \wedge \phi(u^+)|_{t=0}, u^-, \phi(u^+) \vee u^-).$$

As a result,

$$\phi(u) \leqslant u_\phi \leqslant u,$$

which shows that u and u_ϕ have the same zero level set.

To conclude the proof for a general ϕ, just introduce $f(x) = \min(x, \phi(x))$ and $g(x) = \max(x, \phi(x))$ and notice that since ϕ is nondecreasing, $\phi = f \circ g$. So,

$$\{u = 0\} = \{u_f = 0\} = \{(u_g)_f = 0\} = \{u_{f\circ g} = 0\} = \{u_\phi = 0\}.$$

□

Now, to be able to define a real geometrical evolution, we want a more general independence, which is contained in the following

Theorem 2.2. *Let $u = [u_0, u^-, u^+]$. Then, $\{u = 0\} = \{v = 0\}$ with $v = [v_0, v^-, v^+]$ under the (only) assumptions that*

$$\{u_0 = 0\} = \{v_0 = 0\}, \quad \{u^- = 0\} = \{v^- = 0\} \quad \textit{and} \quad \{u^+ = 0\} = \{v^+ = 0\}.$$

Proof. This proof is based on the independence with no obstacles which is proved in [ES91], Theorem 5.1. We assume first that $u^- = v^-$ and $u^+ = v^+$. As in [ES91], we define

$$\forall k \in \mathbb{Z} \setminus \{0\}, \quad E_k = \left\{ x \in \mathbb{R}^n \,\middle|\, u_0 \geqslant \frac{1}{k} \right\}$$

and

$$a_k = \max_{\mathbb{R}^n \setminus E_k} v_0.$$

It is easy to see that

$$\forall k > 0,\ a_1 \geqslant a_2 \geqslant \cdots \to 0 \quad \text{and} \quad a_{-1} \leqslant a_{-2} \leqslant \cdots \to 0.$$

Let us introduce $\phi : [-N, N] \to \mathbb{R}$, piecewise affine and constant at infinity, by

$$\phi\left(\frac{1}{k}\right) = a_k \quad \text{and} \quad \phi(0) = 0.$$

Then, by definition, $\phi(u_0) \geqslant v_0$, $\{\phi = 0\} = \{0\}$ and ϕ is nondecreasing continuous. Thanks to Proposition 2.5, the solution $u_\phi := [\phi(u_0) \wedge u^+, u^-, u^+]$ has the same zero level-set as u, and is bigger than v by comparison principle. Hence

$$\{v \geqslant 0\} \subset \{u_\phi \geqslant 0\} = \{u \geqslant 0\}.$$

We prove the inverse inclusion switching u_0 and v_0.

Now, we assume that $u_0 = v_0$, $u^- = v^-$ and $u^+ \leqslant v^+$. Then, by comparison principle, $u \leqslant v$. We have just seen that there exists $\phi : [-N, N] \to \mathbb{R}$ nondecreasing continuous such that $\phi(u^+) \geqslant v^+$ and $\{\phi = 0\} = \{0\}$. Let $u_\phi = [u_0, u^-, \phi(u^+) \vee u^-]$. We saw that u_ϕ has the same zero set as u. In addition, by comparison, $u_\phi \geqslant v$. As the matter of fact,

$$\{u = 0\} = \{v = 0\} = \{u_\phi = 0\}.$$

If we delete the assumption $u^+ \leqslant v^+$, notice that $[u_0, u^-, u^+]$ and $[u_0, u^-, u^+ \wedge v^+]$ have the same zero set, so do $[u_0, u^-, v^+]$ and $[u_0, u^-, u^+ \wedge v^+]$. Hence $[u_0, u^-, u^+]$ and $[u_0, u^-, v^+]$ have the same zero set.

Of course, changing only u^- yields to the same result.

We deal the general case changing the data (u_0, u^-, u^-) one by one.

□

3.4 Obstacles create fattening

Although the fattening phenomenon may already occur without any obstacle (see [BNP98] for examples and [BSS93, BCL$^+$08] for more general discussion), obstacles can also generate fattening whereas the free evolution is smooth. Consider A a set of three points in $\mathbb{R}^2$ spanning an equilateral triangle and S a sphere enclosing it, centered on the triangle's center. Let $u^- = -1$, $u^+ = d(\cdot, A)$ and $u_0 = d(\cdot, S)$ (d is the signed distance).

It is possible to show (see next section) that the level sets $\{u(\cdot,t) \leqslant \alpha\}$ are minimizing hulls, hence are convex. So, the level set $\{u \leqslant 0\}$ contains the equilateral triangle. On the other hand, the level sets $\{u \leqslant -\delta\}$ behave as if there were no obstacles at all (in Proposition 2.2, one can take $u^+ \equiv 1$ which has the same $-\delta$-set as $d(\cdot, A)$), so they disappear in finite time. As a result, $u = 0$ in the whole triangle, and $\{u = 0\}$ develops non empty interior.

4 Long-time behavior

In this section, we study the behavior of the mean curvature flow only [2] with no forcing term in large times. In particular, we show that for relevant initial conditions, the flow has a limit.

In order to get some monotonicity properties of the flow, we will link our approach to a variational discrete flow built in [Spa11] and [ACN12]. Starting from a set E_0 and obstacle Ω (which corresponds, in our framework, to $E_0 = \{u_0 \geqslant \alpha\}$ and $\Omega = \{u^+ \leqslant \alpha\}$ for all α-level set), these two papers introduce the following minimizing scheme with step h:

$$E_h(t) = T_h^{[t/h]}(E_0)$$

with

$$T_h(E) = \arg\min_{\Omega \subset F} \left[\mathrm{Per}(F) + \frac{1}{h} \int_{F \Delta E} |d_E| \right] \tag{2.10}$$

where d_E is the signed distance to E (negative inside E).

Spadaro introduces the notion of minimizing hull: E is said to be a minimizing hull if $|\partial E| = 0$ (this is not assumed in the definition in [Spa11], but is assumed stating minimizing hull properties).

$$\mathrm{Per}(E) \leqslant \mathrm{Per}(F), \quad \forall F \supset E \text{ satisfying } F \setminus E \text{ is compact.}$$

He shows that if E is a minimizing hull with measure-zero boundary, then for every h, one can define a maximal minimizer in (2.10), still denoted in what follows by $T_h(E)$. Spadaro proves that $T_h(E) \subset E$ and $T_h(E)$ is still a minimizing hull. Moreover, if F satisfies the same assumptions and $F \subset E$, then $T_h(F) \subset T_h(E)$.

We end the general properties of this discrete flow by the

Remark. Let E be a minimizing hull and $h > \tilde{h}$. Then, $T_h(E) \subset T_{\tilde{h}}(E)$.

2. That means $u_t = |\nabla u| \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right)$.

Indeed, Let $F := T_h(E)$ and $\tilde{F} := T_{\tilde{h}}E$. Since E is a minimizing hull, $F, \tilde{F} \subset E$ so $d_E \leqslant 0$ on $F \cup \tilde{F}$. Using the very definition of F and $\tilde{F}$, one can write

$$\mathrm{Per}(F \cap \tilde{F}) + \frac{1}{h}\int_{F\cap\tilde{F}} d_E \geqslant \mathrm{Per}\, F + \frac{1}{h}\int_F d_E$$

$$\mathrm{Per}(F \cup \tilde{F}) + \frac{1}{\tilde{h}}\int_{F\cup\tilde{F}} d_E \geqslant \mathrm{Per}\, \tilde{F} + \frac{1}{\tilde{h}}\int_{\tilde{F}} d_E.$$

Summing, we get

$$\mathrm{Per}(F\cap\tilde{F}) + \mathrm{Per}(F\cup\tilde{F}) + \frac{1}{h}\int_{F\cap\tilde{F}} d_E + \frac{1}{\tilde{h}}\int_{F\cup\tilde{F}} d_E \geqslant \mathrm{Per}\, F + \mathrm{Per}\, \tilde{F} + \frac{1}{h}\int_F d_E + \frac{1}{\tilde{h}}\int_{\tilde{F}} d_E.$$

Since $\mathrm{Per}(F\cap\tilde{F}) + \mathrm{Per}(F\cup\tilde{F}) \leqslant \mathrm{Per}\, F + \mathrm{Per}\, \tilde{F}$, one has

$$\frac{1}{h}\int_{F\cap\tilde{F}} d_E + \frac{1}{\tilde{h}}\int_{F\cup\tilde{F}} d_E \geqslant \frac{1}{h}\int_F d_E + \frac{1}{\tilde{h}}\int_{\tilde{F}} d_E,$$

which means

$$\frac{1}{\tilde{h}}\int_{F\setminus\tilde{F}} d_E \geqslant \frac{1}{h}\int_{F\setminus\tilde{F}} d_E,$$

hence

$$\int_{F\setminus\tilde{F}} d_E \left(\frac{1}{\tilde{h}} - \frac{1}{h}\right) \geqslant 0.$$

Then, since $|\partial E| = 0$, $|F \setminus \tilde{F}| = 0$. By maximality of $\tilde{F}$, one concludes $F \subset \tilde{F}$.

To pass to the limit in h, we want to control the motion speed. First, we compare the constrained and the free motions.

Proposition 2.6. *Let E_0 be a minimizing hull containing Ω. Let E_f be the free evolution of E_0 (E_f solves* (2.10) *with no constraint) and E_c the regular evolution (E_c solves* (2.10) *and is maximal). Then, $E_f \cup \Omega \subset E_c$.*

Proof. Using the definition of E_f and E_c, one can write

$$\mathrm{Per}(E_f \cap E_c) + \int_{E_f\cap E_c} \frac{d_{E_0}}{h} \geqslant \mathrm{Per}(E_f) + \int_{E_f} \frac{d_{E_0}}{h}$$

$$\mathrm{Per}(E_f \cup E_c) + \int_{E_f\cup E_c} \frac{d_{E_0}}{h} \geqslant \mathrm{Per}(E_c) + \int_{E_c} \frac{d_{E_0}}{h}.$$

Summing and using $\mathrm{Per}(E\cap F) + \mathrm{Per}(E \cup F) \leqslant \mathrm{Per}\, E + \mathrm{Per}\, F$, we get

$$\int_{E_c\cap E_f} \frac{d_{E_0}}{h} + \int_{E_c\cup E_f} \frac{d_{E_0}}{h} \geqslant \int_{E_f} \frac{d_{E_0}}{h} + \int_{E_c} \frac{d_{E_0}}{h},$$

which is an equality. We conclude that all the inequalities above are equalities. In particular,

$$\mathrm{Per}(E_f \cup E_c) + \int_{E_f\cup E_c} \frac{d_{E_0}}{h} = \mathrm{Per}(E_c) + \int_{E_c} \frac{d_{E_0}}{h},$$

which shows that $E_f \cup E_c$ is a minimizer of (2.10). Since E_c is a maximal minimizer, one has $E_f \subset E_c$.

One can also notice that by definition, $\Omega \subset E_c$ so $E_f \cup \Omega \subset E_c$. □

Then, it is easy to see that

— A ball $B_R(x_0)$ is a minimizing hull,

— For $h \leqslant \frac{R^2}{4n}$ The free evolution of $B_R(x_0)$ is $B_r(x_0)$ with $r = \frac{R+\sqrt{R^2-4nh}}{2}$.

Thanks to the monotonicity of the flow starting from minimizing hulls ([Spa11], Lemma 3.5) and the last proposition, one can conclude that the evolution E_h of a minimizing hull E_0 contains the free evolution of every ball inside E_0.

More generally, for every function $u_0 : \mathbb{R}^n \to [-1,1]$ whose level sets are minimizing hulls (we always assume this condition in the following) and an obstacle $v : \mathbb{R}^n \to [-1,1]$ with $u_0 \geqslant v$, one wants to define an evolution $u_h : \mathbb{R}^n \times [0,T[\to [-1,1]$ by posing for all $s \in [-1,1]$, $E_s := \{u_0 \leqslant s\}$ and

$$\{u_h(t) \leqslant s\} = (E_s)_h(t).$$

This is well defined (in particular, $\{u_h(t) \leqslant s\} \subset \{u_h(t) \leqslant s'\}$ if $s \leqslant s'$) thanks to the

Proposition 2.7. *Let $\Omega^1 \subset \Omega^2$ be two obstacles and $E^1 \subset E^2$ be two minimizing hulls containing respectively Ω^1 and Ω^2. Then, $E^1_h \subset E^2_h$.*

Proof. Use the definition to write

$$\mathrm{Per}(E^1_h \cap E^2_h) + \int_{E^1_h \cap E^2_h} \frac{d_{E^1}}{h} \geqslant \mathrm{Per}(E^1_h) + \int_{E^1_h} \frac{d_{E^1}}{h},$$

$$\mathrm{Per}(E^1_h \cup E^2_h) + \int_{E^1_h \cup E^2_h} \frac{d_{E^2}}{h} \geqslant \mathrm{Per}(E^2_h) + \int_{E^2_h} \frac{d_{E^2}}{h},$$

Summing and simplifying, we get

$$\int_{E^1_h \cap E^2_h} \frac{d_{E^1}}{h} + \int_{E^1_h \cup E^2_h} \frac{d_{E^2}}{h} \geqslant \int_{E^h_1} \frac{d_{E^1}}{h} + \int_{E^h_2} \frac{d_{E^2}}{h}$$

which can be read

$$\int_{E^1_h \setminus E^2_h} \frac{d_{E^2}}{h} \geqslant \int_{E^1_h \setminus E^2_h} \frac{d_{E^1}}{h}.$$

Since $E_1 \subset E_2$, one has $d_{E_2} \leqslant d_{E_1}$ which shows that the last inequality is in fact an equality, showing as above that $E^1_h \subset E^2_h$. □

One can easily notice the two following points:

— The scheme is invariant by translation (if $\tilde{u}(x) := u(x+z)$ for some $z \in \mathbb{R}^n$, one has $\tilde{u}_h(x) = u_h(x+z)$ where $\tilde{u}_h$ is computed using the obstacle $v(\cdot + z)$).

— Proposition 2.7 gives the following monotonicity. If $u \leqslant \tilde{u}$ are two functions whose level sets are minimizing hulls, $v \geqslant \tilde{v}$ two obstacle functions, then $u_h \leqslant \tilde{u}_h$.

Now, we want to pass to the limit in h in the construction above. We will use the

Proposition 2.8. *If u_0 and v are uniformly continuous (with modulus ω), then the family (u_h) is equicontinuous in space (with modulus ω) and time.*

Proof. — Space continuity. The space continuity is easy to deduce. By continuity and translation invariance, $\tilde{u}_0(x) := u_0(x+z) \leqslant u_0(x) + \omega(|z|)$ and $\tilde{v} = v(\cdot + z) \leqslant v + \omega(|z|)$ so $\tilde{u}_h \leqslant u_h + \omega(|z|)$, which was expected

— Time continuity. Let $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$. Let $r > 0$. By uniform continuity, on $B_r(x)$, $u_h(t) \leqslant u_h(x,t) + \omega(r)$, which means that $A^r := \{u_h \leqslant u_h(x,t) + \omega(r)\}$ contains $B_r(x)$. Thanks to Proposition 2.6, the time evolution of A^r contains the free evolution of $B_r(x)$, as long as the latter exists. That means $u_h(x,t+s) \leqslant u_h(x,t) + \omega(r)$ for $s \leqslant T_r$, extinction time of $B_r(x)$. It is easy to see that this time is controlled, for a sufficiently small h, by $\frac{r^2}{\sqrt{16h}}$.
We proved that for h small enough, u_h is continuous in time with modulus $\tilde{\omega}(T_r) \leqslant \omega(r)$.

□

Corollary 2.1. *Up to a subsequence, the collection $(u_h)_h$ has a limit which is uniformly continuous in space and time.*

Let us denote it by u (we will see that this limit does not depend on the subsequence). We are now able to show the main proposition of this section.

Proposition 2.9. *The function u is a viscosity solution of* (2.2).

Proof. We have just seen that u is uniformly continuous in space and time. In addition, $u \geqslant v$ by construction and the initial conditions are satisfied. We only have to check the fourth point of the definition (we only deal with the subsolution thing, the supersolution one can be treated similarly). Let $(x,t) \in \mathbb{R}^n$. Either $u(x,t) = v(x,t)$ and nothing has to be done, or $u(x,t) > v(x,t)$. In this case, one can directly apply [CMP12], Th. 4.6 or, with a setting closer to ours, [Tho12], Th 3.6.1. See also [EGI12]. □

4.1 The limit is locally minimal

For this section, we deal only with mean curvature motion without forcing term. Thanks to Proposition 2.9, if u_0 has minimizing hull level sets, so does $u(\cdot,t)$. Indeed, $u_h(\cdot,t)$ has thanks to Spadaro's work, and since we have $u_h \to u$ uniformly on compact sets, we have L^1 convergence of the (compact) level sets of u_h to the level sets of u. The minimizing hull property is stable under L^1 convergence of sets (see [Spa11]).

In addition, u is nondecreasing in time (this is true for u_h). As u is uniformly equicontinuous on each compact set, letting t go to $+\infty$ we have a locally uniform convergence to a limit u_∞ which is a solution of

$$|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0$$

with obstacles u^+, u^-, thanks to classical theory of viscosity solutions.

Thanks to [ISZ98], Theorem 3.10, one has the following result.

Proposition 2.10. *Let $s \in u_\infty(\mathbb{R}^n)$ such that $\mathcal{H}^{n-1}(\{u = s\}) < \infty$. Moreover, let $\Omega = \{u^+ \geqslant s\} \cup \{u^- \leqslant s\}$. Then, there exists a relatively open set $U \subset u^{-1}(s)$ with $H^{n-8-\alpha}(u^{-1}(s) \setminus U) = 0$ for all $\alpha > 0$, such that $u^{-1}(s) \setminus \Omega$ is an analytic minimal surface in a neighborhood of each point of U. Moreover, it is stable and stationnary in the varifold sense (classically on U).*

Note in particular that non empty interior can occur for only countable many s.

4.2 Comparison with mean convex hull

In [Spa11], E. Spadaro deals with a so called *mean convex hull*, which is built as a limit in large time of the scheme (2.10). More precisely, he calls a set $\Theta \in \mathbb{R}^n$ a global barrier is

$$\Sigma \text{ minimal hypersurface, } \partial\Sigma \subset \Theta \Longrightarrow \Sigma \subset \Theta.$$

Letting $\mathcal{A}$ denote the family of global barriers, the mean convex hull of an obstacle $\Omega \subset \mathbb{R}^n$ is defined by

$$\Omega^{\mathrm{mc}} := \bigcap_{\Omega \subset \Theta \in \mathcal{A}} \Theta.$$

With $\partial\Omega \in \mathcal{C}^{1,1}$ and $n \leqslant 7$, Spadaro shows that starting from a sphere $S \supset \Omega$, his discrete mean curvature motion converge to Ω^{mc} with $\mathcal{C}^{1,1}$ boundary and that outside $\partial\Omega$, $\partial\Omega^{\mathrm{mc}}$ is a minimal surface.

Let u_0 be any function with zero level set equal to S, $u^+ = d(\cdot, \Omega)$ and $u^- = -1$. Since Spadaro's work is in low dimension, the open set U in Proposition 2.10 is the whole $u^{-1}(s)$. Let us assume that $u^{-1}(0)$ does not fatten. Hence, $\partial\{u \leqslant 0\} = \{u = 0\}$ and $\{u = 0\} \setminus \Omega$ is a minimal hypersurface with boundary in Ω. Using the very definition of the global barrier, we deduce that $\{u \leqslant 0\} \subset \Omega^{mc}$.

Chapter 3

Mean curvature flow with obstacles: existence, uniqueness and regularity of solutions (with M. Novaga)

The results of this chapter constitute a paper which has been accepted for publication in Interfaces and Free Boundaries.

1 Introduction and main results

Mean curvature flow is a prototypical geometric evolution, arising in many models from Physics, Biology and Material Science, as well as in a variety of mathematical problems. For such a reason, this flow has been widely studied in the past years, starting from the pioneristic work of K. Brakke [Bra78] (we refer to [GH86, Hui84, EH89, ES91, CGG91] for a far from complete list of references).

In some models, one needs to include the presence of hard obstacles, which the evolving surface cannot penetrate (see for instance [ESV12] and references therein). This leads to a double obstacle problem for the mean curvature flow, which reads

$$v = H \qquad \text{on } M_t \cap U, \tag{3.1}$$

with constraint

$$M_t \subset \overline{U} \qquad \text{for all } t, \tag{3.2}$$

where v, H denote respectively the normal velocity and d times the mean curvature of the interface M_t, and the open set $U \subset \mathbb{R}^{d+1}$ represents the obstacle. Notice that, due to the presence of obstacles, the evolving interface is in general only of class $C^{1,1}$ in the space variable, differently from the unconstrained case where it is analytic (see [ISZ98]). While the regularity of parabolic obstacle problems is relatively well understood (see [Sha08] and references therein), a satisfactory existence and uniqueness theory for solutions is still missing.

In [ACN12] (see also [Spa11]) the authors approximate such an obstacle problem with an implicit variational scheme introduced in [ATW93, LS95]. As a byproduct, they prove global existence of weak (variational) solutions, and short time existence and uniqueness of regular solutions in the two-dimensional case. In [Mer14] the first author adapts to this setting the theory of viscosity solutions introduced in [CIL92, CGG91], and constructs globally defined continuous (viscosity) solutions.

Let us now state the main results of this chapter.

Theorem 3.1. *Let $M_0 \subset U$ be an initial hypersurface, and assume that both M_0 and ∂U are uniformly of class $C^{1,1}$, with $d(M_0, \partial U) > 0$ outside of a compact subset of $\mathbb{R}^n$. Then there exists $T > 0$ and a unique solution M_t to* (3.1), (3.2) *on $[0, T)$, such that M_t is of class $C^{1,1}$ for all $t \in [0, T)$.*

Notice that Theorem 3.1 extends a result in [ACN12] to dimensions greater than two.

When the hypersurface M_t can be written as the graph of a function $u(\cdot, t) : \mathbb{R}^d \to \mathbb{R}$, equation (3.1) reads

$$u_t = \sqrt{1 + |\nabla u|^2} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right). \tag{3.3}$$

If the obstacles are also graphs, the constraint (3.2) can be written as

$$\psi^- \leqslant u \leqslant \psi^+ , \tag{3.4}$$

where the functions $\psi^\pm : \mathbb{R}^n \to \mathbb{R}$ denote the obstacles.

Theorem 3.2. *Assume that $\psi^\pm \in C^{1,1}(\mathbb{R}^d)$, and let $u_0 \in C^{1,1}(\mathbb{R}^d)$ satisfy* (3.4). *Then there exists a unique (viscosity) solution u of* (3.3), (3.4) *on $\mathbb{R}^d \times [0, +\infty)$, such that*

$$\begin{aligned} \|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} &\leq \max\left(\|\nabla u_0\|_{L^\infty(\mathbb{R}^d)}, \|\nabla \psi^\pm\|_{L^\infty(\mathbb{R}^d)} \right) \\ \|u_t(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} &\leq \left\| \sqrt{1 + |\nabla u_0|^2} \operatorname{div} \left(\frac{\nabla u_0}{\sqrt{1 + |\nabla u_0|^2}} \right) \right\|_{L^\infty(\mathbb{R}^d)} \end{aligned}$$

for all $t > 0$. Moreover u is also of class $C^{1,1}$ uniformly on $[0, +\infty)$.

We observe that Theorem 3.2 extends previous results by Ecker and Huisken [EH89] in the unconstrained case (see also [CN13a]).

Theorem 3.3. *Assume that u_0 and $\psi^\pm$ are Q-periodic, with periodicity cell $Q = [0, L]^d$, for some $L > 0$. Then the solution $u(\cdot, t)$ of* (3.3), (3.4) *is also Q-periodic. Moreover there exists a sequence $t_n \to +\infty$ such that $u(\cdot, t_n)$ converges uniformly as $n \to +\infty$ to a stationary solution to* (3.3), (3.4).

Our strategy will be to approximate the obstacles with "soft obstacles" modeled by a sequence of uniformly bounded forcing terms. Differently from [ACN12], where the existence of regular solution is derived from variational estimates on the approximating scheme, we obtain estimates on the evolving interface, in the spirit of [EH91a, EH91b, CNV11], which are uniform in the forcing terms.

2 Mean curvature flow with a forcing term

2.1 Evolution of geometric quantities

Let M be a complete orientable d-dimensional Riemannian manifold without boundary, let $F(\cdot,t) : M \to \mathbb{R}^{d+1}$ be a smooth family of immersions, and denote by M_t the image $F(M,t)$. Since M_t is orientable, we can write $M_t = \partial E(t)$ where $E(t)$ is a family of open subsets of $\mathbb{R}^{d+1}$ depending smoothly on t. We say that M_t evolves by mean curvature with forcing term k if

$$\frac{d}{dt}F(p,t) = -\big(H(p,t) + k(F(p,t))\big)\,\nu(p,t), \tag{3.5}$$

where $k : \mathbb{R}^{d+1} \to \mathbb{R}$ is a smooth forcing term, ν is the unit normal to M_t pointing outside $E(t)$, and H is (d times) the mean curvature of M_t, with the convention that H is positive whenever $E(t)$ is convex.

We shall compute the evolution of some relevant geometric quantities under the law (3.5). We denote by ∇^S, Δ^S respectively the covariant derivative and the Laplace-Beltrami operator on M. As in [Hui84], the metric on M_t is denoted by $g_{ij}(t)$, it inverse is $g^{ij}(t)$, the scalar product (or any tensors contraction using the metric) on M_t is denoted by $\langle\cdot\,,\,\cdot\rangle$ whereas the ambiant scalar product is $(\cdot\,,\,\cdot)$, the volume element is μ_t, and the second fondamental form is A. In particular we have $A\left(\partial_i, \partial_j\right) = h_{ij}$, where we set for simplicity $\partial_i = \frac{\partial}{\partial x_i}$, and $H = h_{ii}$, using the Einstein notations (we implicitly sum every index which appears twice in an expression). We also denote by $\lambda_1, \ldots, \lambda_d$ the eigenvalues of A.

Notice that, in terms of the parametrization F, we have

$$g_{ij} = \left(\partial_i F\,,\,\partial_j F\right), \qquad h_{ij} = -\left(\partial^2_{ij}F\,,\,\nu\right) \qquad \text{for all } i,j \in \{1,\ldots,d\}. \tag{3.6}$$

Let us recall that the covariant derivatives commute as follow

$$\nabla^S_i\nabla^S_j X^k = \nabla^S_j\nabla^S_i X^k + R^k_{ijh}X^h \tag{3.7}$$

where

$$R^k_{ijh}X^h = (h_{lj}h_{ih} - h_{lh}h_{ij})g^{kl}X^h.$$

In addition, the Gauss Weingarten relations are (as usual, we denote the Christoffel symbols by Γ^k_{ij})

$$\partial^2_{ij}F = \Gamma^k_{ij}\partial_k F - h_{ij}\nu, \quad \partial_j\nu = h_{jl}g^{lm}\partial_m F. \tag{3.8}$$

We also give the Codazzi's equations

$$\forall i,k,l, \quad (\nabla_i h)_{kl} = (\nabla_k h)_{il} = (\nabla_l h)_{ik}. \tag{3.9}$$

Lemma 3.1. *One has (the formula is stated without proof in [Hui84], and is sometimes called Simons' identity)*

$$\Delta^S h_{ij} = \nabla^S_i\nabla^S_j H + Hh_{il}g^{lm}h_{mj} - |A|^2 h_{ij} \tag{3.10}$$

Proof. First, notice that Codazzi's equations imply

$$(\nabla^S_k \nabla^S_i h)_{jl} = (\nabla^S_k \nabla^S_j h)_{il}.$$

Indeed,

$$(\nabla^S_k \nabla^S_i h)_{jl} = \partial_k(\nabla^S_i h_{jl}) + \Gamma^m_{kl} \nabla^S_i h_{jm} + \Gamma^m_{kj} \nabla^S_i h_{lm}$$

and the two quantities are symmetric in i and j.

Then, we just write (we do not use the normal coordinates to see exactly how the terms behave)

$$\begin{aligned}
(\Delta^S h)_{ij} &= g^{kl} \nabla^S_k \nabla^S_l h(\partial_i, \partial_j) = g^{kl} \nabla^S_k \nabla^S_l h_i j \\
&= g^{kl} \nabla^S_k \nabla^S_i h_{jl} \quad \text{(using Codazzi)} \\
&= g^{kl} \left(\nabla^S_i \nabla^S_k h_{jl} + R_{kijm} h_{ml} + R_{kilm} h_{mj} \right) \\
&= g^{kl} \left(\nabla^S_i \nabla^S_j h_{kl} + R_{kijm} g^{ms} h_{sl} + R_{kilm} g^{ms} h_{sj} \right) \quad \text{(using Codazzi again)} \\
&= g^{kl} \left(\nabla^S_i \nabla^S_j h_{kl} + (h_{kj} h_{im} - h_{km} h_{ij}) g^{ms} h_{sl} + (h_{kl} h_{im} - h_{km} h_{il}) g^{ms} h_{sj} \right) \\
&= g^{kl} \nabla^S_i \nabla^S_j h_{kl} - |A|^2 h_{ij} + H h_{im} g^{sm} h_{sj} + g^{kl} g^{sm} h_{kj} h_{sl} h_{mi} - g^{kl} g^{ms} h_{km} h_{il} h_{sj}.
\end{aligned}$$

Since $\nabla_i g^{kl} = 0$ (Ricci's lemma), the first term is exactly $\nabla^S_i \nabla^S_j H$. In addition, switching the indices (k, l) and (s, m), the term $g^{kl} g^{sm} h_{kj} h_{sl} h_{mi} - g^{kl} g^{ms} h_{km} h_{il} h_{sj}$ vanishes. □

Proposition 3.1. *The following equalities hold:*

$$\frac{d}{dt} g_{ij} = -2(H + k) h_{ij} \tag{3.11}$$

$$\frac{d}{dt} \nu = \nabla^S (H + k) \tag{3.12}$$

$$\frac{d}{dt} \mu_t = -H(H + k) \mu_t \tag{3.13}$$

$$\frac{d}{dt} h_{ij} = \Delta^S h_{ij} + \nabla^S_i \nabla^S_j k - 2H h_{il} g^{lm} h_{mj} - k g^{ml} h_{im} h_{jl} + |A|^2 h_{ij} \tag{3.14}$$

$$\frac{d}{dt} H = \Delta^S (H + k) + (H + k)|A|^2 \tag{3.15}$$

$$\frac{d}{dt} |A|^2 = \Delta^S |A|^2 + 2k g^{ij} g^{sl} g^{mn} h_{is} h_{lm} h_{nj} + 2|A|^4 - 2|\nabla^S A|^2 + 2 \left\langle A\, ,\, (\nabla^S)^2 k \right\rangle \tag{3.16}$$

Proof. The proof follows by direct computations as in [Hui84, EH91b]. First, note that since $(\nu\, ,\, \nu) = 1$, we have

$$(\partial_i \nu\, ,\, \nu) = (\partial_t \nu\, ,\, \nu) = 0.$$

Recalling (3.6), we get

$$\frac{d}{dt} g_{ij} = \frac{d}{dt} (\partial_i F\, ,\, \partial_j F) = -(H + k) \left((\partial_i \nu\, ,\, \partial_j F) + (\partial_i F\, ,\, \partial_j \nu) \right) = -2(H + k) h_{ij}$$

Let us notice that given a vector field X tangent to M, one can write $X = \langle X\,,\,\partial_i F\rangle\, g^{ij}\partial_j F$. Since $\partial_t \nu$ is tangent to M_t, we write

$$\begin{aligned}
\frac{d}{dt}\nu &= \left(\frac{d}{dt}\nu\,,\,\partial_i F\right) g^{ij}\partial_j F = -\left(\nu\,,\,\frac{d}{dt}\partial_i F\right) g^{ij}\partial_j F \\
&= (\nu\,,\,\partial_i((H+k)\nu))\, g^{ij}\partial_j F = \partial_i(H+k) g^{ij}\partial_j F = \nabla^S(H+k).
\end{aligned}$$

The evolution of the measure on M_t

$$\mu_t = \sqrt{\det[g]}$$

is given by

$$\begin{aligned}
\frac{d}{dt}\sqrt{\det[g]} &= \frac{\frac{d}{dt}\det[g]}{2\sqrt{\det[g]}} = \frac{\det[g]\cdot \operatorname{Tr}\left(g^{ij}\,\frac{d}{dt}g_{ij}\right)}{2\sqrt{\det[g]}} \\
&= -\sqrt{\det[g]}\cdot (H+k) g^{ij} h_{ji} = -\mu_t H(H+k).
\end{aligned}$$

In order to prove (3.14) we compute, using (3.8),

$$\begin{aligned}
\frac{d}{dt}h_{ij} &= -\frac{d}{dt}\left(\nu\,,\,\partial^2_{ij}F\right) \\
&= -\left(\nabla^S(H+k)\,,\,\partial^2_{ij}F\right) + \left(\partial^2_{ij}(H+k)\nu\,,\,\nu\right) \\
&= -\left(g^{kl}\partial_k(H+k)\partial_l F\,,\,\Gamma^k_{ij}\partial_k F - h_{ij}\nu\right) \\
&\quad +\partial^2_{ij}(H+k) + (H+k)\left(\partial_j\left(h_{im}g^{ml}\partial_l F\right)\,,\,\nu\right) \\
&= \partial^2_{ij}(H+k) - \Gamma^k_{ij}\partial_k(H+k) + (H+k)h_{im}g^{ml}\left(\Gamma^k_{lj}\partial_k F - h_{lj}\nu\,,\,\nu\right) \\
&= \nabla^S_i\nabla^S_j(H+k) - (H+k)h_{il}g^{lm}h_{mj}.
\end{aligned} \tag{3.17}$$

Lemma 3.1 implies (3.14) follows from (3.17). From (3.14) we deduce

$$\begin{aligned}
\frac{d}{dt}H &= \frac{d}{dt}g^{ij}h_{ij} \\
&= 2(H+k)g^{is}h_{sl}g^{lj}h_{ij} + g^{ij}\left(\nabla^S_i\nabla^S_j(H+k) - (H+k)h_{il}g^{lm}h_{mj}\right) \\
&= \Delta^S(H+k) + (H+k)|A|^2,
\end{aligned}$$

which gives (3.15). In addition, we get

$$\begin{aligned}
\frac{d}{dt}|A|^2 &= \frac{d}{dt}\left(g^{ik}g^{jl}h_{ij}h_{kl}\right) \\
&= 2\frac{d}{dt}g^{jl}h_{ij}h_{kl} + 2g^{ik}g^{jl}\frac{d}{dt}h_{ij}h_{kl} \\
&= 2\left(2(H+k)g^{js}h_{st}g^{tl}\right)g^{jl}h_{ij}h_{kl} \\
&\quad + 2g^{ik}g^{jl}\left(\Delta^S h_{ij} + \nabla^S_i\nabla^S_j k - 2Hh_{il}g^{lm}h_{mj} - kg^{ml}h_{im}h_{jl} + |A|^2 h_{ij}\right)h_{kl} \\
&= 2kg^{js}h_{st}g^{tl}g^{jl}h_{ij}h_{kl} + 2g^{ik}g^{jl}\Delta^S h_{ij}h_{kl} + 2|A|^4 + 2\left\langle A\,,\,(\nabla^S)^2 k\right\rangle.
\end{aligned} \tag{3.18}$$

On the other hand, one has

$$\Delta^S|A|^2 = 2\left\langle \Delta^S A\,,\, A\right\rangle + 2|\nabla^S A|^2 = 2g^{pq}g^{mn}h_{pm}\Delta^S h_{qn} + 2|\nabla^S A|^2. \tag{3.19}$$

so that (3.16) follows from (3.19) and (3.18). □

2.2 Higher derivatives of A

In this subsection, we show that the results in [Hui84], Section 7 still hold with a forcing term. More precisely, we show the

Theorem 3.4. *For any $m \in \mathbb{N}$, we have*

$$\begin{aligned}(\partial_t - \Delta^S)|\nabla^m A|^2 &= -2|\nabla^{m+1}A|^2 + \nabla^{m+2}k * \nabla^m A \\ &\quad + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^l A * \nabla^m A + \nabla^i A * \nabla^j A * \nabla^l k * \nabla^m A.\end{aligned}$$

where the notation $*$ denotes any contraction with the metric.

To prove this theorem, we first have to control the derivatives of the Christoffel symbols.

Proposition 3.2. *The Christoffel symbols satisfy*

$$\partial_t \Gamma_{ij}^k = g^{kl}\left(\nabla_i(\partial_t g)_{jl} + \nabla_j(\partial_t g)_{il} - \nabla_l(\partial_t)g_{ij}\right) \tag{3.20}$$

Proof. We simply compute

$$\begin{aligned}2\partial_t\Gamma_{ij}^k &= \partial_t\left(g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)\right) \\ &= \partial_t g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) + g^{kl}\left(\partial_i\partial_t g_{jl} + \partial_j\partial_t g_{il} - \partial_l\partial_t g_{ij}\right) \\ &= 2\partial_t g^{kl}\Gamma_{ij}^m g_{lm} + g^{kl}\left(\partial_i\partial_t g_{jl} + \partial_j\partial_t g_{il} - \partial_l\partial_t g_{ij}\right).\end{aligned}$$

Now, we want to replace the quantities $\partial_\alpha g_{\beta\gamma}$ by $(\nabla^\alpha g)_{\beta\gamma}$ using

$$(\nabla^\alpha g)_{\beta\gamma} = \partial_\alpha g_{\beta\gamma} - \Gamma_{\alpha\beta}^\delta g_{\delta\gamma} - \Gamma_{\alpha\gamma}^\delta g_{\delta\alpha}.$$

That gives

$$\begin{aligned}2\partial_t\Gamma_{ij}^k &= 2\partial_t g^{kl}\Gamma_{ij}^m g_{lm} + g^{kl}\left(\nabla_i(\partial_t g)_{jl} + \nabla_j(\partial_t g)_{il} - \nabla_l(\partial_t)g_{ij}\right) \\ &\quad + g^{kl}\left(\Gamma_{ij}^m\partial_t g_{ml} + \Gamma_{il}^m\partial_t g_{jm} + \Gamma_{ji}^m\partial_t g_{ml} + \Gamma_{jl}^m\partial_t g_{im} - \Gamma_{li}^m\partial_t g_{mj} - \Gamma_{lj}^m\partial_t g_{mi}\right) \\ &= 2\partial_t g^{kl}\Gamma_{ij}^m g_{lm} + g^{kl}\left(\nabla_i(\partial_t g)_{jl} + \nabla_j(\partial_t g)_{il} - \nabla_l(\partial_t)g_{ij}\right) + 2g^{kl}\Gamma_{ij}^m\partial_t g_{ml}.\end{aligned}$$

Differentiating the equation $g^{kl}g_{lm} = \delta_{kl}$, we get that $\partial_t g^{kl}g_{lm} + g^{kl}\partial_t g_{ml} = 0$ which shows (3.20).

To continue, let us state a

Lemma 3.2. *Let Λ and Ξ two tensors such that*

$$(\partial_t - \Delta^S)\Lambda = \Xi.$$

Then, the covariant derivative $\nabla^S\Lambda$ satisfies

$$(\partial_t - \Delta^S)\nabla^S\Lambda = A * A * \nabla^S\Lambda + A * \nabla A * \Lambda + A * k * \Lambda + \nabla^S\Xi$$

where $*$ denotes any contraction using the metric.

Proof. We only prove the result for $(2,0)$ tensors, to avoid heavy notations. Note that since $\partial_t g_{ij} = -2(H+k)h_{ij}$, we have

$$\partial_t\Gamma_{ij}^k = A * \nabla A + A * k$$

One also need to switch Laplacian and covariant derivative (remember that $\nabla_i g_{kl} = 0$, (3.7) and that the curvature tensor $R_m = A * A$):

$$\begin{aligned}
\nabla_i^S(\Delta^S\Lambda)\nabla_i^S(g^{st}\nabla_s^S\nabla_t^S\Lambda) &= g^{st}\nabla_i^S\nabla_s^S\nabla_t^S\Lambda = g^{st}(\nabla_s\nabla_i\nabla_t\Lambda - 2R_m * \nabla^S\Lambda)\\
&= g^{st}(\nabla_s\nabla_t\nabla_i\Lambda - 2\nabla^S(R_m * \Lambda) - 2R * \nabla^S\Lambda)\\
&= \Delta^S(\nabla_i\Lambda) - 4\nabla^S A * A * \Lambda - 4A * A * \nabla^S\Lambda.
\end{aligned}$$

One only has to compute, for any i,

$$\begin{aligned}
\partial_t\nabla_i\Lambda_{jk} &= \partial_t\left(\partial_i\Lambda_{jk} - \Gamma_{ij}^s\Lambda_{sk} - \Gamma_{ik}^s\Lambda_{sj}\right)\\
&= \partial_i\partial_t\Lambda_{jk} - \partial_t(\Gamma_{ij}^s)\Lambda_{sk} - \Gamma_{ij}^s\partial_t(\Lambda_{sk}) - \partial_t(\Gamma_{ik}^s)\Lambda_{sj} - \Gamma_{ik}^s\partial_t(\Lambda_s j)\\
&= \nabla_i(\partial_t\Lambda_{jk}) - \partial_t(\Gamma_{ij}^s)\Lambda_{sk} - \partial_t(\Gamma_{ik}^s)\Lambda_{sj}\\
&= \nabla_i(\Delta^S\Lambda_{jk} + \Xi) - 2A * \nabla A * \Lambda - 2A * k * \Lambda\\
&= \Delta^S\nabla_i\Delta_{jk}^S + \nabla_i\Xi - 6\nabla^S A * A * \Lambda - 4A * A * \nabla^S\Lambda - 2A * k * \Lambda.
\end{aligned}$$

□

Then, we have to show that

$$(\partial_t - \Delta^S)\nabla^m A = \nabla^{m+2}k + \sum_{i+j+l=m} \nabla^i A * \nabla^j A * \nabla^l A + \nabla^i A * \nabla^j A * \nabla^l k. \tag{3.21}$$

We proceed by induction. For $m = 1$, formula (3.14) gives the expected result. Let us assume that the formula is true on $[1, m]$. Then, using Lemma 3.2, we get

$$\begin{aligned}
(\partial_t - \Delta^S)\nabla^{m+1}A &= \nabla\left[\nabla^{m+2}k + \sum_{i+j+l=m} \nabla^i A * \nabla^j A * \nabla^l A + \nabla^i A * \nabla^j A * \nabla^l k\right]\\
&+ A * A * \nabla^{m+1}A + A * \nabla A * \nabla^m A + A * \nabla^m A * k\\
&= \nabla^{m+3}k + \sum_{i+j+l=m} \nabla^{i+1}A * \nabla^j A * \nabla^l A + \nabla^i A * \nabla^{j+1}A * \nabla^l A + \nabla^i A * \nabla^j A * \nabla^{l+1}A\\
&+ \sum_{i+j+l=m} \nabla^{i+1}A * \nabla^j A * \nabla^l k + \nabla^i A * \nabla^{j+1}A * \nabla^l k + \nabla^i A * \nabla^j A * \nabla^{l+1}k\\
&+ A * A * \nabla^{m+1}A + A * \nabla A * \nabla^m A + A * \nabla^m A * k\\
&= \nabla^{m+3}k + \sum_{i+j+l=m+1} \nabla^i A * \nabla^j A * \nabla^l A + \nabla^i A * \nabla^j A * \nabla^l k.
\end{aligned}$$

As a result, we can finish the proof of the theorem. Let us write

$$
\begin{aligned}
(\partial_t - \Delta^S)\frac{|\nabla^m A|^2}{2} &= \big\langle(\partial_t - \Delta^S)\nabla^m A\,,\,\nabla^m A\big\rangle - |\nabla^{m+1}A|^2 \\
&= \left\langle \nabla^{m+2}k + \sum_{i+j+l=m} \nabla^i A * \nabla^j A * \nabla^l A + \nabla^i A * \nabla^j A * \nabla^l k\,,\,\nabla^m A \right\rangle - |\nabla^{m+1}A|^2 \\
&= -|\nabla^{m+1}A|^2 + \nabla^{m+2}k * \nabla^m A + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^l A * \nabla^m A + \nabla^i A * \nabla^j A * \nabla^l k * \nabla^m A.
\end{aligned}
$$

□

2.3 The Monotonicity Formula

We extend Huisken's monotonicity formula [Hui90] to the forced mean curvature flow (3.5) (see also [CNV11, Section 2.2]).

Given a vector field $\omega : M_t \to \mathbb{R}^{d+1}$, we let

$$\omega^\perp = (\omega\,,\,\nu)\,\nu, \qquad \omega^T = \omega - \omega^\perp\,.$$

Letting $X_0 \in \mathbb{R}^{d+1}$ and $t_0 \in \mathbb{R}$, for $(X,t) \in \mathbb{R}^{d+1} \times (t_0, +\infty)$ we define the kernel

$$\rho(X,t) = \frac{1}{(4\pi(t_0-t))^{d/2}} \exp\left(\frac{-|X_0 - X|^2}{4(t_0-t)}\right).$$

A direct computation gives

$$\frac{d\rho}{dt} = -\Delta^S \rho + \rho\left(\frac{(X_0 - X\,,\,(H+k)\nu)}{t_0 - t} - \frac{|(X_0-X)^\perp|^2}{4(t_0-t)^2}\right). \tag{3.22}$$

Proposition 3.3 (Monotonicty Formula)**.**

$$\frac{d}{dt}\int_{M_t} \rho = -\int_{M_t} \rho\left(\left|H + \frac{k}{2} + \frac{(X - X_0\,,\,\nu)}{2(t_0-t)}\right|^2 - \frac{k^2}{4}\right).$$

Proof. Recalling (3.13), we compute

$$
\begin{aligned}
\frac{d}{dt}\int_{M_t}\rho &= \int_{M_t} \frac{d}{dt}\rho - H(H+k)\rho \\
&= \int_{M_t} \rho\left(-\frac{|X-X_0|^2}{4(t_0-t)^2} + \frac{d}{2(t_0-t)} - \frac{(X-X_0\,,\,\nu)}{2(t_0-t)}(H+k) - H(H+k)\right) \\
&= -\int_{M_t} \rho\left(\left|H\nu + \frac{X-X_0}{2(t_0-t)} + \frac{k\nu}{2}\right|^2 - \frac{k^2}{4}\right) + \int_{M_t} \frac{d}{2(t_0-t)}\rho + \int_{M_t} \rho\frac{(X-X_0\,,\,\nu)\,H}{2(t_0-t)}
\end{aligned}
$$

We use the first variation formula: for all vector field $\mathbf{Y}$ on M_t, we have

$$\int_{M_t} \mathrm{div}_{M_t}\,\mathbf{Y} = \int_{M_t} \langle H\nu\,,\,\mathbf{Y}\rangle\,.$$

As a result, with $\mathbf{Y} = \frac{\rho(X-X_0)}{2(t-t_0)}$, we get

$$\frac{d}{dt}\int_{M_t}\rho = -\int_{M_t}\rho\left(\left|H\nu + \frac{X-X_0}{2(t_0-t)} + \frac{k\nu}{2}\right|^2 - \frac{k^2}{4} - \frac{|(X-X_0)^T|^2}{4(t_0-t)^2}\right)$$
$$= -\int_{M_t}\rho\left(\left|H + \frac{(X-X_0\,,\,\nu)}{2(t_0-t)} + \frac{k}{2}\right|^2 - \frac{k^2}{4}\right).$$

□

In a similar way (see [EH89]) one can prove that for all functions $f(X,t)$ defined on M_t, one has

$$\partial_t\int_{M_t}\rho f = \int_{M_t}\left(\frac{df}{dt} - \Delta^S f\right)\rho - \int_{M_t} f\rho\left(\left|H + \frac{(X-X_0\,,\,\nu)}{2(t_0-t)} + \frac{k}{2}\right|^2 - \frac{k^2}{4}\right). \tag{3.23}$$

Indeed, using (3.22)

$$\frac{d}{dt}\int_{M_t}\rho f = \int_{M_t} f\frac{d\rho}{dt} + \frac{df}{dt}\rho - H(H+k)f\rho$$
$$= \int_{M_t} f\left(\frac{d\rho}{dt} - H(H+k)\rho\right) + \frac{df}{dt}\rho$$
$$= \int_{M_t} f\left(-\Delta^S\rho + \rho\left(\frac{(X_0-X\,,\,(H+k)\nu)}{t_0-t} - \frac{1}{4}\frac{|(X_0-X)^\perp|^2}{(t_0-t)^2}\right) - H(H+k)\rho\right) + \frac{df}{dt}\rho$$
$$= \int_{M_t} -\Delta^S f\rho + \left(\rho\left(\frac{(X_0-X\,,\,(H+k)\nu)}{t_0-t} - \frac{1}{4}\frac{|(X_0-X)^\perp|^2}{(t_0-t)^2}\right) - H(H+k)\rho\right) + \frac{df}{dt}\rho$$
$$= \int_{M_t}\rho\left(\frac{d}{dt}f - \Delta^S f\right) - \int f\rho\left(\left|H + \frac{(X-X_0\,,\,\nu)}{2(t_0-t)} + \frac{k}{2}\right|^2 - \frac{k^2}{4}\right).$$

Lemma 3.3. *Let f be defined on M_t and satisfy*

$$\frac{d}{dt}f - \Delta^S f \leqslant a\cdot\nabla^S f \quad \text{on } M_t \tag{3.24}$$

for some vector field a bounded on $[0,t_1]$. Then,

$$\sup_{M_t,\, t\in[0,t_1]} f \leqslant \sup_{M_0} f.$$

Proof. Denote by a_0 the bound on a, $k := \sup_{M_0} f$ and define $f_l = \max(f-l, 0)$. Assumption (3.24) implies

$$\left(\frac{d}{dt} - \Delta^S\right) f_l^2 \leqslant 2f_l a\cdot\nabla^S f_l - 2|\nabla^S f_l|^2$$

which, thanks to Young's inequality, gives

$$\left(\frac{d}{dt} - \Delta^S\right) f_l^2 \leqslant \frac{1}{2}a_0^2 f_l^2.$$

Applying (3.23) to f_l^2, we get

$$\frac{d}{dt}\int f_l^2\rho \leqslant \frac{1}{2}(a_0^2+\|k\|_\infty^2)\int f_l^2\rho. \tag{3.25}$$

Letting $l=\sup_{M_0} f$, so that $f_l\equiv 0$ on M_0, from (3.25) and the Gronwall's Lemma we obtain that $f_l\equiv 0$ on M_t for all $t\in(0,t_1]$, which gives thesis. □

3 Proof of Theorem 3.1

We now prove short time existence for the mean curvature flow with obstacles (3.1), (3.2). Let $M_0=\partial E(0)\subset U$, where we assume that U, $E(0)$ are open sets with boundary uniformly of class $C^{1,1}$. In particular, M_0 satisfies a uniform exterior and interior ball condition, that is, there is $R>0$ such that, for every $x\in M_0$, one can find two open balls B^+ and B^- of radius R which are tangent to M_0 at x and such that $B^+\subset E(0)^c$ and $B^-\subset E(0)$. Let also $\Omega^-:=E(0)\setminus\overline{U}$, and $\Omega^+:=E(0)\cup U$. Notice that $\Omega^\pm$ are open sets with $C^{1,1}$ boundaries, with $\mathrm{dist}(\Omega^-,\partial\Omega^+)>0$. Note that the condition $M_t\subset\overline{U}$ can be rewritten as

$$\Omega^-\subset E(t)\subset\Omega^+.$$

Let also

$$k:=2N(1-\chi_{\Omega^+}-\chi_{\Omega^-})$$

where N is bigger than (d times) the mean curvature of ∂U.

We want to show that equation (3.5), with k as above, has a solution in an interval $[0,T)$. To this purpose, letting ρ_ε be a standard mollifier supported in the ball of radius ε centered at 0, we introduce a smooth regularization $k_\varepsilon=k*\rho_\varepsilon$ of k. Notice that $\|k_\varepsilon\|_\infty=2N$, $k_\varepsilon(x)=-2N$ (resp. $k_\varepsilon(x)=2N$) at every $x\in\Omega^-$ (resp. $x\notin\Omega^+$) such that $\mathrm{dist}(x,\partial U)\geq\varepsilon$, and $k_\varepsilon(x)=0$ at every $x\in U$ such that $\mathrm{dist}(x,\partial U)\geq\varepsilon$.

Using standard arguments (see for instance [EH91b, Theorem 4.1] and [EH91a, Prop. 4.1]) one can show existence of a smooth solution M_t^ε of (3.5), with k replaced by k_ε, on a maximal time interval $[0,T_\varepsilon)$.

Let now

$$\Omega_\varepsilon^-:=\{x\in\Omega^-:\ \mathrm{dist}(x,\partial\Omega^-)>\varepsilon\}$$

and

$$\Omega_\varepsilon^+:=\{x\in\mathbb{R}^n:\ \mathrm{dist}(x,\Omega^+)<\varepsilon\}.$$

The following result follows directly from the definition of k_ε.

Proposition 3.4. *The hypersurfaces $\partial\Omega_\varepsilon^\pm$ are respectively a super and a subsolution of (3.5), with k replaced with k_ε. In particular, by the parabolic comparison principle M_t^ε cannot intersect $\partial\Omega_\varepsilon^\pm$.*

We will show that we can find a time $T>0$ such that for every ε, there exists a smooth solution of (3.5) (with k replaced with k_ε) on $[0,T)$.

The following result will be useful in the sequel. We omit the proof which is a simple ODE argument.

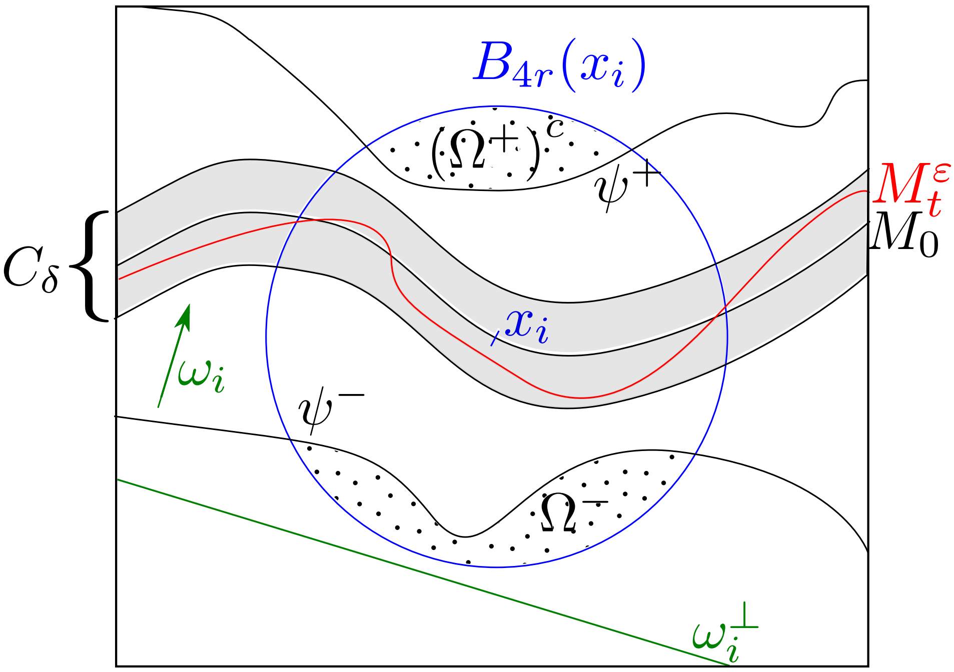

Figure 3.1: Main notations of Proposition 3.5.

Lemma 3.4. *Let $M_0 = \partial B_R(x_0)$ be a ball of radius $R \leq 1$ centered at x_0. Then, the evolution M_t by* (3.5)*, with constant forcing term $k = 2N$, is given by $M_t = B_{R(t)}(x_0)$ with $R(t) \geqslant \sqrt{R^2 - (4N + 2d)t}$. In particular, the solution exists at least on $\left[0, \frac{R^2}{4N+2d}\right)$.*

Proposition 3.5. *There exists $r > 0$, a collection of balls $B_i = B_r(x_i)$ of radius r, and a positive time T_0 such that $M_t^\varepsilon \subset \bigcup_i B_i$ for every $t \in [0, \min(T_0, T_\varepsilon))$. In addition, we can choose the balls B_i in such a way that, for every i, there exists $\omega_i \in \mathbb{R}^{d+1}$ such that $\partial\Omega^\pm \cap B_{4r}(x_i)$ and $M_0 \cap_{4r} (x_i)$ are graphs of some functions $\psi_i^\pm : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ and u_i over $\omega_i^\perp$.*
In particular, one has

$$(\nabla k_\varepsilon\,,\,\omega_i) \geqslant |\nabla k_\varepsilon|/2 \quad \textit{on } B_{2r}(x_i).$$

Most of these notations are summarized in Figure 3.1.

Proof. By assumption, for every $\mathbf{x} \in M_0$ there exist interior and exterior balls $B_x^\pm$ of fixed radius $R \leq 1$. Let $B_x^\pm(t)$ be the evolution of $B_x^\pm$ by (3.5) with forcing term $k = 2N$. By comparison, for every $t \in [0, T_\varepsilon)$, $B_x^+(t) \subset E(t)^c$ and $B_x^-(t) \subset \overline{E}(t)$. Recalling Lemma 3.4, there exists $\delta > 0$ and $T_0 > 0$, independent of ε, such that $M_t^\varepsilon \subset \{d_{M_0} \leqslant \delta\} =: C_\delta$, for all $t \in [0, \min(T_\varepsilon, T_0))$.
We eventually reduce δ, T_0 such that C_δ can be covered with a collection of balls $B_i = B_r(x_i)$, centered at $x_i \in M_0$ and with a radius r such that, for every i, there exists a unit vector $\omega_i \in \mathbb{R}^{d+1}$ satisfying

$$(\omega_i\,,\,\nu^+(x)) \geqslant \frac{1}{2} \quad \text{and} \quad (\omega_i\,,\,\nu^-(y)) \geqslant \frac{1}{2} \quad \text{and} \quad (\omega_i\,,\,\nu(z)) \geqslant \frac{1}{2}$$

for every $x \in \partial\Omega^+ \cap B_{4r}(x_i)$, $y \in \partial\Omega^- \cap B_{4r}(x_i)$ and $z \in M_0 \cap B_{4r}(x_i)$, where $\nu^\pm$ is the outer normal to $\Omega^\pm$ and ν the outer normal to E_0 (recall $M_0 = \partial E_0$). It is possible to satisfy these three conditions because if M_0 touches one of the obstacles (cannot touch both of them because of the distance between Ω^- and $\partial\Omega^+$), they share the same normal whereas if M_0 does not touch any obstacle, we can reduce the ball until the obstacles remain outside of it. The uniformity of r is provided by the uniform $\mathcal{C}^{1,1}$ norm on M_0 as well as the distance between M_0 and ∂U outside of a compact set.

As a result, $\partial\Omega^\pm \cap B_{4r}(x_i)$ and M_0 are graphs of some functions $\psi_i^\pm : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ and u_i over $\omega_i^\perp$ (see Figure 3.1).

Notice also that k is a BV function and Dk is a Radon measure concentrated on ∂U such that

$$(Dk\,,\,\omega_i) \geqslant \frac{|Dk|}{2} \text{ on } B_{4r}(x_i).$$

Then, for every $x \in B_{2r}(x_i)$ and ε sufficiently small (such that $\rho_\varepsilon(x) = 0$ as soon as $|x| \geqslant 2r$), we have

$$\begin{aligned}(\nabla k_\varepsilon\,,\,\omega_i) &= \left(\nabla \int_{\mathbb{R}^{d+1}} k(x-y)\rho_\varepsilon(y)dy\,,\,\omega_i\right)\\ &= \int_{\mathbb{R}^{d+1}} (Dk(x-y)\,,\,\omega_i)\,\rho_\varepsilon(y)dy\\ &\geqslant \int_{\mathbb{R}^{d+1}} \frac{|Dk|(x-y)}{2}\rho_\varepsilon(y)dy\\ &\geqslant \frac{|Dk| * \rho_\varepsilon}{2} \geqslant \frac{|\nabla k_\varepsilon|}{2}.\end{aligned}$$

□

In what follows, we will control the geometric quantities of M_t^ε inside each ball B_i. As in [EH91a], we introduce a localization function ϕ_i as follows: let $\eta_i(x,t) = |x - x_i|^2 + (2d+\Lambda)t$ (Λ is a positive constant that will be fixed later) and, for $R = 2r$, $\phi_i(x,t) = (R^2 - \eta_i(x,t))^+$. We denote by ϕ_i the quantity $\phi_i(\mathbf{x},t)$, where $\mathbf{x} = \mathbf{x}(p,t)$ will be a generic point in M_t. Notice that there exists $T_1 = \frac{r^2}{2d+\Lambda}$ such that for all $t \in [0, \min(T_0, T_1, T_\varepsilon))$,

$$M_t^\varepsilon \subset \bigcup_i \{\phi_i > r^2\}. \tag{3.26}$$

As a result, we have the following

Lemma 3.5. *Let f be a smooth function defined on M_t^ε. Assume that there is a $C > 0$ such that*

$$\phi_i f \leqslant C \text{ on } M_t^\varepsilon \qquad \forall t \leqslant \min(T_\varepsilon, T_1) \text{ and } \forall i \in \mathbb{N}.$$

Then,

$$f \leqslant \alpha C \text{ on } M_t^\varepsilon \qquad \forall t \leqslant \min(T_\varepsilon, T_1),$$

where α depends only on the $C^{1,1}$ norm of M_0.

Lemma 3.6. *Let $v := (\nu\,,\,\omega)^{-1}$. The quantity $v^2\phi^2$ satisfies*

$$\begin{aligned}\left(\frac{d}{dt}-\Delta^S\right)\left(\frac{v^2\phi^2}{2}\right) &\leqslant \frac{1}{2}\left(\nabla^S(v^2\phi^2)\,,\,\frac{\nabla^S\phi^2}{\phi^2}\right)\\ &\quad -\phi^2 v^3\left(\nabla^S k_\varepsilon\,,\,\omega\right)+v^2\phi(2k_\varepsilon\,(\mathbf{x}\,,\,\nu)-\Lambda).\end{aligned}\tag{3.27}$$

Proof. In this proof and the proofs further, we use normal coordinates: we assume that $g_{ij}=\delta_{ij}$ (Kronecker symbol) and that the Christoffel symbols Γ^k_{ij} vanish at the computation point.

We expand the derivatives

$$\left(\frac{d}{dt}-\Delta^S\right)\left(\frac{v^2\phi^2}{2}\right) = v^2\left(\frac{d}{dt}-\Delta^S\right)\frac{\phi^2}{2}+\phi^2\left(\frac{d}{dt}-\Delta^S\right)\frac{v^2}{2}-2\left\langle\nabla^S\frac{\phi^2}{2}\,,\,\nabla^S\frac{v^2}{2}\right\rangle.$$

First term. We start computing

$$\left(\frac{d}{dt}-\Delta^S\right)|\mathbf{x}|^2=-2k_\varepsilon\,(\mathbf{x}\,,\,\nu)-2d.$$

Then,

$$\left(\frac{d}{dt}-\Delta^S\right)\phi^2=2\phi(2k_\varepsilon\,(\mathbf{x}-x_i\,,\,\nu)-\Lambda)-2|\nabla^S|\mathbf{x}|^2|^2.$$

Second term. We are interested in

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\omega\,,\,\nu)^2 &= (\omega\,,\,\nu)\left(\frac{d}{dt}\nu\,,\,\omega\right) &(3.28)\\ &= (\omega\,,\,\nu)\left(\nabla^S(H+k_\varepsilon)\,,\,\omega\right). &(3.29)\end{aligned}$$

So,

$$\frac{1}{2}\frac{d}{dt}(\omega\,,\,\nu)^{-2}=-(\omega\,,\,\nu)^{-3}\left(\nabla^S(H+k_\varepsilon)\,,\,\omega\right).\tag{3.30}$$

On the other hand,

$$\frac{1}{2}\Delta^S((\omega\,,\,\nu)^{-2})=(\omega\,,\,\nu)^{-1}\Delta^S(\omega\,,\,\nu)^{-1}-\left\langle\nabla^S(\omega\,,\,\nu)^{-1}\,,\,\nabla^S(\omega\,,\,\nu)^{-1}\right\rangle.\tag{3.31}$$

Let us note that

$$\partial_{ij}\nu=\partial_i\left(h_{jl}g^{lm}\partial_m F\right)=\partial_i(h_{jl})\delta_{lm}\partial_m F-h_{jl}\delta_{lm}(-h_{im}\nu)=\partial_i(h_{jl})\partial_l F-\lambda_i^2\delta_{ij}\nu.$$

We then get

$$\begin{aligned}\Delta^S(\omega\,,\,\nu)^{-1} &= \partial_{ii}(\omega\,,\,\nu)^{-1}=\partial_i\left(-(\omega\,,\,\partial_i\nu)(\omega\,,\,\nu)^{-2}\right) &(3.32)\\ &= -(\omega\,,\,\partial_{ii}\nu)(\omega\,,\,\nu)^{-2}+2(\omega\,,\,\partial_i\nu)^2(\omega\,,\,\nu)^{-3} &(3.33)\\ &= -(\omega\,,\,\nu)^{-2}\left(\partial_i h_{il}\partial_l F-\lambda_i^2\nu\,,\,\omega\right)+2(\omega\,,\,\nu)^{-3}(\omega\,,\,\lambda_i\partial_i F)^2. &(3.34)\\ &= -(\omega\,,\,\nu)^{-2}(\partial_l h_{ii}\partial_l F\,,\,\omega)+|A|^2(\nu\,,\,\omega)^{-1}+2(\omega\,,\,\nu)^{-3}(\omega\,,\,\lambda_i\partial_i F)^2. &(3.35)\end{aligned}$$

We also have

$$\left\langle \nabla^S (\omega\,,\,\nu)^{-1}\,,\,\nabla^S (\omega\,,\,\nu)^{-1}\right\rangle = (\omega\,,\,\nu)^{-4} (\omega\,,\,\partial_k\nu)\,(\omega\,,\,\partial_k\nu) \tag{3.36}$$

$$= (\omega\,,\,\nu)^{-4} (\omega\,,\,h_{ku}g^{uv}\partial_v F)^2 = (\omega\,,\,\nu)^{-4} (\omega\,,\,\lambda_k\partial_k F)^2\,, \tag{3.37}$$

which leads to

$$\begin{aligned}\left(\frac{d}{dt} - \Delta^S\right)\frac{v^2}{2} \quad &= \quad -v^3\left(\nabla^S(H+k_\varepsilon)\,,\,\omega\right) + v^3\partial_m(h_{ii})\,(\omega\,,\,\partial_m F)\\ &\quad -|A|^2v^2 - 2v^4\lambda_k^2\,(\omega\,,\,\partial_k F)^2 - v^4\,(\omega\,,\,\lambda_k\partial_k F)^2\end{aligned}$$

Third term. We notice, as in [EH91a] that $|\nabla^S\phi^2|^2 = 4\phi^2|\nabla^S(|\mathbf{x}|^2)|^2$ and

$$-\left(\nabla^S(v^2)\,,\,\nabla^S\phi^2\right) = -3\left(v\nabla^S(v)\,,\,\nabla^S\phi^2\right) + \frac{1}{2}\left(\left(\nabla^S(v^2\phi^2)\,,\,\frac{\nabla^S\phi^2}{\phi^2}\right) - v^2\frac{|\nabla^S\phi^2|^2}{\phi^2}\right).$$

Then, Young's inequality gives

$$\begin{aligned}2\left|v\left(\nabla^S v\,,\,\nabla^S\phi^2\right)\right| &\leqslant 2\phi^2|\nabla^S v^2|^2 + \frac{1}{2\phi^2}|\nabla^S\phi^2|^2\\ &\leqslant 2\phi^2|\nabla^S v^2|^2 + 2v^2|\nabla^S|\mathbf{x}|^2|^2.\end{aligned}$$

Hence,

$$-\left(\nabla^S(v^2)\,,\,\nabla^S\phi^2\right) \leqslant -3\phi^2|\nabla^S v^2|^2 - 3v^2|\nabla^S|\mathbf{x}|^2|^2 + \frac{1}{2}\left(\left(\nabla^S(v^2\phi^2)\,,\,\frac{\nabla^S\phi^2}{\phi^2}\right) - v^2\frac{|\nabla^S\phi^2|^2}{\phi^2}\right).$$

Summing the three terms, we get

$$\left(\frac{d}{dt} - \Delta^S\right)\left(\frac{v^2\phi^2}{2}\right) \leqslant \frac{1}{2}\left(\nabla^S(v^2\phi^2)\,,\,\frac{\nabla^S\phi^2}{\phi^2}\right) - \phi^2v^3\left(\nabla^S k_\varepsilon\,,\,\omega\right) + v^2\phi(2k_\varepsilon\,(\mathbf{x}\,,\,\nu) - \Lambda).$$

□

For $\gamma > 0$, we let

$$\psi(v^2) := \frac{\gamma v^2}{1-\gamma v^2}.$$

Lemma 3.7. *For $\varepsilon \leqslant r$, we have*

$$\begin{aligned}\left(\frac{d}{dt} - \Delta^S\right)\frac{\phi^2|A|^2\psi(v^2)}{2} \quad &\leqslant \quad \phi^2\psi(v^2)(-\gamma|A|^4 - 2k_\varepsilon\sum_i\lambda_i^3 - 2\left\langle A\,,\,(\nabla^S)^2k_\varepsilon\right\rangle)\\ &\quad -\phi^2|A|^2v^3\psi'(v^2)\left(\nabla^S k_\varepsilon\,,\,\omega\right) - \phi^2|A|^2\sum_i(\lambda_i\omega^i)^2\frac{2v^4+\gamma v^6}{(1-\gamma v^2)^3}.\end{aligned}$$

Proof. We denote $V = \frac{\phi^2|A|^2\psi(v^2)}{2}$ and compute

$$\begin{aligned}\left(\frac{d}{dt}-\Delta^S\right)\frac{\phi^2|A|^2\psi(v^2)}{2} &= |A|^2\psi(v^2)\left(\frac{d}{dt}-\Delta^S\right)\frac{1}{2}\phi^2+\phi^2\psi(v^2)\left(\frac{d}{dt}-\Delta^S\right)\frac{1}{2}|A|^2\\ &\quad+\phi^2|A|^2\left(\frac{d}{dt}-\Delta^S\right)\frac{1}{2}\psi(v^2)-2\left\langle 1/2\nabla^S|A|^2\,,\,1/2\nabla^S\phi^2\right\rangle\\ &\quad-2\left\langle 1/2\nabla^S|A|^2\,,\,1/2\nabla^S\psi(v^2)\right\rangle-2\left\langle 1/2\nabla^S\phi^2\,,\,1/2\nabla^S\psi(v^2)\right\rangle.\end{aligned}$$

The two first terms have already been computed. Let us consider the third one.

$$\frac{1}{2}\frac{d}{dt}\psi(v^2)=v\frac{dv}{dt}\psi'(v^2)=-v^3\psi'(v^2)\left(\nabla^S(H+k_\varepsilon)\,,\,\omega\right),$$

$$\begin{aligned}\frac{1}{2}\Delta^S\psi(v^2)=\frac{1}{2}\partial_{ii}\psi(v^2)&=\partial_i(v\partial_i v\psi'(v^2))=v\Delta^S v\psi'(v^2)+2v^2|\nabla^S v|^2\psi''(v^2)+|\nabla^S v|^2\psi'(v^2)\\ &=(3|\nabla^S v|^2-v^3(\partial_l(h_{kk})w^l)+v^2|A|^2)\psi'(v^2)+2|\nabla^S v|^2\psi''(v^2).\end{aligned}$$

Hence

$$\left(\frac{d}{dt}-\Delta^S\right)\frac{1}{2}\psi(v^2)=-v^3\psi'(v^2)\left(\nabla^S k_\varepsilon\,,\,\omega\right)-(3|\nabla^S v|^2+v^2|A|^2)\psi'(v^2)-2v^2|\nabla^S v|^2\psi''(v^2).$$

We will want to conclude the proof using the weak maximum principle. So, we want to rewrite the last terms (which are gradient terms) using the gradient of V. Let us expand $\nabla^S V$.

$$\nabla^S\frac{\phi^2|A|^2\psi(v^2)}{2}=\phi^2|A|^2\frac{1}{2}\nabla^S\psi(v^2)+|A|^2\psi(v^2)\frac{1}{2}\nabla^S\phi^2+\phi^2\psi(v^2)\frac{1}{2}\nabla^S|A|^2.$$

So,

$$\begin{aligned}\left|\nabla^S\frac{\phi^2|A|^2\psi(v^2)}{2}\right|^2 &= \phi^4|A|^4\frac{|\nabla^S\psi(v^2)|^2}{4}+|A|^4\psi^2(v^2)\frac{|\nabla^S\phi^2|^2}{4}+\phi^4\psi^2(v^2)\frac{|\nabla^S|A|^2|^2}{4}\\ &\quad+\phi^2|A|^4\psi(v^2)\left\langle\nabla^S\psi(v^2)\,,\,\nabla^S\phi^2\right\rangle+\phi^4|A|^2\psi(v^2)\left\langle\nabla^S\psi(v^2)\,,\,\nabla^S|A|^2\right\rangle\\ &\quad+|A|^2\psi^2(v^2)\phi^2\left\langle\nabla^S\phi^2\,,\,\nabla^S|A|^2\right\rangle.\end{aligned}$$

As a matter of fact,

$$\begin{aligned}\frac{1}{\phi^2|A|^2\psi(v^2)}\left|\nabla^S\frac{\phi^2|A|^2\psi(v^2)}{2}\right|^2 &= \phi^2|A|^2\frac{|\nabla^S\psi(v^2)|^2}{4\psi(v^2)}+|A|^2\psi(v^2)\frac{|\nabla^S\phi^2|^2}{4\phi^2}\\ &\quad+\phi^2\psi(v^2)\frac{|\nabla^S|A|^2|^2}{4|A|^2}+2|A|^2\left\langle\nabla^S\psi(v^2)/2\,,\,\nabla^S\phi^2/2\right\rangle\\ &\quad+2\phi^2\left\langle\nabla^S\psi(v^2)/2\,,\,\nabla^S|A|^2/2\right\rangle+2\psi(v^2)\left\langle\nabla^S\phi^2/2\,,\,\nabla^S|A|^2/2\right\rangle.\end{aligned}$$

We use the last equality to rewrite

$$\begin{aligned}
\left(\frac{d}{dt}-\Delta^S\right)&\frac{\phi^2|A|^2\psi(v^2)}{2}\\
&= |A|^2\psi(v^2)\left(\phi(2k_\varepsilon(\mathbf{x}\,,\,\nu)-\Lambda)-|\nabla^S|x|^2|^2\right)\\
&+\phi^2\psi(v^2)\left(-\left\langle\nabla^S A\,,\,\nabla^S A\right\rangle+|A|^4-2k_\varepsilon g^{js}h_{st}g^{tl}g^{jl}h_{ij}h_{kl}-2\left\langle A\,,\,\nabla^2 k_\varepsilon\right\rangle\right)\\
&+\phi^2|A|^2\left(-v^3\psi'(v^2)\left(\nabla^S k_\varepsilon\,,\,\omega\right)-(3|\nabla^S v|^2+v^2|A|^2)\psi'(v^2)-2v^2|\nabla^S v|^2\psi''(v^2)\right)\\
&-\frac{1}{\phi^2|A|^2\psi(v^2)}\left|\nabla^S\frac{\phi^2|A|^2\psi(v^2)}{2}\right|^2\\
&+\phi^2|A|^2\frac{|\nabla^S\psi(v^2)|^2}{4\psi(v^2)}+|A|^2\psi(v^2)\frac{|\nabla^S\phi^2|^2}{4\phi^2}+\phi^2\psi(v^2)\frac{|\nabla^S|A|^2|^2}{4|A|^2}.
\end{aligned} \tag{3.38}$$

Let us precise some terms:

$$\begin{aligned}
|\nabla^S\phi^2|^2 &= 4\phi^2\cdot|-2\mathbf{x}^T|^2=4\phi^2(4|\mathbf{x}|^2-4\,(\mathbf{x}\,,\,\nu)),\\
|\nabla^S\psi(v^2)|^2 &= \psi'(v^2)^2|\nabla^S v^2|^2=4\psi'(v^2)^2v^6\sum_k(\lambda_k\omega^k)^2,\\
|\nabla^S|A|^2|^2 &= 4\sum_i(\partial_i(h_{ll})\lambda_l)^2,\\
|\nabla^S A|^2 &= \sum_{i,k,l}(\partial_i(h_{km}))^2.
\end{aligned}$$

In addition, we have the obvious estimate

$$|\nabla^S|A|^2|^2\leqslant 4|A|^2|\nabla^S A|^2.$$

So,

$$\begin{aligned}
\phi^2|A|^2&\frac{|\nabla^S\psi(v^2)|^2}{4\psi(v^2)}+|A|^2\psi(v^2)\frac{|\nabla^S\phi^2|^2}{4\phi^2}+\phi^2\psi(v^2)\frac{|\nabla^S|A|^2|^2}{4|A|^2}\\
&\leqslant\phi^2|A|^2\frac{\psi'(v^2)^2v^6\sum_k(\lambda_k\omega^k)^2}{\psi(v^2)}+4|A|^2\psi(v^2)(|\mathbf{x}|^2-(\mathbf{x}\,,\,\nu)^2)+\phi^2\psi(v^2)|\nabla^S A|^2.
\end{aligned}$$

We plug this inequality into (3.38) and obtain

$$\begin{aligned}
\left(\frac{d}{dt}-\Delta^S\right)\frac{\phi^2|A|^2\psi(v^2)}{2}&\leqslant|A|^2\psi(v^2)\left(\phi(2k_\varepsilon(\mathbf{x}\,,\,\nu)-\Lambda)-|\nabla^S|x|^2|^2\right)\\
&+\phi^2\psi(v^2)\left(-\left\langle\nabla^S A\,,\,\nabla^S A\right\rangle+|A|^4-2k_\varepsilon g^{js}h_{st}g^{tl}g^{jl}h_{ij}h_{kl}-2\left\langle A\,,\,\nabla^2 k_\varepsilon\right\rangle\right)\\
&+\phi^2|A|^2\left(-v^3\psi'(v^2)\left(\nabla^S k_\varepsilon\,,\,\omega\right)-(3|\nabla^S v|^2+v^2|A|^2)\psi'(v^2)-2v^2|\nabla^S v|^2\psi''(v^2)\right)\\
&-\frac{1}{\phi^2|A|^2\psi(v^2)}\left|\nabla\frac{\phi^2|A|^2\psi(v^2)}{2}\right|^2\\
&+\phi^2|A|^2\frac{\psi'(v^2)^2v^6\sum_k(\lambda_k\omega^k)^2}{\psi(v^2)}+4|A|^2\psi(v^2)(|\mathbf{x}|^2-(\mathbf{x}\,,\,\nu)^2)+\phi^2\psi(v^2)|\nabla^S A|^2.
\end{aligned}$$

Let us regroup some terms (noting that $|\nabla^S v|^2 = v^4 \sum_i (\lambda_i \omega^i)^2)$, we get

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta^S\right) & \frac{\phi^2|A|^2\psi(v^2)}{2} \\
& \leqslant |A|^2\psi(v^2)\left(\phi(2k_\varepsilon(\mathbf{x}\,,\,\nu) - \Lambda)\right) \\
& + \phi^2|A|^4(\psi(v^2) - v^2\psi'(v^2)) - 2\phi^2\psi(v^2)k_\varepsilon g^{js}h_{st}g^{tl}g^{jl}h_{ij}h_{kl} - 2\phi^2\psi(v^2)\left\langle A\,,\,\nabla^2 k_\varepsilon\right\rangle \\
& - \phi^2|A|^2v^3\psi'(v^2)\left(\nabla^S k_\varepsilon\,,\,\omega\right) - \frac{1}{\phi^2|A|^2\psi(v^2)}\left|\nabla^S\frac{\phi^2|A|^2\psi(v^2)}{2}\right|^2 \\
& + \phi^2|A|^2\sum_i(\lambda_i\omega^i)^2\left(\frac{v^6\psi'(v^2)^2}{\psi(v^2)} - 3v^4\psi'(v^2) - 2v^6\psi''(v^2)\right).
\end{aligned}$$

Then, we note that

$$\frac{v^6\psi'(v^2)^2}{\psi(v^2)} - 3v^4\psi'(v^2) - 2v^6\psi''(v^2) = -\frac{2v^4+\gamma v^6}{(1-\gamma v^2)^3} \leqslant 0$$

and

$$\psi(v^2) - v^2\psi'(v^2) = -\gamma\psi^2(v^2) \leqslant 0.$$

So,

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta^S\right)\frac{\phi^2|A|^2\psi(v^2)}{2} \quad \leqslant \quad & \phi^2\psi(v^2)(-\gamma|A|^4 - 2k_\varepsilon\sum_i\lambda_i^3 - 2\left\langle A\,,\,\nabla^2k_\varepsilon\right\rangle) \\
& -\phi^2|A|^2v^3\psi'(v^2)\left(\nabla^S k_\varepsilon\,,\,\omega\right) - \phi^2|A|^2\sum_i(\lambda_i\omega^i)^2\frac{2v^4+\gamma v^6}{(1-\gamma v^2)^3},
\end{aligned}$$

what was expected. $\square$

We now show that M_t can be locally written as a Lipschitz graph, with Lipschitz constant independent of ε.

Proposition 3.6. *Let $\varepsilon \leqslant r$. Then, for every $t \in [0, \min(T_\varepsilon, T_1))$, $M_t \cap B_i$ can be written as a Lipschitz graph over $\omega_i^\perp$, with Lipschitz constant independent of ε.*

Proof. We want to show that the quantity $(\nu\,,\,\omega_i)$ is bounded from below, or, equivalently, that $v := (\nu\,,\,\omega_i)^{-1}$ is bounded from above on every ball B_i. We want to estimate the quantity $v^2\phi^2$ (we drop the explicit dependence on the index i) using Lemma 3.6.

We choose Λ such that the last term in (3.27) is nonpositive (take for instance $\Lambda = 2NR$). We also have to control

$$v\left(\nabla^S k_\varepsilon\,,\,\omega\right) = (\nu\,,\,\omega)^{-1}\left((\nabla k_\varepsilon\,,\,\omega) - (\nabla k_\varepsilon\,,\,\nu)\,(\nu\,,\,\omega)\right) = (\nu\,,\,\omega)^{-1}\left(\nabla k_\varepsilon\,,\,\omega\right) - \left(\nabla k_\varepsilon\,,\,\nu\right).$$

Proposition 3.5 provides immediately

$$(\nu\,,\,\omega)^{-1}\left(\nabla k_\varepsilon\,,\,\omega\right) - \left(\nabla k_\varepsilon\,,\,\nu\right) \geqslant (\nu\,,\,\omega)^{-1}\frac{|\nabla k_\varepsilon|}{2} - |\nabla k_\varepsilon|$$

which is nonnegative as soon as $(\omega\,,\,\nu) \leqslant \frac{1}{2}$. From Lemma 3.6 and the weak maximum principle (see [PW84]), we obtain that $\|v^2\phi^2\|_\infty(t) \leqslant \max(\|v^2\phi^2\|_\infty(0), 4R^2)$. Thanks to Lemma 3.5, this provides a uniform Lipschitz bound on the whole M_t, for $t \leqslant T_1$. $\square$

We want to prove that T_ε is uniformly bounded below from zero. To this aim, we use the following theorem (which is [Hui84, Th. 8.1] with a forcing term).

Theorem 3.5. *If $T_\varepsilon < T_1$, then the second fundamental form of M_t blows up as $t \to T_\varepsilon$.*

The proof of this result is the same as in [Hui84, Th. 8.1], thanks to the control on the derivatives of A we obtained in Theorem 3.4.

Let us show that $|A|$ cannot blow up.

Proposition 3.7. *For every $\varepsilon \leqslant r$, there exists $C_\varepsilon > 0$ such that*

$$\|A\|_{L^\infty(M_t)} \leqslant C_\varepsilon \qquad \textit{for all } t \in [0, \min(T_\varepsilon, T_1)).$$

Proof. As in [EH91a], we are interested in the evolution of the quantity

$$\frac{\phi^2|A|^2\psi(v^2)}{2}.$$

Notice that

$$|\lambda_i|^3 = |\lambda_i||\lambda_i|^2 \leqslant \frac{1}{2\alpha}\lambda_i^4 + \frac{\alpha}{2}\lambda_i^2.$$

Choosing α such that $\frac{2N}{\alpha} \leqslant \frac{\gamma}{2}$, one can write

$$\left| -2k_\varepsilon \phi^2\psi(v^2)\sum_i \lambda_i^3 \right| \leqslant \phi^2\psi(v^2)\left(\frac{\gamma}{2}|A|^4 + N\alpha|A|^2\right).$$

In addition, as soon as $|A|^2 \geqslant 1$, one has $\left\langle A\,,\, \nabla^2 k_\varepsilon\right\rangle \leqslant |A|^2|\nabla^2 k_\varepsilon|$. One can also notice that as above, $v\left(\nabla^S k_\varepsilon\,,\, \omega\right) \geqslant 0$ as soon as $v \geqslant 2$. On the other hand, if $v \leqslant 2$, one has $v^3\psi'(v^2) = \frac{\psi(v)v}{1-\gamma v^2} \leqslant 4\psi(v)$ for γ sufficiently small.

So, anyway, if $|A| \geqslant 1$,

$$\left(\frac{d}{dt} - \Delta^S\right)\frac{\phi^2|A|^2\psi(v^2)}{2} \leqslant 2N\alpha\frac{\phi^2|A|^2\psi(v^2)}{2} + 4|\nabla^2 k_\varepsilon|\frac{\phi^2|A|^2\psi(v^2)}{2} + 8\frac{\phi^2|A|^2\psi(v^2)}{2}|\nabla^S k_\varepsilon|.$$

Finally, we apply the maximum principle to

$$\tilde{A} := \exp\left[-\left(2N\alpha + 4\|\nabla^2 k_\varepsilon\ \|_\infty + 8\|\nabla k_\varepsilon\|_\infty\right)t\right]\cdot\frac{\phi^2|A|^2\psi(v^2)}{2}$$

which satisfies

$$\left(\frac{d}{dt} - \Delta^S\right)\tilde{A} \leqslant 0.$$

It provides

$$\forall t \leqslant \min(T_\varepsilon, T_1), \quad \|\tilde{A}\|_\infty(t) \leqslant \|\tilde{A}\|_\infty(0)$$

which shows that $\frac{\phi^2|A|^2\psi(v^2)}{2}$ does not blow up.

Using Lemma 3.5 and choosing γ such that $\psi(v^2)$ is bounded and remains far from zero, we know that $|A|$ does not blow up for $t \leqslant T_1$. □

Corollary 3.1. *There exists T_1, depending only on the dimension, $\|k\|_\infty$ and the radius in the ball condition for M_0, such that there exists a solution M_t^ε of the mean curvature flow with forcing term k_ε on $[0, T_1)$.*

The surfaces M_t^ε are uniformly Lipschitz and every $M_t^\varepsilon \cap B_i$ can be written as the graph of some function $u_i^\varepsilon(x,t)$. All the u_i^ε are Lipchitz (in space) with a constant which depends neither on i nor in ε. We want to show that they are also equicontinuous in time.

Proposition 3.8. *The functions u_i^ε are Lipschitz continuous in x and $1/2$-Hölder continuous in t on $B_i \times [0, T_1)$, uniformly with respect to ε and i.*
In addition, they are (classical) solutions of the equation

$$\partial_t u_i^\varepsilon = \sqrt{1+|\nabla u_i^\varepsilon|^2}\,\mathrm{div}\left(\frac{\nabla u_i^\varepsilon}{\sqrt{1+|\nabla u_i^\varepsilon|^2}}\right) - \sqrt{1+|\nabla u_i^\varepsilon|^2}\,k_\varepsilon(x, u_i^\varepsilon). \tag{3.39}$$

Proof. Let δ be fixed (we drop the index ε in what follows), and let $t_0 \in [0, T_1)$. Let $x_0 \in M_t$ and i such that $x_0 \in B_i$. Then, $(\nu(x_0)\,,\,\omega_i)^{-1}$ is bounded above and M_t is the graph of a function u over $\omega_i^\perp$. Then, let $x_1 = x_0 + \delta\omega_i$. Thanks to the Lipschitz condition, there is a ball $B_{1/C\delta}(x_1)$ that does not touch M_t. Evolving by mean curvature with forcing term k_ε, this ball vanishes in a positive time $T_\delta \geqslant \omega(\delta) := \frac{\delta^2}{C^2(2d+1)}$ (note that T_δ does not depend on ε). By comparison principle, for $t \in [t_0, t_0 + \omega(\delta))$, M_t does not go beyond x_1. That is equivalent to say that u is $1/2$-Hölder continuous in time, with a constant independent of ε.

The equation satisfied by u_i^ε is usual. One just has to notice that with the definitions above,

$$\mathrm{div}\left(\frac{\nabla u_i^\varepsilon}{\sqrt{1+|\nabla u_i^\varepsilon|^2}}\right) = -H.$$

□

We now pass to the limit as ε goes to zero. By Proposition 3.8, the family (u_i^ε) is equi-Lipschitz in space and equi-continuous in time on $B_i \times [0, T_1)$. Therefore, by Arzelà–Ascoli's Theorem one can find a sequence $\varepsilon_n \to 0$ and continuous functions u_i such that, for every i, $u_i^{\varepsilon_n} \underset{n\to\infty}{\longrightarrow} u_i$ locally uniformly on $B_i \times [0, T_1)$.

Proposition 3.9. *The functions u_i are viscosity solutions of* (3.3) *on $B_i \times [0, T_1)$, with obstacles $U \cap B_i$ (see Appendix 6).*

Proof. Thanks to Proposition 3.5, every $x \in B_i$ can be decomposed as $x = x' + z\omega_i$ with $z = (x\,,\,\omega_i)$. Then, there exists functions $\psi_i^\pm$ of class $C^{1,1}$ such that

$$U \cap B_i = \{(x', z) \in B_i \,:\, \psi_i^-(x') \leqslant z \leqslant \psi_i^+(x')\}.$$

For simplicity we shall drop the explicit dependence on the index i. Since $u^\varepsilon(x,0) = u_0(x)$ for all ε, and u^{ε_n} converges uniformly to u as $n \to +\infty$, it is clear that $u(x,0) = u_0(x)$. Condition (3.52) immediately follows from Proposition 3.4.

We now check that u is a subsolution of (3.3). Let $(x_0,t_0)\in\mathbb{R}^n\times\mathbb{R}$ and $\varphi\in C^2$ such that $\psi^-(x_0,t_0)<u(x_0,t_0)$ and

$$(u-\varphi)(x_0,t_0)=\max_{|(x,t)-(x_0,t_0)|\leqslant r}(u-\varphi)(x,t).$$

One can change φ so that (x_0,t_0) is a strict maximum point, and $u(x_0,t_0)=\varphi(x_0,t_0)$. Let $2\delta := u(x_0,t_0)-\psi^-(x_0,t_0)$. Thanks to the definition of k_ε, for all $\varepsilon\leqslant\delta$, we have $k_\varepsilon(x,\varphi(x,t))\leqslant 0$ in a small neighborhood V of (x_0,t_0). For ε sufficiently small $u^\varepsilon-\varphi$ attains its maximum in V at $(x_\varepsilon,t_\varepsilon)$, with $(x_\varepsilon,t_\varepsilon)\to(x_0,t_0)$ as $\varepsilon\to 0$. Since u^ε is a classical solution of (3.41), it is also a viscosity solution, therefore

$$\varphi_t-\sqrt{1+|\nabla\varphi|^2}\,\mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right)\leqslant\sqrt{1+|\nabla\varphi|^2}\,k_\varepsilon(x,\varphi)\leqslant 0\qquad\text{at }(x_\varepsilon,t_\varepsilon).$$

Letting $\varepsilon\to 0$ we obtain that u is a subsolution of (3.3). A similar argument shows that u is also a supersolution of (3.3), and this concludes the proof. □

Conclusion of the proof of Theorem 3.1. The result in [PS07, Theorem 4.1] (see also Section 6.4) applies, showing that the functions u_i are of class $C^{1,1}$. As the uniform convergence $u_i^{\varepsilon_n}$ implies the Hausdorff convergence of $M_t^{\varepsilon_n}$ to a limit M_t such that $M_t\cap B_i=\mathrm{graph}(u_i(t))$, we built a $C^{1,1}$ evolution to the mean curvature motion with obstacles on the time interval $[0,T_1)$. Thanks to [ACN12, Theorem 4.8 and Corollary 4.9] this evolution is also unique. This concludes the proof of Theorem 3.1. □

4 Proof of Theorem 3.2

Let $\psi_\varepsilon^\pm$ be smooth functions such that $\psi_\varepsilon^\pm\to\psi^\pm$ as $\varepsilon\to 0$, uniformly in $C^{1,1}(\mathbb{R}^d)$, and let $N>0$ be such that

$$N\geq\left\|\sqrt{1+|\psi_\varepsilon^\pm|^2}\,\mathrm{div}\left(\frac{\psi_\varepsilon^\pm}{\sqrt{1+|\psi_\varepsilon^\pm|^2}}\right)\right\|_{L^\infty(\mathbb{R}^d)}\qquad\text{for all }\varepsilon>0.\tag{3.40}$$

We proceed as in Section 3 and we approximate (3.3), (3.4) with the forced mean curvature equation

$$u_t=\sqrt{1+|\nabla u|^2}\left[\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)+k_\varepsilon(x,u)\right],\tag{3.41}$$

where

$$k_\varepsilon(x,u)=2N\left(\chi\left(\frac{\psi_\varepsilon^-(x)-u}{\varepsilon}\right)-\chi\left(\frac{u-\psi_\varepsilon^+(x)}{\varepsilon}\right)\right),$$

and χ is a smooth increasing function such that $\chi(s)\equiv 0$ for all $s\in(-\infty,0]$, and $\chi(s)\equiv 1$ for all $s\in[1,\infty)$. In particular $\partial_u k_\varepsilon(x,u)\leq 0$ for all (x,u).

Notice that $k_\varepsilon\to g$ as $\varepsilon\to 0$, with

$$k(x,u)=\begin{cases}2N\text{ if }u<\psi^-(x)\\-2N\text{ if }u>\psi^+(x)\\0\text{ elsewhere}\end{cases}.$$

Notice also that

$$
\begin{aligned}
&\frac{\partial k_\varepsilon}{\partial x_k}(x,u)+\frac{\partial k_\varepsilon}{\partial u}(x,u)\frac{\partial \psi_\varepsilon^-}{\partial x_k}=0 \quad \text{if } u<\psi_\varepsilon^+\\
&\frac{\partial k_\varepsilon}{\partial x_k}(x,u)+\frac{\partial k_\varepsilon}{\partial u}(x,u)\frac{\partial \psi_\varepsilon^+}{\partial x_k}=0 \quad \text{if } u>\psi_\varepsilon^- .
\end{aligned}
\tag{3.42}
$$

We denote by u_ε the solution of the approximate problem (3.41), which exists and is smooth for short times.

Proposition 3.10. *The solution u_ε is defined for $t\in[0,+\infty)$, and satisfies the estimates*

$$
\|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\mathbb{R}^d)} \le C \qquad \textit{for all } t\in[0,+\infty) \tag{3.43}
$$

$$
\|u_\varepsilon(\cdot,t)\|_{W^{2,\infty}(\mathbb{R}^d)} \le C(T) \qquad \textit{for all } t\in[0,T]. \tag{3.44}
$$

Proof. Estimate (3.43) follows from Proposition 3.6, choosing $B_i=\mathbb{R}^{d+1}$, $\omega_i=e_{d+1}$ and $\phi\equiv 1$. Estimate (3.44) follows from (3.43) and Proposition 3.7. □

In what follows, we use intrinsic derivatives on the graph $M_t:=\{(x,u_\varepsilon(x,t))\}$, which will be denoted as above by an exponent S. The metric on M_t is

$$
g_{ij}=\delta_{ij}+\partial_i u_\varepsilon \partial_j u
$$

with inverse

$$
g^{ij}=\delta_{ij}-\frac{\partial_i u_\varepsilon \partial_j u_\varepsilon}{1+|\nabla u_\varepsilon|^2}.
$$

The tangential gradient of a function f defined on M_t is given by

$$
(\nabla^S f)^i=g^{ij}\partial_j f=\partial_i f-\frac{\partial_i u_\varepsilon \partial_j u_\varepsilon}{1+|\nabla u_\varepsilon|^2}\partial_j f\,,
$$

so that

$$
\big(\nabla^S f\,,\,\nabla u_\varepsilon\big)=(\nabla f\,,\,\nabla u_\varepsilon)-\frac{|\nabla u_\varepsilon|^2}{1+|\nabla u_\varepsilon|^2}(\nabla f\,,\,\nabla u_\varepsilon)=\frac{1}{1+|\nabla u_\varepsilon|^2}(\nabla f\,,\,\nabla u_\varepsilon)\,, \tag{3.45}
$$

and

$$
\begin{aligned}
|\nabla^S f|^2 &= \left(f_i-(u_\varepsilon)_i\sum_j \frac{(u_\varepsilon)_j f_j}{1+|\nabla u_\varepsilon|^2}\right)^2\\
&=|\nabla f|^2+(u_\varepsilon)_i^2\left(\frac{(\nabla u_\varepsilon\,,\,\nabla f)}{1+|\nabla u_\varepsilon|^2}\right)^2-2\frac{(u_\varepsilon)_i(u_\varepsilon)_j f_i f_j}{1+|\nabla u_\varepsilon|^2}\\
&=|\nabla f|^2+\frac{|\nabla u_\varepsilon|^2}{1+|\nabla u_\varepsilon|^2}\frac{(\nabla u_\varepsilon\,,\,\nabla f)^2}{1+|\nabla u_\varepsilon|^2}-2\frac{(\nabla u_\varepsilon\,,\,\nabla f)^2}{1+|\nabla u_\varepsilon|^2}\\
&=|\nabla f|^2-\frac{(\nabla u_\varepsilon\,,\,\nabla f)^2}{1+|\nabla u_\varepsilon|^2}-\frac{(\nabla u_\varepsilon\,,\,\nabla f)^2}{(1+|\nabla u_\varepsilon|^2)^2}.
\end{aligned}
\tag{3.46}
$$

In addition, the Laplace-Beltrami operator applied to f is

$$
\Delta^S f=g^{ij}f_{ij}=\Delta f-\frac{\partial_i u_\varepsilon \partial_j u_\varepsilon}{1+|\nabla u_\varepsilon|^2}f_{ij}=\Delta f-\frac{\big(\nabla u_\varepsilon \nabla^2 f\,,\,\nabla u_\varepsilon\big)}{1+|\nabla u_\varepsilon|^2}.
$$

Proposition 3.11. *The quantity* $\|(u_\varepsilon)_t^2\|_\infty(t)$ *is nonincreasing in time. In particular,*

$$\|(u_\varepsilon)_t(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq \left\| \sqrt{1+|\nabla u_0|^2}\,\mathrm{div}\left(\frac{\nabla u_0}{\sqrt{1+|\nabla u_0|^2}}\right)\right\|_{L^\infty(\mathbb{R}^d)}.$$

Proof. We compute

$$\frac{d}{dt}\frac{(u_\varepsilon)_t^2}{2} = (u_\varepsilon)_t \left[\sqrt{1+|\nabla u_\varepsilon|^2}\left(\mathrm{div}\left(\frac{\nabla u_\varepsilon}{\sqrt{1+|\nabla u_\varepsilon|^2}}\right) + k_\varepsilon(x,u_\varepsilon\right)\right]_t.$$

Expanding this expression, we get

$$\begin{aligned}\frac{d}{dt}\frac{(u_\varepsilon)_t^2}{2} = (u_\varepsilon)_t &\left[\frac{\nabla(u_\varepsilon)_t\cdot\nabla u_\varepsilon}{\sqrt{1+|\nabla u_\varepsilon|^2}}\left(\mathrm{div}\left(\frac{\nabla u_\varepsilon}{\sqrt{1+|\nabla u_\varepsilon|^2}}\right)+k_\varepsilon\right)\right.\\ &\left.+\sqrt{1+|\nabla u_\varepsilon|^2}\left(\mathrm{div}\left(\frac{(\nabla u_\varepsilon)_t}{\sqrt{1+|\nabla u_\varepsilon|^2}} - \frac{((\nabla u_\varepsilon)_t\cdot\nabla u_\varepsilon)\nabla u_\varepsilon}{(1+|\nabla u_\varepsilon|^2)^{3/2}}\right)+(u_\varepsilon)_t\partial_u k_\varepsilon\right)\right].\end{aligned}$$

Let us compute more explicitly the three terms of the expression above.

$$\begin{aligned}&(u_\varepsilon)_t\frac{(\nabla u_\varepsilon)_t\cdot\nabla u_\varepsilon}{\sqrt{1+|\nabla u_\varepsilon|^2}}\left(\mathrm{div}\left(\frac{\nabla u_\varepsilon}{\sqrt{1+|\nabla u_\varepsilon|^2}}\right)+k_\varepsilon\right)\\ &\quad= \frac{\nabla(\frac{(u_\varepsilon)_t^2}{2})\cdot\nabla u_\varepsilon}{\sqrt{1+|\nabla u_\varepsilon|^2}}\left(\frac{\Delta u}{\sqrt{1+|\nabla u_\varepsilon|^2}} - \frac{(u_\varepsilon)_i\,(\nabla u_\varepsilon\,,\,(\nabla u_\varepsilon)_i)}{(1+|\nabla u_\varepsilon|^2)^{3/2}}+k_\varepsilon\right)\\ &\quad= \nabla(\frac{(u_\varepsilon)_t^2}{2})\cdot\nabla u_\varepsilon\left(\frac{\Delta u_\varepsilon}{1+|\nabla u_\varepsilon|^2} - \frac{\nabla u_\varepsilon\cdot\nabla(\frac{|\nabla u_\varepsilon|^2}{2})}{(1+|\nabla u_\varepsilon|^2)^2}+k_\varepsilon\right),\end{aligned}$$

$$\begin{aligned}(u_\varepsilon)_t\,\mathrm{div}\left(\frac{\nabla(u_\varepsilon)_t}{\sqrt{1+|\nabla u_\varepsilon|^2}}\right) &= (u_\varepsilon)_t\partial_i\left(\frac{(u_\varepsilon)_{ti}}{\sqrt{1+|\nabla u_\varepsilon|^2}}\right)\\ &= \frac{(u_\varepsilon)_t(u_\varepsilon)_{tii}}{\sqrt{1+|\nabla u_\varepsilon|^2}} - \frac{1}{(1+|\nabla u_\varepsilon|^2)^{3/2}}(u_\varepsilon)_t(u_\varepsilon)_{ti}\nabla u_\varepsilon\cdot(\nabla u_\varepsilon)_i\\ &= \frac{(u_\varepsilon)_t\Delta(u_\varepsilon)_t}{\sqrt{1+\nabla u_\varepsilon^2}} - \frac{1}{(1+|\nabla u_\varepsilon|^2)^{3/2}}(u_\varepsilon)_t(u_\varepsilon)_{ti}\partial_i(\frac{|\nabla u_\varepsilon|^2}{2})\\ &= \frac{(u_\varepsilon)_t\Delta(u_\varepsilon)_t}{\sqrt{1+\nabla u_\varepsilon^2}} - \frac{1}{(1+|\nabla u_\varepsilon|^2)^{3/2}}\nabla\left(\frac{(u_\varepsilon)_t^2}{2}\right)\cdot\nabla\left(\frac{|\nabla u_\varepsilon|^2}{2}\right),\end{aligned}$$

and

$$
\begin{aligned}
&(u_\varepsilon)_t \operatorname{div}\left(\frac{((\nabla u_\varepsilon)_t\cdot\nabla u_\varepsilon)\nabla u_\varepsilon}{(1+|\nabla u_\varepsilon|^2)^{3/2}}\right)\\
&\quad=\Delta u_\varepsilon\frac{(\nabla u_\varepsilon\,,\,(u_\varepsilon)_t\nabla(u_\varepsilon)_t)}{(1+|\nabla u_\varepsilon|^2)^{3/2}}+\frac{(u_\varepsilon)_t(u_\varepsilon)_{tij}(u_\varepsilon)_j(u_\varepsilon)_i}{(1+|\nabla u_\varepsilon|^2)^{3/2}}+\frac{((u_\varepsilon)_i\nabla(u_\varepsilon)_i\,,\,(u_\varepsilon)_t\nabla(u_\varepsilon)_t)}{(1+|\nabla u_\varepsilon|^2)^{3/2}}\\
&\qquad-3(u_\varepsilon)_i\frac{((u_\varepsilon)_t\nabla(u_\varepsilon)_t\,,\,\nabla u_\varepsilon)\,(\nabla(u_\varepsilon)_i\,,\,\nabla u_\varepsilon)}{(1+|\nabla u_\varepsilon|^2)^{5/2}}\\
&\quad=\Delta u_\varepsilon\frac{\left(\nabla u_\varepsilon\,,\,\nabla(\frac{(u_\varepsilon)_t^2}{2})\right)}{(1+|\nabla u_\varepsilon|^2)^{3/2}}+\frac{(u_\varepsilon)_t(u_\varepsilon)_{tij}(u_\varepsilon)_j(u_\varepsilon)_i}{(1+|\nabla u_\varepsilon|^2)^{3/2}}+\frac{\left(\nabla(\frac{|\nabla u_\varepsilon|^2}{2})\,,\,\nabla(\frac{(u_\varepsilon)_t^2}{2})\right)}{(1+|\nabla u_\varepsilon|^2)^{3/2}}\\
&\qquad-3\frac{\left(\nabla(\frac{(u_\varepsilon)_t^2}{2})\,,\,\nabla u_\varepsilon\right)\left(\nabla(\frac{|\nabla u_\varepsilon|^2}{2})\,,\,\nabla u_\varepsilon\right)}{(1+|\nabla u_\varepsilon|^2)^{5/2}}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\Delta^S\frac{(u_\varepsilon)_t^2}{2}&=\Delta\frac{(u_\varepsilon)_t^2}{2}-\frac{\left(\nabla u_\varepsilon\,,\,\nabla^2\frac{(u_\varepsilon)_t^2}{2}\nabla u_\varepsilon\right)}{1+|\nabla u_\varepsilon|^2}\\
&=(u_\varepsilon)_t\Delta(u_\varepsilon)_t+|(\nabla u_\varepsilon)_t|^2-\frac{(u_\varepsilon)_i(u_\varepsilon)_j(u_\varepsilon)_t(u_\varepsilon)_{tij}+(u_\varepsilon)_i(u_\varepsilon)_j(u_\varepsilon)_{ti}(u_\varepsilon)_{tj}}{1+|\nabla u_\varepsilon|^2}.
\end{aligned}
$$

We then get

$$
\begin{aligned}
\frac{d}{dt}\frac{(u_\varepsilon)_t^2}{2}&=\frac{\left(\nabla(\frac{(u_\varepsilon)_t^2}{2})\,,\,\nabla u_\varepsilon\right)}{\sqrt{1+|\nabla u_\varepsilon|^2}}k_\varepsilon+\Delta^S\left(\frac{(u_\varepsilon)_t^2}{2}\right)-2\frac{\left(\nabla\left(\frac{(u_\varepsilon)_t^2}{2}\right)\,,\,\nabla\left(\frac{|\nabla u_\varepsilon|^2}{2}\right)\right)}{1+|\nabla u_\varepsilon|^2}\\
&\quad+2\frac{\left(\nabla(\frac{(u_\varepsilon)_t^2}{2})\,,\,\nabla u_\varepsilon\right)\left(\nabla(\frac{|\nabla u_\varepsilon|^2}{2})\,,\,\nabla u_\varepsilon\right)}{(1+|\nabla u_\varepsilon|^2)^2}+\frac{\left(\nabla u_\varepsilon\,,\,(\nabla u_\varepsilon)_t\right)^2}{1+|\nabla u_\varepsilon|^2}-|(\nabla u_\varepsilon)_t|^2+(u_\varepsilon)_t^2\partial_u k_\varepsilon.
\end{aligned}
$$

Note that the last term is nonpositive by definition of k_ε.

In order to apply Lemma 3.3, we have to notice the inequality

$$
-\frac{\left(\nabla u_\varepsilon\,,\,(\nabla u_\varepsilon)_t\right)^2}{1+|\nabla u_\varepsilon|^2}+|(\nabla u_\varepsilon)_t|^2\geqslant 0.
$$

It is then enough to note that, since the solution exists for all times and it is smooth, the term $\nabla(\frac{|\nabla u_\varepsilon|^2}{2})$ is bounded on each $[0,T]$ (the bound depends on T and ε but is enough to apply the lemma). In addition, every factor containing $\nabla((u_\varepsilon)_t^2/2)$ also contains ∇u_ε, hence the assumptions of Lemma 3.3 are satisfied for every $T>0$, and this concludes the proof. □

From Propositions 3.10 and 3.11, we deduce the following result.

Proposition 3.12. *If u_0 is C-Lipschitz in space for some $C>0$, and has bounded mean curvature, then the solution u_ε of the approximate problem* (3.41) *is C-Lipschitz in space* *and Lipschitz in time with constant*

$$\left\| \sqrt{1+|\nabla u_0|^2}\,\mathrm{div}\left(\frac{\nabla u_0}{\sqrt{1+|\nabla u_0|^2}}\right)\right\|_{L^\infty(\mathbb{R}^d)}.$$

Moreover, the following inequalities hold

$$\psi_\varepsilon^-(x) - \varepsilon \le u_\varepsilon(x,t) \le \psi_\varepsilon^+(x) + \varepsilon. \tag{3.47}$$

Proof. The Lipschitz bounds of the solution are clear (it is Proposition 3.10 and 3.11).

In order to prove the second assertion, let us notice that by (3.40) and the definition of k_ε, we have

$$k_\varepsilon(x, \psi_\varepsilon^- - \varepsilon) = 2N \geqslant \left\| \sqrt{1+|\psi_\varepsilon^-|^2}\,\mathrm{div}\left(\frac{\psi_\varepsilon^-}{\sqrt{1+|\psi_\varepsilon^-|^2}}\right)\right\|_{L^\infty(\mathbb{R}^d)},$$

so that $\psi_\varepsilon^- - \varepsilon$ is a subsolution of (3.41). By the parabolic comparison principle (as in Proposition 3.4), we deduce that

$$\psi_\varepsilon^- - \varepsilon \leqslant u_\varepsilon.$$

The same arguments shows the other inequality in (3.47). □

Conclusion of the proof of Theorem 3.2. Since the solutions u_ε are equi-Lipschitz in space and time, they converge uniformly, as $\varepsilon \to 0$, to a limit function u which is also Lipschitz continuous on $\mathbb{R}^d \times [0, +\infty)$.

Equation (3.47) yields

$$\psi^- \leqslant u \leqslant \psi^+,$$

and Proposition 3.9 gives that u is a viscosity solution of (3.50).

Concerning the regularity of u, we proved that $(u_\varepsilon)_t$ and ∇u_ε are bounded on $[0, T]$, for any T in the approximate problem. This gives a bound on the mean curvature of the approximate solution. This bound does not depend on ε and remains true for the viscosity solution. As a result, the exact solution has bounded mean curvature and bounded gradient, which shows that Δu is L^∞ and, by elliptic regularity theory, u is also in $W^{2,p}$ for any $p > 1$, and so $C^{1,\alpha}$ for every $\alpha < 1$ (see [Lun95] for details).

By Theorem 3.6 below, we can also directly apply to the solution u a regularity result by Petrosyan and Shahgholian in [Sha08, PS07]. It follows that u is in fact of class $\mathcal{C}^{1,1}$, and this concludes the proof of Theorem 3.2.

□

5 Proof of Theorem 3.3

Note that the existence and uniqueness proof in appendix gives a periodic solution to (3.3).

We compute the evolution of the area of the graph of u:

$$\frac{d}{dt}\int_Q \sqrt{1+|\nabla u|^2} = \int_Q \frac{(\nabla u_t\,,\,\nabla u)}{\sqrt{1+|\nabla u|^2}} = -\int_Q u_t \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{3.48}$$

Notice that, for almost every $t>0$, $u_t(t,x)=0$ almost everywhere on the contact set. Indeed, for almost every t, u_t exists for almost every $x\in Q$. If $u(x,t)=\psi^\pm(x)$, then $u-\psi^\pm$ reaches an extremum in (x,t), which gives, $u_t(x,t)=0$. In particular, from (3.48) we get

$$\frac{d}{dt}\int_Q \sqrt{1+|\nabla u|^2} = -\int_Q u_t\left(\frac{u_t}{\sqrt{1+|\nabla u|^2}}\right).$$

Integrating this equality in time, we obtain

$$\int_Q \sqrt{1+|\nabla u|^2}\bigg|_0^T = \int_0^T\int_Q -\frac{u_t^2}{\sqrt{1+|\nabla u|^2}}. \tag{3.49}$$

which shows that

$$\int_0^T\int_Q u_t^2$$

is uniformly bounded in T. Indeed, the quantity

$$\int_Q \sqrt{1+|\nabla u|^2}\bigg|_0^T$$

represents the variation of area of the graph of u between $t=0$ and $t=T$. As this area is nonincreasing (thanks to (3.49)), this quantity is uniformly bounded in T. In addition, we recall that ∇u is uniformly bounded in T. As a result $u_t\in L^2(\mathbb{R}^+\times Q)$ so u is in $H^1([0,R)\times Q)$ for every $R>0$.

Since $\|u_t\|_{L^2(Q)}$ is $L^2(\mathbb{R}^+)$, there exists a sequence $t_n\to\infty$ such that

$$\|u_t\|_{L^2(Q)}(t_n)\underset{n\to\infty}{\longrightarrow}0.$$

In addition, $u(t_n)$ is equi Lipschitz and converges uniformly on compact sets to some u_∞ which therefore satisfies in the viscosity sense

$$\sqrt{1+|\nabla u|^2}\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+\nabla u^2}}\right)=0$$

with obstacles $\psi^\pm$ (see Appendix 6).

□

Remark. By [ISZ98], u_∞ is analytic out of the (closed) contact set $\{u_\infty=\psi^\pm\}$.

6 Appendix: on viscosity solutions with obstacles

6.1 Definition of viscosity solution

Given an open subset B of $\mathbb{R}^d$, let u_0, ψ^+ and ψ^- be three Lipschitz functions $B \to \mathbb{R}$ such that

$$\psi^-(x,0) \leqslant u_0(x) \leqslant \psi^+(x,0).$$

We are interested in the viscosity solutions of the equation

$$u_t = \sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right), \qquad u(x,0) = u_0(x), \tag{3.50}$$

with the constraint

$$\psi^-(x) \leqslant u(x,t) \leqslant \psi^+(x). \tag{3.51}$$

Definition 3.1 (see [CIL92, Mer14])**.** *We say that a function $u : B \times [0,T) \to \mathbb{R}$ is a* viscosity subsolution *of* (3.50) *if u satisfies the following conditions:*

– *u is upper semicontinuous;*
– *$u(x,0) \leqslant u_0(x)$;*
–
$$\psi^-(x) \leqslant u(x,t) \leqslant \psi^+(x); \tag{3.52}$$
– *for any $(x_0,t_0) \in \mathbb{R}^n \times \mathbb{R}^+$ and $\varphi \in C^2$ such that $u - \varphi$ has a maximum at (x_0,t_0) and $u(x_0,t_0) > \psi^-(x_0)$,*
$$u_t \leqslant \sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{3.53}$$

Similarly, u is a viscosity supersolution *of* (3.50) *if:*

– *u is lower semicontinuous;*
– *$u(x,0) \geqslant u_0(x)$;*
– (3.52) *holds;*
– *for any $(x_0,t_0) \in \mathbb{R}^n \times \mathbb{R}^+$ and $\varphi \in C^2$ such that $u - \varphi$ has a minimum at (x_0,t_0) and $u(x_0,t_0) < \psi^+(x_0)$,*
$$u_t \geqslant \sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

We say that u is a viscosity solution *of* (3.50) *if it is both a super and a subsolution.*

6.2 Comparison principle

In order to prove uniqueness of continous viscositysolutions of (3.50), we shall prove a comparison principle between solutions following [GGIS91, Theorem 4] (see also [CGG91]).

Proposition 3.13. *If u is a viscosity subsolution of* (3.50) *on $[0,T)$, v is a viscosity supersolution, if $\psi^\pm$ are Lipschitz in space and if $u(x,0) \leqslant v(x,0)$, then $u(x,t) \leqslant v(x,t)$ for all $(x,t) \in \mathbb{R}^n \times [0,T)$.*

Proof. We will check that the proof of [GGIS91, Theorem 2.1] can be extended to the obstacle case. Notice first that the assumptions $(A.1)-(A.3)$ of [GGIS91, Theorem 2.1] are satisfied also in our case. Indeed, $(A.1)$ comes directly from the Lipschitz bound on $\psi^{\pm}$ and the constraint $\psi^- \leqslant u, v \leqslant \psi^+$ whereas $(A.2)$ and $(A.3)$ result from the assumed time zero comparison.

Let us show that [GGIS91, Proposition 2.3] also holds. Indeed, up to Equation (2.9) nothing chenges. To continue the proof, using the same notation of [GGIS91, Proposition 2.3], we have to check that if

$$\sup_V (w - \Psi) > 0,$$

then the supremum is reached in the complementary of the contact set $\{u = \psi^-\} \cup \{v = \psi^+\}$.

Indeed, notice that if $u(x,t) = \psi^-(x)$, then, for all x, y, t, s,

$$u(x,t) - v(y,s) = \psi^-(x) - v(y,s) \leqslant \psi^-(y) + L(|x-y|) - v(y,s) \leqslant L(|x-y|)$$

since $v \geqslant \psi^-$. Hence, if $u(x,t) = \psi^-(x)$, with $K' > L$, we must have $w - \Psi \leqslant 0$, so the supremum of $w - \Psi$ is attained in the complementary of $\{u = \psi^-\}$. One can show similarly that the supremum is reached in the complementary of $\{v = \psi^+\}$. Hence Proposition 2.3 of [GGIS91] holds.

From Proposition 2.4 to Lemma 2.7 of [GGIS91], every result holds without changes.

Concerning the proof of Theorem 2.1 of [GGIS91], the first assumption is

$$\alpha = \lim_{\theta \to 0} \sup \{w(t,x,y), \mid |x-y| \leqslant \theta\} > 0.$$

Then, Proposition 2.4 gives constants δ_0 and γ_0 such that for all $\delta \leqslant \delta_0$, $\gamma \leqslant \gamma_0$ and $\varepsilon > 0$, there holds

$$\Phi(\hat{x}, \hat{y}, \hat{t}) := \sup_{\mathbb{R}^n \times \mathbb{R}^n \times [0,T)} \Phi(x,y,t) > \frac{\alpha}{2}$$

with

$$\Phi(t,x,y) = u(x,t) - v(y,t) - \frac{|x-y|^4}{4\varepsilon} - \delta(|x|^2 + |y|^2) - \frac{\gamma}{T-t}$$

To conclude the proof, we only have to show that the maximum of Φ is once again attained on the complementary of $\{u = \psi^-\} \cup \{v = \psi^+\}$. In the same way as for Proposition 2.3, if $u(x,t) = \psi^-(x)$, we can write

$$\begin{aligned} \Phi(t,x,y) &= u(x,t) - v(y,t) - \frac{|x-y|^4}{4\varepsilon} - \delta(|x|^2 + |y|^2) - \frac{\gamma}{T-t} \\ &\leqslant \psi^-(y) + L|x-y| - v(y,t) \leqslant L|x-y|. \end{aligned}$$

Thanks to Proposition 2.5, $|\hat{x} - \hat{y}| \underset{\varepsilon \to 0}{\longrightarrow} 0$. So, with ε sufficiently small (one can reduce the quantity ε_0 given by Proposition 2.6), Φ has its maximum out of $\{u = \psi^-\}$ (and similarly out of $\{v = \psi^+\}$), which enables the application of Lemma 2.7 and gives a contradiction as in [GGIS91]. $\square$

6.3 Existence

In this subsection, we prove the following result:

Proposition 3.14. *There exists a continuous viscosity solution to* (3.50).

We follow [CIL92] to build a solution by means of the Perron's method. Let us state an obvious but useful proposition and a key lemma for applying Perron's method.

Proposition 3.15. *Let u be a subsolution of the mean curvature motion for graphs (without obstacles) which satisfies $u \leqslant u^+$. Then, $u_{ob} := u \vee u^-$ is a subsolution of* (3.50) *with obtacles (the same happens for v supersolution and $v_{ob} = v \wedge u^+$).*

In the sequel, we shall denote by u^* (resp. u_*) the upper (resp. lower) semicontinuous envelope of a function u.

Lemma 3.8. *Let $\mathcal{F}$ be a family of subsolutions of* (3.50)*. We define*

$$U(x,t) = \sup\{u(x,t) \mid u \in \mathcal{F}\}.$$

Then, U^* is a subsolution of (3.50)*.*

The proof of the proposition and the lemma can be found in [CIL92], Lemma 4.2 (with obvious changes due to the parabolic situation and obstacles).

Construction of barriers In the sequel, to claim that the initial condition is taken by the viscosity solution, we need to build barriers to sandwich the solution. More precisely, we want to build a subsolution w^- such that $(w^-)^*(x,0) = u_0(x)$ and a supersolution w^+ such that $(w^+)_*(x,0) = u_0(x)$. To show this claim, let us begin by a simple fact.

Let

$$g^a_{\alpha,b}(x) = -\sum \alpha_i \frac{(x-a)_i^2}{\sqrt{1+(x-a)_i^2}} + b \tag{3.54}$$

for some $(a,b) \in \mathbb{R}^n \times \mathbb{R}$ and $\alpha_i \geqslant 0$ such that $g(x) \leqslant u_0(x)$. Note in particular that

$$g^a_{\alpha,b}(x) \geqslant -\sum \alpha_i (x-a)_i^2 + b \quad \text{and} \quad H(g^a_{\alpha,b}) \geqslant H(g^a_{\alpha,b})|_{t=0} = -2\sum \alpha_i. \tag{3.55}$$

Then, it is easy to show (using Proposition 3.15) that the function

$$v(x,t) = \left(g^a_{\alpha,b}(x) + \left(2\sum_{i=1}^n \alpha_i + 3M \right) t \right) \vee \psi^-$$

is a subsolution of (3.50). Indeed, the curvature of $g^a_{\alpha,b}$ is smaller than $2\sum \alpha_i$ and its gradient is bounded by 2 (so $\sqrt{1+|\nabla g|^2} \leqslant 3$).

Thanks to Lemma 3.8, the function

$$w^-(x,t) = \left(\sup_{\substack{(\alpha_i),c \\ g^c_{\alpha,b} \leqslant u_0}} \left(g^a_{\alpha,b}(x) - 2\sum_{i=1}^n \alpha_i t - 3Mt \right) \vee \psi^- \right)^*$$

is a subsolution of (3.50) (with obstacles).
It remains to show that $(w^-)^*(x,0) = u_0(x)$. To see this, notice that since u_0 is Lipschitz and $u_0 \geqslant \psi^-$, $u_0(x) = w^-(x,0)$, yielding $u_0(x) \leqslant (w^-)^*(x,0)$. But for all $t \geqslant 0$, $v(x,t) \leqslant u_0(x)$ so $w^-(x,t) \leqslant u_0(x)$. By continuity of u_0, $(w^-)^*(x,t) \leqslant u_0(x)$, which shows that $(w^-)^*(x,0) = u_0(x)$, and w^- is a low barrier for solutions of (3.50).

We build w^+ in the same way.

Perron's method We use the classical Perron's method to build a solution of (3.50) on $[0,T)$ for every $t > 0$. Let us define

$$W(x,t) = \sup\{u(x), \mid u \text{ is a subsolution of (3.50) on } [0,T)\}.$$

Since ψ^- is a subsolution, this set in non empty and W is well defined. Every subsolution is less that ψ^+, so is W.

Thanks to Lemma 3.8, W^* is a subsolution of (3.50) regardless the initial conditions. Applying the comparison principle (Proposition 3.13) to every subsolution u and w^+ gives

$$\forall x,t,\ W(x,t) \leqslant w^+(x,t).$$

Considering the upper-semi-continuous envelopes, we get

$$\forall x,t,\ W^*(x,t) \leqslant (w^+)^*(x,t)$$

which immediately yields to

$$W^*(x,0) = u_0(x).$$

Then, W^* is a subsolution (with initial conditions), hence $W^* = W$ which shows the upper semi-continuity of W.

We want to prove that W is actually a solution of (3.50). In this order, let us prove the following

Lemma 3.9. *Let u be a subsolution of* (3.50)*. If u_* fails to be a supersolution (regardless initial conditions) at some point $(\hat{x},\hat{t})$ then there exists a subsolution u_κ (regardless initial conditions) satisfying $u_\kappa \geqslant u$ and $\sup u_\kappa - u > 0$ and such that $u(x,t) = u_\kappa(x,t)$ for $|x-\hat{x}|, |t-\hat{t}| \leqslant \kappa$.*

Proof. Let us assume that u_* fails to be a supersolution at $(0,1)$. Then there exists $(a,p,X) \in \mathcal{J}^{2,-}u_*(0,1)$ with

$$a + F(p,X) + k(0)\sqrt{1+p^2} < 0.$$

Let us then define

$$u_{\delta,\gamma}(x,t) = u_*(0,1) + \delta + (p\,,\,x) + a(t-1) + \frac{1}{2}\,(Xx\,,\,x) - \gamma(|x|^2 + t - 1).$$

Thanks to the continuity of F and k, $u_{\delta,\gamma}$ is a classical subsolution on $B_r(0,1)$ of $u_t + F(Du,D^2u) + k(x)\sqrt{1+|\nabla u|^2} = 0$ for δ,γ,r sufficiently small. By assumption,

$$u(x,t) \geqslant u_*(x,t) \geqslant u_*(0,1) + a(t-1) + (p\,,\,x) + \frac{1}{2}\,(Xx, x\,,\,+)\,o(|x|^2 + |t-1|).$$

With $\delta = \gamma\frac{r^2+r}{8}$, we get $u(x,t) > u_{\delta,\gamma}(x,t)$ for small r and $|x|, |t-1| \in [\frac{r}{2}, r]$. Reducing again r, we can assume that $u_{\delta,\gamma} < \psi^+$ on B_r. Thanks to Lemma 3.8,

$$\tilde{u}(x,t) = \begin{cases} \max(u(x,t), u_{\delta,\gamma}(x,t)) \text{ if } |x, t-1| < r \\ u(x) \text{ otherwise} \end{cases}$$

is a subsolution of (3.50) (with no initial conditions). □

Finally, this lemma combined with the definition of W proves that W is in fact a solution of (3.50) (the initial conditions were already checked).

6.4 Regularity

Proposition 3.16. *The unique solution u of* (3.50) *is Lipschitz in space, with the same constant as $u_0, \psi^\pm$.*

Proof. We will prove that $u_z(x,t) = u(x+z,t) - L|z|$ is in fact a subsolution of (3.50). The Lipschitz bound is then straightforward (using the comparison principle).

To begin, we notice that $u(x+z,t) - L(|z|) \leqslant u^+(x,t)$ and $u(x+z,0) - L|z| \leqslant u_0(x+z) - L|z| \leqslant u_0(x)$.

Assume now that φ is any smooth function which is greater than u_z with equality at $(\hat{x},\hat{t})$. Then, either, $u_z(\hat{x},\hat{t}) = \psi^-(\hat{x},\hat{t})$ and nothing has to be done, or $u_z(\hat{x},\hat{t}) > \psi^-(\hat{x},\hat{t})$. In the second alternative, one can write

$$u(\hat{x}+t,\hat{t}) > \psi^-(\hat{x}) = \psi^-(\hat{x}+z) + (\psi^-(\hat{x}) - \psi^-(\hat{x}+z),$$

so

$$u(\hat{x}+z,\hat{t}) > \psi^-(\hat{x}+z) + \underbrace{\psi^-(\hat{x}) - \psi^-(\hat{x}+z) + L|z|}_{\geqslant 0} \geqslant u^-(\hat{x}+z,\hat{t}).$$

As u is a subsolution at $(\hat{x}+z,\hat{t})$ and $u(x+z,t) \leqslant \varphi(x,t) + L|z|$ with equality at $(\hat{x}+z,\hat{t})$, one can write with $y = x+z$, $s = t$,

$$u(y,t) \leqslant \varphi(y-z,s) + L|z| := \phi(y,s),$$

with equality at $(\hat{y},\hat{s})$ which gives

$$\phi_t + F(D\phi(\hat{x},\hat{t}), D^2\phi(\hat{x},\hat{t})) \leqslant 0.$$

Since the derivatives of ϕ and φ are the same, we deduce

$$\varphi_t + F(D\varphi, D^2\varphi) \leqslant 0,$$

what was expected. □

Remark. With the same arguments, one can prove that

$$\forall \delta > 0, \quad \forall x,t, \quad |u(x,t+\delta) - u(x,t)| \leqslant \sup_x |u(x,\delta) - u(x,0)|.$$

We now present a general regularity result by Shahgholian [Sha08] which applies to viscosity solutions for parabolic equations with obstacles.

Theorem 3.6 ([PS07], Th. 4.1). *Let $Q^+ := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x| < 1, t \in [0,1)\}$ and $H(u) = F(D^2u, Du) - u_t$ where F is uniformly elliptic. Let u be a continuous viscosity solution of*

$$\begin{aligned} (u-\psi)H(u) &= 0, \\ H(u) &\leqslant 0, \\ u &\geqslant \psi, \end{aligned} \tag{3.56}$$

in Q^+, with boundary data

$$u(x,t) = g(x,t) \geqslant \psi(x,t) \qquad \text{on } \{|x| = 1\} \cup \{t = 0\}. \tag{3.57}$$

Assume that $\psi \in C^{1,1}(Q^+)$ and g is continuous. Then, $u \in C^{1,1}$ on every compact subset of Q^+.

It has to be noticed $H = F - \partial_t$ where $F(D^2u, Du) = -\sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$ satisfies all the assumptions of [Sha08], 1.3. Indeed, the uniform ellipticity is provided by the Lipschitz bound obtained in previous subsection.

Moreover, the viscosity solution u of (3.50) satisfies (3.56) and (3.57) on every cylinder $Q_r^+(x_0) := \{|x-x_0| \leqslant r,\ t \in [t_0, t_0+r)\}$ such that r is choosen sufficiently small in order to have either $Q_r^+(x_0) \cap \{u = \psi^+\} = \emptyset$ or $Q_r^+(x_0) \cap \{u = \psi^-\} = \emptyset$. In the second alternative, change every sign in the equations.

Applying Theorem 3.6 we get a $C^{1,1}$ bound for u on every compact subset of $Q_r^+(x_0)$. To show that u is $C^{1,1}$ in the whole space, just cover $\mathbb{R}^n \times \mathbb{R}^+$ with such $Q_r^+(x_i)$.

6.5 A remark on the forcing term

It is quite simple to show that every result presented in the graph case remains valid with a sufficiently regular forcing term $k(x,u)$ (except the long time behavior which still occurs but is much less meaningful). The corresponding equation rewrites

$$u_t = \sqrt{1+|\nabla u|^2}\left[\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + k(x,u)\right], \qquad u(x,0) = u_0(x), \tag{3.58}$$

Let us mention the most important differences which occur.

Proposition 3.13. One can prove a comparison with the forcing term depending on u adapting Theorem 4.2, and the so called Proposition 4.4 in [GGIS91]. First, it is clear that

$$F(t,x,r,p,X) = \mathrm{Tr}\left[\left(I - \frac{p\otimes p}{1+p^2}\right)X\right] + \sqrt{1+p^2}k(x,r)$$

satisfies the hypothesis of Theorem 4.2. Then, the proof of Proposition 4.4 use only the regularity of F, u and v. Concerning the proof of Theorem 4.2 itself, the process is the same as in Theorem 2.1, and the maximum point is out of the obstacles too.

Existence of a viscosity solution. Every result of Section 6.3 apply with no modification. The only difference is when we construct the barriers: the forcing term has to be involved in their evolution. We suggest to replace v by adding a forcing term:

$$v(x,t) = \left(g^a_{\alpha,b}(x) + \left(2\sum_{i=1}^{n} \alpha_i + 3\|k\|_\infty \right) t \right) \vee \psi^-$$

and take, as before, the sup on this family to build a barrier.

Regularity of the viscosity solution. With the forcing term, finding a time uniform spatial Lipschitz constant for u is hopeless. It grows exponentially. More precisely, we have have the following property (which is essentially [For08], Lemma 2.15)

Proposition 3.17. *Let u be the unique solution of* (3.50) *with forcing term. Then, it is Lipschitz in space with*

$$|u(x,t) - u(y,t)| \leqslant Me^{Lt}|x-y|.$$

Proof. Let $\phi(x,y,t) = Me^{Lt}|x-y|$: we show by contradiction that $u(x,t) - u(y,t) \leqslant \phi(x,y,t)$. Assume that

$$M := \sup_{(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,T)} u(x,t) - u(y,t) - \phi(x,y,t) > 0.$$

Then, as in the proof of uniqueness, we introduce

$$\tilde{M} := \sup_{(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,T)} u(x,t) - u(y,t) - \phi(x,y,t) - \alpha(|x|^2+|y|^2) - \frac{\gamma}{T-t}.$$

If α and γ are small enough, $\tilde{M}$ remains positive and is attained (say in $(\hat{x},\hat{y}\hat{t} < T)$), since the growth at infinity of u and v is at most linear. Since u_0 is Lipschitz, $\hat{t} > 0$. In addition it is clear that $\hat{x} \neq \hat{y}$.

Concerning the obstacles, if $u(\hat{x},\hat{t}) = \psi^-(\hat{x},\hat{t})$, then $u(\hat{x},\hat{t}) - v(\hat{y},\hat{t}) - \phi(\hat{x},\hat{y},\hat{t}) \leqslant \psi^-(\hat{x},\hat{t}) - M|\hat{x}-\hat{y}| - \psi^-(\hat{y},\hat{t}) \leqslant 0$ which shows that $u(\hat{x},\hat{t}) > \psi^-(\hat{x},\hat{t})$. Similarly, $u(\hat{y},\hat{t}) < \psi^+(\hat{y},\hat{t})$.

We then apply Ishii's lemma to $\tilde{u}(x,t) - \tilde{v}(y,t) - \phi(x,y,t) - \frac{\gamma}{T-t}$ with $\tilde{u}(x,t) = u(x,t) - \alpha|x|^2$ and $\tilde{v}(y,t) = v(y,t) - \alpha|y|^2$. We use the following notations

$$\hat{p} = D_x\phi = \frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|}Me^{Lt}|\hat{x}-\hat{y}| = -D_y\phi \neq 0,$$

$$Z = D_x^2\phi = \frac{1}{|\hat{x}-\hat{y}|}Me^{Lt}|\hat{x}-\hat{y}|I + 2\frac{(\hat{x}-\hat{y})\otimes(\hat{x}-\hat{y})}{|\hat{x}-\hat{y}|^2}Me^{Lt}|\hat{x}-\hat{y}|,$$

and

$$A = D^2\phi = \begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix}.$$

The lemma gives, for every β such that $\beta A \leqslant I$, the existence of $\tau_1, \tau_2 \in \mathbb{R}$ and $X, Y \in \mathcal{S}^n$ such that

$$\tau_1 - \tau_2 = \frac{\gamma}{(T-t)^2} + Le^{Lt}A|\hat{x}-\hat{y}|,$$

$$(\tau_1, \hat{p} + \alpha\hat{x}, X + \alpha I) \in \overline{\mathcal{J}}^{2,+} u(\hat{x}, \hat{t}),$$
$$(\tau_2, \hat{p} - \alpha\hat{y}, Y - \alpha I) \in \overline{\mathcal{J}}^{2,-} u(\hat{y}, \hat{t}),$$
$$\frac{-1}{\beta}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leqslant \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leqslant (I - \beta A)^{-1} A.$$

Since u is both a super and a subsolution we get

$$\tau_1 + k(\hat{x}, \hat{t})|\hat{p} + \alpha\hat{x}| + F(\hat{p} + \alpha\hat{x}, X + \alpha I) \leqslant 0, \tag{3.59}$$
$$\tau_2 - k(\hat{y}, \hat{t})|\hat{p} - \alpha\hat{y}| + F(\hat{p} - \alpha\hat{y}, Y - \alpha I) \geqslant 0.$$

$X \leqslant Y$ in the last equation gives

$$-\tau_2 + k(\hat{y}, \hat{t})|\hat{p} - \alpha\hat{y}| - F(\hat{p} - \alpha\hat{y}, X - \alpha I) \leqslant 0. \tag{3.60}$$

Adding 3.60 to 3.59 yields to

$$\frac{\gamma}{(T - \hat{t})^2} + Le^{Lt} A|\hat{x} - \hat{y}| - k(\hat{x}, \hat{t})|\hat{p} + \alpha\hat{x}| + k(\hat{y}, \hat{t})|\hat{p} - \alpha\hat{y}| \\ + F(\hat{p} + \alpha\hat{x}, X + \alpha I) - F(\overline{p} - \alpha\overline{y}, X - \alpha I) \leqslant 0. \tag{3.61}$$

Notice that

$$Le^{L\hat{t}} A|\hat{x} - \hat{y}| - k(\hat{x}, \hat{t})|\hat{p}| + k(\hat{y}, \hat{t})|\hat{p}| \geqslant Le^{L\hat{t}} A|\hat{x} - \hat{y}| - L|\hat{x} - \hat{y}|e^{L\hat{t}} A \geqslant 0. \tag{3.62}$$

Then, (3.61) becomes

$$\frac{\gamma}{(T - \hat{t})^2} + (|\hat{p}| - |\hat{p} + \alpha\hat{x}|)\, k(\hat{x}, \hat{t}) - (|\hat{p}| - |\hat{p} - \alpha\hat{y}|)\, k(\hat{y}, \hat{t}) + \underline{F}(\hat{p} + \alpha\hat{x}, X + \alpha I) - \overline{F}(\hat{p} - \alpha\hat{y}, X - \alpha I) \leqslant 0.$$

Let α go to zero. $\hat{p}$ and X are bounded: one assume they converge and still denote by $\hat{p}, X$ their limit. In addition, $\alpha\hat{x}, \alpha\hat{y} \to 0$ and k is bounded, hence we get

$$\frac{\gamma}{(T - \hat{t})^2} \leqslant 0,$$

which is a contradiction. So, switching x and y if needed,

$$|u(x, t) - u(y, t)| \leqslant Ae^{Lt}|x - y|.$$

$\square$

Chapter 4

A crystalline curve shortening flow (with M. Novaga and P. Pozzi)

1 Introduction

In this chapter, we consider an anisotropic mean curvature motion for planar immersed curves. More precisely, given an initial curve $u_0 : S^1 \to \mathbb{R}^2$ and an anistropy γ, we want to build a family $u(t, x)$ of curves such that for any x, t, the speed of $u(x, t)$ is given by

$$V = \kappa_\gamma \nu$$

where ν is the normal vector of u at x (oriented by the parametrization) and κ_γ the γ-dependent anisotropic mean curvature. We want to prove the existence of such a motion in short time, for smooth and crystalline anisotropies.

The classical (isotropic) mean curvature motion has been widely studied in the past 40 years (with the framework of Geometric Measure Theory [Bra78] as well as using differential geometry [Hui84, EH89, EH91a]) and the behavior of the flow as well as its singularities are fully understood (see [EH91a]). Nonetheless, only a few results are known in the anisotropic case.

The first occurrence of anisotropic curvature flow appeared in [ATW93] with the well known discrete minimizing scheme which approximates curvature motion, with a smooth anisotropy. One year later, in [GL94], Gage and Li presented the anisotropic curvature motion for planar curves, linking it with a homothetical shrinking of the anisotropic unit ball. In 2001, Andrews [And01] extended this work by studying the anisotropic motion for all dimensions in the case of smooth anisotropies in the context of differential geometry (he also studied singularities).

On the other hand, the crystalline case remains widely unknown. In 2 dimensions, Chambolle and Novaga adapted in [CN13b] the scheme of [ATW93] to build a crystalline curvature motion for sets.

Motivated by [CN13b] and inspired by a work of the second author [CN13a] which proposed to pass to the limit in an approximate flow, we extend the results of [CN13b] to a crystalline motion of a planar curve which is only immersed. To do that, we approximate the crystalline anisotropy by a smooth one for which we show a short time existence (in the spirit of [And01]). Then, we show that we can pass to the limit in the anisotropy to obtain a short time existence for a crystalline motion. The proof of this result is organized as follows:

We first introduce the definitions and notation we use in the paper and define what we call anisotropic motion for curves. Then, we work on a smooth approximate motion (that is a anisotropic motion for a smooth anisotropy γ_ε). To define the latter, it is necessary to start from a curve which has a bounded γ_ε-anisotropic curvature. In Lemma 4.1, we show that a curve which has a bounded γ-curvature can be approximated by a curve with bounded γ_ε-curvature.

In Section 4, we study the evolution of the geometric quantities under the flow, and prove, as it happens in the isotropic case, that the curvature must blow up at the first singular time. Since we can show that it does not happen, it provides a uniform bound of the existence time of the approximate flow.

Finally, we pass to the limit in the approximated flow and show that it provides a solution to the crystalline curvature motion in an interval $[0, T)$ where T depends only on the initial conditions.

2 Set up and notation

We consider closed planar curves parametrized by $u : S^1 \times [0, T] \to \mathbb{R}^2$, $u = u(x, t)$. We denote by s the arclength parameter of the curve (thus $\partial_s = \partial_x/|u_x|$), by $\tau = u_x/|u_x| = u_s = (\sin\theta, -\cos\theta)$ its unit tangent and $\nu = (\cos\theta, \sin\theta)$ its unit normal. Recall the classical Frenet formulas

$$u_{ss} = \tau_s = \vec{\kappa} = \kappa\nu, \qquad \nu_s = -\kappa\tau. \tag{4.1}$$

Moreover recall that from $\theta = \theta(s)$ and the expression for ν_s one infers

$$\kappa = \theta_s. \tag{4.2}$$

Anisotropic Length

Definition 4.1. *We call (symmetric) anisotropy a map $\gamma : \mathbb{R}^2 \to [0, \infty)$ which is a norm in $\mathbb{R}^2$, that is*

— *γ is convex,*
— *$\gamma(p) > 0$ for $p \neq 0$,*
— *$\forall \lambda \in \mathbb{R}$, $\gamma(\lambda x) = |\lambda|\gamma(x)$.*

We say that γ is uniformly elliptic if there exists $C > 0$ such that

$$D^2(\gamma^2) \geqslant CI.$$

It is equivalent to ask that the set $\{\gamma \leqslant 1\}$ is uniformly convex (its Euclidean curvature is positive and bounded away from zero). We call this set the Wulff shape *relative to γ and denote it by W_γ.*
On the other hand, γ is said to be crystalline if the set W_γ is a polytope.

The anisotropic length is defined by

$$L_\gamma(u) = \int_{S^1} \gamma(\nu) ds = \int_{S^1} \gamma(u_x^\perp) dx. \tag{4.3}$$

Using the homogeneity properties of γ one obtains (for smooth enough functions)

$$\begin{aligned}\langle L'_\gamma(u), \varphi \rangle &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} L_\gamma(u+\epsilon\varphi) = \int_{S^1} \gamma'(\nu)\cdot \varphi_x^\perp dx = -\int_{S^1} \gamma''(\nu)\nu_s \cdot \varphi^\perp ds \\ &= \int_{S^1} (\gamma''(\nu)\tau\cdot\tau)\kappa\tau\cdot\varphi^\perp ds = -\int_{S^1} (\gamma''(\nu)\tau\cdot\tau)\kappa\nu\cdot\varphi\, ds.\end{aligned}$$

A classical formulation for the anisotropic curve shortening flow (for which Wulff shapes shrink self similarly) is then given by

$$\frac{u_t}{\gamma(\nu)} = (\gamma''(\nu)\tau\cdot\tau)\kappa\nu \tag{4.4}$$

(see for example [GL94]). Note that by setting

$$\phi(\theta) := \gamma(\nu) = \gamma(\cos\theta, \sin\theta) \tag{4.5}$$

a straightforward calculation gives

$$\phi(\theta) + \phi''(\theta) = \gamma''(\nu)\tau\cdot\tau \left(= \frac{\gamma_{p_1p_1}(\cos\theta,\sin\theta)}{\sin^2\theta} \right), \tag{4.6}$$

thus we can write the ACSF as

$$\boxed{u_t = \phi(\theta)(\phi(\theta)+\phi''(\theta))\,\kappa\nu = \psi(\theta)\kappa\nu,} \tag{4.7}$$

where

$$\psi(\theta) := \phi(\theta)(\phi(\theta)+\phi''(\theta)). \tag{4.8}$$

Note that the convexity of γ implies that $\psi \geq 0$.

Definition 4.2. *We say that a curve u is γ-regular if one of these two equivalent conditions hold*

- *Every graphical part M_δ of $u(S^1)$ satisfies the exterior and interior Wulff-shape condition of radius R, for some R. In addition, the edges of M_δ which are parallel to the edges of RW_γ around which $u(S^1)$ lies locally on one side are longer than these edges of RW_γ* [1]
- *There exists a Lipschitz Cahn Hoffman vector field on $u(S^1)$, that is a vector field n such that*

$$\forall x \in S^1, \quad n(u(x)) \in \partial\gamma^\circ(\nu)$$

where ν is a (non unique since u is only Lipschitz) Euclidean normal to u at x.

1. For example, if W_γ is a square with sides of length 1, then a staircase with steps of length $\frac{1}{2}$ satisfies this property with $R=1$ but a square with sides of length $\frac{1}{2}$ does not.

We say that a γ-regular curve has a curvature bounded by C if either the first condition is fulfilled with $R = \frac{1}{C}$ or if $|\partial_x n| \leqslant C|u_x|$.

If γ is smooth, we define the anisotropic curvature *to be*

$$\kappa_\gamma := \gamma''(\nu)\tau \cdot \tau\kappa.$$

In what follows, the strategy is the following: we will approximate γ by a sequence of $\gamma_\varepsilon \geqslant \gamma$ which are smooth and uniformly elliptic. This will allow us to construct an approximate anisotropic motion. Then, we pass to the limit in ε. We first need to approximate the initial curve.

3 Approximation of the initial curve

This subsection is dedicated to prove the

Lemma 4.1. *Let u be a Lipschitz and γ-regular curve whith γ-curvature bounded by C. Then, for all γ_ε smooth and uniformly elliptic such that $\gamma_\varepsilon \geqslant \gamma$ and $\gamma_\varepsilon \to \gamma$ uniformly on compact subsets, and every $C' > C$, there exists $u_\varepsilon \to u$ uniformly such that u_ε has a bounded curvature $|\kappa_{\gamma_\varepsilon}| \leqslant C'$. In addition, the (Euclidian) normal vector $\nu_\varepsilon(x)$ converges to $\nu(x)$ almost everywhere (the latter exists almost everywhere since u is Lipschitz).*

Before the proof, let us state a definition from [CN13b].

Definition 4.3. *Let A be a subset of $\mathbb{R}^2$. We say that A satisfies the inner RW_γ-condition for some $R > 0$ if*

$$\overline{A} = \bigcup_{d^E_\gamma(x) \leqslant -R} (x + RW_\gamma)$$

and for all $r < R$ and $x \in \mathbb{R}^2$, $(x + rW_\gamma) \cap A^c$ is connected.
We say that it satisfies the outer RW_γ-condition if A^c satisfies the outer RW_γ-condition. Finally, we say that it satisfies the RW_γ-condition if it satisfies both inner and outer condition.

This proof is based on [CN13b], Lemma 1. The curve $u(S^1)$ is Lipschitz. Hence, there exists δ_0 such that for every $x_0 \in S^1$, we can find an orientation n and a neighborhood $(x_0 - \delta, x_0 + \delta)$, which we identify with an interval of the real line $\mathbb{R}$, with $\delta \geqslant \delta_0$ such that $u(x_0 - \delta, x_0 + \delta)$ is a graph over $n^\perp$. Without loss of generality, we can assume that u is differentiable at $x_0 \pm \delta$. We denote by Γ_δ the extension of this graph to $\mathbb{R}$ (as the graph of a continuous function with slope $u'(x_0 - \delta)$ if $x \leqslant x_0 - \delta$ and $u'(x_0 + \delta)$ if $x \geqslant x_0 + \delta$). We note M_δ the hypograph of this function (such that $\Gamma_\delta = \partial M_\delta$). Since $\kappa_\gamma(M_\delta) \leqslant C$ and M_δ is a graph, then M_δ satisfies the $\frac{1}{C}W_\gamma$ condition. So, it satifies the $\frac{1}{C'}W_\gamma$ condition too.

We will apply [CN13b, Lemma 1] to M_δ which satisfies the RW_γ-condition. This will provide

$$\tilde{M}_{\delta,\varepsilon} := \bigcup \left\{ (x + RW_{\gamma_\varepsilon}) \mid (x + RW_{\gamma\varepsilon}) \subset \overline{M}_\delta \right\}$$

and

$$M_{\delta,\varepsilon} := \mathbb{R}^2 \setminus \bigcup \left\{ (x + RW_{\gamma_\varepsilon}) \,\middle|\, (x + RW_{\gamma\varepsilon}) \subset \overline{\tilde{M}^c_{\delta,\varepsilon}} \right\}.$$

Note that $\tilde{M}_{\delta,\varepsilon}$ satisfies the inner $\frac{1}{C}W_{\gamma_\varepsilon}$ inner condition and $M_{\delta,\varepsilon}$, the $\frac{1}{C}W_{\gamma_\varepsilon}$ outer condition and that $\partial\tilde{M}_{\delta,\varepsilon}$ and $\partial M_{\delta,\varepsilon}$ both converge to ∂M_δ in Hausdorff distance.

Since M_δ satisfies also the $R'W_\gamma$-condition, the same lemma in [CN13b] provides

$$\hat{M}_{\delta,\varepsilon} := \bigcup \left\{(x + R'W_{\gamma_\varepsilon}) \;\middle|\; (x + R'W_{\gamma\varepsilon}) \subset \overline{M}_\delta\right\}$$

and

$$\check{M}_{\delta,\varepsilon} := \mathbb{R}^2 \setminus \bigcup \left\{(x + R'W_{\gamma_\varepsilon}) \;\middle|\; (x + R'W_{\gamma\varepsilon}) \subset \overline{\hat{M}^c_{\delta,\varepsilon}}\right\}.$$

As before, $\hat{M}_{\delta,\varepsilon}$ satisfies the inner $R'W_{\gamma_\varepsilon}$ inner condition and $\check{M}_{\delta,\varepsilon}$, the $R'W_{\gamma_\varepsilon}$ outer condition and their boundary both converge to ∂M_δ in Hausdorff sense.

To see that the localization does not depend on δ, we need the

Lemma 4.2. *There exists $\delta_1 > 0$ such that for every $\delta, \delta' \leqslant \delta_1$, there exists $\varepsilon_1 > 0$ such that for every $\varepsilon \leqslant \varepsilon_1$,*

$$\hat{M}_{\delta',\varepsilon} = \hat{M}_{\delta,\varepsilon}$$

and

$$\check{M}_{\delta',\varepsilon} = \check{M}_{\delta,\varepsilon}$$

Before proving this lemma, let us conclude the proof of Lemma 4.1. We proved that for each $x_0 \in S^1$, there exists δ_0 such that for every $\delta \leqslant \delta_0$, the construction of $\check{M}_{\delta,\varepsilon}$ does not depend on δ (with the definition above). By compactness, we have a uniform δ_2 (for the whole curve) such that the different

$$\check{M}_{\delta,\varepsilon}$$

which corresponds to every point of $u(S^1)$ can be connected.

Let us now define u_ε. By compactness, one has a finite number of $y_i = u(x_i)$ and $\rho_i > 0$ such that

$$u(S^1) = \bigcup_{i=1}^{p} \partial M_\delta \cap B_{\rho_i}(y_i).$$

For every i, there exists η_i such that

$$\partial M_\delta \cap B_{\rho_i}(y_i) = u([x_i - \eta_i, x_i + \eta_i]),$$

with $x_i - \eta_i < x_{i-1} < x_i < x_{i-1} + \eta_i$. Then, we set $u^i_\varepsilon : [x_i - \eta_i, x_i + \eta_i] \to \mathbb{R}^2$ by

— $u^i_\varepsilon(x_i) = \hat{y}_i$, where $\hat{y}_i \in \check{M}_{\delta,\varepsilon}$ realizes the distance between $u(x_i)$ and $\check{M}_{\delta,\varepsilon}$.

— For $x \neq x_i$, we set $u^i_\varepsilon(x)$ is the point $y \in \check{M}_{\delta,\varepsilon}$ such that

$$d_{\check{M}_{\delta,\varepsilon}}(y, u(x_i)) = d_{M_\delta}(u(x), u(x_i)),$$

whose distances are negative if $x < x_i$ and positive if $x > x_i$.

Then, thanks to the Hausdorff convergence of $\check{M}_{\delta,\varepsilon}$ to M_δ, we have $u^i_\varepsilon \to u_{|[x_i-\eta_i,x_i+\eta_i]}$ in L^∞.

We can now build u. Let φ_i be a partition of unity adapted to $S^1 = \bigcup]x_i-\eta_i, x_i+\eta_i[$. We define

$$u_\varepsilon = \sum \varphi_i u^i_\varepsilon.$$

The function u_ε shares the same regularity as the u^i_ε, it is therefore $\mathcal{C}^{1,1}$. As we clearly have

$$u = \sum \varphi_i u_{|[x_i-\eta_i,x_i+\eta_i]},$$

the fonction u_ε converges to u in $L^\infty(S^1)$.

Let us now show that $\nu_\varepsilon \to \nu$ a.e. Let $x_0 \in S^1$ such that $\partial_x u$ exists around x_0 (the angles are isolated points so this assumption is satisfied almost everywhere), let $y_0 = u(x_0)$ and let $M_\delta(y_0)$ the corresponding graph (such that $u([x_0-\eta, x_0+\eta]) \subset \partial\check{M}_\delta(y_0)$). We also introduce, as before, the corresponding $\hat{M}_{\delta,\varepsilon}$ and $\check{M}_{\delta,\varepsilon}$.

First, we show that for every $y^\varepsilon_0 \to y_0$ with $y^\varepsilon_0 \in \partial\hat{M}_{\delta,\varepsilon}$ the normal vector $\hat{\nu}(y^\varepsilon_0)$ to $\hat{M}_{\delta,\varepsilon}$ at y^ε_0 converges to $\nu(x_0)$. Indeed, either $\hat{M}_{\delta,\varepsilon}$ and M_δ coincide around y_0 and nothing has to be done, or y^ε_0 belongs to an arc of a Wulff shape RW_{γ_ε}. In the last alternative, we just have to notice that the RW_{γ_ε} are convex and lie on one side of ∂M_δ, which forces the tangent line to RW_{γ_ε} at y^ε_0 to converge to $\partial_x u(x_0)$. The same is true with normal vectors.

We now prove that the same result holds for $\check{y}^\varepsilon_0 \in \partial\check{M}_{\delta,\varepsilon}$ with $\check{y}^\varepsilon_0 \to y_0$. Let y^n_0 be a sequence of points belonging to $\check{M}_{\delta,\varepsilon_n}$ where $\varepsilon_n \to 0$ and such that $\check{y}^n_0 \to y_0$. Then, either $\check{y}^n_0 \in \partial M_\delta$ (and nothing has to be done) or $\check{y}^n_0 \in \partial\hat{M}_{\delta,\varepsilon}$ (just apply the first point) or $\check{y}^n_0$ belongs to the boundary of a Wulff shape $RW_{\gamma_{\varepsilon_n}}$ which Hausdorff converges to the Wulff shape RW_γ which (thanks to the RW_γ-condition for M_δ) lies on one side of ∂M_δ and whose boundary contains y_0. Since with this assumption, $\check{y}^n_0 \neq y_0$, the latter is smooth at y_0 and its tangent space coincide with $\operatorname{span}\partial_x u(x_0)$. As a result, we have convergence of normal vectors. Finally, we take $\check{y}^\varepsilon_0 = u_\varepsilon(x_0)$ to conclude the proof.

We will now prove Lemma 4.2. We will need the

Lemma 4.3. *Let $K_i \subset K$ be a set of strict convex compact subsets which are symmetric with respect to the origin and which converge to a set K. Let $\beta \in (0,1)$, $y_i \in \partial K_i$ which converges to $y \in \partial K$ and $\epsilon_i \to 0$. We introduce*

$$L_i := \beta K_i + ((1-\beta)|y_i| + \varepsilon_i)\frac{y_i}{|y_i|}.$$

Then, ∂L_i intersects ∂K_i in two points P_i, Q_i which both converge to y.

Proof. First, let us prove that $d(P_i,Q_i) \to 0$. If not, then there exists $\gamma > 0$ and a sequence (still denoted by i) P_i and Q_i such that $d(P_i,Q_i) \geqslant \gamma$. One can assume that $P_i \to P$ and $Q_i \to Q$ with $P, Q \in K$ and $d(P,Q) \geqslant \gamma$. Noting that

$$L_i \to L := \beta K + (1-\beta)y,$$

one has also $P, Q \in \partial L$ and $L \subset K$. That implies that ∂K and ∂L coincide between P and Q, which is not possible by strict convexity. Since $\partial L \cap \partial K = \{y\}$, the shared limit of P_i and Q_i must be y. □

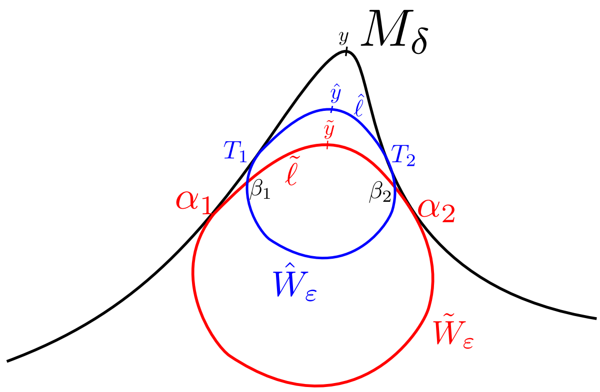

Figure 4.1: Step 1 configuration

We are now able to prove Lemma 4.2. We only prove the coincidence of $\hat{M}_{\delta,\varepsilon}$ (the other equality if proven similarly).

Step 1. The Wulff shape W_γ has no straight edge. We refer to Figure 4.1 to follow this step of the proof.

First, let us assume that the Wulff shape W_γ has no straight edges. Then, W_γ is strictly convex. As a result, RW_γ and $R'W_\gamma$ cannot see their boundary coincide. That implies that M_δ and $R'W_\gamma$ cannot see their boundary coincide either.

Let $y_0 \in M_\delta$ and $\hat{y}$ a point of $\hat{M}_{\delta,\varepsilon}$ which is closer to y_0 than any other point of $\hat{M}_{\delta,\varepsilon}$. Then, either $\hat{y} = y_0$ or $\hat{y}$ belongs to an arc $\hat{\ell}$ of $\hat{W}_\varepsilon$, a translation of $R'W_{\gamma_\varepsilon}$-Wulff shape, and which touches M_δ at two points T_1 and T_2. In the last alternative, let us show that $T_1, T_2 \to y_0$ as $\varepsilon \to 0$.

To this aim, let us introduce $\tilde{y}$ a point of $\tilde{M}_{\delta,\varepsilon}$ which is closer to y_0 than any other point of $\tilde{M}_{\delta,\varepsilon}$. The point $\tilde{y}$ belongs to an arc $\tilde{\ell}$ of $\tilde{W}_\varepsilon$, a translation of the RW_{γ_ε}-Wulff shape, which touches M_δ at two points α_1 and α_2 (for curvature reasons, T_i is between y_0 and α_i: see Figure 4.1).

Then, $\hat{\ell}$ and $\tilde{\ell}$ crosses at two points β_1, β_2 such that β_i is between α_i and T_i (see Figure 4.1).

Then, Lemma 4.3 applied with $K_i = \tilde{W}_\varepsilon$, $L_i = \hat{W}_\varepsilon$ and $y_i = \tilde{y}$ shows that $\beta_1, \beta_2 \to y$. On the other hand, the points T_i stand between β_i and $\hat{y}$ on $\hat{W}_\varepsilon$. That implies that they also converge to y_0. As a result, if δ, δ' are fixed and sufficiently small, we can find ε small enough such that the construction above $\hat{M}_{\delta,\varepsilon}$ and $\hat{M}_{\delta',\varepsilon}$ coincide (one can choose ε such that the corresponding T_i stand on a neighborhood of y where M_δ and $M_{\delta''}$ coincide). This is Lemma 4.2.

Step 2. The point y_0 belongs to a straight line of RW_γ. Now, let us study the straight lines in W. Since the construction above is local, if the Wulff shape RW_{γ_ε} which touches M_δ at y_0 has no straight line around y_0, one still can build $M_{\delta,\varepsilon}$ satisfying the $R'W_{\gamma_\varepsilon}$ condition for δ sufficently small as in Step 1.

So, let us assume that $\hat{W}$ is a $R'W_\gamma$ Wulff shape tangent to M_δ at y_0 (from inside) and

y_0 belongs to a straight line of $\hat{W}$. Then, either y_0 belongs to the interior of a flat edge of $\hat{W}$ which implies, since M_δ satisfies the RW_γ-condition for both inside and outside and since W_γ is symmetric, M_δ must also have a flat edge at y_0, or y_0 belongs to an end of a line, as we assume from now.

Will still have to distinguish two alternatives:

— Assume that y_0 belongs to the end of two (distinct) straight lines of $\hat{W}$ which create an angle, and let, as before, $\hat{y}$ be a point of $\hat{M}_{\delta,\varepsilon}$ which minimizes the distance to y. Once again, either $\hat{y} = y$ or $\hat{y}$ belongs to an arc $\hat{\ell}$ of $\hat{W}'_\varepsilon$, a translation of the approximate Wulff shape $R'W_{\gamma_\varepsilon}$ which touches M_δ at two points α_1 and α_2 (see Figure 4.2). Note that because of the Hausdorff convergence, the curvature of the $R'W_{\gamma_\varepsilon}$ Wulff shape at $\hat{y}$ must tend to $+\infty$ as $\varepsilon \to 0$. As a result, if M_δ has a bounded curvature at y_0, then $M_{\delta,\varepsilon} = M_\delta$ near y_0.

Now, if M_δ has an angle at y_0, either M_δ coincide locally with $\hat{W}$ or not. In the first alternative, for a sufficiently small δ, M_δ is exactly the union of the two lines. As a result, $M_{\delta,\varepsilon}$ does not depend on δ when the latter is small enough, and the statement is proven.

Let us assume now that M_δ coincides with $\hat{W}$ only on one side ℓ of y. Then the approximate $\hat{W}_\varepsilon$ ($\hat{W}_\varepsilon$ is a translated $R'W_{\gamma_\varepsilon}$ Wulff shape which converge Hausdorff to $\hat{W}$) touches M_δ at only one point β (see Figure 4.2).

Thanks to the symmetry of W_γ and the RW_γ outer condition for M_δ, we have $\alpha_1 \in \ell$ and so

$$\alpha_1 = \beta + rn_\ell$$

and

$$\hat{W}'_\varepsilon = \hat{W}_\varepsilon + r_\varepsilon n_\ell.$$

Let us show that $\alpha_1, \alpha_2 \to y$ with $\varepsilon \to 0$. If that were false, there would exist a sequence $\varepsilon_i \to 0$ and $\alpha_2^i \to \alpha \neq y$. Since ∂M_δ does not coincide with the Wulff shape $\hat{W}$, $d(\alpha, \hat{W})) = \eta > 0$. As a result, r cannot go to zero (it is bigger than $\frac{\eta}{2}$ for i large enough). Let $\hat{r}$ be a limit point of r. That would imply that

$$\hat{W} + \hat{r}n_\ell \subset M_\delta$$

which cannot happen because of the angle.

Finally, if M_δ does not coincide at all $\hat{W}$, one has a situation as in Step 1 and we conclude similarly.

— Assume now that y_0 belongs to the end of one straight line of RW_γ. If there is an angle of W_γ at y_0, then we can reproduce the last argument and have the same conclusion. If there is no angle, one has to distinguish between two alternatives: either M_δ coincide with the straight edge or not. In the first case, one can reproduce the proof of the bullet above and get the similar conclusion (note that even if there is no angle, $y_0 + \eta n_\ell$ has no chance to belong to M_δ because y_0 is at the end of the line and M_δ satifies the outer RW_γ condition where W_γ is symmetric).

If M_δ does not coincide with $\hat{W}$ locally, then the Wulff shape $\hat{W}_\varepsilon$ touches M_δ at α_1, α_2 (depending on ε). By contradiction (as in Step 1), we show that $\alpha_{1,2}$ have to tend to y_0 and we have Lemma 4.2.

□

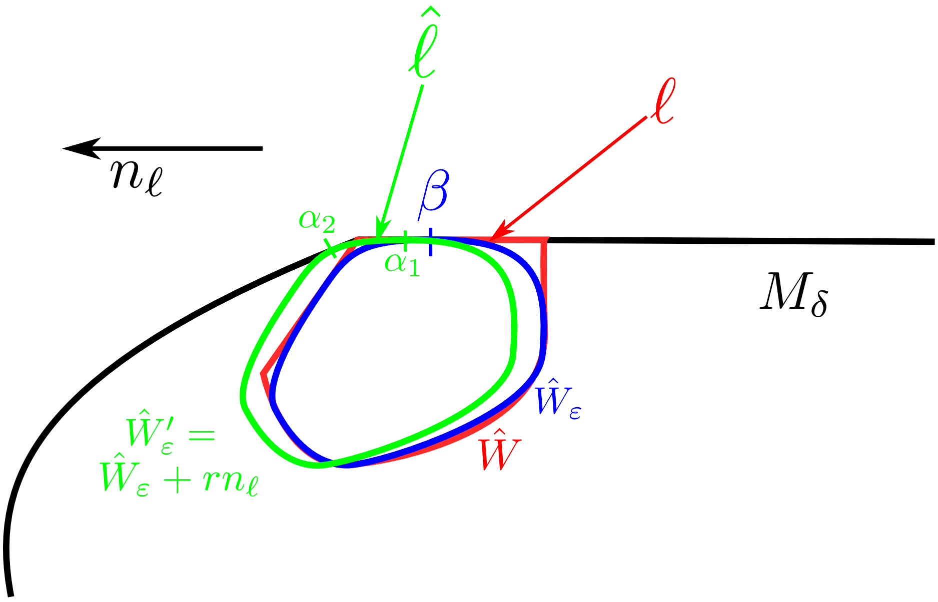

Figure 4.2: Step 2 configuration

4 Smooth motion

In this whole subsection, we replace γ by γ_ε and drop the index ε. Thanks to [Ang90, Th 3.1], there exists a smooth solution to (4.7) on an intervalle $[0, T_\varepsilon)$.

Lemma 4.4. *The following holds*

$$\partial_t \partial_s(\cdot) = \partial_s \partial_t(\cdot) + \psi(\theta)\kappa^2 \partial_s(\cdot) \tag{4.9}$$

$$\tau_t = (\psi(\theta)\kappa)_s \nu \tag{4.10}$$

$$\nu_t = -(\psi(\theta)\kappa)_s \tau \tag{4.11}$$

$$\kappa_t = (\psi(\theta)\kappa)_{ss} + \psi(\theta)\kappa^3 \tag{4.12}$$

$$\theta_t = (\psi(\theta)\kappa)_s. \tag{4.13}$$

Proof. Let $f : S^1 \to \mathbb{R}^2$, $f = f(x,t)$. Then, we compute (we note that the derivatives in x and t can commute)

$$\begin{aligned}
\partial_t \partial_s f = \partial_t \left(\frac{\partial_x f}{|u_x|} \right) &= -\frac{(u_{tx} \, , \, u_x)}{|u_x|^3} \partial_x f + \frac{\partial_t \partial_x f}{|u_x|} \\
&= -\left(\partial_s(\psi(\theta)\kappa\nu) \, , \, \tau \right) \partial_s f + \frac{\partial_x}{|u_x|} \partial_t f \\
&= \psi(\theta)\kappa^2 \partial_s f + \partial_s \partial_t f.
\end{aligned}$$

Applying this formula to the other quantities, we get

$$\tau_t = \partial_t(\partial_s u) = \partial_s(\partial_t u) + \psi(\theta)\kappa^2 \partial_s u = \partial_s(\psi(\theta)\kappa\nu) + \psi(\theta)\kappa^2 \tau = (\psi(\theta)\kappa)_s \nu.$$

We prove the third formula similarly.
Writing $\kappa = (\tau_s\,,\,\nu)$, we get

$$\kappa_t = (\partial_t \tau_s\,,\,\nu) + (\tau_s\,,\,\partial_t \nu)\,.$$

Since τ_s is proportional to ν and ν_t is proportional to τ, the second term vanishes. We obtain

$$\kappa_t = \big(\partial_s((\psi(\theta)\kappa)_s\nu) + \psi(\theta)\kappa^2\tau_s\,,\,\nu\big) = (\psi(\theta)\kappa)_{ss} + \psi(\theta)\kappa^3.$$

Finally, recalling that $\nu = (\cos\theta, \sin\theta)$, one has $\nu_t = -\theta_t(\sin\theta, -\cos\theta)$, which implies the last formula.

□

Remark. For an embedded closed curve moving by ACSF we have that the rate of decrease of the area enclosed by the curve goes like

$$\frac{d}{dt}A(u(\cdot,t)) = -\int_{S^1} \psi\kappa ds.$$

(The proof is based on Gauss theorem + previous Lemma). In particular for the CSF we have that $\frac{d}{dt}A(u(\cdot,t)) = -\int_{S^1}\theta_s ds =$ difference of the normal angles at the meeting point. If $\kappa > 0$ then in the anisotropic case we infer that $\frac{d}{dt}A(u(\cdot,t)) \leq -c\int_{S^1}\theta_s ds$ and therefore an analogous statement holds.

For the evolution of the derivatives of the curvature we have

Lemma 4.5. *For $j \in \mathbb{N}$, $j \geq 1$ we have*

$$\begin{aligned}\partial_t(\partial_s^j\kappa) &= \psi(\theta)(\partial_s^j\kappa)_{ss} + (j+3)\psi'(\theta)\kappa(\partial_s^j\kappa)_s \\ &\quad + P(\psi,\psi',\psi'',\kappa,\kappa_s)\partial_s^j\kappa + P(\psi,\psi',\dots,\psi^{(j+2)},\kappa,\dots,\partial_s^{j-1}\kappa)\end{aligned} \tag{4.14}$$

where $P(\cdot)$ is a polynomial in the given variables and $\psi^{(m)} = \partial_\theta^m\psi$.

Proof. The proof is by induction on j and relies on Lemma 4.4 and the fact that $\psi(\theta)_s = \psi'(\theta)\kappa$. □

Lemma 4.6. *Let $w := \log|u_x|$. There holds*

$$w_t = -\psi(\theta)k^2. \tag{4.15}$$

In particular $\|u_x(t)\|_\infty \leq \|u_x(0)\|_\infty$.

Proof. A direct computation gives

$$w_t = \tau\cdot\partial_s u_t = \tau\cdot\psi(\theta)\kappa\nu_s = -\psi(\theta)k^2.$$

The second statement follows from $\psi \geq 0$. □

Note that if we have a bound on the curvature, then from $-w_t \leq C(\|\kappa\|_\infty, \|\psi\|_\infty)$ we also infer that $|u_x(t)| \geq (\inf_{S^1}|u_x(0)|)e^{-C(\|\kappa\|_\infty,\|\psi\|_\infty)t}$.

Lemma 4.7. *Assume that* (4.7) *has a smooth solution on* $[0, \bar{t}]$, *with* $\bar{t} > 0$. *Then*

$$\max_{S^1\times[0,\bar{t}]} |\partial_s^j \kappa| \le C_j$$

where C_j *depends on* $\bar{t}$, $\|\psi^{(l)}\|_\infty$ *for* $l = 0, \dots, j+2$, C_l *for* $l \le j-1$, $\|\partial_s^j \kappa(0)\|_\infty$, *and* $\max_{S^1\times[0,\bar{t}]} |\kappa|$.

Proof. The proofs goes by induction on j. Let $v = \partial_s^j \kappa$. Then from Lemma 4.5 we know that

$$v_t = \psi(\theta)v_{ss} + (j+3)\psi'(\theta)\kappa v_s + P(\psi, \psi', \psi'', \kappa, \kappa_s)v + P(\psi, \psi', \dots, \psi^{(j+2)}, \kappa, \dots, \partial_s^{j-1}\kappa).$$

(where recall that $\partial_s = \frac{1}{|u_x|}\partial_x$ and $v_{ss} = \frac{1}{|u_x|^2} v_{xx} - \frac{v_x}{|u_x|}\tau \cdot \frac{u_{xx}}{|u_x|^2}$). Together with $\psi(\theta) \ge c > 0$ (the anisotropy is uniformly elliptic), we obtain a parabolic quasilinear equation for which we can apply the arguments given in [Lie05, Thm. 9.5] (cf. [Lie05, ex. 9.5] for the case $j = 1$).

More precisely let us look at the case where $j \ge 2$. Without loss of generality we may assume that there exists a point $(x, t) \in S^1 \times (0, \bar{t}]$ where v attains a positive maximum (if not argue with $-v$). Then v satisfies an equation of type

$$0 = -v_t + \psi(\theta)v_{ss} + a(s, v, v_s)$$

where, in view of the induction hypothesis, we have that $a(s, v, 0) \le c(|v| + 1) \le \alpha|v| + \frac{\beta}{|v|}$ with posistive constants α and β depending on $\|\psi^{(l)}\|_\infty$ for $l = 0, \dots, j+2$, C_l for $l \le j-1$, and $\max_{S^1\times[0,\bar{t}]} |\kappa|$. Set $\lambda = -\alpha - 1$. Suppose $P = (x, t) \in S^1 \times (0, \bar{t}]$ in which $m := e^{\lambda t} v$ attains a positive maximum. Then $m_t = \lambda e^{\lambda t} v + e^{\lambda t} v_t$ and at P we have $m_x = m_t = 0$ (thus $m_s = m_t = v_s = 0$), $m_{xx} \le 0$ (thus $m_{ss} \le 0$, $v_{ss} \le 0$). At P (where $v > 0$) we have

$$0 = -v_t + \psi(\theta)v_{ss} + a(s, v, v_s) \le -v_t + \alpha|v| + \frac{\beta}{|v|} = (\lambda + \alpha)v + \frac{\beta}{v} = -v + \frac{\beta}{v}.$$

Thus $v(P) \le \sqrt{\beta}$ and we infer that

$$\sup_{S^1\times[0,\bar{t}]} v \le e^{(\alpha+1)\bar{t}}(\sqrt{\beta} + \sup_{S^1} v^+(0)).$$

Arguing with $-v$ we get a bound also on v^- and therefore on $|v|$.

For the case $j = 1$ we have that $a(s, v, 0) \le c(|v|^2 + |v| + 1) \le c(|v^2| + |v| + \frac{1}{|v|}) = \Phi(|v|) + \frac{c}{|v|}$, where $\Phi(r) = cr(1 + r)$. Here use the results from [Lie05, ex. 9.5]. $\square$

Proposition 4.1. *Let* T *be the maximal time of existence of* (4.7) *and assume that* $T < \infty$. *Then*

$$\lim_{t\to T} \|\kappa\|_\infty = +\infty.$$

Proof. Assume by contradiction that $|\kappa|$ is uniformly bounded for all $t \in [0,T)$. Then the previous lemmas imply a uniform bound on $|u_x|$, $|u_x|^{-1}$ and $|\partial_s^j \kappa|$. As a result, writing

$$u(s,t) - u(s,t') = \int_t^{t'} \gamma(\nu(s))\gamma''(\nu(x))\tau(s) \cdot \tau(s)\kappa\nu(s),$$

the bound on κ implies that this integral has a limit when $t' \to T$. It remains to show that the convergence of $u(\cdot,t')$ is in fact in $\mathcal{C}^\infty$. We will proceed by induction.

First of all note that for a function $h : S^1 \to \mathbb{R}$ we have that

$$\partial_x^m h - |u_x|^m \partial_s^m h = P(|u_x|, \ldots, \partial_x^{m-1}|u_x|, h, \ldots, \partial_s^{m-1} h).$$

Differentiating the PDE for the length element $z = |u_x|$, namely

$$z_t = -\psi(\theta)\kappa^2 z,$$

we get equation of type

$$(\partial_x^m z)_t = b(x,t)\partial_x^m z + a(x,t)$$

with $|a|$ and $|b|$ uniformly bounded by induction hypothesis. Integrating in time and applying Gronwall's lemma, we infer a uniform bound on $|\partial_x^m |u_x||$ and as a consequence also on $|\partial_x^m u_x|$. Thus we have a $\mathcal{C}^\infty$ convergence and we can extend u past T (Thanks, once again, to [Ang90, Th 3.1]), which gives a contradiction. Thus we have proved that

$$\limsup_{t\to T} \|\kappa\|_{L^\infty} = +\infty.$$

$\square$

Now, let us state a remark which shows, as it happens in the isotropic case, that the blow up rate of the curvature is bounded by below. Even if we will not use it here, it opens the way to a more complete study of singularities.

Remark. Suppose $T < \infty$. Then

$$\liminf_{t\to T} \sqrt{T-t}\|\kappa\|_{L^\infty} \geq \frac{1}{\sqrt{2\alpha}} \tag{4.16}$$

where $\alpha = \max_{S^1} |\psi + \psi''|$.

Proof. Let $w := \kappa^2$. Then from (4.12) we infer that

$$\begin{aligned} w_t &= \psi(\theta)w_{ss} + 2(\psi(\theta) + \psi''(\theta))w^2 + 3\psi'(\theta)w_s\sqrt{w}\,\mathrm{sign}(\kappa) - 2\psi(\theta)(k_s)^2 \\ &\leq \psi(\theta)w_{ss} + 2(\psi(\theta) + \psi''(\theta))w^2 + 3\psi'(\theta)w_s\sqrt{w}\,\mathrm{sign}(\kappa). \end{aligned}$$

Let $M(t) := \max_{S^1} w$. Then evaluating the pde-inequality for w at the point (x,t) for which $M(t) = w(x,t)$ gives, for almost every t,

$$\frac{d}{dt}M(t) \leq 2|\psi + \psi''|M^2(t) \leq 2\alpha M^2(t),$$

where $\alpha = \max_{S^1} |\psi + \psi''|$. The rest follows as in [CN13a, Lemma 2.6]. $\square$

The anisotropic curvature does not blow up in short time. Let recall that $\kappa_\gamma := \kappa(\phi+\phi'')$. We denote by h the quantity $\phi+\phi''$. Then,

$$\begin{aligned}
\partial_t \kappa_\gamma = \partial_t(\kappa h) &= h[(\partial_{ss}(\psi(\theta)\kappa) + \psi\kappa^3] + \kappa h' \partial_s(\psi(\theta)\kappa) \\
&= h[\partial_s(\kappa^2\psi' + \kappa_s\psi) + \kappa^2\psi(\theta)\kappa] + \kappa h'(\kappa^2\psi' + \kappa_s\psi) \\
&= h(3\kappa\kappa_s\psi' + \kappa^3\psi'' + \kappa_{ss}\psi + \kappa^3\psi) + \kappa^3 h'\psi' + \kappa\kappa_s h'\psi
\end{aligned}$$

whereas

$$\partial_{ss}\kappa_\gamma = \partial_{ss}(\kappa h) = \partial_s(\kappa_s h + \kappa^2 h') = \kappa_{ss}h + 3\kappa_s\kappa h' + \kappa^3 h''.$$

Noting that

$$(\partial_t - \psi\partial_{ss})\frac{\kappa_\gamma^2}{2} = \kappa_\gamma \partial_t \kappa_\gamma - \psi\kappa_\gamma\partial_{ss}\kappa_\gamma - \psi(\partial_s\kappa_\gamma)^2$$

we get

$$\begin{aligned}
(\partial_t - \psi\partial_{ss})\frac{\kappa_\gamma^2}{2} &= -\psi\kappa_\gamma\left[\kappa_{ss}h + 3\kappa_s\kappa h' + \kappa^3 h''\right] - \psi(\partial_s\kappa_\gamma)^2 \\
&\quad + \kappa_\gamma\left[h(3\kappa\kappa_s\psi' + \kappa^3\psi'' + \kappa_{ss}\psi + \kappa^3\psi) + \kappa^3 h'\psi' + \kappa\kappa_s h'\psi\right]. \\
&= \kappa_s\kappa_\gamma\kappa(3h\psi' - 2h'\psi) + \kappa_\gamma\kappa^3(h\psi + h\psi'' + h'\psi' - \psi h'') - \psi(\partial_s\kappa_\gamma)^2.
\end{aligned}$$

Now, note that

$$\psi' = (h\phi)' = h'\phi + h\phi',$$

$$\psi'' = h''\phi + 2h'\phi' + h\phi'',$$

we obtain

$$3h\psi' - 2h'\psi = 3h^2\phi' + hh'\phi$$

and

$$h\psi + h\psi'' + h'\psi' - \psi h'' = h^2\phi + hh''\phi + 2hh'\phi' + h^2\phi'' + (h')^2\phi + hh'\phi' - hh''\phi = h^3 + 3hh'\phi' + (h')^2\phi.$$

As a result,

$$(\partial_t - \psi\partial_{ss})\frac{\kappa_\gamma^2}{2} = \kappa_s\kappa_\gamma\kappa(3h^2\phi' + hh'\phi) + \kappa_\gamma\kappa^3(h^3 + 3hh'\phi' + (h')^2\phi) - \psi(\partial_s\kappa_\gamma)^2.$$

Since

$$\partial_s\frac{\kappa_\gamma^2}{2} = \kappa_\gamma(\kappa_s h + \kappa^2 h'),$$

we can write

$$\kappa_s\kappa_\gamma\kappa(3h^2\phi' + hh'\phi) + \kappa^3\kappa_\gamma(3hh'\phi' + (h')^2\phi) = (3\kappa h\phi' + h'\kappa\phi)\partial_s\frac{kg^2}{2}$$

which yields

$$(\partial_t - \psi\partial_{ss})\frac{\kappa_\gamma^2}{2} \leqslant (3\kappa h\phi' + h'\kappa\phi)\partial_s\frac{kg^2}{2} + \kappa_\phi^4.$$

At a maximal point for κ_γ^2, the quantity $\partial_s \kappa_\gamma^2$ vanishes and $\partial_{ss}\kappa_\gamma^2$ is nonpositive. As a result, letting $g := \max_{S^1} \kappa_\gamma^2$, we have $\frac{d}{dt} g \leqslant 2g^2$, which implies

$$g(t) \leqslant \frac{g(0)}{1 - 2tg(0)} \tag{4.17}$$

and, since $g(0) \leqslant (C')^2$, prevent κ_γ from blowing up on the time interval $[0, \frac{1}{2C'^2})$.

4.1 Crystalline curvature flow

We denote by T_0 the quantity $\frac{1}{3C'^2}$. We now pass to the limit in the ε approximation of the flow u^ε we built in the previous subsection. First, thanks to Lemma 4.6 and the approximation lemma 4.1, the collection of u^ε is equi-Lipschitz in space. Thanks to (4.17), it is also equi Lipschitz in time on every compact set of $[0, T)$. Then, thanks to Ascoli Arzela, we can make u^ε converge to some u, up to a subsequence.

Let us show that u is a solution to the crystalline mean curvature motion. More precisely

Theorem 4.1. *Let u be a limit of u_ε and let $\tilde{u}$ be a reparametrization of u by unit arc length (we denote by $L(t)$ the length of $u(S^1)$). Then, there exists a vector field $\tilde{n}$ on $M_t := u([0, L(t)], t)$ such that $\tilde{n} \in L^\infty([0, L(t)] \times [0, T])$, $\tilde{n}$ is Lipschitz in s and satisfies $\gamma(n) = 1$, and such that*

$$\tilde{u}_t^\perp = \gamma(\nu)\,\mathrm{div}\,\tilde{n}\,\nu$$

almost everywhere in (s, t), where $u_t^\perp$ denotes the component of u_t which is normal to the surface.

Before proving this theorem, let us state a standard but useful remark.

Remark. The Cahn-Hoffman vector field n_ε satisfies

$$\partial_s n_\varepsilon = (\partial_s n_\varepsilon \cdot \tau_\varepsilon)\,\tau_\varepsilon.$$

Indeed, if we consider differentiate the identity

$$\gamma(n_\varepsilon) = 1$$

along the curve, we obtain

$$\nabla\gamma(n_\varepsilon) \cdot \partial_s n_\varepsilon = 0.$$

On the other hand,

$$\nabla\gamma(n_\varepsilon) = \nabla\gamma(\nabla\gamma^\circ(\nu_\varepsilon)) = \frac{1}{\gamma_\varepsilon(\nu_\varepsilon)}\nu_\varepsilon$$

which leads to

$$\nu_\varepsilon \cdot \partial_s n_\varepsilon = 0.$$

Proof of Th. 4.1. First, let us say a word on reparametrization. We also denote by $\tilde{u}_\varepsilon$ the reparametrization of u_ε using unit arc length s ($s \in [0, L_\varepsilon(t)]$). Since $u_\varepsilon \to u$ uniformly and u_ε are equi-Lipshitz, we have $L_\varepsilon(t) \to L(t)$.

Let us now introduce $n_\varepsilon := \nabla\gamma_\varepsilon^\circ(\nu_\varepsilon)$ where $\nu_\varepsilon(x,t)$ is the Euclidian normal to $u_\varepsilon(S^1)$. We denote by $\tilde{n}_\varepsilon$ the composition of n_ε with the reparametrization above. We will show that it converges to some $\tilde{n}$ which satisfies the expected properties.

First, let us prove that $\tilde{n}_\varepsilon$ is Lipschitz with respect to s. Let us fix a compact $|0,a] \subset [0,T)$. Thanks to (4.17), the surfaces M_ε^t, for $t \in [0,a]$, have a bounded γ_ε-curvature whose bound does not depend either on ε or in $t \in [0,T]$.

Now, just notice that thanks to the remark before,

$$\kappa_{\gamma_\varepsilon} = \operatorname{div}(\tilde{n}_\varepsilon) = (\partial_s \tilde{n}_\varepsilon \cdot \tau)\,\tau = \partial_s \tilde{n}_\varepsilon,$$

which implies that $\tilde{n}_\varepsilon$ is Lipschitz with a constant which does not depend on ε and $t \in [0,T]$.

Let us now pass to the limit in the equation

$$(\tilde{u}_\varepsilon)_t^\perp = \gamma_\varepsilon(\nu_\varepsilon)\operatorname{div}\tilde{n}_\varepsilon \nu_\varepsilon.$$

Let us notice that $\gamma_\varepsilon \to \gamma$ uniformly whereas $\nu_\varepsilon \to \nu$ almost everywhere. In addition, $\operatorname{div}\tilde{n}_\varepsilon$ is bounded (since $\tilde{n}_\varepsilon$ are equi-Lipschitz) so weakly converge in $L^2(s,t)$ to $\operatorname{div}\tilde{n}$. Similarly, $(\tilde{u}_\varepsilon)_t^\perp$ weakly converges to $\tilde{u}_t^\perp$. As a result, the equation

$$(\tilde{u}_t)^\perp = \gamma(\nu)\operatorname{div}\tilde{n}\;\nu$$

is satisfied distributionnaly and therefore almost everywhere.

□

Remark. Whereas uniqueness is guaranteed in the case of an embedded curve (see [CN13b]), with our framework, it remains an interesting problem.

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Évolutions de courbes et surfaces pour le traitement d'images

L'objectif de cette thèse a été d'étudier différents problèmes apparaissant naturellement en traitement d'images et mettant en jeu des hypersurfaces de l'espace euclidien à n dimensions. Débruiter une image consiste essentiellement à en lisser les lignes. Ce lissage peut apparaître soit comme le résultat d'une minimisation d'une fonctionnelle, soit comme l'application d'un flot régularisant sur les lignes de l'image. Dans ce manuscrit, nous étudions deux exemples de ces deux approches.

- Dans le chapitre 1, on lisse par minimisation et on s'intéresse à la régularité de la solution. Plus précisément, on travaille sur des généralisations de la minimisation proposée par Rudin, Osher et Fatemi qui pénalise la variation totale. On cherche à montrer que sous différentes hypothèses sur le domaine, les conditions d'attaches aux données ainsi que le choix de la variation totale (isotrope, anisotrope,. . .), la continuité de l'image observée se transmet forcément au minimiseur, ce qui montre que le débruitage par minimisation ne vas pas faire apparaître de discontinuité non observée.
- Dans le capitre 2, on étudie le flot par courbure moyenne (éventuellement anisotrope), qui est connu pour avoir un effet régularisant [AGLM93]. On y ajoute des obstacles. L'approche choisie est celle des lignes de niveau : la surface est l'image réciproque de 0 par une fonction qu'on fait évoluer. On démontre existence et unicité d'une fonction solution (de viscosité) de l'équation du mouvement par ligne de niveau et on étudie son asymptotique en temps en la comparant à un mouvement minimisant discret.
- Dans le chapitre 3 (travail en collaboration avec M. Novaga), on précise le résultat du chapitre 2 en étudiant le même problème mais sous forme géométrique (ce qui est nettement plus précis que l'approche ligne de niveau). On suit l'approche de Ecker et Huisken pour montrer qu'il existe une unique solution au mouvement par courbure moyenne avec obstacles en temps court.
- Enfin, dans le dernier chapitre (travail en collaboration avec M. Novaga et P. Pozzi), on fait un premier pas vers l'étude géométrique du mouvement anisotrope (on pourra en particulier traiter les anisotropies cristallines). Uniquement restreints à la dimension deux, on montre, en l'approchant par un mouvement lisse, l'existence d'un mouvement par courbure anisotrope d'une courbe immergée pour un temps petit.

Curve-and-surface Evolutions for Image Processing

The goal of this manuscript was to study several problems which appear in image processing and which involves hypersurfaces of the Euclidian space $\mathbb{R}^n$. Denoising a image basically consists in smoothing its lines. This smoothing can appear either as a minimizer of a suitable functional or result from a regularizing flow on the level sets of the image. In this thesis, we study two examples of these approaches.

- In the first chapter, we smooth by minimization. More precisely, we work on generalizations of the procedure suggested by Rudin Osher and Fatemi, which penalizes the total variation. We prove that under different assumptions on the domain, on the way to link the image to the data and on the choice of the total variation (isotropic, anisotropic,...), the continuity of the source image is preserved by the minimizing procedure.
- In Chapter 2, we study Mean curvature flow and add some obstacles which constraint the evolution. We choose the level-set approach: the surface is the preimage of 0 by a function which therefore satisfies a PDE. We prove existence and uniqueness of a (viscosity) solution for this equation. and study its asymptotic in time using comparison with a discrete minimizing scheme.
- In Chapter 3 (with M. Novaga), we add some information to the result of Chapter 2 by focusing on the geometric formulation of the mean curvature flow with obstacles. We follow the approach by Ecker and Huisken to show that there exists a unique solution of the motion in short times.
- Finally, in the last chapter (with M. Novaga and P. Pozzi), we make a first step towards the understanding of crystalline motion. Restricted to the planar framework, we show (using an approximation by a smooth motion) that there exists a short time of existence for an anisotropic curvature motion of an immersed curve.