NNT : 2016SACLX003

THÈSE DE DOCTORAT
DE
L'UNIVERSITÉ PARIS-SACLAY
PRÉPARÉE À
ÉCOLE POLYTECHNIQUE

ECOLE DOCTORALE N° 574
École doctorale de mathématiques Hadamard

Spécialité de doctorat : Mathématiques fondamentales

Par

Mme Tatiana Zolotareva

Construction de surfaces à courbure moyenne constante et surface minimales par des méthodes perturbatives

Thèse présentée et soutenue à Palaiseau, le 29 janvier 2016

Composition du Jury :

M. Pierre Pansu, Professeur, l'Université Paris-Sud, Président du Jury
M. Laurent Mazet, Chargé de Recherche, CNRS, Rapporteur
M. Tristan Rivière, Professeur, ETH-Zurich, Rapporteur
M. Benoît Daniel, Professeur, l'Université de Lorraine, Examinateur
M. André Neves, Professeur, Imperial College London, Examinateur
M. Frank Pacard, Professeur, École polytechnique, Directeur de thèse

universite PARIS-SACLAY | École doctorale de mathématiques Hadamard (EDMH)

Titre : (en français) Construction de surfaces à courbure moyenne constante et surfaces minimales par des méthodes perturbatives

Mots clés : Courbure moyenne constante, surface minimale, perturbation,

Résumé : Cette thèse s'inscrit dans l'étude des sous-variétés minimales et à courbure moyenne constante et de l'influence de la géométrie de la variété ambiante sur les solutions de ce problème.
Dans le premier chapitre, en suivant les idées de F. Almgren, on propose une généralisation de la notion d'hypersurface à courbure moyenne constante à toutes codimensions. En codimension m-k, on définit les sous-variétés à courbure moyenne constant comme points critiques de la fonctionnelle de k-volume des bords des sous-variétés minimales de dimension k+1. On prouve l'existence dans les variétés riemanniennes compactes de sous-variétés à courbure moyenne constante de codimension quelconque qui sont des perturbations des sphères géodésiques de petit volume.
Dans le deuxième chapitre on s'intéresse aux surfaces minimales à bords libres dans la boule unité de l'espace euclidien de dimension 3, c'est-à-dire aux surfaces minimales plongées dans laboule unité dont le bord rencontre la sphère unité orthogonalement. On démontre l'existence de deux famille géométriquement distinctes de telles surfaces qui sont indexées par un entier n assez grand, qui représente le nombre de composantes connexes du bord de ces surfaces. ous donnons en particulier une deuxième preuve d'un résultat de A. Fraser et R. Schoen concernant l'existence de telles surfaces.
Un des résultats fondamentaux de la théorie des surfaces à courbure moyenne constante est le théorème de Hopf qui affirme que les seules sphères topologiques à courbure moyenne constante dans l'espace euclidien de dimension 3 sont les sphères rondes. Dans le troisième chapitre, on propose une construction dans une variété riemannienne de dimension 3 d'une famille de sphères topologiques à courbure moyenne constante qui ne sont pas convexes et dont la courbure moyenne est très grande.

Title : (en anglais) Construction of constant mean curvature and minimal surfaces by perturbation methods

Keywords : Constant mean curvature, minimal surface, perturbation

Abstract : In the first chapter, following the ideas of F. Almgren, we propose a generalization of the notion of hypersurface with constant mean curvature to all codimensions. In codimension m-k we define constant mean curvature submanifolds as the critical points of the functional of the k-dimensional volume of the boundaries of k+1-dimensional minimal submanifolds. We prove the existence in compact Riemannian manifolds of constant mean curvature submanifolds of arbitrary codimension which are perturbations of geodesic spheres of small volume.
In the second chapter, we consider free boundary minimal surfaces in the unit ball of the three dimensional Euclidean space, i.e. minimal surfaces embedded in the unit ball and which meet the unit sphere orthogonally. We prove the existence of two geometrically distinct families of such surfaces parametrized by an integer n large enough, which represents the number of the boundary components. In particular, we give an independent proof of the result of A. Fraser and R. Schoen concerning the existence of such surfaces.
One of the fundamental results of the theory of constant mean curvature surfaces is the Hopf's theorem which asserts that the only topological spheres with constant mean curvature in the Euclidean 3-space are the round spheres. In the third chapter, we propose a construction in a three dimensional Riemannian manifold of a family of nonconvex topological spheres with large constant mean curvature.

Remerciements

En premier lieu, je voudrais remercier mon directeur de thèse, Frank Pacard, pour avoir guidé ce projet de thèse. Je suis heureuse d'avoir eu la possibilité de travailler avec lui et reconnaissante pour son soutient constant et pour ses conseils grâce auxquels j'ai beaucoup progressé. Je le remercie de m'avoir proposé des problèmes intéressants, et malgré son agenda extrêmement tendu, d'avoir trouvé systématiquement du temps pour discuter avec moi, répondre à toutes mes questions et me donner des indications et des remarques qui ont approfondi beaucoup ma compréhension du sujet. Je remercie également mon directeur de thèse de m'avoir encouragée à participer à de nombreuses conférences, ce qui a été très instructif et m'a permis de rencontrer des gens de la communauté d'analyse géométrique partout dans le monde et de présenter nos résultats.

Je remercie Laurent Mazet et Tristan Rivière d'avoir accepté d'être rapporteurs de ma thèse, pour le temps qu'ils ont consacré pour lire ce manuscrit et pour leurs remarques qui ont aidé à l'améliorer. Je remercie Benoît Daniel, André Neves et Pierre Pansu de me faire l'honneur de participer au jury.

J'aimerais remercier Rafe Mazzeo pour notre collaboration et pour les échanges fructueux pendant nos rencontres à Paris et à Park City qui m'ont été d'une aide importante. Je remercie beaucoup Laurent Hauswirth pour son soutient pendant les trois ans de ma thèse, son intérêt pour mon travail et les nombreuses discussions qui m'ont permis d'étendre ma compréhension des mathématiques.

Je suis reconnaissante pour l'ambiance chaleureuse que j'ai trouvé au CMLS, et je remercie mes collègues pour leurs gentillesse et pour les conversations passionnantes pendant les pauses de midi et les goûters du lundi. J'aimerais remercier François Golse, Jacek Jendrej, Giancarlo Lucchini, Christophe Margerin, , René Mboro, Ivan Moyano et Timofey Zakrevskiy pour leur aide mathématique, aussi bien pour ma thèse que pour des problèmes destinés aux collégiens qui se sont montrés parfois aussi ardus.

Je remercie Pascale Fuseau, Marine Amier, Carole Juppin, Danh Pham Kim, Jean-Luc Bellon et David Delavennant pour leur aide administrative et informatique et pour faire du CMLS un endroit aussi agréable pour travailler.

Finalement, je remercie les amis que j'ai rencontrés pendant la thèse et le M2, grâce auxquels j'ai été heureuse de vivre en France et qui comptent maintenant énormément pour moi. Je remercie ma famille qui, bien que loin, s'intéresse toujours beaucoup à ma vie et mon travail. Je remercie Gwenael d'avoir partagé avec moi les moments heureux et les difficultés et de m'inspirer constamment curiosité et confiance pour l'avenir.

Contents

Introduction

0.1 Brief overview

My Ph.D. research has been focused in 3 main areas: the existence of constant mean curvature higher codimensional submanifolds (in the sense of F. Almgren), the construction of free boundary minimal surfaces in the Euclidean 3-ball and the construction of surfaces with large constant mean curvature in generic Riemannian manifolds. Most of my work techniques come from Riemannian geometry and PDEs analysis: the theory of elliptic operators, perturbation techniques, analysis in weighted function spaces.

The study of minimal and constant mean curvature surfaces has a rich mathematical history. Various approaches and techniques from different branches of mathematics have been used in the investigation of these geometrical objects: calculus of variations, complex analysis, nonlinear PDEs, geometric measure theory, integrable systems, etc. This domain has known quite remarkable developments in the last three decades. A survey of the classical theory of minimal and constant mean curvature surfaces can be found for example in [12], [24], [78], [81], [88], [91]. Even though it would be difficult to mention all the important achievements of this theory, the following section proposes a brief overview of some old and more recent results that are related to the subject of this thesis.

0.1.1 Mean curvature

The mean curvature (function) of a surface in the Euclidean 3-space is a geometric quantity which was first introduced in the 18th century by S. Germain in her study on elasticity, following the works of L. Euler. It is defined as the sum of the principal curvatures of the surface at a point, i.e. the maximum and the minimum values of the curvature of curves on the surface which are obtained as the intersection of the surface with a plane containing the normal to the surface at a given point, as the plane varies. This definition can be generalized to submanifolds of any dimension and codimension in any Riemannian manifold. The mean curvature (vector) of a submanifold Σ is then defined as the trace of the second fundamental form of Σ, i.e. the symmetric bilinear form h_Σ on $T\Sigma$ taking values in the normal bundle of Σ, which is defined by

$$h_\Sigma(X,Y) := \nabla_X Y - \nabla^\Sigma_X Y \in N\Sigma,$$

for all $X, Y \in T\Sigma$, where ∇ is the Levi-Civita connection associated to the ambient metric and ∇^Σ is the Levi-Civita connection on Σ (endowed with the induced metric). Then the mean curvature vector is given by

$$H_\Sigma := \operatorname{Tr} h_\Sigma.$$

In codimension 1, the mean curvature vector is given by the mean curvature (function) times the unit normal.

0.1.2 Minimal surfaces

A hypersurface (or more generally a submanifold) in a Riemannian manifold is said to be minimal if its mean curvature vector vanishes. Let Σ be a compact submanifold (with or without boundary) and Ξ a vector field in the ambient Riemannian manifold (M^{m+1}, g). Let us denote by ξ the flow associated to Ξ, namely

$$\frac{d\xi}{dt}(p,t) = \Xi(\xi(p,t)),$$

for all $p \in M$ and t close to 0, and $\xi(p,0) = p$. For $t \in \mathbb{R}$ close to 0, we define Σ_t to be the image of Σ by $\xi(\cdot, t)$. Then the *first variation formula* of the volume yields

$$\frac{d}{dt}\mathrm{Vol}_m(\Sigma_t)\bigg|_{t=0} = \int_\Sigma g(H_\Sigma, \Xi)\, dvol + \int_{\partial\Sigma} g(\nu_{\partial\Sigma}, \Xi)\, d\sigma, \tag{1}$$

where $\nu_{\partial\Sigma}$ is the conormal to the boundary of Σ (i.e. the unit normal to $\partial\Sigma$ in Σ). In particular, *a compact surface is minimal if and only if it is a critical point of the volume functional with respect to variations preserving its boundary.*

First examples and minimal graph equation

The first example (other than the plane) of a minimal surface in the Euclidean 3-space was the catenoid described by L. Euler [32] in 1744. It is a surface of revolution whose generating curve is (up to scaling) the graph of cosh function and it turns out to be the only minimal surface of revolution in $\mathbb{R}^3$.

Another famous minimal surface in $\mathbb{R}^3$ is the helicoid which was also described by L. Euler but proved to be minimal by J.B. Meusnier [85] in 1776. It is the only ruled minimal surface other than the plane.

In the same work, J.B. Meusnier [85] showed that the mean curvature of a graph of the function u over a domain $\Omega \subset \mathbb{R}^2$ is identically equal to zero if and only if u satisfies the following quasilinear elliptic partial differential equation, formulated by J.L. Lagrange in 1762 in his work on the calculus of variations:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0. \tag{2}$$

Minimal graphs have the property to be area minimizing, they constitute an important class of minimal surfaces. For example, in 1915, S.N. Bernstein [7] proved that the only entire minimal graph in the Euclidean 3-space is the plane.

Weierstrass representation and Plateau's problem

In the 19th century, new increasingly complicated examples of minimal surfaces were provided by A. Enneper [30], H. Scherk [105], H. Schwarz [110], B. Riemann [100] and K. Weierstrass [114]. Many of these minimal surfaces are periodic.

A fundamental contribution to the theory of minimal surfaces was made by the discovery by A. Enneper and K. Weierstrass in 1866 [114], [30] of representation formulas that establish a correspondence between the minimal immersions in $\mathbb{R}^3$ and the so called "Weierstrass data" given by a meromorphic function and a holomorphic 1-form. An important corollary obtained by this approach is the fact that there exist no compact minimal surfaces in $\mathbb{R}^3$. Even though, normally, it is hard to conclude from the Weierstrass data what is the shape of the resulting surface as well as whether the surface is embedded or not, combined with other techniques, Weierstrass representation remains one the most powerful tools in providing examples and classification of minimal surfaces.

One of the most important achievements in the theory of minimal surfaces is the resolution in the beginning of the 20th century of the Plateau's problem named after the mathematician J. Plateau who studied the behavior of soap films. The problem is to show the existence of a least area surface with a given boundary curve. It was solved in 1930 separately by J. Douglas [28] and T. Rado [98] using techniques coming from the calculus of variations.

Topological classification of minimal surfaces

Many aspects of the modern theory of minimal surfaces in $\mathbb{R}^3$ originate from the pioneering work of R. Osserman [91] in 1960s, where questions concerning the analytic and the topological properties of minimal surfaces were stated and for which partial answers have been obtained only recently. A major challenge of this theory has been the classification of the complete embedded minimal surfaces according to their topological type (genus and the number of ends).

For many years it had been conjectured that the only complete minimal surfaces of of finite topological type in $\mathbb{R}^3$ are the catenoid, the helicoid, and the plane. But in 1982, J. Costa [15] discovered a minimal surface that has genus one and two ends asymptotic to catenoids, and one end asymptotic to a plane. A year later, D. Hoffman and W. Meeks [49] proved that this surface is actually embedded. Since then, many other examples have been constructed. D. Hoffman and H. Karcher [46] found a family of minimal deformations of the Costa surface where the planar end becomes catenoidal. A generalization to the case of arbitrary genus was also given by D. Hoffman and W. Meeks [50]. A minimal surface whose end is asymptotic to the end of the helicoid but whose genus is equal to 1 was found by D. Hoffman, H. Karcher and F. Wei [47], [48], while D. Hoffman, M. Weber and M. Wolf [53] proved that it was embedded, which provided an example, different from the helicoid, of a complete embedded minimal surface of finite topology and infinite total curvature. A generalization to the case of arbitrary genus was recently obtained by D. Hoffman, M. Traizet and B. White [52]. A complete minimal topological Möbius strip was discovered by W. Meeks [80].

An important class of minimal surfaces is constituted by complete minimal surfaces with finite total curvature. In 1964, R. Osserman [91] showed that such surfaces have the conformal structure of compact Riemann surfaces with a finite number of points removed and have total curvature equal to an integer multiple of -4π. R. Schoen showed [107] that complete minimal surfaces with finite total curvature can only have planar or catenoidal ends. In particular, when the number of ends is equal to two, the surface must be the catenoid. P. Collin's [13] theorem asserts that if Σ has finite topology and more than one end, then Σ has finite total curvature. F.J. López and A. Ros [71] proved that the only genus zero complete embedded minimal surfaces of finite total curvature are the catenoid and the plane. On the other hand, W. Meeks and H. Rosenberg [82] showed that the only properly embedded minimal surface with only one end is asymptotic to the helicoid. In particular, when the genus is zero, the only possible example is the helicoid.

0.1.3 Constant mean curvature surfaces

In codimension 1, we say that a hypersurface Σ in a Riemannian manifold (M, g) has constant mean curvature (CMC) if the mean curvature function H_Σ is a constant .

Recall that the classical *isoperimetric problem* in (M^{m+1}, g) consists in finding the hypersurface of least m-dimensional volume among all compact hypersurfaces enclosing a given $m+1$-dimensional volume. Solutions to the isoperimetric problem, when they are regular enough, provide examples of constant mean curvature surfaces, unfortunately, the only solution (up to translations) to this problem in $\mathbb{R}^{m+1}$ is the round sphere S^m. More generally, a closed embedded surface has constant mean curvature if and only if its volume is critical among all variations that preserve the volume of the region bounded by the surface.

Indeed, let Σ be an embedded hypersurface that is the boundary of a region Ω in M. We consider the compact hypersurface Σ_t that is the image of Σ under the flow $\xi(\cdot, t)$ associated to a vector field Ξ and let Ω_t be the region in M bounded by Σ_t. Then, given $\lambda > 0$, consider the function

$$\mathcal{E}_\lambda(t) := \mathrm{Vol_m}(\partial\Omega_t) - \lambda\,\mathrm{Vol_{m+1}}(\Omega_t).$$

The first variation of the volume formulae yield

$$\mathcal{E}'_\lambda(0) = -\int_\Sigma (H_\Sigma - \lambda)\ \langle N_\Sigma, \Xi\rangle, \tag{3}$$

where N_Σ is a unit normal to Σ. Therefore, $\Sigma = \partial\Omega$ has constant mean curvature equal to λ if and only if Ω is a critical point of the functional

$$\Omega \mapsto \mathrm{Vol_m}(\partial\Omega) - \lambda\,\mathrm{Vol_{m+1}}(\Omega). \tag{4}$$

Role of the round sphere: Hopf's and Alexandrov's theorems

In 1853, H.J. Jellet [57] proved that the only closed star-shaped constant mean curvature surface in $\mathbb{R}^3$ is the round sphere. In 1956, Hopf [54] showed that the only immersed constant mean curvature topological spheres in the Euclidean 3-space are round spheres and he conjectured that the same is true for any compact immersed orientable constant mean curvature surface in $\mathbb{R}^n$. Two years later A.D. Alexandrov proved [3] that the only compact connected embedded constant mean curvature surface in $\mathbb{R}^3$ is the round sphere.

The proofs of Hopf's and Alexandrov's theorems introduced two powerful tools of investigation of constant mean curvature surfaces. Hopf's proof involves the study of a quadratic differential form, referred to as the Hopf's differential, which is defined for any conformal immersion and which is holomorphic if and only if the immersion has constant mean curvature. Alexandrov's method is referred to as the method of moving planes and uses the maximum principle for elliptic partial differential equations (2).

Further examples of CMC surfaces

Hopf's conjecture was disproven in dimension greater than 3 in 1982 by W.Y. Hsiang [55]. In 1984, immersed constant mean curvature topological tori were constructed by H.C. Wente [115] using some doubly periodic solution to the Sinh-Gordon equation. In 1989 U. Pinkall and I. Sterling [97] classified all constant mean curvature tori immersed in $\mathbb{R}^3$ and A. Bobenko [9] gave an explicit description of the corresponding metrics.

In 1970s H.B. Lawson [69] showed that for each complete constant mean curvature surface in the Euclidean 3-space there is an associated complete minimal "cousin" in the 3-sphere. He introduced a procedure of explicit construction of complete minimal surfaces in S^3 and described two new examples of complete embedded doubly periodic constant mean curvature surfaces in $\mathbb{R}^3$. Lawson's idea was developed by H. Karcher [64] and K. Große-Brauckmann [41] who constructed a large number of new examples.

Noncompact CMC surfaces

For a long time, the only known examples of noncompact CMC surfaces in $\mathbb{R}^3$ have been the cylinder and the one parameter family of *Delaunay surfaces* [23] discovered in 1841. The latter are the only complete, noncompact constant mean curvature surfaces of revolution in $\mathbb{R}^3$ and are generated by rotating roulettes of conics. A roulette of an ellipse gives rise to an embedded constant mean curvature surface referred to as an unduloid, while a roulette of a hyperbola (which is a bit harder to visualize), gives rise to an immersed constant mean curvature surface which is referred to as a nodoid. In the case where the conic is a parabola, one obtains a catenoid whose mean curvature is equal to zero.

The asymptotic behavior of constant mean curvature surfaces was first studied by W. Meeks [79], who proved that any annular end of a complete noncompact Alexandrov-embedded constant mean curvature surface in $\mathbb{R}^3$ is contained in a solid half-cylinder of finite radius. As a byproduct he has proven that there exist no constant mean curvature surfaces with only one end. N. Korevaar, R. Kusner and B. Solomon [65] have proven that an Alexandrov-embedded constant mean curvature surface in $\mathbb{R}^3$ can only have ends asymptotic to Delaunay surfaces. Thanks to this result, R. Kusner, R. Mazzeo and D. Pollack [66] described the structure of the moduli space of complete Alexandrov-embedded noncompact constant mean curvature surfaces with finite topology.

0.1.4 Constant mean curvature surfaces obtained via perturbation techniques

The general idea of perturbation methods is to produce new examples of geometric objects of interest using existing ones as summands and often lead to construction of new non-trivial solutions which would be hard to obtain by different methods. Since 1980s these techniques have played an important role in many areas of geometry and found remarkable applications for example in the study of the topology of smooth 4-manifolds by S. Donaldson [26] and C. Taubes [113] and the study of singular solutions to Yamabe equation by R. Schoen [106].

The first constructions of constant mean curvature surfaces via perturbation methods were obtained in late 1980s in the pioneering work of N. Kapouleas. In [60], the author produces genus 2, compact, constant mean curvature surfaces in $\mathbb{R}^3$ by fusing Wente tori and compact surfaces of higher genus and noncompact surfaces of arbitrary genus and the number of ends great than 3 [58], [59], by gluing together round spheres with pieces of Delaunay surfaces.

Since then, many other examples of compact and noncompact constant mean curvature surfaces were constructed by perturbation methods, including the gluing constructions of R. Mazzeo, F. Pacard and D. Pollack [77], R. Mazzeo and F. Pacard [74], [75], F. Pacard and H. Rosenberg [93], etc.

General scheme of a gluing construction

The results described in chapters 2 and 3 of this thesis rely on perturbation techniques introduced by N. Kapouleas and R. Mazzeo, F. Pacard and D. Pollack. In this paragraph I would like to point out the main ingredients of the constructions described in the works of these authors. In the following example we assume that the ambient space is the Euclidean 3-space, but the ideas described below can be generalized to higher dimensions as well as to the case of a generic Riemannian manifold.

Normal perturbations and the Jacobi operator

Let Σ be a smooth embedded surface with or without boundary in $\mathbb{R}^3$. One way to describe the surfaces nearby Σ is to parametrize them as normal graphs over Σ. More precisely, let N_Σ be a unit normal vector field to Σ and take a $\mathcal{C}^2$ function w on Σ. We denote by $\Sigma(w)$ the surface parametrized by

$$\Sigma \ni p \longmapsto p + w(p)\, N_\Sigma(p) \in \Sigma(w).$$

When the norm of w is small enough, the Taylor expansion of the mean curvature of $\Sigma(w)$ in the powers of w and the partial derivatives of w up to the second order has the form:

$$H(\Sigma(w)) = H_\Sigma + J_\Sigma\, w + Q_\Sigma(w, \nabla w, \nabla^2 w),$$

where H_Σ is the mean curvature of Σ, $J_\Sigma = D_w H|_{w=0}$ is the linearized mean curvature, or the *Jacobi operator* about Σ, and Q_Σ is a nonlinear function of w and the components of the gradient and the Hessian of w, which satisfies:

$$Q_\Sigma(0,0,0) = DQ_\Sigma(0,0,0) = 0.$$

Moreover, the second variation of the area formula gives an explicit expression of the Jacobi operator which reads:

$$J_\Sigma = \Delta_\Sigma + |h_\Sigma|^2,$$

where Δ_Σ is the Laplace-Beltrami operator on Σ and $|h_\Sigma|^2$ is the squared norm of the second fundamental form of Σ.

By definition, a compact surface Σ is *nondegenerate*, if any solution to the problem:

$$J_\Sigma\, w = 0 \quad \text{in} \quad \Sigma \quad \text{and} \quad w = 0 \quad \text{on} \quad \partial\Sigma, \tag{5}$$

is trivial. We say that a noncompact surface Σ is nondegenerate if there are no nontrivial solutions $w \in L^2(\Sigma)$ of (5).

Connected sum construction

From the topological point of view, to perform a connected sum of two surfaces Σ^+ and Σ^- amounts to remove a small disk around a point on each surface, and then to identify the two boundaries of the disks. From the geometric point of view, it amounts to first move the surfaces $\Sigma^\pm$ in $\mathbb{R}^3$, so that they are tangent at a common point p (which can be assumed to be the origin), then to remove a small disk centered at p on each surface, to translate the surfaces away from each other in the direction orthogonal to their common tangent plane at p (which is assumed to be the horizontal plane), and then to perform a connected sum identifying the two circle boundaries with the two boundaries of a "small neck" given by a catenoid scaled down by a small factor $\varepsilon > 0$, and which has been truncated, namely the surface parametrized by

$$C_\varepsilon(s,\phi) = \varepsilon\, (\cosh s\, \cos\phi, \cosh s\, \sin\phi,\, s)\,,$$

for $(s,\phi) \in [-s_\varepsilon, s_\varepsilon] \times S^1$ for some $s_\varepsilon \gg 1$ carefully chosen.

As it has been remarked in the works of R. Mazzeo, F. Pacard and D. Pollack, from the analytical point of view it turns out to be better to deform $\Sigma^\pm$ using the Green's functions associated to the Jacobi operators $J_{\Sigma^\pm}$ to get a better matching with the asymptotic shape of the catenoid. Assume that $\Sigma^\pm$ are nondegenerate surfaces with boundary and let $p \in \Sigma^+ \cap \Sigma^-$ be the point where $\Sigma^\pm$ are tangent. By the Green's function associated to the operator $J_{\Sigma^\pm}$ with a pole at p, we mean the solution to the problem:

$$J_{\Sigma^\pm}\, \Gamma^\pm = -2\pi\, \delta_p \quad \text{in} \quad \Sigma^\pm, \quad \text{and} \quad \Gamma^\pm = 0 \quad \text{on} \quad \partial\Sigma^\pm,$$

where δ_p is the Dirac mass at the point p.

In the neighborhood of p the surfaces $\Sigma^\pm$ can be seen as graphs over their common tangent plane, which we assume to be the horizontal coordinate plane with p being the origin. In Euclidean coordinates in the horizontal plane the functions $\Gamma^\pm$ have the following expansions:

$$\Gamma^\pm(x) = c_0^\pm \pm \log|x| + \mathcal{O}(|x|).$$

On the other hand, the catenoid C_ε can be seen as a bi-graph over $\{x \in \mathbb{R}^2 \,:\, |x| > \varepsilon\}$ of the function

$$G_\varepsilon(x) = -\varepsilon\, \log\frac{\varepsilon}{2} + \log|x| + \mathcal{O}(\varepsilon^3|x|^{-2}).$$

The idea is to translate $\Sigma^\pm$ away from each other by a distance $\pm\, d/2 - \varepsilon c_0^\pm$ respectively and to choose $d > 0$ and a real number r_ε in such a way, that the graph

$$x \mapsto \left(x,\, \pm\, d/2 - \varepsilon\, c_0^\pm + \varepsilon\, \Gamma^\pm(x) + \mathcal{O}(|x|^2)\right), \quad |x| > r_\varepsilon = \varepsilon\, \cosh s_\varepsilon,$$

is "as close as possible" to the graph of the function G_ε in the neighborhood $|x| \sim r_\varepsilon$. Comparing the constant terms, one can see, that we should take:

$$d = 2\varepsilon\, \log \frac{2}{\varepsilon},$$

and comparing the rest of the expansions, we choose r_ε for which the remaining terms (that are of the form $\mathcal{O}(|x|^2) + \mathcal{O}(\varepsilon\, |x|) + \mathcal{O}(\varepsilon^3\, |x|^{-2})$) are minimal, which is the case when

$$r_\varepsilon \sim \varepsilon^{3/4}.$$

We will denote the resulting connected sum by $\mathcal{A}_\varepsilon$. This surface is called the *approximate solution* and its construction can be done in such a way that it depends smoothly on ε.

Perturbation of the approximate solution

The next step consists in perturbing $\mathcal{A}_\varepsilon$ into a constant mean curvature surface, or in other words, this amounts to solve an equation of the form:

$$H(\mathcal{A}_\varepsilon) + J_{\mathcal{A}_\varepsilon}\, w + Q_{\mathcal{A}_\varepsilon}(w) = c, \quad \text{for some } w \in \mathcal{C}^2(\mathcal{A}_\varepsilon), \tag{6}$$

and some constant c (which corresponds to the mean curvature of the surface we are interested in). Remark, that (6) can be written in the form:

$$J_{\mathcal{A}_\varepsilon}\, w = c - H(\mathcal{A}_\varepsilon) - Q_{\mathcal{A}_\varepsilon}(w),$$

which we will try to solve using a fixed point argument. One checks that, as ε tends to 0, the function $c - H(\mathcal{A}_\varepsilon)$ tends to 0 in a suitable topology. If the Jacobi operator $J_{\mathcal{A}_\varepsilon}$ were invertible with inverse uniformly bounded as ε tends to 0, the problem would amount to find a fixed point of the operator

$$w \mapsto J_{\mathcal{A}_\varepsilon}^{-1}\left(H(\mathcal{A}_\varepsilon) + Q_{\mathcal{A}_\varepsilon}(w)\right),$$

in a small closed ball of an appropriate Banach space.

Obstructions

An obstruction to a gluing construction arises when the linearized mean curvature operator $J_{\mathcal{A}_\varepsilon}$ about the approximate solution has small eigenvalues (eigenvalues that tend to 0 as fast as ε tends to 0 or non trivial kernel), which prevents from applying directly the perturbation argument. Small eigenvalues are always expected since the Euclidean catenoid is degenerate in the sense that there are Jacobi fields (solutions to $J_{C_\varepsilon}\, w = 0$) which are defined on the catenoid and tend to 0 fast at the two ends of the catenoid (these arise as the Jacobi fields associated to horizontal translations of the catenoid). The space of eigenfunctions associated to the small eigenvalues of $J_{\mathcal{A}_\varepsilon}$ is usually referred to as the *approximate kernel* of the operator $J_{\mathcal{A}_\varepsilon}$, and will be denoted here by $\mathfrak{K}_\varepsilon$.

Remark that one can construct a family of approximate solutions which depends on a certain number of geometric parameters: for example, one can vary the "size" ε of the "catenoidal neck" and change the points where the connected sum is performed (vary slightly the angle between the axis of the "neck" and the normals to the summands $\Sigma^\pm$). In some cases, one can get rid of small eigenvalues by imposing symmetries on the surface one wants to construct, which also, in general, imposes a unique choice of the free parameters. Many examples of constant mean curvature and minimal surfaces obtained by gluing methods make use of symmetries and therefore are highly symmetric.

In the general case, one can use what is usually referred to as a Lyapunov-Schmidt reduction argument, applying Banach fixed point theorem in the space of functions orthogonal to the approximate kernel $\mathfrak{K}_\varepsilon$. As a result, one obtains a surface whose mean curvature is constant up to an element of $\mathfrak{K}_\varepsilon$. The goal is then to analyze the degrees of freedom in the construction of the family of approximate solutions, and to prove that there is a clever choice of the parameters for which the corresponding term in the approximate kernel vanishes.

Constant mean curvature surfaces in Riemannian manifolds

During the last decades the attention of many researchers was attached to the extension of the classical results stated above to hypersurfaces of higher dimension or to geometries different from the Euclidean space. A lot of progress has been made in the study of constant mean curvature surfaces in simply connected homogeneous 3-manifolds, classified by Thurston according to the dimension of their isometry group. These manifolds include the space forms S^3 and H^3, the products $H^2 \times \mathbb{R}$ or $S^2 \times \mathbb{R}$, the Lie group Sol(3) and many other examples. Techniques coming from complex analysis, harmonic maps, integrable systems, maximum principle and etc. lead to construction of a large number of examples and allowed to obtain important classification results. U. Abresch and H. Rosenberg [1], [2] introduced an analogue of the Hopf's differential, which allowed them to solve the *Hopf's problem* (classification of constant mean curvature surfaces spheres) in some of these geometries, while *Alexandrov's problem* (classification of compact constant mean curvature surfaces surfaces) was studied by B. Daniel and P. Mira [22]. An important progress has been made in the classification of entire minimal graphs (*Bernstein's problem*) in the works of P. Collin and H. Rosenberg [14], B. Daniel and L. Hauswirth [21], I. Fernandez and P. Mira [33], [34]. Solutions of the isoperimetric problem were studied for example in the works of W.T. Hsiang and W.Y. Hsiang [45]. B. Daniel extended the classical Lawson's correspondence to the homogeneous 3-manifolds [19], [20].

Even though a lot of problems remain open, there is by now a rather good understanding of the space of constant mean curvature surfaces in special geometries. In contrast, there exist few results in the case when the ambient manifold is endowed with a "generic" Riemannian metric. In this general setting, even seemingly simple problems have no answer. The existence of closed embedded curve with given constant geodesic curvature remained open for a long time and partial positive answers were given recently in [112] and [109].

Solutions to the isoperimetric problem

Constant mean curvature hypersurfaces in a Riemannian manifold M can be obtained as solutions, when they are regular enough, to the isoperimetric problem. The following fundamental existence theorem results from the works of F. Almgren [4], M. Grüter [42], E. Gonzalez, U. Massari, I. Tamanini [40] and F. Morgan [86]: in an $m+1$-dimensional compact Riemannian manifold M and for all t, $0 < t < \mathrm{Vol}(M)$, there exists a compact region $\Omega \subset M$ whose boundary $\Sigma = \partial\Omega$ minimizes the m-volume among the regions of $m+1$-volume equal to t. Moreover, except for a closed singular set of Hausdorff dimension at most $m-7$, the boundary Σ of any minimizing region is a smooth embedded hypersurface with constant mean curvature. In particular, if $m \leq 6$, then Σ is smooth.

Techniques used in [4], [42], [40] and [86] to investigate the properties of the isoperimetric surfaces come from the geometric measure theory. Another powerful tool is studying the isoperimetric profile of the ambient manifold, more precisely the function

$$I_M : (0, \mathrm{Vol}(M)) \to \mathbb{R}, \quad I_M(t) = \inf \{\mathrm{Vol_m}(\partial\Omega) \, : \, \Omega \subset M \text{ region}, \; \mathrm{Vol_{m+1}}(\Omega) = t\} .$$

The properties of I_M were analyzed in the works of C. Bavard and P. Pansu [6], [96], S. Gallot [38], W.Y. Hsiang [56] and others and were used for example to prove the fact that if Σ_n is a sequence of isoperimetric surfaces in M enclosing volumes $t_n \to t$, then Σ_n converges to an isoperimetric surface enclosing a volume t.

The study of the solutions of the isoperimetric problem remains a very active area of research and a profound description of this theory can be found in works of A. Ros [102]. A drawback of this approach is that one does not control the value of the mean curvature of the surface obtained by this method and very little information is in general available on the geometry of the solution. However, in the case when the volume constraint is small, it has been shown in the works of P. Berard, D. Meyer [8], O. Druet [29] and A. Ros [102] that the solutions are close to geodesic spheres of small radii and concentrate at critical points of the scalar curvature $\mathcal{R}$ of the ambient manifold M.

0.1.5 Constant mean curvature perturbations of geodesic spheres

The proof of Hopf's theorem works in the space forms $\mathbb{R}^3$, S^3, and H^3, while uniqueness Hopf type results were obtained in other special geometries [1], [2], [22]. It is natural to consider the problem of classification of constant mean curvature topological spheres immersed in a generic Riemannian manifold.

In 1990, R. Ye showed [118] that every nondegenerate critical point o of the scalar curvature function $\mathcal{R}$ of a Riemannian manifold (M, g) has a neighborhood that can be foliated by constant mean curvature hypersurfaces which are close to small geodesic spheres centered at o and converge to o when their mean curvature tends to infinity.

I propose the reader to give a closer look to R. Ye's construction. Let $\mathcal{S}_\varepsilon(p)$ denote a geodesic sphere in (M^{m+1}, g), of radius $\varepsilon > 0$ centered at the point p. For ε small enough $\mathcal{S}_\varepsilon(p)$ can be seen as the image by the exponential map of the Euclidean sphere:

$$\Theta \in S^m \subset T_pM \longmapsto \exp_p(\varepsilon\,\Theta) \in \mathcal{S}_\varepsilon(p).$$

The mean curvature of $\mathcal{S}_\varepsilon(p)$ then satisfies:

$$H(\mathcal{S}_\varepsilon(p)) = \frac{m}{\varepsilon} - \frac{\varepsilon}{3}\,\mathrm{Ric}_p(\Theta, \Theta) + \mathcal{O}(\varepsilon^2),$$

where Ric_p is the Ricci tensor of M evaluated at p. Hence, in some sense, $H(\mathcal{S}_\varepsilon(p))$ is close to being constant and it is reasonable to expect that $\mathcal{S}_\varepsilon(p)$ can be deformed into some constant mean curvature surface, at least for ε small enough. Unfortunately, as observed by R. Ye in [118], this is not the case. When ε is small enough, the Jacobi operator about $\mathcal{S}_\varepsilon(p)$ is close to the Jacobi operator of the Euclidean sphere of radius ε which reads $\varepsilon^{-2}\,(\Delta_{S^m} + m)$. This operator has a non trivial $(m+1)$-dimensional kernel consisting of the restrictions to S^m of the coordinate functions:

$$\mathrm{Ker}(\Delta_{S^m} + m) = \mathrm{span}\{\Theta^i,\ i = 1, \ldots, m+1\}.$$

This prevents one from directly applying a perturbation argument. Let $H(w)$ denote the mean curvature of the normal graph of the function w over $\mathcal{S}_\varepsilon(p)$ and let Π be the L^2-orthogonal projection to the space $\mathrm{Ker}(\Delta_{S^m} + m)$ and $\Pi^\perp$ the projection to the corresponding orthogonal complement. Then using Banach fixed point theorem, one can find a function $w_* \in \Pi^\perp\left(\mathcal{C}^{2,\alpha}(S^2)\right)$ such that

$$\Pi^\perp\left(H(w_*)\right) = \frac{m}{\varepsilon}.$$

On the other hand, it turns out that the equation $\Pi\left(H(w_*)\right) = 0$ can be written in the form

$$\langle \nabla^g \mathcal{R}(p), \Theta \rangle = \mathcal{O}(\varepsilon^2), \tag{7}$$

where $\nabla^g \mathcal{R}(p)$ is the gradient of the scalar curvature, calculated with respect to the metric g and evaluated at p. Thus, if o is a nondegenerate critical point of the scalar curvature, for every ε small enough one can find a point p_ε such that (7) holds. In this way, one obtains a constant mean curvature hypersurface which is a normal geodesic graph over the geodesic sphere $\mathcal{S}_\varepsilon(p_\varepsilon)$, where $\mathrm{dist}_g(o, p_\varepsilon) = \mathcal{O}(\varepsilon^2)$.

Later, F. Pacard and X. Xu showed [95] that constant mean curvature topological spheres can also be constructed when the scalar curvature is not a Morse function, which for example covers the case of Einstein or constant scalar curvature manifolds. As a first step, similarly to R. Ye, the authors show that for all $p \in M$, and all $\varepsilon > 0$ small enough, the geodesic sphere $\mathcal{S}_\varepsilon(p)$ can be perturbed to a surface $\Sigma_\varepsilon(p)$ whose mean curvature is close in a particular sense to being constant. More precisely, the mean curvature of $\Sigma_\varepsilon(p)$ (when the surface is parameterized by the unit sphere S^m) satisfies:

$$H(\Sigma_\varepsilon(p)) - \frac{m}{\varepsilon} = \langle A, \Theta \rangle, \quad \Theta \in S^m,$$

for some vector $A \in \mathbb{R}^{m+1}$, where by $\langle \cdot, \cdot \rangle$ we denote the scalar product in $\mathbb{R}^{m+1}$.

Remark that the functions of the form $\langle A, \Theta \rangle$ are exactly the elements of space Ker $(\Delta_{S^m} + m)$. The surfaces $\Sigma_\varepsilon(p)$ are called "*pseudo constant mean curvature spheres*" by F. Pacard and X. Xu and were also studied by S. Nardulli, [90], who refers to them as "*pseudo bubbles*".

To understand the idea of the construction of F. Pacard and X. Xu, let us consider first the following simple example. Let Σ be a compact surface in $\mathbb{R}^{m+1}$ whose mean curvature satisfies:

$$H_\Sigma = \lambda + \langle A, N_\Sigma \rangle, \quad \text{for some fixed vector } A \in \mathbb{R}^{m+1},$$

where N_Σ is the unit normal vector field to Σ and λ is a constant. Take $B \in \mathbb{R}^{m+1}$, and consider the one parameter family of surfaces Σ_t obtained by translation of Σ in the direction B:

$$p \in \Sigma \mapsto p + t\,B \in \Sigma_t.$$

Let Ω and Ω_t be the regions in $\mathbb{R}^{m+1}$ bounded by Σ and Σ_t, and consider the functional

$$\mathcal{E}_\lambda(t) = \mathrm{Area}(\partial\Omega_t) - \lambda\,\mathrm{Vol}(\Omega_t),$$

which appears in the variational characterization (4) of the compact constant mean curvature surfaces. Since translations are isometries in the Euclidean space, we have

$$\mathcal{E}_\lambda'(t) = 0,$$

which implies that

$$\int_\Sigma (H_\Sigma - \lambda)\,\langle B, N_\Sigma \rangle\, dvol_\Sigma = 0, \quad \text{for all} \quad B \in \mathbb{R}^{m+1}.$$

Finally, taking $B = A$, we find:

$$\int_\Sigma |\langle A, N_\Sigma \rangle|^2\, dvol_\Sigma = 0 \quad \Rightarrow \quad A = 0$$

and conclude that Σ is a constant mean curvature surface.

In the Riemannian manifold (M, g) we no longer have isometries given by translations. However, in a small neighborhood of a given point, the expansion of the metric g in geodesic normal coordinates can be seen as a perturbation of the Euclidean metric. Then for ε small enough, one can apply an idea similar to the one described above to the pseudo CMC spheres $\Sigma_\varepsilon(p)$ whose mean curvature satisfies:

$$H(\Sigma_\varepsilon(p)) = \frac{m}{\varepsilon} + \langle A, \Theta \rangle.$$

Let $\Omega_\varepsilon(p)$ denote the region in M bounded by $\Sigma_\varepsilon(p)$. F. Pacard and X. Xu show that if p is a critical point of the functional

$$\mathcal{E}_\varepsilon(p) = \mathrm{Vol_m}(\partial\Omega_\varepsilon(p)) - \frac{m}{\varepsilon}\,\mathrm{Vol}_{m+1}(\Omega_\varepsilon(p)), \tag{8}$$

then the mean curvature of $\Sigma_\varepsilon(p)$ is constant.

Finally, remark that the scalar curvature $\mathcal{R}$ of the ambient manifold M appears in the expression of the volume of the geodesic spheres, more precisely, we have:

$$\mathrm{Vol}_m\left(\mathcal{S}_\varepsilon(p)\right) = \varepsilon^m \,\mathrm{Vol}(S^m)\left(1 - \frac{\varepsilon^2}{2(m+1)}\mathcal{R}(p) + \mathcal{O}(\varepsilon^4)\right).$$

Using the fact that $\Sigma_\varepsilon(p)$ are constructed as small perturbations of geodesic spheres, F. Pacard and X. Xu obtain that

$$\varepsilon^{-m}\mathcal{E}_\varepsilon(p) = c_0 + c_1\,\varepsilon^2\,\mathcal{R}(p) + \mathcal{O}(\varepsilon^4),$$

for some constants c_0 and c_1 independent of ε.

The results of R. Ye and F. Pacard and X. Xu outlined above play a key role in the constructions described in the chapters 1 and 3 of this thesis.

Other examples of surfaces with large constant mean curvature

H. Rosenberg [103] has shown that if Σ is a closed surface with sufficiently large constant mean curvature H in a compact Riemannian 3-manifold M, then Σ separates M into two connected components and the distance between Σ and a point of the component of $M \setminus \Sigma$ towards which the mean curvature vector is pointed is bounded above by a constant times $1/H$.

The result of H. Rosenberg implies that a constant mean curvature surface should look like a small tube around some set and classification of such sets has been a challenging problem. R. Mazzeo and F. Pacard [76] showed the existence of constant mean curvature surfaces obtained by perturbation of geodesic tubes about nondegenerate closed geodesics. A. Butscher and R. Mazzeo [10] proposed a construction (under some symmetry assumptions on the metric) of a family of compact constant mean curvature surfaces condensing along a geodesic segment, passing through a nondegenerate critical point of the scalar curvature, obtained by gluing together a large number of geodesic spheres. For some special noncompact 3-manifolds they also prove the existence of one-ended constant mean curvature surfaces condensing to geodesic rays. F. Mahmoudi, R. Mazzeo and F. Pacard proved in [72] the existence of constant mean curvature surfaces with large mean curvature which are close to small geodesic tubes around embedded minimal submanifolds.

0.1.6 Constant mean curvature surfaces via doubling constructions

Examples of surfaces with small constant mean curvature in Riemannian manifolds have been obtained via perturbation techniques, which are referred to, following N. Kapouleas, as "doubling constructions". In this paragraph, I would like to explain the main ideas behind this method, since similar techniques will be applied in the results described in the chapter 2 of this thesis.

The idea is to construct a constant mean curvature surface by performing a connected sum of two copies of a given oriented, embedded compact minimal surface Λ. More precisely, assuming that Λ is nondegenerate, one can apply the implicit function theorem to obtain, for any ε close to 0, a constant mean curvature surface Λ_ε with mean curvature ε. It is easy to check that $\Lambda_{\pm\varepsilon}$ are normal graphs over Λ for some functions of the form $\pm\varepsilon\,\psi + \mathcal{O}(\varepsilon^2)$, where $\psi \in \mathcal{C}^\infty(\Sigma)$ solves $J_\Lambda \psi = 1$. In a recent unpublished paper F. Pacard and T. Sun [94] show that one can perform a connected sum between Λ_ε and $\Lambda_{-\varepsilon}$ at any nondegenerate point of the function ψ to produce new constant mean curvature surfaces with $H = \varepsilon$.

In the case where there is a group of isometries acting on M and the Jacobi operator of Λ has a nontrivial kernel, the fact whether it is possible or not to carry out a doubling construction depends on what is called the "neck configuration", more precisely the set of points $p_1, \dots, p_k \in \Lambda$ where the connected sum is performed and the "size of the necks". To understand this, it is convenient to use the Green's function method, namely, to understand for which choice of $p_1, \dots, p_k$, there exists a solution to the problem:

$$J_\Lambda\, \Gamma = -2\pi\,(\delta_{p_1} + \cdots + \delta_{p_k})\,, \tag{9}$$

and then to glue the graph of the function Γ together with k "necks" given by catenoids centered at p_i and scaled by a small factor.

For example, consider the case where the initial minimal surface Λ is the equatorial sphere S^2 in the 3-sphere S^3. The corresponding Jacobi operator $\Delta_{S^2} + 2$ has a nontrivial kernel given by restrictions to S^2 of the coordinate functions. This prevents from solving (9) for the number of points $k = 1$. On the other hand, introducing symmetry with respect to one of the coordinate axis and taking $k = 2$ and p_1, p_2 antipodal points on S^2, one can find a solution to (9).

A. Butscher and F. Pacard [11] showed that when Λ is the minimal Clifford torus in S^3 and the Jacobi operator is given by $2\,(\Delta + 2)$, surfaces with small constant mean curvature can be produced by performing connected sums at the points of a lattice which contains $2\pi\mathbb{Z}^2$ and which is not included in $\{(x, y) \in \mathbb{R}^2 \,:\, x \pm y = 0 \text{ mod } [2\pi]\}$.

Remark that the constructions described in [94] and [11] produce for all ε small enough constant mean curvature surfaces of mean curvature $H = \varepsilon$, and the number of "necks" in the construction is bounded independently of ε, while the size of the "necks" is given by a constant times ε.

On the other hand, N. Kapouleas and S.D. Yang [63] showed that the doubling construction technique can be applied to produce new examples of minimal surfaces, but this time a certain relation has to be satisfied between the number of points where the connected sum is performed and the size of the "necks". The authors prove the existence of minimal surfaces in S^3 obtained by doubling the minimal Clifford torus performing the connected sum at the points of the square lattice $l_n := \frac{2\pi}{n}\,\mathbb{Z}^2$ for n large enough, where the Clifford torus T^2 is identified with $\mathbb{R}^2/2\pi\mathbb{Z}^2$. To understand why the construction works only for large numbers n, consider the Green's function Γ_n with poles at the vertices of l_n:

$$(\Delta + 2)\;\Gamma_n = -2\pi \sum_{p_i \in l_n} \delta_{p_i}, \tag{10}$$

where the non-trivial kernel of the operator $\Delta + 2$ is eliminated using the symmetry of the lattice. Next, one can introduce a function G, such that $\Gamma_n(x) = G(nx)$. Then (10) is equivalent to

$$\Delta G + \frac{2}{n^2} G = -2\pi\, \delta_{p_0} \quad \text{in} \quad T^2.$$

The solution can be found by induction, putting $G = c + \sum_{j=0}^{\infty} \left(\frac{2}{n^2}\right)^j G_j$, where the functions G_j solve

$$\Delta G_0 = -2\pi\, \delta_{p_0} + \frac{1}{2\pi} \quad \text{and} \quad \Delta G_j = -G_{j-1}, \quad \text{with} \quad \int_{T^2} G_j = 0.$$

A direct calculation then shows that $c = -\frac{n^2}{2\pi}$ and in local coordinates in the neighborhood of $p_i \in l_n$ the function Γ_n has the expansion

$$\Gamma_n(x) = -\frac{n^2}{2\pi} - \log n - \log|x| + \ldots, \tag{11}$$

where by ... we denote some function bounded independently of n in some neighborhood of the origin. On the other hand, as it has been observed in the subsection 0.1.4, a catenoid centered at p_i and scaled by the factor ε can be seen as a bi-graph over T^2 of the function

$$G_\varepsilon = -\varepsilon\, \log\frac{\varepsilon}{2} + \varepsilon\, \log|x| + \mathcal{O}(\varepsilon^3\, |x|^{-2}). \tag{12}$$

In order to perform connected sums of the graphs of the functions $\mp\Gamma_n$ and $\pm G_\varepsilon$, one needs the largest terms in the expansions (11) and (12) to match exactly. We can make the logarithmic terms match by multiplying Γ_n by the factor ε. On the other hand, the constant terms coincide, when

$$\log 2/\varepsilon = \frac{n^2}{2\pi} + \log n + \ldots$$

which defines the relation between the "neck size" ε and the number of "necks" given by the integer n. Finally, as in the case of constructions described in the subsection 0.1.4, one applies a perturbation argument to deform the constructed connected sum into a minimal surface. Other examples of minimal surfaces in S^3 were constructed by N. Kapouleas by doubling the equatorial sphere [61] and by D. Wiygul by stacking (tripling, quadrupling, etc.) Clifford tori.

0.2 Chapter 1: Higher codimension isoperimetric problems

In the first chapter, which is a work in collaboration with R. Mazzeo and F. Pacard, we propose a generalization of the classical notion of a constant mean curvature hypersurface to submanifolds of arbitrary codimension. We also prove the existence in compact Riemannian manifolds of constant mean curvature (in the sense that we introduce) submanifolds which are small perturbations of geodesic spheres of small volume.

Let K be an embedded submanifold of a compact Riemannian manifold (M^{m+1}, g). Recall, that by definition (given in the subsection 0.1.1) the mean curvature vector of K is defined as the trace of its second fundamental form:

$$h_K(X,Y) = \pi_{NK} \nabla_X Y, \quad \text{for} \quad \mathrm{X,Y} \in TK, \quad \text{and} \quad H_K = \operatorname{Tr} h_K,$$

where ∇ is the Levi-Civita connection on M associated to the metric g and π_{NK} is the fibrewise orthogonal projection $T_K M \to NK$ on the normal bundle of K. In codimension 1, when $\dim(NK) = 1$, we say that K has constant mean curvature if the mean curvature function $g(H_K, N_K)$ is constant. There are many possible extensions of the notion of constant mean curvature to higher codimensions, for example, asking that H_K is parallel, or H_K is harmonic. We propose a different, strictly variational definition, building on the ideas of F. Almgren [5].

Critical points of the area functional subject to the volume constraint produce constant mean curvature surfaces. F. Almgren [5] generalized the classical isoperimetric problem to higher codimensional submanifolds, which amounts to solve the following minmax problem:

$$\max_{K\,:\,\mathrm{Vol}_k(K)=c} \left(\min_{Q\,:\,\partial Q=K} \mathrm{Vol}_{k+1}(Q) \right).$$

where c is a constant. Moreover, F. Almgren proves that the solutions in the Euclidean space $\mathbb{R}^{m+1}$ are round spheres. The existence result in an arbitrary Riemannian manifold as well as regularity properties of the solutions were obtained in the work of F. Morgan and M.C. Salavessa [87].

Recall that constant mean curvature hypersurfaces can also be understood as boundaries of the critical points of the functional

$$\Omega \mapsto \mathrm{Vol}(\partial\Omega) - \lambda\, \mathrm{Vol}(\Omega), \tag{13}$$

where λ is a constant which corresponds to the value of the mean curvature.

In arbitrary codimension, we will say that a k-dimensional submanifold K has constant mean curvature if it is a boundary of a smooth (k+1)-dimensional submanifold Q which is a critical point of the functional

$$Q \mapsto \mathrm{Vol}(\partial Q) - \lambda\, \mathrm{Vol}(Q).$$

These critical points are characterized by the fact that Q is a minimal submanifold such that the mean curvature vector of the boundary $K = \partial Q$ satisfies:

$$H_K = \lambda\, n,$$

where n is a unit vector field normal to K and tangent to Q, in other words, if K has constant mean curvature in Q and H_K has no components orthogonal to Q. A k-dimensional sphere $S^k = S^k \times \{0\}$, which is the boundary of a $(k+1)$-dimensional ball B^{k+1} in the Euclidean space $\mathbb{R}^{m+1}$, $k < m$, is an example of a constant mean curvature (in the given sense) submanifold of codimension $m + 1 - k$.

We have proved the existence of constant mean curvature submanifolds of arbitrary codimension in generic Riemannian manifolds in the above sense. Our result is a generalization of the works of R. Ye [118] and F. Pacard and X. Xu [95] on constant mean curvature spheres described in the subsection 0.1.5. We construct constant mean curvature spheres of arbitrary codimension near nondegenerate critical points of the *partial scalar curvature* function which is defined on the Grassmannian bundle of the ambient manifold. For any $(k+1)$-dimensional subspace $\Pi_p \subset T_pM$, we define the partial scalar curvature:

$$\mathcal{R}_{k+1}(\Pi_p) := -\sum_{i,j=1}^{k+1} \langle R(E_i, E_j)E_i, E_j\rangle,$$

where $E_1, \dots, E_{k+1}$ is any orthonormal basis for Π_p. Note that $\mathcal{R}_{m+1}(T_pM)$ is the standard scalar curvature at p, while $\mathcal{R}_2(\Pi_p)$ is twice the sectional curvature of the 2-plane Π_p.

We define a k-dimensional geodesic sphere $\mathcal{S}^k_\varepsilon(\Pi_p)$ in M as the image by the exponential map of a sphere of radius ε in T_pM:

$$\mathcal{S}^k_\varepsilon(\Pi_p) := \Big\{ \exp_p\left(\varepsilon \sum_{i=1}^{k+1} \Theta^i\, E_i\right),\ \Theta \in S^k \Big\}.$$

We prove the following result:

Theorem 0.2.1. *If Π_p is a nondegenerate critical point of $\mathcal{R}_{k+1}$, then for all ε sufficiently small, there exists a constant mean curvature submanifold $K_\varepsilon(\Pi_p)$ which is a normal graph over $\mathcal{S}^k_\varepsilon(\tilde{\Pi}_{\tilde{p}})$ by some section with $\mathcal{C}^{2,\alpha}$ norm bounded by $c\varepsilon^3$, and* $\operatorname{dist}(\tilde{\Pi}_{\tilde{p}}, \Pi_p) \le c\,\varepsilon^2$.

Let me give some details of our construction.

Study of the Jacobi operator in arbitrary codimension

Let Σ be an embedded submanifold of M, either closed or with boundary. By definition, the Jacobi operator J_Σ about Σ is the differential of the mean curvature with respect to normal perturbations of Σ. Let $\Phi \in \mathcal{C}^2(\Sigma, N\Sigma)$ be a normal vector field to Σ with $\|\Phi\|_{\mathcal{C}^0}$ sufficiently small. If $\partial\Sigma \neq \emptyset$, we require also $\Phi = 0$ on $\partial\Sigma$. Then

$$\Sigma_\Phi := \big\{ \exp_q(\Phi(q)),\ q \in \Sigma \big\}$$

is an embedded submanifold. We define:

$$J_\Sigma(\Phi) := \nabla_{\frac{\partial}{\partial s}} H(\Sigma_{s\,\Phi})\Big|_{s=0}.$$

Let $\pi_{N\Sigma}$ and $\pi_{T\Sigma}$ be the orthogonal projections to the normal and the tangent bundles of Σ. We denote the operators $\pi_{N\Sigma} \circ J_\Sigma$ and $\pi_{T\Sigma} \circ J_\Sigma$ by J^N_Σ and J^T_Σ respectively. The expression of the operator J^N_Σ is given by a standard formula [68]:

$$J^N_\Sigma = -\Delta^N_\Sigma + \mathrm{Ric}^N_\Sigma + \mathfrak{H}^{(2)}_\Sigma, \tag{14}$$

where Δ^N_Σ is the (positive definite) connection Laplacian on sections of $N\Sigma$, and the other two terms are the following symmetric endomorphisms of $N\Sigma$: the orthogonal projection $\mathrm{Ric}^N_\Sigma = \pi_{N\Sigma} \circ \mathrm{Ric}_\Sigma$ of the partial Ricci curvature Ric_Σ defined by

$$g(\mathrm{Ric}_\Sigma X, Y) := \sum_{i=1}^{\dim\Sigma} g\left(R(E_i, X)E_i, Y\right), \quad \text{for all} \quad X, Y \in T_\Sigma M,$$

(note that the curvature tensor appearing on the right is the one on all of M, and is not the curvature tensor for Σ); the square of the shape operator defined by

$$\mathfrak{H}^{(2)}_{\Sigma}(X) := \sum_{i,j=1}^{\dim \Sigma} \langle h_{\Sigma}(E_i, E_j), X\rangle h_{\Sigma}(E_i, E_j), \quad \text{for all} \quad X \in T_{\Sigma}M.$$

In general, $J_{\Sigma}(\Phi)$ has a nontrivial component $J^{T}_{\Sigma}(\Phi)$ which is parallel to Σ. However, when Σ is a minimal submanifold, we have $J^{T}_{\Sigma} = 0$. We will see that in the linearized problem that we consider in our work this term is also canceled, so we do not need to make it explicit.

Linearized problem

Now, let K^k, $k \leq m$ be a constant mean curvature submanifold in M^{m+1}. By definition, there exists a minimal submanifold Q such that

$$K = \partial Q \quad \text{and} \quad H_K = \lambda\, n,$$

where n is a unit normal to K in Q. We would like now to perturb K into another CMC submanifold with the same value of the mean curvature. Take a vector field $\Phi \in NK$ and consider the submanifold $K_{\Phi} = \left\{ \exp_p(\Phi),\ p \in K \right\}$. By definition, K_{Φ} has constant mean curvature if and only if

$$K_{\Phi} = \partial Q_{\Phi} \quad \text{and} \quad H(K_{\Phi}) = \lambda\, n_{\Phi},$$

where Q_{Φ} is a minimal submanifold, and n_{Φ} is a unit normal to K_{Φ} in Q_{Φ}. We see that Q_{Φ} and n_{Φ} depend on the extension of the vector field Φ defined on $K = \partial Q$ to the interior of Q. Therefore, unlike the case of codimension 1, the problem is no longer local, which a priori gives rise to supplementary difficulties. First of all, we prove the following result:

Lemma 0.2.1. *If Q is a nondegenerate minimal submanifold, there is a smooth mapping $\Phi \mapsto Q_{\Phi}$ from a neighborhood of 0 in $\mathcal{C}^{2,\alpha}(K, NK)$ into the space of $(k+1)$-dimensional minimal submanifolds of M with $\mathcal{C}^{2,\alpha}$ boundary, such that $Q_0 = Q$ and $\partial Q_{\Phi} = K_{\Phi}$. Moreover, Q_{Φ} is a geodesic graph over Q for some vector field $U_{\Phi} \in T_Q M$ such that $U_{\Phi}\big|_K = \Phi$.*

Secondly, we consider the functional

$$\mathcal{H}(s) := H(K_{s\Phi}) - \lambda\, n_{s\Phi},$$

where $n_{s\Phi}$ is a unit normal to $K_{s\Phi}$ in the minimal submanifold $Q_{s\Phi}$ and consider the linear operator

$$L_Q := \nabla_{\frac{\partial}{\partial s}} \mathcal{H}(s)\Big|_{s=0}.$$

Notice that $\pi_{TK} \circ L_Q = 0$. Indeed, let T be a tangent vector field to $K_{s\Phi}$. Then since $\mathcal{H}(s) \perp K_{s\Phi}$ and $\mathcal{H}(0) = 0$, we find

$$g(\mathcal{H}'(0), T) + g(\mathcal{H}(0), T'(0)) = 0.$$

Notation 0.2.1. *We introduce the operator*

$$D_Q := \pi_{NK} \circ \nabla_{\frac{\partial}{\partial s}} n_{s\Phi}\Big|_{s=0}.$$

Let U_Φ be the extension in Q of the vector field $\Phi \in NK$ defined in Lemma 0.2.1, and $\bar{U}_\Phi$ the part of U_Φ linear in Φ. We denote by $[\cdot]^\perp$ the orthogonal projection $T_K M \to NK \cap N_K Q$. We have the following result:

Proposition 0.2.1. *The operator D_Q satisfies*

$$D_Q = \left[\nabla_n \bar{U}_\Phi\big|_K\right]^\perp,$$

while the operator L_Q is given by

$$L_Q(\Phi) = \pi_{NK} \circ J_K(\Phi) - \lambda\, D_Q(\Phi).$$

Linearization for $K = S^k \subset \mathbb{R}^{m+1}$

As a next step, we apply the analysis described above to the case where $K = S^k = \partial B^{k+1}$ in the Euclidean space $\mathbb{R}^{m+1}$, $k < m$. Take $\Phi \in \mathcal{C}^{2,\alpha}(S^k, NS^k)$ and decompose

$$\Phi = [\Phi]^\perp - \phi\, \Theta, \quad \text{where} \quad \Theta \in S^k, \quad [\Phi]^\perp \in NS^k \cap NB^{k+1}.$$

Then the operator $\pi_{NS^k} \circ J_{S^k}$ acts separately on these components. Using (14) and Proposition 0.2.1, we find:

$$\pi_{NS^k} \circ J_{S^k}(\Phi) = \Delta_{S^k}\, [\Phi]^\perp - (\Delta_{S^k} + k)\, \phi\, \Theta,$$

while the operator $D_{B^{k+1}}$ acts only the component $[\Phi]^\perp$ of Φ and is given by the Dirichlet-to-Neumann operator, more precisely by,

$$D_{B^{k+1}}(\Phi) = \frac{\partial}{\partial n}\, U_\Phi, \quad \text{where} \quad \Delta_{B^{k+1}}\, U_\Phi = 0 \quad \text{and} \quad U_\Phi|_{S^k} = [\Phi]^\perp,$$

where $n = -\Theta$ is the unit normal to S^k in B^{k+1}.

Kernel of the linearized operator

Analyzing the properties of the operators given above, we find that the linear operator $L_{B^{k+1}}$ has a non-trivial kernel. More precisely, we have:

$$(\Delta_{S^k} + k)\ \phi = 0 \Rightarrow \phi \in \operatorname{span}\{\Theta^j,\ j = 1, \dots, k+1\} \quad \text{and}$$

$$(\Delta_{S^k} - k\, D_{B^{k+1}})\, [\Phi]^\perp = 0 \Rightarrow [\Phi]^\perp \in \operatorname{span}\left\{\left(c^j_\mu\, \Theta^j + d_\mu\right) E_\mu,\ j = 1, \dots, k+1\right\},$$

where $E_\mu,\ \mu = k+2, \dots, m+1$ is an orthonormal basis of NB^{k+1}.

Construction of constant mean curvature submanifolds

Let Π_p be a $(k+1)$-dimensional plane in T_pM and consider the k-dimensional geodesic sphere $\mathcal{S}^k_\varepsilon(\Pi_p)$ defined in (14), which is the boundary of a $(k+1)$-dimensional geodesic ball which we denote by $\mathcal{B}^{k+1}_\varepsilon(\Pi_p)$. We find:

$$H(\mathcal{B}^{k+1}_\varepsilon(\Pi_p)) = \mathcal{O}(\varepsilon) \quad \text{and} \quad H(\mathcal{S}^k_\varepsilon(\Pi_p)) - \frac{k}{\varepsilon}\, n = \mathcal{O}(\varepsilon),$$

where n is a unit normal to $\mathcal{S}^k_\varepsilon(\Pi_p)$ in $\mathcal{B}^{k+1}_\varepsilon(\Pi_p)$. So, in some sense, $\mathcal{S}^k_\varepsilon(\Pi_p)$ is close to being a constant mean curvature submanifold when ε is small enough. Unfortunately, similarly to the case of codimension 1 described in the subsection 0.1.5, we cannot directly apply a perturbation argument to deform $\mathcal{S}^k_\varepsilon(\Pi_p)$ into a CMC submanifold, since for ε small enough the operator $L_{\mathcal{B}^{k+1}_\varepsilon(\Pi_p)}$ is close to $\varepsilon^{-2}\, L_{B^{k+1}}$ which, according to the previous paragraph, has a nontrivial kernel.

We show then that for ε small enough, $\mathcal{S}^k_\varepsilon(\Pi_p)$ can be deformed into a submanifold which is a generalization to higher codimensions of a "pseudo constant mean curvature sphere", or a "pseudo bubble" described in [95] and [90] and which we discuss in the subsection 0.1.5. We prove that for all Π_p and all ε small enough, there exists a minimal submanifold $Q_\varepsilon(\Pi_p)$, which is a small perturbation of $\mathcal{B}^{k+1}_\varepsilon(\Pi_p)$, whose boundary $K_\varepsilon(\Pi_p)$ is a normal graph over $\mathcal{S}^k_\varepsilon(\Pi_p)$ such that

$$H(K_\varepsilon(\Pi_p)) - \frac{k}{\varepsilon}\, n = \langle \vec{a}, \Theta\rangle\, n + \sum_{\mu=k+2}^{m+1} \left(\langle \vec{c}_\mu, \Theta\rangle + d_\mu\right)\, N_\mu,$$

for some $\vec{a}, \vec{c}_\mu \in \mathbb{R}^{k+1}$ and $d_\mu \in \mathbb{R}$. Here by n we denote a unit normal to $K_\varepsilon(\Pi_p)$ in $Q_\varepsilon(\Pi_p)$ and by $N_\mu,\ \mu = k+2, \ldots, m+1$ an orthonormal basis of $NQ_\varepsilon(\Pi_p)$.

Choice of the parameters

We consider the functional

$$\mathcal{E}_\varepsilon(\Pi_p) := \mathrm{Vol}_k\left(\partial Q_\varepsilon(\Pi_p)\right) - \frac{k}{\varepsilon}\mathrm{Vol}_{k+1}\left(Q_\varepsilon(\Pi_p)\right),$$

defined on the Grassmannian bundle of TM, which is a generalization to higher codimension of the functional (8) defined by F. Pacard and X. Xu [95]. We show that if Π_p is a critical point of $\mathcal{E}_\varepsilon$, then $K_\varepsilon(\Pi_p)$ is a constant mean curvature submanifold. Remark, that in the construction of $K_\varepsilon(\Pi_p)$ we have $m+1$ degrees of freedom which correspond to infinitesimal translations of the plane Π_p, and $(k+1)\times(m-k)$ degrees of freedom which correspond to rotations of Π_p in T_pM which transform Π_p to a plane orthogonal to Π_p. Thus, the number of degrees of freedom matches exactly the number of equations in the system

$$\vec{a} = 0, \quad \vec{c}_\mu = 0, \quad d_\mu = 0, \quad \mu = k+2, \ldots, m+1.$$

Finally, we obtain:

$$\varepsilon^{-k}\, \mathcal{E}_\varepsilon(\Pi_p) = c_0 + c_1\, \varepsilon^2\, \mathcal{R}_{k+1}(\Pi_p) + \mathcal{O}(\varepsilon^4),$$

for some constants c_0 and c_1 independent of ε, where $\mathcal{R}_{k+1}$ is the partial scalar curvature of the manifold M. Thus, if Π_p is a nondegenerate critical point of $\mathcal{R}_{k+1}$, then there exists a plane $\tilde{\Pi}_{\tilde{p}}$ which is a critical point of $\mathcal{E}_\varepsilon$ and such that $\operatorname{dist}\left(\Pi_p, \tilde{\Pi}_{\tilde{p}}\right) \leq c\,\varepsilon^2$.

0.3 Chapter 2 : Free boundary minimal surfaces in the unit 3-ball

In the second chapter, which is a work in collaboration with A. Folha and F. Pacard, we are interested in the existence of minimal surfaces embedded in the Euclidean unit 3-ball B^3 which meet the unit sphere S^2, the boundary of B^3, orthogonally. Such surfaces arise as critical points of the area functional among the surfaces embedded in B^3 whose boundaries lie on S^2, and, following [36], are referred to as *free boundary minimal surfaces*. We prove the existence of two geometrically distinct families of free boundary minimal surfaces in B^3, parametrized by an integer n large enough which represents the number of boundary components.

Obviously, horizontal unit disks obtained by the intersection of the planes passing through the origin with the unit 3-ball are examples of such free boundary minimal surfaces. J.C. Nitsche [88] showed that these are the only free boundary solutions of topological disk type.

A. Fraser, M. Li [35] have formulated the conjecture that the only free boundary minimal surface of topological type of an annulus which is properly embedded in B^3 is the critical catenoid:

$$(s, \phi) \in \mathbb{R} \times S^1 \mapsto \frac{1}{s_* \cosh s_*}(\cosh s\, \cos\phi, \cosh s \sin\phi, s), \quad \text{where} \quad s_* \tanh s_* = 1.$$

In a recent paper, A. Fraser and R. Schoen [37] proved the existence of free boundary minimal surfaces Σ_n in B^3 which have genus 0 and n boundary components for all $n \geq 3$. These surfaces emerge in the study of maximizing metrics for the first eigenvalue of the Dirichlet-to-Neumann operator on compact 2-manifolds. Let (M^2, g) be a compact 2-dimensional Riemannian manifold with nonempty boundary. Then the Dirichlet-to-Neumann operator $D_g : \mathcal{C}^\infty(\partial M) \to C^\infty(\partial M)$ is defined by:

$$D_g u = \left.\frac{\partial \hat{u}}{\partial \nu}\right|_M, \quad \text{where} \quad \Delta_g \hat{u} = 0 \quad \text{in} \quad M, \quad \hat{u}|_{\partial M} = u,$$

ν being a unit normal to ∂M in M. D_g is a non-negative self-adjoint operator with discrete spectrum, referred to as *Steklov eigenvalues*. Let $\sigma_1(g)$ denote the first Stekloff eigenvalue associated to a metric g. The authors prove that when M has genus 0 and n boundary components, there exists a metric g_{max} for which the maximum of $\sigma_1(g)\,\text{length}_g(\partial M)$ is achieved. Moreover, the corresponding eigenfunctions define a minimal free boundary embedding of M in B^3. They remark that when n tends to infinity, the corresponding free boundary minimal surface Σ_n converges on compact sets of B^3 to a double copy of the equatorial disk.

In our work, give an independent construction of Σ_n for n large enough using perturbation techniques. We also prove for all n large enough the existence of free boundary minimal surfaces $\tilde{\Sigma}_n$ in B^3 which have genus 1 and n boundary components, which is a new result that doesn't follow from [37].

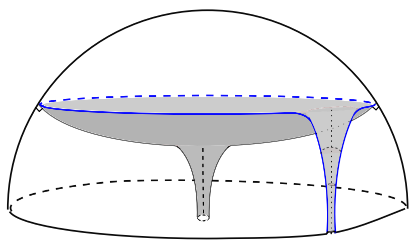
To state our main theorem, let us identify $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$ and denote by $\mathfrak{S}_n$ the group of isometries generated by

$$(z,t) \mapsto (\bar{z}, t), \quad (z,t) \mapsto (z, -t), \quad \text{and} \quad (z,t) \mapsto (z \cdot e^{\frac{2\pi i}{n}}, t).$$

Theorem 0.3.1. *There exists $n_0 \geq 0$ such that for each $n \geq n_0$ there exists a genus 0 free boundary minimal surface Σ_n and a genus 1 free boundary minimal surface $\tilde{\Sigma}_n$ which are both embedded in B^3 and meet S^2 orthogonally along n closed curves.*

Both surfaces are invariant under the action of the group $\mathfrak{S}_n$ and, as n tends to infinity, the sequence Σ_n converges to a double copy of the unit horizontal (open) disk, uniformly on compacts of B^3, while the sequence $\tilde{\Sigma}_n$ converges to a double copy of the unit horizontal (open) punctured disk, uniformly on compacts of $B^3 \setminus \{0\}$.

Our proof is in the spirit of the proof of existence of minimal surfaces in S^3 by doubling by N. Kapouleas [61], N. Kapouleas and S.D. Yang [63], and D. Wiygul [117]. Similarly to [63], our constructions work only for a large number n of boundary components. The resulting surfaces have the structure of two nearby parallel horizontal disks joined by n boundary bridges which are close to scaled down copies of half-catenoids arranged periodically along the unit horizontal great circle of S^2, and a "neck" which is close to a scaled down catenoid with vertical axis centered at the origin.



Let me give here a brief description of our construction of the family of genus 1 free boundary minimal surfaces $\tilde{\Sigma}_n$. We will see that an analogous proof with several simplifications gives the existence of a family of genus 0 free boundary minimal surfaces Σ_n.

Parametrization of the unit ball

First of all, we parametrize a neighborhood of the horizontal disk $D^2 \times \{0\}$ in the unit ball B^3 by a region in the unit cylinder $D^2 \times \mathbb{R}$ in the following way:

$$X : D^2 \times \mathbb{R} \to B^3, \quad X(z,t) = A(z,t)\,(z, B(z)\sinh t)\,,$$

where

$$B(z) = \frac{1+|z|^2}{2} \quad \text{and} \quad A(z,t) = \frac{1}{1+B(z)(\cosh t - 1)}.$$

In this parametrization, the boundary $|z| = 1$ of the unit cylinder corresponds to the boundary S^2 of the unit ball. Moreover, each leaf $t = t_0$ is a constant mean curvature spherical cap with $H = 2\sinh t_0$ which meets the boundary of the ball orthogonally (when $t = 0$ we obtain the unit disk $D^2 \times \{0\}$).

Moreover, we find that the pull-back metric $X^* g_{eucl}$ in $D^2 \times \mathbb{R}$ has the form

$$X^* g_{eucl} = A^2(z,t)\left(|dz|^2 + B^2(z)\, dt^2\right). \tag{15}$$

Vertical graphs in B^3

An important role in our construction is played by vertical graphs over the horizontal disk $D^2 \times \{0\}$ in B^3 parametrized by

$$z \in D^2 \mapsto X(z, w(z)) \in B^3 \quad \text{for} \quad w \in \mathcal{C}^2(D^2). \tag{16}$$

We find that the Taylor expansion of the mean curvature of such graphs in powers of w and derivatives of w has the form:

$$H_{gr}(w) = L_{gr}\, w + Q_{gr}(w, \nabla w, \nabla^2 w), \tag{17}$$

where

$$L_{gr} \cdot := \Delta(B\, \cdot\,) = \Delta\left(\frac{1+|z|^2}{2}\,\cdot\right)$$

is the linearized mean curvature operator and Q_{gr} is a smooth nonlinear function which satisfies $Q_{gr}(0,0,0) = DQ_{gr}(0,0,0) = D^2Q_{gr}(0,0,0) = 0$. Moreover, the graph (16) meets the boundary of the ball orthogonally if w satisfies the homogeneous Neumann boundary condition:

$$\partial_r w|_{r=1} = 0.$$

Remark that L_{gr} has a non-trivial kernel which consists of the functions $\frac{2x_1}{1+|z|^2}$ and $\frac{2x_2}{1+|z|^2}$, and corresponds to tilting the unit disk $D^2 \times \{0\}$ in B^3. The kernel can be eliminated by imposing invariance under the action of a group of rotations around the vertical axis.

Green's Function

According to A. Fraser and R. Schoen, the surfaces that we would like to construct should have the structure of connected sums of two nearby copies of the unit disk with small "bridges" that are close to half-catenoids located symmetrically on the unit circle S^1 and a small "neck" close to a catenoid centered at the origin.

As before, to get a better matching with the asymptotics of the catenoid, we first deform $D^2 \times \{0\}$ using a suitable Green's function and then perform the connected sum. We place the poles of the Green's function at $z = 0$ and at the n-th roots of unity $z_m = e^{\frac{2\pi m i}{n}}$, $m = 1, \ldots, n$, and look for the solution of the problem:

$$\begin{cases} \Delta(B\Gamma_n) = c_0\, \delta_0 & \text{in} \quad D^2, \\ \partial_r \Gamma_n = \sum_{m=1}^{n} c_n\, \delta_{z_n} & \text{on} \quad S^1, \end{cases} \tag{18}$$

for some constants $c_0, c_1, \ldots, c_n$. If we assume that Γ_n is invariant under rotations by the angle $\frac{2\pi}{n}$, then solving (18) is equivalent to finding a function G such that $\Gamma_n(z) = G(z^n)/B(z)$ and:

$$\begin{cases} \Delta G = \tilde{c}_0 \, \delta_0 & \text{in} \quad D^2, \\ \partial_r G - \frac{1}{n} G = \tilde{c}_1 \, \delta_1 & \text{on} \quad S^1. \end{cases} \tag{19}$$

for some constants $\tilde{c}_0, \tilde{c}_1$. A solution to (19) can be found explicitly. We decompose G as a sum of two functions, one of which has a singularity at $z = 0$, and the other a singularity at $z = 1$. More precisely, we find that

$$G_0(z) := -\log|z| - n$$

satisfies $\Delta G_0 = 0$ in $D^2 \setminus \{0\}$ and $\partial_r G_0 - \frac{1}{n} G_0 = 0$ on S^1. On the other hand, the function

$$G_1(z) := -\frac{n}{2} + \sum_{k=0}^{\infty} \frac{1}{n^k} \operatorname{Re} H_k(z), \quad \text{where} \quad H_k(z) := \sum_{j=1}^{\infty} \frac{z^j}{j^{k+1}},$$

is harmonic in D^2, and, since

$$H_0(z) = -\log(1 - z),$$

and

$$\partial_r \left(\operatorname{Re} H_k\right) = \operatorname{Re} H_{k-1} \quad \text{on} \quad S^1, \quad \forall k \in \mathbb{N},$$

G_1 satisfies:

$$\partial_r G_1 - \frac{1}{n} G_1 = 0 \quad \text{on} \quad S^1 \setminus \{1\}.$$

Finally, we put:

$$\Gamma_n(z) := \frac{1}{B(z)} \left(\tau_0 \, G_0(z^n) + \tau_1 \, G_1(z^n)\right),$$

where the coefficients $\tau_0, \tau_1 \in \mathbb{R}$ are carefully chosen when we "glue" the graph of Γ_n with a "catenoidal neck" and "half-catenoidal bridges". We find that Γ_n has the following expansion in the neighborhoods of $z = 0$ and $z = z_m$:

$$\Gamma_n(z) = \begin{cases} -n\left(2\tau_0 + \frac{\tau_1}{2}\right) - 2\, n\, \tau_0 \log|z| + \mathcal{O}(|z|^2 \, \log|z|), \quad \text{as } |z| \to 0 \\ \\ -n\,(2\tau_0 + \frac{\tau_1}{2}) + \tau_1 \, \log n - \tau_1 \, \log|z - z_m| \\ +\mathcal{O}(|z - z_m| \, \log|z - z_m|), \quad \text{as } |z - z_m| \to 0. \end{cases} \tag{20}$$

"Half-catenoidal bridges" and "catenoidal neck"

The role of boundary "bridges" connecting two "copies" of the unit disk could be played by minimal stripes obtained by the intersection of Euclidean catenoids centered at the n-th roots of unity with the unit sphere S^2. The difficulty of this approach is the fact that those stripes do not meet S^2 orthogonally. We prefer to embed "catenoidal bridges" in B^3 orthogonally to S^2 but loosing the minimality condition.

Let $\mathbb{C}_- := \{\zeta \in \mathbb{C} \,:\, \mathrm{Re}(\zeta) \leq 0\}$ be a half-plane and consider the conformal mappings

$$\lambda_m \,:\, \zeta \in \mathbb{C}_- \mapsto e^{\frac{2\pi i m}{n}} \frac{1+\zeta}{1-\zeta} \in \bar{D}^2$$

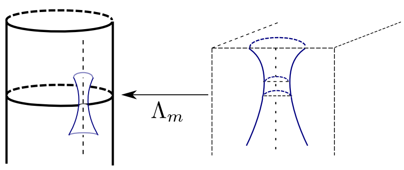
which provide local diffeomorphisms from a neighborhood of $\zeta = 0$ in $\mathbb{C}_-$ to neighborhoods of $z = z_m$ in the unit disk. Then we introduce the mappings

$$\Lambda_m \,:\, (\zeta, \tau) \in \mathbb{C}_- \times \mathbb{R} \mapsto (\lambda_m(\zeta), 2\tau) \in \bar{D}^2 \times \mathbb{R},$$

and for $\varepsilon \in (0,1)$ parametrize the m-th "catenoidal bridge" $C_{\varepsilon,m}$ in B^3 by

$$(\sigma, \theta) \in [-\sigma_\varepsilon, \sigma_\varepsilon] \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \mapsto X \circ \Lambda_m \left(\frac{\varepsilon}{2} \cosh \sigma \, e^{i\theta}, \frac{\varepsilon}{2}\, \sigma\right) \in C_{\varepsilon,m},$$

for some $\sigma_\varepsilon \gg 1$. Since the restriction of $X \circ \Lambda_m$ to horizontal planes is conformal, $C_{\varepsilon,m}$ meets the boundary of the unit ball orthogonally and is close to the truncated Euclidean half-catenoid scaled by the factor ε and centered at $z = z_m$.



On the other hand, in the neighborhood of $z = 0$, the metric $X^* g_{eucl}$ given by (15), is close to the metric $|dz|^2 + \frac{1}{4} dt^2$. For $\tilde{\varepsilon} \in (0,1)$ we define the surface

$$X^{cat}_{\tilde{\varepsilon}} \,:\, (s, \phi) \in (-s_{\tilde{\varepsilon}}, s_{\tilde{\varepsilon}}) \times S^1 \mapsto \left(\tilde{\varepsilon} \cosh s \, e^{i\phi}, 2\, \tilde{\varepsilon} s\right) \in \mathbb{R}^3, \quad \text{for some} \quad s_{\tilde{\varepsilon}} \gg 1,$$

that is minimal with respect to $|dz|^2 + \frac{1}{4}\, dt^2$. Then the surface $C_{\tilde{\varepsilon},0}$ parametrized by

$$(s, \phi) \in (-s_{\tilde{\varepsilon}}, s_{\tilde{\varepsilon}}) \times S^1 \mapsto X \circ X^{cat}_{\tilde{\varepsilon}}(s, \phi) \in C_{\tilde{\varepsilon},0},$$

is close to a truncated catenoid scaled by the factor $\tilde{\varepsilon}$.

Matching

Taking the change of variables $z = \lambda_m\left(\frac{\varepsilon}{2} \cosh \sigma \, e^{i\theta}\right)$ away from $\sigma = 0$, we can see the "half-catenoidal bridge" $C_{\varepsilon,m}$ as a bi-graph over $\{z \in D^2 \,:\, |z - z_m| > \varepsilon\}$ for the function

$$G_{\varepsilon,m} = -\varepsilon \, \log \frac{\varepsilon}{2} + \varepsilon \, \log |z - z_m| + \mathcal{O}(\varepsilon^3 \, |z - z_m|^{-2}). \tag{21}$$

On the other hand, taking the change of variables $z = \tilde{\varepsilon} \cosh s \, e^{i\phi}$ away from $s = 0$, we can see the "catenoidal neck" $C_{\tilde{\varepsilon},0}$ as a bi-graph over $\{z \in D^2 \,:\, |z| > \tilde{\varepsilon}\}$ of the function

$$G_{\tilde{\varepsilon},0} = -\tilde{\varepsilon} \, \log \frac{\tilde{\varepsilon}}{2} + \tilde{\varepsilon} \, \log |z| + \mathcal{O}(\tilde{\varepsilon}^3 \, |z|^{-2}). \tag{22}$$

Comparing (21) and (22) with the expansions of the function Γ_n in the neighborhoods of $z = z_m$ and $z = 0$ given by (20), we find that ε, $\tilde{\varepsilon}$, as well as the coefficients τ_1 and τ_2, can be expressed as functions of the number n of boundary components. Comparing the logarithmic terms, we find:

$$\tau_1 = \varepsilon \quad \text{and} \quad n\,\tau_0 = \tilde{\varepsilon},$$

while the comparison of the constant terms gives:

$$\tilde{\varepsilon} \sim \varepsilon \quad \text{and} \quad n \sim \log(1/\varepsilon).$$

Finally, we shall effectuate the "gluing" in the regions where the resting terms are minimized, that is when

$$|z| = \tilde{\varepsilon} \cosh s_{\tilde{\varepsilon}} \sim \varepsilon^{\frac{1}{2}} \quad \text{and} \quad |z - z_m| \sim \varepsilon \cosh \sigma_\varepsilon \sim \varepsilon^{\frac{2}{3}}.$$

Remark 0.3.1. *These computations, together with the fact that constant functions are not in the kernel of the linearized mean curvature operator L_{gr} give an idea why our construction works only for n large enough.*

Perturbation argument

At this stage, using the connected sum construction, we obtain for all n large enough a genus 1 surface that is embedded in B^3 and meets ∂B^3 orthogonally along n boundary components. We will denote this surface by $\mathcal{A}_n$ and refer to it as *approximate solution.* The next step is to perturb $\mathcal{A}_n$ into a minimal surface. Take a vector field Ξ in $\bar{B}^3$ transverse to $\mathcal{A}_n$ and let $\xi : \bar{B}^3 \times (0,1) \to \bar{B}^3$ be the associated flow:

$$\frac{d\xi}{dt} = \Xi(\xi(\cdot,t)), \quad \xi(p,0) = p, \quad \text{for} \quad p \in \bar{B}^3.$$

We shall choose Ξ in such a way that the surfaces $\mathcal{A}_{n,t} := \xi_t(\mathcal{A}_n)$ are embedded in B^3 for all t small enough and meet S^2 orthogonally along $\partial\mathcal{A}_{n,t}$. Finally, for $w \in \mathcal{C}^{2,\alpha}(\mathcal{A}_n)$, we consider the surface $\mathcal{A}_n(w)$ parametrized by

$$p \in \mathcal{A}_n \mapsto \xi(p, w(p)) \in \mathcal{A}_n(w).$$

Then $\mathcal{A}_n(w)$ meets the boundary of the unit ball orthogonally when w satisfies the homogeneous Neumann boundary condition on ∂A_n:

$$g_n(\nabla^{g_n} w, N_{\partial\mathcal{A}_n}) = 0,$$

where g_n is the metric induced on $\mathcal{A}_n$ from the Euclidean metric and $N_{\partial\mathcal{A}_n}$ is a unit normal to $\partial\mathcal{A}_n$ in $\mathcal{A}_n$.

As in the construction described in the subsection 0.1.4, the Taylor expansion of the mean curvature of $\mathcal{A}_n(w)$ in powers of w and derivatives of w has the form:

$$H(\mathcal{A}_n(w)) = H(\mathcal{A}_n) + \mathcal{L}_n\, w + \mathcal{Q}_n(w, \nabla w, \nabla^2 w),$$

where $H(\mathcal{A}_n)$ is the mean curvature of the approximate solution, $\mathcal{L}_n$ is the linearized mean curvature operator about $\mathcal{A}_n$, and $\mathcal{Q}_n$ is a smooth non-linear function, which satisfies $\mathcal{Q}_n(0,0,0) = D\mathcal{Q}_n(0,0,0) = 0$. Our goal then is to solve the equation:

$$\mathcal{L}_n\, w = -H(\mathcal{A}_n) - \mathcal{Q}_n(w).$$

Working in appropriate functional spaces, we show that the norm of $H(\mathcal{A}_n)$ tends to 0 when n tends to infinity as e^{-na} for some $a > 1$. Then choosing an appropriate transverse vector field Ξ, we show that the operator $\mathcal{L}_n$ has a right inverse which is not uniformly bounded in n, but explodes as $e^{n\nu}$, for $\nu \ll 1$ that we choose. The result follows from Banach fixed point theorem for contracting mappings applied to $w \mapsto -\mathcal{L}_n^{-1}\left(H(\mathcal{A}_n) + \mathcal{Q}_n(w)\right)$.

Linear analysis

In conclusion, let me say a few words about the properties of the linear operator $\mathcal{L}_n$. We show that the problem of small eigenvalues (eigenvalues that tend to 0 as fast as n tends to infinity) can be solved by imposing symmetries to the constructed surface. More precisely, in the regions of the "half-catenoidal bridges", $\mathcal{L}_n$ is close to the Jacobi operator about the half-catenoid. Then the small eigenvalues generated by rotations and translations of the catenoid are eliminated by imposing the symmetries $w(\sigma,\theta) = w(-\sigma,\theta) = w(\sigma, 2\pi-\theta)$ and the homogeneous Neumann boundary condition $\partial_\theta w|_{\theta\in\{\frac{\pi}{2},\frac{3\pi}{2}\}} = 0$. All the other small eigenvalues are eliminated by imposing invariance under rotations by the angle $\frac{2\pi}{n}$. Finally, in the region where $\mathcal{A}_n$ is parametrized by a domain of the unit disk, $\mathcal{L}_n$ is close to the operator L_{gr} defined in (17). So, we are interested in the solutions of the linear problem:

$$\begin{cases} \Delta(Bw) = f & \text{in} \quad D^2 \setminus \{0\} \\ \partial_r w = 0 & \text{on} \quad S^1 \setminus \{z_1, \ldots, z_n\}. \end{cases} \tag{23}$$

After the change of variables $z \mapsto z^n$, we find that (23) is equivalent to a Poisson's equation with homogeneous Robin boundary data:

$$\begin{cases} \Delta\, W = F & \text{in} \quad D^2 \setminus \{0\} \\ \partial_r W - \frac{1}{n}\, W = 0 & \text{on} \quad S^1 \setminus \{1\}. \end{cases} \tag{24}$$

A solution to (24) can be found using the fact that for $n \geq 2$ the associated operator has no bounded kernel. On the other hand, when n tends to infinity, (24) converges to the Poisson's equation with homogeneous Neumann boundary data, which, in its turn, has a nontrivial kernel. This (together with the presence of the eigenvalues generated by the dilation of the catenoid) explains why the inverse operator "slightly" explodes when n tends to infinity. However, this does not prevent us from applying Banach fixed point theorem in a ball the radius of which tends to 0 when n tends to infinity much faster than the norm of $\mathcal{L}_n^{-1}$ explodes.

0.4 Chapter 3: Nonconvex constant mean curvature surfaces in Riemannian manifolds

In the third chapter, we propose a construction in a Riemannian 3-manifold (M, g) of a family of nonconvex topological spheres with large constant mean curvature.

These surfaces are obtained as connected sums of two geodesic spheres of equal small radii tangent at one point. Our construction is possible when the scalar curvature function $\mathcal{R}$ of the ambient manifold has a critical point o_{cr} such that the Hessian of $\mathcal{R}$ at o_{cr} has a simple nonzero eigenvalue.

More precisely, let (M, g) be a smooth 3-dimensional Riemannian manifold. Let $o_{cr} \in M$ be a critical point of the scalar curvature function $\mathcal{R}$, $\lambda \neq 0$ a simple non-zero eigenvalue of $\mathrm{Hess}_{o_{cr}}\mathcal{R}$, and $v_\lambda \in T_{o_{cr}}M$ the associated unit eigenvector. Take $\varepsilon \in \mathbb{R}_+$ small enough and consider the union $\mathfrak{S}_\#(\varepsilon, o_{cr}, v_\lambda)$ of two geodesic spheres of radius ε tangent at o_{cr}, with centers located symmetrically with respect to o_{cr} on the geodesic passing through o_{cr} with velocity v_λ. Our result reads:

Theorem 0.4.1. *There exist $\varepsilon_* \in (0,1)$ and for every $\varepsilon \in (0, \varepsilon_*)$ a surface $\mathfrak{S}_\varepsilon$ of constant mean curvature equal to $\frac{2}{\varepsilon}$ such that the Hausdorff distance between $\mathfrak{S}_\varepsilon$ and $\mathfrak{S}_\#(\varepsilon, o_{cr}, v_\lambda)$ is bounded by a constant times ε^2. The surface $\mathfrak{S}_\varepsilon$ is embedded if $\lambda < 0$ and immersed if $\lambda > 0$.*

Let me give a short description of our construction.

Pseudo constant mean curvature spheres

Our goal is to construct a constant mean curvature surface which is close to a connected sum of two geodesic spheres in M. The first difficulty then is the fact that the summands in our construction do not have constant mean curvature. Let $\mathcal{S}_\varepsilon(p)$ be a geodesic sphere of radius ε. The first step would be to perturb $\mathcal{S}_\varepsilon(p)$ into some CMC surface. Unfortunately, as observed R. Ye [118], F. Pacard and X. Xu [95] and S. Nardulli [90], in general, this is not possible, since the Jacobi operator about $\mathcal{S}_\varepsilon(p)$ is close to the operator $\varepsilon^{-2}(\Delta_{S^2} + 2)$ which has a nontrivial kernel. As it is explained in the subsection 0.1.5, in some sense, the best we can do is to perturb $\mathcal{S}_\varepsilon(p)$ into a pseudo constant mean curvature sphere $\Sigma_\varepsilon(p)$ whose mean curvature is constant up to an element of $\mathrm{Ker}(\Delta_{S^2} + 2)$:

$$H(\Sigma_\varepsilon(p)) - \frac{2}{\varepsilon} = \langle A, \Theta \rangle, \quad \Theta \in S^2,$$

for some vector $A \in \mathbb{R}^3$. Here and below we denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^3$. Moreover, one can explicitly calculate [118]:

$$A = -\frac{2\,\pi\,\varepsilon^2}{15}\,\nabla^g \mathcal{R}(p) + \mathcal{O}(\varepsilon^4).$$

“Catenoidal neck”

The next step is to describe surfaces that can play the role of the “neck” in the connected sum construction. As it is pointed out in the subsection 0.1.4, in the Euclidean space this role is usually played by an element of the family catenoids:

$$C_\eta \,:\, (s,\phi) \in [-s_\eta, s_\eta] \times S^1 \mapsto (\eta \, \cosh s \, \cos\phi, \eta \, \cosh s \, \sin\phi, \eta \, s)\,,$$

where the parameter $\eta \in (0,1)$ is referred to as the “neck size”. In the Riemannian case, one can use geometric properties of the catenoid, given an embedding of C_η in M which, at least for η small enough, is close to the identity in some chosen coordinates.

Consider Λ a smooth embedded surface in M, and let N_Λ denote a unit normal to Λ. Given $q \in \Lambda$, then the mapping

$$F_\Lambda(q', z) = \exp_{q'}\left(z\, N_\Lambda(q')\right)$$

defines a diffeomorphism from a neighborhood of $(q,0)$ in $\Lambda \times \mathbb{R}$ to a neighborhood of q in M. Now let $(y^1, y^2) \mapsto \zeta(y^1, y^2)$ be local coordinates on Λ with the origin at q. The mapping

$$F_{\Lambda,q}(y^1, y^2, z) = F_\Lambda(\zeta(y^1, y^2), z)$$

defines a local diffeomorphism from a neighborhood of 0 in $\mathbb{R}^3$ to a neighborhood of q in M and is referred to as *Fermi coordinates*. We call the surface $\mathfrak{C}_{\eta,q}$:

$$(s,\phi) \in [-s_\eta, s_\eta] \times S^1 \mapsto F_{\Lambda,q}\,(\eta \, \cosh s \, \cos\phi, \eta \, \cosh s \, \sin\phi, \eta \, s) \in \mathfrak{C}_{\eta,q}$$

a “catenoidal neck”.

In our case, it is convenient to take Λ to be a geodesic disk of small radius. More precisely, first we fix the “axis” of the “neck”, i.e. a minimizing geodesic γ in M and a point $q_0 \in \gamma$, and then consider the geodesic disk Λ centered at q_0 and orthogonal to γ. For all $q \in \Lambda$ we have a “catenoidal neck” $\mathfrak{C}_{\eta,q}$, with its “axis” “parallel” to γ.

Connected sum construction in M

Because of the absence of isometries in a generic Riemannian manifold M (namely the absence of translations and rotations), we cannot apply directly the procedure described for the euclidean space in the paragraph 0.1.4 to perform a connected sum of two surfaces in M. We could imagine an analogous procedure, if for a given family of surfaces, parametrized, for example, by their location in M, we could assign to all $d > 0$ small enough, a pair of elements the distance between which (in the sense of the distance between closed disjoint sets in the metric space M) is equal to d.

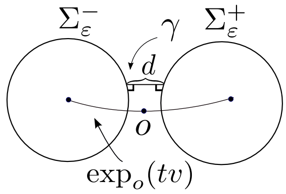
Take $d < \varepsilon$. It is easy to choose a pair of geodesic spheres of radius ε the distance between which is equal to d. For this it is sufficient to fix a point $o \in M$, a vector $v \in T_oM$ with $\|v\|_g = 1$ and to place the centers of the spheres at $\exp_o\left(\pm \left(\varepsilon + \frac{d}{2}\right) v\right)$.

In the next result we show that an analogous procedure also works for a family of pseudo CMC spheres since the last ones are small perturbations of geodesic spheres. More precisely, consider the family

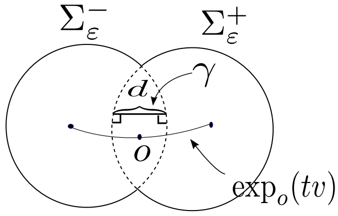
$$\Sigma^{\pm}_{\varepsilon,t} = \Sigma_{\varepsilon}\left(\exp_o(\pm\, tv)\right).$$

Then we have

Lemma 0.4.1. *For all $d \in (0, 1/2\,\varepsilon)$ there exists a unique $t \in (\varepsilon, 2\varepsilon)$ such that the distance between the surfaces $\Sigma^{\pm}_{\varepsilon} := \Sigma^{\pm}_{\varepsilon,t}$ is equal to d and is realized by a unique geodesic γ, a priori different from $t \mapsto \exp_o(tv)$.*

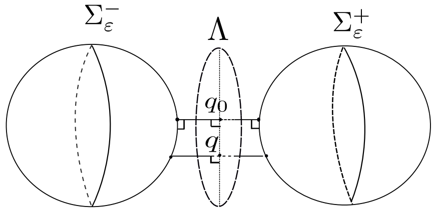


Remark 0.4.1. *We will see that we also need to perform a connected sum of a pair of intersecting "pseudo bubbles". In this case, we cannot talk about the "distance" between the surfaces, but prove that there exist two intersecting pseudo constant mean curvature spheres $\Sigma^{\pm}_{\varepsilon}$, such that the interior of $\Sigma^+_{\varepsilon} \cap \Sigma^-_{\varepsilon}$ is crossed by a unique minimizing geodesic γ of length d which intersects $\Sigma^{\pm}_{\varepsilon}$ orthogonally.*



In what follows, we will have two connected sums, one embedded and one immersed, and we will see in the final argument that, depending on the sign of the eigenvalue $\lambda \neq 0$ of the Hessian of the scalar curvature $\mathcal{R}$, one of these connected sums can be perturbed into a constant mean curvature surface.

Next, we describe the "gluing procedure" between two pseudo CMC spheres and a "catenoidal neck". Let $\gamma : [0, 1] \to M$ be the geodesic that realizes the distance between $\Sigma^{\pm}$ and consider a small geodesic disc Λ orthogonal to γ at $q_0 = \gamma(1/2)$. For all $\eta > 0$ small enough and all $q \in \Lambda$ there is a "catenoidal neck" $C_{\eta,q}$ which is an embedding of the euclidean catenoid via Fermi coordinates. Notice that when $q = q_0$, the "axis" of the "neck" is orthogonal to $\Sigma^{\pm}_{\varepsilon}$, and otherwise we create a small angle between the normals to $\Sigma^{\pm}_{\varepsilon}$ and the "axis".



Using the ideas of R. Mazzeo, F. Pacard and D. Pollack [77], to get a better matching with the asymptotics of the catenoid, first we deform $\Sigma_\varepsilon^\pm$ with Green's functions associated to their Jacobi operators and then perform the connected sum.

Green's functions

Let $J_{\Sigma_\varepsilon^\pm}$ be the Jacobi operators about the pseudo CMC spheres $\Sigma_\varepsilon^\pm$. We would like to define and to study the Green's functions $\Gamma^\pm$ associated to $J_{\Sigma_\varepsilon^\pm}$ with poles at $p^\pm \in \Sigma_\varepsilon^\pm$, namely, the solutions to the problem:

$$J_{\Sigma_\varepsilon^\pm}\,\Gamma^\pm = -2\pi\,\delta_{p^\pm} \quad \text{in} \quad \Sigma_\varepsilon^\pm \tag{25}$$

where $\delta_{p^\pm}$ are the Dirac masses at $p^\pm$. On the other hand, parametrizing $\Sigma_\varepsilon^\pm$ by the Euclidean sphere S^2, and we find that $J_{\Sigma_\varepsilon^\pm}$ satisfy:

$$J_{\Sigma_\varepsilon^\pm} = \varepsilon^{-2}\,(\Delta_{S^2}+2) + L, \quad \text{where} \quad \|L\,u\|_{\mathcal{C}^{0,\alpha}(S^2)} \le c\,\|u\|_{\mathcal{C}^{2,\alpha}(S^2)}.$$

Unfortunately, the fact that the operator $\Delta_{S^2}+2$ has a nontrivial kernel prevents us from finding directly $\Gamma^\pm$ and getting reasonable estimates which would be uniform in ε when ε tends to 0. However, instead of (25), we can solve the problem:

$$J_{\Sigma_\varepsilon^\pm}\,\Gamma^\pm = -2\pi\,\delta_{p^\pm} + \langle B^\pm, \Theta\rangle, \tag{26}$$

for some vectors $B^\pm \in \mathbb{R}^3$. Moreover, an explicit computation gives

$$B^\pm = \frac{\varepsilon^{-2}}{2}\,\Theta(p^\pm) + \mathcal{O}(1).$$

Approximate solution

Let the poles of the Green's functions $p^\pm$ be the points of intersection of the geodesic γ with the pseudo bubbles. We can parametrize $\Sigma_\varepsilon^\pm$ in the neighborhood of $p^\pm$ as normal graphs over the geodesic disc Λ. In geodesic normal coordinates in Λ centered at q we have:

$$\Gamma^\pm(y) = c^\pm \pm \log|y| + \mathcal{O}(|y|\log|y|). \tag{27}$$

On the other hand, the "catenoidal neck" $\mathfrak{C}_{\eta,q}$ can be seen as a bi-graph over Λ of the function G_η:

$$G_\eta = -\eta\,\log\frac{\eta}{2} + \eta\,\log|y| + \mathcal{O}\left(\eta^3\,|y|^{-2}\right). \tag{28}$$

Finally, comparing (27) and (28), we show that for all $o \in M$, $v \in T_oM$ with $\|v\|_g = 1$, $\eta > 0$, and $q \in \Lambda$ with η and $\mathrm{dist}(q, q_0)$ small enough, one can choose the "distance" $d(o, v, \eta, q)$ between $\Sigma_\varepsilon^\pm$ in such a way that the constant terms in the expansions of $\eta\,\Gamma^\pm$ and $G_{\eta,q}$ match exactly. This distance satisfies $|d - 2\,\eta\log(2/\eta)| \le c\,\eta$. Next, we "glue" the graphs together with the help of a cut-off function at $|y| \sim \eta^{\frac{3}{4}}$.

At this stage, for ε small enough, the resulting surface, which we will denote by $\mathcal{A}_\varepsilon$, depends on 8 geometric parameters: the point $o \in M$ (3 degrees of freedom), the unit vector $v \in T_oM$ (2 degrees of freedom), the "neck size" η (1 degree of freedom), and the "location" q of the "axis" of the "neck" in the geodesic disk Λ (2 degrees of freedom).

Perturbation argument

As a next step, we shall perturb $\mathcal{A}_\varepsilon$ into a constant mean curvature surface provided ε is chosen small enough. Let Ξ be a smooth vector field defined in a neighborhood of $\mathcal{A}_\varepsilon$ in M and transverse to $\mathcal{A}_\varepsilon$, and let $\xi : M \times (0,1) \to M$ be the associated flow:

$$\frac{d\xi}{dt} = \Xi(\xi(\cdot,t)), \quad \xi(p,0) = p, \quad \forall p \in M.$$

Take a function $w \in \mathcal{C}^2(\mathcal{A}_\varepsilon)$ small enough and let us denote by $\mathcal{A}_\varepsilon(w)$ the surfaces parametrized by $p \in \mathcal{A}_\varepsilon \mapsto \xi(p, w(p)) \in \mathcal{A}_\varepsilon(w)$. Then, the expansion of the mean curvature of $\mathcal{A}_\varepsilon(w)$ in powers of w and derivatives of w can be written in the form:

$$H(\mathcal{A}_\varepsilon(w)) = H(\mathcal{A}_\varepsilon) + \mathcal{L}_\varepsilon\, w + \mathcal{Q}_\varepsilon(w, \nabla w, \nabla^2 w),$$

where $H(\mathcal{A}_\varepsilon)$ is the mean curvature of the approximate solution, $\mathcal{L}_\varepsilon$ is the linearized mean curvature operator about $\mathcal{A}_\varepsilon$ and $\mathcal{Q}_\varepsilon$ is a non-linear function which satisfies:

$$\mathcal{Q}_\varepsilon(0,0,0) = D\mathcal{Q}_\varepsilon(0,0,0) = 0.$$

Our goal is to solve the equation:

$$\mathcal{L}_\varepsilon\, w = \frac{2}{\varepsilon} - H(\mathcal{A}_\varepsilon) - \mathcal{Q}_\varepsilon(w, \nabla w, \nabla^2 w).$$

If $\mathcal{L}_\varepsilon$ were an invertible linear operator with its inverse uniformly bounded in ε when ε tends to 0, we could apply Banach fixed point theorem to $w \mapsto \mathcal{L}_\varepsilon^{-1}\left(\frac{2}{\varepsilon} - H(\mathcal{A}_\varepsilon) - \mathcal{Q}_\varepsilon(w)\right)$ in a ball of an appropriate Banach space, where the radius of the ball would be determined by the norm of $H(\mathcal{A}_\varepsilon) - \frac{2}{\varepsilon}$.

First, we study the mean curvature of $\mathcal{A}_\varepsilon$. We will assume that η and $\mathrm{dist}(q, q_0)$ are bounded by a suitable power of ε. Using the minimality of the catenoid in the Euclidean space, we show that in the "neck region" the L^∞ norm of $H(\mathcal{A}_\varepsilon)$ tends to 0 when ε tends to 0. On the other hand, will make $H(\mathcal{A}_\varepsilon) - \frac{2}{\varepsilon}$ very small in a suitable topology which will take in the account the fact that the area of "neck region" tends to 0 much faster than the area of the rest of $\mathcal{A}_\varepsilon$ when ε tends to 0.

Let $\chi^\pm \in \mathcal{C}^\infty(\mathcal{A}_\varepsilon)$ be some cut-off functions supported in the regions parametrized by the pseudo CMC spheres. Then, using the expression for the mean curvature of $\Sigma_\varepsilon^\pm$ and the equation satisfied by the Green's functions, we find:

$$H(\mathcal{A}_\varepsilon) - \frac{2}{\varepsilon} = \chi^+ \langle C^+, \Theta\rangle + \chi^- \langle C^-, \Theta\rangle + \mathcal{H}_\varepsilon,$$

where $C^\pm \in \mathbb{R}^3$ satisfy:

$$C^\pm = -\frac{2\pi\,\varepsilon^2}{15}\,\nabla\mathcal{R}(o^\pm) + \frac{\varepsilon^{-2}\,\eta}{2}\,\Theta(p^\pm) + \mathcal{O}(\varepsilon^4), \tag{29}$$

and $\mathcal{H}_\varepsilon$ tends very fast to 0 when ε tends to 0 in the appropriate topology. Here $o^\pm$ are the "centers" of the pseudo bubbles $\Sigma_\varepsilon^\pm$ and $p^\pm$ are the poles of the Green's functions.

Next, we study the properties of the linear operator $\mathcal{L}_\varepsilon$. We remark the presence of small eigenvalues of $\mathcal{L}_\varepsilon$ (eigenvalues that tend to 0 as fast as ε tends to 0) which can be identified knowing the structure of $\mathcal{L}_\varepsilon$.

We find that in the regions parametrized by pseudo CMC spheres, $\mathcal{L}_\varepsilon$ is close to the operator $\Delta_{S^2} + 2$ which has a 3 - dimensional kernel consisting of the coordinate functions $\Theta^1, \Theta^2, \Theta^3$.

On the other hand, in the "catenoidal neck" region, $\mathcal{L}_\varepsilon$ is close to the Jacobi operator about the Euclidean catenoid. Recall, that the Euclidean catenoid is degenerate and in particular has 2 nontrivial Jacobi fields given by the functions $\frac{\cos\phi}{\cosh s}$ and $\frac{\sin\phi}{\cosh s}$ which decay at infinity and correspond to the horizontal translations of the catenoid.

We denote by $\mathfrak{K}_\varepsilon$ the space of eigenfunctions of $\mathcal{L}_\varepsilon$ corresponding to small eigenvalues, and following N. Kapouleas, we refer to it as *approximate kernel* of $\mathcal{L}_\varepsilon$. Using the ideas described above, we find that the dimension of $\mathfrak{K}_\varepsilon$ is equal to 8, matching exactly the number of free parameters in our construction.

We will use a Lyapunov-Schmidt reduction argument, applying Banach fixed point theorem in the space of functions orthogonal to $\mathfrak{K}_\varepsilon$, to perturb $\mathcal{A}_\varepsilon$ into a surface whose mean curvature H satisfies:

$$H - \frac{2}{\varepsilon} = \sum_{i=1}^{n} A^i\,\Phi_i, \quad \text{where} \quad \mathfrak{K}_\varepsilon = \mathrm{span}\{\Phi_i,\ i = 1, \dots, 8\}, \tag{30}$$

provided ε is chosen small enough.

Choice of the parameters

In the final argument, we will explain how to choose the 8 geometric parameters appropriately to ensure that $H = \frac{2}{\varepsilon}$. The coefficients A^i in (30) depend continuously on $o \in M$, $v \in T_oM$, $\eta \in (0,1)$ and $a \in \mathbb{R}^2$. We show that for ε small enough the system of equations

$$A^i = 0,\ i = 1, \dots, 8, \tag{31}$$

can be written in the form $(Id + F)(\cdot) = 0$ for a function F bounded uniformly on o, v, η, a, and apply the Schauder's fixed point theorem in a ball of $\mathbb{R}^8$ in order to find a solution.

First 6 equations. We define the basis $\{\Phi_i\}$ of the approximate kernel $\mathfrak{K}_\varepsilon$ in such a way that its first 6 elements are close to the elements of

$$\mathrm{Ker}(\Delta_{S^2} + 2) = \mathrm{span}\{\Theta^1, \Theta^2, \Theta^3\},$$

defined in the regions parametrized by the pseudo bubbles. So, using the structure of the mean curvature of $\mathcal{A}_\varepsilon$, given by (29), we write the first 6 equations in (31) in the form:

$$-\frac{2\,\pi\,\varepsilon^2}{15}\,\nabla^g\mathcal{R}(o^\pm) + \frac{1}{2}\eta\,\varepsilon^{-2}\,\Theta(p^\pm) + \varepsilon^4 F(o, v, \eta, a) = 0. \tag{32}$$

With a slight abuse of notations, we can write:

$$o^{\pm} = o \pm \varepsilon\, v + \mathcal{O}(\varepsilon^2), \qquad \begin{array}{ll} p^{\pm} = \mp v + \mathcal{O}(\varepsilon^2), & \text{when } \mathcal{A}_\varepsilon \text{ is embedded} \\ p^{\pm} = \pm v + \mathcal{O}(\varepsilon^2), & \text{when } \mathcal{A}_\varepsilon \text{ is immersed.} \end{array}$$

First, let us assume first that $\mathcal{A}_\varepsilon$ is embedded and take o_{cr} a critical point of the scalar curvature: $\nabla^g \mathcal{R}(o_{cr}) = 0$. Then (32) is equivalent to the following 6 equations:

$$\begin{cases} -\frac{2\pi\,\varepsilon^2}{15}\,\mathrm{Hess}_{o_{cr}}\mathcal{R}(o - o_{cr} - \varepsilon\, v) - \frac{1}{2}\,\eta\,\varepsilon^{-2}\, v + \varepsilon^4\, F(o, v, \eta, a) = 0, \\ -\frac{2\pi\,\varepsilon^2}{15}\,\mathrm{Hess}_{o_{cr}}\mathcal{R}(o - o_{cr} + \varepsilon\, v) + \frac{1}{2}\,\eta\,\varepsilon^{-2}\, v + \varepsilon^4\, F(o, v, \eta, a) = 0. \end{cases} \tag{33}$$

This gives

$$o - o_{cr} + \varepsilon^2\, F(o, v, \eta, a) = 0.$$

Next, let v_λ be the unit eigenvector corresponding to a simple eigenvalue $\lambda \neq 0$ of $\mathrm{Hess}_{o_{cr}}\mathcal{R}$. If we write

$$v = v_\lambda + \tilde{v},$$

then the projection of (33) to v_λ gives:

$$\eta + \frac{4\,\pi\,\lambda\,\varepsilon^5}{15} + \varepsilon^6\, F(o, v, \eta, a) = 0.$$

In particular we see that since $\eta > 0$, the solution exists only if $\lambda < 0$. In the case where $\lambda > 0$, we should choose the immersed approximate solution.

Finally, projecting (33) on the subspace of $\mathbb{R}^3$ orthogonal to v_λ, gives the equation:

$$(\mathrm{Hess}_{o_{cr}} - \lambda\,\mathrm{Id})\ \tilde{v} + \varepsilon\, F(o, v, \eta, a) = 0,$$

which can be rewritten in the form:

$$\tilde{v} + \varepsilon\, F(o, v, \eta, a) = 0.$$

Thus if the solution of (31) exists, it satisfies:

$$\mathrm{dist}_g(o, o_{cr}) \leq c\,\varepsilon^2, \quad \angle(v, v_\lambda) \leq c\,\varepsilon, \quad \left|\eta - \mathrm{sign}(\lambda)\,\lambda\,\varepsilon^5\right| \leq c\,\varepsilon^6.$$

Last 2 equations. To understand the structure of the last two equations, we project (31) on the functions Φ_7 and Φ_8 supported in the neck region and close to $\frac{\cos\phi}{\cosh s}$ and $\frac{\sin\phi}{\cosh s}$. Taking the change of coordinates

$$y = \eta\, \cosh s(\cos\phi, \sin\phi),$$

we find that away from $s = 0$, we have $\Phi_{6+i} \approx \frac{\eta\, y^i}{|y|^2}$, $i = 1, 2$.

To explain why these equations can be written in the form $(\mathrm{Id} + F)(\cdot) = 0$, we propose to consider the following example.

Let P_0 be the horizontal plane in R^3 and C_η the catenoid scaled by the factor η with vertical axis centered at the origin. Recall, that C_η can be written as a bi-graph over $\{y \in P_0 \, : \, |y| > \eta\}$ of the function

$$G_\eta(y) = \log \frac{2}{\eta} + \eta \, \log |y| + \mathcal{O}(\eta^3 \, |y|^{-2}).$$

On the other hand, let $P^\pm$ be two planes parametrized as graphs over P_0 of the affine functions

$$u^\pm(y) = \pm \, \log \frac{2}{\eta} + c_1^\pm \, y^1 + c_2^\pm \, y^2.$$

Take $\rho > 0$ and let $D^2(\rho)$ be a unit disk in P_0 of radius ρ centered at the origin. We denote by χ a cut-of function which satisfies

$$\chi \equiv 0 \quad \text{in} \quad D^2(\rho/2) \quad \text{and} \quad \chi \equiv 1 \quad \text{in} \quad P_0 \setminus D^2(\rho).$$

Finally, we remark that the mean curvature of the surface parametrized by

$$\big(y, \chi(y) \, u^\pm(y) \pm (1 - \chi(y)) \, G_\eta(y)\big)$$

is equal to 0 everywhere but $D^2(\rho) \setminus D^2(\rho/2)$. On the other hand, for η small enough, in this region the largest terms in the projection of the mean curvature to $\frac{y^i}{|y|^2}$ are given by

$$\begin{aligned} \int_{D^2(\rho)\setminus D^2(\frac{\rho}{2})} \Delta \left(\chi \, \left(u^+ - u^-\right)\right) \frac{y^i}{|y|^2} &= \int_{\partial D^2(\rho)} \partial_r \left(\chi \, \left(u^+ - u^-\right)\right)\big|_{r=2\rho} \, \frac{y^i}{|y|^2} \, d\phi \qquad (34) \\ & \int_{\partial D^2(\rho)} \left(\chi \, \left(u^+ - u^-\right)\right) \, \partial_r \left(\frac{y^i}{|y|^2}\right)\bigg|_{r=2\rho} d\phi \\ &= \frac{4\pi}{\rho} \, \left(c_i^+ - c_i^-\right). \end{aligned}$$

In particular, we see that the largest terms in this projection are determined by the slopes of the planes $P^\pm$.

Let us go back to our construction. We can see the regions of the approximate solution where we effectuate the gluing, as graphs over the geodesic disc Λ of the functions

$$u^\pm(y) = u^\pm(q) + \nabla u^\pm(q) \, y + \mathcal{O}(|y|^2), \qquad \left(u^\pm(q_0) = \pm\frac{d}{2}, \quad \nabla u^\pm(q_0) = 0\right).$$

In particular, the role of the slopes will be played by $\nabla u^\pm(q)$ which are nontrivial when $q \neq q_0$, or in other words, when we vary the angle between the "axis" of the "neck" and the normals to the pseudo bubbles in the connected sum construction. A calculation similar to (34) gives the equations:

$$\frac{\nabla u^+(q) - \nabla u^-(q)}{\rho} + F(o, v, \eta, a) = 0,$$

which, using the fact that u^+ has a local maximum and u^- a local minimum at q_0, can be written in the form:

$$\mathrm{Hess}_{q_0}\left(u^+ - u^-\right)(q - q_0) + \rho\, F(o, v, \eta, a) = 0.$$

In our construction, we take $\rho = \varepsilon^4$ and roughly speaking we obtain $\mathrm{dist}(q, q_0) \leq c\,\varepsilon^4$.

Finally, by Schauder's fixed point theorem, there exists a set of parameters (o, v, η, a) for which $A^i(o, v, \eta, a) = 0$. The corresponding surface, which we denote by $\mathfrak{S}_\varepsilon$, has constant mean curvature and is embedded when $\lambda < 0$ and immersed with self-intersections when $\lambda > 0$. Finally, let $\mathfrak{S}_\#(\varepsilon, o_{cr}, v_\lambda)$ be the union of two geodesic spheres $\mathcal{S}_\varepsilon(\exp_{o_{cr}}(\pm\,\varepsilon v_\lambda))$ tangent at o_{cr}. Then the Hausdorff distance between $\mathfrak{S}_\varepsilon$ and $\mathfrak{S}_\#(\varepsilon, o_{cr}, v_\lambda)$ is bounded by a constant times ε^2.

0.5 Perspectives of future work

In conclusion, I would like to say a few words about further possible developments of our results.

The proposed definition of constant mean curvature submanifolds of arbitrary codimension introduces a whole new class of geometrical objects which are worth studying, even in $\mathbb{R}^3$. A curve in $\mathbb{R}^3$ has constant mean curvature according to the above definition, if it is the boundary of a minimal surface and if its extrinsic curvature is equal to the geodesic curvature and is constant. Following the work of J.C.C. Nitsche on free boundary minimal surfaces [88], one can prove that the only compact constant mean curvature curve that bounds a topological disk, is a circle S^1 bounding a flat disk D^2. Unfortunately, further topological classification appears to be a difficult task. It is likely that there exist no constant mean curvature compact curves bounding a minimal surfaces of topological type of an annulus, but we have no proof of this fact. There are many examples of noncompact minimal surfaces bounded by constant mean curvature curves: minimal surfaces bounded by straight lines (the examples of Riemann in $\mathbb{R}^3$ and B. Daniel [18] in $\mathbb{H}^3$), a portion of the plane contained between two parallel lines or a portion of the helicoid contained between two parallel helices. To give further evidence that this notion of constant mean curvature surfaces is the right one for codim > 1, we would like to prove the existence of singly periodic examples paralleling the construction of Delaunay surfaces (which can be obtained by gluing infinitely many spheres arranged along an axis using small "catenoidal necks"), which could be obtained by gluing techniques from flat disks and pieces of helicoids contained between two straight lines.

We hope that the second construction can be generalized to prove the existence of higher genus free boundary minimal surfaces in B^3.

Finally, the third construction should generalize to the case of any finite number of spheres which would give examples of constant mean curvature surfaces of arbitrary genus in Riemannian manifolds.

0.6 Introduction aux résultats de thèse en français

Chapitre 1: Problèmes isopérimétriques en codimension quelconque

Dans le premier chapitre, qui est un travail en collaboration avec R. Mazzeo et F. Pacard, on propose une généralisation de la notion classique d'hypersurface à courbure moyenne constante à des sous-variétés de codimension quelconque. On prouve également l'existence dans des variétés riemanniennes compactes de sous-variétés à courbure moyenne constante (dans le sens introduit) qui sont des perturbations des sphères géodésiques de petit volume.

Soit K une sous-variété plongée d'une variété riemannienne compacte (M^{m+1}, g). Par définition, le vecteur courbure moyenne de K est défini comme la trace de la seconde forme fondamentale de K:

$$h_K(X,Y) = \pi_{NK}\nabla_X Y, \quad \text{pour} \quad \mathrm{X,Y} \in TK, \quad \text{et} \quad H_K = \operatorname{Tr} h_K,$$

où ∇ est la connexion de Levi-Civita sur M associée à la métrique g et π_{NK} est la projection orthogonale $T_K M \to NK$ sur le fibré normale de K. En codimension 1, lorsque $\dim(NK) = 1$, on dit que K est à courbure moyenne constante quand la fonction courbure moyenne $g(H_K, N_K)$ est constante. Il existe déjà plusieurs extensions de la notion de courbure moyenne constante en codimension supérieure, parmi lesquelles, celles qui demandent que le champ de vecteurs H_K soit parallèle où harmonique. En s'appuyant sur les idées de Almgren [5], on propose d'adapter une définition directement variationnelle.

F. Almgren [5] a généralisé le problème isopérimétrique classique aux sous-variétés de codimension quelconque, ce qui consiste à résoudre le problème minmax suivant :

$$\max_{K\,:\,\mathrm{Vol}_k(K)=c} \left(\min_{Q\,:\,\partial Q = K} \mathrm{Vol}_{k+1}(Q) \right),$$

où c est une constante. Almgren prouve que les solutions dans l'espace euclidien $\mathbb{R}^{m+1}$ sont les sphères rondes, tandis que les questions d'existence et de régularité des solutions dans des variétés riemanniennes ont été étudiées dans le travail de F. Morgan et M.C. Salavessa [87].

Les hypersurfaces à courbure moyenne constante peuvent être vues comme bords des points critiques de la fonctionnelle

$$\Omega \mapsto \mathrm{Vol}(\partial\Omega) - \lambda\,\mathrm{Vol}(\Omega), \tag{35}$$

où λ est une constante qui correspond à la valeur de la courbure moyenne.

En codimension supérieure, on dit qu'une sous-variété K de dimension $k < m$ est à courbure moyenne constante si K est un bord d'une sous-variété Q qui est point critique de la fonctionnelle

$$Q \mapsto \mathrm{Vol}(\partial Q) - \lambda\,\mathrm{Vol}(Q).$$

Ces points critiques sont caractérisés par le fait que Q est une sous-variété minimale telle que la courbure moyenne du bord $K = \partial Q$ vérifie :

$$H_K = \lambda\, n,$$

où n est une normale unitaire de K dans Q, autrement dit si K est à courbure moyenne constante dans Q et si H_K n'a pas de composantes orthogonales à Q. Dans l'espace euclidien $\mathbb{R}^{m+1}$, la sphère $S^k = S^k \times \{0\}$ de dimension $k < m$ (le bord de la boule B^{k+1} de dimension $k+1$), est un exemple de sous-variété à courbure moyenne constante de codimension $m+1-k$.

Dans notre travail, on propose une généralisation du théorème de R. Ye [118] qui prouve l'existence de familles d'hypersurfaces à CMC qui sont des petites perturbations des sphères géodésiques centrées aux points critiques non-dégénérés de la fonction courbure scalaire, et du résultat plus récent de F. Pacard et X. Xu [95] qui construisent de telles familles dans le cas où la courbure scalaire n'est pas une fonction de Morse. Par analogie, on obtient des familles de sous-variétés à CMC associées aux points critiques non-dégénérés d'un invariant géométrique qu'on appelle *courbure scalaire partielle*, défini sur le fibré grassmannien de la variété ambiante. Pour tout sous-espace $\Pi_p \subset T_pM$ de dimension $(k+1)$, on définit :

$$\mathcal{R}_{k+1}(\Pi_p) := -\sum_{i,j=1}^{k+1} \langle R(E_i, E_j)E_i, E_j\rangle,$$

où $E_1, \ldots, E_{k+1}$ est une base orthonormée de Π_p. On remarque que $\mathcal{R}_{m+1}(T_pM)$ est égale à la courbure moyenne standard en p, tandis que $\mathcal{R}_2(\Pi_p)$ est le double de la courbure sectionnelle du plan Π_p.

On définit la sphère géodésique $\mathcal{S}^k_\varepsilon(\Pi_p)$ associée au plan Π_p comme l'image par l'application exponentielle de la sphère de rayon ε centrée en 0 dans Π_p :

$$\mathcal{S}^k_\varepsilon(\Pi_p) := \Big\{ \exp_p\Big(\varepsilon \sum_{i=1}^{k+1} \Theta^i\, E_i\Big),\ \Theta \in S^k \Big\}.$$

On prouve le résultat suivant :

Théorème 0.6.1. *Soit Π_p un point critique non-dégénéré de $\mathcal{R}_{k+1}$, alors pour tout ε suffisamment petit, il existe une sous-variété à courbure moyenne constante $K_\varepsilon(\Pi_p)$ qui est un graphe normal sur la sphère géodésique $\mathcal{S}^k_\varepsilon(\tilde{\Pi}_{\tilde{p}})$ d'une section de norme $\mathcal{C}^{2,\alpha}$ bornée par $c\varepsilon^3$, et* $\operatorname{dist}(\tilde{\Pi}_{\tilde{p}}, \Pi_p) \leq c\,\varepsilon^2$.

Voici une courte description de notre construction.

L'étude de l'opérateur de Jacobi en codimension quelconque

Soit Σ une sous-variété plongée dans M, fermée ou à bord. L'opérateur de Jacobi J_Σ de Σ est défini comme différentielle de la fonctionnelle courbure moyenne par rapport aux perturbations normales de Σ. Soit $\Phi \in \mathcal{C}^2(\Sigma, N\Sigma)$ un champ de vecteurs normal à Σ avec la norme $\|\Phi\|_{\mathcal{C}^0}$ suffisamment petite. Si $\partial\Sigma \neq \emptyset$, on demande également $\Phi = 0$ sur $\partial\Sigma$. Alors,

$$\Sigma_\Phi := \big\{ \exp_q(\Phi(q)),\ q \in \Sigma \big\}$$

est une sous-variété plongée. On définit :

$$J_\Sigma(\Phi) := \nabla_{\frac{\partial}{\partial s}} H(\Sigma_{s\,\Phi})\Big|_{s=0}.$$

Soient $\pi_{N\Sigma}$ et $\pi_{T\Sigma}$ les projections orthogonales sur les fibrés normal et tangent de Σ. On utilise les notations J_Σ^N et J_Σ^T pour les opérateurs $\pi_{N\Sigma} \circ J_\Sigma$ et $\pi_{T\Sigma} \circ J_\Sigma$ respectivement. L'expression explicite de J_Σ^N est donnée par la formule standard [68] :

$$J_\Sigma^N = -\Delta_\Sigma^N + \mathrm{Ric}_\Sigma^N + \mathfrak{H}_\Sigma^{(2)}, \tag{36}$$

où Δ_Σ^N est le Laplacien (défini positif) qui agit sur les sections de $N\Sigma$, et les deux autres termes sont des endomorphismes symétriques suivants de $N\Sigma$: la projection orthogonale $\mathrm{Ric}_\Sigma^N = \pi_{N\Sigma} \circ \mathrm{Ric}_\Sigma$ de la courbure partielle de Ricci de Σ définie par

$$g(\mathrm{Ric}_\Sigma X, Y) := \sum_{i=1}^{\dim \Sigma} g\left(R(E_i, X)E_i, Y\right), \quad \text{pour tous} \quad X, Y \in T_\Sigma M,$$

(ici le tenseur de courbure R est celui associé à la variété ambiante M); le carré de l'opérateur de forme défini par

$$\mathfrak{H}_\Sigma^{(2)}(X) := \sum_{i,j=1}^{\dim \Sigma} \langle h_\Sigma(E_i, E_j), X\rangle h_\Sigma(E_i, E_j), \quad \text{pour tous} \quad X \in T_\Sigma M.$$

En générale, $J_\Sigma(\Phi)$ a une composante non-triviale $J_\Sigma^T(\Phi)$ parallèle à Σ. Néanmoins, quand Σ est une sous-variété minimale, on a $J_\Sigma^T = 0$. On verra que dans le problème linéarisé qu'on étudie ce terme parallèle disparaît également, et pour cette raison, on n'a pas besoin de le rendre explicite.

Problème linéarisé

Soit K^k, $k \leq m$ une sous-variété fermée à courbure moyenne constante plongée dans M^{m+1}. Par définition, il existe une sous-variété minimale Q telle que

$$K = \partial Q \quad \text{et} \quad H_K = \lambda\, n,$$

où n est une normale unitaire de K dans Q. On aimerait perturber K en une autre sous-variété dont la courbure moyenne est à nouveau égale à λ. Plus précisément, on cherche un champ de vecteurs $\Phi \in NK$ tel que la sous-variété $K_\Phi = \left\{ \exp_p(\Phi(p)),\ p \in K \right\}$ vérifie

$$K_\Phi = \partial Q_\Phi \quad \text{et} \quad H(K_\Phi) = \lambda\, n_\Phi,$$

où Q_Φ est une sous-variété minimale, et n_Φ est une normale unitaire de K_Φ dans Q_Φ. On voit que la construction de Q_Φ et n_Φ dépend de l'extension à l'intérieur de Q du champ de vecteurs Φ défini sur $K = \partial Q$. En conséquence, contrairement au cas de codimension 1, le problème n'est plus locale, ce qui à priori engendre des difficultés supplémentaires. En premier lieu, on prouve le résultat suivant :

Lemme 0.6.1. *Soit Q une sous-variété minimale non-dégénérée. Alors il existe une application lisse $\Phi \mapsto Q_\Phi$ d'un voisinage de 0 dans $\mathcal{C}^{2,\alpha}(K, NK)$ sur l'espace des sous-variétés minimales de M, telle que $Q_0 = Q$ et $\partial Q_\Phi = K_\Phi$. De plus, Q_Φ est un graphe géodésique sur Q d'un champ de vecteurs $U_\Phi \in T_Q M$ tel que $U_\Phi|_K = \Phi$.*

L'étape suivante est de considérer la fonctionnelle

$$s \in \mathbb{R} \mapsto \mathcal{H}(s) := H(K_{s\Phi}) - \lambda\, n_{s\Phi},$$

où $n_{s\Phi}$ est une normale unitaire de $K_{s\Phi}$ dans la sous-variété minimale $Q_{s\Phi}$, et l'opérateur linéarisé

$$L_Q := \nabla_{\frac{\partial}{\partial s}} \mathcal{H}(s)\Big|_{s=0}.$$

On remarque que $\pi_{TK} \circ L_Q = 0$. En effet, soit T un champ de vecteurs tangent à $K_{s\Phi}$. Or $\mathcal{H}(s) \perp K_{s\Phi}$ et $\mathcal{H}(0) = 0$, donc

$$g(\mathcal{H}'(0), T) + g(\mathcal{H}(0), T'(0)) = 0.$$

Notation 0.6.1. *On introduit l'opérateur*

$$D_Q := \pi_{NK} \circ \nabla_{\frac{\partial}{\partial s}} n_{s\Phi}\Big|_{s=0}.$$

Soient U_Φ l'extension dans Q du champ de vecteurs Φ définie par Lemme (0.6.1) et $\bar{U}_\Phi$ sa partie linéaire en Φ. On note par $[\cdot]^\perp$ la projection orthogonale $T_K M \to NK \cap N_K Q$. On obtient le résultat suivant :

Proposition 0.6.1. *L'opérateur D_Q vérifie*

$$D_Q = \left[\nabla_n \bar{U}_\Phi \big|_K \right]^\perp,$$

tandis que l'opérateur L_Q est donné par

$$L_Q(\Phi) = \pi_{NK} \circ J_K(\Phi) - \lambda\, D_Q(\Phi).$$

Linéarisation pour $K = S^k \subset \mathbb{R}^{m+1}$

On applique l'analyse décrite ci-dessus au cas particulier où $K = S^k = \partial B^{k+1}$ dans l'espace euclidien $\mathbb{R}^{m+1}$, $k < m$. Soit $\Phi \in \mathcal{C}^{2,\alpha}(S^k, NS^k)$. On décompose

$$\Phi = [\Phi]^\perp - \phi\, \Theta, \quad \text{où} \quad \Theta \in S^k, \quad [\Phi]^\perp \in NS^k \cap N_{S^k} B^{k+1}.$$

Alors, un calcul explicite montre que l'opérateur $\pi_{NS^k} \circ J_{S^k}$ vérifie

$$\pi_{NS^k} \circ J_{S^k}(\Phi) = \Delta_{S^k} [\Phi]^\perp - (\Delta_{S^k} + k)\, \phi\, \Theta,$$

tandis que $D_{B^{k+1}}$ est donné par l'opérateur Dirichlet-to-Neumann :

$$D_{B^{k+1}}(\Phi) = \frac{\partial \hat{U}_\Phi}{\partial n},$$

où $\hat{U}_\Phi \in NQ$, $\Delta_{B^{k+1}} \hat{U}_\Phi = 0$ et $\hat{U}_\Phi \Big|_{S^k} = [\Phi]^\perp$.

Noyau de l'opérateur linéarisé

Une analyse simple des propriétés de $L_{B^{k+1}}$ montre que cet opérateur a un noyau non-trivial, plus précisément, on trouve que :

$$(\Delta_{S^k} + k)\ \phi = 0 \Rightarrow \phi \in \operatorname{span}\{\Theta^j,\ j = 1, \ldots, k+1\} \quad \text{et}$$

$$(\Delta_{S^k} - k\, D_{B^{k+1}})\, [\Phi]^\perp = 0 \Rightarrow [\Phi]^\perp \in \operatorname{span}\left\{\left(c_\mu^j\, \Theta^j + d_\mu\right) E_\mu,\ j = 1, \ldots, k+1\right\},$$

où $E_\mu,\ \mu = k+2, \ldots, m+1$ une base orthonormée de NB^{k+1}.

Argument perturbatif

Soit $\Pi_p \subset T_pM$ un plan de dimension $k+1$ et $\mathcal{S}_\varepsilon^k(\Pi_p) = \partial \mathcal{B}_\varepsilon^{k+1}(\Pi_p)$ la sphère géodésique, le bord de la boule géodésique, associées à Π_p. On trouve :

$$H(\mathcal{B}_\varepsilon^{k+1}(\Pi_p)) = \mathcal{O}(\varepsilon) \quad \text{et} \quad H(\mathcal{S}_\varepsilon^k(\Pi_p)) - \frac{k}{\varepsilon}\, n = \mathcal{O}(\varepsilon),$$

où n est une normale unitaire de $\mathcal{S}_\varepsilon^k(\Pi_p)$ dans $\mathcal{B}_\varepsilon^{k+1}(\Pi_p)$. C'est donc naturel, au moins pour ε suffisamment petit, de s'attendre à pouvoir perturber $\mathcal{S}_\varepsilon^k(\Pi_p)$ en une sous-variété à courbure moyenne constante. Malheureusement, l'opérateur $L_{\mathcal{B}_\varepsilon^{k+1}(\Pi_p)}$ est proche de l'opérateur $\varepsilon^{-2}\, L_{B^{k+1}}$ qui, selon le paragraphe précédent, a un noyau non-trivial, ce qui nous empêche de directement appliquer un argument perturbatif.

Néanmoins, on peut utiliser la méthode de réduction de Lyapunov-Schmidt et perturber $\mathcal{S}_\varepsilon^k(\Pi_p)$ en une sous-variété qui est à courbure moyenne constante à un terme de $\operatorname{Ker}(L_Q)$ près. Plus précisément, on prouve que pour tout $\Pi_p \in G_{k+1}(TM)$ et tout ε assez petit, il existe une sous-variété minimale $Q_\varepsilon(\Pi_p)$ qui est une petite perturbation de $\mathcal{B}_\varepsilon^{k+1}(\Pi_p)$, et dont le bord $K_\varepsilon(\Pi_p)$ est un graphe normal sur $\mathcal{S}_\varepsilon^k(\Pi_p)$ tel que

$$H(K_\varepsilon(\Pi_p)) - \frac{k}{\varepsilon}\, n = \langle \vec{a}, \Theta \rangle\, n + \sum_{\mu=k+2}^{m+1} \left(\langle \vec{c}_\mu, \Theta \rangle + d_\mu\right)\, N_\mu,$$

avec $\vec{a}, \vec{c}_\mu \in \mathbb{R}^{k+1}$ et $d_\mu \in \mathbb{R}$. Ici, les champs de vecteurs $n, N_{k+2}, \ldots, N_{m+1}$ forment une base orthonormée de $NK_\varepsilon(\Pi_p)$ et n désigne une normale unitaire de $K_\varepsilon(\Pi_p)$ dans $Q_\varepsilon(\Pi_p)$.

Choix des paramètres

On considère la fonctionnelle

$$\mathcal{E}_\varepsilon(\Pi_p) := \operatorname{Vol}_k\left(\partial Q_\varepsilon(\Pi_p)\right) - \frac{k}{\varepsilon} \operatorname{Vol}_{k+1}\left(Q_\varepsilon(\Pi_p)\right),$$

définie sur le fibré grassmannien de TM et introduite dans le cas de codimension 1 par F. Pacard et X. Xu [95]. On montre que si Π_p est point critique de $\mathcal{E}_\varepsilon$, alors $K_\varepsilon(\Pi_p)$ est une sous-variété à courbure moyenne constante. On remarque que dans la construction de $K_\varepsilon(\Pi_p)$

on a $m+1$ degrés de liberté qui correspondent aux translations infinitésimales du plan Π_p, et $(k+1)\times(m-k)$ degrés de liberté qui correspondent aux rotations de Π_p dans T_pM qui transforment Π_p en plans orthogonales à Π_p. Ainsi, le nombre de degrés de liberté correspond au nombre d'équations dans le système

$$\vec{a}=0, \quad \vec{c}_\mu = 0, \quad d_\mu = 0, \quad \mu = k+2,\dots,m+1.$$

Enfin, on obtient

$$\varepsilon^{-k}\,\mathcal{E}_\varepsilon(\Pi_p) = c_0 + c_1\,\varepsilon^2\,\mathcal{R}_{k+1}(\Pi_p) + \mathcal{O}(\varepsilon^4),$$

où $\mathcal{R}_{k+1}$ est la fonction courbure scalaire partielle de M et où les constantes c_0, c_1 ne dépendent pas de ε. En particulier, pour tout point critique non-dégénéré Π_p de $\mathcal{R}_{k+1}$, il existe un point critique $\tilde{\Pi}_{\tilde{p}}$ de $\mathcal{E}_\varepsilon$ tel que $\operatorname{dist}\left(\Pi_p, \tilde{\Pi}_{\tilde{p}}\right) \le c\,\varepsilon^2$, d'où suit notre résultat.

Chapitre 2 : Surfaces à bords libres dans la boule unité euclidienne B^3

Dans le deuxième chapitre, qui est un travail en collaboration avec A. Folha et F. Pacard, on s'intéresse à l'existence des surfaces minimales plongées proprement dans la boule unité euclidienne B^3 qui rencontrent la sphère unité S^2, le bord de B^3, de manière orthogonale. Ces surfaces apparaissent comme des points critiques de la fonctionnelle d'aire parmi les surfaces plongées dans B^3 dont les bords varient sur S^2, et, d'après [36], sont appelées *surfaces minimales à bords libres*. On prouve l'existence de deux familles géométriquement distinctes de telles surfaces, paramétrées par un entier n suffisamment grand qui représente le nombre de composantes connexes du bord.

Les disques équatoriaux obtenus par l'intersection des plans passants par l'origine avec la boule unité, sont des exemples de surfaces minimales à bords libres. J.C. Nitsche [88] a montré que ce sont les seules solutions du type topologique disque.

A. Fraser, M. Li [35] ont formulé la conjecture selon laquelle modulo les isométries, la seule surface minimale à bords libres du type topologique anneau plongée proprement dans B^3 est le caténoïd critique :

$$(s,\phi)\in\mathbb{R}\times S^1 \;\mapsto\; \frac{1}{s_*\cosh s_*}(\cosh s\,\cos\phi, \cosh s\sin\phi, s), \quad \text{où} \quad s_*\tanh s_* = 1.$$

Dans un travail récent, A. Fraser and R. Schoen [37] ont prouvé l'existence pour tous $n\ge 3$ de surfaces minimales à bords libres dans B^3 qui ont un genre 0 et n composantes connexes de bord. Ces surfaces émergent dans l'étude des métriques maximisantes pour la première valeur propre de l'opérateur Dirichlet-to-Neumann sur les variétés compactes de dimension 2. Plus précisément, soit (M^2, g) une variété riemannienne à bord. Pour $u\in\mathcal{C}^\infty(\partial M)$, soit $\hat{u}$ l'extension harmonique (calculée par rapport à la métrique g) de u dans M. Alors l'opérateur Dirichlet-to-Neumann est défini par :

$$D_g(u) = \left.\frac{\partial\hat{u}}{\partial\nu}\right|_{\partial M},$$

où ν est une normale unitaire de ∂M dans M. D_g est un opérateur auto-adjoint à spectre discret, appelé *valeurs propres de Steklov*. Soit $\sigma_1(g)$ la première valeur propre de Steklov associée à la métrique g. A Fraser et R. Schoen montrent que si M a un genre 0 et n composantes connexes de bord, il existe une métrique g_{max} pour laquelle le maximum de $\sigma_1(g)\,\mathrm{length}_g(\partial M)$ est atteint. De plus, les fonctions propres correspondantes définissent un plongement minimal à bords libres Σ_n de M dans B^3. Ils remarquent que quand n tend vers l'infinie, la suite Σ_n converge sur les compacts de B^3 vers une copie double d'un disque équatorial.

Dans notre travail, on propose une construction indépendante de Σ_n pour n assez grand utilisant des techniques perturbatives. On prouve également l'existence de surfaces minimales à bords libres $\tilde{\Sigma}_n$ dans B^3 qui ont un genre 1 et n composantes connexes de bord pour n assez grand, ce qui un nouveau résultat qui ne découle pas de [37].

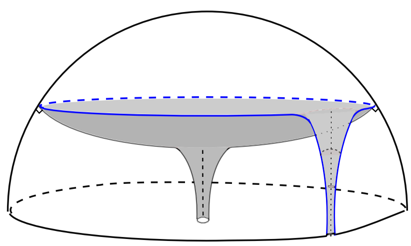
Avant d'énoncer notre théorème, on identifie $\mathbb{R}^3$ avec $\mathbb{C} \times \mathbb{R}$ et introduit un groupe d'isométries $\mathfrak{S}_n$ généré par

$$(z,t) \mapsto (\bar{z},t), \quad (z,t) \mapsto (z,-t), \quad \text{et} \quad (z,t) \mapsto (z \cdot e^{\frac{2\pi i}{n}}, t).$$

Théorème 0.6.2. *Il existe $n_0 \geq 0$ tel que pour tout $n \geq n_0$ il existent une surface minimale à bords libres Σ_n de genre 0 et une surface minimale à bords libres $\tilde{\Sigma}_n$ de genre 1 plongées dans B^3 qui rencontrent S^2 de manière orthogonale le long de n courbes fermées.*

Les deux surfaces sont invariantes par l'action du groupe $\mathfrak{S}_n$ et, quand n tend vers l'infinie, la suite Σ_n converge vers une copie double du disque horizontal (ouvert), uniformément sur les compacts de B^3, tandis que la suite $\tilde{\Sigma}_n$ converge vers une copie double du disque épointé, uniformément sur les compacts de $B^3 \setminus \{0\}$.

Notre preuve est dans l'esprit des preuves d'existence de surfaces minimales dans S^3 par dédoublement proposées par N. Kapouleas [61], N. Kapouleas et S.D. Yang [63], et D. Wiygul [117]. Comme dans [63], notre construction marche quand le nombre n de composantes connexes du bord est assez grand. Les surfaces qu'on obtient ont la structure de deux disques horizontaux parallèles connectés par n "demi-ponts" caténoïdaux arrangés le long du cercle horizontal de S^2 de manière périodique et par un "cou" caténoïdal à axe vertical centré à l'origine.



Voici une courte description de notre construction d'une famille de surfaces minimales à bords libres de genre 1, et on verra qu'une preuve analogue avec quelques simplifications donne l'existence d'une famille de surfaces minimales à bords libres de genre 0.

Paramétrisation de la boule unité

Premièrement, on paramètre un voisinage du disque horizontal $D^2 \times \{0\}$ dans la boule B^3 par une région du cylindre $D^2 \times \mathbb{R}$ de manière suivante :

$$X \,:\, D^2 \times \mathbb{R} \to B^3, \quad X(z,t) = A(z,t)\,(z, B(z)\,\sinh t)\,,$$

où

$$B(z) = \frac{1+|z|^2}{2} \quad \text{et} \quad A(z,t) = \frac{1}{1+B(z)(\cosh t - 1)}.$$

Dans cette paramétrisation, le bord $|z| = 1$ du cylindre correspond au bord S^2 de la boule unité. De plus, chaque feuille $t = t_0$ est une calotte sphérique à courbure moyenne constante $H = 2\sinh t_0$ qui rencontre le bord de la boule de manière orthogonale (quand $t = 0$ on obtient le disque unité $D^2 \times \{0\}$).

On trouve que la métrique $X^* g_{eucl}$ dans $D^2 \times \mathbb{R}$ est de forme

$$X^* g_{eucl} = A^2(z,t)\left(|dz|^2 + B^2(z)\,dt^2\right). \tag{37}$$

Graphes verticaux dans B^3

Un rôle important dans notre construction est joué par les graphes verticaux sur le disque $D^2 \times \{0\} \subset B^3$ paramétrés par

$$z \in D^2 \mapsto X(z, w(z)) \in B^3 \quad \text{pour} \quad w \in \mathcal{C}^2(D^2). \tag{38}$$

Le développement limité de la courbure moyenne d'un tel graphe en puissances de w et de dérivées de w jusqu'au deuxième ordre s'écrit sous la forme :

$$H_{gr}(w) = L_{gr}\, w + Q_{gr}(w, \nabla w, \nabla^2 w), \tag{39}$$

où

$$L_{gr} := \Delta(B\,\cdot\,) = \Delta\left(\frac{1+|z|^2}{2}\,\cdot\right)$$

est l'opérateur de courbure moyenne linéarisé et Q_{gr} est une fonction non-linéaire lisse qui vérifie $Q_{gr}(0,0,0) = DQ_{gr}(0,0,0) = D^2 Q_{gr}(0,0,0) = 0$. De plus, le graphe (38) rencontre le bord de la boule de manière orthogonale si w vérifie la condition homogène de Neumann au bord :

$$\partial_r w|_{r=1} = 0.$$

On remarque que l'opérateur L_{gr} a un noyau non-trivial composé des fonctions $\frac{2x_1}{1+|z|^2}$ et $\frac{2x_2}{1+|z|^2}$ ce qui correspond à pencher le disque $D^2 \times \{0\}$ dans B^3. Ce noyau peut être éliminé en imposant l'invariance par l'action d'un groupe de rotations autours de l'axe verticale.

Fonction de Green

D'après A. Fraser et R. Schoen, les surfaces qu'on aimerait construire devraient être proches de sommes connexes de deux copies du disque horizontal avec des petits "demi-ponts" proches de demi-caténoïdes placés de manière periodique sur le cercle unité horizontal et un petit "cou" proche du caténoïde centré à l'origine.

En utilisant les idées R. Mazzeo, F. Pacard et D. Pollack [77], pour obtenir une meilleure correspondance avec le comportement (logarithmique) du caténoïde à l'infinie, on perturbe d'abord le disque $D^2 \times \{0\}$ à l'aide d'une fonction de Green associée à l'opérateur L_{gr}, puis effectue la somme connexe. Soit $z_m = e^{\frac{2\pi m i}{n}}$, $m = 1, \ldots, n$ les n^{e} racines de l'unité. On cherche une solution du problème :

$$\begin{cases} \Delta(B\Gamma_n) = c_0\, \delta_0 & \text{dans} \quad D^2, \\ \partial_r \Gamma_n = \sum_{m=1}^n c_n\, \delta_{z_n} & \text{sur} \quad S^1, \end{cases} \tag{40}$$

avec des constantes $c_0, c_1, \ldots, c_n$. Si on suppose que la fonction Γ_n est invariante par les rotations d'angle $\frac{2\pi}{n}$, alors résoudre (40) est équivalent à trouver une fonction G telle que $\Gamma_n(z) = G(z^n)/B(z)$,

$$\begin{cases} \Delta\, G = \tilde{c}_0\, \delta_0 & \text{dans} \quad D^2, \\ \partial_r G - \frac{1}{n}\, G = \tilde{c}_1\, \delta_1 & \text{sur} \quad S^1, \end{cases} \tag{41}$$

avec des constantes $\tilde{c}_0$ et $\tilde{c}_1$. Une solution de (41) peut être trouvée explicitement. On décompose G en somme de deux fonctions, dont une a une singularité en $z = 0$, et l'autre a une singularité en $z = 1$. Plus précisément, la fonction

$$G_0(z) := -\log|z| - n$$

vérifie $\Delta G_0 = 0$ dans $D^2 \setminus \{0\}$ et $\partial_r G_0 - \frac{1}{n} G_0 = 0$ sur S^1. D'un autre coté, la fonction

$$G_1(z) := -\frac{n}{2} + \sum_{k=0}^{\infty} \frac{1}{n^k}\, \operatorname{Re} H_k(z), \quad \text{où} \quad H_k(z) := \sum_{j=1}^{\infty} \frac{z^j}{j^{k+1}},$$

est harmonique dans D^2, et, puisque

$$H_0(z) = -\log(1 - z),$$

et

$$\partial_r\, (\operatorname{Re} H_k) = \operatorname{Re} H_{k-1} \quad \text{on} \quad S^1, \quad \forall k \in \mathbb{N},$$

G_1 vérifie :

$$\partial_r G_1 - \frac{1}{n} G_1 = 0 \quad \text{on} \quad S^1 \setminus \{0\}.$$

Enfin, on écrit notre fonction de Green sous la forme :

$$\Gamma_n(z) := \frac{1}{B(z)}\, (\tau_0\, G_0(z^n) + \tau_1\, G_1(z^n))\,,$$

où les coefficients $\tau_0, \tau_1 \in \mathbb{R}$ sont soigneusement choisis à l'étape où on "recolle" le graphe de Γ_n avec le "cou" et les "demi-ponts" caténoïdaux. Le développement limité de Γ_n aux voisinages de $z=0$ et $z=z_m$ s'écrit sous la forme :

$$\Gamma_n(z) = \begin{cases} -n\left(2\tau_0 + \frac{\tau_1}{2}\right) - 2\,n\,\tau_0 \log|z| + \mathcal{O}(|z|^2 \log|z|), \quad \text{quand } |z| \to 0 \\ \\ -n\,(2\tau_0 + \frac{\tau_1}{2}) + \tau_1 \log n - \tau_1 \log|z - z_m| \\ +\mathcal{O}(|z-z_m| \log|z-z_m|), \quad \text{quand } |z - z_m| \to 0. \end{cases} \tag{42}$$

"Demi-ponts" et "cou" catenoïdaux

Le rôle des "demi-ponts" caténoïdaux qui vont dans notre construction joindre les deux "copies" du disque unité, pourrait être joué par des bandes obtenus par l'intersection des caténoïdes euclidiens centrés aux n^{e} racines de l'unité avec la sphère S^2. La difficulté de cette approche est que ces bandes ne sont pas orthogonales à S^2. On préfère de plonger les demi-caténoïdes dans B^3 orthogonalement à S^2, en perdant la propriété de minimalité.

Soit $\mathbb{C}_- := \{\zeta \in \mathbb{C} \,:\, \mathrm{Re}(\zeta) \leq 0\}$ un demi-plan. On considère les applications

$$\lambda_m \,:\, \zeta \in \mathbb{C}_- \mapsto e^{\frac{2\pi i m}{n}} \frac{1+\zeta}{1-\zeta} \in \bar{D}^2$$

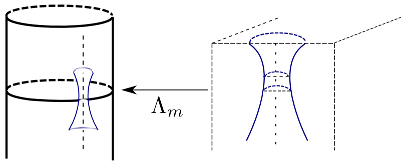
qui définissent des difféomorphismes locales entre le voisinage de $\zeta = 0$ dans $\mathbb{C}_-$ et les voisinages de $z = z_m$ dans le disque unité. Puis, on introduit les applications

$$\Lambda_m \,:\, (\zeta, \tau) \in \mathbb{C}_- \times \mathbb{R} \mapsto (\lambda_m(\zeta), 2\tau) \in \bar{D}^2 \times \mathbb{R}.$$

Pour $\varepsilon \in (0,1)$ on paramètre le m^{e} "demi-pont" caténoïdal $C_{\varepsilon,m}$ dans B^3 par

$$(\sigma, \theta) \in [-\sigma_\varepsilon, \sigma_\varepsilon] \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \mapsto X \circ \Lambda_m\left(\frac{\varepsilon}{2} \cosh\sigma\, e^{i\theta}, \frac{\varepsilon}{2}\,\sigma\right) \in C_{\varepsilon,m},$$

pour $\sigma_\varepsilon \gg 1$ choisi plus tard. Alors, $C_{\varepsilon,m}$ est proche d'un demi-caténoïde centré en $z = z_m$, dilaté par $\varepsilon > 0$, et tronqué à la hauteur σ_ε. De plus, puisque la restriction de $X \circ \Lambda_m$ aux plans horizontaux est conforme, $C_{\varepsilon,m}$ rencontre ∂B^3 de manière orthogonale.



D'en autre coté, dans le voisinage de $z = 0$, la métrique $X^* g_{eucl}$ donnée par (37) es proche de la métrique $|dz|^2 + \frac{1}{4}dt^2$. Pour $\tilde{\varepsilon} \in (0,1)$ et $s_{\tilde{\varepsilon}} \gg 1$ choisis plus tard, on considère la surface paramétrée par

$$X^{cat}_{\tilde{\varepsilon}} \,:\, (s, \phi) \in (-s_{\tilde{\varepsilon}}, s_{\tilde{\varepsilon}}) \times S^1 \mapsto \left(\tilde{\varepsilon}\cosh s\, e^{i\phi}, 2\,\tilde{\varepsilon} s\right) \in \mathbb{R}^3,$$

minimale par rapport à la métrique $|dz|^2 + \frac{1}{4}\,dt^2$. Alors, la surface $C_{\tilde{\varepsilon},0}$:

$$(s, \phi) \in (-s_{\tilde{\varepsilon}}, s_{\tilde{\varepsilon}}) \times S^1 \mapsto X \circ X^{cat}_{\tilde{\varepsilon}}(s, \phi) \in C_{\tilde{\varepsilon},0},$$

est proche d'un caténoïde dilaté par $\tilde{\varepsilon}$ et tronqué à la hauteur $s_{\tilde{\varepsilon}}$.

"Matching"

Avec le changement de variables $z = \lambda_m \left(\frac{\varepsilon}{2} \cosh \sigma \, e^{i\theta}\right)$ dans des régions où $\sigma \neq 0$, on peut considérer le "demi-pont" caténoïdal $C_{\varepsilon,m}$ comme bi-graphe sur $\{z \in D^2 \,:\, |z - z_m| > \varepsilon\}$ de la fonction

$$G_{\varepsilon,m} = -\varepsilon \log \frac{\varepsilon}{2} + \varepsilon \log |z - z_m| + \mathcal{O}(\varepsilon^3 \, |z - z_m|^{-2}). \tag{43}$$

D'un autre coté, avec le changement de variables $z = \tilde{\varepsilon} \cosh s \, e^{i\phi}$ dans des régions où $s \neq 0$, on peut considérer le "cou" caténoïdal $C_{\tilde{\varepsilon},0}$ comme bi-graphe sur $\{z \in D^2 \,:\, |z| > \tilde{\varepsilon}\}$ de la fonction

$$G_{\tilde{\varepsilon},0} = -\tilde{\varepsilon} \log \frac{\tilde{\varepsilon}}{2} + \tilde{\varepsilon} \log |z| + \mathcal{O}(\tilde{\varepsilon}^3 \, |z|^{-2}). \tag{44}$$

En comparant les expressions (43) et (44) avec le développement limité de la fonction de Green Γ_n aux voisinages de $z = z_m$ et $z = 0$ (42), on trouve que ε, $\tilde{\varepsilon}$, ainsi que les coefficients τ_1 et τ_2, sont exprimés en fonction du nombre n de composantes connexes du bord de la surface qu'on construit. En comparant les termes logarithmiques, on trouve :

$$\tau_1 = \varepsilon \quad \text{and} \quad n \, \tau_0 = \tilde{\varepsilon},$$

tandis que en comparant les termes constants on trouve :

$$\tilde{\varepsilon} \sim \varepsilon \quad \text{and} \quad n \sim \log(1/\varepsilon).$$

Enfin, en comparant les termes restants on conclut que le "recollement" devrait être effectué dans des régions où

$$|z| = \tilde{\varepsilon} \cosh s_{\tilde{\varepsilon}} \sim \varepsilon^{\frac{1}{2}} \quad \text{et} \quad |z - z_m| \sim \varepsilon \cosh \sigma_{\varepsilon} \sim \varepsilon^{\frac{2}{3}}.$$

Remark 0.6.1. *Ces calculs, avec le fait que les fonctions constantes n'appartiennent pas au noyau de l'opérateur linéarisé L_{gr} donnent une idée pourquoi notre construction marche uniquement pour n assez grand.*

Perturbation

À cette stade, pour tout n assez grand, on obtient via la somme connexe une surface de genre 1 plongée dans B^3 et orthogonale à ∂B^3 le long de n composantes connexes de bord. On notera cette surface par $\mathcal{A}_n$ et l'appellera *solution approchée.* La prochaine étape est alors de perturber $\mathcal{A}_n$ en une surface minimale à bords libres.

Soient Ξ un champ de vecteurs dans $\bar{B}^3$ transverse à $\mathcal{A}_n$ et $\xi \,:\, \bar{B}^3 \times (0,1) \to \bar{B}^3$ le flot associé :

$$\frac{d\xi}{dt} = \Xi(\xi(\cdot, t)), \quad \xi(p, 0) = p, \quad p \in \bar{B}^3.$$

On choisi Ξ de telle façon que pour tout t suffisamment petit, la surface $\mathcal{A}_{n,t} := \xi_t(\mathcal{A}_n)$ soit plongée dans B^3 et rencontre S^2 orthogonalement le long de $\partial \mathcal{A}_{n,t}$.

Enfin, pour $w \in \mathcal{C}^{2,\alpha}(\mathcal{A}_n)$, on introduit la surface $\mathcal{A}_n(w)$ dans B^3 paramétrée par

$$p \in \mathcal{A}_n \mapsto \xi(p, w(p)) \in \mathcal{A}_n(w).$$

Alors, $\mathcal{A}_n(w)$ rencontre S^2 de manière orthogonale quand w vérifie sur ∂A_n la condition homogène de Neumann au bord :

$$g_n(\nabla^{g_n} w, N_{\partial \mathcal{A}_n}) = 0,$$

où g_n est la métrique induite sur $\mathcal{A}_n$ de la métrique euclidienne et $N_{\partial \mathcal{A}_n}$ est une normale unitaire de $\partial \mathcal{A}_n$ dans $\mathcal{A}_n$.

Le développement limité de la courbure moyenne de $\mathcal{A}_n(w)$ en puissances de w et des dérivées de w s'écrit sous la forme :

$$H(\mathcal{A}_n(w)) = H(\mathcal{A}_n) + \mathcal{L}_n\, w + \mathcal{Q}_n(w, \nabla w, \nabla^2 w),$$

où $H(\mathcal{A}_n)$ est la courbure moyenne de la solution approchée, $\mathcal{L}_n$ est l'opérateur de courbure moyenne linéarisé défini sur $\mathcal{A}_n$, et $\mathcal{Q}_n$ une fonction non-linéaire lisse, qui vérifie $\mathcal{Q}_n(0,0,0) = D\mathcal{Q}_n(0,0,0) = 0$. Notre but alors est de résoudre l'équation :

$$\mathcal{L}_n\, w = -H(\mathcal{A}_n) - \mathcal{Q}_n(w).$$

On montre que dans une topologie adaptée, la norme de $H(\mathcal{A}_n)$ tend vers 0 quand n tend vers l'infinie comme e^{-na} pour une constante $a > 1$. De plus, on montre que pour un choix convenable du champ de vecteur Ξ, l'opérateur $\mathcal{L}_n$ est inversible. Son inverse ne sera pas uniformément borné en n, mais explosera comme $e^{n\nu}$, pour $\nu \ll 1$ qu'on choisit. Alors, le résultat découlera du théorème de point fixe de Banach pour les applications contractantes, appliqué à l'application $w \mapsto -\mathcal{L}_n^{-1}\left(H(\mathcal{A}_n) + \mathcal{Q}_n(w)\right)$.

Analyse linéaire

En conclusion, disons quelques mots sur les propriétés de l'opérateur linéaire $\mathcal{L}_n$. On montre que le problème de petites valeurs propres (de valeurs propres qui tendent vers 0 aussi vite que n tend vers l'infinie) peut être résolu en imposant l'invariance par rapport à un groupe d'isométries de la surface qu'on construit. Plus précisément, dans les régions de "demi-ponts", l'opérateur $\mathcal{L}_n$ est proche de l'opérateur de Jacobi du demi-caténoïde. Les petites valeurs propres qui correspondent aux rotations et translations du catenoid sont éliminées en imposant les symétries $w(\sigma, \theta) = w(-\sigma, \theta) = w(\sigma, 2\pi - \theta)$ ainsi que la condition homogène de Neumann au bord $\partial_\theta w|_{\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}} = 0$. Les autres petites valeurs propres sont éliminées en imposant l'invariance par la rotation d'angle $\frac{2\pi}{n}$. Enfin dans la région de $\mathcal{A}_n$ paramétrée par un domaine du disque unité, l'opérateur $\mathcal{L}_n$ est proche de l'opérateur L_{gr} (39). On s'intéresse alors aux solutions du problème :

$$\begin{cases} \Delta(Bw) = f & \text{dans} \quad D^2 \setminus \{0\} \\ \partial_r w = 0 & \text{sur} \quad S^1 \setminus \{z_1, \dots, z_n\}. \end{cases} \tag{45}$$

qui est équivalent, après le changement de variables $z \mapsto z^n$, à l'équation de Poisson avec la condition de Robin au bord :

$$\begin{cases} \Delta\, W = F & \text{dans} \quad D^2 \setminus \{0\} \\ \partial_r W - \frac{1}{n} W = 0 & \text{sur} \quad S^1 \setminus \{1\}. \end{cases} \tag{46}$$

Une solution de (46) peut être trouvée en utilisant que pour tout $n \geq 2$ le noyau l'opérateur associé est trivial. D'un autre coté, quand n tend vers l'infinie, (46) converge vers l'équation de Poisson avec la condition homogène de Neumann au bord, qui, à son tour, a un noyau non-trivial. Ça explique (ainsi que la présence des valeurs propores engendrées par les dilations du caténoïde) pourquoi la solution de (46), et en conséquence, la norme de $\mathcal{L}_n^{-1}$, explosent "légèrement" quand n tend vers l'infinie. Cependant, on obtient le résultat en appliquant le théorème de point fixe de Banach dans une boule d'un espace fonctionnel dont le rayon tend vers 0 beaucoup plus vite que la norme de $\mathcal{L}_n^{-1}$ explose.

Chapitre 3: Surfaces non-convexes à courbure moyenne constante dans des variétés riemanniennes de dimension 3

Dans le troisième chapitre, on prouve l'existence dans une variété riemannienne générique de dimension 3 d'une famille de sphères topologiques non-convexes dont la courbure moyenne est grande.

Ces surfaces sont obtenues comme sommes connexes de deux sphères géodésiques de rayons identiques tangentes en un point. Notre construction est possible quand la fonction courbure scalaire $\mathcal{R}$ de la variété ambiante a un point critique o_{cr}, tel que la hessienne de $\mathcal{R}$ en o_{cr} a une valeur propre simple non-nulle.

Plus précisément, soit (M, g) une variété riemannienne lisse de dimension 3. Soient $o_{cr} \in M$ un point critique de la courbure scalaire $\mathcal{R}$, $\lambda \neq 0$ une valeur propre simple de $\mathrm{Hess}_{o_{cr}}\mathcal{R}$ et $v_\lambda \in T_{o_{cr}}M$ le vecteur propre associé. Pour $\varepsilon \in \mathbb{R}_+$ suffisamment petit, on considère l'union $\mathfrak{S}_\#(\varepsilon, o_{cr}, v_\lambda)$ de deux sphères géodésiques de rayon ε tangentes en o_{cr}, dont les centres sont placés de manière symétrique par rapport à o_{cr} sur la géodésique qui passe par o_{cr} avec la vitesse égale à v_λ. On prouve le résultat suivant :

Théorème 0.6.3. *Il existe $\varepsilon_* \in (0,1)$ tel que pour tous $\varepsilon \in (0, \varepsilon_*)$ il existe une surface $\mathfrak{S}_\varepsilon$ à courbure moyenne constante égale à $\frac{2}{\varepsilon}$ telle que la distance de Hausdorff entre $\mathfrak{S}_\varepsilon$ et $\mathfrak{S}_\#(\varepsilon, o_{cr}, v_\lambda)$ est bornée par une constante fois ε^2. La surface $\mathfrak{S}_\varepsilon$ est plongée si $\lambda < 0$ et immergée si $\lambda > 0$.*

Voici une courte description de notre construction.

Pseudo bulles

Rappelons, que dans l'espace euclidien d'effectuer la somme connexe de deux surfaces $\Sigma^\pm$, de point de vue topologique, signifie de faire $\Sigma^\pm$ tangentes en un point, les translater légèrement dans la direction de leur normale commune, puis enlever des petits disques autours des points où on aimerait effectuer la somme connexe, et identifier les bords de ces disques avec les bords d'un petit "cou". R. Mazzeo, F. Pacard et D. Pollack [77] ont montré que si les surfaces $\Sigma^\pm$ sont à courbure moyenne constante, alors leurs somme connexe peut être perturbée, quand la taille du "cou" est assez petite, en une surface qui a la même valeur de la courbure moyenne.

Dans notre cas, on aimerait effectuer la somme connexe de deux sphères géodésiques dont la courbure moyenne n'est pas constante, mais proche d'être constante quand le rayon des sphères est petit. Soit $\mathcal{S}_\varepsilon(p)$ une sphère géodésique de rayon ε centrée en $p \in M$. La courbure moyenne de $\mathcal{S}_\varepsilon(p)$ étant donnée par :

$$H(\mathcal{S}_\varepsilon(p)) = \frac{2}{\varepsilon} + \mathcal{O}(\varepsilon),$$

la première étape serait de perturber $\mathcal{S}_\varepsilon(p)$ en une surface à CMC. Malheureusement, d'après l'observation de R. Ye [118], en générale, ce n'est pas possible à cause de la présence de petites valeurs propres (de valeurs propres qui tendent vers 0 aussi vite que ε tend vers 0) de l'opérateur de Jacobi de $\mathcal{S}_\varepsilon(p)$. En effet, ce dernier est proche de l'opérateur de Jacobi de la sphère euclidienne de rayon ε qui s'écrit sous la forme : $\varepsilon^{-2}\,(\Delta_{S^2}+2)$ et admet un noyau composé des restrictions sur S^2 des fonctions coordonnées :

$$\mathrm{Ker}(\Delta_{S^2}+2) = \{\Theta^1, \Theta^2, \Theta^3\}.$$

Alors, dans un certain sens, le mieux qu'on puisse faire est de perturber $\mathcal{S}_\varepsilon(p)$ en une surface $\Sigma_\varepsilon(p)$ dont la courbure moyenne est constante modulo un élément de $\mathrm{Ker}(\Delta_{S^2}+2)$. Plus précisément, soit $\langle\cdot,\cdot\rangle$ le produit scalaire dans $\mathbb{R}^3$. Alors la courbure moyenne de $\Sigma_\varepsilon(p)$ vérifiera :

$$H(\Sigma_\varepsilon(p))(\Theta) - \frac{2}{\varepsilon} = \langle A, \Theta\rangle, \quad \Theta \in S^2,$$

pour un vecteur $A \in \mathbb{R}^3$. Un calcul explicite [118] montre que

$$A = -\frac{2\,\pi\,\varepsilon^2}{15}\,\nabla^g \mathcal{R}(p) + \mathcal{O}(\varepsilon^4).$$

D'après Naridulli [90], on appelle les surfaces $\Sigma_\varepsilon(p)$ *pseudo bulles*.

"Cou" caténoïdal

La prochaine étape est de trouver une surface qui jouerait le rôle du "cou" dans la construction de la somme connexe. Dans l'espace euclidien ce rôle est la plus part de temps joué par un élément de la famille de caténoïdes :

$$C_\eta\,:\,(s,\phi) \in [-s_\eta, s_\eta] \times S^1 \mapsto (\eta\,\cosh s\,\cos\phi, \eta\,\cosh s\,\sin\phi, \eta\,s)\,,$$

où le paramètre $\eta \in (0,1)$ est appelé "la taille du cou". Dans le cas riemannien, on peut utiliser les propriétés géométriques du caténoïde étant donné, au moins pour η suffisamment petit, un plongement de C_η dans M qui est proche de l'identité dans les coordonnées choisies.

Soient Λ une surface lisse plongée dans M et N_Λ une normale unitaire de Λ. Pour $q \in \Lambda$ l'application

$$F_\Lambda(q', z) = \exp_{q'}\left(z\,N_\Lambda(q')\right)$$

définit un difféomorphisme d'un voisinage de $(q,0)$ dans $\Lambda \times \mathbb{R}$ sur un voisinage de q dans M. Soient $(y^1,y^2) \mapsto \zeta(y^1,y^2)$ des coordonnées locales sur Λ à l'origine en q. Alors l'application

$$F_{\Lambda,q}(y^1,y^2,z) = F_\Lambda(\zeta(y^1,y^2), z)$$

définit un difféomorphisme d'un voisinage de 0 dans $\mathbb{R}^3$ sur un voisinage de q dans M, qu'on appelle *coordonnées de Fermi.* On paramètre le "cou" caténoïdal $\mathfrak{C}_{\eta,q}$ par

$$(s, \phi) \in [-s_\eta, s_\eta] \times S^1 \mapsto F_{\Lambda,q}\,(\eta\,\cosh s\,\cos\phi, \eta\,\cosh s\,\sin\phi, \eta\, s) \in \mathfrak{C}_{\eta,q}.$$

Dans notre cas, il sera pertinent de prendre pour Λ un disque géodésique de rayon suffisamment petit. Plus précisément, on fixe un point $q_0 \in M$ et une géodésique minimisante γ qui passe par q_0 et qu'on appelle "l'axe du cou", puis on plonge un caténoïde de "taille du cou" η dans M via coordonnées de Fermi associées au disque géodésique Λ de rayon ε centré en q_0 et orthogonale à γ.

On verra dans l'argument finale qu'on doit introduire dans notre construction des degrés de liberté supplémentaires et pour cette raison, on considère une famille de "cous" caténoïdaux $\mathfrak{C}_{\eta,q}$ paramétrés par la position $q \in \Lambda$ de leurs "axes" parallèles à γ.

Somme connexe dans une variété riemannienne

Dans une variété riemannienne, à cause de l'absence d'isométries (notamment, l'absence de translations et de rotations), on ne peut pas effectuer la somme connexe entre deux surfaces de la même façon que dans l'espace euclidien. Néanmoins, on pourrait imaginer une procédure analogue si, étant donné une famille de surfaces, paramétrées par exemple par leur positions dans M, on pouvait choisir, pour tout $d > 0$ suffisamment petit, une unique paire d'éléments la distance entre lesquelles (dans le sens de distance entre deux ensembles disjoints dans l'espace métrique M) est égale à d.

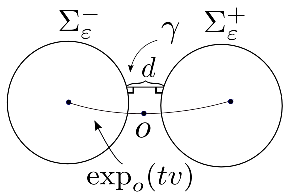
Soit $d \in (0, 1/2\,\varepsilon)$. Un exercice simple est de choisir deux sphères géodésiques de rayon ε à distance d. Pour ça, il suffit de fixer un point $o \in M$ et un vecteur $v \in T_oM$, et de placer les centres des sphères en $\exp_o\left(\pm\,\left(\varepsilon + \frac{d}{2}\right) v\right)$.

On montre qu'une procédure analogue existe pour une famille de pseudo bulles, en utilisant le fait que ces dernières sont des petites perturbations des sphères géodésiques. Plus précisément, on considère la famille

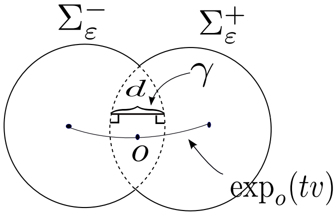
$$\Sigma^{\pm}_{\varepsilon,t} = \Sigma_\varepsilon\,(\exp_o(\pm\, tv))$$

et obtient le résultat suivant

Lemme 0.6.2. *Pour tout $d \in (0, 1/2\,\varepsilon)$ il existe unique $t \in (\varepsilon, 2\varepsilon)$, tel que la distance entre les surfaces $\Sigma^{\pm}_{\varepsilon} := \Sigma^{\pm}_{\varepsilon,t}$ est égale à d, réalisée par une unique géodésique γ, à priori différente de $t \mapsto \exp_o(tv)$.*

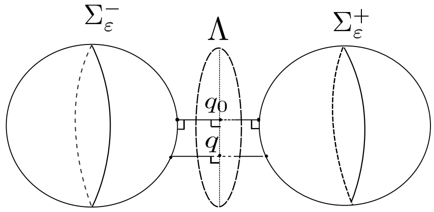


Remark 0.6.2. *On aura également besoin d'effectuer une somme connexe d'une paire de pseudo bulles qui ont une intersection non-vide. Dans ce cas là, on ne peut pas parler de la "distance" entre les surfaces, mais on montre que pour tout $d \in (0, \varepsilon)$, il existe une unique paire de pseudo bulles $\Sigma_\varepsilon^\pm$, dont l'intersection est traversée par une unique géodésique minimisante γ de longueur d qui intersecte $\Sigma_\varepsilon^\pm$ orthogonalement.*



À partir de maintenant, on aura deux sommes connexes, une plongée et l'autre immergée, et on verra dans l'argument finale que, en fonction du signe de la valeur propre $\lambda \neq 0$ de la hessienne de la fonction courbure scalaire $\mathcal{R}$, une de ces sommes connexes peut être perturbée en une surface à courbure moyenne constante.

L'étape suivante est de décrire une procédure de "recollement" entre une paire de pseudo bulles $\Sigma_\varepsilon^\pm$ et un "cou" caténoïdal à "axe" parallèle à la géodésique γ qui réalise la "distance" entre les pseudo bulles. Plus précisément, soit Λ le disque géodésique centré en $q_0 = \gamma\left(\frac{1}{2}\right)$ et orthogonale à γ. Alors, on considère la famille de "cous" $\mathfrak{C}_{\eta,q}$ paramétrée par la "taille du cou" η et la position $q \in \Lambda$ de "l'axe" dans Λ. On remarque que par construction, quand $q = q_0$, "l'axe du cou" est orthogonale à $\Sigma_\varepsilon^\pm$, tandis qu'en variant q, on varie l'angle entre "l'axe" et les normales à $\Sigma_\varepsilon^\pm$.



Fonctions de Green

Selon R. Mazzeo, F. Pacard et D. Pollack [77], de point de vue analytique, pour obtenir une meilleure correspondance avec le comportement asymptotique (logarithmique) d'une caténoïde, c'est mieux de d'abord perturber les surfaces $\Sigma_\varepsilon^\pm$ à l'aide des fonctions de Green associées à leurs opérateurs de Jacobi, puis effectuer la somme connexe.

Soient $J_{\Sigma_\varepsilon^\pm}$ les opérateurs de Jacobi des pseudo bulles $\Sigma_\varepsilon^\pm$. On aimerait définir et étudier les solutions du problème :

$$J_{\Sigma_\varepsilon^\pm}\, \Gamma^\pm = -2\pi\, \delta_{p^\pm}, \tag{47}$$

où $\delta_{p^\pm}$ sont les masses de Dirac en $p^\pm \in \Sigma_\varepsilon^\pm$. D'un autre coté, en paramétrant $\Sigma_\varepsilon^\pm$ par la sphère euclidienne S^2, on trouve que les opérateurs $J_{\Sigma_\varepsilon^\pm}$ s'écrivent sous la forme :

$$J_{\Sigma_\varepsilon^\pm} = \varepsilon^{-2}\,(\Delta_{S^2} + 2) + L, \quad \text{où} \quad \|L\,u\|_{\mathcal{C}^{0,\alpha}(S^2)} \leq c\,\|u\|_{\mathcal{C}^{2,\alpha}(S^2)}.$$

Malheureusement, le fait que l'opérateur $\Delta_{S^2} + 2$ admet un noyau non-trivial nous empêche de trouver directement $\Gamma^{\pm}$ et d'obtenir des estimations raisonnables qui seraient uniformes en ε quand ε tend vers 0. Néanmoins, au lieu de résoudre (47) on peut trouver une solution du problème :

$$J_{\Sigma_\varepsilon^\pm} \Gamma^{\pm} = -2\pi\, \delta_{p^\pm} + \langle B^\pm, \Theta \rangle, \tag{48}$$

pour des vecteurs $B^\pm \in \mathbb{R}^3$. Un calcul explicite donne :

$$B^\pm = \frac{\varepsilon^{-2}}{2}\, \Theta(p^\pm) + \mathcal{O}(1)$$

Solution approchée

On choisit les pôles $p^\pm$ des fonctions $\Gamma^\pm$ comme les points les de l'intersection de "l'axe du cou" γ avec $\Sigma_\varepsilon^\pm$. Alors, aux voisinages de $p^\pm$, les surfaces $\Sigma_\varepsilon^\pm$ peuvent être paramétrées comme graphes normaux sur le disque Λ. Dans les coordonnées normales géodésiques centrés en $q \in \Lambda$, on trouve :

$$\Gamma^\pm(y) = c^\pm \pm \log|y| + \mathcal{O}(|y| \log|y|). \tag{49}$$

D'un autre coté, le "cou" caténoïdal $\mathfrak{C}_{\eta,q}$ peut être vu comme bi-graphe sur le disque géodésique Λ d'une fonction G_η, qui vérifie au voisinage de q :

$$G_\eta = -\eta \log \frac{\eta}{2} + \eta \log|y| + \mathcal{O}\left(\eta^3\, |y|^{-2}\right). \tag{50}$$

On montre que pour tous $o \in M$, $v \in T_oM$ avec $\|v\|_g = 1$, $\eta \in (0,1)$, et $q \in \Lambda$ avec η et $\operatorname{dist}(q, q_0)$ suffisamment petits, on peut choisir la distance d entre les pseudo bulles de telle façon que les termes constants dans les développements limités de $\eta\,\Gamma^\pm$ et $\pm G_\eta$ coïncident exactement, ce qui permet de "recoller" les graphes de $\eta\,\Gamma^\pm$ et le "cou" $\mathfrak{C}_{\eta,q}$ à l'aide d'une fonction troncature.

À cette stade, pour tout ε suffisamment petit, on obtient une surface, qu'on notera $\mathcal{A}_\varepsilon$ et qu'on appellera *solution approchée*, qui dépend de huit paramètres géométriques : le point $o \in M$ (trois degrés de liberté), le vecteur unitaire $v \in T_oM$ (deux degrés de liberté), la "taille" du cou η (un degré de liberté), et la position q de "l'axe du cou" dans le disque géodésique Λ (deux degrés de liberté).

Argument perturbatif

Le prochaine étape est de perturber pour ε suffisamment petit la solution approchée $\mathcal{A}_\varepsilon$ en une surface à courbure moyenne constante. Soient Ξ un champ de vecteurs lisse défini dans un voisinage de $\mathcal{A}_\varepsilon$ dans M et transverse à $\mathcal{A}_\varepsilon$, et $\xi : M \times (0,1) \to M$ le flot associé :

$$\frac{d\xi}{dt} = \Xi(\xi(\cdot, t)), \quad \xi(p, 0) = p, \quad \forall p \in M.$$

Pour $w \in \mathcal{C}^2(\mathcal{A}_\varepsilon)$ suffisamment petit, soit $\mathcal{A}_\varepsilon(w)$ la surface paramétrée par $p \in \mathcal{A}_\varepsilon \mapsto \xi(p, w(p)) \in \mathcal{A}_\varepsilon(w)$. Alors, le développement limité de la courbure moyenne de $\mathcal{A}_\varepsilon(w)$ en puissances de w et les dérivées de w s'écrit sous la forme :

$$H(\mathcal{A}_\varepsilon(w)) = H(\mathcal{A}_\varepsilon) + \mathcal{L}_\varepsilon\, w + \mathcal{Q}_\varepsilon(w, \nabla w, \nabla^2 w),$$

où $H(\mathcal{A}_\varepsilon)$ est la courbure moyenne de la solution approchée, $\mathcal{L}_\varepsilon$ est l'opérateur de courbure moyenne linéarisé et $\mathcal{Q}_\varepsilon$ est une fonction non-linéaire lisse qui vérifie :

$$\mathcal{Q}_\varepsilon(0,0,0) = D\mathcal{Q}_\varepsilon(0,0,0) = 0.$$

Notre but est de trouver une solution de l'équation :

$$\mathcal{L}_\varepsilon\, w = \frac{2}{\varepsilon} - H(\mathcal{A}_\varepsilon) - \mathcal{Q}_\varepsilon(w, \nabla w, \nabla^2 w).$$

Si $\mathcal{L}_\varepsilon$ était inversible avec l'inverse uniformément borné en ε quand ε tend vers 0, alors on pourrait appliquer le théorème de point fixe de Banach à l'application $w \mapsto \mathcal{L}_\varepsilon^{-1}\left(\frac{2}{\varepsilon} - H(\mathcal{A}_\varepsilon) - \mathcal{Q}_\varepsilon(w)\right)$ dans une boule d'un espace de Banach adapté, à condition que la norme de $H(\mathcal{A}_\varepsilon) - \frac{2}{\varepsilon}$ tend vers 0 quand ε tend vers 0.

On étudie d'abord la courbure moyenne de $\mathcal{A}_\varepsilon$. On suppose que η et $\mathrm{dist}(q, q_0)$ soient bornés par une puissance de ε adapté. En utilisant le fait que le caténoïd est minimal dans l'espace euclidien, on trouve que dans la région du "cou", la norme L^∞ de $H(\mathcal{A}_\varepsilon)$ tend vers 0 quand ε tend vers 0. D'un autre coté, on rend la norme de $H(\mathcal{A}_\varepsilon) - \frac{2}{\varepsilon}$ très petite dans une topologie adaptée qui prend en compte que l'aire de la région caténoïdale tend vers 0 beaucoup plus vite que l'aire du reste de $\mathcal{A}_\varepsilon$.

D'un autre coté, en dehors du "cou", $\mathcal{A}_\varepsilon$ est paramétrée comme graphe sur $\Sigma_\varepsilon^\pm$ des fonctions de Green. Soient $\chi^\pm$ des fonctions troncatures à support dans les régions paramétrées par les pseudo bulles. Alors, en utilisant l'expression de la courbure moyenne de $\Sigma_\varepsilon^\pm$ ainsi que les équations vérifiées par les fonctions de Green, on obtient :

$$H(\mathcal{A}_\varepsilon) - \frac{2}{\varepsilon} = \mathcal{H}_\varepsilon + \chi^+ \langle C^+, \Theta\rangle + \chi^- \langle C^-, \Theta\rangle,$$

où la norme de $\mathcal{H}_\varepsilon$ tend très vite vers 0 quand ε tend vers 0 dans un espace fonctionnel adapté et $\langle C^\pm, \Theta\rangle \in \mathrm{Ker}(\Delta_{S^2} + 2)$. Plus précisément, on trouve :

$$C^\pm = -\frac{2\pi\,\varepsilon^2}{15}\,\nabla\mathcal{R}(o^\pm) + \frac{\varepsilon^{-2}\,\eta}{2}\,\Theta(p^\pm) + \mathcal{O}(\varepsilon^4), \tag{51}$$

où $o^\pm$ sont les "centres" des pseudo bulles $\Sigma_\varepsilon^\pm$ et $p^\pm$ sont les pôles des fonctions de Green.

Ensuite, on s'intéresse aux propriétés de l'opérateur linéaire $\mathcal{L}_\varepsilon$. On constate la présence de petites valeurs propres de $\mathcal{L}_\varepsilon$ (de valeurs propres qui tendent vers 0 aussi vite que ε tend vers 0) qui peuvent être identifiées en utilisant la structure de $\mathcal{L}_\varepsilon$.

Plus précisément, dans les régions de $\mathcal{A}_\varepsilon$ paramétrées par les pseudo bulles, l'opérateur $\mathcal{L}_\varepsilon$ est proche de l'opérateur $\varepsilon^{-2}\,(\Delta_{S^2} + 2)$ qui admet un noyau de dimension trois composé des fonctions coordonnées $\Theta^1, \Theta^2, \Theta^3$.

D'un autre coté, dans la région du "cou", $\mathcal{L}_\varepsilon$ est proche de l'opérateur de Jacobi du ceténoïde euclidien. Ce dernier est une surface dégénérée, et en particulier admet deux champs de Jacobi engendrés par les translations horizontales du caténoïde et donnés par les fonctions $\frac{\cos\phi}{\cosh s}$ et $\frac{\sin\phi}{\cosh s}$ qui décroissent très vite à l'infinie.

On note par $\mathfrak{K}_\varepsilon$ l'espace des fonctions propres de $\mathcal{L}_\varepsilon$ associées aux petites valeurs propres. D'après N. Kapouleas, on appelle $\mathfrak{K}_\varepsilon$ *noyau approché*. En utilisant les idées décrites ci-dessus, on montre que la dimension de $\mathfrak{K}_\varepsilon$ est égale à huit, ce qui correspond au nombre de paramètres libres dans notre construction.

Dans l'étape suivante, on utilise la méthode de réduction de Lyapunov-Schmidt, qui consiste à appliquer, pour ε est suffisamment petit, le théorème de point fixe de Banach dans un espace des fonctions orthogonales à $\mathfrak{K}_\varepsilon$ pour perturber la solution approchée $\mathcal{A}_\varepsilon$ en une surface dont la courbure moyenne vérifie :

$$H - \frac{2}{\varepsilon} = \sum_{i=1}^{8} A^i \, \Phi_i, \tag{52}$$

où $\mathfrak{K}_\varepsilon = \mathrm{span}\{\Phi_i, \ i = 1, \dots, 8\}$.

Choix des paramètres

Dans l'argument finale, on explique comment choisir les huit paramètres géométriques pour avoir $H = \frac{2}{\varepsilon}$. Les coefficients A^i dans (52) dépendent de manière continue de $o \in M$, $v \in T_oM$, $\eta \in (0,1)$ et $q \in \Lambda$. On montre que pour ε suffisamment petit, le système d'équations

$$A^i = 0, \quad i = 1, \dots, 8, \tag{53}$$

peut s'écrire sous la forme $(Id + F)(\cdot) = 0$ pour une fonction F bornée uniformément en o, v, η, q, et trouve une solution dans une boule de $\mathbb{R}^8$ grâce au théorème de point fixe de Schauder.

Six premières équations On définit la base $\{\Phi_i\}$ du noyau approché $\mathfrak{K}_\varepsilon$ de telle façon que les six premières fonctions soient proches aux éléments de

$$\mathrm{Ker}(\Delta_{S^2} + 2) = \mathrm{span}\{\Theta^1, \Theta^2, \Theta^3\},$$

définis dans les régions paramétrées par les pseudo bulles. On projette la courbure moyenne de notre surface perturbée sur ces six premières fonctions et remarque que, l'impact de la perturbation étant négligeable, les plus grands termes dans cette projection viennent de la courbure moyenne de la solution approchée $\mathcal{A}_\varepsilon$. D'après (51), les premières six équations de (53) vont s'écrire sous la forme :

$$-\frac{2\,\pi\,\varepsilon^2}{15}\,\nabla^g \mathcal{R}(o^\pm) + \frac{1}{2}\eta\,\varepsilon^{-2}\,\Theta(p^\pm) + \varepsilon^4\,F(o, v, \eta, a) = 0. \tag{54}$$

Avec un léger abus de notation, on peut écrire :

$$o^\pm = o \pm \varepsilon\, v + \mathcal{O}(\varepsilon^2), \qquad \begin{array}{ll} p^\pm = \mp v + \mathcal{O}(\varepsilon^2), & \text{si } \mathcal{A}_\varepsilon \text{ est plongée} \\ p^\pm = \pm v + \mathcal{O}(\varepsilon^2), & \text{si } \mathcal{A}_\varepsilon \text{ est immergée.} \end{array}$$

D'abord, supposons que la surface $\mathcal{A}_\varepsilon$ est plongée. Soit o_{cr} un point critique de la courbure scalaire : $\nabla^g \mathcal{R}(o_{cr}) = 0$. Alors, (54) est équivalent à :

$$\begin{cases} -\frac{2\pi\,\varepsilon^2}{15}\,\mathrm{Hess}_{o_{cr}}\mathcal{R}(o - o_{cr} - \varepsilon\, v) - \frac{1}{2}\,\eta\,\varepsilon^{-2}\,v + \varepsilon^4\,F(o,v,\eta,a) = 0, \\ -\frac{2\pi\,\varepsilon^2}{15}\,\mathrm{Hess}_{o_{cr}}\mathcal{R}(o - o_{cr} + \varepsilon\, v) + \frac{1}{2}\,\eta\,\varepsilon^{-2}\,v + \varepsilon^4\,F(o,v,\eta,a) = 0. \end{cases} \tag{55}$$

Ça implique

$$o - o_{cr} + \varepsilon^2\,F(o,v,\eta,a) = 0.$$

Maintenant, soit v_λ le vecteur propre associé à une valeur propre simple $\lambda \neq 0$ de $\mathrm{Hess}_{o_{cr}}\mathcal{R}$. On écrit

$$v = v_\lambda + \tilde{v},$$

et trouve que la projection de (55) sur v_λ donne :

$$\eta + \frac{4\,\pi\,\lambda\,\varepsilon^5}{15} + \varepsilon^6\,F(o,v,\eta,a) = 0.$$

En particulier, on voit que puisque $\eta > 0$, la solution existe si et seulement si $\lambda < 0$. D'un autre coté, si $\lambda > 0$, on obtient une solution en supposant que $\mathcal{A}_\varepsilon$ est immergée.

Enfin, en projetant (55) sur le sous-espace de $\mathbb{R}^3$ orthogonale à v_λ, on trouve le système :

$$(\mathrm{Hess}_{o_{cr}} - \lambda\,\mathrm{Id})\;\tilde{v} + \varepsilon\,F(o,v,\eta,a) = 0,$$

qu'on peut écrire sous la forme :

$$\tilde{v} + \varepsilon\,F(o,v,\eta,a) = 0.$$

On conclut, qu'une solution de (53) doit vérifier :

$$\mathrm{dist}_g(o,o_{cr}) \leq c\,\varepsilon^2, \quad \angle(v,v_\lambda) \leq c\,\varepsilon, \quad \left|\eta - \mathrm{sign}(\lambda)\,\lambda\,\varepsilon^5\right| \leq c\,\varepsilon^6.$$

Deux dernières équations. Pour comprendre la structure des deux dernières équations, on projette (53) sur les éléments Φ_7 et Φ_8 de $\mathfrak{K}_\varepsilon$ définis dans la région du "cou" et proches des fonctions $\frac{\cos\phi}{\cosh s}$ et $\frac{\sin\phi}{\cosh s}$. En effectuant le changement de variables

$$y = \eta\,\cosh s(\cos\phi, \sin\phi),$$

dans une région où $s \neq 0$, on trouve $\Phi_{6+i} \approx \frac{\eta\,y^i}{|y|^2}$, $i = 1,2$.

Pour expliquer pourquoi ces deux dernières équations s'écrivent sous la forme $(\mathrm{Id} + F)(\cdot) = 0$, on propose de considérer l'exemple suivant.

Soient P_0 le plan horizontal dans R^3, C_η le caténoïde vertical dilaté par $\eta \ll 1$, et P^+, P^- deux plans obtenus comme copies de P_0 légèrement écartées et légèrement penchées. Soient $\rho \in \mathbb{R}$ et $D^2(\rho) := \{x \in \mathbb{R}^2 \,:\, |x| < \rho\}$ le disque de rayon ρ dans P_0. Soit χ une fonction troncature, telle que

$$\chi \equiv 0 \quad \text{dans} \quad D^2(\rho/2) \quad \text{et} \quad \chi \equiv 1 \quad \text{dans} \quad \mathbb{R}^2 \setminus D^2(\rho).$$

Soit y^1, y^2 des coordonnées dans P_0. On peut voir le caténoïde comme bi-graphe sur $\{y \in P_0 : |y| > \eta\}$ de la fonction

$$G_\eta = \eta \log \frac{2}{\eta} + \eta \log |y| + \mathcal{O}(\eta^3 |y|^{-2})$$

et paramétrer $P^\pm$ comme graphes sur P_0 des fonctions affines

$$u^\pm = \pm \eta \log \frac{2}{\eta} + c_1^\pm y^1 + c_2^\pm y^2.$$

Enfin, on considère la surface “recollée”, paramétrée par

$$y \mapsto \Big(y, (1 - \chi(y))\, G_\eta(y) + \chi(y)\, u^\pm(y)\Big).$$

On trouve que la courbure moyenne de cette surface est nulle partout, sauf dans l'anneau $D^2(\rho) \setminus D^2\left(\frac{\rho}{2}\right)$. D'un autre coté, pour η suffisamment petit et $P^\pm$ suffisamment proches de P_0, la courbure moyenne sera proche de $\pm\Delta\,(\chi\, u^\pm)$. Le calcul

$$\begin{aligned} \int_{D^2(\rho)\setminus D^2(\frac{\rho}{2})} \Delta\Big(\chi\, \left(u^+ - u^-\right)\Big) \frac{y^i}{|y|^2} &= \int_{\partial D^2(\rho)} \partial_r\Big(\chi\, \left(u^+ - u^-\right)\Big) \left.\frac{y^i}{|y|^2}\right|_{r=\rho} d\phi \qquad (56) \\ &- \int_{\partial D^2(\rho)} \left(\chi\, \left(u^+ - u^-\right)\right)\, \partial_r \left.\left(\frac{y^i}{|y|^2}\right)\right|_{r=\rho} d\phi \\ &= \frac{4\pi}{\rho} \left(c_i^+ - c_i^-\right). \end{aligned}$$

montre que les plus grands termes dans la projection de la courbure moyenne sur $\frac{y^i}{|y|^2}$ sont déterminés par les pentes $c_i^\pm$ des plans $P^\pm$.

Revenons maintenant à notre problème. Par construction, dans les régions où on effectue le recollement, on peut voir notre surface comme un graphe normale sur le disque Λ d'une fonction

$$u^\pm(y) = u^\pm(q) + \nabla u^\pm(q)\, y + \mathcal{O}(|y|^2), \qquad \left(u^\pm(q_0) = \pm\frac{d}{2}, \quad \nabla u^\pm(q_0) = 0\right).$$

On voit que la “pente” $\nabla u^\pm(q)$ apparaît quand on varie la position q de “l'axe du cou” dans Λ. Alors, les plus grands termes dans la projection de la courbure moyen sur $\Phi_{7,8}$ sont donnés par la projection de $\Delta(\chi^+\, u^+ - \chi^-\, u^-)$ sur $\frac{y^i}{|y|^2}$. Un calcul similaire à (56) montre que les deux dernières équations s'écrivent sous la forme :

$$\frac{\nabla u^+(q) - \nabla u^-(q)}{\rho} + F(o, v, \eta, a) = 0,$$

ou, puisque q_0 est un maximum locale pour u^+ et un minimum locale pour u^-, sous la forme

$$\mathrm{Hess}_{q_0} \left(u^+ - u^-\right) (q - q_0) + \rho\, F(o, v, \eta, a) = 0.$$

Dans notre construction on prend $\rho = \varepsilon^4$ et obtient que la solution doit vérifier $\mathrm{dist}(q, q_0) \leq c\,\varepsilon^4$.

Enfin, le théorème de point fixe de Schauder nous donne l'existence pour tout ε suffisamment petit, de (o, v, η, a) tels que $A^i(o, v, \eta, a) = 0$. La surface correspondante, qu'on note $\mathfrak{S}_\varepsilon$ (plongée quand $\lambda < 0$ et immergée quand $\lambda > 0$) est à courbure moyenne constante. Enfin, si $\mathfrak{S}_\#(\varepsilon, o_{cr}, v_\lambda)$ désigne l'union de deux sphères géodésiques $\mathcal{S}_\varepsilon(\exp_{o_{cr}}(\pm\,\varepsilon v_\lambda))$ tangentes en o_{cr}, alors, la distance de Hausdorff entre $\mathfrak{S}_\varepsilon$ et $\mathfrak{S}_\#(\varepsilon, o_{cr}, v_\lambda)$ est bornée par une constante fois ε^2.

Chapter 1

Higher codimension isoperimetric problems

1.1 Introduction and statement of the result

Constant mean curvature (CMC) hypersurfaces are critical points of the area functional subject to a volume constraint. Examples include sufficiently smooth solutions to the isoperimetric problem. If K is an embedded submanifold in a Riemannian manifold (M^{m+1}, g), then its mean curvature vector H_K is the trace of its second fundamental form. When K is a hypersurface, then we say that K has CMC if this vector has constant length, and this is the only sensible definition in this case. However, when $\operatorname{codim} K > 1$, it is less obvious how to formulate the CMC condition, since there is more than one way one might regard the mean curvature vector as being constant. One definition that has perhaps received the most attention is to require that H_K be parallel. This is quite restrictive, and for that reason, not very satisfactory.

We propose a different, and directly variational definition building on the ideas of F. Almgren [5]. The classical isoperimetric problem amounts to find hypersurfaces K of least m-dimensional volume enclosing a region of prescribed $m+1$ dimensional volume. F. Almgren generalized the isoperimetric problem in higher codimension by defining the volume enclosed by S as the infimum of volumes of $(m+1)$-dimensional submanifolds Q with $\partial Q = S$.

In this chapter, which is a work in collaboration with R. Mazzeo and F. Pacard, extending the standard characterization of CMC hypersurfaces, we propose to define constant mean curvature submanifolds to be boundaries of submanifolds which are critical for a certain energy functional. Roughly speaking, we say that K has constant mean curvature if $K = \partial Q$ where Q is minimal, K has CMC in Q, and H_K has no component orthogonal to Q.

Our goal is to show that generic metrics on any compact manifold admit "small" CMC submanifolds in this sense. The result proved here is a generalization of the theorem by R. Ye [118] described in the subsection 0.1.5, which proves the existence of families of CMC hypersurfaces that are small perturbations of geodesic spheres centered at nondegenerate critical points of the scalar curvature function $\mathcal{R}$ of the ambient manifold M. The more recent paper [95] by F. Pacard and X. Xu obtains such families of CMC hypersurfaces when the scalar curvature is not a Morse function; in that case, these hypersurfaces are centered near critical points of a different curvature invariant.

Let us now introduce the relevant curvature function. For any $(k+1)$-dimensional subspace $\Pi_p \subset T_pM$, define the partial scalar curvature

$$\mathcal{R}_{k+1}(\Pi_p) := -\sum_{i,j=1}^{k+1} g(R(E_i, E_j)E_i, E_j),$$

where $E_1, \ldots, E_{k+1}$ is any orthonormal basis for Π_p. Note that $\mathcal{R}_{m+1}(T_pM)$ is the standard scalar curvature at p, while $\mathcal{R}_2(\Pi_p)$ is twice the sectional curvature of the 2-plane Π_p. The Grassmannian bundle $G_{k+1}(TM)$ is the fiber bundle over M with fiber at $p \in M$ the Grassmannian of all $(k+1)$-planes in T_pM. We regard $\mathcal{R}_{k+1}$ as a smooth function on $G_{k+1}(TM)$.

We denote by $\mathcal{S}^k_\varepsilon(\Pi_p)$ and $\mathcal{B}^{k+1}_\varepsilon(\Pi_p)$ the images of the sphere and ball of radius ε in Π_p under the exponential map $\exp_p$, $p \in M$. We can now state our main result.

Theorem 1.1.1. *If Π_p is a nondegenerate critical point of $\mathcal{R}_{k+1}$, then for all ε sufficiently small, there exists a CMC submanifold $K_\varepsilon(\Pi_p)$ which is a normal graph over $\mathcal{S}^k_\varepsilon(\tilde{\Pi}_{\tilde{p}})$ by some section with $\mathcal{C}^{2,\alpha}$ norm bounded by $c\varepsilon^3$ and* $\operatorname{dist}(\tilde{\Pi}_{\tilde{p}}, \Pi_p) \leq c\,\varepsilon^2$.

Our construction of CMC submanifolds generalizes the method introduced in [95], and can also be carried out in certain cases when the partial scalar curvature has degenerate critical points, for example when (M, g) has constant partial scalar curvature.

Theorem 1.1.2. *There exists $\varepsilon_0 > 0$ and a smooth function*

$$\Psi : G_{k+1}(TM) \times (0, \varepsilon_0) \longrightarrow \mathbb{R},$$

defined in (1.9) *below, such that if $\varepsilon \in (0, \varepsilon_0)$, and Π_p is a critical point of $\Psi(\cdot, \varepsilon)$, then there exists an embedded k-dimensional submanifold $K_\varepsilon(\Pi_p)$ with constant mean curvature equal to k/ε. This submanifold is a normal graph over the geodesic sphere $\mathcal{S}^k_\varepsilon(\Pi_p)$ with respect to a section with $\mathcal{C}^{2,\alpha}$ bounded by $c\,\varepsilon^3$.*

The function Ψ is essentially just the associated energy functional restricted to a particular finite dimensional set of approximately CMC submanifolds.

Existence of CMC submanifolds also follows from the work of F. Morgan and M.C. Salavessa [87] as smooth solutions to the higher codimension isoperimetric problem defined by F. Almgren. Observe that these solutions should correspond to points where $\mathcal{R}_{k+1}$ has a local maximum as in [90].

1.2 Outline of the chapter

The outline of this chapter is as follows. We first give a more careful description of our proposed definition of constant mean curvature and its relationship to the associated energy functional.

We introduce the linearization and the second variation of this energy, then compute these operators in detail for the round sphere $S^k \subset \mathbb{R}^{m+1}$, $k \leq m$. The construction of "small" solutions of the CMC problem concentrating around critical points of the function Ψ proceeds in stages. We construct a family of approximate solutions, then solve the problem up to a finite dimensional defect. This defect depends on certain parameters in the approximate solution, and in the last step we employ a variational argument to choose the parameters appropriately to solve the exact problem. Certain long technical calculations are relegated to the appendices.

1.3 Preliminaries

In this section we begin by setting notations and recalling some standard formulæ. This is followed by the introduction of a variational notion of constant mean curvature for closed submanifolds of arbitrary codimension. We compute the first and the second variations of the associated energy functional, and then explain what these look like for round spheres (of arbitrary codimension) in $\mathbb{R}^{m+1}$.

1.3.1 Mean curvature vector

Let (M^{m+1}, g) be a compact smooth Riemannian manifold. We write ∇^Σ for the induced connection on any embedded submanifold Σ, and reserve ∇ for the full Levi-Civita connection on M.

The second fundamental form of Σ is the symmetric bilinear form on $T\Sigma$ taking values in the normal bundle $N\Sigma$ defined by

$$h_\Sigma(X, Y) := \nabla_X Y - \nabla^\Sigma_X Y = \pi_{N\Sigma} \nabla_X Y, \quad X, Y \in T\Sigma;$$

here $\pi_{N\Sigma}$ is the fibrewise orthogonal projection $T_\Sigma M \to N\Sigma$. The trace of h_Σ is a section of $N\Sigma$, and is called the mean curvature vector field

$$H_\Sigma := \operatorname{tr}^g h_\Sigma = \sum_{i=1}^{\dim \Sigma} h_\Sigma(E_i, E_i),$$

where $\{E_i\}$ is any orthonormal basis for $T\Sigma$. By definition, Σ is minimal provided $H_\Sigma \equiv 0$.

1.3.2 Constant mean curvature in higher codimension

Let us now specialize to the case where $Q^{k+1} \subset M$ is a smooth, compact submanifold with boundary, and $\partial Q =: K$. The normal bundle NK decomposes as an orthogonal direct sum

$$NK = NK^\perp \oplus NK^\|,$$

where $NK^\| = NK \cap TQ$ has rank 1, and $NK^\perp = NK \cap NQ$ has rank $m - k$. We shall write n for the inward pointing unit normal to K in Q. Thus if $\Phi \in NK$, then

$$\Phi = [\Phi]^\perp + [\Phi]^\| = [\Phi]^\perp + \phi\, n$$

for some scalar function ϕ.

Definition 1.3.1. *The closed submanifold $K \subset M$ is said to have constant mean curvature if $K = \partial Q$ where Q is minimal in M, K has constant mean curvature in Q, and the Q-normal component $[H_K]^\perp \in NK^\perp$ vanishes.*

A key motivation is that this definition is variational, where the relevant energy is given by

$$\mathcal{E}_{h_0}(Q) := \mathrm{Vol}_k(\partial Q) - h_0 \, \mathrm{Vol}_{k+1}(Q), \tag{1.1}$$

where h_0 is a constant.

Proposition 1.3.1. *The submanifold $K = \partial Q$ has constant mean curvature h_0 (in the sense of Definition 1.3.1) if and only if*

$$D\mathcal{E}_{h_0}|_Q = 0.$$

The meaning of the differential here is the usual one. Let Ξ be a smooth vector field on M and denote by ξ its associated flow. For t small, write $Q_t = \xi(Q,t)$ and $K_t := \partial Q_t = \xi(K,t)$. The requirement in the Proposition is then that for any smooth vector field Ξ,

$$\frac{d}{dt}\mathcal{E}_{h_0}(Q_t)\bigg|_{t=0} = 0.$$

The proof is standard. The classical first variation formula (see Appendix 1) states that

$$\frac{d}{dt}\mathrm{Vol}(K_t)\bigg|_{t=0} = -\int_K g(H_K, \Xi)\, \mathrm{dvol}_K,$$

and

$$\frac{d}{dt}\,\mathrm{Vol}(Q_t)\bigg|_{t=0} = -\int_Q g(H_Q, \Xi)\, \mathrm{dvol}_Q - \int_K g(n, \Xi)\, \mathrm{dvol}_K\,.$$

It follows directly from this that

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{E}_{h_0}(Q_t) = 0,$$

for all vector fields Ξ if and only if $H_K = h_0\, n$ and $H_Q \equiv 0$, as claimed.

The definition above coincides with the standard meaning of CMC when K is a hypersurface in M which is the boundary of a region Q. Notice that $K^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^{m+1}$ and K has CMC as a hypersurface in $\mathbb{R}^{k+1}$, then it has CMC in the sense of Definition 1.3.1. In particular, round sphere $S^k \subset \mathbb{R}^{m+1}$ has CMC in this sense.

A similar result has been obtained in [87] for stationary submanifolds for the isoperimetric problem in higher codimension.

1.3.3 Jacobi operator in higher codimension

Let us now study the differential of the mean curvature operator, which is known as the Jacobi operator. For this subsection, we revert to considering an arbitrary submanifold Σ, either closed or with boundary, and shall now recall the expression for this operator.

The Jacobi operator J_Σ is the differential of the mean curvature vector field with respect to normal perturbations of Σ. To describe this more carefully, consider the exponential map exp from an ε-neighborhood of the zero section in $T_\Sigma M$ into M. Since $\exp_*|_{\{v=0\}} = \mathrm{Id}$, if $\Phi \in \mathcal{C}^2(\Sigma; N\Sigma)$ has $||\Phi||_{\mathcal{C}^0}$ sufficiently small, then

$$\Sigma_\Phi := \{\exp_q(\Phi(q)) : q \in \Sigma\}$$

is an embedded submanifold. We shall denote the family of submanifolds $\Sigma_{s\Phi}$ by Σ_s, and their mean curvature vector fields by H_s. We also write $F_s : \Sigma \to \Sigma_s$ for the map $q \mapsto \exp_q(s\Phi(q))$. By definition,

$$J_\Sigma(\Phi) = \nabla_{\partial/\partial s} H_s\big|_{s=0} .$$

When $\partial\Sigma \neq \emptyset$, we also require that $\Phi = 0$ on $\partial\Sigma$. The operator $\pi_{N\Sigma} \circ J_\Sigma$ will be denoted J^N_Σ. We recall in Appendix 1 the proof of the standard formula

$$J^N_\Sigma = -\Delta^N_\Sigma + \mathrm{Ric}^N_\Sigma + \mathfrak{H}^{(2)}_\Sigma, \tag{1.2}$$

where Δ^N_Σ is the (positive definite) connection Laplacian on sections of $N\Sigma$,

$$\forall \Phi \in N\Sigma, \quad \Delta^N_\Sigma \Phi = \sum_{i=1}^{\dim(\Sigma)} \nabla^N_{E_i} \nabla^N_{E_i} \Phi - \nabla^N_{\nabla^\Sigma_{E_i} E_i} \Phi,$$

where $\nabla^N_X Y = \pi_{N_\Sigma} \circ \nabla_X Y$ and the other two terms are the following symmetric endomorphisms of $N\Sigma$:

(i) The orthogonal projection $\mathrm{Ric}^N_\Sigma = \pi_{N_\Sigma} \circ \mathrm{Ric}_\Sigma$ on the normal bundle of Σ of the partial Ricci curvature Ric_Σ, defined by

$$\begin{aligned} g\left(\mathrm{Ric}_\Sigma X, Y\right) &:= -\mathrm{tr}^{\,g}\, g\left(R(\cdot, X)\cdot, Y\right)\\ &= -\sum_{i=1}^{\dim \Sigma} g\left(R(E_i, X)E_i, Y\right), \quad \text{for all} \quad X, Y \in TM, \end{aligned} \tag{1.3}$$

(note that the curvature tensor appearing on the right is the one on all of M, and is not the curvature tensor for Σ);

(ii) the square of the shape operator, defined by

$$\mathfrak{H}^{(2)}_\Sigma(X) := \sum_{i,j=1}^{\dim \Sigma} g\left(h(E_i, E_j), X\right)\, h(E_i, E_j), \quad \text{for all} \quad X \in TM. \tag{1.4}$$

In general, $J_\Sigma(\Phi) \neq J^N_\Sigma(\Phi)$ since $J_\Sigma(\Phi)$ has a nontrivial component $J^T_\Sigma(\Phi)$ which is parallel to Σ; as we show later, that part is canceled in our final formula so we do not need to make it explicit. Note, however, that $J^T_\Sigma(\Phi)$ vanishes when Σ is minimal. Indeed, writing the mean curvature vector field to $\Sigma_{s\Phi}$ in the form

$$H_s = \sum_\nu g\left(H_s, N_\nu(s)\right)\, N_\nu(s),$$

where $N_\nu(s)$, $\nu = \dim\Sigma + 1, \ldots, m+1$ is a local orthonormal frame for $N\Sigma_{s\Phi}$ we find

$$[J_\Sigma(\Phi)]^T = \sum_\nu \Big[\Big(g\left(\nabla_{\partial/\partial s} H_s\big|_{s=0}, N_\nu(0)\right) + g\left(H_\Sigma, \nabla_{\partial/\partial s}\big|_{s=0} N_\nu(s)\right)\Big) N_\nu(0)$$

$$+ g\left(H_\Sigma, N_\nu(0)\right) \nabla_{\partial/\partial s}\big|_{s=0} N_\nu \Big]^T = \sum_\nu g\left(H_\Sigma, N_\nu(0)\right) \Big[\nabla_{\partial/\partial s} N_\nu(s)\big|_{s=0}\Big]^T,$$

and if $H_\Sigma = 0$, we have $J^T_\Sigma = 0$.

1.3.4 Linearization about a constant mean curvature submanifold

Let Q be a smooth compact minimal submanifold with a boundary K such that

$$H_K = h_0\, n$$

where n is a unit normal to K in Q and h_0 is a constant. We set

$$\mathcal{C}^{2,\alpha}_0(NQ) := \{V \in \mathcal{C}^{2,\alpha}(NQ) : V|_K = 0\}.$$

With this notation in mind, we have the:

Definition 1.3.2. *The minimal submanifold Q is nondegenerate if*

$$J_Q : \mathcal{C}^{2,\alpha}_0(NQ) \longrightarrow \mathcal{C}^{0,\alpha}(NQ),$$

is invertible.

Lemma 1.3.1. *If Q is nondegenerate, then there is a smooth mapping $\Phi \mapsto Q_\Phi$ from a neighborhood of 0 in $\mathcal{C}^{2,\alpha}(NK)$ into the space of $(k+1)$-dimensional minimal submanifolds of M with $\mathcal{C}^{2,\alpha}$ boundary, such that Q_0 is the initial submanifold Q and $\partial Q_\Phi = K_\Phi$.*

Proof. Fix a continuous linear extension operator

$$\mathcal{C}^{2,\alpha}(NK) \ni \Phi \mapsto V_\Phi \in \mathcal{C}^{2,\alpha}(T_Q M).$$

Thus V_Φ is a vector field along Q which restricts to Φ on K. Without loss of generality, we can assume that $V_\Phi \in TQ$ if $[\Phi]^\perp = 0$ and $V_\Phi \in NQ$ when $[\Phi]^\parallel = 0$. Next, let W be a $\mathcal{C}^{2,\alpha}$ section of NQ which vanishes on K. If both $||\Phi||_{2,\alpha}$ and $||W||_{2,\alpha}$ are sufficiently small, then $\exp_Q(V_\Phi + W)$ is an embedded $\mathcal{C}^{2,\alpha}$ submanifold Q_U with $U = V_\Phi + W$, and $K_\Phi := \partial Q_U$.

Denoting the mean curvature vector of Q_U by $H(\Phi, W)$, we find

$$D_W H|_{(0,0)}(W) = J_Q W.$$

Since Q is minimal, $D_W H|_{(0,0)}(W)$ takes values in NQ, whereas $H(\Phi, W) \in NQ_U \subset T_{Q_U}M$, so we cannot directly apply the implicit function theorem. To remedy this, first let $\widetilde{H}(\Phi, W)$ be the parallel transport of $H(\Phi, W)$ along the geodesic $s \mapsto \exp_q(sU(q))$, from $s = 1$ to $s = 0$. Parallel transport preserves regularity (this reduces to the standard result on smooth dependence on initial conditions for the solutions of a family of ODE's), so $\widetilde{H}(\Phi, W)$ is a $\mathcal{C}^{0,\alpha}$ section of $T_Q M$. Now define

$$\widehat{H}(\Phi, W) := \pi_{NQ} \circ \widetilde{H}(\Phi, W),$$

where $\pi_{NQ} : T_Q M \to NQ$ is the orthogonal projection. Since $H(\Phi, W) \in N_{Q_U}M$ and since $||U||_{\mathcal{C}^1}$ is small, $\widetilde{H}(\Phi, W)$ lies in the nullspace of π_{NQ} at any $q \in Q$ if and only if it actually vanishes. Thus it is enough to look for solutions of $\widehat{H}(\Phi, W) = 0$. Notice that $D_W \widehat{H}|_{(0,0)} = J_Q$. We can now apply the implicit function theorem to conclude the existence of a $\mathcal{C}^{2,\alpha}$ map $\Phi \mapsto W(\Phi)$ such that $H(\Phi, W(\Phi)) = \widehat{H}(\Phi, W(\Phi)) \equiv 0$ for all small Φ.

$\square$

We henceforth denote by Q_Φ the minimal submanifold $\exp_Q(V_\Phi + W(\Phi))$. Observe that when $[\Phi]^\perp = 0$, the submanifold parametrized by $\exp_Q(V_{[\Phi]^\parallel})$ is $\mathcal{O}(\|\Phi\|^2_{\mathcal{C}^{2,\alpha}})$ close to Q_Φ; this is easy to check when $\Phi := \phi\, n$ where ϕ is small. Therefore, in this 'tangential' case, we conclude that

$$U_\Phi = V_{[\Phi]^\parallel} + \mathcal{O}(\|\Phi\|^2_{\mathcal{C}^{2,\alpha}}).$$

Next, when $[\Phi]^\parallel = 0$, we define $Z_{[\Phi]^\perp}$ as the solution of

$$J_Q\, Z_{[\Phi]^\perp} = 0, \qquad Z_{[\Phi]^\perp}\Big|_K = \Phi^\perp,$$

and it is easy to check that the submanifold parametrized by $\exp_Q(Z_{[\Phi]^\perp})$ is also $\mathcal{O}(\|\Phi\|^2_{\mathcal{C}^{2,\alpha}})$ close to Q_Φ. We summarize all this in the

Lemma 1.3.2. *When $\|\Phi\|_{\mathcal{C}^{2,\alpha}}$ is small, we have the decomposition*

$$U_\Phi = V_{[\Phi]^\parallel} + Z_{[\Phi]^\perp} + \mathcal{O}(\|\Phi\|^2_{\mathcal{C}^{2,\alpha}}).$$

Now consider the energy $\mathcal{E}_{h_0}$ along a one-parameter family $s \mapsto Q_s := Q_{s\Phi}$ of minimal submanifolds with boundaries $K_s := \partial Q_s = K_{s\Phi}$. By the formulæ of the last subsection,

$$\frac{d}{ds}\mathcal{E}_{h_0}(Q_s) = -\int_{K_s} g(H_s - h_0\, n_s, \partial/\partial s)\, \mathrm{dvol}_{K_s},$$

where H_s is the mean curvature of K_s and n_s is the inward pointing unit normal to K_s in Q_s. Note that this first variation of energy is localized to the boundary; the interior terms vanish because of the minimality of the Q_s. Our task is to compute

$$\frac{d^2}{ds^2}\mathcal{E}_{h_0}(Q_s)\Big|_{s=0},$$

when Q is critical for $\mathcal{E}_{h_0}$.

Parametrize both K_s and Q_s by $y \mapsto F_s(y) := \exp_y(U_{s\Phi}(y))$ (with $y \in K$ or $y \in Q$, respectively). As before, choose a smooth local orthonormal frame E_α for TK, so that $(F_s)_* E_\alpha = E_\alpha(s)$ is a local (non-orthonormal) frame for $TK_{s\Phi}$. We then include n_s, the unit inward normal to K_s in Q_s. Moreover, we extend n_s to a vector $\bar{n}_s \in TQ_s$ so that it satisfies $\nabla^{Q_s}_{\bar{n}_s} \bar{n}_s = 0$. We supplement this to a complete local frame for $T_{Q_s}M$ (at least near points of K_s) by adding a local orthonormal frame $N_\mu(s) \in NQ_s$. Here we let the indexes $\alpha, \beta, \ldots$ run from 1 to k while $\mu, \nu, \ldots$ run from $k+2$ to $m+1$.

Notation 1.3.1. *Set $\mathcal{H}_s = H(K_s) - h_0\, n_s$. We also write*

$$L_Q = \nabla_{\partial/\partial s} \mathcal{H}_s \big|_{s=0} .$$

Note that we can decompose $\mathcal{H}'(0)$ into $\mathcal{H}'(0)^{N_K} + \mathcal{H}'(0)^{T_K}$, its components perpendicular and parallel to K. Since $\mathcal{H}(s) \perp K_s$, we have that $\langle \mathcal{H}(s), E_\alpha(s) \rangle = 0$, so

$$\langle \mathcal{H}'(0), E_\alpha \rangle + \langle \mathcal{H}(0), E'_\alpha(0) \rangle = 0.$$

Since $\mathcal{H}(0) = 0$, we obtain $\pi_{TK} \circ L_Q = 0$.

Next decompose $\Phi = [\Phi]^\perp + \phi\, n$ into parts perpendicular and parallel to Q (along K). Note that we can choose the vector field U_Φ extending Φ and defined in Lemma 1.3.1 so that its component tangent to Q lies in the span of $\bar{n}$. More precisely, we have a decomposition $U_\Phi = [U_\Phi]^\perp + u_\phi\, \bar{n}$ locally near K_Φ, where $[U_\Phi]^\perp\big|_K = [\Phi]^\perp$ and $u_\phi\big|_K = \phi$.

To see that $E'_\alpha(0) = \nabla_{E_\alpha} \Phi$, choose a curve $c(t)$ in K with $c(0) = p$, $c'(0) = E_\alpha$ and define $G(t,s) = \exp_{c(t)}(s\Phi(c(t)))$; we then obtain that

$$\nabla_{\partial/\partial s} E_\alpha \big|_{s=0} = \nabla_{\partial/\partial s} \nabla_{\partial/\partial t} G(t,s) \big|_{s=t=0} = \nabla_{\partial/\partial t} \Phi(c(t)) \big|_{t=0} = \nabla_{E_\alpha} \Phi,$$

as claimed. To compute $n'(0)$, observe that $(F_s)_*(n(0))$ is always tangent to Q_s and transverse, but not necessarily a unit normal, to K_s. We can adjust it, using the Gram-Schmidt process, to get that

$$n_s = \Big((F_s)_*(n(0)) - \sum c_\alpha E_\alpha(s) \Big) \Big/ \Big| ((F_s)_*(n(0)) - \sum c_\alpha E_\alpha(s) \Big| ,$$

where

$$c_\alpha(s) = \langle E_\alpha(s), (F_s)_* n(0) \rangle / |E_\alpha(s)|^2.$$

Arguing as before, take a curve $d(t)$ in Q such that $d(0) = p$ and $d'(0) = n$ and define $\tilde{G}(t,s) = \exp_{d(t)}(U_{s\Phi}(d(t)))$. Note that $U_{s\Phi} = s(V_{[\Phi]^\parallel} + Z_{[\Phi]^\perp}) + \mathcal{O}(s^2 \|\Phi\|^2_{\mathcal{C}^{2,\alpha}})$. We get

$$\nabla_{\partial/\partial s} (F_s)_* n(0) \big|_{s=0} = \nabla_{\partial/\partial s} \nabla_{\partial/\partial t} \tilde{G}(t,s) \Big|_{t=s=0} = \nabla_n (V_{[\Phi]^\parallel} + Z_{[\Phi]^\perp})$$

and since $c_\alpha(0) = 0$, we obtain

$$[n'(0)]^\perp = \Big[\nabla_n V_{[\Phi]^\parallel} + \nabla_n Z_{[\Phi]^\perp} \Big] \Big|_K^\perp = \Big[\nabla_n^\perp Z_{\Phi^\perp} + \phi\, \nabla_n^\perp \bar{n} \Big] \Big|_K .$$

Finally, the component $[n'(0)]^\parallel = 0$. Combining these calculations gives the

Proposition 1.3.2. *If Q is critical for $\mathcal{E}_{h_0}$, then*

$$L_Q\,\Phi = (\pi_{NK} \circ J_K - h_0\,D_Q)\;\Phi,$$

where

$$D_Q\Phi = \Big[\nabla^{\perp}_{n} Z_{\Phi} + \phi\,\nabla^{\perp}_{n}\bar{n}\Big]\Big|_{K}.$$

1.3.5 Linearization about the Euclidean sphere of higher codimension

We conclude this section by discussing the precise form of this linearization, and its nullspace, when

$$K = S^k \times \{0\} \subset Q = B^{k+1} \times \{0\} \subset \mathbb{R}^{m+1},$$

since this is our basic model later. It is easy to see that B^{k+1} is critical for $\mathcal{E}_k$.

The unit inward normal to S^k in B^{k+1} is $n_{S^k}(\Theta) = -\Theta$. If $\Phi \in \mathcal{C}^{2,\alpha}(NS^k)$, then

$$\Phi = [\Phi]^{\perp} - \phi\,\Theta,$$

where the first term on the right is perpendicular to B^{k+1}. The operator $J^N_{S^k}$ acts on these two components separately, via $J^{\perp}_{S^k}$ and $J^{||}_{S^k}$, respectively.

The first of these operators acts on sections of the trivial bundle of rank $m-k$. Obviously, $\mathrm{Ric}^N_{S^k} = 0$, cf. (1.3), and $(\mathfrak{H}^{(2)}_{S^k})^{\perp} = 0$ as well, so

$$J^{\perp}_{S^k} = \Delta_{S^k}$$

acting on $(m-k)$-tuples of functions. Its eigenvalues are $\ell(k+\ell-1)$. The operator $D_{B^{k+1}}$ also acts on sections of the trivial bundle $NB^{k+1}\big|_{S^k}$. In fact, since $J_{B^{k+1}} = \Delta_{B^{k+1}}$, this operator is simply the standard Dirichlet-to-Neumann operator for the Laplacian (acting on $\mathbb{R}^{m-k}$-valued functions). Its eigenfunctions are the restrictions to $r=1$ of the homogeneous harmonic polynomials $P(x)$, $x = r\Theta$, $\Theta \in S^k$. If P is homogeneous of order ℓ, then $P(x) = r^{\ell}P(\Theta)$, so $D_{B^{k+1}}P(\Theta) = -\ell P(\Theta)$ (recall we are using the inward-pointing normal). Combining these two operators, we see that $\Delta_{S^k} - kD_{B^{k+1}}$ has eigenvalues $-\ell(k+\ell-1)+k\ell = -\ell(\ell-1)$, hence

$$\begin{aligned}&\Big(J^{\perp}_{S^k} - kD_{B^{k+1}}\Big)\,[\Phi]^{\perp} = 0\\ &\qquad\qquad \Rightarrow\quad [\Phi]^{\perp} \in \mathrm{span}\{\,(a_\mu + b_\mu\Theta^j)E_\mu,\quad j=1,\dots,k+1,\ \ \mu = k+2,\dots,m+1\},\end{aligned}$$

where E_μ, $\mu = k+2,\dots,m+1$ is an orthonormal basis for $NB^{k+1} = \mathbb{R}^{m-k}$.

The remaining part is

$$J^{||}_{S^k} = \Delta_{S^k} + k,$$

since $\mathrm{Ric}_{S^k} = 0$ and $\mathfrak{H}^{(2)}_{S^k} = k\,\mathrm{Id}$. Thus

$$J^{||}_{S^k}(\phi\,\Theta) = J^{||}_{S^k}(\phi)\,\Theta = 0 \;\Rightarrow\; \phi \in \mathrm{span}\,\{\Theta^1,\dots,\Theta^{k+1}\}.$$

We have now shown that the nullspace $\mathcal{K}$ of $L_{B^{k+1}}$ splits as $\mathcal{K}^{\perp} \oplus \mathcal{K}^{||}$. The first of these summands is comprised by infinitesimal translations in $\mathbb{R}^{m-k}$ and rotations in the $j\mu$ planes (now $j \le k+1$); the second summand corresponds to infinitesimal translations in $\mathbb{R}^{k+1}$.

1.4 Construction of constant mean curvature submanifolds

We now turn to the main task of this paper, which is to construct small constant mean curvature submanifolds concentrated near the critical points of $\mathcal{R}_{k+1}$. The first step is to define a family of approximate solutions, i.e. a family of pairs $(Q_\varepsilon, K_\varepsilon)$ where Q_ε is minimal and has nearly CMC boundary. We then use a variational argument to perturb this to a minimal submanifold with exactly CMC boundary.

1.4.1 Approximate solutions

We adopt all the notations used earlier. Thus we fix $\Pi_p \in G_{k+1}(TM)$ and an orthonormal basis E_i, $1 \le i \le m+1$ of T_pM, where E_j, $1 \le j \le k+1$ span Π_p and E_μ, $\mu \ge k+2$, span $\Pi_p^\perp$. This induces a Riemann normal coordinate system $(x^1, \ldots, x^{m+1})$ near p, and it is standard that

$$g_{ij}(x) = g(\partial_{x^i}, \partial_{x^j}) = \delta_{ij} + \frac{1}{3}\sum_{k,\ell}(R_p)_{ikj\ell}\, x^k x^\ell + \mathcal{O}(|x|^3), \tag{1.5}$$

where δ is the Euclidean metric.

Rescaling

In terms of the map $F_\varepsilon : T_pM \to M$, $F_\varepsilon(v) = \exp_p(\varepsilon v)$, used earlier, define the metric

$$g_\varepsilon = \varepsilon^{-2} F_\varepsilon^* g$$

on T_pM, or equivalently, work in the rescaled coordinates $y^j = x^j/\varepsilon$. In either case,

$$g_\varepsilon = |dy|^2 + \varepsilon^2 h_\varepsilon(y, dy), \tag{1.6}$$

where h_ε is family of smooth symmetric two-tensors depending smoothly on $\varepsilon \in [0, \varepsilon_0]$. The mean curvature vectors H^g and H^{g_ε} with respect to g and g_ε satisfy

$$\varepsilon^2 H^g = (F_\varepsilon)_* H^{g_\varepsilon}, \quad \text{and} \qquad \|H^{g_\varepsilon}\|_{g_\varepsilon} = \varepsilon\, \|H^g\|_g.$$

Let $B^{k+1} = B^{k+1}(\Pi_p) \subset \Pi_p$ be the unit ball and $S^k = S^k(\Pi_p) = \partial B^{k+1}$, and denote their images under F_ε by $\mathcal{B}_\varepsilon^{k+1}(\Pi_p)$ and $\mathcal{S}_\varepsilon^{k+1}(\Pi_p)$. These have parametrizations

$$S^k \ni \Theta \longmapsto \exp_p^g\left(\varepsilon \sum_{j=1}^{k+1} \Theta^j\, E_j\right), \quad B^{k+1} \ni y \longmapsto \exp_p^g\left(\varepsilon \sum_{j=1}^{k+1} y^j\, E_j\right).$$

In the lemmas (1.4.1) and (1.4.2) below we give the expansion of the mean curvature of $\mathcal{B}_\varepsilon^{k+1}(\Pi_p)$ and $\mathcal{S}_\varepsilon^k(\Pi_p)$ in terms of ε. To this end we introduce two supplementary curvature invariants which are restrictions of the Ricci curvature of the ambient manifold M:

Notation 1.4.1.

$$\mathcal{R}ic_{k+1}(\Pi_p)(v_1,v_2) = -\sum_{i=1}^{k+1} g_p(R_p(E_i,v_1)E_i,v_2), \qquad v_1,v_2\in\Pi_p$$

$$\mathcal{R}ic^{\perp}_{k+1}(\Pi_p)(v,N) = -\sum_{i=1}^{k+1} g_p(R_p(E_i,v)E_i,N), \qquad v\in\Pi_p,\ N\in\Pi_p^{\perp}.$$

Note that

$$\mathcal{R}ic^{\perp}_{k+1}(\Pi_p) = \left[\mathrm{Ric}^{N}_{\mathcal{B}^{k+1}_{\varepsilon}(\Pi_p)}\right]_p.$$

Moreover, here and below, we write $\mathcal{O}(\varepsilon^k)$ for a function with $\mathcal{C}^{0,\alpha}$ norm bounded by $C\varepsilon^k$.

Lemma 1.4.1. *The mean curvature of the geodesic ball $\mathcal{B}^{k+1}_{\varepsilon}(\Pi_p)$*

$$H^g(\mathcal{B}^{k+1}_{\varepsilon}(\Pi_p))(y) = \sum_{\mu=k+2}^{m+1}\left(\frac{2\,\varepsilon}{3}\,\mathcal{R}ic^{\perp}_{k+1}(\Pi_p)(y,E_\mu)+\mathcal{O}(\varepsilon^2)\right)\mathcal{N}_\mu, \qquad y\in B^{k+1}$$

where $\mathcal{N}_\mu$, $k+2\le\mu\le m+1$ is an orthonormal basis of $N\mathcal{B}^{k+1}_{\varepsilon}(\Pi_p)$.

Proof. Recall that

$$H^g(\mathcal{B}^{k+1}_{\varepsilon}(\Pi_p)) = \frac{1}{\varepsilon^2}\,(F_\varepsilon)_*\,H^{g_\varepsilon}(B^{k+1}).$$

We denote $\mathcal{N}^{\varepsilon}_{\mu}$, $k+1<\mu<m+1$ the orthonormal basis of the normal bundle of B^{k+1} with respect to the metric g_ε obtained by applying the Gram-Schmidt process to the vectors $E_i(p)$, $1\le i\le m+1$. Remark that

$$g_\varepsilon(\mathcal{N}^{\varepsilon}_{\mu},E_\nu) = \delta_{\mu\nu}+\mathcal{O}(\varepsilon^2), \quad \mu=k+1,\dots,m+1.$$

Remark, that the vector fields $\mathcal{N}_\mu = \frac{1}{\varepsilon}\,(F_\varepsilon)_*(\mathcal{N}^{\varepsilon}_{\mu})$ form an orthonormal basis of $N\mathcal{B}^{k+1}_{\varepsilon}(\Pi_p)$ with respect to the metric g.

The Christoffel symbols corresponding to the metric g_ε are:

$$\begin{aligned}
(\Gamma^{g_\varepsilon})^{\ell}_{ij}(y) &= \frac{1}{2}\,g^{\ell q}_{\varepsilon}\big(\partial_{y^j}(g_\varepsilon)_{iq}+\partial_{y^i}(g_\varepsilon)_{jq}-\partial_{y^q}(g_\varepsilon)_{ij}\big)\\
&= \delta^{\ell q}\,\frac{\varepsilon^2}{6}\,y^p\,\left(R_{ijqp}+R_{ipqj}+R_{jiqp}+R_{jpqi}-R_{iqjp}-R_{ipjq}\right)+\mathcal{O}(\varepsilon^3)\\
&= -\frac{\varepsilon^2}{3}\,(R_{ipj\ell}+R_{i\ell jp})\,y^p+\mathcal{O}(\varepsilon^3),
\end{aligned}$$

whence

$$g_\varepsilon(\nabla^{g_\varepsilon}_{\partial_{y^i}}\partial_{y^j},\mathcal{N}^{\varepsilon}_{\mu}) = (\Gamma^{g_\varepsilon})^{\mu}_{ij}+\mathcal{O}(\varepsilon^4)$$

Taking the trace in the indexes $i,j=1,\dots,k+1$ with respect to g_ε gives the result.

□

Lemma 1.4.2. *The mean curvature of the geodesic sphere $\mathcal{S}^k_\varepsilon(\Pi_p)$ satisfies*

$$H^g(\mathcal{S}^k_\varepsilon(\Pi_p)) = \left(\frac{k}{\varepsilon} - \frac{\varepsilon}{3}\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) + \mathcal{O}(\varepsilon^2)\right) n_{\mathcal{S}}$$

$$+ \sum_{\mu=k+2}^{m+1}\left(\frac{2\,\varepsilon}{3}\mathcal{R}ic^{\perp}_{k+1}(\Pi_p)(\Theta,E_\mu) + \mathcal{O}(\varepsilon^2)\right)\mathcal{N}_\mu, \qquad \Theta \in S^k,$$

where $n_{\mathcal{S}}$ is a unit normal vector field to $\mathcal{S}^k_\varepsilon(\Pi_p)$ in $\mathcal{B}^{k+1}_\varepsilon(\Pi_p)$ with respect to the metric g.

Proof. The proof is similar to that of the previous lemma, but with several changes. Let $u^1,\ldots,u^k \mapsto \Theta(u^1,\ldots,u^k)$ be a local parametrization of $S^k \subset \Pi_p$. The tangent bundle TS^k is spanned by the vector fields $\Theta_\alpha = \partial_{u^\alpha}\Theta$, $\alpha = 1,\ldots,k$. We remark that

$$H^g(\mathcal{S}^k_\varepsilon(\Pi_p)) = \frac{1}{\varepsilon^2}\,(F_\varepsilon)_*\,H^g_\varepsilon(S^k).$$

By Gauss's lemma,

$$g\left((F_\varepsilon)_*\Theta_\alpha,(F_\varepsilon)_*\Theta\right)(F_\varepsilon(\Theta)) = g_p(\Theta_\alpha,\Theta) = 0,$$

for $\alpha = 1,\ldots,k$, hence, we put $n_{\mathcal{S}} := -\frac{1}{\varepsilon}\,(F_\varepsilon)_*\Theta$. We have

$$\nabla^{g_\varepsilon}_{\partial_{u^\alpha}}\partial_{u^\beta} = \partial_{u^\alpha}\,\partial_{u^\beta}\Theta + (\Gamma^{g_\varepsilon})^\ell_{ij}(\Theta_\alpha)^i(\Theta_\beta)^j\,E_\ell,$$

$\alpha,\beta = 1,\ldots,k$; $i,j,\ell = 1,\ldots,m+1$. Since he vector field $\partial_{u^\alpha}\,\partial_{u^\beta}\Theta$ is tangent to $B^{k+1}(\Theta)$, we find

$$g_\varepsilon\left(\nabla^{g_\varepsilon}_{\partial_{u^\alpha}}\partial_{u^\beta},\mathcal{N}^\varepsilon_\mu\right) = (\Gamma^{g_\varepsilon})^\mu_{ab}\,(\Theta_\alpha)^a\,(\Theta_\beta)^b + \mathcal{O}(\varepsilon^3).$$

Taking trace in the indexes α,β with respect to the metric induced on S^k from g_ε we get

$$g_\varepsilon(H^{g_\varepsilon}(S^k),\mathcal{N}^\varepsilon_\mu) = \frac{2\,\varepsilon^2}{3}\,\mathcal{R}ic^{\perp}_{k+1}(\Pi_p)(\Theta,E_\mu) + \mathcal{O}(\varepsilon^3).$$

In order to find $[H^{g_\varepsilon}(S^k(\Pi_p))]^{||}$, recall the standard fact that if $\Sigma \subset M$ is an oriented hypersurface with unit inward pointing normal N_Σ, and if Σ_z is the family of hypersurfaces defined by

$$\Sigma \times \mathbb{R}(q,z) \mapsto \exp_q(zN_\Sigma(q)) \in \Sigma_z,$$

with induced metric g_z, then

$$|H_\Sigma| = -\frac{d}{dz}\;\log\sqrt{\det g_z}.$$

In our case, considering $S^k = \partial B^{k+1}$ with metric g_ε, let $g_{\varepsilon z}$ be the induced metrics on the Euclidean sphere of radius $1-z$. Then,

$$\det g_{\varepsilon z} = (1-z)^{2k}\det g^S\left(1 - \frac{\varepsilon^2(1-z)^2}{3}\mathcal{R}ic_{S^k}(\Pi_p)(\Theta,\Theta) + \mathcal{O}(\varepsilon^3)\right),$$

where g^S is the standard spherical metric on $S^k(\Pi_p)$. From this we deduce that

$$g_\varepsilon\left(H^{g_\varepsilon}(S^k), -\Theta\right) = k - \frac{\varepsilon^2}{3}\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) + \mathcal{O}(\varepsilon^3).$$

this completes the proof.

$\square$

Proposition 1.4.1. *Fix $\Pi_p \in G_{k+1}(TM)$. Then for $\varepsilon > 0$ small enough, there exists a minimal submanifold $Q_\varepsilon(\Pi_p)$ which is a small perturbation of $\mathcal{B}_\varepsilon^{k+1}(\Pi_p)$, whose boundary $K_\varepsilon(\Pi_p) = \partial Q_\varepsilon(\Pi_p)$ is a normal graph over $S_\varepsilon^k(\Pi_p)$ and whose mean curvature vector field satisfies*

$$H^g(K_\varepsilon(\Pi_p))(\Theta) - \frac{k}{\varepsilon}\,n_K = \langle \vec{a}, \Theta\rangle\, n_K + \sum_{\mu=k+2}^{m+1} \left(\langle \vec{c}_\mu, \Theta\rangle + d_\mu\right) N_\mu, \tag{1.7}$$

for some constant vectors $\vec{a} = \vec{a}(\varepsilon,\Pi_p)$, $\vec{c}_\mu = \vec{c}_\mu(\varepsilon,\Pi_p) \in \mathbb{R}^{k+1}$ and constants $d_\mu = d_\mu(\varepsilon,\Pi_p) \in \mathbb{R}$ and where by $\langle\cdot,\cdot\rangle$ we denote the scalar product in $\mathbb{R}^{m+1}$. Here n_K is a normal vector field to $K_\varepsilon(\Pi_p)$ in $Q_\varepsilon(\Pi_p)$ and N_μ, $\mu = k+2,\dots,m+1$ form an orthonormal basis of $[NK_\varepsilon(\Pi_p)]^\perp$.

Proof. Take a vector field $\Phi \in \mathcal{C}^{2,\alpha}(T_pM)$ defined along the unit sphere $S^k(\Pi_p)$, such that

$$\Phi(\Theta) = -\phi(\Theta)\,\Theta + \sum_{\mu=k+2}^{m+1} \Phi^\mu(\Theta)\,E_\mu,$$

and write

$$S_\Phi^k = \left\{\Theta + \Phi(\Theta),\ \Theta \in S^k\right\}.$$

Then there exists a submanifold $B_{\varepsilon,\Phi}^{k+1}$ such that $\partial B_{\varepsilon,\Phi}^{k+1} = S_\Phi^k$ and which is minimal with respect to g_ε. The proof of this fact is almost the same as the proof of the Lemma (1.3.1); the only difference is that we use a "perturbed" metric and the starting submanifold is no longer minimal. Let V_Φ be a linear extension of Φ in B^{k+1} and take

$$W \in \mathcal{C}^{2,\alpha}(T_pM), \quad W = \sum_{\mu=k+2}^{m+1} W^\mu\,E_\mu, \quad W|_{S^k} = 0.$$

We put $U(y) := V_\Phi(y) + W(y)$ and let $H(\varepsilon,\Phi,W)$ denote the mean curvature with respect to the metric g_ε of the submanifold

$$B_U^{k+1} := \left\{y + U(y),\ y \in B^{k+1}\right\}.$$

Note that $H(0,0,0) = 0$ and

$$D_3H|_{(0,0,0)} = J_{B^{k+1}} = \Delta_{B^{k+1}}.$$

We can then apply the implicit function theorem to $\hat{H}(\varepsilon, \Phi, W) = \pi \circ H(\varepsilon, \Phi, W)$, where π is the orthogonal projection onto the subspace of T_pM spanned by E_μ, $k+2 \le \mu \le m+1$. Then for ε and $\|\Phi\|_{\mathcal{C}^{2,\alpha}}$ small enough, there exists a mapping $(\varepsilon, \Phi) \mapsto W(\varepsilon, \Phi)$ such that

$$\hat{H}(\varepsilon, \Phi, W(\varepsilon, \Phi)) = 0 \quad \text{and} \quad H(\varepsilon, \Phi, W(\varepsilon, \Phi)) = 0.$$

Moreover, we can write

$$U_{\varepsilon,\Phi} = V_\Phi + W(\varepsilon, \Phi) = V_\phi + Z_\Phi + W_\varepsilon + \mathcal{O}(\|\varepsilon^3\|) + \mathcal{O}(\varepsilon^2\| \Phi\|) + \mathcal{O}(\|\Phi^2\|),$$

where $V_\phi(y) = -\phi(y/\|y\|)\, y$, the vector field Z_Φ is the harmonic extension of Φ in B^{k+1} and W_ε satisfies

$$\Delta_{B^{k+1}} W_\varepsilon^\mu = -\frac{2\,\varepsilon^2}{3}\, \mathcal{R}ic^\perp_{k+1}(\Pi_p)(y, E_\mu), \quad W_\varepsilon = 0 \quad \text{on} \quad S^k.$$

Remark 1.4.1. *A simple calculation shows that*

$$W_\varepsilon(y) = \frac{\varepsilon^2}{3}\,\frac{1}{k+3}\,(1 - |y|^2) \sum_{\mu=k+2}^{m+1} \mathcal{R}ic^\perp_{k+1}(\Pi_p)(y, E_\mu)\, E_\mu.$$

As a next step, we calculate the mean curvature of S^k_Φ with respect to the metric g_ε. First note that the vector fields

$$\tau_\alpha = (1-\phi)\,\Theta_\alpha - \partial_{u_\alpha}\phi\,\Theta + \sum_{\mu=k+2}^{m+1} \partial_{u_\alpha}\Phi^\mu\, E_\mu$$

locally frame TS^k_Φ, while

$$\Theta_\Phi = \Theta + \frac{1}{1-\phi}\,\nabla_{S^k}\phi, \quad \text{and} \quad (E_\mu)_\Phi = E_\mu - \frac{1}{1-\phi}\,\nabla_{S^k}\Phi^\mu$$

are a local basis for the normal bundle of S^k_Φ with respect to the Euclidean metric. Applying the Gram-Schmidt process with respect to the metric g_ε to these local frames yields the unit normal to S^k_Φ in $B^{k+1}_{\varepsilon,\Phi}$, which we denote n^ε_Φ, and the orthonormal frame $(\mathcal{N}_\Phi)^\varepsilon_\mu$ for the normal bundle of $B^{k+1}_{\varepsilon,\Phi}$ along S^k_Φ with respect to g_ε. It is clear that

$$\langle n^\varepsilon_\Phi, -\Theta_\Phi/|\Theta_\Phi|_{g_{eucl}}\rangle_{g_\varepsilon} = 1 + \mathcal{O}(\varepsilon^2), \quad \text{and} \quad \langle (\mathcal{N}_\mu)^\varepsilon_\Phi, (E_\mu)_\Phi/|(E_\mu)_\Phi|_{g_{eucl}}\rangle_{g_\varepsilon} = 1 + \mathcal{O}(\varepsilon^2),$$

and $n^\varepsilon_0 = -\Theta$ and $(\mathcal{N}_\mu)^\varepsilon_0 = \mathcal{N}^\varepsilon_\mu$. We can then write

$$H^{g_\varepsilon}(S^k_\Phi) - k\, n^\varepsilon_\Phi = \left(g_\varepsilon\left(H^{g_\varepsilon}(S^k_\Phi), n^\varepsilon_\Phi\right) - k\right) n^\varepsilon_\Phi + \sum_{\mu=k+2}^{m+1} g_\varepsilon\left(H^{g_\varepsilon}(S^k_\Phi), (\mathcal{N}_\Phi)^\varepsilon_\mu\right)\, (\mathcal{N}_\Phi)^\varepsilon_\mu.$$

Notation 1.4.2. *We let $\mathcal{L}_{\Pi_p}(\Phi)$ denote any second order linear differential operator acting on Φ. The coefficients of $\mathcal{L}_{\Pi_p}(\Phi)$ may depend on $\Pi_p \in G_{k+1}(TM)$ and $\varepsilon \in (0,1)$, but for all $j \in \mathbb{N}$ there exists a constant $C_j > 0$ independent of Π_p and ε such that*

$$\|\mathcal{L}_{\Pi_p}(\Phi)\|_{\mathcal{C}^{j,\alpha}(S^k)} \le C_j\, \|\Phi\|_{\mathcal{C}^{j+2,\alpha}(NS^k)}.$$

Similarly, for $\ell \in \mathbb{N}$, $\mathcal{Q}^{\ell}_{\Pi_p}(\Phi)$ denotes some nonlinear operator in Φ, depending also on Π_p and ε, such that $\mathcal{Q}^{\ell}_{\Pi_p}(0) = 0$ and which has the following properties. The coefficients of the Taylor expansion of $\mathcal{Q}^{\ell}_{\Pi_p}(\Phi)$ in powers of the components of Φ and its derivatives satisfy that for any $j \geq 0$, there exists a constant $C_j > 0$, independent of $\Pi_p \in G_{k+1}(TM)$ and $\varepsilon \in (0,1)$,

$$\|\mathcal{Q}^{\ell}_{\Pi_p}(\Phi_1) - \mathcal{Q}^{\ell}_{\Pi_p}(\Phi_2)\|_{\mathcal{C}^{j,\alpha}(S^k)} \leq C_j \left(\|\Phi_1\|_{\mathcal{C}^{j+2,\alpha}(NS^k)} + \|\Phi_2\|_{\mathcal{C}^{j+k,\alpha}(NS^k)} \right)^{\ell-1} \|\Phi_1 - \Phi_2\|_{\mathcal{C}^{j+k,\alpha}(NS^k)}$$

provided $\|\Phi_i\|_{\mathcal{C}^1(NS^k)} \leq 1$, $i = 1, 2$.

Using the fact that the Christoffel symbols associated to the metric g_ε are of order $\mathcal{O}(\varepsilon^2)$, we obtain

$$\begin{aligned} g_\varepsilon \left(H^{g_\varepsilon}(S^k_\Phi), n^\varepsilon_\Phi \right) - k &= -\frac{\varepsilon^2}{3} \mathcal{R}ic_{k+1}(\Pi_p)(\Theta, \Theta) + J^{\|}_{S^k} \phi \\ &\quad + \mathcal{O}(\varepsilon^3) + \varepsilon^2 \mathcal{L}_{\Pi_p}(\Phi) + \mathcal{Q}^2_{\Pi_p}(\Phi), \\ g_\varepsilon \left(H^{g_\varepsilon}(S^k_\Phi), (\mathcal{N}_\Phi)^\varepsilon_\mu \right) &= \frac{2\varepsilon^2}{3} \mathcal{R}ic^{\perp}_{k+1}(\Pi_p)(\Theta, E_\mu) + L^{\perp}_{B^{k+1}} \Phi^\mu \\ &\quad + \mathcal{O}(\varepsilon^3) + \varepsilon^2 \mathcal{L}_{\Pi_p}(\Phi) + \mathcal{Q}^2_{\Pi_p}(\Phi). \end{aligned}$$

As before, we let $\mathcal{K}^{\|}$ and $\mathcal{K}^{\perp}$ be the null-spaces of the operators

$$J^{\|}_{S^k} = \Delta_{S^k} + k \quad \text{and} \quad L^{\perp}_{B^{k+1}} = \Delta_{S^k} - D_{B^{k+1}},$$

and write $\mathcal{P}^{\|}$ and $\mathcal{P}^{\perp}$ for the L^2 orthogonal complements of $\mathcal{K}^{\|}$ and $\mathcal{K}^{\perp}$ in $\mathcal{C}^{2,\alpha}(S^k)$. Define the space

$$\mathfrak{E} := \mathbb{R}^{k+1} \times (\mathbb{R}^{k+1} \oplus \mathbb{R})^{m-k} \times \mathcal{P}^{\|} \times (\mathcal{P}^{\perp})^{m-k}. \tag{1.8}$$

There exists an operator

$$\mathcal{G} : (\mathcal{C}^{0,\alpha}(S^k))^{m-k} \longrightarrow \mathfrak{E},$$

such that

$$\mathcal{G}(f_0, f_1, \ldots, f_{m-k}) = \left(\vec{a}(\Pi_p, f), \vec{c}_\mu(\Pi_p, f), d_\mu(\Pi_p, f), \phi(\Pi_p, f), \Phi^{\perp}(\Pi_p, f) \right)$$

is the solution to

$$\begin{cases} J^{\|}_{S^k} \phi = \langle \vec{a}, \Theta \rangle + f_0, \\ L^{\perp}_{B^{k+1}} \Phi^\mu = \langle \vec{c}_\mu, \Theta \rangle + d_\mu + f_{\mu-k}. \end{cases}$$

Applying a standard fixed point theorem for contraction mappings, we find that there exist $c > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and $\Pi_p \in G_{k+1}(TM)$ there is a unique element

$$\left(\vec{a}(\varepsilon, \Pi_p), \vec{c}_\mu(\varepsilon, \Pi_p), d_\mu(\varepsilon, \Pi_p), \phi(\varepsilon, \Pi_p), \Phi^{\perp}(\varepsilon, \Pi_p) \right),$$

in a closed ball of radius $c\,\varepsilon^2$ centered at 0 in $\mathfrak{E}$ (for some constant $c > 0$) such that

$$H^{g_\varepsilon}(S^k_\Phi) = -k\, n^\varepsilon_\Phi + \langle \vec{a}, \Theta\rangle\, n^\varepsilon_\Phi + \sum_{\mu=k+2}^{m+1} \left(\langle \vec{c}_\mu, \Theta\rangle + d_\mu\right)\, (\mathcal{N}_\Phi)^\varepsilon_\mu.$$

Finally, to finish the proof we put

$$n_K = \frac{1}{\varepsilon}\,(F_\varepsilon)_*\, n^\varepsilon_\Phi \quad \text{and} \quad N_\mu = \frac{1}{\varepsilon}\,(F_\varepsilon)_*\,(\mathcal{N}_\mu)^\varepsilon_\Phi$$

and $K_\varepsilon(\Pi_p) := F_\varepsilon(S^k_{\Phi(\varepsilon,\Pi_p)})$, $\;Q_\varepsilon(\Pi_p) := F_\varepsilon(B^{k+1}_{\varepsilon,\Phi(\varepsilon,\Pi_p)})$.

□

Remark 1.4.2. *Using the fact that*

$$\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) \in \mathcal{P}^{\|} \quad \textit{and} \quad \mathcal{R}ic^\perp_{k+1}(\Pi_p)(\Theta, E_\mu) \in \mathcal{K}^\perp,$$

and decomposing

$$\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) = \sum_{a=1}^{k+1} \mathcal{R}ic_{k+1}(\Pi_p)_{aa}\,(\Theta^a)^2 + \sum_{a\neq b=1}^{k+1} \mathcal{R}ic_{k+1}(\Pi_p)_{ab}\,\Theta^a\,\Theta^b,$$

one can easily verify that the vector field Φ_{ε,Π_p} obtained in Proposition 1.4.1, satisfies

$$\phi_{\varepsilon,\Pi_p} = \frac{\varepsilon^2}{3}\left(\frac{2}{k(k+2)}\,\mathcal{R}_{k+1}(\Pi_p) - \frac{1}{k+2}\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\right) + \mathcal{O}(\varepsilon^3),$$

$$[\Phi]^\perp_{\varepsilon,\Pi_p} = \mathcal{O}(\varepsilon^3).$$

1.4.2 Variational argument

We now employ a variational argument to prove that one can choose $\Pi_p \in G_k(M)$ in such a way that the submanifold $K_\varepsilon(\Pi_p)$ obtained in the previous Proposition has constant mean curvature.

To state our result, we introduce the following restrictions of the Riemann tensor of M:

Notation 1.4.3.

$$\begin{aligned} R_{k+1}(\Pi_p)(v_1,v_2,v_3,v_4) &= g_p(R_p(v_1,v_2)v_3,v_4), \qquad && v_1,v_2,v_3,v_4 \in \Pi_p,\\ R^\perp_{k+1}(\Pi_p)(v_1,v_2,v_3,N) &= g_p(R_p(v_1,v_2)v_3,N), \qquad && v_1,v_2,v_3 \in \Pi_p, \quad N \in \Pi_p^\perp, \end{aligned}$$

Finally, introduce the function $\mathbf{r}$ *on* $G_{k+1}(TM)$*:*

$$\begin{aligned} \mathbf{r}(\Pi_p) \;&= \tfrac{1}{36(k+5)}\Big[8\,\|\mathcal{R}ic_{k+1}(\Pi_p)\|^2 - 18 \sum_{i,j,\ell=1}^{k+1} \nabla_{E_i}\nabla_{E_i}\, g(R(E_j,E_\ell)E_j,E_\ell)\big|_p \\ &\quad -3\,\|R_{k+1}(\Pi_p)\|^2 + 5\,\mathcal{R}_{k+1}(\Pi_p)^2 + 24\,\tfrac{k+1}{k+3}\,\|\mathcal{R}ic^\perp_{k+1}(\Pi_p)\|^2 + 12\,\|R^\perp_{k+1}(\Pi_p)\|^2\Big] \\ &\quad +\frac{\varepsilon^4}{18}\,\frac{1}{(k+2)(k+3)}\Big[\tfrac{k+6}{k}\,\mathcal{R}^2_{k+1}(\Pi_p) - 2\,\|\mathcal{R}ic_{k+1}(\Pi_p)\|^2\Big] \end{aligned}$$

Now consider the energy $\mathcal{E}_\varepsilon$ restricted to this finite dimensional space of submanifolds,

$$\mathcal{E}_\varepsilon(\Pi_p) := \mathrm{Vol}_k(K_\varepsilon(\Pi_p)) - \frac{k}{\varepsilon}\,\mathrm{Vol}_{k+1}(Q_\varepsilon(\Pi_p)),$$

which is a function on $G_{k+1}(TM)$. Tracing through the construction of $K_\varepsilon(\Pi_p)$ one obtains the relationship of this function to the curvature functions defined above.

Lemma 1.4.3. *There is an expansion*

$$\frac{(k+1)\,\mathcal{E}_\varepsilon(\Pi_p)}{\varepsilon^k\,\mathrm{Vol}(S^k)} = \left(1 - \frac{\varepsilon^2}{2(k+3)}\,\mathcal{R}_{k+1}(\Pi_p) + \frac{\varepsilon^4}{2(k+3)}\,\mathbf{r}(\Pi_p) + \mathcal{O}(\varepsilon^5)\right).$$

Proof. The proof is a technical calculation, contained in the Appendix.

□

The main result of this section is the following proposition

Proposition 1.4.2. *If Π_p is a critical point of $\mathcal{E}_\varepsilon$, then $K_\varepsilon(\Pi_p)$ has constant mean curvature.*

Remark 1.4.3. *Theorems* (1.1.1) *and* (1.1.2) *are Corollaries of Proposition* (1.4.2). *Indeed, if we define*

$$\Psi(\varepsilon,\Pi_p) = 2\,\varepsilon^{-2}\,(k+3)\left(1-(k+1)\frac{\mathcal{E}_\varepsilon(\Pi_p)}{\varepsilon^k\mathrm{Vol}(S^k)}\right); \tag{1.9}$$

then for any $j \geq 0$, there exists a constant C_j which is independent of ε such that

$$\|\Psi(\varepsilon,\cdot) - \mathcal{R}_{k+1}(\cdot) + \varepsilon^2\mathbf{r}(\cdot)\|_{\mathcal{C}^j(G_{k+1}(TM))} \leq C_j\,\varepsilon^3.$$

Proof of the Proposition. Let Π_p be a critical point of $\mathcal{E}_\varepsilon$. We show that the parameters $\vec{a}$, $\vec{c}$ and d must then necessarily vanish. We do this by considering various types of perturbations of Π_p.

First consider the perturbations in $G_{k+1}(M)$ which correspond to parallel translations of Π_p. In other words, we suppose that the family of planes $\Pi_{\exp_p(t\xi)}$ in $G_{k+1}(M)$ are parallel translates of Π_p along the geodesic $\exp_p(t\xi)$.

The submanifold $K_\varepsilon(\Pi_{\exp_p(t\xi)})$ is a normal graph over $K_\varepsilon(\Pi_p)$ by a vector field $\Psi_{\varepsilon,\Pi_p,\xi,t}$ which depends smoothly on t. This defines a vector field on $K_\varepsilon(\Pi_p)$ by

$$Z_{\varepsilon,\Pi_p,\xi} = \partial_t \Psi_{\varepsilon,\Pi_p,\xi,t}\big|_{t=0}\,.$$

The first variation of the volume formula yields

$$\begin{aligned}
0 \;&= D\mathcal{E}_\varepsilon|_{\Pi_p}(\xi)\\
&= \int_{K_\varepsilon(\Pi_p)} \left(g(H(K_\varepsilon(\Pi_p)), Z_{\varepsilon,\Pi_p,\xi}) - \frac{k}{\varepsilon}\,g(n, Z_{\varepsilon,\Pi_p,\xi})\right) dvol_{K_\varepsilon(\Pi_p)}\\
&-\frac{k}{\varepsilon}\int_{Q_\varepsilon(\Pi_p)} g(H(Q_\varepsilon(\Pi_p)), Z_{\varepsilon,\Pi_p,\xi})\; dvol_{Q_\varepsilon(\Pi_p)},
\end{aligned} \tag{1.10}$$

and then the construction of $Q_\varepsilon(\Pi_p)$ and $K_\varepsilon(\Pi_p)$ gives that

$$\int\limits_{K_\varepsilon(\Pi_p)} \Big(\langle\vec{a},\Theta\rangle g(n, Z_{\varepsilon,\Pi_p,\xi}) + \sum_{\mu=k+2}^{m+1} (\langle\vec{c}_\mu,\Theta\rangle + d_\mu)\, g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu)\Big) = 0.$$

Let Ξ be the vector field obtained by parallel transport of ξ along geodesics issuing from p, and suppose that c is a constant independent of ε and ξ. Then

$$\|Z_{\varepsilon,\Pi_p,\xi} - \Xi\|_g \le c\,\varepsilon^2\,\|\xi\|.$$

By construction of $K_\varepsilon(\Pi_p)$, we have

$$\|n + \frac{1}{\varepsilon}\,(F_\varepsilon)_*\Theta\|_g \le c\,\varepsilon^2, \quad \text{and} \quad \|N_\mu - \frac{1}{\varepsilon}\,(F_\varepsilon)_* E_\mu\|_g \le c\,\varepsilon^2.$$

Now take $\xi \in \Pi_p \subset TM_p$, so that

$$g(n, Z_{\varepsilon,\Pi_p,\xi}) = g\left(-\frac{1}{\varepsilon}\,(F_\varepsilon)_*\Theta + \left(n + \frac{1}{\varepsilon}\,(F)_*\Theta\right),\ \Xi + \big(Z_{\varepsilon,\Pi_p,\xi} - \Xi\big)\right),$$

and

$$g(N_\mu, Z_{\varepsilon,\Pi_p,\xi}) = g\left(\frac{1}{\varepsilon}\,(F_\varepsilon)_* E_\mu + \left(N_\mu - \frac{1}{\varepsilon}\,(F_\varepsilon)_*\vec{E}_\mu,\ \Xi + \big(Z_{\varepsilon,\Pi_p,\xi} - \Xi\big)\right)\right).$$

We conclude that

$$\big|g(n, Z_{\varepsilon,\Pi_p,\xi}) + g_p(\xi,\Theta)\big| \le c\,\varepsilon^2\|\xi\|, \quad \text{and} \quad \big|g(N_\mu, Z_{\varepsilon,\Pi_p,\xi})\big| \le c\,\varepsilon^2\|\xi\|,$$

hence

$$\begin{aligned}
\int\limits_{K_\varepsilon(\Pi_p)} \langle\vec{a},\Theta\rangle\, g_p(\xi,\Theta) \le \Big| &\int\limits_{K_\varepsilon(\Pi_p)} \langle\vec{a},\Theta\rangle\, g_p(\xi,\Theta) \\
&+ \int\limits_{K_\varepsilon(\Pi_p)} \langle\vec{a},\Theta\rangle\, g(Z_{\varepsilon,\Pi_p,\xi}, n) \\
&+ \sum_{\mu=k+2}^{m+1} \int\limits_{K_\varepsilon(\Pi_p)} (\langle\vec{c}_\mu,\Theta\rangle + d_\mu)\, g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu)\Big| \\
&\le c\,\varepsilon^2\,\|\xi\|\Big(\int\limits_{K_\varepsilon(\Pi_p)} |\langle\vec{a},\Theta\rangle| + \sum_{\mu=k+2}^{m+1} \int\limits_{K_\varepsilon(\Pi_p)} |\langle\vec{c}_\mu,\Theta\rangle + d_\mu|\Big).
\end{aligned}$$

Now taking $\xi = \sum_{i=1}^{k+1} a^i\, E_i$ we obtain

$$\int\limits_{K_\varepsilon(\Pi_p)} \langle\vec{a},\Theta\rangle^2 \le c\,\varepsilon^2\,\|\vec{a}\| \left(\int\limits_{K_\varepsilon(\Pi_p)} |\langle\vec{a},\Theta\rangle| + \sum_{\mu=k+2}^{m+1} \int\limits_{K_\varepsilon(\Pi_p)} |\langle\vec{c}_\mu,\Theta\rangle + d_\mu|\right).$$

In Euclidean space, we have the equality

$$\mathrm{Vol}_k(S^k)\|v\|^2 = (k+1)\int_{S^k} \langle v, \Theta\rangle^2, \quad \text{for all } \ v \in \mathbb{R}^{k+1}.$$

By the expansion of the induced metric, we obtain for ε small enough

$$\frac{1}{2}\ \mathrm{Vol}_k(S^k)\,\varepsilon^k\,\|v\|^2 \le (k+1)\int_{K_\varepsilon(\Pi_p)} \langle v, \Theta\rangle^2.$$

Also, since $\mathrm{Vol}_k(K_\varepsilon(\Pi_p)) = \mathcal{O}(\varepsilon^k)$, we deduce

$$\|\vec{a}\| \le c\,\varepsilon^2\Big(\|\vec{a}\| + \sum_{\mu=k+2}^{m+1}(\|\vec{c}_\mu\| + |d_\mu|)\Big). \tag{1.11}$$

Now move p in the direction of a vector $\xi \in \Pi_p^\perp$ to get

$$\big|g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu) - g_p(\xi, E_\mu)\big| \le c\,\varepsilon^2\|\xi\|, \quad \text{and} \quad |g(n, Z_{\varepsilon,\Pi_p,\xi})| \le c\,\varepsilon^2\|\xi\|.$$

We can write

$$\begin{aligned}
\sum_{\mu=k+2}^{m+1}\int_{K_\varepsilon(\Pi_p)} (\langle \vec{c}_\mu, \Theta\rangle + d_\mu)\ g_p(\xi, E_\mu) \le \Big| &\sum_{\mu=k+2}^{m+1}\int_{K_\varepsilon(\Pi_p)} (\langle \vec{c}_\mu, \Theta\rangle + d_\mu)\ g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu) \\
&- \sum_{\mu=k+2}^{m+1}\int_{K_\varepsilon(\Pi_p)} (\langle \vec{c}_\mu, \Theta\rangle + d_\mu)\ g_p(\xi, E_\mu) \\
&+ \int_{K_\varepsilon(\Pi_p)} \langle \vec{a}, \Theta\rangle\, g(Z_{\varepsilon,\Pi_p,\xi}, n)\Big| \\
&\le c\,\varepsilon^2\,\|\xi\|\int_{K_\varepsilon(\Pi_p)}\Big(|\langle \vec{a}, \Theta\rangle| + \sum_{\mu=k+2}^{m+1}|\langle \vec{c}_\mu, \Theta\rangle + d_\mu|\Big).
\end{aligned}$$

Taking $\xi = d_\nu\,E_\nu$ gives

$$\begin{aligned}
\int_{K_\varepsilon(\Pi_p)} d_\nu\ \langle \vec{c}_\nu, \Theta\rangle + {d_\nu}^2 \quad &\le c\,\varepsilon^2\,|d_\nu|\Big(\textstyle\int_{K_\varepsilon(\Pi_p)} |\langle \vec{a}, \Theta\rangle| \\
&+ \sum_{\mu=k+2}^{m+1}\int_{K_\varepsilon(\Pi_p)} |\langle \vec{c}_\mu, \Theta\rangle + d_\mu|\Big).
\end{aligned} \tag{1.12}$$

Next consider a perturbation of Π_p by a one-parameter family of rotations of Π_p in T_pM generated by an $(m+1)\times(m+1)$ skew matrix A. Then

$$D\mathcal{E}_\varepsilon|_{\Pi_p}(A) = \frac{d}{dt}\Big|_{t=0}\mathcal{E}_\varepsilon((I + tA + O(t^2))\Pi_p) = \frac{d}{dt}\Big|_{t=0}\mathcal{E}(A_t(K_\varepsilon(\Pi_p))),$$

where, in geodesic normal coordinates

$$A_t(x) = x + tAx + \mathcal{O}(t^2).$$

The coordinates of the vector field associated to this flow are

$$Z_{\varepsilon,\Pi_p,\xi}(x) = \left.\frac{d}{dt}\right|_{t=0} A_t(x) = Ax.$$

Considering only matrices $A \in \mathfrak{o}(m)$ such that $A : \Pi_p \to \Pi_p^{\perp}$, we obtain

$$\big|g(Z_{\varepsilon,\Pi_p,\xi}, n)\big| \le c\,\varepsilon^2 \|A\Theta\|, \quad \text{and} \quad \big|g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu) - \langle A\Theta, E_\mu\rangle\big| \le c\,\varepsilon^2 \|A\Theta\|.$$

This gives the

$$\begin{aligned}
&\sum_{\mu=k+2}^{m+1} \int_{K_\varepsilon(\Pi_p)} (\langle \vec{c}_\mu, \Theta\rangle + d_\mu)\,\langle A\Theta, E_\mu\rangle \\
&\quad \le \Big| \sum_{\mu=k+2}^{m+1} \int_{K_\varepsilon(\Pi_p)} (\langle \vec{c}_\mu, \Theta\rangle + d_\mu)\, g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu) \\
&\quad - \sum_{\mu=k+2}^{m+1} \int_{K_\varepsilon(\Pi_p)} (\langle \vec{c}_\mu, \Theta\rangle + d_\mu)\,\langle A\Theta, E_\mu\rangle \\
&\quad + \int_{K_\varepsilon(\Pi_p)} \langle \vec{a}, \Theta\rangle\, g(Z_{\varepsilon,\Pi_p,\xi}, n)\Big| \\
&\quad \le c\,\varepsilon^2 \int_{K_\varepsilon(\Pi_p)} \left(\|A\Theta\|\; |\langle \vec{a}, \Theta\rangle| + \sum_{\mu=k+2}^{m+1} \|A\Theta\|\; |\langle \vec{c}_\mu, \Theta\rangle + d_\mu| \right).
\end{aligned}$$

Let C_ν be the $(m-k)\times(k+1)$ matrix with column ν equal to the vector $\vec{c}_\nu \in \mathbb{R}^{k+1}$, and all other columns equal to 0. Then if

$$A = \begin{pmatrix} 0 & -C_\nu^T \\ C_\nu & 0 \end{pmatrix},$$

we get

$$\begin{aligned}
\int\limits_{K_\varepsilon(\Pi_p)} \langle \vec{c}_\nu, \rangle^2 + \langle \vec{c}_\nu, \Theta\rangle d_\nu \quad &\le C\,\varepsilon^2 \Big(\int\limits_{K_\varepsilon(\Pi_p)} |\langle \vec{c}_\nu, \Theta\rangle|\; |g_p(\vec{a}, \Theta)| \\
&\quad + \sum_{\mu=k+2}^{m+1} \int\limits_{K_\varepsilon(\Pi_p)} |\langle \vec{c}_\nu, \Theta\rangle|\, |\langle \vec{c}_\mu, \Theta\rangle + d_\mu| \Big)
\end{aligned} \tag{1.13}$$

Adding (1.12) and (1.13) now gives

$$\begin{aligned}
\int_{K_\varepsilon(\Pi_p)} |d_\nu + \langle \vec{c}_\nu, \Theta\rangle|^2 \le c\,\varepsilon^2 \Big(&\int_{K_\varepsilon(\Pi_p)} (|d_\nu| + |\langle \vec{c}_\nu, \Theta\rangle|)\, |\langle \vec{a}, \Theta\rangle| \\
&+ \sum_{\mu=k+2}^{m+1} (|d_\nu| + |\langle \vec{c}_\nu, \Theta\rangle|)\, |\langle \vec{c}_\mu, \Theta\rangle + d_\mu| \Big).
\end{aligned}$$

In Euclidean space, if $v \in \mathbb{R}^{k+1}$ and $\alpha \in \mathbb{R}$ are arbitrary, then

$$\int_{S^k} |\alpha + \langle v, \Theta\rangle|^2 = \left(\alpha^2 + \frac{1}{k+1}\|v\|^2\right) \mathrm{Vol}_k(S^k).$$

Using once again the decomposition of the induced metric on $K_\varepsilon(\Pi_p)$ we find for ε small enough

$$\frac{1}{2(k+1)}\varepsilon^k \,\mathrm{Vol}_k(S^k)\left(\alpha^2 + \|v\|^2\right) \leq \int_{K_\varepsilon(\Pi_p)} |\alpha + \langle v, \Theta\rangle|^2 . \tag{1.14}$$

which gives

$$\|\vec{c}_\nu\|^2 \;+\; |d_\nu|^2 \;\leq\; c\,\frac{1}{\varepsilon^{k-2}}\left(\|\vec{c}_\nu\| + |d_\nu|\right)\Big(\int_{K_\varepsilon(\Pi_p)} |\langle \vec{a}, \Theta\rangle| \;+\; \sum_{\mu=k+2}^{m+1}\int_{K_\varepsilon(\Pi_p)} |\langle \vec{c}_\mu, \Theta\rangle \;+\; d_\mu|\Big).$$

Since $\mathrm{Vol}_k(K_\varepsilon(\Pi_p)) = \mathcal{O}(\varepsilon^k)$, we get

$$\|\vec{c}_\nu\| + |d_\nu| \leq c\,\varepsilon^2\left(\|\vec{a}\| + \sum_{\mu=k+2}^{m+1}(\|\vec{c}_\mu\| + |d_\mu|)\right). \tag{1.15}$$

Adding (1.11) and (1.15) gives

$$\left(\|\vec{a}\| + \sum_{\mu=k+2}^{m+1}(\|\vec{c}_\mu\| + |d_\mu|)\right) \leq c\,\varepsilon^2\left(\|\vec{a}\| + \sum_{\mu=k+2}^{m+1}(\|\vec{c}_\mu\| + |d_\mu|)\right),$$

which implies finally that $\|\vec{a}\| = 0$, $\|\vec{c}_\mu\| = 0$ and $|d_\mu| = 0$, $k+1 \leq \mu$.

We conclude that if Π_p is a critical point of the functional $\mathcal{E}_\varepsilon$, then the manifold $K_\varepsilon(\Pi_p)$ is a constant mean curvature submanifold of M. □

1.5 Appendix 1

Mean curvature of submanifolds: Let $\Sigma^k \subset M^{m+1}$ be an embedded submanifold. Let $x^1, \ldots, x^k$ be local coordinates on Σ and

$$E_\alpha = \partial_{x_\alpha}$$

the corresponding coordinate vector fields. Suppose that $E_{k+1}, \ldots, E_{m+1}$ is a local frame for $N\Sigma$. This gives local coordinates transverse to Σ by

$$p \in \Sigma \longmapsto \exp_p\left(\sum_{j=k+1}^{m+1} x^j\, E_j\right).$$

We make the convention that Greek indexes run from 1 to k, while Latin indexes run from $k+1$ to $m+1$. The induced metric on Σ has coefficients $\bar{g}_{\alpha\beta}$, while

$$\bar{h}^i_{\alpha\beta} := \Gamma^i_{\alpha\beta} = g(\nabla_{E_\alpha} E_\beta, E_i)$$

are the coefficients of the second fundamental form. We also record the Christoffel symbols

$$\Gamma^j_{\alpha i} = g(\nabla_{E_\alpha} E_i, E_j).$$

The following result is standard, cf. [72] for a proof.

Lemma 1.5.1. *If* $X = \sum_{j=k+1}^{m+1} x^j\, E_j$, *then*

$$\begin{aligned}
g_{\alpha\beta} &= \bar g_{\alpha\beta} - 2\,\bar g(\bar h_{\alpha\beta}, X) + g(R(E_\alpha, X)E_\beta, X) + g(\nabla_{E_\alpha} X, \nabla_{E_\beta} X) + \mathcal{O}(|x|^3)\\
&= \bar g_{\alpha\beta} - 2\,\bar h^i_{\alpha\beta}\, x^i + \Big(g(R(E_\alpha, E_i)E_\beta, E_j) + g^{\gamma\gamma'}\,\bar h^i_{\alpha\gamma}\,\bar h^j_{\gamma'\beta} + \Gamma^i_{\alpha\ell}\,\Gamma^j_{\ell\beta}\Big)\; x^i\, x^j + \mathcal{O}(|x|^3)\\
g_{\alpha j} &= -\Gamma^i_{\alpha j}\, x^i + \mathcal{O}(|x|^2)\\
g_{ij} &= \delta_{ij} + \frac{1}{3}\, g(R(E_i, E_\ell)E_j, E_{\ell'})\, x^\ell\, x^{\ell'} + \mathcal{O}(|x|^3).
\end{aligned}$$

Let Φ be a smooth section of $N\Sigma$ and consider the normal graph $\Sigma_\Phi = \{\exp_p(\Phi(p)) : p \in \Sigma\}$. Now let us use the previous lemma to expand the metric and volume form on Σ_Φ. To state this result properly, introduce ∇^N, the induced connection on $N\Sigma$,

$$\nabla^N \Phi = \pi_{N\Sigma} \circ \nabla\Phi.$$

Using the definitions of §2, we find that

Lemma 1.5.2.

$$\begin{aligned}
\mathrm{Vol_k}(\Sigma_\Phi) &= \mathrm{Vol}_k(\Sigma) - \int_\Sigma g(H(\Sigma), \Phi)\, \mathrm{dvol}_\Sigma\\
&+ \frac{1}{2}\int_\Sigma \big(|\nabla^N\Phi|^2_g - g((\mathrm{Ric}_\Sigma + \mathfrak{H}^2_\Sigma)\,\Phi, \Phi)\big)\ \mathrm{dvol}_\Sigma\\
&+ \frac{1}{2}\int_\Sigma (g(H(\Sigma), \Phi))^2\, \mathrm{dvol}_\Sigma + ...
\end{aligned}$$

Proof. First of all we expand the induced metric on Σ_Φ. Using the result of the previous Lemma, we find

$$\begin{aligned}
(\bar g_\Phi)_{\alpha\beta} &= \bar g_{\alpha\beta} - 2\, g(\bar h_{\alpha\beta}, \Phi) + g(R(E_\alpha, \Phi)\, E_\beta, \Phi) + g(\nabla_{E_\alpha}\Phi, \nabla_{E_\beta}\Phi) + \ldots\\
&= \bar g_{\alpha\beta} - 2\, g(\bar h_{\alpha\beta}, \Phi) + g(R(E_\alpha, \Phi)\, E_\beta, \Phi)\\
&+ \bar g^{\gamma\gamma'} g(\bar h_{\alpha\gamma}, \Phi)\, g(\bar h_{\gamma\beta}, \Phi) + g(\nabla^N_{E_\alpha}\Phi, \nabla^N_{E_\beta}\Phi) + \ldots
\end{aligned}$$

Next we use the well known expansion

$$\sqrt{\det(I+A)} = 1 + \frac{1}{2}\mathrm{Tr}\, A + \frac{1}{8}\,(\mathrm{Tr} A)^2 - \frac{1}{4}\,\big(\mathrm{Tr}(A^2)\big) + ...$$

to find
$$\sqrt{\det \bar g_\Phi} = \big(1 - g(H(\Sigma),\Phi) + \tfrac{1}{2}\left(|\nabla^N\Phi|_g^2 - g((\mathrm{Ric}_\Sigma + (\mathfrak{H})^2_\Sigma)\,\Phi,\Phi)\right.$$
$$\left. + \,(g(H(\Sigma),\Phi))^2\right) + \ldots\big)\,\sqrt{\det \bar g}.$$
This completes the proof.

□

From this we obtain the first and second variations of the volume functional,

$$D_\Phi \mathrm{Vol_k}(\Sigma_\Phi)|_\Phi \Psi = -\int_\Sigma g(H(\Sigma_\Phi),\Psi)\,\mathrm{dvol}_{\Sigma_\Phi}, \tag{1.16}$$

and
$$\begin{aligned} D^2_\Phi \mathrm{Vol_k}(\Sigma_\Phi)|_{\Phi=0}(\Psi,\Psi) &= \int_\Sigma \left(|\nabla^N\Psi|^2 - g((\mathrm{Ric}_\Sigma + \mathfrak{H}^2_\Sigma)\,\Psi,\Psi)\right)\,\mathrm{dvol}_\Sigma \\ &+ \int_\Sigma (g(H(\Sigma),\Psi))^2\,\mathrm{dvol}_\Sigma. \end{aligned}$$

On the other hand, differentiating (1.16) once more gives

$$\begin{aligned} D^2_\Phi \mathrm{Vol_k}(\Sigma_\Phi)|_{\Phi=0}(\Psi,\Psi) &= -\int_\Sigma g(D_\Phi H(\Sigma_\Phi)|_{\Phi=0}\Psi,\Psi)\,\mathrm{dvol}_\Sigma \\ &+ \int_\Sigma (g(H(\Sigma),\Psi))^2\,\mathrm{dvol}_K. \end{aligned}$$

Comparing the two formulæ implies that the orthogonal projection of the Jacobi operator to $N\Sigma$ equals
$$J^N_\Sigma := D_\Phi H(\Sigma_\Phi)|_{\Phi=0} = \Delta^N_g + \mathrm{Ric}^N_\Sigma + \mathfrak{H}^2_\Sigma\,,$$

1.6 Appendix 2

We give here the proof of Lemma 1.4.3, namely the proof of the formula
$$\frac{(k+1)\,\mathcal{E}_\varepsilon(\Pi_p)}{\varepsilon^k\,\mathrm{Vol}(S^k)} = \left(1 - \frac{\varepsilon^2}{2(k+3)}\mathcal{R}_{k+1}(\Pi_p) + \frac{\varepsilon^4}{2(k+3)}\,\mathbf{r}(\Pi_p) + \mathcal{O}(\varepsilon^5)\right)$$
and find the expression of the function r. Let $K_\varepsilon(\Pi_p)$ be the constant mean curvature submanifold constructed in Proposition (1.4.1) and denote by $F : T_pM \longrightarrow M$ the exponential map. Recall that
$$K_\varepsilon(\Pi_p) = F(S^k_{\varepsilon,\Phi}),$$
where $S^k_{\varepsilon,\Phi}$ is a submanifold of T_pM parametrized by $\left\{\varepsilon\,(1-\phi)\,\Theta + \varepsilon\,\Phi^\perp,\ \Theta \in S^k\right\}$. It follows from the proof of that proposition that

$$\phi(\Theta) = \frac{\varepsilon^2}{3}\left(\frac{2}{k(k+2)}\mathcal{R}_{k+1}(\Pi_p) - \frac{1}{k+2}\mathcal{R}ic(\Pi_p)(\Theta,\Theta)\right) + \mathcal{O}(\varepsilon^3),$$

$$\Phi^\perp = \mathcal{O}(\varepsilon^3).$$

Also consider the minimal submanifold

$$Q_\varepsilon(\Pi_p) = F(B^{k+1}_{\varepsilon,\Phi}),$$

where $B^{k+1}_{\varepsilon,\Phi} = \left\{\varepsilon\, y + \varepsilon\, U_\Phi(y),\ y \in B^{k+1}\right\}$ and recall that

$$U_\Phi(y) = \phi\,(y/\|y\|) + W_\varepsilon(y) + \mathcal{O}(\varepsilon^3),$$

$$W_\varepsilon(y) = \frac{\varepsilon^2}{(k+3)} \sum_{\mu=k+2}^{m+1} \sum_{i=1}^{k+1} \mathcal{R}ic^\perp(\Pi_p)_{i\mu}(|y|^2 - 1)\, y^i\, E_\mu.$$

We shall calculate the volume forms of $S^k_{\varepsilon,\Phi}$ and $B^{k+1}_{\varepsilon,\Phi}$ with respect to F^*g. First of all, recall that in the neighborhood of $x = 0$ we have

$$\begin{aligned}(F^*g)_{ij} = \delta_{ij} &+ \frac{1}{3}\, g_p(R_p(x, E_i)x, E_j) + \frac{1}{6}\, g_p(\nabla_x R_p(x, E_i)x, E_j) \\ &+ \frac{1}{20}\, g_p(\nabla_x \nabla_x R_p(x, E_i)x, E_j) \\ &+ \sum_{\ell=1}^{m+1} \frac{2}{45}\, g_p(R_p(x, E_i)x, E_\ell)\, g_p(R_p(x, E_j)x, E_\ell) + \mathcal{O}_p(|x|^5),\end{aligned}$$

where R_p is the curvature tensor of M at the point p, cf. [108].

Volume of the CMC sphere

We first find the expansion of the metric induced on $S^k_{\varepsilon,\Phi}$. To this end we express the tangent vector fields to $S^k_{\varepsilon,\Phi}$ in terms of the vector fields $\Theta_\alpha,\ \alpha = 1, \ldots, k$ tangent to the unit sphere S^k:

$$\tau_\alpha = \varepsilon\,(1 - \phi(\Theta))\,\Theta_\alpha - \varepsilon\, \partial_\alpha\, \phi\, \Theta + \sum_{\mu=k+2}^{m+1} \varepsilon\, \partial_\alpha \Phi^\mu\, E_\mu, \quad \alpha = 1, \ldots, k.$$

The metric coefficients then satisfy

$$\begin{aligned}g^K_{\alpha\beta} = \varepsilon^2(1-\phi)^2\, g^S_{\alpha\beta} &+ \varepsilon^2\, \partial_\alpha\phi\, \partial_\beta\phi + \frac{\varepsilon^4}{3}\,(1-\phi)^4\, g_p(R_p(\Theta, \Theta_\alpha)\Theta, \Theta_\beta) \\ &+ \frac{\varepsilon^5}{6}\, g_p(\nabla_\Theta R_p(\Theta, \Theta_\alpha)\Theta, \Theta_\beta) + \frac{\varepsilon^6}{20}\, g_p(\nabla_\Theta \nabla_\Theta R_p(\Theta, \Theta_\alpha)\Theta, \Theta_\beta) \\ &+ \sum_{l=1}^{k+1} \frac{2\varepsilon^6}{45}\, g_p(R_p(\Theta, \Theta_\alpha)\Theta, E_l)\, g_p(R_p(\Theta, \Theta_\beta)\Theta, E_l) \\ &+ \sum_{\mu=k+2}^{m+1} \frac{2\varepsilon^6}{45}\, g_p(R_p(\Theta, \Theta_\alpha)\Theta, E_\mu)\, g_p(R_p(\Theta, \Theta_\beta)\Theta, E_\mu) + \mathcal{O}(\varepsilon^7).\end{aligned}$$

Using

$$\sqrt{\det(I+A)} = 1 + \frac{1}{2}\mathrm{tr}A + \frac{1}{8}(\mathrm{tr}A)^2 - \frac{1}{4}\mathrm{tr}(A^2) + \mathcal{O}(|A|^3),$$

we get

$$\begin{aligned}
\varepsilon^{-k}\,\frac{\sqrt{\det g^K}}{\sqrt{\det g^S}} &= 1 - k\phi + \frac{k(k-1)}{2}\phi^2 + \frac{1}{2}\,|\nabla_{S^k}\phi|^2 \\
&- \frac{\varepsilon^2}{6}(1-(k+2)\phi)\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) - \frac{\varepsilon^3}{12}\,\nabla_\Theta\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) \\
&- \frac{\varepsilon^4}{40}\nabla^2_\Theta\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) + \frac{\varepsilon^4}{72}\,(\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta))^2 \\
&- \frac{\varepsilon^4}{180}\sum_{i,j=1}^{k+1} g_p(R_p(\Theta,E_i)\Theta,E_j)^2 \\
&+ \frac{\varepsilon^4}{45}\sum_{i=1}^{k+1}\sum_{\mu=k+2}^{m+1} g_p(R_p(\Theta,E_i)\Theta,E_\mu)^2 + \mathcal{O}_p(\varepsilon^5).
\end{aligned}$$

Volume of the minimal ball

Now let us calculate the volume element of $Q_\varepsilon(\Pi_p)$. The tangent vectors to $B^{k+1}_{\varepsilon,\Phi}$ are given by

$$T_i(y) = \varepsilon\,(1-u(y))\,E_i + \varepsilon\,\partial_{y^i}\,u(y)\,y + \varepsilon\sum_{\mu=k+2}^{m+1}\partial_{y^i}\,W^\mu_\varepsilon(y)\,E_\mu + \mathcal{O}_p(\varepsilon^4),$$

where $u(y) = \phi(y/|y|)$. The corresponding metric coefficients have then the expansion

$$
\begin{aligned}
\varepsilon^{-2}\, g^Q_{ij} = {} & (1-u)^2\,\delta_{ij} + (1-u)\left(\partial_{y^i} u\, y_j + \partial_{y^j} u\, y^i\right) + |y|^2 \partial_{y^i} u\, \partial_{y^j} u + \sum_{\mu=k+2}^{m+1} \partial_{y_i} W^\mu_\varepsilon\, \partial_{y_j} W^\mu_\varepsilon \\
& + \frac{\varepsilon^2}{3}\,(1-u)^4\, g_p(R_p(y,E_i)y,E_j) + \frac{\varepsilon^2}{3} \sum_{\mu=k+2}^{m+1} \Big[W^\mu_\varepsilon\, g_p(R_p(E_\mu,E_i)y,E_j) \\
& + W^\mu_\varepsilon\, g_p(R_p(y,E_i)E_\mu,E_j) + \partial_{y^i} W^\mu_\varepsilon g_p(R_p(y,E_\mu)y,E_j) + \partial_{y^j} W^\mu_\varepsilon\, g_p(R_p(y,E_i)y,E_\mu) \Big] \\
& + \frac{\varepsilon^3}{6}\, g_p(\nabla_y R_p(y,E_i)y,E_j) + \frac{\varepsilon^4}{20}\, g_p(\nabla_y \nabla_y R_p(y,E_i)y,E_j) \\
& + \frac{2\varepsilon^4}{45} \sum_{l=1}^{k+1} g_p(R_p(y,E_i)y,E_l)\, g_p(R_p(y,E_i)y,E_l) \\
& + \frac{2\varepsilon^4}{45} \sum_{\mu=k+2}^{m+1} g_p(R_p(y,E_i)y,E_\mu)\, g_p(R_p(y,E_i)y,E_\mu) + \mathcal{O}(\varepsilon^5).
\end{aligned}
$$

Using the fact $\langle \nabla u, y\rangle = 0$ and the fact that for the matrix $A_{ij} = y^i\,\partial_{y^j} u + y^j\,\partial_{y^i} u$ we have $\frac{1}{4}\,\mathrm{tr}(A^2) = \frac{1}{2}\,|y|^2\,|\nabla u|^2$, we calculate the volume element of $Q_\varepsilon(\Pi_p)$:

$$
\begin{aligned}
\varepsilon^{-(k+1)}\,\sqrt{\det g^Q} = {} & 1 - (k+1)u + \frac{k(k+1)}{2}\,u^2 + \sum_{\mu=k+2}^{m+1} \frac{1}{2} |\nabla_{S^k} W^\mu_\varepsilon|^2 \\
& - \frac{\varepsilon^2}{6}\,(1-(k+3)\,u)\,\mathcal{R}ic_{k+1}(\Pi_p)(y,y) \\
& + \frac{\varepsilon^2}{3} \sum_{i=1}^{k+1} \sum_{\mu=k+2}^{m+1} \Big[W^\mu_\varepsilon\, g_p(R_p(y,E_i,E_\mu,E_i) + \partial_{y^i} W^\mu_\varepsilon\, g_p(R_p(y,E_i)y,E_\mu) \Big] \\
& - \frac{\varepsilon^3}{12}\,\nabla_y\,\mathcal{R}ic_{k+1}(\Pi_p)(y,y) - \frac{\varepsilon^4}{40}\,\nabla^2_y\,\mathcal{R}ic_{k+1}(\Pi_p)(y,y) \\
& + \frac{\varepsilon^4}{72}\,\left(\mathcal{R}ic_{k+1}(\Pi_p)(y,y)\right)^2 - \frac{\varepsilon^4}{180} \sum_{i,j=1}^{k+1} g_p(R_p(y,E_i)y,E_j)^2 \\
& + \frac{\varepsilon^4}{45} \sum_{i=1}^{k+1} \sum_{\mu=k+2}^{m+1} g_p(R_p(y,E_i)y,E_\mu)^2 + \mathcal{O}_p(\varepsilon^5).
\end{aligned}
$$

Expansion of the energy functional

Collecting the results obtained above, we find

$$
\begin{aligned}
&\varepsilon^{-k}\Big(\mathrm{Vol}(K_\varepsilon(\Pi_p)) - \frac{k}{\varepsilon}\,\mathrm{Vol}(Q_\varepsilon(\Pi_p))\Big)\\
&= \frac{1}{k+1}\,\mathrm{Vol}(S^k) - \frac{\varepsilon^2}{2(k+3)}\int_{S^k} \mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\,d\sigma + \frac{\varepsilon^2}{3}\int_{S^k} \mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\,\phi\,d\sigma\\
&+ \frac{5\,\varepsilon^4}{k+5}\int_{S^k}\Big[-\frac{1}{40}\,\nabla^2_\Theta\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) + \frac{1}{72}\,\left(\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\right)^2\\
&- \frac{1}{180}\sum_{i,j=1}^{k+1} g_p(R_p(\Theta,E_i)\Theta,E_j)^2 + \frac{1}{45}\sum_{i=1}^{k+1}\sum_{\mu=k+2}^{m+1} g_p(R_p(\Theta,E_i)\Theta,E_\mu)^2\Big]\,d\sigma\\
&+ \sum_{\mu=k+2}^{m+1}\int_{B^{k+1}}\Big[\frac{k}{2}\,W^\mu_\varepsilon\,\Delta_{B^{k+1}}\,W^\mu_\varepsilon\Big] - \frac{\varepsilon^2\,k}{3}\sum_{i=1}^{k+1}\Big(W^\mu_\varepsilon\,g_p(R_p(\Theta,E_i,E_\mu,E_i) + \partial_{y_i}W^\mu_\varepsilon\,R_p(\Theta,E_i,\Theta,E_\mu)\Big)\,dy\\
&- \frac{1}{2}\int_{S^k}\phi\,\Delta_{S^k}\,\phi - \frac{k}{2}\int_{S^k}\phi^2\,d\sigma + \mathcal{O}(\varepsilon^5).
\end{aligned}
$$

We now recall some identities. First,

$$
\int_{S^k}(\Theta^i)^2\,d\sigma = \frac{1}{k+1}\,\mathrm{Vol}(S^k),
$$

$$
\int_{S^k}(\Theta^i)^4\,d\sigma = 3\int_{S^k}(\Theta^i\,\Theta^j)^2\,d\sigma = \frac{3}{(k+1)(k+3)}\,\mathrm{Vol}(S^k);
$$

and second, if $a_{ijpq}\in\mathbb{R}$ $i,j,p,q = 1,\dots,k+1$, then

$$
\begin{aligned}
\sum_{p,q,l,n=1}^{k+1}\int_{S^k} a_{pqln}\,\Theta^p\,\Theta^q\,\Theta^l\,\Theta^n\,d\sigma &= \frac{3}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\sum_{i=1}^{k+1} a_{pppp}\\
&+ \frac{1}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\sum_{q\neq p=1}^{k+1}(a_{ppqq} + a_{pqpq} + a_{pqqp})\\
&= \frac{1}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\sum_{p,q=1}^{k+1}(a_{ppqq} + a_{pqpq} + a_{pqqp})\,;
\end{aligned}
$$

and develop each term:

$$\int_{S^k} \mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\, d\sigma = \sum_{i,j=1}^{k+1} \int_{S^k} \mathcal{R}ic_{k+1}(\Pi_p)(E_i,E_j)\, \Theta^k\, \Theta^l\, d\sigma$$

$$= \sum_{i=1}^{k+1} \mathcal{R}ic_{k+1}(\Pi_p)(E_i,E_i)\, (\Theta^i)^2\, d\sigma$$

$$= \frac{1}{k+1}\, \mathrm{Vol}(S^k)\, \mathcal{R}_{k+1}(\Pi_p);$$

$$\int_{S^k} (\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta))^2 d\sigma = \frac{1}{(k+1)(k+3)}\, \mathrm{Vol}(S^k)\, \Big(2 \sum_{i,j=1}^{k+1} (\mathcal{R}ic_{k+1}(\Pi_p)(E_i,E_j))^2\Big)$$

$$+ \sum_{i,j=1}^{k+1} \mathcal{R}ic_{k+1}(\Pi_p)(E_i,E_i)\, \mathcal{R}ic_{k+1}(\Pi_p)(E_j,E_j)$$

$$= \frac{1}{(k+1)(k+3)}\, \mathrm{Vol}(S^k)\, \Big(2\, \|\mathcal{R}ic_{k+1}(\Pi_p)\|^2 + \mathcal{R}_{k+1}(\Pi_p)^2\Big)\,;$$

$$\sum_{i,j=1}^{k+1} \int_{S^k} g_p(R_p(\Theta,E_i)\Theta,E_j)^2\, d\sigma$$

$$= \frac{1}{(k+1)(k+3)} \mathrm{Vol}(S^k) \sum_{i,j,p,q=1}^{k+1} \big(R_{ipjq}^2 + R_{ipjp}\, R_{iqjq} + R_{ipjq}\, R_{iqjp}\big)$$

$$= \frac{1}{(k+1)(k+3)}\, \mathrm{Vol}(S^k)\, \left(\|\mathcal{R}ic_{k+1}(\Pi_p)\|^2 + \frac{3}{2}\, \|R_{k+1}(\Pi_p)\|^2\right);$$

(we use here that $R_{ijpq}^2 = (R_{ipjq} - R_{iqjp})^2 = R_{ipjq}^2 + R_{iqjp}^2 - 2\, R_{ipjq}\, R_{iqjp}$);

$$\sum_{i=1}^{k+1} \sum_{\mu=k+2}^{m+1} \int_{S^k} g_p(R_p(\Theta,E_i)\Theta,E_\mu)^2\, d\sigma$$

$$= \frac{1}{(k+1)(k+3)}\, \mathrm{Vol}(S^k)\, \left(\left\|\mathcal{R}ic_{k+1}^{\perp}(\Pi_p)\right\|^2 + \frac{3}{2}\, \left\|R_{k+1}^{\perp}(\Pi_p)\right\|^2\right);$$

$$\int_{S^k} \nabla^2_\Theta \, \mathcal{R}ic_{k+1}(\Theta,\Theta)\, d\sigma$$

$$= \frac{1}{(k+1)(k+3)} \sum_{i,j=1}^{k+1} \Big(\nabla^2_{E_i} \, \mathcal{R}ic_{k+1}(\Pi_p)(E_j,E_j) + 2\, \nabla_{E_i} \nabla_{E_j} \, \mathcal{R}ic_{k+1}(E_i,E_j) \Big)$$

$$= \frac{2}{(k+1)(k+3)} \, \mathrm{Vol}(S^k) \, \sum_{i,j,\ell=1}^{k+1} \nabla_{E_i} \nabla_{E_i} g(R(E_k,E_l)E_k,E_l)\big|_p \, ;$$

$$\sum_{\mu=k+2}^{m+1} \int_{B^{k+1}} W^\mu_\varepsilon \, \Delta_{B^{k+1}} W^\mu_\varepsilon \, dy$$

$$= -\frac{2\,\varepsilon^4}{9} \, \frac{1}{k+3} \sum_{\mu=k+2}^{m+1} \int_{B^{k+1}} \sum_{j=1}^{k+1} \Big(\mathcal{R}ic^\perp_{k+1}(E_j,E_\mu) \Big)^2 \, (y^j)^2 \, (1-|y|^2) \, dy$$

$$= -\frac{2\,\varepsilon^4}{9} \, \frac{1}{(k+3)(k+1)} \, \mathrm{Vol}(S^k) \, \|\mathcal{R}ic^\perp_{k+1}\|^2 \left(\frac{1}{k+3} - \frac{1}{k+5} \right)$$

$$= -\frac{\varepsilon^4}{9} \, \frac{4}{(k+1)(k+3)^2(k+5)} \mathrm{Vol}(S^k) \, \|\mathcal{R}ic^\perp_{k+1}\|^2;$$

$$\sum_{\mu=k+2}^{m+1} \int_{B^{k+1}} W^\mu_\varepsilon \sum_{i,p=1}^{k+1} R_{ipi\mu} \, y^p \, dy = -\frac{\varepsilon^2}{3} \, \frac{1}{(k+3)} \int_{B^{k+1}} \sum_{\mu=k+2}^{m+1} \sum_{j=1}^{k+1} \Big(\mathcal{R}ic^\perp_{k+1}(E_j,E_\mu) \Big)^2 \, (y^j)^2 (1-|y|^2) \, dy$$

$$= -\frac{\varepsilon^2}{3} \, \frac{2}{(k+1)(k+3)^2(k+5)} \mathrm{Vol}(S^k) \, \|\mathcal{R}ic^\perp_{k+1}\|^2;$$

and

$$
\begin{aligned}
&\sum_{\mu=k+2}^{m+1} \int_{B^{k+1}} \partial_{y^i} \, W^\mu R_{piq\mu} \, y^p \, y^q \, dy \\
&\quad = \frac{\varepsilon^2}{3} \, \frac{1}{(k+3)} \sum_{\mu=k+2}^{m+1} \int_{B^{k+1}} \sum_{i,p,q=1}^{k+1} \Big(\mathcal{R}ic(\Pi_p)^{\perp}_{i\mu} \, R_{piq\mu} \, y^p \, y^q \, (1-|y|^2) \\
&\quad - \, 2 \sum_{j=1}^{k+1} \mathcal{R}ic(\Pi_p)^{\perp}_{j\mu} \, R_{piq\mu} \, y^j \, y^i \, y^p \, y^q \Big) \, dy \\
&\quad = \frac{\varepsilon^2}{3} \, \frac{2}{(k+1)(k+3)^2(k+5)} \mathrm{Vol}(S^k) \Big[- \|\mathcal{R}ic^{\perp}_{k+1}\|^2 \\
&\quad - \sum_{p,q=1}^{k+1} \sum_{\mu=k+2}^{m+1} \Big(\mathcal{R}ic(\Pi_p)^{\perp}_{p\mu} \, R_{qpq\mu} + \mathcal{R}ic(\Pi_p)^{\perp}_{q\mu} \, R_{ppq\mu} + \mathcal{R}ic(\Pi_p)^{\perp}_{q\mu} \, R_{qpp\mu} \Big) \Big] \\
&\quad = - \frac{2 \, \varepsilon^2}{3} \, \frac{1}{(k+1)(k+3)^2(k+5)} \, \mathrm{Vol}(S^k) \, \|\mathcal{R}ic^{\perp}_{k+1}\|^2 .
\end{aligned}
$$

This gives finally

$$
\begin{aligned}
\frac{(k+1) \, \mathcal{E}(\Pi_p)}{\varepsilon^k \, \mathrm{Vol}(S^k)} &= 1 - \frac{\varepsilon^2}{2} \frac{1}{k+3} \, \mathcal{R}_{k+1}(\Pi_p) \\
&\quad + \frac{\varepsilon^4}{72} \, \frac{1}{(k+3)(k+5)} \Big[8 \, \|\mathcal{R}ic_{k+1}(\Pi_p)\|^2 - 18 \sum_{i,j,\ell=1}^{k+1} \nabla_{E_i} \nabla_{E_i} \, g(R(E_j, E_\ell)E_j, E_\ell)|_p \\
&\quad - 3 \, \|R_{k+1}(\Pi_p)\|^2 + 5 \, \mathcal{R}_{k+1}(\Pi_p)^2 + 24 \, \frac{k+1}{k+3} \, \|\mathcal{R}ic^{\perp}_{k+1}(\Pi_p)\|^2 + 12 \, \|R^{\perp}_{k+1}(\Pi_p)\|^2 \Big] \\
&\quad + \frac{\varepsilon^4}{18} \, \frac{1}{(k+2)(k+3)} \Big[\frac{k+6}{k} \, \mathcal{R}^2_{k+1}(\Pi_p) - 2 \, \|\mathcal{R}ic_{k+1}(\Pi_p)\|^2 \Big] + \mathcal{O}(\varepsilon^5) \\
&\quad = 1 - \frac{\varepsilon^2}{2(k+3)} \, \mathcal{R}_{k+1}(\Pi_p) + \frac{\varepsilon^4}{2(k+3)} \, \mathbf{r}(\Pi_p) + \mathcal{O}(\varepsilon^5).
\end{aligned}
$$

Chapter 2

Free boundary minimal surfaces in the unit 3-ball

2.1 Introduction and statement of the result

In this chapter, which is a work in collaboration with A. Folha and F. Pacard, we are interested in minimal surfaces which are embedded in the Euclidean 3-dimensional unit open ball B^3 and which meet S^2, the boundary of B^3, orthogonally. Following [36], we refer to such minimal surfaces as *free boundary minimal surfaces.*

Obviously, unit disks obtained as the intersection of B^3 with planes passing through the origin, are examples of free boundary minimal surfaces. Moreover, these are the only free boundary minimal surfaces in B^3 of topological disk type [89].

The so called *critical catenoid* parametrized by

$$(s, \theta) \mapsto \frac{1}{s_* \cosh s_*} (\cosh s \cos\theta, \cosh s \sin\theta, s), \quad \text{where} \quad s_* \tanh s_* = 1,$$

is another example of a free boundary minimal surface. A. Fraser and M. Li conjectured that up to congruences this it is the only free boundary minimal annulus embedded in B^3 [35].

Free boundary minimal surfaces arise as critical points of the area among surfaces embedded in B^3 whose boundaries lie on S^2 but are free to vary on S^2. The fact that the area is critical for variations of the boundary of the surface which are tangent to S^2 translates into the fact that the minimal surface meets S^2 orthogonally.

In a recent paper [37], A. Fraser and R. Schoen have proved the existence of free boundary minimal surfaces Σ_n in B^3 which have genus 0 and n boundary components, for all $n \geq 3$. For large n, these surfaces can be understood as connected sums of two nearby parallel horizontal disks joined by n boundary "bridges" which are close to scaled down copies of half catenoids arranged periodically along the unit horizontal great circle of S^2. Furthermore, as n tends to infinity, these free boundary minimal surfaces converge on compact subsets of B^3 to the horizontal unit disk taken with multiplicity two.

We give in this chapter another independent construction of Σ_n for n large enough. Our proof is very different from the proof of A. Fraser and R. Schoen and is more in the spirit of the proof of the existence of minimal surfaces in S^3 by doubling constructions by N. Kapouleas [61] and N. Kapouleas and S.D. Yang [63]. We also prove the existence of free boundary minimal surfaces in B^3 which have genus 1 and n boundary components for all n large enough.

To state our result precisely, we define P_n to be the regular polygon with n-sides, which is included in the horizontal plane $\mathbb{R}^2 \times \{0\}$ and whose vertices are given by

$$\left(\cos\left(\frac{2\pi m}{n}\right), \sin\left(\frac{2\pi m}{n}\right), 0\right) \in \mathbb{R}^3, \qquad \text{for} \qquad m = 1, \ldots, n.$$

We define $\mathfrak{S}_n \subset O(3)$ to be the subgroup of isometries of $\mathbb{R}^3$ which is generated by the orthogonal symmetry with respect to the horizontal plane, the orthogonal symmetry with respect to the horizontal coordinate axis Ox_1 and the rotations around the vertical axis which leave P_n globally invariant.

Our result reads:

Theorem 2.1.1. *There exists $n_0 \geq 0$ such that, for each $n \geq n_0$, there exists a genus 0 free boundary minimal surface Σ_n and a genus 1 free boundary minimal surface $\tilde{\Sigma}_n$ which are both embedded in B^3 and meet S^2 orthogonally along n closed curves.*

Both surfaces are invariant under the action of the elements of $\mathfrak{S}_n$ and, as n tends to infinity, the sequence Σ_n converges to a double copy of the unit horizontal (open) disk, uniformly on compacts of B^3 while the sequence $\tilde{\Sigma}_n$ converges to a double copy of the unit horizontal (open) punctured disk, uniformly on compacts of $B^3 \setminus \{0\}$.

Even though we do not have a proof of this fact, it is very likely that (up to the action of an isometry of $\mathbb{R}^3$) the surfaces Σ_n coincide with the surfaces already constructed by R. Schoen and A. Fraser. In contrast, the existence of $\tilde{\Sigma}_n$ is new and does not follow from the results in [37]. The parametrization of the free boundary minimal surfaces we construct is not explicit, nevertheless our construction being based on small perturbations of explicitly designed surfaces, it has the advantage to give a rather precise description of the surfaces Σ_n and $\tilde{\Sigma}_n$. Naturally, the main drawback is that the existence of the free boundary minimal surfaces is only guaranteed when n, the number of boundary curves, is large enough.

2.2 Outline of the chapter

Remark 2.2.1. *Through out the chapter we explain in details the construction of the genus 1 free boundary minimal surfaces $\tilde{\Sigma}_n$ with n boundary components for n large enough. One will see that the same proof (with several simplifications) gives the existence of the genus 0 free boundary surfaces Σ_n with n boundary components.*

In section 2.3, we study the mean curvature of surfaces embedded in B^3 which are graphs over the horizontal disk $D^2 \times \{0\}$. In section 2.4, we analyze harmonic functions which are defined on the unit punctured disk in the Euclidean 2-plane and have log-type singularities at the punctures. In sections 2.5, 2.6, and 2.7 for every $n \in \mathbb{N}$ large enough, we construct a genus 1 surface $\mathcal{A}_n$ embedded in B^3 which meets $S^2 = \partial B^3$ orthogonally along n boundary curves, and such that the mean curvature of $\mathcal{A}_n$ tends to 0 in a suitable topology when n tends to infinity. We refer to $\mathcal{A}_n$ as *approximate solution*. In section 2.8, we consider the surfaces embedded in B^3 with genus 1 and n boundary components which are obtained as perturbations of $\mathcal{A}_n$ and meet the sphere S^2 orthogonally. In section 2.9, we study the properties of the linearized mean curvature operator about the approximate solution $\mathcal{A}_n$ and finally, in the section 2.10, we explain a fixed point argument that allows us for n large enough to perturb $\mathcal{A}_n$ into a free boundary minimal surface $\tilde{\Sigma}_n$ satisfying Theorem 2.1.1.

2.3 Mean curvature operator for graphs in the unit 3-ball

We are interested in surfaces embedded in B^3 which are graphs over the horizontal disk $D^2 \times \{0\}$. To define these precisely, we identify $\mathbb{R}^2 \times \{0\}$ with the complex plane $\mathbb{C}$, and introduce the following parametrization of the unit 3-ball:

$$X(\psi, \phi, x_3) := \frac{1}{\cosh x_3 + \cos\psi} \left(\sin\psi \, e^{i\phi}, \sinh x_3\right),$$

where $\psi \in (0, \pi/2)$, $\phi \in S^1$ and $x_3 \in \mathbb{R}$. The horizontal disk $D^2 \times \{0\}$ corresponds to $x_3 = 0$ in this parametrization and the unit sphere S^2 corresponds to $\psi = \pi/2$. Also, the leaf $x_3 = x_3^0$ is a constant mean curvature surface (in fact it is a spherical cap) with mean curvature given by

$$H = 2 \sinh x_3^0,$$

(we agree that the mean curvature is the *sum* of the principal curvatures, not the average) moreover, this leaf meets S^2 orthogonally.

In these coordinates, the expression of the Euclidean metric is given by

$$X^* g_{eucl} = \frac{1}{(\cosh x_3 + \cos\psi)^2} \left(d\psi^2 + (\sin\psi)^2 \, d\phi^2 + dx_3^2\right).$$

We introduce the coordinate

$$z = \frac{\sin\psi}{1 + \cos\psi} \, e^{i\phi},$$

which belongs to the unit disk $D^2 \subset \mathbb{C}$. We then define $\mathcal{X}$ by the identity

$$\mathcal{X}(z, x_3) = X(\psi, \phi, x_3),$$

where z and (ψ, ϕ) are related as above. Then

$$\mathcal{X}(z, x_3) = A(z, x_3)(z, B(z) \sinh x_3),$$

where the functions B and A are explicitly given by

$$B(z) = \frac{1}{2}\left(1 + |z|^2\right), \quad A(z, x_3) := \frac{1}{1 + B(z)(\cosh x_3 - 1)}.$$

Let $|dz|^2 = dz\, d\bar{z}$ be the Euclidean metric on D^2, then the pull-black metric in $D^2 \times \mathbb{R}$ is given by

$$\mathcal{X}^* g_{eucl} = A^2(z, x_3) \, \left(|dz|^2 + B^2(z)\, dx_3^2 \right). \tag{2.1}$$

Take a function $u \in \mathcal{C}^2(D^2)$. In the next result, we compute the mean curvature of the graphs:

$$z \in D^2 \; \mapsto \; \mathcal{X}(z, u(z)) \in B^3. \tag{2.2}$$

We have:

Lemma 2.3.1. *The mean curvature of the surface parametrized by* (2.2) *is given by*

$$H_{gr}(u) = \frac{1}{A^3(u)\, B} \operatorname{div} \left(\frac{A^2(u)\, B^2\, \nabla u}{\sqrt{1 + B^2\, |\nabla u|^2}} \right) + 2\, \sqrt{1 + B^2\, |\nabla u|^2}\, \sinh u, \tag{2.3}$$

where by definition $A(u) = A(\cdot, u)$. In this expression, the gradient of u, the divergence and the norm of ∇u are computed with respect to the metric $|dz|^2$ on D^2.

Proof. The area form of the vertical graph $z = x_1 + i\, x_2 \mapsto (z, u(z))$ with respect to the pull-back metric $\mathcal{X}^* g_{eucl}$ is given by

$$da := A^2(u)\, \sqrt{1 + B^2\, |\nabla u|^2}\, dx_1\, dx_2.$$

The differential of the area functional at u is given by

$$D\mathrm{Area}|_u\,(v) = \iint_{D^2} \left(\frac{A^2(u)\, B^2\, \nabla u \cdot \nabla v}{\sqrt{1 + B^2\, |\nabla u|^2}} + 2A(u)\, \partial_{x_3} A(u) \sqrt{1 + B^2 |\nabla u|^2}\, v \right) dx_1\, dx_2.$$

Since

$$\partial_{x_3} A = -A^2\, B\, \sinh x_3,$$

we conclude that

$$\begin{aligned} &D\mathrm{Area}|_u\,(v) = \\ &- \iint_{D^2} \left(\operatorname{div} \left(\frac{A^2(u)\, B^2 \nabla u}{\sqrt{1 + B^2\, |\nabla u|^2}} \right) + 2\, A^3(u)\, B\, \sqrt{1 + B^2\, |\nabla u|^2}\, \sinh u \right) v\, dx_1\, dx_2. \end{aligned}$$

Observe that the unit normal vector with respect to $\mathcal{X}^* g_{eucl}$ is given by

$$N_{gr} := \frac{1}{A(u)}\, \frac{1}{\sqrt{1 + B^2\, |\nabla u|^2}} \left(-B\, \nabla u + \frac{1}{B}\, \partial_{x_3} \right),$$

and hence

$$\mathcal{X}^* g_{eucl}(N_{gr}, \partial_{x_3}) = \frac{A(u)\, B}{\sqrt{1 + B^2\, |\nabla u|^2}},$$

so that

$$\mathcal{X}^* g_{eucl}(N_{gr}, \partial_{x_3})\, da = A^3(u)\, B,$$

and the result follows from the first variation of the area formula

$$D\mathrm{Area}_{|u}(v) = -\iint_{D^2} H_{gr}(u)\, g_{eucl}(N, \partial_{x_3})\, v\, da.$$

$\square$

Let us denote by Δ the (flat) Laplacian on D^2 and by ∇u and $\nabla^2 u$ the gradient and the Hessian of u with respect to the Euclidean metric $|dz|^2$ on D^2. Finally, consider the polar coordinates (r, ϕ) in D^2.

Corollary 2.3.1. *The graph of the function u meets the sphere $S^2 = \partial B^3$ orthogonally at the boundary when u satisfies homogeneous Neumann boundary condition:*

$$\partial_r u|_{r=1} = 0.$$

Moreover, we can rewrite (2.3) *in the form*

$$H_{gr}(u) = L_{gr}\, u + Q_{gr}(u, \nabla u, \nabla^2 u),$$

where L_{gr} is the linearized mean curvature operator which reads

$$L_{gr}\, u = \Delta(Bu) = \Delta\left(\frac{1+|z|^2}{2}\, u\right), \tag{2.4}$$

and $Q_{gr}(\cdot,\cdot,\cdot)$ is a smooth nonlinear function that satisfies

$$Q_{gr}(0,0,0) = 0, \quad DQ_{gr}(0,0,0) = 0, \quad D^2 Q_{gr}(0,0,0) = 0$$

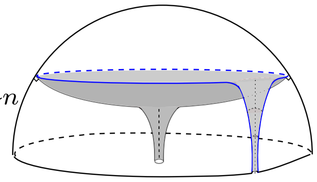
and whose Taylor expansion is affine in $\nabla^2 w$ and at least quadratic in ∇w.

Proof. It is easy to verify that if $\partial_r u|_{r=1} = 0$, then the tangent vector $\frac{\partial}{\partial_r}\mathcal{X}(r\, e^{i\phi}, u(r\, e^{i\phi}))\Big|_{r=1}$ is collinear to $\mathcal{X}(e^{i\phi}, u(e^{i\phi}))$, which implies that the graph of u meets the boundary of B^3 orthogonally. The expression for the mean curvature follows from a careful analysis of (2.3). $\square$

Remark 2.3.1. *The operator L_{gr} in D^2 with homogeneous Neumann boundary data has a kernel which consists of the functions $\frac{2\,x_1}{1+|z|^2}, \frac{2\,x_2}{1+|z|^2}$ and corresponds to tilting the unit disk $D^2 \times \{0\}$ in B^3. The kernel can be eliminated by imposing invariance under the action of a group of rotations around the vertical axis.*

2.4 Harmonic functions with singularities defined on the unit disk

According to A. Fraser and R. Schoen, the surfaces that we would like to construct should have the structure of connected sums of two “nearby copies” of the unit disk $D^2 \times \{0\}$ with small “bridges” which are close to truncated scaled down half-catenoids centered at the n-th roots of unity $z_m = e^{\frac{2\pi i m}{n}}$, and a small “neck” which is close to a truncated scaled down catenoid centered at the origin.



Recall that the catenoid has logarithmic behavior at infinity. According to the ideas of R. Mazzeo, F. Pacard, and D. Pollack [77], it turns out that from the analytical point of view, to get a better matching with the asymptotics of the catenoid, it is better first to deform the unit disk using a Green's function associated to the operator L_{gr} with poles at $z = 0$ and $z = z_m$, $m = 1, \ldots, n$, and then perform the connected sum. We use the notation D^2_* for the unit open punctured disk $D^2 \setminus \{0\}$. Our goal now is to understand the solutions of the problem

$$\begin{cases} L_{gr}\,\Gamma_n = 0 \quad \text{in} \quad D^2_*, \\ \partial_r \Gamma_n = 0 \quad \text{on} \quad \partial D^2 \setminus \{z_1, \ldots, z_n\} \end{cases} \tag{2.5}$$

which have log-type singularities at $z = z_m$ and $z = 0$. Let G_n be a solution of

$$\begin{cases} \Delta G_n = 0 \quad \text{in} \quad D^2_*, \\ \partial_r G_n - \frac{1}{n} G_n = 0 \quad \text{on} \quad \partial D^2 \setminus \{1\} \end{cases} \tag{2.6}$$

which has logarithmic growth at $z = 1$ and $z = 0$. Then $\Gamma_n(z) = G_n(z^n)/B(z)$ is a solution to (2.5) invariant under rotations by the angle $\frac{2\pi}{n}$. A solution of (2.6) can be constructed explicitly. We define in $\bar{D}^2 \setminus \{1\}$ the function

$$G^1_n(z) := -\frac{n}{2} + \mathrm{Re}\left(\sum_{j=1}^{\infty} \frac{n\, z^j}{nj - 1} \right). \tag{2.7}$$

Writing

$$\frac{1}{nj - 1} = \sum_{k=0}^{\infty} \frac{1}{(nj)^{k+1}},$$

we see that we also have the expression

$$G^1_n(z) := -\frac{n}{2} + \mathrm{Re}\left(\sum_{k=0}^{\infty} \frac{H_k(z)}{n^k} \right),$$

where, for all $k \in \mathbb{N}$, the function H_k is given by

$$H_k(z) := \sum_{j=1}^{\infty} \frac{z^j}{j^{k+1}}. \tag{2.8}$$

Observe, that

$$H_0(z) = -\ln(1 - z). \tag{2.9}$$

Obviously, G_n^1 is harmonic in the open unit disk. Making use of (2.8) we see that for all $k \geq 1$

$$\partial_r (\operatorname{Re} H_k) = \operatorname{Re} H_{k-1}$$

on ∂D^2, while it follows from (2.9) that

$$\partial_r (\operatorname{Re} H_0) = -\frac{1}{2}$$

again on $\partial D^2 \setminus \{1\}$. Therefore, we conclude from (2.4) that

$$\partial_r G_n^1 - \frac{1}{n} G_n^1 = 0 \quad \text{on} \quad \partial D^2 \setminus \{1\}.$$

We also define in $\bar{D}^2_*$, the function G_n^0 by

$$G_n^0(z) := -n - \log|z|. \tag{2.10}$$

Again G_n^0 is harmonic in D^2_* and we have

$$\partial_r G_n^0 - \frac{1}{n} G_n^0 = 0 \quad \text{on} \quad \partial D^2.$$

To complete this paragraph, we define

$$\Gamma_n^1(z) := \frac{1}{B(z)} G_n^1(z^n) \qquad \text{and} \qquad \Gamma_n^0(z) := \frac{1}{B(z)} G_n^0(z^n). \tag{2.11}$$

By construction, $L_{gr}\, \Gamma_n^1 = 0$ in D^2 and $\partial_r \Gamma_n^1 = 0$ on ∂D^2 away from the n-th roots of unity; while $L_{gr}\, \Gamma_n^0 = 0$ in D^2_* and $\partial_r \Gamma_n^0 = 0$ on ∂D^2.

2.5 "Half-catenoidal bridges"

One of the options could be to construct the "bridges" in B^3 as minimal stripes obtained by the intersection of Euclidean catenoids centered at $z = z_m$ with the unit sphere. The difficulty of this approach is that those stripes do not meet S^2 orthogonally. We prefer to find a free boundary embedding of the half-catenoids in B^3, but loosing the minimality condition.

We use the notation $\mathbb{C}_-$ for the half-plane $\{\zeta \in \mathbb{C} \,|\, \operatorname{Re}(\zeta) \leq 0\}$. For $m = 1, \ldots, n$ consider the conformal mappings

$$\lambda_m : \zeta \in \mathbb{C}_- \longmapsto e^{\frac{2i\pi m}{n}} \frac{1+\zeta}{1-\zeta} \in \bar{D}^2 \setminus \{-e^{\frac{2i\pi m}{n}}\}, \tag{2.12}$$

which transform a half-disk in $\mathbb{C}_-$ centered at $\zeta = 0$ and of radius $\rho < 1$ to a domain obtained by the intersection of the unit disk $\bar{D}^2$ with the disk of radius $\frac{2\rho}{1-\rho^2}$ centered at $\frac{1+\rho^2}{1-\rho^2} e^{\frac{2i\pi m}{n}}$. Let $(\zeta = \xi_1 + i\xi_2, \xi_3)$ be coordinates in $\mathbb{C}_- \times \mathbb{R}$. We define the mapping

$$\Lambda_m : (\zeta, \xi_3) \in \mathbb{C}_- \times \mathbb{R} \longmapsto (\lambda_m(\zeta), 2\,\xi_3) \in \bar{D}^2 \setminus \{-e^{\frac{2i\pi m}{n}}\} \times \mathbb{R}. \tag{2.13}$$

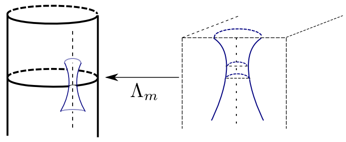
Take $\varepsilon \in (0,1)$ and consider the half-catenoid $C_{\varepsilon/2}$ in $\mathbb{C}_- \times \mathbb{R}$, parametrized by

$$X^{cat}_{\varepsilon/2} : (\sigma, \theta) \in \mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \mapsto \left(\frac{\varepsilon}{2} \cosh \sigma \, e^{i\theta}, \frac{\varepsilon}{2}\sigma\right) \in \mathbb{C}_- \times \mathbb{R}. \tag{2.14}$$

Then we parametrize the m-th "catenoidal bridge" $\mathfrak{C}_{\varepsilon,m}$ by

$$(\sigma, \theta) \in [-\sigma_\varepsilon, \sigma_\varepsilon] \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \mapsto \mathcal{X} \circ \Lambda_m \circ X^{cat}_{\varepsilon/2}(\sigma, \theta) \subset \bar{B}^3,$$

where the value of σ_ε will be made precise in the subsection 2.7. Using the facts that the half-catenoid meets the boundary of the half-space orthogonally and that the restriction of the mapping $\mathcal{X} \circ \Lambda_m$ to horizontal planes is conformal, one can check that $\mathfrak{C}_{\varepsilon,m}$ meets $S^2 = \partial B^3$ orthogonally at the boundary.



2.6 "Catenoidal neck"

Again, one of the possibilities could be to use as the "catenoidal neck" the standard Euclidean catenoid embedded in B^3 and centered at the origin. We choose an alternative construction, changing slightly the value of the mean curvature but simplifying the perturbation argument. Remark that in a neighborhood of $(z, x_3) = 0$ the metric $\mathcal{X}^* g_{eucl} = A^2(dz^2 + B^2\, dx_3^2)$ is close to $\tilde{g}_{eucl} := dz^2 + \frac{1}{4} dx_3^2$. Fixing $\tilde{\varepsilon} \in (0,1)$ we introduce the surface $\tilde{C}_{\tilde{\varepsilon}}$ parametrized by

$$\tilde{X}^{cat}_{\tilde{\varepsilon}} \,:\, (s, \phi) \in [-s_{\tilde{\varepsilon}}, s_{\tilde{\varepsilon}}] \times S^1 \,\mapsto\, (\tilde{\varepsilon} \cosh s \, e^{i\phi}, 2\,\tilde{\varepsilon} s) \in D^2 \times \mathbb{R}, \tag{2.15}$$

where the value of $s_{\tilde{\varepsilon}}$ will be made precise in the subsection 2.7. Then $\tilde{C}_{\tilde{\varepsilon}}$ is minimal with respect to $\tilde{g}_{eucl}$. Finally, we parametrize the "catenoidal neck" $\mathfrak{C}_{\tilde{\varepsilon},0}$ by

$$\mathcal{X} \circ \tilde{X}^{cat}_{\tilde{\varepsilon}} : [-s_{\tilde{\varepsilon}}, s_{\tilde{\varepsilon}}] \times S^1 \longrightarrow B^3.$$

2.7 Approximate solution

In this section we describe the "gluing procedure" between a graph over the horizontal disk of a suitable Green's function and the catenoidal "bridges" and "neck". Let the functions Γ^0_n and Γ^1_n be defined as in (2.11). As a fist step, for some $\tau_0, \tau_1 \in \mathbb{R}$, we find the expansion of the function

$$\mathcal{G}_n(z) := \tau_0\, \Gamma^0_n + \tau_1\, \Gamma^1_n$$

in the neighborhood of $z = z_0$ and $z = z_m$.

Notation 2.7.1. *We denote by $\hat{c}$ any positive constant independent of the choice of $\tau_1, \tau_0, n, \varepsilon$ and $\tilde{\varepsilon}$. Let u and v be two functions, then we write $v = \hat{\mathcal{O}}(u)$ when $z \to \tilde{z}$, if there exists $\hat{c} > 0$ (independent of $\tau_0, \tau_1, n, \varepsilon$ and $\tilde{\varepsilon}$) such that $|v| \leq \hat{c}\,|u|$ in some neighborhood of $\tilde{z}$.*

Consider the functions G_n^0 and G_n^1 defined in (2.10) and (2.7). We introduce the function

$$f_n(z) := \sum_{k=0}^{\infty} \frac{H_k(z^n)}{n^k} = \sum_{k=0}^{\infty} \frac{1}{n^k} \sum_{j=1}^{\infty} \frac{z^{nj}}{j^{k+1}},$$

where $G_n^1(z^n) = -\frac{n}{2} + \operatorname{Re} f_n(z)$. It is easy to verify that

$$\frac{\partial f_n}{\partial z}(z) = -\frac{d}{dz}\log(1 - z^n) + \frac{1}{z} f_n(z),$$

which yields

$$\frac{d}{dz}\left(\frac{f_n}{z}\right) = -\frac{n z^{n-2}}{z^n - 1}.$$

A straightforward calculation shows that the function

$$z \mapsto h_n(z) := -\frac{n z^{n-2}}{z^n - 1} + \frac{1}{z_m(z - z_m)}$$

is continuous in a small enough neighborhood of $z = z_m$ and that

$$|h_n(z_m)| \leq \hat{c}\, n.$$

Thus, in a neighborhood of $z = z_m$ we have:

$$\frac{f_n(z)}{z} + \frac{1}{z_m}\log(z - z_m) = \frac{1}{z_m}\lim_{z \to z_m}\left(f_n(z) + \log(z - z_m)\right) + \int_z^{z_m} h_n(z)dz,$$

where the integral is taken along the segment of the straight line passing from z to z_m and by log we mean the principal value of complex logarithm defined in the unit disk deprived of a segment of a straight line which doesn't pass through any of the n-th roots of unity. We obtain:

$$\sum_{k=1}^{\infty} \frac{1}{n^k} H_k(z_m^n) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{n^k\, j^{k+1}} \leq \frac{\pi^2}{6(n-1)}.$$

Moreover,

$$\operatorname{Re}\lim_{z \to z_m}\left(-\log(1 - z^n) + \log(z - z_m)\right) = -\log|n\, z_m^{n-1}| = -\log n.$$

In a small enough neighborhood of $z = z_m$ and we obtain:

$$G_n^1(z^n) = -\frac{n}{2} + c_{gr}(n) - \log|z - z_m| + \hat{\mathcal{O}}(|z - z_m|\log|z - z_m|) + \hat{\mathcal{O}}(n|z - z_m|),$$

where $|c_{gr}(n)| \leq \hat{c}\,\log n$. This yields

$$\mathcal{G}_n(z) = \begin{cases} -n\left(\tau_0 + \frac{\tau_1}{2}\right) + \tau_1\, c_{gr}(n) - \tau_1 \log|z - z_m| \; + \hat{\mathcal{O}}\Big((|\tau_0| + |\tau_1|)n|z - z_m|\Big) \\ +\hat{\mathcal{O}}\Big((|\tau_0| + |\tau_1|)|z - z_m| \log|z - z_m|\Big), \text{ as } |z - z_m| \to 0; \\ \\ -2n\left(\tau_0 + \frac{\tau_1}{2}\right) - 2\,\tau_0 n\, \log|z| + \hat{\mathcal{O}}\Big((|\tau_0| + |\tau_1|)|z|^2\Big), \text{ as } |z| \to 0. \end{cases} \tag{2.16}$$

On the other hand, let the mappings $\lambda_m : \mathbb{C}_- \to \bar{D}^2$ and $\Lambda_m : \mathbb{C}_- \times \mathbb{R} \to \bar{D}^2 \times \mathbb{R}$ be defined as in (2.12) and (2.13). In a neighborhood of $z = z_m$ taking the change of coordinates

$$(\sigma, \theta) \in [-\sigma_\varepsilon, \sigma_\varepsilon] \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \mapsto z = \lambda_m\left(\frac{\varepsilon}{2}\cosh\sigma\, e^{i\theta}\right) \in \bar{D}^2,$$

we can see the image by the mapping Λ_m of the truncated half-catenoid (parametrized in $\mathbb{C}_- \times \mathbb{R}$ by (2.14)) as bi-graph over $\bar{D}^2 \setminus \lambda_m(\{\zeta \in \mathbb{C}_- : |\zeta| > \varepsilon/2\})$ of the function

$$G^m_{cat}(z) := -\varepsilon \log\frac{\varepsilon}{2} + \varepsilon \log|z - z_m| + \hat{\mathcal{O}}\left(\varepsilon^3\, |z - z_m|^{-2}\right). \tag{2.17}$$

Similarly, in a neighborhood of $z = 0$, taking the change of coordinates

$$(s, \phi) \in [-s_{\tilde{\varepsilon}}, s_{\tilde{\varepsilon}}] \times S^1 \mapsto z = \tilde{\varepsilon}\cosh s\, e^{i\phi} \in D^2,$$

we can see the surface embedded in the unit cylinder and parametrized by (2.15) as a bi-graph over $\{z \in D^2 : |z| > \tilde{\varepsilon}\}$ of the function

$$G^0_{cat}(z) := -2\,\tilde{\varepsilon}\log\frac{\tilde{\varepsilon}}{2} + 2\,\tilde{\varepsilon}\log|z| + \hat{\mathcal{O}}\left(\tilde{\varepsilon}^3\, |z|^{-2}\right). \tag{2.18}$$

The next step is to choose the parameters τ_1, τ_0, ε, and $\tilde{\varepsilon}$ in such a way that the leading terms in (2.16) match exactly the leading terms in (2.17) and (2.18). More precisely, comparing the logarithmic terms, we take $\tau_1 = \varepsilon$ and $n\tau_0 = \tilde{\varepsilon}$. On the other hand, the constant terms match if

$$-\tilde{\varepsilon} - \frac{\varepsilon n}{2} = \tilde{\varepsilon}\log\frac{\tilde{\varepsilon}}{2} \quad \text{and} \quad -\tilde{\varepsilon} - \frac{\varepsilon n}{2} + \varepsilon\, c_{gr}(n) = \varepsilon\log\frac{\varepsilon}{2}.$$

This gives us the relation

$$\log\frac{\varepsilon}{\tilde{\varepsilon}} + \frac{\tilde{\varepsilon}}{\varepsilon} - \frac{n}{2}\frac{\varepsilon}{\tilde{\varepsilon}} = -\frac{n}{2} + c_{gr}(n) + 1,$$

which yields

$$\frac{\varepsilon}{\tilde{\varepsilon}} = g_n^{-1}\Big(-\frac{n}{2} + c_{gr}(n) + 1\Big) := d(n) \in [1/2, 1],$$

where $g_n(t) : t \in (0, +\infty) \mapsto \log t - \frac{n}{2}\,t + \frac{1}{t} \in (-\infty, \infty)$. This gives a unique correspondence between ε, $\tilde{\varepsilon}$ and n, which satisfy

$$\varepsilon \sim \tilde{\varepsilon}, \quad n \sim \log\varepsilon.$$

Finally, comparing the remaining terms we see that we should truncate the summands and effectuate the connected sum in the regions where

$$|z - z_m| \sim \varepsilon^{2/3} \quad \text{and} \quad |z| \sim \varepsilon^{1/2}.$$

Remark 2.7.1. *These estimates together with the fact that constant functions are not in the kernel of the operator L_{gr} give an idea why our construction works only for large numbers n.*

Finally, we perform a connected sum in such a way that the resulting genus 1 surface is embedded in B^3 and meets S^2 orthogonally at the boundary. Moreover, since all the parameters in our constructions are expressed as functions of the number of boundary components n, we denote our surface by $\mathcal{A}_n$ and refer to it as *approximate solution.* Here is a more detailed description of $\mathcal{A}_n$.

Notation 2.7.2. *We introduce the cut-off functions $\bar{\eta}, \eta \in \mathcal{C}^\infty(0,1)$ defined by*

$$\begin{aligned} \bar{\eta}(t) &= 1 \quad \textit{for} \quad t < \tfrac{1}{2}\,\varepsilon^{2/3}, \quad \bar{\eta}(t) = 0 \quad \textit{for} \quad t > 2\,\varepsilon^{2/3}, \\ \eta(t) &= 1 \quad \textit{for} \quad t < \tfrac{1}{2}\,\varepsilon^{1/2}, \quad \eta \equiv 0 \qquad \textit{for} \quad t > 2\,\varepsilon^{1/2}, \end{aligned}$$

and the functions $\eta_0, \eta_m \in \mathcal{C}^\infty(D^2)$ defined by

$$\eta_0(z) := \eta(|z|), \quad \eta_m(z) := \bar{\eta}\left(\left|\lambda_m^{-1}(z)\right|\right). \tag{2.19}$$

1) We parametrize the *graph regions* $\Omega_{gr}^{\pm}$ of $\mathcal{A}_n$ as vertical graphs:

$$z \in D_{gr} \longmapsto \mathcal{X}\left(z, \mp \mathcal{G}_n\right),$$

where $\mathcal{G}_n$ is defined in (2.16) and D_{gr} is a subdomain of $\bar{D}^2$ defined by

$$D_{gr} := \left\{z \in D^2 \;:\; |z| > 2\,\varepsilon^{1/2}\right\} \bigcap_{m=1}^{n} \lambda_m\left(\left\{\zeta \in \mathbb{C}_- \;:\; |\zeta| > 2\,\varepsilon^{2/3}\right\}\right). \tag{2.20}$$

2) As in the subsection 2.6, we parametrize the *"catenoidal neck" region* Ω_{cat}^0 by:

$$(s,\phi) \in [-s_{\tilde{\varepsilon}}, s_{\tilde{\varepsilon}}] \times S^1 \longmapsto \mathcal{X} \circ \tilde{X}_{\tilde{\varepsilon}}^{cat}(s,\phi) = \mathcal{X}\left(\tilde{\varepsilon}\cosh s\, e^{i\phi}, 2\,\tilde{\varepsilon} s\right),$$

where $s_{\tilde{\varepsilon}}$ satisfies $\tilde{\varepsilon}\cosh s_{\tilde{\varepsilon}} = \frac{1}{2}\varepsilon^{\frac{1}{2}}$.

3) The *"half-catenoidal bridges" regions* Ω_{cat}^m are parametrized as the images by the mappings $\mathcal{X} \circ \Lambda_m$ of the truncated half-catenoid $C_{\varepsilon/2} \subset \mathbb{C}_- \times \mathbb{R}$:

$$(\sigma,\theta) \in [-\sigma_\varepsilon, \sigma_\varepsilon] \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \longmapsto \mathcal{X} \circ \Lambda_m \circ X_{\varepsilon/2}^{cat}(\sigma,\theta) = \mathcal{X} \circ \Lambda_m\left(\frac{\varepsilon}{2}\cosh\sigma\, e^{i\theta}, \frac{\varepsilon}{2}s\right),$$

where σ_ε satisfies $\varepsilon\cosh\sigma_\varepsilon = \varepsilon^{\frac{2}{3}}$.

4) Finally, in the *gluing regions* $\Omega_{glu}^{i,\pm}$, we interpolate between $\Omega_{gr}^{\pm}$ and the catenoidal regions and parametrize $\mathcal{A}_n$ by:

$$z \in D_{glu}^i \longmapsto \mathcal{X}\left(z, \pm\,\upsilon_i(z)\right),$$

where the functions υ_i are defined by

$$\upsilon_i := \left(\eta_i\, G_{cat}^i - (1-\eta_i)\,\mathcal{G}_n\right), \quad i = 0, 1, \ldots, n, \tag{2.21}$$

and D^i_{glu} are the subdomains of the unit disk defined by

$$D^0_{glu} := \left\{ z \in D^2 \,:\, \frac{1}{2}\,\varepsilon^{1/2} < |z| < 2\,\varepsilon^{1/2} \right\} \subset D^2, \tag{2.22}$$
$$D^m_{glu} := \lambda_m \left(\left\{ \zeta \in \mathbb{C}_- \,:\, \frac{1}{2}\,\varepsilon^{2/3} < |\zeta| < 2\,\varepsilon^{2/3} \right\} \right) \subset D^2, \quad m = 1, \dots, n.$$

2.8 Perturbation argument

The next step is to show that the approximate solution $\mathcal{A}_n$ can be perturbed, at least for n large enough, into some free boundary minimal surface. To this end, we describe all genus 1 surfaces embedded in B^3 which are close to $\mathcal{A}_n$ and meet S^2 orthogonally at the boundary. Let Ξ be a vector field in $\bar{B}^3$ transverse to $\mathcal{A}_n$ and ξ be the associated flow:

$$\frac{d\xi}{dt} = \Xi\,(\xi(\cdot, t)) \quad \text{and} \quad \xi(p, 0) = p \quad \forall p \in \bar{B}^3.$$

We shall choose Ξ in such a way that that for all t small enough the surface $\mathcal{A}_{n,t} := \xi(\mathcal{A}_n, t)$ is embedded in B^3 and meets S^2 orthogonally along $\partial\mathcal{A}_{n,t} = \xi_t(\partial\mathcal{A}_n)$. Take $w \in \mathcal{C}^2(\mathcal{A}_n)$ and let $\mathcal{A}_n(w)$ be the surface parametrized by

$$p \in \mathcal{A}_n \mapsto \xi(p, w(p)) \in \mathcal{A}_n(w).$$

Then $\mathcal{A}_n(w)$ meets S^2 orthogonally at the boundary if w satisfies the homogeneous Neumann condition on $\partial\mathcal{A}_n$:

$$g_n(\nabla^{g_n} w, N_{\partial\mathcal{A}_n}) = 0,$$

where g_n is the metric induced on $\mathcal{A}_n$ from the Euclidean metric, ∇^{g_n} is the gradient calculated with respect to g_n, and $N_{\partial\mathcal{A}_n}$ is a unit normal to $\partial\mathcal{A}_n$ in $\mathcal{A}_n$.

The expansion of the mean curvature of $\mathcal{A}_n(w)$ in powers of w and derivatives of w up to the second order has the form:

$$H(\mathcal{A}_n(w)) = H(\mathcal{A}_n) + \mathcal{L}_n\, w + \mathcal{Q}_n(w, \nabla w, \nabla^2 w),$$

where $H(\mathcal{A}_n)$ is the mean curvature of the approximate solution, $\mathcal{L}_n$ is the linearized mean curvature operator about $\mathcal{A}_n$, and $\mathcal{Q}_n$ is a smooth nonlinear function of w and the components of the gradient and the Hessian of w. Below, we explain an appropriate choice of the vector field Ξ and study the properties of the function $H(\mathcal{A}_n)$ and the operators $\mathcal{L}_n$ and $\mathcal{Q}_n$ in appropriate function spaces.

2.8.1 Choice of the transverse vector field

In this paragraph we describe explicitly the parametrization of the perturbed surfaces $\mathcal{A}_n(w)$, which implicitly explains the choice of the transverse vector field Ξ.

1) In the graph regions $\Omega_{gr}^{\pm}$ we parametrize $\mathcal{A}_n(w)$ as a vertical bi-graph over the subdomain $D_{gr} \subset D^2$, defined as in (2.20):

$$z \in D_{gr} \mapsto \mathcal{X}\left(z, \mp\left[\mathcal{G}_n(z) + w(z)\right]\right).$$

2) In the "catenoidal neck" region, we parametrize $\mathcal{A}_n(w)$ as an image in B^3 of a normal graph over the surface $\tilde{C}_{\tilde{\varepsilon}}$ defined in the section 2.6, more precisely we put

$$(s, \phi) \in [-s_\varepsilon, s_\varepsilon] \times S^1 \mapsto \mathcal{X}\left(\tilde{X}_{\tilde{\varepsilon}}^{cat}(s, \phi) + \frac{w}{2}\,\tilde{N}_{cat}(s, \phi)\right) \subset \mathcal{A}_n(w)$$

where (s, ϕ) are cylindrical coordinates in Ω_{cat}^0 and

$$\tilde{N}_{cat}(s, \phi) = \left(-\frac{1}{\cosh s}\, e^{i\phi}, 2\,\tanh s\right), \tag{2.23}$$

is a unit normal to $\tilde{C}_{\tilde{\varepsilon}} \subset D^2 \times \mathbb{R}$ with respect to the metric $\tilde{g}_{eucl} = dz^2 + \frac{1}{4}\,dx_3^2$.

3) In the same manner, in the "half-catenoidal bridges" regions, we parametrize $\mathcal{A}_n(w)$ as an image in B^3 of a normal graph over the half-catenoid:

$$(\sigma, \theta) \in [-\sigma_\varepsilon, \sigma_\varepsilon] \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \mapsto \mathcal{X} \circ \Lambda_m \left(X_{\varepsilon/2}^{cat}(\sigma, \theta) + \frac{w}{2}\, N_{cat}(\sigma, \theta)\right) \in \mathcal{A}_n(w)$$

where (σ, θ) are half-cylindrical coordinates in Ω_{cat}^m and

$$N_{cat}(\sigma, \theta) = \left(-\frac{1}{\cosh \sigma}\, e^{i\theta}, \tanh \sigma\right), \tag{2.24}$$

is a unit normal to the Euclidean half-catenoid $C_{\varepsilon/2}$.

4) Finally, in the gluing regions $\Omega_{glu}^{i,\pm}$ we interpolate smoothly between the parametrizations described above. Consider the functions v_i defined in (2.21). We introduce the function $\bar{v}$ in $\mathbb{C}_-$ defined by $\bar{v}(\zeta) := v_m(\lambda_m(\zeta))$, $m = 1, \ldots, n$. Finally, let $\eta,\ \bar{\eta} \in \mathcal{C}^\infty(\mathbb{R})$ be the cut-off functions defined in (2.19). Then in $\Omega_{glu}^{i,\pm}$ we parametrize $\mathcal{A}_n(w)$ by

$$\begin{aligned} z \in D^2,\ \tfrac{1}{2}\,\varepsilon^{1/2} < |z| < 2\,\varepsilon^{1/2} \quad &\mapsto \quad \mathcal{X}\Big(\left(z,\, v_0(z)\right) + w\,\mathcal{V}_0(z)\Big), \\ \zeta \in \mathbb{C}_-,\ \tfrac{1}{2}\,\varepsilon^{2/3} < |\zeta| < 2\,\varepsilon^{2/3} \quad &\mapsto \quad \mathcal{X} \circ \Lambda_m\Big(\left(\zeta,\, \bar{v}(\zeta)\right) + \frac{w}{2}\,\bar{\mathcal{V}}(\zeta)\Big), \end{aligned}$$

where the vector fields $\mathcal{V}_0$ and $\bar{\mathcal{V}}$ are defined in $D^2 \times \mathbb{R}$ and $\mathbb{C}_- \times \mathbb{R}$ by

$$\mathcal{V}_0 := \eta\,\frac{1}{2}\,\tilde{N}_{cat} \pm (1 - \eta)\ (0, 0, 1) \quad \text{and} \quad \bar{\mathcal{V}} := \bar{\eta}\, N_{cat} \pm (1 - \bar{\eta})\ (0, 0, 1). \tag{2.25}$$

Remark 2.8.1. *The surface $\mathcal{A}_n(w)$ is invariant under the action of the group $\mathfrak{S}_n$ and meets the unit sphere $S^2 = \partial B^3$ orthogonally at the boundary.*

2.8.2 Function spaces

In this paragraph we define the function spaces we will work in.

Definition 2.8.1 (Weight function). *We introduce the weight function*

$$\gamma \,:\, p \in \mathcal{A}_n \,\mapsto\, \prod_{i=0}^{n} |p - p_i|, \quad \textit{where} \quad p_0 = 0, \quad p_m = \left(\cos\frac{2\pi m}{n}, \sin\frac{2\pi m}{n}, 0\right),$$

where by $|\cdot|$ we denote the Euclidean distance in $\mathbb{R}^3$.

Definition 2.8.2 (Weighted Hölder spaces). *Let g_n be the metric induced on $\mathcal{A}_n$ from the Euclidean metric and take $\nu \in \mathbb{R}$. We denote by $\mathcal{E}^{k,\alpha}_{\nu,n}$ the subspace of functions $w \in \mathcal{C}^{k,\alpha}(\mathcal{A}_n)$*

1) *endowed with the norm*

$$\|w\|_{\mathcal{E}^{k,\alpha}_{n,\nu}} := \sup_{\mathcal{A}_n} |\gamma^{-\nu} w| + \sum_{\ell=1}^{k} \sup_{\mathcal{A}_n} \|\gamma^{\ell-\nu}\nabla^\ell w\|_{g_n} + \sup_{p,p'\in\mathcal{A}_n} \left| \frac{\gamma^{k+\alpha-\nu}(p)\,\nabla^k w(p) - \gamma^{k+\alpha-\nu}(p')\ \nabla^k w(p')}{d_{g_n}(p,p')^\alpha} \right|, \tag{2.26}$$

2) *invariant under the action of the group $\mathfrak{S}_n$,*

3) *and for $k = 2$ satisfying homogeneous Neumann boundary condition*

$$g_n(\nabla^{g_n} w, N_{\partial\mathcal{A}_n}) = 0.$$

Remark 2.8.2. *Recall that in different regions of $\mathcal{A}_n$ we work with different coordinate systems. For $r, \rho \in (0,1]$ consider the following subdomains of the unit disk*

$$D^0_r := \left\{ z \in D^2 \,:\, |z| < r \right\} \quad \textit{and} \quad D^m_\rho := \lambda_m\big(\{\zeta \in \mathbb{C}_- \,:\, |z| < \rho\}\big)$$

$$A^0_r := \left\{ z \in \bar{D}^2 \,:\, r < |z| < 2\,r \right\} \quad \textit{and} \quad A^m_\rho := \lambda_m\big(\{\zeta \in \mathbb{C}_- \,:\, \rho < |\zeta| < 2\,\rho\}\big)$$

and finally

$$D_{r,\rho} := \Big(A^0_r \bigcup_{m=1}^{n} A^m_\rho\Big) \setminus \Big(D^0_r \bigcup_{m=1}^{n} D^m_\rho\Big).$$

We introduce the function

$$\gamma_D : z \in \bar{D}^2 \mapsto |z| \prod_{m=1}^{n} |z - z_m| = |z||z^n - 1|.$$

Then the norm (2.26) is equivalent to the norm defined by

$$\sup_{s\in[-s_\varepsilon, s_\varepsilon-1]} \big\|(\tilde{\varepsilon}\cosh s)^{-\nu}\, w\big\|_{\mathcal{C}^{k,\alpha}([s,s+1]\times S^1)}$$

$$+ \sup_{\sigma\in[-\sigma_\varepsilon,\sigma_\varepsilon-1]} \Big\| \left(\frac{\varepsilon}{2}\cosh\sigma\right)^{-\nu} w\Big\|_{\mathcal{C}^{k,\alpha}\left([\sigma,\sigma+1]\times\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\right)}$$

$$+ \sup_{\rho\in[\rho_\varepsilon,1/2],\ r\in[r_\varepsilon,1/2]} \|\gamma_D^{-\nu}\, w\|_{\mathcal{C}^{k,\alpha}\left(D_{r,\rho},\gamma_D^{-2}\,|dz|^2\right)},$$

where $\rho_\varepsilon = 1/2\,\varepsilon^{2/3}$ and $r_\varepsilon = 1/2\,\varepsilon^{1/2}$. Observe that, in the last term we use the singular metric $\gamma_D^{-2}\,|dz|^2$ to calculate the gradient and the Hessian of the function. Finally, the homogeneous Neumann boundary condition on the function w on the boundaries of the "half-catenoidal bridges" reads:

$$\partial_\theta w|_{\left\{\frac{\pi}{2},\frac{3\pi}{2}\right\}} = 0.$$

2.8.3 Mean curvature of the approximate solutions

In this section, we show that in a suitable topology $H(\mathcal{A}_n)$ tends to 0, when n tends to infinity. To this end, we obtain L^∞ estimates for $H(\mathcal{A}_n)$ in different regions of $\mathcal{A}_n$. In the graph and gluing regions we use the result of Lemma 2.3.1 for the mean curvature of vertical graphs. On the other hand, in the catenoidal regions, our task amounts to calculate the mean curvature of the catenoid embedded in B^3 via a diffeomorphism which can be seen as a perturbation of the identity. More precisely, we have the following result:

Proposition 2.8.1. *For all $\beta \in (0,1)$ and for all $k \in \mathbb{N}\cup\{0\}$ there exist constants $\hat{c}_k > 0$ and $\hat{c} > 0$ independent of n such that the mean curvature of the surface $\mathcal{A}_n$ satisfies*

$$\left|\gamma^k\,\nabla^k H(\mathcal{A}_n)\right| \le \hat{c}_k \begin{cases} e^{-n(3-\beta)}\,\gamma^{-4} & \textit{in}\quad \Omega^{\pm}_{gr}\bigcup\limits_{i=0}^{n}\Omega^{i,\pm}_{glu} \\ e^{-n(1-\beta)}\,\gamma^{-1} & \textit{in}\quad \bigcup\limits_{m=1}^{n}\Omega^{m}_{cat} \\ e^{-n(1-\beta)} & \textit{in}\quad \Omega^{0}_{cat} \end{cases} \tag{2.27}$$

$$\|H(\mathcal{A}_n)\|_{\mathcal{E}^{0,\alpha}_{n,\nu-2}} \le \hat{c}\,e^{-n\left(\frac{5}{3}-2\nu\right)}. \tag{2.28}$$

Proof. **Graph and gluing regions:**

According to Lemma 2.3.1 and Corollary 2.3.1, the mean curvature of the graph $\mathcal{X}\,(z,u(z)))$ for $u\in\mathcal{C}^2(D^2)$ with $\mathcal{C}^1$ norm small enough satisfies:

$$H_{gr}(u) = \Delta\,(Bu) + Q_{gr}(u,\nabla u,\nabla^2 u).$$

In the regions $\Omega^{\pm}_{gr}$ we take $u = \mp\mathcal{G}_n$. Then (2.27) follows from the fact that $\Delta\,(B\,\mathcal{G}_n) = 0$, the properties of the operator Q_{gr} described in Corollary 2.3.1, and the estimates

$$|\mathcal{G}_n(z)| \le \hat{c}_0\,\varepsilon\left(|\log\varepsilon| + |\log|z|| + \sum_{m=1}^{n}|\log|z-z_m||\right),$$

$$\left|\nabla^k\mathcal{G}_n(z)\right| \le \hat{c}_k\,\varepsilon\left(\frac{1}{|z|^k} + \sum_{m=1}^{n}\frac{1}{|z-z_m|^k}\right),$$

for $\hat{c}_i > 0$ independent of ε.

In the gluing regions $\Omega^{i,\pm}_{glu}$ we take

$$u = \pm v_i = \pm \left(\eta_i \, G^i_{cat} - (1-\eta_i)\, \mathcal{G}_n\right), \quad i = 0, \dots, n.$$

For $k \in \mathbb{N} \cup \{0\}$ and for all $\beta \in (0,1)$ we have $\nabla^k \eta_i = \hat{\mathcal{O}}(|z-z_i|^{-k})$, and

$$\nabla^k \mathcal{G}_n \sim \nabla^k G^i_{cat} = \hat{\mathcal{O}}(\varepsilon\, |z-z_i|^{-k}), \quad \nabla^k \left(\mathcal{G}_n - G^i_{cat}\right) = \hat{\mathcal{O}}(\varepsilon^{1-\beta} |z-z_i|^{1-k}),$$

where $z_0 = 0,\ z_m = e^{\frac{2\pi i m}{n}}$. A direct calculation gives

$$|H(\mathcal{A}_n)| \leq \hat{c}\, \varepsilon^{3-\beta}\, |z-z_i|^{-4}.$$

The estimates for the derivatives of $H(\mathcal{A}_n)$ follow from structure of the smooth function $(u, \nabla u, \nabla^2 u) \mapsto Q_{gr}(u)$ and the estimates for $\nabla^k \mathcal{G}_n$ and $\nabla^k v_i$.

"Half-catenoidal" regions:

In Ω^m_{cat} the proof amounts to calculate the mean curvature of $C_{\varepsilon/2} \subset \mathbb{C}_- \times \mathbb{R}$ with respect to the ambient pull-back metric

$$\begin{aligned}
(\mathcal{X} \circ \Lambda_m)^* g_{eucl}(\zeta, \xi_3) &= A^2(\Lambda_m(\zeta, \xi_3)) \Big(\frac{4\, |d\zeta|^2}{|1-\zeta|^4} + 4\, B^2(\Lambda_m(\zeta,\xi_3))\, d\xi_3^2 \Big) \\
&= \frac{4}{\Big(|1-\zeta|^2 + (1+|\zeta|^2)(\cosh(2\xi_3) - 1)\Big)^2} \Big(|d\zeta|^2 + (1+|\zeta|^2)^2\, d\xi_3^2 \Big) \\
&= a^2(\zeta, \xi_3) \left(|d\zeta|^2 + b^2(\zeta)\, d\xi_3^2 \right),
\end{aligned}$$

where $a(\zeta, \xi_3) = \dfrac{2}{|1-\zeta|^2 + (1+|\zeta|^2)(\cosh(2\xi_3) - 1)}$ and $b(\zeta) = 1 + |\zeta|^2$.

Using the notations 2.7.1, we can express the metric induced on $C_{\varepsilon/2}$ in the form:

$$\mathfrak{g}_\varepsilon(\sigma, \theta) = \varepsilon^2 \cosh^2 \sigma (d\sigma^2 + d\phi^2) + \mathcal{O}(\varepsilon^3 \cosh^3 \sigma).$$

Let ∇ be the Levi-Civita connection associated to the metric $(\mathcal{X} \circ \Lambda_m)^* g_{eucl}$. We have the following estimates for the Christoffel symbols in a neighborhood of $(\zeta, \xi_3) = 0$:

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \tfrac{1}{a} \tfrac{\partial a}{\partial \xi_1} = \mathcal{O}(1), \quad \Gamma^2_{11} = -\Gamma^2_{22} = -\Gamma^1_{12} = -\tfrac{1}{a} \tfrac{\partial a}{\partial \xi_2} = \mathcal{O}(1),$$

$$\Gamma^1_{13} = \Gamma^2_{23} = \Gamma^3_{33} = \tfrac{1}{a} \tfrac{\partial a}{\partial \xi_3} = \mathcal{O}(|\xi_3|), \quad \Gamma^3_{11} = \Gamma^3_{22} = -\tfrac{1}{ab^2} \tfrac{\partial a}{\partial \xi_3} = \mathcal{O}(|\xi_3|),$$

$$\Gamma^1_{33} = -(\tfrac{b^2}{a} \tfrac{\partial a}{\partial \xi_1} + b \tfrac{\partial b}{\partial \xi_1}) = \mathcal{O}(1), \quad \Gamma^2_{33} = -(\tfrac{b^2}{a} \tfrac{\partial a}{\partial \xi_2} + b \tfrac{\partial b}{\partial \xi_2}) = \mathcal{O}(1),$$

$$\Gamma^3_{13} = \tfrac{1}{a} \tfrac{\partial a}{\partial \xi_1} + \tfrac{1}{b} \tfrac{\partial b}{\partial \xi_1} = \mathcal{O}(1), \quad \Gamma^3_{23} = \tfrac{1}{a} \tfrac{\partial a}{\partial \xi_2} + \tfrac{1}{b} \tfrac{\partial b}{\partial \xi_2} = \mathcal{O}(1),$$

$$\Gamma^1_{23} = \Gamma^2_{13} = \Gamma^3_{12} = 0.$$

Using that $\nabla_{\partial_k}\partial_\ell = \partial_\ell\,\partial_k\,X^{cat}_{\varepsilon/2} + \left[\partial_\ell X^{cat}_{\varepsilon/2}\right]^i \left[\partial_k X^{cat}_{\varepsilon/2}\right]^j \Gamma^q_{ij}\,\partial_q$, we find:

$$\left|\left[\nabla_{\partial_k}\partial_\ell - \partial_k\,\partial_\ell\,X^{cat}_{\varepsilon/2}\right]^i (\sigma,\theta)\right| \leq \hat{c}\,\varepsilon^2\cosh^2\sigma, \quad i = 1,2$$

$$\left|\left[\nabla_{\partial_k}\partial_\ell - \partial_k\,\partial_\ell\,X^{cat}_{\varepsilon/2}\right]^3 (\sigma,\theta)\right| \leq \hat{c}\,\varepsilon^2\cosh\sigma,$$

for a constant $\hat{c}$ independent of ε, where $\partial_k, \partial_\ell$ denote either ∂_σ or ∂_θ and $i,j,q = 1,\dots,3$. On the other hand, the unit outward normal to $C_{\varepsilon/2}$ with respect to the metric $(\mathcal{X}\circ\Lambda_m)^* g_{eucl}$ reads

$$\mathfrak{N}(\sigma,\theta) = \frac{1}{a\sqrt{\frac{b^2}{\cosh^2\sigma} + \tanh^2\sigma}}\left(-\frac{b}{\cosh\sigma}\,e^{i\theta}, \frac{1}{b}\tanh\sigma\right).$$

Thus, the second fundamental form $(\mathfrak{h}_\varepsilon)_{k\ell} = (\mathcal{X}\circ\Lambda_m)^* g_{eucl}(\nabla_{\partial_k}\partial_\ell, \mathfrak{N})$ satisfies

$$\mathfrak{h}_\varepsilon(\sigma,\theta) = \varepsilon(-d\sigma^2 + d\phi^2) + \hat{\mathcal{O}}(\varepsilon^2\,\cosh\sigma).$$

This yields: $|H(\mathcal{A}_n)(\sigma,\theta)| = \left|\mathrm{tr}\left(\mathfrak{g}_\varepsilon^{-1}\mathfrak{h}_\varepsilon\right)(\sigma,\theta)\right| \leq \dfrac{\hat{c}}{\cosh\sigma}$ for a constant $\hat{c}$ independent of ε.

"Catenoidal neck" region:

In order to calculate the mean curvature in the "catenoidal neck" region, we need to calculate the mean curvature of the surface $\tilde{C}_\varepsilon \subset D^2\times\mathbb{R}$ with respect to the ambient pull-back metric

$$\mathcal{X}^* g_{eucl}(z,x_3) = A^2(z,x_3)\left(|dz|^2 + B^2(z)dx_3^2\right).$$

The proof repeats the one given above for the mean curvature of the "half-catenoidal bridges" up to changing several estimates. The first fundamental form in this region satisfies:

$$\tilde{\mathfrak{g}}_\varepsilon(s,\phi) = \tilde{\varepsilon}^2\cosh^2 s(ds^2 + d\phi^2) + \hat{\mathcal{O}}(\varepsilon^{4-\beta}\,\cosh^4 s),$$

while the Christoffel symbols satisfy:

$$\tilde{\Gamma}^1_{11} = -\tilde{\Gamma}^1_{22} = \tilde{\Gamma}^2_{12} = \tfrac{1}{A}\tfrac{\partial A}{\partial x_1} = \mathcal{O}(|z||x_3|^2), \quad \tilde{\Gamma}^2_{11} = -\tilde{\Gamma}^2_{22} = -\tilde{\Gamma}^1_{12} = -\tfrac{1}{A}\tfrac{\partial A}{\partial x_2} = \mathcal{O}(|z||x_3|^2),$$

$$\tilde{\Gamma}^1_{13} = \tilde{\Gamma}^2_{23} = \tilde{\Gamma}^3_{33} = \tfrac{1}{A}\tfrac{\partial A}{\partial x_3} = \mathcal{O}(|x_3|), \quad \tilde{\Gamma}^3_{11} = \tilde{\Gamma}^3_{22} = -\tfrac{1}{AB^2}\tfrac{\partial A}{\partial x_3} = \mathcal{O}(|x_3|),$$

$$\tilde{\Gamma}^1_{33} = -(\tfrac{B^2}{A}\tfrac{\partial A}{\partial x_1} + B\tfrac{\partial B}{\partial x_1}) = \mathcal{O}(|z|), \quad \tilde{\Gamma}^2_{33} = -(\tfrac{B^2}{A}\tfrac{\partial A}{\partial x_2} + B\tfrac{\partial B}{\partial x_2}) = \mathcal{O}(|z|),$$

$$\tilde{\Gamma}^3_{13} = \tfrac{1}{A}\tfrac{\partial A}{\partial x_1} + \tfrac{1}{B}\tfrac{\partial B}{\partial x_1} = \mathcal{O}(|z|), \quad \tilde{\Gamma}^3_{23} = \tfrac{1}{A}\tfrac{\partial A}{\partial x_2} + \tfrac{1}{B}\tfrac{\partial B}{\partial x_2} = \mathcal{O}(|z|),$$

$$\tilde{\Gamma}^1_{23} = \tilde{\Gamma}^2_{13} = \tilde{\Gamma}^3_{12} = 0,$$

which yields for all $\beta \in (0,1)$

$$\left| \left[\tilde{\nabla}_{\partial_k} \partial_\ell - \partial_k \, \partial_\ell \, \tilde{X}^{cat}_{\tilde{\varepsilon}} \right]^i (s,\phi) \right| \leq \hat{c}\, \varepsilon^{3-\beta} \cosh^3 s, \quad i = 1,2$$

$$\left| \left[\tilde{\nabla}_{\partial_k} \partial_\ell - \partial_k \, \partial_\ell \, \tilde{X}^{cat}_{\tilde{\varepsilon}} \right]^3 (s,\phi) \right| \leq \hat{c}\, \varepsilon^{3-\beta} \cosh^2 s,$$

for a constant $\hat{c}$ independent of ε, where $\tilde{\nabla}$ is the Levi-Civita connection associated to the metric $\mathcal{X}^* g_{eucl}$. The second fundamental form then satisfies

$$\tilde{\mathfrak{h}}_\varepsilon(s,\phi) = \tilde{\varepsilon}\,(-ds^2 + d\phi^2) + \hat{\mathcal{O}}(\varepsilon^{3-\beta} \cosh^2 s).$$

Finally, we obtain for all $\beta \in (0,1)$

$$|H(\mathcal{A}_n)| = \left| \mathrm{tr} \left(\tilde{\mathfrak{g}}_{\tilde{\varepsilon}}^{-1} \tilde{\mathfrak{h}}_\varepsilon \right) \right| \leq \hat{c}\, \varepsilon^{1-\beta}.$$

The estimates for the derivatives of the mean curvature in the catenoidal regions follow from the estimates for the mappings X^{cat}_{ε}, $\tilde{X}^{cat}_{\tilde{\varepsilon}}$, $\mathcal{X}$ and Λ_m.

□

2.8.4 Mean curvature of the perturbed surfaces

Take a function $w \in \mathcal{E}^{2,\alpha}_{n,\nu}$ and let $\mathcal{A}_n(w)$ be the perturbed surface defined in the beginning of the section. Recall that the Taylor expansion of the mean curvature of $\mathcal{A}_n(w)$ in powers of w and its derivatives has the form:

$$H(\mathcal{A}_n(w)) = H(\mathcal{A}_n) + \mathcal{L}_n\, w + \mathcal{Q}_n(w).$$

In this subsection, we analyze the properties of the linear operator $\mathcal{L}_n$ and the nonlinear function $\mathcal{Q}_n(w) := \mathcal{Q}_n(w, \nabla w, \nabla^2 w)$ separately in different regions of $\mathcal{A}_n$.

We start by studying the properties of $\mathcal{L}_n$ and $\mathcal{Q}_n$ in the regions where $\mathcal{A}_n$ is parametrized as a vertical graph over a subdomain of the unit disk. We obtain the following result. Let the domain $D_{r,\rho} \subset \bar{D}^2$ and the function $\gamma_D \in \mathcal{C}^\infty(\bar{D}^2)$ be defined as in Remark 2.8.2.

Proposition 2.8.2. *For all $\beta \in (0,1)$ the linearized mean curvature operator $\mathcal{L}_n$ in the regions $\Omega^{\pm}_{gr} \bigcup_{i=0}^{n} \Omega^{i,\pm}_{glu}$ can be expressed in the form:*

$$\mathcal{L}_n = L_{gr} + e^{-n(2-\beta)}\, \gamma^{-4}\, \hat{L},$$

where $L_{gr} = \Delta\,(B\cdot\,)$ and $\hat{L}$ is a linear partial differential of second order which satisfies

$$\|\hat{L}\, w\|_{\mathcal{C}^{0,\alpha}\left(D_{r,\rho},\, \gamma_D^{-2}\, |dz|^2\right)} \leq C\, \|w\|_{\mathcal{C}^{2,\alpha}\left(D_{r,\rho},\, \gamma_D^{-2}\, |dz|^2\right)}, \quad \forall r, \rho \in (0,1), \tag{2.29}$$

for a constant C is independent of r, ρ and ε. If in addition $\|\gamma^{-1}\, w\|_{\mathcal{E}^{1,\alpha}_{n,\nu}} < 1$, the non-linear function $\mathcal{Q}_n$ satisfies

$$\mathcal{Q}_n(w) = e^{-n(1-\beta)}\, \gamma^{-4}\, Q^2(w) + e^{n\,\beta}\, \gamma^{-4}\, Q^3(w),$$

$$\|Q^k(w_1) - Q^k(w_2)\|_{\mathcal{C}^{0,\alpha}\left(D_{r,\rho},\gamma_D^{-2}|dz|^2\right)} \leq C \max_{i=1,2}\left\{\|w_i\|^{k-1}_{\mathcal{C}^{2,\alpha}(D_{r,\rho},|dz|^2)}\right\}$$
$$\times \|w_1 - w_2\|_{\mathcal{C}^{2,\alpha}\left(D_{r,\rho},\gamma_D^{-2}|dz|^2\right)}$$

for a constant $C > 0$ independent of ε, r and ρ.

Proof. In the graph region $\Omega^{\pm}_{gr}$ the surface $\mathcal{A}_n(w)$ is parametrized as a vertical bi-graph over $D_{gr} \subset D^2 \times \{0\}$ of the function $(\mathcal{G}_n + w)$ and, by Corollary 2.3.1, we have

$$H\left(\mathcal{A}_n(w)\right) = H_{gr}\left(\mathcal{G}_n + w\right),$$

where

$$\mathcal{H}_{gr}(\mathcal{G}_n + w) = H(\mathcal{A}_n) + \Delta(B\,w) + Q_{gr}\left(\mathcal{G}_n + w\right) - Q_{gr}(\mathcal{G}_n). \tag{2.30}$$

Developing this expression and using the structure of the operator Q_{gr}, described in Corollary 2.3.1, we obtain the desired properties of $\mathcal{L}_n$ and $\mathcal{Q}_n$.

In the gluing regions $\Omega^{i,\pm}_{glu}$ the result is a consequence of the following lemma which is a simple generalization of a classical result already used for example in [77] and [93].

Lemma 2.8.1. *Let Σ be a smooth surface in $\mathbb{R}^3$. Take $w \in \mathcal{C}^2_{loc}(\Sigma)$ and V_1 and V_2 two smooth vector fields on Σ. Let $\mathcal{H}^i(w)$ be the mean curvature of the surfaces parametrized by*

$$\Sigma \ni p \mapsto p + w(p)\,V_i(p) \in \mathbb{R}^3, \quad i = 1, 2.$$

Then the following relation holds:

$$D\mathcal{H}^2\big|_{w=0}(v) = D\mathcal{H}^1\big|_{w=0}(\tau\,v) + \nabla H \cdot \mathcal{T}$$

where $\tau = \|V_2^N\|/\|V_1^N\|$ and $\mathcal{T} = V_2^T - \tau\,V_1^T$, and where V_i^N and V_i^T are the orthogonal projections of V_i on the normal and the tangent bundles of Σ.

Proof of Lemma 2.8.1. The proof consists of applying the implicit function theorem to the equation

$$p + t\,V_1(p) = q + s\,V_2(q), \quad p, q \in \Sigma, \quad t, s \in \mathbb{R}.$$

Expressing locally p and t as functions of q and s:

$$p = \Phi(q, s) \quad \text{and} \quad t = \Psi(q, s),$$

with $\Phi(q, 0) = q$ and $\Psi(q, 0) = 0$, one obtains

$$\partial_s\Psi(\cdot, 0)[V_1]^N = [V_2]^N \quad \text{and} \quad \partial_s\Phi(\cdot, 0) = [V_2]^T - \partial_s\Psi(\cdot, s)[V_1]^T.$$

On the other hand, differentiating the identity

$$\mathcal{H}^1(\Psi(q, w(q)))(\Phi(q, w(q))) = \mathcal{H}^2(w)(q)$$

with respect to w at $w = 0$ yields

$$D\mathcal{H}^1\big|_{w=0}(\partial_s\Psi(\cdot, 0)\,v) + \nabla H \cdot \partial_s\Phi\,v = D\mathcal{H}^2\big|_{w=0}(v).$$

□

In order to calculate the mean curvature in the regions $\Omega^{0,\pm}_{glu}$, we apply the result of Lemma 2.8.1 to the surface $\tilde{C}_{\tilde{\varepsilon}}$ embedded in $D^2 \times \mathbb{R}$ and parametrized by

$$(\tilde{\varepsilon} \cosh s \, e^{i\phi}, 2\,\tilde{\varepsilon}\, s),$$

where $\tilde{\varepsilon}\, \cosh s \sim \varepsilon^{\frac{1}{2}}$ and $\phi \in S^1$. We take

$$V_1 := (0,0,1) \quad \text{and} \quad V_2 := \mathcal{V}_0 = \frac{1}{2}\, \eta_0\, \tilde{N}_{cat} \pm (1-\eta_0)\,(0,0,1),$$

where the vector field $\tilde{N}$ is a unit normal to $\tilde{C}_{\tilde{\varepsilon}}$ with respect to $\tilde{g}_{eucl}$. Finally, we calculate the mean curvature and the orthonormal projections with respect to the metric $\mathcal{X}^* g_{eucl} = |dz|^2 + \frac{1}{4}\, dx_3^2 + \hat{\mathcal{O}}(\varepsilon)$. We find $|\nabla H(\mathcal{A}_n)| = \hat{\mathcal{O}}(\varepsilon^{1/2})$ and

$$[(0,0,1)]^N = \frac{1}{2} + \hat{\mathcal{O}}(\varepsilon), \quad [(0,0,1)]^T = \hat{\mathcal{O}}(\varepsilon), \quad \left[\tilde{N}_{cat}\right]^N = 1 + \hat{\mathcal{O}}(\varepsilon), \quad \left[\tilde{N}_{cat}\right]^T = \hat{\mathcal{O}}(\varepsilon)$$

which yields $\tau = 1 + \hat{\mathcal{O}}(\varepsilon)$, $\mathcal{T} = \hat{\mathcal{O}}(\varepsilon^{1/2})$ and the desired properties of the operator $\mathcal{L}_n$.

Similarly, in the regions $\Omega^{m,\pm}_{glu}$ we apply Lemma 2.8.1 to the surface embedded in $\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and parametrized by

$$\left(\frac{\varepsilon}{2} \cosh \sigma \, e^{i\theta},\ \frac{\varepsilon}{2}\, \sigma\right),$$

where $\varepsilon\, \cosh \sigma \sim \varepsilon^{\frac{2}{3}}$ and $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. We take

$$V_1 = \frac{1}{2}\,(0,0,1) \quad \text{and} \quad V_2 = \frac{1}{2}\, \bar{\mathcal{V}} = \frac{1}{2}\big(\bar{\eta}\, N_{cat} \pm (1-\bar{\eta})\,(0,0,1)\big),$$

and calculate the mean curvature and the orthogonal projections with respect to the metric $(\mathcal{X} \circ \Lambda_m)^* g_{eucl} = 4\, g_{eucl} + \hat{\mathcal{O}}(\varepsilon^{2/3})$.

□

In the following two results, we show that in the "catenoidal regions" the properties of $H(\mathcal{A}_n(w))$ can be obtained using the properties of the normal graphs over the catenoid in $\mathbb{R}^3$ scaled by a small factor. We start by analyzing of the mean curvature in the "catenoidal neck" region Ω^0_{cat}.

Proposition 2.8.3. *For all $\beta \in (0,1)$ the linearized mean curvature operator $\mathcal{L}_n$ in the "catenoidal neck" region Ω^0_{cat} can be expressed in the form*

$$\mathcal{L}_n = \frac{1}{\tilde{\varepsilon}^2 \cosh^2 s} \left(\partial_s^2 + \partial_\phi^2 + \frac{2}{\cosh^2 s}\right) + e^{n\,\beta}\, \hat{L},$$

where $\hat{L}$ is the second order partial differential operator which satisfies

$$\|\hat{L}\, w\|_{\mathcal{C}^{0,\alpha}((s,s+1)\times S^1)} \leq C\, \|w\|_{\mathcal{C}^{2,\alpha}((s,s+1)\times S^1)}, \quad \forall s \in (-s_\varepsilon, s_\varepsilon - 1)$$

for a constant C independent of s and ε. If in addition $\|\gamma^{-1}\, w\|_{\mathcal{E}^{1,\alpha}_{n,\nu}} < 1$, the nonlinear term $\mathcal{Q}_n(w)$ can be expressed in the form

$$\mathcal{Q}_n(w) = \frac{1}{\varepsilon^3 \cosh^4 s} Q^2_{cat}(w) + \frac{1}{\varepsilon^4 \cosh^4 s} Q^3_{cat}(w), \quad \textit{where}$$

$$\left\| Q^k_{cat}(w_1) - Q^k_{cat}(w_2) \right\|_{\mathcal{C}^{0,\alpha}((s,s+1)\times S^1)} \leq C \max_{i=1,2} \left\{ \|w_i\|^{k-1}_{\mathcal{C}^{2,\alpha}((s,s+1)\times S^1)} \right\}$$
$$\times \|w_1 - w_2\|_{\mathcal{C}^{2,\alpha}((s,s+1)\times S^1)},$$

for a constant C independent of s and ε.

Proof. Recall that the region $\Omega^0_{cat} \subset \mathcal{A}_n$ can be seen as the image by the mapping $\mathcal{X}$ of the normal (with respect to the metric $\tilde{g}_{eucl} = dz^2 + \frac{1}{4}dx_3^2$) graph over the surface $\tilde{C}_{\tilde{\varepsilon}} \subset D^2 \times \mathbb{R}$:

$$(s, \phi) \in (-s_\varepsilon, s_\varepsilon) \times S^1 \;\mapsto\; \tilde{X}^{cat}_{\tilde{\varepsilon}}(s, \phi) + \frac{w(s,\phi)}{2}\, \tilde{N}^{cat}(s, \phi) \in \tilde{C}_{\tilde{\varepsilon}}\left(\frac{w}{2}\right).$$

On the other hand, we notice that calculating the mean curvature of $\tilde{C}_{\tilde{\varepsilon}}\left(\frac{w}{2}\right)$ with respect to the metric $\tilde{g}_{eucl}$ is equivalent to calculating the mean curvature of a normal graph about the Euclidean catenoid scaled by the factor $\tilde{\varepsilon}$. Then, a standard computation which we postpone to the appendix, gives

$$H_{cat}\left(\frac{w}{2}\right) = \frac{1}{\tilde{\varepsilon}^2 \cosh^2 s}\left(\partial_s^2 + \partial_\phi^2 + \frac{2}{\cosh^2 s}\right)\frac{w}{2} + \frac{1}{\varepsilon^3 \cosh^4 s} Q^2_{cat}(w) + \frac{1}{\varepsilon^4 \cosh^4 s} Q^3_{cat}(w).$$

Secondly, we use the fact the pull-back metric $\mathcal{X}^* g_{eucl}$ can be seen as a perturbation of the metric $\tilde{g}_{eucl}$:

$$\mathcal{X}^* g_{eucl}(z, x_3) = (1 + \mathcal{O}(x_3^2))\, \tilde{g}_{eucl} + \mathcal{O}(|z|^2)\, dx_3.$$

Calculating the mean curvature with respect to $\mathcal{X}^* g_{eucl}$ corresponds to adding to $H_{cat}(w)$ an initial mean curvature term equal to $H(\mathcal{A}_n)$ and some smaller linear and nonlinear terms. Since the nonlinear part has the same properties as when we calculation the mean curvature with respect to $\tilde{g}_{eucl}$, we only have to understand the behavior of the additional linear terms. This can be achieved by a direction computation which can be also found in the Appendix. □

Finally, the same ideas can be applied to analyze the properties of the operators $\mathcal{L}_n$ and $\mathcal{Q}_n$ in the "half-catenoidal bridge" regions.

Proposition 2.8.4. *The linearized mean curvature operator $\mathcal{L}_n$ in the "half-catenoidal bridge" region Ω^m_{cat} can be expressed in the form:*

$$\mathcal{L}_n = \frac{1}{\varepsilon^2 \cosh^2 \sigma}\left(\partial_\sigma^2 + \partial_\theta^2 + \frac{2}{\cosh^2 \sigma}\right) + \frac{1}{\varepsilon \cosh \sigma}\,\hat{L},$$

where $\hat{L}$ is the second order partial differential operator which satisfies:

$$\|\hat{L}\, w\|_{\mathcal{C}^{0,\alpha}((\sigma,\sigma+1)\times[\frac{\pi}{2},\frac{3\pi}{2}])} \leq C\, \|w\|_{\mathcal{C}^{2,\alpha}((\sigma,\sigma+1)\times[\frac{\pi}{2},\frac{3\pi}{2}])}, \quad \forall \sigma \in [-\sigma_\varepsilon, \sigma_\varepsilon - 1]$$

for a constant C independent of σ and ε. If in addition $\|\gamma^{-1}\, w\|_{\mathcal{E}^{1,\alpha}_{n,\nu}} < 1$, the nonlinear term $\mathcal{Q}_n(w)$ can be expressed in the form

$$\mathcal{Q}_n(w) = \frac{1}{\varepsilon^3 \cosh^4 \sigma} Q^2_{cat}(w) + \frac{1}{\varepsilon^4 \cosh^4 \sigma} Q^3_{cat}(w), \quad \textit{where}$$

$$\begin{aligned}\left\| Q^k_{cat}(w_1) - Q^k_{cat}(w_2) \right\|_{\mathcal{C}^{0,\alpha}\left((\sigma,\sigma+1)\times\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\right)} &\leq C \max_{i=1,2} \left\{ \|w_i\|^{k-1}_{\mathcal{C}^{2,\alpha}\left((\sigma,\sigma+1)\times\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\right)} \right\} \\ &\times \|w_1 - w_2\|_{\mathcal{C}^{2,\alpha}\left((\sigma,\sigma+1)\times\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\right)},\end{aligned}$$

for a constant C independent of σ and ε.

Proof. The proof follows from the same argument as the proof of Proposition 2.8.3 if one remarks that $\mathcal{A}_n(w)$ is obtained as the image by the mapping $\mathcal{X} \circ \Lambda_m$ of a normal (with respect to g_{eucl}) graph about the Euclidean catenoid $C_{\varepsilon/2}$ scaled by the factor $\frac{\varepsilon}{2}$ and that the metric $(\mathcal{X} \circ \Lambda_m)^*\, g_{eucl}$ can be seen as a small perturbation of the Euclidean metric scaled by the factor 4:

$$(\mathcal{X} \circ \Lambda_m)^* g_{eucl}(\zeta, \xi_3) = (4 + \mathcal{O}(|\zeta|))\, g_{eucl} + \mathcal{O}(|\zeta|^2)\, d\xi_3^2.$$

□

2.9 Linear analysis

Recall, that our goal is to solve

$$H(\mathcal{A}_n(w)) = 0 \tag{2.31}$$

for some $w \in \mathcal{E}^{2,\alpha}_{n,\nu}$. Using the notations introduced in the previous section, we can write this equation in the form

$$\mathcal{L}_n\, w = -H(\mathcal{A}_n) - \mathcal{Q}_n(w).$$

Since $H(\mathcal{A}_n)$ tends to 0 in a suitable topology when n tends to infinity, we hope for n large enough to find a solution using a fixed point argument. To this end, we need to show that the operator $\mathcal{L}_n$ has a right inverse in suitable function spaces and study its norm when n tends to infinity. In this section, we show that the properties of $\mathcal{L}_n$ can be deduced from the properties of the operator L_{gr} defied in the noncompact domain D^2_* with homogeneous Neumann boundary condition on $S^1 \setminus \{z_1, \ldots, z_n\}$ together with the properties of the Jacobi operator about the Euclidean catenoid defined in the infinite unit cylinder.

2.9.1 Linear analysis on the punctured disk

We now analyze the operator L_{gr} in the unit punctured disk D^2_* subject to homogeneous Neumann boundary data on $S^1 \setminus \{z_1, \ldots, z_n\}$:

$$\begin{cases} \Delta(Bw) = f \quad \text{in} \quad D^2 \setminus \{0\}, \\ \partial_r w = 0 \quad \text{on} \quad S^1 \setminus \{z_1, \ldots, z_n\}. \end{cases} \tag{2.32}$$

Since $\partial_r B|_{r=1} = 1$, this is equivalent to studying the problem:

$$\begin{cases} \Delta\, w = f \quad \text{in} \quad D^2 \setminus \{0\}, \\ \partial_r w - w = 0 \quad \text{on} \quad S^1 \setminus \{z_1, \ldots, z_n\}, \end{cases} \tag{2.33}$$

where f is a given function whose regularity and properties will be stated shortly. We assume that the functions f and w are invariant under rotations by the angle $\frac{2\pi}{n}$. With this assumption, the operator associated to (2.33) does not have any bounded kernel for $n \geq 2$ and hence, the solvability of (2.33) follows from classical arguments [67]. For example, if $f \in \mathcal{C}^{0,\alpha}(\bar{D}^2)$ we get the existence of a solution $w \in \mathcal{C}^{2,\alpha}(\bar{D}^2)$ to (2.33). Moreover,

$$\|w\|_{\mathcal{C}^{2,\alpha}(\bar{D}^2)} \leq C \left(\|w\|_{\mathcal{C}^0(\bar{D}^2)} + \|f\|_{\mathcal{C}^{0,\alpha}(\bar{D}^2)} \right).$$

We need to understand what happens if we allow f to have singularities at 0 and/or z_m, $m = 1, \ldots, n$.

Definition 2.9.1 (Weighted Hölder spaces in the punctured disk)**.** *Take the function*

$$\gamma_D : z \mapsto |z||z^n - 1|$$

and $\nu \in \mathbb{R}$. The space $\mathcal{C}^{k,\alpha}_\nu(D^2_*)$ is defined to be the space of functions $u \in \mathcal{C}^{k,\alpha}_{loc}(D^2_*)$ for which the following norm is finite

$$\|u\|_{\mathcal{C}^{k,\alpha}_\nu(D^2_*)} \quad := \quad \|\gamma_D^{-\nu}\, u\|_{\mathcal{C}^{k,\alpha}(D_*, \gamma_D^{-2}\, |dz|^2)} < \infty.$$

Notation 2.9.1. *Let $\bar{\chi} \in \mathcal{C}^\infty(\mathbb{C}_-)$ be a cut-off function, which is radial and satisfies*

$$\bar{\chi}(\zeta) \equiv 0 \quad \textit{for} \quad |\zeta| < \frac{1}{5}, \quad \textit{and} \quad \bar{\chi}(\zeta) \equiv 1 \quad \textit{for} \quad |\zeta| > \frac{2}{5}.$$

Then we consider the conformal mapping $\lambda : \mathbb{C}_- \to \bar{D}^2$,

$$\lambda(\zeta) := \frac{1+\zeta}{1-\zeta} \in \bar{D}^2,$$

and introduce the functions $\chi, \chi_n \in \mathcal{C}^\infty(\bar{D}^2)$, defined by

$$\chi(z) = \bar{\chi}(\lambda^{-1}(z)), \quad \chi_n(z) = \chi(z^n).$$

Notice, that $\chi \equiv 1$ in a neighborhood of 1 and $\chi_n \equiv 1$ in neighborhoods $z = z_m$, $m = 1, \ldots, n$. Moreover, we have

$$\partial_r \chi|_{r=1} = 0 \quad \textit{and} \quad \partial_r \chi_n|_{r=1} = 0.$$

Definition 2.9.2 (Deficiency space)**.** *We define the deficiency space*

$$\mathfrak{D}_n = \operatorname{span}\{1, \chi_n\},$$

Proposition 2.9.1. *Assume that $\nu \in (0,1)$. Then there exists a constant $C > 0$ and for all n large enough and all $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(D^2_*)$ such that $f(z) = f(\bar{z}) = f(z \cdot z_m)$, there exist a unique function $\psi \in \mathcal{C}^{2,\alpha}_{\nu}(D^2_*)$ and unique constants $\hat{c}_0$ and $\hat{c}_1$ such that*

$$w := \psi + \hat{c}_0 + \hat{c}_1\, \chi_n$$

is a solution to

$$\begin{cases} \Delta w = f & \text{in} \quad D^2_*, \\ \partial_r w - w = 0 & \text{on} \quad S^1 \setminus \{z_1, \dots, z_n\}, \end{cases}$$

that satisfies $w(z) = w(\bar{z}) = w(z \cdot z_m)$ and

$$\|w\|_{\mathcal{C}^{2,\alpha}_{\nu}(D^2_*)\oplus\mathfrak{D}_n} \leq C\, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(D^2_*)}. \tag{2.34}$$

Before starting the proof of Proposition 2.9.1, we notice that instead of the problem (2.33), we can consider an equivalent problem defined in $\bar{D}^2 \setminus \{0, 1\}$. Since we assume that $f(z) = f(\bar{z}) = f(z \cdot z_m)$, there exists a function F such that

$$F(z^n) = \frac{1}{n^2}\, |z|^{2-2n} f(z),$$

and if W is a solution to

$$\begin{cases} \Delta W = F & \text{in} \quad D^2 \setminus \{0\}, \\ \partial_r W - \frac{1}{n} W = 0 & \text{on} \quad S^1 \setminus \{1\}, \end{cases} \tag{2.35}$$

then

$$w(z) = W(z^n)$$

satisfies (2.33) and $w(z) = w(\bar{z}) = w(z \cdot z_m)$.

We proceed as follows. First, we analyze the existence and the properties of the solution of (2.35) in weighted L^∞ spaces. In Lemma 2.9.1 given below, we consider the case where the function F only has a singularity at $z = 0$. Next, in Lemma 2.9.2, we consider the general case where F has singularities at $z = 0$ and $z = 1$. In order to finish the proof of Proposition 2.9.1, we go back to the problem (2.33) and obtain regularity properties for the solution in $\bar{D}^2 \setminus \{0, z_1, \dots, z_n\}$ using the Schauder's elliptic regularity theory in Hölder weighted spaces. Finally, this provides us the solution to the problem (2.32).

First, let us assume that the function F in (2.35) only has a singularity at $z = 0$. We have the following result.

Lemma 2.9.1. *Assume $\nu_0 \in (0,1)$. There exists a constant $C_0 > 0$ (which depends on ν_0) and for all $n \geq 2$ and all functions F, such that $|z|^{-\nu_0+2} F \in L^\infty(D^2)$, there exist a unique function Ψ and a unique constant c_0^* such that*

$$W := \Psi + n\, c_0^*$$

is a solution to (2.35) and

$$\| \, |z|^{-\nu_0} \, \Psi \, \|_{L^\infty(D^2)} + |c_0^*| \leq C_0 \, \| \, |z|^{-\nu_0+2} \, F \, \|_{L^\infty(D^2)}. \tag{2.36}$$

Proof of Lemma 2.9.1. First, let us assume that F is radial. In this case, (2.35) reduces to a second order ordinary differential equation which can be solved explicitly:

$$\Psi_0^{rad}(r) = \int_0^r \frac{1}{s} \int_0^s t\, F(t)\, dt\, ds, \quad W_0^{rad} = \Psi_0^{rad} + n\, c_0^*,$$

$$c_0^* = -\int_0^1 s\, F(s)\, ds + \frac{1}{n} \int_0^r \frac{1}{s} \int_0^s t\, F(t)\, dt\, ds. \tag{2.37}$$

With little work, one checks that the result is indeed correct in this special case.

Furthermore, we claim that, if we restrict our attention to the space of functions for which

$$\int_{S^1} F(re^{i\phi})\, r\, d\phi = 0, \quad \forall r \in (0,1),$$

then there exists a function Ψ_0^{mean} such that

$$\| |z|^{-\nu_0}\, \Psi_0^{mean} \|_{L^\infty(D^2)} \leq C_0 \, \| |z|^{-\nu_0+2} F \|_{L^\infty(D^2)}.$$

We construct Ψ_0^{mean} as a uniform limit of solutions to the Poisson's equation in annuli with mixed boundary data. More precisely, for any $R \in (0,1)$ we put $A_R = \{z \in D^2 \,:\, |z| > R\}$ and remark that for $n \geq 2$ the operator associated to the problem:

$$\begin{cases} \Delta \Psi_R = F \quad \text{in} \quad A_R, \\ \left(\partial_r \Psi_R - \frac{1}{n} \Psi_R\right)\big|_{r=1} = 0, \quad \Psi_R|_{r=R} = 0 \end{cases} \tag{2.38}$$

has no kernel. This yields (cf. [67]) the existence of a solution Ψ_R to (2.38) which satisfies:

$$\|\Psi_R\|_{L^\infty(A_R)} \leq c(R)\, \|F\|_{L^\infty(A_R)},$$

for some constant $c(R)$ which depends on R. Next, we show that this inequality can be rewritten in the form:

$$\| |z|^{-\nu_0} \Psi_R \|_{L^\infty(A_R)} \leq C_0 \, \| |z|^{-\nu_0+2} F \|_{L^\infty(A_R)}, \tag{2.39}$$

for a constant $C_0 > 0$, which, this time, is independent of R. The last fact is proven by contradiction, using the following classical argument (cf. [93]). Assume that $C_0 = C_0(R)$ depends on R and that there exists a sequence $R_j \to 0$ such that $C_0(R_j) \to \infty$. We put:

$$\Psi_j := \frac{1}{C_0(R_j)}\, \Psi_{R_j}, \quad F_j := \frac{1}{C_0(R_j)}\, F, \quad A_j = A_{R_j}.$$

Then

$$\Delta\,\Psi_j = F_j, \quad \||z|^{2-\nu_0} F_j\|_{L^\infty(A_j)} \underset{j\to\infty}{\to} 0.$$

Next, by linearity we can assume that $\big\|\,|z|^{-\nu_0+2}\,F\big\|_{L^\infty(A_R)} = 1$, then we have

$$\||z|^{-\nu_0}\,\Psi_j\|_{L^\infty(A_j)} \leq 1.$$

Furthermore, by assumption there exists a sequence of points $z_j \in A_j$ such that

$$\Psi_j(z_j) = |z_j|^{\nu_0}.$$

Several situations can occur:

1. If the sequence $(z_j) \subset A_j$ admits a subsequence converging to a point $z_\infty \in D^2_*$, then by Schauder's elliptic estimates on the gradient of Ψ_j and by Arzelà-Ascoli theorem, the sequence (Ψ_j) admits a subsequence converging uniformly on compact sets to a solution Ψ_∞ of

$$\begin{cases} \Delta\,\Psi_\infty = 0 \quad \text{in} \quad D^2_*, \\ \partial_r \Psi_\infty - \frac{1}{n}\Psi_\infty = 0 \quad \text{on} \quad S^1, \end{cases} \tag{2.40}$$

such that $|\Psi_\infty(z)| \leq |z|^{\nu_0}$. This implies $\Psi_\infty = 0$ and contradicts the fact that $\Psi_\infty\,(z_\infty) = |z_\infty|^{\nu_0} \neq 0$.

2. If the sequence (z_j) admits a subsequence converging to 0, while $\frac{R_j}{|z_j|} \underset{j\to\infty}{\to} 0$, we put $\Phi_j(z) := \Psi_j\,(z\,|z_j|)\ |z_j|^{-\nu_0}$ and verify that

$$\|\,|z|^{-\nu_0}\Phi_j\,\|_{L^\infty(A_j)} \leq 1, \quad \Phi_j\,(z_j/|z_j|) = 1.$$

Then the sequence (Φ_j) admits a subsequence converging uniformly on compact sets to a solution Φ_∞ of the problem

$$\Delta\,\Phi_\infty = 0, \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\},$$

which satisfies $|\Phi_\infty(z)| \leq |z|^{\nu_0}$. This yields $\Phi_\infty \equiv 0$ and gives a contradiction with the fact that $\Phi_j\left(\frac{z_j}{|z_j|}\right) = 1$ for all j.

3. If the sequence (z_j) admits a subsequence converging to 0, while $\frac{R_j}{|z_j|} \underset{j\to\infty}{\to} a < 1$, then the sequence (Φ_j) admits a subsequence converging on compact sets to a solution Φ_∞ of the problem

$$\begin{cases} \Delta\,\Phi_\infty = 0 \quad \text{in} \quad \{z \in \mathbb{R}^2\,:\,|z| > a\}, \\ \Phi_\infty|_{|z|=a} = 0, \end{cases}$$

such that $|\Phi_\infty| \leq |z|^{\nu_0}$. On the other hand, decomposing Φ_∞ in Fourier series, we see that for $n \geq 2$ the problem has no non-trivial solution. This once again implies $\Phi_\infty \equiv 0$ and gives a contradiction.

4. Finally, the case when sequences (z_j) admits a subsequence converging to 0, while $\frac{R_j}{|z_j|} \underset{j\to\infty}{\to} 1$ doesn't happen. Indeed, for every j we have

$$\begin{cases} \Delta \Psi_j = F_j \quad \text{in} \quad \{z \in D^2 \,:\, R_j < |z| < 2\,R_j\}, \\ \Psi_j|_{r=R_j} = 0. \end{cases}$$

Moreover, $|F_j| \leq R_j^{\nu_0 - 2}$ and $|\Psi_j| \leq R_j^{\nu_0}$. By elliptic regularity, we have $|\nabla \Psi_j| \leq \hat{c}\, R_j^{\nu_0-1}$ in the subsets of $\{z \in D^2 \,:\, R_j < |z| < 2\,R_j\}$. This implies that in a neighborhood of $|z| = R_j$, we have

$$|\Psi_j| \leq \hat{c}\, R_j^{\nu_0 - 1}\,(|z| - R_j)\,,$$

for a constant $\hat{c}$ independent of ε. At $z = z_j$ this yields $1 - \frac{R_j}{|z_j|} \geq \hat{c}$, which is not possible for j large enough.

Therefore, (2.39) implies that by the elliptic regularity theory we have a uniform bound on the gradient of Ψ_R on compact sets of D^2_*, and by Arzelà-Ascoli theorem, there exists a subsequence of (Ψ_R) converging uniformly on compact sets to a solution of (2.35).

Finally, let as prove that the constant in (2.36) does not depend on n. By construction, for all $n \geq 2$ the solution W can be written in the form:

$$W = \Psi^{rad} + \Psi^{mean} + c_0^*\, n,$$

where Ψ^{rad} is radial and $\int_{S^1} \Psi^{mean}(re^{i\phi})r\, d\phi = 0, \quad \forall r \in (0,1)$. The fact that

$$\|\, |z|^{-\nu_0}\, \Psi^{rad}\,\|_{L^\infty(D^2)} + |c_0^*| \leq C_0\, \|\, |z|^{2-\nu_0}\, F\,\|_{L_\infty(D^2)}$$

for a constant C_0 independent of n follows from the explicit expression given by (2.37). We use once again an argument by contradiction to prove the inequality for Ψ^{mean}. Assume by linearity that $\|\, |z|^{2-\nu_0}\, F\,\|_{L_\infty(D^2)} = 1$ and that there exists a sequence $(n_k) \in \mathbb{N}$ and a sequence of functions (Ψ_k) which satisfy (2.35) and a sequence of points $(z_k) \in D^2_*$ such that

$$\|\, |z|^{-\nu_0} \Psi_k\,\|_{L^\infty(D^2)} \leq C_0(n_k) \quad \text{and} \quad \Psi_k(z_k) = C_0(n_k)\,|z_k|^{\nu_0}.$$

Put $\Psi_k := \frac{1}{C_0(n_k)}\,\Psi_k$ and assume that (z_k) admits a subsequence converging to a point $z_\infty \in D^2_*$. Then, by the Elliptic estimates for the gradient and by Arzelà-Ascoli theorem, the sequence (Ψ_k) admits a subsequence converging uniformly on compact sets of D^2_* to a solution Ψ_∞ of the problem:

$$\Delta \Psi_\infty = 0 \quad \text{in} \quad D^2_*, \quad \partial_r \Psi_\infty = 0 \quad \text{on} \quad S^1,$$

which satisfies $|\Psi_\infty(z)| \leq |z|^{\nu_0}$. Considering the expansion of Ψ_∞ in Fourier series and using the fact that Ψ_∞ has no radial part, we find $\Psi_\infty \equiv 0$, which contradicts the fact $\Psi_\infty(z_\infty) = |z_\infty|^{\nu_0}$. In the case where z_k converges to 0, we put

$$\Phi_k(z) := \Psi_k(|z_k|z)|z_k|^{-\nu_0}.$$

Then $\Phi_k(z_k/|z_k|) = 1$ and (Φ_k) admits a subsequence converging uniformly on compact sets to a function Φ_∞ which is harmonic in $\mathbb{R}^2 \setminus \{0\}$. Using that Φ_∞ has no radial part and that $|\Phi_\infty(z)| \leq |z|^{\nu_0}$, we obtain a contradiction.

□

Corollary 2.9.1. *For all $\nu_0 \in (0,1)$, can write* (2.36) *in the form:*

$$\| \, |z|^{-\nu_0+k} \, \nabla^k \Psi \, \|_{L^\infty(D^2)} \leq C_0^k(\nu_0) \, \| \, |z|^{2-\nu_0} \, F \, \|_{L^\infty(D^2)}, \quad k = 0, 1, 2,$$

$$|c_0^*| \leq C_0^1(\nu_0) \, \| \, |z|^{2-\nu_0} \, F \, \|_{L^\infty(D^2)},$$

where

$$C_0^k(\nu_0) \leq C \, \nu_0^{k-2}$$

for a constant C independent of ν_0.

Proof. The inequality for the radial part Ψ^{rad} and the constant c_0^* follows directly from (2.37). On the other hand, Ψ^{mean} is bounded by a constant does not depend on ν_0. The estimates for the derivatives follow from the the Schauder's elliptic estimates in weighted spaces. To see this, it is sufficient to apply in the neighborhood of $z = 0$ the classical Schauder's estimates in the annuli of inner radius R and exterior radius $2R$ to the function $\Psi(R\cdot)$. $\square$

As a next step, we assume that F can have singularities at $z = 0$ and $z = 1$. Let $\chi \in \mathcal{C}^\infty(D^2)$ be the cut-off function defined in Notation 2.9.1. We have the following result.

Lemma 2.9.2. *Assume $\nu_0, \nu_1 \in (0,1)$. There exists a constant $C_1 > 0$ (which depends on ν_0 and ν_1) and for all n large enough and all functions F, such that $|z|^{-\nu_0+2}|z-1|^{-\nu_1+2} F \in L^\infty(D^2)$ there exist a unique function Ψ and unique constants c_0^* and c_1^* such that*

$$W := \Psi + n \, c_0^* + c_1^* \, \chi$$

is a solution to (2.35) and satisfies

$$\| \, |z|^{-\nu_0}|z-1|^{-\nu_1} \Psi \, \|_{L^\infty(D^2)} + |c_0^*| + |c_1^*| < C_1 \, \| \, |z|^{-\nu_0+2}|z-1|^{-\nu_1+2} F \, \|_{L^\infty(D^2)}.$$

Proof of Lemma 2.9.2. Let the mapping $\lambda : \mathbb{C}_- \to \bar{D}^2$ be defined as in Notation 2.9.1. We decompose

$$F = F_0 + F_1 = (1-\chi) \, F + \chi \, F,$$

and define the function

$$\overline{F}_1 \, : \zeta \mapsto \frac{4}{|1-\zeta|^2} \, F_1(\lambda(\zeta)).$$

Notice that $|\zeta|^{-\nu_1+2} \, \overline{F}_1 \in L^\infty(\mathbb{C}_-)$ and

$$\left\| \, |\zeta|^{-\nu_1+2} \, \overline{F}_1 \right\|_{L^\infty(\mathbb{C}_-)} \leq C \, \left\| \, |z|^{-\nu_0+2} \, |z-1|^{-\nu_1+2} \, F \right\|_{L^\infty(D^2)}.$$

Let $\overline{W}_1$ be solution of

$$\begin{cases} \Delta \overline{W}_1 = \overline{F}_1(\zeta) \quad \text{in} \quad \{\zeta \in \mathbb{C}_- : |\zeta| < \frac{2}{5}\}, \\ \partial_{\xi_1} \, \overline{W}_1 \big|_{\xi_1 = 0} = 0, \\ \overline{W}_1 \big|_{|\zeta| = \frac{2}{5}} = 0, \end{cases}$$

then the function

$$W_1(z) := \overline{W}_1(\lambda^{-1}(z))$$

satisfies:

$$\Delta W_1 = F_1 \quad \text{in} \quad \operatorname{supp}(\chi) \setminus \{1\} \quad \text{and} \quad \partial_r W_1|_{r=1} = 0.$$

Moreover, we have:

$$\big\| \, |z-1|^{-\nu_1} \, W_1 \big\|_{L^\infty(\operatorname{supp}(\chi)\setminus\{1\})} \leq C \, \| \, |z|^{-\nu_0+2} \, |z-1|^{-\nu_1+2} \, F \, \|_{L^\infty(D^2)}.$$

The existence and the properties of $\overline{W}_1$ follow from Lemma 2.9.1. More precisely, we extend $\overline{F}_1$ by symmetry with respect to the coordinate axis $\xi_1 = 0$ to $\{\zeta \in \mathbb{C} : |\zeta| < \frac{2}{5}\}$ and consider the problem

$$\begin{cases} \Delta \overline{W}_1 = \overline{F}_1 \quad \text{in} \quad \{\zeta \in \mathbb{C} : |\zeta| < \frac{2}{5}\}, \\ \overline{W}_1\big|_{|\zeta|=\frac{2}{5}} = 0. \end{cases} \tag{2.41}$$

If $\overline{F}_1$ is radial, the solution of (2.41) is found explicitly and can be written in the form

$$\overline{W}_1^{rad} = \overline{\Psi}_1^{rad}(\rho) + c_1^*, \tag{2.42}$$

where

$$\overline{\Psi}_1^{rad}(\rho) = \int_0^\rho \frac{1}{s} \int_0^s t \, \overline{F}_1(t) \, dt \, ds \quad \text{and} \quad c_1^* = -\int_0^a \frac{1}{s} \int_0^s t \, \overline{F}_1(t) \, dt \, ds.$$

Moreover, there exists a constant $C > 0$ such that

$$\| \, |\zeta|^{-\nu_1} \, \overline{\Psi}_1^{rad} \, \|_{L^\infty(D^2(\frac{2}{5}))} + |c_1^*| \leq C \, \| \, |\zeta|^{-\nu_1+2} \, \overline{F}_1 \, \|_{L^\infty(D^2(\frac{2}{5}))}.$$

On the other hand, if

$$\int_{S^1} \overline{F}_1(\rho, \theta) \, \rho \, d\theta = 0, \quad \text{for all} \quad \rho \in (0, 2/5),$$

we prove, using the same argument as in Lemma 2.9.1, the existence of $\overline{\Psi}_1^{mean}$ which is a solution to (2.41) and satisfies:

$$\| \, |\zeta|^{-\nu_1} \, \overline{\Psi}_1^{mean} \, \|_{L^\infty(D^2(\frac{2}{5}))} \leq C \, \| \, |\zeta|^{-\nu_1+2} \overline{F}_1 \, \|_{L^\infty(D^2)}.$$

Finally, we put

$$\overline{W}_1 = \overline{W}^{rad} + \overline{\Psi}_1^{mean}.$$

The function

$$\chi \, W_1 = \chi \, \overline{W}_1(\lambda^{-1}(\cdot))$$

can be extended by zero to the entire open unit disk D^2. We have:

$$\begin{cases} \Delta(\chi \, W_1) = F_1 + 2\nabla\chi \, \nabla W_1 + W_1 \, \Delta \, \chi \quad \text{in} \quad D^2, \\ \partial_r(\chi \, W_1) = 0 \quad \text{on} \quad S^1 \setminus \{1\}. \end{cases}$$

The function $\nabla\chi\,\nabla W_1 + W_1\,\Delta\,\chi$ has compact support in D^2_*, so according to Lemma 2.9.1 we can find a function W_0 which satisfies:

$$\begin{cases} \Delta\, W_0 = F_0 - 2\nabla\chi\,\nabla W_1 - W_1\,\Delta\,\chi \quad \text{in} \quad D^2_*, \\ \partial_r W_0 - \frac{1}{n} W_0 = 0 \quad \text{on} \quad S^1 \setminus \{1\}. \end{cases}$$

By the elliptic regularity, we have:

$$\|\nabla\chi\,\nabla W_1\,\|_{L^\infty(D^2)} \leq C\,\|\,|z-1|^{-\nu_1+2}\,F_1\|_{L^\infty(D^2)},$$

for some positive constant C. This yields:

$$\|\,|z|^{-\nu_0+2}\,(F_0 - 2\nabla\chi\,\nabla W_1 - W_1\,\Delta\,\chi)\,\|_{L^\infty(D^2)} \leq C\,\|\,|z|^{-\nu_0+2}|z-1|^{-\nu_1+2}\,F\,\|_{L^\infty(D^2)}.$$

So, we can write $W_0 = \Psi_0 + c_0^*\,n$, where

$$\|\,|z|^{-\nu_0}\,\Psi_0\,\|_{L^\infty(D^2)} + |c_0^*| \leq C\,\|\,|z|^{-\nu_0+2}|z-1|^{-\nu_1+2}F\|_{L^\infty(D^2)}.$$

The function $W_{app} := W_0 + \chi\,W_1$ satisfies:

$$\begin{cases} \Delta\, W_{app} = F \quad \text{in} \quad D^2_*, \\ \partial_r\, W_{app} - \frac{1}{n} W_{app} = -\frac{1}{n}\,\chi\,W_1 \quad \text{on} \quad S^1 \setminus \{1\}. \end{cases}$$

Notice, that we can write

$$W_{app} = \Psi + c_0^*\,n + c_1^*\,\chi$$

(changing if necessary the values of the constants c_0^* and c_1^*), and for example writing

$$W_{app} = [(1-\chi)\,\Psi_0 + \chi\,\Psi_1 + \chi\,(\Psi_0 - \,\Psi_0(1))] + \chi\,\Psi_0(1) + \chi\,c_1^* + c_0^*\,n.$$

In order to find the exact solution, we define the function

$$h_n(z) := \frac{|z|^2 - 1}{2n}\,\chi\,W_1(z),$$

which satisfies:

$$\begin{cases} \Delta h_n = \frac{|z|^2-1}{2n}\,\Delta(\chi\,W_1) + \frac{2r}{n}\partial_r\,(\chi\,W_1) + \frac{2}{n}\,\chi\,W_1, \\ \partial_r h_n - \frac{1}{n} h_n\big|_{r=1} = \frac{1}{n}\chi\,W_1. \end{cases}$$

Then, we introduce the function

$$W_n := W_{app} + h_n,$$

and verify that

$$\begin{cases} \big\|\,|z|^{-\nu_0+2}\,|z-1|^{-\nu_1+2}\,(\Delta W_n - F)\big\|_{L^\infty(D^2)} \leq \frac{\hat{c}}{n}\,\big\|\,|z|^{-\nu_0+2}\,|z-1|^{-\nu_1+2}\,F\,\big\|_{L^\infty(D^2)}, \\ \partial_r W_n - \frac{1}{n}\,W_n\big|_{r=1} = 0, \end{cases}$$

for a constant $\hat{c}$ independent of n. Next, consider the operator

$$\mathfrak{R}_n \,:\, F \in |z|^{\nu_0-2}\,|z-1|^{\nu_1-2}\,L^\infty(D^2) \;\mapsto\; \Delta\, W_n(F) - \mathrm{Id}(F) \in |z|^{\nu_0-2}\,|z-1|^{\nu_1-2}\,L^\infty(D^2).$$

Then $\|\mathfrak{R}_n\| \leq \frac{\hat{c}}{n}$ for a constant $\hat{c}$ independent of n, and thus, for n large enough, the operator $\mathrm{Id} + \mathfrak{R}_n$ is invertible. Finally, the function

$$W := W_n\left((\mathrm{Id} + \mathfrak{R}_n)^{-1}(F)\right)$$

satisfies (2.35) and can be written in the form:

$$W = \Psi + c_0^*\, n + c_1^*\, \chi,$$

where

$$\|\, |z|^{-\nu_0}\,|z-1|^{-\nu_1}\,\Psi\,\|_{L^\infty(D^2)} + |c_0^*| + |c_1^*| \leq C_1\,\|\,|z|^{2-\nu_0}\,|z-1|^{2-\nu_1}\,F\,\|_{L^\infty(D^2)}.$$

Finally, we prove that the constant C_1 does not depend on n. Again, for the radial part of Ψ and the deficiency terms the inequality follows from the explicit expressions given by (2.37) and (2.42). For the remaining part Ψ^{mean}, such that $\int_{S^1} \Psi^{mean}(r\phi)\, r\, d\phi = 0, \quad \forall r \in (0,1)$ the proof follows from an argument by contradiction analogous to the one described in Lemma 2.9.1.

□

Now we go back to the problem (2.33) and analyze the solutions for $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(D^2_*)$.

Proof of Proposition 2.9.1. Consider the function F defined by the relation

$$f(z) = n^2\,|z|^{2n-2}\,F(z^n).$$

If $f \in C^{0,\alpha}_{\nu-2}(D^2_*)$, then

$$F(z) \in |z|^{\nu/n-2}\,|z-1|^{\nu-2}\,L^\infty(D^2),$$

and

$$\|\,|z|^{-\nu/n+2}|z-1|^{-\nu+2}\,F\,\|_{L^\infty(D^2)} \leq C\,\frac{1}{n^2}\,\|\,\gamma_D^{-\nu+2}\,f\,\|_{L^\infty(D^2)}$$

for a constant C independent of n. Furthermore, by Lemma 2.9.2, for all n large enough, there exits a function W which satisfies

$$\begin{cases} \Delta W = F & \text{in} \quad D^2 \setminus \{0\}, \\ \partial_r W - \frac{1}{n}\,W = 0 & \text{on} \quad S^1 \setminus \{1\}, \end{cases}$$

and can be decomposed as

$$W = \Psi + c_0^*\, n + c_1^*\, \chi,$$

where

$$\| |z|^{-\nu/n}\,|z-1|^{-\nu}\,\Psi\|_{L^\infty(D^2)} \leq C\,n^2\,\| |z|^{2-\nu/n}\,|z-1|^{2-\nu}\,F\|_{L^\infty(D^2)} \leq C\,\|\,\gamma_D^{-\nu+2}\,f\,\|_{L^\infty(D^2)}$$

$$|c_0^*| \leq C\,n\,\| |z|^{2-\nu/n}\,|z-1|^{2-\nu}\,F\|_{L^\infty(D^2)} \leq C\,\frac{1}{n}\,\|\,\gamma_D^{-\nu+2}\,f\,\|_{L^\infty(D^2)}$$

$$|c_1^*| \leq C\;n^2\,\| |z|^{2-\nu/n}\,|z-1|^{2-\nu}\,F\|_{L^\infty(D^2)} \leq C\,\|\,\gamma_D^{-\nu+2}\,f\,\|_{L^\infty(D^2)}$$

for a constant $C > 0$ independent of n. Then the function $w(z) := W(z^n)$ satisfies (2.33) and can be decomposed as

$$w(z) = \psi(z) + \hat{c}_0 + \hat{c}_1\,\chi_n, \tag{2.43}$$

with

$$\|\gamma_D^{-\nu}\,\psi\|_{L^\infty(D^2)} + |\hat{c}_0| + |\hat{c}_1| \le C\,\|\gamma_D^{2-\nu}\,f\|_{L^\infty(D^2)}. \tag{2.44}$$

Finally, in (2.43) we have $\psi \in \mathcal{C}^{2,\alpha}_\nu(D^2)$ which follows from the Schauder's elliptic estimates in weighted spaces. To see this, it is sufficient to apply in the neighborhood of $z = 0$ the classical Schauder's estimates to the functions $\psi(R, \cdot\,)$ and $\psi(\lambda_m(R\cdot\,))$ in the domains $\big\{z \in D^2 \,:\, R < |z| < 2R\big\}$ and $\lambda_m\left(\{\zeta \in \mathbb{C}_- \,:\, R < |\zeta| < 2R\}\right)$ for all $R \in (0, \frac{1}{2})$.

□

Corollary 2.9.2. *From Proposition 2.9.1 we deduce the properties of the solutions to the problem* (2.32)*.*

2.9.2 Linear analysis on the half-catenoid

Consider the problem:

$$\begin{cases} \left(\partial^2_\sigma + \partial^2_\theta + \frac{2}{\cosh^2\sigma}\right)\,w = f \quad \text{in} \quad \mathbb{R}\times\left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \\ \partial_\theta w = 0 \quad \text{on} \quad \mathbb{R}\times\{\frac{\pi}{2}, \frac{3\pi}{2}\}. \end{cases} \tag{2.45}$$

Lemma 2.9.3. *Assume that $\delta \in (-1, 0)$. The subspace of $(\cosh\sigma)^\delta\mathcal{C}^{2,\alpha}\left(\mathbb{R}\times\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right)$ which is invariant by $(\sigma, \theta) \mapsto (\sigma, 2\pi - \theta)$ and $(\sigma, \theta) \mapsto (-\sigma, \theta)$ and solves*

$$\begin{cases} \left(\partial^2_\sigma + \partial^2_\theta + \frac{2}{\cosh^2\sigma}\right)\,w = 0 \quad \textit{in} \quad \mathbb{R}\times\left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \\ \partial_\theta w = 0 \quad \textit{on} \quad \mathbb{R}\times\{\frac{\pi}{2}, \frac{3\pi}{2}\}, \end{cases}$$

is trivial.

Proof. We decompose w in Fourier series

$$w(\sigma, \theta) = \sum_{j\in\mathbb{Z}} \omega_j(\sigma)e^{ij\theta},$$

then the functions w_j are solutions of the ordinary equations

$$\left(\partial^2_\sigma - j^2 + \frac{2}{\cosh^2\sigma}\right)w_j = 0.$$

These solutions are asymptotic either to $(\cosh\sigma)^j$ or to $(\cosh\sigma)^{-j}$. By hypothesis, the solution is bounded by a constant times $(\cosh\sigma)^\delta$ and $|\delta| < 1$, so the solution has to be asymptotic to $(\cosh\sigma)^{-j}$, and then the solution is bounded. On the other hand, $-j^2 + \dfrac{2}{\cosh^2\sigma} \le 0$, so the maximum principle assures that $w_j = 0$, for all $j \ge 2$.

Observe that the imposed symmetry $(\sigma,\theta)\mapsto(\sigma,-\theta)$ and the boundary condition imply $w_1=0$. When $j=0$, ω_0 is the solution of the ordinary equation

$$\left(\partial_\sigma^2+\frac{2}{\cosh^2\sigma}\right)w_0=0.$$

By direct computations, we can see that $\tanh\sigma$ and $\sigma\tanh\sigma-1$ are two independent solutions neither of which belongs $(\cosh\sigma)^\delta\mathcal{C}^{2,\alpha}(\mathbb{R}\times\left[\frac{\pi}{2},\frac{3\pi}{2}\right])$.

□

The next step is to prove that, under some assumptions, there exists a right inverse of the problem (2.45) and it is bounded.

Proposition 2.9.2. *Assume that* $\delta\in(-1,0)$. *Then there exists a constant* C *and for all* $f\in(\cosh\sigma)^\delta\mathcal{C}^{0,\alpha}(\mathbb{R}\times\left[\frac{\pi}{2},\frac{3\pi}{2}\right])$ *such that* $f(\sigma,\theta)=f(-\sigma,\theta)=f(\sigma,2\pi-\theta)$ *there exists a unique constant* $\hat{d}_1$ *and a unique function* $v\in(\cosh\sigma)^\delta\mathcal{C}^{2,\alpha}(\mathbb{R}\times\left[\frac{\pi}{2},\frac{3\pi}{2}\right])$ *such that*

$$w=v+\hat{d}_1$$

solves (2.45), $w(\sigma,\theta)=w(-\sigma,\theta)=w(\sigma,2\pi-\theta)$, *and*

$$\|(\cosh\sigma)^{-\delta}\,v\|_{\mathcal{C}^{2,\alpha}\left(\mathbb{R}\times\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\right)}+|\hat{d}_1|\le C\,\|(\cosh\sigma)^{-\delta}f\|_{\mathcal{C}^{0,\alpha}\left(\mathbb{R}\times\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\right)}. \tag{2.46}$$

Proof. Let us extend the function f by symmetry with respect to the coordinate axis $\xi_1=0$ to the entire unit cylinder $\mathbb{R}\times S^1$. Then there exists a function w which satisfies:

$$\left(\partial_\sigma^2+\partial_\theta^2+\frac{2}{\cosh^2\sigma}\right)w=f\quad\text{in}\quad\mathbb{R}\times S^1,$$

where $w=v+\hat{d}_1$ and

$$\|(\cosh\sigma)^{-\delta}v\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}\times S^1)}+|\hat{d}_1|\le C\,\|(\cosh\sigma)^{-\delta}f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}\times S^1)}.$$

The proof of this fact is classic, and can be found for instance in [77], but, for the sake of completeness, we give here the details. Decompose f in Fourier series in θ:

$$f(\sigma,\theta)=\sum_{j\in\mathbb{Z}}f_j(\sigma)\,e^{ij\theta},$$

then for all $t\in\mathbb{R}$ and $|j|\ge 2$, let v_j^t be a solution to

$$\left(\frac{d^2}{d\sigma^2}-j^2+\frac{2}{\cosh^2\sigma}\right)v_j^t=f_j\quad\text{in}\quad|\sigma|<t,\quad v_j^t(\pm t)=0,\quad j\ge 2,$$

obtained by the maximum principal and the method of sub- and supersolutions, taking $\frac{1}{j^2-2-\delta}(\cosh\sigma)^\delta$ as a barrier function (using that $(\delta^2-j^2)\cosh^2 s+2+\delta-\delta^2\le -j^2+2+\delta$).

Taking a sum over $|j| > 2$, we obtain a function v_t which by the Schauder's elliptic theory satisfies

$$\|(\cosh\sigma)^{-\delta} v_t\|_{\mathcal{C}^{2,\alpha}((-t,t)\times S^1)} \leq C\|(\cosh\sigma)^{-\delta} f\|_{\mathcal{C}^{0,\alpha}((-t,t)\times S^1)},$$

for a constant C independent of t. Finally, the sequence v_t admits a subsequence which converges uniformly on compact sets of $\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ as t tends to infinity to a solution $\hat{v}$ of (2.45), such that (2.46) is satisfied.

Notice that, by construction, $f(\sigma,\theta) = f(\sigma,-\theta) = f(\sigma,\pi-\theta)$, so f has no Fourier mode 1 and we only need to treat the case when $f = f_0(s)$. We construct a solution explicitly:

$$w_0(\sigma) = \tanh\sigma \int_0^\sigma (1 - t\,\tanh t)\,dt - (1 - \sigma\tanh\sigma)\int_0^\sigma \tanh t\,dt.$$

Remark that for $|f_j(\sigma)| \leq (\cosh\sigma)^{\delta}$ there exist constants c, d and $\hat{d}_1$ such that

$$\omega_0 + d\,(1 - s\tanh s) = v_0 + \hat{d}_1 \quad \text{and}$$

$$\|(\cosh\sigma)^{-\delta}\, v_0\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}\times S^1)} + |\hat{d}_1| \leq \|(\cosh\sigma)^{-\delta} f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}\times S^1)}.$$

The estimates for derivatives of w_0 are obtained by Schauder's theory. Finally, we put

$$v = v_0 + \hat{v} \quad \text{and} \quad w = v + \hat{d}_1,$$

and, by symmetry, $\partial_\theta\, w|_{\{\frac{\pi}{2},\frac{3\pi}{2}\}} = 0$.

□

2.9.3 Linear analysis on the catenoid

In this subsection we consider the equation

$$\left(\partial_s^2 + \partial_\phi^2 + \frac{2}{\cosh^2 s}\right) w = f \quad \text{in} \quad \mathbb{R}\times S^1. \tag{2.47}$$

Proposition 2.9.3. *Assume that $\delta \in (-1,0)$. Then given $f \in (\cosh s)^{\delta}\mathcal{C}(\mathbb{R}\times S^1)$, such that $f(s,\phi) = f(-s,\phi) = f(s,-\phi) = f(s,\phi+\frac{2\pi}{n})$, there exists a unique constant $\hat{d}_0$ and a unique function $v \in (\cosh s)^{\delta}\mathcal{C}^{2,\alpha}(\mathbb{R}\times S^1)$ such that*

$$w = v + \hat{d}_0$$

solves (2.47), *$w(s,\phi) = w(-s,\phi) = w(s,-\phi) = w(s,\phi+\frac{2\pi}{n})$ and*

$$\|\,(\cosh s)^{-\delta}\, v\,\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}\times S^1)} + |\hat{d}_0| \leq C\,\|\,(\cosh s)^{-\delta}\, f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}\times S^1)}. \tag{2.48}$$

Proof of Proposition 2.9.3. Consider the function $F : (s,\phi) \mapsto \frac{1}{n^2} f(\frac{s}{n}, \frac{\phi}{n})$ and let W be a solution of

$$\left(\partial_s^2 + \partial_\phi^2 + \frac{2}{n^2 \cosh^2 \frac{s}{n}}\right) W = F, \tag{2.49}$$

then the function

$$w(s,\phi) = W(ns, n\phi)$$

satisfies (2.47) and $w(s,\phi) = w(-s,\phi) = w(s,-\phi) = w(s,\phi + \frac{2\pi}{n})$. The existence of W with desired properties is a consequence of the following two lemmas.

Lemma 2.9.4. *Assume that $\delta \in (-1,0)$. The subspace of $(\cosh \frac{s}{n})^\delta L^\infty\left(\mathbb{R} \times S^1\right)$ which is invariant by $(s,\phi) \mapsto (s,-\phi)$ and $(s,\phi) \mapsto (-s,\phi)$ and solves*

$$\left(\partial_s^2 - j^2 + \frac{2}{n^2 \cosh^2 \frac{s}{n}}\right) W = 0 \quad in \quad \mathbb{R} \times S^1,$$

is trivial.

Proof. The proof is analogous to the proof of the lemma (2.9.3) and uses the maximum principal for the Fourier modes $j \geq 1$ and treats explicitly the case $j = 0$.

□

Lemma 2.9.5. *Assume that $\delta \in (-1,0)$. Then there exists a constant C and for all functions $F \in \left(\cosh \frac{s}{n}\right)^\delta L^\infty\left(\mathbb{R} \times S^1\right)$ such that $F(s,\phi) = F(-s,\phi) = F(s,-\phi)$, there exists a unique constant $\hat{d}_0$ and a unique function $V \in \left(\cosh \frac{s}{n}\right)^\delta L^\infty(\mathbb{R} \times S^1)$ such that*

$$W = V + \hat{d}_0$$

solves (2.49) *and*

$$\left\| \left(\cosh \frac{s}{n}\right)^{-\delta} V \right\|_{L^\infty(\mathbb{R}\times S^1)} + |\hat{d}_0| \leq C \left\| \left(\cosh \frac{s}{n}\right)^{-\delta} F \right\|_{L^\infty(\mathbb{R}\times S^1)}. \tag{2.50}$$

Proof of Lemma 2.9.5. We decompose both F and W in Fourier series

$$F = \sum_{j\in\mathbb{Z}} F_j(s)\, e^{ij\phi}, \quad \text{and} \quad W = \sum_{j\in\mathbb{Z}} W_j(s)\, e^{ij\phi}.$$

For all $t \in \mathbb{R}$ and $|j| \geq 1$, there exists a function V_j^t that satisfies

$$\left(\partial_s^2 - j^2 + \frac{2}{n^2 \cosh^2 \frac{s}{n}}\right) V_j^t = F_j, \quad V_j^t(\pm t) = 0.$$

This follows from the maximum principle if we take $\dfrac{1}{j^2 n^2 - 2 - \delta} (\cosh \frac{s}{n})^\delta$ as a barrier function. When t tends to infinity, we can choose a subsequence of functions converging uniformly on compact sets of $\mathbb{R}\times S^1$ to a solution of (2.49) which satisfies (2.50). For $j=0$ we find the solution explicitly

$$W_0(s)=\tanh\frac{s}{n}\int_0^{\frac{s}{n}}(1-t\tanh t)\,F(nt)\,dt-\Big(1-\frac{s}{n}\,\tanh\frac{s}{n}\Big)\int_0^{\frac{s}{n}}\tanh t\,F(nt)\,dt.$$

As in Proposition 2.9.2 there exists a function $V_0\in\left(\cosh\frac{s}{n}\right)^{\delta}L^\infty(\mathbb{R}\times S^1)$ and a constant $\hat{d}_0$, such that the function $W_0=V_0+\hat{d}_0$ satisfies (2.50).

□

Finally, we put $v(s,\phi)=V(ns,n\phi)$. By the Schauder's theory, $v\in(\cosh s)^{-\delta}\,\mathcal{C}^{2,\alpha}(\mathbb{R}\times S^1)$ and

$$\|(\cosh s)^{-\delta}v\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}\times S^1)}\le C\,\|(\cosh s)^{-\delta}f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}\times S^1)}.$$

□

2.9.4 Gluing the parametrices together

In this subsection we construct a solution of the linear problem

$$\mathcal{L}_n\,w=f$$

by gluing together solutions to linear problems in the punctured disk $\bar{D}^2\setminus\{0,z_1,\dots,z_n\}$, the cylinder $\mathbb{R}\times S^1$ and the half-cylinder $\mathbb{R}\times\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$ obtained in the subsections 2.9.1, 2.9.2 and 2.9.3. The main result of this subsection reads

Proposition 2.9.4. *There exist constants $C>0$ and $\ell\in\mathbb{N}$ and for all $f\in\mathcal{E}^{0,\alpha}_{n,\nu-2}$ and all n large enough, there exists a function $w\in\mathcal{E}^{2,\alpha}_{n,\nu}$ which satisfies*

$$\mathcal{L}_n\,w=f\quad\textit{and}\quad\|w\|_{\mathcal{E}^{2,\alpha}_{n,\nu}}\le\hat{c}\,e^{\ell\,\nu\,n}\,\|f\|_{\mathcal{E}^{0,\alpha}_{n,\nu-2}}.$$

Proof. The proof consists of 6 steps. Let us use the decomposition of $\mathcal{A}_n$ in graph, catenoidal and gluing regions as it is described in the subsection 2.7. In Step 1, we show that a function $f\in\mathcal{E}^{0,\alpha}_{n,\nu-2}$ can be written as a sum $f=f^++f^-$, where

$$f^+\equiv 0\quad\text{in}\quad\Omega^-_{gr}\bigcup_{i=0}^{n}\Omega^{i,-}_{glu}\quad\text{and}\quad f^-\equiv 0\quad\text{in}\quad\Omega^+_{gr}\bigcup_{i=0}^{n}\Omega^{i,+}_{glu}.$$

Then we show that there exist diffeomorphisms $\mathfrak{Y}^\pm$ from a subdomain of $\bar{D}^2\setminus\{0,z_1,\dots,z_n\}$ to some regions in $\mathcal{A}_n$ such that the function

$$f^+\circ\mathfrak{Y}^+=f^-\circ\mathfrak{Y}^-:=\breve{f}$$

can be extended to the entire $\bar{D}^2\setminus\{0,z_1,\dots,z_n\}$ in such a way that $\breve{f}\in\mathcal{C}^{0,\alpha}_{\nu-2}(D^2_*)$.

In Step 2, using the results of Proposition 2.9.1, we find a solution $\breve{w}_{gr}$ to the problem

$$\begin{cases}\Delta\,(B\,\breve{w}_{gr})=\breve{f} & \text{in}\quad D^2\setminus\{0\},\\ \partial_r w_{gr}=0 & \text{in}\quad S^1\setminus\{z_1,\dots,z_n\},\end{cases}$$

and show that truncating $\breve{w}_{gr} \circ \mathfrak{Y}^{-1}$ in the neighborhood of the punctures, we obtain an approximate solution to the equation $\mathcal{L}_n\, w = f$ in the graph and gluing regions.

In Step 3, we show that in the "catenoidal neck" and "half-catenoidal bridge" regions, the error has a specific form and can be compensated using linear analysis in the cylinder $\mathbb{R} \times S^1$ and the half-cylinder $\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ described in Propositions 2.9.2 and 2.9.3.

In Step 4, we combine the solutions obtained in Steps 2 and 3 to obtain an approximate solution to our problem in $\mathcal{A}_n$ by truncating the terms that decay at infinity and gluing together the deficiency terms.

Finally, in Step 5, we find an exact solution using a perturbation argument.

Step 1 : Decomposition of the function f

Take a function $f \in \mathcal{E}^{0,\alpha}_{n,\nu-2}$ and consider the cut-off function $\vartheta \in \mathcal{C}^\infty(\mathbb{R})$ which satisfies

$$\vartheta(t) \equiv 1 \quad \text{for} \quad t > 1 \quad \text{and} \quad \vartheta(t) \equiv 0 \quad \text{for} \quad t < -1.$$

Let (s, ϕ) be the cylindrical coordinates in the "catenoidal neck". We write

$$f = f^+ + f^-,$$

where while in the half-cylindrical coordinates (σ, θ) in regions of the "catenoidal bridges" we put

$$f^+(s, \phi) = \vartheta(s)\, f(s, \phi) \quad \text{and} \quad f^-(s, \phi) = (1 - \vartheta(s))\, f(s, \phi).$$

In the same manner, let (σ, θ) be half-cylindrical coordinates in region of one of the "half-catenoidal" bridges. We put

$$f^+(\sigma, \theta) = \vartheta(\sigma)\, f(\sigma, \theta) \quad \text{and} \quad f^-(\sigma, \theta) = (1 - \vartheta(\sigma))\, f(\sigma, \theta).$$

Next, let us explain the construction of diffeomorphisms $\mathfrak{Y}^\pm$ from some neighborhoods of $\mathcal{A}_n$ to a subdomain of $\bar{D}^2 \setminus \{0, z_1, \ldots, z_n\}$.

Take real numbers $s_n > 0$ and $\sigma_n > 0$ that satisfy

$$\tilde{\varepsilon}\, \cosh s_n = \frac{1}{4n} \quad \text{and} \quad \frac{\varepsilon}{2}\, \cosh \sigma_n = \frac{1}{4n}.$$

Let us denote by $\mathfrak{x}_0$ the map that corresponds to the parametrization of the "catenoidal region" in $\mathcal{A}_n$ by cylindrical coordinates. We also put $\mathcal{A}^0_n := \mathfrak{x}_0\left([-s_n, s_n] \times S^1\right)$.

In the same manner, we denote by $\mathfrak{x}_m$ the map which corresponds to the parametrization of the "half-catenoidal bridge" by half-cylindrical coordinates, and put $\mathcal{A}^m_n := \mathfrak{x}_m\left([-\sigma_n, \sigma_n] \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right)$.

On the other hand, we introduce cylindrical coordinates in a neighborhood of $z = 0$ in $D^2 \setminus \{0\}$ via the one of the mappings

$$\mathfrak{z}^{0,+}_{\varepsilon} \,:\, (s, \phi) \in (-\infty, s_n) \times S^1 \mapsto \frac{\tilde{\varepsilon}}{2} e^s\, e^{i\phi} \in D^2,$$

$$\mathfrak{z}^{0,-}_{\varepsilon} \,:\quad (s, \phi) \in (-s_n, +\infty) \mapsto \frac{\tilde{\varepsilon}}{2}\, e^{-s}\, e^{i\phi} \in D^2.$$

In the same manner, we introduce half-cylindrical coordinates in a neighborhood of $z = z_m$ in $\bar{D}^2 \setminus \{z_m\}$ via one of the mappings

$$\mathfrak{z}_\varepsilon^{m,+} \,:\, (\sigma, \theta) \in (-\infty, \sigma_n) \mapsto \lambda_m \left(\frac{\varepsilon}{4}\, e^{\sigma} e^{i\theta} \right) \in \bar{D}^2 \quad \text{or}$$

$$\mathfrak{z}_\varepsilon^{m,-} \,:\, (\sigma, \theta) \in (-\sigma_n, +\infty) \mapsto \lambda_m \left(\frac{\varepsilon}{4}\, e^{-\sigma}\, e^{i\theta} \right) \in \bar{D}^2.$$

Finally, we introduce the mappings

$$\mathfrak{z}_\varepsilon^{0} \,:\, (s, \phi) \in (-s_n, s_n) \times S^1 \mapsto \tilde{\varepsilon}\, \cosh s\, e^{i\phi} \in D^2,$$

$$\mathfrak{z}_\varepsilon^{m} \,:\, (\sigma, \theta) \in (-\sigma_n, \sigma_n) \times \left[\frac{\pi}{2}, \frac{3\pi}{2} \right] \mapsto \lambda_m \left(\frac{\varepsilon}{2}\, \cosh \sigma\, e^{i\theta} \right) \in D^2.$$

Consider the following regions of the unit disk:

$$D_{o,n} := \left\{ z \in D^2 \,:\, \frac{\tilde{\varepsilon}}{2}\, e^{-s_n} \leq |z| \leq \frac{\tilde{\varepsilon}}{2}\, e^{s_n} \right\} \quad \text{and} \quad D_{m,n} := \left\{ z \in D^2 \,:\, \frac{\varepsilon}{4}\, e^{-\sigma_n} \leq |\lambda_m(z)| \leq \frac{\varepsilon}{4}\, e^{\sigma_n} \right\},$$

and remark that the mappings

$$\mathfrak{x}_i \circ \left(\mathfrak{z}_\varepsilon^{i,\pm} \right)^{-1}$$

provide diffeomorphisms from $D_{i,n}$ to $\mathcal{A}_n^i$. On the other hand, let

$$\mathfrak{p}^{\pm} \,:\, \Omega_{gr}^{\pm} \bigcup_{i=0}^{n} \Omega_{glu}^{i,\pm} \subset \mathcal{A}_n \longrightarrow D^2$$

denote the vertical projections of the graph and to the unit disk. By construction, we have

$$\mathfrak{p}^{\pm} \circ \mathfrak{x}_0(s, \phi) = \mathfrak{z}_\varepsilon^{0}(s, \phi) = \tilde{\varepsilon}\, \cosh s\, e^{i\phi} \quad \text{in} \quad \Omega_{gr}^{\pm} \cup \Omega_{glu}^{0,\pm},$$

$$\mathfrak{p}^{\pm} \circ \mathfrak{x}_m(\sigma, \theta) = \mathfrak{z}_\varepsilon^{m}(\sigma, \theta) = \lambda_m \left(\frac{\varepsilon}{2}\, \cosh \sigma\, e^{i\theta} \right) \quad \text{in} \quad \Omega_{gr}^{\pm} \cup \Omega_{glu}^{m,\pm}.$$

We can write

$$\mathfrak{p}^{\pm} \circ \mathfrak{x}_0 \circ \left(\mathfrak{z}_\varepsilon^{0,\pm} \right)^{-1} = \mathrm{Id} + \Phi_\varepsilon^{0,\pm} \quad : D_{0,n} \to D_{0,n}, \quad \text{where} \quad \left| \Phi_\varepsilon^{0,\pm}(z) \right| \leq c\, \tfrac{\varepsilon^2}{|z|},$$

$$\mathfrak{p}^{\pm} \circ \mathfrak{x}_m \circ \left(\mathfrak{z}_\varepsilon^{m,\pm} \right)^{-1} = \mathrm{Id} + \Phi_\varepsilon^{m,\pm} \quad : D_{m,n} \to D_{m,n}, \quad \text{where} \quad \left| \Phi_\varepsilon^{m,\pm}(z) \right| \leq c\, \tfrac{\varepsilon^2}{|z - z_m|}.$$

for a constant c independent of ε. Consider the domain

$$D_{\#}^2 := D^2 \setminus \left(\left\{ z \in D^2 \,:\, |z| < \frac{\tilde{\varepsilon}}{2}\, e^{-s_n} \right\} \bigcup_{m=1}^{n} \left\{ z \in D^2 \,:\, |\lambda_m(z)| < \frac{\varepsilon}{4}\, e^{-\sigma_n} \right\} \right)$$

of the unit disk and the cut-off functions $\mu_{i,n} \in \mathcal{C}^\infty(D^2)$, $i = 0, \ldots, n$, such that

$$\mu_{0,n}(z) \equiv 1 \quad \text{for} \quad |z| \leq \tfrac{1}{8n} \quad \text{and} \quad \mu_{0,n} \equiv 0 \quad \text{for} \quad |z| \geq \tfrac{1}{4n};$$

$$\mu_{m,n}(z) \equiv 1 \quad \text{for} \quad \left| \lambda_m^{-1}(z) \right| \leq \tfrac{1}{8n} \quad \text{and} \quad \mu_{m,n} \equiv 0 \quad \text{for} \quad \left| \lambda_m^{-1}(z) \right| > \tfrac{1}{4n}.$$

Then we define the diffeomorphism $\mathfrak{Y}^+$ from $D^2_\#$ to its image in $\mathcal{A}_n$ by

$$\mathfrak{Y}^+(z) := \begin{cases} \mathfrak{x}_0 \circ \left(\mathfrak{z}_\varepsilon^{0,+}\right)^{-1}(z), & \text{for} \quad |z| \le \frac{1}{8n} \\ \mathfrak{x}_m \circ \left(\mathfrak{z}_\varepsilon^{m,+}\right)^{-1}(z), & \text{for} \quad \left|\lambda_m^{-1}(z)\right| \le \frac{1}{8n}, \\ (\mathfrak{p}^+)^{-1} \circ \left(\mathrm{Id} + \mu_{i,n}\, \Phi_\varepsilon^{i,+}\right)(z), & \text{for} \quad z \in \operatorname{supp}\left(\nabla \mu_{i,n}\right), \\ (\mathfrak{p}^+)^{-1}(z) & \text{elsewhere.} \end{cases}$$

Finally, we check that the function

$$\breve{f} := f^+ \circ \mathfrak{Y}^+ = f^- \circ \mathfrak{Y}^-,$$

defined in $D^2_\#$ can be extended by 0 to the entire punctured disk $\bar{D}^2 \setminus \{0, z_1, \dots, z_n\}$ and moreover, we have $\breve{f} \in \mathcal{C}^{0,\alpha}_{\nu-2}(D^2_*)$.

Step 2: Contribution of the linear analysis on the punctured disk

The function $\breve{f} \in \mathcal{C}^{0,\alpha}_{\nu-2}(D^2_*)$ satisfies $\breve{f}(z) = \breve{f}(\bar{z}) = \breve{f}(z \cdot z_m)$, so, by Corollary 2.9.1, there exists a function $\breve{w}_{gr} \in \mathcal{C}^{2,\alpha}_\nu(D^2_*) \oplus \mathfrak{D}_n$, which is a solution to

$$\begin{cases} \Delta\left(B\, \breve{w}_{gr}\right) = \breve{f} & \text{in} \quad D^2 \setminus \{0\}, \\ \partial_r \breve{w} = 0 & \text{in} \quad S^1 \setminus \{z_1, \dots, z_n\}, \end{cases} \tag{2.51}$$

where $\breve{w}_{gr} = \breve{\psi}_{gr} + \hat{c}_0 + \hat{c}_1\, \chi_n$ with

$$\|\breve{\psi}_{gr}\|_{\mathcal{C}^{2,\alpha}_\nu(D_*)} + |\hat{c}_0| + |\hat{c}_1| \le C\, \|\breve{f}\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(D^2_*)}.$$

Next, we show that the functions

$$w^\pm_{gr} := \breve{w}_{gr} \circ \left(\mathfrak{Y}^\pm\right)^{-1},$$

(defined locally in $\mathcal{A}_n$) are approximate solutions to $\mathcal{L}_n\, w = f$ in the regions $\Omega^\pm_{gr} \cup \Omega^{i,\pm}_{glu}$, $i = 0, 1, \dots, n$. Indeed, by Proposition 2.8.2, for all $u \in \mathcal{E}^{2,\alpha}_{n,\nu}$ we have in $\Omega^\pm_{gr} \cup \Omega^{i,\pm}_{glu}$

$$\left|\gamma^{2-\nu}\left(\mathcal{L}_n\, u - L_{gr}\, u\right)\right| \le C\, \varepsilon^2\, \gamma^{-2}\, \|u\|_{\mathcal{E}^{2,\alpha}_{n,\nu}}.$$

Moreover, by construction,

$$\left|\gamma^{2-\nu}\left(L_{gr}\, w^\pm_{gr} - f\right)\right| \le C\, \left|\gamma^{2-\nu}\left(L_{gr} \circ \mathfrak{Y}^\pm - L_{gr}\right)\, w^\pm_{gr}\right| \le C\, \varepsilon^2\, \gamma^{-2}\, \|f\|_{\mathcal{E}^{0,\alpha}_{n,\nu-2}}.$$

This yields

$$\left\|\mathcal{L}_n\, w^\pm_{gr} - f\right\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\Omega^\pm_{gr} \cup \Omega^{i,\pm}_{glu})} \le c\, \varepsilon^{\frac{2}{3}}\, \|f\|_{\mathcal{E}^{0,\alpha}_{n,\nu-2}}, \quad i = 0, 1, \dots, n,$$

where for $\Omega \subset \mathcal{A}_n$, we denote by $\mathcal{C}^{0,\alpha}_{\nu-2}(\Omega)$ the restriction of $\mathcal{E}^{0,\alpha}_{n,\nu-2}$ to Ω.

On the other hand, since the function $\breve{\psi}_{gr}$ in the decomposition of $\breve{w}_{gr}$ decays in the neighborhood of the punctures in $\bar{D}^2 \setminus \{0, z_1, \dots, z_n\}$, we can extend $\breve{\psi}_{gr} \circ (\mathfrak{Y}^\pm)^{-1}$ to the entire surface $\mathcal{A}_n$ in a natural way using a suitable cut-off function which we define below.

Notation 2.9.2. *Let us choose real numbers s_g and S_g, such that the gluing region Ω^+_{glu} of $\mathcal{A}_n$ is parametrized by $[s_g, S_g] \times S^1 \subset \mathbb{R} \times S^1$. By symmetry, the region Ω^-_{glu} is parametrized by $[-S_g, -s_g] \times S^1$. We introduce the cut-off function $\xi_0 \in \mathcal{C}^\infty\left(D^2\right)$ which is radial and satisfies*

$$\xi_0(z) \equiv 1 \quad \textit{for} \quad |z| > \frac{\tilde{\varepsilon}}{2}\, e^{-s_g} \quad \textit{and} \quad \xi_0(z) \equiv 0 \quad \textit{for} \quad |z| < \frac{\tilde{\varepsilon}}{2}\, e^{-S_g}.$$

In the same manner, let the gluing region $\Omega^{m,+}_{glu}$ be parametrized by $[\sigma_g, \Sigma_g] \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Then we introduce the cut-off function $\xi_m \in \mathcal{C}^\infty\left(D^2\right)$ which satisfies

$$\xi_m(z) \equiv 1 \quad \textit{when} \quad \left|\lambda_m^{-1}(z)\right| \geq \frac{\varepsilon}{4}\, e^{-\sigma_g} \quad \textit{and} \quad \xi_m(z) \equiv 0 \quad \textit{when} \quad \left|\lambda_m^{-1}(z)\right| < \frac{\varepsilon}{4}\, e^{-\Sigma_g}.$$

Finally, we define the function $\xi := \prod\limits_{i=0}^{n} \xi_i$.

We put

$$\psi^\pm_{gr} := \left(\xi\, \breve{\psi}\right) \circ \left(\mathfrak{Y}^\pm\right)^{-1} \in \mathcal{E}^{2,\alpha}_{n,\nu},$$

and since

$$\left|\psi^\mp_{gr}\right| \leq c\, \varepsilon^\nu\, \gamma^{-\nu} \quad \text{in} \quad \Omega^\pm_{gr} \cup \Omega^{i,\pm}_{glu} \cup \Omega^i_{cat},$$

we find

$$\left|\gamma^2\, \mathcal{L}_n\left(\psi^+_{gr} + \psi^-_{gr}\right)\right| \leq c\, \varepsilon^\nu\, \gamma^{-\nu} \quad \text{in} \quad \mathcal{A}_n. \tag{2.52}$$

Step 3 : Contribution of the linear analysis on the catenoids

Notation 2.9.3. *Let $\eta_i \in \mathcal{C}^\infty(\mathcal{A}_n)$ be the cut-off functions, which are invariant under the action of the group $\mathfrak{S}_n$ and satisfy*

$$\eta_i \equiv 1 \quad \textit{in} \quad \Omega^i_{cat} \quad \textit{and} \quad \eta_i \equiv 0 \quad \textit{in} \quad \mathcal{A}_\varepsilon \setminus \left(\Omega^i_{cat} \cup \Omega^{i,+}_{glu} \cup \Omega^{i,-}_{glu}\right).$$

Next, consider the functions

$$h_i := \gamma^2\, \eta_i\, \left(\mathcal{L}_n\left(\psi^+_{gr} + \psi^-_{gr}\right) - f\right) \in \mathcal{E}^{0,\alpha}_{n,\nu-2}.$$

Taking the cylindrical coordinates (s, ϕ), we can extend the function $h_0(s,\phi)$ by 0 to the entire cylinder $\mathbb{R} \times S^1$. In the same manner, we can extend the function $h_m(s,\sigma)$ to the entire half-cylinder $\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. By the invariance by the action of the group $\mathfrak{S}_n$, in half-cylindrical coordinates all the functions h_m coincide and we can omit the index m in what follows.

By (2.52), we have

$$h_0 \in (\cosh s)^{-\nu}\, \mathcal{C}^{0,\alpha}(\mathbb{R} \times S^1) \quad \text{and} \quad h \in (\cosh \sigma)^{-\nu}\, \mathcal{C}^{0,\alpha}\left(\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right).$$

Moreover, there exists a constant $C > 0$ independent of ε such that

$$\|(\cosh s)^\nu\, h_0\|_{\mathcal{C}^{0,\alpha}((\mathbb{R}\times S^1)} \leq C\, \varepsilon^\nu\, \|f\|_{\mathcal{E}^{0,\alpha}_{\nu,n}},$$

$$\|(\cosh \sigma)^\nu\, h\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}\times[\frac{\pi}{2},\frac{3\pi}{2}])} \leq C\, \varepsilon^\nu\, \|f\|_{\mathcal{E}^{0,\alpha}_{\nu,n}}.$$

Using the results of the subsection 2.9.2, 2.9.3 we find functions $\breve{w}^0_{cat}$ and $\breve{w}^1_{cat}$, such that

$$\begin{cases} \left(\partial^2_\sigma + \partial^2_\theta + \frac{2}{\cosh^2\sigma}\right)\breve{w}^1_{cat} = h \quad \text{in} \quad \mathbb{R}\times\left[\frac{\pi}{2},\frac{3\pi}{2}\right], \\ \partial_\theta \breve{w}^1_{cat}\big|_{\{\frac{\pi}{2},\frac{3\pi}{2}\}} = 0, \end{cases}$$

$$\frac{1}{2}\left(\partial^2_s + \partial^2_\phi + \frac{2}{\cosh^2 s}\right)\breve{w}^0_{cat} = h_0 \quad \text{in} \quad \mathbb{R}\times S^1.$$

We can write

$$\breve{w}^0_{cat} = \breve{v}^0_{cat} + \hat{d}_0, \quad \text{and} \quad \breve{w}^1_{cat} = \breve{v}^1_{cat} + \hat{d}_1,$$

where

$$\|(\cosh s)^\nu\,\breve{v}^0_{cat}\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}\times S^1)} + |\hat{d}_0| \le C\,\|(\cosh s)^\nu h_0\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}\times S^1)},$$

$$\|(\cosh\sigma)^\nu\,\breve{v}^1_{cat}\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}\times[\frac{\pi}{2},\frac{3\pi}{2}])} + |\hat{d}_1| \le C\,\|(\cosh\sigma)^\nu h\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}\times[\frac{\pi}{2},\frac{3\pi}{2}])},$$

and

$$\breve{v}^0_{cat}(s,\phi) = \breve{v}^0_{cat}(-s,\phi) = \breve{v}^0_{cat}(s,-\phi) = \breve{v}^0_{cat}\left(s,\phi+2\pi/n\right),$$

$$\breve{v}^1_{cat}(\sigma,\theta) = \breve{v}^1_{cat}(-\sigma,\theta) = \breve{v}^1_{cat}(\sigma,2\pi-\theta).$$

Notice that since the functions $\breve{v}^0_{cat}$ and $\breve{v}^1_{cat}$ have exponential decay at infinity, we can extend them in a natural way to the entire surface $\mathcal{A}_n$ with the help of suitable cut-off functions. We put

$$v^0_{cat} := \eta_0\,\breve{v}^0_{cat} \quad \text{and} \quad v^m_{cat} := \eta_m\,\breve{v}^1_{cat},$$

where η_i are defined as in Notation 2.9.3.

Step 4: Approximate solution to the linear equation

First of all, we show that we can extend the deficiency terms $\hat{c}_0\,n + \hat{c}_1\,\chi_n$ coming from the linear analysis in the punctured disk, and the deficiency terms $\hat{d}_0$ and $\hat{d}_1$ coming from the linear analysis about the catenoid and the half-catenoids, to the entire surface $\mathcal{A}_n$ by gluing them together in the regions $\Omega^{i,\pm}_{glu}$, $i = 0,1,\ldots,n$. Consider the functions

$$u_0(s) := 1 - s\,\tanh s, \quad u_1(\sigma) := 1 - \sigma\,\tanh\sigma,$$

that can be locally considered as functions on $\mathcal{A}_n$ in some small neighborhoods including the "catenoidal neck" and the "half-catenoidal bridges" respectively. Also, let Γ^0_n and Γ^1_n be the Green's functions defined in (2.11). We introduce on $\mathcal{A}_n$ the function

$$\begin{aligned}\kappa := \eta_0\,(a_0\,u_0 + \hat{d}_0) + \sum_{m=1}^{n}\eta_m\,(a_1\,u_1 + \hat{d}_1) \\ + (1 - \sum_{i=0}^{n}\eta_i)\left(b_0\,\Gamma^0_n + b_1\,\Gamma^1_n + \hat{c}_0 + \hat{c}_1\,\chi_n\right),\end{aligned} \tag{2.53}$$

where η_i are the cut-off functions defined in Notation 2.9.3 and show that one can choose the constants a_0, a_1, b_0 and b_1 in such a way that

$$\left\|\kappa - b_0\,\Gamma_n^0 - b_1\,\Gamma_n^1 - \hat{c}_0 - \hat{c}_1\,\chi_n\right\|_{\mathcal{C}^{2,\alpha}_\nu(\Omega^{i,\pm}_{glu})} \ll \|f\|_{\mathcal{E}^{0,\alpha}_{n,\nu-2}}. \tag{2.54}$$

First, remark that, by construction, given in Notation 2.9.1, we have $\chi_n \equiv 0$ in Ω^0_{glu} and $\chi_n \equiv 1$ in Ω^m_{glu}, $m = 1, \ldots, n$. Moreover, $\Omega^{0,\pm}_{glu}$, we obtain

$$\Gamma_n^0(\tilde{\varepsilon}\,\cosh s\, e^{i\phi}) = \begin{cases} -2n + 2n\,\log\tilde{\varepsilon} - 2n\,s + \hat{\mathcal{O}}(\varepsilon), & s > 0, \\ -2n - 2n\,\log\tilde{\varepsilon} + 2n\;s + \hat{\mathcal{O}}(\varepsilon), & s < 0, \end{cases}$$

where the meaning of the symbol $\hat{\mathcal{O}}$ is explained in 2.7.1. On the other hand, we have

$$u_0(s) = \begin{cases} 1 - s + \hat{\mathcal{O}}(\varepsilon), & s > 0 \\ 1 + s + \hat{\mathcal{O}}(\varepsilon), & s < 0, \end{cases}$$

and

$$\Gamma_n^1(\tilde{\varepsilon}\cosh s\, e^{i\phi}) = -\frac{n}{2} + \hat{\mathcal{O}}(\varepsilon).$$

Comparing linear terms in the the first and the third terms in (2.53), we find the first equation on a_0, a_1, b_0, b_1:

$$a_0 = 2n\,b_0, \quad -\frac{b_1\,n}{2} - 2n\,b_0 + 2n\,b_0\,\log\tilde{\varepsilon} + \hat{c}_0 = a_0 + \hat{d}_0. \tag{2.55}$$

Similarly, in $\Omega^{m,\pm}_{glu}$, using the notations of the subsection 2.4 we obtain

$$\Gamma_n^1\left(\lambda_m\left(\frac{\varepsilon}{2}\cosh\sigma\, e^{i\theta}\right)\right) = \begin{cases} -\dfrac{n}{2} + c_{gr}(n) - \log\frac{\varepsilon}{2} - \sigma + \hat{\mathcal{O}}(\varepsilon^{2/3-\beta}), & \sigma > 0, \\ -\dfrac{n}{2} + c_{gr}(n) - \log\frac{\varepsilon}{2} + \sigma + \hat{\mathcal{O}}(\varepsilon^{2/3-\beta}), & \sigma < 0 \end{cases}$$

$$u_1(\sigma) = \begin{cases} 1 - \sigma + \hat{\mathcal{O}}(\varepsilon^{2/3}), & \sigma > 0, \\ 1 + \sigma + \hat{\mathcal{O}}(\varepsilon^{2/3}) & \sigma < 0, \end{cases}$$

and

$$\Gamma_n^{0,\pm}(s,\phi) = -n + \hat{\mathcal{O}}(\varepsilon^{2/3}).$$

This gives us the second equation:

$$a_1 = b_1, \quad b_1\left(c_{gr}(n) - \frac{n}{2} - \log\frac{\varepsilon}{2}\right) - b_0\,n + \hat{c}_0 + \hat{c}_1 = a_1 + \hat{d}_0. \tag{2.56}$$

Then the system (2.55) and (2.56) has a unique solution.

Next, let the functions ψ_{gr}, v^i_{cat} and κ be defined as above. We introduce the function

$$w_{app} := \psi^+_{gr} + \psi^-_{gr} + \sum_{i=0}^{n} v^i_{cat} + \kappa.$$

Then $w_{app} \in \mathcal{E}^{2,\alpha}_{n,\nu}$ and it follows from the previous estimates that

$$\|\mathcal{L}_n\, w_{app} - f\|_{\mathcal{E}^{0,\alpha}_{n,\nu-2}} \le c\, \varepsilon^{\frac{2\nu}{3}}\, \|f\|_{\mathcal{E}^{0,\alpha}_{n,\nu-2}},$$

for a constant c independent of ε. Moreover, there exist constants $C > 0$ and $\ell \in \mathbb{N}$, such that for all $n \ge 2$ and all $f \in \mathcal{E}^{0,\alpha}_{n,\nu-2}$

$$\|w_{app}\|_{\mathcal{E}^{2,\alpha}_{n,\nu}} \le C\, \varepsilon^{-\ell\nu}\, \|f\|_{\mathcal{E}^{0,\alpha}_{n,\nu-2}}.$$

Step 5: Exact solution to the linear equation

Consider the operator

$$\mathcal{R}_{app} \,:\, f \in \mathcal{E}^{0,\alpha}_{n,\nu-2} \longmapsto \mathcal{L}_n\, w_{app}(f) - \mathrm{Id}(f) \in \mathcal{E}^{0,\alpha}_{n,\nu-2}.$$

Then $\|\mathcal{R}_{app}\| \ll 1$ and the operator $\mathrm{Id} + \mathcal{R}_{app}$ is an invertible. Finally, we put

$$w(f) := w_{app}\left((\mathrm{Id} + \mathcal{R}_{app})^{-1}\, f \right).$$

and verify that

$$\mathcal{L}_n\, w(f) = f, \quad \text{and} \quad \|w(f)\|_{\mathcal{E}^{2,\alpha}_{n,\nu}} \le C\, \varepsilon^{-\ell\nu}\, \|f\|_{\mathcal{E}^{0,\alpha}_{n,\nu-2}}.$$

□

2.10 Nonlinear argument

Proposition 2.10.1. *There exists $n_0 \in \mathbb{N}$, such that for all $n \ge n_0$, there exist $\nu, \alpha \in (0,1)$ and a function $w_n \in \mathcal{E}^{2,\alpha}_{n,\nu}$ such that $\mathcal{A}_n(w_n)$ is a free boundary minimal surface in B^3.*

Proof. Our goal is to solve the equation

$$\mathcal{L}_n\, w = -\mathcal{H}_n - \mathcal{Q}_n(w) \quad \text{for some } w \in \mathcal{E}^{2,\alpha}_{n,\nu}.$$

By Propositions 2.8.1 and 2.8.2, 2.8.4, 2.8.3 and 2.9.4, there exist constants $C > 0$ and $\ell \in \mathbb{N}$ independent of n, such that

$$\|\mathcal{L}_n^{-1}\, \mathcal{H}_n\|_{\mathcal{E}^{2,\alpha}_{n,\nu-2}} \le C\, e^{-n(5/3 - \ell\,\nu)} := r_n,$$

$$\|\mathcal{L}_n^{-1}\, \mathcal{Q}_n(w)\|_{\mathcal{E}^{2,\alpha}_{n,\nu-2}} \le C\, e^{-n(1/3 - \ell\,\nu)}\, \|w\|_{\mathcal{E}^{2,\alpha}_{n,\nu}},$$

$$\|\mathcal{L}_n^{-1}\, \left(\mathcal{Q}_n(w_1) - \mathcal{Q}_n(w_2)\right)\|_{\mathcal{E}^{2,\alpha}_{n,\nu-2}} \le \frac{1}{2}\, \|w_1 - w_2\|_{\mathcal{E}^{2,\alpha}_{n,\nu}},$$

for $\|w\| \leq r_n$ and α, ν, β and n large enough. Theorem 2.1.1 then follows from Banach fixed point theorem for contracting mappings applied to the mapping

$$w \in \mathcal{E}^{2,\alpha}_{n,\nu} \longmapsto -\mathcal{L}_n^{-1}\left(\mathcal{H}_n + \mathcal{Q}_n(w)\right) \in \mathcal{E}^{2,\alpha}_{n,\nu}$$

in the ball of radius $2\, r_n$.

$\square$

2.11 Appendix

Let $\tilde{C}_{\tilde{\varepsilon}}$ be the surface in $\mathbb{R}^3$ parametrized as in (2.15). Then

Lemma 2.11.1. *The mean curvature of $\tilde{C}_{\tilde{\varepsilon}}(w)$ with respect to the metric $\tilde{g}_{eucl} = dz^2 + \frac{1}{4}dx_3^2$ satisfies*

$$H_{cat}(w) = \frac{1}{\tilde{\varepsilon}^2 \cosh^2 s}\left(\partial_s^2 + \partial_\phi^2 + \frac{2}{\cosh^2 s}\right)\frac{w}{2} + \frac{1}{\varepsilon^3 \cosh^4 s}\, Q^2_{cat}(w) + \frac{1}{\varepsilon^4 \cosh^4 s}\, Q^3_{cat}(w),$$

where $Q^k_{cat}(w)$ are a nonlinear functions of w and the components of the gradient and the Hessian of w calculated with respect to the metric $ds^2 + d\phi^2$. Moreover, for all $s \in \mathbb{R}$

$$\left\|Q^k_{cat}(w_1) - Q^k_{cat}(w_2)\right\|_{\mathcal{C}^{0,\alpha}([s,s+1]\times S^1)} \leq C \max_{i=1,2}\left\{\|w_i\|^{k-1}_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)}\right\} \qquad (2.57)$$

$$\times\, \|w_1 - w_2\|_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)},$$

for a constant C independent of s and ε and $\left\|\frac{w_i}{\varepsilon \cosh s}\right\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}\times S^1)} < 1$.

Proof. Let us denote by $Q^k_{cat}(w)$ any nonlinear function satisfying the property (2.57). The tangent vectors to $\tilde{C}_{\tilde{\varepsilon}}(w)$ are given by

$$T_s(w) = T_s + \partial_s w\, \tilde{N}^{cat} + w\, \partial_s \tilde{N}^{cat}, \quad T_\phi(w) = T_\phi + \partial_\phi w\, \tilde{N}^{cat} + w\, \partial_\phi \tilde{N}^{cat},$$

where $T_s = \left(\tilde{\varepsilon}\, \sinh s\, e^{i\phi}, \tilde{\varepsilon}\right)$, $T_\phi = \left(i\, \tilde{\varepsilon} \cosh s\, e^{i\phi}, 0\right)$. Let

$$g^{cat} = \tilde{\varepsilon}^2\, \cosh^2 s(ds^2 + d\phi^2), \quad h^{cat} = \tilde{\varepsilon}(-ds^2 + d\phi^2)$$

be the first and the second fundamental forms of the standard Euclidean catenoid scaled by a factor $\tilde{\varepsilon}$. The induced metric on $\tilde{C}_{\tilde{\varepsilon}}(w)$ can be written in the form

$$g^{cat}(w) = g^{cat} - \frac{w}{2}\, h^{cat} + Q^2_{cat}(w).$$

We look for a normal (with respect to $\tilde{g}_{eucl}$) vector field to $\tilde{C}_{\tilde{\varepsilon}}(w)$ in the form

$$\tilde{N}^\sharp(w) = \tilde{N}^{cat} + a_s(w)\, T_s + a_\phi(w)\, T_\phi.$$

Then the equations

$$\tilde g_{eucl}(\tilde N^\sharp(w), T_s(w)) = 0, \quad \tilde g_{eucl}(\tilde N^\sharp(w), T_\phi(w)) = 0$$

yield: $a_k(w) = -\dfrac{\partial_k w}{2\,\tilde\varepsilon^2 \cosh^2 s} + \dfrac{1}{\varepsilon^3 \cosh^4 s}\, Q^2_{cat}(w)$. We find

$$\begin{aligned}\tilde N(w) := \tilde N^\sharp(w)/\|\tilde N^\sharp(w)\|_{\tilde g_{eucl}} = \tilde N^{cat} - \frac{\partial_s w}{2\,\tilde\varepsilon^2 \cosh^2 s}\, T_s - \frac{\partial_\phi w}{2\,\tilde\varepsilon^2 \cosh^2 s}\, T_\phi, \\ + \frac{1}{\varepsilon^2 \cosh^2 s} Q^2_{cat}(w)\, \tilde N^{cat} + \frac{1}{\varepsilon^2 \cosh^3 s}\, Q^2_{cat}(w)\, T,\end{aligned}$$

where T is a unit tangent vector. Since $\tilde g_{eucl}$ is a scalar metric, the second fundamental form satisfies

$$\tilde h^{cat}(w)_{k\ell} = \tilde g_{eucl}\left(\partial_k\partial_\ell \tilde X^{cat}_{\tilde\varepsilon}, \tilde N(w)\right),$$

$$\partial_k\partial_\ell \tilde X^{cat}_{\tilde\varepsilon}(w) = \partial_k\partial_\ell X^{cat}_{\tilde\varepsilon} + \frac{1}{2}\left(\partial_k w\, \partial_\ell \tilde N^{cat} + \partial_k w\, \partial_\ell \tilde N^{cat} + \partial_k\partial_\ell w\, \tilde N^{cat} + w\, \partial_k\partial_\ell \tilde N^{cat}\right),$$

$$\tilde h^{cat}(w) = h^{cat}_{\tilde\varepsilon} + \text{Hess}\,\frac{w}{2} - \frac{w}{2\cosh^2 s}\,\text{Id} + \frac{1}{2}\tanh s \begin{pmatrix} -\partial_s w & \partial_\phi w \\ \partial_\phi w & \partial_s w \end{pmatrix} + \frac{1}{\varepsilon \cosh^2 s} Q^2_{cat}(w).$$

Finally, the result of the lemma follows by taking trace with respect to the metric $g^{cat}(w)$.

□

Now, let give the details of the proof of Proposition 2.8.3. By $\hat L$ we denote any bounded linear operator from $\mathcal C^{2,\alpha}(\mathbb R \times S^1)$ in $\mathcal C^{0,\alpha}(\mathbb R \times S^1)$.

The metric induced on $S_n(w)$ from $\mathcal X^* g_{eucl}$ can be written in the form

$$\tilde{\mathfrak g}(w) = g^{cat}(w) + \left(\tilde{\mathfrak g} - g^{cat}\right) + \varepsilon^{3-\beta} \cosh^2 s\, \hat L\, w + \text{small nonlinear terms}.$$

We look for a normal (with respect to he metric $\mathcal X^* g_{eucl}$) vector field to $S_n(w)$ in the form

$$\mathfrak N^\sharp(w) = \tilde N(w) + \tilde a_s(w)\, T_s(w) + \tilde a_\phi(w)\, T_\phi(w).$$

Then it follows from the equations

$$\mathcal X^* g_{eucl}(\mathfrak N^\sharp(w), T_s(w)) = 0, \quad \mathcal X^* g_{eucl}(\mathfrak N^\sharp(w), T_\phi(w)) = 0,$$

that $\tilde a_s(w)$ and $\tilde a_\phi(w)$ satisfy

$$\begin{pmatrix} \tilde a_s(w) \\ \tilde a_\phi(w) \end{pmatrix} = -A^2(w) \left(B^2(w) - \frac14\right) (\tilde{\mathfrak g}(w))^{-1} \begin{pmatrix} dx_3^2(\tilde N(w), T_s(w)) \\ dx_3^2(\tilde N(w), T_\phi(w)) \end{pmatrix}.$$

We obtain

$$\begin{aligned}\mathfrak N(w) &:= \mathfrak N^\sharp(w)/\|\mathfrak N^\sharp(w)\|_{\mathcal X^* g_{eucl}} \\ &= \tilde N(w) + \left(\mathfrak N(0) - \tilde N\right) + \varepsilon \cosh s\, \hat L\, w + \text{small nonlinear terms}.\end{aligned}$$

Let $\tilde{\nabla}(w)$ be the Levi-Civita connection corresponding to the metric $\mathcal{X}^* g_{eucl}$ and taken along $S_n(w)$, then we have

$$\tilde{\nabla}_{\partial_k} \partial_\ell(w) = \partial_k \partial_\ell \tilde{X}^{cat}_{\tilde{\varepsilon}}(w) + \left(\tilde{\nabla}_{\partial_k} \partial_\ell - \partial_k \partial_\ell \tilde{X}^{cat}_{\tilde{\varepsilon}}\right) + \varepsilon^2 \cosh^2 s\, \hat{L}\, w \\ + \text{small nonlinear terms.}$$

The second fundamental form $\mathfrak{h}_{k\ell}(w) = \mathcal{X}^* g_{eucl} \left(\tilde{\nabla}_{\partial_k} \partial_\ell(w), \mathfrak{N}(w)\right)$ satisfies

$$\mathfrak{h}(w) = \tilde{h}^{cat}(w) + \left(\mathfrak{h} - h^{cat}\right) + \varepsilon^{2-\beta} \cosh^2 s\, \hat{L}\, w + \text{ small nonlinear terms,}$$

and, finally, taking the trace with respect to the metric $\tilde{\mathfrak{g}}(w)$ and using the results of Proposition 2.8.1 we obtain

$$\mathcal{H}(w) = \mathcal{H}_n + H_{cat}(w) + \varepsilon^{-\beta}\, \hat{L}\, w + \text{nonlinear terms.}$$

Chapter 3

Nonconvex constant mean curvature spheres in Riemannian 3-manifolds

3.1 Introduction and the statement of the result

In Euclidean 3-space, Hopf's Theorem (1950s) asserts that round spheres are the only topological spheres whose mean curvature is constant.

In 1990, R. Ye [118] proved the existence of embedded constant mean curvature spheres in any Riemannian manifold whose scalar curvature function has nondegenerate critical points. More precisely, let (M, g) be a Riemannian manifold and let $\mathcal{S}_\varepsilon(p)$ denote the geodesic sphere of radius $\varepsilon > 0$ centered at $p \in M$. Given a nondegenerate critical point $o_{cr} \in M$ of the scalar curvature function $\mathcal{R}$ on M, there exists a neighborhood of o_{cr} which is foliated by constant mean curvature topological spheres Σ_ε for $\varepsilon \in (0, \varepsilon_*)$. Each leaf Σ_ε of this foliation is a normal geodesic graph for some function $w = \mathcal{O}(\varepsilon^3)$ over the geodesic sphere $\mathcal{S}_\varepsilon(o_\varepsilon)$ centered at a point $p_\varepsilon \in M$ that satisfies $\mathrm{dist}_g(o_\varepsilon, o_{cr}) = \mathcal{O}(\varepsilon^2)$.

In [95] F. Pacard and X. Xu generalized the result of R. Ye to the case where the scalar curvature of the ambient compact manifold is not a Morse function (which includes the case of manifolds endowed with constant scalar curvature metrics), constructing topological spheres with large constant mean curvature but loosing the foliation property.

In this chapter, we prove the existence in "generic" Riemannian 3-manifolds of families of topological spheres that have large constant mean curvature but are not convex. Such surface can be obtained by perturbing a connected sum of two tangent geodesic spheres of small radii whose centers are lined up along a geodesic that passes through a critical point o_{cr} of the scalar curvature function $\mathcal{R}$ with velocity equal to a unit eigenvector associated to a simple nonzero eigenvalue of the Hessian of $\mathcal{R}$ at o_{cr}.

More precisely, let (M, g) be a Riemannian 3-dimensional manifold. We assume that we are given $o_{cr} \in M$, a critical point of the scalar curvature function $\mathcal{R}$, for which $\mathrm{Hess}\,\mathcal{R}$ at o_{cr} has a simple nonzero eigenvalue λ. Let v_λ be a unit eigenvector of associated to λ. Take $\varepsilon > 0$ small enough and consider the union $\mathfrak{S}_\#(\varepsilon, o_{cr}, v_\lambda)$ of two geodesic spheres of radius ε

tangent at o_{cr}, with centers located symmetrically with respect to o_{cr} on the geodesic passing through o_{cr} with velocity v_λ. With these notations, our result reads:

Theorem 3.1.1. *There exists $\varepsilon_* > 0$ and, for all $\varepsilon \in (0, \varepsilon_*)$, a surface $\mathfrak{S}_\varepsilon$ of constant mean curvature equal to $2/\varepsilon$ such that the Hausdorff distance between $\mathfrak{S}_\varepsilon$ and $\mathfrak{S}_\#(\varepsilon, o_{cr}, v_\lambda)$ is bounded by a constant times ε^2. The surface $\mathfrak{S}_\varepsilon$ is embedded if $\lambda < 0$ and immersed if $\lambda > 0$.*

The existence of nonconvex topological spheres of large constant mean curvature in Riemannian manifolds has already been considered by A. Butcher and R. Mazzeo [10] under some symmetry assumption on the metric g. These authors prove the existence of families of constant mean curvature topological spheres obtained by gluing together a large number of geodesic spheres of small radius which are tangent and arranged along a geodesic segment γ passing through a nondegenerate critical point of the scalar curvature function $\mathcal{R}$, provided the metric is rotationally symmetric in a tubular neighborhood of γ (i.e. only depends on the distance to γ).

Our result is reminiscent of a result of N. Kapouleas [62], where the existence of "slowly rotating drops" or "gyrostatic equilibria" (equilibria of rotating liquid masses) is proven. In this work, the problem amounts to find embedded surfaces in $\mathbb{R}^3$ whose mean curvature is given by

$$H = 1 + c\, d^2, \tag{3.1}$$

where d is the distance to the axis of rotation (say the vertical axis). From the point of view of physics, one can consider two drops of liquid of small radii arranged symmetrically with respect to the origin along an horizontal straight line passing through the origin. One can imagine that these two drops are connected together by a small liquid bridge near the origin. If there is no rotation, the two drops will merge under the action of capillarity while, if some small rotation is imposed to the system, the centrifugal force induced by the rotation will counterbalance the capillarity force and somehow prevent the drops to merge.

In our construction, it is the gradient of the scalar curvature that plays the role of the centrifugal force in [62]. However, the fact that we are working in a setting without any symmetry induces a lot of technical complications which we will explain.

3.2 Outline of the chapter

In Section 3.3, we provide some classical results in differential geometry which we use in this work. For example, we recall the expansion of the metric in some special coordinate systems and the expression of the mean curvature of normal graphs over a given surface.

In Section 3.4, we explain the construction of a family of surfaces, which we refer to as *approximate solutions*, whose mean curvature is close to $2/\varepsilon$ (in a sense to be made precise) and which depend on 8 geometric parameters. Let us briefly describe this construction here in the case where one tries to construct embedded constant mean curvature topological spheres that are mentioned in the above Theorem.

We start with $p \in M$ and ε small enough. We will see that the mean curvature of $\mathcal{S}_\varepsilon(p)$, the geodesic sphere of radius ε centered at p, is given by

$$H(\mathcal{S}_\varepsilon(p)) = \frac{2}{\varepsilon} + \mathcal{O}(\varepsilon),$$

and hence it is, in some sense, close to being constant. It is reasonable to expect that $\mathcal{S}_\varepsilon(p)$ can be perturbed into some constant mean curvature surface, at least for ε small enough. Unfortunately, as observed by R. Ye in [118], this is not the case. Indeed, when ε is small enough, the Jacobi operator about $\mathcal{S}_\varepsilon(p)$ is close to the Jacobi operator about the Euclidean sphere of radius ε which reads $\varepsilon^{-2}\,(\Delta_{S^2} + 2)$. This operator has a non trivial 3-dimensional kernel $\mathfrak{K}_0$ which prevents one from applying directly a perturbation argument to deform $\mathcal{S}_\varepsilon(p)$ into a constant mean curvature surface. The best one can do is to perturb $\mathcal{S}_\varepsilon(p)$ into a surface $\Sigma_\varepsilon(p)$ whose mean curvature is constant up to an element of $\mathfrak{K}_0$ and, with slight abuse of notations, we can write

$$H(\Sigma_\varepsilon(p)) - \frac{2}{\varepsilon} \in \mathfrak{K}_0.$$

These surfaces are called *pseudo CMC spheres* by F. Pacard and X. Xu [95] or *pseudo bubbles* by S. Nardulli [90]. They are, in some sense, the closer we can approach a constant mean curvature surface when we start from a geodesic sphere of small radius.

We now fix a point $o \in M$, a unit vector $v \in T_oM$, $\varepsilon > 0$ small enough and a parameter $d \in (0,1)$ also very small (say $d \ll \varepsilon$). We consider a pair of "pseudo CMC spheres" $\Sigma_\varepsilon^\pm := \Sigma_\varepsilon\left(\exp_o(\pm tv)\right)$, where the parameter $t > 0$ is chosen in such a way that the distance between $\Sigma_\varepsilon^\pm$ is exactly equal to d. We let γ be the geodesic of length d which realizes the distance between the two surfaces.

In the next step, we perform a connected sum of Σ_ε^+ and Σ_ε^- using a "catenoidal neck" that looks like a Euclidean catenoid which has been scaled down by a factor $\eta \ll 1$ and whose "axis" is "parallel" to γ (we hope that the rough picture is clear and we will make these notions precise later on). Observe that we have two degrees of freedom in choosing the axis of the catenoidal neck "parallel" to γ. We will see that a certain relation needs to be satisfied between the distance d and the size of the catenoidal neck η. At this stage, for all ε small enough, the resulting surface which will be denoted by $\mathcal{A}_\varepsilon$ depends on the choice of 8 parameters: the point $o \in M$ (3 degrees of freedom), the unit vector $v \in T_oM$ (two degrees of freedom), the size of the neck η (one degree of freedom) and the location of the neck parallel to γ (two degrees of freedom).

In sections 3.5, 3.6 and 3.7, we prove that it is possible to perturb $\mathcal{A}_\varepsilon$ into a constant mean curvature surface provided ε is chosen small enough. This goes through a careful study of the Jacobi operator about $\mathcal{A}_\varepsilon$ and the identification of its "small" eigenvalues (i.e. eigenvalues which tend to 0 fast as ε tends to 0). It turns out that $\mathfrak{K}_\varepsilon$, the space of eigenfunctions of the Jacobi operator about $\mathcal{A}_\varepsilon$ associated to these small eigenvalues, is 8 dimensional, matching exactly the number of free parameters in our construction. Following N. Kapouleas, this space will be called *approximate kernel*.

In the last section, we will use a Lyapunov-Schmidt reduction argument, applying Banach fixed point theorem in the space of functions orthogonal to $\mathfrak{K}_\varepsilon$, to perturb $\mathcal{A}_\varepsilon$ into a surface whose mean curvature H satisfies

$$H - \frac{2}{\varepsilon} \in \mathfrak{K}_\varepsilon,$$

provided ε is chosen small enough. Observe that the surface we have constructed still depends on 8 parameters. In the final argument, we will explain how to choose the 8 parameters appropriately to ensure that $H = \frac{2}{\varepsilon}$. The corresponding surface $\mathfrak{S}_\varepsilon$ will be the constant mean curvature surface we are looking for and we will see that it is at Hausdorff distance at most $c\,\varepsilon^2$ from $\mathfrak{S}_\#(\varepsilon, o_{cr}, v_\lambda)$ for some constant $c > 0$.

3.3 Preliminaries

In this section, we collect some classical results in differential geometry and introduce the main notations and geometric objects that we use in the chapter. We refer to [12], [68] and [108] for further details.

Remark 3.3.1.

1. *Throughout this chapter, we will assume that the manifold M is compact. In reality, when M is not compact, given $o_{cr} \in M$ a critical point of the scalar curvature $\mathcal{R}$, we will only work in a geodesic disk in M of some bounded radius centered at o_{cr}.*

2. *By $\varepsilon_* \in (0,1)$ we will denote a constant which will vary from result to result but can be chosen uniformly for all results.*

3.3.1 Normal Geodesic Coordinates

For $p \in M$, we denote by $\exp_p$ the exponential map defined on T_pM and associated to the metric g. We fix an orthonormal frame E_1, E_2, E_3 of the tangent bundle TM and consider in a neighborhood of $p \in M$ normal geodesic coordinates centered at p. The mapping

$$\zeta_p(x) := \exp_p\left(\sum_{i=1}^{3} x^i\, E_i(p)\right),$$

where $x = (x^1, x^2, x^3) \in \mathbb{R}^3$ gives us a local diffeomorphism between a neighborhood of 0 in $\mathbb{R}^3$ and a neighborhood of p in M. In these coordinates, the metric g has the following expansion in powers of x, [108]:

$$\begin{aligned} g(\partial_{x^i}, \partial_{x^j})(x) = \delta_{ij} &+ \frac{1}{3}\, g(R_p(E_i, E_k)E_j, E_\ell)\, x^k\, x^\ell \\ &+ \frac{1}{6}\, g(\nabla_{E_m} R_p(E_i, E_k)E_j, E_l)\, x^k\, x^l\, x^m + \mathcal{O}(|x|^4), \end{aligned} \tag{3.2}$$

where R_p is the Riemann curvature tensor of (M, g) evaluated at the point p.

Notation 3.3.1. *In the following, $\mathcal{O}(|x|^k)$ denotes a smooth function whose i-th partial derivatives are bounded by a constant times $|x|^{k-i}$ in a fixed neighborhood of $p \in M$, the bounds being uniform in p.*

3.3.2 Change of coordinates

It will be necessary in some of our computations to compare coordinates of a point in normal geodesic coordinate systems centered at two different points. Assume that $o \in M$ is fixed and take $p \in M$ in a small neighborhood of o. We fix an orthonormal basis $E_1(o), E_2(o), E_3(o)$ of T_oM and denote by x the geodesic normal coordinates centered at o.

Let $\tilde{E}_1(p), \tilde{E}_2(p), \tilde{E}_3(p)$ be an orthonormal basis of T_pM obtained by the Gram-Schmidt orthogonalization process starting from the basis $\partial_{x^1}(p), \partial_{x^2}(p), \partial_{x^3}(p)$ of T_pM. Given q close enough to p, we denote by $\gamma_{pq}(s)$ the minimizing geodesic starting at time $s = 0$ at p and ending at q at time $s = 1$. We set $v := \gamma'_{pq}(0) \in T_pM$. The coordinates of $\gamma_{pq}(s)$ in the normal coordinate system centered at p are given by

$$z^k(s) = v^k s, \quad k = 1, 2, 3,$$

while, in the normal coordinate system centered at o, its coordinates can be expanded in powers of s as

$$x^k(s) = \sum_{i=0}^{\infty} a_i^k \, s^i,$$

where

$$a_0^k = x^k(p) \quad \text{and} \quad a_1^k = v^k + \mathcal{O}(|v||x(p)|^2).$$

Putting this information into the geodesics equation

$$\frac{d^2x^k}{ds^2} + \Gamma_{ij}^k(x) \, \frac{dx^i}{ds} \, \frac{dx^j}{ds} = 0, \tag{3.3}$$

evaluated at p and using the fact that $\Gamma_{ij}^k(x(p)) = \mathcal{O}(|x(p)|)$, we conclude that $a_2^k = \mathcal{O}\left(|x(p)| \, |v|^2\right)$. Finally, differentiating (3.3) ℓ times and evaluating it at p gives the expression of the coefficients $a_{2+\ell}^k$ for all ℓ. Using that $x^k(p) = z^k(o)$ and $v^k = z^k(q)$, we obtain the expansion

$$x^k(q) = z^k(q) - z^k(o) + \mathcal{O}\left(|z(o)|^2 |z(q)| + |z(o)||z(q)|^2\right).$$

3.3.3 Fermi coordinates

Let S be an oriented surface in M and N_S a unit normal vector field to S. Take $q_0 \in S$, then the mapping F_S given by

$$F_S(q, z) := \exp_q(z \, N_S(q))$$

defines a local diffeomorphism from a neighborhood of $(q_0, 0)$ in $S \times \mathbb{R}$ into a neighborhood of q_0 in M. The coordinates associated to F_S are called *Fermi coordinates*.

Given $x \mapsto \zeta_{q_0}^S(x)$ local coordinates on S, for x in a neighborhood of 0 in $\mathbb{R}^2$, we define

$$F_{S,q_0}(x, z) := F_S\left(\zeta_{q_0}^S(x), z\right). \tag{3.4}$$

Lemma 3.3.1 (Expansion of the metric in Fermi coordinates). *There exists a tubular neighborhood of the surface S in which the metric g can be written as*

$$g = \sum_{i,j=1}^{2} (g_z)_{ij}\, dx_i\, dx_j + dz^2,$$

where g_z is the induced metric on the surface $S_z = \{F_S(q, z) \in M \,:\, q \in S\}$ parallel to S and has the following expansion in powers of z:

$$g_z = g_S - 2\,z\,h_S + z^2\,k_S + \mathcal{O}(z^3),$$

where g_S and h_S denote respectively the first and the second fundamental forms of S and where k_S is defined by

$$k_S = h_S \otimes h_S + g(R(N_S, \cdot)N_S, \cdot).$$

Proof. The coordinate vector fields corresponding to F_{S,q_0} are denoted by

$$X_0 := (F_{S,q_0})_*(\partial_z) \quad \text{and} \quad X_i := (F_{S,q_0})_*(\partial_{x^i}),\ i = 1, 2.$$

The curve $z \mapsto F_{S,q_0}(x, z)$ is a geodesic and $X_0(x, 0) = N_S(x)$. This implies $g(X_0, X_0) = 1$ and $\nabla_{X_0} X_0 = 0$. Furthermore, using that X_0 and X_i are coordinate vector fields and therefore commute, we get

$$\begin{aligned}
\partial_z g(X_0, X_i) &= g(\nabla_{X_0} X_0, X_i) + g(\nabla_{X_0} X_i, X_0) \\
&= g(\nabla_{X_0} X_i, X_0) = g(\nabla_{X_i} X_0, X_0) = \frac{1}{2}\,\partial_{x^i} g(X_0, X_0) = 0.
\end{aligned}$$

Therefore, $g(X_0, X_i)(z) = g(X_0, X_i)(0) = 0$. This yields the decomposition of the ambient metric g.

Notice that, by definition, $g_z(x, 0) = g_S(x)$. Let us calculate the next terms in the expansion of g_z in powers of z. We have

$$\partial_z g(X_i, X_j)(x, 0) = g(\nabla_{X_i} X_0, X_j)(x, 0) + g(\nabla_{X_j} X_0, X_i)(x, 0) = -2\,(h_S)_{ij}(x).$$

Furthermore, using the fact that $[X_i, X_j] = 0$ we get $\nabla_{X_0} \nabla_{X_i} = R(X_0, X_i) + \nabla_{X_i} \nabla_{X_0}$ and

$$\begin{aligned}
\partial_z^2 g(X_i, X_j) &= g(\nabla_{X_0} \nabla_{X_i} X_0, X_j) + g(\nabla_{X_0} \nabla_{X_j} X_0, X_i) + 2g(\nabla_{X_i} X_0, \nabla_{X_k} X_0) \\
&= 2\,g(R(X_0, X_i)X_0, X_j) + 2\,g(\nabla_{X_i} X_0, \nabla_{X_j} X_0).
\end{aligned}$$

This yields $\partial_z^2 g(x, 0) = 2\,g(R(N_S, \cdot)N_S, \cdot)(x, 0) + 2\,g(\nabla_\cdot N_S, \nabla_\cdot N_S)(x, 0) = k_S(x)$ and finishes the proof.

$\square$

3.3.4 Mean curvature of normal graphs

We keep the notations of the previous subsection and we recall here the proof proposed in [93] (see also [94]) of a formula for the mean curvature of normal graphs about a given surface in a Riemannian manifold.

Proposition 3.3.1. *Assume that we are given a function $u \in \mathcal{C}^2(S)$ which is small enough and has compact support. Then the mean curvature of the $S(u)$, the normal graph of u over S, namely*

$$S(u) := \left\{ \exp_q(u(q)\, N_S(q)) \; : \; q \in S \right\},$$

is given by

$$H_S(u) = \mathrm{div}_{g_u} \left(\frac{\nabla^{g_u} u}{\sqrt{1 + |\nabla^{g_u} u|^2_{g_u}}} \right) - \frac{1}{2} \sqrt{1 + |\nabla^{g_u} u|^2_{g_u}}\, \mathrm{Tr}_{g_u} \dot{g}_u + \frac{1}{2} \frac{\dot{g}_u\, (\nabla^{g_u} u, \nabla^{g_u} u)}{\sqrt{1 + |\nabla^{g_u} u|^2_{g_u}}}, \qquad (3.5)$$

where $\dot{g}_z := \partial_z g_z$.

Proof. The induced metric $g_S(u)$ on $S(u)$ reads

$$g_S(u) = g_u + du \otimes du,$$

and hence we get

$$\det g_S(u) = \left(1 + |\nabla^{g_u} u|^2\right) \, \det g_u,$$

and the volume of $S(u)$ is given by

$$\mathrm{Vol}(S(u)) = \int_S \sqrt{1 + |\nabla^{g_u} u|^2}\, dvol_{g_u}.$$

Computing the differential of this functional with respect to u, we obtain

$$\begin{aligned} D_u \, \mathrm{Vol}(S(u))|_u \, v = & \int_S \frac{1}{\sqrt{1 + |\nabla^{g_u} u|^2_{g_u}}}\, g_u(\nabla^{g_u} u, \nabla^{g_u} v)\, dvol_{g_u} \\ & - \frac{1}{2} \int_S \frac{1}{\sqrt{1 + |\nabla^{g_u} u|^2_{g_u}}}\, \dot{g}_u \left(\nabla^{g_u} u, \nabla^{g_u} u\right) \, v\, dvol_{g_u} \\ & + \frac{1}{2} \int_S \sqrt{1 + |\nabla^{g_u} u|^2_{g_u}}\, \mathrm{Tr}_{g_u} \dot{g}_u \, v\, dvol_{g_u}. \end{aligned}$$

Integrating by parts the first term gives

$$\begin{aligned} D_u \, \mathrm{Vol}(S(u))|_u \, v = & - \int_S \mathrm{div}_{g_u} \left(\frac{\nabla^{g_u} u}{\sqrt{1 + |\nabla^{g_u} u|^2_{g_u}}} \right) \, v\, dvol_{g_u} \\ & - \frac{1}{2} \int_S \frac{1}{\sqrt{1 + |\nabla^{g_u} u|^2_{g_u}}}\, \dot{g}_u \left(\nabla^{g_u} u, \nabla^{g_u} u\right) \, v\, dvol_{g_u} \\ & + \frac{1}{2} \int_S \sqrt{1 + |\nabla^{g_u} u|^2_{g_u}}\, \mathrm{Tr}_{g_u} \dot{g}_u \, v\, dvol_{g_u}. \end{aligned}$$

Next, observe that the unit normal to $S(u)$ can be written in the form

$$N_S(u) = \frac{1}{\sqrt{1 + |\nabla^{g_u} u|^2_{g_u}}} (X_0 - \nabla^{g_u} u),$$

and hence

$$dvol_{g_u} = g(N_S(u), X_0)\, dvol_{g_S}.$$

The result than follows from the first variation formula, when S is deformed using the vector field $v\, X_0$

$$D_u \mathrm{Vol}(S(u))|_u \; v = - \int_S H_S(u)\, g(N_S(u), X_0)\, v\, dvol_{g_S}.$$

□

As a consequence, we get the:

Corollary 3.3.1. *The expression* (3.5) *can be expanded in powers of* u *and the derivatives of* u *up to the second order as*

$$H_S(u) = H_S + J_S\, u + Q_S(u, \nabla u, \nabla^2 u), \tag{3.6}$$

where H_S *is the mean curvature of* S, J_S *is the Jacobi operator about* S *given explicitly by*

$$J_S := \Delta_S + \mathrm{Tr}_{g_S}(h_S \otimes h_S) + \mathrm{Ric}(N_S, N_S), \tag{3.7}$$

where Δ_S *is the Laplace-Beltrami operator on* S *and* Ric *is the Ricci tensor of* (M, g)*. Finally,* Q_S *is a smooth function of* u, ∇u *and* $\nabla^2 u$*, which satisfies*

$$Q_S(0,0,0) = 0, \quad DQ_S(0,0,0) = 0.$$

Observe that the Taylor expansion of Q_S *is affine in* $\nabla^2 u$ *and at least quadratic in* ∇u.

Proof. The result follows directly from a careful examination of the terms in (3.5). The expression for the Jacobi operator can be obtained from

$$D_u H_S(u)|_{u=0} = \Delta_{g_u} + \frac{1}{2} \left(\mathrm{Tr}_{g_u}(\dot g_u \otimes \dot g_u) - \mathrm{Tr}_{g_u} \ddot g_u \right),$$

and the expansion $g_z = g_S - 2z\, h_S + z^2\, k_S + \mathcal{O}(z^3)$ given by Lemma 2.1.

□

3.3.5 Mean curvature of transverse graphs

Assume that we are given $\tilde N_S$, a smooth vector field transverse (but not necessarily normal nor unitary) to the surface S. Given a function $u \in \mathcal{C}^2(S)$ which is small enough and which has compact support, we define, as above, $S(u)$ to be the normal graph over S for the function u and we also consider the surface $\tilde S(u)$ parametrized by

$$p \mapsto \exp_p \left(u(p)\, \tilde N_S(p) \right),$$

for $p \in S$. The following result [77], [93] gives the relation between the mean curvature function $\tilde H_S(u)$ of $\tilde S(u)$ and the mean curvature function $H_S(u)$ of $S(u)$.

Proposition 3.3.2. *The mean curvature of $\tilde{S}(u)$ can be written in the form*

$$\tilde{H}_S(u) = H_S + J_S\left(g(N_S, \tilde{N}_S)\, u\right) + g(\nabla H_S, \tilde{N}_S)\, u + \tilde{Q}_S(u),$$

where J_S is the Jacobi operator about the surface S and $\tilde{Q}_S$ is a nonlinear operator which enjoys the same properties as the operator Q_S described in Corollary 3.3.1.

Proof. For $s \in \mathbb{R}$ close to 0 and $q \in S$ one can apply the implicit function theorem to

$$\exp_p(t\, N_S(p)) = \exp_q(s\, \tilde{N}_S(q)),$$

to express $p \in S$ and $t \in \mathbb{R}$ as functions of q and s:

$$p = \Psi(q, s) \quad \text{and} \quad t = \psi(q, s),$$

with $\Psi(q, 0) = q$ and $\psi(q, 0) = 0$. Using the fact that $(\exp)_*|_{v=0} = \mathrm{Id}$, one checks that

$$\frac{\partial \Psi}{\partial s}(\cdot, 0) = \tilde{N}_S^T, \quad \text{and} \quad \frac{\partial \psi}{\partial s}(\cdot, 0) = g(\tilde{N}_S, N_S),$$

where $\tilde{N}_S^T$ is the projection of $\tilde{N}_S$ on the tangent bundle of S. On the other hand, differentiating the identity

$$H_S(\psi(\cdot, u))(\Psi(\cdot, u)) = \tilde{H}_S(u)(\cdot)$$

with respect to u at $u = 0$ yields

$$DH_S(u)|_{u=0}\left(\frac{\partial \psi}{\partial s}(\cdot, 0)\, v\right) + g\left(\nabla H_S, \frac{\partial \Psi}{\partial s}(\cdot, 0)\right) v = D\tilde{H}_S(u)\Big|_{u=0}(v),$$

which completes half of the result. The fact that the structure of nonlinear terms is preserved follows from the fact that Ψ and ψ are local diffeomorphisms with $\psi(\cdot, 0) = 0$.

□

3.4 Construction of the approximate solution

3.4.1 Blowing up the metric

Throughout the chapter it will be easier for us to work with a rescaled metric on M. To this aim, given $\varepsilon \in (0, 1)$, we define

$$g_\varepsilon := \frac{1}{\varepsilon^2}\, g. \tag{3.8}$$

Notation 3.4.1. *The symbol $\mathcal{O}(\varepsilon^k)$ will denote a smooth function whose derivatives are bounded by a constant (depending on the number of derivatives) times ε^k in a fixed neighborhood of a given point $p \in M$, the bounds being uniform in p.*

In normal coordinates (x^1, x^2, x^3) (associated to the metric g_ε), we have the expansion

$$(g_\varepsilon)_{ij}(x) := \delta_{ij} + \frac{\varepsilon^2}{3}\, g(R_p(E_i, E_k)E_j, E_\ell)\, x^k\, x^\ell \tag{3.9}$$
$$+ \frac{\varepsilon^3}{6}\, g(\nabla_{E_m} R_p(E_i, E_k)E_j, E_l)\, x^k\, x^l\, x^m + \mathcal{O}(\varepsilon^4),$$

where E_1, E_2, E_3 is an orthonormal (with respect to g) frame of TM. Now let $E_i^\varepsilon := \varepsilon\, E_i$ be a frame orthonormal with respect to g_ε. We introduce the mapping

$$\zeta_{\varepsilon,p}(x) := \exp_p\left(\sum_{i=1}^{3} x^i\, E_i^\varepsilon\right), \tag{3.10}$$

where $x \in \mathbb{R}^3$, $|x| \leq 1$. Then there exists $\varepsilon_* \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_*)$, $\zeta_{\varepsilon,p}$ is a diffeomorphism and the mapping $(\varepsilon, p) \mapsto \zeta_{\varepsilon,p}$ depends smoothly on $\varepsilon \in (0, \varepsilon_*)$ and $p \in M$.

With these definitions, the geodesic sphere $\mathcal{S}_\varepsilon(p)$ in (M, g) of radius ε centered at p can be seen as the image by $\zeta_{\varepsilon,p}$ of the Euclidean unit sphere S^2.

Jacobi operator in the blown up metric

Again, assume that $S \subset M$ is an orientable surface and let N_S and $N_S^\varepsilon = \varepsilon\, N_S$ be unit normal (with respect to the metrics g and g_ε respectively) vector fields on S. Given a function $u \in \mathcal{C}^{2,\alpha}(S)$ small enough, we define the surfaces $S(u)$ and $S_\varepsilon(u)$ by

$$p \in S \mapsto \exp_p(u(p)\, N_S(p)) \in S(u) \quad \text{and} \quad p \in S \mapsto \exp_p\left(u(p)\, N_S^\varepsilon(p)\right) \in S_\varepsilon(u).$$

Let J^g and $J_S^{g_\varepsilon}$ denote the Jacobi operators about S calculated with respect to the metrics g and g_ε respectively. Then we have

$$\varepsilon\, H^g\left(S(u)\right) = H^{g_\varepsilon}\left(S(u)\right) = H^{g_\varepsilon}\left(S_\varepsilon\left(\frac{u}{\varepsilon}\right)\right), \tag{3.11}$$

which yields

$$J_S^{g_\varepsilon}\, u = \varepsilon^2\, J_S^g\, u.$$

3.4.2 Pseudo CMC spheres

Jacobi operator about the Euclidean sphere

The Jacobi operator about the Euclidean unit sphere S^2 reads

$$J_{S^2} = \Delta_{S^2} + 2.$$

This operator has a nontrivial kernel spanned by the restrictions to S^2 of the coordinate functions:

$$\mathrm{Ker}(J_{S^2}) = \mathrm{span}\{\Theta^1, \Theta^2, \Theta^3\},$$

where $\Theta \in \mathbb{R}^3$, $\sum_{i=1}^3 \Theta^i = 1$. Let Π and $\Pi^\perp$ denote the L^2 orthogonal projections onto $\mathrm{Ker}(J_{S^2})$ and the orthogonal complement of $\mathrm{Ker}(J_{S^2})$ respectively. Then the operator

$$\Delta_{S^2} + 2 \,:\, \Pi^\perp\left(\mathcal{C}^{2,\alpha}(S^2)\right) \longrightarrow \Pi^\perp\left(\mathcal{C}^{0,\alpha}(S^2)\right),$$

is invertible. So, for every function $f \in \mathcal{C}^{0,\alpha}(S^2)$ we can find unique function $u \in \Pi^\perp(\mathcal{C}^{2,\alpha}(S^2))$ and vector $A \in \mathbb{R}^3$ such that

$$(\Delta_{S^2} + 2)\, u = f - \langle A, \Theta\rangle,$$

where

$$A = \int_{S^2} f\, \Theta \, dvol_{S^2}$$

and where by $\langle \cdot, \cdot \rangle$ we denote the scalar product in $\mathbb{R}^3$.

Construction of pseudo CMC spheres

A key ingredient in our construction is the following result which is already available in the works of S. Nardulli [90] and F. Pacard and X. Xu [95].

Let $\mathcal{S}_\varepsilon(o)$ be the geodesic sphere in (M, g) of radius ε small enough centered at $o \in M$. We identify the metric $g_\varepsilon = \frac{1}{\varepsilon^2} g$ with the pull-back metric $(\zeta_{\varepsilon,o})^* g_\varepsilon = g_{eucl} + \mathcal{O}(\varepsilon^2)$ in $\mathbb{R}^3$. Then the mean curvature of $\mathcal{S}_\varepsilon(o)$ satisfies

$$H^{g_\varepsilon}(\mathcal{S}_\varepsilon(o)) = H^{g_\varepsilon}(S^2) = 2 + \mathcal{O}(\varepsilon^2).$$

We would like to perturb $\mathcal{S}_\varepsilon(o)$ into a constant mean curvature surface with mean curvature equal to 2. To this end, we take a function $u \in \mathcal{C}^{2,\alpha}(S^2)$ and let $S^2(u)$ be the normal (with respect to the Euclidean metric) graph over S^2:

$$S^2(u) := \{(1-u)\,\Theta,\ \Theta \in S^2\}.$$

We consider the function $K(\varepsilon, u) := H^{g_\varepsilon}(S(u)) - 2$. Obviously, we have

$$K(0,0) = 0 \quad \text{and} \quad D_u K(0,0) = \Delta_{S^2} + 2.$$

We would like to apply the Implicit Function Theorem to the equation $K(\varepsilon, u) = 0$, unfortunately, as we have seen, the operator $J_{S^2} = \Delta_{S^2} + 2$ has a nontrivial kernel $\mathrm{Ker}(J_{S^2})$ which prevents us from doing so. However, one can certainly invert the operator J_{S^2} in the space of functions orthogonal to $\mathrm{Ker}(J_{S^2})$. This implies that it is possible to apply the implicit function theorem to solve

$$H^{g_\varepsilon}(S^2(u)) - 2 \in \mathrm{Ker}(J_{S^2}),$$

for all $\varepsilon > 0$ close enough to 0.

Proposition 3.4.1. *There exists $\varepsilon_* > 0$ such that for all $\varepsilon \in (0, \varepsilon_*)$ and all $o \in M$ there exists a surface $\Sigma_\varepsilon(o)$ parametrized by*

$$\Theta \in S^2 \mapsto \zeta_{\varepsilon,o}\big((1 + u_{\varepsilon,o})\,\Theta\big) \in \Sigma_\varepsilon(o), \tag{3.12}$$

with $u_{\varepsilon,o} \in \mathcal{C}^{2,\alpha}(S^2)$, $\|u_{\varepsilon,o}\|_{\mathcal{C}^{2,\alpha}(S^2)} \leq C\,\varepsilon^2$, *and such that the mean curvature of* $\Sigma_\varepsilon(o)$ *satisfies*

$$\varepsilon\, H^g(\Sigma_\varepsilon(o)) = H^{g_\varepsilon}(\Sigma_\varepsilon(o)) = 2 + \langle A_{\varepsilon,o}, \Theta\rangle, \tag{3.13}$$

where $A_{\varepsilon,o} \in \mathbb{R}^3$ *and*

$$\left| A_{\varepsilon,o} + \frac{2\pi\,\varepsilon^3}{15}\,\nabla^g \mathcal{R}(o)\right| \leq C\,\varepsilon^5,$$

for a constant C *independent of* $\varepsilon \in (0,\varepsilon_*)$ *and* $o \in M$. *Moreover,* $u_{\varepsilon,o}$ *and* $A_{\varepsilon,o}$ *depend smoothly on* $\varepsilon \in (0,\varepsilon_*)$ *and* $o \in M$.

Proof. First, let x^1, x^2, x^3 be geodesic normal coordinates at o associated to the metric g_ε. We have the expansion:

$$(g_\varepsilon)_{ij} = \delta_{ij} + \frac{\varepsilon^2}{3} R_{ikjl}\, x^k\, x^l + \frac{\varepsilon^3}{6} R_{ikjl,m} x^k\, x^l\, x^m + \mathcal{O}(\varepsilon^4),$$

where R_{ikjl} and $R_{ikjl,m}$ are the components of the Riemann curvature tensor R and the tensor ∇R in an orthonormal (with respect to g) basis of T_oM. Then we calculate

$$H^{g_\varepsilon}(S^2) = 2 - \frac{\varepsilon^2}{3}\,\mathrm{Ric}_{ij}\,\Theta^i\Theta^j - \frac{\varepsilon^3}{4}\,\mathrm{Ric}_{ij,k}\,\Theta^i\Theta^j\Theta^k + \mathcal{O}(\varepsilon^4), \tag{3.14}$$

where Ric_{ij} and $\mathrm{Ric}_{ij,k}$ are the components of the Ricci tensor Ric and the tensor ∇Ric in an orthonormal (with respect to g) basis of T_oM. We postpone the proof of this claim to the Appendix.

Next, take a function $u \in \mathcal{C}^{2,\alpha}(S^2)$ and consider the surface $S^2(u)$ which is a normal (with respect to the Euclidean metric) graph about S^2. Remark that this is equivalent to take a normal geodesic graph about $\mathcal{S}_\varepsilon(o)$ since by Gauss lemma $(\exp_o)_*(\Theta)$ is a normal vector field to $\mathcal{S}_\varepsilon(o)$. Take $A \in \mathbb{R}^3$ and consider the function

$$K(\varepsilon, u, A) = H^{g_\varepsilon}(S^2(u)) - 2 - \langle A, \Theta\rangle,$$

defined for $\varepsilon \in [0,1)$. Then K is linear in A and we have

$$K(0,0,0) = 0 \quad \text{and} \quad D_uK|_{(0,0,0)} = \Delta_{S^2} + 2.$$

By the Implicit Function theorem, for ε small enough there exists a function $u_{\varepsilon,o} \in \Pi^\perp\left(\mathcal{C}^{2,\alpha}(S^2)\right)$ and a vector $A_{\varepsilon,o} \in \mathbb{R}^3$ such that

$$K(\varepsilon, u_{\varepsilon,o}, A_{\varepsilon,o}) = H^{g_\varepsilon}(S(u_{\varepsilon,o})) - 2 - \langle A_{\varepsilon,o}, \Theta\rangle = 0.$$

A straightforward calculation gives $u_{\varepsilon,o} = \mathcal{O}(\varepsilon^2)$. Moreover, we have

$$A_{\varepsilon,o} = \int_{S^2} \left(H^{g_\varepsilon}\left(S^2(u_{\varepsilon,o})\right) - 2\right)\Theta\, dvol_{S^2}.$$

Using Bianchi identity (see Appendix for the proof), we find

$$\Pi\left(\frac{\varepsilon^3}{4}\,\mathrm{Ric}_{ij,k}\,\Theta^i\Theta^j\Theta^k\right) = \frac{2\pi\,\varepsilon^3}{15}\,\nabla^g\mathcal{R}(o). \tag{3.15}$$

This, together with the identities

$$\Pi\left(\Theta^i\,\Theta^j\right) = \Pi\left(\Theta^i\Theta^j\Theta^k\Theta^l\right) = 0, \quad \Pi^{\perp}\left(\Theta^i\Theta^j\Theta^k\right) = 0 \quad i,j,k,l = 1,2,3,$$

$$\Pi\left((\Delta_{S^2}+2)u_{\varepsilon,o}\right) = 0,$$

yields

$$\left| A_{\varepsilon,o} + \frac{2\pi\,\varepsilon^3}{15}\,\nabla^g\mathcal{R}(o)\right| \leq C\,\varepsilon^5,$$

for a constant C independent of ε and o. Finally, the surface $\Sigma_\varepsilon(o) := \zeta_{\varepsilon,o}(S^2(u_{\varepsilon,o}))$ satisfies the claim in the proposition.

□

Jacobi operator about pseudo CMC spheres

In what follows, we will often omit the index g_ε, when it is clear that the computations are done with respect to this metric. Since geodesic spheres $\mathcal{S}_\varepsilon(o)$ and pseudo CMC spheres $\Sigma_\varepsilon(o)$ constructed in the previous paragraph are parametrized by the Euclidean unit sphere S^2, we identify from now on the function spaces $\mathcal{C}^{k,\alpha}(\mathcal{S}_\varepsilon(o))$ and $\mathcal{C}^{k,\alpha}(\Sigma_\varepsilon(o))$ with $\mathcal{C}^{k,\alpha}(S^2)$.

According to the proof of Proposition 3.4.1, using a perturbation argument together with 3.4.2, we express the Jacobi operator (calculated with respect to the metric g_ε) about $\mathcal{S}_\varepsilon(o)$ in the form:

$$J_{\mathcal{S}_\varepsilon(o)} = \Delta_{S^2} + 2 + \varepsilon^2\,L_{\varepsilon,o},$$

where by $L_{\varepsilon,o}$ we will denote any linear operator on S^2 whose coefficients depend smoothly on ε and o and that satisfies the property

$$\|L_{\varepsilon,o}\,u\|_{\mathcal{C}^{0,\alpha}(S^2)} \leq C\,\|u\|_{\mathcal{C}^{2,\alpha}(S^2)}. \tag{3.16}$$

We show that an analogous result also holds for the Jacobi operator about $\Sigma_\varepsilon(o)$.

Notation 3.4.2. *We denote by $J_{\varepsilon,o}$ the Jacobi operator (calculated with respect to the metric g_ε) about $\Sigma_\varepsilon(o)$.*

Notice that the geodesics in M issued from o can be extended in a unique manner until their intersection with $\Sigma_\varepsilon(o)$. The unit (with respect to g_ε) tangent vectors to these geodesics form a $\mathcal{C}^{2,\alpha}$ vector field on $\Sigma_\varepsilon(o)$ which we denote by $\Upsilon_{\varepsilon,o}$. Remark that the mapping $(\varepsilon,o)\mapsto\Upsilon_{\varepsilon,o}$ is smooth in $\varepsilon\in(0,\varepsilon_*)$ and $o\in M$. Let $\hat{J}_{\varepsilon,o}$ be the linearized mean curvature operator which arises when $\Sigma_\varepsilon(o)$ is perturbed in the direction $\Upsilon_{\varepsilon,o}$. Clearly,

$$\hat{J}_{\varepsilon,o} = \Delta_{S^2} + 2 + \varepsilon^2\,L_{\varepsilon,o}.$$

On the other hand, let $N_{\varepsilon,o}$ be a unit normal (with respect to g_ε) to $\Sigma_\varepsilon(o)$, then

$$\left|g_\varepsilon\left(\Upsilon_{\varepsilon,o}, N_{\varepsilon,o}\right) - 1\right| \leq C\,\varepsilon^2$$

for a constant $C>0$ independent of ε and o. Thus, by Proposition 3.3.2, we obtain

$$J_{\varepsilon,o} = \Delta_{S^2} + 2 + \varepsilon^2 L_{\varepsilon,o}.$$

Lemma 3.4.1. *For all $o \in M$, $\varepsilon \in (0, \varepsilon_*)$ and all $f \in \mathcal{C}^{0,\alpha}(S^2)$ there exist unique function $u \in \Pi^\perp\left(\mathcal{C}^{2,\alpha}(S^2)\right)$ and vector $A \in \mathbb{R}^3$ such that*

$$J_{\varepsilon,o}\, u = f - \langle A, \Theta\rangle.$$

Moreover,

$$\|u\|_{\mathcal{C}^{2,\alpha}(S^2)} \leq C\, \|f\|_{\mathcal{C}^{0,\alpha}(S^2)},$$

$$\left| A - \int_{S^2} \Theta\, f\, dvol_{S^2} \right| \leq C\, \varepsilon^2\, \|f\|_{\mathcal{C}^{0,\alpha}(S^2)},$$

where the constant $C > 0$ is independent of $\varepsilon \in (0, \varepsilon_*)$ and $o \in M$, and the mapping $f \mapsto (u, A)$ depends smoothly on ε and o.

Proof. According to the results stated in the subsection 3.4.2, there exist $u_{app} \in \Pi^\perp\left(\mathcal{C}^{2,\alpha}(S^2)\right)$ and $A_{app} \in \mathbb{R}^3$ such that

$$(\Delta_{S^2} + 2)\, u_{app} = f - \langle A_{app}, \Theta\rangle.$$

Consider the mapping

$$\mathfrak{R} \,:\, f \in \mathcal{C}^{0,\alpha}(S^2) \;\mapsto\; J_{\varepsilon,o}\, u_{app}(f) + \langle A_{app}(f), \Theta\rangle - f \in \mathcal{C}^{0,\alpha}(S^2).$$

Then $\|\mathfrak{R}\| \leq c\, \varepsilon^2$ for a constant c independent of ε. Thus, for ε small enough, there exists an inverse operator $(\mathrm{Id} + \mathfrak{R})^{-1}$ and we put

$$u(f) := u_{app}\left((\mathrm{Id} + \mathfrak{R})^{-1}\, f\right) \quad \text{and} \quad A(f) := A_{app}\left((\mathrm{Id} + \mathfrak{R})^{-1}\, f\right),$$

which satisfy

$$J_{\varepsilon,o}\, u(f) = f - \langle A(f), \Theta\rangle.$$

□

3.4.3 Green's function for the Jacobi operator about a pseudo CMC sphere

Given $o \in M$ and $\varepsilon \in (0, \varepsilon_*)$, let $\Sigma_\varepsilon(o)$ be the pseudo CMC sphere defined in the subsection 3.4.2, and take $p \in \Sigma_\varepsilon(o)$. We would like to define and study the Green's function $\Gamma_{\varepsilon,o,p}$ associated to the operator $J_{\varepsilon,o}$ with a pole at p. In principle, we should be looking for the solution of the problem

$$J_{\varepsilon,o}\, \Gamma_{\varepsilon,o,p} = -2\, \pi\, \delta_p, \tag{3.17}$$

where δ_p is the Dirac mass on $\Sigma_\varepsilon(o)$ supported at p. Unfortunately, the presence of small eigenvalues (and potentially the presence of a nontrivial kernel) for the operator $J_{\varepsilon,o}$ prevents us from finding directly $\Gamma_{\varepsilon,o,p}$ and getting reasonable estimates which would be uniform in ε as this parameter tends to 0. Therefore, in view of the properties of the operator $J_{\varepsilon,o}$ that are described in the previous paragraph, instead of (3.17), we can consider the problem

$$J_{\varepsilon,o}\, \Gamma_{\varepsilon,o,p} = -2\, \pi\, \delta_p + \langle B, \Theta\rangle, \quad \text{for some} \quad B \in \mathbb{R}^3.$$

Let $g_{\varepsilon,o}$ and $d_{\varepsilon,o}$ be the metric and the intrinsic geodesic distance induced on $\Sigma_\varepsilon(o)$ by the metric g_ε. We have the following result:

Proposition 3.4.2. *There exist a function $\Gamma_{\varepsilon,o,p}$ defined on $\Sigma_\varepsilon(o)\backslash\{p\}$ and a vector $B_{\varepsilon,o,p} \in \mathbb{R}^3$ which satisfy*

$$J_{\varepsilon,o}\,\Gamma_{\varepsilon,o,p} = -2\pi\,\delta_p + \langle B_{\varepsilon,o,p}, \Theta\rangle. \tag{3.18}$$

Moreover,

$$B_{\varepsilon,o,p} = \frac{1}{2}\,\Theta(p) + \mathcal{O}(\varepsilon^2),$$

and for $\ell = 0,1,2$ there exists a constant $C_\ell > 0$ such that

$$\left|(d_{\varepsilon,o}(p,\cdot))^\ell\;\nabla^\ell\left(\Gamma_{\varepsilon,o,p} + \log d_{\varepsilon,o}(p,\cdot) - \gamma_{\varepsilon,o,p}\right)\right|_{g_{\varepsilon,o}} \leq C_\ell\;(d_{\varepsilon,o}(p,\cdot))^2\;|\log d_{\varepsilon,o}(p,\cdot)|\,,$$

(the estimate being uniform in ε, o and p) where $\gamma_{\varepsilon,o,p}$ is a function which is affine in normal geodesic coordinates in $\Sigma_\varepsilon(o)$ defined in the neighborhood of p and the mapping $(\varepsilon,o,p)\mapsto(\Gamma_{\varepsilon,o,p},\gamma_{\varepsilon,o,p},B_{\varepsilon,o,p})$ is smooth in ε, o and p.

Proof. First of all, notice that according to Corollary 3.3.1, the Jacobi operator about $\Sigma_\varepsilon(o)$ satisfies

$$J_{\varepsilon,o} = \Delta_{\Sigma_\varepsilon(o)} + \mathrm{tr}_{g_{\varepsilon,o}}\,(h_{\varepsilon,o}\otimes h_{\varepsilon,o}) + \mathrm{Ric}_g(N_{\varepsilon,o},N_{\varepsilon,o}),$$

where $h_{\varepsilon,o}$ is the second fundamental form on $\Sigma_\varepsilon(o)$ and $N_{\varepsilon,o}$ is a unit normal to $\Sigma_{\varepsilon,o}$ calculated with respect to the metric g_ε. So, the potential in this expression is a $\mathcal{C}^{2,\alpha}$ function on $\Sigma_\varepsilon(o)$, bounded by a constant independent of ε.

Secondly, we work in geodesic normal coordinates on $\Sigma_\varepsilon(o)$ centered at the point p. In these coordinates, we have

$$\Delta_{\Sigma_\varepsilon(o)} = \sum_{i=1}^{2}\partial^2_{x^i} + \sum_{i,j=1}^{2}\mathcal{O}(|x|^2)\,\partial_{x^i}\,\partial_{x^j} + \sum_{i=1}^{2}\mathcal{O}(|x|)\partial_{x^i},$$

where the functions $\mathcal{O}(|x|^k)$ are defined as in Notation 3.3.1. Remark that

$$J_{\varepsilon,o}\left(-\log d_{\varepsilon,o}(p,\cdot)\right) = -2\pi\,\delta_p + F,$$

where $F = F_0 + F_1\,\log d_{\varepsilon,o}(p,\cdot)$ with $F_1 \in \mathcal{C}^{2,\alpha}(\Sigma_\varepsilon(o))$ and $\left|\nabla^k F_0\right| \leq (d_{\varepsilon,o}(p,\cdot))^{-k}$, $k = 0,1,2$.

Let $\mu_{\varepsilon,o,p} \in \mathcal{C}^{2,\alpha}(\Sigma_\varepsilon(o))$ be a cut-off function that is identically equal to 1 in the exterior of the geodesic disk in $\Sigma_\varepsilon(o)$ of radius $\frac{1}{2}$ centered at p and to 0 in the geodesic disk of radius $\frac{1}{4}$ centered at p. Now put

$$f(x) := (1-\mu_{\varepsilon,o,p}(x))\,F(x). \tag{3.19}$$

For $r_0 > 0$ we use the notations $D^2_*(r_0) = \left\{x\in\mathbb{R}^2 \,:\, 0<|x|<r_0\right\}$ and $S^1(r_0) = \partial D^2_*(r_0)$. Finally, let $r\,e^{i\phi}$ be complex polar coordinates in $D^2_*(r_0)$. We prove the following lemma:

Lemma 3.4.2. *Let f be the function defined in* (3.19). *Then there exist unique function $\hat{v}$ such that $|\nabla^\ell\hat{v}| \leq c\,r^{2-\ell}$ and constants $\tilde{c}_0$, c_i, $i = 0,1,2$ such that*

$$v = \hat{v} + c_0 + \tilde{c}_0\,r^2\,\log r + c_1\,r\,e^{i\phi} + c_2\,r^2\,\log r\,e^{2i\phi}$$

satisfies

$$\begin{cases} \Delta_{eucl}\,v = f & in \quad D^2_*\left(\frac{1}{2}\right), \\ v = 0 & on \quad S^1\left(\frac{1}{2}\right). \end{cases} \tag{3.20}$$

Proof of Lemma 3.1. We decompose the functions v and f in Fourier series:

$$v = \sum_{j\in\mathbb{Z}} v_j(r)\, e^{i\phi j}, \quad f = \sum_{j\in\mathbb{Z}} f_j(r)\, e^{i\phi j}.$$

For all $0 < \rho < \frac{1}{2}$ let $v_{j,\rho}$ be the solution of the ordinary differential equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{j^2}{r^2}\right) v_{j,\rho}(r) = f_j(r), \quad v_{j,\rho}(\rho) = v_{j,\rho}\left(\frac{1}{2}\right) = 0.$$

For $|j| > 2$ we find $v_{j,\rho}$ using $\frac{r^2}{j^2-4}\,|f_j|$ as a supersolution. Putting

$$v_\rho := \sum_{|j|\geq 3} v_{j,\rho}(r) e^{i\phi j} \quad \text{and} \quad A_\rho := \left\{x \in \mathbb{R}^2 \,:\, \rho < |x| < \frac{1}{2}\right\},$$

we find

$$\|\, r^{-2}\, v_\rho \,\|_{L^\infty(A_\rho)} \leq C \left\| \sum_{|j|\geq 3} f_j(r)\, e^{i\phi j} \right\|_{L^\infty\left(D^2_*(\frac{1}{2})\right)} \tag{3.21}$$

for a constant C independent of ρ. By elliptic regularity theory, (3.21) implies the existence of a uniform bound on the gradient of v_ρ and thus, by the Arzelà-Ascoli theorem, there exists a subsequence of functions converging uniformly on compact sets to a solution $\hat{v}$ of (3.20).

Solutions in Fourier modes $|j| \leq 2$ can be constructed explicitly. We obtained

$$v_0(r) := \int_0^r \frac{1}{z} \int_0^z t\, f_0(t)\, dt + d_0;$$

$$v_1(r) := -r^{-1} \int_0^r \frac{t^2\, f_1(t)}{2}\, dt + r \int_0^r \frac{f_1(t)}{2}\, dt + d_1\, r;$$

$$v_2(r) := -\frac{1}{r^2} \int_1^r \frac{t^3\, f_2(t)}{4}\, dt + r^2 \int_1^r \frac{f_2(t)}{4t}\, dt + d_2\, r^2.$$

A direct calculation then shows that for a suitable choice of the constants d_i, $i = 0, 1, 2$ the function $v := \hat{v} + v_0 + v_1 + v_2$ has the right properties.

□

Now we put $u := (1 - \mu_{\varepsilon,o,p})\, v$ and $\tilde{F} := J_{\varepsilon,o}\, u - F$. Then $\tilde{F} \in \mathcal{C}^{0,\alpha}(\Sigma_\varepsilon(o))$ and $\|\tilde{F}\|_{L^\infty(\Sigma_\varepsilon(o))} \leq C$ for a constant C independent of ε, o and p. By Lemma 3.4.1, there exist a function $\tilde{u} \in \mathcal{C}^{2,\alpha}(\Sigma_\varepsilon(o))$ and a vector $B_{\varepsilon,o,p} \in \mathbb{R}^3$, such that

$$J_{\varepsilon,o}\, \tilde{u} = \tilde{F} - \langle B_{\varepsilon,o,p}, \Theta\rangle,$$

where

$$B_{\varepsilon,o,p} = \int_{S^2} \Theta^i\, \tilde{F}\, dvol_{S^2} + \mathcal{O}(\varepsilon^2).$$

Finally, we put

$$\Gamma_{\varepsilon,o,p} := -\log d_{\varepsilon,o}(p,\cdot) - u + \tilde{u}.$$

Next, applying (in the sense of distributions) both sides of the expression (3.18) to Θ, we obtain

$$\int_{\Sigma_\varepsilon(o)} (J_{\varepsilon,o}\,\Gamma_{\varepsilon,o,p})\,\Theta\,dvol^{g_\varepsilon}_{\Sigma_\varepsilon(o)} = -2\pi\,\Theta(p) + \int_{\Sigma_\varepsilon(o)} \Theta\,\langle B_{\varepsilon,o,p},\Theta\rangle\,dvol^{g_\varepsilon}_{\Sigma_\varepsilon(o)}.$$

Integrating the first term by parts and using the fact that $J_{\varepsilon,o}$ is a self-adjoint operator, we obtain

$$\begin{aligned}\int_{\Sigma_\varepsilon(o)} (J_{\varepsilon,o}\,\Gamma_{\varepsilon,o,p})\;\Theta^i\,dvol^{g_\varepsilon}_{\Sigma_\varepsilon(o)} &= \int_{\Sigma_\varepsilon(o)} \left(J_{\varepsilon,o}\,\Theta^i\right)\Gamma_{\varepsilon,o,p}\,dvol^{g_\varepsilon}_{\Sigma_\varepsilon(o)}\\ &= \int_{S^2} \left(\varepsilon^2\,L_{\varepsilon,o}\,\Theta^i\right)\Gamma_{\varepsilon,o,p}\,(1+\mathcal{O}(\varepsilon^2))\,dvol_{S^2} = \mathcal{O}(\varepsilon^2).\end{aligned}$$

On the other hand,

$$\int_{\Sigma_\varepsilon(o)} \Theta^i\Theta^j\,\mathrm{dvol}^{g_\varepsilon}_{\Sigma_\varepsilon(o)} = (1+\mathcal{O}(\varepsilon^2))\int_{S^2}\Theta^i\,\Theta^j\,dvol_{S^2} = 4\pi\,\delta_{ij} + \mathcal{O}(\varepsilon^2).$$

This yields $B_{\varepsilon,o,p} = \frac{1}{2}\,\Theta(p) + \mathcal{O}(\varepsilon^2)$.

□

3.4.4 Pseudo CMC spheres as summands in a gluing construction

In the Euclidean space to perform a connected sum of two surfaces $\Sigma^\pm$ amounts to make $\Sigma^\pm$ tangent at a common point, then to translate the surfaces slightly away from each other in the direction orthogonal to their common tangent plane, remove small disks around the points where the surfaces are tangent and "replace" these disks by a small neck. In particular, it was shown in [77]), that if $\Sigma^\pm$ have CMC, one can perturb a connected sum of $\Sigma^\pm$ into a surface with the same value of the mean curvature.

Unfortunately, we cannot apply this construction directly in a generic Riemannian manifold M because of the absence of isometries (namely, the absence of translations and rotations). Instead, we given family of surfaces parametrized by their location in M, we should define a procedure that allows to associate to a number $d > 0$ small enough, a pair of surfaces the distance between which is equal to d.

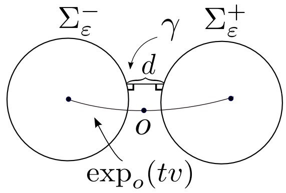
Take $d \in (0, 1/2\,\varepsilon)$. It is easy to choose a pair of geodesic spheres of radius ε the distance between which is equal to d. For this it is sufficient to fix a point $o \in M$, a vector $v \in T_oM$ with $\|v\|_g = 1$ and to place the centers of the spheres at $\exp_o\left(\pm\,\left(\varepsilon + \frac{d}{2}\right)v\right)$.

In the next result we show that an analogous procedure also works for a family of pseudo CMC spheres since the last ones are small perturbations of geodesic spheres. More precisely, consider the family

$$\Sigma^\pm_{\varepsilon,t} = \Sigma_\varepsilon\left(\exp_o(\pm\,tv)\right).$$

We prove

Lemma 3.4.3. *For all $d \in (0, 1/2\,\varepsilon)$ there exists a unique $t \in (\varepsilon, 2\varepsilon)$ such that the distance (calculated with respect to the metric g) between the surfaces $\Sigma_\varepsilon^\pm := \Sigma_{\varepsilon,t}^\pm$ is equal to d and is realized by a unique geodesic γ, a priori different from $t \mapsto \exp_o(tv)$.*



Moreover, the mapping $(\varepsilon, o, v, d) \mapsto \gamma$ is $\mathcal{C}^1$ in $\varepsilon \in (0, \varepsilon_*)$, $o \in M$, $v \in T_oM$ and $d \in (0, 1/2\,\varepsilon)$.

Proof. For the sake of convenience we work with the rescaled metric g_ε. Take an orthonormal (with respect to g_ε) basis $E_1^\varepsilon, E_2^\varepsilon, E_3^\varepsilon$ of T_oM, such that $E_3^\varepsilon = \varepsilon\, v$ and let x^1, x^2, x^3 be the corresponding normal geodesic coordinates. We denote by $\bar{x}$ the 2-vector (x^1, x^2).

Notation 3.4.3. *In what follows we denote by $\bar{x} \mapsto F_{\varepsilon,o,v,t}(\bar{x})$ any $\mathcal{C}^{2,\alpha}$ function which depends in a $\mathcal{C}^1$ manner on ε, o, v, t and whose derivatives are bounded independently of ε, o, v, t.*

Take a pair of points $p^+ \in \Sigma_{\varepsilon,t}^+$ and $p^- \in \Sigma_{\varepsilon,t}^-$ of coordinates x_+ and x_- in the neighborhoods the intersection of $\Sigma_{\varepsilon,t}^\pm$ with the geodesic $t \mapsto \exp_o(tv)$. Using the information about the structure of the surfaces $\Sigma_{\varepsilon,t}^\pm$ given in Proposition 3.4.1, the formula for the change of coordinates given in the subsection 3.3.2, and the fact that the Christoffel symbols associated to the metric $g_\varepsilon = g_{eucl} + \mathcal{O}(\varepsilon^2)$ satisfy $(\Gamma_\varepsilon)_{ij}^k = \mathcal{O}(\varepsilon^2)$, we can write

$$x_\pm^3 = \pm\left(\frac{t}{\varepsilon} - \sqrt{1 - |\bar{x}|^2}\right) + \varepsilon^2\, F_{\varepsilon,o,v,t}(\bar{x}_\pm).$$

Let $\gamma(p^+, p^+)$ be the minimizing geodesic connecting p^+ and p^-. The unit outward normals to $\Sigma_{\varepsilon,t}^\pm$ at $p^\pm$ can be written in the form

$$N_\pm(\bar{x}) = -\sum_{i=1}^2 x_\pm^i\, \partial_i + \sqrt{1 - |\bar{x}|^2}\, \partial_3 + \varepsilon^2\, F_{\varepsilon,o,v,t}(\bar{x}_\pm).$$

On the other hand, the unit tangent vectors to $\gamma(p^+, p^-)$ at $p^\pm$ can be written in the form

$$T_\pm(\bar{x}) = \pm \sum_{k=1}^3 \frac{(x_-^k - x_+^k)}{|x_+ - x_-|}\, \partial_k + \varepsilon^2\, F_{\varepsilon,o,v,t}(\bar{x}_+, \bar{x}_-).$$

Then the system of equations $N_\pm = T_\pm$ can be written in the form

$$G_{o,v,t}(\varepsilon, \bar{x}_+, \bar{x}_-) = \hat{G}(\bar{x}_+, \bar{x}_-) + \varepsilon^2\, F_{\varepsilon,o,v,t}(\bar{x}_+, \bar{x}_-) = 0,$$

where $\hat{G} : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ is defined by

$$\begin{cases} \hat{G}^i(\bar{x}_+, \bar{x}_-) & = & x_+^i + \frac{(x_+^i - x_-^i)}{|x^+ - x^-|}, \quad i = 1, 2, \\ \hat{G}^{2+i}(\bar{x}_+, \bar{x}_-) & = & x_-^i + \frac{(x_+^i - x_-^i)}{|x^+ - x^-|}, \quad i = 1, 2. \end{cases}$$

We have $G_{o,v,t}(0,0,0) = 0$ and $D_{(\bar{x}_+,\bar{x}_-)}G_{o,v,t}\big|_{(0,0,0)}$ is invertible. So, by the Implicit Function Theorem, there exists $\varepsilon_* \in (0,1)$ and for all $\varepsilon \in (0,\varepsilon_*)$ there exist $\mathcal{C}^1$ functions $(\varepsilon, o, v, t) \mapsto x_\pm(\varepsilon, o, v, t)$ such that

$$G_{o,v,t}(x_+(\varepsilon, o, v, t), x_-(\varepsilon, o, v, t), \varepsilon) = 0.$$

Moreover, the solution satisfies

$$\bar{x}_\pm(\varepsilon, o, v, t) = \varepsilon^2 F_{\varepsilon,o,v,t}(\bar{x}_+, \bar{x}_-),$$

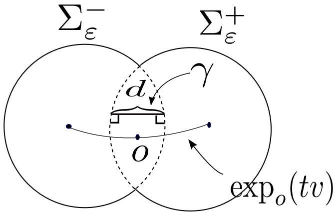
which yields the uniqueness of $\bar{x}_\pm(\varepsilon, o, v, t)$. We denote the points of coordinates $(\bar{x}_\pm, x^3_\pm)$ by $p_0^\pm$ and put

$$d(t) := \mathrm{dist}_g(p_0^+, p_0^-) = 2\,(t-\varepsilon) + \mathcal{O}(\varepsilon^2).$$

Since $d'(t) > 0$, we can express t as a $\mathcal{C}^1$ function of d.

□

Remark 3.4.1. *We will see in the final argument developed in the section 3.7 that we also need to perform a connected sum of two intersecting pseudo CMC spheres. In this case we cannot talk about the distance between the surfaces, but we prove (the proof is exactly the same as the proof of Lemma 3.4.3) that for all $d \in (0, 1/2\,\varepsilon)$ there exists $t \in (0,\varepsilon)$ such that the interior of $\Sigma^+_{\varepsilon,t} \cap \Sigma^-_{\varepsilon,t}$ is crossed by a unique minimizing geodesic γ of length d that intersects $\Sigma^\pm_{\varepsilon,t}$ orthogonally. Moreover, the mapping $(\varepsilon, o, v, d) \mapsto \gamma$ is $\mathcal{C}^1$.*



In what follows we will construct two connected sums (and two approximate solutions), one embedded and one immersed with self-intersections, and we will see that depending on the sign of the eigenvalue $\lambda \neq 0$ of the Hessian of the scalar curvature, we will show that one of these connected sums can be deformed into a constant mean curvature surface.

3.4.5 "Catenoidal neck"

Now that we can construct a pair of pseudo CMC spheres at a fixed "distance" we would like to insert a small neck between them. In $\mathbb{R}^3$ the role of the neck is usually played by a catenoid scaled by a small factor $\eta > 0$ referred to as the "neck size":

$$C_\eta : (s, \phi) \in [-s_\eta, s_\eta] \times S^1 \mapsto (\eta \cosh s\, \cos\phi, \eta \cosh s\, \sin\phi, \eta s)\,, \tag{3.22}$$

for some $s_\eta \gg 1$ carefully chosen in the gluing argument [77]. In a Riemannian manifold (M, g), in order to use the geometric properties of the catenoid, we shall embed of C_η in M taking coordinates that are close to cylindrical coordinates in $\mathbb{R}^3$.

Using the results of the subsection 3.4.4, let us fix $\varepsilon \in (0, \varepsilon_*)$, $o \in M$, $v \in T_oM$ with $\|v\|_g = 1$, and $d \in (0, 1/2\,\varepsilon)$ and consider a pair of pseudo CMC spheres $\Sigma_\varepsilon^\pm(o, v, d)$ (either disjoint or intersecting) the "distance" between which is equal to d. Let γ be the minimizing geodesic with $\gamma(0) = p_0^- \in \Sigma_\varepsilon^-$ and $\gamma(1) = p_0^+ \in \Sigma_\varepsilon^+$, where $\mathrm{dist}_g(p_0^+, p_0^-) = d$.

Next, take $q_0 := \gamma\left(\frac{1}{2}\right)$ and consider the geodesic disk Λ centered at q_0 and orthogonal to γ. Remark, that we can choose an orthonormal (with respect to the metric g_ε) basis $E_1^\varepsilon, E_2^\varepsilon, E_3^\varepsilon$ of $T_{q_0}M$ that depends in a $\mathcal{C}^1$ manner on ε, o, v, d and such that $E_3 = \gamma'(1/2)/\|\gamma'(1/2)\|_{g_\varepsilon}$. For this it is sufficient to fix an orthonormal frame tangent of S^2 (which will provide an orthonormal frame tangent of Σ_ε^- that is $\mathcal{C}^1$ in ε, o, v, d) and to take the parallel transport of this frame along γ from p_0^- to q_0. The geodesic disc Λ is then parametrized by

$$x \in D^2 \mapsto \exp_{q_0}\left(x^1\, E_1^\varepsilon + x^2\, E_2^\varepsilon\right) \in \Lambda.$$

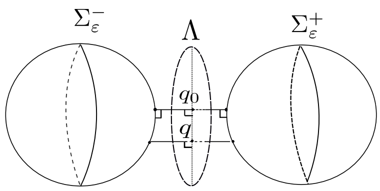
Consider the mapping $(x^1, x^2, z) \mapsto F_{\Lambda, q_0}^\varepsilon(x^1, x^2, z)$ from a neighborhood of 0 in $\mathbb{R}^3$ to a neighborhood of q_0 in M which defines Fermi coordinates associated to the surface Λ (see 3.3.3, where the normal to Λ is unit with respect to the metric g_ε). We refer to the surface

$$\mathfrak{C}_{\eta,0} := F_{\Lambda, q_0}^\varepsilon(C_\eta)$$

as the initial position of the "catenoidal neck" with "axis" γ.

Remark 3.4.2 (Varying the position of the "axis").

For the reasons that will be explained in the section 3.7, we need to introduce extra freedom and to this end consider a 2-parameter family of "catenoidal necks" with their "axis" parallel to γ. More precisely, take a point $q \in \Lambda$ of coordinates $(x^1, x^2) = (a^1, a^2)$. Then, let (y^1, y^2) be normal geodesic coordinates centered at q (as before, we work with the metric g_ε and choose an orthonormal frame tangent to Λ which is $\mathcal{C}^1$ in ε, o, v, d), and $(y^1, y^2, z) \mapsto F_{\Lambda, q}^\varepsilon(y^1, y^2, z)$ the corresponding Fermi coordinates.



Then, we have a two parameter family of "catenoidal necks" given by

$$\mathfrak{C}_{\eta,a} := F_{\Lambda, q}^\varepsilon(C_\eta).$$

3.4.6 Gluing the summands together

In this subsection, we explain how we can "glue together" a pair of pseudo CMC surfaces $\Sigma_\varepsilon^\pm$ with a "catenoidal neck" $\mathfrak{C}_{\eta,a}$, where the construction these surfaces depends in $\mathcal{C}^1$ manner on $\varepsilon \in (0, \varepsilon_*), o \in M, v \in T_pM, a \in \mathbb{R}^2$ and $d \in \mathbb{R}$. Using the ideas introduced in [77], in order to get a better matching with the asymptotics of the catenoid, we first deform $\Sigma_\varepsilon^\pm$ using Green's functions associated to their Jacobi operators, and then perform the connected sum.

We notice also, that a certain relation between the parameters should be satisfied. More precisely, we leave $\varepsilon, o, v, \eta, a$ as free parameters and express d as a function of $\varepsilon, o, v, \eta, a$.

Assumptions on the parameters

For the reasons explained in the section 3.7, we need a certain restriction on the choice of the parameters $o \in M$, $v \in T_oM$, with $\|v\|_g = 1$, $\eta \in (0,1)$ and $|a| \in \mathbb{R}^2$. Let $o_{cr} \in M$ be a critical point of the scalar curvature function $\mathcal{R}$ of the ambient manifold M and v_λ the unit eigenvector associated to a simple eigenvalue $\lambda \neq 0$ of the Hessian of $\mathcal{R}$ at o_{cr}.

Assumption 3.4.1. *We assume that there exists a (possibly large) constant C_* independent of $\varepsilon, o, v, \eta, a$ such that*

$$\mathrm{dist}_g(o, o_{cr}) \le C_*\,\varepsilon^2, \quad \angle(v, v_\lambda) \le C_*\,\varepsilon, \quad \eta \le C_*\varepsilon^4 \quad \textit{and} \quad |a| \le C_*\eta^{3/4},$$

where by $\angle(v, v_\lambda)$ we mean the angle between v_λ and the result of the parallel transport of v along the minimizing geodesic from o to o_{cr}.

$\Sigma_\varepsilon^\pm$ as normal graphs over Λ

With the notations introduced in the previous subsections the pseudo CMC spheres $\Sigma_\varepsilon^\pm$ can be seen in the neighborhood of $p_0^\pm$ as normal geodesic graphs over the geodesic disk Λ. Let N_Λ^ε be a unit normal (with respect to the metric g_ε) on Λ. In what follows let us assume that the "distance" d between $\Sigma_\varepsilon^\pm$ is computed using the metric g_ε. Then:

Lemma 3.4.4. *In the neighborhood of $p_0^\pm \in \Sigma_\varepsilon^\pm$ the surfaces $\Sigma_\varepsilon^\pm$ can be parametrized by*

$$\mathcal{F}_\Lambda^\pm \,:\, q' \in \Lambda \mapsto \exp_{q'}\left(u^\pm(q')\, N_\Lambda^\varepsilon\right), \tag{3.23}$$

where $u^\pm \in \mathcal{C}^{2,\alpha}(\Lambda)$ are convex functions such that

$$u^\pm(q_0) = \pm\frac{d}{2} \quad \textit{and} \quad \nabla u^\pm(q_0) = 0.$$

Take a point $q \in \Lambda$, such that $\mathrm{dist}_{g_\varepsilon}(q, q_0) = |a|$. Then in normal geodesic coordinates on Λ centered at q we have the expansions:

$$u^\pm(y) = u^\pm(q) + \nabla u^\pm(q)\cdot y + \mathcal{O}(|y|^2), \tag{3.24}$$

where $u^\pm(q) = \pm\frac{d}{2} + \mathcal{O}(|a|^2)$ and $|\nabla u^\pm(q)| = \mathcal{O}(|a|)$.

Proof. The proof follows from the construction of surfaces $\Sigma_\varepsilon^\pm$ and Λ.

□

Transverse vector fields on $\Sigma_\varepsilon^\pm$

Remark that in the neighborhood of $q_0 \in \Lambda$, normal geodesic graphs over Λ can be seen as transverse geodesic graphs over $\Sigma_\varepsilon^\pm$. Indeed, the geodesics issued from Λ with velocity vectors orthogonal to Λ can be extended until their intersection with $\Sigma_\varepsilon^\pm$. The unit (with respect to g_ε) tangent vectors to theses geodesics then form $\mathcal{C}^{2,\alpha}$ vector fields on $\Sigma_\varepsilon^\pm$ which we denote by $K_\pm$. (By construction $K_\pm$ are $\mathcal{C}^1$ in ε, o, v, and d).

Let $N_\pm$ be unit normals (with respect to g_ε) on $\Sigma_\varepsilon^\pm$, which we take outward when the surfaces $\Sigma_\varepsilon^\pm$ are disjoint, and inward when $\Sigma_\varepsilon^\pm$ intersect. We use the notation $p_q^\pm := \mathcal{F}_\Lambda^\pm(q)$ for $q \in \Lambda$. Then for all $\psi^\pm \in L^\infty(\Sigma_\varepsilon^\pm)$ small enough defined in the neighborhood of $p_q^\pm$ in $\Sigma_\varepsilon^\pm$, the transverse geodesic graph over $\Sigma_\varepsilon^\pm$ parametrized by

$$p \in \Sigma^\pm \mapsto \exp_p\left(\psi^\pm(p)\, K_\pm(p)\right)$$

coincides with the normal geodesic graph over Λ parametrized by

$$q' \in \Lambda \mapsto \exp_{q'}\Big(\left(u^\pm - \psi^\pm \circ \mathcal{F}_\Lambda^\pm\right)(q')\, N_\Lambda(q')\Big),$$

where N_Λ is a unit normal to Λ with respect to g_ε.

Lemma 3.4.5. *In a neighborhood of $p_q^\pm \in \Sigma_\varepsilon^\pm$ of radius $c|a|$ for some $c > 0$ independent of $\varepsilon, o, v, \eta, a$, we have*

$$|g_\varepsilon\left(K_\pm, N_\pm\right)| = 1 + \mathcal{O}\left(|a|^2\right)$$

Proof. First of all, when $q = q_0$, then $p_q^\pm = p_0^\pm$ and the result follows from the fact that

$$K_\pm(p_0^\pm) = N_\pm(p_0^\pm) \quad \text{and}$$
$$\partial_{x^i} g_\varepsilon\left(K_\pm, N_\pm\right)(p_0^\pm) = g_\varepsilon(\nabla_{\partial_{x^i}} K_\pm, K_\pm)(p_0) + g_\varepsilon(\nabla_{\partial_{x^i}} N_\pm, N_\pm)(p_0) = 0.$$

When $q \neq q_0$, the result follows from the fact that the distance in $\Sigma_\varepsilon^\pm$ between $p_0^\pm$ and $p_q^\pm$ is bounded by a constant (independent of $\varepsilon, o, v, \eta, a$) times $|a|$.

□

Notation 3.4.4. *We denote by $\Sigma_\varepsilon^\pm(\rho)$ the region in $\Sigma_\varepsilon^\pm$ obtained as the image by $\mathcal{F}_\Lambda^\pm$ of geodesic disc in Λ of radius ρ centered at q. We denote by $\Sigma_\varepsilon^\pm(\rho_1, \rho_2)$ the image by $\mathcal{F}_\Lambda^\pm$ of the geodesic annulus of inner radius ρ_1 and outer radius ρ_2 centered at q. Finally, we use the notation $\left(\Sigma_\varepsilon^\pm(\rho)\right)^c = \Sigma_\varepsilon^\pm \setminus \Sigma_\varepsilon^\pm(\rho)$.*

Let $\mu^\pm \in \mathcal{C}^{2,\alpha}(\Sigma_\varepsilon^\pm)$ be cut-off functions such that $\mu^\pm \equiv 0$ in $\Sigma_\varepsilon^\pm(1/4)$ and $\mu^\pm \equiv 1$ in $\left(\Sigma_\varepsilon^\pm(1/2)\right)^c$. We introduce the vector fields

$$\tilde{N}_\pm := \mu^\pm\, N_\pm + (1 - \mu^\pm)\, K_\pm. \tag{3.25}$$

Graphs of Green's functions

Notation 3.4.5. *We denote by $J_\pm$ the Jacobi operators about $\Sigma_\varepsilon^\pm$ calculated with respect to the metric g_ε. We also denote by $\tilde{J}_\pm$ the linearized mean curvature operators which arise when $\Sigma_\varepsilon^\pm$ are perturbed in the direction of the vector fields $\tilde{N}_\pm$ defined in* (3.25).

Let $d_{\Sigma_\varepsilon^\pm}$ be the geodesic distances induced by the metric g_ε on the surfaces $\Sigma_\varepsilon^\pm$. We use the notation $d_\pm := d_{\Sigma_\varepsilon^\pm}\left(p_q^\pm, \cdot\right)$.

We study the Green's functions associated to the operators $\tilde{J}^\pm$ with poles at $p_q^\pm$. More precisely, we have the following result:

Proposition 3.4.3. *There exist functions $\Gamma^\pm$ defined on $\Sigma_\varepsilon^\pm \setminus \{p_q^\pm\}$, vectors $B^\pm \in \mathbb{R}^3$ and constants $\varkappa^\pm > 0$ such that*

$$\tilde{J}_\pm \Gamma^\pm = -\varkappa^\pm \delta_{p_q^\pm} + \langle B^\pm, \Theta\rangle. \tag{3.26}$$

Moreover,

$$B^\pm = \frac{1}{2}\Theta\left(p_q^\pm\right) + \mathcal{O}(\varepsilon^2),$$

and there exist constants $c^\pm \in \mathbb{R}$ and $C_\ell > 0$, $\ell = 0, 1, 2$ such that

$$\left|(d_\pm)^\ell \nabla^\ell \left(\Gamma^\pm + \log(d_\pm) - c^\pm\right)\right| \leq C_\ell\ (d_\pm)\ |\log(d_\pm)|,$$

where the mapping $(\varepsilon, o, v, \eta, a) \mapsto (\Gamma^\pm, B^\pm, c^\pm)$ is $\mathcal{C}^1$.

Proof. By Proposition 3.3.2, the operators $J_\pm$ and $\tilde{J}_\pm$ are conjugate, more precisely

$$\tilde{J}_\pm = J_\pm\left(g_\varepsilon(\tilde{N}_\pm, N_\pm)\ \cdot\right) + V^\pm$$

where $V^\pm \in \mathcal{C}^{2,\alpha}(\Sigma_\varepsilon^\pm)$ is a potential bounded by a constant independent of $\varepsilon, o, v, \eta, a$. The result follows then from Proposition 3.4.2 and the estimates on the scalar product $g_\varepsilon(K_\pm, N_\pm)$ obtained in Lemma 3.4.5.

□

Let the mappings $\mathcal{F}_\Lambda^\pm : \Lambda \to \Sigma_\varepsilon^\pm$ be defined as in (3.23) and consider the functions

$$\Gamma_\Lambda^\pm := \Gamma^\pm \circ \mathcal{F}_\Lambda^\pm. \tag{3.27}$$

In the following result we compare the behavior of $\Gamma_\Lambda^\pm$ in a neighborhood of q in $\Lambda \setminus \{q\}$ and the asymptotic behavior of the "catenoidal neck" $\mathfrak{C}_{\eta,a}$.

Lemma 3.4.6. *In the geodesic normal coordinates on Λ centered at q the functions $\Gamma_\Lambda^\pm$ have the expansions*

$$\Gamma_\Lambda^\pm(y) = c_0^\pm + \log|y| + \mathcal{O}\left(|y|\ \log|y|\right). \tag{3.28}$$

On the other hand, the "catenoidal neck" $\mathfrak{C}_{\eta,a}$ can be seen as a normal geodesic bi-graph over Λ of the function

$$G_\eta := \eta \log\frac{2}{\eta} + \eta\log|y| + \mathcal{O}\left(\eta^3\,|y|^{-2}\right), \tag{3.29}$$

where the mapping $(\varepsilon, o, v, d, \eta, a) \mapsto (\Gamma_\Lambda^\pm, G_\eta)$ is $\mathcal{C}^1$.

Proof. Using Lemma 3.3.1 and the fact that the second fundamental form of Λ is bounded by a constant times ε^2, in Fermi coordinates in M given by the mapping $F^{\varepsilon}_{\Lambda,q}$ we have

$$(g_\varepsilon)_{ij} = \delta_{ij} + \mathcal{O}(\varepsilon^2).$$

Let $d_{\Sigma^\pm_\varepsilon}$ and d_Λ be intrinsic geodesic distances on $\Sigma^\pm_\varepsilon$ and Λ associated to g_ε. The result then follows from Proposition 3.4.3 and the fact that in a neighborhood of $q \in \Lambda$ small enough

$$d_\Lambda(q, q') = d_{\Sigma^\pm_\varepsilon}(p^\pm_q, p^\pm_{q'}) + \mathcal{O}\left(\varepsilon^2\, d_{\Sigma^\pm_\varepsilon}(p^\pm_q, p^\pm_{q'})\right).$$

The expansion of the function G_η follows from the change of coordinates

$$(y^1, y^2) = \eta \cosh s(\cos\phi, \sin\phi).$$

□

Matching

Using the results of the previous paragraph, we see that in order to "glue" together the graphs of the Green's functions $\Gamma^\pm$ over $\Sigma^\pm_\varepsilon$ with a "catenoidal neck" $\mathfrak{C}_{\eta,a}$ we need the expansions of the functions $u^\pm - \eta\,\Gamma^\pm_\Lambda$ and $\pm\, G_\eta$ to be close in some neighborhood of $q \in \Lambda$.

1) **Adjusting the "distance":** Let us first assume that $\Sigma^\pm_\varepsilon$ are disjoint. In order to match the constant terms in the expansions obtained in Lemma 3.4.6 we first "translate" the "catenoid" in the direction orthogonal to Λ by the constant

$$\eta\, \log\frac{\eta}{2} + u^+(q) - \eta\, c^+_0.$$

More precisely, this means that we parametrize the upper and the lower parts of the neck as graphs of the functions

$$G^+_\eta(y) := u^+(q) - \eta\, c^+_0 + \eta\, \log|y| + \mathcal{O}\left(\eta^3|y|^{-2}\right), \tag{3.30}$$

$$G^-_\eta(y) := 2\eta\, \log\frac{\eta}{2} + u^+(q) - \eta\, c^+_0 - \eta\, \log|y| + \mathcal{O}\left(\eta^3|y|^{-2}\right). \tag{3.31}$$

Next, in order to match the constant terms in the expansions of Γ^-_Λ and G^-_η, we need to have

$$2\,\eta\, \log\frac{\eta}{2} + u^+(q) - \eta\, c^+_0 = u^-(q) - \eta\, c^-_0. \tag{3.32}$$

We can rewrite this equation in the form $\mathcal{D}(d, o, v, \eta, a) = 0$, where

$$\mathcal{D}(d, o, v, \eta, a) = d + 2\,\eta\, \log\frac{\eta}{2} + \eta(c^-_0 - c^+_0) + \mathcal{O}\left(\eta^{3/2}\right).$$

Take $o_0 \in M$, $v_0 \in T_oM$, with $\|v_0\|_g = 1$, $\eta_0 > 0$ and $a_0 \in \mathbb{R}^2$ satisfying the assumption 3.4.1. Then we can find $d_0 \in (0,1)$, such that

$$\mathcal{D}(d_0, o_0, v_0, \eta_0, a_0) = 0.$$

By the Implicit Function theorem, there exists a neighborhood of o_0, v_0, η_0, a_0, where $(o, v, \eta, a) \mapsto d(o, v, \eta, a)$ is a $\mathcal{C}^1$ function and $\mathcal{D}(d(o, v, \eta, a), o, v, \eta, a) = 0$. Remark, that we have

$$|d - 2\,\eta\, \log(2/\eta)| \leq C\,\eta,$$

for a constant C independent of $\varepsilon, \eta, a, o, v$ when ε is small enough.

2) **Choosing the gluing region**: The difference between $u^{\pm} - \eta\, \Gamma^{\pm}_{\Lambda}$ and $G^{\pm}_{\eta}$ is now a function of the form

$$\mathcal{O}\left(\eta |y|\, \log |y| + |a||y| + \eta^3\, |y|^{-2} + |y|^2\right).$$

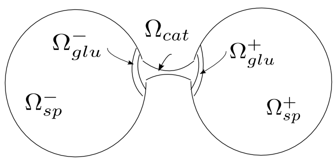
We perform the "gluing" in the region where this difference is minimized, namely, when

$$|y| = \eta^{\frac{3}{4}} =: r_\eta. \tag{3.33}$$

Parametrization of the resulting surface

In this paragraph we summarize the results given above and describe the parametrization of the resulting connected sum, which we denote by $\mathcal{A}_\varepsilon$ and refer to as *approximate solution.* We divide $\mathcal{A}_\varepsilon$ into 5 regions:

$$\mathcal{A}_\varepsilon = \Omega^+_{sp} \cup \Omega^+_{glu} \cup \Omega_{cat} \cup \Omega^-_{glu} \cup \Omega^-_{sp}.$$



1) The *"spherical regions"* $\Omega^{\pm}_{sp}$ are parametrized as transverse transverse graphs over the pseudo CMC spheres $\Sigma^{\pm}_{\varepsilon}$ of functions the $\eta\,\Gamma^{\pm}$ given by Proposition 3.4.3, when the surfaces are perturbed in the direction of the vector fields $\tilde{N}_{\pm}$ defined in (3.25).

2) The *"catenoidal region"* Ω_{cat} is parametrized as a Euclidean catenoid scaled by the factor η, truncated and embedded in M via Fermi coordinates, as it is explained in the subsection 3.4.5:

$$(s, \phi) \in [-s_\eta, s_\eta] \times S^1 \mapsto F^{\varepsilon}_{\Lambda,q}\left(\eta \cosh s\, \cos\phi, \eta \cosh s\, \sin\phi, \eta s + \eta\, c\right),$$

where $\eta \cosh s_\eta = \frac{1}{2} r_\eta$ and the constant c is defined in the paragraph "Matching" and depends on $\varepsilon, o, v, \eta, a$ but is bounded independently of $\varepsilon, o, v, \eta, a$.

3) Finally, in the *"gluing regions"* $\Omega^{\pm}_{glu}$ we interpolate smoothly between $\Omega^{\pm}_{sp}$ and Ω_{cat}.

Notation 3.4.6. *We introduce a cut-off function $\chi \in \mathcal{C}^{2,\alpha}(\Lambda)$ which is radial in geodesic normal coordinates in Λ centered at q and is identically equal to 1 when $r > r_\eta$ and to 0 when $r < r_\eta/2$. (By construction, χ depends continuously on $\varepsilon, o, v, \eta, a$).*

We parametrize the regions $\Omega^{\pm}_{glu}$ as normal graphs over the geodesic annulus in Λ of inner radius $r_\eta/2$ and outer radius r_η of the functions

$$\mathcal{G}^{\pm} := \chi(u^{\pm} - \eta\,\Gamma^{\pm}_{\Lambda}) + (1-\chi)\,G^{\pm}_{\eta}, \quad \text{when } \Sigma^{\pm}_{\varepsilon} \text{ are disjoint,} \tag{3.34}$$

$$\mathcal{G}^{\pm} := \chi(u^{\mp} - \eta\,\Gamma^{\mp}_{\Lambda}) + (1-\chi)\,G^{\pm}_{\eta}, \quad \text{when } \Sigma^{\pm}_{\varepsilon} \text{ intersect.} \tag{3.35}$$

Remark 3.4.3. *The surface $\mathcal{A}_\varepsilon$ is either embedded or immersed with self-intersections and depends on $\varepsilon \in (0, \varepsilon_*)$ and on 8 geometric parameters: the point $o \in M$, the unit vector $v \in T_oM$, the "neck size" $\eta > 0$ and the "location" of the neck $a \in \mathbb{R}^2$ that satisfy the assumptions 3.4.1. We write $\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon(o, v, \eta, a)$.*

3.5 Perturbation of the approximate solution

We would like to show that the surface $\mathcal{A}_\varepsilon$ can be perturbed at least for ε small enough into some CMC surface. To this end, we describe in this section the surfaces in M obtained as small deformations of $\mathcal{A}_\varepsilon$. Let Ξ be a vector field in M defined in the neighborhood of $\mathcal{A}_\varepsilon$ and transverse to $\mathcal{A}_\varepsilon$, and let ξ be the associated flow:

$$\frac{d\xi}{dt} = \Xi\left(\xi(\cdot, t)\right), \quad \text{and} \quad \xi(p, 0) = p.$$

Take a function $w \in \mathcal{C}^{2,\alpha}(\mathcal{A}_\varepsilon)$ and consider the surface $\mathcal{A}_\varepsilon(w)$ parametrized by:

$$p \in \mathcal{A}_\varepsilon \mapsto \xi\left(p, w(p)\right) \in \mathcal{A}_\varepsilon(w).$$

The expansion of the mean curvature of $\mathcal{A}_\varepsilon(w)$ in the powers of w and derivatives of w up to the second order has the form:

$$H^{g_\varepsilon}\left(\mathcal{A}_\varepsilon(w)\right) = H^{g_\varepsilon}\left(\mathcal{A}_\varepsilon\right) + \mathcal{L}_\varepsilon\, w + \mathcal{Q}_\varepsilon(w), \tag{3.36}$$

where $H(\mathcal{A}_\varepsilon)$ is the mean curvature of the approximate solution $\mathcal{A}_\varepsilon$, $\mathcal{L}_\varepsilon$ is a linearized mean curvature operator about $\mathcal{A}_\varepsilon$, and $\mathcal{Q}_\varepsilon$ is a nonlinear function in w and the components of the gradient and the Hessian of w. In the results described below we explain an appropriate choice of the vector field Ξ and study the properties of the function $H(\mathcal{A}_\varepsilon)$ and the operators $\mathcal{L}_\varepsilon$ and $\mathcal{Q}_\varepsilon$ in appropriate function spaces.

3.5.1 Choice of the transverse vector field

In this subsection we describe explicitly the parametrization of the perturbed surfaces $\mathcal{A}_\varepsilon(w)$ which will explain implicitly the choice of the transverse vector field Ξ.

1) First, we describe the region of $\mathcal{A}_\varepsilon(w)$ parametrized by the "spherical regions" $\Omega^{\pm}_{sp}$ of $\mathcal{A}_\varepsilon$. Let $\tilde{N}_\pm$ be the vector fields defined on $\Sigma^{\pm}_{\varepsilon}$ by (3.25). By construction, in $\left(\Sigma^{\pm}_{\varepsilon}(1/2)\right)^c$ these vectors coincide with unit normals to $\Sigma^{\pm}_{\varepsilon}$, while in $\Sigma^{\pm}_{\varepsilon}(1/4)$, the parallel transport of $\tilde{N}_\pm$ to Λ coincides with a unit normal to Λ. We parametrize $\mathcal{A}_\varepsilon(w)$ by

$$p \in \left(\Sigma_\varepsilon^\pm(r_\eta)\right)^c \mapsto \exp_p\Big(\left(\eta\,\Gamma^\pm + w\right)(p)\,\tilde N_\pm(p)\Big) \in \mathcal{A}_\varepsilon(w).$$

2) In the "catenoidal region" we parametrize $\mathcal{A}_\varepsilon(w)$ as an image in M of a normal (with respect to the Euclidean metric) graph over the Euclidean catenoid:

$$(s,\phi) \in [-s_\eta, s_\eta] \times S^1 \mapsto F^\varepsilon_{\Lambda,q}\Big(C_\eta + w\,N_{cat}\Big) \in \mathcal{A}_\varepsilon(w),$$

where

$$N_{cat}(s,\phi) = \left(-\frac{1}{\cosh s}\,\cos\phi, -\frac{1}{\cosh s}\,\sin\phi,\,\tanh s\right)$$

is a unit normal (with respect to the Euclidean metric) to the catenoid C_η and the mapping $F^\varepsilon_{\Lambda,q}$ defines, as in (3.4), Fermi coordinates centered at $q \in \Lambda$.

3) Finally, we have the region parametrized by the "gluing regions" $\Omega^\pm_{glu}$ where we interpolate smoothly between N_{cat} and a unit normal to Λ. We parametrize $\mathcal{A}_\varepsilon(w)$ by

$$y \in \mathbb{R}^2,\ r_\eta/2 < |y| < r_\eta \ \mapsto\ F^\varepsilon_{\Lambda,q}\Big(\left(y, \mathcal{G}^\pm(y)\right) + w(y)\,\Upsilon^\pm(y)\Big),$$

where the functions $\mathcal{G}^\pm$ are defined in (3.34) and the vector fields $\Upsilon^\pm$ are defined in $\mathbb{R}^3$ by

$$\Upsilon^\pm = (1-\chi)N_{cat} \pm \chi(0,0,1). \tag{3.37}$$

3.5.2 Function spaces

Notation 3.5.1. *Let $\chi \in C^{2,\alpha}(\Lambda)$ be the cut-off function defined as in Notation 3.4.6. We define the function $\chi^+ \in \mathcal{C}^{2,\alpha}(\mathcal{A}_\varepsilon)$, such that $\chi^+ \equiv 1$ in Ω^+_{sp}, $\chi^+ \equiv \chi$ in Ω^+_{glu}, and $\chi^+ \equiv 0$ in $\mathcal{A}_\varepsilon \setminus \left(\Omega^+_{sp} \cup \Omega^-_{sp}\right)$. In the same manner, we define the function $\chi^- \in \mathcal{C}^{2,\alpha}(\mathcal{A}_\varepsilon)$.*

Notation 3.5.2 (Weight function)**.** *We introduce a weight function $\vartheta \in \mathcal{C}^\infty(\mathcal{A}_\varepsilon)$ that interpolates smoothly between the distance to points $p^\pm_q$ in $\Sigma^\pm_\varepsilon$ and the function $s \mapsto \eta\cosh s$ defined in the "catenoidal neck" region. More precisely, we put*

$$\vartheta := \chi^+\,d_+ + \chi^-\,d_- + (1-\chi^+-\chi^-)\,\eta\,\cosh s$$

where $d_\pm$ is the distance to $p^\pm_q$ in $\Sigma^\pm_\varepsilon$ associated to the metric g_ε.

Definition 3.5.1 (Weighted Hölder spaces)**.** *Take $\nu \in \mathbb{R}$. We say that a function w belongs to the space $\mathcal{C}^{k,\alpha}_\nu(\mathcal{A}_\varepsilon)$, $k = 0,1,2$ if the following norm is finite*

$$\begin{aligned}\|w\|_{\mathcal{C}^{k,\alpha}_\nu} = \sup_{\mathcal{A}_\varepsilon}|\vartheta^{-\nu}w| + \sum_{\ell=1}^k \sup_{\mathcal{A}_\varepsilon}\|\vartheta^{\ell-\nu}\nabla^\ell w\|_{g_\varepsilon} \\ + \sup_{p,p'\in\mathcal{A}_\varepsilon}\left|\frac{\vartheta^{k+\alpha-\nu}(p)\,\nabla^k w(p) - \vartheta^{k+\alpha-\nu}(p')\,\nabla^k w(p')}{d^\varepsilon_{\mathcal{A}_\varepsilon}(p,p')^\alpha}\right|.\end{aligned} \tag{3.38}$$

For a region $\Omega \subset \mathcal{A}_\varepsilon$, we denote the restriction of $\mathcal{C}^{k,\alpha}_\nu$ to Ω by $\mathcal{C}^{k,\alpha}_\nu(\Omega)$.

Remark 3.5.1. *Let us adopt the notations 3.4.4. Then the norm* (3.38) *is equivalent to*

$$\|w\|_{\mathcal{C}^{k,\alpha}_\nu} = \sup_{s\in[-s_\eta, s_\eta-1]} \|(\eta \cosh s)^{-\nu}\, w\|_{\mathcal{C}^{k,\alpha}\left([s,s+1]\times S^1,\ ds^2+d\phi^2\right)}$$

$$+ \|w\|_{\mathcal{C}^{k,\alpha}\left(\left(\Sigma^+_\varepsilon(1/2)\right)^c\right)} + \|w\|_{\mathcal{C}^{k,\alpha}\left(\left(\Sigma^-_\varepsilon(1/2)\right)^c\right)}$$

$$+ \sup_{\rho\in[r_\eta,1/4]} \|d_+^{-\nu}\, w\|_{\mathcal{C}^{k,\alpha}\left(\Sigma^+_\varepsilon(\rho,2\rho),\ d_+^{-2} g_+\right)} + \sup_{\rho\in[r_\eta,1/4]} \|d_-^{-\nu}\, w\|_{\mathcal{C}^{k,\alpha}\left(\Sigma^-_\varepsilon(\rho,2\rho),\ d_-^{-2} g_-\right)},$$

where $g_\pm$ are the metrics induced on $\Sigma^\pm_\varepsilon$ from g_ε. Notice, that in the last two terms we use singular metrics $d_\pm^{-2}\, g_\pm$ to calculate the gradient and the Hessian of the function.

3.5.3 Mean curvature of the approximate solutions

In this subsection we analyze the mean curvature of the approximate solution $\mathcal{A}_\varepsilon$.

First of all, using the fact that the catenoid is minimal with respect to the Euclidean metric and that $g_\varepsilon = g_{eucl} + \mathcal{O}(\varepsilon^2)$, we show that in the "catenoidal region" the mean curvature of $\mathcal{A}_\varepsilon$ tends to 0 in L^∞ norm as ε tends to 0. More precisely, we have the following result:

Proposition 3.5.1. *For all $\varepsilon \in (0, \varepsilon_*)$ and all o, v, η, a satisfying the assumptions 3.4.1, the mean curvature of the surface $\mathcal{A}_\varepsilon(o, v, \eta, q)$ in the catenoidal region Ω_{cat} satisfies for $k = 0, 1, 2$*

$$\left|\nabla^k\, H^{g_\varepsilon}(\mathcal{A}_\varepsilon)(s,\phi)\right| \le C_k\, \varepsilon^2,$$

for a constant $C_k > 0$ independent of $\varepsilon, o, v, \eta, a$ and where the derivatives ∇^k are calculated with respect to the metric $ds^2 + d\phi^2$.

Proof. In Ω_{cat} the surface $\mathcal{A}_\varepsilon$ is parametrized as the image of the Euclidean catenoid

$$C_\eta : (s,\phi) \in \mathbb{R}\times S^1 \mapsto (\eta\, \cosh s\, \cos\phi, \eta\, \cosh s\, \sin\phi, \eta\, s + \eta\, c)\,,$$

where $\eta\, \cosh s_\eta = \frac{1}{2}\, r_\eta$ by the mapping $F^\varepsilon_{\Lambda,q}$ which defines Fermi coordinates at q associated to the geodesic disk Λ. Our task is to find the mean curvature of C_η with respect to the ambient pull-back metric $\left(F^\varepsilon_{\Lambda,q}\right)^*\, g_{eucl}$. We have

$$\left[\left(F^\varepsilon_{\Lambda,q}\right)^*\, g_{eucl}\right]_{ij}(y,z) = \delta_{ij} + \mathcal{O}\left(\varepsilon^2\left(|y|^2 + |z|\right)\right),$$

which follows from Lemma 3.3.1 and the fact that the second fundamental form of the geodesic disk Λ calculated with respect to the metric g_ε satisfies

$$h_\Lambda = \mathcal{O}(\varepsilon^2).$$

Then the induced metric on the catenoid can be written in the form

$$g^{cat}_\varepsilon = \eta^2\, \cosh^2 s(ds^2 + d\phi^2) + \mathcal{O}(\varepsilon^2\, \eta^3\, \cosh^3 s).$$

Let N_{cat} be a unit normal to C_η with respect to g_{eucl}, then the unit normal with respect to the pullback metric satisfies

$$N_{cat}^\varepsilon = N_{cat} + \mathcal{O}(\varepsilon^2 \eta \cosh s).$$

Next, since the Christoffel symbols corresponding to the Levi-Civita connection associated to the metric $\left(F_{\Lambda,q}^\varepsilon\right)^* g_{eucl}$ satisfy $\Gamma_{ij}^k = \mathcal{O}(\varepsilon^2)$, we calculate the second fundamental form:

$$h_\varepsilon^{cat} = \eta(-ds^2 + d\phi^2) + \mathcal{O}(\varepsilon^2 \eta^2 \cosh^2 s).$$

Finally, we deduce the estimates for the mean curvature:

$$H^{g_\varepsilon}(\mathcal{A}_\varepsilon) = \mathrm{tr}\left((g_\varepsilon^{cat})^{-1}(h_\varepsilon^{cat})\right) = \mathcal{O}(\varepsilon^2).$$

□

Next, we estimate the mean curvature of $\mathcal{A}_\varepsilon$ in the "spherical regions", applying the formula for the mean curvature of transverse geodesic graphs given by Proposition 3.3.2. Parametrizing $\Sigma_\varepsilon^\pm$ by the Euclidean unit sphere S^2, we obtain an expression for $H^{g_\varepsilon}(\mathcal{A}_\varepsilon) - 2$ where we distinguish two types of terms: terms controlled by $\eta^3 \, (d_\pm)^{-4}$ and terms which belong to $\mathrm{Ker}\,(\Delta_{S^2} + 2)$.

As before, let $\mathcal{R}$ be the scalar curvature function of M and assume that $o^\pm \in M$ are the "centers" of the pseudo CMC spheres $\Sigma_\varepsilon^\pm = \Sigma_\varepsilon(o^\pm)$. Then, we have:

Proposition 3.5.2. *For all $\varepsilon \in (0, \varepsilon_*)$ and all o, v, η, q satisfying the assumption 3.4.1, the mean curvature of $\mathcal{A}_\varepsilon(o, v, \eta, q)$ in $\Omega_{sp}^\pm$ can be written in the form*

$$H^{g_\varepsilon}(\mathcal{A}_\varepsilon) - 2 = \mathcal{H}_0 + \langle C^\pm, \Theta\rangle$$

where

$$C^\pm := -\frac{2\pi\,\varepsilon^3}{15}\,\nabla^g \mathcal{R}(o^\pm) + \frac{\eta}{2}\,\Theta(p_q^\pm) + \mathcal{O}(\varepsilon^5) \in \mathbb{R}^3, \tag{3.39}$$

and

$$|\nabla^k \mathcal{H}_0| \le C_k\, \eta^3\, (d_\pm)^{-4-k},$$

for a constant $C_k > 0$ independent of $\varepsilon, o, v, \eta, a$ and where the derivatives ∇^k are calculated with respect to the metric $g_\pm$ induced on $\Sigma_\varepsilon^\pm$ from g_ε.

Proof. For simplicity we will omit the index g_ε in the expression for the mean curvature, keeping in mind that all the calculations are done with respect to this rescaled metric. By construction, in $\Omega_{sp}^\pm$ the surface $\mathcal{A}_\varepsilon$ can be seen as a transverse graph over the pseudo CMC spheres $\Sigma_\varepsilon^\pm$ of the functions $\eta\,\Gamma^\pm$. Hence, by Proposition 3.3.2,

$$H(\mathcal{A}_\varepsilon) = H(\Sigma_\varepsilon^\pm) + \eta\,\tilde{J}_\pm\,\Gamma^\pm + Q_\pm\left(\eta\,\Gamma^\pm\right),$$

where $\tilde{J}_\pm$ are linearized mean curvature operators and $Q_\pm$ are non-linear, smooth, quadratically vanishing functions. By Proposition 3.4.3,

$$\eta\,\tilde{J}_\pm\,\Gamma^\pm = \eta\,\langle B^\pm,\,\Theta\rangle, \quad \text{where} \quad B^\pm = \frac{1}{2}\,\Theta(p_q^\pm) + \mathcal{O}(\varepsilon^2).$$

On the other hand, by Proposition 3.4.1, we have

$$H(\Sigma_\varepsilon^\pm) = 2 + \langle A_{\varepsilon,o^\pm}, \Theta\rangle, \quad \text{where} \quad A_{\varepsilon,o^\pm} = -\frac{2\pi\,\varepsilon^3}{15}\,\left(\nabla^g\mathcal{R}(o^\pm)\right) + \mathcal{O}(\varepsilon^5).$$

On the other hand, using the structure of the nonlinear terms given by Corollary 3.3.1, we find that the leading terms in $Q_\pm\,(\eta\,\Gamma^\pm)$ are controlled by $\eta^3\left|\nabla^2\Gamma^\pm\,|\nabla\Gamma^\pm|^2\right| \le C\,\eta^3\,\left(d_q^\pm\right)^{-4}$. The estimates for the derivatives of the mean curvature follow from the estimates for the derivatives of $\Gamma^\pm$ and the fact that $(u, \nabla u, \nabla^2 u) \mapsto Q_\pm(u)$ is a smooth function.

□

Finally, we find that the mean curvature of $\mathcal{A}_\varepsilon$ in the "gluing regions" is bounded in L^∞ norm. However, we distinguish two types of terms: terms bounded independently of $\varepsilon, o, v, \eta, a$ and terms for which the estimates depend on the constant C_* appearing in the assumption 3.4.1. We have the following result:

Proposition 3.5.3. *For all $\varepsilon \in (0, \varepsilon_*)$ and all o, v, η, a satisfying the assumptions 3.4.1, the mean curvature of $\mathcal{A}_\varepsilon(\varepsilon, o, v, \eta, a)$ in $\Omega_{glu}^\pm$ can be written in the form*

$$H^{g_\varepsilon}(\mathcal{A}_\varepsilon) = \mathcal{H}_0 + \mathcal{H}_1,$$

where $\operatorname{supp}(\mathcal{H}_1) \subset \Omega_{glu}^+ \cup \Omega_{glu}^-$ *and*

$$|\nabla^k\,\mathcal{H}_0| \le C_k\,r_\eta^{-k}, \qquad |\nabla^k\,\mathcal{H}_1| \le C_k\,C_*\,r_\eta^{-k},$$

for a constant $C_k > 0$ independent of $\varepsilon, o, v, \eta, a$.

Proof. In the gluing regions, the surface $\mathcal{A}_\varepsilon$ is parametrized as normal graphs over the geodesic disc Λ of the functions $\mathcal{G}^\pm$, defined in (3.34).

Let us denote by $H_\Lambda(u)$ the mean curvature of the normal graph over Λ of the function u, calculated with respect to the metric g_ε. On the other hand, we denote by $H_\pm(v)$ the mean curvature of the transverse graphs over $\Sigma_\varepsilon^\pm$ of the function v, when the surfaces are perturbed in the direction of the vector fields $\tilde{N}_\pm$. By Corollary 3.3.1,

$$H_\Lambda(\mathcal{G}^\pm) = H(\Lambda) + J_\Lambda\,\mathcal{G}^\pm + Q_\Lambda(\mathcal{G}^\pm),$$

where J_Λ is the Jacobi operator about Λ, which by Corollary 3.3.1 satisfies

$$J_\Lambda = \Delta_\Lambda + V_\Lambda,$$

where Δ_Λ is the Laplace-Beltrami operator on Λ, calculated with respect to the metric induced from g_ε, V_Λ is a potential bounded by a constant times ε^2, and Q_Λ is a smooth nonlinear quadratically vanishing function. On the other hand, by results of the subsection 3.4.6,

$$H_\Lambda\left(u^\pm - \eta\,\Gamma_\Lambda^\pm\right) = H_\pm\left(\eta\,\Gamma^\pm\right),$$

thus, we can write

$$
\begin{aligned}
H(\mathcal{A}_\varepsilon) &= \chi^\pm H_\pm\left(\eta\,\Gamma^\pm\right) + (1-\chi^\pm)\, H_\Lambda(G^\pm_\eta)\\
&+ 2\,(\nabla_\Lambda\chi)\;\nabla_\Lambda\left(u^\pm - \eta\,\Gamma_\Lambda - G^\pm_\eta\right)\\
&+ (\Delta_\Lambda\chi)\;\left(u^\pm - \eta\,\Gamma^\pm_\Lambda - G^\pm_\eta\right)\\
&+ Q_\Lambda(\mathcal{G}^\pm) - \chi\, Q_\Lambda(u^\pm - \eta\,\Gamma^\pm_\Lambda) - (1-\chi)\, Q_\Lambda(G^\pm_\eta),
\end{aligned}
$$

where ∇_Λ is the gradient associated to the metric g_ε. By Lemma 3.4.4, in $\Omega^\pm_{glu}$, we can have:

$$
u^\pm - \eta\,\Gamma^\pm_\Lambda - G^\pm_\eta = \nabla u^\pm \cdot y + v^\pm, \quad \text{where} \quad |\nabla^k v^\pm| \le C_k\, |y|^{2-k},
$$

for a constant $C_k > 0$ independent of $\varepsilon, o, v, \eta, a$. This yields the desired expression for the mean curvature in $\Omega^\pm_{glu}$, where we put

$$
\mathcal{H}_1 := \Delta_{eucl}\left(\chi\,\nabla\left(u^+ - u^-\right)\cdot y\right), \tag{3.40}
$$

and combining all the other terms in $\mathcal{H}_0$ we verify that

$$
|\nabla^k\,\mathcal{H}_0| \le C_k\, r_\eta^{-k}, \quad |\nabla^k\,\mathcal{H}_1| \le C_k\, C_*\, r_\eta^{-k}
$$

for a constant C_k independent of $\varepsilon, o, v, \eta, a$ for ε small enough.

□

Remark 3.5.2. *By product of this proof is the presence of the term* $\mathcal{H}_1$ (3.40) *in the expression of the mean curvature of the approximate solution* $\mathcal{A}_\varepsilon$*, which plays an important role in the fixed point argument developed in the section 3.7.*

Corollary 3.5.1. *For all* $\varepsilon \in (0, \varepsilon_*)$ *and all* o, v, η, a *satisfying the assumption 3.4.1, the mean curvature of the approximate solution* $\mathcal{A}_\varepsilon$ *can be written in the form*

$$
H^{g_\varepsilon}(\mathcal{A}_\varepsilon) - 2 = \mathcal{H} + \chi^+\langle C^+, \Theta\rangle + \chi^-\,\langle C^-, \Theta\rangle,
$$

where

$$
\|\mathcal{H}\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)} \le C\,\eta^{3/2-\ell\,\nu}
$$

for constants $C > 0$ *and* $\ell \in \mathbb{N}$ *independent of* $\varepsilon, o, v, \eta, a$.

3.5.4 Mean curvature of the perturbed surfaces

Let $\mathcal{A}_\varepsilon(w)$ be the surface obtained as a perturbation of the approximate solution $\mathcal{A}_\varepsilon$ for $w \in \mathcal{C}^{2,\alpha}(\mathcal{A}_\varepsilon)$ as it is described in the beginning of the section. Recall that we can express the mean curvature of $\mathcal{A}_\varepsilon(w)$ in the form

$$
H^{g_\varepsilon}\left(\mathcal{A}_\varepsilon(w)\right) = H^{g_\varepsilon}\left(\mathcal{A}_\varepsilon\right) + \mathcal{L}_\varepsilon\, w + \mathcal{Q}_\varepsilon(w).
$$

In this subsection we analyze the properties of the operators $\mathcal{L}_\varepsilon$ and $\mathcal{Q}_\varepsilon$ in appropriate function spaces. We start by studying these properties in the "spherical" and "gluing" regions, parametrized as transverse graphs over subdomains in the pseudo CMC spheres $\Sigma^\pm_\varepsilon$.

Proposition 3.5.4. *For all $\beta \in (0,1)$ and ε small enough, the linearized mean curvature operator $\mathcal{L}_\varepsilon$ restricted to $\Omega^{\pm}_{sp} \cup \Omega^{\pm}_{glu}$ can be expressed in the form*

$$\mathcal{L}_\varepsilon = \tilde{J}_\pm + \eta^{2-\beta}\,\vartheta^{-4}\,\hat{L}_\pm,$$

where $\tilde{J}_\pm$ are the linearized mean curvature operators about $\Sigma^\pm_\varepsilon$ defined in (3.4.5), are $\hat{L}_\pm$ are linear partial differential operators that satisfy (with the notations 3.4.4):

$$\|\hat{L}_\pm\, w\|_{\mathcal{C}^{0,\alpha}\left(\left(\Sigma^\pm_\varepsilon(1/2)\right)^c,\ g_\pm\right)} \leq C\,\|w\|_{\mathcal{C}^{2,\alpha}\left(\left(\Sigma^\pm_\varepsilon(1/2),\right)^c,\ g_\pm\right)}$$

$$\|\hat{L}_\pm\, w\|_{\mathcal{C}^{0,\alpha}\left(\Sigma^\pm_\varepsilon(\rho,2\rho),\ \vartheta^{-2}\,g_\pm\right)} \leq C\,\|w\|_{\mathcal{C}^{2,\alpha}\left(\Sigma^\pm_\varepsilon(\rho,2\rho),\ \vartheta^{-2}\,g_\pm\right)}, \quad \forall \rho \in (r_\eta, 1/4)$$

for a constant $C > 0$ independent of $\varepsilon, o, v, \eta, a$ and ρ. If in addition $\|\vartheta^{-1}\,w\|_{\mathcal{C}^{1,\alpha}_\nu(\mathcal{A}_\varepsilon)} < 1$, then the nonlinear function $\mathcal{Q}_\varepsilon(w)$ for all $\beta \in (0,1)$ can be expressed in the form

$$\mathcal{Q}_\varepsilon(w) = \frac{\eta^{1-\beta}}{\vartheta^4}\,Q^2_\pm(w) + \frac{\eta^{-\beta}}{\vartheta^4}\,Q^3_\pm(w),$$

where

$$\left\|Q^k_\pm(w_1) - Q^k_\pm(w_2)\right\|_{\mathcal{C}^{0,\alpha}\left(\left(\Sigma^\pm_\varepsilon(1/2)\right)^c,\ g_\pm\right)} \leq C\,\max_{i=1,2}\left\{\|w_i\|^{k-1}_{\mathcal{C}^{2,\alpha}\left(\left(\Sigma^\pm_\varepsilon(1/2)\right)^c,\ g_\pm\right)}\right\} \times \|w_1 - w_2\|_{\mathcal{C}^{2,\alpha}\left(\left(\Sigma^\pm_\varepsilon(1/2)\right)^c,\ g_\pm\right)}$$

$$\left\|Q^k_\pm(w_1) - Q^k_\pm(w_2)\right\|_{\mathcal{C}^{0,\alpha}\left(\Sigma^\pm_\varepsilon(\rho,2\rho),\ \vartheta^{-2}g_\pm\right)} \leq C\,\max_{i=1,2}\left\{\|w_i\|^{k-1}_{\mathcal{C}^{2,\alpha}\left(\Sigma^\pm_\varepsilon(\rho,2\rho),\ \vartheta^{-2}g_\pm\right)}\right\} \times \|w_1 - w_2\|_{\mathcal{C}^{2,\alpha}\left(\Sigma^\pm_\varepsilon(\rho,2\rho),\ \vartheta^{-2}g_\pm\right)},$$

for a constant $C > 0$ independent of $\varepsilon, o, v, \eta, a$ and ρ.

Proof. Again, for simplicity we will omit the index g_ε. First, consider the regions of $\mathcal{A}_n(w)$ parametrized as transverse graphs over $\Sigma^\pm_\varepsilon$ of the functions $\Gamma^\pm + w$. By Proposition 3.3.2,

$$H(\mathcal{A}_\varepsilon(w)) = H(\Sigma^\pm_\varepsilon) + \tilde{J}_\pm\left(\eta\,\Gamma^\pm + w\right) + Q_\pm\left(\eta\,\Gamma^\pm + w\right).$$

Then the properties of $\mathcal{L}_\varepsilon$ and $\mathcal{Q}_\varepsilon$ in this region follow from the properties of the nonlinear function $Q_\pm$ described in Corollary 3.3.1.

Recall that in the "gluing regions", the surface $\mathcal{A}_\varepsilon(w)$ is parametrized by some subdomains of the geodesic disc Λ. Let y^1, y^2 be normal geodesic coordinates in Λ centered at the point q. By construction, we obtain $\mathcal{A}_\varepsilon(w)$ by first taking normal graphs over Λ of the functions $\mathcal{G}^\pm$, and then perturbing them in the direction of some vector fields $\Upsilon^\pm$ which satisfy

$$\left|\nabla^k\,g_\varepsilon(\Upsilon^\pm, N_\Lambda)\right| \leq C_k\,\eta^2\,|y|^{-2-k}$$

for a constant C_k independent of $\varepsilon, o, v, \eta, a$. We check that the corresponding linearized mean curvature operator can be written in the form

$$\Delta_{\mathbb{R}^2} + \eta^2 \, |y|^{-4} \, \hat{L}.$$

Moreover, the same is true for the operators $\tilde{J}_\pm$ defined on $\Sigma_\varepsilon^\pm$, when we parametrize the last surfaces as transverse graphs over Λ. This, together with the properties of the nonlinear terms, yield the properties of $\mathcal{L}_\varepsilon$ and $\mathcal{Q}_\varepsilon$ in $\Omega_{glu}^\pm$.

□

In the next result, we show that the properties of the operators $\mathcal{L}_\varepsilon$ and $\mathcal{Q}_\varepsilon$ in the "catenoidal region" can be deduced from the properties of the mean curvature of normal graphs about the Euclidean catenoid.

Proposition 3.5.5. *For ε small enough, the linearized mean curvature operator about $\mathcal{A}_\varepsilon$ restricted to the region Ω_{cat} can be expressed in the form*

$$\mathcal{L}_\varepsilon = \frac{1}{\eta^2 \cosh^2 s} \left(\partial_s^2 + \partial_\phi^2 + \frac{2}{\cosh^2 s} \right) + \frac{\varepsilon^2}{\eta \cosh s} \, \hat{L}_{cat} \, w,$$

where $J_{cat} = \frac{1}{\eta^2 \cosh^2 s} \left(\partial_s^2 + \partial_\phi^2 + \frac{2}{\cosh^2 s} \right)$ is the Jacobi operator about the Euclidean catenoid scaled by the factor η and $\hat{L}_{cat}$ is a linear partial differential operator such that

$$\| \hat{L}_{cat} \, w \|_{\mathcal{C}^{0,\alpha}([s,s+1] \times S^1)} \le C \, \|w\|_{\mathcal{C}^{2,\alpha}([s,s+1] \times S^1)},$$

for all $s \in [-s_\eta, s_\eta - 1]$ and a constant C independent of $\varepsilon, o, v, \eta, q$ and s. If in addition $\|\vartheta^{-1} \, w\|_{\mathcal{C}^{1,\alpha}_\nu(\mathcal{A}_\varepsilon)} < 1$, then the nonlinear function $\mathcal{Q}_\varepsilon(w)$ can be written in the form

$$\mathcal{Q}_\varepsilon(w) = \frac{1}{\eta^3 \, \cosh^4 s} \, Q_{cat}^2(w) + \frac{1}{\eta^4 \, \cosh^4 s} \, Q_{cat}^3 \, w,$$

where

$$\left\| Q_{cat}^k(w_1) - Q_{cat}^k(w_2) \right\|_{\mathcal{C}^{0,\alpha}([s,s+1] \times S^1)} \le C \max_{i=1,2} \left\{ \|w_i\|^{k-1}_{\mathcal{C}^{2,\alpha}([s,s+1] \times S^1)} \right\} \times \|w_1 - w_2\|_{\mathcal{C}^{2,\alpha}([s,s+1] \times S^1)} \tag{3.41}$$

for a constant C independent of $\varepsilon, o, v, \eta, a$ and s.

Proof. The proof is based on the fact that by construction, the region Ω_{cat} of the surface $\mathcal{A}_\varepsilon(w)$ can be seen as the image by the mapping $F^\varepsilon_{\Lambda,q}$ of the normal (with respect to the Euclidean metric) graph:

$$C_\eta(w) \; : \; (s, \phi) \in [-s_\eta, s_\eta] \times S^1 \mapsto C_\eta(s, \phi) + w \, N_{cat}(s, \phi) \in \mathbb{R}^3$$

over the Euclidean catenoid C_η. The computation of the mean curvature of $C_\eta(w)$ with respect to g_{eucl} is classic and we postpone it to appendix. On the other hand, we use the fact that the pull-back metric $\left(F^\varepsilon_{\Lambda,q} \right)^* g_{eucl}$ can be seen as a perturbation of the euclidean metric:

$$\left(F^\varepsilon_{\Lambda,q} \right)^* g_{eucl}(y, z) = g_{eucl} + \mathcal{O}\left(\varepsilon^2 \left(|y|^2 + |z| \right) \right).$$

To calculate the mean curvature of $C_\eta(w)$ with respect to $\left(F^\varepsilon_{\Lambda,q}\right)^* g_{eucl}$ amounts to add an initial mean curvature term which is equal to $H^{g_\varepsilon}(\mathcal{A}_\varepsilon)$ and some smaller linear and nonlinear terms. Since the nonlinear terms have the same properties as the ones that appear from the computations in the Euclidean space, we only need to understand how changes the linearized mean curvature operator. This is a straight forward computation which can be also found in the Appendix.

□

3.6 Linear analysis

Ideally, our goal is to solve the equation

$$H^{g_\varepsilon}\left(\mathcal{A}_\varepsilon(w)\right) = 2,$$

for some $w \in \mathcal{C}^{2,\alpha}_\nu(\mathcal{A}_\varepsilon)$, or equivalently

$$-\mathcal{L}_\varepsilon\, w = H^{g_\varepsilon}(\mathcal{A}_\varepsilon) - 2 + \mathcal{Q}_\varepsilon(w). \tag{3.42}$$

We hope to find a solution using a fixed point argument and to this end we would like to find a right inverse of the linear operator $\mathcal{L}_\varepsilon$ and study its properties when ε tends to 0. Unfortunately, it turns out that we do not get estimates uniformly bounded in ε, namely because of the presence of small eigenvalues of $\mathcal{L}_\varepsilon$ (eigenvalues that tend to 0 as fast as ε tends to 0), which can be identified knowing the structure of this operator.

By Proposition 3.5.4 and the results of the section 3.4.2, in the "spherical regions" of $\mathcal{A}_\varepsilon$ parametrized by a region of the Euclidean unit 2-sphere, $\mathcal{L}_\varepsilon$ is close to the operator $\Delta_{S^2} + 2$, which has a 3-dimensional kernel that consists of the coordinate functions Θ^i, $i = 1, 2, 3$.

On the other hand, by Proposition 3.5.5, in the "catenoidal region" of $\mathcal{A}_\varepsilon$ the operator $\mathcal{L}_\varepsilon$ is close to the Jacobi operator about the Euclidean catenoid. Recall that due to the isometries in the Euclidean space and the dilation, the catenoid is degenerate. In particular, its Jacobi fields corresponding to horizontal translations are given by the functions

$$\xi_1 = \frac{\cos\phi}{\cosh s} \quad \text{and} \quad \frac{\sin\phi}{\cosh s}$$

which decay very fast at infinity, and hence generate small eigenvalues of $\mathcal{L}_\varepsilon$.

The idea is instead of solving

$$\mathcal{L}_\varepsilon\, w = f,$$

to solve the problem

$$\mathcal{L}_\varepsilon\, w - f \in \mathfrak{K}_\varepsilon, \tag{3.43}$$

where $\mathfrak{K}_\varepsilon$ is a finite (actually 8) dimensional space, which we will define in the subsection 3.6.3 and will refer to as *approximate kernel.* For ε small enough, we find a solution to (3.43) using linear analysis about 2 noncompact domains, namely the punctured sphere $\Sigma^\pm_\varepsilon \setminus \{p^\pm_q\}$ and the infinite catenoid.

3.6.1 Linear analysis on a punctured CMC sphere

Let Σ be a pseudo CMC sphere in M. For $p \in \Sigma$, we use the notation $\Sigma_* := \Sigma \setminus \{p\}$. In this subsection, we analyze the properties of the Jacobi operator J_Σ about Σ_*.

Definition 3.6.1. *We adopt the notations 3.4.4, taking Σ instead of $\Sigma_\varepsilon^\pm$ and p instead of $p_q^\pm$. Then, let $\mu_{sp} \in \mathcal{C}^{2,\alpha}(\Sigma)$ be a function such that $\mu_{sp} \equiv 0$ in $\Sigma(1/4)$ and $\mu \equiv 1$ in $(\Sigma(1/2))^c$.*

We introduce the deficiency space

$$\mathcal{D}_{sp} := \mathrm{span}\{1, \mu_{sp}\}.$$

We have the following result:

Proposition 3.6.1. *Assume that $\nu \in (0,1)$. Then there exists a constant C and for all $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\Sigma_*)$ there exist unique $\psi_{sp} \in \mathcal{C}^{2,\alpha}_{\nu}(\Sigma_*)$, c^1_{sp}, $c^2_{sp} \in \mathbb{R}$, and $B_{sp} \in \mathbb{R}^3$ such that*

$$w_{sp} = \psi_{sp} + c^1_{sp}\, \mu_{sp} + c^2_{sp}$$

satisfies

$$J_\Sigma\, w_{sp} = f - \langle B_{sp}, \Theta \rangle. \tag{3.44}$$

Moreover,

$$\|w_{sp}\|_{\mathcal{C}^{2,\alpha}_{\nu}(\Sigma_*)\oplus \mathcal{D}_{sp}} \leq C\, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\Sigma_*)}, \tag{3.45}$$

$$|B_{sp}| \leq C\, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\Sigma_*)}. \tag{3.46}$$

Before we proceed to the proof of Proposition 3.6.1, let us recall the proof of the following classical result [72]:

Lemma 3.6.1. *Assume that $\nu \in (0,1)$. Then there exists a constant $C > 0$ and for all $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(D^2_*)$ there exist unique $\psi \in \mathcal{C}^{2,\alpha}_{\nu}(D^2_*)$ and $c_* \in \mathbb{R}$ such that $w := c_* + \psi$ satisfies*

$$\begin{cases} \Delta\, w = f & \text{in} \quad D^2_*, \\ w = 0 & \text{in} \quad S^1. \end{cases} \tag{3.47}$$

and

$$\|\psi\|_{\mathcal{C}^{2,\alpha}_{\nu}(D^2_*)} + |c_*| \leq C\, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(D^2_*)}.$$

Proof of Lemma 3.6.1. We construct the solution as a limit of the solutions w_ρ to the Poisson's equation with homogeneous Dirichlet boundary data in $A_\rho := \{y \in \mathbb{R}^2 \,:\, \rho < |y| < 1\}$.

Let (r, ϕ) be polar coordinates in D^2_*. We decompose the functions w_ρ and f in Fourier series in the angular variable ϕ:

$$w_\rho = \sum_{j\in\mathbb{Z}} w_{j,\rho}(r)\, e^{i\phi j} \quad \text{and} \quad f = \sum_{j\in\mathbb{Z}} f_j(r)\, e^{i\phi j}.$$

By linearity, we may assume that $|f| \le r^{\nu-2}$ and therefore $|f_j(r)| \le r^{\nu-2}$. The function $w_{j,\rho}$ then satisfies

$$\left(\partial_r^2 + \frac{1}{r}\,\partial_r - \frac{j^2}{r^2}\right) w_{j,\rho} = f_j \quad \text{in} \quad (\rho, 1), \quad w_{j,\rho}(1) = w_{j,\rho}(\rho) = 0.$$

Notice that for $|j| \ge 1$ the function $\frac{r^\nu}{j^2-\nu^2}$ is a supersolution to our problem and hence

$$|w_{j,\rho}(r)| \le \frac{r^\nu}{j^2 - \nu^2}.$$

We put $w_\rho = \sum_{|j|\ge 1} w_{j,\rho}$, and obtain

$$\|r^{-\nu}\, w_\rho\|_{L^\infty(A_\rho)} \le C\, \|r^{2-\nu}\, f\|_{L^\infty(A_\rho)},$$

for a constant C independent of ρ. By the Schauder's estimates, this yields a uniform bound on the gradient of w_ρ and hence, by the Arzelà-Ascoli theorem, w_ρ converges uniformly on compact sets to a solution of (3.47). Moreover, we have

$$\|w\|_{\mathcal{C}^{2,\alpha}_\nu(D^2_*)} \le C\, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(D^2_*)},$$

which follows from the Schauder's estimates in weighted Hölder spaces obtained from the standard Schauder's estimates [39] on concentric annuli of inner radius R and outer radius $2R$ applied to $w(R\,\cdot)$.

For $j = 0$ we find the solution explicitly:

$$w_0 = \int_0^r \frac{1}{z} \int_0^z t\, f_0(t)\, dt\, dz + c_*, \quad c_* = \int_0^1 \frac{1}{z} \int_0^z t\, f_0(t)\, dt\, dz.$$

□

Proof of Proposition 3.6.1. In geodesic normal coordinates in Σ centered at p, we have

$$J_\Sigma = \Delta + \hat{L}_\Sigma,$$

where

$$\hat{L}_\Sigma = \mathcal{O}(|x|^2) \sum_{i,j=1}^{3} \partial_{x^i}\partial_{x^j} + \mathcal{O}(|x|)\partial_{x^i} + \mathcal{O}(1),$$

and Δ is the flat Laplacian in $\mathbb{R}^2$. First, let w_1 be the solution of the problem

$$\begin{cases} \Delta\, w_1 = (1 - \mu_{sp})\, f & \text{in} \quad D^2_*\left(\frac{1}{2}\right), \\ w_1 = 0 & \text{on} \quad S^1\left(\frac{1}{2}\right), \end{cases}$$

given by Lemma (3.6.1). Then we can write $w_1 = c^1_{sp} + \psi_1$, where

$$\|\psi_1\|_{\mathcal{C}^{2,\alpha}_\nu(D^2_*)} + |c^1_{sp}| \le C\, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(D^2_*)}.$$

Consider the function

$$\tilde{f} := \mu_{sp}\, f + 2\,\nabla\mu_{sp}\,\nabla\, w_1 + \Delta\mu_{sp}\, w_1 - \hat{L}_\Sigma\, \left((1-\mu_{sp})w_1\right) \in \mathcal{C}^{0,\alpha}(\Sigma),$$

which has compact support in Σ and satisfies

$$\|\tilde{f}\|_{\mathcal{C}^{0,\alpha}(\Sigma)} \leq C\, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\Sigma_*)}.$$

According to Proposition 3.4.1, there exist a function $w_2 \in \mathcal{C}^{2,\alpha}(\Sigma)$ and a vector $B_{sp} \in \mathbb{R}^3$ such that

$$J_\Sigma\, w_2 = \tilde{f} - \langle B_{sp}, \Theta\rangle,$$

where

$$\|w_2\|_{\mathcal{C}^{2,\alpha}(\Sigma)} \leq C\, \|\tilde{f}\|_{\mathcal{C}^{0,\alpha}(\Sigma)},$$

$$\left| B_{sp} - \int_{S^2} \tilde{f}\,\Theta\, dvol_{S^2} \right| \leq C\,\varepsilon^2\, \|\tilde{f}\|_{\mathcal{C}^{0,\alpha}(\Sigma)}.$$

We can write $w_2 = \psi_2 + c^2_{sp}$, where

$$\|\psi_2\|_{\mathcal{C}^{2,\alpha}_{\nu}(\Sigma_*)} + |c^2_{sp}| \leq C\, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\Sigma_*)}.$$

Finally, the function $w_{sp} := (1-\mu_{sp})w_1 + w_2$ satisfies the claim of the theorem.

□

Remark 3.6.1. *In what follows we apply the result of Proposition 3.6.1 to the punctured pseudo CMC spheres $\Sigma^\pm_\varepsilon \setminus \{p^\pm_q\}$, but instead of the Jacobi operator we will take the linearized mean curvature operator $\tilde{J}_\pm$ that arises when the surfaces are perturbed in the direction of the vector field $\tilde{N}_\pm$ defined in 3.25, using that these two operators are conjugate.*

3.6.2 Linear analysis on the Euclidean catenoid

Lemma 3.6.2. *Assume that $\delta \in (-1,0)$. The subspace of $(\cosh s)^\delta \mathcal{C}^{2,\alpha}\left(\mathbb{R}\times S^1\right)$ which solves*

$$\left(\partial^2_s + \partial^2_\phi + \frac{2}{\cosh^2 s}\right) w = 0$$

is 2 dimensional and is spanned by the functions

$$\xi_1 = \frac{\cos\phi}{\cosh s} \quad \text{and} \quad \xi_2 = \frac{\sin\phi}{\cosh s}.$$

Proof. We decompose w in Fourier series

$$w(s,\phi) = \sum_{j\in\mathbb{Z}} \omega_j(s) e^{ij\phi},$$

then the functions w_j are solutions of the ordinary equations

$$\left(\partial_s^2 - j^2 + \frac{2}{\cosh^2 s}\right) w_j = 0.$$

These solutions are asymptotic either to $(\cosh s)^j$ or to $(\cosh s)^{-j}$. By hypothesis, the solution is bounded by a constant times $(\cosh s)^\delta$ and $|\delta| < 1$, so the solution has to be asymptotic to $(\cosh s)^{-j}$, and then the solution is bounded. On the other hand, $-j^2 + \dfrac{2}{\cosh^2 s} \leq 0$, so the maximum principle assures that $w_j = 0$, for all $|j| \geq 2$.

By a direct computation we find that for $|j| = 1$ the space of solutions is spanned by the functions $\frac{1}{\cosh s}$ and $\sinh s + \frac{s}{\cosh s}$ while the functions $\tanh s$ and $1 - s\tanh s$ are two independent solutions when $j = 0$. Among these four functions only the first one belongs to $(\cosh s)^\delta\,\mathcal{C}^{2,\alpha}(\mathbb{R} \times S^1)$.

$\square$

Definition 3.6.2. *Let $\mu_{cat} \in \mathcal{C}^\infty(\mathbb{R} \times S^1)$ be a cut-off function which is identically equal to 0 in $(-\infty, 1) \times S^1$ and to 1 in $(2, +\infty) \times S^1$. We introduce the deficiency space*

$$\mathcal{D}_{cat} := \operatorname{span}\{\mu_{cat}, s\,\mu_{cat}\}.$$

Proposition 3.6.2. *Assume that $\delta \in (-1, 0)$. Then there exists a constant C such that for all $h \in (\cosh s)^\delta \mathcal{C}^{0,\alpha}(\mathbb{R} \times S^1)$ there exist unique function $v_{cat} \in (\cosh s)^\delta \mathcal{C}^{2,\alpha}(\mathbb{R} \times S^1)$ and constants c^i_{cat}, B^i_{cat}, $i = 1, 2$ such that*

$$w_{cat} = v_{cat} + \mu_{cat}\left(c^1_{cat} + c^2_{cat}\, s\right)$$

satisfies

$$\left(\partial_s^2 + \partial_\phi^2 + \frac{2}{\cosh^2 s}\right) w_{cat} = h - B^1_{cat}\,\xi_1 - B^2_{cat}\,\xi_2.$$

Moreover, we have

$$B^i_{cat} = \int_{\mathbb{R}\times S^1} h\,\xi_i\, ds\, d\phi \quad \textit{and} \quad \int_{\mathbb{R}\times S^1} w\,\xi_i = 0,\ i = 1, 2, \tag{3.48}$$

and

$$\|(\cosh s)^{-\delta}\, v_{cat}\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}\times S^1)} + |c^2_{cat}| + |c^2_{cat}| \leq C\,\|(\cosh s)^{-\delta} h\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}\times S^1)}. \tag{3.49}$$

Proof. We borrow the techniques of the proof from [77] and [93], but for the sake of completeness we give here the details. Decompose h in Fourier series in ϕ:

$$h(s, \phi) = \sum_{j\in\mathbb{Z}} h_j(s)\, e^{i\phi j}.$$

Then for every $t \in \mathbb{R}$ and $|j| \geq 2$, let v^t_j be a solution of

$$\left(\frac{d^2}{ds^2} - j^2 + \frac{2}{\cosh^2 s}\right) v^t_j = h_j \quad \text{in} \quad |s| < t, \quad v^t_j(\pm t) = 0,$$

which can be obtained by the maximum principal and the method of sub- and supersolutions, taking $\dfrac{1}{j^2-2-\delta}\,(\cosh s)^{\delta}$ as a barrier function. Taking a sum over $|j|>2$ of $v_j^t\, e^{i\phi j}$, we obtain a function v_t which by Schauder's elliptic theory, satisfies

$$\|(\cosh s)^{-\delta} v_t\|_{\mathcal{C}^{2,\alpha}((-t,t)\times S^1)} \leq C\, \|(\cosh s)^{-\delta} h\|_{\mathcal{C}^{0,\alpha}((-t,t)\times S^1)}, \tag{3.50}$$

for a constant C independent of t. When t tends to infinity, by the Arzelà-Ascoli theorem the sequence v_t admits a subsequence that converges on compact sets to a function v such that

$$\left(\partial_s^2+\partial_\phi^2+\frac{2}{\cosh^2 s}\right) v = \sum_{|j|\geq 2} h_j(r)\, e^{i\phi j}.$$

For $j=1$ and $j=0$ we construct the solutions explicitly. We use the notation

$$\tilde h_{\pm 1} := h_{\pm 1} - \frac{2\pi}{\cosh s}\int_{\mathbb{R}} \frac{h_{\pm 1}(t)}{\cosh t}\, dt.$$

Then by the variation of the constant method we obtain

$$v_{\pm 1} = \frac{1}{\cosh s}\int_0^s \left(\sinh t + \frac{t}{\cosh t}\right) \tilde h_{\pm 1}(t)\, dt - \left(\sinh s + \frac{s}{\cosh s}\right)\int_{-\infty}^{s} \frac{\tilde h_{\pm 1}(t)}{\cosh t}\, dt.$$

A simple verification, using $\displaystyle\int_{\mathbb{R}} \frac{\tilde h_{\pm 1(t)}}{\cosh t}\, dt = 0$, gives

$$\left\|(\cosh s)^{-\delta}\, v_{\pm 1}\right\|_{L^\infty(\mathbb{R}\times S^1)} \leq c\, \left\|(\cosh s)^{-\delta}\, h\right\|_{L^\infty(\mathbb{R}\times S^1)}.$$

Finally, we take

$$w_0(s) = \tanh s \int_0^s (1-t\,\tanh t)\, h_0(t)\, dt - (1-s\,\tanh s)\int_0^s \tanh t\, h_0(t)\, dt$$

and notice that there exist constants $\tilde c_1, \tilde c_2$ and c^1_{cat} and c^2_{cat} such that

$$w_0 + \tilde c_1\,(1-s\tanh s) + \tilde c_2\,\tanh s = v_0 + \mu_{cat}\left(c^1_{cat} + c^2_{cat} s\right),$$

where

$$\|(\cosh s)^{-\delta}\, v_0\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}\times S^1)} + |c^1_{cat}| + |c^2_{cat}| \leq \|(\cosh s)^{-\delta} h\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}\times S^1)}.$$

The estimates for derivatives of w_0 are obtained by the Schauder's theory. Finally, we put $v_{cat} = v + v_0 + v_1\, e^{i\phi} + v_{-1}\, e^{-i\phi}$.

□

3.6.3 Gluing the parametrices together

In this subsection we construct a right inverse of the operator $\mathcal{L}_\varepsilon$ in some appropriate function spaces.

Notation 3.6.1. *Let us adopt the notations 3.4.4 and introduce a cut-off function* $\chi_0 \in \mathcal{C}^{2,\alpha}(\mathcal{A}_\varepsilon)$ *such that* $\chi_0 \equiv 0$ *in regions parametrized by* $(\Sigma_\varepsilon^\pm(1/2))^c$ *and* $\chi_0 \equiv 1$ *in the union of* Ω_{cat} *and the regions parametrized by* $\Sigma_\varepsilon(r_\eta/2, 1/4)$.

Let $\chi^\pm$ be the cut-off functions defined in Notation 3.5.1.

Definition 3.6.3 (Approximate kernel)**.** *Let us introduce the functions*

$$\Phi_j := \chi^+ \, \Theta^j, \quad \Phi_{3+j} := \chi^- \, \Theta^j, \; j = 1,2,3, \quad \Phi_7 := \frac{\chi_0}{\eta^2 \cosh^2 s} \, \xi_1, \quad \Phi_8 := \frac{\chi_0}{\eta^2 \cosh^2 s} \, \xi_2.$$

We define the space

$$\mathfrak{K}_\varepsilon = \operatorname{span}\Big\{ \Phi_j, \; j = 1, \dots 8 \Big\} \tag{3.51}$$

to be the approximate kernel of the operator $\mathcal{L}_\varepsilon$.

Proposition 3.6.3. *Assume that* $\nu \in (0,1)$. *Then for all* $\varepsilon \in (0, \varepsilon_*)$, *all* o, v, η, a *satisfying the assumptions 3.4.1 and for all* $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)$ *there exists a function* $w \in \mathcal{C}^{2,\alpha}_{\nu}(\mathcal{A}_\varepsilon)$ *and a vector* $A \in \mathbb{R}^8$ *such that*

$$\mathcal{L}_\varepsilon \, w = f - \sum_{i=1}^{8} A^i \, \Phi_i. \tag{3.52}$$

Moreover, we have

$$\|w\|_{\mathcal{C}^{2,\alpha}_{\nu}(\mathcal{A}_\varepsilon)} \le C \, \eta^{-\ell \nu} \, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)} \quad \text{and} \quad |A| \le C \, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)}$$

for constants $C > 0$ *and* $\ell \in \mathbb{N}$ *independent of* $\varepsilon, o, v, \eta, a$.

Proof. The proof consists of constructing an an approximate solution of (3.52) by gluing together solutions of linear problems on the punctured CMC spheres $\Sigma_\varepsilon^\pm \setminus \{p_q^\pm\}$ and the Euclidean catenoid C_η obtained by Propositions 3.6.1 and 3.6.2, and then applying a perturbation argument to find an exact solution. We do this in 5 steps.

In Step 1, we show that a function $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)$ can be decomposed as a sum $f = f^+ + f^-$, where

$$f^+ \equiv 0 \quad \text{in} \quad \Omega^-_{glu} \cup \Omega^-_{sp} \quad \text{and} \quad f^- \equiv 0 \quad \text{in} \quad \Omega^+_{glu} \cup \Omega^+_{sp}.$$

Then we find diffeomorphisms $\Psi^\pm$ from some regions in $\Sigma_\varepsilon^\pm \setminus \{p_q^\pm\}$ to a region in $\mathcal{A}_\varepsilon$, such that the functions $\breve{f}^\pm := f^\pm \circ \Psi^\pm$ can be extended up to the punctures in such a way that

$$\breve{f}^\pm \in \mathcal{C}^{0,\alpha}_{\nu-2}\left(\Sigma_\varepsilon^\pm \setminus \{p_q^\pm\}\right).$$

In Step 2, using the results of the subsection 3.6.1, we find solutions to the linear problems

$$\tilde{J}_{\pm}\, \breve{w}^{\pm}_{sp} = \breve{f}^{\pm} - \langle B^{\pm}_{sp}, \Theta\rangle,$$

and show that the functions $\breve{w}^{\pm}_{sp} \circ (\Psi^{\pm})^{-1}$ provide an approximate solution to (3.52) in the regions $\Omega^{\pm}_{sp} \cup \Omega^{\pm}_{glu}$ of $\mathcal{A}_\varepsilon$.

In Step 3, we explain that in the catenoidal region Ω_{cat} the error is of special form and can be corrected using the linear analysis on the Euclidean catenoid described in the subsection 3.6.2. Here again, we solve a linear problem in a noncompact domain, namely, in $\mathbb{R} \times S^1$.

In Step 4, we combine the solutions obtained in Steps 2 and 3 to obtain an approximate solution to (3.52) in $\mathcal{A}_\varepsilon$, truncating the terms that decay at infinity and gluing together the deficiency terms.

Finally, in Step 5, we find an exact solution by applying a perturbation argument.

Step 1: Decomposition of the function f

We introduce the cut-off function $\mathfrak{X} \in \mathcal{C}^\infty(\mathcal{A}_\varepsilon)$ such that in the region Ω_{cat} endowed with cylindrical coordinates (s, ϕ) we have

$$\mathfrak{X} \equiv 0 \quad \text{for} \quad s < -1 \quad \text{and} \quad \mathfrak{X} \equiv 1 \quad \text{for} \quad s > 1$$

and extend $\mathfrak{X}$ to the entire surface $\mathcal{A}_\varepsilon$ by 0 and by 1. Then for $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)$, we can write

$$f = \mathfrak{X}\, f + (1 - \mathfrak{X})\, f = f^+ + f^-.$$

Next, we explain now the construction of $\Psi^\pm$.

Notation 3.6.2. *Let us use Notation 3.4.4 and also denote by $\mathcal{A}^{\pm}_c(\rho)$ the parts of $\mathcal{A}_\varepsilon$ parametrized by $(\Sigma^{\pm}_\varepsilon(\rho))^c$ and put*

$$\mathcal{A}^0(\rho) := \mathcal{A}_\varepsilon \setminus \left(\mathcal{A}^+_c(\rho) \cup \mathcal{A}^-_c(\rho)\right).$$

Next, consider the mapping that provides cylindrical coordinates (s, ϕ) in $\mathcal{A}^0(\frac{1}{4})$:

$$\zeta^0_{cyl} \,:\, (s, \phi) \in [\,-\,\hat{s}, \hat{s}] \times S^1 \mapsto \zeta^0_{cyl}(s, \phi) \in \mathcal{A}^0\,(1/4)\,, \quad \eta\, \cosh \hat{s} = 1/4.$$

On the other hand, one can define cylindrical coordinates in $\Sigma^+_\varepsilon(1/4)$ via the mapping

$$\zeta^+_{cyl} \,:\, (s, \phi) \in [\,-\,\infty, \hat{s}] \times S^1 \mapsto \mathcal{F}^+_\Lambda \left(\frac{\eta}{2} e^s\,(\cos\phi, \sin\phi)\right) \in \Sigma^+\,(1/4)\,.$$

In the same manner, one defines cylindrical coordinates in $\Sigma^-_\varepsilon(1/4)$ via the mapping

$$\zeta^-_{cyl} \,:\, (s, \phi) \in [\,-\,\hat{s}, +\infty] \times S^1 \mapsto \mathcal{F}^-_\Lambda \left(\frac{\eta}{2}\, e^{-s}\, (\cos\phi, \sin\phi)\right).$$

Remark that the mappings

$$\zeta^0_{cyl} \circ \left(\zeta^{\pm}_{cyl}\right)^{-1} \,:\, \Sigma^{\pm}_\varepsilon \left(e^{-\hat{s}}, 1/4\right) \longrightarrow \mathcal{A}^0\,(1/4)$$

locally provide the diffeomorphisms we are looking for.

On the other hand, by construction, the regions $\Omega^{\pm}_{sp} \cup \Omega^{\pm}_{glu} \subset \mathcal{A}_\varepsilon$ are parametrized as transverse graphs over some regions in $\Sigma^{\pm}_{\varepsilon}$ and we denote the corresponding mappings by

$$\zeta^{\pm}_{sp} \,:\, \left(\Sigma^{\pm}_{\varepsilon}\,(1/2\, r_\eta)\right)^c \longrightarrow \Omega^{\pm}_{sp} \cup \Omega^{\pm}_{glu} \subset \mathcal{A}_\varepsilon,$$

where r_η describes the size of the gluing region (3.33).

By construction, in $\Sigma^{\pm}_{\varepsilon}(1/2\, r_\eta, 1/4)$

$$\zeta^0_{cyl} = \zeta^{\pm}_{sp} \circ \mathcal{F}^{\pm}_{\Lambda}\,(\eta\, \cosh s\,(\cos\phi, \sin\phi))\,.$$

Thus, the mappings

$$\Phi^{\pm} := \left(\zeta^{\pm}_{sp}\right)^{-1} \circ \zeta^0_{cyl} \circ \left(\zeta^{\pm}_{cyl}\right)^{-1} \,:\, \Sigma^{\pm}_{\varepsilon}\,(1/2\, r_\eta, 1/4) \longrightarrow \Sigma^{\pm}_{\varepsilon}\,(1/2\, r_\eta, 1/4)\,,$$

satisfy

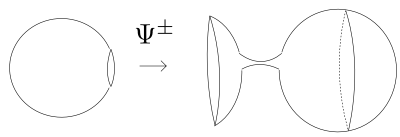
$$\left|\left(\Phi^{+} - \mathrm{Id}\right)(s,\phi)\right| \leq c\,\eta\, e^{-s} \quad \text{for} \quad s \in (s_\eta, \hat{s}), \qquad \left|\left(\Phi^{-} - \mathrm{Id}\right)(s,\phi)\right| \leq c\,\eta\, e^{s} \quad \text{for} \quad s \in (-\hat{s}, -s_\eta),$$

where $s_\eta > 0$ is defined by $\eta\, \cosh s_\eta = \frac{1}{2}\, r_\eta$. With the help of some cut-off functions supported in $\Sigma^{\pm}_{\varepsilon}\,(1/5, 2/5)$ we glue $\Phi^{\pm}$ with the identity, extending it to:

$$\hat{\Phi}^{\pm} \,:\, \left(\Sigma^{\pm}_{\varepsilon}\,(1/2\, r_\eta)\right)^c \longrightarrow \left(\Sigma^{\pm}_{\varepsilon}\,(1/2\, r_\eta)\right)^c.$$

Finally, we define the mappings $\Psi^{\pm}_{\varepsilon} : \left(\Sigma^{\pm}(e^{-\hat{s}})\right) \to \mathcal{A}_\varepsilon \setminus \mathcal{A}^{\mp}_{c}(\frac{1}{4})$ by

$$\Psi^{\pm} = \begin{cases} \zeta^0_{cyl} \circ \left(\zeta^{\pm}_{cyl}\right) & \text{in} \quad \Sigma^{\pm}_{\varepsilon}\left(e^{-\hat{s}}, 1/4\right) \\ \zeta^{\pm}_{sp} \circ \hat{\Phi}^{\pm} & \text{in} \quad \left(\Sigma^{\pm}_{\varepsilon}\,(1/4)\right)^c \end{cases}$$



and verify that the functions $\breve{f}^{\pm} := f^{\pm} \circ \Psi^{\pm}$ can be extended by 0 to the entire punctured CMC spheres $\Sigma^{\pm}_{\varepsilon} \setminus \{p^{\pm}_{q}\}$ and satisfy $\breve{f}^{\pm} \in \mathcal{C}^{0,\alpha}_{\nu-2}\left(\Sigma^{\pm}_{\varepsilon} \setminus \{p^{\pm}_{q}\}\right)$.

Step 2 : Contribution of the linear analysis about the pseudo CMC spheres

Making use of Proposition 3.6.1 we find functions $\breve{w}^{\pm}_{sp} \in \mathcal{C}^{2,\alpha}_{\nu}\left(\Sigma^{\pm}_{\varepsilon} \setminus \{p^{\pm}_{q}\}\right) \oplus \mathcal{D}_{sp}$ and vectors $B^{\pm}_{sp} \in \mathbb{R}^3$ which satisfy

$$\tilde{J}_{\pm}\, \breve{w}^{\pm}_{sp} = \breve{f}^{\pm} - \langle B^{\pm}_{sp}, \Theta\rangle.$$

Moreover, $\breve{w}^{\pm}_{sp}$ can be decomposed as

$$\breve{w}^{\pm}_{sp} = \breve{\psi}^{\pm}_{sp} + c^{1,\pm}_{sp} + c^{2,\pm}_{sp}\, \mu^{\pm}_{sp}, \tag{3.53}$$

where

$$\|\breve{\psi}^{\pm}_{sp}\|_{\mathcal{C}^{2,\alpha}_{\nu}(\Sigma^{\pm}_{\varepsilon}\setminus\{p^{\pm}_{q}\})} + |c^{1,\pm}_{sp}| + |c^{2,\pm}_{sp}| \leq C\, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)} \quad \text{and} \quad \left|B^{\pm}_{sp}\right| \leq C\, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)}.$$

Next, we show that in $\Omega^{\pm}_{sp} \cup \Omega^{\pm}_{glu}$, the functions

$$w^{\pm}_{sp} := \breve{w}^{\pm}_{sp} \circ \left(\Psi^{\pm}\right)^{-1}$$

provide approximate solutions to (3.52). This follows from the fact that in these regions the operator $\mathcal{L}_\varepsilon$ is close to the linearized mean curvature operator $\tilde{J}_{\pm}$ about $\Sigma^{\pm}_{\varepsilon}$ and the fact that the mapping $\Phi^{\pm} = (\zeta^{\pm}_{sp})^{-1} \circ \Psi^{\pm}$, defined in Step 1, is close to identity.

Indeed, according to Propositions 3.5.4 and 3.5.5, there exists a constant $C > 0$, such that for all $u \in \mathcal{C}^{2,\alpha}_{\nu}(\mathcal{A}_\varepsilon)$, we have

$$\left|\mathcal{L}_\varepsilon \, u - \tilde{J}_{\pm} \, u\right| \leq C \, \eta^2 \, \vartheta^{-4} \, \|u\|_{\mathcal{C}^{2,\alpha}_{\nu}(\mathcal{A}_\varepsilon)}.$$

Moreover, by the definition of the diffeomorphisms $\Psi^{\pm}$ and $\Phi^{\pm}$, in $\Omega^{\pm}_{sp} \cup \Omega^{\pm}_{glu}$, we find

$$\left|\tilde{J}_{\pm} \, w^{\pm}_{sp} - f + \chi^{\pm} \, \langle B^{\pm}_{sp}, \Theta\rangle\right| \leq C \left|\left(\tilde{J}_{\pm} \circ \Phi^{\pm} - \tilde{J}_{\pm}\right) \left(w^{\pm}_{sp}\right)\right| \leq C \, \eta^2 \, \vartheta^{-4} \, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)}.$$

This yields

$$\left\|\mathcal{L}_\varepsilon \, w^{\pm}_{sp} - f - \chi^{\pm} \, \langle B^{\pm}_{sp}, \Theta\rangle\right\|_{\mathcal{C}^{0,\alpha}_{\nu-2}\left(\Omega^{\pm}_{sp} \cup \Omega^{\pm}_{glu}\right)} \leq C \, \eta^{\frac{1}{2}} \, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)}. \tag{3.54}$$

Next, we remark that the functions $\breve{\psi}^{\pm}_{sp} \in \mathcal{C}^{2,\alpha}_{\nu}\left(\Sigma^{\pm}_{\varepsilon} \setminus \{p^{\pm}_{q}\}\right)$ in the decomposition (3.53) decay in the neighborhood of $p^{\pm}_{q}$ as a power of the distance to $p^{\pm}_{q}$. For the moment, we leave the deficiency terms aside and show that we can naturally extend

$$\psi^{\pm}_{sp} := \breve{\psi}^{\pm}_{sp} \circ \left(\Psi^{\pm}\right)^{-1}$$

to the entire surface $\mathcal{A}_\varepsilon$ by truncating them in the "opposite" gluing region $\Omega^{\mp}_{glu}$ with the help of an appropriate cut-off function. More precisely, we introduce the function

$$\psi_{sp} := \left(1 - \chi^{-}\right) \, \psi^{+}_{sp} + (1 - \chi)^{+} \, \psi^{-}_{\eta} \quad \in \mathcal{C}^{2,\alpha}_{\nu}(\mathcal{A}_\varepsilon). \tag{3.55}$$

Then the estimates

$$\left|\nabla^k \, \psi^{\pm}_{sp}(s, \phi)\right| \leq c \, \eta^{\nu} \, (\cosh s)^{-\nu} \, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)} \quad \text{in} \quad \Omega^{\mp}_{glu} \cup \Omega_{cat},$$

together with the fact, that by Proposition 3.5.5 we have

$$\left|\eta^2 \, \cosh^2 s \, \left(\mathcal{L}_\varepsilon - \tilde{J}_{\pm}\right) \, u\right| \leq c \, (\cosh s)^{-2} \, \|u\|_{\mathcal{C}^{2,\alpha}_{\nu}(\mathcal{A}_\varepsilon)} \quad \text{in} \quad \Omega_{cat}$$

yield

$$\left|\eta^2 \cosh^2 s \Big(\mathcal{L}_\varepsilon \, \psi_{sp} - f - \chi^{\pm} \, \langle B^{\pm}_{sp}, \Theta\rangle\Big)\right| \leq c \, \eta^{\nu} \, (\cosh s)^{-\nu} \, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)}. \tag{3.56}$$

Step 3: Contribution of the linear analysis about the catenoid

Let the function $\chi_0 \in \mathcal{C}^\infty(\Lambda)$ be given as in Notation 3.6.1 and consider the function

$$h := \chi_0 \, \eta^2 \cosh^2 \left(\mathcal{L}_\varepsilon \, \psi_{sp} - f - \chi^+ \langle B^+_{sp}, \Theta \rangle - \chi^- \langle B^-_{sp}, \Theta \rangle \right)$$

which, by (3.56), satisfies

$$\|(\cosh s)^\nu \, h\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}\times S^1)} \le C \, \eta^\nu \, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)}.$$

By Proposition 3.6.2, (where we take $\delta = -\nu$), there exists a function

$$\breve{w}_{cat} \in (\cosh s)^{-\nu} \mathcal{C}^{2,\alpha} \oplus \mathcal{D}_{cat}$$

and constants B^1_{cat} and B^2_{cat} such that

$$\left(\partial_s^2 + \partial_\phi^2 + \frac{2}{\cosh^2 s} \right) \breve{w}_{cat} = h - B^1_{cat} \, \xi_1 - B^2_{cat} \, \xi_2, \quad \text{and} \quad B^i_{cat} = \int_{\mathbb{R}\times S^1} h \, \xi_i \, ds \, d\phi.$$

Moreover, we have $\breve{w}_{cat} = \breve{v}_{cat} + \mu_{cat} \left(c^1_{cat} + c^2_{cat} \, s \right)$ and

$$\|(\cosh s)^\nu \, \breve{v}_{cat}\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}\times S^1)} + |c^1_{cat}| + |c^2_{cat}| \le C \, \eta^\nu \, \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)}, \quad \left| B^i_{cat} \right| \le C \, \eta^\nu \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)}.$$

Once again, since $\breve{v}_{cat}$ has exponential decay at $\pm\infty$, we can naturally extend it to the entire surface $\mathcal{A}_\varepsilon$ by truncating for s large enough with the help of some cut-off function. We introduce:

$$v_{cat} := \chi_0 \, \breve{v}_{cat}, \tag{3.57}$$

where χ_0 is defined as in Notation 3.6.1.

Step 4: Approximate solution to 3.52

Now we need to understand how to extend to $\mathcal{A}_\varepsilon$ the deficiency terms $c^{1,\pm}_{sp} + c^{2,\pm}_{sp} \, \mu^\pm_{sp}$ coming from the linear analysis about the punctured CMC spheres $\Sigma^\pm_\varepsilon \setminus \{p^\pm_q\}$, and the deficiency terms $\mu_{cat}(c^1_{cat} + c^2_{cat} \, s)$ coming from the linear analysis about the Euclidean catenoid.

Let $\Gamma^\pm$ be the Green's functions associated to the operators $\tilde{J}_\pm$ with poles at $p^\pm_q$ defined in Lemma 3.4.3, and consider the functions

$$s \mapsto 1 - s \tanh s \quad \text{and} \quad s \mapsto \tanh s$$

which are the Jacobi fields corresponding to dilation and vertical translation of the Euclidean catenoid. We introduce the function

$$\kappa := \chi^+ \left(c^{1,+}_{sp} + c^{2,+}_{sp} \mu^+_{sp} + k_+ \, \Gamma^+ \right) + \chi^- \left(c^{1,-}_{sp} + c^{2,-}_{sp} \, \mu^-_{sp} + k_- \, \Gamma^- \right) \tag{3.58}$$
$$+ (1 - \chi^+ - \chi^-) \left(\mu_{cat} \left(c^1_{cat} + c^2_{cat} \, s \right) + k_0 \, (1 - s \tanh s) + k_1 \, \tanh s \right),$$

where k_0, k_1 and $k_\pm$ are constants which we would like to choose in such a way that κ is close to $c_{sp}^{1,\pm} + c_{sp}^{2,\pm}\,\mu_{sp}^{\pm} + k_\pm\,\Gamma^\pm$ in $\Omega_{glu}^{\pm}$. In this region we have

$$1 - s\,\tanh s = 1 \mp s + \mathcal{O}(\eta^{\frac{1}{2}}), \quad \tanh s = \pm 1 + \mathcal{O}(\eta^{\frac{1}{2}}).$$

Moreover, with the change of coordinates $\eta\cosh s = |y|$, we find

$$\Gamma^\pm(s,\phi) = -\log\frac{\eta}{2} \mp s + c_0^\pm + \mathcal{O}(\eta^{1/2}).$$

Finally, we choose k_0, k_1 and k_+ and k_- to be the unique solution of the system

$$\begin{cases} k_+ = -k_0 + c_{cat}^2, & k_0 + k_1 + c_{cat}^1 = c_{sp}^{1,+} + c_{sp}^{2,+} + k_+\,\log\frac{\eta}{2} + k_+\,c_0^+; \\ k_- = k_0 & k_0 - k_1 + c_{cat}^1 = c_{sp}^{1,-} + c_{sp}^{2,-} + k_-\,\log\frac{\eta}{2} + k_-\,c_0^-; \end{cases}$$

and since $|c_{cat}^i|$ and $|c_{sp}^{i,\pm}|$ are bounded by $\|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)}$, we obtain

$$\left\|\kappa - \left(c_{sp}^{1,\pm} + c_{sp}^{2,\pm}\,\mu_{sp}^{\pm} + k^\pm\Gamma_\Lambda^\pm\right)\right\|_{\mathcal{C}^{2,\alpha}_{\nu-2}(\Omega_{glu}^{\pm})} \le c\,\eta^{1/2}\,\|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)}.$$

Next, consider the function

$$w_{app} := \psi_{sp} + v_{cat} + \kappa,$$

where ψ_{sp} and v_{cat} are given by (3.55) and (3.57) and the vector $A_{app} \in \mathbb{R}^8$ defined by

$$A_{app}^j = (B_{sp}^+)^j, \quad A_{app}^{j+3} = (B_{sp}^-)^j, \quad j = 1,2,3, \quad A_{app}^{7,8} = B_{cat}^{1,2}.$$

By results described in Step 2, Step 3 and Step 4, (w_{app}, A_{app}) satisfies

$$\left\|\mathcal{L}_\varepsilon\,w_{app} + \sum_{i=1}^{8} A_{app}^i\,\Phi_i - f\right\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)} \le c\,\eta^{\nu/4}\,\|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)}, \tag{3.59}$$

for a constant c independent of η. Moreover, there exist constants $\ell \in \mathbb{N}$ and $C > 0$ independent of η and ε, such that

$$\|w_{app}\|_{\mathcal{C}^{2,\alpha}_{\nu}} \le C\,\eta^{-\ell\,\nu}\|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)} \quad \text{and} \quad |A_{app}| \le C\,\|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)}.$$

Step 5: Exact solution

Consider the mappings

$$f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon) \mapsto (w_{app}(f), A_{app}(f)) \in \mathcal{C}^{2,\alpha}_{\nu}(\mathcal{A}_\varepsilon) \oplus \mathbb{R}^8, \quad \text{and}$$

$$\mathfrak{R} : \mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon) \mapsto \mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon), \quad \mathfrak{R}(f) := \mathcal{L}_\varepsilon\,w_{app}(f) + \sum_{i=1}^{8} A_{app}^i(f)\,\Phi_i - \mathrm{Id}(f).$$

Then $\|\mathfrak{R}\| \leq c\,\eta^{\frac{\nu}{4}}$, so $\mathrm{Id} + \mathfrak{R}$ is invertible for η small enough. Finally, we put

$$w(f) = w_{app}\left((\mathrm{Id} + \mathfrak{R})^{-1}(f)\right) \quad \text{and} \quad A^i(f) = A^i_{app}\left((\mathrm{Id} + \mathfrak{R})^{-1}(f)\right), \tag{3.60}$$

which yields

$$\mathcal{L}_\varepsilon\, w(f) + \sum_{i=1}^{8} A^i(f)\, \Phi_i = f. \tag{3.61}$$

Moreover,

$$\|w(f)\|_{\mathcal{C}^{2,\alpha}_\nu(\mathcal{A}_\varepsilon)} \leq C\,\eta^{-\ell\,\nu} \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)} \quad \text{and} \quad |A(f)| \leq C\,\|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)},$$

for $C > 0$ and $l \in \mathbb{N}$ independent of $\varepsilon, o, v, \eta, a$. This finishes the proof of Proposition 3.6.3.

$\square$

3.7 Nonlinear argument

Using the notations introduced in the section 3.5, ideally our goal is to solve the equation

$$\mathcal{L}_\varepsilon\, w = 2 - H^{g_\varepsilon}(\mathcal{A}_\varepsilon) - \mathcal{Q}_\varepsilon(w).$$

If the linear operator $\mathcal{L}_\varepsilon : \mathcal{C}^{2,\alpha}_\nu(\mathcal{A}_\varepsilon) \to \mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)$ were invertible with inverse uniformly bounded when ε tends to 0, we could use Banach fixed point theorem for contracting mappings in a ball of $\mathcal{C}^{2,\alpha}_\nu(\mathcal{A}_\varepsilon)$, where the radius of the ball would be defined by a constant times $\|\mathcal{L}_\varepsilon^{-1}\|\,\|H^{g_\varepsilon}(\mathcal{A}_\varepsilon) - 2\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)}$.

However, this is not the case and according to Proposition 3.6.3, for all $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)$, we only can solve the problem

$$\mathcal{L}_\varepsilon\, w - f \in \mathfrak{K}_\varepsilon,$$

where $\mathfrak{K}_\varepsilon = \mathrm{span}\,\{\Phi_i,\, i = 1, \ldots, 8\}$ is an 8-dimensional space which we refer to as the approximate kernel of $\mathcal{L}_\varepsilon$. In the subsection 3.7.1 we explain the Lyapunov-Schmidt reduction argument, which consists in applying Banach fixed point theorem in the space nearly orthogonal to $\mathfrak{K}_\varepsilon$. Then, we obtain a surface the mean curvature of which is constant up to a element of the form $\sum_{i=1}^{8} A^i\,\Phi_i$. Finally, in the subsection 3.7.2, we show that we can choose the parameters o, v, η, a in our construction in such a way that $A^i = 0$, $i = 1, \ldots, 8$.

3.7.1 Lyapunov-Schmidt reduction argument

Our goal is to solve the problem

$$H^{g_\varepsilon}(\mathcal{A}_\varepsilon) - 2 + \mathcal{L}_\varepsilon\, w + \mathcal{Q}_\varepsilon(w) \in \mathfrak{K}_\varepsilon. \tag{3.62}$$

As we have mentioned above, the radius of the ball in which we hope to carry out a fixed point argument depends on the norm of the function $H(\mathcal{A}_\varepsilon) - 2$. By Corollary 3.5.1, we have

$$H^{g_\varepsilon}(\mathcal{A}_\varepsilon) = 2 + \mathcal{H} + \chi^+\langle C^+, \Theta\rangle + \chi^-\langle C^-, \Theta\rangle$$

where

$$\chi^{\pm}\,\langle C^{\pm},\Theta\rangle \in \mathfrak{K}_\varepsilon, \quad \text{and} \quad \|\mathcal{H}\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)} \leq C\,\eta^{\frac{3}{2}-\ell\,\nu}$$

for constants $C>0$ and $\ell\in\mathbb{R}$ independent of $\varepsilon, o, v, \eta, a$ and C_* appearing in the assumption 3.4.1. So, (3.62) can be reformulated as

$$\mathcal{H} + \mathcal{L}_\varepsilon\, w + \mathcal{Q}_\varepsilon(w) \in \mathfrak{K}_\varepsilon.$$

Proposition 3.7.1. *For all $\varepsilon\in(0,\varepsilon_*)$ and all o, v, η, a satisfying the assumptions 3.4.1, there exists a function $w_*\in\mathcal{C}^{2,\alpha}_\nu(\mathcal{A}_\varepsilon)$ and a vector $A_*\in\mathbb{R}^8$ such that the mean curvature of $\mathcal{A}_\varepsilon(w_*)$ is constant up to an element of the approximate kernel of the operator $\mathcal{L}_\varepsilon$. More precisely,*

$$H(\mathcal{A}_\varepsilon(w_*)) = 2 + \sum_{i=1}^{8} A_*^i\,\Phi_i, \quad \Phi_i\in\mathfrak{K}_\varepsilon,\ i=1,\dots,8. \tag{3.63}$$

Moreover,

$$\|w_*\|_{\mathcal{C}^{2,\alpha}_\nu(\mathcal{A}_\varepsilon)} \leq C\,\eta^{\frac{3}{2}-\ell\nu}$$

for $C>0$ and $\ell\in\mathbb{N}$ independent of $\varepsilon, o, v, \eta, a$.

Proof. Consider the mapping

$$\mathfrak{G}\,:\,\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)\to\mathcal{C}^{2,\alpha}_\nu(\mathcal{A}_\varepsilon)$$

defined in (3.60), which to a function $f\in\mathcal{C}^{0,\alpha}_{\nu-2}(\mathcal{A}_\varepsilon)$ associate the solution $w(f)$ of the equation

$$\mathcal{L}_\varepsilon\, w(f) = f - \sum_{i=1}^{8} A^i(f)\,\Phi_i, \quad \Phi_i\in\mathfrak{K}_\varepsilon.$$

Next, consider the mapping

$$w\in\mathcal{C}^{2,\alpha}_\nu(\mathcal{A}_\varepsilon)\mapsto\mathfrak{G}(-\mathcal{H}-\mathcal{Q}_\varepsilon(w)).$$

According to Corollary 3.5.1, there exist $\ell\in\mathbb{N}$ and $C>0$ independent of η, such that

$$\|\mathfrak{G}\,(\mathcal{H})\|_{\mathcal{C}^{2,\alpha}_\nu(\mathcal{A}_\varepsilon)} \leq C\,\eta^{\frac{3}{2}-\ell\,\nu} =: r_*.$$

On the other hand, by Propositions 3.5.4 and 3.5.5, with the assumptions 3.4.1, we obtain

$$\big\|\mathfrak{G}\big(\mathcal{Q}_\varepsilon(w)\big)\big\|_i \leq C\,\eta^{\frac{1}{2}-\ell\,\nu}\,\|w\|_{\mathcal{C}^{2,\alpha}_\nu(\mathcal{A})},$$

$$\big\|\mathfrak{G}\,(\mathcal{Q}_\varepsilon(w_1)) - \mathfrak{G}^i\,(\mathcal{Q}_\varepsilon(w_2))\big\|_i \leq \frac{1}{2}\,\|w_1-w_2\|_{\mathcal{C}^{2,\alpha}_\nu(\mathcal{A}_\varepsilon)},$$

for ν and η small enough and $\|w\|_{\mathcal{C}^{2,\alpha}_\nu(\mathcal{A}_\varepsilon)}\leq r_*$, $\|w_i\|_{\mathcal{C}^{2,\alpha}_\nu(\mathcal{A}_\varepsilon)}\leq r_*$. By Banach fixed point theorem for contracting mappings, there exists a function w_* and a vector $A_*\in\mathbb{R}^8$ in the ball of $\mathcal{C}^{2,\alpha}_\nu(\mathcal{A}_\varepsilon)\times\mathbb{R}^8$ of radius $2\,r_*$, which satisfy

$$H(\mathcal{A}_\varepsilon(w_*)) - 2 - \sum_{i=1}^{8} A_*^i\,\Phi_i = 0. \tag{3.64}$$

□

3.7.2 Choice of the parameters

We have now constructed a family $\mathcal{A}_\varepsilon(w_*)(o, v, \eta, a)$ of surfaces in M whose mean curvature is constant up to an element of an 8 dimensional space $\mathcal{K}_\varepsilon$. To summarize, these surfaces are obtained as small perturbations of connected sums of two pseudo CMC spheres $\Sigma_\varepsilon^\pm$ whose "centers" are located symmetrically on the geodesic passing through the point $o \in M$ with velocity vector $v \in T_oM$, $\|v\|_g = 1$ and a small neck parametrized by the "neck size" η and the "location" $a = (a^1, a^2) \in \mathbb{R}^2$ of its axis in a 2-dimensional geodesic disk Λ.

Next, let o_{cr} be a critical point of the scalar curvature function $\mathcal{R}$, $\lambda \neq 0$ a simple eigenvalue of $\mathrm{Hess}_{o_{cr}}\mathcal{R}$, and v_λ the corresponding unit eigenvector. According to the results of the section 3.4, we have two families of surfaces which satisfy (3.64), the first family embedded and the second immersed with self intersections. In the following result we show that for all ε small enough we can find a set of parameters (o, v, η, a) in a neighborhood of $(o_{cr}, v_\lambda, 0, 0)$ for which $\mathcal{A}_\varepsilon(w_*)(o, v, \eta, a)$ has constant mean curvature and is embedded or immersed with self intersections depending on the sign of λ.

Proposition 3.7.2. *There exists $\varepsilon_* > 0$ such that for all $\varepsilon \in (0, \varepsilon_*)$ there exist $o_\varepsilon \in M$, $v_\varepsilon \in T_oM$ with $\|v_\varepsilon\|_g = 1$, $\eta_\varepsilon \in \mathbb{R}_+$ and $a_\varepsilon \in \mathbb{R}^2$, such that*

$$H^g\Big(\mathcal{A}_\varepsilon(w_*)(o_\varepsilon, v_\varepsilon, \eta_\varepsilon, a_\varepsilon)\Big) = \frac{2}{\varepsilon}$$

and

$$\mathrm{dist}_g(o_\varepsilon, o_{cr}) \leq C_*\,\varepsilon^2, \quad \angle(v_\varepsilon, v_\lambda) \leq C_*\,\varepsilon, \quad \left|\eta_\varepsilon - \mathrm{sign}(\lambda)\,\frac{4\pi\lambda\,\varepsilon^4}{15}\right| \leq C_*\,\varepsilon^5 \quad \textit{and} \quad |a_\varepsilon| \leq C_*\varepsilon^3$$

for a constant C_* independent of ε.

Proof. We would like to use the Schauder's Fixed point Theorem in a ball of $\mathbb{R}^8$ to solve the system of equations

$$A_*^i(o, v, \eta, a) = 0, \quad i = 1, \ldots, 8. \tag{3.65}$$

To this end, we would like to show that the system (3.65) can be written in the form

$$(\mathrm{Id} + F)\,(\cdot) = 0 \tag{3.66}$$

where by F we denote any continuous function bounded uniformly in ε and o, v, η, a for ε small enough.

Step 1

Let us consider the first 6 equations. By Corollary 3.5.1 and Proposition 3.7.1, the constants A_*^i, $i = 1, \ldots, 6$ are given by the components of the vectors $C^\pm \in \mathbb{R}^3$ plus some terms bounded by a constant times $\eta^{3/2-\ell\nu}$. Thus, the equations can be written in the form

$$-\frac{2\pi\,\varepsilon^3}{15}\nabla^g\,\mathcal{R}(o^\pm) + \frac{1}{2}\,\eta\,\Theta(p_q^\pm) + \varepsilon^5 F(o, v, \eta, a) = 0 \tag{3.67}$$

where $o^{\pm}$ correspond to the "centers" of the pseudo CMC spheres $\Sigma_{\varepsilon}^{\pm}$ and $p_q^{\pm}$ to the poles of the Green's functions that we used in construction of the approximate solution.

Since $o_{cr} \in M$ is a critical point of the scalar curvature $\mathcal{R}$, we have $\nabla^g \mathcal{R}(o_{cr}) = 0$. Take an orthonormal (with respect to the metric g) basis E_1, E_2, E_3 of $T_{o_{cr}}M$, such that $E_1 = v_\lambda$ is the unit eigenvector associated to the eigenvalue λ. Let x_0^1, x_0^2, x_0^3 be the coordinates of the point o in corresponding geodesic normal coordinates centered at o_{cr}.

By construction, the points $o^{\pm}$ lie on the geodesic passing through the point $o \in M$ with velocity vector $v \in T_oM$, $\|v\|_g = 1$. By the assumption 3.4.1, the coordinates $x_{\pm}$ of $o^{\pm}$ satisfy

$$x_{\pm} = x_0 \pm \varepsilon\, v + \mathcal{O}(\varepsilon^2).$$

We also have

$$\Theta(p_q^{\pm}) = \mp v + \mathcal{O}(\varepsilon^2) \quad \text{when } \mathcal{A}_\varepsilon \text{ is embedded;}$$

$$\Theta(p_q^{\pm}) = \pm v + \mathcal{O}(\varepsilon^2) \quad \text{when } \mathcal{A}_\varepsilon \text{ has self-intersections.}$$

Assume first, that $\mathcal{A}_\varepsilon$ is embedded. Putting expressions found above into (3.67) and projecting the equations to the direction v_λ, we obtain

$$\begin{cases} \left(-\frac{\varepsilon^4\pi}{15}\,\lambda - \frac{1}{2}\,\eta\right)\sqrt{1-(v^2)^2-(v^3)^2} - \frac{\varepsilon^3\pi}{15}\,\lambda\, x_0^1 + \varepsilon^5\, F_1(x_0, v, \eta, a) = 0, \\ \left(\frac{\varepsilon^4\pi}{15}\,\lambda + \frac{1}{2}\,\eta\right)\sqrt{1-(v^2)^2-(v^3)^2} - \frac{\varepsilon^3\pi}{15}\,\lambda\, x_0^1 + \varepsilon^5\, F_0(x_0, v, \eta, a) = 0. \end{cases}$$

This can be written in the form

$$\begin{cases} -\frac{\varepsilon^4\pi}{15}\,\lambda - \frac{1}{2}\,\eta + \varepsilon^5\, F_1(x_0, v, \eta, a) = 0, \\ x_0^1 + \varepsilon^2\, F_2(x_0, v, \eta, a) = 0, \end{cases}$$

and in particular, we see that since $\eta > 0$, a solution exists only when $\lambda < 0$. When $\lambda > 0$, we need to take the immersed version of the approximate solution $\mathcal{A}_\varepsilon$.

On the other hand, projecting (3.67) on the subspace of $\mathbb{R}^3$ orthogonal to v_λ, we obtain

$$\begin{cases} \frac{\varepsilon^3\pi}{15}\,\left(\mathrm{Hess}_{o_{cr}}\,\mathcal{R} - \lambda\,\mathrm{Id}\right)\begin{pmatrix} 0 \\ x_0^2 + \varepsilon\, v^2 \\ x_0^3 + c\,\varepsilon\, v^3 \end{pmatrix} + \varepsilon^5\, F_{3,4}(x_0, v, \eta, a) = 0 \\ \frac{\varepsilon^3\pi}{15}\,\left(\mathrm{Hess}_{\hat{0}}\,\mathcal{R} - \lambda\,\mathrm{Id}\right)\begin{pmatrix} 0 \\ x_0^2 - \varepsilon\, v^2 \\ x_0^3 - c\,\varepsilon\, v^3 \end{pmatrix} + \varepsilon^5\, F_{5,6}(x_0, v, \eta, a) = 0 \end{cases}$$

We can rewrite this in the form

$$\begin{cases} x_0^i = \varepsilon^2\, F_i(x_0, v, \eta, a), & i = 1, 2, 3, \\ v^j = \varepsilon\, F_{3+j}(x_0, v, \eta, a), & j = 1, 2, \\ \eta = \mathrm{sign}(\lambda)\frac{4\,\pi\,\lambda\varepsilon^4}{15} + \varepsilon^5\, F_6(x_0, v, \eta, a), \end{cases} \tag{3.68}$$

where F_j are continuous functions bounded by a constant independent of ε, o, v, η and a.

Step 2

The last two equations can be obtained by taking the L^2 orthogonal projection of

$$H\left(\mathcal{A}_\varepsilon(w_*)\right) - 2 = \sum_{i=1}^{8} A_*^i\, \Phi_i$$

to Φ_7 and Φ_8. To explain why these equation can be written in the form (3.66), we propose to consider the following example. Taking the change of coordinates

$$y = \eta\, \cosh s(\cos\phi, \sin\phi),$$

we find that away from $s = 0$, we have $\Phi_{6+i} \approx \frac{\eta\, y^i}{|y|^2}$, $i = 1, 2$. Let P_0 be the horizontal plane in R^3 and C_η the catenoid scaled by the factor η with vertical axis centered at the origin. Recall, that C_η can be written as a bi-graph over $\{y \in P_0 \,:\, |y| > \eta\}$ of the function

$$G_\eta(y) = \log\frac{2}{\eta} + \eta\, \log|y| + \mathcal{O}(\eta^3\, |y|^{-2}).$$

On the other hand, let $P^\pm$ be two planes parametrized as graphs over P_0 of the affine functions

$$u^\pm(y) = \pm\, \log\frac{2}{\eta} + c_1^\pm\, y^1 + c_2^\pm\, y^2.$$

Take $\rho > 0$ and let $D^2(\rho)$ be a unit disk in P_0 of radius ρ centered at the origin. We denote by χ a cut-of function which satisfies

$$\chi \equiv 0 \quad \text{in} \quad D^2(\rho/2) \quad \text{and} \quad \chi \equiv 1 \quad \text{in} \quad P_0 \setminus D^2(\rho).$$

Finally, we remark that the mean curvature of the surface parametrized by

$$\big(y,\ \chi(y)\, u^\pm(y) \pm (1 - \chi(y))\, G_\eta(y)\big)$$

is equal to 0 everywhere but $D^2(\rho) \setminus D^2(\rho/2)$. On the other hand, for η small enough, in this region the largest terms in the projection of the mean curvature to $\frac{y^i}{|y|^2}$ are given by

$$\begin{aligned}\int_{D^2(\rho)\setminus D^2(\frac{\rho}{2})} \Delta\left(\chi\, \left(u^+ - u^-\right)\right) \frac{y^i}{|y|^2} &= \int_{\partial D^2(\rho)} \partial_r\left(\chi\, \left(u^+ - u^-\right)\right)\Big|_{r=2\rho} \frac{y^i}{|y|^2}\, d\phi \\ & \int_{\partial D^2(\rho)} \left(\chi\, \left(u^+ - u^-\right)\right)\, \partial_r\left(\frac{y^i}{|y|^2}\right)\Bigg|_{r=2\rho} d\phi \\ &= \frac{4\pi}{\rho}\, \left(c_i^+ - c_i^-\right).\end{aligned} \tag{3.69}$$

In particular, we see that the largest terms in this projection are determined by the slopes of the planes $P^\pm$.

Let us go back to our construction. The influence of the perturbation being negligible, the "slopes" in our case will be determined by $\nabla u^{\pm}(q)$, where $u^{\pm}$ are the function which appear when we parametrize of the pseudo CMC spheres $\Sigma_\varepsilon^{\pm}$ as normal graphs over Λ. More precisely, we proceed as follows. The equation

$$H\left(\mathcal{A}_\varepsilon(w)\right) = 2 + \sum_{i=1}^{b} A_*^i\,\Phi_i$$

can be written in the form

$$\mathcal{H} + \mathcal{L}_\varepsilon\, w_* + \mathcal{Q}_\varepsilon(w_*) - \sum_{i=1}^{6} A_*^i\,\Phi_i = A_*^7\,\frac{\cos\phi}{\eta^2\,\cosh^3 s} + A_*^8\,\frac{\sin\phi}{\eta^2\cosh^3 s}.$$

We multiply this expression by $\eta^2\,\cosh^2 s\,\Phi_{7,8}$ and integrate in $[-s_*, s_*]\times S^1$ for s_* large enough. By Propositions 3.5.4 and 3.5.5, we have

$$\eta^2\,\cosh^2 s\,\mathcal{L}_\varepsilon = \left(\partial_s^2 + \partial_\phi^2 + \frac{2}{\cosh^2 s}\right) + \varepsilon^2\,\hat{L}_1 + \frac{1}{\cosh^2 s}\,\hat{L}_2,$$

where $\hat{L}_1$ and $\hat{L}_2$ linear partial differential operators with coefficients bounded independently of $\varepsilon, o, v, \eta, a$, with support in Ω_{cat} and $\mathcal{A}_\varepsilon\setminus\Omega_{cat}$ respectively. Integrating by parts and using the estimates for w_* and $\mathcal{Q}_\varepsilon(w_*)$ given in Proposition 3.7.1, we find

$$\int_{[-s_*,s_*]\times S^1}\left(\mathcal{L}_\varepsilon\,w_* + \mathcal{Q}_\varepsilon(w_*) - \sum_{i=1}^{6} A_*^i\,\Phi_i\right)\frac{\cos\phi}{\cosh s}\,\eta^2\,\cosh^2 s\,ds\,d\phi \le C\,\eta^{2-\ell\,\nu},$$

for some $C>0$ and $\ell\in\mathbb{N}$ independent of the choice of $\varepsilon, o, v, \eta, a$. On the other hand, making use of the results of Propositions 3.5.1 and 3.5.2 we obtain

$$\left|\int_{[-s_*,s_*]\times S^1}\mathcal{H}_0\frac{\cos\phi}{\cosh s}\,\eta^2\,\cosh^2 s\,ds\,d\phi\right| \le C\,\eta,$$

for some $C>0$ independent of $\varepsilon, o, v, \eta, a$. Finally, we find

$$\begin{aligned}
&\int_{[-s_*,s_*]\times S^1}\mathcal{H}_1\frac{\cos\phi}{\cosh s}\,\eta^2\,\cosh^2 s\,ds\,d\phi\\
=&\int_{D^2(r_\eta)\setminus D^2(r_\eta/2)}\Delta\Big(\chi\,\left(\nabla u^+(q) - \nabla u^-(q)\right)^i\,y^i\Big)\,\frac{\eta y^i}{|y|^2}\,dy^1\,dy^2\\
=&\frac{4\pi\,\eta}{r_\eta}\left(\nabla u^+(q) - \nabla u^-(q)\right).
\end{aligned}$$

Thus, the system of equations $A_{7,8}=0$ can be written in the form

$$\nabla(u^+ - u^-)(q) + \eta^{3/4}\,F_{7,8}(x_0, v, \eta, a) = 0. \tag{3.70}$$

Furthermore, (a^1, a^2) are the coordinates of the point q in the normal geodesic coordinates (associated to the metric g_ε) centered at the point q_0, where q_0 is a local minimum of the function u^+ and a local maximum of u^-. We can write (3.70) in the form

$$\text{Hess}_{q_0}\left(u^+ - u^-\right)\begin{pmatrix} a^1 \\ a^2 \end{pmatrix} + \eta^{3/4}\, F_{7,8}(x_0, v, \eta, a) = 0, \tag{3.71}$$

and since, by construction, $\text{Hess}_{q_0}(u^+ - u^-)$ is invertible, we find the equations

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \varepsilon^3\, F_{7,8}(x_0, v, \eta, a) = 0.$$

Step 3

By the Schauder's Fixed Point Theorem for every ε small enough there exists a solution $o_\varepsilon \in M$, $v_\varepsilon \in T_{o_\varepsilon}M$ with $\|v_\varepsilon\|_g = 1$, $\eta_\varepsilon \in \mathbb{R}_+$ and $a_\varepsilon \in \mathbb{R}^2$ of (3.65) in a ball of $\mathbb{R}^8$, such that

$$\text{dist}_g(o_\varepsilon, o_{cr}) \leq C_*\,\varepsilon^2, \quad \angle(v_\varepsilon, v_\lambda) \leq C_*\,\varepsilon, \quad |a_\varepsilon| \leq C_*\,\varepsilon^3, \quad \left|\eta_\varepsilon - \text{sign}(\lambda)\frac{4\,\pi\varepsilon^4}{15}\,\lambda\right| \leq C_*\,\varepsilon^5.$$

Put $\mathfrak{S}_\varepsilon = \mathcal{A}_\varepsilon(o_\varepsilon, v_\varepsilon, \eta_\varepsilon, a_\varepsilon)$ and let $\mathfrak{S}_\#(\varepsilon, o_{cr}, v_\lambda)$ be the union of two geodesic spheres of radius ε, with their centeres located symmetrically from o_{cr} on the geodesic passing through the point o_{cr} with velocity vector given by v_λ. Then we have

$$H^g(\mathfrak{S}_\varepsilon) = \frac{2}{\varepsilon}, \quad \text{dist}_H(\mathfrak{S}_\varepsilon, \mathfrak{S}_\#(\varepsilon, o_{cr}, v_\lambda)) \leq c\,\varepsilon^2,$$

where by dist_H we denote the Hausdorff distance. Finally, $\mathfrak{S}_\varepsilon$ is embedded when $\lambda < 0$ and immersed when $\lambda > 0$.

$\square$

3.8 Appendix 1

First, let us find the mean curvature of the unit Euclidean sphere S^2 with respect to the metric

$$(g_\varepsilon)_{ij} = \delta_{ij} + \frac{\varepsilon^2}{3}\,(R_p)_{iljl}\,x^k\,x^\ell + \frac{\varepsilon^3}{6}\,(R_p)_{ikjl,m}\,x^k\,x^l\,x^m + \mathcal{O}_p(\varepsilon^5).$$

Recall the standard fact that if $\Sigma \subset M$ is an oriented hypersurface with unit inward pointing normal N_Σ, and if Σ_z is the family of hypersurfaces defined by

$$\Sigma \times \mathbb{R} \ni (q, z) \mapsto \exp_q(zN_\Sigma(q)) \in \Sigma_z$$

with induced metric g_z, then

$$|H_\Sigma| = -\frac{d}{dz}\,\log\sqrt{\det g_z}\bigg|_{z=0}.$$

In our case, considering S^2 with metric g_ε, let $g_{\varepsilon z}$ be the induced metrics on the Euclidean sphere of radius $1-z$. Then if g_{sp} denote the metric induced on S^2 from the euclidean metric, then it follows from (3.2) that

$$\begin{aligned} H^{g_\varepsilon}(S^2) &= \operatorname{Tr}\left(g_{sp}^{-1}\ \dot g_z|_{z=0}\right) \\ &= 2 - \frac{\varepsilon^2}{3}\operatorname{Ric}_p(\Theta,\Theta) - \frac{\varepsilon^3}{4}\nabla_\Theta \operatorname{Ric}_p(\Theta,\Theta) + \mathcal{O}_p(\varepsilon^4) \end{aligned}$$

Next, let us prove, that if Π is the L^2-orthogonal projection to the space of the restrictions to the unit sphere $S^2 \subset \mathbb{R}^3$ of coordinate functions. Then

$$\Pi\left(\frac{1}{4}\nabla_\Theta \operatorname{Ric}_p(\Theta,\Theta)\right) = \frac{2\pi}{15}\nabla R(p).$$

Let x^1, x^2, x^3 be geodesic normal coordinates centered at the point $p \in M$. Then $\nabla_{\partial x^i}\partial_{x^j}|_p = 0$, so $\nabla_{\partial_{x^i}}\operatorname{Ric}_p(\partial_{x^j},\partial_{x^k}) = \nabla_{\partial_{x^i}}(\operatorname{Ric}_p(\partial_{x^j},\partial_{x^k}))$. We use the notation $R_{ij,k} = \nabla_k \operatorname{Ric}_{ij}(0)$ and let $d\sigma$ be the volume element of S^2. Then

$$\Pi\left(\nabla_\Theta \operatorname{Ric}_p(\Theta,\Theta)\right) = \left(R_{ij,k}\int_{S^2}\Theta^i\,\Theta^j\,\Theta^k\,\Theta^\ell\,d\sigma\right)\Theta^l$$

On the other hand,

$$\begin{aligned} \left(R_{ij,k}\int_{S^2}\Theta^i\,\Theta^j\,\Theta^k\,\Theta^\ell\,d\sigma\right) &= R_{\ell\ell,\ell}\int_{S^2}(\Theta^\ell)^4 d\sigma + \sum_{j,k\neq\ell} R_{\ell j,k}\int_{S^2}(\Theta^\ell)^2\,\Theta^j\,\Theta^k\,d\sigma \\ &+ \sum_{i,k\neq\ell} R_{i\ell,k}\int_{S^2}(\Theta^\ell)^2\,\Theta^i\,\Theta^k\,d\sigma + \sum_{i,j\neq\ell} R_{ij,\ell}\int_{S^2}(\Theta^\ell)^2\,\Theta^i\,\Theta^j\,d\sigma \\ &= R_{\ell\ell,\ell}\int_{S^2}(\Theta^1)^4\,d\sigma + \sum_{i,j}(R_{\ell j,j} + R_{i\ell,i} + R_{ii,\ell})\int_{S^2}(\Theta^1)^2\,(\Theta^2)^2\,d\sigma \\ &- 3\,R_{\ell\ell,\ell}\int_{S^2}(\Theta^1)^2\,(\Theta^2)^2\,d\sigma = \frac{2\,\pi \operatorname{Vol}(S^2)}{15}\nabla_\ell R(0), \end{aligned}$$

where we used

$$\frac{1}{3}\int_{S^2}(\Theta^1)^4\,d\sigma = \int_{S^2}(\Theta^1)^2\,(\Theta^2)^2\,d\sigma = \frac{\operatorname{Vol}(\mathrm{S}^2)}{30}.$$

Finally, the second Bianchi identity

$$\nabla_m R_{ij\ell k} + \nabla_\ell R_{ijkm} + \nabla_k R_{ijm\ell} = 0$$

at the point p contracted twice with respect to the indexes m, j and i, ℓ yields

$$\nabla_\ell R = 2\sum_j R_{jk,j}.$$

3.9 Appendix 2

Let C_η denote the Euclidean catenoid scaled by the factor η. Take $w \in \mathcal{C}^{2,\alpha}(\mathbb{R}) \times S^1$ with $\|\frac{w}{\cosh s}\|_{\mathcal{C}^{1,\alpha}} < 1$. And consider the normal (with respect to the Euclidean metric) graph

$$C_\eta(w) \,:\, (s,\phi) \in \mathbb{R} \times S^1 \mapsto C_\eta(s,\phi) + w\, N_{cat}(s,\phi).$$

Lemma 3.9.1. *The mean curvature of $C_\eta(w)$ with respect to the Euclidean metric satisfies*

$$\begin{aligned} H_{cat}(w) = \frac{1}{\eta^2 \cosh^2 s} \left(\partial_s^2 + \partial_\phi^2 + \frac{2}{\cosh^2 s} \right) w &+ \frac{1}{\eta^3 \cosh^4 s}\, Q^2_{cat}(w) \\ &+ \frac{1}{\eta^4 \cosh^4 s}\, Q^3_{cat}(w) \end{aligned}$$

where $Q^k_{cat}(w)$ are a non-linear functions of w ∇w and $\nabla^2 w$, such that for all $s \in [-s_\eta, s_\eta - 1]$

$$\begin{aligned} \left\| Q^k_{cat}(w_1) - Q^k_{cat}(w_2) \right\|_{\mathcal{C}^{0,\alpha}\left([s,s+1]\times S^1\right)} \leq C \max_{i=1,2} \left\{ \left(\|w_i\|_{\mathcal{C}^{2,\alpha}\left([s,s+1]\times S^1\right)} \right)^{k-1} \right\} \\ \times \|w_1 - w_2\|_{\mathcal{C}^{2,\alpha}\left([s,s+1]\times S^1\right)} \end{aligned} \tag{3.72}$$

for a constant C independent of s, η and ε and $\left\|\frac{w_i}{\varepsilon \cosh s}\right\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}\times S^1)} < 1$.

Proof of Lemma 3.9.1. Let us denote by $Q^k_{cat}(w)$ any non-linear function satisfying the property (3.72). The tangent vectors to $C_\eta(w)$ are given by

$$T_s(w) = T_s + \partial_s w\, N^{cat} + w\, \partial_s N^{cat}, \quad T_\phi(w) = T_\phi + \partial_\phi w\, N^{cat} + w\, \partial_\phi N^{cat}$$

where $T_s = \left(\eta \sinh s\, e^{i\phi}, \eta\right)$, $T_\phi = \left(i\, \eta \cosh s\, e^{i\phi}, 0\right)$. Let

$$g^{cat} = \eta^2 \cosh^2 s (ds^2 + d\phi^2), \quad h^{cat} = \eta(-ds^2 + d\phi^2)$$

be the first and the second fundamental forms of the standard Euclidean catenoid scaled by a factor η. The induced metric on $C_\eta(w)$ can be written in the form

$$g^{cat}(w) = g^{cat} - \frac{w}{2}\, h^{cat} + Q^2_{cat}(w).$$

We look for a normal vector field to $C_\eta(w)$ in the form

$$N^\sharp(w) = N^{cat} + a_s(w)\, T_s + a_\phi(w)\, T_\phi.$$

Then the equations

$$g_{eucl}(N^\sharp(w), T_s(w)) = 0, \quad \tilde{g}_{eucl}(N^\sharp(w), T_\phi(w)) = 0$$

yield

$$a_k(w) = -\frac{\partial_k w}{2\, \eta^2 \cosh^2 s} + \frac{1}{\eta^3 \cosh^4 s}\, Q^2_{cat}(w).$$

We find

$$N(w) := N^\sharp(w)/\|N^\sharp(w)\|_{g_{eucl}} = N^{cat} - \frac{\partial_s w}{2\,\eta^2\cosh^2 s}\,T_s - \frac{\partial_\phi w}{2\,\eta^2\cosh^2 s}\,T_\phi$$
$$+ \frac{1}{\eta^2\cosh^2 s}Q_2''(w)\,N^{cat} + \frac{1}{\eta^2\cosh^3 s}\,Q^2_{cat}(w)\,T,$$

where T is a unit tangent vector. Since

$$\partial_k\partial_\ell\tilde{C}_\eta(w) = \partial_k\partial_\ell C_\eta + \frac{1}{2}\left(\partial_k w\,\partial_\ell N^{cat} + \partial_k w\,\partial_\ell N^{cat} + \partial_k\partial_\ell w\,N^{cat} + w\,\partial_k\partial_\ell N^{cat}\right),$$

we find that the second fundamental form satisfy

$$h^{cat}(w) = h^{cat} + \operatorname{Hess}\frac{w}{2} - \frac{w}{2\cosh^2 s}\,\mathrm{Id} + \frac{1}{2}\tanh s\begin{pmatrix} -\partial_s w & \partial_\phi w \\ \partial_\phi w & \partial_s w \end{pmatrix} + \frac{1}{\eta\cosh^2 s}Q^2_{cat}(w).$$

Finally, the result of the lemma follows by taking trace with respect to the metric $g^{cat}(w)$. □

Now, let us give more details on the proof of Proposition 3.5.5. We need to calculate the mean curvature of $C_\eta(w)$ with respect to the metric

$$\left(F^\varepsilon_{\Lambda,q}\right)^* g_{eucl}(y^1,y^2,z) = g_{eucl} + \mathcal{O}\left(\varepsilon^2\left(|y|^2+|z|\right)\right).$$

Let us denote by $\hat{L}$ any bounded linear operator from $\mathcal{C}^{2,\alpha}(\mathbb{R}\times S^1)$ in $\mathcal{C}^{0,\alpha}(\mathbb{R}\times S^1)$.

The metric induced on $\mathcal{A}_\varepsilon(w)$ can be written in the form

$$\mathfrak{g}(w) = g^{cat}(w) + \left(\mathfrak{g}(0) - g^{cat}\right) + \varepsilon^2\,\eta^2\,\cosh^2 s\,\hat{L}\,w + \text{small nonlinear terms.}$$

Again, we look for a normal vector field in $\mathcal{A}_\varepsilon(w)$ written in the form

$$\mathfrak{N}^\sharp(w) = N(w) + \tilde{a}_s(w)\,T_s(w) + \tilde{a}_\phi(w)\,T_\phi(w).$$

It follows from the equations

$$\left(F^\varepsilon_{\Lambda,q}\right)^* g_{eucl}(\mathfrak{N}^\sharp(w),T_s(w)) = 0, \quad \left(F^\varepsilon_{\Lambda,q}\right)^* g_{eucl}(\mathfrak{N}^\sharp(w),T_\phi(w)) = 0$$

that

$$\mathfrak{N}(w) := \mathfrak{N}^\sharp(w)/\|\mathfrak{N}^\sharp(w)\|_{\left(F^\varepsilon_{\Lambda,q}\right)^* g_{eucl}}$$
$$= N(w) + (\mathfrak{N}(0) - N_{cat}) + \varepsilon^2\,\hat{L}\,w + \text{small non-linear terms.}$$

Let $\nabla^\varepsilon(w)$ be the Levi-Civita connection corresponding to the metric $\left(F^\varepsilon_{\Lambda,q}\right)^* g_{eucl}$ and taken along $\mathcal{A}(w)$, then we have

$$\nabla^\varepsilon_{\partial_k}\partial_\ell(w) = \partial_k\partial_\ell C_\eta(w) + \left(\nabla^\varepsilon_{\partial_k}\partial_\ell(0) - \partial_k\partial_\ell C_\eta\right) + \varepsilon\,\eta\,\cosh^2 s\,\hat{L}\,w$$
$$+ \text{small nonlinear terms.}$$

The second fundamental form $\mathfrak{h}_{k\ell}(w) = \left(F^{\varepsilon}_{\Lambda,q}\right)^* g_{eucl}\left(\nabla^{\varepsilon}_{\partial_k}\partial_\ell(w), \mathfrak{N}(w)\right)$ satisfies

$$\mathfrak{h}(w) = h^{cat}(w) + \left(\mathfrak{h}(0) - h^{cat}\right) + \varepsilon\,\eta\,\cosh^2 s\,\hat{L}\,w + \text{ small non-linear terms.}$$

and, finally, the result follows when we take the trace with respect to the metric $\left(F^{\varepsilon}_{\Lambda,q}\right)^* g_{eucl}$.

Bibliography

[1] U. Abresch and H. Rosenberg. *A Hopf Differential for constant mean curvature surfaces in $S^2 \times R$ and $H^2 \times R$.* Acta Math., 193(2004) 2, 141-174

[2] U. Abresch and H. Rosenberg. *Generalized Hopf Differentials.* Math. Contemp. Vol. 28, (2005), 1-28.

[3] A.D. Alexandrov. *Uniqueness theorems for surfaces in the large.* V. Vestnik Leningrad Univ. Mat. Mekh. Astronom. 13 (1958).

[4] F. Almgren. *Optimal Isoperimetric inequalities.* Bulletin of the AMS, Vol. 13, 2, (1985), 123-126.

[5] F. Almgren. *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints* (Vol. 165). American Mathematical Soc., (1976).

[6] C. Bavard, P. Pansu. *Sur le volume minimal de R^2* . Annales scientifiques de l'École Normale Supérieure (Vol. 19, No. 4, pp. 479-490). (1986).

[7] S.N. Bernstein. *Uber ein geometrisches theorem und seine anwendung auf die partiellen differentialglechungen vom auf die partiellen differentialglechungen vom elliptischen typus.* Math. Z., 26:551-558, (1927). MR1544873, Zbl JFM 53.0670.01

[8] P. Berard and D. Meyer, *Inégalities isopériméetriques et applications*, Ann. Sci. Éc. Norm. Supér., IV. Sér. 15, (1982) 513-541

[9] A.I. Bobenko. *All constant mean curvature tori in R^3, S^3, H^3 in terms of theta functions.* Math. Ann. 290 (1991), 209–245

[10] A. Butscher and R. Mazzeo. *Constant mean curvature hypersurfaces condensing to geodesic segments and rays in riemannian manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 11, no. 3, 653–706, (2012).

[11] A. Butscher and F. Pacard. *Doubling constant mean curvature tori in S^3*, Annali della Scuola Normale Superiore di Pisa–Classe di Scienze, 5(4), 611-638, (2006).

[12] T.H. Colding and W.P. Minicozzi. *Minimal surfaces (Vol. 4)*, Courant Institute of Mathemetical Sciences, (1999).

[13] P. Collin. *Topology and curvature of properly embedded minimal surfaces in R^3*. Annals of Math. (2) 145 (1997), no. 1

[14] P. Collin and H. Rosenberg. *Construction of harmonic diffeomorphisms and minimal graphs*, Ann. of Math., to appear (2007)

[15] C.J. Costa. *Example of a complete minimal immersion in R^3 of genus one and three embedded ends*. Bol. Soc. Bras. Mat, 15, 47-54, (1984).

[16] C.J. Costa. *Uniqueness of minimal surfaces embedded in R^3, with total curvature 12π*. Journal of Differential Geometry, 30(3), 597-618, (1989).

[17] C.J. Costa. *Classification of complete minimal surfaces in R^3 with total curvature 12π*. Inventiones mathematicae, 105(1), 273-303, (1991).

[18] B. Daniel. *Minimal disks bounded by three straight lines in Euclidean space and trinoids in hyperbolic space*. J. Differential Geometry 72 (2006), no. 3, 467-508.

[19] B. Daniel. *Isometric immersions into 3-dimensional homogeneous manifolds*. Comment. Math. Helv. 82 (2007), no. 1, 87–131.

[20] B. Daniel. *Isometric immersions into $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ and applications to minimal surfaces*. Trans. Amer. Math. Soc. 361 (2009), no. 12, 6255–6282

[21] B. Daniel and L. Hauswirth. *Half-space theorem, embedded annuli and minimal graphs in the Heisenberg group*, Proc. London Math. Soc(3) 98, 445-470, (2009).

[22] B. Daniel and P. Mira. *Existence and uniqueness of constant mean curvature spheres in* Sol(3). American Journal of MathematicsVolume 135, Number 3, (2013).

[23] C. Delaunay. *Sur la surface de révolution, dont la courbure moyenne est constante*. Journal de mathématiques 6, 309–320, (1841).

[24] U. Dierkes, S. Hildebrandt, A. Küster and O. Wohlrab. *Minimal Surfaces, Vol. I, II*. Grundlehren Math. Wiss, 295, 296, (1992).

[25] U. Dierkes, S. Hildebrandt and H. Lewy. *On the analyticity of minimal surfaces at movable boundaries of prescribed length*, J. Reine Angew. Math. 379, 100-114, (1987).

[26] S. Donaldson. *Connections, cohomology and the intersection forms of 4-manifolds*. J. Diff. Geom., 24, pp.275-341, (1986).

[27] J. Dorfmeister, F. Pedit and H. Wu. *Weierstrass type representation of harmonic maps into symmetric spaces* , Comm. Anal. Geom. 6, no. 4, 633–668, (1998).

[28] J. Douglas. *Solutions of the problem of Plateau*. Transactions of the American Mathematical Society 33 (1931), 263-321

[29] O. Druet. *Sharp local isoperimetric inequalities involving the scalar curvature*. Proceedings of the American Mathematical Society, 130, 8, 2351-2361, (2002).

[30] A. Enneper. *Analytisch-geometrische Untersuchungen.* Z. Math. und Phys., 9:96-125, 1864

[31] M. Espinar and H. Rosenberg. *Complete constant mean curvature surfaces in homogeneous spaces.* Comment. Math. Helv. 86, no. 3, 659–674, (2011).

[32] L. Euler. *Methodus inveniendi lineas curvas maximi minimive propietate gaudeates sive solutio problematis isoperimetrici latissimo sensu accepti*, Harvard Univ. Press, Cambridge, MA, 1969. Opera omnia(1), 24, Fussli, Turici, 1952. A source book in mathematics, partially translated by D. J. Struik.

[33] I. Fernández and P. Mira. *Harmonic maps and constant mean curvature surfaces in $H^2 \times R$.* American Journal of Mathematics, 129, 1145-1181, (2007).

[34] I. Fernández and P. Mira. *Holomorphic quadratic differentials and the Bernstein problem in Heisenberg space.* Transactions of the American Mathematical Society, 361, 5737-5752, (2009).

[35] A. Fraser and M. Li. *Compactness of the space of embedded minimal surfaces with free boundary in three-manifolds with nonnegative Ricci curvature and convex boundary.* J. Differential Geom. 96, no. 2, 183–200, (2014).

[36] A. Fraser and R. Schoen. *The first Stekloff eigenvalue, conformal geometry and minimal surfaces* . Adv. Math. 226, no. 5, 4011-4030, (2011).

[37] A. Fraser and R. Schoen. *Sharp eigenvalue bounds and minimal surfaces in the ball.* arXiv:1209.3789v2[MathDG].

[38] S. Gallot, *Inégalités isopérimétriques et analytiques sur les variétés riemanniennes.* Astérisque, 163(164), 31-91. (1988).

[39] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer, 2015.

[40] E. Gonzalez, U. Massari and I. Tamanini. *On the regularity of boundaries of sets minimizing perimeter with a volume constraint.* Indiana University Mathematics Journal, 32(1), 25-37, (1983).

[41] K. Große-Brauckmann. *New surfaces of constant mean curvature.* Math. Z. 214, no. 4, 527–565, (1993).

[42] M. Grüter. *Boundary regularity for solutions of a partitioning problem.* Archive for Rational Mechanics and Analysis, 97(3), 261-270, (1987).

[43] L. Hauswirth. *Minimal surfaces of Riemann type in three dimensional product manifolds.* Pacific J. Math., 224 , no.1, 91–117 (2006).

[44] L. Hauswrith and H. Rosenberg. *Minimal surfaces of finite total curvature in $H^2 \times R$* . Mat. Contemp. 31, 65–80, (2006).

[45] W.T. Hsiang and W.Y. Hsiang. *On the uniqueness of isoperimetric solutions and imbedded soap bubbles in noncompact symmetric spaces* . I. Invent. Math., 98 (1), (1989), 39–58

[46] D. Hoffman and H. Karcher. *Complete embedded minimal surfaces of finite total curvature.* Geometry, V, 5–93, Encyclopaedia Math. Sci., 90, Springer, Berlin, (1997).

[47] D. Hoffman, H. Karcher and F.S. Wei. *Adding handles to the helicoid.* Bulletin of The American Mathematical Society, 29(1), 77-84, (1993).

[48] D. Hoffman, H. Karcher and F. S. Wei. *The genus one helicoid and the minimal surfaces that led to its discovery.* Global analysis in modern mathematics (Orono, ME, (1991); Waltham, MA, (1992)), 119–170, Publish or Perish, Houston, TX, (1993).

[49] D. Hoffman and W.H. Meeks. *A complete embedded minimal surface in $\mathbb{R}^3$ with genus one and three ends.* Journal of differential geometry, 21(1), 109-127, (1985).

[50] D. Hoffman and W.H. Meeks. *Embedded minimal surfaces of finite topology.* Annals of Mathematics, 1-34, (1990).

[51] D. Hoffman and W.H. Meeks. *Limits of minimal surfaces and Scherk's Fifth Surface.* Archive for Rational Mechanics and Analysis, 111(2), 181-195, (1990).

[52] D. Hoffman, M. Traizet, and B. White. *Helicoidal minimal surfaces of prescribed genus.* I. arXiv preprint arXiv:1304.5861.

[53] M. Weber, D. Hoffman and M. Wolf. *An embedded genus-one helicoid.* Annals of mathematics, 347-448, (2009).

[54] H. Hopf. *Lectures on differential geometry in the large.* Notes by John Gray, Stanford University, 1956.

[55] W.Y. Hsiang. *Generalized rotional hypersurfaces of constant mean curvature in the Euclidean spaces.* I, J. Differential Geometry 17 (1982) 337-356.

[56] W.Y. Hsiang,, *A symmetry theorem on isoperimetric regions.* PAM-409 (1988), UC Berkeley.

[57] J.H. Jellet. *Sur la Surface dont la Courbure Moyenne est Constant.* J. Math. Pures Appl., 18 (1853), 163–167.

[58] N. Kapouleas. *Complete constant mean curvature surfaces in Euclidean three-space.* Annals of Math. 131 (1990), 239–330.

[59] N. Kapouleas. *Constant mean curvature surfaces in Euclidean spaces.* Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 481–490.

[60] N. Kapouleas. *Constant mean curvature surfaces constructed by fusing Wente tori.* Invent. Math. 119 (1995) 443-518

[61] N. Kapouleas. *Minimal surfaces in the round three-sphere by doubling the equatorial two-sphere.* (2014) I. arXiv preprint arXiv:1409.0226.

[62] N. Kapouleas, *Slowly rotating drops* , Communications in mathematical physics, 129(1), 1990.

[63] N. Kapouleas and S.D. Yang. *Minimal surfaces in the three-sphere by doubling the Clifford torus* . Amer. J. Math. 132, no. 2,257-295, (2010).

[64] H. Karcher. *The triply periodic minimal surfaces of Alan Schoen and their constant mean curvature companions.* Manuscripta Math. 64 (1989), no. 3, 291–357.

[65] N. Korevaar, R. Kusner and B. Solomon. *The structure of complete embedded surfaces with constant mean curvature.* J. Diff. Geom. 30 (1989) 465–503.

[66] R. Kusner, R. Mazzeo, D. Pollack, *The moduli space of complete embedded constant mean curvature surfaces.* Geometric and Functional Analysis GAFA, 6(1), 120-137. (1996)

[67] O. A. Ladyzhenskaya, N. N. Ural'ceva *Equations aux dérivées partielles de type elliptique* . – 1968.

[68] H.B. Lawson. *Lectures on minimal submanifolds.* Berkeley CA. 1980.

[69] H.B. Lawson. *Complete minimal surfaces in S^3.* Ann. of Math. (2) 92 (1970), 335–374.

[70] H. Liebmann. *Uber die Verbiegung der geschlossenen Flächen positier Krümmung* . Math.Ann. 53(1900) 81-112.

[71] F. J. López and A. Ros. *On embedded complete minimal surfaces of genus zero.* Journal of Differential Geometry, 33(1), 293-300, (1991).

[72] F. Mahmoudi, R. Mazzeo and F. Pacard. *Constant mean curvature hypersurfaces condensing along a submanifold.* Geom. Funct. Anal. 16, no 4, (2006), 924-958.

[73] R. Mazzeo. *Recent advances in the global theory of constant mean curvature surfaces.* Noncompact problems at the intersection of geometry, analysis, and topology, Contemp. Math., vol. 350, Amer. Math. Soc., Providence, RI, 2004, pp. 179–199.

[74] R. Mazzeo and F. Pacard. *Constant mean curvature surfaces with Delaunay ends.* Comm. Anal. Geom. 9 (2001), no. 1, 169–237.

[75] R. Mazzeo and F. Pacard. *Bifurcating nodoids.* Topology and geometry: commemorating SISTAG, Contemp. Math., vol. 314, Amer. Math. Soc., Providence, RI, 2002, pp. 169–186.

[76] R. Mazzeo and F. Pacard. *Foliations by constant mean curvature tubes.* Comm. Anal. Geom. 13 (2005), no. 4, 633–670.

[77] R. Mazzeo, F. Pacard and D. Pollack. *Connected sums of constant mean curvature surfaces in Euclidean 3 space.* J. Reine Angew. Math. 536 (2001), 115–165.

[78] W.H. Meeks. *Geometric results in classical minimal surface theory.* Surveys in Differential Geometry, 8(1), (2003).

[79] W.H. Meeks. *The topology and geometry of embedded surfaces of constant mean curvature.* J. Diff. Geom. 27 (1988) 539–552.

[80] W.H. Meeks. *The classification of complete minimal surfaces in R^3 with total curvature greater than -8π.* Duke Mathematical Journal, 48(3), 523-535, (1981).

[81] W.H. Meeks, J. Pérez, *A survey on classical minimal surface theory.* (Vol. 60) American Mathematical Society, (2012).

[82] W.H. Meeks and H. Rosenberg. *The uniqueness of the helicoid and the asymptotic geometry of properly embedded minimal surfaces with finite topology.* Annals of Math. 161, no. 2 (2005).

[83] W.H. Meeks, J. Perez and A. Ros *Properly embedded minimal planar domains with infinite topology are Riemann minimal examples.* Current Developments in Mathematics Volume 2008 (2009), 281-346.

[84] W.H. Meeks, L. Simon and S.T. Yau. *Embedded minimal surfaces, exotic spheres and manifolds with positive Ricci curvature.* Ann. of Math. (2) 116 (1982), 621-659.

[85] J.B. Meusnier. *Mémoire sur la courbure des surfaces.* Mém. Mathé m. Phys. Acad. Sci. Paris, prés. par div. Savans, 10:477-510, 1785. Presented in 1776.

[86] F. Morgan. *Regularity of isoperimetric hypersurfaces in Riemannian manifolds.* Transactions of the American Mathematical Society, 355(12), 5041-5052, (2003).

[87] F. Morgan and I. Salavessa. *The isoperimetric problem in higher codimension.* arXiv preprint arXiv:1203.3304, 2012.

[88] J.C. Nitsche. *Lectures on minimal surfaces: vol. 1.* Cambridge university press, (1989).

[89] J.C. Nitsche. *On the boundary regularity of surfaces of least area in euclidan spaces.* Continuum Mechanics and related problems of analysis, Moscow 1972, 375-377

[90] S. Nardulli. *The isoperimetric profile of a smooth Riemannian manifold for small volumes.* Annals of Global Analysis and Geometry, 36(2), 111-131, (2009).

[91] R. Osserman. *A survey on Minimal Surfaces.* Dover Publications, New York, 2nd edition, (1986).

[92] F. Pacard. *Connected sum construction in geometry and nonlinear analysis* http://www.cmls.polytechnique.fr/perso/pacard.frank/Publications/Lecture-Part-I.pdf

[93] F. Pacard and H. Rosenberg. *Attaching handles to Delaunay nodoöids.* Pacific J. Math. 266 (2013), no. 1, 129–183.

[94] F. Pacard and T. Sun. *Doubling constructions for constant mean curvature surfaces in Riemannian manifolds.* unpublished.

[95] F. Pacard and X. Xu. *Constant mean curvature spheres in Riemannian manifolds.* Manuscripta Mathematica, 128, (2009), 275-295.

[96] P. Pansu, *Sur la régularité du profil isopérimétrique des surfaces riemanniennes compactes.* In Annales de l'institut Fourier (Vol. 48, No. 1, pp. 247-264). (1998).

[97] U. Pinkall and I. Sterling. *On the classification of constant mean curvature tori.* Ann. of Math. 130 (1989), 407–451.

[98] T. Radó. *On Plateau's problem.* Ann. Of Math. (2) (The Annals of Mathematics, Vol. 31, No. 3) 31 (3): 457–469 (1930).

[99] J. Ratzkin. *An end-to-end gluing construction for surfaces of constant mean curvature.* (Doctoral dissertation, University of Washington - 2001).

[100] B. Riemann. *Oeuvres mathématiques de Riemann.* Gauthiers-Villards, Paris 1898

[101] M. Ritoré. *Examples of constant mean curvature surfaces obtained from harmonic maps to the two sphere.* Math. Z. 226 (1997), 127–146.

[102] A. Ros. *The isoperimetric problem.* Global Theory of Minimal Surfaces, Clay Mathematics Proceedings, D. Hoffman Edt, AMS (2005).

[103] H. Rosenberg. *Constant mean curvature surfaces in homogeneously regular 3-manifolds.* Bulletin of the Australian Mathematical Society, 74(02), 227-238, (2006).

[104] J. Sacks and K. Uhlenbeck. *The existence of minimal immersions of 2-spheres.* Ann. of Math.(2) 113 (1981), 1-24.

[105] H.F. Scherk. *Bemerkungen über die kleinste Fläache innerhalb gegebener Grenzen.* J. R. Angew. Math., 13:185-208.

[106] R. Schoen. *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation.* Communications on pure and applied mathematics, 41(3), 317-392. (1988).

[107] R. Schoen. . *Uniqueness, symmetry, and embeddedness of minimal surfaces.* Journal of Differential Geometry, 18(4), 791-809, (1983).

[108] R. Schoen and S.T. Yau. *Existence of incompressible minimal surfaces and the topology of 3-dimensional manifolds with nonnegative scalar curvature.* Ann. of Math. (2) 110 (1979), 127-142

[109] M. Schneider. *Closed Magnetic Geodesics on* S^2 . J. Differential Geom. 87:2 (2011), 343–388. MR 2788659

[110] H.A. Schwarz. *Gesammelte Mathematische Abhandlungen.* Springer, Berlin, 1933.

[111] L. Simon. *Entire solutions of the minimal surface equation.* J. Differential Geometry 30 (1989), 643–688

[112] T. Sun. *A note on constant geodesic curvature curves on surfaces.* Annales de l'I.H.P. Analyse non linéaire, 26 no. 5 (2009), p. 1569-1584.

[113] C. Taubes. *Self-dual connections on manifolds with indefinite intersection matrix.* J. Diff. Geom., 19, (1984), pp. 517-560.

[114] K. Weierstrass. *Über die Flächen deren mittlere Krümmung überall gleich null ist* . Monatsber. Berliner Akad., 612-625, (1866).

[115] H.C. Wente. *Counterexample to a conjecture of H. Hopf.* Pacific J. Math. 121 (1986) 193-242.

[116] B. White. *The space of minimal submanifolds for varying Riemannian metrics.* Indiana Univ. Math. J. 40 (1991), no. 1,161-200.

[117] D. Wiygul. *Minimal surfaces in the 3-sphere by doubling the Clifford torus over rectangular lattices. arXiv preprint arXiv:1312.4619, (2013).*

[118] R. Ye. Foliation by constant mean curvature spheres*. Pacific J. Math. 147 (1991), no. 2, 381-396.*