

A non-Archimedean Montel's theorem

Rita Rodriguez Vazquez

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par

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Théorème de Montel non-Archimédien

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Introduction

Cette thèse est dédiée à l'étude des propriétés de compacité de familles d'applications analytiques entre espaces analytiques définis sur un corps métrisé non-archimédien.

Nous travaillerons dans le contexte des espaces analytiques développés par Berkovich dans [**Ber90, Ber93**]. La raison principale pour ceci est que les espaces analytiques au sens de Berkovich possèdent des bonnes propriétés topologiques: ils sont localement compacts et localement connexes par arcs, ce qui les rend plus adaptés à des arguments de nature analytique.



Avant de discuter les résultats que nous avons obtenus, regardons tout d'abord le cas des variétés analytiques complexes.

La compacité des familles d'applications holomorphes apparait en lien avec la notion d'hyperbolicité au sens de Kobayashi [**Kob67**], qui est l'un des concepts fondamentaux en géométrie complexe. Nous renvoyons à [**Kob98**] et [**Lan87**] pour des monographies détaillées sur le sujet ou aux surveys plus récents [**Voi03, DR11**]. Cette notion a motivé de nombreux travaux dans les dernières décennies ainsi que de profondes conjectures.

Rappelons que la semi-distance de Kobayashi est une généralisation aux espaces analytiques complexes quelconques de la métrique de Poincaré ρ sur le disque unité $\mathbb{D}$, et qu'elle est définie comme suit. Soit X un espace analytique complexe connexe. Une chaîne de Kobayashi reliant deux points x et y dans X est la donnée d'une famille finie d'applications holomorphes $f_l : \mathbb{D} \to X$ et de points $z_l, w_l \in \mathbb{D}$, $l = 1, \cdots, m$ tels que $f_1(z_1) = x$, $f_l(w_l) = f_{l+1}(z_{l+1})$ pour $l = 1, \cdots, m-1$ et $f_m(w_m) = y$. La semi-distance de Kobayashi entre x et y est alors donnée par

$$d_X(x, y) = \inf \sum_{l=1}^{m} \rho(w_l, z_l) \ ,$$

où l'infimum est pris parmi toutes les chaînes de Kobayashi reliant x à y.

L'espace X est dit (Kobayashi) hyperbolique lorsque d_X est une distance, c'est-à-dire lorsque $d_X(x, y) = 0$ si et seulement si $x = y$. Une

surface de Riemann est hyperbolique si et seulement si elle est uniformisée par le disque unité, et une variété contenant une courbe entière n'est jamais hyperbolique, comme par exemple tout tore complexe.

C'est un fait remarquable et fondamental dû à Brody [**Bro78**] que les espaces analytiques complexes compacts hyperboliques peuvent être caractérisés de la façon suivante:

THÉORÈME 1. *Soit X un espace analytique complexe compact lisse muni d'une métrique hermitienne. Les conditions suivantes sont alors équivalentes.*

(1) *L'espace X est Kobayashi hyperbolique.*
(2) *La dérivée d'une application holomorphe du disque dans X est bornée sur tout compact* K *de* $\mathbb{D}$ *par une constante ne dépendant que de X et de* K.
(3) *L'espace X ne contient pas de courbe entière.*
(4) *La famille* $\mathrm{Hol}(\mathbb{D}, X)$ *est normale.*

Rappelons qu'une famille d'applications analytiques $f_n : \mathbb{D} \to X$ est dite normale si elle est équicontinue, ce qui revient par le théorème d'Ascoli-Arzelà à dire que quitte à extraire une sous-suite, la suite d'applications f_n converge uniformément sur tout compact vers une application holomorphe $g : \mathbb{D} \to X$.

Le contenu essentiel du théorème ci-dessus est l'implication (3)⇒ (1) dont la démontration procède comme suit. Si X est un espace compact non hyperbolique alors il existe une suite d'applications holomorphes $f_n : \mathbb{D} \to X$ telles que $|df_n(0)| \to \infty$. Le lemme dit de Zalcman [**Zal75**] nous permet alors de trouver une suite de points $z_n \to 0$ et de réels $\rho_n \to 0$ tels que la suite renormalisée $g_n(z) = f_n(z_n + \rho_n z)$ définisse une famille d'applications holomorphes $g_n : \mathbb{D}(0, n) \to X$ avec $|dg_n(0)| = 1$ et de dérivée uniformément bornée sur tout compact. L'équicontinuité des g_n nous permet alors de produire une courbe entière.

Les espaces complexes analytiques hyperboliques jouissent de propriétés remarquables concernant la compacité des espaces d'applications holomorphes à valeurs dans eux.

Le théorème de De Franchis généralisé par [**Sam66, KO75, Nog92**] assure ainsi qu'il n'existe qu'une quantité finie d'applications méromorphes surjectives d'une variété compacte dans une variété compacte Kobayashi hyperbolique.

Le théorème de Montel classique affirme que la famille des applications holomorphes d'une variété complexe à valeurs soit dans un domaine borné de $\mathbb{C}^N$, soit dans $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, qui sont tous les deux hyperboliques, est normale, signifiant qu'une telle famille est relativement compacte pour la topologie de la convergence uniforme locale [**Mon07**].

Mentionnons pour conclure cette discussion une conjecture dûe à Lang qui fait le pont entre géométrie complexe et des questions plus arithmétiques, énonçant que toute variété projective lisse hyperbolique X définie sur un corps de nombres k ne contient qu'une quantité finie de points k-rationnels [**Lan74**].

Fixons maintenant un corps valué complet k non-archimédien et algébriquement clos et non-trivialement valué.

Nous allons considérer des espaces analytiques définis sur k au sens de Berkovich. Rappelons brièvement que ce sont des espaces localement modelés sur des espaces dits affinoïdes, qui sont des espaces compacts isomorphes à des sous-espaces analytiques d'un polydisque fermé. Nous nous intéresserons surtout à des espaces analytiques obtenus comme analytifications au sens de Berkovich des variétés algébriques définies sur un corps non-archimédien. L'ensemble des points fermés d'une telle variété forme un sous-ensemble dense et d'intérieur vide dans son analytification, désignés comme points rigides.

W. Cherry a initié l'étude de l'hyperbolicité dans le cadre non-archimédien dans [**Che96**]. Il a défini une semi-distance d_{CK} sur l'ensemble des points fermés d'un espace k-analytique en termes de chaînes de Kobayashi. Soient x, y deux points rigides d'un espace k-analytique connexe X. Comme dans le cas complexe, une chaîne reliant x à y est la donnée d'une famille finie d'applications analytiques $f_l : \mathbb{D} \to X$ et de points rigides z_l, w_l dans le disque ouvert $\mathbb{D}$, $l = 1, \cdots, m$ tels que $f_1(z_1) = x$, $f_l(w_l) = f_{l+1}(z_{l+1})$ pour $l = 1, \cdots, m-1$ et $f_m(w_m) = y$. La semi-distance de Cherry-Kobayashi entre x et y est alors donnée par

$$d_{\mathrm{CK}}(x, y) = \inf \sum_{l=1}^{m} |w_l - z_l| \ ,$$

où l'infimum est pris parmi toutes les chaînes reliant x à y. Nous disons qu'un espace k-analytique est Cherry hyperbolique lorsque d_{CK} est une vraie distance. Cette semi-distance définit la distance standard dans le cas du disque unité, et est comme dans le cas complexe contractée par les applications analytiques.

Dans une série d'articles [**Che93, Che96, ACW08, Che94**], W. Cherry a étudié en détail le comportement de cette semi-distance et l'existence de courbes entières dans le cas des variétés abéliennes et des courbes projectives, en exploitant de manière essentielle la théorie de la réduction pour ces variétés. Il a obtenu des résultats surprenants qui contrastent avec le cas complexe.

THÉORÈME 2. *Soit X une variété abélienne.*

(1) *Toute application analytique de la droite affine à valeurs dans X est constante.*

(2) *La variété X est Cherry hyperbolique.*

Notons néanmoins que ceci n'entraîne pas de contrexemple non-archimédien au Théorème 1. Inspiré par ses résultats, Cherry a même proposé la conjecture remarquable suivante dans [**Che94, Che96**]:

CONJECTURE 1. *Une variété projective lisse définie sur un corps non-archimédien est Cherry hyperbolique si et seulement si elle ne contient pas de courbe rationnelle.*

Cette conjecture est spécifique aux corps non-archimédiens, mais il existe néanmoins des conjectures analogues dans le cas complexe. Mentionnons par exemple la conjecture suivante dûe à Lang:

CONJECTURE 2. *Une variété projective complexe X lisse est Kobayashi hyperbolique si et seulement si toute application holomorphe d'une variété abélienne dans X est constante.*

La Conjecture 1 reste largement ouverte bien que Cherry ait traité le cas des courbes, des variétés abéliennes et certaines larges classes de surfaces algébriques compactes dans [**Che93**, §VII.3]. Le premier cas à regarder serait celui des surfaces de type général.

✵

Un de nos objectifs est de comprendre la semi-distance de Cherry-Kobayashi sur un espace analytique X en termes de la compacité des applications analytiques du disque dans X en s'inspirant de l'équivalence entre les propriétés (1) et (3) du Théorème 1. Pour celà, nous allons tout d'abord dégager une notion de normalité pour des familles d'applications analytiques à valeurs dans un compact.

Nous faisons cependant face immédiatement au problème suivant. Les points rigides d'une variété X sont denses, et donc tout point non rigide $x_0 \in X$ peut être approximé par une suite de points rigides x_n. En particulier, la suite d'applications analytiques constantes $f_n \equiv x_n$ converge vers l'application constante $f \equiv x_0$ qui n'est pas analytique car x_0 n'est pas un k-point.

Pour cette raison, nous posons:

DÉFINITION. *Soit X un espace k-analytique sans bord et Y un espace k-analytique compact.*

Une famille d'applications analytiques $\mathcal{F}$ de X dans Y est normale en un point $x \in X$ s'il existe un voisinage $V \ni x$ sur lequel toute suite $\{f_n\}$ dans $\mathcal{F}$ possède une sous-suite $\{f_{n_j}\}$ qui converge ponctuellement vers une application continue.

La famille $\mathcal{F}$ est normale si elle est normale en tout point $x \in X$.

L'hypothèse pour une variété analytique d'être sans bord est une hypothèse non triviale, mais elle est vérifiée par tous les analytifiés

des variétés algébriques et tous les sous-espaces analytiques fermés des polydisques ouverts. Remarquons que cette hypothèse est nécessaire. En effet, il a été remarqué dans [**FKT12**, §4.2] que la suite d'applications analytiques du disque unité fermé $\bar{\mathbb{D}}$ dans lui-même $f_n(z) = z^{2^{n!}}$ converge ponctuellement sur tout $\bar{\mathbb{D}}$, mais que la limite f n'est pas continue. Le point de Gauss, qui est l'unique point du bord de $\bar{\mathbb{D}}$, est fixé par f, tandis que tout le disque ouvert $\mathbb{D}$ est envoyé sur 0 par f.

✵

Notre premier résultat est une version du théorème de Montel pour des applications analytiques à valeurs dans un domaine borné de l'espace affine qui généralise en dimension quelconque [**FKT12**, Theorem 2.1].

Théorème A. *Soit X un espace k-analytique réduit, sans bord et σ-compact, et Y un espace strictement k-affinoïde.*

Alors, la famille des applications analytiques de X dans Y est normale.

Expliquons notre méthode de démonstration. On se ramène tout d'abord au cas des polydisques de telle sorte que l'on supposera dans la suite de la discussion que $X = \mathbb{D}^r$ et $Y = \bar{\mathbb{D}}^s$. Notre idée centrale est alors de munir l'espace des applications analytiques de $\mathbb{D}^r$ dans $\bar{\mathbb{D}}^s$ d'une structure de spectre analytique d'une k-algèbre de Banach adéquate. Le résultat découle de la compacité séquentielle de cet espace.

Pour comprendre notre construction, considérons l'espace des applications polynomiales de l'espace affine $\mathbb{A}^{r,\mathrm{an}} \to \mathbb{A}^{s,\mathrm{an}}$ de degré au plus δ^r avec $\delta \geq 1$ et envoyant le polydisque ouvert $\mathbb{D}^r$ dans le polydisque fermé $\bar{\mathbb{D}}^s$. Une telle application f est donnée par s polynômes $P_l(T_1, \ldots, T_r) = \sum_{|I| \leq \delta} a_{l,I} T^I$, où l'on pose $|I| = |(i_1, \ldots, i_r) := \max i_j$, dont les coefficients vérifient $|a_{l,I}| \leq 1$. On voit donc que f détermine un point rigide d'un polydisque unité fermé de dimension $s(\delta+1)^r$.

Toute suite d'applications polynomiales $f_n : \mathbb{A}^{r,\mathrm{an}} \to \mathbb{A}^{s,\mathrm{an}}$ comme ci-dessus fournit donc une suite de points rigides α_n dans $\bar{\mathbb{D}}^{s(\delta+1)^r}$, dont nous pouvons extraire une sous-suite α_{n_j} convergente vers un point $\alpha \in \bar{\mathbb{D}}^{s(\delta+1)^r}$, car les polydisques fermés sont séquentiellement compacts par [**Poi13**]. Un petit argument montre alors que la sous-suite f_{n_j} converge ponctuellement en tout point. Finalement, nous calculons l'application limite explicitement en termes du point α et démontrons qu'elle est continue.

La construction précédente dans le cas des applications polynomiales de degré δ suggère que l'espace des applications analytiques de $\mathbb{D}^r$ dans $\bar{\mathbb{D}}^s$ peut être paramétré par un polydisque de *dimension infinie*. Pour donner un sens précis à cet énoncé, nous construisons une k-algèbre de Banach $\mathcal{T}_\infty^{r,s}$ qui est un analogue de l'algèbre de Tate en une quantité

dénombrable de variables. Nous montrons que son spectre analytique noté $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ satisfait la propriété universelle suivante.

Notons $X(k)$ l'ensemble des points rigides d'un espace k-analytique X.

THÉORÈME B. *Il existe une application analytique*

$$\Phi : \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s) \times \mathbb{D}^r \to \bar{\mathbb{D}}^s$$

satisfaisant la propriété universelle suivante.

Soit W le spectre analytique d'une k-algebra de Banach ou un espace k-analytique bon. Pour toute application analytique $F : W \times \mathbb{D}^r \to \bar{\mathbb{D}}^s$ il existe une unique application analytique $G : W \to \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ telle que $F(x, z) = \Phi(G(x), z)$ pour tout $x \in W(k)$ et tout $z \in \mathbb{D}^r(k)$.

Les points rigides de $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ sont donc en correspondance avec les applications analytiques $\mathbb{D}^r \to \bar{\mathbb{D}}^s$, et il s'avère que tout point non-rigide de $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ définit une application continue $\mathbb{D}^r \to \bar{\mathbb{D}}^s$. Le Théorème A résulte alors de la compacité séquentielle de l'espace $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$, ce qui découle d'une adaptation à ce cadre de dimension infinie de [**Poi13**, Proposition 5.2].

Nous construisons plus généralement l'espace des applications analytiques d'un polydisque ouvert vers un affinoide quelconque Y. L'espace résultant $\mathrm{Mor}(\mathbb{D}^r, Y)$ est le spectre analytique d'un quotient de $\mathcal{T}_\infty^{r,s}$ par un idéal fermé qui safisfait une propriété universelle analogue.

Revenons aux limites continues fournies par le Théorème A. Nous établissons que toute application $f : \mathbb{D}^r \to \bar{\mathbb{D}}^s$ obtenue comme limite d'une suite d'applications analytiques $f_n : \mathbb{D}^r \to \bar{\mathbb{D}}^s$ se relève en une application analytique après un changement de base convenable.

L'énoncé précis utilise une section de l'application de changement de base introduite dans [**Poi13**]. Pour toute extension complète K/k, et tout espace k-analytique X, notons par X_K son extension des scalaires. Celle-ci est munie d'une projection canonique $\pi_{K/k} : X_K \to X$. C'est un fait fondamental que tout k-point de X peut être relevé à X_K, et Poineau a démontré que ce relèvement s'étend en une application continue $\sigma_{K/k} : X \to X_K$ telle que $\pi_{K/k} \circ \sigma_{K/k} = \mathrm{id}$.

COROLLAIRE C. *Soit X un espace k-analytique réduit, sans bord et σ-compact, et Y un espace k-affinoïde.*

Soit $f_n : X \to Y$ une suite d'applications analytiques convergeant ponctuellement vers une application continue $f : X \to Y$. Alors, il existe une extension complète K/k et une application K-analytique $F : X_K \to Y_K$ telles que

$$f = \pi_{K/k} \circ F \circ \sigma_{K/k} \ .$$

Ce résultat suggère la définition suivante. Une application continue $f : X \to Y$ entre espaces analytiques est dite faiblement analytique si

elle est localement de la forme $f = \pi_{K/k} \circ F \circ \sigma_{K/k}$ pour une extension complète K de k et une application K-analytique F. Ces applications partagent plusieurs propriétés caractéristiques des applications analytiques, que nous discutons dans §3. En particulier, nous démontrons une version du principe des zéros isolés pour ces applications.

Cette définition technique présente quelques difficultés, car il n'est pas clair si les applications faiblement analytiques peuvent être relevées sur tout X en une application analytique, et pas seulement localement au voisinage de tout point. Nous proposons la conjecture suivante:

CONJECTURE 3. *Soit X un espace k-analytique sans bord et Y un espace k-affinoide. Soit $f : X \to Y$ une application faiblement analytique. Alors, il existe une extension complète K/k et une application K-analytique $F : X_K \to Y_K$ telles que $f = \pi_{K/k} \circ F \circ \sigma_{K/k}$.*

Cette conjecture permettrait d'étendre le Théorème A à toute suite d'applications faiblement analytiques.

✵

Dans le Chapitre 4, nous nous sommes intéressée aux applications possibles de notre version du théorème de Montel à la dynamique des endomorphismes de $\mathbb{P}^{N,\mathrm{an}}$ de degré au moins 2, et plus particulièrement à des problèmes de nature globale. La première question que nous avons abordée est la définition dans ce cadre des ensembles de Fatou (où la dynamique est régulière) et de Julia (où elle est chaotique).

La première idée est de définir l'ensemble de Fatou $F_{\mathrm{norm}}(f)$ d'un endomorphisme de l'espace projectif $f : \mathbb{P}^{N,\mathrm{an}} \to \mathbb{P}^{N,\mathrm{an}}$ comme le lieu de normalité de la famille de ses itérés. Dans le cas complexe, cet ensemble possède plusieurs caractérisations, notamment comme complémentaire du support du courant de Green. Dans le cas non-archimédien, Kawaguchi et Silverman [**KS07, KS09**] ont associé à f une fonction de Green G_f définie sur $\mathbb{A}^{N+1,\mathrm{an}}$. Si $\rho : \mathbb{A}^{N+1,\mathrm{an}} \to \mathbb{P}^{N,\mathrm{an}}$ désigne la projection canonique, il est alors naturel d'introduire l'ensemble $F_{\mathrm{harm}}(f)$ de f comme l'ensemble des $z \in \mathbb{P}^{N,\mathrm{an}}$ possédant un voisinage U tel que G_f est fortement pluriharmonique au sens de [**CL11**] sur $\rho^{-1}(U)$.

CONJECTURE 4. *Pour tout endomorphisme f de l'espace projectif de degré au moins 2, nous avons $F_{\mathrm{norm}}(f) = F_{\mathrm{harm}}(f)$.*

La dynamique des endomorphismes de la droite projective a été étudiée par de nombreux auteurs, et on peut trouver une démonstration de la conjecture ci-dessus dans [**FKT12**, Theorem 5.4] qui repose sur les travaux précédents de Rivera-Letelier et sur la construction de la mesure d'équilibre due à Favre-Rivera-Letelier et Baker-Rumely.

Mentionnons que l'on peut également étudier l'ensemble $\mathrm{Equi}(f)$ des points rigides au voisinage desquels les itérés de f sont équicontinues

par rapport à la métrique chordale. Ce point de vue est exploré en dimension 1 par H'sia [**Hsi00**], et l'on peut montrer que l'intersection de l'ensemble de Fatou avec les points rigides coïncide avec $\mathrm{Equi}(f)$, voir [**FKT12**, Theorem C]. En dimension quelconque, Kawaguchi et Silverman ont donné une caractérisation de $\mathrm{Equi}(f)$ en termes de la fonction de Green.

Nous démontrons à la Proposition 4.4.4 que $F_{\mathrm{harm}}(f)$ peut être caractérisé comme le lieu où les itérés f^n se relèvent en des applications analytiques à valeurs dans un compact fixe de $\mathbb{A}^{N+1,\mathrm{an}} \setminus \{0\}$. Il s'ensuit alors du Théorème A que $F_{\mathrm{harm}}(f)$ est contenu dans $F_{\mathrm{norm}}(f)$.

Nous nous intéressons ensuite aux propriétés géométriques de l'ensemble de Fatou harmonique. Rappelons que dans le cas complexe, Ueda a démontré que toute composante connexe de l'ensemble de Fatou peut être plongée hyperboliquement au sens de Kobayashi dans $\mathbb{P}^N_{\mathbb{C}}$ [**Ued94**]. En adaptant ses arguments au cas non-archimédien, on obtient:

Théorème D. *Soit $f : \mathbb{P}^{N,\mathrm{an}} \to \mathbb{P}^{N,\mathrm{an}}$ un endomorphisme non inversible, Ω une composante connexe de $F_{\mathrm{harm}}(f)$ et U un ouvert de $\mathbb{P}^{1,\mathrm{an}}$.*

Alors, toute suite d'applications analytiques $g_n : U \to \Omega$ possède une sous-suite g_{n_j} qui converge ponctuellement vers une application continue $g : U \to \mathbb{P}^{N,\mathrm{an}}$.

Notre preuve utilise de façon essentielle l'hypothèse que U est un ouvert de la droite projective, bien qu'il soit vraisemblable que le résultat reste valide pour tout espace analytique sans bord et σ-compact.

Comme conséquence au Théorème A, nous obtenons:

Théorème E. *Soit $f : \mathbb{P}^{N,\mathrm{an}} \to \mathbb{P}^{N,\mathrm{an}}$ un endomorphisme non inversible et Ω une composante connexe de $F_{\mathrm{harm}}(f)$. Toute application analytique $\mathbb{A}^{1,\mathrm{an}} \setminus \{0\} \to \Omega$ est constante.*

Un autre résultat basique portant sur la géométrie des composantes de Fatou complexes est que celles-ci sont Stein [**Ued94, FS95, Mae04**]. La démonstration repose sur la théorie du pluripotentiel, et la caractérisation du complémentaire de l'ensemble de Fatou comme support du courant de Green. Sur un corps non-archimédien, la notion d'espace analytique Stein a été introduite dans [**Kie67**] en termes d'exhaustion par un certain type d'espaces affinoides, ce qui peut être vérifié dans le cas sans bord avec des critères cohomologiques. Nous proposons la question suivante:

Question. *Les composantes connexes de Fatou d'un endomorphisme de $\mathbb{P}^{N,\mathrm{an}}$ sont-elles Stein?*

Si l'on s'attend à ce que la démonstration suive son analogue complexe, on a besoin de développer une caractérisation pluripotentielle

des domaines Stein de l'espace projectif. Nous ne disposons cependant pas à l'heure actuelle d'une théorie du pluripotentiel sur un corps non-archimédien arbitraire en dimension supérieure. Ducros et Chambert-Loir ont proposé une définition de courant dans un contexte assez général dans [**CLD12**], mais certains ingrédients importants, comme la régularisation des fonctions plurisousharmoniques, ne sont pas encore disponibles alors qu'ils jouent un rôle crucial dans le cas complexe.

✵

Au chapitre 5, nous revenons au problème initial de cette thèse portant sur l'étude de la semi-distance d_{CK}. Nous démontrons tout d'abord une version non-archimédienne du Théorème 1 et caractérisons les variétés projectives lisses pour lesquelles d_{CK} est non dégénérée en un sens adéquat.

Rappelons que l'espace projectif $\mathbb{P}^N_k$ est naturellement muni de la distance sphérique $d_{\mathbb{P}}$. Notre premier résultat est le suivant:

Théorème F. *Soit X une variété projective lisse définie sur un corps complet non-archimédien algébriquement clos de caractéristique résiduelle nulle. Les conditions suivantes sont équivalentes:*

i) *Tout point rigide $x \in X$ possède un voisinage U tel que les semi-distances $\min\{1, d_{\mathrm{CK}}\}$ et $d_{\mathbb{P}}$ sont équivalentes sur $U(k)$.*
ii) *La semi-distance d_{CK} est non dégénérée et définit la même topologie sur l'ensemble des points rigides de X que la distance projective $d_{\mathbb{P}}$.*
iii) *La dérivée de Fubini-Study de toute application analytique de $\mathbb{D}$ dans X est localement uniformément bornée au voisinage de tout point rigide.*

Le contenu principal de cet énoncé réside dans l'implication ii) $\Rightarrow$ iii), qui résulte de l'adaptation du lemme de Zalcman au cas non-archimédien. Nous produisons ainsi une suite d'applications analytiques $g_n : \mathbb{D}(0; n) \to X$ dont la dérivée sphérique est uniformément bornée sur tout compact de $\mathbb{A}^{1,\mathrm{an}}$. Une différence majeure avec le cas complexe est que cette suite ne converge a priori pas vers une courbe entière $\mathbb{A}^{1,\mathrm{an}} \to X$.

Remarquons que dans le cas complexe l'assertion ii) est vérifiée si et seulement si X est Kobayashi hyperbolique par un théorème de Barth [**Lan87**, I, §2].

Notons de plus que l'hypothèse de caractéristique résiduelle nulle sur k est nécessaire pour borner les dérivées, et nous l'utilisons pour l'implication iii) $\Rightarrow$ i).

Nous observons enfin que la condition iii) implique que la famille $\mathrm{Mor}_k(\mathbb{D}, X)$ d'applications analytiques du disque ouvert dans X est normale au voisinage de tout point rigide. Il est vraisemblable que

l'implication réciproque soit aussi valide. La conjecture 1 montre que si la famille $\mathrm{Mor}_k(\mathbb{D}, X)$ est normale au voisinage de tout point rigide, alors X est Cherry hyperbolique, ce qui est une condition légèrement plus faible que ii).

Pour les courbes projectives, on obtient le théorème suivant, qui donne un analogue précis du Théorème 1:

THÉORÈME G. *Soit k un corps complet non-archimédien et algébriquement clos de caractéristique résiduelle nulle. Soit X une courbe projective lisse sur k. Les conditions suivantes sont équivalentes:*

i) *La courbe X est de genre strictement positif.*
ii) *Tout point rigide $x \in X$ possède un voisinage U tel que les semi-distances $\min\{1, d_{\mathrm{CK}}\}$ et $d_{\mathbb{P}}$ sont équivalentes sur $U(k)$.*
iii) *La dérivée de Fubini-Study de toute application analytique de $\mathbb{D}$ dans X est uniformément bornée au voisinage de tout point rigide.*
iv) *La famille des applications analytiques de $\mathbb{D}$ dans X est normale.*
v) *La courbe X est Cherry hyperbolique.*

✵

Au dernier chapitre, nous étudions de près la normalité des familles d'applications analytiques à valeurs dans une courbe algébrique (lisse et irréductible) fixée.

Rappelons que pour une telle courbe X, il existe une unique courbe projective $\bar{X}$ lisse contenant X comme ouvert Zariski dense. On définit alors la caratéristique d'Euler de X comme la quantité

$$\chi(X) = 2 - 2g - \#(\bar{X} \setminus X) \ ,$$

où g dénote le genre de $\bar{X}$.

Dans le cas complexe, les courbes algébriques lisses Kobayashi-hyperboliques sont exactement celles de caractéristique d'Euler négative, et pour celles-ci la famille $\mathrm{Hol}(Y, X)$ est normale pour un espace analytique Y quelconque. Nous démontrons ici un analogue non-archimédien.

Pour celà, nous adaptons notre définition de famille normale lorsque le but est non compact de la manière suivante.

Soit U un espace k-analytique sans bord. Une famille $\mathcal{F}$ d'applications analytiques de U dans X est normale si pour toute suite $f_n \in \mathcal{F}$ il existe une sous-suite f_{n_j} qui converge ponctuellement sur U vers une application continue $f_\infty : U \to \bar{X}$.

Nous montrons:

THÉORÈME H. *Soit k un corps non-archimédien complet et algébriquement clos dont le corps résiduel $\tilde{k}$ est de caractéristique zéro et dénombrable. Soit X une courbe algébrique irréductible lisse sur k.*

Alors, la caractéristique d'Euler $\chi(X)$ de X est strictement négative si et seulement si pour toute courbe lisse connexe et sans bord U, la famille $\mathrm{Mor}_k(U, X)$ est normale.

En outre, toute suite d'applications analytiques $f_n : U \to X$ avec $\chi(X) < 0$ admet une sous-suite f_{n_j} qui converge ponctuellement vers une application continue $f_\infty : U \to \bar{X}$ tel que soit $f_\infty(U) \subset X$ ou f_∞ est une constante dans $\bar{X} \setminus X$.

Pour mémoire notons que $\chi(X) \leq 0$ si et seulement si $\mathrm{Mor}_k(\mathbb{D}, X)$ est normale, le cas projectif étant une conséquence du Théorème G.

L'une des implications du Théorème H a déjà été essentiellement démontrée dans [**FKT12**]. Si la caractéristique d'Euler de X est positive ou nulle, alors il existe une courbe k-analytique sans bord U telle que la famille $\mathrm{Mor}_k(U, X)$ n'est pas normale. Quand X est la droite affine ou la droite projective, alors on peut prendre U égal au le disque unité. Dans le cas où X est la droite affine épointée, il suffit de considérer une couronne ouverte. Il ne reste qu'à traiter le cas où X est une courbe elliptique, pour lequel on peut prendre $U = X$.

La partie centrale de la démonstration est l'implication suivante: la famille $\mathrm{Mor}_k(U, X)$ est normale pour toute courbe lisse analytique connexe sans bord U dès que $\chi(X) < 0$. La preuve se divise en plusieurs étapes: on s'occupe tout d'abord du cas où X est une courbe projective puis on traite le cas algébrique général.

Théorème I. *Soit k un corps non-archimédien complet et algébriquement clos dont le corps résiduel $\tilde{k}$ est de caractéristique zéro. Supposons que X est une courbe algébrique irréductible lisse de caractéristique d'Euler négative dont le squelette $\mathsf{S}^{\mathrm{an}}(X)$ n'est pas un singleton.*

Soit U une courbe k-analytique lisse connexe sans bord. Alors, il existe un recouvrement $(\bar{X}_i)$ de $\bar{X}$ par des domaines affinoides et un recouvrement ouvert (U_j) de U localement fini par des tubes basiques tel que pour toute application analytique $f : U \to X$ et tout U_j, l'image $f(U_j)$ est contenue dans un domaine affinoide $\bar{X}_i$.

Notons que ce dernier théorème montre une forme d'équicontinuité pour les applications de U dans X.

Dans le cas où $\mathsf{S}^{\mathrm{an}}(X)$ n'est pas un singleton, il implique avec Théorème A une version plus forte du Théorème H. Par exemple, toute limite ainsi obtenue est une application faiblement analytique.

Lorsque $\mathsf{S}^{\mathrm{an}}(X)$ est un singleton, la courbe X est projective et à bonne réduction au sens où elle admet un modèle lisse sur k°. Dans ce cas nos arguments utilisent de façon cruciale l'hypothèse que le corps résiduel $\tilde{k}$ est dénombrable, et nous ne pouvons pas assurer que la limite soit faiblement analytique.

Les deux théorèmes ci-dessus sont démontrés en examinant de près le comportement des applications analytiques entre courbes, et en particulier sur leurs squelettes respectifs.

✵

Il serait intéressant de clarifier la différence entre la notion de normalité et celle d'équicontinuité. Rappelons que lorsque Y est compact, une famille d'applications analytiques $\mathcal{F}$ de X dans Y est équicontinue en un point $x \in X$ si pour tout recouvrement ouvert fini (Y_i) de Y il existe un voisinage U_x de x tel que pour toute application $f \in \mathcal{F}$ il existe un indice i tel que $f(U_x) \subseteq Y_i$. Une famille équicontinue est nécessairement normale, mais nous ne savons pas si la réciproque est aussi valide.

Si ces deux notions étaient équivalentes, alors les ensembles de Fatou harmonique et normal coïncideraient. On obtiendrait également une forme plus forte du Théorème H.

✵

Cette thèse est organisée comme suit.

Tout d'abord, nous donnons quelques rappels de la géométrie de Berkovich dans §1, ainsi que des résultats de [**Poi13**]. Le chapitre 2 est consacré à la paramétrisation de l'espace des applications analytiques de $\mathbb{D}^r$ dans un espace affinoide Y, i.e. à la construction de l'espace $\mathrm{Mor}(\mathbb{D}^r, Y)$, ainsi qu'à l'étude de ses propriétés. La démonstration du Théorème A se trouve dans §2.5. Les applications faiblement analytiques sont étudiées dans §3.

Dans §4 nous nous occupons de la dynamique des endomorphismes de $\mathbb{P}^{N,\mathrm{an}}$. La preuve du Théorème D se trouve dans §4.5.

Les chapitres §5 et 6 sont dédiés à la recherche d'une notion d'espace k-analytique hyperbolique. Dans §5 nous reprenons les travaux de Cherry et démontrons les Théorèmes F et G. La démonstration des Théorèmes H et I se trouve dans le chapitre 6.

CHAPTER 1

Some facts on Berkovich spaces

Throughout this dissertation, k is a field endowed with a non-Archimedean complete absolute value $|.|$. We will *always* assume that k is non trivially valued. Except if explicitely mentioned we shall always assume k to be algebraically closed.

We write $|k^\times| = \{|x| : x \in k^\times\} \subseteq \mathbb{R}_+$ for its value group and $k^\circ = \{x \in k : |x| \leq 1\}$ for its ring of integers. The latter is a local ring with maximal ideal $k^{\circ\circ} = \{x \in k : |x| < 1\}$. The *residue field of* k is $\tilde{k} = k^\circ/k^{\circ\circ}$.

In this chapter we make no assumptions on the characteristic of the field k.

The basic reference for the construction of Berkovich analytic spaces is Berkovich's original text [**Ber90**], see also [**Tem15**] for a more recent survey. The structure of k-analytic curves is detailed in [**Duc14**], and we shall recall some of these facts in §1.4. Of crucial importance for us are the base change morphism and the continuous section introduced by Poineau [**Poi13**] which we discuss in §1.7.

1.1. Analytic spaces

1.1.1. Analytic spectrum of a Banach k-algebra. Let $\mathcal{A}$ be a commutative k-algebra with identity. A norm on $\mathcal{A}$ is a function $\|.\| : \mathcal{A} \to \mathbb{R}_+$ satisfying the following properties:

i) $\|f\| = 0$ if and only if $f = 0$;
ii) $\|f + g\| \leq \|f\| + \|g\|$;
iii) $\|fg\| \leq \|f\|\|g\|$

for all $f, g \in \mathcal{A}$. A Banach k-algebra is a normed k-algebra $(\mathcal{A}, \|.\|)$ which is complete with respect to the norm $\|.\|$.

Pick any ideal $I \subset \mathcal{A}$ and denote by $\pi : \mathcal{A} \to \mathcal{A}/I$ the usual morphism. The quotient seminorm on $\mathcal{A}/I$ is defined as follows. For any $f \in \mathcal{A}/I$, we set $\|f\| = \inf\{\|g\| : \pi(g) = f\}$. If the ideal I is closed, then this seminorm is an actual norm.

Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach k-algebras and denote by $\|.\|_\mathcal{A}$ and $\|.\|_\mathcal{B}$ their respective Banach norms. On the tensor product $\mathcal{A} \otimes_k \mathcal{B}$ we have the seminorm that associates to every $f \in \mathcal{A} \otimes_k \mathcal{B}$ the quantity

$$||f|| = \inf \max \|a_i\|_\mathcal{A} \cdot \|b_i\|_\mathcal{B} \ ,$$

where the infimum is taken over all the possible expressions of f of the form $f = \sum_i a_i \otimes b_i$ with $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$. The seminorm $||.||$ induces the *tensor norm* on the quotient $\mathcal{A} \otimes_k \mathcal{B}/\{||f|| = 0\}$, whose completion is a Banach k-algebra satisfying a suitable natural universal property. This algebra is called the *complete tensor product* of $\mathcal{A}$ and $\mathcal{B}$ and we denote it by $\mathcal{A}\hat{\otimes}_k\mathcal{B}$, see [**BGR84**, §2.1.7].

Recall that a morphism of Banach k-algebras $\varphi : \mathcal{B} \to \mathcal{A}$ is bounded if there exists a positive constant C such that $\|\varphi(f)\| \leq C\|f\|$ for every $f \in \mathcal{B}$. A bounded morphism φ is admissible if the residue norm on $\mathcal{B}/\ker(\varphi)$ is equivalent to the restriction to the image of φ of the norm on $\mathcal{A}$.

The analytic spectrum $\mathcal{M}(\mathcal{A})$ of a Banach k-algebra $\mathcal{A}$ is the set of all mutiplicative seminorms on $\mathcal{A}$ that are bounded by the norm $\|.\|$ on $\mathcal{A}$. Given $f \in \mathcal{A}$, its image under a seminorm $x \in \mathcal{M}(\mathcal{A})$ is denoted by $|f(x)| \in \mathbb{R}_+$. The set $\mathcal{M}(\mathcal{A})$ is endowed with the weakest topology such that all the functions of the form $x \mapsto |f(x)|$ with $f \in \mathcal{A}$ are continuous. The resulting topological space is nonempty, compact and Hausdorff [**Ber90**, Theorem 1.2.1].

Given a point $x \in \mathcal{M}(\mathcal{A})$, the fraction field of $\mathcal{A}/\mathrm{Ker}(x)$ naturally inherits from x an absolute value extending the one on k. Its completion is the *complete residue field at* x and denoted by $\mathcal{H}(x)$.

Recall that the spectral radius of $f \in \mathcal{A}$ is defined by

$$\rho(f) = \lim_{n\to\infty} \|f^n\|^{1/n},$$

and the supremum seminorm on $\mathcal{A}$ is defined by $|f|_{\sup} := \sup\{|f(x)| : x \in \mathcal{M}(\mathcal{A})\}$ for $f \in \mathcal{A}$. The spectral radius and the supremum seminorm agree [**Ber90**, Theorem 1.3.1].

The set $\mathcal{A}^\circ = \{f \in \mathcal{A} : \rho(f) \leq 1\}$ is a subring of $\mathcal{A}$ and $\mathcal{A}^{\circ\circ} = \{f \in \mathcal{A} : \rho(f) < 1\}$ an ideal. The reduction of $\mathcal{A}$ is defined as the quotient $\widetilde{\mathcal{A}} := \mathcal{A}^\circ/\mathcal{A}^{\circ\circ}$ and the reduction of $X = \mathcal{M}(\mathcal{A})$ is $\widetilde{X} = \mathrm{Spec}(\widetilde{A})$.

A character on $\mathcal{A}$ is a bounded homomorphism $\mathcal{A} \to K$, where K is any complete extension of k. Two characters $\chi_1 : \mathcal{A} \to K_1$ and $\chi_2 : \mathcal{A} \to K_2$ are equivalent if there exists a character $\chi : \mathcal{A} \to L$ and inclusions $i_1 : L \to K_1$ and $i_2 : L \to K_2$ such that $i_1 \circ \chi = \chi_1$ and $i_2 \circ \chi = \chi_2$.

Composing the character $\mathcal{A} \to K$ with the norm on K gives rise to a seminorm on $\mathcal{A}$ that is bounded, and thus corresponds to a point $x \in \mathcal{M}(\mathcal{A})$. Equivalent characters give rise to the same point. Conversely, every point $x \in \mathcal{M}(\mathcal{A})$ induces a character $\chi_x : \mathcal{A} \to \mathcal{H}(x)$ in a natural way. Any other character $\mathcal{A} \to K$ giving rise to x can be decomposed as $\mathcal{A} \to \mathcal{H}(x) \hookrightarrow K$.

A morphism between the analytic spectra of two k-Banach algebras $\varphi^\sharp : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{B})$ is by definition one induced by a bounded morphism of Banach k-algebras $\varphi : \mathcal{B} \to \mathcal{A}$. The morphism $\varphi^\sharp$ is continuous.

The fibre of a morphism $\varphi^\sharp : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{B})$ over a point $y \in \mathcal{M}(\mathcal{B})$ is isomorphic to $\mathcal{M}(\mathcal{A}\hat{\otimes}_{\mathcal{B}}\mathcal{H}(y))$. Indeed, let $y \in \mathcal{M}(\mathcal{B})$ and let $\chi_y : \mathcal{B} \to \mathcal{H}(y)$ be the associated character. By definition, a point $x \in \mathcal{M}(\mathcal{A})$ is mapped to y if and only if the composition $\mathcal{B} \xrightarrow{\varphi} \mathcal{A} \to \mathcal{H}(x)$ factors through $\mathcal{H}(y)$, which is equivalent to the character χ_x factorizing through the $\mathcal{B}$-algebra morphism $\mathcal{A}\hat{\otimes}_{\mathcal{B}}\mathcal{H}(y) \to \mathcal{H}(x)$. Pick $x \in \mathcal{M}(\mathcal{A}\hat{\otimes}_{\mathcal{B}}\mathcal{H}(y))$ and let $\mathcal{A}\hat{\otimes}_{\mathcal{B}}\mathcal{H}(y) \to \mathcal{H}(x)$ be the associated character. The latter is equivalent to the data of morphisms $\mathcal{H}(y) \to \mathcal{H}(x)$ and $\mathcal{A} \to \mathcal{H}(x)$ such that the composition $\mathcal{B} \xrightarrow{\varphi} \mathcal{A} \to \mathcal{H}(x)$ equals $\mathcal{B} \to \mathcal{H}(y) \to \mathcal{H}(x)$. In other words, the image of x in $\mathcal{M}(\mathcal{A})$ is mapped to y by φ.

1.1.2. Affinoid spaces. Given a positive integer N and an N-tuple of positive real numbers $r = (r_1, \cdots, r_N)$, we denote by $k\{r^{-1}T\}$ the set of power series $f = \sum_I a_I T^I$, $I = (i_1, \cdots, i_N)$, with coefficients $a_I \in k$ such that $|a_I|r^I \to 0$ as $|I| := i_1 + \cdots + i_N$ tends to infinity. The norm $\| \sum_I a_I T^I \| = \max_I |a_I| r^I$ makes $k\{r^{-1}T\}$ into a Banach k-algebra. When $r = (1, \cdots, 1)$, this algebra is called the Tate algebra and we denote it by $\mathcal{T}_n$.

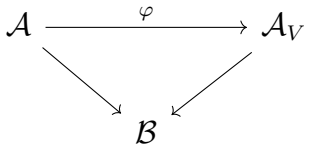
A Banach k-algebra $\mathcal{A}$ is called *affinoid* if there exists an admissible surjective morphism of k-algebras $\varphi : k\{r^{-1}T\} \to \mathcal{A}$. If $r_i \in |k^\times|$ for all i, then $\mathcal{A}$ is said to be strictly affinoid. It is a fundamental fact that all k-affinoid algebras are noetherian and that all their ideals are closed, see [**Ber90**, Proposition 2.1.3]. Notice that the fact that the epimorphism φ is admissible implies that $\mathcal{A}$ and $k\{r^{-1}T\}/\ker(\varphi)$ endowed with the residue norm are isomorphic as Banach algebras.

The analytic spectrum $X = \mathcal{M}(\mathcal{A})$ of a k-affinoid algebra $\mathcal{A}$ is called a k-affinoid space. When $\mathcal{A}$ is strictly affinoid, one says that X is strictly affinoid.

The affinoid space X naturally carries a sheaf of analytic functions $\mathcal{O}_X$, see [**Ber90**, §2.3], whose construction relies on the notion of affinoid domain.

Definition 1.1.1. *Let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space and V a closed subset of X. One says that V is an affinoid domain of X if there exists a bounded homomorphism of k-affinoid algebras $\varphi : \mathcal{A} \to \mathcal{A}_V$ inducing a morphism whose image is contained in V satisfying the following universal property. For every bounded homomorphism of affinoid k-algebras $\mathcal{A} \to \mathcal{B}$ such that the image of $\mathcal{M}(\mathcal{B})$ in X is contained in V, there exists a unique bounded homomorphism $\mathcal{A}_V \to \mathcal{B}$*

such that the following diagram commutes:



The construction of the sheaf $\mathcal{O}_X$ is based on Tate's Acyclicity Theorem [**BGR84**, §8.2]:

THEOREM 1.1.2. *Let V be an affinoid domain of a k-affinoid space X. Let $\{V_i\}$ be a finite covering of V by affinoid domains.*
Then the sequence induced by the restriction

$$\mathcal{A}_V \longrightarrow \prod \mathcal{A}_{V_i} \rightrightarrows \prod \mathcal{A}_{V_i \cap V_j}$$

is exact.

Let V be a finite union of affinoid domains of X. Given a finite covering (V_i) of V by affinoid domains, set $\mathcal{A}_V = \ker\left(\prod \mathcal{A}_{V_i} \to \prod \mathcal{A}_{V_i \cap V_j}\right)$. It is a commutative Banach k-algebra that does not depend on the choice of the covering (V_i) [**Ber90**, Corollary 2.2.6].

In view of this result, for every open set $U \subset X$, one sets

$$\mathcal{O}_X(U) = \varprojlim \mathcal{A}_V \ ,$$

where the limit is taken over all closed subsets $V \subset U$ that are a finite union of affinoid domains of X.

For any point x in a k-affinoid space X, the stalk $\mathcal{O}_{X,x}$ is a local k-algebra. Its maximal ideal $\mathfrak{m}_x$ consists of all the functions $f \in \mathcal{O}_{X,x}$ that do not vanish at x. The stalk $\mathcal{O}_{X,x}$ inherits from x an absolute value extending the one on k, and the completion of $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is precisely $\mathcal{H}(x)$, the complete residue field at x.

When $\mathcal{H}(x)$ is a finite extension of k (or equivalently when $\mathcal{H}(x) = k$, since k is supposed to be algebraically closed), one says that x is *rigid*. The set $X(k)$ of rigid points of a strictly k-affinoid space X is dense in X.

EXAMPLE 1.1.3. *The closed polydisk of dimension N and polyradius $r = (r_1, \cdots, r_N) \in (\mathbb{R}_*^+)^N$ is defined to be $\bar{\mathbb{D}}^N(r) := \mathcal{M}(k\{r^{-1}T\})$. The Gauss point $x_g \in \bar{\mathbb{D}}^N$ is the point associated to the norm*

$$|(\sum a_I T^I)(x_g)| := \max |a_I|.$$

When $r = (1, \cdots, 1)$ we just write $\bar{\mathbb{D}}^N$, and when $N = 1$ we denote it by $\bar{\mathbb{D}}$.

EXAMPLE 1.1.4. *Pick any real numbers $r \leq R$. The closed annulus is the affinoid space $A[r, R] := \mathcal{M}(k\{R^{-1}T, rS\}/(ST - 1))$. It can be identified with the closed subset of the closed disk $\bar{\mathbb{D}}(R)$ consisting of the points $x \in \bar{\mathbb{D}}(R)$ with $r \leq |T(x)| \leq R$.*

1.1.3. General analytic spaces. General analytic spaces are locally ringed spaces $(X, \mathcal{O}_X)$ obtained by gluing together affinoid spaces. Difficulties arise in the gluing construction due to the fact that affinoid spaces are compact, and we refer to [**Ber90, Ber93**] for the precise (and delicate) definition.

In the following, we shall however only deal with *good* analytic spaces which is formed by the subcategory of analytic spaces that are locally ringed spaces modelled on affinoid spaces. Let us make this definition precise.

A (good) k-analytic space X is a locally ringed space endowed with a k-analytic atlas, which is the data of an open cover $(U_i)_{i\in I}$ of X and an open immersion φ_i on each U_i into a k-affinoid space such that the induced morphism of locally ringed spaces $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ satisfies the following property. For every affinoid domains $V_i \subset \varphi_i(U_i \cap U_j)$ and $V_j \subset \varphi_j(U_i \cap U_j)$ with $\varphi_j \circ \varphi_i^{-1}(V_i)$ contained in the topological interior of V_j in $\varphi_j(U_i \cap U_j)$, the restriction of $\varphi_j \circ \varphi_i^{-1}$ to V_i is an isomorphism of k-affinoid spaces.

Denote by $\mathcal{O}_X$ the structure sheaf of X. The complete residue field at a point $x \in X$ is the defined as the completion of the quotient $\mathcal{O}_{X,x}/\mathfrak{m}_x$, as in the affinoid case. Rigid points in X are those such that $\mathcal{H}(x)$ is a finite extension of k (or equivalently when $\mathcal{H}(x) = k$, since k is supposed to be algebraically closed). The set $X(k)$ of rigid points of X is dense in X.

Example 1.1.5. *The open polydisk of dimension N and polyradius $r = (r_1, \dots, r_N) \in (\mathbb{R}_*^+)^N$ is the set*

$$\mathbb{D}_k^N(r) = \{x \in \bar{\mathbb{D}}^N(r) : |T_i(x)| < r_i, i = 1, \dots, N\}.$$

It can be naturally endowed with a structure of good analytic space by writing it as the increasing union of N-dimensional polydisks $\bar{\mathbb{D}}_k^N(\rho)$ whose radii $\rho = (\rho_1, \cdots, \rho_N) \in (|k^\times|)^N$ satisfy $\rho_i < r_i$ for all $i = 1, \dots, N$.

Example 1.1.6. *Pick any real numbers $r < R$. The open annulus is the set*

$$A(r, R) = \{x \in A[r, R] : r < |T(x)| < R\}.$$

It can be naturally endowed with a structure of good analytic space by writing it as an increasing union of closed annuli.

1.1.4. Topological properties of analytic spaces. Every k-analytic space is locally compact, and it is a theorem of Berkovich that it is locally path-connected [**Ber90**, Theorem 3.2.1].

Analytic spaces also satisfy the following remarkable property that their topology can be characterized by sequences, although they are not metrizable in general (for instance when $\tilde{k}$ is uncountable). The precise statement is due to Poineau and relies on the following notion.

A topological space X is Fréchet-Urysohn if every subset $A \subseteq X$ is sequential, that is for every point x in the closure of A there exists a sequence of points $a_n \in A$ converging to x.

THEOREM 1.1.7 ([**Poi13**]). *Every k-analytic space X is a Fréchet-Urysohn space.*

1.1.5. Analytic maps. An analytic map between k-affinoid spaces $\varphi^\sharp : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{B})$ is a *closed immersion* when the underlying homomorphism $\varphi : \mathcal{B} \to \mathcal{A}$ is surjective and admissible.

Let X and Y be (good) k-analytic spaces. A continuous map $f : X \to Y$ is analytic if there exists an atlas $\{(U_i, \varphi_i)\}$ of X and an atlas $\{(V_j, \psi_j)\}$ of Y such that for every pair of indexes i, j, the composition $\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \psi_j(V_j)$ satisfies the following property. For every affinoid domains $W \subset \varphi_i(U_i)$ and $Z \subset \psi_j(V_j)$ with $\psi_j \circ f \circ \varphi_i^{-1}(W)$ contained in the topological interior of Z in $\psi_j(V_j)$, the restriction of $\varphi_j \circ f \circ \varphi_i^{-1}$ to W is an morphism of k-affinoid spaces, i.e. is induced by a bounded morphism between the underlying k-affinoid algebras.

We shall denote by $\mathrm{Mor}_k(X, Y)$ the set of all analytic maps from X to Y.

EXAMPLE 1.1.8. *An analytic map $f : \bar{\mathbb{D}}^N \to \bar{\mathbb{D}}^M$ is given by a bounded homomorphism between the corresponding k-affinoid algebras. It is thus of the form $f = (f_1, \ldots, f_M)$, where $f_l = \sum_{I \in \mathbb{N}^N} a_{l,I} T^I$ with $|a_{l,I}| \leq 1$ and $|a_{l,I}| \to 0$ as $|I| \to \infty$ for all $l = 1, \ldots, M$.*

EXAMPLE 1.1.9. *An analytic map $f : \mathbb{D}^N \to \bar{\mathbb{D}}^M$ is a continuous map such that the restriction to $\bar{\mathbb{D}}^N(\rho)$ for every $\rho < 1$ is given by a bounded homomorphism of k-affinoid algebras. It follows that f is of the form $f = (f_1, \ldots, f_M)$, where $f_l = \sum_{I \in \mathbb{N}^N} a_{l,I} T^I$ with the only requirement that the coefficients $a_{l,I}$ are of norm at most one, for every $l = 1, \ldots, M$.*

We may now extend the definition of an affinoid domain to a general k-analytic space X. Let Y be a k-affinoid space and $f : Y \to X$ an analytic map that is an homeomorphism on its image. If for every analytic map $g : Z \to X$ with $g(Z) \subseteq f(Y)$ there exists a unique analytic map $h : Z \to Y$ such that $g = f \circ h$, then the image of f in X is called an affinoid domain. Affinoid domains form a basis for the topology of X.

1.2. Analytification of algebraic varieties

1.2.1. General construction. A fundamental class of good analytic spaces are the analytifications of algebraic varieties. To every algebraic variety X over k one can associate a k-analytic space X^{an} in a functorial way. We refer to [**Ber90**, §3.4] for a detailed construction.

In the case of an affine variety $X = \mathrm{Spec}(A)$, where A is a finitely generated k-algebra, then the set X^{an} consists of all the multiplicative seminorms on A whose restriction to k coincides with the norm on k. This set is endowed with the weakest topology such that all the maps of the form $x \in X^{\mathrm{an}} \mapsto |f(x)|$ with $f \in A$ are continuous. Fix an embedding of X into some affine space $\mathbb{A}^N$. Then the intersections with the open polydisks $X^{\mathrm{an}} \cap \mathbb{D}^N(r)$ with $r > 0$ define an analytic atlas on X^{an}.

Observe that any k-point $x \in X$ corresponds to a morphism of k-algebras $A \to k$ and its composition with the norm on k defines a rigid point in X^{an}. Since k is algebraically closed, one obtains in this way an identification of the set of closed points in X with the set of rigid points in X^{an}.

Let X be a general algebraic variety and fix an affine open cover. The analytification of a general algebraic variety X is obtained by glueing together the analytification of its affine charts in natural way. Analytifications of algebraic varieties are good analytic spaces, and closed points are in natural bijection with rigid points as in the affine case.

1.2.2. The analytic affine line. As described above, the analytic affine line $\mathbb{A}^{1,\mathrm{an}}$ is the set of bounded seminorms on the polynomial ring $k[T]$.

The points in $\mathbb{A}^{1,\mathrm{an}}$ can be explicitly described as follows [**Ber90**, §1.4.4].

Pick $a \in k$ and $r \in \mathbb{R}_+$ and denote by $\bar{B}(a;r)$ the closed ball in k centered at a and of radius r. To $\bar{B}(a;r)$ one can associate a point $\eta_{a,r} \in \mathbb{A}^{1,\mathrm{an}}$ by setting $|P(\eta_{a,r})| := \sup_{|y-a|\leq r} |P(y)|$ for every polynomial $P \in k[T]$. If $r = 0$, then $\eta_{a,0}$ corresponds to evaluating polynomials in $a \in k$.

More generally, any decreasing sequence of closed balls $\bar{B}(a_i;r_i)$ in k defines a sequence of points η_{a_i,r_i} that converges in $\mathbb{A}^{1,\mathrm{an}}$ to a point $\eta \in \mathbb{A}^{1,\mathrm{an}}$ sending any polynomial $P \in k[T]$ to $|P(\eta)| = \lim_i |P(\eta_{a_i,r_i})|$. Observe that such a sequence of balls might have empty intersection, in which case $\lim_i r_i = r > 0$ since k is complete.

It is a key fact due to Berkovich [**Ber90**, §1.4.4] that any point in $\mathbb{A}^{1,\mathrm{an}}$ comes from a decreasing sequence of closed balls in k.

Suppose that $x = \lim \eta_{a_i,r_i}$ and set $\bar{B} = \cap_i \bar{B}(a_i;r_i)$. V. Berkovich introduced the following terminology.

i) The point x is of type I if and only if $\bar{B} = \{a\}$, with $a \in k$.
ii) The point x is of type II if and only if $\bar{B} = \bar{B}(a;r)$ with $r \in |k^\times|$.

iii) The point x is ot type III if and only if $\bar{B} = \bar{B}(a;r)$ with $r \notin |k^\times|$.
iv) The point x is of type IV if and only if $\bar{B} = \emptyset$.

Every point in $\mathbb{A}^{1,\mathrm{an}}$ falls into one of these four types.

The analytic affine line carries a tree structure, which will be discussed in §4.2.

Let us describe a basis of neighbourhoods of a point $x \in \mathbb{A}^{1,\mathrm{an}}$ according to its type.

If x is of type I or IV, then x admits a fundamental family of neighbourhoods that are open disks.

If $x = \eta_{a,r}$ is of type II, then it has a fundamental family of neighbourhoods of the form $\mathbb{D}(a;R) \setminus \sqcup_{1\leq i\leq n}\bar{\mathbb{D}}(a_i;r_i)$ with $R > r > r_i$, $|a_i - a| \leq r$ and $|a_i - a_j| = r$ whenever $i \neq j$. In particular, the complement of x in this neighbourhood is a disjoint union of open disks and finitely many open annuli.

Finally, if $x = \eta_{a,r}$ is a type III point, then open annuli $\mathbb{D}(a;R_1) \setminus \bar{\mathbb{D}}(a;R_2)$ with $R_1 > r > R_2$ form a fundamental family of neighbourhoods of x.

The analytic projective line $\mathbb{P}^{1,\mathrm{an}}$ is the one-point compactification of $\mathbb{A}^{1,\mathrm{an}}$. An open (resp. closed) disk in $\mathbb{P}^{1,\mathrm{an}}$ is either an open (resp. closed) disk in $\mathbb{A}^{1,\mathrm{an}}$ or the complement of a closed (resp. open) disk in $\mathbb{A}^{1,\mathrm{an}}$. Connected affinoid domains in $\mathbb{P}^{1,\mathrm{an}}$ are the complement of finitely many open disks.

1.3. Reduction, boundary and interior

1.3.1. Reduction. Recall from §1.1.1 the definition of the reduction of the analytic spectrum of a Banach algebra. The reduction of the closed polydisk $\bar{\mathbb{D}}_k^N$ is the affine space $\mathbb{A}_{\tilde{k}}^N$. Observe that Noether's normalization Lemma [**BGR84**, Corollary 6.1.2/2] implies that for any strictly k-affinoid algebra $\mathcal{A}$, the reduction $\widetilde{\mathcal{A}}$ is a finitely generated $\tilde{k}$-algebra. The reduction of the strictly k-affinoid space $X = \mathcal{M}(\mathcal{A})$ is thus the affine variety $\widetilde{X} = \mathrm{Spec}(\widetilde{A})$ defined over the residue field $\tilde{k}$.

1.3.2. Boundary. Let us recall the notions of boundary and interior of a good k-analytic space X from [**Ber90**, §3.1]. We refer to [**Ber93**, §1.5.4] for a discussion in the case of general Berkovich spaces.

A point x in an affinoid space X lies in the *interior* of X if there exists a closed immersion $\varphi : X \to \bar{\mathbb{D}}^N(r)$ for some polyradius r and some integer N such that $\varphi(x)$ lies in the open polydisk $\mathbb{D}^N(r)$.

If X is a good analytic space, a point x belongs to its interior if it admits an affinoid neighbourhood U such that x belongs to the interior of U. We let $\mathrm{Int}(X)$ be the open set consisting of all the interior points

in X. Its complement $\partial(X)$ is called the boundary of X. It is a closed subset of X.

EXAMPLE 1.3.1. *The analytification of any algebraic variety has empty boundary.*

In the remainder of this section, we explain how to compute the interior of a strictly k-affinoid space $X = \mathcal{M}(\mathcal{A})$.

The reduction map $\mathrm{red} : X \to \widetilde{X}$ is defined as follows. Every bounded morphism of Banach k-algebras $\mathcal{A} \to \mathcal{B}$ induces a morphism between their reductions $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$. In particular, from the character $\chi_x : \mathcal{A} \to \mathcal{H}(x)$ associated to a point $x \in X$ we obtain a $\tilde{k}$-algebra morphism $\widetilde{\chi}_x : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{H}(x)}$. We set $\mathrm{red}(x) := \mathrm{Ker}(\widetilde{\chi}_x)$. This map is anticontinuous for the Zariski topology, meaning that the inverse image of a closed set is an open set.

Let $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$ be strictly k-affinoid spaces. Any analytic map $f : X \to Y$ induces the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{red}}\downarrow & & \downarrow{\scriptstyle \mathrm{red}} \\ \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{Y} \end{array} \tag{1.1}$$

Let us verify the commutativity. Denote by $f^\sharp$ and $\tilde{f}^\sharp$ the dual morphisms of f and $\tilde{f}$ respectively. Pick any $x \in X$. On the one hand, by definition we have that $\mathrm{red} \circ f(x)$ is the kernel of the composition $\widetilde{B} \to \widetilde{A} \to \widetilde{H(x)}$. On the other hand, we see that $\tilde{f} \circ \mathrm{red}(x) = \tilde{f}(\ker(\widetilde{\chi}_x)) = \left(\tilde{f}^\sharp\right)^{-1}(\ker(\widetilde{\chi}_x))$, which is precisely $\ker(\widetilde{\chi}_x \circ \tilde{f})$, and thus the diagram commutes.

LEMMA 1.3.2. *Let X be a strictly k-affinoid space. Then,*

$$\mathrm{Int}(X) = \{x \in X : \mathrm{red}(x) \text{ is a closed point}\}.$$

PROOF. Let $\varphi : X \to \bar{\mathbb{D}}^N$ be a closed immersion. By (1.1) we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \bar{\mathbb{D}}^N \\ {\scriptstyle \mathrm{red}}\downarrow & & \downarrow{\scriptstyle \mathrm{red}} \\ \widetilde{X} & \xrightarrow{\widetilde{\varphi}} & \mathbb{A}^N_{\tilde{k}} \end{array}$$

Let $\mathcal{A}$ be the underlying affinoid algebra of X and pick any $x \in X$. If the reduction $\tilde{x} = \mathrm{red}(x)$ of x is a closed point then so is $\widetilde{\varphi}(\tilde{x})$. The inverse image of $\widetilde{\varphi}(\tilde{x})$ is isomorphic to an open polydisk. Up to composing φ with an automorphism of $\bar{\mathbb{D}}^N$, we may assume that $\mathrm{red}^{-1}(\widetilde{\varphi}(\tilde{x}))$ is isomorphic to $\mathbb{D}^N$. the commutativity of the diagram implies that $\varphi(x)$ lies in $\mathbb{D}^N$.

Pick a point $x \in \mathrm{Int}(X)$. By [**Ber90**, Proposition 2.5.2], the image of the morphism of $\tilde{k}$-algebras $\widetilde{\chi}_x : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{H}(x)}$ induced by χ_x is integral over $\tilde{k}$. This implies that $\tilde{\chi}_x(\widetilde{\mathcal{A}}) \simeq \widetilde{\mathcal{A}}/\mathrm{Ker}(\widetilde{\chi}_x)$ is a field. Thus, $\tilde{x}$ is a closed point of $\widetilde{X}$. □

Recall that a morphism of k-affinoid spaces $f : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{B})$ is finite if $\mathcal{A}$ is a finite Banach $\mathcal{B}$-module.

Proposition 1.3.3. *Let $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$ be k-affinoid spaces, and let $f : X \to Y$ be a finite morphism. Then,* $\mathrm{Int}(X) = f^{-1}(\mathrm{Int}(Y))$.

This result is a consequence from [**Ber90**, Proposition 2.5.8] and [**Ber90**, Corollary 2.5.13]. Here we give a proof in the strictly k-affinoid case.

Proof. We prove the result only in the strictly affinoid case. In order to adapt this proof to the general one, one needs to use Temkin's graded reduction of affinoid algebras ([**Tem00, Tem04**]).

The morphism $f : X \to Y$ induces the following commutative diagram, cf. (1.1):

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \mathrm{red}}\downarrow & & \downarrow{\scriptstyle \mathrm{red}} \\ \mathrm{Spec}(\widetilde{\mathcal{A}}) & \xrightarrow{\tilde{f}} & \mathrm{Spec}(\widetilde{\mathcal{B}}) \end{array}$$

Let x be a point in $\mathrm{Int}(X)$. By Lemma 1.3.2, its image $f(x)$ belongs to $\mathrm{Int}(Y)$.

Let now $x \in X$ be such that $f(x) = y$ lies in $\mathrm{Int}(Y)$. By the previous lemma, we have to show that $\mathrm{red}(x)$ is a closed point of $\widetilde{X}$. Consider the ring homomorphism $\varphi : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{A}}$ inducing $\tilde{f}$. It induces a morphism $\varphi' : \widetilde{\mathcal{B}}/\ker(\widetilde{\chi}_y) \to \widetilde{\mathcal{A}}/\ker(\widetilde{\chi}_x)$, as the diagram above is commutative. Observe that φ is integral, since it is finite ([**BGR84**, Theorem 6.3.5/1]), and thus φ' is also integral. As $y \in \mathrm{Int}(Y)$, by Lemma 1.3.2 the quotient $\widetilde{\mathcal{B}}/\ker(\widetilde{\chi}_y)$ is a field. This implies that $\widetilde{\mathcal{A}}/\ker(\chi_x)$ is a field and thus that $\mathrm{red}(x)$ is a closed point. □

1.4. Smooth analytic curves

In this section, we recall some facts on the structure of smooth analytic curves. Our main references are [**Ber90**, §4] and [**Duc14**].

1.4.1. First properties. Recall that a k-analytic curve X is a k-analytic space that is Hausdorff and of pure dimension 1. Throughout this section, X will denote a smooth analytic curve over k.

Points in a k-analytic curve can be classified as follows, see [**Duc14**, §3.3]. Let $x \in X$ and let $\mathcal{H}(x)$ be its complete residue field. Then one says that:

i) The point x is of type I if $\mathcal{H}(x) \simeq k$;
ii) The point x is of type II if the reduction $\widetilde{\mathcal{H}(x)}$ has transcendence degree 1 over $\tilde{k}$ and $|\mathcal{H}(x)| = |k|$;
iii) The point x is of type III if $\widetilde{\mathcal{H}(x)} \simeq \tilde{k}$ and the value group $|\mathcal{H}(x)^\times|$ is generated by $|k^\times|$ and some real number $r \notin |k^\times|$;
iv) The point x is of type IV if $\widetilde{\mathcal{H}(x)} \simeq \tilde{k}$, $|\mathcal{H}(x)| = |k|$ and $\mathcal{H}(x)$ is a non-trivial extension of k.

On $\mathbb{A}^{1,\mathrm{an}}$, this classification of the points in an analytic curve agrees with the one introduced above.

Let x be a type II point in a k-analytic curve X. We define the genus $g(x)$ of the point x as the genus of the unique smooth projective curve C over $\tilde{k}$ whose field of rational functions is isomorphic to $\widetilde{\mathcal{H}(x)}$. The set of points in X with positive genus is a closed discrete subset of X by [**Duc14**, Théorème 4.4.17].

The following fundamental topological result will be used in the sequel, see [**Duc14**, Théorème 4.5.10]:

THEOREM 1.4.1. *Every k-analytic curve is paracompact.*

1.4.2. Graph structure. Following the terminology of [**Duc14**], we say that a locally compact Hausdorff topological space X is a graph if it admits a fundamental basis of open sets U satisfying the following properties:

i) For every pair of points $x, y \in U$, there exists a unique closed subset $[x, y] \subset U$ homeomorphic to a segment of endpoints x and y.
ii) The boundary of U in X is finite.

A graph X is a tree if X itself satisfies property i).

It follows from the definition that graphs are locally path-connected and that trees are path-connected.

It is a fundamental fact that every k-analytic curve is a graph [**Duc14**, Théorème 3.5.1].

Let X be a k-analytic curve. Given a point $x \in X$, the tangent space T_xX at x is defined as the set of connected components of $U \setminus \{x\}$ where U is an open neighborhood of x which is a tree. It can be also defined as the set of paths leaving from x modulo the relation having a common initial segment proving that the definition does not depend on the choice of U. Given any tangent direction $\vec{v} \in T_xX$, we denote by $U(\vec{v})$ the open subset of points $y \in X \setminus \{x\}$ such that there exists a path starting from y and abuting at x in the direction of $\vec{v}$.

1.4.3. Skeleton of an analytic curve. Recall the definition of the skeleton of an analytic curve:

DEFINITION 1.4.2. *The skeleton of a curve X is the set of all points $x \in X$ having no neighbourhood isomorphic to an open disk. It will be denoted by $\mathsf{S}^{\mathrm{an}}(X)$.*

The only smooth projective curve with empty skeleton is $\mathbb{P}^{1,\mathrm{an}}$, see [**Duc14**, §5.4.8]. In the non-compact case, examples of curves with empty skeleton include the open disk and the affine line. The skeleton of an open annulus $A(\rho, 1)$ is the segment consisting of the points $\eta_{0,r}$ for $\rho < r < 1$.

By definition, $\mathsf{S}^{\mathrm{an}}(X)$ is a closed subset of X and its complement is a disjoint union of open disks. The skeleton of X is a closed locally finite subgraph of X. If X is projective, then $\mathsf{S}^{\mathrm{an}}(X)$ is compact.

For any curve X with nonempty skeleton there is a retraction map $r_X : X \to \mathsf{S}^{\mathrm{an}}(X)$, defined as follows. Every point in the skeleton is fixed by r_X. For every point $x \in X \setminus \mathsf{S}^{\mathrm{an}}(X)$, denote by U_x the maximal open neighbourhood of x that is isomorphic to $\mathbb{D}$. Then, $r_X(x)$ is the unique point in the topological boundary of U_x in X. The retraction map is continuous.

Let us now introduce the notion of nodes of a curve X following [**Duc14**, Lemme 6.2.3], which is a subset of its skeleton. In order to define it, recall that we say that a tangent direction $\vec{v} \in T_x X$ at a type II or type III point $x \in X$ is *discal* if $U(\vec{v})$ is a disc.

DEFINITION 1.4.3. *Let X be a smooth analytic curve over k. A type II point $x \in \mathsf{S}^{\mathrm{an}}(X)$ is a* node *if one of the following conditions is satisfied:*

i) *The point x has positive genus;*
ii) *There exist three distinct tangent directions at x that are non-discal;*
iii) *The point x belongs to the boundary of X.*

The set of all nodes of a curve X will be denoted by $\mathsf{N}(X)$. Observe that it contains every branching point in $\mathsf{S}^{\mathrm{an}}(X)$, which is discrete and closed as $\mathsf{S}^{\mathrm{an}}(X)$ is a locally finite graph. Since the boundary of X is finite and the set of points $x \in X$ of positive genus is closed and discrete [**Duc14**, Théorème 4.4.7], then $\mathsf{N}(X)$ is also discrete and closed. The complement of $\mathsf{N}(X)$ in $\mathsf{S}^{\mathrm{an}}(X)$ is by definition a disjoint union of open segments. It follows that if $\mathsf{N}(X)$ is nonempty, then $X \setminus \mathsf{N}(X)$ is a disjoint union of infinitely many open disks, finitely many open annuli, and finitely many punctured disks.

1.4.4. Smooth projective curves. Let us describe in more detail the structure of smooth projective curves. Recall that in this case, the skeleton is compact and the set of nodes finite.

Let X be a smooth irreducible projective curve. Its genus g is encoded in the topology of the skeleton and in the points of positive

genus as follows, see [**Duc14**, §5.2.6] and [**Ber90**, §4.3]. Denote by b the first Betti number of $\mathsf{S}^{\mathrm{an}}(X)$. Then, one has the equality

$$g = b + \sum_{x \in X_{[2]}} g(x) \ , \tag{1.2}$$

where $X_{[2]}$ is the set of type II points in X. Notice that the sum is finite, since a smooth projective curve X has only finitely many points of positive genus.

We remark that the endpoints of the skeleton are nodes:

LEMMA 1.4.4. *Let X be a smooth projective curve. Let η be an endpoint of $\mathsf{S}^{\mathrm{an}}(X)$. Then η is a node and has positive genus.*

This result helps us describe all the possibilities for a smooth projective curve X in terms of its skeleton and its nodes, see [**Duc14**, §5.4.12].

i) If X has empty skeleton, then it is isomorphic to $\mathbb{P}^{1,\mathrm{an}}$.
ii) If $\mathsf{S}^{\mathrm{an}}(X)$ is nonempty and X has no nodes, then $\mathsf{S}^{\mathrm{an}}(X)$ has no endpoints by Lemma 1.4.4, and hence the skeleton must be homeomorphic to a circle. It follows from (1.2) that X has genus 1. In that case, we say that X is a Tate curve, i.e. the analytification of an elliptic curve whose j-invariant is not integral.
iii) If $\mathsf{N}(X)$ is nonempty, then it follows from (1.2) that X has positive genus.

The particular case where X has only one node η_X deserves a more thorough description. We distinguish two possibilities for the geometry of X based on its skeleton, which will be used in §6:

i) The skeleton of X consists only of the point η_X. In that case, we say that X has good reduction. By (1.2), the genus of η_X equals the genus of the curve X. In particular, if $g(\eta_X) = 1$ then the curve X is the analytification of an elliptic curve with bad reduction, i.e. whose j-invariant is integral.
ii) There is at least one loop in $\mathsf{S}^{\mathrm{an}}(X)$ passing through η_X. By (1.2), X has genus at least 2.

PROOF OF LEMMA 1.4.4. Let η be an endpoint of the skeleton of X. If $\mathsf{S}^{\mathrm{an}}(X) = \{\eta\}$, then η is a node, since otherwise it has an open neighbourhood that is isomorphic to an open disk. As X is boundaryless and $\mathsf{S}^{\mathrm{an}}(X)$ has no branching points, we conclude that η has positive genus.

We may assume that $\{\eta\}$ is strictly contained in $\mathsf{S}^{\mathrm{an}}(X)$. As $\eta \in \mathsf{S}^{\mathrm{an}}(X)$, there exists a tangent direction $\vec{v} \in T_\eta X$ such that $U(\vec{v}) \cap \mathsf{S}^{\mathrm{an}}(X)$ is non-empty, and so $\vec{v}$ is non-discal. Being an endpoint of $\mathsf{S}^{\mathrm{an}}(X)$, all the other tangent directions at η are discal. Since X is projective, the

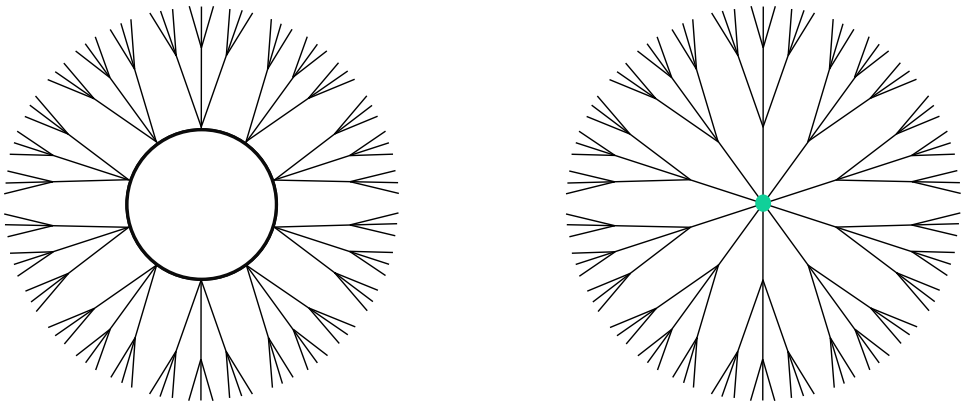


FIGURE 1. Elliptic curves

only remaining possibility for η to be a node is to have positive genus, since η is not a branching point of the skeleton.

Suppose by contradiction that $g(\eta) = 0$. By [**Duc14**, Théorème 4.5.4], then point η has a simply connected open neighbourhood V. Moreover, this theorem states that V is isomorphic to an open annulus if η is a type III point. If η is a type II point, the same result implies that we may reduce V such that it is also isomorphic to an open annulus, since η is an endpoint of $\mathsf{S}^{\mathrm{an}}(X)$.

In fact, as every tangent direction $\vec{v}' \in T_\eta X$ different from $\vec{v}$ is discal, we may assume that every $U(\vec{v}')$ is contained in V. After maybe reducing V, we may assume that he topological boundary of V in X is a single point in $\mathsf{S}^{\mathrm{an}}(X)$ and that no point in V has positive genus. By [**Duc14**, Proposition 5.1.18], V is isomorphic to an open disk, contradicting the fact that $\eta \in \mathsf{S}^{\mathrm{an}}(X)$. □

We have the following description of curves of positive genus:

LEMMA 1.4.5. *Let X be a smooth irreducible projective curve with non-empty set of nodes. Then $\mathsf{S}^{\mathrm{an}}(X)$ can be decomposed as the disjoint union of $\mathsf{N}(X)$ and open segments $I_1, \dots, I_a$, with each I_j isomorphic to a real segment $(1, R_j)$ with $R_j \in |k^\times|$, for $1 \leq j \leq a$. The complement of $\mathsf{N}(X)$ in X is a disjoint union of infinitely many open unit disks and annuli $A(1, R_1), \dots, A(1, R_a)$.*

Moreover, if $\mathsf{N}(X)$ consists of a single node η_X, then the closure $\bar{I}_j = I_j \cup \{\eta_X\}$ of each I_j in X is a circle.

PROOF. Let X be a smooth projective curve with non-empty set of nodes. Suppose that $\mathsf{S}^{\mathrm{an}}(X) \setminus \mathsf{N}(X)$ contains a loop C. As the skeleton of X is connected and every branching point in $\mathsf{S}^{\mathrm{an}}(X)$ is a

node, we conclude that $C = \mathsf{S}^{\mathrm{an}}(X)$, contradicting the fact that $\mathsf{N}(X)$ is nonempty.

Moreover, since X is projective there are only finitely many nodes, which are all type II points. As a consequence, the set $\mathsf{S}^{\mathrm{an}}(X) \setminus \mathsf{N}(X)$ consists of finitely many open segments I_j isomorphic to real segments $(1, R_j)$ with $R_j \in |k^\times|$, for $1 \leq j \leq a$. It follows that $X \setminus \mathsf{N}(X)$ consists of a disjoint union of open disks and the open annuli $A(1, R_1), \ldots, A(1, R_a)$.

Assume now that X has only one node η_X. If $\mathsf{S}^{\mathrm{an}}(X)$ consists only of the point η_X, then the complement of $N(X)$ is a disjoint union of open disks. Otherwise, the exists at least one loop in $\mathsf{S}^{\mathrm{an}}(X)$ passing through η_X by Lemma 1.4.4. Necessarily, the closure of each segment I_j in X is $I_j \cup \{\eta_X\}$, which is homeomorphic to a circle. □

The following lemma will be essential in the subsequent chapters, specially the proof of Theorem H. It is a particular case of [**Duc14**, Proposition 6.1.2].

LEMMA 1.4.6. *Let X be a smooth irreducible projective curve over k and $x \in X$ a rigid point. Every open neigbourhood U of x has an open subset $V \subseteq U$ such that $X \setminus V$ is an affinoid domain of X.*

1.5. Distinguished morphisms

Of special interest for us are analytic maps induced by a distinguished morphism of affinoid algebras, which are extensively discussed in [**BGR84**, §6.4.3]. Let $\mathcal{A}$ be a k-affinoid algebra.

DEFINITION 1.5.1. *A surjective morphism $\varphi : \mathcal{T}_N \to \mathcal{A}$ is called* distinguished *if the quotient norm $|.|_\varphi$ induced by φ agrees with the supremum norm on $\mathcal{A}$. We say that $\mathcal{A}$ is distinguished if such an epimorphism exists.*

It can be shown that over an algebraically closed field k, every reduced algebra (i.e. without non-trivial nilpotents) is distinguished [**BGR84**, Theorem 6.4.3/1].

From the definition one obtains the following useful result:

PROPOSITION 1.5.2. *Let X be a k-affinoid space and let $X \to \bar{\mathbb{D}}^N$ be a closed immersion induced by a distinguished morphism of Banach algebras. Then, every analytic map on X with values in a polydisk $\bar{\mathbb{D}}^M$ extends to an analytic map $\bar{\mathbb{D}}^N \to \bar{\mathbb{D}}^M$.*

PROOF. Let $\mathcal{A}$ be the underlying affinoid algebra of X. Pick an analytic map $f : X \to \bar{\mathbb{D}}^M$, which by definition is given by elements $f_1, \ldots, f_M \in \mathcal{A}$ with $|f_i|_{\sup} \leq 1$. Fix a distinguished epimorphism $\mathcal{T}_N \to \mathcal{A}$. For $l = 1, \ldots, M$, we may lift f_l to an element g_l in $\mathcal{T}_N$ having the same norm. The resulting analytic map $g = (g_1, \ldots, g_M) : \bar{\mathbb{D}}^N \to \bar{\mathbb{D}}^M$ agrees with f on the affinoid space X. □

The key property of distinguished affinoid algebras is that the reduction commutes with the quotient, see [**BGR84**, Corollary 6.4.3/5]:

THEOREM 1.5.3. *Let $\mathcal{T}_N \to \mathcal{A}$ be a distinguished morphism of strictly k-affinoid algebras. Then the reduction $\widetilde{\mathcal{A}}$ is isomorphic to the quotient $\widetilde{\mathcal{T}}_N/\widetilde{\ker(\varphi)}$.*

1.6. Basic tubes

We introduce the following terminology.

DEFINITION 1.6.1. *A k-analytic space X is called a basic tube if there exists a reduced equidimensional strictly k-affinoid space $\hat{X}$ and a closed point $\tilde{x}$ in its reduction such that X is isomorphic to* $\mathrm{red}^{-1}(\tilde{x})$.

By convention, a basic tube is reduced.

THEOREM 1.6.2. *A basic tube is connected.*

The fact that any basic tube over an algebraically closed field is connected is a deep theorem due to [**Bos77**], which was generalized to arbitrary base fields in [**Poi14**].

EXAMPLE 1.6.3. *Let $a_1, \cdots, a_m$ be type II points in $\mathbb{P}^{1,\mathrm{an}}$. Then every connected component of $\mathbb{P}^{1,\mathrm{an}} \setminus \{a_1, \cdots, a_m\}$ is a basic tube, cf. §1.2.2.*

PROPOSITION 1.6.4. *A k-analytic space X is a basic tube if and only if it is isomorphic to a connected component of the interior of some equidimensional strictly k-affinoid space.*

REMARK 1.6.5. *Every good reduced boundaryless k-analytic space has a basis of open neighbourhooods that are basic tubes.*

PROOF. Let V be any connected component of the interior of an equidimensional strictly k-affinoid space $\hat{X}$. By Lemma 1.3.2, $\mathrm{red}(V)$ is contained in the set of closed points of the reduction of $\hat{X}$. If $\mathrm{red}(V)$ contains at least two distinct points, then V can be written as a disjoint union of nonempty open sets, contradicting the connectednes. Hence, $\mathrm{red}(V)$ is a singleton.

Let conversely $X = \mathrm{red}^{-1}(\tilde{x})$ be a basic tube, where $\tilde{x}$ is a closed point in the reduction of an equidimensional strictly k-affinoid space $\hat{X}$. Clearly, X is contained in some connected component V of $\mathrm{Int}(\hat{X})$. The previous argument shows that $\mathrm{red}(V) = \{\tilde{x}\}$. □

Recall that a topological space is σ-compact if it is the union of countably many compact subspaces. For instance, open Berkovich polydisks or the analytification of an algebraic variety are σ-compact spaces. Observe that there exist simple examples of k-analytic spaces which are not σ-compact, e.g. the closed unit disk of dimension $N \geq 2$ with the Gauss point removed over a base field k with uncountable reduction $\tilde{k}$.

PROPOSITION 1.6.6. *For every basic tube X there exist a strictly k-affinoid space $\hat{X}$ and a distinguished closed immersion into some closed polydisk $\hat{X} \to \bar{\mathbb{D}}^N$ such that X is isomorphic to $\hat{X} \cap \mathbb{D}^N$.*

In particular, X is boundaryless and σ-compact.

PROOF. Let $\hat{X} = \mathcal{M}(\mathcal{A})$ be an equidimensional reduced k-affinoid space and let $\tilde{x}$ be a closed point in its reduction such that $\mathrm{red}^{-1}(\tilde{x})$ is isomorphic to X. Recall from §1.5 that there exists a distinguished closed immersion $\varphi : \hat{X} \to \bar{\mathbb{D}}^N$, as k is algebraically closed and $\mathcal{A}$ is reduced. Hence, $\widetilde{\mathcal{A}}$ is isomorphic to $\tilde{k}[T_1, \cdots, T_N]/\widetilde{\ker(\varphi)}$ by Theorem 1.5.3.

The induced morphism $\mathrm{Spec}(\widetilde{\mathcal{A}}) \to \mathbb{A}^N_{\tilde{k}}$ is a closed immersion by [**BGR84**, Proposition 6.4.3/3], since φ is distinguished. We may assume that $\tilde{x}$ is mapped to 0. We conclude that x is mapped to a point in $\mathrm{red}^{-1}(0)$, which is isomorphic to $\mathbb{D}^N$. □

1.7. Universal points and base changes

Given a k-affinoid algebra $\mathcal{A}$ and a complete extension K of k, the K-algebra $\mathcal{A}\hat{\otimes}_k K$ is in fact K-affinoid. One defines the scalar extension of the k-affinoid space $X = \mathcal{M}(\mathcal{A})$ by K as the K-affinoid space $X_K := \mathcal{M}(\mathcal{A}\hat{\otimes}_k K)$. The natural morphism $\mathcal{A} \to \mathcal{A}\hat{\otimes}_k K$ induces a base change morphism $\pi_{K/k} : X_K \to X$ which is continuous and surjective. This construction can be done similarly for general k-analytic spaces.

Recall the following definition from [**Ber90, Poi13**]:

DEFINITION 1.7.1. *Let X be a k-analytic space. A point x in X is* universal *if for every complete extension K of k the tensor norm on $\mathcal{H}(x)\hat{\otimes}_k K$ is multiplicative.*

The key feature of universal points is that they can be canonically lifted to any scalar extension. To explain this fact we may suppose that X is an affinoid space with underlying algebra $\mathcal{A}$. Pick any universal point $x \in X$ and fix any complete extension K of k. The k-algebra morphism $\mathcal{A} \to \mathcal{H}(x)$ corresponding to the point x induces a K-algebra morphism $\mathcal{A}\hat{\otimes}_k K \to \mathcal{H}(x)\hat{\otimes}_k K$.

Since x is universal, the tensor norm on $\mathcal{H}(x)\hat{\otimes}_k K$ is multiplicative, and so the composition of $\mathcal{A}\hat{\otimes}_k K \to \mathcal{H}(x)\hat{\otimes}_k K$ with the tensor norm defines a point in X_K. The point in X_K obtained by these means is denoted by $\sigma_{K/k}(x)$.

Observe that if $x \in X$ is rigid, then so is $\sigma_{K/k}(x)$, and that $\sigma_{K/k}$ is a section of $\pi_{K/k}$ on the set of universal points of X.

THEOREM 1.7.2 ([**Poi13**]). *Let k be an algebraically closed complete field and X a k-analytic space. Then, every point $x \in X$ is universal, and the map $\sigma_{K/k} : X \to X_K$ is continuous.*

Let us describe the map $\sigma_{K/k}$ on $\mathbb{A}_k^{1,\mathrm{an}}$ explicitly. Recall from §1.2.2 that every closed ball $\bar{B}(a;r)$ in k defines a point $\eta_{a,r} \in \mathbb{A}^{1,\mathrm{an}}$ (of type I, II or III) whose corresponding seminorm satisfies $|P(\eta_{a,r})| = \sup_{y\in\bar{B}(a;r)} |P(y)|$ for any polynomial $P \in k[T]$. Then, $\sigma_{K/k}(x)$ is the point in $\mathbb{A}_K^{1,\mathrm{an}}$ given by the closed ball in K centered at $a \in k$ and of radius r. The same procedure applies when x is a type IV point, given by a decreasing sequence of closed balls in k with empty intersection.

Notice that $\sigma_{K/k}$ preserves the type of type I and II points, whereas type III points can be sent to type II or type III points and type IV points to type II, III or IV points.

We conclude this section by recalling the following construction.

LEMMA 1.7.3. *Let X be a k-analytic space and x a point in X. Then for every complete extension K of $\mathcal{H}(x)$, the fibre $\pi_{K/k}^{-1}(x)$ contains a rigid point.*

PROOF. Pick a point $x \in X$. We may suppose $K = \mathcal{H}(x)$. Since the statement is local at x, we may replace X by any affinoid domain of X containing x. Denote by $\mathcal{A}$ the underlying k-affinoid algebra. Consider the character $\chi_x : \mathcal{A} \to \mathcal{H}(x)$. The morphism $\mathcal{A}\hat{\otimes}_k\mathcal{H}(x) \to \mathcal{H}(x)$ sending $f \otimes a$ to $\chi_x(f) \cdot a$ is by definition a rigid point in $X_{\mathcal{H}(x)}$ lying over x. □

We shall denote by $\tau(x) \in X_{\mathcal{H}(x)}$ the rigid point lying over $x \in X$ obtained in the previous proof. This point $\tau(x)$ is not to be confused with $\sigma_{K/k}(x)$.

CHAPTER 2

Parametrization of the space of analytic maps

Our goal in this chapter is to prove the following non-Archimedean analogue of Montel's theorem:

THEOREM A. *Let k be a non-Archimedean complete field that is non trivially valued and X a good, reduced, σ-compact, boundaryless strictly k-analytic space. Let Y be a strictly k-affinoid space.*

Then, every sequence of analytic maps $f_n : X \to Y$ admits a pointwise converging subsequence whose limit is continuous.

Observe that the previous theorem only supposes the base field k to be complete and non trivially valued. However, the key arguments of the proof are done under the assumption that k is algebraically closed, and the result is thereupon generalized to an arbitrary base field. We shall therefore assume troughout this chapter that k is algebraically closed. The case of an arbitrary complete non-Archimedean field k is treated in §2.5.2.

It is crucial in Theorem A to assume that X has no boundary. Indeed, as pointed out in [**FKT12**, §4.2], the sequence of analytic maps from the closed unit disk $\bar{\mathbb{D}}$ to itself $f_n(z) = z^{2^{n!}}$ converges pointwise everywhere, but the limit map f is not continuous. The Gauss point x_g is a fixed point of f, but f maps the whole of $\mathbb{D}$ to 0.

Let us clarify the remaining assumptions on X. Our proof first deals with the case where X is an open polydisk. We need the space X to be reduced in order to treat the case where X is a basic tube, as this ensures the existence of a distinguished closed immersion into some closed polydisk, which allows us to reduce to the polydisk case by Proposition 1.5.2. Finally, if X is a σ-compact space, then we may cover it by countably many basic tubes and extract diagonally.

This chapter is structured as follows. In §2.1 we give a proof of Theorem A in the simpler case of sequences of polynomial maps $\mathbb{A}^{r,\mathrm{an}} \to \mathbb{A}^{s,\mathrm{an}}$ of uniformly bounded degree sending the open polydisk $\mathbb{D}^r$ to $\bar{\mathbb{D}}^s$. The reason for this is that the arguments used in this case are at the core of the proof of Theorem A and motivate the subsequent technical constructions needed to deal with general analytic maps. The idea is to interpret any such polynomial map as a rigid point of a closed polydisk of the adequate dimension, which is determined by the degree of the polynomial. Specifically, if the polynomial maps have degree at most δ^r with $\delta \in \mathbb{N}^*$, the dimension of this closed polydisk is $s(\delta+1)^r$.

By these means, we parametrize the space of such polynomial maps by the closed polydisk $\bar{\mathbb{D}}^{s(\delta+1)^r}$. Any such sequence $f_n : \mathbb{A}^{r,\mathrm{an}} \to \mathbb{A}^{s,\mathrm{an}}$ then provides a sequence of rigid points α_n in $\bar{\mathbb{D}}^{s(\delta+1)^r}$, and we may extract a subsequence α_{n_j} converging to some point α, since closed polydisks are sequentially compact by [**Poi13**]. We infer that the subsequence f_{n_j} is converging. Finally, we compute its limit map explicitly in terms of α and show that it is continuous.

In light of the construction in the case of polynomial maps of bounded degree, it is apparent that the space of analytic maps from an open polydisk $\mathbb{D}^r$ to a closed polydisk $\bar{\mathbb{D}}^s$ should be parametrized by an *infinite dimensional* polydisk. To this end, we construct a k-Banach algebra $\mathcal{T}_\infty^{r,s}$ that is an analogue of the Tate algebra in countably many variables. Its analytic spectrum $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ satisfies the following universal property.

THEOREM 2.0.1. *There exists an analytic map* $\Phi : \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s) \times \mathbb{D}^r \to \bar{\mathbb{D}}^s$ *satisfying the following universal property. Let* W *be the analytic spectrum of a Banach* k*-algebra or any good* k*-analytic space. For any analytic map* $F : W \times \mathbb{D}^r \to \bar{\mathbb{D}}^s$ *there exists a unique morphism* $G : W \to \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ *such that* $F(x, z) = \Phi(G(x), z)$ *for all* $x \in W(k)$ *and* $z \in \mathbb{D}^r(k)$.

In fact, this construction may be generalized to analytic maps having an arbitrary affinoid space Y as a target. The resulting space $\mathrm{Mor}(\mathbb{D}^r, Y)$ is defined as the analytic spectrum of a quotient of $\mathcal{T}_\infty^{r,s}$ and satisfies the same universal property, cf Theorem B.

Then, Theorem A is a consequence of the following fact:

THEOREM 2.0.2. *The space* $\mathrm{Mor}(\mathbb{D}^r, Y)$ *is Fréchet-Urysohn.*

Further properties of the space $\mathrm{Mor}(\mathbb{D}^r, Y)$ are stated in Theorem 2.3.1. It specifies how to associate a continuous map $\mathbb{D}^r \to Y$ to every point in $\mathrm{Mor}(\mathbb{D}^r, Y)$ in a continuous way.

Finally, in §2.7 we discuss the convergence of sequences of analytic maps defined on a k-affinoid space X, which has nonempty boundary, with values in any k-affinoid space Y. We establish that when the residue field of k is countable, every sequence $f_n : X \to Y$ has a converging subsequence, but the limit map may fail to be continuous.

2.1. The polynomial case

As a first step in proving Theorem A, we deal with the case of sequences of polynomial maps of bounded degree.

Throughout this section, we fix integers $r, s, \delta > 0$.
The result we aim to show is the following:

THEOREM 2.1.1. *Let k be an algebraically closed non-Archimedean complete field. Let $f_n : \mathbb{A}^{r,\mathrm{an}} \to \mathbb{A}^{s,\mathrm{an}}$ be a sequence of polynomial maps of uniformly bounded degree satisfying $f_n(\mathbb{D}^r) \subset \bar{\mathbb{D}}^s$. Then, there exists a subsequence that is converging pointwise to a continuous map $f : \mathbb{A}^{r,\mathrm{an}} \to \mathbb{A}^{s,\mathrm{an}}$.*

2.1.1. Parametrization of polynomial maps of uniformly bounded degree. In order to prove this theorem, we reinterpret polynomial maps between analytic affine spaces as rigid points in a closed polydisk.

Given a multi-index $I = (i_1, \cdots, i_r)$, denote by $|I| = \max_j i_j$.

Every polynomial map $f : \mathbb{A}^{r,\mathrm{an}} \to \mathbb{A}^{s,\mathrm{an}}$ of degree at most δ^r where $\delta \in \mathbb{N}^*$ satisfying $f(\mathbb{D}^r) \subseteq \bar{\mathbb{D}}^s$ is of the form

$$f = (f_1, \cdots, f_s) = \left(\sum_{|I| \leq \delta} a_{1,I} T^I, \cdots, \sum_{|I| \leq \delta} a_{s,I} T^I \right),$$

with $|a_{l,I}| \leq 1$. Thus, the point

$$\alpha = \alpha(f) := \Big((a_{1,I})_{|I| \leq \delta}, \cdots, (a_{s,I})_{|I| \leq \delta} \Big) \tag{2.1}$$

can be realized as rigid point in the (Berkovich) analytic space $\bar{\mathbb{D}}^{s(\delta+1)^r}$.

Additionally, to every not-necessarily rigid point α in $\bar{\mathbb{D}}^{s(\delta+1)^r}$ we shall associate a continuous map

$$P_\alpha = P_\alpha^{r,s} : \mathbb{A}^{r,\mathrm{an}} \to \mathbb{A}^{s,\mathrm{an}}$$

as follows. Consider first the analytic map $\Phi : \bar{\mathbb{D}}^{s(\delta+1)^r} \times \mathbb{A}^{r,\mathrm{an}} \to \mathbb{A}^{s,\mathrm{an}}$, given by the k-algebra morphism

$$\begin{array}{rcl} k[T_1, \ldots, T_s] & \to & k[T_1, \ldots, T_r]\{(a_{1,I})_{|I| \leq \delta}, \cdots, (a_{s,I})_{|I| \leq \delta}\} \\ T_l & \mapsto & \displaystyle\sum_{|I| \leq \delta} a_{l,I} T^I. \end{array}$$

Next, consider the projection $\pi_1 : \bar{\mathbb{D}}_k^{s(\delta+1)^r} \times \mathbb{A}^{r,\mathrm{an}} \to \bar{\mathbb{D}}^{s(\delta+1)^r}$. The fibre over the point $\alpha \in \bar{\mathbb{D}}^{s(\delta+1)^r}$ is isomorphic to $\mathbb{A}^{r,\mathrm{an}}_{\mathcal{H}(\alpha)}$ (cf. §1.1.1). Recall that the point $\alpha \in \bar{\mathbb{D}}^{s(\delta+1)^r}$ is associated to the character $\chi_\alpha : k\{(a_{1,I})_{|I| \leq \delta}, \ldots, (a_{s,I})_{|I| \leq \delta}\} \to \mathcal{H}(\alpha)$. Set $K := \mathcal{H}(\alpha)$. The inclusion $\iota_K : \mathbb{A}_K^{r,\mathrm{an}} \to \bar{\mathbb{D}}_k^{s(\delta+1)^r} \times \mathbb{A}_k^{r,\mathrm{an}}$ is given by

$$\begin{array}{rcl} k[T_1, \cdots, T_r]\{(a_{1,I})_{|I| \leq \delta}, \ldots, (a_{s,I})_{|I| \leq \delta}\} & \to & K[T_1, \ldots, T_r] \\ T_i & \mapsto & T_i \\ a_{l,I} & \mapsto & \chi_\alpha(a_{l,I}) \ . \end{array}$$

Recall that $\sigma_{K/k} : \mathbb{A}_k^{r,\mathrm{an}} \to \mathbb{A}_K^{r,\mathrm{an}}$ denotes the continuous section of the base change morphism described in §1.7. Finally, for every $z \in \mathbb{A}^{r,\mathrm{an}}$ we set:

$$P_\alpha(z) = \Phi \circ \iota_K \circ \sigma_{K/k}(z) \ . \tag{2.2}$$

The map $P_\alpha : \mathbb{A}^{r,\mathrm{an}} \to \mathbb{A}^{s,\mathrm{an}}$ is clearly continuous. Explicitely, given a polynomial $g = \sum_J g_J T^J \in k[T_1, \cdots, T_s]$ and a point $z \in \mathbb{A}^{r,\mathrm{an}}$, we have

$$|g(P_\alpha(z))| = \left| \left(\sum_{J \in \mathbb{N}^s} g_J \prod_{l=1}^{s} \Big(\sum_{|I| \leq \delta} \chi_\alpha(a_{l,I}) T^I \Big)^{j_l} \right) \sigma_{K/k}(z) \right| . \tag{2.3}$$

To emphasize the fact that $\bar{\mathbb{D}}^{s(\delta+1)^r}$ parametrizes polynomial maps of degree δ, we shall denote it from now on by $\mathrm{Mor}_\delta^{r,s}$. For r, s and $\delta \in \mathbb{N}$ fixed, we have thus constructed a map

$$\begin{aligned} \mathrm{Ev} : \mathrm{Mor}_\delta^{r,s} &\to \mathcal{C}^0(\mathbb{A}^{r,\mathrm{an}}, \mathbb{A}^{s,\mathrm{an}}) \\ \alpha &\mapsto \mathrm{Ev}(\alpha) := P_\alpha . \end{aligned}$$

2.1.2. Proof of Theorem 2.1.1. Consider a sequence of polynomial maps $f_n : \mathbb{A}^{r,\mathrm{an}} \to \mathbb{A}^{s,\mathrm{an}}$ of degree at most $\delta \in \mathbb{N}$ satisfying $f_n(\mathbb{D}^r) \subset \bar{\mathbb{D}}^s$.

For every $n \in \mathbb{N}$, let α_n be the rigid point in the polydisk $\bar{\mathbb{D}}^{s(\delta+1)^r}$ corresponding to the mapping f_n, as constructed above. The polydisk $\bar{\mathbb{D}}^{s(\delta+1)^r}$ is sequentially compact by Theorem 1.1.7, therefore we may find a subsequence $\{\alpha_{n_j}\}_{n_j}$ converging to some point $\alpha \in \bar{\mathbb{D}}^{s(\delta+1)^r}$. Recall that this limit point defines a continuous map $\mathrm{Ev}(\alpha) : \mathbb{A}^{r,\mathrm{an}} \to \mathbb{A}^{s,\mathrm{an}}$.

It remains to verify that $\mathrm{Ev}(\alpha)$ is the pointwise limit of the subsequence $\{f_{n_j}\}$. Observe that this is equivalent to checking that for every $z \in \mathbb{A}^{r,\mathrm{an}}$ and every polynomial $g \in k[T_1, \ldots, T_s]$, the sequence of real numbers $\{|g(f_{n_j}(z))|\}_{n \in \mathbb{N}}$ converges to $|g(\mathrm{Ev}(\alpha)(z))|$.

If z is a non-rigid point in $\mathbb{A}^{r,\mathrm{an}}$, we make a base change by $\mathcal{H}(z)$ and take a rigid point $x \in \mathbb{D}^r_{\mathcal{H}(z)}$ lying over z (see Lemma 1.7.3). The maps f_{n_j} induce analytic maps $\mathbb{A}^{r,\mathrm{an}}_{\mathcal{H}(z)} \to \mathbb{A}^{s,\mathrm{an}}_{\mathcal{H}(z)}$ and g defines an analytic function on $\mathbb{A}^{r,\mathrm{an}}_{\mathcal{H}(z)}$. By definition,

$$|g(f_{n_j}(z))| = |g(f_{n_j}(\pi_{\mathcal{H}(z)/k}(x))| = |g(f_{n_j}(x))| ,$$

so that $|g(f_{n_j}(z))|$ converges if and only if $|g(f_{n_j}(x))|$ converges.

Similarly, $\mathrm{Ev}(\alpha)$ defines a continuous map $\mathbb{A}^{r,\mathrm{an}}_{\mathcal{H}(z)} \to \mathbb{A}^{s,\mathrm{an}}_{\mathcal{H}(z)}$. Indeed, recall from (2.2) that $\mathrm{Ev}(\alpha) = \Phi \circ \iota_{\mathcal{H}(\alpha)} \circ \sigma_{\mathcal{H}(\alpha)/k}$. As Φ is k-analytic, it induces a $\mathcal{H}(z)$-analytic map $\bar{\mathbb{D}}^{s(\delta+1)^r}_{\mathcal{H}(z)} \times \mathbb{A}^{r,\mathrm{an}}_{\mathcal{H}(z)} \to \mathbb{A}^{s,\mathrm{an}}_{\mathcal{H}(z)}$ that we shall also denote by Φ.

Denote by L the complete residue field $\mathcal{H}(\sigma_{\mathcal{H}(z)/k}(\alpha))$, which is a complete extension of $\mathcal{H}(z)$. Moreover, we claim that it is also a complete extension of $\mathcal{H}(\alpha)$. In order to see this, notice that $\ker(\alpha) = k\{a_{l,I}\}_{|I| \leq \delta, 1 \leq l \leq s} \cap \ker(\sigma_{\mathcal{H}(z)/k}(\alpha))$. Thus, we have inclusions

$$k\{a_{l,I}\}_{|I| \leq \delta, 1 \leq l \leq s} / \ker(\alpha) \subset \mathcal{H}(z)\{a_{l,I}\}_{|I| \leq \delta, 1 \leq l \leq s} / \ker(\sigma_{\mathcal{H}(z)/k}(\alpha)) ,$$

and so $\mathcal{H}(\sigma_{\mathcal{H}(z)/k}(\alpha))$ is a complete extension of $\mathcal{H}(\alpha)$.

Consider next the inclusion $\iota_L : \mathbb{A}^{r,\mathrm{an}}_L \to \bar{\mathbb{D}}^{s(\delta+1)^r}_{\mathcal{H}(z)} \times \mathbb{A}^{r,\mathrm{an}}_{\mathcal{H}(z)}$ given by the inclusion of the fibre of the first projection over the point $\sigma_{\mathcal{H}(z)/k}(\alpha)$.

We obtain that $\mathrm{Ev}(\alpha)$ induces the continuous map $\Phi \circ \iota_L \circ \sigma_{L/\mathcal{H}(z)}$. By construction, we see that

$$|g(\mathrm{Ev}(\alpha)(z))| = |g(\mathrm{Ev}(\alpha)(\pi_{\mathcal{H}(z)/k}(x))| = |g(\mathrm{Ev}(\alpha)(x))| \ .$$

We may thus assume that z is rigid. Let $g = \sum_{J\in\mathbb{N}^s} g_J T^J$ be a polynomial of degree d. Denoting $f_{n_j} = (f_1^{(n_j)}, \ldots, f_s^{(n_j)})$, we have:

$$\begin{aligned}\Big|g(f_{n_j}(z))\Big| &= \left|\sum_{|J|\leq d} g_J \prod_{l=1}^{s} \Big(f_l^{(n_j)}(z)\Big)^{j_l}\right| = \\ &= \left|\sum_{|J|\leq d} g_J \prod_{l=1}^{s} \left(\sum_{|I|\leq\delta} a_{l,I}^{(n_j)} z^I\right)^{j_l}\right| = (*) \ .\end{aligned}$$

Taking the polynomial in $s(\delta+1)^r$-variables

$$R := \sum_{|J|\leq d} g_J \prod_{l=1}^{s} \left(\sum_{|I|\leq\delta} S_{l,I} z^I\right)^{j_l} \in k\Big[\{S_{l,I}\}_{1\leq l\leq s, |I|\leq\delta}\Big] \ , \tag{2.4}$$

one sees that $(*) = |R(\alpha_{n_j})|$, and so $|R(\alpha_{n_j})| \to |R(\alpha)|$ as n tends to infinity since $\alpha_{n_j} \to \alpha$. Moreover, it is clear from (2.3) that $R(\alpha) = g(\mathrm{Ev}(\alpha)(z))$, and so the sequence $\Big|g(f_{n_j}(z))\Big|$ converges to $|g(\mathrm{Ev}(\alpha)(z))|$, concluding the proof.

2.1.3. Remarks on the map Ev.

1. The assignment

$$(\alpha, z) \mapsto \mathrm{Ev}(\alpha)(z)$$

does not define a continuous map on $|\mathrm{Mor}_\delta^{r,s}| \times |\mathbb{A}^{r,\mathrm{an}}|$. This phenomenon already appears when $r = s = \delta = 1$.

Indeed, suppose by contradiction that there exists a continuous map $\varphi : |\bar{\mathbb{D}}^2| \times |\mathbb{A}^{1,\mathrm{an}}| \to |\mathbb{A}^{1,\mathrm{an}}|$ such that $\varphi\left((\alpha_0, \alpha_1), z\right) = \alpha_0 + \alpha_1 z$ for any $\alpha_0, \alpha_1, z \in k$ and $|z| \leq 1$. Pick any sequence of points $\zeta_n \in k$ such that $|\zeta_n| = 1$ and $|\zeta_n - \zeta_m| = 1$ for $n \neq m$. Both the sequences $\{\zeta_n\}$ and $\{-\zeta_n\}$ converge to the Gauss point x_g. We compute:

$$\lim_n \varphi\left((\zeta_n, 1), \zeta_n\right) = \lim_n \varphi\left((\zeta_n, 1), -\zeta_n\right) = \varphi\left((x_g, 1), x_g\right) = x_g \ .$$

However, we have that $\varphi\left((\zeta_n, 1), -\zeta_n\right) = 0$ for all n, contradicting the continuity of φ.

2. In general, the map

$$\begin{aligned}\mathrm{Ev} : \mathrm{Mor}_\delta^{r,s} &\to \mathcal{C}^0(\mathbb{A}^{r,\mathrm{an}}, \mathbb{A}^{s,\mathrm{an}}) \\ \alpha &\mapsto \mathrm{Ev}(\alpha)\end{aligned}$$

is not injective. This already occurs in the case $r = s = 1$ for affine maps.

The space $\mathrm{Mor}_1^{1,1}$ is naturally isomorphic to the polydisk $\bar{\mathbb{D}}^2$. Denote by p_0 and p_1 the first and second projections $\mathrm{Mor}_1^{1,1} \to \mathrm{Mor}_0^{1,1}$. Pick two

points $\alpha, \alpha' \in \mathrm{Mor}_1^{1,1}$ such that $p_0(\alpha) = p_0(\alpha') = x_g \in \bar{\mathbb{D}}$. As seen in §1.1.1, the fibre $p_0^{-1}(x_g)$ is naturally homeomorphic to $\bar{\mathbb{D}}_{\mathcal{H}(x_g)}$, and so the points α and α' correspond to points $\alpha_1, \alpha_1' \in \bar{\mathbb{D}}_{\mathcal{H}(x_g)}$ respectively. Write $K = \mathcal{H}(x_g)$ for simplicity, and recall that K is a non-trivial extension of k that contains the field of rational functions in one variable $k(S)$ as a dense subset. Assume that both α_1 and α_1' are the rigid points in $\bar{\mathbb{D}}_K$ given by $\alpha_1 = Q(S) = q_0 + q_1 S + q_2 S^2$ and $\alpha_1' = Q'(S) = q_0 + q_1 S + q_2' S^2$, with $q_2 \neq q_2'$ and $|q_2| = |q_2'|$.

We claim that $\mathrm{Ev}(\alpha) = \mathrm{Ev}(\alpha')$. It suffices to check that they agree on the set of rigid points. Indeed, pick any $z \in \mathbb{A}^{1,\mathrm{an}}(k)$. Following Berkovich's classification of the points in the disk (cf. §1.2.2), the point $\mathrm{Ev}(\alpha)(z)$ corresponds to the closed ball in k centered at zq_0 and of radius $\max\{|1 + q_1 z|, |q_2 z|\}$. Since $|q_2| = |q_2'|$, we conclude that $\mathrm{Ev}(\alpha)(z) = \mathrm{Ev}(\alpha')(z)$.

3. It is a direct consequence of (2.3) that for every fixed $\alpha \in \mathrm{Mor}_\delta^{r,s}$, the map $z \mapsto \mathrm{Ev}(\alpha)(z)$ can be expressed as

$$\mathrm{Ev}(\alpha)(z) = \pi_{K/k} \circ F_\alpha \circ \sigma_{K/k}(z) \ ,$$

where $K = \mathcal{H}(\alpha)$ and $F_\alpha : \mathbb{A}_K^{r,\mathrm{an}} \to \mathbb{A}_K^{s,\mathrm{an}}$ is the polynomial map

$$F_\alpha = (F_1, \cdots, F_s) = \left(\sum_{|I| \leq \delta} \chi_\alpha(a_{1,I}) T^I, \cdots, \sum_{|I| \leq \delta} \chi_\alpha(a_{s,I}) T^I \right) \ .$$

Observe that the coefficients of F_α define a rigid point

$$\beta := \Big((\chi_\alpha(a_{1,I}))_{|I| \leq \delta}, \cdots, (\chi_\alpha(a_{s,I}))_{|I| \leq \delta} \Big) \in \mathrm{Mor}_{\delta,K}^{r,s}$$

and that it satisfies $\pi_{K/k}(\beta) = \alpha$.

2.2. Construction of the space $\mathrm{Mor}(\mathbb{D}^r, Y)$

Our aim is to parametrize in a suitable sense the set of all analytic maps from the polydisk $\mathbb{D}^r$ to a strictly affinoid space Y. We shall realize this space as the analytic spectrum of a k-Banach algebra $\mathcal{T}_\infty^{r,Y}$.

The first step is to construct the Banach k-algebra $\mathcal{T}_\infty^{r,Y}$. In the case where Y is a unit polydisk, this algebra is a sort of Tate algebra in countably many variables. In §2.2.3, we show that its analytic spectrum satisfies an appropriate universal property and specify in which sense it parametrizes the set of analytic maps $\mathbb{D}^r \to Y$.

2.2.1. Construction of the Banach k-algebra $\mathcal{T}_\infty^{r,s}$. We shall first concentrate on the case where $Y = \bar{\mathbb{D}}^s$.

Pick some integer $\delta \in \mathbb{N}^*$. Recall from §2.1 that the set of all polynomial maps $P : \mathbb{A}^{r,\mathrm{an}} \to \mathbb{A}^{s,\mathrm{an}}$ of degree at most δ such that $P(\mathbb{D}^r) \subset \bar{\mathbb{D}}^s$ can be endowed with a natural structure of affinoid space whose affinoid algebra is the Tate algebra $k\{a_{1,I}, \cdots, a_{s,I}\}_{|I| \leq \delta} = k\{a_{l,I}\}_{|I| \leq \delta, 1 \leq l \leq s}$. We

denote this space by $\mathrm{Mor}_\delta^{r,s}$. It is isomorphic as a k-analytic space to the unit polydisk $\bar{\mathbb{D}}^{s(\delta+1)^r}$.

Observe that for any given $\delta \in \mathbb{N}^*$ there exists a natural truncation map $\mathrm{pr}_\delta : \mathrm{Mor}_{\delta+1}^{r,s} \to \mathrm{Mor}_\delta^{r,s}$, which is a surjective analytic map dual to the inclusion of Tate algebras $k\{a_{l,I}\}_{|I|\leq\delta,1\leq l\leq s} \subset k\{a_{l,I}\}_{|I|\leq\delta+1,1\leq l\leq s}$. These inclusions are isometric and we may so consider the inductive limit of this directed system. It is a normed k-algebra that we denote by $\mathcal{T}^{r,s}$.

In order to describe the elements of $\mathcal{T}^{r,s}$ and its norm, we introduce the set $\mathcal{S}$ of all maps $\mathrm{M} : \{1,\dots,s\} \times \mathbb{N}^r \to \mathbb{N}$ having finite support and set $|\mathrm{M}| = \sum_{l,I} \mathrm{M}(l,I)$ for every $\mathrm{M} \in \mathcal{S}$. We define $\mathcal{S}_\delta$ as the subset of $\mathcal{S}$ consisting of all $\mathrm{M} \in \mathcal{S}$ such that $\mathrm{M}(l,I) = 0$ for all $|I| \geq \delta+1$. Observe that no such set $\mathcal{S}_\delta$ is finite. Given $a = \Big((a_{1,I})_{|I|\leq\delta}, \dots, (a_{s,I})_{|I|\leq\delta}\Big)$ and $\mathrm{M} \in \mathcal{S}$, we write

$$a^{\mathrm{M}} = \prod_{1\leq l\leq s, I\in\mathbb{N}^r} a_{l,I}^{\mathrm{M}(l,I)}.$$

The k-algebra $\mathcal{T}^{r,s}$ consists of all power series that are of the form

$$\sum_{\mathrm{M}\in\mathcal{S}_\delta} g_{\mathrm{M}} \cdot a^{\mathrm{M}},$$

for some $\delta \in \mathbb{N}$ and whose coefficients $g_{\mathrm{M}} \in k$ are such that $|g_{\mathrm{M}}| \to 0$ as $|\mathrm{M}| \to \infty$.

Let us describe the norm on $\mathcal{T}^{r,s}$. Observe that by the definition of $\mathcal{S}_\delta$, every element $\sum_{\mathrm{M}\in\mathcal{S}_\delta} g_{\mathrm{M}} \cdot a^{\mathrm{M}} \in \mathcal{T}^{r,s}$ belongs to the Tate algebra $k\{a_{l,I}\}_{|I|\leq\delta,1\leq l\leq s}$, and we may associate to $\sum_{\mathrm{M}\in\mathcal{S}_\delta} g_{\mathrm{M}} \cdot a^{\mathrm{M}}$ the norm on $k\{a_{l,I}\}_{|I|\leq\delta,1\leq l\leq s}$. Since the inclusions of $k\{a_{l,I}\}_{|I|\leq\delta,1\leq l\leq s}$ in $k\{a_{l,I}\}_{|I|\leq\delta+1,1\leq l\leq s}$ are isometric, this norm is well-defined.

Remark 2.2.1. *The k-algebra $\mathcal{T}^{r,s}$ is not complete. Take for instance $r = s = 1$ and consider the sequence $f_n = \sum_{i=1}^n g_i \cdot a_i \in \mathcal{T}^{1,1}$. This is a Cauchy sequence as soon as the coefficients $g_i \in k$ are such that $|g_i| \to 0$ when $i \to \infty$, but it does not have any limit in $\mathcal{T}^{1,1}$.*

The completion $\mathcal{T}_\infty^{r,s}$ of $\mathcal{T}^{r,s}$ is the Banach k-algebra consisting of all power series

$$\sum_{\mathrm{M}\in\mathcal{S}} g_{\mathrm{M}} \cdot a^{\mathrm{M}}$$

such that $|g_{\mathrm{M}}|$ tends to zero with respect to the filter of cofinite subsets, i.e. such that for all $\epsilon > 0$ the set of $\mathrm{M} \in \mathcal{S}$ such that $|g_{\mathrm{M}}| > \epsilon$ is finite.

Definition 2.2.2. *The space $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ is the analytic spectrum of the Banach algebra $\mathcal{T}_\infty^{r,s}$.*

In particular, $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ is compact, because it is the analytic spectrum of the k-Banach algebra $\mathcal{T}_\infty^{r,s}$.

For every $\delta \in \mathbb{N}$, the isometric inclusion $k\{a_{l,I}\}_{|I|\leq\delta,1\leq l\leq s} \subset \mathcal{T}_\infty^{r,s}$ defines a natural surjective continuous map $\mathrm{Pr}_\delta^\infty : \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s) \to \mathrm{Mor}_\delta^{r,s}$.

We may as well consider the inverse limit of all the spaces $\mathrm{Mor}_\delta^{r,s}$, induced by the truncation maps $\mathrm{pr}_\delta : \mathrm{Mor}_{\delta+1}^{r,s} \to \mathrm{Mor}_\delta^{r,s}$. These maps verify the equality $\mathrm{pr}_\delta \circ \mathrm{Pr}_{\delta+1}^\infty = \mathrm{Pr}_\delta^\infty$ and induce a continuous map $\varphi : \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s) \to \varprojlim_\delta \mathrm{Mor}_\delta^{r,s}$.

We shall consider the inclusions $i_\delta : \mathrm{Mor}_\delta^{r,s} \to \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ given by the bounded morphism $\mathcal{T}_\infty^{r,s} \to k\{a_{l,I}\}_{|I|\leq\delta, 1\leq l\leq s}$, sending $a_{l,I}$ to itself if $|I| \leq \delta$ and to 0 otherwise. These are closed immersions.

PROPOSITION 2.2.3. *The map $\varphi : \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s) \to \varprojlim_\delta \mathrm{Mor}_\delta^{r,s}$ is a homeomorphism.*

PROOF. The inverse limit $\varprojlim_\delta \mathrm{Mor}_\delta^{r,s}$ is compact by Tychonoff.

Let us show that $\varphi : \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s) \to \varprojlim_\delta \mathrm{Mor}_\delta^{r,s}$ is bijective.

Fix $\delta \geq 0$. Let $\pi_\delta : \varprojlim_\delta \mathrm{Mor}_\delta^{r,s} \to \mathrm{Mor}_\delta$ be the natural map and $\mathrm{pr}_\delta : \mathrm{Mor}_{\delta+1}^{r,s} \to \mathrm{Mor}_\delta^{r,s}$ the truncation map. We know that $\mathrm{Pr}_\delta^\infty = \pi_\delta \circ \varphi : \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s) \to \mathrm{Mor}_\delta^{r,s}$. Pick a point $y \in \varprojlim_\delta \mathrm{Mor}_\delta^{r,s}$ and consider $\pi_\delta(y) \in \mathrm{Mor}_\delta^{r,s}$. Consider the set K_δ consisting of all the points $\alpha \in \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ such that $\mathrm{Pr}_\delta^\infty(\alpha) = \pi_\delta(y)$. The closed immersion $i_\delta : \mathrm{Mor}_\delta^{r,s} \to \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ constructed above is a section of $\mathrm{Pr}_\delta^\infty$. Thus, the map $\mathrm{Pr}_\delta^\infty$ is surjective and the subset K_δ is non-empty. Clearly, we have that $K_{\delta+1} \subseteq K_\delta$. Every K_δ is compact and so the intersection $\cap_{\delta\geq 0} K_\delta$ is nonempty. This shows that φ is surjective.

For the injectivity, let α, α' be two points in $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ having the same image in $\varprojlim_\delta \mathrm{Mor}_\delta^{r,s}$. We have to check that $|g(\alpha)| = |g(\alpha')|$ for every $g \in \mathcal{T}_\infty^{r,s}$, that by density reduces to the case where $g \in \mathcal{T}^{r,s}$. We know that $\mathrm{Pr}_\delta^\infty(\alpha) = \mathrm{Pr}_\delta^\infty(\alpha') \in \mathrm{Mor}_\delta^{r,s}$ for all δ. Given $g \in \mathcal{T}^{r,s}$ observe that it lies in $k\{a_{l,I}\}_{|I|\leq\delta, 1\leq l\leq s}$ for some $\delta \geq 0$. Thus,

$$|g(\alpha)| = |g(\mathrm{Pr}_\delta^\infty(\alpha))| = |g(\mathrm{Pr}_\delta^\infty(\alpha'))| = |g(\alpha')| \ ,$$

concluding the proof. □

Recall from §1 the definition of the complete residue field $\mathcal{H}(\alpha)$ of a point $\alpha \in \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$. We say that α is rigid when $\mathcal{H}(\alpha) = k$. To simplify notation, we write $\alpha_\delta = \mathrm{Pr}_\delta^\infty(\alpha)$.

PROPOSITION 2.2.4. *Let α be a point in $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$. For every $\delta \in \mathbb{N}$, the inclusion of Banach k-algebras $k\{a_{l,I}\}_{1\leq l\leq s, |I|\leq\delta} \subset \mathcal{T}_\infty^{r,s}$ induces an extension of valued fields $\mathcal{H}(\alpha)/\mathcal{H}(\alpha_\delta)$.*

The complete residue field $\mathcal{H}(\alpha)$ is isomorphic to the completion of the inductive limit of valued fields $\varinjlim_\delta \mathcal{H}(\alpha_\delta)$.

PROOF. A point $\alpha \in \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ corresponds to a seminorm on the k-algebra $\mathcal{T}_\infty^{r,s}$, whose restriction to $k\{a_{l,I}\}_{|I|\leq\delta, 1\leq l\leq s}$ is the seminorm α_δ. The kernel of α_δ is the intersection of $k\{a_{l,I}\}_{|I|\leq\delta, 1\leq l\leq s}$ with $\ker(\alpha)$. This induces inclusions

$$k\{a_{l,I}\}_{|I|\leq\delta, 1\leq l\leq s}/\ker(\alpha_\delta) \subset \mathcal{T}_\infty^{r,s}/\ker(\alpha). \tag{2.5}$$

It follows that there are inclusions $\mathcal{H}(\alpha_\delta) \subset \mathcal{H}(\alpha)$, and thus the direct limit of the $\mathcal{H}(\alpha_\delta)$ is naturally contained in $\mathcal{H}(\alpha)$. In order to show that $\mathcal{H}(\alpha)$ is isometrically isomorphic to the completion of $\varinjlim_\delta \mathcal{H}(\alpha_\delta)$, it suffices to show that $\varinjlim_\delta \mathcal{H}(\alpha_\delta)$ is dense in $\mathcal{H}(\alpha)$.

Consider the field $K := \varinjlim_\delta \mathrm{Frac}\left(k\{a_{l,I}\}_{|I|\leq\delta, 1\leq l\leq s}/\ker(\alpha_\delta)\right)$. It is clear that K is contained in $\varinjlim_\delta \mathcal{H}(\alpha_\delta)$. By (2.5) and by the definition of $\mathcal{T}^{r,s}_\infty$, we also know that K is dense in $\mathrm{Frac}\,(\mathcal{T}^{r,s}_\infty/\ker(\alpha))$. The latter is by definition dense in $\mathcal{H}(\alpha)$, which proves that $\varinjlim_\delta \mathcal{H}(\alpha_\delta)$ is dense in $\mathcal{H}(\alpha)$. □

Proposition 2.2.5. *The set of rigid points in* $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ *is dense.*

Proof. Pick any point $\alpha \in \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$. For every $\delta \in \mathbb{N}$, pick a sequence of rigid points $\alpha_n^{(\delta)} \in \mathrm{Mor}^{r,s}_\delta$ converging to α_δ. By Proposition 2.2.3 and Proposition 2.2.4, a point in $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ is rigid if and only if for every $\delta \in \mathbb{N}$ its projection to $\mathrm{Mor}^{r,s}_\delta$ is rigid.

We may view each point $\alpha_n^{(\delta)}$ as a rigid point in $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ via de map $i_\delta : \mathrm{Mor}^{r,s}_\delta \to \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ constructed above. We claim that α lies in the closure of the set $\{\alpha_n^{(\delta)}\}_{n,\delta}$. Indeed, fix an open neighbourhood U of α. It is a finite intersection of open sets of the form $\{\beta \in \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s) : |g(\beta)| - |g(\alpha)|| \leq r\}$ for some $r \leq 1$. Since $\mathcal{T}^{r,s}$ is dense in $\mathcal{T}^{r,s}_\infty$, we may assume that $g \in \mathcal{T}^{r,s}$. Thus, for sufficiently large δ one has that $|g(\alpha_\delta)| = |g(\alpha)|$. Moreover, by construction we have $|g(\alpha_n^{(\delta)})| \overset{n\to\infty}{\to} |g(\alpha_\delta)|$. It follows that for $\delta, n \gg 0$, the points $\alpha_n^{(\delta)}$ belong to U. □

2.2.2. Construction of the Banach k-algebra $\mathcal{T}^{r,Y}_\infty$. We now generalize the construction of the infinite dimensional Tate algebra $\mathcal{T}^{r,s}_\infty$ to the case where the target Y is any strictly k-affinoid space. We obtain a Banach k-algebra $\mathcal{T}^{r,Y}_\infty$ that is homeomorphic to a quotient of $\mathcal{T}^{r,s}_\infty$ by some closed ideal.

Let Y be a strictly k-affinoid space with underlying algebra $\mathcal{B}$. Fix an admissible epimorphism $\mathcal{T}_s \to \mathcal{B}$ giving rise to a closed immersion $Y \to \bar{\mathbb{D}}^s$ and such that $\mathcal{B}$ is isomorphic as an affinoid algebra to $\mathcal{T}_s/\mathcal{J}$ for some closed ideal $\mathcal{J}$. Recall that an analytic map $f : \mathbb{D}^r \to \bar{\mathbb{D}}^s$ is given in coordinates by

$$f = \left(\sum_{I\in\mathbb{N}^r} a_{1,I}T^I, \ldots, \sum_{I\in\mathbb{N}^r} a_{s,I}T^I\right), \quad |a_{l,I}| \leq 1\ ,$$

and that an analytic map $\mathbb{D}^r \to Y$ is an analytic map $f : \mathbb{D}^r \to \bar{\mathbb{D}}^s$ such that $\varphi \circ f \equiv 0$ for all $\varphi \in \mathcal{J}$. Explicitly, for every $\varphi = \sum_{J\in\mathbb{N}^s} \varphi_J T^J$ one

has:

$$\begin{aligned}\varphi \circ f &= \sum_{J\in\mathbb{N}^s} \varphi_J \left(\sum_{I\in\mathbb{N}^r} a_{1,I}T^I, \ldots, \sum_{I\in\mathbb{N}^r} a_{s,I}T^I \right)^J \\ &= \sum_{J\in\mathbb{N}^s} \varphi_J \prod_{1\leq l\leq s} \left(\sum_{I\in\mathbb{N}^r} a_{l,I}T^I \right)^{j_l} .\end{aligned}$$

Further developing the formulae, one obtains:

$$\begin{aligned}\varphi \circ f &= \sum_{J\in\mathbb{N}^s} \varphi_J \prod_{1\leq l\leq s} \left(a_{l,(0,\ldots,0)} \right)^{j_l} + \cdots \\ &= \sum_{K\in\mathbb{N}^r} B_{\varphi,K}(a_{l,I})T^K ,\end{aligned}$$

where $B_{\varphi,K}(a_{l,I})$ is a series in the $a_{l,I}$'s whose coefficients are determined by those of φ. In fact, in the expression of $B_{\varphi,K}(a_{l,I})$ intervene only the $a_{l,I}$ such that the multiindex $I = (i_1, \ldots, i_r)$ satisfies $i_n \leq k_n$ for all $n = 1, \ldots, r$ and $B_{\varphi,K}(a_{l,I})$ is linear in φ. That is, only finitely many coefficients $a_{l,I}$ appear in the expression of $B_{\varphi,K}(a_{l,I})$, and so $B_{\varphi,K}(a_{l,I}) \in \mathcal{T}_\infty^{r,s}$. Nevertheless, notice that $B_{\varphi,K}(a_{l,I})$ is not a polynomial in general.

Denote by $\mathcal{I} := \overline{(B_{\varphi,K}(a_{l,I}))_K}$ the closure of the ideal in $\mathcal{T}_\infty^{r,s}$ generated by all the series $B_{\varphi,K}(a_{l,I})$, for $\varphi \in \mathcal{J}$ and $K \in \mathbb{N}^r$. The quotient $\mathcal{T}_\infty^{r,Y} := \mathcal{T}_\infty^{r,s}/\mathcal{I}$ endowed with the quotient norm is a Banach algebra, and we set:

$$\mathrm{Mor}(\mathbb{D}^r, Y) = \mathcal{M}\left(\mathcal{T}_\infty^{r,Y}\right) .$$

2.2.3. Universal property of the space $\mathrm{Mor}(\mathbb{D}^r, Y)$**.** We now prove that the space $\mathrm{Mor}(\mathbb{D}^r, Y)$ satisfies the following natural universal property.

THEOREM B. *There exists an analytic map* $\Phi : \mathrm{Mor}(\mathbb{D}^r, Y) \times \mathbb{D}^r \to Y$ *satisfying the following universal property. Let* W *be the analytic spectrum of a Banach* k*-algebra or any good* k*-analytic space. Then, for any analytic map* $F : W \times \mathbb{D}^r \to Y$ *there exists a unique morphism* $G : W \to \mathrm{Mor}(\mathbb{D}^r, Y)$ *such that* $F(x, z) = \Phi(G(x), z)$ *for all* $x \in W(k)$ *and* $z \in \mathbb{D}^r(k)$*.*

REMARK 2.2.6. *As a consequence of the previous theorem, the space* $\mathcal{M}\left(\mathcal{T}_\infty^{r,Y}\right)$ *does not depend on the presentation of* Y*.*

Recall from §1.1.1 that a morphism between the spectra of two Banach k-algebras is by definition a continuous map induced by a bounded morphism between the underlying algebras. In the same fashion, an analytic map from a good k-analytic space W into $\mathrm{Mor}(\mathbb{D}^r, Y)$ is given by an affinoid covering $\{W_i\}$ of W and analytic maps $W_i \to \mathrm{Mor}(\mathbb{D}^r, Y)$, which are induced by bounded morphisms of Banach k-algebras and are compatible with the restrictions.

Being the analytic spectrum of a k-Banach algebra, there is a natural notion of complete residue field of any point in $\mathrm{Mor}(\mathbb{D}^r, Y)$. Recall that a point $\alpha \in \mathrm{Mor}(\mathbb{D}^r, Y)$ is rigid if and only if its complete residue field $\mathcal{H}(\alpha)$ is equal to k. When α is rigid, $\Phi(\alpha, \cdot)$ defines an analytic map from $\mathbb{D}^r$ to Y. The previous theorem shows in particular that the set of analytic maps from $\mathbb{D}^r$ to Y is in bijection with the set $\{\Phi(\alpha, \cdot) : \alpha \in \mathrm{Mor}(\mathbb{D}^r, Y)(k)\}$, hence with the set of rigid points in $\mathrm{Mor}(\mathbb{D}^r, Y)$.

Proof of Theorem B. Let Y be a strictly k-affinoid space with underlying affinoid algebra $\mathcal{B}$. Fix an admissible epimorphism $\mathcal{T}_s \to \mathcal{B}$ such that $\mathcal{B}$ is isomorphic as an affinoid algebra to $\mathcal{T}_s/\mathcal{J}$ for some ideal $\mathcal{J}$.

As a first step, let us construct the analytic map $\Phi : \mathrm{Mor}(\mathbb{D}^r, Y) \times \mathbb{D}^r \to Y$. The assignment

$$(S_1, \ldots, S_s) \mapsto \left(\sum_{I \in \mathbb{N}^r} a_{1,I} T^I, \ldots, \sum_{I \in \mathbb{N}^r} a_{s,I} T^I \right)$$

defines a bounded morphism of Banach k-algebras $\psi : k\{S_1, \ldots, S_s\} \to \mathcal{T}_\infty^{r,s}\{\rho^{-1}T_1, \ldots, \rho^{-1}T_r\}$ for every positive $\rho < 1$, and thus an analytic map $\Psi : \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s) \times \mathbb{D}^r \to \bar{\mathbb{D}}^s$.

Pick any $\varphi \in \mathcal{J}$. We see that $\varphi \circ \Psi = \sum_{K \in \mathbb{N}^r} B_{\varphi,K}(a_{l,I}) T^K$, and so it follows from the definition of the Banach algebra $\mathcal{T}_\infty^{r,Y}$ that Ψ induces an analytic map $\Phi : \mathrm{Mor}(\mathbb{D}^r, Y) \times \mathbb{D}^r \to Y$.

We now prove the universal property. Suppose first that W is the aalytic spectrum of a Banach k-algebra $\mathcal{A}$. Let $F : W \times \mathbb{D}_k^r \to Y$ be an analytic map, which is given by an analytic map $f : W \times \mathbb{D}^r \to \bar{\mathbb{D}}^s$ induced by some bounded homomorphism of Banach k-algebras

$$(S_1, \ldots, S_s) \mapsto \left(\sum_{I \in \mathbb{N}^r} b_{1,I} T^I, \ldots, \sum_{I \in \mathbb{N}^r} b_{s,I} T^I \right) ,$$

where $b_{l,I} \in \mathcal{A}$ are such that $\sup_{l,I} |b_{l,I}(x)| \leq 1$ for all $x \in W$ and such that for every element $\varphi = \sum_{J \in \mathbb{N}^s} \varphi_J T^J$ in the ideal $\mathcal{J} \subset \mathcal{T}_s$ we have $\varphi \circ f \equiv 0$. In the notation of the preceding section,

$$\begin{aligned} 0 \equiv \varphi \circ f &= \sum_{J \in \mathbb{N}^s} \varphi_J \left(\sum_{I \in \mathbb{N}^r} b_{1,I} T^I, \ldots, \sum_{I \in \mathbb{N}^r} b_{s,I} T^I \right)^J \\ &= \sum_{K \in \mathbb{N}^r} B_{\varphi,K}(b_{l,I}) T^K \ . \end{aligned} \tag{2.6}$$

Consider the analytic map $g : W \to \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ given by $a_{l,I} \mapsto b_{l,I}$ for all $I \in \mathbb{N}^r$ and all $1 \leq l \leq s$. By (2.6), the composition $B_{\varphi,K}(a_{l,I}) \circ g \equiv 0$ for every $K \in \mathbb{N}^r$ and every $\varphi \in \mathcal{J}$. Thus, the map g induces an analytic map $G : W \to \mathrm{Mor}(\mathbb{D}^r, Y)$.

A rigid point $x \in W$ together with a rigid point $z \in \mathbb{D}^r$ defines a rigid point in the product $W \times \mathbb{D}^r$, and by construction we have $F(x,z) = \Phi(G(x),z)$.

Conversely, let $H : W \to \mathrm{Mor}(\mathbb{D}^r, Y)$ be an analytic map induced by the map $h : W \to \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ sending $a_{l,I}$ to some $c_{l,I} \in \mathcal{A}$ and satisfying $F(x,z) = \Phi(h(x),z)$ for all $x \in W(k)$ and all $z \in \mathbb{D}^r(k)$. For every fixed $x \in W(k)$, consider the analytic map $z \in \mathbb{D}^r \mapsto \Phi(h(x),z)$. By hypothesis, it agrees with the map $z \in \mathbb{D}^r \mapsto \Phi(g(x),z)$, and so $b_{l,I}(x) = c_{l,I}(x)$ for every $I \in \mathbb{N}^r$ and $1 \leq l \leq s$. As the equalities hold for every rigid $x \in W$, we conclude that $h = g$, and so $H = G$.

Let now W be an arbitrary good k-analytic space. Let $\{W_i\}$ be an affinoid covering of W inducing an analytic map $F : W \times \mathbb{D}^r \to Y$. By the previous case, for every affinoid domain W_i of W there exists a unique analytic map $G_i : W_i \to \mathrm{Mor}(\mathbb{D}^r, Y)$, induced by a bounded morphism of Banach algebras, such that $F(x,z) = \Phi(G_i(x),z)$ for all $x \in W_i(k)$ and $z \in \mathbb{D}^r(k)$. By construction, the maps G_i agree on the intersections $W_i \cap W_j$ and are compatible with the restrictions. □

2.3. Points of $\mathrm{Mor}(\mathbb{D}^r, Y)$ as continuous maps $\mathbb{D}^r \to Y$

The following theorem specifies in which sense the points of the space $\mathrm{Mor}(\mathbb{D}^r, Y)$ correspond to continuous maps from $\mathbb{D}^r$ to Y. As before, k denotes a complete algebraically closed field.

Recall that $\Phi : \mathrm{Mor}(\mathbb{D}^r, Y) \times \mathbb{D}^r \to Y$ denotes the analytic map from Theorem B.

THEOREM 2.3.1. *Let Y be a strictly k-affinoid space. There exists a map* Ev *from* $\mathrm{Mor}(\mathbb{D}^r, Y)$ *to the space of continuous functions* $\mathcal{C}^0(\mathbb{D}^r, Y)$ *such that the following holds:*

i) *The map* $\mathrm{Ev}(\alpha)$ *is analytic if and only if* α *is a rigid point in* $\mathrm{Mor}(\mathbb{D}^r, Y)$. *In that case, the map* $\mathrm{Ev}(\alpha)$ *is precisely* $\Phi(\alpha, \cdot)$.
ii) *For any fixed* $z \in \mathbb{D}^r$, *the assignment* $\alpha \in \mathrm{Mor}(\mathbb{D}^r, Y) \mapsto \mathrm{Ev}(\alpha)(z)$ *is a continuous map.*

PROOF. Let $\mathcal{B}$ be the underlying affinoid algebra of Y and fix an admissible surjection $\mathcal{T}_s \to \mathcal{B}$ with kernel $\mathcal{J} \subset \mathcal{T}_s$.

The map $\mathrm{Ev} : \mathrm{Mor}(\mathbb{D}^r, Y) \to \mathcal{C}^0(\mathbb{D}^r, Y)$ is given as follows. Fix a point $\alpha \in \mathrm{Mor}(\mathbb{D}^r, Y)$ and consider the first projection $\pi_1 : \mathrm{Mor}(\mathbb{D}^r, Y) \times \mathbb{D}^r \to \mathrm{Mor}(\mathbb{D}^r, Y)$. The fibre $\pi_1^{-1}(\alpha)$ is canonically isomorphic to $\mathbb{D}^r_{\mathcal{H}(\alpha)}$ (cf. §1.1.1). We can thus consider the inclusion map $\iota_{\mathcal{H}(\alpha)} : \mathbb{D}^r_{\mathcal{H}(\alpha)} \to \mathrm{Mor}(\mathbb{D}^r, Y) \times \mathbb{D}^r$, given by

$$\begin{array}{rcl} \mathcal{T}_\infty^{r,Y}\{\rho^{-1}T_1, \ldots, \rho^{-1}T_r\} & \to & \mathcal{H}(\alpha)\{\rho^{-1}T_1, \ldots, \rho^{-1}T_r\} \\ T_i & \mapsto & T_i \\ a \in \mathcal{T}_\infty^{r,Y} & \mapsto & \chi_\alpha(a) \end{array} \tag{2.7}$$

for $\rho < 1$, where $\chi_\alpha : \mathcal{T}_\infty^{r,Y} \to \mathcal{H}(\alpha)$ denotes the character associated to the point α. Let $\sigma_{\mathcal{H}(\alpha)/k} : \mathbb{D}^r \to \mathbb{D}^r_{\mathcal{H}(\alpha)}$ be the continuous map discussed in §1.7. Let $\Phi : \mathrm{Mor}(\mathbb{D}^r, Y) \times \mathbb{D}^r \to Y$ be the analytic map from Theorem B. We set:

$$\mathrm{Ev}(\alpha) = \Phi \circ \iota_{\mathcal{H}(\alpha)} \circ \sigma_{\mathcal{H}(\alpha)/k} \ .$$

Clearly, $\mathrm{Ev}(\alpha)$ is a continuous map from $\mathbb{D}^r$ to Y.

It will be useful in the remaining of the proof to compute $\mathrm{Ev}(\alpha)(z)$ explicitely for any point $z \in \mathbb{D}^r$. The map $\mathrm{Ev}(\alpha)$ is given by a continuous map $f : \mathbb{D}^r \to \bar{\mathbb{D}}^s$ such that $\varphi \circ f \equiv 0$ for every element φ in the ideal $\mathcal{J} \subset \mathcal{T}_s$ defining the affinoid space Y. For any $z \in \mathbb{D}^r$ and for any $g = \sum_{J \in \mathbb{N}^s} g_J S^J$ in $k\{S_1, \ldots, S_s\}$, we have

$$|g(\mathrm{Ev}(\alpha)(z))| = \left| \sum_J g_J \prod_{l=1}^{s} \Big(\sum_I \chi_\alpha(a_{l,I}) \cdot T^I \Big)^{j_l} (\sigma_{\mathcal{H}(\alpha)/k}(z)) \right| . \tag{2.8}$$

Pick a rigid point $\alpha \in \mathrm{Mor}(\mathbb{D}^r, Y)$, i.e. such that $\mathcal{H}(\alpha) = k$. In this situation, the fibre $\pi_1^{-1}(\alpha)$ is homeomorphic to $\mathbb{D}^r$, and so $\iota_{\mathcal{H}(\alpha)}$ is in fact an analytic map between k-analytic spaces, and the map $\sigma_{\mathcal{H}(\alpha)/k}$ is the identity on $\mathbb{D}^r$. Then, for every $z \in \mathbb{D}^r$ the pair (α, z) defines a point in $\mathrm{Mor}(\mathbb{D}^r, Y) \times \mathbb{D}^r$, and so $\iota_k(z) = (\alpha, z)$. Thus, $\mathrm{Ev}(\alpha) = \Phi(\alpha, \cdot)$ is analytic.

Suppose conversely that $\mathrm{Ev}(\alpha)$ is analytic. It follows from (2.8) that the map $\mathrm{Ev}(\alpha)$ can be decomposed as $\mathrm{Ev}(\alpha) = \pi_{K/k} \circ F \circ \sigma_{K/k}$, where $F : \mathbb{D}^r \to \bar{\mathbb{D}}^s$ is the K-analytic map

$$F(z) = (\sum_I \chi_\alpha(a_{1,I}) \cdot z^I, \ldots, \sum_I \chi_\alpha(a_{s,I}) \cdot z^I).$$

It suffices to treat the case $s = 1$. Since $\mathrm{Ev}(\alpha)$ is analytic, we may find coefficients $b_I \in k$ bounded by 1 such that $\mathrm{Ev}(\alpha)(z) = \sum_{I \in \mathbb{N}^r} b_I z^I$ for every $z \in \mathbb{D}^r(k)$. Notice that the equality $\pi_{K/k}\left(\sum_{I \in \mathbb{N}^r} \chi_\alpha(a_I) \cdot z^I\right) = \sum_{I \in \mathbb{N}^r} b_I z^I \in k$ implies that $\sum_{I \in \mathbb{N}^r} \chi_\alpha(a_I) \cdot z^I \in k$, as k is algebraically closed.

Suppose by contradiction that α is not a rigid point and consider the equation

$$\sum_{I \in \mathbb{N}^r} b_I z^I = \sum_{I \in \mathbb{N}^r} \chi_\alpha(a_I) \cdot z^I \ , \tag{2.9}$$

where we may assume that every $\chi_\alpha(a_I)$ is either 0 or does not belong to k. Since α is not rigid, not all of them are zero. We may consider the nonempty set $M \subseteq \mathbb{N}^r$ consisting of all the multi-indices $I \in \mathbb{N}^r$ such that $\chi_\alpha(a_I) \notin k$. Let P be the Newton polytope of M, i.e. the convex hull of the union of all upper-quadrants $I + \mathbb{R}^r_+$ with $I \in M$. It is a non-compact polytope in $\mathbb{R}^r_+$ whose extremal points all belong to M.

Pick any extremal point p of P, and take any hyperplane in $\mathbb{R}^r$ with integer coefficients $H = \{\beta_1 x_1 + \ldots + \beta_r x_r = \beta_0\}$ intersecting the

polytope P exactly at the point p. In other words, we have

$$i_1\beta_1 + \ldots + i_r\beta_r > \beta_0 \tag{2.10}$$

for every $I \in \mathbb{N}^r$ distinct from p intervening in (2.9). Fix any $\lambda \in k$ with $|\lambda| < 1$ and consider the rigid point $z = (\lambda^{\beta_1}, \ldots, \lambda^{\beta_r}) \in \mathbb{D}^r$. Then,

$$\begin{aligned}\sum_{I\in\mathbb{N}^r} b_I z^I &= \sum_{I\in\mathbb{N}^r} b_I(\lambda^{\beta_1},\ldots,\lambda^{\beta_r})^I = \sum_{I\in\mathbb{N}^r} b_I\lambda^{i_1\beta_1+\ldots+i_r\beta_r} = \\ &= b_p\lambda^{\beta_0} + \sum_{I\in\mathbb{N}^r, I\neq p} b_I\lambda^{i_1\beta_1+\ldots+i_r\beta_r} = b_p\lambda^{\beta_0} + O(\lambda^{\beta_0}) \ ,\end{aligned}$$

where the last equality follows from (2.10). It follows that $b_p\lambda^{\beta_0} + O(\lambda^{\beta_0}) = \chi_\alpha(a_p)\lambda^{\beta_0} + O(\lambda^{\beta_0})$, and hence $\chi_\alpha(a_p) = b_p \in k$. Repeating this procedure at every vertex of the polytope P, we conclude that $\chi_\alpha(a_I) \in k$ for every $I \in \mathrm{P}$, contradicting the fact that α is not rigid.

Let us now prove the continuity statement. Fix a point $z \in \mathbb{D}^r$. It suffices to check that for any sequence of points $\{\alpha_n\} \subset \mathrm{Mor}(\mathbb{D}^r, Y)$ converging to some $\alpha \in \mathrm{Mor}(\mathbb{D}^r, Y)$, we have $\mathrm{Ev}(\alpha_n)(z) \to \mathrm{Ev}(\alpha)(z)$.

Consider the second projection $\pi_2 : \mathrm{Mor}(\mathbb{D}^r, Y) \times \mathbb{D}^r \to \mathbb{D}^r$. The fibre $\pi_2^{-1}(z)$ is isomorphic to the space $\mathrm{Mor}(\mathbb{D}^r, Y)_{\mathcal{H}(z)}$. The inclusion map $\iota_{\mathcal{H}(z)} : \mathrm{Mor}(\mathbb{D}^r, Y)_{\mathcal{H}(z)} \to \mathrm{Mor}(\mathbb{D}^r, Y) \times \mathbb{D}^r$ is given by

$$\begin{aligned}\mathcal{T}_\infty^{r,Y}\{\rho^{-1}T_1,\ldots,\rho^{-1}T_r\} &\to \mathcal{T}_\infty^{r,Y}\hat{\otimes}_k\mathcal{H}(z) \\ T_i &\mapsto \chi_z(T_i) \\ a_{l,I} &\mapsto a_{l,I}\end{aligned}$$

for any $\rho < 1$, where $\chi_z : k\{\rho^{-1}T_1,\ldots,\rho^{-1}T_r\} \to \mathcal{H}(z)$ denotes the character associated to the point z. Pick some converging power series $g = \sum_{I\in\mathbb{N}^r} g_I T^I$ in $\mathcal{T}_\infty^{r,Y}\{\rho^{-1}T_1,\ldots,\rho^{-1}T_r\}$ and compute:

$$\begin{aligned}\left|g\left(\iota_{\mathcal{H}(z)}\circ\sigma_{\mathcal{H}(z)/k}(\alpha)\right)\right| &= \left|\Big(\sum_{I\in\mathbb{N}^r} g_I\cdot\chi_z(T)^I\Big)\left(\sigma_{\mathcal{H}(z)/k}(\alpha)\right)\right| \\ &= \max_{I\in\mathbb{N}^r}|g_I(\alpha)|\cdot\left|\chi_z(T)^I\right|_{\mathcal{H}(z)} = \max_{I\in\mathbb{N}^r}|\chi_\alpha(g_I)|_{\mathcal{H}(\alpha)}\cdot\left|T^I(z)\right| \\ &= \left|\Big(\sum_{I\in\mathbb{N}^r}\chi_\alpha(g_I)\cdot T^I\Big)\left(\sigma_{\mathcal{H}(\alpha)/k}(z)\right)\right| = \left|g\left(\iota_{\mathcal{H}(\alpha)}\circ\sigma_{\mathcal{H}(\alpha)/k}(z)\right)\right| \ .\end{aligned} \tag{2.11}$$

That is, for all fixed $z \in \mathbb{D}^r$ and $\alpha \in \mathrm{Mor}(\mathbb{D}^r, Y)$,

$$\iota_{\mathcal{H}(\alpha)}\circ\sigma_{\mathcal{H}(\alpha)/k}(z) = \iota_{\mathcal{H}(z)}\circ\sigma_{\mathcal{H}(z)/k}(\alpha) \ .$$

Consider the continuous map $\Psi(z) : \mathrm{Mor}(\mathbb{D}^r, Y) \to Y$, defined as the composition $\Psi(z) = \Phi\circ\iota_{\mathcal{H}(z)}\circ\sigma_{\mathcal{H}(z)/k}$. For every fixed $\alpha \in \mathrm{Mor}(\mathbb{D}^r, Y)$ and every fixed $z \in \mathbb{D}^r$, we have

$$\Psi(z)(\alpha) = \mathrm{Ev}(\alpha)(z).$$

If α_n is a sequence of points in $\mathrm{Mor}(\mathbb{D}^r, Y)$ converging to α, then the continuity of $\Psi(z)$ implies that $\Psi(z)(\alpha_n)$ converges to $\Psi(z)(\alpha)$ as n goes

to infinity, and so we see that

$$\mathrm{Ev}(\alpha_n)(z) \overset{n\to\infty}{\longrightarrow} \mathrm{Ev}(\alpha)(z) \ ,$$

concluding the proof. □

2.4. The space $\mathrm{Mor}(\mathbb{D}^r, Y)$ is Fréchet-Urysohn.

We prove Theorem 2.0.2, which states that the space $\mathrm{Mor}(\mathbb{D}^r, Y)$ is Fréchet-Urysohn. This technical result is a key step in the proof of Theorem A. Recall that the field k is complete and algebraically closed.

We follow Poineau's proof of the fact that analytic spaces are Fréchet-Urysohn [**Poi13**, Proposition 5.2], which in turn relies on [**Poi13**, Théorème 4.22].

Recall that a subset Γ of the analytic spectrum of a k-Banach algebra $(\mathcal{A}, \|.\|)$ is a boundary if for every $g \in \mathcal{A}$ there exists some $x \in \Gamma$ such that $|g(x)| = \|g\|$. A closed boundary is called the *Shilov boundary* if it is the smallest closed subset Γ of $\mathcal{M}(\mathcal{A})$ satisfying this property. Since we have excluded the trivially valued case and the norm on $\mathcal{T}_\infty^{r,s}$ is multiplicative, there exists a Shilov boundary in $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ by [**EMN04**, Theorem C].

In the following we deal with subfields l of k that are of *countable type* over the prime subfield k_p of k, i.e. such that l has a dense k_p-vector subspace of countable dimension.

The following proposition is an infinite dimensional analogue of [**Poi13**, Théorème 4.22].

Proposition 2.4.1. *For every point α in $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ there exists a subfield l of k that is of countable type over the prime subfield k_p of k and satisfying the following property.*

Let l' be any subfield of k with $l \subset l' \subset k$ and let $\pi^\infty_{k/l'} : \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s) \to \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)_{l'}$ be the base change morphism. Then α is the unique point in the Shilov boundary of the fibre $(\pi^\infty_{k/l'})^{-1}(\pi^\infty_{k/l'}(\alpha))$.

Proof of Proposition 2.4.1. The space $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ is the projective limit of $\mathrm{Mor}_\delta^{r,s}$ with the morphisms $\mathrm{Pr}^\infty_{\delta,k} : \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s) \to \mathrm{Mor}_\delta^{r,s}$ for $\delta \in \mathbb{N}^*$ (cf. Proposition 2.2.3). A point α in $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ is thus determined by a sequence $(\alpha_\delta)_{\delta\geq 0}$, where each α_δ lies in $\mathrm{Mor}_\delta^{r,s}$ and satisfies $\mathrm{pr}_{\delta+1}(\alpha_{\delta+1}) = \alpha_\delta$ for the projections $\mathrm{pr}_{\delta+1} : \mathrm{Mor}_{\delta+1}^{r,s} \to \mathrm{Mor}_\delta^{r,s}$.

To every α_δ we apply [**Poi13**, Théorème 4.22]. We obtain a field $l^\delta \subset k$ that is of countable type over the prime subfield k_p of k and such that for any subfield $l^\delta \subset l' \subset k$ the point α_δ is the only point in the Shilov boundary of $(\pi^\delta_{k/l'})^{-1}(\pi^\delta_{k/l'}(\alpha_\delta))$, where $\pi^\delta_{k/l'} : \mathrm{Mor}_\delta^{r,s} \to \mathrm{Mor}_{\delta,l'}^{r,s}$ denotes the base change morphism.

Let l be the subfield of k generated by all the l^δ. By construction, l is of countable type over k_p. We may assume in addition that l is algebraically closed.

The equality $\pi^{\delta}_{k/l'} \circ \mathrm{Pr}^{\infty}_{\delta,k} = \mathrm{Pr}^{\infty}_{\delta,l'} \circ \pi^{\infty}_{k/l'}$ implies that $\mathrm{Pr}^{\infty}_{\delta,k}$ maps the fibre $(\pi^{\infty}_{k/l'})^{-1}(\pi^{\infty}_{k/l'}(\alpha))$ to the fibre $(\pi^{\delta}_{k/l'})^{-1}(\pi^{\delta}_{k/l'}(\alpha_\delta))$. We show that α belongs to the Shilov boundary of $(\pi^{\infty}_{k/l'})^{-1}(\pi^{\infty}_{k/l'}(\alpha))$. Pick an element $g \in \mathcal{T}^{r,s}_{\infty}$. As $\mathcal{T}^{r,s}$ is dense in $\mathcal{T}^{r,s}_{\infty}$, we may assume that g lies in $k\{a_{l,I}\}_{|I|\leq\delta,1\leq l\leq s}$ for some $\delta \geq 0$. Thus, $|g(\alpha)| = |g(\alpha_\delta)|$, which is the maximum value of g, since α_δ belongs to the Shilov boundary of $(\pi^{\delta}_{k/l'})^{-1}(\pi^{\delta}_{k/l'}(\alpha_\delta))$.

Pick a point $\beta \in (\pi^{\infty}_{k/l'})^{-1}(\pi^{\infty}_{k/l'}(\alpha))$ different from α, i.e. such that $\beta_\delta \neq \alpha_\delta$ for some $\delta \geq 0$. As α_δ is the unique point in the Shilov boundary of $(\pi^{\delta}_{k/l'})^{-1}(\pi^{\delta}_{k/l'}(\alpha_\delta))$, we may find some $g \in k\{a_{l,I}\}_{|I|\leq\delta}$ such that

$$|g(\beta)| = |g(\beta_\delta)| < |g(\alpha_\delta)| = |g(\alpha)|,$$

showing that α is the unique point in the Shilov boundary of the space $(\pi^{\infty}_{k/l'})^{-1}(\pi^{\infty}_{k/l'}(\alpha))$. □

Proof of Theorem 2.0.2. Since $\mathrm{Mor}(\mathbb{D}^r, Y)$ is a closed subset of $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$, it suffices to prove that the space $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ is Fréchet-Urysohn.

Let A be any subset of $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ and let α be a point in the closure of A. Let l be the subfield of k associated to α from Proposition 2.4.1. Let $l \subset l' \subset k$ be any subfield of k that is of countable type over l. Every polydisk $\mathrm{Mor}^{r,s}_{\delta,l'}$ is first countable, and as a consequence so is the countable product of all the $\mathrm{Mor}^{r,s}_{\delta,l'}$. The space $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)_{l'}$ is a subspace of the product $\prod_\delta \mathrm{Mor}^{r,s}_{\delta,l'}$ by Proposition 2.2.3, and thus is first countable.

Copying Poineau's proof of [**Poi13**, Proposition 5.2] and using Proposition 2.4.1, we may find a sequence of points α_n in A converging to α. □

2.5. Montel's theorem

This section is devoted to the proof of Theorem A. We first apply the results and constructions from the previous sections to prove the case where the base field k is algebraically closed and next we generalize this argument to an arbitrary non-Archimedean complete field.

2.5.1. Proof of Theorem A in the algebraically closed case. Let k be an algebraically closed complete non-Archimedean field.

Let X be a good, reduced, σ-compact k-analytic space without boundary and Y a strictly k-affinoid space. Pick a sequence of analytic maps $f_n : X \to Y$. We claim that there exists a subsequence that is pointwise converging to a continuous map.

Assume first that $X = \mathbb{D}^r$. In this case, each analytic map f_n corresponds to a rigid point α_n in $\mathrm{Mor}(\mathbb{D}^r, Y)$ by Theorem B. Since the space $\mathrm{Mor}(\mathbb{D}^r, Y)$ is compact and Fréchet-Urysohn by Theorem 2.0.2,

we may find a converging subsequence α_{n_j} converging to some point $\alpha \in \mathrm{Mor}(\mathbb{D}^r, Y)$. The continuous map $\mathrm{Ev}(\alpha) : \mathbb{D}^r \to Y$ is the limit map of the subsequence f_{n_j} by Theorem 2.3.1.

Suppose now that X is a basic tube in the sense of §1.6. Let $\hat{X}$ be a strictly k-affinoid space and $\hat{X} \to \bar{\mathbb{D}}^r$ a distinguished closed immersion such that X is isomorphic to $\hat{X} \cap \mathbb{D}^r$ (cf. Proposition 1.6.6). We may thus write X as a growing countable union of affinoid spaces $X = \bigcup_{0<\rho<1} X_\rho$. Moreover, since k is algebraically closed, we may take every ρ in $|k^\times|$. As the affinoid algebra corresponding to $\hat{X}$ is isomorphic to the quotient of the Tate algebra $\mathcal{T}_r$ by some closed ideal I, we may assume that the affinoid algebra $\mathcal{A}_\rho$ of each X_ρ is of the form $k\{\rho^{-1}T_1, \ldots, \rho^{-1}T_r\}$ modulo the ideal generated by I. In particular, we have distinguished closed immersions $\varphi_\rho : X_\rho \to \bar{\mathbb{D}}^r(\rho)$.

Let $f_n : X \to \bar{\mathbb{D}}^s$ be a sequence of analytic maps. Fix $0 < \rho < 1$, $\rho \in |k^\times|$. We may apply Proposition 1.5.2 to the restriction of f_n to X_ρ to obtain an analytic map $g_n^{(\rho)} : \bar{\mathbb{D}}^r(\rho) \to \bar{\mathbb{D}}^s$ extending $f_n|_{X_\rho}$. Indeed, $f_n|_{X_\rho}$ is given by elements $f_1^{(n)}, \ldots, f_s^{(n)} \in \mathcal{A}_\rho$ of norm at most 1. As we have a distinguished epimorphism $k\{\rho^{-1}T_1, \ldots, \rho^{-1}T_r\} \to \mathcal{A}_\rho$, we may lift each $f_l^{(n)}$, $l = 1, \ldots, s$, to an element in $k\{\rho^{-1}T_1, \ldots, \rho^{-1}T_r\}$ having the same norm. These define analytic maps $g_n^{(\rho)} : \bar{\mathbb{D}}^r(\rho) \to \bar{\mathbb{D}}^s$ satisfying $g_n^{(\rho)} \circ \varphi_\rho = f_n|_{X_\rho}$ for all $n \in \mathbb{N}$. We now apply the previous case to the restricted sequence $\{g_n^{(\rho)}|_{\mathbb{D}^r(\rho)}\}_n$. We conclude by a diagonal extraction argument.

Consider now X as in the theorem. Being σ-compact, X is the union of countably many compact sets K_n. Since it is a good analytic space without boundary, each compact set K_n is included in a finite union of open sets, each isomorphic to a basic tube. It follows that X is a countable union of basic tubes U_m. By the previous case, on every open set U_m there exists a subsequence converging pointwise, and extracting diagonally we may find a subsequence $\{f_{n_j}\}$ converging pointwise on the whole X. The limit is continuous on every U_m and hence on X since they are open. □

2.5.2. Proof of Theorem A over an arbitrary base field. Let K be a completed algebraic closure of k, and X_K, Y_K be the scalar extensions of X and Y respectively, see §1.7.

Pick a sequence $f_n : X \to Y$ of analytic maps and consider the analytic maps $F_n : X_K \to Y_K$ induced by the base change. The following diagram commutes:

$$\begin{array}{ccc} X_K & \xrightarrow{F_n} & Y_K \\ {\scriptstyle \pi_{K/k}}\downarrow & & \downarrow{\scriptstyle \pi_{K/k}} \\ X & \xrightarrow{f_n} & Y \end{array}$$

Observe that the analytic space X_K is good and σ-compact, since the preimage $\pi_{K/k}^{-1}(U)$ of an affinoid domain U of X is an affinoid domain in X_K. It follows directly from the definition of the interior that X_K is boundaryless ([**Ber90**, Proposition 3.1.3]). Thus, by the algebraically closed case of Theorem A proved above, we may assume that F_n is pointwise converging to a continuous map $F : X_K \to Y_K$. Pick a point $z \in X$. As $\pi_{K/k}$ is surjective, we may choose a point $z' \in \pi_{K/k}^{-1}(z)$. It follows that $f_n(z) = f_n(\pi_{K/k}(z')) = \pi_{K/k} \circ F_n(z')$, which tends to $\pi_{K/k} \circ F(z') =: f(z)$ as n goes to infinity. The limit map f is well-defined. Indeed, if z', z'' are two points in $\pi_{K/k}^{-1}(z)$, then

$$\begin{aligned} & \lim_n \pi_{K/k} \circ F_n(z') = \lim_n f_n(\pi_{K/k}(z')) = \\ = \ & \lim_n f_n(\pi_{K/k}(z'')) = \lim_n \pi_{K/k} \circ F_n(z'') \ . \end{aligned}$$

It remains to check that f is continuous. Let A be any closed (hence compact) subset of Y. By continuity, the set $F^{-1}\left(\pi_{K/k}^{-1}(A)\right)$ is closed. Recall that the map $\pi_{K/k} : X_K \to X$ is proper. Since X_K and X are locally compact, then $\pi_{K/k}$ is closed. As a consequence, $f^{-1}(A) = \pi_{K/k}\left(F^{-1} \circ \pi_{K/k}^{-1}(A)\right)$ is closed. □

2.6. Analytic properties of pointwise limits of analytic maps

Continuous maps of the form $\mathrm{Ev}(\alpha) : \mathbb{D}^r \to Y$ are very special, as they exhibit properties that are distinctive of analytic maps. We shall prove that they lift to analytic maps after a suitable base change and that the graph of $\mathrm{Ev}(\alpha)$ is well-defined in the analytic product $\mathbb{D}^r \times Y$ and not just in the topological product $|\mathbb{D}^r| \times |Y|$.

THEOREM 2.6.1. *Let k be a complete non-Archimedean field that is algebraically closed.*

Let α be a point in $\mathrm{Mor}(\mathbb{D}^r, Y)$. Then there exists a closed subset Γ_α of $\mathbb{D}^r \times Y$ such that the first projection $\pi_1 : \Gamma_\alpha \to \mathbb{D}^r$ is a homeomorphism and such that for every $z \in \mathbb{D}^r$ the image of $\Gamma_\alpha \cap \pi_1^{-1}(z)$ under the second projection is the point $\mathrm{Ev}(\alpha)(z) \in Y$.

Moreover, there exist a complete extension K of k and a K-analytic map $F_\alpha : \mathbb{D}_K^r \to Y_K$ such that $\mathrm{Ev}(\alpha) = \pi_{K/k} \circ F_\alpha \circ \sigma_{K/k}$.

We deduce the following result:

COROLLARY C. *Let k be a non-Archimedean algebraically closed complete field and X a good, reduced, boundaryless strictly k-analytic space. Let Y be a k-affinoid space. Let $f_n : X \to Y$ be a sequence of analytic maps converging pointwise to a continuous map f.*

Then, for any point $x \in X$ one can find an affinoid neighbourhood Z of x, a complete extension K/k and a K-analytic map $F : Z_K \to Y_K$ such that

$$f|_Z = \pi_{K/k} \circ F \circ \sigma_{K/k} \ .$$

PROOF OF THEOREM 2.6.1. Fix a point $\alpha \in \mathrm{Mor}(\mathbb{D}^r, Y)$ and denote by $\mathcal{H}(\alpha)$ its complete residue field. We define Γ_α as the image of a continuous map $\psi : \mathbb{D}^r \to \mathbb{D}^r \times Y$, that we construct as follows.

Let $\iota_{\mathcal{H}(\alpha)} : \mathbb{D}^r_{\mathcal{H}(\alpha)} \to \mathrm{Mor}(\mathbb{D}^r, Y) \times \mathbb{D}^r$ be the inclusion map defined in (2.7). Let $\Upsilon : \mathrm{Mor}(\mathbb{D}^r, Y) \times \mathbb{D}^r \to \mathbb{D}^r \times Y$ be the analytic map induced by

$$\begin{aligned} k\{\rho^{-1}T_1, \ldots, \rho^{-1}T_r\}\{S_1, \ldots, S_s\} &\to \mathcal{T}_\infty^{r,s}\{\rho^{-1}T_1, \ldots, \rho^{-1}T_r\} \\ T_i &\mapsto T_i \\ S_l &\mapsto \sum_I a_{l,I}T^I. \end{aligned}$$

Let $\sigma_{\mathcal{H}(\alpha)/k} : \mathbb{D}^r \to \mathbb{D}^r_{\mathcal{H}(\alpha)}$. We set $\psi = \Upsilon \circ \iota_{\mathcal{H}(\alpha)} \circ \sigma_{\mathcal{H}(\alpha)}$. Explicitly, ψ is induced by the analytic map $\Psi : \mathbb{D}^r \to \mathbb{D}^r \times \bar{\mathbb{D}}^s$ that maps any $z \in \mathbb{D}^r$ to the seminorm sending every $g \in \mathcal{T}_s\{\rho^{-1}T_1, \ldots, \rho^{-1}T_r\}$, which is of the form $g = \sum_{J \in \mathbb{N}^s} g_J S^J$ with $g_J \in k\{\rho^{-1}T_1, \ldots, \rho^{-1}T_r\}$ are such that $|g_J| \to 0$ as $|J| \to 0$, to the following real number:

$$|g(\Psi(z))| = \left| \sum_J g_J \prod_{l=1}^{s} \Big(\sum_I \chi_\alpha(a_{l,I}) \cdot T^I \Big)^{j_l} (\sigma_{\mathcal{H}(\alpha)/k}(z)) \right|. \tag{2.12}$$

Consider the projections π_1 and π_2 on $\mathbb{D}^r \times Y$ to the first and second factor respectively. It is an immediate consequence of the previous computation and (2.8) that

$$\pi_2(\psi(z)) = \mathrm{Ev}(\alpha)(z).$$

If no variables S_l appear in the expression of $g \in \mathcal{T}_s\{\rho^{-1}T_1, \ldots, \rho^{-1}T_r\}$, then g lies in the algebra $k\{\rho^{-1}T_1, \ldots, \rho^{-1}T_r\}$. Thus, by (2.12) we see that $|g(\Psi(z))| = |g(z)|$, and so

$$\pi_1(\psi(z)) = z \ .$$

It remains to check that the image Γ_α of ψ is a closed subset of $\mathbb{D}^r \times Y$. Let z_n be a sequence of points in $\mathbb{D}^r$ such that $\psi(z_n)$ converges to some point x in $\mathbb{D}^r \times Y$. As $\pi_1(\psi(z_n)) = z_n$, we see that z_n converges to $\pi_1(x) \in \mathbb{D}^r$, and by continuity of ψ we have that $x = \psi(\pi_1(x))$ lies in Γ_α. The set Γ_α is so sequentially closed, and hence closed.

Consider now the continuous map $\mathrm{Ev}(\alpha) : \mathbb{D}^r \to Y$. Let K be the complete residue field $\mathcal{H}(\alpha)$. Consider the $\mathcal{H}(\alpha)$-analytic map

$$F_\alpha = \left(\sum_{I \in \mathbb{N}^r} \chi_\alpha(a_{1,I}) \cdot T^I, \ldots, \sum_{I \in \mathbb{N}^r} \chi_\alpha(a_{s,I}) \cdot T^I \right) \ .$$

A direct computation together with (2.8) shows that $\mathrm{Ev}(\alpha) = \pi_{\mathcal{H}(\alpha)/k} \circ F_\alpha \circ \sigma_{\mathcal{H}(\alpha)/k}$. □

PROOF OF COROLLARY C. Suppose first that $X = \mathbb{D}^r$. Each analytic map f_n is of the form $f_n = \mathrm{Ev}(\alpha_n)$ for some rigid point $\alpha_n \in \mathrm{Mor}(\mathbb{D}^r, Y)$ by Theorem 2.3.1. It was shown in Proposition 2.2.3

that the space $\mathrm{Mor}(\mathbb{D}^r, Y)$ is Fréchet-Urysohn so that we may assume that α_n converges to some point $\alpha \in \mathrm{Mor}(\mathbb{D}^r, Y)$. The limit map f is precisely $\mathrm{Ev}(\alpha)$ (cf. Theorem 2.3.1) and we conclude by Theorem 2.6.1.

Let now X be any good, boundaryless, reduced k-analytic space. Pick a point $x \in X$ and an affinoid neigbourhood Z of x containing x in its interior. Fix a distinguished closed immersion of Z into some closed unit polydisk $\bar{\mathbb{D}}^r$. For every n we may find an analytic map $\hat{f}_n : \bar{\mathbb{D}}^r \to \bar{\mathbb{D}}^s$ such that $\hat{f}_n|_Z = f_n$ by Proposition 1.5.2. We now apply the previous case to the restriction of $\hat{f}_n$ to $\mathbb{D}^r$, concluding the proof. □

2.7. Fields with countable residue field

We observe in this section that part of the assertion of Theorem A extends to maps between any k-affinoid spaces when the residue field of k is countable. Specifically, we do not exclude source spaces with boundary and show that one may always extract an everywhere converging subsequence.

Recall that the boundary of an affinoid space can be written as a finite union of affinoid spaces defined over some extension of k, see [**Duc12**, Lemma 3.1]. Here we shall only use the following observation. Consider the closed N-dimensional polydisk $\bar{\mathbb{D}}^N$, and denote by $p_i : \bar{\mathbb{D}}^N \to \bar{\mathbb{D}}$ the projection to the i-th coordinate. Recall that the boundary of $\bar{\mathbb{D}}$ consists only of the Gauss point. It follows from Lemma 1.3.2 that $p_i^{-1}(x_g)$ is contained in the boundary of $\bar{\mathbb{D}}^N$ for every $i = 1, \cdots, N$. Let now x be a point in $\partial\bar{\mathbb{D}}^N$ and consider the commutative diagram:

$$\begin{array}{ccc} \bar{\mathbb{D}}^N & \xrightarrow{p_i} & \bar{\mathbb{D}} \\ {\scriptstyle \mathrm{red}}\downarrow & & \downarrow{\scriptstyle \mathrm{red}} \\ \mathbb{A}^N_{\tilde{k}} & \xrightarrow{\tilde{p_i}} & \mathbb{A}^1_{\tilde{k}} \end{array}$$

Suppose that $p_i(x) \neq x_g$ for all i. By Lemma 1.3.2, $\widetilde{p_i(x)}$ is a closed point in $\mathbb{A}^1_{\tilde{k}}$ corresponding to some maximal ideal $\langle T_i - \zeta_i \rangle \subset \tilde{k}[T_i]$ for every $i = 1, \ldots, N$. The commutativity of the diagram implies that the ideal $\langle T_1 - \zeta_1, \ldots, T_N - \zeta_N \rangle$ of $\tilde{k}[T_1, \ldots, T_N]$ is contained in the prime ideal corresponding to $\mathrm{red}(x)$. As a consequence, $\mathrm{red}(x) \in \mathbb{A}^N_{\tilde{k}}$ is closed, contradicting the fact that x belongs to $\partial\bar{\mathbb{D}}^N$.

The boundary of $\bar{\mathbb{D}}^N$ is thus equal to the union $p_1^{-1}(x_g) \cup \ldots \cup p_N^{-1}(x_g)$. Observe that each fibre $p_i^{-1}(x_g)$ is isomorphic to $\bar{\mathbb{D}}^{N-1}_{\mathcal{H}(x_g)}$.

Proposition 2.7.1. *Suppose k is a non-Archimedean complete valued field that is algebraically closed and such that $\tilde{k}$ is countable. Let X and Y be k-affinoid spaces and assume that X is reduced and distinguished. Then, every sequence of analytic maps $f_n : X \to Y$ has an everywhere pointwise converging subsequence.*

PROOF. We may assume $X = \bar{\mathbb{D}}^r$, $Y = \bar{\mathbb{D}}^s$ as in the proof of Theorem A. The set of connected components of the interior of $\bar{\mathbb{D}}^r$ is in bijection with the set of $\tilde{k}$-points on its reduction $\mathbb{A}^r_{\tilde{k}}$ and hence is countable.

We now argue inductively on r. When $r = 1$, then the boundary of $\bar{\mathbb{D}}$ consists of a single point, namely the Gauss point. We may therefore apply Theorem A to each of the (countably many) components of the interior of $\bar{\mathbb{D}}$ and apply a diagonal extraction argument to conclude.

Assume now that the statement holds for the polydisk of dimension $r-1$ defined over *any* complete valued field with countable residue field, and pick a sequence of analytic maps $f_n : \bar{\mathbb{D}}^r \to \bar{\mathbb{D}}^s$. As before, we apply Theorem A to each of the (countably many) components of the interior of $\bar{\mathbb{D}}^r$ so that we may suppose that f_n converges pointwise on the interior of $\bar{\mathbb{D}}^r$.

The boundary of $\bar{\mathbb{D}}^r$ is the union of r unit polydisks of dimension $r-1$ defined over the field $\mathcal{H}(x_g)$ by our previous discussion. On each of these we may apply the induction hypothesis, as the field $\widetilde{\mathcal{H}(x_g)}$ is isomorphic to $\tilde{k}(T)$, which is countable. This concludes the proof. □

CHAPTER 3

Weakly analytic maps

In this chapter we study the continuous maps that are locally pointwise limits of analytic maps, as obtained in Corollary C. Since they can be locally lifted to an analytic map over some field extension K of k, they share several properties with analytic maps, so that they deserve the name of weakly analytic maps.

As before, k is any complete non trivially valued non-Archimedean valued field which is algebraically closed.

3.1. Definition and first properties

We begin with a definition.

DEFINITION 3.1.1. *Let X and Y be any two good k-analytic spaces, and let $f : X \to Y$ be a continuous map.*

We say that f is weakly analytic if for every point $x \in X$ there exist an affinoid neighbourhood U of x, a complete field extension K/k and an analytic map $F : U_K \to Y_K$ such that $f_{|U} = \pi_{K/k} \circ F \circ \sigma_{K/k}$.

It will be convenient to denote by $\mathrm{WA}(X, Y)$ the set of all weakly analytic maps from X to Y.

Clearly, the set $\mathrm{Mor}_k(X, Y)$ of analytic maps from X to Y is a subset of $\mathrm{WA}(X, Y)$. It is also a strict subset if Y has dimension at least 1, since any constant map is weakly analytic, but it is analytic only if the constant is a rigid point.

PROPOSITION 3.1.2. *Let X be a basic tube and Y be a k-affinoid space. Let $f : X \to Y$ be a continuous map. The following two conditions are equivalent.*

i) *For any point $x \in X$ there exist an affinoid neighbourhood Z of x and a sequence of analytic maps $f_n : Z \to Y$ pointwise converging to $f|_Z$.*
ii) *For any point $x \in X$ there exist an affinoid neighbourhood Z of x, a complete extension K of k and an analytic map $F : Z_K \to Y_K$ such that $f|_Z = \pi_{K/k} \circ F \circ \sigma_{K/k}$.*

A consequence of the previous result is that when X has no boundary, a continuous map $f : X \to Y$ is weakly analytic whenever for every point $x \in X$ there exists a basic tube U containing x and a sequence of analytic maps f_n from U to Y that converge pointwise to f.

PROOF. The implication i) $\Rightarrow$ ii) is precisely Corollary C, since basic tubes are boundaryless.

Suppose that ii) is satisfied. Choosing a closed immersion $Y \to \bar{\mathbb{D}}^s$, we may assume $Y = \bar{\mathbb{D}}^s$. Pick a point $x \in X$ and an affinoid neighbourhood Z of x such that there exists a complete extension K/k and a K-analytic map $F : Z_K \to \bar{\mathbb{D}}^s_K$ such that $f|_Z = \pi_{K/k} \circ F \circ \sigma_{K/k}$. Fix a closed immersion of Z_K into some closed polydisk $\mathbb{D}^r_K$. By Proposition 1.5.2, we may find an analytic map $\hat{F} : \mathbb{D}^r_K \to \bar{\mathbb{D}}^s_K$ that agrees with F on $Z_K \cap \mathbb{D}^r_K$. By Theorem 2.3.1, there exists a rigid point $a \in \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)_K$ such that $\hat{F} = \Phi(a, \cdot)$. The point $\alpha = \pi^{\infty}_{K/k}(a)$ in $\mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ is not rigid in general, but we may find rigid points $\alpha_n \in \mathrm{Mor}(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ converging to α by Proposition 2.2.5, since k is assumed to be non trivially valued. The analytic maps $\mathrm{Ev}(\alpha_n)$ converge pointwise to $\mathrm{Ev}(\alpha) : \mathbb{D}^r \to \bar{\mathbb{D}}^s$ by Theorem 2.3.1, and by construction we have that $\mathrm{Ev}(\alpha) = \pi_{K/k} \circ \hat{F} \circ \sigma_{K/k}$, see Theorem 2.6.1. □

3.2. Rigidity of weakly analytic maps

We prove here the following statement

PROPOSITION 3.2.1. *Suppose $f : X \to Y$ is a weakly analytic map, where Y is a curve. If x is a rigid point that is mapped to a non-rigid point by f, then f is locally constant near x.*

PROOF. Let $x \in X$ be a rigid point such that $y = f(x)$ is not rigid. Since this is a local statement, we may replace X and Y by affinoid neighbourhoods of x and y respectively. In particular, we may assume that $X = \bar{\mathbb{D}}^r$ and $x = 0$. After maybe reducing X, there exists an extension K of k and a K-analytic map $F : X_K \to Y_K$ such that $f = \pi_{K/k} \circ F \circ \sigma_{K/k}$. Observe that $F(x)$ is a rigid point of Y_K.

Suppose first that $Y = \bar{\mathbb{D}}$. The fact that y is not rigid means that y has positive diameter, i.e.

$$\inf_{a \in k^\circ} |(T-a)(y)| = r > 0 \ .$$

By continuity, we can find a polyradius $\epsilon > 0$ such that every rigid point z in $\mathbb{D}^r_K(0;\epsilon)$ satisfies $|F(z) - F(0)|_K < r$, where $|.|_K$ denotes the absolute value on K. Pick a point $a \in k^\circ$. For every rigid point $z \in \mathbb{D}^r_K(0;\epsilon)$, we get

$$\begin{aligned}
|(T-a)(y)| &= \max\{|F(z)-F(0)|_K, |(T-a)(y)|\} \\
&= \max\left\{|F(z)-F(0)|_K, |(T-a)(\pi_{K/k}\circ F(0))|\right\} \\
&= \max\{|F(z)-F(0)|_K, |F(0)-a|_K\} \\
&= |F(z)-a|_K \\
&= |(T-a)(\pi_{K/k}\circ F(z))| \ .
\end{aligned}$$

Thus, F maps the polydisk $\mathbb{D}^r_K(0;\epsilon)$ into the fibre $\pi^{-1}_{K/k}(y)$. As

$$\sigma_{K/k}(\mathbb{D}^r(0;\epsilon)) \subseteq \mathbb{D}^r_K(0;\epsilon),$$

we conclude that f is locally constant near 0.

For Y any affinoid space of dimension 1 there exists a finite morphism $\varphi : Y \to \bar{\mathbb{D}}$ by Noether's Lemma. By what precedes, the composition $\varphi \circ f$ is locally constant near 0, and by finiteness so is f. □

EXAMPLE 3.2.2. *The previous result does not hold if Y has dimension greater than 2. Consider for instance the weakly analytic map $f : \bar{\mathbb{D}} \to \bar{\mathbb{D}}^2$ given by $f = \pi_{K/k} \circ F \circ \sigma_{K/k}$, where $K = \mathcal{H}(x_g)$ and $F(z) = (x_g, z)$. No rigid point in $\bar{\mathbb{D}}$ has rigid image under f, but f is not locally constant at these points.*

3.3. Weakly analytic maps from curves

PROPOSITION 3.3.1. *Let $f : X \to Y$ be a weakly analytic map, where X is a curve. If there exists a converging sequence of rigid points of X whose images under f are rigid points, then f is analytic.*

REMARK 3.3.2. *Let X be a k-affinoid space. Let $f : X \to \bar{\mathbb{D}}^s$ be a continuous map such that there exists a complete extension K/k such that $f = \pi_{K/k} \circ F \circ \sigma_{K/k}$ for some K-analytic map F. We may assume that the extension K/k is of countable type* [**BGR84**, §2.7].

Indeed, let $\mathcal{A}$ be the underlying k-affinoid algebra of X and fix an epimorphism $k\{r^{-1}T\} \to \mathcal{A}$ such that $\mathcal{A}$ is isomorphic as a Banach algebra to $k\{r^{-1}T\}/I$ for some closed ideal $I \subset k\{r^{-1}T\}$. Extending scalars, we see that $\mathcal{A}_K$ is isomorphic to $K\{r^{-1}T\}/I$ as a K-affinoid algebra. The map F is then determined by elements $F_1, \cdots, F_s \in \mathcal{A}_K$ with $|F_l|_{\sup} \leq 1$, and hence the expression of F contains at most countably many elements of K.

PROOF. Pick any sequence $x_n \in X(k)$ such that $f(x_n)$ are also rigid, and assume that $\lim_n x_n = x$. Here x may be non-rigid. We may replace X by some affinoid neighbourhood of x and assume that $f = \pi_{K/k} \circ F \circ \sigma_{K/k}$ for some complete extension K/k of countable type and some K-analytic map F. Observe that $f(x_n) = F(x_n) \in Y(k)$. We may as well replace Y by an affinoid neighbourhood of $f(x)$ and embed it in some polydisk $\bar{\mathbb{D}}^s$.

Let $\mathcal{A}$ be the underlying k-affinoid algebra of X. The map F is then determined by elements $F_1, \cdots, F_s$ in the K-affinoid algebra $\mathcal{A}_K$ with $|F_l|_{\sup} \leq 1$. Pick any real number $\alpha > 1$. By [**BGR84**, Proposition 2.7.2/3] there is an α-cartesian Schauder basis $\{e_j\}_{j\in\mathbb{N}}$ of K, and we may choose $e_0 = 1$ by [**BGR84**, Proposition 2.6.2./3].

Fix a distinguished epimorphism $\mathcal{T}_M \to \mathcal{A}_K$ and lift every F_l to an element G_l in $\mathcal{T}_M$. Then for every $l = 1, \cdots, s$ we can develop $G_l = \sum_I a^l_I T^I$ with $a^l_I \in K$ and such that $|a^l_I|_K \to 0$ as $|I|$ goes to

infinity. Using the Schauder basis we may find elements $a^l_{I,j} \in k$ such that $a^l_I = \sum_j a^l_{I,j} e_j$ and satisfying

$$|a^l_{I,j}|_k \leq \max_j |a^l_{I,j}|_k \leq \alpha |a^l_I|_K \ .$$

Since $\alpha |a^l_I|_K \to 0$ as as $|I|$ goes to infinity, the series $A^j_l = \sum_I a^l_{I,j} T^I$ defines an element in $\mathcal{T}_M$. Thus, we obtain a converging power series $G_l = \sum_j \left(\sum_I a^l_{I,j} T^I \right) e_j$. Recall that $F_l(x_n) \in k$ for all n, and so $G_l(x_n) \in k$. We infer that for $j \geq 1$ and for all n, $A^j_l(x_n) = 0$. Each of these A^j_l defines in turn an analytic map on X that vanishes at every x_n, and hence is constant equal to zero on X by the principle of isolated zeros. It follows that $F_{l|_X} = A^0_l$ for every $1 \leq i \leq s$, thus they are defined over k. □

We observe that the previous result does not hold in higher dimension.

EXAMPLE 3.3.3. *Let $\zeta_n \in k$, $|\zeta_n| = 1$, $|\zeta_n - \zeta_m| = 1$ for $n \neq m$. Let f be the weakly analytic map obtained as the limit of the sequence $f_n : \mathbb{D}^2 \to \bar{\mathbb{D}}^1$, given by on the rigid points by $f_n(z_1, z_2) = \zeta_n z_1 + z_2$. The map f is not analytic, since the rigid point $(\lambda, 0) \in \mathbb{D}^2$, $0 < |\lambda| < 1$, is mapped to the point in $\bar{\mathbb{D}}$ corresponding to the closed ball $\bar{B}(0; |\lambda|)$. However, the set $\{0\} \times \mathbb{D}^1(k)$ is mapped to the set of rigid points.*

A consequence of the previous result is the following statement that can be viewed as the principle of isolated zeroes for weakly analytic maps.

PROPOSITION 3.3.4. *Let $f : X \to Y$ be a non constant weakly analytic map where X is a curve without boundary. Then the fibre of any rigid point in Y contains no accumulation point.*

PROOF. Let $y \in Y(k)$ and suppose there exist points $x_n \in X$ converging to a point x and such that $f(x_n) = y$ for all n. In this situation, we may assume $Y = \bar{\mathbb{D}}^s$, $y = (0, \cdots, 0)$ and replace X with some affinoid neighbourhood of x such that f lifts to a K-analytic map F over some complete extension K/k. This map F is given by some elements $F_1, \cdots, F_s$ in the underlying affinoid algebra of X_K of norm at most 1.

The point y is rigid and so it has only one preimage under $\pi_{K/k}$. Thus,

$$(0, \cdots, 0) = f(x_n) = F \circ \sigma_{K/k}(x_n) \in \bar{\mathbb{D}}^s_K$$

for all n. Since X is a curve and F is non-constant (otherwise f would be so), $F^{-1}(0)$ is included in the set of rigid points of X. It follows that every $\sigma_{K/k}(x_n)$ is rigid. Each component F_i of F defines an analytic map between the curves X_K and $\bar{\mathbb{D}}_K$ and admits a sequence of zeros with an accumulation point $\sigma_{K/k}(x)$. It follows that every F_i is identically zero, hence so is f. □

3.4. A conjecture on weakly analytic maps

On basic tubes, we conjecture that weakly analytic maps can be globally lifted to analytic maps.

CONJECTURE 5. *Let Y be a k-affinoid space and X a basic tube. Let $f : X \to Y$ be a weakly analytic map. Then, there exist a complete extension K/k and $F : X_K \to Y_K$ analytic such $f = \pi_{K/k} \circ F \circ \sigma_{K/k}$.*

Notice that a weakly analytic map can be locally lifted to an analytic map over some complete extension of k. Conjecture 5 means that this can be done globally.

REMARK 3.4.1. *In the case when X and Y are polydisks, Conjecture 5 amounts to saying that the map* Ev *is surjective onto the set* $\mathrm{WA}(X, Y)$.

The map Ev becomes closed by Theorem 2.3.1 for the topology of the pointwise convergence, and so $\mathrm{WA}(X, Y)$ becomes Fréchet-Urysohn for this topology.

Observe that if Conjecture 5 holds, then using Theorem 2.0.2 we have:

THEOREM 3.4.2. *Suppose that Conjecture 5 holds.*

Let X be a boundaryless σ-compact k-analytic space and Y a k-affinoid space. Then, every sequence of weakly analytic maps $f_n : X \to Y$ admits a subsequence that is pointwise converging to a weakly analytic map $f : X \to Y$.

As a consequence, we have:

COROLLARY 3.4.3. *Suppose that Conjecture 5 holds. Let X be a boundaryless σ-compact k-analytic space and Y a k-affinoid space. Let $\{f_n\} \subset \mathrm{WA}(X, Y)$ be a sequence converging to some continuous map f. Then, f is weakly analytic.*

CHAPTER 4

Dynamics of endomorphisms of the projective space

In this chapter, we give applications of Theorem A to the dynamics of an endomorphism f of the projective space $\mathbb{P}^{N,\mathrm{an}}$ of degree at least 2, and attach to f two different notions of Fatou sets.

The first definition relies on the following notion of normal families, whose motivation comes from Theorem A.

DEFINITION 4.0.1. *We say that a family of analytic maps $\mathcal{F}$ from a boundaryless k-analytic space X into a compact space Y is* normal *at a point $x \in X$ if for every sequence $\{f_n\}$ in $\mathcal{F}$ there exists a neighbourhood $V \ni x$ and a subsequence f_{n_j} that is pointwise converging on V to some continuous map $f : V \to Y$.*

We may then define the *normal Fatou set* $F_{\mathrm{norm}}(f)$ of f as the normality locus of the family of the iterates $\{f^n\}$.

Next, we define the *harmonic Fatou set* $F_{\mathrm{harm}}(f)$ as the set where the non-Archimedean Green function G_f of f introduced by Kawaguchi-Silverman is strongly pluriharmonic in the sense of [**CL11**].

In the complex setting, the definition of the Fatou set as the locus of normality of the family of the iterates agrees with the one in terms of the pluriharmonicity locus of the Green function. It is thus natural to compare $F_{\mathrm{harm}}(f)$ and $F_{\mathrm{norm}}(f)$ in the non-Archimedean setting. In dimension 1, it was shown in [**FKT12**, Theorem 5.4] that both Fatou sets agree on the rigid points. We shall obtain as a consequence of Proposition 4.4.4 and Theorem A that $F_{\mathrm{harm}}(f) \subseteq F_{\mathrm{norm}}(f)$.

We focus our attention on the hyperbolicity properties of harmonic Fatou components. Specifically, we give an analogue to a theorem by Ueda [**Ued94**] stating that every Fatou component of a non-invertible endomorphism of $\mathbb{P}^N_{\mathbb{C}}$ is hyperbolically embedded in $\mathbb{P}^N_{\mathbb{C}}$. To do so, we adapt our above definition of normal families to a non-compact target as follows.

Let X be any boundaryless k-analytic space and $Y \subset Z$ a relatively compact subset of a k-analytic space Z. We say that a family of analytic maps $\mathcal{F} \subset \mathrm{Mor}_k(X, Y)$ is normal at a point $x \in X$ if there exists an open neighbourhood $V \ni x$ such that every sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence f_{n_j} that is pointwise converging on V to a continuous map $f : V \to Z$.

THEOREM D. *Let Ω be a connected component of the harmonic Fatou set $F_{\mathrm{harm}}(f)$ of an endomorphism $f : \mathbb{P}^{N,\mathrm{an}} \to \mathbb{P}^{N,\mathrm{an}}$ of degree at least 2. Let U be a connected open subset of $\mathbb{P}^{1,\mathrm{an}}$. Then the family $\mathrm{Mor}_k(U, \Omega)$ is normal.*

As a consequence of Theorem D we obtain the following Picard-type result:

THEOREM E. *Let Ω be a connected component of the harmonic Fatou set $F_{\mathrm{harm}}(f)$ of an endomorphism $f : \mathbb{P}^{N,\mathrm{an}} \to \mathbb{P}^{N,\mathrm{an}}$ of degree $d \geq 2$. Then every analytic map from $\mathbb{A}^{1,\mathrm{an}} \setminus \{0\}$ to Ω is constant.*

This chapter is structured as follows. In §4.1 we discuss strongly pluriharmonic functions in the sense of [**CL11**] and in §4.2 we review the one-dimensional case. We recall the construction of the Green function of a non-invertible endomorphism from [**KS07, KS09**] in section 4.3, and next we introduce our notions of Fatou sets in §4.4. Finally, Theorem D is proved in §4.5 and Theorem E in §4.6.

4.1. Strongly pluriharmonic functions

We recall the definition from [**CL11**]:

DEFINITION 4.1.1. *Let X be any boundaryless k-analytic space. A continuous function $u : X \to \mathbb{R}$ is strongly pluriharmonic if for every $x \in X$ there exist an open neighbourhood U of x, a sequence of invertible analytic functions h_n on U and real numbers b_n such that*

$$u = \lim_{n \to +\infty} b_n \cdot \log |h_n|$$

locally uniformly on U.

Strongly pluriharmonic functions form a sheaf by definition.

Harmonic functions have been widely studied in dimension 1. Baker-Rumely [**BR10**], Favre-Rivera Letelier [**FRL10**], and Thuillier [**Thu05**] have defined non-Archimedean analogues of the Laplacian on $\mathbb{P}^{1,\mathrm{an}}$ and on general analytic curves respectively.

If X is an analytic curve, strongly harmonic functions are harmonic in the sense of Thuillier. It is not known whether the converse holds, see [**CL11**, Remark 2.4.6]. However, if X is a connected open subset of $\mathbb{P}^{1,\mathrm{an}}$, then all definitions agree by [**BR10**, Corollary 7.32].

Observe that over $\mathbb{C}$, pluriharmonic functions are in fact locally the logarithm of the norm of an invertible analytic function, whereas this is not true in the non-Archimedean setting. Counterexamples appear already for curves, see [**CL11**, §2.3].

REMARK 4.1.2. *Let X be any boundaryless k-analytic space. The set of all strongly pluriharmonic functions on X forms a $\mathbb{R}$-vector space.*

4.2. Harmonic functions on open subsets of $\mathbb{P}^{1,\mathrm{an}}$

The Berkovich projective line $\mathbb{P}^{1,\mathrm{an}}$ is the one-point compactification of the analytic affine line, see [**Ber90**, §4.2]. The classification of the points in the analytic affine line given in §1.2.2 extends naturally to $\mathbb{P}^{1,\mathrm{an}}$.

It is a fundamental fact that the Berkovich projective line carries a tree structure. It is obtained by patching together one-dimensional line segments in such a way that it contains no loop. We refer to [**Jon15**, §2] for a precise definition. Suffice it to say that for any two points $x, y \in \mathbb{P}^{1,\mathrm{an}}$ there exists a closed subset $[x, y] \subset \mathbb{P}^{1,\mathrm{an}}$ containing x and y that can be endowed with a partial order making it isomorphic to the real closed unit interval $[0, 1]$ or to $\{0\}$. These ordered sets are required to satisfy a suitable set of axioms. For instance, for any triple x, y, z there exists a unique point w such that $[z, x] \cap [y, x] = [w, x]$ and $[z, y] \cap [x, y] = [w, y]$. Any subset of the form $[x, y]$ is called a segment.

As a consequence, $\mathbb{P}^{1,\mathrm{an}}$ is uniquely path-connected, meaning that given any two distinct points $x, y \in \mathbb{P}^{1,\mathrm{an}}$ the image of every injective continuous map γ from the real unit interval $[0, 1]$ into $\mathbb{P}^{1,\mathrm{an}}$ with $\gamma(0) = x$ and $\gamma(1) = y$ is isomorphic to the segment $[x, y]$.

A nonempty closed subset $\Gamma \subseteq \mathbb{P}^{1,\mathrm{an}}$ is called a subtree if it is connected. An endpoint of Γ is a point $x \in \Gamma$ such that $\Gamma \setminus \{x\}$ either remains connected or is empty. For every subtree Γ of $\mathbb{P}^{1,\mathrm{an}}$ there is a canonical retraction $r_\Gamma : \mathbb{P}^{1,\mathrm{an}} \to \Gamma$, which sends a point $x \in \mathbb{P}^{1,\mathrm{an}}$ to the unique point in Γ such that the intersection of the segment $[x, r_\Gamma(x)]$ with Γ consists only of the point $r_\Gamma(x)$.

A *strict* finite subtree Γ of $\mathbb{P}^{1,\mathrm{an}}$ is the convex hull of finitely many type II points $x_1, \ldots, x_n$. As a set, it is the union of all the paths $[x_i, x_j]$, $i = 1, \ldots, n$.

Recall from §1.2.2 that a disk in $\mathbb{P}^{1,\mathrm{an}}$ is by definition either a disk in $\mathbb{A}^{1,\mathrm{an}}$ or the complement of a disk in $\mathbb{A}^{1,\mathrm{an}}$. Basic tubes in $\mathbb{P}^{1,\mathrm{an}}$ are *strict simple domains* in the terminology of [**BR10**]. They are either $\mathbb{P}^{1,\mathrm{an}}$ or strict open disks in $\mathbb{P}^{1,\mathrm{an}}$ with a finite number of strict closed disks of $\mathbb{P}^{1,\mathrm{an}}$ removed, cf. §1.6. In particular, basic tubes different from $\mathbb{P}^{1,\mathrm{an}}$ and strict open disks can be obtained as an inverse image $r_\Gamma^{-1}(\Gamma^0)$, where Γ is a strict finite subtree of $\mathbb{P}^{1,\mathrm{an}}$ and Γ^0 the open subset of Γ consisting of Γ with its endpoints removed.

Similarly, every connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}$ is either a closed disk or a closed disk in $\mathbb{P}^{1,\mathrm{an}}$ with a finite number of open disks of $\mathbb{P}^{1,\mathrm{an}}$ removed. In particular, an affinoid subset of the form $\bar{\mathbb{D}}(a; r) \setminus \bigcup_{i=1}^n \mathbb{D}(a_i; r_i)$ is homeomorphic to the Laurent domain of underlying affinoid algebra

$$k\{r^{-1}(T - a), r_1 S_1, \ldots, r_n S_n\}/(S_1(T - a_1) - 1, \ldots, S_n(T - a_n) - 1).$$

Given a subset $W \subset \mathbb{P}^{1,\mathrm{an}}$, denote by $\overline{W}$ its closure and by $\partial_{\mathrm{top}}W$ its topological boundary. If W is a basic tube strictly contained in $\mathbb{P}^{1,\mathrm{an}}$, then $\partial_{\mathrm{top}}W$ consists of a finite set of type II points.

PROPOSITION 4.2.1. *Let U be a proper connected open subset of $\mathbb{P}^{1,\mathrm{an}}$. Then there exist an increasing sequence W_m of basic tubes of $\mathbb{P}^{1,\mathrm{an}}$ exhausting U and a sequence of strictly affinoid subspaces X_m of $\mathbb{P}^{1,\mathrm{an}}$ satisfying*

$$\overline{W}_m \subset X_m \subset W_{m+1} \subset U$$

for every $m \in \mathbb{N}^*$.

The proof makes extensive use of the tree structure of $\mathbb{P}^{1,\mathrm{an}}$. Recall from §1.4 the construction of the tangent space $T_x\mathbb{P}^{1,\mathrm{an}}$ at a point $x \in \mathbb{P}^{1,\mathrm{an}}$.

PROOF. By [**BR10**, Corollary 7.11] there exists a sequence of basic tubes W_m exhausting U and such that $\overline{W}_m \subset W_{m+1} \subset U$ for every $m \in \mathbb{N}^*$. The idea is to take the set of points in the topological boundary of U, which is a compact subset of $\mathbb{P}^{1,\mathrm{an}}$, and to consider their convex hull. Denote by D the intersection of this set with U. The set D can be then expressed as a countable growing union of finite $\mathbb{R}$-trees D_m, whose endpoints are of type II. The skeleton of $\overline{W}_m$ is precisely D_m.

Fix a positive integer $m > 0$. As we have assumed that U is strictly contained in $\mathbb{P}^{1,\mathrm{an}}$, the topological boundary of W_m is a non-empty finite set of type II points of $\mathbb{P}^{1,\mathrm{an}}$. The convex hull Γ_m of $\partial_{\mathrm{top}}W_m$ is thus a subgraph of $\mathbb{P}^{1,\mathrm{an}}$ with finitely many endpoints.

If W_m is an open disk, we set X_m to be the closed disk of same centre and same radius as W_m. Otherwise, consider the following strict finite subtree Γ of $\mathbb{P}^{1,\mathrm{an}}$. Let Γ_m^0 be the open subset of Γ_m consisting of Γ_m with its endpoints removed. Pick a point x in $\Gamma_m \setminus \Gamma_m^0$. There are at most finitely many tangent directions at x containing points of the complement in U and not contained in Γ_m. For every such tangent direction, attach a segment to Γ_m in that direction and in such a way that it is contained in W_{m+1} and such that its endpoint is a type II point. If no such tangent direction exists, lengthen that edge ending at x such that the new endpoint is again of type II and belongs to W_{m+1}. Denote by Γ the strict finite subtree obtained this way. Observe that all the boundary points of Γ_m are contained in Γ^0.

Let $r_\Gamma : \mathbb{P}^{1,\mathrm{an}} \to \Gamma$ be the natural retraction map. The basic tube W_m is precisely $r_\Gamma^{-1}(\Gamma_m^0)$. Setting $X_m = r_\Gamma^{-1}(\Gamma_m)$, clearly one has $\overline{W}_m \subset X_m \subset W_{m+1}$. Let $x_{i_1}, \ldots, x_{i_m}$ be the endpoints of Γ_m, where $x_{i_j} = \eta_{a_{i_j}, r_{i_j}}$ are of type II. The set X_m is homeomorphic to $\mathbb{P}^{1,\mathrm{an}}$ minus the strict open disks $\mathbb{D}(a_{i_j}; r_{i_j})$, $j = 1, \ldots, m$, and is thus strictly affinoid. □

The following lemma will be essential for the proof of Theorem D.

PROPOSITION 4.2.2. *Let U be a basic tube in $\mathbb{P}^{1,\mathrm{an}}$. There exists a positive constant C depending only on U such that for every harmonic function $g : U \to \mathbb{R}$ there exists an analytic function $h : U \to \mathbb{A}^{1,\mathrm{an}} \setminus \{0\}$ such that*

$$\sup_U \Big| g - \log|h| \Big| \leq C.$$

PROOF. If U is either $\mathbb{P}^{1,\mathrm{an}}$ or $\mathbb{D}$, the assertion is trivial, because every harmonic function on $\mathbb{D}$ or on $\mathbb{P}^{1,\mathrm{an}}$ is constant by [**BR10**, Proposition 7.12]. We may thus assume that U is of the form $\mathbb{D} \setminus \cup_{i=1}^m \overline{\mathbb{D}}(a_i, r_i)$ with $r_i \in |k^\times|$, $0 < r_i < 1$ and $|a_i| < 1$ for $i = 1, \ldots, m$. The topological boundary of U consists of $m+1$ type II points.

By the Poisson formula [**BR10**, Proposition 7.23], we may find real numbers $c_0, \ldots, c_m$ with $\sum_{i=1}^m c_i = 0$ such that for all $z \in U$

$$g(z) = c_0 + \sum_{i=1}^m c_i \cdot \log|(T - a_i)(z)|.$$

Pick non-zero integers $n_1, \ldots, n_m$ such that $|c_i - n_i| < 1$ and $b \in k$ such that $|\log|b| - c_0| < 1$. Consider the map $h : U \to \mathbb{A}^{1,\mathrm{an}} \setminus \{0\}$,

$$h(z) = b \prod_{i=1}^m (T - a_i)^{n_i}(z).$$

Since $a_i \notin U$, the function $\log|h|$ is harmonic on U and we have

$$\sup_U |g - \log|h|| \leq |c_0 - \log|b|| + \sum_{i=1}^m |c_i - n_i| \cdot \sup_U \log|(T - a_i)(z)|.$$

The functions $\log|(T - a_i)(z)|$ are bounded on U and it follows that the right-hand side of the inequality is bounded. □

4.3. Green functions after Kawaguchi-Silverman

Consider an endomorphism of the N-dimensional projective analytic space $f : \mathbb{P}_k^{N,\mathrm{an}} \to \mathbb{P}_k^{N,\mathrm{an}}$ of degree $d \geq 2$. Denote by f^n its n-th iterate.

Fixing homogeneous coordinates, such a map can be written as $f = [F_0 : \cdots : F_N]$, with F_i homogeneous polynomials of degree d without non-trivial common zeros.

Denote by $\rho : \mathbb{A}^{N+1,\mathrm{an}} \setminus \{0\} \to \mathbb{P}^{N,\mathrm{an}}$ the natural projection map. An endomorphism f of $\mathbb{P}^{N,\mathrm{an}}$ can be lifted to a map $F : \mathbb{A}^{N+1,\mathrm{an}} \to \mathbb{A}^{N+1,\mathrm{an}}$ such that $\rho \circ F = f \circ \rho$. One can take for instance $F = (F_0, \cdots, F_N)$. In the sequel, we will always choose lifts of f such that all the coefficients of the F_i's lie in k° and at least one of them has norm 1.

Given $T_0, \ldots, T_N$ affine coordinates of $\mathbb{A}^{N+1,\mathrm{an}}$ and a point $z \in \mathbb{A}^{N+1,\mathrm{an}}$, we define its norm as $|z| = \max_{0 \leq i \leq N} |T_i(z)|$. Similarly, we set $|F(z)| = \max_{0 \leq i \leq N} |F_i(z)|$. With these norms in hand, we may now define the Green function associated to f following Kawaguchi and Silverman [**KS07, KS09**], see [**Sib99**] for the complex case.

PROPOSITION-DEFINITION 4.3.1. *The sequence of functions*

$$G_n(z) = \frac{1}{d^n} \log |F^n(z)|$$

converges uniformly on $\mathbb{A}^{N+1,\mathrm{an}}$.

One defines the dynamical Green function associated to f *as* $G_f(z) = \lim_{n\to\infty} G_n$.

REMARK 4.3.2. *The Green function does not depend on the choice of the lift of* f *up to a constant. Indeed, let* $F = (F_0, \dots, F_N)$ *and* $\widetilde{F} = (\widetilde{F}_0, \dots, \widetilde{F}_N)$ *be two different lifts of* f. *We may find a nonzero* $\lambda \in k$ *such that* $\widetilde{F} = \lambda \cdot F$. *Since both* F *and* $\widetilde{F}$ *are given by homogeneous polynomials of degree* d, *for every* $n \in \mathbb{N}$ *we have that* $\widetilde{F}^n(z) = \lambda^{d^n} \cdot F^n(z)$. *Thus,*

$$\lim_{n\to\infty} \frac{1}{d^n} \log |\widetilde{F}^n(z)| = \log |\lambda| + \lim_{n\to\infty} \frac{1}{d^n} \log |F^n(z)| \ .$$

PROOF. Let us show that the limit $\lim_n G_n$ exists. The inequality $|F(z)| \leq |z|^d$ is clear. Since the polynomials F_i have no common zeros other than the origin, by the homogeneous Nullstellensatz we may find a positive integer s such that the homogeneous polynomial $T_i^s \in k[T_0, \dots, T_N]$ belongs to the ideal generated by $F_0, \dots, F_N$ for every $i = 0, \dots, N$. That is, for every i there are homogeneous polynomials $\lambda_j^i \in k[T_0, \dots, T_N]$ such that $T_i^s = \sum_{j=0}^N \lambda_j^i F_j$. For any $z \in \mathbb{A}^{N+1,\mathrm{an}}$, we have:

$$|z|^s = \max_{0\leq i\leq N} |z_i|^s \leq \max_{0\leq i,j\leq N} |\lambda_j^i(z) F_j(z)| \leq \max_{0\leq i,j\leq N} C|z|^{s-d} \cdot \max_{0\leq j\leq N} |F_j(z)|$$

for some positive constant C depending only on the polynomials λ_j^i. Hence, for all z we have that

$$C \cdot |z|^d \leq |F(z)| \leq |z|^d, \tag{4.1}$$

and so

$$C \cdot |F^n(z)|^d \leq |F^{n+1}(z)| \leq |F^n(z)|^d.$$

Set $C_1 = |\log C|$. Taking logarithms, one obtains

$$|G_{n+1} - G_n| \leq \frac{C_1}{d^n}. \tag{4.2}$$

By the ultrametric inequality, $|G_{n+j} - G_n| \leq \frac{C_1}{d^n}$ for all $j \geq 0$ and for all n, and so the limit $G_f = \lim_{n\to\infty} G_n$ exists. □

REMARK 4.3.3. *Letting* j *go to infinity in* (4.2), *one obtains the inequality*

$$|G_f - G_n| \leq \frac{C_1}{d^n}. \tag{4.3}$$

THEOREM 4.3.4 ([**KS07**]). i) *The function* G_f *is continuous.*
ii) *For every* $\lambda \in k^\times$ *and for every* $z \in \mathbb{A}^{N+1,\mathrm{an}}$ *we have that*

$$G_f(\lambda \cdot z) = G_f(z) + \log |\lambda| \ .$$

iii) *There exists a positive constant C such that*

$$\sup_{z\in\mathbb{A}^{N+1,\mathrm{an}}} |G_f(z) - \log|z|| \leq C.$$

PROOF. i) The continuity of the Green function follows directly from the fact that G_f is the uniform limit of the functions G_n.

ii) Pick any $\lambda \in k^\times$ and any $z \in \mathbb{A}^{N+1,\mathrm{an}}$. As every polynomial F_i is homogeneous of degree d, we have that $F^n(\lambda \cdot z) = \lambda^{d^n} F(z)$ and the assertion follows.

iii) By (4.1), for any $z \in \mathbb{A}^{N+1,\mathrm{an}}$ we have that $|G_1(z) - \log|z|| \leq C_1$. Thus, (4.3) implies that

$$|G_f(z) - \log|z|| \leq |G_f(z) - G_1(z)| + |G_1(z) - \log|z|| \leq \frac{C_1}{d} + C_1 \ ,$$

concluding the proof. □

REMARK 4.3.5. *It is a direct consequence of the definition that $G_f = G_{f^n}$ for every $n \in \mathbb{N}$.*

4.4. Fatou and Julia sets

4.4.1. Dimension 1. Let us first discuss the one-dimensional situation, both in the complex and in the non-Archimedean setting.

Recall that there are several characterizations of the Fatou and Julia sets of an endomorphism f of $\mathbb{P}^1_{\mathbb{C}}$. The Fatou set $F(f)$ can be defined as the normality locus of the family of the iterates of f, and the Julia set $J(f)$ as its complement. Equivalently, one can set $J(f)$ to be the support of unique measure of maximal entropy, also referred to as the equilibrium measure, see [**Sib99**], or as the closure of the repelling periodic points.

Some of these equivalences have a non-Archimedean counterpart. There is a well-defined notion of the canonical measure of an endomorphism f on $\mathbb{P}^{1,\mathrm{an}}$ (see [**FRL04, FRL06**] and [**BR10**, §10.1]), and so one sets $J(f)$ to be its support and $F(f)$ its complement. Using a similar definition of normality as ours, it can be shown that this notion of Fatou set agrees with the normality locus of the family of the iterates of f [**FKT12**, Theorem 5.4].

One may as well consider the Fatou and Julia sets in restriction to the set of rigid points of $\mathbb{P}^{1,\mathrm{an}}$, see [**Sil07**] for a survey on the topic. However, notice that if f is a map with good reduction, i.e. if the reduction $\tilde{f}$ of f is a selfmap of $\mathbb{P}^1_{\tilde{k}}$ of the same degree as f, then its Julia set contains no rigid points [**Sil07**, Theorem 2.17].

We mention the following two characterizations of the intersections of $J(f)$ and $F(f)$ with $\mathbb{P}^{1,\mathrm{an}}(k)$. It was shown in [**FKT12**, Theorem C] that the intersection of the Fatou set $F(f)$ with the set of rigid

points in $\mathbb{P}^{1,\mathrm{an}}$ agrees with the set of rigid points where the sequence of the iterates f^n is equicontinuous with respect to the chordal metric on $\mathbb{P}^{1,\mathrm{an}}(k)$.

In [**Oku12**, Theorem 2] it is shown that the set of rigid points in the Julia set is precisely the closure of the repelling periodic points under the technical assumption that the Lyapunov exponent of f is strictly positive.

4.4.2. Higher dimensions. The Fatou set of a non-invertible complex endomorphism f of $\mathbb{P}^N_{\mathbb{C}}$ for $N \geq 2$ is defined as the normality locus of the family of the iterates. Its complement is the support of the Green current, which is the unique positive closed $(1,1)$-current that is forward invariant by f, see [**Sib99**, Théorème 1.6.5] for a proof. There are several possible definitions for the Julia set of f, see [**Sib99**, Définition 3.31]. We define the Julia set of f as the complement of the Fatou set.

We now explore the non-Archimedean higher dimensional case. We consider two different Fatou sets of f:

DEFINITION 4.4.1. *The* normal Fatou set $F_{\mathrm{norm}}(f)$ *of an endomorphism* $f : \mathbb{P}^{N,\mathrm{an}} \to \mathbb{P}^{N,\mathrm{an}}$ *of degree at least 2 is the set of all points* $z \in \mathbb{P}^{N,\mathrm{an}}$ *where the family* $\{f^n\}$ *is normal.*

The normal Julia set $J_{\mathrm{norm}}(f)$ *is the complement of* $F_{\mathrm{norm}}(f)$.

DEFINITION 4.4.2. *Let* $\rho : \mathbb{A}^{N+1,\mathrm{an}} \to \mathbb{P}^{N,\mathrm{an}}$ *be the usual map. We define the harmonic Fatou set* $F_{\mathrm{harm}}(f)$ *of* f *as the set of points* $z \in \mathbb{P}^{N,\mathrm{an}}$ *having a neighbourhood* U *such that the Green function* G_f *is strongly pluriharmonic on* $\rho^{-1}(U)$.

The harmonic Julia set $J_{\mathrm{harm}}(f)$ *is the complement of* $F_{\mathrm{harm}}(f)$.

It follows directly from the definitions that both Fatou sets $F_{\mathrm{norm}}(f)$ and $F_{\mathrm{harm}}(f)$ are open and totally invariant.

The set $J_{\mathrm{harm}}(f)$ is always nonempty. Indeed, Chambert-Loir has constructed a natural invariant probability measure μ_f on $\mathbb{P}^{N,\mathrm{an}}$ and shown that its support is contained in the complement of the locus where G_f is strongly pluriharmonic, see [**CL11**, Proposition 2.4.4]. In other words, the support of μ_f is included in the harmonic Julia set of f.

We do not know whether the Fatou set is always non-empty.

EXAMPLE 4.4.3. *Let* $z \in \mathbb{P}^{N,\mathrm{an}}$ *be any rigid fixed point for* f *such that the eigenvalues of its differential* $Df(z)$ *are all of norm at most* 1. *Then, we may find an arbitrarily small open neighbourhood* U *of* z *which is* f*-invariant, i.e. such that* $f(U) \subseteq U$. *After maybe reducing* U, *we may assume that* $U \subset \{z_0 = 1, |z_i| < 2, i = 1, \cdots, N\}$. *We thus*

have:

$$\begin{aligned} G_n &= \frac{1}{d^n}\log|(F_0^n,\cdots,F_N^n)| \\ &= \frac{1}{d^n}\log|F_0^n| + \frac{1}{d^n}\log\max_{1\leq i\leq N}\left|\frac{F_i^n}{F_0^n}\right| . \end{aligned}$$

The second term converges uniformly to 0. *On the open set* $\rho^{-1}(U)$, *the function* G_f *is thus the uniform limit of the sequence* $\frac{1}{d^n}\log|F_0^n|$, *hence strongly pluriharmonic. Hence* z *belongs to the harmonic Fatou set.*

In dimension 1, it follows from the Woods Hole formula that any rational map admits at least one indifferent fixed point p, i.e. such that $|f'(p)| = 1$. We observe that the same result holds for any polynomial map $f : \mathbb{A}^{2,\mathrm{an}} \to \mathbb{A}^{2,\mathrm{an}}$ that extends to an endomorphism of $\mathbb{P}^{2,\mathrm{an}}$ so that $F_{\mathrm{harm}}(f) \neq \emptyset$ in this case.

In [**KS09**], the authors define the Fatou set of an endomorphism of the N-th projective space $\mathbb{P}_k^N$ as the equicontinuity locus of the family of iterates, which they prove to be the same as the locus where it is locally uniformly Lipschitz. However, the definition of the Fatou set in terms of equicontinuity presents some difficulties already in dimension one. Indeed, let k be a field of characteristic $p > 0$ and consider the polynomial $f(z) = pz^2 + cz$, with $|c| = 1$. Then, the family of the iterates f^n is normal at the Gauss point, but it is not equicontinuous at x_g, see [**BR10**, Example 10.53]

4.4.3. Comparison between F_{norm} and F_{harm}. We expect our two notions of Fatou sets to coincide.

Conjecture 6. *For every non-invertible endomorphism* f *of the projective space, we have that* $F_{\mathrm{norm}}(f) = F_{\mathrm{harm}}(f)$.

In dimension 1, the equality follows from [**FKT12**, Theorem 5.4], and we are able to prove one inclusion in general. Our argument relies on the following result which gives a characterization of $F_{\mathrm{harm}}(f)$ in terms of a sort of equicontinuity property for the iterates of f. Its proof follows its complex counterpart.

Recall that the norm of an analytic map selfmap $F = (F_0, \ldots, F_N)$ of $\mathbb{A}^{N+1,\mathrm{an}}$ at a point $z \in \mathbb{A}^{N+1,\mathrm{an}}$ is $|F(z)| = \max_{0\leq i\leq N}|F_i(z)|$.

Proposition 4.4.4. *Let* $f : \mathbb{P}^{N,\mathrm{an}} \to \mathbb{P}^{N,\mathrm{an}}$ *be an endomorphism of degree* $d \geq 2$ *and* U *a basic tube in* $\mathbb{P}^{N,\mathrm{an}}$.

The Green function G_f *is strongly pluriharmonic on the open set* $\rho^{-1}(U) \subset \mathbb{A}^{N+1,\mathrm{an}} \setminus \{0\}$ *if and only if for every* $n \in \mathbb{N}$ *there exists a lift* F_n *of* f^n *on* U *and a positive constant* C_1 *such that* $e^{-C_1} \leq |F_n| \leq e^{C_1}$ *on* $\rho^{-1}(U)$ *for all* $n \in \mathbb{N}$.

This result together with Theorem A implies the following:

COROLLARY 4.4.5. *The harmonic Fatou set $F_{\mathrm{harm}}(f)$ is contained in $F_{\mathrm{normal}}(f)$.*

PROOF OF PROPOSITION 4.4.4. Pick any lift $F = (F_0, \cdots, F_N)$ of f, where $F_i \in k[T_0, \cdots, T_N]$ are homogeneous polynomials of degree d without nontrivial common zeros. We may assume that $\sup_{\mathbb{D}} |F(z)| = 1$. Recall from (4.3) that there exists a positive constant C_1 such that $|G_f - G_n| \leq \frac{C_1}{d^n}$ for all $n \in \mathbb{N}$.

Let U be a basic tube on which G_f is strongly pluriharmonic. Let $h_n \in \mathcal{O}^{\times}_{\mathbb{A}^{N+1}}(U)$ and let b_n be non-zero real numbers such that G_f is the uniform limit of the sequence $b_n \cdot \log|h_n|$. After maybe extracting a subsequence and renumbering it, we may assume that

$$|G_f - b_n \cdot \log|h_n|| \leq \frac{C_1}{d^n} \quad \forall n \gg 0$$

on U. Thus, we have

$$\begin{aligned} \left|\frac{1}{d^n}\log|F^n| - b_n \cdot \log|h_n|\right| &= \left|\frac{1}{d^n}\log\left(\frac{|F^n|}{|h_n|^{b_n \cdot d^n}}\right)\right| \\ &\leq \max\left\{|G_f - b_n \cdot \log|h_n||, |G_f - G_n|\right\} \\ &\leq \frac{C_1}{d^n}\ . \end{aligned}$$

So we see that for $n \gg 0$

$$e^{-C_1} \leq \frac{|F^n|}{|h_n|^{b_n \cdot d^n}} \leq e^{C_1}. \tag{4.4}$$

Since the functions h_n have no zeros on U, each $F_n := \frac{F^n}{h_n^{b_n \cdot d^n}}$ is a lift of f^n.

Assume conversely that on U, for every $n \in \mathbb{N}$ there exists a lift F_n of f^n such that $e^{-C_1} \leq |F_n| \leq e^{C_1}$ for some positive constant C_1. Then, for every $n \in \mathbb{N}$ we may choose a non-vanishing function h_n on U such that $F^n = h_n \cdot F_n$. It follows that

$$G_n = \frac{1}{d^n}\log|F^n| = \frac{1}{d^n}\log|h_n| + \frac{1}{d^n}\log|F_n|\ .$$

The second term converges uniformly to 0. On the open set $\rho^{-1}(U)$, the function G_f is thus the uniform limit of the sequence $\frac{1}{d^n}\log|h_n|$, hence strongly pluriharmonic. □

REMARK 4.4.6. *For any polynomial map $f : \mathbb{A}^{2,\mathrm{an}} \to \mathbb{A}^{2,\mathrm{an}}$ of degree $d \geq 2$ that extends to an endomorphism $\bar{f}$ of $\mathbb{P}^{2,\mathrm{an}}$, one has $F_{\mathrm{harm}}(\bar{f}) = F_{\mathrm{normal}}(\bar{f})$.*

4.5. Hyperbolicity of the Fatou components

4.5.1. The complex case: tautness and hyperbolic embeddings. Recall that $\mathrm{Mor}_k(X, Y)$ denotes the set of analytic maps from X to Y.

For a non-necessarily compact target, we set:

DEFINITION 4.5.1. *Let Ω be a relatively compact subset of an analytic space Y and let U be a basic tube.*

The family $\mathrm{Mor}_k(U,\Omega)$ *is said to be* normal *if for every sequence of analytic maps $\{f_n\} \subset \mathrm{Mor}_k(U,\Omega)$ there exists a subsequence f_{n_j} that is pointwise converging to a continuous map $f : U \to Y$.*

In order to motivate our definition of normal families of maps with non-compact target, let us recall the complex setting.

In the complex setting, our definition of normality for a non-compact target corresponds to the family $\mathrm{Hol}(U,\Omega)$ being relatively compact in $\mathrm{Hol}(U,Y)$. The complex definition of normality for a non-compact target is slightly different, since it allows for a sequence to be compactly divergent [**Kob98**, §I.3].

Recall that a relatively compact complex subspace X of a complex space Y is tautly imbedded if every sequence of holomorphic maps $\{f_n\}$ on the open unit disk with values in X satisfies one of the following conditions:

(1) There exists a subsequence $\{f_{n_j}\}$ converging to a holomorphic map $f : \mathbb{D} \to Y$;
(2) For every compact subset K of $\mathbb{D}$ and every compact subset L of Y, there exists a integer $N \in \mathbb{N}$ such that $f_n(\mathrm{K}) \cap \mathrm{L} = \emptyset$ for every $n \geq N$.

A complex compact space is taut if it is tautly imbedded in itself. It is shown in [**Wu67**] that every taut domain in $\mathbb{C}^n$ is pseudoconvex.

In [**KK73, Kie73**], it is shown that X is tautly imbedded in Y if and only if X is hyperbolically imbedded in Y.

4.5.2. Proof of Theorem D. Let Ω be a connected component of the harmonic Fatou set $F_{\mathrm{harm}}(f)$ of an endomorphism $f : \mathbb{P}^{N,\mathrm{an}} \to \mathbb{P}^{N,\mathrm{an}}$ of degree at least 2. Let U be any connected open subset of $\mathbb{P}^{1,\mathrm{an}}$. Our aim is to show that the family $\mathrm{Mor}_k(U,\Omega)$ is normal.

The projective space $\mathbb{P}^{N,\mathrm{an}}$ can be covered by $N+1$ charts $V_0,\dots,V_N$ analytically isomorphic to $\bar{\mathbb{D}}^N$. For every $i = 0,\cdots,N$, let $s_i : \{z \in \mathbb{P}^{N,\mathrm{an}} : z_i \neq 0\} \to \mathbb{A}^{N+1,\mathrm{an}}$ be the analytic local section of ρ sending the point $z = [z_0 : \ldots : z_N]$ to $(\frac{z_0}{z_i},\dots,\frac{z_{i-1}}{z_i},1,\frac{z_{i+1}}{z_i},\dots,\frac{z_N}{z_i})$.

Let $g : U \to \Omega$ be an analytic map. We claim that for any compact subset $\mathrm{K} \subset U$ the map $g_{|\mathrm{K}}$ admits a lift to $\rho^{-1}(\Omega)$.

Suppose first that U is not the whole $\mathbb{P}^{1,\mathrm{an}}$. By Proposition 4.2.1, there exists a sequence of basic tubes W_m exhausting U and a sequence of affinoid subspaces X_m satisfying

$$\overline{W}_m \subset X_m \subset U \ .$$

Pick any compact subset $\mathrm{K} \subset U$. For m sufficiently large, K is contained in some X_m. Fix $m \in \mathbb{N}^*$. Cover X_m by sets $U_i^{(m)} =$

$g^{-1}(V_i)\cap X_m$ with $0\leq i\leq N$. On every $U_{ij}^{(m)}=g^{-1}(V_i)\cap g^{-1}(V_j)\cap X_m$, we know that $\rho\circ s_i\circ g=\rho\circ s_j\circ g$, and thus $s_i\circ g=\varphi_{ij}^{(m)}\cdot(s_j\circ g)$ for some $\varphi_{ij}^{(m)}\in\mathcal{O}^\times(U_{ij}^{(m)})$. Since X_m is an affinoid subspace of $\mathbb{P}^{1,\mathrm{an}}$ we have that $H^1(X_m,\mathcal{O}^\times)=0$ by [**Put80**]. We may thus find $\varphi_i\in\mathcal{O}^\times(U_i^{(m)})$ and $\varphi_j\in\mathcal{O}^\times(U_j^{(m)})$ such that $\varphi_{ij}^{(m)}=\frac{\varphi_i^{(m)}}{\varphi_j^{(m)}}$. On X_m, consider the following local lifts of g:

$$\widehat{g_i}^m:U_i^{(m)}\to\rho^{-1}(\Omega),\quad \widehat{g_i}^m=\frac{s_i\circ g}{\varphi_i^{(m)}}.$$

It follows that $\widehat{g_i}^m=\widehat{g_j}^m$ on $U_{ij}^{(m)}$, and hence we have a lift $\widehat{g}^m:X_m\to\rho^{-1}(\Omega)$ of g as required.

By definition of the harmonic Fatou set, the Green function G_f of f is strongly pluriharmonic on $\rho^{-1}(\Omega)$, and thus $G_f\circ\widehat{g}^m$ is harmonic on X_m.

Let $g_n:U\to\Omega$ be a sequence of analytic maps. For every X_m consider the lifts $\widehat{g_n}^m:X_m\to\rho^{-1}(\Omega)$ of the restriction of g_n to X_m constructed above.

Fix a sufficiently large real number $C>0$ and consider the set $\mathrm{M}=\{z\in\mathbb{A}^{N+1,\mathrm{an}}\setminus\{0\}:\frac{1}{C}\leq|G_f(z)|\leq C\}$. By Theorem 4.3.4, the set M is compact. By Proposition 4.2.2, for every n and every m there exists an analytic map $h_n^m:W_m\to\mathbb{A}^{1,\mathrm{an}}\setminus\{0\}$ such that

$$\sup_{W_m}\left|G_f\circ\widehat{g_n}^m-\log|h_n^m|\right|\leq C.$$

We set $\widetilde{g_n}^m=\frac{\widehat{g_n}^m}{h_n^m}$. Each $\widetilde{g_n}^m:W_m\to\rho^{-1}(\Omega)$ is a lift of g_n and its image lies in the compact M. By Theorem A, there exists a subsequence of $\widetilde{g_n}^m$ converging pointwise to a continuous map. By a diagonal extraction argument, we conclude that the family $\mathrm{Mor}_k(U,\Omega)$ is normal.

The case $U=\mathbb{P}^{1,\mathrm{an}}$ follows by writing $\mathbb{P}^{1,\mathrm{an}}$ as a finite union open disks. □

4.6. Curves in Fatou sets

The aim of this section is to prove Theorem E, which states that harmonic Fatou components do not contain any non trivial image of the punctured affine line.

4.6.1. No algebraic curves in the Fatou set. We briefly observe the following fact that follows almost directly from the work of Chambert-Loir.

Proposition 4.6.1. *Suppose that C is an algebraic curve in $\mathbb{P}^{N,\mathrm{an}}$, and let $f:\mathbb{P}^{N,\mathrm{an}}\to\mathbb{P}^{N,\mathrm{an}}$ be any endomorphism of degree at least 2. Then the harmonic Fatou set of f cannot contain a Zariski open subset of C.*

In particular, a Fatou component contains no complete algebraic curve. This supports the conjectural fact that any Fatou component should be Stein (in the sense of [**Kie67**]). Over the complex numbers, this result is proved in [**Ued94, FS95, Mae04**], but the proof relies on pluripotential techniques which are not available at the moment over a non-Archimedean field.

Proof of Proposition 4.6.1. Since the result is not central to our studies, we shall only give a sketch of proof, which relies on special metrizations of line bundles. We refer to [**CL11**, §2] for a detailed exposition of these notions. Choose a homogeneous lift $F = (F_0, \cdots, F_N)$ of f to $\mathbb{A}_k^{N+1,\mathrm{an}} \setminus \{0\}$, and consider the associated Green function $G_f = \lim_n \frac{1}{d^n} \log |F^n|$. The function G_f induces a continuous and semi-positive metrization $|\cdot|_F$ in the sense of Zhang on the tautological line-bundle $\mathcal{O}(1)$ on $\mathbb{P}^{N,\mathrm{an}}$, see [**CL11**, §2.1].

Pick any algebraic curve C in $\mathbb{P}^{N,\mathrm{an}}$. The restriction of the metrized line bundle $(\mathcal{O}(1), |\cdot|_F)$ to C is again continuous and semi-positive. We may thus consider its curvature, see [**Thu05**, Proposition 4.2.3]. It is a positive measure μ_C on the Berkovich analytification of C of mass $\deg_C(\mathcal{O}(1))$ which does not charge any rigid point, see [**Thu05**, §4.2.1]. The support of μ_C is contained in $J_{\mathrm{harm}}(f)$, which implies the result. □

4.6.2. No image of $\mathbb{A}^{1,\mathrm{an}} \setminus \{0\}$ in the harmonic Fatou set. Our main result is:

Proposition 4.6.2. *Let Ω be an open subset of $\mathbb{P}^{N,\mathrm{an}}$.*

If the family of analytic maps $\mathrm{Mor}_k(\mathbb{A}^{1,\mathrm{an}} \setminus \{0\}, \Omega)$ is normal, then every analytic map $\mathbb{A}^{1,\mathrm{an}} \setminus \{0\} \to \Omega$ is constant.

As a direct application, we obtain:

Proof of Theorem E. It follows from Theorem D and Proposition 4.6.2. □

Example 4.6.3. *Complex Hénon maps constitute an important class of automorphisms of the complex affine plane $\mathbb{A}^2_{\mathbb{C}}$. These exhibit very different dynamical properties from endomorphisms of the projective space. For instance, consider the map Hénon map*

$$f : (z, w) \mapsto (z^2 - aw, z) \ , \quad |a| < 1 \ .$$

The point $(0, 0)$ is an attracting fixed point for f, and thus lies in the Fatou set, i.e. the set where the iterates of f form locally a normal family. However, its basin of attraction is biholomorphic to $\mathbb{C}^2$, see e.g. [**FsS95, Ued08**]. *The proof relies on the fact that f is an automorphism and on the existence of a local analytic diffeomorphism ϕ defined in a neighbourhood of the origin conjugating f to a linear contracting map, see* [**Ste57**]. *Such a domain is usually referred to as a Fatou-Bieberbach domain.*

An analogous construction gives rise to the same phenomenon over a non-Archimedean complete field k of any characteristic. Suppose f is a Hénon map as above with coefficient $|a| < 1$. We may then define the Fatou set as the normality locus (in the sense of §4.5) of the iterates of f. The basin of attraction of $(0,0)$ is then a component of the Fatou set of f and the same proof as in the complex case shows that it is analytically isomorphic to $\mathbb{A}^2_k$.

As a first step in proving Proposition 4.6.2, we deal with a simpler particular case, that of entire curves.

Proof of the particular case of entire curves. Let Ω be any open subset of $\mathbb{P}^{N,\mathrm{an}}$ and assume that the family $\mathrm{Mor}_k(\mathbb{A}^{1,\mathrm{an}}, \Omega)$ is normal. Suppose that there exists a non-constant analytic map $g : \mathbb{A}^{1,\mathrm{an}} \to \Omega$. Consider the sequence of analytic maps from $\mathbb{A}^{1,\mathrm{an}}$ into Ω given by $f_n(z) = g(z^n)$. By normality there is a subsequence $\{f_{n_j}\}$ that is pointwise converging to a continuous map $f : \mathbb{A}^{1,\mathrm{an}} \to \mathbb{P}^{N,\mathrm{an}}$.

The Gauss point x_g is fixed by all the maps $z \mapsto z^n$, and so $f(x_g) = g(x_g)$. For every integer $m > 0$ let $z_m = \eta_{0,1-\frac{1}{m}} \in \mathbb{A}^{1,\mathrm{an}}$. Since every z_m lies in the open unit disk $\mathbb{D}$, we have that

$$f(z_m) = \lim_{n_j \to \infty} f_{n_j}(z_m) = \lim_{n_j \to \infty} g\left((z_m)^{n_j}\right) = g(0)$$

for all m. The continuity of f implies that the $f(z_m)$ tend to $f(x_g)$ as m goes to infinity. It follows that $g(x_g) = g(0)$ is a rigid point of Ω. As the source $\mathbb{A}^{1,\mathrm{an}}$ is one-dimensional, g must be constant. □

In order to prove Proposition 4.6.2, we need to recall some basic topological facts. Recall from §4.2 that given a point $x \in \mathbb{P}^{1,\mathrm{an}}$, we denote by $U(\vec{v})$ the connected component of $\mathbb{P}^{1,\mathrm{an}} \setminus \{x\}$ corresponding to the tangent direction $\vec{v} \in T_x\mathbb{P}^{1,\mathrm{an}}$.

Let $g : U \subseteq \mathbb{P}^{1,\mathrm{an}} \to \mathbb{P}^{1,\mathrm{an}}$ be a non-constant analytic map. For every point $x \in U$, the map g induces a tangent map $dg(x)$ between T_xU and $T_{g(x)}\mathbb{P}^{1,\mathrm{an}}$. Let $\vec{v}$ be a tangent direction at x that is mapped to $\vec{v}' \in T_{g(x)}\mathbb{P}^{1,\mathrm{an}}$ by $dg(x)$. Then either $g(U(\vec{v})) = U(\vec{v}')$ or $g(U(\vec{v})) = \mathbb{P}^{1,\mathrm{an}}$. This follows from the fact that the map g is open [**BR10**, Corollary 9.10].

Of special interest for us is the case when x is a type II point. Assume for simplicity that both x and $g(x)$ are the Gauss point. The space $T_{x_g}\mathbb{P}^{1,\mathrm{an}}$ is isomorphic to $\mathbb{P}^1_{\tilde{k}}$, and the tangent map $dg(x) : \mathbb{P}^1_{\tilde{k}} \to \mathbb{P}^1_{\tilde{k}}$ and can be described as follows. In homogeneous coordinates g can be written as $g = [G_0 : G_1]$ with $G_0, G_1 \in \mathcal{O}(\mathbb{A}^{1,\mathrm{an}})$ without common zeros by [**FvdP04**, Theorem 2.7.6], where all the coefficients of G_0 and G_1 are of norm less or equal than one and least one has norm one. Thus, we may consider the reduction map of g, which is a non-constant rational map from $\mathbb{P}^1_{\tilde{k}}$ to itself, and hence surjective. One can show that $dg(x)$ is given by the reduction of g [**BR10**, Corollary 9.25].

PROOF OF PROPOSITION 4.6.2. Suppose that $\mathrm{Mor}_k(\mathbb{A}^{1,\mathrm{an}} \setminus \{0\}, \Omega)$ is normal. We first deal with the case where Ω is contained in $\mathbb{P}^{1,\mathrm{an}}$. Let $g : \mathbb{A}^{1,\mathrm{an}} \setminus \{0\} \to \mathbb{P}^{1,\mathrm{an}}$ be a non-constant analytic map. We may assume that it is of the form $g = [G_0 : G_1]$ with $G_i : \mathbb{A}^{1,\mathrm{an}} \setminus \{0\} \to \mathbb{A}^{1,\mathrm{an}}$ analytic without common zeros by [**FvdP04**, Theorem 2.7.6]. Our goal is to construct a sequence of analytic maps from $\mathbb{A}^{1,\mathrm{an}} \setminus \{0\}$ to itself such that the composition with g gives a sequence $g_n : \mathbb{A}^{1,\mathrm{an}} \setminus \{0\} \to \Omega$ that admits no converging subsequence with continuous limit.

Suppose first that there exists a type II point in $\mathbb{P}^{1,\mathrm{an}}$ having infinitely many preimages in the segment $T = \{\eta_{0,r} \in \mathbb{A}^{1,\mathrm{an}} : 0 < r < \infty\}$. Composing with an automorphism of $\mathbb{P}^{1,\mathrm{an}}$, we may assume that this point is the Gauss point. Let thus $\{\eta_{0,r_n}\}$ be a sequence of preimages of x_g.

Denote by V_n the compact set containing η_{0,r_n} consisting of $\mathbb{A}^{1,\mathrm{an}} \setminus \{0\}$ minus the open sets $U(\vec{v}_0)$ and $U(\vec{v}_\infty)$, where $\vec{v}_0$ and $\vec{v}_\infty$ are the tangent directions at η_{0,r_n} pointing at 0 and at infinity respectively. As $dg(\eta_{0,r_n})$ is surjective, we deduce that $g(V_n)$ avoids at most two tangent directions at x_g. After maybe extracting a subsequence, we may find a connected component B of $\mathbb{P}^{1,\mathrm{an}} \setminus \{x_g\}$ that is contained in $g(V_n)$ for all $n \gg 0$. As a consequence, we may pick a rigid point a_0 in B and rigid points $x_n \in V_n$ such that $g(x_n) = a_0$ for every $n \in \mathbb{N}$.

Consider the sequence in $\mathrm{Mor}_k(\mathbb{A}^{1,\mathrm{an}} \setminus \{0\}, \mathbb{P}^{1,\mathrm{an}})$ given by $g_n(z) = g(x_{n!} z^{n!})$. By normality, we may assume that g_n converges to a continuous map g_∞. The Gauss point x_g is fixed by g_∞, as $g_n(x_g) = x_g$ for all $n \in \mathbb{N}$. For every fixed $n \in \mathbb{N}$ and every $m \leq n$, the map g_n sends the set of all the m-th roots of unity R_m to a_0, and so g_∞ maps every R_m to a_0. For every $m \in \mathbb{N}$ pick a point $\zeta_m \in R_m$ such that $\zeta_m \to x_g$ as m tends to infinity. We have that

$$g_\infty(x_g) = \lim_{m \to \infty} g_\infty(\zeta_m) = a_0 \ ,$$

contradicting the continuity.

Suppose next that every type II point in $\mathbb{P}^{1,\mathrm{an}}$ has at most finitely many preimages in the segment T. Pick a sequence of type II points $\{\eta_{0,r_n}\}$ with $r_n \to +\infty$ as n goes to infinity. By compactness, we may assume that the points $g(\eta_{0,r_n})$ converge to some point $y_\infty \in \mathbb{P}^{1,\mathrm{an}}$. We claim that the points $g(\eta_{0,r})$ converge to a point y_∞ as r tends to infinity. To see this, fix a basic tube V containing y_∞. Recall that $\partial_{\mathrm{top}} V$ is a finite set of type II points. By assumption, $g(\eta_{0,r})$ does not belong to $\partial_{\mathrm{top}} V$ for sufficiently large r. For $n \gg 0$ we have that $g(\eta_{0,r_n})$ lies in V. Thus, $g(\eta_{0,r})$ must belong to V for $r \gg 0$.

Pick any $r \in \mathbb{R}_+$ and consider the tangent direction $\vec{v}$ at $\eta_{0,r}$ pointing towards infinity. We may assume that $g(U(\vec{v}))$ avoids at most one rigid point in $\mathbb{P}^{1,\mathrm{an}}$, as otherwise Picard's Big theorem [**CR04**] asserts that g admits an analytic extension at infinity and we conclude by the case of

entire curves. After maybe varying the r_n, we may find a rigid point $a_0 \in \mathbb{P}^{1,\mathrm{an}}$ and rigid points x_n with $|x_n| = r_n$ such that $g(x_n) = a_0$ for all n.

Consider the sequence $g_n(z) = g(x_{n!}z^{n!})$ and assume that it admits a continuous limit g_∞. Our previous argument shows that g_∞ maps every set R_m to a_0. The points $g_n(x_g)$ converge to y_∞ by our claim, and hence g_∞ is not continuous.

Assume now that Ω is an open subset of $\mathbb{P}^{N,\mathrm{an}}$. Let $g : \mathbb{A}^{1,\mathrm{an}} \setminus \{0\} \to \Omega$ be a non-constant analytic map. This map can be written in homogeneous coordinates as $g = [G_0 : \ldots : G_N]$, with $G_i \in \mathcal{O}^\times(\mathbb{A}^{1,\mathrm{an}} \setminus \{0\})$. As g is not constant we may assume that G_0 is non-constant and that G_1 is not a scalar multiple of G_0. We may assume by [**FvdP04**, Theorem 2.7.6] that G_0 and G_1 have no common zeros. As a consequence, the map defined on the image of g by

$$\pi : [G_0(z) : \ldots : G_N(z)] \mapsto [G_0(z) : G_1(z)]$$

is well-defined and analytic. By construction $\pi \circ g$ is non-constant and analytic. By the previous case we may find $x_n \in k^\times$ such that no subsequence of $\{\pi \circ g(x_{n!}z^{n!})\}$ has a continuous limit, and thus neither $\{g(x_{n!}z^{n!})\}$. □

CHAPTER 5

Cherry hyperbolicity

In [**Che96**], the author translates the definition of Kobayashi chains on complex spaces [**Kob67**] to the set of rigid points of a Berkovich space X, which gives rise to a semi distance d_{CK} on $X(k)$.

In this chapter, we continue the study initiated by W. Cherry of this semi distance. We give a non-Archimedean version of Zalcman's reparametrization lemma [**Zal75**], which we apply to characterize smooth projective varieties defined over a field of residue characteristic zero for which d_{CK} is an actual distance, with special interest in the case of curves.

5.1. Cherry's notion of hyperbolicity

Recall the definition of the Cherry-Kobayashi semi distance [**Che96**]:

Definition 5.1.1. *Let x, y be rigid points in a k-analytic space X. A Kobayashi chain joining x and y is a finite set of analytic maps $f_l : \bar{\mathbb{D}} \to X$ and points $z_l, w_l \in \bar{\mathbb{D}}(k)$, $l = 1, \cdots, m$ such that $f_1(z_1) = x$, $f_l(w_l) = f_{l+1}(z_{l+1})$ for $l = 1, \cdots, m-1$ and $f_m(w_m) = y$. The* Cherry-Kobayashi semi distance on X *is defined by*

$$d_{\mathrm{CK}}(x, y) = \inf \sum_{l=1}^{m} |w_l - z_l| \ ,$$

where the infimum is taken over all Kobayashi chains joining x and y. If there is no Kobayashi chain joining x and y, we set $d_{\mathrm{CK}}(x, y) = \infty$.

Observe that the group of analytic automorphisms of $\bar{\mathbb{D}}$ is the set of series of the form $\sum_{n\geq 0} a_n T^n$ such that $|a_1| = 1$ and such that $\max_n |a_n| \leq 1$, which are isometries for the distance $|.|$. Thus, d_{CK} is invariant under automorphisms of $\bar{\mathbb{D}}$. Up to composition by such an automorphism we may suppose that $z_l = 0$, for all l.

On the closed disk $\bar{\mathbb{D}}$, the Cherry-Kobayashi semi distance agrees with the distance induced by the norm on k.

Remark 5.1.2. *Not every point pair of points in an analytic space X can be joined by a Kobayashi chain. Assume for instance that X has dimension at least two and take a point $x \in X$ such that the transcendance degree of $\widetilde{\mathcal{H}(x)}$ over $\tilde{k}$ greater than one. Then, no analytic map $\mathbb{D} \to X$ avoids the point x.*

DEFINITION 5.1.3. *A k-analytic space X is* Cherry hyperbolic *if d_{CK} is an actual distance on $X(k)$, which might take the value ∞.*

As $\mathbb{A}^{1,\mathrm{an}}$ can be written as a union of disks of whose radii tend to infinity, we see that d_{CK} is exactly zero on $\mathbb{A}^{1,\mathrm{an}}$, just like over $\mathbb{C}$.

The semi distance d_{CK} shares an important property with its complex counterpart: analytic maps are distance decreasing with respect to d_{CK}. As a consequence, the fact that d_{CK} is 0 on the whole $\mathbb{A}^{1,\mathrm{an}}$ implies that Cherry hyperbolicity is stronger than Brody hyperbolicity, i.e. the non-existence of entire curves.

Recall from §1.4 that the skeleton $\mathsf{S}^{\mathrm{an}}(X)$ of a curve X is the set of points not having a neighbourhood that is isomorphic to an open disk. Let X be an elliptic curve and pick any two distinct rigid points $x, y \in X$. If x and y belong to the same connected component of $X \setminus \mathsf{S}^{\mathrm{an}}(X)$, then $d_{\mathrm{CK}}(x, y)$ agrees with the distance on the disk, and hence $d_{\mathrm{CK}}(x, y) \neq 0$. Otherwise, they cannot be joined by a Kobayashi chain and thus $d_{\mathrm{CK}}(x, y) = \infty$. Thus, elliptic curves are Cherry hyperbolic. The same argument shows that a projective curve X is Cherry-hyperbolic if and only if it has strictly positive genus.

In a series of papers [**Che93, Che96, ACW08, Che94**], Cherry studied in detail the behaviour of d_{CK} and the existence of entire curves in the case of abelian varieties and of projective curves, making extensive use of the reduction theory available for these varieties. His results contrast with the complex case.

THEOREM 5.1.4. *Let X be an abelian variety.*

1. *Every analytic map $\mathbb{A}^{1,\mathrm{an}} \to X$ is constant.*
2. *The variety X is Cherry hyperbolic.*

We refer to [**Che96**] for further details on the Cherry-Kobayashi semi distance.

5.2. Alternate definition of the Kobayashi semi distance

Let us briefly comment on the following alternate definition of an analogue of the Kobayashi semi distance on the set of rigid points of a k-analytic space X, which is very natural. In the same notation as above, for any $x, y \in X(k)$ we set

$$d(x, y) = \inf \max_{1 \leq l \leq m} |w_l - z_l| \ ,$$

where the infimum is taken over all Kobayashi chains joining x and y. As before, if there is no Kobayashi chain joining x and y we set $d_{\mathrm{CK}}(x, y) = \infty$. The obtained semi distance d satisfies the ultrametric inequality.

REMARK 5.2.1. *Let X be a smooth analytic curve, and pick any two rigid points x, y. Observe that there exists a Kobayashi chain joining x and y if and only if they belong to the same connected component of $X \setminus \mathsf{S}^{\mathrm{an}}(X)$. In this case, the chain consists of a single analytic map $f : \bar{\mathbb{D}} \to X$, and as a consequence we have that $d = d_{\mathrm{CK}}$.*

Clearly, $d \leq d_{\mathrm{CK}}$ for any analytic space $X(k)$. In general, these semidistances are not equivalent, as shown in the following example.

Indeed, pick a closed unit disk and take two rigid points $x, y \in \bar{\mathbb{D}}$ with $|x - y| = 1$. Attach to $\bar{\mathbb{D}}$ two irreducible components X_1 and Y isomorphic to $\bar{\mathbb{D}}$, one passing through x and the other through y. For every integer $n \geq 3$, take rigid points $x_1^{(n)} \in X_1$, $y_n \in Y$ such that $|x - x_1^{(n)}| = |y - y_n| = \frac{1}{n}$. Attach a closed disk to X_1 passing through $x_1^{(n)}$. Denote this new irreducible component by $X_2^{(n)}$ and pick a rigid point $x_2^{(n)} \in X_2^{(n)}$ with $|x_1^{(n)} - x_2^{(n)}| = \frac{1}{n}$. Repeat this procedure as to obtain irreducible components $X_l^{(n)} \simeq \bar{\mathbb{D}}$ and rigid points $x_l^{(n)} \in X_l^{(n)}$ for $1 \leq l \leq n$ with $X_n^{(n)} = Y$, $x_{n-1}^{(n)} = y_n$ and $x_n^{(n)} = y$. Denote by $x_0^{(n)} := x$ for every $n \geq 3$. Observe that $|x_l^{(n)} - x_{l-1}^{(n)}| = \frac{1}{n}$ for every $l = 1, \ldots, n$.

For every n, the points $x = x_0^{(n)}, x_1^{(n)}, \ldots, x_n^{(n)} = y$ form a Kobayashi chain joining x and y. We see that

$$d(x, y) = \inf_{n \geq 3} |x_1^{(n)} - x_0^{(n)}| = \inf_{n \geq 3} \frac{1}{n} = 0 \ ,$$

whereas

$$d_{\mathrm{CK}}(x, y) = \inf_{n \geq 3} \sum_{l=1}^{n} |x_l^{(n)} - x_{l-1}^{(n)}| = \inf_{n \geq 3} n \cdot \frac{1}{n} = 1 \ .$$

In the sequel, we shall build on W. Cherry's work and only consider the semi distance d_{CK}.

5.3. Zalcman's reparametrization lemma

We give a non-Archimedean version of Zalcman's lemma [**Zal75**]. After renormalizing, we obtain a sequence of analytic maps whose Fubini-Study derivative at 0 does not vanish. However, and contrary to the complex case, this does not assure that this sequence is converging.

5.3.1. Fubini-Study derivative. We fix once and for all homogeneous coordinates on $\mathbb{P}^{N,\mathrm{an}}$ and $\mathbb{P}^{1,\mathrm{an}}$.

DEFINITION 5.3.1. *Let Ω be any open subset of $\mathbb{A}^{1,\mathrm{an}}$. Consider an analytic map $f : \Omega \to \mathbb{P}^{N,\mathrm{an}}$ and choose coordinates $f = [f_0 : \cdots : f_N]$. The Fubini-Study derivative of f at a point $z \in \Omega$, is*

$$|f'(z)| = \max\{1, |z|^2\} \frac{\max |(f_i' f_j - f_j' f_i)(z)|}{\max |f_i(z)|^2} \ .$$

Observe that, by construction, the function $z \in \Omega \mapsto |f'(z)|$ is continuous.

Next we prove some properties of the Fubini-Study derivative.

LEMMA 5.3.2. *For every analytic map $f : \mathbb{D} \to \mathbb{P}^{N,\mathrm{an}}$ given in homogeneous coordinates by $f = [f_0 : \cdots : f_N]$, we have*

$$|f'(z)| \leq \frac{\max\{|f_i'(z)|\}}{\max\{|f_i(z)|\}} \ .$$

The proof is trivial.

Since rigid points are dense, for any analytic map $f : \Omega \to \mathbb{P}^{N,\mathrm{an}}$ we have:

$$\sup_{z\in\Omega} |f'(z)| = \sup_{z\in\Omega(k)} |f'(z)| \ .$$

A direct computation shows:

LEMMA 5.3.3. *Let Ω, Ω' be open subsets of $\mathbb{A}^{1,\mathrm{an}}$. Consider analytic maps $g : \Omega \to \mathbb{P}^{1,\mathrm{an}}$, and $f : \Omega' \to \mathbb{P}^{N,\mathrm{an}}$ with $g(\Omega) \subseteq \Omega'$. Then, we have*

$$|(f \circ g)'(z)| = |f'(g(z))| \cdot |g'(z)| \ .$$

COROLLARY 5.3.4. *For every $g \in \mathrm{PGL}(2, k^\circ)$ and every analytic map $f : \Omega \to \mathbb{P}^{1,\mathrm{an}}_k$, we have $|(f \circ g)'(z)| = |f'(z)|$.*

PROOF. We have to prove that $|g'(z)| = 1$ for every $g \in \mathrm{PGL}(2, k^\circ)$. A simple calculation shows that it holds for every g of the form $z \mapsto az$, $z \mapsto z + b$ and $z \mapsto 1/z$, with $|a| = 1$ and $|b| \leq 1$. Since $\mathrm{PGL}(2, k^\circ)$ is generated by all the maps of this form, the assertion is proved. □

5.3.2. Diameter function. It will be useful in the sequel to estimate the size of the image of a disk. The required tool is the diameter function.

There are diameter functions on $\mathbb{A}^{1,\mathrm{an}}$ and on $\mathbb{P}^{1,\mathrm{an}}$. Recall from §1.2.2 that any point $x \in \mathbb{A}^{1,\mathrm{an}}$ is uniquely determined by a decreasing sequence of disks $\{\bar{B}(a_i; r_i)\}$ in k. The diameter function is defined as $\mathrm{diam}_{\mathbb{A}}(x) = \lim r_i$. On $\mathbb{P}^{1,\mathrm{an}}$, one sets

$$\mathrm{diam}(x) = \frac{\mathrm{diam}_{\mathbb{A}}(x)}{\max\{1, |x|^2\}} \ .$$

In both cases, a point has zero diameter if and only if it is rigid.

Observe that $\mathrm{diam}_{\mathbb{A}}(x) = \inf_{c\in k} |(T - c)(x)|$. We refer to [**BR10**, §2.7] for further details on the diameter function on $\mathbb{P}^{1,\mathrm{an}}$.

We now extend these definitions to any dimension.

DEFINITION 5.3.5. *For any $x \in \mathbb{A}^{N,\mathrm{an}}$, we set:*

$$\mathrm{diam}_{\mathbb{A}}(x) := \max_{1\leq i\leq N} \inf_{c_i\in k} |(T_i - c_i)(x)| = \max_{1\leq i\leq N} \mathrm{diam}_{\mathbb{A}}\pi_i(x) \ ,$$

where $\pi_i : \mathbb{A}^{N,\mathrm{an}} \to \mathbb{A}^{1,\mathrm{an}}$ is the usual projection to the i-th coordinate.

DEFINITION 5.3.6. *Let $x \in \mathbb{P}^{N,\mathrm{an}}$. Choose an affine chart isomorphic to $\mathbb{A}^{N,\mathrm{an}}$ at x. We may assume $x = [1 : x_1 : \cdots : x_N]$. Write $|x_i| = |T_i(x)|$. We set:*

$$\mathrm{diam}(x) := \frac{\mathrm{diam}_{\mathbb{A}}(x)}{\max\{1, |x_i|^2\}} = \frac{\max_{1\leq i\leq N} \mathrm{diam}_{\mathbb{A}}\pi_i(x)}{\max\{1, |x_i|^2\}} \ .$$

It is clear from the definitions that $\mathrm{diam}_{\mathbb{A}}$ and diam are zero exactly on the rigid points.

LEMMA 5.3.7. *The function* $\mathrm{diam} : \mathbb{P}^{N,\mathrm{an}} \to \mathbb{R}_{\geq 0}$ *is invariant under the action of* $\mathrm{PGL}(N+1, k^\circ)$.

PROOF. The function diam is clearly invariant under translations of the form $T_i \mapsto T_i - a_i$, $a_i \in k^\circ$.

It follows directly from the definition that for $a_1, \ldots, a_N \in k$ with $|a_i| = 1$,

$$\mathrm{diam}(x_1, \cdots, x_N) = \mathrm{diam}(a_1 x_1, \cdots, a_N x_N).$$

Finally, consider maps of the form

$$\varphi : [1 : x_1 : \cdots : x_N] \mapsto [x_i : x_1 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_N].$$

Clearly, we have $\mathrm{diam}(\varphi(x)) = \mathrm{diam}(x)$. All these transformations generate $\mathrm{PGL}(N+1, k^\circ)$. □

LEMMA 5.3.8. *Assume that* $\mathrm{char}(\tilde{k}) = 0$*, and consider an analytic map $f : \mathbb{D} \to \mathbb{P}^{N,\mathrm{an}}$. Then for every $z \in \mathbb{D}$, we have*

$$\mathrm{diam}(f(z)) \leq \mathrm{diam}(z) \cdot |f'(z)| \ .$$

Moreover, for $N = 1$ we have an equality.

REMARK 5.3.9. *The previous lemma does not hold if* $\mathrm{char}(\tilde{k}) = p > 0$*. In fact, there are maps with small Fubini-Study derivative and whose image is arbitrarily big. Take for instance the sequence $f_n : z \mapsto c_n z^{p^n}$, with $|c_n| = (p^n)^{p^n}$. Denote by $\eta_{0,\epsilon}$ the point in $\mathbb{D}$ associated to the closed ball $\bar{B}(0;\epsilon)$. A direct computation shows that*

$$|f_n'(\eta_{0,\epsilon})| = \frac{|c_n|\epsilon^{p^n-1}}{p^n \max\{1, |c_n|\epsilon^{p^n}\}^2} \ .$$

It follows that

$$\sup_n \sup_{z\in\mathbb{D}} |f_n'(z)| = 1 \ .$$

If $\epsilon < p^{-n}$, then the Fubini-Study derivative is $|f_n'(\eta_{0,\epsilon})| = (p^n\epsilon)^{p^n-1}$. Thus, we see that $\mathrm{diam}(f_n(\eta_{0,\epsilon}))$ *cannot be bounded away from 1 uniformly in ϵ and $n \in \mathbb{N}$.*

PROOF OF LEMMA 5.3.8. Let us first consider the case $N = 1$. By continuity, it suffices to consider points of type II and III. More so, we may assume that $z = \eta_{0,r} \in \mathbb{D}$ (i.e. z is associated to the closed ball $\bar{B}(0;r) \subset k$) and that $f(z) = \eta_{0,R}$, since both the diameter function and the Fubini-Study derivative are invariant under the action

of $\mathrm{PGL}(2, k^\circ)$ by Lemmas 5.3.4 and 5.3.7. Let $f(z) = \sum_{i\geq 0} a_i z^i$ be the series development of f. It follows from the definitions that the equality is equivalent to $\mathrm{diam}_{\mathbb{A}}(f(\eta_{0,r})) = \mathrm{diam}(\eta_{0,r}) \cdot |f'(\eta_{0,r})|$. We have

$$\mathrm{diam}_{\mathbb{A}}(f(\eta_{0,r})) = \max_{i\geq 1} |a_i| r^i = r \cdot \max_{i\geq 1} |a_i| \cdot |i| r^{i-1} = r \cdot |f'(\eta_{0,r})| \ ,$$

concluding the proof for $N = 1$.

Consider now the general case. We may choose homogeneous coordinates $[z_0, \cdots, z_N]$ in $\mathbb{P}^{N,\mathrm{an}}$ such that the inverse image under f of the hyperplane $H_\infty = \{z_0 = 0\}$ is a discrete subset of $\mathbb{D}$. Denote by Γ its convex hull in $\mathbb{D}$. It suffices to prove the result for points $z \in \mathbb{D}$ lying outside Γ. On a neighbourhood U of z contained in $\mathbb{D} \setminus \Gamma$, the map f can be expressed as a map $f : U \to \mathbb{A}^{N,\mathrm{an}}$, i.e. $f = [1 : f_1, \cdots : f_N]$.

By the previous case, we know that

$$\begin{aligned} \mathrm{diam}_{\mathbb{A}}(f(z)) &= \max_{1\leq i\leq n} \mathrm{diam}(z) \cdot |(\pi_i \circ f)'(z)| \\ &= \mathrm{diam}(z) \max_{1\leq i\leq n} |f_i'(z)| \ . \end{aligned}$$

By Lemma 5.3.2, we see that

$$\mathrm{diam}(f(z)) = \mathrm{diam}(z) \frac{\max_{1\leq i\leq n} |f_i'(z)|}{\max\{1, |f_i(z)|^2\}} \leq \mathrm{diam}(z) \cdot |f'(z)| \ ,$$

proving the assertion. □

5.3.3. Zalcman's reparametrization lemma. We notice that the complex Zalcman's lemma holds in the non-Archimedean setting. We follow the proof found in [**Ber06**]. Notice that our result does not imply that the reparametrized sequence is converging.

Proposition 5.3.10. *Let X be a smooth projective variety defined over an algebraically closed complete non-Archimedean field k.*

Suppose that there exists a sequence of analytic maps $f_n : \mathbb{D} \to X$ whose Fubini-Study derivative is not locally uniformly bounded in a neighbourhood of some rigid point $z_0 \in \mathbb{D}$. Then, we can find a sequence of rigid points $z_n \to z_0$ and a sequence $k \ni \rho_n \to 0$ such that the rescaled sequence $g_n(z) := f_n(z_n + \rho_n z)$ satisfies the following properties:

i) *Each g_n is defined on the open disk of radius n;*
ii) *The Fubini-Study derivatives of the maps g_n are uniformly bounded on any compact subset of $\mathbb{A}_k^{1,\mathrm{an}}$;*
iii) *For every $n \in \mathbb{N}$ we have $|g_n'(0)| = 1$.*

The proof relies on the following technical result, whose proof we transpose directly to the non-Archimedean setting.

Lemma 5.3.11 (Gromov). *Let $\varphi : \bar{\mathbb{D}}(0; R) \to \mathbb{R}_+$ be a locally bounded function, and fix $\epsilon > 0$ and $\tau > 1$. Then, for every $a \in \bar{\mathbb{D}}(0; R)(k)$ such that $\varphi(a) > 0$, there is $b \in \bar{\mathbb{D}}(0; R)(k)$ satisfying:*

i) $|a - b| \leq \frac{\tau}{\epsilon(\tau-1)\varphi(a)}$

ii) $\varphi(b) \geq \varphi(a)$
iii) *If* $x \in \bar{\mathbb{D}}(0;R)(k)$ *is such that* $|x-b| \leq \frac{1}{\epsilon\varphi(b)}$, *then* $\varphi(x) \leq \tau\varphi(b)$.

PROOF. Suppose we can find a point $a \in \bar{\mathbb{D}}(0;R)(k)$ such that every $b \in \bar{\mathbb{D}}(0;R)(k)$ fails to satisfy one of the three conditions. In particular, so does a. As a itself obviously satisfies i) and ii), there must exist a rigid point a_1 such that $|a_1 - a| \leq \frac{1}{\epsilon\varphi(a)}$ and $\varphi(a_1) > \tau\varphi(a)$. We will show by induction that we can construct a Cauchy sequence of rigid points along which φ is not bounded.

Suppose that we have constructed $a_1, \cdots, a_n \in \bar{\mathbb{D}}(0;R)(k)$ satisfying $|a_i - a| \leq \frac{1}{\epsilon\varphi(a)}$ and $\varphi(a_i) > \tau^i\varphi(a)$. In particular, a_n satisfies i) and ii) and hence not iii). We then find a_{n+1} satisfying $|a_{n+1} - a_n| \leq \frac{1}{\tau^n\epsilon\varphi(a)}$ and $\varphi(a_{n+1}) > \tau\varphi(a_n) > \tau^{n+1}\varphi(a)$. The ultrametric inequality now shows that $|a_{n+1} - a| \leq \frac{1}{\epsilon\varphi(a)}$ and that $|a_{n+j} - a_n| \leq \frac{1}{\tau^n\epsilon\varphi(a)}$ for every positive integer j. Thus, $\{a_n\}$ is a Cauchy sequence and must converge to some rigid point α, but we have shown that φ is not bounded at α. □

PROOF OF PROPOSITION 5.3.10. We may suppose $z_0 = 0$, and $X = \mathbb{P}_k^{N,\mathrm{an}}$.

Pick a sequence of rigid points $a_n \to 0$ such that $|f_n'(a_n)| \geq n^3$. For every a_n, we now apply Lemma 5.3.11 chosing $\epsilon = 1/n$, $\tau_n = 1 + \frac{1}{n}$ and $\varphi = |f_n'|$ and obtain a sequence $z_n \in \mathbb{D}(k)$ satisfying:

i) $|a_n - z_n| \leq \frac{n^2+n}{|f_n'(a_n)|} \leq \frac{2}{n}$;
ii) $|f_n'(z_n)| \geq |f_n'(a_n)| \geq n^3$;
iii) If $x \in \bar{\mathbb{D}}(k)$ is such that $|x - z_n| \leq \frac{n}{|f_n'(z_n)|}$, then

$$|f_n'(x)| \leq (1 + \frac{1}{n})|f_n'(z_n)| \ .$$

It is clear that $z_n \to 0$. Now set $r_n = \frac{1}{|f_n'(z_n)|}$, and pick $\rho_n \in k$ with $|\rho_n| = r_n$. We see that $r_n \leq \frac{1}{n^3}$, and hence $\rho_n \to 0$. Each map $g_n(z) := f_n(z_n + \rho_n z)$ is hence defined on $\mathbb{D}(0;n)$. Fix some $R > 0$ and pick $z \in \bar{\mathbb{D}}(R)$. We compute using Lemma 5.3.3:

$$\begin{aligned} |g_n'(z)| &\leq R^2 \cdot r_n \cdot |f_n'(z_n + \rho_n z)| \leq \\ &\leq R^2 \cdot r_n(1 + \frac{1}{n})|f_n'(z_n)| = R^2(1 + \frac{1}{n}) \ . \end{aligned}$$

The Fubini-Study derivative of the maps g_n is thus uniformly bounded on compact sets.

Clearly, $|g_n'(0)| = |\rho_n| \cdot |f_n'(z_n)| = 1$ for all $n \in \mathbb{N}$. □

5.3.4. The projective distance. For any two points given in homogeneous coordinates by $x = [x_0, \cdots, x_N]$, $y = [y_0, \cdots, y_N] \in \mathbb{P}_k^N$,

one defines their *projective distance* as

$$d_{\mathbb{P}}(x,y) = \frac{\max|x_i y_j - x_j y_i|}{\max|x_i| \max|y_j|} \ .$$

By continuity, one can extend this definition to the whole Berkovich projective space $\mathbb{P}^{N,\mathrm{an}}$. This leads to the definition of discs in the projective space. We denote by $B_{d_{\mathbb{P}}}(x;R)$ the open polydisk for the projective distance centered at x and of radius R.

Observe that $d_{\mathbb{P}}(x,y) \leq 1$ for every $x, y \in \mathbb{P}^{N,\mathrm{an}}$. Hence, if $R \geq 1$ one has that $B_{d_{\mathbb{P}}}(x;R) = \mathbb{P}^{N,\mathrm{an}}$.

LEMMA 5.3.12. *Let $f : \mathbb{D} \to \mathbb{P}^{N,\mathrm{an}}$ be an analytic map. Then,*

$$\sup_{x\in\mathbb{D}} |f'(x)| \leq \sup_{x,y\in\mathbb{D}} \frac{d_{\mathbb{P}}(f(x),f(y))}{|x-y|} \ .$$

PROOF. Fix some point rigid point $x \in \mathbb{D}$. Using the Taylor series of f, we have that for y close to x,

$$\begin{vmatrix} f_i(x) & f_i(y) \\ f_j(x) & f_j(y) \end{vmatrix} = \begin{vmatrix} f_i(x) & f_i(x) + f_i'(x)(x-y) + O((x-y)^2) \\ f_j(x) & f_j(x) + f_j'(x)(x-y) + O((x-y)^2) \end{vmatrix} =$$
$$= |x-y| \begin{vmatrix} f_i(x) & f_i'(x) \\ f_j(x) & f_j'(x) \end{vmatrix} + O((x-y)^2) \ .$$

This means that

$$|f'(x)| = \lim_{y\to x} \frac{d_{\mathbb{P}}(f(x),f(y))}{|x-y|} \ ,$$

and thus $|f'(x)| \leq \sup_{y\in\mathbb{D}} \frac{d_{\mathbb{P}}(f(x),f(y))}{|x-y|}$. □

5.3.5. Topology induced by the projective distance. Let X be a projective variety defined over a field k of zero characteristic that is algebraically closed. Fixing an embedding of X into some projective space, we obtain a distance function on $X(k)$ induced by the pull-back of the Fubini-Study distance. It is a fundamental fact that any two embeddings $X \to \mathbb{P}^{N,\mathrm{an}}$ and $X \to \mathbb{P}^{M,\mathrm{an}}$ induce equivalent distances on $X(k)$, see e.g. [**Gri15**, Proposition 4.3].

Our aim is to prove that on the set of rigid points of X the topology induced by $d_{\mathbb{P}}$ agrees with the Berkovich topology.

PROPOSITION 5.3.13. *Let X smooth projective variety and fix an embedding $X \to \mathbb{P}^{N,\mathrm{an}}$.*

Then the Berkovich topology on the set of rigid points of X agrees with the one induced by the projective distance $d_{\mathbb{P}}$.

To this end, we consider first the affinoid case.

Recall that given two points $z = (z_1, \ldots, z_N), w = (w_1, \ldots, w_N)$ in $\bar{\mathbb{D}}^N(k)$, the usual distance is given by

$$d_{\mathbb{D}}(z,w) = \max_{1\leq i\leq N} |z_i - w_i| \ .$$

LEMMA 5.3.14. *Let X be a strictly k-affinoid space and fix a closed immersion $X \to \bar{\mathbb{D}}^N$. The Berkovich topology on the set of rigid points of X agrees with the one induced by the usual distance $d_{\mathbb{D}}$.*

PROOF. Pick a rigid point $x \in X$ and fix some positive number ϵ. The open ball $B_{d_{\mathbb{D}}}(x;\epsilon)$ for the distance $d_{\mathbb{D}}$ centered at x of radius ϵ can be expressed as the following finite intersection:

$$B_{d_{\mathbb{D}}}(x;\epsilon) = \bigcap_{i=1}^{N} \{z \in X(k) : |(T_i - x_i)(z)| < \epsilon\} \ .$$

For every $1 \leq i \leq N$, the set $\{z \in X(k) : |(T_i - x_i)(z)| < \epsilon\}$ is an open set for the Berkovich topology.

Conversely, pick any Berkovich open set U in $X(k)$. We may assume that U is a finite intersection of sets of the form $\{x \in X(k) : r_i < |f_i(x)| < s_i\}$ for some analytic function $f_i \in \mathcal{O}(\bar{\mathbb{D}}^N)$ and some positive real numbers r_i and s_i. Recall that for any $z, w \in X(k)$ the following inequality holds:

$$|f_i(z) - f_i(w)| \leq \|f_i\| d_{\mathbb{D}}(z,w) \ ,$$

where $\|.\|$ denotes the norm on the Tate algebra $\mathcal{T}_N$. As a consequence,

$$U \cap X(k) = \bigcap_i \bigcup_{x \in U \cap X(k)} B_{d_{\mathbb{D}}} \left(x; \ \frac{\min\{||f_i(x)| - r| \, , ||f_i(x)| - s|\}}{\|f_i\|} \right) \ ,$$

and the result follows. □

PROOF OF PROPOSITION 5.3.13. Pick a rigid point $x \in X$ and fix some positive real number $\epsilon < 1$. The open ball $B_{d_{\mathbb{P}}}(x;\epsilon)$ for the projective distance can be expressed as a finite intersection of open sets for the Berkovich topology as follows:

$$B_{d_{\mathbb{P}}}(x;\epsilon) = \bigcap_{\substack{0 \leq i,j \leq N \\ i \neq j}} \{y \in X(k) : |(x_i T_j - x_j T_i)(y)| < \epsilon\} \ .$$

The converse follows from the fact that the projective space can be covered by a finite number of Berkovich polydisks. □

Given a projective variety X, we may consider the semi distance $d'_{\mathrm{CK}} := \min\{1, d_{\mathrm{CK}}\}$. We now compare it with the projective distance.

PROPOSITION 5.3.15. *Let X be a smooth projective variety. For any rigid point $x \in X$, there exists an open neighbourhood U of x and a positive constant C such that $d'_{\mathrm{CK}} \leq C d_{\mathbb{P}}$ on $U(k)$.*

PROOF. Denote by M the dimension of X. Pick a rigid point $x \in X$ and fix an analytic map $\varphi : \mathbb{D}^M \to X$ that is an isomorphism on its image, sending some $z \in \mathbb{D}^M$ to x. Set $U := \varphi(\mathbb{D}^M)$. Embed X in some projective space $\mathbb{P}^{N,\mathrm{an}}$. By Proposition 5.3.13, we may choose a positive number ϵ such that $B_{d_{\mathbb{P}}}(x;\epsilon) \cap X(k)$ is contained in U.

After maybe reducing the polydisk $\mathbb{D}^M$, we may assume that U is contained in some fixed unit polydisk $\bar{\mathbb{D}}^N \subset \mathbb{P}^{N,\mathrm{an}}$. Thus, the projective distance agrees with the usual distance on U. Notice that the map φ is given by $\varphi = (\varphi_1, \ldots, \varphi_N)$, where every $\varphi_i \in \mathcal{T}_M$ has coefficients bounded by 1, for $1 \leq i \leq N$. Given any rigid point $y \in \bar{\mathbb{D}}^N$ with $y = \varphi(w)$ for some $w \in \mathbb{D}^M$, we have that

$$d_{\mathbb{P}}(x, y) = d_{\mathbb{P}}(\varphi(z), \varphi(w)) = \max_{1 \leq i \leq N} |\varphi_i(z) - \varphi_i(w)| \ .$$

For distinct $z, w \in \mathbb{D}^M(k)$, consider the real-valued function

$$\Theta(z, w) = \frac{d_{\mathbb{P}}(\varphi(z), \varphi(w))}{d'_{\mathrm{CK}}(z, w)} \ .$$

This function is strictly positive. By the previous equation, we know that $\Theta(z, w) = \frac{\max_{i \leq N} |\varphi_i(z) - \varphi_i(w)|}{\max_{j \leq M} |z_j - w_j|}$. The Taylor series development of each component φ_i implies that for any $z, w \in \mathbb{D}^M(k)$, we may write $\varphi_i(w) - \varphi_i(z) = \sum_{j=1}^{M} (z_j - w_j) \partial_j \varphi_i(z) + O(\sum_{1 \leq j \leq M} |z_j - w_j|^2)$, where $\partial_j \varphi_i(z)$ denotes the partial derivative of φ_i with respect to the j-th component. Using this observation, we may extend the function Θ continuously to the diagonal by setting

$$\Theta(z, z) = \lim_{w \to z} \Theta(z, w) = \max_{\substack{1 \leq i \leq N \\ 1 \leq j \leq M}} |\partial_j \varphi_i(z)| \ .$$

As φ is an isomorphism on its image, not all the partial derivatives $\partial_j \varphi_i$ are zero at the same time, and so Θ is strictly positive on the whole $\mathbb{D}^M(k) \times \mathbb{D}^M(k)$. We may so find a positive constant C such that $\Theta(z, w) \geq C$ for every $z, w \in \mathbb{D}^M(k)$. As the Cherry-Kobayashi semi distance contracts analytic maps, we see that $d_{\mathrm{CK}}(\varphi(z), \varphi(w)) \leq d_{\mathrm{CK}}(z, w)$. Thus, $d'_{\mathrm{CK}}(\varphi(z), \varphi(w)) \leq C d_{\mathbb{P}}(\varphi(z), \varphi(w))$. □

5.4. Royden's length function

Royden's length function on the tangent bundle of a complex manifold has the particularity that the semi distance it defines on the manifold is precisely the Kobayashi semi distance. This enables us to translate the notion of Kobayashi hyperbolicity into infinitesimal terms. We refer to [**Roy71**] for further details.

One can adapt this definition to Berkovich spaces as follows. Recall that the tangent space of a smooth analytic space X at a point x is the set of all derivations on the local ring $\mathcal{O}_{X,x}$.

Definition 5.4.1. *Let X be a smooth analytic space over k. For every $x \in X(k)$, Royden's length function is defined for every $\vec{v} \in T_x X$ as*

$$|\vec{v}|_{\mathrm{Roy}} := \inf \left\{ \frac{1}{|\lambda|} : \exists f : \mathbb{D} \to X \ \textit{analytic}, f(0) = x, f'(0) = \lambda \vec{v} \right\} .$$

The following result holds for fields k of arbitrary characteristic:

PROPOSITION 5.4.2. *Let X be a smooth projective variety over some non-Archimedean field k.*

If Royden's function is such that $|\vec{v}|_{\mathrm{Roy}} = 0$ if and only if $\vec{v} = 0$, then X contains no entire curve.

PROOF. Suppose there exists an entire curve $f : \mathbb{A}^{1,\mathrm{an}} \to X$. We may suppose that f is not constant at 0, and hence $\vec{v} = f'(0) \in T_{f(0)}X$ is a non-zero vector. Denoting by $m_n : \mathbb{D} \to \mathbb{D}(0; n)$ the homothety of ratio n, the sequence $f_n := f \circ m_n : \mathbb{D} \to X$ is such that $f_n'(0) = \lambda_n \vec{v}$, with $|\lambda_n| = n$. Hence, $|\vec{v}|_{\mathrm{Roy}} = 0$. □

PROPOSITION 5.4.3. *Let X be a smooth projective variety defined over a complete non-Archimedean field k of characteristic zero.*

Assume that every rigid point admits a neighbourhood on which the Fubini-Study derivative of $\mathrm{Mor}_k(\mathbb{D}, X)$ is uniformly bounded. Then, Royden's function is such that $|\vec{v}|_{\mathrm{Roy}} = 0$ if and only if $\vec{v} = 0$.

PROOF. Suppose that there exists a point $x \in X$ and some nonzero $\vec{v} \in T_xX$ such that $|\vec{v}|_{\mathrm{Roy}} = 0$. Then there is a sequence $f_n : \mathbb{D} \to X$ fixing the origin and such that $f_n'(0) = R_n v$, with $v \neq 0$ and $|R_n| \to +\infty$. Hence, the Fubini-Study derivative explodes at 0. □

5.5. Further notions of hyperbolicity

In an attempt to obtain hyperbolicity results analogous to complex ones, we may consider other notions of hyperbolicity that arise naturally from the Cherry-Kobayashi semi distance:

DEFINITION 5.5.1. *Let X be a smooth projective variety defined over an algebraically closed non-Archimedean complete field. The variety X is strongly Cherry hyperbolic if the semi distance d_{CK} defines the same topology as the projective distance on rigid points.*

We shall see in Theorem F that this notion is stronger than that of Cherry hyperbolicity. If X is a hyperbolic complex analytic space, Barth showed that the Kobayashi metric defines the topology of X, see [**Lan87**, Theorem §I.2.3].

5.5.1. Smooth projective varieties of arbitrary dimension. Recall that $d'_{\mathrm{CK}} := \min\{1, d_{\mathrm{CK}}\}$. We have the following equivalences for general smooth projective varieties:

THEOREM F. *Let X be a smooth projective variety defined over an algebraically closed non-Archimedean complete field k of residue characteristic zero. The following conditions are equivalent:*

i) *Every rigid point $x \in X$ has a neighbourhood U such that the semi distances d'_{CK} and $d_{\mathbb{P}}$ are equivalent on $U(k)$.*
ii) *The variety X is strongly Cherry hyperbolic.*

iii) *For every rigid point $z \in \mathbb{D}$ there exists a neighbourhood U and a positive constant C such that*

$$\sup_{f \in \mathrm{Mor}_k(\mathbb{D},X)} \sup_{z \in U} |f'(z)| \leq C \ .$$

REMARK 5.5.2. *Residue characteristic 0 is used for the implications iii) $\Rightarrow$ i) and iii) $\Rightarrow$ ii).*

PROOF OF THEOREM F. i) $\Rightarrow$ ii): Let us first show that X is Cherry hyperbolic. Pick any point $x \in X(k)$. We shall prove that $d_{\mathrm{CK}}(x,y) > 0$ for all $y \in X(k)$ different from x.

By Corollary 5.3.13, the topology induced by $d_{\mathbb{P}}$ agrees with the Berkovich topology. Our assumption i) thus implies the existence of $\epsilon > 0$ and a constant $C > 0$ such that $d'_{\mathrm{CK}}(x_1,x_2) \geq C d_{\mathbb{P}}(x_1,x_2)$ whenever $\max\{d_{\mathbb{P}}(x,x_1), d_{\mathbb{P}}(x,x_2)\} \leq \epsilon$.

If y is such that $d_{\mathbb{P}}(x,y) \leq \epsilon$, then $d'_{\mathrm{CK}}(x,y) \geq C d_{\mathbb{P}}(x,y) > 0$ as required. Suppose now that $d_{\mathbb{P}}(x,y) > \epsilon$, and pick any Kobayashi chain joining x and y. We get a finite set of analytic maps $f_l : \bar{\mathbb{D}} \to X$ and points $z_l \in \bar{\mathbb{D}}(k)$, $l = 1, \cdots, m$ such that $f_1(0) = x$, $f_l(z_l) = f_{l+1}(0)$ for $l = 1, \cdots, m-1$ and $f_m(z_m) = y$. We shall prove that $|z_l| \geq C\epsilon/4$ for some l, which proves that $d_{\mathrm{CK}}(x,y) \geq C\epsilon/4$.

For each l, consider the function $d_l(t) := d_{\mathbb{P}}(x, f_l(t))$. This is a continuous function on the whole disk $\bar{\mathbb{D}}$. Since all closed disks $\bar{\mathbb{D}}(0;|z_l|)$ are connected, the subset of the real line $\cup_{l=1}^m d_l(\bar{\mathbb{D}}(0;|z_l|))$ is also connected. As $d_1(0) = 0$ and $d_m(z_m) > \epsilon$, we may find an integer l and a point $\tau \in \bar{\mathbb{D}}(0;|z_l|)$ such that $d_l(\tau) = \epsilon/2$. By density of rigid points in the open disk, we may find $t \in \bar{\mathbb{D}}(0;|z_l|)(k)$ such that $d_l(t) \in (\epsilon/4, 3\epsilon/4)$. We get that $|z_l| \geq |t| \geq d_{\mathrm{CK}}(x, f_l(t)) \geq C\epsilon/4$.

We now prove that the two topologies induced by d'_{CK} and $d_{\mathbb{P}}$ are the same. This amounts to checking that converging sequences for one topology are converging for the other one. Suppose first that $d_{\mathbb{P}}(x_n,x) \to 0$. Then for sufficiently large n we have that x_n lies in a neighbourhood of x where d'_{CK} is equivalent to $d_{\mathbb{P}}$, hence $d'_{\mathrm{CK}}(x_n,x) \to 0$.

Suppose next that $d'_{\mathrm{CK}}(x_n,x) \to 0$. Our arguments above show that for n sufficiently large x_n belongs to a neighbourhood of x on which d'_{CK} is equivalent to $d_{\mathbb{P}}$, so that again $d_{\mathbb{P}}(x_n,x) \to 0$.

ii) $\Rightarrow$ iii): Fix an embedding of X in some analytic projective space $\mathbb{P}^{N,\mathrm{an}}$. Suppose that the Fubini-Study derivative explodes at some point of $\mathbb{D}$. We apply Proposition 5.3.10 to find a sequence of analytic maps $g_n : \mathbb{D}(0;n) \to X$ satisfying $|g'_n(0)| = 1$ and with uniformly bounded Fubini-Study derivative on compact subsets of $\mathbb{A}^{1,\mathrm{an}}$.

Denote by $d_n = \mathrm{diam}(g_n(\mathbb{D}))$. If d_n tends to zero as n goes to infinity, then after maybe extracting a subsequence all the $g_n(\mathbb{D})$ are contained in some fixed ball of $\mathbb{P}^{N,\mathrm{an}}$. Schwarz' lemma implies that the derivative at zero is strictly smaller than 1, contradicting the fact that $|g'_n(0)| = 1$. Thus, we may assume that there exists some $\epsilon > 0$

such that $d_n > \epsilon$ for every $n \in \mathbb{N}$. In particular, for every n there are rigid points $w_n, z_m \in \mathbb{D}$ such that $d_{\mathbb{P}}(g_n(w_n), g_n(z_n)) \geq \frac{\epsilon}{2}$. However, $d_{\mathrm{CK}}(g_n(w_n), g_n(z_n)) \leq d_{\mathrm{CK}}(w_n, z_n) \leq \frac{1}{n}$, since g_n is defined on $\mathbb{D}(0; n)$, and so the distances d_{CK} and $d_{\mathbb{P}}$ cannot be equivalent.

iii) $\Rightarrow$ ii): Suppose that the Fubini-Study derivative of all the analytic maps $f : \mathbb{D} \to X$ is uniformly bounded on some open disk $\mathbb{D}(0; r)$ by some positive constant C. It suffices to show that given any rigid points x_n, x in X such that $d_{\mathrm{CK}}(x_n, x)$ tends to 0, $d_{\mathbb{P}}(x_n, x) \to 0$ as n goes to infinity.

For every n, consider a Kobayashi chain $f_l^n : \mathbb{D} \to X$, $l = 1, \cdots, N_n$, joining x_n and x of length $r_n < r$ and such that $r_n \to 0$. By Lemma 5.3.8, we see that $f_l^n(\mathbb{D}(0; r_n)) \subseteq \mathbb{D}^N(f_l^n(0); r_n C)$. For every fixed $n \in \mathbb{N}^*$, these polydisks have nonempty intersection by definition of Kobayashi chain and have the same radius, and so they must be the same. Hence, $d_{\mathbb{P}}(x_n, x)$ tends to 0.

iii) $\Rightarrow$ i): Let $r > 0$ and $C > 0$ be constants such that

$$\sup_{\mathrm{Mor}_k(\mathbb{D},X)} \sup_{\mathbb{D}(0;r)} |f'(z)| \leq C \ .$$

After maybe reducing the radius r, we may assume that $d'_{\mathrm{CK}} \leq d_{\mathbb{P}}$ on $\mathbb{D}(0; r)$ by Proposition 5.3.15. Pick any two rigid points $x, y \in X$. Assume that $d_{\mathrm{CK}}(x, y) \leq \frac{r}{2}$. Let C_n be a sequence of Kobayashi chains joining x and y of length d_n and such that $\lim_n d_n = d_{\mathrm{CK}}(x, y)$. Recall that each chain C_n is given by analytic maps $f_1^{(n)}, \ldots, f_{N_n}^{(n)} : \mathbb{D} \to X$ and rigid points $z_1^{(n)}, \ldots, z_{N_n}^{(n)}$ in $\mathbb{D}$ satisfying the appropritate equalities. For sufficiently large n we may assume that

$$d_{\mathrm{CK}}(x, y) \leq d_n < 2 \cdot d_{\mathrm{CK}}(x, y) = r \ ,$$

and in particular we see that $|z_l^{(n)}| < r$ for every $n \gg 0$ and every $1 \leq l \leq N_n$. As a consequence of Lemma 5.3.8, we see that $f_l^{(n)}(\bar{\mathbb{D}}(0; |z_l^{(n)}|)) \subseteq \mathbb{D}^N(f_l^{(n)}(0); |z_l^{(n)}|C)$. Thus, for every $n \in \mathbb{N}$ we have

$$d_{\mathbb{P}}(x, y) \leq C \cdot d_n < 2C \cdot d_{\mathrm{CK}}(x, y) \ ,$$

concluding the proof. □

Proposition 5.5.3. *Let X be a smooth projective variety defined over an algebraically closed non-Archimedean complete field k of residue characteristic zero. If the Fubini-Study derivative of $\mathrm{Mor}_k(\mathbb{D}, X)$ is uniformly bounded in a neighbourhood of every rigid point, then the family $\mathrm{Mor}_k(\mathbb{D}, X)$ is normal at every rigid point.*

Proof. Embed X in some projective space $\mathbb{P}^{N,\mathrm{an}}$. It suffices to prove the assertion for $z = 0$ in $\mathbb{D}$. Assume first that U is a neighbourhood of 0 on which there exists a positive constant C such that

$$\sup_{\mathrm{Mor}_k(\mathbb{D},X)} \sup_{U} |f'(z)| \leq C < +\infty \ .$$

Pick any sequence of analytic maps $f_n : \mathbb{D} \to X$. Since $\mathbb{P}^{N,\mathrm{an}}$ can be covered by a finite number of closed polydisks isomorphic to $\bar{\mathbb{D}}^N$, we see that, after maybe to extracting a subsequence and rescaling the image, the points $f_n(0)$ converge to a point in $\bar{\mathbb{D}}^N(0;\frac{1}{2})$, as $\mathbb{P}^{N,\mathrm{an}}$ is sequentially compact [**Poi13**].

Now let $\eta_{0,r} \in U$ be the point corresponding to the closed all $\bar{B}(0;r)$ in k. It follows from Lemma 5.3.8 that $\mathrm{diam} f_n(\eta_{0,r}) \leq r \cdot C$ for all n. Choose $r > 0$ such that $r \leq \frac{1}{2C}$ and set $U' = \mathbb{D}(0;r)$. By continuity, $f_n(U') \subseteq \bar{\mathbb{D}}^N(0;\frac{1}{2})$, and by Theorem A there exists a subsequence converging on U' to a continuous map. □

5.5.2. Smooth projective curves. In the case of smooth projective curves, we have the following characterization:

THEOREM G. *Let X be a smooth projective curve defined over an algebraically closed field of residue characteristic zero. The following conditions are equivalent:*

i) *The curve X has positive genus.*
ii) *Every rigid point $x \in X$ has a neighbourhood U such that the semi distances d'_{CK} and $d_{\mathbb{P}}$ are equivalent on $U(k)$.*
iii) *The Fubini-Study derivative of $\mathrm{Mor}_k(\mathbb{D}, X)$ is uniformly bounded in a neighbourhood of every rigid point.*
iv) *The family $\mathrm{Mor}_k(\mathbb{D}, X)$ is normal.*
v) *The curve X is Cherry hyperbolic.*

The equivalence between i) and v) was proved in [**Che96**]. We provide a new proof of the fact that every smooth projective curve with positive genus is Cherry hyperbolic.

PROOF. iv) ⇒ i): Pick any $|\lambda| > 1$ and consider the sequence of analytic maps $f_n(z) = (\lambda z)^n$ from $\mathbb{D}$ to $\mathbb{P}^{1,\mathrm{an}}$. As explained at the beginning of this chapter, no subsequence of f_n has a continuous limit, and thus the family $\mathrm{Mor}_k(\mathbb{D}, \mathbb{P}^{1,\mathrm{an}})$ is not normal.

i) ⇒ iv): Assume that X has positive genus. Recall from §1.4 that if X is a smooth projective curve with positive genus, then $\mathsf{S}^{\mathrm{an}}(X)$ is nonempty. The set $X \setminus \mathsf{S}^{\mathrm{an}}(X)$ is a disjoint union of infinitely many open disks.

Let $f_n : \mathbb{D} \to X$ be a sequence of analytic maps. By [**Ber90**, Theorem 4.5.3], the image of each map f_n does not intersect $\mathsf{S}^{\mathrm{an}}(X)$.

If the image of infinitely many maps f_n is contained in the same connected component of $X \setminus \mathsf{S}^{\mathrm{an}}(X)$, then we may find a subsequence f_{n_j} avoiding some fixed connected component of the complement of $\mathsf{S}^{\mathrm{an}}(X)$. Hence, by Lemma 1.4.6 the maps f_{n_j} take values in a fixed affinoid domain of X and so they converge pointwise to some continuous map by Theorem A.

Assume next that at most finitely many $f_n(\mathbb{D})$ are contained in the same connected component of $X \setminus \mathsf{S}^{\mathrm{an}}(X)$. If $\mathsf{S}^{\mathrm{an}}(X)$ consists only of

one point η_X, this means that the sequence f_n converges pointwise to the constant map η_X. Suppose otherwise $\mathsf{S}^{\mathrm{an}}(X)$ is not a singleton. Denoting by $r_X : X \to \mathsf{S}^{\mathrm{an}}(X)$ the usual retraction map, we consider the composition $y_n := r_X \circ f_n$. Thus, each map y_n is constant. By compactness of $\mathsf{S}^{\mathrm{an}}(X)$ we may find a subsequence $\{y_{n_j}\}$ converging to some point $y \in \mathsf{S}^{\mathrm{an}}(X)$.

Fix an open neighbourhood $V \ni y$. By [**Duc14**, Théorème 4.5.4], we are reduced to the following possibilities for V. If y is a type III point, then V is isomorphic to an open annulus whose skeleton is contained in $\mathsf{S}^{\mathrm{an}}(X)$. Otherwise, if y has type II then $V \setminus \{y\}$ is the disjoint union of infinitely many open disks and finitely many open annuli. In particular, the intersection of the skeleton of V and $\mathsf{S}^{\mathrm{an}}(X)$ is nonempty. Pick any $z \in \mathbb{D}$. For sufficiently large n_j, the points $f_{n_j}(z)$ lie in V. Thus, the subsequence $\{f_{n_j}\}$ converges pointwise to the constant map $f \equiv y$.

i) $\Rightarrow$ iii): Let X be a curve of positive genus. We show that the Fubini-Study derivative of every map from $\mathbb{D}$ to X is bounded. Let $f : \mathbb{D} \to X$ be an analytic map. By [**Ber90**, Theorem 4.5.3], the image of f is contained in some connected component of the complement of $\mathsf{S}^{\mathrm{an}}(X)$, i.e. in some open subset V of X analytically isomorphic to $\mathbb{D}$. This implies that the Fubini-Study derivative of f is bounded by 1 on the whole disk.

iii) $\Leftrightarrow$ ii): This equivalence was shown in Theorem F.

ii) $\Rightarrow$ v): This implication follows from Theorem F. $\square$

CHAPTER 6

Compactness of spaces of analytic maps with values in an algebraic curve

Throughout this chapter, the base field k is a non-Archimedean complete algebraically closed field. Our main results, stated below, assume furthermore that k has zero residue characteristic.

Our aim is to give a characterization of smooth (irreducible) algebraic curves having negative Euler characteristic in terms of normality properties of suitable families of analytic maps.

Suppose that X is a smooth (irreducible) algebraic curve. One can then find a unique open embedding of X in a smooth projective curve $\bar{X}$ such that $\bar{X} \setminus X$ is a finite set of k-points. The Euler characteristic of X is by definition

$$\chi(X) = 2 - 2g - \#(\bar{X} \setminus X) \ ,$$

where g denotes the genus of $\bar{X}$.

Pick any boundaryless k-analytic space U. A family $\mathcal{F}$ of analytic maps $f : U \to X$ is said to be normal if and only if for any sequence $f_n \in \mathcal{F}$ there exists a subsequence f_{n_j} converging pointwise on U to a continuous function $f_\infty : U \to \bar{X}$. This definition is analogous to the one used in §4.

Our main theorem is:

THEOREM H. *Suppose that k is a complete non-Archimedean algebraically closed field of zero residue characteristic whose residue field is countable. Let X be a smooth irreducible algebraic curve over k.*

Then, the Euler characteristic $\chi(X)$ of X is negative if and only if for every smooth connected boundaryless analytic curve U the family $\mathrm{Mor}_k(U, X)$ is normal.

More precisely, any sequence of analytic maps $f_n : U \to X$ admits a subsequence f_{n_j} that converges pointwise to a continuous map $f_\infty : U \to \bar{X}$ such that either $f_\infty(U) \subset X$ or f_∞ is constant equal to a point in $\bar{X} \setminus X$.

One implication was basically already noticed in [**FKT12**]. When the Euler characteristic of X is non-negative, then we may find a smooth boundaryless curve such that the family $\mathrm{Mor}_k(U, X)$ is not normal. When X is the projective or the affine line, one can take U to be the unit disk. When X is the punctured affine line any open

annulus works. In §6.3, we extend these arguments to any elliptic curve, in which case we may take $U = X$.

The core of the proof lies in the forward implication: the family $\mathrm{Mor}_k(U, X)$ is normal as soon as $\chi(X) < 0$. We consider first the case where the skeleton of X is not too small and next the general case.

THEOREM I. *Suppose that k is a complete non-Archimedean algebraically closed field of zero residue characteristic. Let X be a smooth irreducible algebraic curve of negative Euler characteristic whose skeleton $\mathsf{S}^{\mathrm{an}}(X)$ is not a singleton.*

Let U be a smooth connected boundaryless analytic curve. Then there exists a finite affinoid cover $(\bar{X}_i)$ of $\bar{X}$ and a locally finite cover (U_j) of U by basic tubes such that for every analytic map $f : U \to X$ and every j the image $f(U_j)$ is contained in some affinoid $\bar{X}_i$.

Moreover, the affinoid cover $(\bar{X}_i)$ is independent of U.

This result together with Theorem A implies a stronger form of Theorem H when $\mathsf{S}^{\mathrm{an}}(X)$ is not a singleton. For instance, any limit map is weakly analytic in this case.

When the skeleton of X is reduced to a point, then X is a projective curve and admits a smooth model over k°. In other words, it is a curve with good reduction.

When X is a complete curve having good reduction, our arguments rely crucially on the countability assumption on the residue field $\tilde{k}$ and we are no longer able to ensure that limit maps are weakly analytic.

The two above stated theorems are proved by closely examining the behaviour of analytic maps between curves, in particular on the skeleta, and thus rely heavily on the reduction theory of curves which is explained in detail in [**Duc14**].

6.1. Geometry of basic tubes of dimension 1

Recall from §1.6 the definition of a basic tube. Any connected component of the interior of an equidimensional strictly affinoid space is a basic tube.

Recall from §1.4 that the skeleton $\mathsf{S}^{\mathrm{an}}(X)$ of a curve X is the set of points in X not having an open neighbourhood that is isomorphic to an open disk. In particular, a one-dimensional basic tube has empty skeleton if and only if it is isomorphic to an open disk, see [**Duc14**, Proposition 5.1.18]. An open annulus $A(R, 1)$ with $R \in |k^\times|$ is a basic tube, and its skeleton is isomorphic to the open real segment $(R, 1)$.

A point x in the skeleton of a smooth analytic curve X is a node if it satisfies one of the following conditions: the point x is a branching point of the skeleton, x has positive genus or it belongs to the boundary of X.

An important class of basic tubes of dimension one are star-shaped domains:

DEFINITION 6.1.1. *A basic tube U of dimension* 1 *is called a star-shaped domain if it is simply connected and contains exactly one node η_U.*

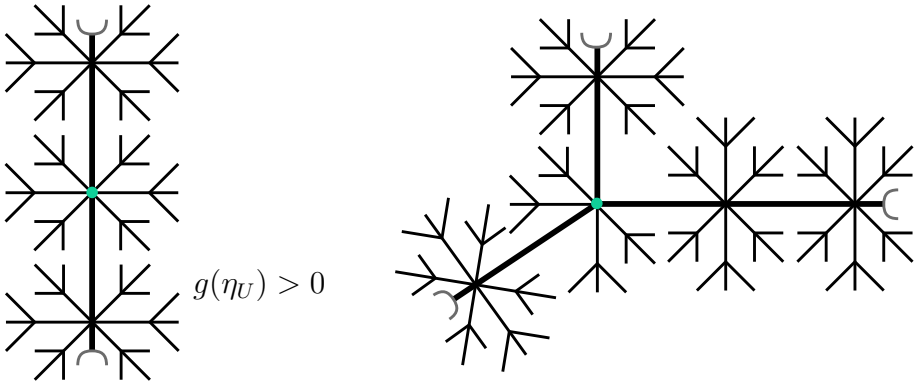


FIGURE 1. Star-shaped domains

Let us describe the geometry of star-shaped domains in more detail. Let U be a star-shaped domain. Since basic tubes are boundaryless, the point η_U has positive genus or it is a branching point of $\mathsf{S}^{\mathrm{an}}(U)$. The skeleton of U can be decomposed in a disjoint union of $\{\eta_U\}$ and finitely many open segments. Thus, every connected component of $U \setminus \{\eta_U\}$ is either isomorphic to an open disk or to an open annulus. The latter correspond to the non-discal tangent directions in $T_{\eta_U}U$.

As a consequence, a star-shaped domain U determines the following data:

i) The residue curve at η_U, which is the unique smooth projective curve C_U over $\tilde{k}$ such that $\tilde{k}(C_U) \simeq \widetilde{\mathcal{H}(\eta_U)}$. The curve C_U has genus $g(U)$;
ii) A reduced divisor D_U on C_U whose support is the set of non-discal directions at η_U;
iii) For every non-discal direction $\vec{v} \in T_{\eta_U}U$ a real number $\rho \in |k^\times|$ of norm less than 1 such that the open set $U(\vec{v})$ is isomorphic to the open annulus $A(\rho, 1)$.

6.2. Analytic maps between curves

The following result will be systematically used in the sequel, see [**Duc14**, Lemme 6.2.4]:

LEMMA 6.2.1. *Let X be a smooth projective curve over k and U a basic tube of dimension* 1.

Let $f : U \to X$ a non-constant analytic map. Let $z \in U$ be a type II or III point and consider the tangent map $df(z) : T_zU \to T_{f(z)}X$. If a tangent direction $\vec{v} \in T_zU$ is discal, then so is $df(z)(\vec{v})$.

As an immediate consequence, we have:

LEMMA 6.2.2. *Let X be a smooth projective curve and $f : \mathbb{D} \to X$ an analytic map. Then the image of f is contained in some connected component of $X \setminus \mathsf{S}^{\mathrm{an}}(X)$.*

Another important fact that will be used throughout the present chapter is the following proposition:

PROPOSITION 6.2.3. *Let U be a basic tube of dimension 1 that is not analytically isomorphic to the unit disk. Let X be a smooth projective curve and $f : U \to X$ an analytic map. If a point $z \in \mathsf{S}^{\mathrm{an}}(U) \setminus \mathsf{N}(U)$ is such that $f(z)$ lies in the skeleton of X, then the connected component of $\mathsf{S}^{\mathrm{an}}(U) \setminus \mathsf{N}(U)$ containing z is mapped to $\mathsf{S}^{\mathrm{an}}(X)$.*

The proof relies on the following Lemma [**Duc14**, Lemme 6.2.5]:

LEMMA 6.2.4. *Let X be a smooth projective curve over k and U a basic tube of dimension 1.*

Let $f : U \to X$ a non-constant analytic map. If $f(z)$ is a node in $\mathsf{S}^{\mathrm{an}}(X)$, then z is a node in $\mathsf{S}^{\mathrm{an}}(U)$.

PROOF OF PROPOSITION 6.2.3. Let $z \in \mathsf{S}^{\mathrm{an}}(U)$ be a non-nodal point such that $f(z) \in \mathsf{S}^{\mathrm{an}}(X)$. By Lemma 6.2.4, the point $f(z)$ is not a node. In particular, $f(z)$ cannot be an endpoint of the skeleton by Lemma 1.4.4, and so both z and $f(z)$ have exactly two non-discal tangent directions.

Consider the complement of $\mathsf{N}(U)$ in $\mathsf{S}^{\mathrm{an}}(U)$, and let I be the connected component containing the point z. Suppose by contradiction that not the whole I is mapped to the skeleton of X. In this case, we may find a point $z' \in I$ such that $f(z') \in \mathsf{S}^{\mathrm{an}}(X)$ and such that a non-discal direction $\vec{v} \in T_{z'}U$ is mapped to some discal direction at $f(z')$. The tangent map $df(z') : T_{z'}U \to T_{f(z')}X$ is surjective, and so there is a discal direction at z' that is mapped to a non-discal direction at $f(z')$. This contradicts Lemma 6.2.1. □

We shall use the following version of Hurwitz's theorem during the proof of Theorem H:

PROPOSITION 6.2.5. *Let U be a boundaryless connected curve over k and X a k-affinoid space. Let Z be any closed analytic subset of X.*

Suppose that f_n is a sequence of analytic maps from U to $X \setminus Z$ converging pointwise to a continuous map g.

Then, we have either that $g(U) \cap Z = \emptyset$ or $g(U) \subset Z$.

PROOF. The set Z is the zero locus of some analytic function φ in the affinoid algebra of X with $|\varphi|_{\sup} \leq 1$. Since the zeros of φ form a

finite subset of X of rigid points, we may assume that X is the closed unit disk and that Z is the origin by replacing f_n by $\varphi \circ f_n$.

In this case our assumption ensures that the functions $h_n = \log|f_n| : X \to \mathbb{R}_-$ are harmonic. It follows from [**Thu05**, Proposition 3.1.2] that either h_n converges uniformly to $-\infty$ on compact subsets and so $g(U) \subset \{0\}$, or any limit map of the sequence h_n is still harmonic, in which case one necessarily has $g(U) \cap \{0\} = \emptyset$. □

6.3. Curves with non-negative Euler characteristic

In this section we prove one of the implications of Theorem H.

PROPOSITION 6.3.1. *Let X be a curve with non-negative Euler characteristic. Then there exists a one-dimensional basic tube U such that the family $\mathrm{Mor}_k(U, X)$ is not normal.*

Our proof follows [**FKT12**, Theorem 5.4] and uses an equidistribution result for non-Archimedean elliptic curves from [**Pet09**].

PROOF. Recall that a smooth algebraic curve satisfies $\chi(X) \geq 0$ if and only if it is isomorphic to one of the following models:

(1) $\mathbb{P}^{1,\mathrm{an}}$, $\mathbb{A}^{1,\mathrm{an}}$, $\mathbb{A}^{1,\mathrm{an}} \setminus \{0\}$;
(2) an elliptic curve.

We claim that for any $\rho > 1$ the family of analytic maps

$$\mathrm{Mor}_k\left(A\left(\frac{1}{\rho}, \rho\right), \mathbb{A}^{1,\mathrm{an}} \setminus \{0\}\right)$$

is not normal. To see this, consider the sequence $f_n(z) = z^n$ from $A(\frac{1}{\rho}, \rho)$ to $\mathbb{A}^{1,\mathrm{an}} \setminus \{0\}$. Observe that $f_n(x_g) = x_g$ for all $n \in \mathbb{N}$, whereas $f_n(z) \to 0$ for any $|z| < 1$. It follows that no limit map of f_n can be continuous at the Gauss point. As a consequence of the definition of normality given at the begining of this chapter for families of maps whose target is a smooth algebraic curve, this proves the proposition for all cases in the first item.

Suppose now that X is an elliptic curve. Consider the map $f : X \to X$ induced by the multiplication by 2. Pick any point $x_0 \in \mathsf{S}^{\mathrm{an}}(X)$, and suppose by contradiction that the family of the iterates $\{f^n\}$ is normal on a neighborhood U of x_0. Assume that the subsequence f^{n_j} converges on U to a continuous function $g : U \to X$.

Choose any fixed rigid point $y \in X$ for f. By [**Pet09**, Theorem 1] the sequence of probability measures $4^{-n}(f^n)^*\delta_y$ converges to a probability measure μ whose support is equal to $\mathsf{S}^{\mathrm{an}}(X)$, hence contains x_0. We may thus find a sequence of rigid points $y_n \to x_0$ such that $f^n(y_n) = y$. Observe that $y_m \in U$ for sufficiently large m, thus $g(y_m) = \lim_{n_j} f^{n_j}(y_m) = y$, for all $m \in \mathbb{N}$.

But f leaves the skeleton of X invariant, hence $y = g(x_0) \in \mathsf{S}^{\mathrm{an}}(X)$ which gives a contradiction. □

6.4. Analytic maps on special domains

In this section, we study the normality of the family of analytic maps taking values in a smooth projective curve X having only one node that do not have good reduction, i.e. whose skeleton contains points different from the node. Our discussion is based on the study of three fundamental families of one-dimensional basic tubes: open disks, open annuli and star-shaped domains.

6.4.1. Analytic maps avoiding a type II point.

PROPOSITION 6.4.1. *Let X be a smooth irreducible projective curve and U any smooth connected curve. Let $\mathcal{F}$ be a family of analytic maps from U to X.*

If there exists a type II point $\eta \in X$ such that $\eta \notin f(U)$ for every $f \in \mathcal{F}$, then there exists an affinoid covering (X_1, X_2) of X such that for every $f \in \mathcal{F}$, the image $f(U)$ is contained either in X_1 or in X_2.

Moreover, the affinoid cover (X_i) is independent of U.

COROLLARY 6.4.2. *Let X be a smooth irreducible projective curve of genus at least 2 and U be an open disk or an open annulus. Then, there exists a finite affinoid cover (X_i) of X such that the image of every analytic map $f : U \to X$ is contained in some affinoid X_i.*

Moreover, the affinoid cover (X_i) is independent of U.

COROLLARY 6.4.3. *Let X be a smooth irreducible projective curve having at least two nodes and let U be any smooth connected boundaryless curve. Then, there exists a finite affinoid cover (X_i) of X and a locally finite open cover (U_j) of U by basic tubes such that for every analytic map $f : U \to X$ and every element of the cover U_j, the image $f(U_j)$ is contained in some affinoid X_i.*

Moreover, the affinoid cover (X_i) is independent of U.

PROOF OF PROPOSITION 6.4.1. Let X be any smooth irreducible projective curve and U any smooth connected curve. Let $\mathcal{F}$ be a family of analytic maps in $\mathrm{Mor}_k(U, X)$ whose images avoid some type II point $\eta \in X$.

The image of every $f \in \mathcal{F}$ is contained in some connected component of $X \setminus \{\eta\}$. We may thus pick any two distinct connected components B_1, B_2 of $X \setminus \{\eta\}$. Then, $X_1 := X \setminus B_1$ and $X_2 = X \setminus B_2$ are affinoid domains of X by Lemma 1.4.6, and (X_1, X_2) is a cover of X safisfying the required property. □

PROOF OF COROLLARY 6.4.2. Let X be a smooth irreducible projective curve of genus at least two. The curve X contains at least one node η by (1.2). On the other hand, if U is an open disk or an open annulus, then it has no nodes. By Lemma 6.2.4, every analytic map $f : U \to X$ avoids η. The result follows from Proposition 6.4.1. □

Proof of Corollary 6.4.3. Let U be any smooth irreducible boundaryless curve. Recall that its set of nodes is discrete. Consider a locally finite open cover $(U_j)_{j\in J}$ of U, where each U_j is either an open disk, an open annulus or a star-shaped domain. In particular, every basic tube U_j contains at most one node.

Let X be a smooth projective curve with at least two nodes, and denote $\mathsf{N}(X) = \{\eta_1, \dots, \eta_a\}$. For every $1 \leq l \leq a$, let B_l^1 and B_l^2 be two distinct connected components of $X \setminus \{\eta_l\}$ that are isomorphic to an open disk. The sets $X_l^1 = X \setminus B_l^1$ and $X_l^2 = X \setminus B_l^2$ are affinoid domains of X by Lemma 1.4.6. The sets $(X_l^i)_{\substack{i=1,2 \\ 1\leq l\leq a}}$ form a finite affinoid cover of X.

Fix some U_j. For every fixed $1 \leq l \leq a$, consider the family of analytic maps $\mathcal{F}_{l,j} = \{f : U \to X \text{ analytic } : \eta_l \notin f(U_j)\}$. By Lemma 6.2.4, we have that

$$\bigcup_{\substack{1\leq l\leq a \\ j\in J}} \mathcal{F}_{l,j} = \mathrm{Mor}_k(U, X) \ .$$

We conclude by applying Proposition 6.4.1 to every family $\mathcal{F}_{l,j}$. □

6.4.2. Analytic maps into curves having only one node.

Proposition 6.4.4. *Let X be a smooth irreducible projective curve over k of genus at least 2 having a unique node η_X. Assume further that the first Betti number of the skeleton of X is at least 1. Let U be a smooth connected boundaryless curve.*

Then there exists a finite affinoid cover $(X_i)_{i\in I}$ of X and a locally finite open cover $(U_j)_{j\in J}$ of U by basic tubes such that for every analytic map $f : U \to X$ and every $j \in J$, the set $f(U_j)$ is contained in some affinoid domain X_i.

Moreover, the affinoid cover (X_i) is independent of U.

Proof. Let $b := b_1(\mathsf{S}^{\mathrm{an}}(X)) > 1$ be the first Betti number of the skeleton of X. As $b \geq 1$ and X has only one node, the skeleton of X consists of b loops $C_1, \dots, C_b$ passing through η_X, and so there are exactly $2b$ non-discal tangent directions at η_X.

The curve X may be decomposed as a disjoint union of $\{\eta_X\}$, open annuli $A_1, \dots, A_b$ and infinitely many open disks by Lemma 1.4.5. Fix some $1 \leq i \leq b$. Pick an isomorphism $\varphi_i : A(R_i, 1) \to A_i$ with $R_i < 1$ and such that $\lim_{r\to 1} \varphi_i(\eta_{0,r}) = \lim_{r\to R_i} \varphi_i(\eta_{0,r}) = \eta_X$. Consider the type II point $x_i := \varphi_i(\eta_{0,\sqrt{R_i}})$, which lies on the loop C_i. Pick any connected component B_i of $X \setminus \{x_i\}$ isomorphic to an open disk. The set $X_i := X \setminus B_i$ is an affinoid domain of X by Lemma 1.4.6 and contains the point η_X. The family $(X_i)_{1\leq i\leq b}$ forms an affinoid cover of X.

Since U is paracompact (cf. Theorem 1.4.1), it suffices to show that for every point $z \in U$ there exists an open neighbourhood V_0 of z

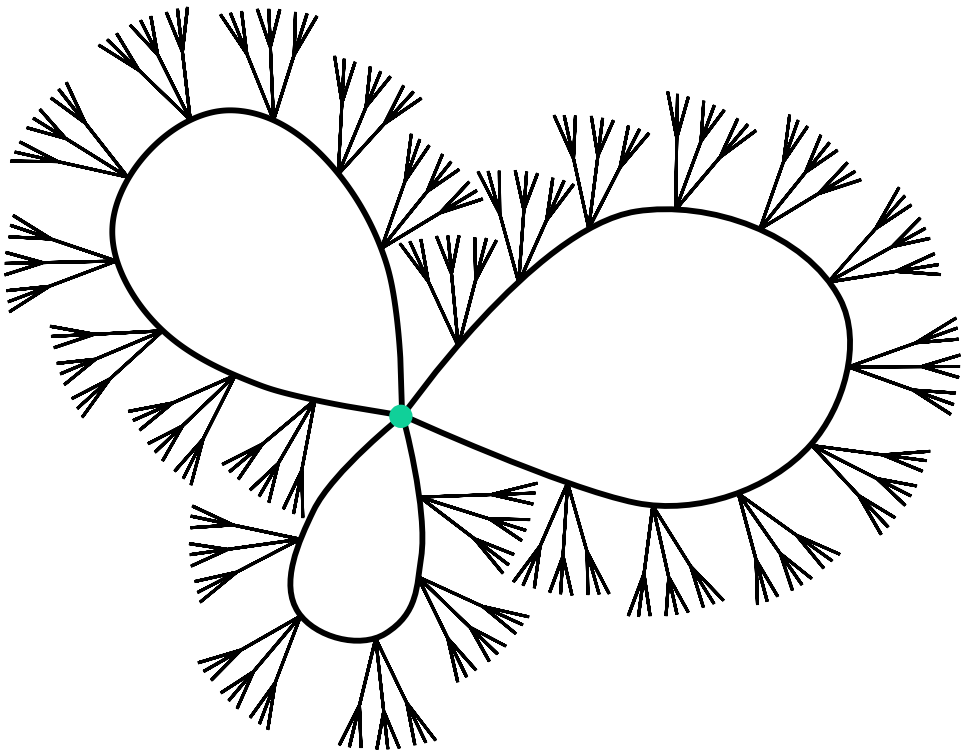


FIGURE 2. Curve with only one node and skeleton with first Betti number 3

such that for every analytic map $f : U \to X$, there exists some affinoid domain X_i in the cover of X such that $f(V_0) \subset X_i$. Moreover, since the cases of disks and annuli have been treated separately in Corollary 6.4.2, obtaining an affinoid cover of X similar to (X_i), we may assume that $z \in \mathsf{N}(U)$. We aim to construct a star-shaped domain $V_0 \subset U$ containing z and such that every analytic map $f : V_0 \to X$ sends V_0 to some affinoid domain X_i.

Let V be the connected component of the complement of $\mathsf{N}(U) \setminus \{z\}$ in U containing z. It is a star-shaped domain in U containing z whose only node η_V is precisely z. Let $\{\vec{v}_1, \ldots, \vec{v}_a\}$ be the set of non-discal directions at η_V. Fix some $1 \leq j \leq a$ and let I_j be the connected component of $\mathsf{S}^{\mathrm{an}}(V) \setminus \{\eta_V\}$ corresponding to the direction $\vec{v}_j$. It is isomorphic to an open segment. The set $U(\vec{v}_j)$ is isomorphic to an open annulus whose skeleton is precisely I_j. We fix an isomorphism $\psi_j : A(\rho_j, 1) \to U(\vec{v}_j)$ with $\rho_j < 1$ and such that ψ_j extends continuously to the Gauss point x_g satisfying $\psi_j(x_g) = \eta_V$.

Pick any analytic map $f : U \to X$. If $f(V)$ avoids the node η_X, then it is contained in some annulus A_i or in some connected component of the complement of η_X that is isomorphic to an open disk, hence in some affinoid domain X_i.

Assume otherwise that f maps the unique node $z = \eta_V \in V$ to η_X.

Consider the star-shaped domain V_0 contained in V obtained by reducing every segment I_j in such a way that the resulting segment I_j^0 is isomorphic to $(\rho_j^0, 1)$ with $\rho_j^0 \geq \max_{1\leq j\leq a} \sqrt{\rho_j}$. We claim that x_i does not belong to $f(I_j^0)$ and that $f(V_0) \subseteq X_i$ for some index i.

Recall that the tangent map $df(\eta_V) : T_{\eta_V} V \to T_{\eta_X} X$ is a rational map on the residue curve at η_V with values in the residue curve at η_X, which is surjective. The preimage of every non-discal direction at η_X consists only of non-discal directions at η_V by Lemma 6.2.1. We may thus choose j such that $df(\eta_V)(\vec{v}_j) \in T_{\eta_X} X$ is non-discal.

The restriction of f to $U(\vec{v}_j)$ takes values in some annulus $A_i \subset X$, and so we may consider the composition $F_{i,j} = \varphi_i^{-1} \circ f \circ \psi_j : A(\rho_j, 1) \to A(R_i, 1)$. Since $f(\eta_V) = \eta_X$, we see that $\lim_{r\to 1} F_{i,j}(\eta_{0,r}) = x_g$. Additionally, $F_{i,j}(\mathsf{S}^{\mathrm{an}}(A(\rho_j, 1)) \subseteq \mathsf{S}^{\mathrm{an}}(A(R_i, 1))$ by Proposition 6.2.3.

The map $F_{i,j}$ can be expanded into a Laurent series $F_{i,j}(z) = \sum_{n\in\mathbb{Z}} a_n z^n$. Consider the real function $\theta_{i,j}(r) := \max_{n\in\mathbb{Z}}\{\log|a_n| + nr\}$, defined on the open real interval $(\log \rho_j, 0)$. Since $F_{i,j}$ is an analytic function on an open annulus without zeroes, there exists an integer $n_0 \in \mathbb{Z}$ such that the function $\theta_{i,j}$ is of the form $\theta_{i,j}(r) = \log|a_{n_0}| + n_0 r$. As $F_{i,j}$ extends continuously to the Gauss point in $A(\rho_j, 1)$ with $\lim_{r\to 1} F_{i,j}(\eta_{0,r}) = x_g$, we see that $|a_{n_0}| = 1$. It follows that $\theta_{i,j}$ extends continuously to the origin $0 \in \mathbb{R}$ with $\theta_{i,j}(0) = 0$.

Observe that $\theta_{i,j}(r) \geq \log R_i$ for every $r \in (\log \rho_j, 0)$ by the definition of $F_{i,j}$. It follows that the graph of $\theta_{i,j}$ lies above the linear function $r \in (\log \rho_j, 0) \mapsto \frac{\log R_i}{\log \rho_j} r$. In particular, we see that $n_0 \leq \frac{\log R_i}{\log \rho_j}$. We conclude that $\theta_{i,j}(r) > \frac{1}{2} \log R_i$ as soon as $r > \log \sqrt{\rho_j}$. Notice that this condition does not depend on R_i.

Pick any $1 > \rho_j^0 > \max_{1\leq j\leq a} \sqrt{\rho_j}$ and reduce the segment I_j into a segment I_j^0 such that the corresponding open subset $U(\vec{v}_j^0) \subset V$ is isomorphic to $\psi_j(A(\rho_j^0, 1))$. The previous calculations show that the image under f of the segment I_j^0 covers at most half the loop C_i at η_X starting with the direction $df(\eta_U)(\vec{v}_j)$, avoiding the point $x_i \in X$. It follows that $f(U(\vec{v}_j^0)) \subseteq X_i$.

We may carry over this procedure to every non-discal direction at η_V, imposing that $\rho_j^0 > \max_{1\leq j\leq a} \sqrt{\rho_j}$ for every $j = 1, \ldots, a$. Let V_0 be the resulting star-shaped domain in U, which contains the point z. We conclude that the restriction of any $f : U \to X$ to V_0 takes values in some affinoid domain X_i. □

6.5. Proof of Theorem I

From now on, we will suppose that the base field k has zero residue characteristic.

6.5.1. De Franchis theorem. At several stages of the proofs of Theorems I and H, we shall need the following (slight) improvement of the original De Franchis theorem that applies to nonproper curves. We refer for example to [**Tsu79**] for a purely algebraic proof in a much more general context in arbitrary dimension.

THEOREM 6.5.1. *Let $\tilde{k}$ be any algebraically closed field of characteristic zero. Let X and Y be two smooth algebraic curves defined over $\tilde{k}$. Suppose that $\chi(X) < 0$, with X not necessarily proper. Then the set of regular maps from Y to X is finite.*

6.5.2. The compact case. Let X be a smooth projective curve of genus at least 2 not having good reduction. Recall that the latter condition means that its skeleton consists of more than one point. Since $g(X) \geq 2$, then we know that $\mathsf{N}(X)$ is nonempty.

Let U be any smooth boundaryless curve. If $\mathsf{N}(X)$ consists of more than one point, then we conclude by Corollary 6.4.3. Otherwise, we are exactly in the situation of Proposition 6.4.4. □

6.5.3. General algebraic case. Let X be a smooth irreducible algebraic curve with negative Euler characteristic whose skeleton is not a singleton, and let $\bar{X}$ be the unique smooth projective curve such that there exists an open embedding $X \to \bar{X}$ with $\bar{X} \setminus X$ a finite set of rigid points. Let U be any smooth connected boundaryless curve.

Our aim is to construct a finite affinoid cover $(\bar{X}_i)$ of $\bar{X}$ and a locally finite cover (U_j) of U such that for every analytic map $f : U \to X$ there exists some $\bar{X}_i$ with $f(U_j) \subseteq \bar{X}_i$.

Recall that the non-proper algebraic curves X with negative Euler characteristic are $\mathbb{P}^{1,\mathrm{an}}$ with at least three rigid points removed, and elliptic curves and curves with genus at least 2 with finitely many rigid points removed.

The case of $\mathbb{P}^{1,\mathrm{an}} \setminus \{0, 1, \infty\}$ is treated in [**FKT12**, Proposition 3.2]. If X is such that $\bar{X}$ does not have good reduction and its genus is greater than 1, then we are reduced to the projective case, which has already been treated.

Therefore, it only remains to address the cases where X is either an elliptic curve or a projective curve with good reduction $\bar{X}$ with a rigid point removed. The curve X has exactly one node η_X, which is a branching point of the skeleton in the case where $\bar{X}$ has bad reduction and a point of positive genus if $\bar{X}$ has good reduction. We shall make no distinction in the genus of $\bar{X}$ when dealing with the good reduction case.

If $\bar{X}$ is an elliptic curve with bad reduction, then the skeleton of X consists of a loop C passing through η_X and the segment joining η_X and the unique point in $\bar{X} \setminus X$ with its endpoint removed. We are in a situation similar to that of Proposition 6.4.4. We obtain the following affinoid cover $(\bar{X})_{i=1,2}$ of X. Let $x \in \mathsf{S}^{\mathrm{an}}(X)$ be any type II point on the loop $C \subset \mathsf{S}^{\mathrm{an}}(X)$ different from η_X and pick any connected component B_1 of $X \setminus \{x\}$ that is isomorphic to an open disk. We set $\bar{X}_1 := \bar{X} \setminus B_1$. Set $\bar{X}_2 := \bar{X} \setminus B_2$, where B_2 is a connected component of $X \setminus \{\eta_X\}$ isomorphic to a disk. We obtain a finite open cover of U by basic tubes U_j having at most one node.

Assume now that $\bar{X}$ is a curve with good reduction and genus at least 1. There is only one non-discal direction $\vec{w}$ at η_X, and the set $U(\vec{w})$ is isomorphic to a punctured disk. Pick any analytic map $f : U \to X$.

Consider the following affinoid cover $(\bar{X}_1, \bar{X}_2)$ of $\bar{X}$. Pick an open subset B_1 of $\bar{X}$ isomorphic to an open disk and containing the unique point in $\bar{X} \setminus X$. Assume further that $B_1 \cap X$ is strictly contained in $U(\vec{w})$ and set $\bar{X}_1 := \bar{X} \setminus B_1$. Let B_2 be a connected component of $X \setminus \{\eta_X\}$ isomorphic to a disk and set $\bar{X}_2 := \bar{X} \setminus B_2$.

If f avoids the point η_X, then $f(U)$ is clearly contained either in $\bar{X}_1$ or in $\bar{X}_2$. We may thus assume that there exists some $\eta_j \in \mathsf{N}(U)$ such that $f(\eta_j) = \eta_X$.

Let $U_j \subset U$ be the connected component of the complement of $\mathsf{N}(U) \setminus \{\eta_j\}$ in U containing η_j. The tangent map $df(\eta_j)$ is surjective, and by Lemma 6.2.1 every preimage of $\vec{w}$ is non-discal. We may thus pick $\vec{v} \in T_{\eta_j} U$ non-discal such that $df(\eta_j)(\vec{v}) = \vec{w}$. Fix isomorphisms $\varphi : \mathbb{D} \setminus \{0\} \to U(\vec{w})$ and $\psi : A(\rho, 1) \to U(\vec{v})$ with $\rho < 1$. Assume further that both extend continuously to the Gauss point, with $\lim_{r \to 1} \varphi(\eta_{0,r}) = \eta_X$ and $\lim_{r \to 1} \psi(\eta_{0,r}) = \eta_X$.

The composition $F = \varphi^{-1} \circ f \circ \psi$ is so an analytic map on an open annulus with values in the punctured disk. Write $F(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ and consider the real function $\theta(r) = \max_{n \in \mathbb{Z}} \{\log |a_n| + nr\}$ on the open interval $(\log \rho, 0)$. Since F has no zeros, there exists an integer n_0 such that $\theta(r) = \log |a_{n_0}| + n_0 r$ for all $r \in (\log \rho, 0)$. Moreover, $|a_{n_0}| = 1$, as $F(x_g) = x_g$.

Consider the tangent map $df(\eta_j) : T_{\eta_j} U_j \to T_{\eta_X} X$. Both $T_{\eta_j} U_j$ and $T_{\eta_X} X$ are isomorphic to smooth projective curves C_{U_j} and C_X respectively over $\tilde{k}$ with a finite number of marked points, corresponding to the non-discal directions at η_j and η_X respectively. In particular, C_X is a curve with one marked point and genus at least 1. The inverse image under $df(\eta_j)$ of the marked point in C_X is contained in the set of marked points in C_{U_j} by Lemma 6.2.1. Applying Theorem 6.5.1 to the curve C_X with the marked point, we obtain that there are only finitely many possibilities for the tangent map $df(\eta_j)$. As a consequence, the degree at every marked point in C_{U_j} of the rational map $df(\eta_j)$ is

bounded. At the marked point corresponding to $\vec{v} \in T_{\eta_j} U_j$, this degree is precisely n_0, the slope of θ, which is thus bounded.

Hence, after maybe reducing the basic tube U_j we see that $f(U_j) \subset \bar{X}_1$. Repeating this procedure at every node of U, we obtain a locally finite open cover (U_j) of U consisting of open disks, open annuli and star-shaped domains satisfying the required property. □

REMARK 6.5.2. *The previous arguments in the case where $\bar{X}$ has good reduction apply verbatim to $\mathbb{P}^{1,\mathrm{an}} \setminus \{0, 1, \infty\}$, since its skeleton is a tripod joining the points $0, 1$ and ∞ and $\mathsf{N}(\mathbb{P}^{1,\mathrm{an}} \setminus \{0, 1, \infty\}) = \{x_g\}$.*

6.6. Proof of Theorem H

Recall that the base field k is assumed to have zero residue characteristic.

6.6.1. Curves having good reduction. Recall that the skeleton of a smooth projective curve X with good reduction consists of a single point η_X whose genus equals that of X.

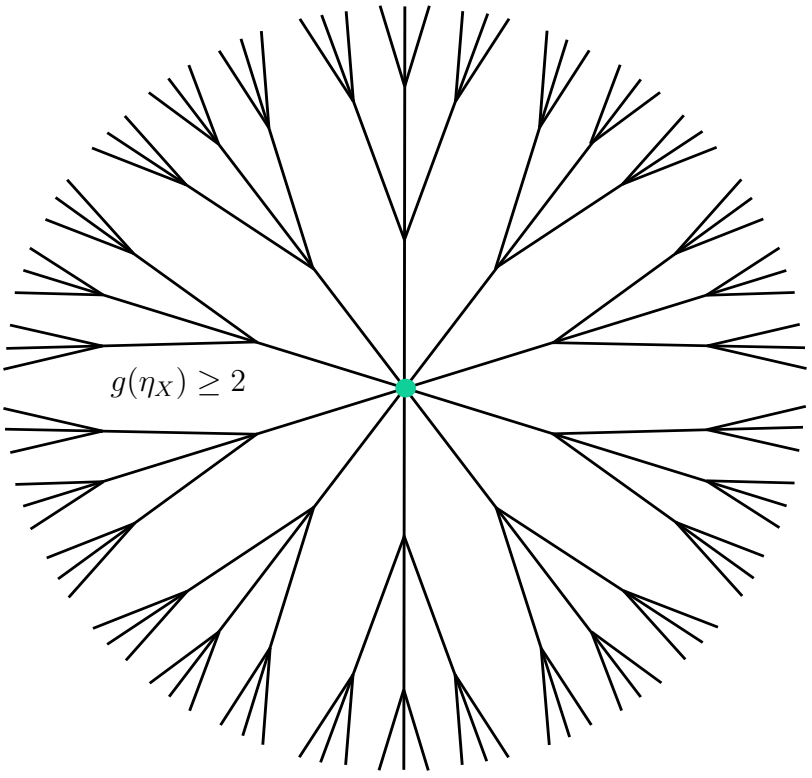


FIGURE 3. Curve with good reduction

Our previous arguments do not apply in the case of smooth projectve curves having good reduction, and we therefore treat this case separately.

PROPOSITION 6.6.1. *Let k be an algebraically closed complete field of zero residue characteristic. Assume that the residue field is countable. Let X be a smooth irreducible projective curve over k with good reduction and of genus at least 2, and let U be a star-shaped domain. Then the family $\mathrm{Mor}_k(U, X)$ is normal.*

We shall need for the proof the following weaker version of [**CTT16**, Lemma 3.6.8]

LEMMA 6.6.2. *Let k be an algebraically closed field of zero residue characteristic. Let U be the complement of finitely many closed disks in D. Let $f : U \to \mathbb{D}$ be an analytic map.Let $I \subset U$ be an interval.*

Then there exists a finite subdivision of I into smaller intervals I_j such that $\mathrm{diam}(f(z)) = a_j \mathrm{diam}(z)^{n_j}$ for every $z \in I_j$.

PROOF OF PROPOSITION 6.6.1. Let $f_n : U \to X$ be a sequence of analytic maps. We reduce to the case where the only node η_U in U is mapped by every f_n to the only node η_X in X by Proposition 6.4.1.

The formula (1.2) assures that $g(\eta_X) = g(X) \geq 2$. For every $n \in \mathbb{N}$, the tangent map $df_n(\eta_U) : T_{\eta_U} U \to T_{\eta_X} X$ is a rational map between the residue curves at η_U and η_X. By Theorem 6.5.1 there are only finitely many such nonconstant maps, as the residue curve at η_X has genus greater than 1. We may thus assume that all the tangent maps $df_n(\eta_U)$ are equal. Let d be the degree of the $df_n(\eta_U)$.

We treat every connected component of $U \setminus \{\eta_U\}$ separately. Pick any tangent direction $\vec{v}$ at η_U. The image of $U(\vec{v})$ under every f_n is contained in some fixed connected component V of $X \setminus \{\eta_X\}$, as all the tangent maps $df(\eta_U)$ agree. Thus, $f_n(U(\vec{v}))$ is contained in some affinoid domain of X for every n. Theorem A implies that there exists a subsequence f_{n_j} converging on $U(\vec{v})$ to some continuous map f_∞. Since $\tilde{k}$ is countable, we may extract diagonally at every tangent direction at η_U and obtain a limit map $f_\infty : U \to X$ that is continuous on $U \setminus \{\eta_U\}$.

Observe that $f_\infty(\eta_U) = \eta_X$. It remains to check that f_∞ is continuous at η_U. In order to do so, it suffices to verify that for every sequence of points $z_m \in U$ converging to η_U we have $f_\infty(z_m) \overset{m\to\infty}{\to} \eta_X$. If the points z_m belong to infinitely many different connected components of $U \setminus \{\eta_U\}$, then their images $f_\infty(z_m)$ belong to infinitely many distinct connected components of $X \setminus \{\eta_X\}$ and we conclude.

We may thus assume that all the points z_m belong to $U(\vec{v})$ for some fixed tangent direction $\vec{v} \in T_{\eta_U} U$. Fix an isomorphism $\psi : Y \to U(\vec{v})$, where Y is an open disk or an open annulus depending on whether $\vec{v}$ is discal or not. Assume that ψ extends continuously to the Gauss point x_g with $\lim_{r\to 1} \varphi(\eta_{0,r}) = \eta_U$. We may assume that for every $m \in \mathbb{N}$ there exists some $0 \leq r < 1$ such that $z_m = \psi(\eta_{0,r})$. Fix an isomorphism $\varphi : \mathbb{D} \to V$ that extends continuously to x_g with $\lim_{r\to 1} \varphi(\eta_{0,r}) = \eta_X$. Checking the continuity of f_∞ at η_U amounts to showing that $\lim_{r\to 1} \varphi^{-1} \circ f_\infty \circ \psi(\eta_{0,r}) = x_g$.

For every fixed $n \in \mathbb{N} \cup \{\infty\}$, set $F_n = \varphi^{-1} \circ f_n \circ \psi : Y \to \mathbb{D}$. The series development $F_n(z) = a_0^{(n)} + \sum_{i \neq 0} a_i^{(n)} z^i$ is such that $|a_i^{(n)}| \leq 1$ for all $i \in \mathbb{Z}$, $|a_d^{(n)}| = 1$ and $|a_i^{(n)}| < 1$ for $i \leq d$, where d denotes the degree of $df_n(\eta_U)$. Consider the segments $I = \{\eta_{0,r} : 0 \leq r < 1\} \subset \mathbb{D}$ and $l_n = I \cap F_n^{-1}(I)$. Notice that $\operatorname{diam}(F_n(\eta_{0,r_n})) = |a_0^{(n)}|$, where $r_n = \inf\{r : \eta_{0,r} \in l_n\}$.

We distinguish two cases. Assume first that there exists a positive real number $R < 1$ such that $|a_0^{(n)}| \leq R$ for infinitely many indices $n \in \mathbb{N}$. After maybe extracting a subsequence, we may reduce Y as to obtain an annulus Y_1 centered at 0 containing the point x_g in its topological boundary and such that every map F_n avoids the disk $\bar{\mathbb{D}}(0; R)$. Moreover, the skeleton $\mathsf{S}^{\mathrm{an}}(Y_1) \subseteq \bigcap_n l_n$ is mapped to the segment I. For sufficiently large r, Lema 6.6.2 implies that $F_n(\eta_{0,r}) = \eta_{0,rd}$, and so we see that $\lim_{r \to 1} F_\infty(\eta_{0,r}) = x_g$.

Suppose next that $|a_0^{(n)}| \to 1$. In this case, there exists an open annulus $A \subseteq U(\vec{v})$ whose topological boundary contains η_U such that the restriction of f_∞ to A is the constant map η_X. Indeed, after extracting a subsequence we have that $F_n(\eta_{0,r_n}) = \eta_{0,r_n d} = \eta_{0,|a_0^{(n)}|}$ by Lemma 6.6.2. As $|a_0(n)| \to 1$, then $r_n \to \frac{1}{d}$. Pick any $r > \frac{1}{d}$. For $n \gg 0$, we may assume that $r > r_n$. Then, $F_n(\eta_{0,r}) = \eta_{0,rd}$, and since $rd > r_n d = |a_0^{(n)}| \to 1$, we see that $F_n(\eta_{0,r}) \overset{n \to \infty}{\to} x_g$. Thus, $F_\infty \equiv x_g$ on the open annulus $A(\frac{1}{d}, 1)$. □

6.6.2. Proof of Theorem H. Assume first that X is a projective curve of genus at least 2 with good reduction. Denote by η_X its only node. Recall that $X \setminus \{\eta_X\}$ is a countable disjoint union of open disks. Since the open disk and the open annulus have been treated separatedly in Corollary 6.4.2, we may assume that U has nodes. The set $\mathsf{N}(U)$ is discrete and consequently we may find a locally finite cover (U_j) of U by basic tubes being either an open disk, an open annulus or a star-shaped domain.

Fix some basic tube U_j. If the set of nodes of U_j is empty, then U_j is either a disk or an annulus, and we may apply Corollary 6.4.2 and Theorem A to extract a subsequence converging pointwise on U_j to some continuous map. Suppose now that U_j contains one node η_j, i.e. it is a star-shaped domain. Then we apply Proposition 6.6.1 to extract a subsequence that is pointwise converging on U_j to some continuous map. This procedure may be repeated for every open set U_j, and extracting diagonally we obtain a subsequence f_{n_j} that converges on U to some continuous map $f_\infty : U \to X$. This concludes the proof in the case of curves having good reduction.

Suppose now that X is a smooth algebraic curve with negative Euler characteristic whose skeleton is not a single point. Let $\bar{X}$ be the smooth

projective curve such that X can be embedded in $\bar{X}$ and $\bar{X} \setminus X$ is a finite set of rigid points.

Let U be any smooth connected boundaryless curve and pick any sequence of analytic maps $f_n : U \to X$. By Theorem I we may find a locally finite open cover U_j of U by basic tubes, and a finite k-affinoid cover $\bar{X}_i$ of $\bar{X}$ such that for every $n \in \mathbb{N}$ and every j one has $f_n(U_j) \subseteq \bar{X}_i$ for some i. By Theorem A and a diagonal extraction argument, we may extract a subsequence converging pointwise to some continuous map $f_\infty : U \to \bar{X}$.

By Proposition 6.2.5, for each index j either we have $f_\infty(U_j) \subset X$ or $f_\infty|_{U_j}$ is constant equal to some point in $\bar{X} \setminus X$. If $f_\infty(U)$ is not included in X, then by continuity and connectedness we conclude that f_∞ is constant equal to some point in $\bar{X} \setminus X$ as required. □

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Titre : Théorème de Montel non-Archimédien

Mots Clefs : Famille normale, géométrie de Berkovich, hyperbolicité, systèmes dynamiques.

Résumé : Cette thèse est dédiée à l'étude des propriétés de compacité de familles d'applications analytiques entre espaces analytiques définis sur un corps métrisé non-Archimédien k. Nous travaillons dans le contexte des espaces analytiques développés par Berkovich pour exploiter leur topologie modérée. Une de nos motivations est le désir d'introduire une notion naturelle d'hyperbolicité au sens de Kobayashi dans ce cadre.

Nous démontrons d'abord un analogue au théorème de Montel pour des applications analytiques à valeurs dans un domaine borné de l'espace affine. Afin de ceci faire, nous paramétrisons l'espace des applications analytiques d'un polydisque ouvert dans un polydique fermé par le spectre analytique d'une k-algèbre de Banach adéquate. Le résultat découle alors de la compacité séquentielle de cet espace.

Nos résultats mènent naturellement à une définition de famille normale, et nous introduisons ensuite deux ensembles de Fatou associés à un endomorphisme de l'espace projectif. Nous montrons que les composantes de Fatou se comportent comme dans le cas complexe et ne contiennent pas d'image non-triviale de la droite affine épointée.

Ensuite, nous appliquons notre notion de normalité à l'étude de l'hyperbolicité dans le cadre non-Archimédien. Nous reprenons les travaux de W. Cherry et démontrons plusieurs caractérisations des variétés projectives lisses pour lesquelles la semi-distance de Cherry-Kobayashi sur l'ensemble des points rigides définit la topologie usuelle. Nous obtenons finalement une caractérisation des courbes algébriques lisses X de caractéristique d'Euler négative en termes de la normalité de certaines familles d'applications analytiques à valeurs dans X.

Title : A non-Archimedean Montel's theorem

Key words : Normal family, Berkovich geometry, hyperbolicity, dynamical systems.

Abstract : This thesis is devoted to the study of compactness properties of spaces of analytic maps between analytic spaces defined over a non-Archimedean metrized field k. We work in the theory of analytic spaces as developed by Berkovich to fully exploit their tame topology. One of our motivations is the desire to introduce a natural notion of Kobayashi hyperbolicity in this setting.

We first prove an analogue of Montel's theorem for analytic maps taking values in a bounded domain of the affine space. In order to do so, we parametrize the space of analytic maps from an open polydisk to a closed one by the analytic spectrum of a suitable Banach k-algebra. Our result then follows from the sequential compactness of this space. Our results naturally lead to a definition of normal families, and we subsequently introduce two notions of Fatou sets attached to an endomorphism of the projective space. We show that Fatou components behave like in the complex case and cannot contain non trivial images of the punctured affine line.

Thereafter, we apply our normality notion to the study of hyperbolicity in the non-Archimedean setting. We pursue the work of W. Cherry and prove various characterizations of smooth projective varieties whose Cherry-Kobayashi semi distance on the set of rigid points defines the classical topology. We finally obtain a characterization of smooth algebraic curves X of negative Euler characteristic in terms of the normality of certain families of analytic maps taking values in X.